BACHET'S DUPLICATION FORMULA

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1. Introduction.

The purpose of this note is to derive the so-called Bachet's duplication formula. Let c be an integer. Consider the equation

$$y^2 - x^3 = c$$

We are interested in finding rational solutions to this diophantine equation; that is, we would like to find all pairs $(x, y) \in \mathbb{Q}^2$ such that $y^2 - x^3 = c$ holds. Of course, the existence and the number of solutions will depend on c. However, there is a striking result, discovered by Bachet in 1621 that states that if (x, y) is a rational solution to $y^2 - x^3 = c$, then

$$\left(\frac{x^4 - 8cx}{4y^2}, \frac{-x^6 - 20cx^3 + 8c^2}{8y^3}\right)$$

is another rational solution to $y^2 - x^3 = c$. This is referred to as *Bachet's duplication formula.* While at first glance this may seem like a fortunate coincidence in algebra, there is a strong geometrical interpretation of the duplication formula. In fact, the geometric insight leads to the proof.

2. Proof of Bachet's Duplication Formula.

Suppose $P = (x_0, y_0)$ is a rational point on $y^2 - x^3 = c$. We construct tangent line to this cubic curve at P, and this line will intersect the curve at one more point Q.

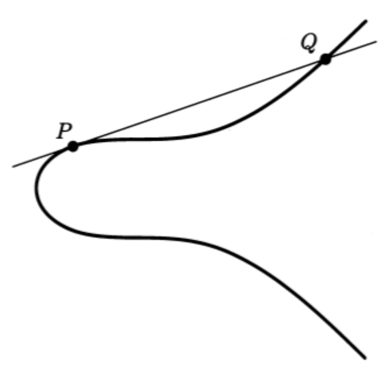

Figure 1. The tangent line through $P = (x_0, y_0)$

To find the coordinates of the point of $Q = (x_1, y_1)$, we first parametrize the line joining P and Q. The slope of this line is the derivative dy/dx at (x_0, y_0). We have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-3x^2}{2y} = \frac{3x^2}{2y}$$

by Implicit Function Theorem applied to $F(x,y) = y^2 - x^3$. Thus, the equation of the tangent line through P is:

$$y - y_0 = \frac{3x_0^2}{2y_0}(x - x_0) \quad \Longrightarrow \quad y = \frac{3x_0^2}{2y_0}(x - x_0) + y_0$$

To find the intersection points of the cubic curve and the line, we substitute the expression above for y in $y^2 - x^3 = c$ to get:

$$\left(\frac{3x_0^2}{2y_0}(x - x_0) + y_0\right)^2 - x^3 = c$$

which can be rearranged to look like this:

$$x^3 - \frac{9x_0^4}{4y_0^2}x^2 + Mx + N = 0$$

where M and N are some constants, which we do not need to calculate explicitly. The key point is that x_0 is a double root of the above cubic equation. But the x-coordinate of Q, namely x_1 must also satisfy the above equation. Now the sum of the roots of a cubic equation is the negative of the coefficient in front of x^2. So,

$$x_1 + x_0 + x_0 = \frac{9x_0^4}{4y_0^2} \quad \Longrightarrow \quad x_1 = \frac{9x_0^4}{4y_0^2} - 2x_0 = \frac{9x_0^4 - 8x_0y_0^2}{4y_0^2}$$

Using $y_0^2 - x_0^3 = c$, we can rewrite x_1 as

$$x_1 = \frac{x_0^4 + 8x_0^4 - 8x_0y_0^2}{4y_0^2} = \frac{x_0^4 + 8x_0(x_0^3 - y_0^2)}{4y_0^2} = \frac{x_0^4 - 8cx_0}{4y_0^2} \quad \Longrightarrow \quad \boxed{x_1 = \frac{x_0^4 - 8cx_0}{4y_0^2}}$$

We can now compute the y-coordinate of Q using the equation of the tangent line:

$$\begin{aligned}
y_1 &= \frac{3x_0^2}{2y_0}(x_1 - x_0) + y_0 = \frac{3x_0^2}{2y_0}\left(\frac{x_0^4 - 8cx_0}{4y_0^2} - x_0\right) + y_0 \\
&= \frac{3x_0^6 - 24cx_0^3 - 12x_0^3y_0^2 + 8y_0^4}{8y_0^3} \\
&= \frac{-x_0^6 - 24cx_0^3 + (4x_0^3y_0^2 - 4x_0^6) + (8x_0^6 - 16x_0^3y_0^2 + 8y_0^4)}{8y_0^3} \\
&= \frac{-x_0^6 - 24cx_0^3 + 4x_0^3(y_0^2 - x_0^3) + 8(y_0^2 - x_0^3)^2}{8y_0^3} \\
&= \frac{-x_0^6 - 24cx_0^3 + 4cx_0^3 + 8c^2}{8y_0^4} = \frac{-x_0^6 - 20cx_0^3 + 8c^2}{8y_0^3} \\
&\therefore \boxed{y = \frac{-x_0^6 - 20cx_0^3 + 8c^2}{8y_0^3}}
\end{aligned}$$

We finally conclude that

$$Q = (x_1, y_1) = \left(\frac{x_0^4 - 8cx_0}{4y_0^2}, \frac{-x_0^6 - 20cx_0^3 + 8c^2}{8y_0^3}\right)$$