Feb 4th. Shen-Ning. Rigid Analytic Geometry.

§ Tate Algebra

Let k be a non-arch. local field complete (with arch. valuation)

$\rightsquigarrow k[\![x_1, \dots, x_n]\!]$.

(Tate Algebra) $T_n(k) = k\langle x_1, \dots, x_n\rangle := \{ \sum_{v_1,\dots,v_n \geq 0} a_{v_1,\dots,v_n} x_1^{v_1} \cdots x_n^{v_n} \mid$ $a_{v_1,\dots,v_n} \in k$ s.t. $|a_{v_1,\dots,v_n}| \to 0$ as $\sum v_i \to \infty$ $\}$.

- $|f| := \max_v |a_v|$ Gauss norm.
- $T_n(k)$ is complete k-Banach space (complete k-vector space) wrt $|f|$, containing $k[\![x_1, \dots, x_n]\!]$ as a dense subset.
- $\mathring{T}_n := \{ f \in T_n \mid |f| \leq 1 \}$ ring of integer.

  $\cup$
- $\check{T}_n := \{ f \in T_n \mid |f| < 1 \}$ maximal ideal.

$\rightsquigarrow \tilde{T}_n := \mathring{T}_n / \check{T}_n$ residue field.

$\simeq k[x_1, \dots, x_n]$.

Prop: $f \in T_n$, $|f| = 1$, is a unit iff $|f(0)| = 1$. and $|f - f(0)| < 1$.

$\Leftrightarrow$ f is a unit iff $\tilde{f}$ is a unit.

reduction in residue field.

Theorem: Every k-algebra homomorphism $\phi : T_n \to T_m$ is a contraction. i.e. $|\phi(f)| \le |f|$ $\forall f \in T_n$.

Pf: Suppose not. $\exists f \in T_n$ s.t. $|\phi(f)| > |f|$.

Assume $|\phi(f)| = 1$ (by multiply by a constant).
choose $c \in k$ s.t. $|c| = 1$. $c + \phi(f)$ is not a unit in T_m.

Define $g := c + f$ is a unit in T_n
$\because$ $|g| = 1$
$|g - g(0)| = |f - f(0)| < 1$.

$\Rightarrow$ $\phi(g) = c + \phi(f)$ is a unit. contradiction!
(image of a unit is a unit).

$\Rightarrow |\phi(f)| \le |f|$ $\square$.

K: valuation field.

$\rightarrow \mathbb{B}^n(K) := \{(x_1, \dots, x_n) \in K^n \mid \max|x_v| \le 1\}$ unit ball

$f \in T_n$ induces a map $\mathbb{B}^n(\bar{k}) \to \bar{k}$ via evaluation
$(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$.

Prop: The image is exactly those functions $f : \mathbb{B}^n(\bar{k}) \to \bar{k}$ such that
(1) Have a power series expansion over $\bar{k}$ converges on $\mathbb{B}^n(\bar{k})$
(2) Maps $\mathbb{B}^n(k)$ to k.

- If $L \subset \bar{k}$, finite, algebraic extension of K, then $f(\mathbb{B}^n(L)) \subseteq L$.

- Define sup norm $\sup_{x \in \mathbb{B}^n(\bar{k})} f(x) \leq |f|$.

Prop: (Maximal modulo Principle).
$f \in T_n$, $\exists\, x \in \mathbb{B}^n(\bar{k})$ s.t. $|f(x)| = |f|$.

Pf: WLOG, Assume $|f| = 1 \Rightarrow f \neq 0$.
$\Rightarrow \exists\ (x_1, \dots, x_n) \in \mathbb{B}^n(\bar{k})$ s.t. $f(x_1, \dots, x_n) \neq 0$.
$\Rightarrow |f(x_1, \dots, x_n)| = 1$
$\square$.

Cor. If $f \in T_n$ vanish for all points on the unit ball $\mathbb{B}^n(\bar{k})$ then $f \equiv 0$.
$\rightsquigarrow T_n \underset{\text{injection}}{\hookrightarrow} \{\text{maps}: \mathbb{B}^n(\bar{k}) \longrightarrow \bar{k}\}$

- Gauss norm coinsides with supnorm.
- 1-1 correspondence:

$$\mathbb{B}^n(\bar{k}) \overset{1-1}{\longleftrightarrow} \mathrm{Hom}_k(T_n, \bar{k}) \overset{1-1}{\longleftrightarrow} \underbrace{\mathrm{Max}_k\, T_n}_{\text{maximal ideal}}.$$

Def^n: $g = \sum_{v=0}^{\infty} g_v(x_1, \dots, x_{n-1}) x_n^v \in T_n$ is x_n-distinguished at degree s iff
- g_s is a unit in T_{n-1}
- $|g_s| = |g|$, $|g_s| > |g_v|$ $\forall v > s$

Thm (Weierstrass Division Theorem)
Let $g \in T_n$ be x_n-distinguished of deg s.
Then $\forall f \in T_n$, $\exists!$ $q \in T_n$, $r \in T_n[x_n]$ with
$\deg r < s$, s.t. $f = qg + r$.

$\Rightarrow$ $|f| = \max\{|q||g|, |r|\}$.

Thm (Weierstrass Preparation Theorem).
Let $g \in T_n$, x_n-distinguished of deg s, $\exists!$
monic polynomial $\omega \in T_{n-1}[x_n]$ of degree s
and a unit $e \in T_n$ s.t. $g = \omega \cdot e$ where ω
is x_n-distinguished and $|\omega| = 1$.

Pf: Apply WDT to x_n^s

$$x_n^s = qg + r \qquad q \in T_n,\ r \in T_{n-1}[x_n]$$
$$\deg r < s.$$
$$|r| \le 1.$$

$\omega = x_n^s - r$.

$\Rightarrow$ $|\omega| = 1$ x_n-distinguished of order/degree s.

claim: $\underline{q \text{ is a unit.}}$ $\Rightarrow$ $\omega = x_n^s - r = qg$
$\downarrow$ $\quad \exists e$ s.t. $\omega \cdot e = g$.
why? $\square$.

Cor: (Noether Normalization).
For any proper ideal $a \subseteq T_n$, there is a k-alg
mono. $T_d \to T_n$ for some $d \in \mathbb{N}$ s.t.
$$T_d \hookrightarrow T_n \twoheadrightarrow T_n/a$$
is finite.
- d = krull dim. of T_n/a

Cor: Let $m \subseteq T_n$ maximal ideal, then T_n/m is a finite extension over k.

- $\mathbb{B}^n(\bar{k}) \longrightarrow \mathrm{Max}\, T_n$ (maximal ideal of T_n)
  $x \longmapsto m_x = \{f \in T_n \mid f(x) = 0\}$.
  This map is surjective.

- T_n is noetherian.

- T_n is factorial ($\Rightarrow$ normal)

- T_n is jacobson.
  $\hookrightarrow$ $a \trianglelefteq T_n$ ideal $a = \underbrace{\bigcap_{\substack{m \supseteq a \\ \text{maximal}}} m}$
  intersection of all maximal ideals containing a.

- $\forall\, m \subset T_n$, maximal ideal is of height n.
  & can be generated by n elements.
  ($\Rightarrow$ krull dim. of $T_n = n$).

§ Affinoid Algebra.

$$T_n \longleftrightarrow \{\mathbb{B}^n(\bar{k}) \longrightarrow \bar{k}\}.$$

$$\triangle$$

$$a \rightsquigarrow V(a) = \{x \in \mathbb{B}^n(\bar{k}) \mid f(x) = 0 \ \ \forall f \in a\}.$$

$A := T_n/a$. functions on $V(a)$.

Def^n: A k-alg A is called an affinoid k-algebra if $\exists\, \alpha : T_n \twoheadrightarrow A$ for some n.

⇝ Define category of affinoid algebra
morphism = k-morphism

- Product: complete tensor product (tensor prod. ⇝ take completion)

- Noetherian, Jacobson, Noether normal.

- suppose $\alpha: T_n \twoheadrightarrow A$ then it induces a norm, called residue norm:

$$|\alpha(f)|_\alpha := \inf_{a \in \ker \alpha} |f - a|$$

- sup norm (semi-norm) $\quad |f|_{sup} = \sup_{x \in Max(A)} |f(x)|$.

- $\varphi: B \to A$ morphism between affinoid algebras.
$\Rightarrow |\varphi(b)|_{sup} \le |b|_{sup} \quad \forall b \in B$.

- $f \in A$, $|f|_{sup} \le |f|_\alpha$

- $|f|_{sup} = 0 \iff f$ is nilpotent.

Thm. (Maximal Principle).
A - k-affinoid algebra $\Rightarrow \exists\ x \in Max\,A$ s.t. $|f(x)| = |f|_{sup}$.
$\ni f$

Remark: (1) All residue norms are equivalent.
(2) $|\cdot|_{sup}$ is equivalent to residue norm for reduced affinoid algebra.
↑
use def^n of sup norm
⇒ sup norm is a norm
⇒ ring is reduced

§ Affinoid space

A = k-affinoid alg.

$\{\text{elt. of } A\} \longleftrightarrow \{\text{fcns on Max } A\}$

$f \longmapsto \bar{f} \in A/m \hookrightarrow \bar{k}$

$\rightsquigarrow \mathrm{Sp}\, A = (\underset{\text{space}}{\underset{\Downarrow}{\mathrm{Max}\, A}}, \underset{\text{fcn. on space}}{\underset{\Downarrow}{A}})$

is called affinoid k-space associated to A.

- Zariski Topology. $a \triangleleft A \rightarrow V(a)$ ↳ closed.
- $D_f = \{x \in \mathrm{Sp}\, A \mid f(x) \neq 0\}$ $f \in A$. forms a basis of Zariski open subsets of $\mathrm{Sp}A$.
- $Y \subset \mathrm{Sp}(A) \rightsquigarrow \underset{\text{ideal}}{\mathrm{id}(Y)} = \{f \in A \mid f(y) = 0 \ \forall y \in Y\} = \bigcap_{y \in Y} M_y$
- $V(\mathrm{id}(Y)) = \bar{Y}$

Thm. (Hilberts Nullstellensatz)
Let A = affinoid k-alg. $a \triangleleft A$.
$\Rightarrow \mathrm{id}(V(a)) = \mathrm{rad}\, a$ ↳ radical of a

(on affinoid case we have valuations

$\sigma: B \rightarrow A$ affinoid k-alg. homomorphism

$\rightsquigarrow \ {}^a\sigma: \mathrm{Sp}\, A \rightarrow \mathrm{Sp}\, B$

$m \longmapsto \sigma^{-1}(m)$

morphism of affinoid k-space.

Prop: For two affinoid k-spaces X, Y over Z, the fibre product $X \times_Z Y$ exists as affinoid k-space.

$$\begin{tikzcd} & X\times_Z Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

§ Affinoid subdomain.

$X = \mathrm{Sp}\, A$ — affinoid k-space.

$\rightsquigarrow$ Canonical Topology : top generated by

$X(f, \varepsilon) = \{ x \in X \mid |f(x)| \leq \varepsilon \}$. $\quad \varepsilon \in \mathbb{R}^{>0}$

$\hookrightarrow$ basis of open subsets.

Notation: $X(f) := X(f, 1)$

$X(f_1, \ldots, f_r) := X(f_1) \cap \cdots \cap X(f_r)$

Prop: The following sets are open wrt canonical topology:

Fix a function f:

$\{ x \in \mathrm{Sp}\, A \mid f(x) \neq 0 \}$

$\{ x \in \mathrm{Sp}\, A \mid |f(x)| \leq \varepsilon \}$ $\qquad$ need lemma from

$\{ x \in \mathrm{Sp}\, A \mid |f(x)| = \varepsilon \}$ $\qquad$ (Bosch)

$\{ x \in \mathrm{Sp}\, A \mid |f(x)| \geq \varepsilon \}$

Def$^{\underline{n}}$: $X = \mathrm{Sp}\, A$

(1) $X(f_1, \ldots, f_r) = \{ x \in X \mid |f_i(x)| \leq 1 \}$

Weierstrass domain.

(2) $X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \{ x \in X \mid |f_i(x)| \leq 1, \ |g_j(x)| \geq 1 \}$

Laurent domain.

(3) $X\left(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}\right) = \{ x \in X \mid |f_i(x)| \leq |f_0(x)| \}$

s.t. $f_0, \ldots, f_r \in A$ without common zero.

Rational Domain.

— All the above 3 are open (by definition)

Def^n: Let $X = Sp\,A$. A subset $U \subset X$ is called an affinoid subdomain of X if there $\exists$ a morphism of affinoid k-spaces $i : X' \longrightarrow X$ s.t.

(1) $i(X') \subset U$

(2) Universal Property:
Any morphism of k-spaces $\varphi : Y \to X$ with $\varphi(Y) \subset U$ admits a unique factorization through $X' \xrightarrow{i} X$ via $\varphi' : Y \longrightarrow X'$.

$\Rightarrow$ (1) $i : X' \longrightarrow X$ is a injection and satisfies $i(X') = U$.

$\Rightarrow X' \xrightarrow{\sim} U$ bijective as sets.

(2) For all $x \in X'$, $n \in \mathbb{N}$.

$$A/m^n_{i(x)} \xrightarrow{\sim} A'/m^n_x \qquad \begin{pmatrix} X' = Sp(A') \\ X = Sp(A) \end{pmatrix}$$

(3). $x \in X'$, $m_x = m_{i(x)} \cdot A'$.

Prop: Weierstrass, Laurant & Rational Domain in X are examples of open affinoid subdomains.
↳ wrt canonical topology.

$$\underset{\substack{\| \\ SpA'}}{i : X'} \longrightarrow \underset{\substack{\| \\ SpA}}{X} \rightsquigarrow i^* : A \longrightarrow A'$$

WD : $i^* : A \longrightarrow A\langle f\rangle := A\langle x_1, \dots, x_r\rangle / (x_1 - f_1, \dots, x_r - f_r)$

LD : $i^* : A \longrightarrow A\langle f, g^{-1}\rangle := \dfrac{A\langle x_1, \dots, x_r, y_1, \dots, y_s\rangle}{(x_1 - f_1, \dots, x_r - f_r, 1 - y_1 g_1 \dots)}$

RD : $i^* : A \longrightarrow A\langle \frac{f}{f_0}\rangle$.

- $V \subset X$ affinoid subdomain
  $U \subset V$ affinoid sub
  $\Rightarrow U \subset X$ affinoid subdomain of X.

- $\varphi : Y \to X$ morphism of affinoid k-spaces.
  $X' \hookrightarrow X$ affinoid subdomain.

$\Rightarrow$ $Y' = \varphi^{-1}(X')$ is an affinoid subdomain of X.

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & X' \arrow[d] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

- $\varphi^{-1}(X(f)) = Y(\varphi^*(f))$ WD $\xrightarrow[\text{back}]{\text{pull}}$ WD

- $\varphi^{-1}(X(f, g^{-1})) = Y(\varphi^*(f), \varphi^*(g)^{-1})$

- $\varphi^{-1}(X(\frac{h}{h_0})) = Y\left(\frac{\varphi^*(h)}{\varphi^*(h_0)}\right)$

- If U, V are two affinoid subdomain of X, then $U \cap V$ also affinoid subdomain of X.

Prop: Let $U \to X$ morphism of affinoid k-spaces. define U as affinoid subdomain of X. Then
- U is open in X
- The canonical topology of X restricted to U is the same as the canonical top. of U.

($\Rightarrow$ Affinoid subdomain forms a basis for the canonical top.)

Thm: (Gerriten - Granert).
Let X = affinoid k-space $U \subset X$ affinoid subdomain
then U is a finite union of rational domain.