1 Galatic setup

1.1 Units

Computer simulations are sensitive to rounding errors due to the lack of infinite precision when representing decimal numbers. Really small numbers as well as really big ones tend to have bigger errors than those close to the unity, as can be seen on Figure 1.

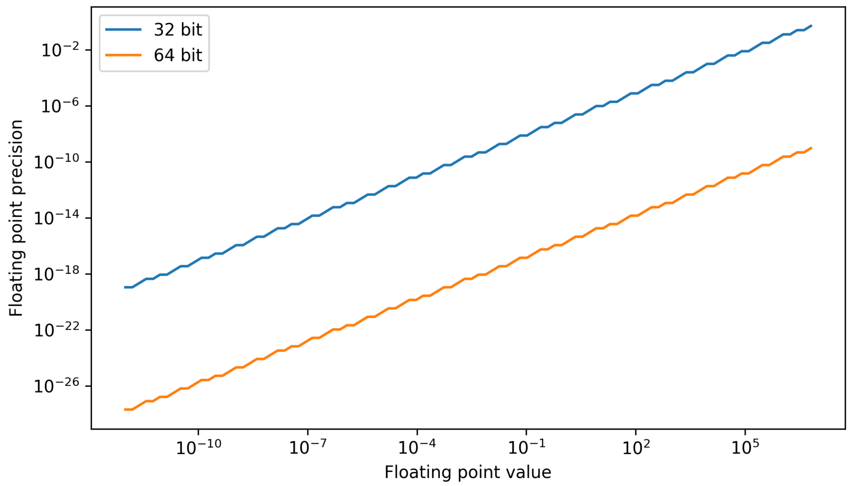


FIGURE 1: Floating point precision for different values, for a 32 bit and 64 bit holders.

Under the International System of Units, distances are measured on meters, times on seconds, and mass on kilograms, nevertheless black holes are too heavy to be measured on kilograms, galaxies sizes too big to be quantified on meters, and time scales too large for a second. Because of that, the following units will be used throughout this document:

TABLE 1: Natural units

Physical property	unit
Length	1 kilo-parsec (kpc)
Mass	10^5 solar masses ($10^5\ M_\odot$)
Time	1 giga-year (Gyr)

Universal gravitational constant

First quantified by Henry Cavendish the gravitational constant has a value of $G_0 = 6.67408 \times 10^{-11}$ on SI units of $\mathrm{m^3 s^{-2} kg^{-1}}$. With the units of length, mass and time on Table 1, the constant of gravity to be used is given by:

$$G = G_0 \left(\frac{1\ \mathrm{kpc}^3}{(3.0857 \times 10^{19})^3\ \mathrm{m}^3} \right) \left(\frac{(3.154 \times 10^{16})^2\ \mathrm{s}^2}{1\ \mathrm{Gyr}^2} \right) \left(\frac{1.98847 \times 10^{35}\ \mathrm{kg}}{10^5 M_\theta} \right) = 0.4493 \frac{\mathrm{kpc}^3}{\mathrm{Gyr}^2 10^5 M_\odot} \quad (1)$$

Hubble constant

The hubble constant is frequently used as $H_0 = 67.66 \pm 0.42\ \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ [1], stating the speed of an astronomical body on kms^{-1} at a distance of 1 Mpc. Nevertheless, the hubble constant has units of 1/time, thus, taking into account the units on Table 1 one gets:

$$H = H_0 \left(\frac{1\ \mathrm{kpc}}{3.0857 \times 10^{16}\ \mathrm{km}} \right) \left(\frac{3.154 \times 10^{16}\ \mathrm{s}}{1\ \mathrm{Gyr}} \right) \left(\frac{1\ \mathrm{Mpc}}{1000\ \mathrm{kpc}} \right) = 6.916 \times 10^{-2}\ \mathrm{Gyr}^{-1} \quad (2)$$

1.2 Mass distributions

The host galaxy has two mass distributions that are superimposed, one for dark matter and the other one for all the luminous or baryonic matter.

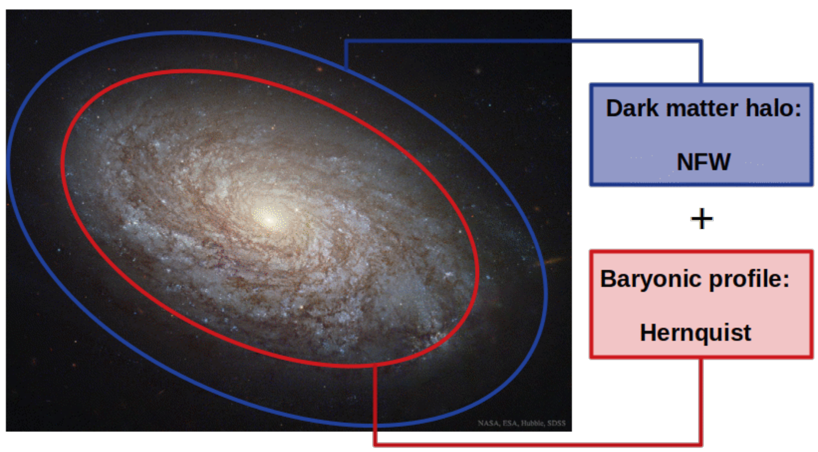


FIGURE 2: NGC4414 galaxy #TODO: cite

The dark matter halo used on the host, follows a NFW (Navarro–Frenk–White) profile, while the baryonic mass distribution of the galaxy depends on the redshift z. For high redshifts a gaseous disk with constant density followed by a density dependance with $r^{-2.2}$ is used, while for $z \approx 0$ a Hernquist model is applied [2].

1.3 Dark matter halo

For a dark matter halo following a NFW profile, the density at some distance r is given by the formula:

$$\rho_{\mathrm{DM}}(r) = \frac{\rho_0^{\mathrm{DM}}}{\frac{r}{R_s}\left(1 + \frac{r}{R_s}\right)^2} \quad (3)$$

Where R_s and ρ_0^{DM} are constants for a given dark matter halo. Using the density, the cumulative mass $M_{\text{DM}}(r)$ within some radius r is given by the integral of the density over a volume, since Equation 3 is spherically symmetrical, the only dependance of the integral is with distance. On Equation 4 the r'^2 comes from the Jacobian of spherical coordinates, and the 4π from the solid angle.

$$M_{DM}(r) = \int_0^r 4\pi r'^2 \rho_{\text{DM}}(r')dr' = 4\pi\rho_0^{\text{DM}} R_s^3 \left[\ln\left(\frac{R_s + r}{R_s}\right) - \frac{r}{R_s + r}\right] \tag{4}$$

Since the mass of dark matter of a single galaxy diverges for $r \to \infty$ there is a radius called R_{vir}, at which the density of the NFW profile is 200 times the critical density ρ_{crit}, the minimum density for an expanding universe `#TODO: cite`.

$$R_{\text{vir}} = 200\rho_{\text{crit}} = 200\left(\frac{3H^2}{8\pi G}\right) = 0.2541 \text{ kpc} \tag{5}$$

Considering a concentration parameter $c(M_h, z)$ of dark matter in the halo, the following relation holds for the viral radius R_{vir} and the scale radius R_s:

$$R_{\text{vir}} = c(M_h, z)R_s \tag{6}$$

Where the concentration parameter, dependence with the dark matter halo mass (M_h) and redshift is given by:

$$c(M_h, z) = c_0(z)\left(\frac{M_h}{10^{13}M_\theta}\right)^{\alpha(z)} \tag{7}$$

where $\alpha(z)$ and $c_0(z)$ were fitted using simulation data to the following functions [2]:

$$c_0(z) = \frac{4.58}{2}\left[\left(\frac{1+z}{2.24}\right)^{0.107} + \left(\frac{1+z}{2.24}\right)^{-1.29}\right] \tag{8}$$

$$\alpha(z) = -0.0965 \exp\left(-\frac{z}{4.06}\right) \tag{9}$$

For a fixed halo mass, as time passes (smaller redshift), concentration of dark matter will increase, as can be shown on Figure 3. Because of Equation 6 the value of R_{vir} will increase as well, this equation can be used to obtain the value of ρ_0^{DM} by evaluating Equation 4 at R_{vir}.

$$M_h \equiv M_{\text{DM}}(R_{\text{vir}}) = 4\pi\rho_0^{\text{DM}} R_s^3 \left[\ln\left(\frac{R_s + c(M_h, z)R_s}{R_s}\right) - \frac{c(M_h, z)R_s}{R_s + c(M_h, z)R_s}\right] \tag{10}$$

$$\rho_0^{\text{DM}} = \frac{M_h}{4\pi R_s^3 \left[\ln\left(1 + c(M_h, z)\right) - \frac{c(M_h, z)}{1 + c(M_h, z)}\right]} \tag{11}$$

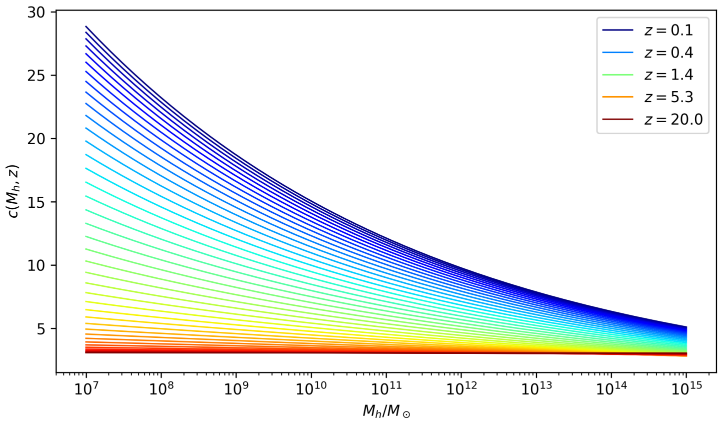


FIGURE 3: Dark matter concentration parameter as a function of the halo mass and the redshift.

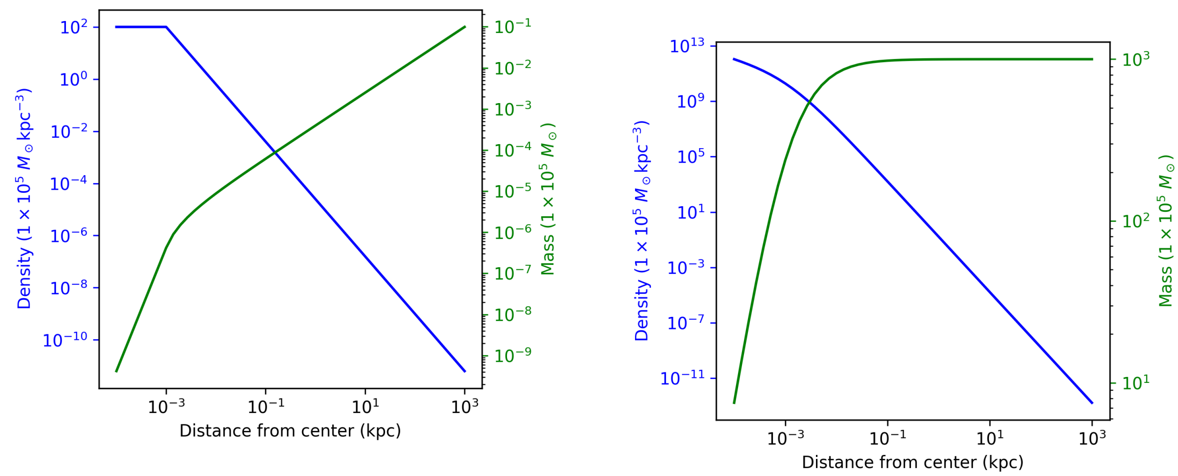
By considering a fixed concentration parameter $c(M_h, z) \equiv 4$, one obtains $R_s = 0.0635$ kpc, and $\rho_0^{\mathrm{DM}} = 383457\ 10^5 M_\odot \mathrm{kpc}^{-3}$.

Baryonic profile

For high redshift the baryonic profile resembles that of a gaseous galaxy with a constant density for distances lower than $r_0 = 1 \times 10^{-3}$ kpc, followed by an $r^{-2.2}$ profile for greater distances as in Figure 4a.

$$\rho_B(r) = \begin{cases} \rho_0^B & \text{if } r < r_0 \\ \rho_0^B \left(\dfrac{r_0}{r}\right)^{2.2} & \text{if } r \geq r_0 \end{cases} \tag{12}$$

$$M_B(r) = \begin{cases} \dfrac{4}{3}\pi \rho_0^B r^3 & \text{if } r < r_0 \\ 4\pi \rho_0^B r_0^{2.2} \left(r^{0.8} - r_0^{0.8}\right) + \dfrac{4}{3}\pi \rho_0^B r_0^3 & \text{if } r \geq r_0 \end{cases} \tag{13}$$



(A) Constant density profile for $r < 1$ pc, followed by a fall as $r^{-2.2}$ for $z \gg 0$.

(B) Hernquist profile for mass distribution at $z \approx 0$.

FIGURE 4: Baryonic mass distributions for $z \gg 0$ and $z \approx 0$.

At low redshifts, stars have already been formed, and the baryonic profile is modeled as a Hernquist profile with half-mass radius $R_{1/2} = 0.01 R_{\mathrm{vir}}$, as in Figure 4b. The half-mass radius, as the name implies, is the distance at which the cumulative mass is half the total mass [3].

$$\rho_B(r) = \frac{M_T^B \mathcal{R}_s}{2\pi r (r + \mathcal{R}_s)^3} \tag{14}$$

$$M_B(r) = \frac{M_T^B r^2}{(r + \mathcal{R}_s)^2} \tag{15}$$

$$R_{1/2} = \left(1 + \sqrt{2}\right) \mathcal{R}_s \tag{16}$$

1.4 Dynamical friction

As the black hole travels through the galaxy, dark matter, stars and gaseous materials from the medium interact with the black hole adding a drag force due to friction. Drag force is different in nature depending on it source, collisionless components, such as dark matter and stars, apply a drag force to the black hole that follows the standard Chandrasekhar formula [2].

$$a_{\mathrm{DF}}^{\mathrm{DM}}(r, v) = -4\pi G^2 M_\bullet \rho(r) \times \ln \Lambda \left(\operatorname{erf}(X) - \frac{2}{\sqrt{\pi}} X e^{-X^2} \right) \frac{\vec{v}}{v^3} \tag{17}$$

$$X \equiv \frac{|v|}{\sqrt{2}\sigma_{\mathrm{DM}}} \qquad \text{with } \sigma_{\mathrm{DM}}(r) = \sqrt{\frac{G M_{\mathrm{DM}}}{2 R_{\mathrm{vir}}}} \tag{18}$$

Gas on the other hand is collisional, special care must be taken since gas can cool behind a passing object, such as a black hole [2]. A hybrid model for the drag force was proposed by Choksi, in which both subsonic and supersonic velocities are possible. To do so, a mach number was defined as:

$$\mathcal{M} \equiv \frac{v}{c_s} \tag{19}$$

where v is the speed of the black hole, and c_s is the local sound speed, which depends on local temperature, nevertheless it can be approximated to an isothermal halo as:

$$c_s \approx 1.8(1+z)^{1/2} \left(\frac{M_h}{10^7 M_\odot} \right)^{1/3} \left(\frac{\Omega_M h^2}{0.14} \right) \text{ kms}^{-1} \tag{20}$$

where Ω_M and h depend on the cosmology chosen. By knowing $\mathcal{M}$, the acceleration caused by gas can be written as:

$$a_{\text{DF}}^{\text{gas}}(r, \vec{v}) = -4\pi G^2 M_\bullet \rho_{\text{gas}}(r) \times f(\mathcal{M}) \frac{\vec{v}}{v^3} \tag{21}$$

with

$$f(\mathcal{M}) = \begin{cases} 0.5 \ln \Lambda \left[\text{erf}\left(\frac{\mathcal{M}}{\sqrt{2}} \right) - \sqrt{\frac{2}{\pi}} \mathcal{M} e^{-\mathcal{M}^2/2} \right] & \text{if } \mathcal{M} \leq 0.8 \\ 1.5 \ln \Lambda \left[\text{erf}\left(\frac{\mathcal{M}}{\sqrt{2}} \right) - \sqrt{\frac{2}{\pi}} \mathcal{M} e^{-\mathcal{M}^2/2} \right] & \text{if } 0.8 < \mathcal{M} \leq 1.7 \\ 0.5 \ln \left(1 - \mathcal{M}^{-2} \right) + \ln \Lambda & \text{if } \mathcal{M} > 1.7 \end{cases} \tag{22}$$

finally, the gas density is taken as a power-law profile, $\rho_{\text{gas}} = \rho_0^{\text{gas}} r^{-n}$, were $1 \leq n \leq 3$ [2].

TABLE 2: Constants used in the simulation

Constant	**Value (unit)**
n	2
h	0.678
Ω_M	0.309
$\ln \Lambda$	2.3
M_h	1×10^3 $(10^5 M_\odot)$
$M_\bullet$	1 $(10^5 M_\odot)$
M_T^B	158 $(10^5 M_\odot)$
R_s	1.3 (kpc)
$\mathcal{R}_s$	0.14 (kpc)
ρ_0^{DM}	1700 $(10^5 M_\odot \text{kpc}^{-3})$
ρ_0^{gas}	0.01 $(10^5 M_\odot \text{kpc}^{-3})$
f_b	baryon fraction $\Omega_b / \Omega_M = f_{\text{gas}} + f_{\text{gal}}$