B_{dR} and deRham Galois reps

§1. Motivation

p - fixed prime
$K/\mathbb{Q}_p$ finite extension

$G_K = \mathrm{Gal}(\bar{K}/K)$

(all that follows is valid when K is a p-adic field, i.e. a field of char 0 complete w.r.t. a fixed discrete valuation that has a perfect residue field of char $p>0$)

Historically, the first class of "good" p-adic representations of G_K were those of Hodge-Tate type; they were discovered by Serre and Tate in their study of p-adic reps arising from abelian varieties with good reduction over p-adic fields. They were concerned with finding a p-adic analogue of the classical Hodge decomposition

$$\mathbb{C} \otimes_{\mathbb{Q}} H^n_{top}(X(\mathbb{C}), \mathbb{Q}) \simeq \bigoplus_{k+l=n} H^k(X, \Omega^l_X)$$

for smooth proper $X/\mathbb{C}$.

Let X/K be a smooth proper scheme.

Tate discovered that, in the special case of abelian varieties with good reduction, even though $H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ are mysterious, they become much simpler after we apply a drastic operation

$$V \rightsquigarrow \mathbb{C}_p \otimes_{\mathbb{Q}_p} V \circlearrowleft G_K$$

$$g(c \otimes v) = g(c) \otimes g(v).$$

$\mathrm{Rep}_{\mathbb{C}_p}(G_K)$ - category of $\mathbb{C}_p$-representations of G_K
abelian category with evident notions of tensor product, direct sum and exact sequence

Thm (Faltings)

X/K smooth proper scheme

There exists a canonical isomorphism

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p) \simeq \bigoplus_q \left(\mathbb{C}_p(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K}) \right)$$

in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$ where the G_K-action on RHS through each $\mathbb{C}_p(-q)$. In particular, non-canonically

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p) \simeq \bigoplus_q \mathbb{C}_p(-q)^{h^{n-q,q}}$$

in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$ with $h^{k,l} = \dim_K H^k(X, \Omega^l_{X/K})$.

This is a basic example of comparison isomorphism relating one p-adic cohomology to another.

Remarkable theorem for 2 reasons:

1° $\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ as $\mathbb{C}_p$-rep space of G_K is a direct sum of extremely simple pieces

2° We are able to recover $H^{n-q}(X, \Omega^q_{X/K})$ from LHS by operations that make sense on all objects in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$.

The real importance of this thm is revealed when we consider an arbitrary $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ admitting isomorphism as in Faltings' thm

$$(*) \quad W \simeq \bigoplus_q \mathbb{C}_p(-q)^{h_q}.$$

Such a direct sum decomposition is non-canonical (in the sense that the individual lines $\mathbb{C}_p(-q)$ appearing in the direct sum decomposition are generally not uniquely determined within W when $h_q > 1$). (However, for such W there is a canonical decomposition ....)

$(*) \iff W$ is a Hodge-Tate rep

$V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is HT if $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ is HT.

As suggested, the HT property interacts nicely w.r.t. taking duals and direct sums, and to express how the HT property interacts with tensorial and other operations, it is useful to introduce some terminology.

Def: A $\mathbb{Z}$-graded vector space over K is an K-vector space D equipped with direct sum decomposition $\bigoplus_q D_q$ for K-subspaces $D_q \subseteq D$ $\quad (\mathrm{gr}^q(D) := D_q)$?
Morphisms $T : D \to D'$ between graded K-vector spaces are K-linear maps that respect the grading.

$\mathrm{Gr}_K \quad \mathrm{Gr}_{K,f} \leftarrow$ full sub-category of finite dimensional objects

abelian category
evident notions of kernel, cokernel, exact seq.

Def: The HT ring of K is the $\mathbb{C}_p$-algebra

$$B_{HT} := \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_p(q)$$

in which multiplication is defined via the natural maps

$$\mathbb{C}_p(q) \otimes_{\mathbb{C}_p} \mathbb{C}_p(q') \simeq \mathbb{C}_p(q+q').$$

B_{HT} is graded $\mathbb{C}_p$-algebra in the sense that its graded pieces are $\mathbb{C}_p$-subspaces w.r.t which multiplication is additive in degrees and the G_K-action respects the gradings and the ring structure (and is semilinear over $\mathbb{C}_p$).

Concretely:

$$B_{HT} \xrightarrow[\sim]{\text{choice of a basis of } \mathbb{Z}_p(1)} \mathbb{C}_p[t, t^{-1}]$$

with the evident grading (by monomials in t) and G_K-action via

$$g(t^i) = \chi_{cyc}(g)^i \, t^i \qquad i \in \mathbb{Z},\ g \in G_K.$$

Def: The covariant functor

$$\underline{D}_K : \operatorname{Rep}_{\mathbb{Q}_p}(G_K) \to Gr_K$$

$$\underline{D}_K(W) = \bigoplus_q \left(\mathbb{C}_p(q) \otimes_{\mathbb{Q}_p} W \right)^{G_K}$$

$$= \left(B_{HT} \otimes_{\mathbb{Q}_p} W \right)^{G_K}$$

Properties:

- left exact
- exact on short exact seq. of HT objects
- insensitive to replacing K with a finite extn or restricting to inertia subgp (i.e. replacing K with $\hat{K}^{un}$). That is, the natural map

$W \in Rep_{\mathbb{Q}_p}(G_K)$: $K' \otimes_K \underline{D}_K(W) \to \underline{D}_{K'}(W)$ in $Gr_{K'}$ is isomorphism.

The latter is very desirable, but the insensitivity of HT property to finite (possibly ramified) extensions is a bad feature, indicating that HT property is not sufficiently fine (e.g. to distinguish between good reduction and potentially good reduction for elliptic curves).

The multiplicative structure on B_{HT} defines a ~~natural~~ B_{HT}-linear composite morphism

$$\gamma^{B_{HT}}_W : B_{HT} \otimes_K \underline{D}_K(W) \hookrightarrow B_{HT} \otimes_K (B_{HT} \otimes_{\mathbb{Q}_p} W) \to B_{HT} \otimes_{\mathbb{Q}_p} W$$

that respect Galois action.

Let $Rep_{HT}(G_K) \subseteq Rep_{\mathbb{Q}_p}(G_K)$ be the full subcategory of objects that are HT.

It is stable under tensor product, duality, subreps and quotients (and the formation of D_{HT} naturally commutes with finite extensions of K as well as with scalar extensions to $\hat{K}^{un}$.) ← later

$$D_{HT} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Gr}_{K,f}$$

$$D_{HT}(V) = \left(B_{HT} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$$

The comparison morphism

reasonable regularity conditions/property of B_{HT} — injective, $\dim_K D_{HT}(V) \leq \dim_{\mathbb{Q}_p}(V)$

$$\gamma_V : B_{HT} \otimes_K D_{HT}(V) \to B_{HT} \otimes_{\mathbb{Q}_p} V$$

for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is an isomorphism precisely when V is Hodge-Tate and $D_{HT} : \mathrm{Rep}_{HT}(G_K) \to \mathrm{Gr}_{K,f}$ is faithfull, but NOT fully faithfull functor

(terminology: B_{HT} - admissible)

Example:

Faltings thm can be rewritten as:

X/K smooth proper

$V := H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ is in $\mathrm{Rep}_{HT}(G_K)$ with

$$D_{HT}(V) \simeq H^n_{Hodge}(X/K) := \bigoplus_q H^{n-q}(X, \Omega^q_{X/K})$$

and the comparison morphism takes the form of B_{HT} - linear G_K-equivariant isomorphism

$$B_{HT} \otimes_K H^n_{Hodge}(X/K) \simeq B_{HT} \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p).$$

Goal: To improve D_{HT} to get a fully faithfull functor from a nice category of p-adic reps of G_K into a category of semilinear algebra objects by

1° refining B_{HT} to a ring with more structure (going beyond mere grading with a compatible Galois action)

2° introduce a target semilinear algebra category that is richer than $G_{K,f}$.

In the view of the comparison morphism in Faltings then the grading on B_{HT} is closely related to the grading on the Hodge cohomology, so to motivate how we should refine B_{HT}, we can get a clue from the refinement of $H^n_{Hodge}(X)$ by the algebraic deRham cohomology.

$$H^n_{Hodge}(X) = \bigoplus_{k+l=n} H^k(X, \Omega^l_{X/K})$$

$H^n(X) := H^n_{dR}(X/K)$ are finite dim. K vector spaces endowed with a natural decreasing Hodge filtration

$$H^n(X) = Fil^0(H^n(X)) \supseteq Fil^1(\cdot) \supseteq \dots \supseteq Fil^{n+1}(\cdot) = 0$$

and

$$Fil^q(H^n(X)) / Fil^{q+1}(H^n(X)) = H^{n-q}(X, \Omega^q_{X/K}).$$

as $\mathrm{char}(K) = 0$.

Def: A filtered module over a comm. ring R is an R-module endowed with a decreasing filtration $\{Fil^i(M)\}_{i \in \mathbb{Z}}$.

- if $\bigcup \mathrm{Fil}^i(M) = M$ ⇝ exhaustive
- If $\bigcap \mathrm{Fil}^i(M) = 0$ ⇝ separated
- the associated graded module $gr^\bullet(M) = \bigoplus_i \left(\mathrm{Fil}^i(M) / \mathrm{Fil}^{i+1}(M) \right)$

- filtered ring R — exhaustive and sep. $\{R^i\}$ additive subgps s.t. $1 \in R^0$, $R^i . R^j \subseteq R^{i+j}$

$$gr^\bullet(R) = \bigoplus_i R^i / R^{i+1}$$

<u>Example</u>: R dvr with max. id $\mathfrak{m}$, residue field k

- filtered K-algebra A — $A = \mathrm{Frac}(R)$, $\{\mathfrak{m}^i\}$, t k-basis of $\mathfrak{m}/\mathfrak{m}^2$

$$gr^\bullet(A) \simeq k[t, t^{-1}]$$

<u>filtered ring structure + filtered pieces are K-subspaces</u>

<u>Idea</u>: To replace graded $\mathbb{C}_p$-algebra B_{HT} with a filtered K-algebra B_{dR} endowed with a G_K-action respecting filtration such that:

(1) B_{dR} is <u>$(\mathbb{Q}_p, G_K)$-regular</u> with $B_{dR}^{G_K} = K$.

$\mathrm{Frac}(B_{dR})^{G_K} = B_{dR}^{G_K}$
and every non zero $b \in B$ whose $\mathbb{Q}_p$ linear span is G_K-stable is a unit in B_{dR}

(2) $\mathrm{Fil}^0(B_{dR}) / \mathrm{Fil}^1(B_{dR}) \simeq \mathbb{C}_p$ as rings with G_K-action

(3) There is a canonical G_K-inv isomorphism

$gr^\bullet(B_{dR}) \simeq B_{HT}$ as graded $\mathbb{C}_p$-algebras

Given such B_{dR}

consider the assoc. functor on $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Fil}_F$

$$D_{dR}(V) = \left(B_{dR} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$$
$$\alpha_V^{B_{dR}} : B_{dR} \otimes_K D_{dR}(V) \to B_{dR} \otimes_{\mathbb{Q}_p} V$$

This has a functorial filtration via

$$\mathrm{Fil}^i(D_{dR}(V)) = \left(\mathrm{Fil}^i(B_{dR}) \otimes_{\mathbb{Q}_p} V\right)^{G_K}$$

and it is exhaustive and separated.

By left exactness of $(\cdot)^{G_K}$ there is injective map

$$\mathrm{gr}^\bullet(D_{dR}(V)) \hookrightarrow \left(\mathrm{gr}^\bullet(B_{dR}) \otimes_{\mathbb{Q}_p} V\right)^{G_K} = \left(B_{HT} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$$
$$= D_{HT}(V)$$

of graded K-vector spaces, so if V is B_{dR}-admissible
$(\alpha_V^{B_{dR}} : B_{dR} \otimes_K D_{dR}(V) \to B_{dR} \otimes_{\mathbb{Q}_p} V)$

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{dR}(V) = \dim_K \mathrm{gr}^\bullet(D_{dR}(V)) \le \dim_K D_{HT}(V) \le \dim_{\mathbb{Q}_p} V$$

so V is necessarily HT.

· Further aims:

(4) B_{dR} should lead to a refinement of Faltings comparison thm between p-adic étale and Hodge cohomology, by using de Rham cohomology instead:

for smooth proper X/K

$H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ should be B_{dR}-admissible with a natural isomorphism

$$D_{dR}(H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)) \simeq H^n_{dR}(X/K)$$

$B_{dR} \otimes_K H^n_{dR}(X/K) \simeq B_{dR} \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$

whose induced isomorphism between graded K-vector spaces is Faltings comparison isomorphism between p-adic étale and Hodge cohomologies.

Conj: mention and discuss. Fontaine-Mazur Conjecture

(5) Inspired by the description $B_{HT} \simeq \mathbb{C}_p[t, t^{-1}]$, we are led to seek a complete DVR B^+_{dR} over K (with maximal ideal denoted $\mathfrak{m}$) endowed with a G_K-action such that the residue field is naturally G_K-equivariantly isomorphic to $\mathbb{C}_p$ and the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ isomorphic to $\mathbb{C}_p(1)$:

B^+_{dR} complete DVR/K

$B^+_{dR}/\mathfrak{m} \simeq \mathbb{C}_p$

$\mathfrak{m}/\mathfrak{m}^2 \simeq \mathbb{C}_p(1)$ in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$

Since $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq (\mathfrak{m}/\mathfrak{m}^2)^{\otimes i}$ in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$

If $B_{dR} = \mathrm{Frac}(B^+_{dR})$ we would have

$\mathrm{gr}^\bullet(B_{dR}) \simeq B_{HT}$ as graded $\mathbb{C}_p$-alg with G_K-action.

A naive guess that will not work is to take

$$B_{dR}^{+} = \prod_{q \geq 0} \mathbb{C}_p(q) \simeq \mathbb{C}_p[[t]]$$

$$g\left(\sum a_n t^n\right) = \sum g(a_n) \chi_{cyc}(g)^n t^n$$

but this does not lead to a new concept since the product decomposition canonically defines a splitting of the filtration on $\mathfrak{m}^i/\mathfrak{m}^j$.

B_{dR}^{+} will be isomorphic to $\mathbb{C}_p[[t]]$ but only as abstract rings and there is no such isomorphism compatible with the Galois action!

Rough Idea of the Construction:

to imitate the procedure of constructing Witt vectors:

k - ~~perfect field of char p~~ $\rightsquigarrow W(k)$
a complete DVR with unif. p and residue field k

$\mathbb{C}_p = \mathcal{O}_{\mathbb{C}_p}[\frac{1}{p}]$

Apply Witt style construction to $\mathcal{O}_{\mathbb{C}_p}$

make a certain height 1 valuation ring of equicharacteristic p whose $\mathrm{Frac}(R)$ is alg. closed (hence perfect) such that

natural $R \circlearrowleft G_K$ action

natural surj. $\theta : W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}$ G_K-equiv.

$\theta_{\mathbb{Q}}\ \underbrace{W(R)[\tfrac{1}{p}]} \to \mathbb{C}_p$

if $\ker\theta_{\mathbb{Q}}$ is principal replace with its $\ker\theta_{\mathbb{Q}}$-adic completion to get a complete DVR B_{dR}^{+} having residue field $\mathbb{C}_p$.