Lecture 3

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, \ .)$.

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, .)$.

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, \ .)$.

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, \ .)$.

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, \ .)$.

- Construction of the universal cover $\widetilde{X}_x$: The points of : Homotopy classes of paths starting from x.
  The projection π : For each point $\widetilde{y} \in \widetilde{X}_x$ pick a path $f : [0,1] \to X$ with $f(0) = x$ representing $\widetilde{X}_y$ and put $\pi(\widetilde{X}_y) = f(1) = y$. (well-defined ?)
  The topology on $\widetilde{X}_x$: Take the following set $\widetilde{U}$ as a basis of open neighbourhoods of a point $\widetilde{y}$: start from a simply connected nbhd U of $\pi(\widetilde{X}_x) = y$ and if $f : [0,1] \to X$ is a path representing $\widetilde{y}$, define $\widetilde{U}_{\widetilde{y}}$ to be the set of homotopy classes of paths obtained by composing the homotopy class of f with the homotopy class of some path $g : [0,1] \to X$ with $g(0) = y$ and $g([0,1]) \subseteq U$.
- Proof of the theorem 2.3.5 : For a connected and locally simply connected topological space X and a base point $x \in X$ the functor Fib_x is representable by a cover $\widetilde{X}_x \to X$, i.e. $\mathrm{Fib}_x(.) \cong \mathrm{Hom}(\widetilde{X}_x, \ .)$.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- **Lemma 2.4.2** : The space $\widetilde{X}_x$ is connected.
- **Proposition 2.4.3** : The cover $\pi : \widetilde{X}_x \to X$ is Galois.
  For proof, we need two lemmas :
- **Lemma 2.4.4** : A cover of a simply connected and locally path-connected space is trivial.
  Proof :...
- **Corollary 2.4.5** : Let X be a locally simply connected space. Given two covers $p : Y \to X$ and $q : Z \to Y$, their composite $q \circ p : Z \to X$ is again a cover of X.
- *Proof of Theorem 2.3.4* : Let X be connected and locally simply connected top. space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of covers of X with the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Gaois covers to coset spaces of normal subgroups.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X, x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X, y) \xrightarrow{\sim} \pi_1(X, x)$. Thus we get the dependence of $\pi_1(X, .)$ on the base point.
  λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X, x)$.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X,x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X,y) \xrightarrow{\sim} \pi_1(X,x)$. Thus we get the dependence of $\pi_1(X,.)$ on the base point.
  λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X,x)$.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X,x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X,y) \xrightarrow{\sim} \pi_1(X,x)$. Thus we get the dependence of $\pi_1(X,.)$ on the base point.
  λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X,x)$.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X,x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X,y) \xrightarrow{\sim} \pi_1(X,x)$. Thus we get the dependence of $\pi_1(X,.)$ on the base point.
  λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X,x)$.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X,x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X,y) \xrightarrow{\sim} \pi_1(X,x)$. Thus we get the dependence of $\pi_1(X,.)$ on the base point. λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X,x)$.

- *Proof of Proposition 2.4.6 :* There is a natural isomorphism $\mathrm{Aut}(\widetilde{X}_x|X)^{op} \cong \pi_1(X,x)$.
- **Dependence of the fundamental group on the choice of the base point :** Assume X is a path-connected and locally simply connected space. Pick two base points $x, y \in X$.
- **Proposition 2.4.7** : There is a bijection between homotopy classes of path joining x to y and isomorphisms $\widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ in the category of covers of X.
- A cover isomorphism $\lambda : \widetilde{X}_y \xrightarrow{\sim} \widetilde{X}_x$ induces a group isomorphism $\mathrm{Aut}(\widetilde{X}_y|X) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_x|X)$ by the map $\phi \mapsto \lambda \circ \phi\lambda^{-1}$. Via Prop. 2.4.6, it corresponds to an ismorphism $\lambda^{op} : \pi_1(X,y) \xrightarrow{\sim} \pi_1(X,x)$. Thus we get the dependence of $\pi_1(X,.)$ on the base point.
  λ^{op} is uniquely determined up to an inner automorphism of $\pi_1(X,x)$.

- Any cover isomorphic to some $\pi_1(X, x)$ is called a *universal cover* of X.
- **Proposition 2.4.9** Let X be a path-connected and locally simply connected space. A cover $\widetilde{X} \to X$ is universal iff it is simply connected.

- Any cover isomorphic to some $\pi_1(X, x)$ is called a *universal cover* of X.
- **Proposition 2.4.9** Let X be a path-connected and locally simply connected space. A cover $\widetilde{X} \to X$ is universal iff it is simply connected.