AN OVERVIEW OF SERRE'S p-ADIC MODULAR FORMS

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The arithmetic theory of modular forms has two main themes that are intertwined. One is the theory of congruences of modular forms and the other is the theory of Galois representations. The first theme is classical, dating back to Ramanujan, with subsequent important contributions from Serre, Swinnerton-Dyer, Atkin, Ribet, Hida etc, and the developments in the latter theme started with the work of Deligne, Eichler, Shimura, Serre and others. It opened up new frontiers in the last few decades of the twentieth century with work of Hida, Mazur, Taylor, Wiles, etc. This expository article is based on a series of lectures given at IISER Pune in June 2014, in the 'Workshop on p-adic aspects of modular forms'. The subject of the lectures was the classical approach of Serre [Se] in defining p-adic families of modular forms. Of course, the geometric approach as developed by Katz, Dwork, Coleman, Mazur, culminating in the theory of overconvergent modular forms is central to the study of p-adic modular forms today, but we shall not discuss this and refer the interested reader to [K1], [K-M].

1. Notation and Preliminaries

Throughout, p will denote a prime number ≥ 5. The field $\bar{\mathbb{Q}}_p$ denotes a fixed algebraic closure of $\mathbb{Q}_p$ and $\bar{\mathbb{Z}}_p$ will be the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p$. Fix an embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$; this takes $\bar{\mathbb{Z}}$ into $\bar{\mathbb{Z}}_p$. The p-adic norm on the field $\mathbb{Q}_p$ is denoted by $|\ |_p$, normalized so that $|\ p\ |_p= 1/p$, and ord_p will denote the corresponding discrete valuation. The field $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$.

1.1. **Modular forms.** We shall refer to [Mi] or [Ki] for more details on the theory of modular forms. Let N be an integer ≥ 1. We shall consider the following three modular subgroups of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathrm{SL}_2(\mathbb{Z}) \supset \Gamma_0(N) \supset \Gamma_1(N) \supset \Gamma(N)$$

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$$
\begin{aligned}
\Gamma_0(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \bmod N\right\} \\
\Gamma_1(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \bmod N,\, a, d \equiv 1 \text{ mod } N\right\} \\
\Gamma(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \text{ and } b \equiv \ 0 \text{ mod } N,\ a \text{ and } d \equiv \ 1 \bmod N\right\}.
\end{aligned}
$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some N is called a *congruence subgroup*; its *level* is the least number N with this property.

Let $\mathrm{GL}_2^+(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$ be the subgroup consisting of matrices with positive determinant and let $\mathbb{H}$ be the upper half plane, on which there is the usual action of $\mathrm{GL}_2^+(\mathbb{R})$ via fractional transformation. Given an integer k, the weight k-action of $\mathrm{GL}_2^+(\mathbb{R})$ on the complex vector space of complex valued functions on $\mathbb{H}$, denoted $f_{|_k}(\gamma)(z)$ is defined as follows. Let $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, then

$$
f_{|_k}(\gamma)(z) := \det(\gamma)^{k/2}(cz+d)^{-k} f(\gamma.z),\ z \in \mathbb{H}. \tag{1}
$$

Definition 1.2. A weakly modular function of weight k for a congruence subgroup Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfying $f_{|_k}(\gamma) = f$ for all $\gamma \in \Gamma$.

Consider the extended upper half plane

$$
\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}. \tag{2}
$$

The action of of $\mathrm{SL}_2(\mathbb{Z})$ then extends to an action on $\mathbb{H}^*$ and the *finite* set of *cusps* for the congruence subgroup Γ is the set of Γ-orbits of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

Definition 1.3. A modular form (resp. cusp form) of integer weight k and level N is a weakly modular function $f : \mathbb{H} \to \mathbb{C}$ for the congruence subgroup $\Gamma_1(N)$, that is holomorphic on $\mathbb{H}$ and for which

$$
\lim_{y \to \infty} f_{|_k}(\gamma)(iy) \tag{3}
$$

is finite (resp. vanishes) for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Given a modular form of weight k and level N, the invariance property of f for $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives $f(z+1) = f(z)$. This allows for a Fourier expansion (called the q-expansion) $f(\tau) = \sum_{n=-\infty}^{\infty} c_n(f) q^n$, where $q = e^{2\pi i \tau}$, whereas the condition (3) with $\gamma = 1$ guarantees $c_n(f) = 0$ for $n < 0$ (resp. for $n \leq 0$ if f is a cusp form). It is an important fact that the Fourier coefficients of f actually generate a number field called the *Hecke field* of f. We refer the reader to [Mi] for details on modular forms.

The complex vector space of modular forms of weight k and level N is denoted $\mathcal{M}_k(N)$ and it contains the subspace $\mathcal{S}_k(N)$ of cusp forms of weight k and level N. These are finite dimensional vector spaces. We shall always assume that $k \geq 1$

and for the purposes of this article, that $N = 1$. The space $\mathcal{M}_k(N) = 0$ if $k < 0$ and $\mathcal{M}_0(N)$ is just the space of constant functions on the upper half plane, and $\mathcal{S}_0(N) = 0$. Also $\mathcal{M}_k(1) = 0$ if k is odd or if $0 < k < 4$. The weight k-action of $\Gamma_0(N)$ preserves $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$. In addition, the space of cusp forms comes equipped with a positive definite Hermitian inner product, called the Petersson inner product. For $M \mid N$, we have an inclusion $\mathcal{S}_k(M) \subset \mathcal{S}_k(N)$. Another way to embed $\mathcal{S}_k(M)$ into $\mathcal{S}_k(N)$ is via the multiplication-by-d map $f(z) \mapsto d^{k-1}f(dz)$, where d is any divisor of N/M. The *old subspace* of level N in $\mathcal{S}_k(N)$, denoted $\mathcal{S}_k(N)^{\text{old}}$, is the subspace generated by the images of cusp forms of all levels M dividing N by both the inclusion and the multiplication-by-d maps, for all divisors d of N/M. Its orthogonal complement with respect to the Petersson inner product is the new subspace and is denoted $\mathcal{S}_k(N)^{\text{new}}$. A newform is said to be *normalized* if its first Fourier coefficient $a_1 = 1$.

1.4. **Hecke algebras.** There are the operators called the *Hecke operators* that act on modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, as well as those for congruence subgroups, and which preserve the space of cusp forms. For the group $\Gamma_1(N)$, there are two classes of Hecke operators. These are usually denoted by T_n, $n \in \mathbb{N}$, $(n, N) = 1$), and the diamond operators $\langle d \rangle$. Following Emerton [E2], we consider the operators S_l for primes $l \nmid N$ defined by $S_l = \langle l \rangle l^{k-2}$. The operators S_l preserve the space of cusp forms. For primes p dividing the level N, there are also the operators U_p. There is a double coset description of the Hecke operators and in fact the Hecke operators T_m can be defined for any positive integer m prime to N. The action of these operators on modular forms can be made explicit in terms of the Fourier coefficients.

Definition 1.5. The Hecke algebra $\mathfrak{h}_k(N)$ for the given weight k and level N (or simply $\mathfrak{h}_k$ when the level N is understood), is the $\mathbb{Z}$-subalgebra of $\mathrm{End}(\mathcal{M}_k(N))$ generated by the operators lS_l and T_l as l ranges over the primes not dividing the level N. This coincides with the algebra generated by the collection of operators T_m.

Definition 1.6. A modular form that is a simultaneous eigenform for all the Hecke operators is called an *eigenform*. A *newform* is an element in $\mathcal{S}_k(N)^{\text{new}}$ that is an eigenform.

Here are some examples of modular forms.

- The form
$$\Delta(q) = q\prod_m (1 - q^m)^{24} = \sum_{n \geq 1} \tau(n)q^n, \tag{4}$$
where $n \mapsto \tau(n)$ is the Ramanujan τ-function, is a holomorphic cusp form of weight 12 and level 1, with q-expansion $q - 24q^2 + 252q^3 - 1472q^4 \cdots$.

- Let k be an even integer and consider the summation function $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. Define the functions

$$\begin{aligned} G_k &= -B_k/2k + \Sigma_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \\ E_k &= (-2k/B_k) G_k = 1 - \tfrac{2k}{B_k} \Sigma_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \end{aligned} \tag{5}$$

where B_k is the k^{th} Bernoulli number. When $k \geq 4$, G_k and E_k are modular forms of weight k (even relative to $\mathrm{SL}_2(\mathbb{Z})$), called the Eisenstein series.

- The form

$$\omega(q) := q \prod_n (1-q^m)^2 (1-q^{11m})^2 = \sum_{n \geq 1} a_n q^n, \tag{6}$$

is a cusp form of weight 2 and level 11. The Fourier coefficients are given by $\{p \mapsto a_p\}$ where a_p is related to $N_E(p)$, the number of rational points modulo p on the elliptic curve $E:\ y^2 + y = x^3 - x^2$, and $N_E(p) = p + 1 - a_p$.

1.7. **Modular curves.** The quotient $\mathbb{H}^*/\mathrm{SL}_2(\mathbb{Z})$ denoted $X_0(1)$ is the *modular curve of level 1.* The analogously defined quotient for a congruence subgroup of level N is denoted $X_0(N)$. It is a compact Riemann surface and is obtained from the corresponding quotient of $\mathbb{H}$ (called the *open modular curve* and denoted $Y_0(N)$) by adding the cusps. The points of $Y_0(1)(\mathbb{C})$ classify isomorphism classes of elliptic curves over $\mathbb{C}$. For $\tau \in \mathbb{H}$, set L_τ to be the lattice $\mathbb{Z} + \mathbb{Z}\tau$, and let E_τ denote the elliptic curve $\mathbb{C}/(L_\tau)$. The j-invariant associated to a lattice provides isomorphisms $Y_0(1) \simeq \mathbb{A}^1$ and $X_0(1) \simeq \mathbb{P}^1$. The corresponding quotients for $\Gamma_1(N)$ are denoted respectively by $X_1(N)$ and $Y_1(N)$, and for a general modular group Γ by X_Γ and Y_Γ, respectively. These are 'moduli spaces'; for instance $Y_0(N)$ classifies isomorphism classes of pairs (E, H) where $E/\mathbb{C}$ is an elliptic curve and H is a cyclic subgroup of E of order N ('level structure'). Similarly, $Y_1(N)$ classifies pairs (E, P) where $E/\mathbb{C}$ is an elliptic curve and $P \in E$ is a point of exact order N. The weight k modular forms for $\Gamma_1(N)$ are interpreted as global sections of a certain line bundle on $Y_1(N)$ while the cusp forms of weight k have an interpretation as global sections of a certain invertible sheaf on $X_1(N)$. Viewed in this optic, a classical modular form f of weight k and level 1 over $\mathbb{C}$ is a rule that attaches to a pair (E, ω) consisting of an elliptic curve E and a non-zero regular differential ω on E, a complex number $f(E, \omega)$ depending only on the isomorphism class of the pair (E, ω), such that for all $\lambda \in \mathbb{C}^*$ we have $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$ and which "behaves well in families" (see [K1] for a precise definition).

1.8. **Congruences.**

Definition 1.9. Two modular forms f and g are congruent modulo m, where m is an integer ≥ 2, if their corresponding Fourier coefficients are congruent modulo m. We write $f \equiv g \bmod m$ to denote the congruence.

For example, $\Delta \equiv \omega \bmod 11$. Recall that by the classical Kummer congruence, we have

$$B_k/k \equiv B_h/h \bmod p \text{ whenever } h \equiv k \bmod (p-1). \tag{7}$$

Here p is a prime, and k, h are positive even integers not divisible by $p-1$. In fact, we have

$$(1-p^{h-1})B_h/h \equiv (1-p^{k-1})B_k/k \bmod p^a \tag{8}$$

whenever $h \equiv k \bmod \phi(p^{a+1})$, where ϕ denotes the Euler ϕ-function. Consider the Eisenstein series

$$\begin{aligned} G_k &= -B_k/2k + \textstyle\sum_{n=1}^{\infty} \{\sum_{d|n} d^{k-1}\} q^n \\ G_{k+p-1} &= -B_{k+p-1}/2(k+p-1) + \textstyle\sum_{n=1}^{\infty} \{\sum_{d|n} d^{k-1} d^{p-1}\} q^n \\ G_{k+\phi(p^r)} &= -B_{k+\phi(p^r)}/2(k+\phi(p^r)) + \textstyle\sum_{n=1}^{\infty} \{\sum_{d|n} d^{k-1+\phi(p^r)}\} q^n. \end{aligned}$$

Using (8) along with Fermat's theorem (resp. Euler's theorem), we see that the non constant Fourier coefficients of G_k and G_{k+p-1} (resp. G_k and $G_{k+\phi(p^r)}$) are congruent modulo p (resp. p^r). The classical Kummer congruence then guarantees the analogous result for the constant coefficients.

If $k \mid (p-1)$, then by the theorem of Clausen–von-Stadt, we have $\mathrm{ord}_p(B_k/k) = -1 - \mathrm{ord}_p(k)$, hence $\mathrm{ord}_p(k/B_k) \geq 1$, and

$$E_k \equiv 1 \bmod p \text{ if } k \equiv 0 \bmod (p-1).$$

In fact, we also have

$$E_k \equiv 1 \bmod p^m \text{ if } k \equiv 0 \bmod (p-1)p^{m-1}. \tag{9}$$

1.10. **Graded algebra of modular forms.** We recall some results due to Swinnerton-Dyer on the reduction of modular forms modulo p. Let $k \in \mathbb{Z}$. For a modular form f of weight k with q-expansion $f = \Sigma\, a_n q^n$, where $a_n \in \mathbb{Q}$, and are p-integers, the reduction modulo p of f is denoted $\bar{f}$, and is an element of $\mathbb{F}_p[[q]]$. The set of such power series is denoted $\tilde{\mathcal{M}}_k$; this is a vector subspace of $\mathbb{F}_p[[q]]$, and we put $\tilde{\mathcal{M}} := \sum_k \tilde{\mathcal{M}}_k$. Similarly, we set $\mathcal{M} = \bigoplus_k \mathcal{M}_k$, to denote the graded $\mathbb{Q}$-algebra where $\mathcal{M}_k$ is the subspace of modular forms of weight k. The series P, Q and R of weight 2, 4 and 6 respectively, are defined by

$$\begin{aligned} P = \quad E_2 &= 1 - 24 \textstyle\sum \sigma_1(n) q^n \\ Q = \quad E_4 &= 1 + 240 \textstyle\sum \sigma_3(n) q^n \\ R = \quad E_6 &= 1 - 504 \textstyle\sum \sigma_5(n) q^n. \end{aligned} \tag{10}$$

For $p \geq 5$, the elements Q and R generate $\mathcal{M}$ and hence $\tilde{\mathcal{M}}$, as well. Indeed, $\mathcal{M} \simeq \mathbb{Q}[Q, R]$, with Q, R being algebraically independent. Any $f \in \mathcal{M}_k$ can be written uniquely as a finite sum

$$f = \sum a_{m,n}\, Q^m R^n, \;\; a_{m,n} \in \mathbb{Q},$$

where (m, n) are pairs of positive integers such that $4m + 6n = k$. For instance,

$$\Delta = \frac{1}{1728}(Q^3 - R^2). \tag{11}$$

For $p \geq 5$, we have

$$\tilde{\mathcal{M}} = \mathbb{F}_p[Q, R]/ < \tilde{A} - 1 >$$

where $A(Q, R) = E_{p-1}$ is the polynomial expression for E_{p-1}. Thus $\tilde{\mathcal{M}}$ is the affine algebra of a smooth algebraic curve $Y/\mathbb{F}_p$. For example, when $p = 11$, $Y = \operatorname{Spec} \tilde{\mathcal{M}}$ is a curve of genus 0, and for $p = 13$, Y is a curve of genus 1.

2. p-adic modular forms

Serre defines the following valuation on the formal power series ring $\mathbb{Q}_p[[q]]$. If $f = \sum a_n q^n$, then set

$$\operatorname{ord}_p(f) = \inf \operatorname{ord}_p(a_n).$$

If $\operatorname{ord}_p(f) \geq 0$, then $f \in \mathbb{Z}_p[[q]]$ and if $\operatorname{ord}_p(f) \geq m$, then $\tilde{f} \equiv 0 \bmod p^m$. A p-adic modular form is then defined as follows.

Definition 2.1. Let $\mathcal{M}_k$ be the space of modular forms of level 1 and weight k for $\Gamma_1(N)$. A q-expansion

$$f = \sum_{n=0}^{\infty} a_n q^n \;\in\; \mathbb{Q}_p[[q]]$$

is a p-adic modular form (in the sense of Serre) if there exists a sequence of classical modular forms $f_i \in \mathcal{M}_{k_i}$ for $\Gamma_1(N)$ such that

$$\operatorname{ord}_p(f - f_i) \to \infty \text{ as } i \to \infty.$$

Concretely, this means that the Fourier coefficients of f_i tend uniformly to those of f. Note that this says nothing about the weight of the p-adic modular form f. To do this, we set (recall that we are assuming $p \geq 5$)

$$\begin{aligned} X_m &= \mathbb{Z}/(p-1)p^{m-1}\mathbb{Z} \\ X &= \varprojlim X_m \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p. \end{aligned}$$

The group X is the *weight space* for p-adic modular forms and can be identified with the group $\operatorname{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ of continuos characters of $\mathbb{Z}_p^\times$ into $\mathbb{Z}_p$. If $k \in X$, then we write $k = (s, u)$, where $s \in \mathbb{Z}_p$, $u \in \mathbb{Z}/(p-1)\mathbb{Z}$. If v is the corresponding element in $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, then we write $v = v_1 \cdot v_2$ with $v_1^{p-1} = 1$ and $v_2 \equiv 1 \bmod p$; further $v^k = v_1^s v_2^u$. An element $k \in X$ is *even* if it belongs to the subgroup $2X$; this just means that the component u of k is an even element of $\mathbb{Z}/(p-1)\mathbb{Z}$. Further, the natural map $\mathbb{Z} \to X$ is injective, and we thus view the integers as a dense subgroup of X. The following theorem enables one to define the weight of a p-adic modular form.

Theorem 2.2. *(Serre) Let m be an integer ≥ 1. Suppose that f and f' are two modular forms with rational coefficients, of weights k and k' respectively. Assume that $f \neq 0$ and that*

$$\mathrm{ord}_p(f - f') \geq \mathrm{ord}_p(f) + m.$$

Then $k' \equiv k \bmod (p-1) \cdot p^{m-1}$.

The reader is referred to [Se, Théorème 1] for the detailed proof, we just outline the key ideas mentioning that the assumption $p \geq 5$ is needed here. Put $\tilde{\mathcal{M}}^0 = \bigcup_k \tilde{\mathcal{M}}_k$, where k varies over the positive integers divisible by $(p-1)$. The key fact that is needed is that $\tilde{\mathcal{M}}^0$ (which is an $\mathbb{F}_p$-algebra and an integral domain), is integrally closed in its quotient field. The case $m = 1$ is easy and so we may assume that $m \geq 2$. Let $h = k' - k$, and put $r = \mathrm{ord}_p(h) + 1$. We may further assume $h \geq 4$, since by utilizing the congruence (9) we could replace f' by $f' E_{(p-1)p^n}$ for a large enough integer n. The theorem then reduces to proving that $r \geq m$. Using the Eisenstein series E_h and the operator $\theta = q \cdot d/dq$ on $\tilde{\mathcal{M}}$, along with some delicate but standard calculations in $\tilde{\mathcal{M}}$, one finds an element $\tilde{\phi}$ in the fraction field of $\tilde{\mathcal{M}}^0$. This element is seen to be integral over $\tilde{\mathcal{M}}^0$, and does not lie in $\tilde{\mathcal{M}}^0$, when $r < m$ and $m \geq 2$. This contradicts the fact that $\tilde{\mathcal{M}}^0$ is integrally closed, hence $r \geq m$ and the theorem follows.

For example, consider the modular form Δ of weight 12, and the modular form ω of weight 2 ((6), (11)) for the prime $p = 11$. We have $12 \equiv 2 \bmod (11-1)$.

Theorem 2.3. *(Serre) Let f be a p-adic modular form $f \neq 0$, and let (f_i) be a sequence of modular forms of weight (k_i) with rational coefficients and limit f. The k_i's then tend to a limit k in the weight space X; this limit depends only on f and not on the chosen sequence f_i.*

By hypothesis, $\mathrm{ord}_p(f - f_i) \to \infty$. Setting $k = \lim_i k_i$ we see that $k \in X$ satisfies the required properties. That it is independent of the chosen sequence k_i follows from Theorem 2.2.

Serre [Se, Cor.2] also proves the following.

Proposition 2.4. *Suppose $f_i = \sum_{n=0}^{\infty} a_{i,n} q^n$ is a sequence of p-adic modular forms of weights k_i such that*

(i) *for $n \geq 1$, the $a_{i,n}$ converge uniformly to some $a_n \in \mathbb{Q}_p$, and*

(ii) *the weights k_i converge to some k in X.*

Then $a_{0,n}$ converges to an element $a_0 \in \mathbb{Q}_p$ and the series

$$f = a_0 + a_1 q + \cdots + a_n q^n + \cdots$$

is a p-adic modular form of weight k.

Example 2.5. Define

$$\sigma^*_{k-1}(n) = \sum_{\substack{d|n \\ \gcd(d,p)=1}} d^{k-1} \in \mathbb{Z}_p.$$

Assume k is even and choose a sequence of even integers $k_i \geq 4$ tending to infinity in the archimedean sense and tending to k, p-adically. Then, we have that for a positive integer d coprime to p, $d^{k_i-1} \to d^{k-1}$ in the p-adic norm and

$$\sigma_{k_i-1}(n) \to \sigma^*_{k-1}(n) \in \mathbb{Z}_p,$$

this convergence being uniform for all $n \geq 1$. By the above results, it then follows that the Eisenstein series G_{k_i} converge to a p-adic modular form of weight k, called the p-adic Eisenstein series of weight k, and

$$G^*_k = \left(\lim_i B_{k_i}/2k_i\right) + \sum_{n=1}^{\infty} \sigma^*_{k-1}(n) q^n. \tag{12}$$

We remark in passing that the p-adic Eisenstein series of weight 2 is a p-adic modular form even though the classical weight two Eisenstein series is not a classical modular form.

2.6. **The p-adic Riemann-zeta function.** Recall that $B_{k_i}/2k_i = \frac{1}{2}\zeta(1-k_i)$, where $\zeta(s)$ is the classical Riemann-zeta function.

We denote the constant term of the p-adic Eisenstein series G^*_k by $\frac{1}{2}\zeta^*(1-k)$, where $k \neq 0$ is an even element of X. Then $\frac{1}{2}\zeta^*(1-k)$ is the p-adic limit of $\frac{1}{2}\zeta(1-k_i)$. This defines a function

$$(1-k) \mapsto \zeta^*(1-k) \tag{13}$$

on the odd elements $(1-k)$ of $X \setminus 1$. This function is continuous and is essentially the p-adic zeta function of Kubota-Leopoldt, denoted $\mathcal{L}_p$ [Iw]. We have

$$\zeta^*(1-k) = (1-p^{k-1})\zeta(1-k)$$

which is the imprimitive ζ function with the Euler factor at p removed.

$$G^*_k = a_0 + \sum_{n=1}^{\infty} \sigma^*_{k-1}(n) q^n = \frac{1}{2}\zeta^*(1-k) + \sum_{n=1}^{\infty} \sigma^*_{k-1}(n) q^n.$$

Additionally, the series G^*_k itself depends continuously on k.

We largely follow Iwasawa's book [Iw]. Suppose that χ is a Dirichlet character and let $L(s,\chi)$ be the classical Dirichlet L-function of a complex variable s. The values $L(s,\chi)$ are algebraic numbers for negative integers $s \leq 0$, and hence can be viewed as elements of $\mathbb{C}_p$. The Kubota-Leopoldt p-adic zeta function p-adically interpolates these values in the following sense. The function $\mathcal{L}_p$ is viewed as a continuous function of $s \in \mathbb{Z}_p,\ s \neq 1$ and can be evaluated on Dirichlet characters so that $\mathcal{L}_p(s,\chi) \in \mathbb{C}_p$. It has the property that

$$\mathcal{L}_p(s,\chi) = (1 - \chi(p)p^{-s})L(s,\chi) \tag{14}$$

for integers $s \leq 0,\ s \equiv 1 \bmod (p-1)$. With these notations, Iwasawa showed the following result. Suppose that χ is any character distinct from w^{-1}, where w is the Teichmüller character, and let $k \in X,\ k = (s,u)$ be an odd element, then the function

$$\begin{aligned} \zeta' : X &\to \mathbb{Z}_p \\ (s,u) &\mapsto \mathcal{L}_p(s, w^{1-u}), \end{aligned}$$

is continuous on X, and $\zeta'(1-k) = (1-p^{k-1})\zeta(1-k)$. Since $\mid k_i \mid \to \infty$, we have $\lim_{i\to\infty}(1-p^{k_i-1}) = 1$, and $\zeta'(1-k) = \zeta^*(1-k)$, where ζ^* is defined in (13). In particular, we see that $\zeta' = \mathcal{L}_p$ as functions on X.

We have

Theorem 2.7. [Se] *If (s,u) is an odd element of X, $(s,u) \neq 1$, then $\zeta^*(s,u) = \mathcal{L}_p(s, w^{1-u})$, where $\mathcal{L}_p(s,\chi)$ is the p-adic L-function.*

3. Iwasawa algebra

In this section, we define the Iwasawa algebra of a profinite group and consider its other interpretations. This is then used in the context of p-adic modular forms to show how one can recover the classical Kummer congruences. We first recall the definition of the Iwasawa algebra of $\mathcal{G}$.

3.1. **Iwasawa algebra as a group completion.**

Definition 3.2. The Iwasawa algebra of $\mathcal{G}$ over $\mathbb{Z}_p$ is denoted $\Lambda(\mathcal{G})$ and is defined as

$$\Lambda(\mathcal{G}) = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{U}]$$

where $\mathcal{U}$ varies over open normal subgroups of $\mathcal{G}$ and the inverse limit is taken with respect to the natural maps.

We shall largely be interested in the case when $\mathcal{G} = \mathbb{Z}_p^\times$; then we have

$$\mathcal{G} \simeq U_1 \times \Delta, \tag{15}$$

where $U_1 \simeq \mathbb{Z}_p$ and $\Delta \simeq \mathbb{Z}/(p-1)$. The Iwasawa algebra $\Lambda = \mathbb{Z}_p[[U_1]]$ is a regular local ring and may be identified with the power series $\mathbb{Z}_p[[T]]$, the isomorphism is non-canonical and sends a generator of U_1 to $(1+T)$. It is a compact $\mathbb{Z}_p$-algebra with respect to the topology defined by the powers of the maximal ideal. We denote by U_n the subgroup of $\mathcal{G}$ consisting of elements u such that $u \equiv 1 \bmod p^n$.

3.3. **Measure theoretic interpretation and power series.** The Iwasawa algebra also has an interpretation in terms of p-adic measures [Wa]. A $\mathbb{C}_p$-valued measure on $\mathcal{G}$ is a function on the set of compact open subsets of $\mathcal{G}$ that is additive on disjoint unions. Let $C(\mathcal{G}, \mathbb{C}_p)$ denote the space of continuous functions from $\mathcal{G}$ to $\mathbb{C}_p$. Such measures are in bijection with continuous linear maps $C(\mathcal{G}, \mathbb{C}_p) \to \mathbb{C}_p$. Given $f \in C(\mathcal{G}, \mathbb{C}_p)$ and an element λ in $\Lambda(\mathcal{G})$, the corresponding measure is denoted $d\lambda$, and we have the value

$$\lambda(f) = \int_{\mathcal{G}} f d\lambda \in \mathbb{C}_p.$$

The set of all O-valued measures, where O is the ring of integers in a finite extension of $\mathbb{Q}_p$, is similarly defined, and we denote it by M_O. Under the operations of addition and convolution, the set M_O forms a ring. This ring is isomorphic to $O[[T]]$, the isomorphism being given by the Mahler transform

$$\begin{array}{rcl} M_O & \to & O[[T]] \\ \lambda & \mapsto & \hat{\lambda}(T) = \int_{\mathbb{Z}_p} (1+T)^x \, d\lambda(x) = \sum_{m\geq 0} \left(\int_{\mathbb{Z}_p} \binom{x}{m} d\lambda(x) \right) T^m. \end{array}$$

The power series on the right is called the power series associated to the measure λ.

The map $\phi : \mathbb{Z}_p \to U_1$ given by $s \mapsto (1+p)^s$ gives a topological group isomorphism, noting that $1+p$ is a particular choice of a topological generator of U_1. Given $u \in U_1$, let f_u denote the element in $C(\mathbb{Z}_p, \mathbb{Z}_p)$ defined by $s \mapsto u^s$. The $\mathbb{Z}_p$-module L generated by such elements f_u is in fact a subalgebra of $C(\mathbb{Z}_p, \mathbb{Z}_p)$. Let $\bar{L}$ denote its closure (in the uniform convergence topology). It is not difficult to see [Se, 4.1(b)] that if $f \in \bar{L},\ n \geq 0$, then

$$s \equiv s' \bmod p^n \Leftrightarrow f(s) \equiv f(s') \bmod p^{n+1}. \tag{16}$$

By a classical result of Mahler, any $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ can be written uniquely as

$$f(x) = \sum_{n\geq 0} a_n \binom{x}{n},$$

where $a_n \in \mathbb{C}_p$ and $a_n \to 0$ as $n \to \infty$.

Fixing now a choice of a topological generator u of U_1, we have the map

$$\begin{array}{rcl} M : \Lambda(U_1) & \to & \mathbb{Z}_p[[T]] \\ \lambda & \mapsto & \int_{\mathbb{Z}_p} f_u \, d\lambda = \int_{\mathbb{Z}_p} (1+T)^s d\lambda, \end{array}$$

where the last integral is really $\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p}\binom{s}{n}d\lambda\right)T^n$. This gives another interpretation of the identification mentioned above, and amounts to the change of variable $T=(u-1)$.

Suppose $\mathcal{O}_p$ is the set of elements x in $\mathbb{C}_p$ such that $|x|_p \leq 1$. Serre also interprets the Iwasawa algebra over $\mathcal{O}_p$ (see [C]) as the set of functions f in $C(\mathbb{Z}_p, \mathbb{C}_p)$ that are of the form

$$f(x) = F(\phi(x)-1) \text{ with } F \in \mathcal{O}_p[[T]], \tag{17}$$

This doesn't depend on the choice of a generator of U_1 in the definition of the isomorphism ϕ. Similarly, if $g = \sum a_n T^n$ is an element of $\mathbb{Z}_p[[T]]$, and $\varepsilon(g)$ in $C(\mathbb{Z}_p, \mathbb{Z}_p)$ is defined as the function

$$s \mapsto g(u^s-1) = \sum_n a_n(u^s-1)^n, \tag{18}$$

then it can be shown [Se, §4] that ε gives an isomorphism

$$\Lambda \simeq \bar{L} \tag{19}$$

which is the identity on $\mathbb{Z}_p[U_1]$. In this context, Serre shows [Se, Théorème 13] that an element $f \in C(\mathbb{Z}_p, \mathbb{Z}_p)$ belongs to the Iwasawa algebra Λ if and only if there are p-adic integers b_n $(n \in \mathbb{Z},\ n \geq 1)$ such that

$$f(s) = \sum_{n=0}^{\infty} b_n p^n s^n/n! \text{ for } s \in \mathbb{Z}_p, \tag{20}$$

and

$$\frac{(\sum_{i=1}^n c_{in} b_i)}{n!} \text{ is a } p-\text{adic integer.} \tag{21}$$

Here the c_{in}'s are defined by

$$\sum_{i=1}^{n} c_{in} x^i = n!\binom{x}{n}.$$

In particular, it follows (see [Se, Corollaire, p. 241]) that if $f \in \Lambda$ and we consider the corresponding coefficients b_n (cf. (20)), then

$$b_n \equiv b_{n+p-1} \bmod p \text{ for all } n \geq 1. \tag{22}$$

3.4. **Iwasawa algebra and interpolation.** There is yet another way of viewing elements in the Iwasawa algebra $\Lambda(U_1)$, which is related to interpolation data (see [Se, §2]). Given a sequence of elements $\underline{b} = (b_0, b_1, \cdots)$ of elements of $\mathbb{C}_p$, by abuse of language, one says that $\underline{b}$ *belongs to the Iwasawa algebra* if there exists a function $f : \mathbb{Z}_p \to \mathbb{C}_p$ as in (17) such that $f(k) = b_k$ for each $k \geq 0$. This amounts to saying that there exists a formal power series $F \in \mathcal{O}_p[[T]]$ such that $b_k = F(u^k - 1)$ for a topological generator u of U_1 and all $k \geq 0$. Let

$$c_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} b_j.$$

By Mahler's criterion, in order that there exist a continuous interpolating function $f : \mathbb{Z}_p \to \mathbb{C}_p$ such that $f(k) = b_k$, for every integer $k \geq 0$, it is necessary and sufficient that $\lim_{n\to\infty} \mid c_n \mid = 0$. The function f is then defined by the series

$$f(x) = \sum_{n=0}^{\infty} c_n \binom{x}{n}.$$

If the *coefficients of interpolation,* c_n, satisfy the congruence

$$c_n \equiv 0 \bmod p^n \mathcal{O}_p \text{ for } n \geq 0, \tag{23}$$

then there exists an analytic function f in the open disc $\{x \in \mathbb{C}_p \mid \|x\| < R\}$ of $\mathbb{C}_p$ with $R = p^{\frac{p-2}{p-1}}$ and such that $f(k) = b_k$ for $k \geq 0$.

Indeed, let $S_n^{(m)}$, $0 \leq m \leq n$ be the Stirling numbers given by the identity

$$X(X-1)\dots(X-(n-1)) = \sum_{m=0}^{n} S_n^{(m)} X^m;$$

recall that there is a recurrence formula

$$S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)} \ (1 \leq m \leq n)$$

with $S_0^{(0)} = 1$, $S_n^{(0)} = 0$ and $S_n^{(n)} = 1$ for $n \geq 1$. Put

$$a_m = \sum_{n=m}^{\infty} S_n^{(m)} c_n / n!,$$

then the interpolating function f admits a Taylor expansion $f(x) = \sum_{m=0}^{\infty} a_m x^m$ with radius of convergence $\geq R$, and such that $f(k) = b_k$.

3.5. **Recovering the Kummer congruences.** We return to the p-adic Eisenstein series G_k^* considered in §2. Let $k \in X$ be an even element, $k \neq 0$ and write $k = (s, u)$ with $s \in \mathbb{Z}_p$, $u \in \mathbb{Z}/(p-1)\mathbb{Z}$, and $G_k^* = G_{s,u}^*$. Writing G_k^* in terms of its Fourier expansion, and denoting the n-th Fourier coefficient ($n \geq 1$) by $a_n(G_{s,u}^*)$, we have

$$\begin{aligned} G_k^* &= \textstyle\sum_{n\geq 0} a_n(G_{s,u}^*) \\ a_0(G_{s,u}^*) &= \tfrac{1}{2}\,\zeta^*(1-s,1-u) \\ a_n(G_{s,u}^*) &= \sigma_{k-1}^*(n) = \sum_{\substack{d\mid n \\ \gcd(d,p)=1}} d^{k-1}. \end{aligned} \tag{24}$$

Note that the decomposition (15) allows us to express a p-adic unit $d \in \mathbb{Z}_p^\times$ as $d = \omega(d)\langle d\rangle$, where $\omega(d)^{p-1} = 1$ and $\langle d\rangle \in U_1$. Hence

$$a_n(G_{s,u}^*) = \sum_{\substack{d\mid n \\ \gcd(d,p)=1}} d^{-1}\omega(d)^k\langle d\rangle^k = \sum_{\substack{d\mid n \\ \gcd(d,p)=1}} d^{-1}\omega(d)^u\langle d\rangle^s.$$

Thus, for fixed $n \geq 1$ and $u \in \mathbb{Z}/(p-1)\mathbb{Z}$, we may consider the function $s \mapsto a_n(G_{s,u}^*)$ as an element of L and hence as an element of the Iwasawa algebra Λ via $L \subset \bar{L} \simeq \Lambda$ (see Section 3.3 and (19) there). It then follows, thanks to a result of Iwasawa [Se, Théorème 16] that if u is an even element of $\mathbb{Z}/(p-1)\mathbb{Z}$, then the function $s \mapsto a_0(G_{s,u}^*)$ is also an element of Λ. Thus the p-adic modular form G_k^* can also be viewed as having Fourier coefficients in the Iwasawa algebra. The Kummer congruences then follow from (16).

4. The case of totally real fields

Let K be an abelian, totally real, number field of degree r. Recall that the Riemann zeta function $\zeta_K(s)$ is defined by

$$\zeta_K(s) = \sum \mathrm{N}\mathfrak{a}^{-s} = \prod (1-\mathrm{N}\mathfrak{p}^{-s})^{-1} \text{ for } \mathrm{Re}(s) > 1,$$

where $\mathfrak{a}$ (resp. $\mathfrak{p}$) varies over the set of nonzero ideals (resp. the nonzero prime ideals) of the ring of integers $\mathcal{O}_K$ of K and N denotes the norm. This function may be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. Recall that there are r characters $\chi_1, \cdots, \chi_r$ such that $\zeta_K(s) = \prod_{j=1}^{r} L(s,\chi_j)$. We denote the *different ideal* in $\mathcal{O}_K$ by $\mathfrak{d}$ and by d the discriminant of K, remarking that the absolute value of the discriminant is the norm of the different, i.e. $\mid \mathrm{N}\mathfrak{d} \mid = d$.

Without delving into the details (see [Se, §5]), we briefly indicate the existence of the corresponding p-adic modular form in this case.

Let S be the set of primes of K lying above p. We set

$$\zeta_K^{(p)}(s) = \zeta_K(s) \prod_{\mathfrak{p}\in S} (1-\mathrm{N}\mathfrak{p}^{-s}) \text{ for } \mathrm{Re}(s) > 1.$$

Let k be an even integer ≥ 2. First, one associates to k a modular form g_k of weight rk such that the constant term $a_0(g_k)$ satisfies

$$a_0(g_k) = 2^{-r}\zeta_K^{(p)}(1-k),$$

and the n-th Fourier coefficient is given by

$$a_n(g_k) = \sum_{\substack{\mathrm{Tr}(x)=n \\ x\in\mathfrak{d}^{-1} \\ x>>0}} \sum_{\substack{\mathfrak{a}|x\mathfrak{d} \\ (\mathfrak{a},p)=1}} (\mathrm{N}(\mathfrak{a}))^{k-1}, \ \ n \geq 1.$$

Here the sum varies over totally positive elements x in K such that $\mathrm{trace}(x) = n$ and $\mathfrak{a}|x\mathfrak{d}$.

As before, for an element $k \in X$ one then chooses a sequence of even integers $k_i \geq 4$ such that $|k_i| \to \infty$ and $k_i \to k$ in X. Then the forms g_{k_i} have a limit g_k^* which is a p-adic modular form of weight rk, and is independent of the chosen sequence $\{k_i\}$. We have

$$a_0(g_k^*) = 2^{-r}\zeta_K^*(1-k) := 2^{-r}\lim_{i\to\infty}\zeta_K^{(p)}(1-k_i),$$

and

$$a_n(g_k^*) = \sum_{\substack{\mathrm{Tr}(x)=n \\ x\in\mathfrak{d}^{-1} \\ x>>0}} \sum_{\substack{\mathfrak{a}|x\mathfrak{d} \\ (\mathfrak{a},p)=1}} (\mathrm{N}(\mathfrak{a}))^{k-1}, \ \ n \geq 1.$$

The following results are proved in [Se] (see also [C]):

- If $k \geq 1$ is even and $rk \neq 0 \bmod (p-1)$, then $\zeta_K(1-k)$ is p-integral.
- If $k \geq 1$ is even, and $rk \equiv 0 \bmod (p-1)$, then $prk \cdot \zeta_K(1-k)$ is p-integral.
- If k is an even integer ≥ 2, then
$$\zeta_K^*(1-k) = \zeta_K(1-k)\prod_{\mathfrak{p}\in S}\left(1-\mathrm{N}\mathfrak{p}^{k-1}\right).$$
- The function ζ_K^* is again the p-adic zeta function of Kubota-Leopoldt that interpolates values of ζ_K.
- If $k = (s,u) \in X$ with $s \in \mathbb{Z}_p$ and u an even element of $\mathbb{Z}/(p-1)\mathbb{Z}$, we write $\zeta_K^*(1-s,1-u)$ for $\zeta_K^*(1-k)$. If $ru \neq 0$ then the function $s \mapsto \zeta_K^*(1-s,1-u)$ belongs to the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$. It is holomorphic in a disc strictly larger than the unit disc.
- $\zeta_K^*(1-s,1-u) = \zeta_K^*(1-s,1-u')$ whenever $u \equiv u' \bmod m$, where m is the degree $[K(\mu_p):K]$.
- If $k = (s,u)$ as above, and u is an even element of $\mathbb{Z}/(p-1)\mathbb{Z}$ with $ru = 0$, then the function $s \mapsto \zeta_K^*(1-s,1-u)$ is of the form $h(T)/((1+T)^r-1)$, for $h \in \Lambda$.

5. Galois representations

In this section we recall the Galois representations associated to modular forms, and to the $\underline{\Lambda}$-adic forms.

5.1. **Galois representations for modular forms.** Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ be a normalized Hecke eigenform which is a newform of weight $k \geq 2$, and level N. By results of Eichler-Shimura, Deligne, for every prime l, there is an associated Galois representation

$$\rho_{l,f} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K), \tag{25}$$

where K is a finite extension of $\mathbb{Q}_l$. As f is a Hecke eigenform, there is a ring homomorphism

$$\lambda : \mathfrak{h}_k \to \mathbb{C}$$

such that $T(f) = \lambda(T)f$ for all elements T in the Hecke algebra $\mathfrak{h}_k$. Every eigenvalue of a Hecke operator is an algebraic integer.

This representation has the following properties:

(i) If Σ denotes the set of primes dividing lN, then $\rho_{l,f}$ is unramified outside Σ.

(ii) For each $p \nmid Nl$, we have

$$\begin{aligned} \mathrm{Tr}\rho_{l,f}(\mathrm{Frob}_p) &= a_p \\ \det\rho_{l,f}(\mathrm{Frob}_p) &= \chi(p)p^{k-1}; \end{aligned}$$

here Frob_p denotes the Frobenius endomorphism at the prime p and χ is the cyclotomic character.

(iii) For $p \nmid Nl$, the matrix $\rho_{f,l}(\mathrm{Frob_p})$ has characteristic polynomial (called the *Hecke polynomial*) equal to

$$X^2 - \lambda(T_p)X + \lambda(pS_p).$$

If $k = 1$, then by results of Deligne-Serre, there is an irreducible degree two complex representation

$$\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

such that ρ_f has finite image.

5.2. **$\underline{\Lambda}$-adic forms and big Galois representations.** The theory of $\underline{\Lambda}$-adic forms was introduced and studied by Hida ([H1], [H2]). We fix an integer N such that $p \nmid N$. Let $\Lambda = \mathbb{Z}_p[[U_1]]$ be an Iwasawa algebra and let $\underline{\Lambda}$ be a finite integral extension of Λ. Consider the natural inclusion

$$\varepsilon : (1 + p\mathbb{Z}_p) \to \bar{\mathbb{Q}}_p^*$$

Then for every $k \in \mathbb{Z}_p$, the homomorphism $\varepsilon^k : 1 + p\mathbb{Z}_p \to \bar{\mathbb{Q}}_p^*$ induces a $\mathbb{Z}_p$-algebra homomorphism $\varepsilon^k : \Lambda \to \bar{\mathbb{Q}}_p$. Suppose $k \geq 1$ is an integer and let $\phi \in \mathrm{Hom}(\underline{\Lambda}, \bar{\mathbb{Q}}_p)$ be a $\mathbb{Z}_p$-algebra homomorphism such that $\phi \mid_\Lambda = \varepsilon^k$. We have the specialisation maps

$$\eta_k : \underline{\Lambda} \to \bar{\mathbb{Q}}_p^* \tag{26}$$

corresponding to such ϕ.

Definition 5.3. A $\underline{\Lambda}$-adic form $\mathbf{f}$ of level N is a formal q-expansion

$$\mathbf{f} = \sum_{n=0}^{\infty} a_n(\mathbf{f}) q^n \in \underline{\Lambda}[[q]] \tag{27}$$

such that for all specialisations η_k as above, the corresponding specialisations f_k of $\mathbf{f}$ give rise to classical modular (cusp) forms of weight k for the congruence subgroup $\Gamma_1(Np^r)$, $r \geq 1$. The $\underline{\Lambda}$-adic form is said to be a newform(resp. an eigenform) if each specialisation is a newform (resp. an eigenform).

Hida proved that there is a complete, local Noetherian domain $\underline{\Lambda}$ which is finite flat over Λ and such that any $\underline{\Lambda}$-adic form $\mathbf{f}$ gives rise to a 'big' Galois representation

$$\rho_{\mathbf{f}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\underline{\Lambda})$$

which is continuous in a suitable sense. Further, it has the property that for any ϕ as above, the composite

$$\rho_{\mathbf{f}} \circ \phi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\underline{\Lambda}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$$

corresponds to the Galois representation induced by the associated modular form f_k obtained by specialisation of $\mathbf{f}$.

Thus a $\underline{\Lambda}$-adic form is a family of classical forms of varying weights of level Np^r with $r \geq 1$ with isomorphic residual q-expansions modulo the maximal ideal of $\underline{\Lambda}$.

A Hecke eigenform f is *ordinary* if its p-th Fourier coefficient a_p (and at the same time the U_p eigenvalue) is a p-adic unit. Hida utilized the study of congruences of eigenforms and showed that any ordinary eigenform can be placed as a member of an $\underline{\Lambda}$-adic form in the above sense. Hida's construction makes essential use of the ordinary p-adic Hecke algebra $\mathfrak{h}_k^{\mathrm{ord}} \subset \mathfrak{h}_k$, the maximal ring direct summand on which U_p is invertible. In other words, if we write $e = \lim_{n\to\infty} U_p^{n!}$ under the p-adic topology of $\mathfrak{h}_k$, then e is idempotent and we have $\mathfrak{h}_k^{\mathrm{ord}} = e\mathfrak{h}_k$. As constructed in [H1] and [H2], the unique "big" ordinary Hecke algebra $\mathfrak{h}^{\mathrm{ord}}$ is characterized by the following two properties usually referred to as Control theorems:

(C1) $\mathfrak{h}^{\mathrm{ord}}$ is free of finite rank over $\underline{\Lambda}$,

(C2) When $k \geq 2$, $\eta_k(\mathfrak{h}^{\mathrm{ord}}) \simeq \mathfrak{h}_k^{\mathrm{ord}}$ for η_k given by (26).

The space of all $\underline{\Lambda}$-adic modular forms is free of finite rank over $\underline{\Lambda}$ and moreover is the $\underline{\Lambda}$-dual of the "big" ordinary Hecke algebra $\mathfrak{h}^{\mathrm{ord}}$. Thus, all the structural properties of the space of $\underline{\Lambda}$-adic modular forms mirror the structural properties of $\mathfrak{h}^{\mathrm{ord}}$. In conclusion, Hida theory gives many examples of p-adic families of modular forms. The p-adic Eisenstein series considered in §2 are the simplest example of such a p-adic family and in fact give rise to a Λ-adic form where $\Lambda = \mathbb{Z}_p[[T]]$, as discussed in §3. There is also a $\underline{\Lambda}$-adic form whose weight two specialization gives the elliptic curve $X_0(11)$ and has the modular form Δ at weight 12 (cf. [EPW, §5.3]).

5.4. **Further Vistas.** In this last brief subsection, we sketch the contours of the theory of p-adic modular forms stemming from the work of Katz [K1] and others. This affords a geometric point of view for the theory. Katz and Dwork developed an equivalent definition of p-adic modular forms as sections of line bundles over certain p-adic rigid analytic spaces related to modular curves. Recall the modular curves $Y_i(N)$, $i = 0, 1$, and $X_i(N)$ from 1.7. There exist integral models $\mathcal{Y}_i(N)$ and $\mathcal{X}_i(N)$, $i = 0, 1$, over $\mathbb{Z}[1/N]$ such that $\mathcal{Y}_i(N)$ is a smooth curve over $\operatorname{Spec}\mathbb{Z}[1/N]$, and has the property that $\mathcal{Y}_i(N) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to $Y_i(N)$ as complex manifolds. Further, for $N \geq 5$, it represents the functor which sends a $\mathbb{Z}[1/N]$-scheme S to the set of isomorphism classes (E, P) where E is an elliptic curve over S and $P \in E(S) = \mathrm{Hom}(S, E)$ is a point of exact order N. There is an invertible sheaf $\bar{\omega}$ over $\mathcal{Y}_i(N)$ (which canonically extends to its compactification $\mathcal{X}_i(N)$. If $N \geq 5$, and R is a $\mathbb{Z}[1/N]$-algebra, a modular form over R of weight k and level N is a global section of $\bar{\omega}^{\otimes k}$ over $X_1(N) \times_{\mathbb{Z}[1/N]} R$. The R-module of such modular forms is denoted $\mathcal{M}_k(\Gamma_1(N); R)$ and we have $\mathcal{M}_k(\Gamma_1(N); \mathbb{C}) = \mathcal{M}_k(\Gamma_1(N))$, the classical space of modular forms of weight k. An element f in $\mathcal{M}_k(\Gamma_1(N); R)$ has a q-expansion with the Fourier coefficients in R.

For $N \leq 4$, the above functor is not representable, and one gets around this problem by working with $X_1(M)$ for $M \geq 5$ and $N \mid M$. One then defines a modular form of weight k and level N to be a modular form of weight k and level M that is invariant under the action of the quotient $\Gamma_1(N)/\Gamma_1(M)$. These are the Katz' algebraic modular forms over R. For $R = \mathbb{Q}_p$ we write $\mathcal{M}_k(\Gamma_1(N); \mathbb{Q}_p)$ for the corresponding space of modular forms. This does not give a satisfactory theory of p-adic modular forms however, as it does not reflect the p-adic congruences between modular forms. The Hecke operators are defined in this geometric setting as well. To define p-adic modular forms, Katz begins by considering the integral modular curves $\mathcal{X}_1(N)$ and the modular curve for the group $\Gamma_1(N) \cap \Gamma_0(p)$, denoted $\mathcal{X}_{\Gamma,p}(N)$. The latter is again a moduli space that classifies elliptic curves with additional 'level structure data'. The 'cusps' correspond to pairs where the elliptic curve is a Tate elliptic curve $\mathbb{G}_m/q^{\mathbb{Z}}$ (recall that $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[T, T^{-1}]$) with different level structures (for instance ζ_N).

Considering $\mathbb{F}_p$ as a $\mathbb{Z}[1/N]$-algebra and applying base change to $\mathbb{F}_p$, we get the corresponding 'reduced' modular curves $\bar{\mathcal{X}}_1(N)$ and $\bar{\mathcal{X}}_{\Gamma,p}(N)$ over $\mathbb{F}_p$. These curves are also moduli spaces for elliptic curves with suitable level structures. The mod-p-geometry of these curves is well-understood and one first defines mod-p modular forms of level $\Gamma_1(N)$ (resp. of level $(\Gamma_1(N) \cap \Gamma_0(p))$, as sections of an invertible sheaf on these spaces. The "Hasse invariant" is an example of a mod-p-modular form which has weight $p-1$. The curve $\bar{\mathcal{X}}_{\Gamma,p}(N)$ has the 'supersingular locus' and the 'ordinary locus', and the Hasse invariant vanishes precisely on the supersingular locus. One then considers the inverse image of the complement of the supersingular locus in $\bar{\mathcal{X}}_{\Gamma,p}(N)$ under the reduction map $h : \mathcal{X}_{\Gamma,p}(N) \to \bar{\mathcal{X}}_{\Gamma,p}(N)$. This is not an algebraic variety over $\mathbb{Q}_p$ but is a rigid analytic p-adic variety. There is a p-adic rigid analytic invertible sheaf on this rigid analytic variety and the p-adic modular forms of weight k are defined to be the global sections of the k-fold tensor product of this sheaf. Omitting the supersingular points amounts to removing small open disks ("'supersingular disks") on the classical modular curve over $\mathbb{Z}_p$. It turns out to be interesting to study the sections that can be extended to small contiguous areas of these disks, and translates to imposing conditions on the growth of the coefficients of Laurent series expansions associated to these sections. Such forms are called overconvergent modular forms. For an excellent introductory survey on the p-adic geometry of modular curves see [E1]

For $R = \mathbb{C}_p$ and $N = 1$, Katz's algebraic modular forms of weight k over $\mathbb{C}_p$ may be viewed as a rule that attaches to a pair (E, ω) consisting of an elliptic curve E and a non-zero regular differential ω on E defined over $\mathbb{C}_p$ a number $f(E, \omega) \in \mathbb{C}_p$ depending only on the isomorphism class of the pair (E, ω), such that for all $\lambda \in \mathbb{C}^*$ we have $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$, and which behaves well in families. We usually refer to such modular forms as classical. A p-adic modular form of weight k on the other hand is such a rule defined only on those pairs (E, ω) for which E has ordinary reduction. Thus, any classical modular form gives rise to a p-adic modular form, but the converse is far from true in the sense that there are many p-adic modular forms which are not classical. For instance, the "weight 2 Eisenstein series of level 1" given by the q-expansion P in (10) is not a classical modular form, but is a q-expansion of a p-adic modular form for every p (see [K1]). It is worth pointing out that over a p-adically complete and separated $\mathbb{Z}_p$-algebra, Serre's definition of p-adic modular forms central to this article and the alluded Katz's algebraic definition of p-adic modular forms from [K1] amount to the same space (see Proposition A1.6 in [K2]). The space of all p-adic modular forms of a fixed weight k is an infinite dimensional p-adic Banach space lacking a good theory of Hecke eigenforms. To deal with this issue, one needs to consider its refinement, namely the space of *overconvergent* modular forms. The q-expansion P in (10), while a p-adic modular form for every p, is not overconvergent for any prime p (see [CGJ]).

The idea behind constructing families of modular forms is the following: Given an eigenform f for $\Gamma_1(Np)$, find a p-adic family of eigenforms passing through f. This amounts to asking for eigenforms (of possibly different level, but with weights congruent to that of f) which p-adically converge to f. Hida's theory solves this problem for ordinary modular forms. Coleman's study of the theory of overconvergent modular forms ([C]) and the theory of the eigencurve solve this for an eigenform f of positive slope (which means that the p-adic valuation of the p-th Fourier coefficient a_p is positive).

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