FONTAINE RINGS AND p-ADIC L-FUNCTIONS

Contents

1. One variable p-adic functions

In this section, we denote L a closed subfield of $\mathbf{C}_p$.

1.1. **Functions on $\mathbf{Z}_p$.** Let $\mathscr{C}^0(\mathbf{Z}_p, L)$ be the space of continuous function from $\mathbf{Z}_p$ to L. Since $\mathbf{Z}_p$ is compact, every continuous function on $\mathbf{Z}_p$ is bounded. This allows us to define a valuation $v_{\mathscr{C}^0}$ on $\mathscr{C}^0(\mathbf{Z}_p, L)$ by $v_{\mathscr{C}^0}(\phi) = \inf_{x\in\mathbf{Z}_p}(\phi(x))$, which makes $\mathscr{C}^0(\mathbf{Z}_p, L)$ a L-Banach space.

If $n \in \mathbf{N}$, let $\binom{x}{n}$ be the polynomial defined by

$$\binom{x}{n} = \begin{cases} 1 & \text{if } n = 0, \\ \frac{x(x-1)\cdots(x-n+1)}{n!} & \text{if } n \geq 1. \end{cases}$$

Theorem 1.1. *(Mahler) [REF] $\{\binom{x}{n}, n\in \mathbf{N}\}$ forms a Banach basis of $\mathscr{C}^0(\mathbf{Z}_p, L)$.*

If $h \in \mathbf{N}$, let $\mathrm{LA}_h(\mathbf{Z}_p, L)$ be the space of functions from $\mathbf{Z}_p$ to L which is analytic on $a + p^h\mathbf{Z}_p$ for all $a \in \mathbf{Z}_p$, that is, if $\phi \in \mathrm{LA}_h(\mathbf{Z}_p, L)$, $x_0 \in \mathbf{Z}_p$ and $x \in x_0 + p^h\mathbf{Z}_p$, then ϕ can be written as the form

$$\phi(x) = \sum_{k=0}^{\infty} a_k(\phi, x_0)(x - x_0)^k,$$

where $a_k(x_0, \phi)$ is a sequence in L such that $\nu_p(a_k(\phi, x_0)) + kh$ tends to $+\infty$ as k tends to $+\infty$. We endow $\mathrm{LA}_h(\mathbf{Z}_p, L)$ a valuation v_{LA_h} defined by

$$v_{\mathrm{LA}_h}(\phi) = \inf_{x_0\in\mathbf{Z}_p} \inf_{k\in\mathbf{N}} \nu_p(a_k(\phi, x_0)) + kh,$$

which makes $\mathrm{LA}_h(\mathbf{Z}_p, L)$ a L-Banach space. One can show that $v_{\mathrm{LA}_h}(\phi) = \inf\limits_{a\in S} \inf\limits_{k\in\mathbf{N}} \nu_p(\phi, a_k(a)) + kh$ where S is a representative of $\mathbf{Z}_p/p^h\mathbf{Z}_p$.

We denote $\mathrm{LA}(\mathbf{Z}_p, L)$ the space of locally analytic functions on $\mathbf{Z}_p$. Since $\mathbf{Z}_p$ is compact, it is a inductive limit of $\mathrm{LA}_h(\mathbf{Z}_p, L)$, $h \in \mathbf{N}$, and we endow it with the inductive limit topology.

Theorem 1.2. *(Amice)* $\{[\frac{n}{p^h}]!\binom{x}{n}, n \in \mathbf{N}\}$ *forms a Banach basis of* $\mathrm{LA}_h(\mathbf{Z}_p, L)$.

Theorem 1.3. *The function* $\phi = \sum_{n=0}^{+\infty} a_n(\phi)\binom{x}{n} \in \mathscr{C}^0(\mathbf{Z}_p, L)$ *is in* $\mathrm{LA}(\mathbf{Z}_p, L)$ *if and only if* $\liminf \frac{1}{n}\nu_p(a_n(\phi)) > 0$.

$n \to \infty$

A function $\phi : \mathbf{Z}_p \to L$ is differentiable at $x_0 \in \mathbf{Z}_p$ if $\lim_{h \to 0} \frac{\phi(x_0+h)-\phi(x_0)}{h}$ exists. The limit is denoted by $\phi'(x_0)$. A function is said to be differentiable of order 1 if it is differentiable at all $x_0 \in \mathbf{Z}_p$. We say that a function is differentiable of order k if its differentiation is of order $k-1$.

If $r \geq 0$, we say that $\phi : \mathbf{Z}_p \to L$ is of class $\mathscr{C}^r$ if there exists functions $\phi^{(j)} : \mathbf{Z}_p \to L$ for $0 \leq j \leq [r]$, such that, if we define $\varepsilon_{\phi,r} : \mathbf{Z}_p \times \mathbf{Z}_p \to L$ and $C_{\phi,r} : \mathbf{N} \to \mathbb{R} \cup \{+\infty\}$ by

$$\varepsilon_{\phi,r}(x,y) = \phi(x+y) - \sum_{j=0}^{[r]} \phi^{(j)}(x)\frac{y^j}{j!} \quad \text{and} \quad C_{\phi,r}(h) = \inf_{x \in \mathbf{Z}_p, y \in p^h\mathbf{Z}_p} \nu_p(\varepsilon_{\phi,r}(x,y)) - rh,$$

then $C_{\phi,r}(h)$ tends to $+\infty$ as h tends to $+\infty$.

We denote $\mathscr{C}^r(\mathbf{Z}_p, L)$ the set of functions $\phi : \mathbf{Z}_p \to L$ of class $\mathscr{C}^r$. We endow $\mathscr{C}^r(\mathbf{Z}_p, L)$ the valuation $v_{\mathscr{C}^r}$ defined by

$$v_{\mathscr{C}^r}(\phi) = \inf\left(\inf_{0 \leq j \leq [r],\, x \in \mathbf{Z}_p} \nu_p\Big(\frac{\phi^{(j)}(x)}{j!}\Big), \inf_{x,y \in \mathbf{Z}_p} \nu_p\big(\varepsilon_{\phi,r}(x,y) - r\nu_p(y)\big)\right),$$

which makes it a L-Banach space.

Proposition 1.4. *If* $h \in \mathbf{N}$, *and if* $r \geq 0$, *then* $\mathrm{LA}_h(\mathbf{Z}_p, L) \subset \mathscr{C}^r(\mathbf{Z}_p, L)$. *Moreover, if* $\phi \in \mathrm{LA}_h(\mathbf{Z}_p, L)$, *then*

$$v_{\mathscr{C}^r}(\phi) \geq v_{\mathrm{LA}_h}(\phi) - rh.$$

Proof. See [Col10, proposition I.5.7]. □

If $i \in \mathbf{N}$, we denote $l(i)$ the least integer n such that $p^n > i$, then we ~~We~~ have

$$l(0) = 0 \quad \text{and} \quad l(i) = [\frac{\log i}{\log p}] + 1, \text{ if } i \geq 1.$$

Theorem 1.5. *(Mahler) The function* $\phi = \sum_{n=0}^{+\infty} a_n(\phi)\binom{x}{n} \in \mathscr{C}^0(\mathbf{Z}_p, L)$ *is in* $\mathscr{C}^r(\mathbf{Z}_p, L)$, $r \geq 0$ *if and only if* $\nu_p(a_n(\phi)) - rl(n) \to +\infty$ *as* $n \to +\infty$. *Moreover, the valuation* $v'_{\mathscr{C}^r}$ *defined on* $\mathscr{C}^r(\mathbf{Z}_p, L)$ *by the formula*

$$v'_{\mathscr{C}^r}(\phi) = \inf_{n \in \mathbf{N}}\Big(\nu_p(a_n(\phi)) - rl(n)\Big)$$

is equivalent to the valuation $v_{\mathscr{C}^r}$.

Proof. See [Col10, proposition I.5.18]. □

Corollary 1.6. $p^{[rl(n)]}\binom{x}{n}$, $n \in \mathbf{N}$ *forms a Banach basis of* $\mathscr{C}^r(\mathbf{Z}_p, L)$.

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1.2. **Distributions on $\mathbf{Z}_p$.** A continuous distribution on $\mathbf{Z}_p$ is a continuous linear function on $\mathrm{LA}(\mathbf{Z}_p, L)$, that is, a linear function on $\mathrm{LA}(\mathbf{Z}_p, L)$ whose restriction to $\mathrm{LA}_h(\mathbf{Z}_p, L)$ is continuous. We denote $\mathscr{D}(\mathbf{Z}_p, L)$ the set of continuous distributions on $\mathbf{Z}_p$ with values in L and endow $\mathscr{D}(\mathbf{Z}_p, L)$ with the Fréchet topology defined by the family of valuations v_{LA_h}, $h \in \mathbf{N}$. which take values in .

For a continuous distribution μ, we associate it with the formal series:

$$\mathscr{A}_\mu(T) = \sum_{n=0}^{+\infty} \int_{\mathbf{Z}_p} (1+T)^x \mu = \sum_{n=0}^{+\infty} T^n \int_{\mathbf{Z}_p} \binom{x}{n} \mu,$$

which is called the Amice transform of μ.

Lemma 1.7. *If $\mu \in \mathscr{D}(\mathbf{Z}_p, L)$ and if $\nu_p(x) > 0$, then $\int_{\mathbf{Z}_p} (1+z)^x \mu(x) = \mathscr{A}_\mu(z)$.*

Let $\mathscr{R}^+$ be the ring of power series $f = \sum_{n=0}^{\infty} a_n T^n$ with coefficients in L, which is convergent if $\nu_p(T) > 0$.

that We say an element $f = \sum_{n=0}^{\infty} a_n T^n \in \mathscr{R}^+$ is of order r if $\nu_p(a_n) + rl(n)$ is bounded. We denote $\mathscr{R}_h^+$ the subset of $\mathscr{R}^+$ of elements of order r, and we endow $\mathscr{R}_r^+$ the valuation v_r defined by $v_r(f) = \inf_{n \in \mathbf{N}} \nu_p(a_n) + rl(n)$, which makes it a L-Banach space. We endow $\mathscr{R}^+$ the Fréchet topology defined by the family of valuations v_r.

Theorem 1.8. *The map $\mu \mapsto \mathscr{A}_u$ is an isomorphism of* Fréchet *space from $\mathscr{D}(\mathbf{Z}_p, L)$ to $\mathscr{R}^+$.*

Proof. See [Col10, Theorem II.2.2]. ☐

If $r \geq 0$, we say a continuous distribution μ on $\mathbf{Z}_p$ is of order r if it can be extended by continuity to $\mathscr{C}^r$. We denote $\mathscr{D}_r(\mathbf{Z}_p, L)$ the set of distributions of order r, which is equipped with a valuation $v_{\mathscr{D}_r}$ defined by

$$v_{\mathscr{D}_r}(\mu) = \inf_{f \in \mathscr{C}^r(\mathbf{Z}_p, L) - \{0\}} \left(\nu_p\Big(\int_{\mathbf{Z}_p} f\mu\Big) - v_{\mathscr{C}^r}(f) \right),$$

which gives $\mathscr{D}_r(\mathbf{Z}_p, L)$ the dual topology of $\mathscr{C}^r(\mathbf{Z}_p, L)$.

A distribution is said to be tempered if there exist $r \in \mathbb{R}^+$ such that it is of order r. We denote $\mathscr{D}_{temp}(\mathbf{Z}_p, L)$ the space of tempered distributions.

Proposition 1.9. *The map $\mu \mapsto \mathscr{A}_u$ induces an isometry from $\mathscr{D}_r(\mathbf{Z}_p, L)$ equipped with valuation $v_{\mathscr{D}_r}$ to $\mathscr{R}_r^+$ equipped with valuation v_r.* ?

A distribution of order 0 is called the measure. By definition, $\mathscr{D}_0(\mathbf{Z}_p, L)$ is the topological dual of the space of continuous functions. By proposition 1.9, we have a one-one correspondence from a measure to a power series of bounded coefficients.

1.3. **Operations on the distributions.**

1. Harr measure: $\mu(\mathbf{Z}_p) = 1$ and μ is invariant by translation. We must have $\mu(i + p^n\mathbf{Z}_p) = \frac{1}{p^n}$ which is not bounded. Hence there exists no Harr measure on $\mathbf{Z}_p$.
2. Dirac measure: For $a \in \mathbf{Z}_p$, we define δ_a the Dirac measure associated to $f(a)$. The Amice transform of δ_a is $\mathscr{A}_{\delta_a}(T) = (1+T)^a$.
3. Multiplication by a function: If μ is a distribution on $\mathbf{Z}_p$ and f is a locally analytic function on $\mathbf{Z}_p$, we define the distribution $f\mu$ by $\int_{\mathbf{Z}_p} \phi(f\mu) = \int_{\mathbf{Z}_p} (f\phi)\mu$.
   - Multiplication by x: We have $x \cdot \binom{x}{n} = ((x-n)+n)\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}$, and hence we have

$$\mathscr{A}_{x\mu}(T) = \partial \mathscr{A}_\mu \quad \text{where } \partial = (1+T)\frac{d}{dT}.$$

when

- Multiplication by z^x ~~if~~ $\nu_p(z-1)>0$: By lemma 1.7, if $\nu_p(y-1)>0$, and if λ is a continuous distribution on $\mathbf{Z}_p$, then $\int_{\mathbf{Z}_p} y^x\lambda(x)=\mathscr{A}_\lambda(y-1)$. Applying this to $\lambda=z^x\mu$, we obtain $\mathscr{A}_\lambda(y-1)=\mathscr{A}_\mu(yz-1)$. We ~~hence~~ have the formula
$$\mathscr{A}_{z^x\mu}(T)=\mathscr{A}_\mu((1+T)z-1).$$
then the

4. Restriciton to compact open subset: If X is a compact open subset of $\mathbf{Z}_p$, then characteristic function 1_X is continuous on $\mathbf{Z}_p$. If μ is a distribution on $\mathbf{Z}_p$, the measure $1_X\mu$ is the restriction of μ to X and is denoted by $\mathrm{Res}_X(\mu)$. In particular for $n\in\mathbf{N}$ and $a\in\mathbf{Z}_p$, we have $1_{a+p^n\mathbf{Z}_p}(x)=p^{-n}\sum_{z^{p^n}=1}z^{-a}z^x$, hence
$$\mathscr{A}_{\mathrm{Res}_{a+p^n\mathbf{Z}_p}(\mu)}(T)=p^{-n}\sum_{z^{p^n}=1}z^{-a}\mathscr{A}_\mu((1+T)z-1).$$
5. Derivation of distribution: If $\mu\in\mathscr{D}(\mathbf{Z}_p,L)$, we define $d\mu$ by
$$\int_{\mathbf{Z}_p}\phi(x)d\mu=\int_{\mathbf{Z}_p}\phi'(x)\mu,\quad\text{and therefore}\quad\mathscr{A}_{d\mu}(T)=\log(1+T)\cdot\mathscr{A}_\mu(T).$$
6. Actions of $\mathbf{Z}_p^*$, φ and ψ:
   - If $a\in\mathbf{Z}_p^*$, and if $\mu\in\mathscr{D}(\mathbf{Z}_p,L)$, we define $\sigma_a(\mu)\in\mathscr{D}(\mathbf{Z}_p,L)$ by
   $$\int_{\mathbf{Z}_p}\phi(x)\sigma_a(\mu)=\int_{\mathbf{Z}_p}\phi(ax)\mu,\text{ and therefore }\mathscr{A}_{\sigma_a(\mu)}(T)=\mathscr{A}_\mu((1+T)^a-1).$$
   - φ acts on distribution μ by
   $$\int_{\mathbf{Z}_p}\phi(x)\varphi(\mu)=\int_{\mathbf{Z}_p}\phi(px)\mu,\text{ and therefore }\mathscr{A}_{\varphi(\mu)}(T)=\mathscr{A}_\mu((1+T)^p-1).$$
   - If μ is a distribution on $\mathbf{Z}_p$, we denote $\psi(\mu)$ the distribution on $\mathbf{Z}_p$ defined by
   $$\int_{\mathbf{Z}_p}\phi(x)\psi(\mu)=\int_{p\mathbf{Z}_p}\phi(p^{-1}x)\mu\quad\text{and therefore}\quad\mathscr{A}_{\psi(\mu)}=\psi(\mathscr{A}_\mu),$$
   where $\psi:\mathscr{R}^+\to\mathscr{R}^+$ is defined by $\psi(F)((1+T)^p-1)=\frac{1}{p}\sum_{\zeta^p=1}F((1+T)\zeta-1)$.

   The action of $\mathbf{Z}_p^*$, φ and ψ satisfy the relations:
   (a) $\psi\circ\phi=$id.
   (b) $\psi\circ\sigma_a=\sigma_a\circ\psi$ and $\varphi\circ\sigma_a=\sigma_a\circ\psi$ if $a\in\mathbf{Z}_p^*$.
   (c) $\psi(\mathscr{A}_\mu)=0$ if and only if μ has support on $\mathbf{Z}_p^*$, and $\mathscr{A}_{\mathrm{Res}_{\mathbf{Z}_p^*}(\mu)}=(1-\varphi\psi)\mathscr{A}_\mu$.
7. Convolution of distribution: If λ and μ are two distributions on $\mathbf{Z}_p$, we define the convolution $\lambda*\mu$ by
$$\int_{\mathbf{Z}_p}\phi\cdot\lambda*\mu=\int_{\mathbf{Z}_p}\Big(\int_{\mathbf{Z}_p}\phi(x+y)\mu(x)\Big)\lambda(y).$$
Take $\phi(x)$ the function $x\mapsto z^x$, where $\nu_p(z-1)>0$, then we have $\mathscr{A}_{\lambda*\mu}(z)=\mathscr{A}_\lambda(z)\mathscr{A}_\mu(z)$. Hence we deduce $\mathscr{A}_{\lambda*\mu}=\mathscr{A}_\lambda\cdot\mathscr{A}_\mu$.

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2. Kubota-Leopoldt p-adic L-functions

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2.1. **Riemann zeta function.** Let $\zeta(s)=\sum_{n=1}^{+\infty}n^{-s}=\prod_{p:\boxed{prime}}(1-p^{-s})^{-1}$ the Riemann zeta function. Let $\Gamma(s)=\int_{t=0}^{+\infty}e^{-t}t^s\frac{dt}{t}$ the Gamma function, which is holomorphic on $Re(s)>0$ and satisfies the functional equation $\Gamma(s+1)=s\Gamma(s)$, ~~therefore~~ it can be extended to a meromorphic function on $\mathbb{C}$.

be and thus

Recall we have:

Lemma 2.1. *If $Re(s) > 1$, then*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1}{e^t - 1} t^s \frac{t}{dt}.$$

Proposition 2.2. *If f is a $\mathscr{C}^\infty$ function on $\mathbb{R}^+$ which decreases rapidly at infinite, then the function*

$$L(f, s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f(t) t^s \frac{dt}{t}$$

defined on $Re(s) > 0$ admits a holomorphic extension to $\mathbb{C}$ and if $n \in \mathbf{N}$, then $L(f, -n) = (-1)^n f^{(n)}(0)$.

Apply ~~the~~ P proposition 2.2 to $f_0(t) = \frac{t}{e^t - 1}$. Let $\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!}$ be the Taylor expension of f_0 at 0, where B_n ~~are~~ is the n th Bernoulli number. We have, in particular

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = \frac{-1}{30} \cdots.$$

Since $f_0(t) - f_0(-t) = -t$, we have $B_{2k+1} = 0$ if $k \geq 1$.

Theorem 2.3.

i) The function ζ has a meromorphic continuation to $\mathbb{C}$, which has a simple pole at $s = 1$ with residue 1.

FONT *ii) If $n \in \mathbf{Q}$* N *, then $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$, ~~In~~* and in *particular $\zeta(-n) \in \mathbf{Q}$.*

2.2. **Kummer congruence.** If $a \in \mathbb{R}_+^*$, by applying proposition 2.2 to the function $f_a(t) = \frac{1}{e^t - 1} - \frac{a}{e^{at} - 1}$, which is $\mathscr{C}^\infty$ (removed the pole at $t = 0$) on $\mathbb{R}^+$ and decreases rapidly at infinity, we have

Corollary 2.4. *If $a \in \mathbb{R}_+^*$, the function $(1 - a^{1-s})\zeta(s) = L(f_a, s)$ has an analytic continuation on $\mathbb{C}$, and if $n \in \mathbf{N}$, then $(1 - a^{1+n})\zeta(-n) = (-1)^n f_a^{(n)}(0)$. In particular, if $a \in \mathbf{Q}$, then $(1 - a^{1+n})\zeta(-n) \in \mathbf{Q}$.*

For a continuous distribution μ, we associate it with the formal series:

$$\mathscr{L}_\mu(t) = \sum_{n=0}^{+\infty} \int_{\mathbf{Z}_p} e^{tx} \mu = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \int_{\mathbf{Z}_p} x^n \mu,$$

which is called the Laplace transform of μ.

We have $\mathscr{L}_\mu(t) = \mathscr{A}_\mu(e^t - 1)$.

Proposition 2.5. *If $a \in \mathbf{Z}_p^*$, there exists a measure μ_a whose Laplace transform is $f_a(t)$. Moreover $v_{\mathscr{D}_0}(\mu_a) \geq 0$ and if $n \in \mathbf{N}$, then $\int_{\mathbf{Z}_p} x^n \mu_a = (-1)^n (1 - a^{1+n})\zeta(-n)$.*

Proof. To show the existance of μ_a, it suffices to prove the coefficients of series obtained by replace e^t by $1 + T$ (Amice transform of μ_a) is bounded by proposition 1.9. Since $(1 + T)^a - 1$ is of the form $aT(1 + Tg(T))$ where $g(T) = \sum_{n=2}^{+\infty} \frac{1}{a} \binom{a}{n} T^{n-2} \in \mathbf{Z}_p[[T]]$, we have

$$\frac{1}{T} - \frac{a}{(1 + T)^a - 1} = \sum_{n=1}^{+\infty} (-T)^{n-1} g^n \in \mathbf{Z}_p[[T]].$$

Since the coefficients are in $\mathbf{Z}_p$, we have $v_{\mathscr{D}_0}(\mu_a) \geq 0$. Moreover, we have $\int_{\mathbf{Z}_p} x^n \mu_a = \mathscr{L}_{\mu_a}^{(n)}(0) = f_a^{(n)}(0)$. □

Corollary 2.6. *(Kummer congruences) If $a \in \mathbf{Z}_p^*$ and $k \geq 1$, if n_1 and n_2 are two integers $\geq k$ such that $n_1 \equiv n_2 \bmod (p-1)p^{k-1}$, then*

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$$\nu_p\Big((1-a^{1+n_1})\zeta(-n_1) - (1-a^{1+n_2})\zeta(-n_2)\Big) \geq k.$$

Proof. ~~Since we~~ Suppose $n_1 \geq k$ and $n_2 \geq k$, we have $\nu_p(x^{n_1}) \geq k$ and $\nu_p(x^{n_2}) \geq k$ if $x \in p\mathbf{Z}_p$. On the other hand, since the order of $(\mathbf{Z}/p^k\mathbf{Z})^*$ is $(p-1)p^{k-1}$, and ~~we suppose~~ $n_1 \equiv n_2 \bmod (p-1)p^{k-1}$, we have $x^{n_1} - x^{n_2} \in p^k\mathbf{Z}_p$ if $x \in \mathbf{Z}_p^*$. To sum up, we have $\nu_p(x^{n_1} - x^{n_2}) \geq k$ if $x \in \mathbf{Z}_p$ and hence $v_{\mathscr{C}^0}(x^{n_1} - x^{n_2}) \geq k$. Since $v_{\mathscr{D}_0}(\mu_a) \geq 0$, which implies Too colloquial

by assumption

$$\nu_p((1-a^{1+n_1})\zeta(-n_1) - (1-a^{1+n_2})\zeta(-n_2)) = \nu_p(\int_{\mathbf{Z}_p}(x^{n_1} - x^{n_2})\mu_a(x)) \geq k.$$

□

Proposition 2.7. *If $a \in \mathbf{Z}_p^*$, then*

i) $\psi(\mu_a) = \mu_a$.
ii) $\mathrm{Res}_{\mathbf{Z}_p^*}(\mu_a) = (1-\varphi)\mu_a$
iii) $\int_{\mathbf{Z}_p^*} x^n\mu_a = (1-p^n)\int_{\mathbf{Z}_p} x^n\mu_a$ *for all $n \in \mathbf{N}$.*

Proof. Let $F(T) = \psi(\frac{1}{T})$. By definition, we have

i) →

$$\begin{aligned} F((1+T)^p - 1) =& \frac{1}{p}\sum_{\zeta^p=1}\frac{1}{(1+T)\zeta - 1} = \frac{-1}{p}\sum_{\zeta^p=1}\sum_{n=0}^{+\infty}((1+T)\zeta)^n \\ =& -\sum_{n=0}^{+\infty}(1+T)^{pm} = \frac{1}{(1+T)^p - 1}. \end{aligned}$$

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Hence we have $\psi(\frac{1}{T}) = \frac{1}{T}$. On the other hand, ~~we know that~~ since the Amice transform of μ_a is $\frac{1}{T} - \frac{a}{(1+T)^a - 1} = \frac{1}{T} - a\sigma_a(\frac{1}{T})$, ~~and~~ the action of ψ commutes with σ_a., and ~~By~~ $\psi(\mathscr{A}_\mu) = \mathscr{A}_{\psi(\mu)}$ if μ is a distribution, we ~~deduce~~ obtain i).

ii) follows from i) since we have $\mathrm{Res}_{\mathbf{Z}_p^*}(\mu) = (1-\varphi\psi)\mu$ if μ is a distribution. iii) follows ii) and $\int_{\mathbf{Z}_p} x^n\varphi(\mu) = \int_{\mathbf{Z}_p}(px)^n\mu$. □

Corollary 2.8. *Let $a \in \mathbf{N} - \{1\}$ be prime to p. Let $k \geq 1$. If n_1 and n_2 are two integers $\geq k$ such that $n_1 \equiv n_2 \bmod (p-1)p^{k-1}$, then*

$$\nu_p((1-a^{1+n_1})(1-p^{n_1})\zeta(-n_1) - (1-a^{1+n_2})(1-p^{n_2})\zeta(-n_2)) \geq k.$$

By corollary 2.8, ~~we know that~~ the function $n \mapsto (1-p^n)\zeta(-n)$ is continuous under p-adic topology. To have a uniform formula, we put $q = 4$ if $p = 2$ and $q = p$ if $p \neq 2$. We denote ϕ the Euler totient function, thus ~~we have~~ and $\phi(q) = 2$ if $q = 4$ and $\phi(q) = p - 1$ if $p \neq 2$. ← cases

Theorem 2.9. *If $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, there exist an unique function $\zeta_{p,i}$ continuous on $\mathbf{Z}_p$ (resp. $\mathbf{Z}_p - \{1\}$) if $i \neq 1$ (resp. $i = 1$) such that the function $(s-1)\zeta_{p,i}$ is analytic on $\mathbf{Z}_p$ (resp. $i + 2\mathbf{Z}_p$ if $p = 2$) and one has $\zeta_{p,i}(-n) = (1-p^n)\zeta(-n)$ if $n \in \mathbf{N}$ verified ? $-n \equiv i \bmod p - 1$.*

Remark 2.10. $\zeta_{p,i}$ is called the i-th branch of Kubota-Leopoldt zeta function. If i is even, then $\zeta_{p,i}$ is identically zero since $\zeta(-n) = 0$ if $n \geq 2$ is even.

2.3. **p-adic Mellin transform and Leopoldt's Γ-transform.** We denote Δ the group of roots of unity of $\mathbf{Q}_p^*$. Therefore Δ is a cyclic group of order $\phi(q)$ and $\mathbf{Z}_p^*$ is disjoint union of $\varepsilon + q\mathbf{Z}_p$ with $\varepsilon \in \Delta$. We denote $\omega : \mathbf{Z}_p \to \Delta \cup \{0\}$ the function defined by $\omega(x) = 0$ if $x \in p\mathbf{Z}_p$, and $x - \omega(x) \in q\mathbf{Z}_p$, if $x \in \mathbf{Z}_p^*$. If $x \in \mathbf{Z}_p^*$, we define $\langle x \rangle \in 1 + q\mathbf{Z}_p$ by $\langle x \rangle = x\omega(x)^{-1}$.

Proposition 2.11. *If $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, the function $x \mapsto \omega(x)^i\langle x\rangle^s$ is a locally analytic function on $\mathbf{Z}_p$. Moreover, we have*

i) $\omega(x)^i\langle x\rangle^n = x^n$ if $n \equiv i$ mod $\phi(q)$ and If $x \in \mathbf{Z}_p^*$

ii) $\omega(x)^i\langle x\rangle^s = \lim\limits_{\substack{n\to s\\ n\equiv i \bmod \phi(q)}} x^n$ for $x \in \mathbf{Z}_p$.

Proof. Note that we have $\omega(x)^i\langle x\rangle^s = 0$ on $p\mathbf{Z}_p$ and

$$\omega(x)^i\langle x\rangle^s = \varepsilon^i\left(\frac{x}{\varepsilon}\right)^s = \sum_{n=0}^{+\infty}\binom{s}{n}\varepsilon^{i-n}(x-\varepsilon)^n,$$

if $x \in \varepsilon + q\mathbf{Z}_p$ and $\varepsilon \in \Delta$, thus the function is locally analytic.

Since the order of Δ is $\phi(q)$, we have $\omega(x)^n = \omega(x)^i$ if $n \equiv i$ mod $\phi(q)$, i) and ii) follow~~s~~. $\square$

If $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, we defined the i-th branch of the Mellin transform of a continuous distribution μ by the formula

$$\mathrm{Mel}_{i,\mu}(s) = \int_{\mathbf{Z}_p}\omega(x)^i\langle x\rangle^s\mu(x) = \int_{\mathbf{Z}_p^*}\omega(x)^i\langle x\rangle^s\mu(x)$$

the second equality is because $\omega(x) = 0$ if $x \in p\mathbf{Z}_p$. On the other hand, we have $\mathrm{Mel}_{i,\mu}(n) = \int_{\mathbf{Z}_p^*} x^n\mu$ if $n \equiv i$ mod $\psi(q)$.

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Let u be a topological generator of multiplicative group of $1+q\mathbf{Z}_p$, and let $\theta : 1+q\mathbf{Z}_p \to \mathbf{Z}_p$ the homomorphism which sends x to $\frac{\log x}{\log u}$. This homomorphism is analytic and its inverse also.
it is
If f is a locally analytic function (resp. continuous) function on $1+q\mathbf{Z}_p$, the function θ^*f defined by $\theta^*\phi(x) = \phi(\theta(x))$ is locally analytic (resp. continuous) on $\mathbf{Z}_p$.

If μ is a distribution support on $1+q\mathbf{Z}_p$, we define a distribution $\theta_*\mu$ on $\mathbf{Z}_p$ by the formula

$$\int_{\mathbf{Z}_p}\phi(\theta_*\mu) = \int_{1+q\mathbf{Z}_p}(\theta^*\phi)\mu.$$

In particular, θ_* sends measure to measure.

Lemma 2.12. *If X is a open compact subset of $\mathbf{Z}_p$, If $\alpha \in \mathbf{Z}_p^*$, and if μ is a continuous distribution on $\mathbf{Z}_p$, then*

$$\mathrm{Res}_X(\sigma_\alpha(\mu)) = \sigma_\alpha(\mathrm{Res}_{\alpha^{-1}X}(\mu))$$

Proof. Since we have $1_X(\alpha x) = 1_{\alpha^{-1}X}(x)$ if $X \subset \mathbf{Z}_p$, we deduce the formula

$$\begin{aligned}\int_{\mathbf{Z}_p}\phi(x)\mathrm{Res}_X(\sigma_\alpha(\mu)) &= \int_{\mathbf{Z}_p}1_X(x)\phi(x)\sigma_\alpha\mu = \int_{\mathbf{Z}_p}1_X(\alpha x)\phi(\alpha x)\mu(x)\\ &= \int_{\mathbf{Z}_p}\phi(\alpha x)(1_{\alpha^{-1}X}(x)\mu(x)) = \int_{\mathbf{Z}_p}\phi(\alpha x)\mathrm{Res}_{\alpha^{-1}X}(\mu)\\ &= \int_{\mathbf{Z}_p}\phi(x)\sigma_\alpha(\mathrm{Res}_{\alpha^{-1}X}(\mu)),\end{aligned}$$

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which proves the lemma. $\square$

Definition 2.13. If μ is a distribution on $\mathbf{Z}_p^*$ and if $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, we define $\Gamma_\mu^{(i)}$ the i-th branch of the Γ-transform of μ by

$$\Gamma_\mu^{(i)} = \boxed{\theta_*}\mathrm{Res}_{1+q\mathbf{Z}_p}(\sum_{\varepsilon\in\Delta} \varepsilon^{-i}\sigma_\varepsilon(\mu)) = \theta_*(\sum_{\varepsilon\in\Delta} \varepsilon^{-i}\sigma_\varepsilon(\mathrm{Res}_{\varepsilon^{-1}+q\mathbf{Z}_p}(\mu))),$$

where the second equality ~~is~~ follows from the above lemma. Moreover, it is clear that if μ is a measure on $\mathbf{Z}_p^*$, then $\Gamma_\mu^{(i)}$ is a measure on $\mathbf{Z}_p$, and we have $v_{\mathscr{D}_0}(\Gamma_\mu^{(i)}) \geq v_{\mathscr{D}_0}(\mu)$.

Proposition 2.14. *If μ is a continuous distribution on $\mathbf{Z}_p^*$ and $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, then*

$$\mathrm{Mel}_{i,\mu}(s) = \int_{\mathbf{Z}_p^*} \omega(x)^i \langle x\rangle^s \mu(x) = \int_{\mathbf{Z}_p} u^{sy}\Gamma_\mu^{(i)}(y) = \mathscr{A}_{\Gamma_\mu^{(i)}}(u^s-1)$$

Proof. The first equality is by the definition of Mellin transform and the third equality is by the definition of Amice transform. If $y = \theta(x) = \frac{\log x}{\log u}$, we have $u^{sy} = \exp(s\log x) = \langle x\rangle^s$ and

$$\int_{\mathbf{Z}_p} u^{sy}\Gamma_\mu^{(i)}(y) = \int_{1+q\mathbf{Z}_p} \langle x\rangle^s \sum_{\varepsilon\in\Delta}\varepsilon^{-i}\sigma_\varepsilon(\mathrm{Res}_{\varepsilon^{-1}+q\mathbf{Z}_p}(\mu)).$$

Using the fact that $\omega(x) = \varepsilon^{-1}$ if $x \in \varepsilon^{-1}+q\mathbf{Z}_p$ and $\langle \varepsilon x\rangle = \langle x\rangle$, we obtain

$$\int_{\mathbf{Z}_p} u^{sy}\Gamma_\mu^{(i)}(y) = \sum_{\varepsilon\in\Delta}\int_{\varepsilon^{-1}+q\mathbf{Z}_p} \omega(x)^i\langle x\rangle^s\mu(x),$$

and the proposition follows from that $\mathbf{Z}_p^*$ is the disjoint union of $\varepsilon + q\mathbf{Z}_p$ for $\varepsilon\in\Delta$. □

Corollary 2.15.

i) If μ is a continuous distribution and $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, the function $\mathrm{Mel}_{i,\mu}(s)$ is a analytic function of s and even u^s-1.

ii) If μ is a measure verified $v_{\mathscr{D}_0}(\mu) \geq 0$, and if $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, then there exists $g_{i,\mu} \in \mathscr{O}_L[[T]]$ such that $\mathrm{Mel}_{i,\mu}(s) = g_{i,\mu}(u^s-1)$.

2.4. **Construction of the Kubota-Leopoldt zeta function.** If $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$ and $a \in \mathbf{Z}_p^*$ such that $\langle a\rangle \neq 1$, we define the function $g_{a,i}$ on $\mathbf{Z}_p$ by the formula

$$g_{a,i}(s) = \frac{1}{1-\omega(a)^{1-i}\langle a\rangle^{1-s}}\mathrm{Mel}_{-i,\mu_a}(-s) = \frac{1}{1-\omega(a)^{1-i}\langle a\rangle^{1-s}}\int_{\mathbf{Z}_p^*}\omega(a)^{-i}\langle a\rangle^{-s}\mu_a.$$

By corollary 2.15, $\mathrm{Mel}_{-i,\mu_a}(-s)$ is an analytic function of s. On the other hand, if $\omega(a)^{1-i}\neq 1$, the function $s\mapsto 1-\omega(a)^{1-i}\langle a\rangle^{1-s}$ is an nonzero analytic function on $\mathbf{Z}_p$ since $\langle a\rangle^s \in 1+q\mathbf{Z}_p$ and $\omega(a)^{1-i}\in\Delta-\{1\}$, therefore $\omega(a)^{1-i}\notin 1+q\mathbf{Z}_p$ and if $\omega(a)^{1-i}=1$, the function $1-\langle a\rangle^{1-s}$ vanishes only at $s=1$. We deduce that $g_{a,i}$ is a function continuous on $\mathbf{Z}_p-\{1\}$ and even on $\mathbf{Z}_p$ if $\omega(a)^{1-i}\neq 1$.

Moreover, if $-n\equiv i \bmod \phi(q)$, we have $\omega(a)^{1-i} = \omega(a)^{1+n}$ and $\omega(x)^{-i} = \omega(x)^n$ if $x\in\mathbf{Z}_p^*$. Therefore

$$g_{a,i}(-n) = \frac{1}{1-\omega(a)^{1+n}\langle a\rangle^{1+n}}\int_{\mathbf{Z}_p^*}\omega(x)^n\langle a\rangle^n\mu_a(x) = \frac{1}{1-a^{1+n}}\int_{\mathbf{Z}_p^*}x^n\mu_a(x) = (-1)^n(1-p^n)\zeta(-n)$$

does not depend on the choice of a. If a and a' two elements of $\mathbf{Z}_p^*$, the function $g_{a,i}-g_{a',i}$ is a quotient of analytic functions on $\mathbf{Z}_p$ vanishing at infinite many points, which implies it identical zero and the function $g_{a,i}$ is independent of choice of a. Thus we set $\zeta_{p,i} = g_{a,i}$ for any a satisfies $\langle a\rangle\neq 1$ and $\omega(a)^{1-i}\neq 1$ if $i\neq 1$ to construct Kubota-Leopoldt zeta function.

Let $F_n = \mathbf{Q}_p(\varepsilon_{p^n})$ and $F_\infty = \cup F_n$. The norm N_{F_{n+1}/F_n} ~~induced~~ induces a homomorphism from $\mu_{p^{n+1}}$ to μ_{p^n}, where μ_{p^n} be the set of p^n-th roots of unity in F_n. We denote the projective limit of μ_{p^n} with respect to N_{F_{n+1}/F_n} by μ_{p^∞} (the Tate module), which is a compact $\mathbf{Z}_p$-module.

The following theorem is due to Mazur and Wiles: [REF] ?

Theorem 2.16. *If $i \in (\mathbf{Z}/(p-1)\mathbf{Z})^*$ is odd and if $s \in \mathbf{Z}_p$, then the following two conditions are equivalent:*

i) $\zeta_{p,i}(s) = 0$;

ii) There exists an element $u \in \mu_{p^\infty}$ which is not killed by a power of p such that $\sigma \in \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ acts by the formula

$$\sigma(u) = \omega(\chi_{cycl}(\sigma))^i \langle \chi_{cycl}(\sigma) \rangle^s \cdot u.$$

2.5. **The residue at $s = 1$ and the p-adic zeta function.** The formal power series $\frac{\log(1+T)}{T}$ converges on open unit disk, thus it is an Amice transform of an unique distribution μ_{KL}. The Laplace transform of μ_{KL} is $\frac{t}{e^t - 1} = f_0(t)$ and

$$\int_{\mathbf{Z}_p} x^n \mu_{KL} = (-1)^{n-1} n \zeta(1-n)$$

Lemma 2.17. $\int_{a+p^n\mathbf{Z}_p} \mu_{KL} = \frac{1}{p^n}$

Proof. Since $\int_{a+p^n\mathbf{Z}_p} \mu_{KL} = \frac{1}{p^n} \sum_{\varepsilon^{p^n}=1} \varepsilon^{-a} \mathscr{A}_{\mu_{KL}}(\varepsilon - 1)$ and since $\log \varepsilon = 0$ if ε is a roots of unity of order power of p, all terms of the sum is zero except for the term corresponding to $\varepsilon = 1$, we get the result. □

Proposition 2.18. *We have*

i) $\psi(\mu_{KL}) = p^{-1}\mu_{KL}$

ii) $\mathrm{Res}_{\mathbf{Z}_p^*}(\mu_{KL}) = (1 - p^{-1}\varphi)\mu_{KL}$

iii) $\int_{\mathbf{Z}_p^*} \mu_{KL} = (-1)^{n-1} n(1 - p^{n-1})\zeta(1-n)$ *if* $n \in \mathbf{N}$.

Proof. i) follows from the formula $\psi(\frac{1}{T}) = \frac{1}{T}$ (c.f. proposition 2.7) and $\varphi(\log(1+T)) = p\log(1+T)$ and $\psi(\varphi(a)b) = a\psi(b)$. The rest can be deduced from proposition 2.7. □

Theorem 2.19. *The p-adic zeta function $\zeta_{p,1}$ has a simple pole at $s = 1$ with residue $1 - \frac{1}{p}$.*

Proof. According to the above, we can define the function $\zeta_{p,i}$, if $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$ by the formula

$$\zeta_{p,i}(s) = \frac{(-1)^{i-1}}{s-1} \mathrm{Mel}_{1-i,\mu_{KL}}(1-s) = \frac{(-1)^{i-1}}{s-1} \int_{\mathbf{Z}_p^*} \omega^{1-i} \langle x \rangle^{1-s} \mu_{KL}(x).$$

Indeed, the function is analytic on $\mathbf{Z}_p - \{1\}$ by above formula, and take the same value $\zeta_{p,i}(-n) = (1-p^n)\zeta(-n)$ if $n \in \mathbf{N}$ satisfies $-n \equiv i \bmod p-1$. Moreover,

$$\begin{aligned}\lim_{s\to 1}(s-1)\zeta_{p,i}(s) &= \int_{\mathbf{Z}_p^*} \omega(x)^{1-i} \mu_{KL}(x) \\ &= \sum_{\alpha\in\Delta} \omega(\alpha)^{1-i} \int_{\alpha+p\mathbf{Z}_p} \mu_{KL}(x) = \begin{cases} 1 - \frac{1}{p} & \text{if } i = 1 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

□

2.6. **Dirichlet L-function.** For χ a Dirichlet character of conductor $D > 1$ and ~~if~~ $n \in \mathbf{Z}$, we define the Gauss sum $G(\chi)$ by ~~the formula~~

$$G(\chi) = \sum_{a \bmod D} \chi(a) e^{2\pi i \frac{a}{D}}.$$

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Let

$$L(\chi, s) = \sum_{n=1}^{\cancel{+}\infty} \frac{\chi(n)}{n^s} = \prod_{p:prime} (1 - \chi(p)p^{-s})^{-1}, \quad \cancel{\text{For}}\ \mathrm{Re(s)} \geq 1,$$

be the Dirichlet L-function attached to χ. By the formula

$$\chi(n) = \frac{1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) e^{2\pi i \frac{nb}{D}},$$

we obtain

$$L(\chi, s) = \frac{1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) \frac{e^{2\pi i \frac{nb}{D}}}{n^s}.$$

Using the formula $\int_0^{+\infty} e^{-nt} t^s \frac{dt}{t} = \frac{\Gamma(s)}{n^s}$ and ~~put~~ with $\varepsilon_D := e^{\frac{2\pi i}{D}}$, we obtain

$$\begin{aligned} L(\chi, s) =& \frac{1}{G(\chi^{-1})} \frac{1}{\Gamma(s)} \sum_{b \bmod D} \chi^{-1}(b) \int_0^{+\infty} \sum_{n=1}^{+\infty} \varepsilon_D^{nb} e^{-nt} t^s \frac{dt}{t} \\ =& \frac{1}{G(\chi^{-1})} \frac{1}{\Gamma(s)} \int_0^{+\infty} \sum_{b \bmod D} \frac{\chi^{-1}(b)}{\varepsilon_D^{-b} e^t - 1} t^s \frac{dt}{t}. \end{aligned}$$

In particular, proposition 2.2 implies that $L(\chi, s)$ can be extended to a holomorphic function on $\mathbb{C}$. Moreover, $L(\chi, -n) = (\frac{d}{dt})^n \mathscr{L}_\chi(t) \mid_{t=0}$ where

$$\mathscr{L}_\chi(t) = \frac{-1}{G(\chi^{-1})} \sum_{b \bmod D} \frac{\chi^{-1}(b)}{\varepsilon_D^b e^t - 1}.$$

2.7. **p-adic L-function attached to Dirichlet character.** Let χ be a Dirichlet character of conductor $D > 1$ prime to p. If $\chi^{-1}(b) \neq 0$, then ε_D^b is a roots of unity of order prime to p and distinct from 1, this implies $\nu_p(\varepsilon_D^b - 1) = 0$. We deduce that the power series

$$F_\chi(T) = \frac{-1}{G(\chi^{-1})} \sum_{b \bmod D} \frac{\chi^{-1}(b)}{(1+T)\varepsilon_D^b - 1} = \frac{1}{G(\chi)^{-1}} \sum_{b \bmod D} \chi^{-1}(b) \sum_{n=0}^{+\infty} \frac{\varepsilon_D^{nb}}{(\varepsilon_D^b - 1)^{n+1}} T^n$$

is of bounded coefficients (since $\nu_p(G(\chi)G(\chi^{-1})) = \nu_p(D) = 0$) and hence ∃ an Amice transform of a measure μ_χ on $\mathbf{Z}_p$ whose Laplace transform $F_\chi(e^t - 1) = \mathcal{L}_\chi(t)$. We have $\int_{\mathbf{Z}_p} x^n \mu_\chi = \mathcal{L}_\chi^{(n)}(0) = L(\chi, -n)$ and $v_{\mathscr{D}_0}(\mu_\chi) \geq 0$.

Definition 2.20. We define the p-adic L-function associated to χ by the Mellin transform of μ_χ, that is, the function $\beta \mapsto L_p(\chi \otimes \beta)$ defined by

$$L_p(\chi \otimes \beta) = \int_{\mathbf{Z}_p^*} \beta(x) \mu_\chi(x).$$

where β is a locally analytic character on $\mathbf{Z}_p^*$. On the other hand, if $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, we put

$$L_{p,i}(\chi, s) = L_p(\chi \otimes (\omega^{-i}(x)\langle x \rangle^{-s})) = \int_{\mathbf{Z}_p^*} \omega^{-i} \langle x \rangle^{-s} \mu_\chi(x).$$

Proposition 2.21. *If $i \in \mathbf{Z}/\phi(q)\mathbf{Z}$, the function $L_{p,i}(\chi, s)$ is an analytic function on $\mathbf{Z}_p$ and we have $L_{p,i}(\chi, -n) = (1 - \chi(p)p^n)L(\chi, -n)$ if $n \in \mathbf{N}$ satisfying $-n \equiv i$ mod $\phi(q)$.*

Proof. The fact that $L_{p,i}(\chi, s)$ is an analytic function on $\mathbf{Z}_p$ follows from corollary 2.15. On the other hand, we have

$$\sum_{\eta^p=1} \frac{1}{(1+T)\varepsilon_D \eta - 1} = p\frac{1}{(1+T)^p\varepsilon_D^{pb} - 1},$$

and thus ~~thus we deduce~~ the Amice transform of μ_χ restriction to $\mathbf{Z}_p^*$ is

$$\frac{-1}{G(\chi)} \sum_{b \bmod D} \frac{\chi^{-1}(b)}{(1+T)\varepsilon_D^b - 1} - \frac{\chi^{-1}(b)}{(1+T)^p\varepsilon_D^{pb} - 1},$$

which can be written as $\mathscr{A}_{\mu_\chi}(T) - \chi(p)\mathscr{A}_{\mu_\chi}((1+T)^p - 1)$. Hence we deduce the formula

$$\mathscr{L}_{\mathrm{Res}_{\mathbf{Z}_p^*}(\mu_\chi)}(t) = \mathscr{L}_{\mu_\chi}(t) - \chi(p)\mathscr{L}_{\mu_\chi}(pt) \quad \text{and} \quad \int_{\mathbf{Z}_p^*} x^n \mu_\chi = (1 - \chi(p))L(\chi, -n),$$

and the proposition follows. □

2.8. **Behavior at $s = 1$ of Dirichlet L-function.** By section 2.6, we have

$$\begin{aligned} L(\chi, 1) =& \frac{1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) \sum_{n=0}^{+\infty} \frac{\varepsilon_D^{nb}}{n} \\ =& \frac{-1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) \log(1 - \varepsilon_D^b). \end{aligned}$$

We will establish the p-adic analogy of this formula by calculating $\int_{\mathbf{Z}_p^*} x^{-1}\mu_\chi$. To do this, we will calculate the Amice transform of $x^{-1}\mu_\chi$ and then restrict it to $\mathbf{Z}_p^*$.

Proposition 2.22. *The Amice transform of $x^{-1}\mu_\chi$ is*

$$\mathscr{A}_{x^{-1}\mu_\chi}(T) = \frac{-1}{G(\chi)^{-1}} \sum_{b \bmod D} \chi^{-1}(b) \log((1+T)\varepsilon_D^b - 1).$$

Proof. If μ is a distribution, the relation of Amice transform of μ and $x^{-1}\mu$ is give by

$$(1+T)\frac{d}{dT}\mathscr{A}_{x^{-1}\mu}(T) = \mathscr{A}_\mu(T).$$

Apply the operator $(1+T)\frac{d}{dT}$ on the right hand side of the equality in the proposition we obtain

$$\frac{-1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) \frac{(1+T)\varepsilon_D^b}{(1+T)\varepsilon_D^b - 1} = \frac{-1}{G(\chi^{-1})} \sum_{b \bmod D} \chi^{-1}(b) \Big(\frac{1}{(1+T)\varepsilon_D^b - 1} + 1\Big)$$

which is equal to $\mathscr{A}_{\mu_\chi}$ since $\sum_{b \bmod D} \chi^{-1}(b) = 0$. We deduce that the two elements have the same image by $(1+T)\frac{d}{dT}$ and therefore differs by a locally constant function. To conclude, we must verify that the right hand side is given by a series which converges on the open unit disk. Since ~~we have~~

$$\log((1+T)\varepsilon_D^b - 1) = \log(\varepsilon_D^b - 1) + \log(1 + \frac{\varepsilon_D^b T}{\varepsilon_D^b - 1}) = \log(\varepsilon_D^b - 1) + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (\frac{\varepsilon_D^b T}{\varepsilon_D^b - 1})^n,$$

and we suppose $(D,p)=1$, we have $\nu_p(\varepsilon_D^b-1)=0$, and hence the series converges on open unit disk. $\square$

Lemma 2.23. *The Amice transform of the restriction of $x^{-1}\mu_\chi$ to $\mathbf{Z}_p^*$ is defined by*

$$\begin{aligned}\mathscr{A}_{\mathrm{Res}_{\mathbf{Z}_p^*}(x^{-1}\mu_\chi)}(T)=&\frac{-1}{G(\chi^{-1})}\sum_{b \bmod D}\chi^{-1}(b)\Big(\log((1+T)\varepsilon_D^b-1)-\frac{1}{p}\log((1+T)^p\varepsilon_D^{pb}-1)\Big)\\ =&\mathscr{A}_{x^{-1}\mu_\chi}(T)-\frac{\chi(p)}{p}\mathscr{A}_{x^{-1}\mu_\chi}((1+T)^p-1).\end{aligned}$$

Proof. Use the formula for Amice transform of $\mathrm{Res}_{\mathbf{Z}_p^*}$. $\square$

By taking $T=0$ ~~to~~ in the above formula, we obtain

$$L_{p,1}(\chi,1)=L_p(\chi\otimes x^{-1})=\int_{\mathbf{Z}_p^*}x^{-1}\mu_\chi=\frac{-1}{G(\chi^{-1})}(1-\frac{\chi(p)}{p})\sum_{b \bmod D}\chi^{-1}(b)\log(\varepsilon_D^b-1).$$

which differs from the complex L-function ~~case~~ by an Euler factor.

2.9. **Twist by a character of conductor power of p.** Let χ be a Dirichlet character of conductor D prime to p and β be a Dirichlet character of conductor p^k. We denote $\chi\otimes\beta$ to be the Dirichlet character of conductor Dp^k defined by $(\chi\otimes\beta)(a)=\chi(a)\beta(a)$, where χ and β are viewed as characters mod Dp^k via the projections from $(\mathbf{Z}/Dp^k\mathbf{Z})^*$ to $(\mathbf{Z}/D\mathbf{Z})^*$ and $(\mathbf{Z}/p^k\mathbf{Z})^*$.

Lemma 2.24. *Let $k\geq 1$, β a Dirichlet character of conductor p^k and μ a continuous distribution on $\mathbf{Z}_p$~~.~~, ~~Then~~ then we have*

$$\int_{\mathbf{Z}_p}\beta(x)(1+T)^x\mu(x)=\frac{1}{G(\beta)^{-1}}\sum_{c \bmod p^k}\beta^{-1}(c)\mathscr{A}_\mu((1+T)\varepsilon_{p^k}^c-1).$$

Proof. We have

$$\begin{aligned}\int_{\mathbf{Z}_p}\beta(x)(1+T)^x\mu(x)&=\sum_{a \bmod p^k}\beta(a)\int_{a+p^k\mathbf{Z}_p}(1+T)^x\mu\\ &=\sum_{a \bmod p^k}\beta(a)\Big(\frac{1}{p^k}\sum_{\eta^{p^k}=1}\eta^{-a}\mathscr{A}_\mu((1+T)\eta-1)\Big)\\ &=\sum_{\eta^{p^k}=1}\mathscr{A}_\mu((1+T)\eta-1)\Big(\frac{1}{p^k}\sum_{a \bmod p^k}\beta(a)\eta^{-a}\Big);\end{aligned}$$

~~The~~ and the lemma follows from the identity

$$\frac{1}{p^k}\beta^{-1}(-c)G(\beta)=\frac{\beta^{-1}(c)}{G(\beta^{-1})}.$$

$\square$

Proposition 2.25. *If μ is a measure on $\mathbf{Z}_p$ with Amice transform of the form*

$$\mathscr{A}_\mu(T)=\frac{1}{G(\chi^{-1})}\sum_{b \bmod D}\chi^{-1}(b)F((1+T)\varepsilon_D^b-1)$$

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and if β is a Dirichlet character of conductor p^k with $k \geq 1$, then

$$\int_{\mathbf{Z}_p} \beta(x)(1+T)^x \mu(x) = \frac{1}{G((\chi \otimes \beta)^{-1})} \sum_{a \bmod Dp^k} (\chi \otimes \beta)^{-1}(a) F((1+T)\varepsilon_D^b \varepsilon_{p^k}^c - 1).$$

Proof. By the preceding lemma we have

$$\int_{\mathbf{Z}_p} \beta(x)(1+T)^x \mu(x) = \frac{-1}{G(\chi^{-1})G(\beta^{-1})} \sum_{b \bmod D} \sum_{c \bmod p^k} \chi^{-1}(b)\beta^{-1}(c) F((1+T)\varepsilon_D^b \varepsilon_{p^k}^c - 1).$$

Using the fact that every element of $\mathbf{Z}/Dp^n\mathbf{Z}$ can be written uniquely ~~as~~ of the form $Dc + p^k b$, where $b \in \mathbf{Z}/D\mathbf{Z}$ and $c \in \mathbf{Z}/p^k\mathbf{Z}$, we have the following formulae

$$\begin{aligned}
\varepsilon_{Dp^n}^a &= \varepsilon_D^b \varepsilon_{p^k}^c \\
(\chi \otimes \beta)^{-1}(a) &= \chi^{-1}(p^k)\beta^{-1}(D)\chi^{-1}(b)\beta^{-1}(c) \\
G((\chi \otimes \beta)^{-1}) &= \sum_{a \bmod Dp^k} (\chi \otimes \beta)^{-1}(a)\varepsilon_{Dp^k}^a \\
&= \chi^{-1}(p^k)\beta^{-1}(D)\Big(\sum_{b \bmod D} \chi^{-1}(b)\varepsilon_D^b\Big)\Big(\sum_{c \bmod p^k} \beta^{-1}(c)\varepsilon_{p^k}^c\Big) \\
&= \chi^{-1}(p^k)\beta^{-1}(D)G(\chi^{-1})G(\beta^{-1})
\end{aligned}$$

and the conclusion follows. □

Proposition 2.26. *If β is a non-trivial Dirichlet character of conductor a power of p and if $n \in \mathbf{N}$, then $L_p(\chi \otimes (x^n\beta)) = L(\chi \otimes \beta, -n)$*

Proof. By the ~~preceding~~ 2.25 proposition and the formula for the Amice transform of μ_χ, ~~we have~~ the Amice transform of $\beta\mu_\chi$ is

$$\frac{-1}{G((\chi \otimes \beta)^{-1})} \sum_{x \bmod Dp^n} \frac{(\chi \otimes \beta)^{-1}(x)}{(1+T)\varepsilon_{Dp^n}^x - 1}$$

and thus its Laplace transform is the function $\mathscr{L}_{\chi\otimes\beta}(t)$. □

3. (φ,Γ)-modules and p-adic representations

Throughout this article, k will denote a finite field of characteristic $p > 0$, so if $W(k)$ denotes the ring of Witt vectors over k, then $F = W(k)[\frac{1}{p}]$ is a finite unramified extension of $\mathbf{Q}_p$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure $\mathbf{Q}_p$, let K be a totally ramified extension of F, and let $G_K = Gal(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K. Let μ_{p^n} be the group of p^n-th roots of unity; for every n, we will choose a generator $\varepsilon^{(n)}$ of μ_{p^n} with the additional requirement that $(\varepsilon^{(n)})^p = \varepsilon^{(n-1)}$, This makes $\varprojlim \varepsilon^{(n)}$ into a generator $\varprojlim \mu_{p^n} \simeq \mathbf{Z}_p(1)$. We set $K_n = K(\mu_{p^n})$ and $K_\infty = \bigcup_{n\geq 0} K_n$. Recall that the cyclotomic character $\chi : G_K \to \mathbf{Z}_p^*$ is defined by the relation: $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for all $g \in G_K$. The kernel of the cyclotomic character is $H_K = \mathrm{Gal}(\mathbf{Q}_p/K_\infty)$, and χ therefore identifies $\Gamma_K = G_K/H_K$ with an open subgroup of $\mathbf{Z}_p^*$.

3.1. **The field $\widetilde{\mathbf{E}}$ and its subrings.** Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ for the p-adic topology and let

$$\widetilde{\mathbf{E}} = \varprojlim \mathbf{C}_p = \{(x^{(0)}, x^{(1)}, ...) \mid (x^{(n+1)})^p = x^{(n)}$$

and let $\widetilde{\mathbf{E}}^+$ be the set of $x \in \widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathscr{O}_{\mathbf{C}_p}$. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\widetilde{\mathbf{E}}$, we define the sum $x+y$ and their product xy by

$$(x+y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \quad \textit{and} \quad (xy)^{(i)} = x^{(i)} y^{(i)},$$

which makes $\widetilde{\mathbf{E}}$ an algebraically closed field of characteristic p. If $x = (x^{(n)}) \in \widetilde{\mathbf{E}}$, let $\nu_E(x) = \nu_p(x^{(0)})$. This is a valuation on $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{E}}$ is complete for this valuation; the ring of integers of $\widetilde{\mathbf{E}}$ is $\widetilde{\mathbf{E}}^+$. If $\mathfrak{a}$ is an ideal of $\mathscr{O}_{\mathbf{C}_p}$ contains p and contained in maximal ideal of $\mathscr{O}_{\mathbf{C}_p}$, the $\widetilde{\mathbf{E}}^+$ is identified with the projective limit of A_n, where if $n \in N$, we put $A_n = \mathscr{O}_{\mathbf{C}_p}/\mathfrak{a}$ and the transition amp from A_{n+1} to A_n is given by $x \mapsto x^p$.

Let $\varepsilon = (1, \varepsilon^{(1)}, ..., \varepsilon^{(n)}, ...)$ be an element of $\widetilde{\mathbf{E}}$ such that $\varepsilon^{(1)} \neq 1$, this implies that $\varepsilon^{(n)}$ is an primitive p^n-th roots of unity if $n \geq 1$. Let $\overline{\pi} = \varepsilon - 1$, we have $\nu_E(\overline{\pi}) = \frac{p}{p-1}$ and denotes $\mathbf{E}_{\mathbf{Q}_p}$ the subfield $\mathbf{F}_p((\overline{\pi}))$ of $\widetilde{\mathbf{E}}$. We denote $\mathbf{E}$ the separable closure of $\mathbf{E}_{\mathbf{Q}_p}$ in $\widetilde{\mathbf{E}}$ and $\mathbf{E}^+$ (resp. $\mathfrak{m}_{\mathbf{E}}$) the ring of integers (resp. the maximal ideal of $\mathbf{E}^+$).

By ramification theory, if K is a finite extension of $\mathbf{Q}_p$, then for all $\eta > 0$, there exists $n_\eta \in N$ such that if $n \geq n_\eta$, and if $\tau \in \Gamma_{K_n}$, then $\nu_p(\tau(x) - x) \geq \frac{1}{p} - \eta$. In particular if $\mathfrak{a}$ is an ideal of $\mathscr{O}_{\mathbf{C}_p}$ defined by $\mathfrak{a} = \{x \in \mathscr{O}_{\mathbf{C}_p} \mid \nu_p(x) \geq \frac{1}{p}\}$, then $N_{K_{n+1}/K_n}(x) - x^p \in \mathfrak{a}$ if n is large enough and $x \in \mathscr{O}_{K_{n+1}}$. This allows us to construct a map ι_K from the projective limit $\varprojlim \mathscr{O}_{K_n}$ of $\mathscr{O}_{K_n}$ with respect to norm map to $\widetilde{\mathbf{E}}^+$ (field of norm), such that $u = (u^{(n)})_{n \in \mathbf{N}}$ associates to $\iota_K(u) = (x^{(n)})_{n \in \mathbf{N}}$, where $x^{(n)}$ is the image of $u^{(n)}$ in $\mathscr{O}_{\mathbf{C}_p}/\mathfrak{a}$ if n large enough. Hence we have the following proposition:

Proposition 3.1. *If K is a finite extension of $\mathbf{Q}_p$, then ι_K induces bijection from $\varprojlim \mathscr{O}_{K_n}$ to the ring of integers $\mathbf{E}_K^+$ of $\mathbf{E}_K = \mathbf{E}^{H_K}$.*

By this proposition, one can show that $\mathbf{E}_K$ is a finite separable extension of $\mathbf{E}_{\mathbf{Q}_p}$ of degree $[H_{\mathbf{Q}_p} : H_K] = [K_\infty : \mathbf{Q}_p(\mu_{p^\infty})]$ and one can identify $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)$ with H_K.

Remark 3.2.

i) If F is a finite unramified extension of $\mathbf{Q}_p$ with residue field k_F, the field $\mathbf{E}_F$ is the composition of k_F and $\mathbf{E}_{\mathbf{Q}_p}$, that is, $k_F((\overline{\pi}))$.
ii) If K is a finite extension of $\mathbf{Q}_p$ and $F = K \cap \mathbf{Q}_p^{nr}$ it maximal unramified subfield, then $\mathbf{E}_K$ is an extension of $\mathbf{E}_F$ of degree $[K_\infty : F_\infty]$ which is equal to $[K_n : F_n]$ for n large enough.

3.2. **The field $\widetilde{\mathbf{B}}$ and its subrings.** Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ (resp. $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$) the Witt vectors with coefficients in $\widetilde{\mathbf{E}}$ (resp. $\widetilde{\mathbf{E}}^+$). By construction, we have $\widetilde{\mathbf{A}}/p\widetilde{\mathbf{A}} = \widetilde{\mathbf{E}}$ (resp. $\widetilde{\mathbf{A}}^+/p\widetilde{\mathbf{A}}^+ = \widetilde{\mathbf{E}}^+$). Let

$$\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p] = \{\sum_{k \gg -\infty} p^k [x_k] \mid x_k \in \widetilde{\mathbf{E}}\} \quad (\text{resp. } \widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] = \{\sum_{k \gg -\infty} p^k [x_k] \mid x_k \in \widetilde{\mathbf{E}}^+\}),$$

where $[x] \in \widetilde{\mathbf{A}}$ is the Teichmüller lift of $x \in \widetilde{\mathbf{E}}$ (resp. $\widetilde{\mathbf{E}}^+$).

We endows $\widetilde{\mathbf{A}}$ (resp. $\widetilde{\mathbf{A}}^+$) the topology by taking the collection of open sets $\{[\overline{\pi}]^k \widetilde{\mathbf{A}}^+ + p^n \widetilde{\mathbf{A}}\}_{k,n \geq 0}$ (resp. $\{([\overline{\pi}]^k + p^n)\widetilde{\mathbf{A}}^+\}_{k,n \geq 0}$) as family of neighborhoods of 0 and endow $\widetilde{\mathbf{B}} =$

$\cup_{n\in\mathbf{N}} p^{-n}\widetilde{\mathbf{A}}$ (resp. $\widetilde{\mathbf{B}}^{+}$) the inductive limit topology. The action of $G_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{E}}$ can be extended by continuity to $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ which commutes with the Frobenius action φ.

For F finite unramified extension over $\mathbf{Q}_p$, let $\pi = [\varepsilon]-1$, we define $\mathbf{A}_F$ the closure of $\mathscr{O}_F[[\pi,\pi^{-1}]]$ in $\widetilde{\mathbf{A}}$ by the above topology, thus

$$\mathbf{A}_F = \{\sum_{k\in\mathbf{Z}} a_k\pi^n \mid a_n \in \mathscr{O}_F, \lim_{k\to-\infty} \nu_p(a_k) = +\infty\}$$

which is a complete discrete valuation ring with residual field $\mathbf{E}_F$ and the Galois action and Frobenius action is defined by

$$\varphi(\pi) = (1+\pi)^p - 1 \quad \text{and} \quad g(\pi) = (1+\pi)^{\chi(g)} - 1 \quad g \in G_F,$$

and its fraction field $\mathbf{B}_F = \mathbf{A}_F[\frac{1}{p}]$ is stable by actions of φ and G_F.

Let $\mathbf{B}$ be the completion for the p-adic topology of the maximal unramified extension of $\mathbf{B}_F$ in $\widetilde{\mathbf{B}}$ and $\mathbf{A} = \mathbf{B}\cap\widetilde{\mathbf{A}}$. We have $\mathbf{B} = \mathbf{A}[\frac{1}{p}]$ and $\mathbf{A}$ are a complete discrete valuation ring with fractional field $\mathbf{B}$ and residual field $\mathbf{E}$. We then define $\mathbf{B}^{+} = \mathbf{B}\cap\widetilde{\mathbf{B}}^{+}$ and $\mathbf{A}^{+} = \mathbf{A}\cap\widetilde{\mathbf{A}}^{+}$. These rings are endowed with an action of Galois and a Frobenius deduced from those on $\widetilde{\mathbf{E}}$.

If K is a finite extension of $\mathbf{Q}_p$, we put $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{A}_K[1/p]$, this makes $\mathbf{A}_K$ a complete discrete valuation ring with residual field $\mathbf{E}_K$ and fraction field $\mathbf{B}_K = \mathbf{A}_K[1/p]$. On the other hand, when $K = F$, the definitions of $\mathbf{A}_F$ and $\mathbf{B}_F$ coincide with previous definitions. We put $\mathbf{A}_F^{+} = (\mathbf{A}^{+})^{H_F}$ and $\mathbf{B}_F^{+} = (\mathbf{B}^{+})^{H_F}$ then by using fields of norm above, we can show that $\mathbf{A}_F^{+} = \mathscr{O}_F[[\pi]]$ and $\mathbf{B}_F^{+} = F[[\pi]]$.

If L is a finite extension of K, $\mathbf{B}_L$ is an unramified extension of $\mathbf{B}_K$ of degree $[L_\infty : K_\infty]$. If L/K is Galois extension, then the extension $\widetilde{\mathbf{B}}_L/\widetilde{\mathbf{B}}_K$ and $\mathbf{B}_L/\mathbf{B}_K$ is Galois with Galois group $\mathrm{Gal}(\widetilde{\mathbf{B}}_L/\widetilde{\mathbf{B}}_K) = \mathrm{Gal}(\mathbf{B}_L/\mathbf{B}_K) = \mathrm{Gal}(\mathbf{E}_L/\mathbf{E}_K) = \mathrm{Gal}(L_\infty/K_\infty) = H_K/H_L$.

Remark 3.3.

i) If $\overline{\pi}_K$ is a uniformizer of $\mathbf{E}_K$, let π_K be any lifting of $\overline{\pi}_K$ in $\mathbf{A}_K$. Then,

$$\mathbf{A}_K = \{\sum_{k\in\mathbf{Z}} a_k\pi_K^k \mid a_k \in \mathscr{O}_{F'}, \lim_{k\to-\infty} \nu_p(a_k) = +\infty\}$$

where F' is the maximal unramified extension of F contained in K_∞.

ii) In the above construction, the correspondence $R \longrightarrow \widetilde{R}$ is obtained by making φ bijective and then complete, where $R = \{\mathbf{E}_K, \mathbf{E}, \mathbf{A}_K, \mathbf{A}, \mathbf{B}_K, \mathbf{B}\}$.

3.3. **(φ,Γ)-module and Galois representations.** A p-adic representation V is a finite dimensional $\mathbf{Q}_p$-vector space with a continuous linear action of G_K. It is easy to see that there is always a $\mathbf{Z}_p$-lattice of V which is stable by the action of G_K, and such lattices will be denoted by T (called a $\mathbf{Z}_p$-representation). The main strategy due to Fontaine for studying p-adic representations of a group G is to construct topological $\mathbf{Q}_p$-algebras B (period rings), endowed with an action of G and some additional structures so that if V is a p-adic representation, then

$$D_B(V) = (B\otimes_{\mathbf{Q}_p} V)^G$$

is a B^G-module which inherits these structures, and so that the functor $V \mapsto D_B(V)$ gives interesting invariants of V. We say that a p-adic representation V of G is B-admissible if we have $B\otimes_{\mathbf{Q}_p} V \simeq B^d$ as $B[G]$-modules.

Definition 3.4. If K is a finite extension of $\mathbf{Q}_p$, we say

i) A (φ,Γ)-module over $\mathbf{A}_K$ (resp. $\mathbf{B}_K$) is a $\mathbf{A}_K$-module of finite type (resp. a finite dimensional $\mathbf{B}_K$-vector space) equipped with a Γ_K-action and a Frobenius action φ which commutes with Γ_K.
ii) A (φ,Γ)-module D over $\mathbf{A}_K$ is *étale* if $\varphi(D)$ generates D as an $\mathbf{A}_K$-module. A (φ,Γ)-module D over $\mathbf{B}_K$ is *étale* if it has an $\mathbf{A}_K$-lattice which is *étale*, equivalently, there exists a basis $\{e_1,...,e_d\}$ over $\mathbf{B}_K$, such that the matrix of φ in terms of the basis is in $GL_d(\mathbf{A}_K)$.

If K is a finite extension of $\mathbf{Q}_p$ and V is a $\mathbf{Z}_p$-representation (resp. p-adic representation) of G_K, we put

$$D(V) = (\mathbf{A}\otimes_{\mathbf{Z}_p} V)^{H_K} \quad (\text{resp.} \quad D(V) = (\mathbf{B}\otimes_{\mathbf{Q}_p} V)^{H_K})$$

Since the action of φ commutes with G_K, $D(V)$ is a equipped with a Frobenius action φ which commutes with the residual action $G_K/H_K = \Gamma_K$. This make $D(V)$ a (φ,Γ)-module over $\mathbf{A}_K$ (resp. $\mathbf{B}_K$).

On the other hand, if V is a $\mathbf{Z}_p$-representation (resp. a p-adic representation) of G_K, then $(\mathbf{A}\otimes_{\mathbf{A}_K} D(V))^{\varphi=1}$ (resp. $(\mathbf{B}\otimes_{\mathbf{A}_K} D(V))^{\varphi=1}$) is canonically isomorphism to V as a representation of G_K. In other words, V is determined by the (φ,Γ)-module $D(V)$.

Theorem 3.5. *(Fontaine) The correspondence*

$$V \longmapsto D(V) = (\mathbf{A}\otimes_{\mathbf{Z}_p} V)^{H_K}$$

is an equivalence of $\otimes$ categories from the category of $\mathbf{Z}_p$-representations (resp. p-adic representation) of G_K to the category of étale (φ,Γ)-module over $\mathbf{A}_K$ (resp. $\mathbf{B}_K$), and its inverse functor is

$$D \longmapsto V(D) = (\mathbf{A}\otimes_{\mathbf{A}_K} D)^{\varphi=1}.$$

4. (φ,Γ)-modules and Galois cohomology

4.1. **The complex $C_{\varphi,\gamma}(K,V)$.** Let K be an finite extension of $\mathbf{Q}_p$ such that Γ_K is isomorphic to $\mathbf{Z}_p$ (i.e. contains $\mathbf{Q}_p(\mu_p)$ if $p\geq 3$ or three quadratic ramified extensions of $\mathbb{Q}_2$ if $p=2$) and γ is a generator of Γ_K. If V is a $\mathbf{Z}_p$-representation or p-adic representation of G_K and $f: D(V)\to D(V)$ is a $\mathbf{Z}_p$-linear map commutes with action of Γ, we denote $C_{f,\gamma}(K,V)$ the complex

$$0 \longrightarrow D(V) \longrightarrow D(V)\oplus D(V) \longrightarrow D(V) \longrightarrow 0$$

where maps $D(V)$ to $D(V)\oplus D(V)$ and $D(V)\oplus D(V)$ to $D(V)$ respectively by

$$x\mapsto ((f-1)x, (\gamma-1)x) \quad \textit{and} \quad (a,b)\mapsto (\gamma-1)a-(f-1)b$$

we denote $Z^i(C_{f,\gamma}(K,V))$ (resp. $B^i(C_{f,\gamma}(K,V))$, resp. $H^i(C_{f,\gamma}(K,V)) = \frac{Z^i(C_{f,\gamma}(K,V))}{B^i(C_{f,\gamma}(K,V))}$) the i-th cocycles (resp. coboundaries, resp. cohomologies) of complex $C_{f,\gamma}(K,V)$.

The $C_{f,\gamma}(K,V)$ canonically and functorially identified with the Galois cohomology group $H^i(K,V)$ (c.f. [Her98]). The following proposition gives the case of H^1.

Let $\Lambda_K = \mathbf{Z}_p[[\Gamma_K]]$ the complete group algebra of Γ_K. Since Γ_K acts continuously on $D(V)$, we can view $D(V)$ as a Λ_K-module. On the other hand, Γ_K is pro-cyclic, if γ is a generator of Γ_K and γ' is any element of Γ_K, then the element $\frac{\gamma'-1}{\gamma-1}$ of $\mathrm{Frac}(\Lambda_K)$ is indeed in Λ_K. Moreover, the G_K action factors through Γ_K on $D(V)$, so the expression $\frac{\sigma-1}{\gamma-1}y$ make sense if $y\in D(V)$, $\sigma\in G_K$ and γ is a generator of Γ_K.

Proposition 4.1.

i) If $(x,y)\in Z^1(C_{\varphi,\gamma}(K,V))$ and $b\in \mathbf{A}\otimes_{\mathbf{Z}_p} V$ is a solution of $(\varphi-1)b=x$, then $\sigma\mapsto c_{x,y}(\sigma)=\frac{\sigma-1}{\gamma-1}y-(\sigma-1)b$ is a cocycle of G_K with values in V.

ii) The map sends $(x,y)\in Z^1(C_{\varphi,\gamma}(K,V))$ to the class of $c_{x,y}$ in $H^1(K,V)$ induces an isomorphism $\iota_{\varphi,\gamma}$ of $H^1(C_{\varphi,\gamma}(K,V))$ to $H^1(K,V)$.

Proof. It clear that $\sigma\mapsto c_{x,y}(\sigma)$ is a cocycle by definition. On the other hand, we have

$$(\varphi-1)(c_{x,y}(\sigma))=\frac{\sigma-1}{\gamma-1}((\varphi-1)y)-(\sigma-1)x=0$$

since $(\gamma-1)x=(\varphi-1)y$. Hence $c_{x,y}(\sigma)\in(\mathbf{A}\otimes_{\mathbf{Z}_p}V)^{\varphi=1}=V$. This proves (i).

To prove (ii), suppose the image of $c_{x,y}$ in $H^1(K,V)$ is zero, there exist $z\in V$ such that

$$\frac{\sigma-1}{\gamma-1}y-(\sigma-1)(b+z)=0\quad\forall\sigma\in G_K.$$

We deduce that $b+z$ is stable by H_K and therefore belongs to $D(V)$. Take $\sigma=\gamma$, we have $y=(\gamma-1)(b+z)$ and hence $x=(\varphi-1)(b+z)$, which implies $(x,y)\in B^1(C_{\varphi,\gamma}(K,V))$ and the injectivity of $\iota_{\varphi,\gamma}$ follows.

To prove the surjectivity, let $c\in H^1(K,V)$ and V' an extension of $\mathbf{Z}_p$ by V corresponding to c. That is, an exact sequence

$$0\longrightarrow V\longrightarrow V'\longrightarrow \mathbf{Z}_p\longrightarrow 0$$

such that $e\in V'$ sends to $1\in\mathbf{Z}_p$ and $\sigma(e)=e+c_\sigma$, where $\sigma\mapsto c_\sigma$ is the cocycle of G_K represents c. Apply functor D, we get

$$0\longrightarrow D(V)\longrightarrow D(V')\longrightarrow D(\mathbf{Z}_p)\longrightarrow 0,$$

let $\widetilde{e}\in D(V')$ element maps to $1\in\mathbf{Z}_p=D(\mathbf{Z}_p)$ and x,y elements of $D(V)$ defined by $x=(\varphi-1)\widetilde{e}$ and $y=(\gamma-1)\widetilde{e}$. Since γ and φ commute, (x,y) is belongs to $Z^1(C_{\varphi,\gamma}(K,V))$. On the other hand, $b=\widetilde{e}-e\in\mathbf{A}\otimes_{\mathbf{Z}_p}V$ satisfies $(\varphi-1)b=x$, so we have

$$c_{x,y}(\sigma)=\frac{\sigma-1}{\gamma-1}y-(\sigma-1)b=(\sigma-1)(\widetilde{e}-b)=(\sigma-1)e=c_\sigma.$$

From this, we deduce the surjectivity of $\iota_{\varphi,\gamma}$. □

If γ' is another generator of Γ_K, then $\frac{\gamma-1}{\gamma'-1}\in\mathrm{Frac}(\Gamma_K)$ is indeed a unit in Γ_K and the diagram

$$\begin{array}{ccccccccc}
C_{\varphi,\gamma}(K,V):0 & \longrightarrow & D(V) & \longrightarrow & D(V)\oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \frac{\gamma-1}{\gamma'-1}} & & \Big\downarrow{\scriptstyle \frac{\gamma-1}{\gamma'-1}\oplus id} & & \Big\downarrow{\scriptstyle id} & & \\
C_{\varphi,\gamma'}(K,V):0 & \longrightarrow & D(V) & \longrightarrow & D(V)\oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0
\end{array}$$

is commutative. It hence induces via cohomology an isomorphism $\iota_{\gamma,\gamma'}$ from $H^1(C_{\varphi,\gamma}(K,V))$ to $H^1(C_{\varphi,\gamma'}(K,V))$.

Since we assume Γ_K is torsion free, we have $\chi(\gamma)\in 1+p\mathbf{Z}_p$ for $\gamma\in\Gamma_K$, then there exists $k\geq 1$ such that $\log_p(\chi(\Gamma))\in p^k\mathbf{Z}_p^*$ and we'll write $\log_p^0(\gamma)=\log_p(\chi(\gamma))/p^k$. The following lemma shows that $\log_p^0(\gamma)\iota_{\varphi,\gamma}$ does not depend on the choice of generator γ of Γ_K.

Lemma 4.2. *If γ and γ' are two generators of Γ_K, then the isomorphisms $\log_p^0(\gamma)\iota_{\varphi,\gamma}$ and $\log_p^0(\gamma')\iota_{\varphi,\gamma'}\circ\iota_{\gamma,\gamma'}$ from $H^1(C_{\varphi,\gamma}(K,V))$ to $H^1(K,V)$ are equal.*

Proof. If $(x,y) \in Z^1(C_{\varphi,\gamma}(K,V))$. Let b (resp. b') be element of $\mathbf{A} \otimes_{\mathbf{Z}_p} V$ verifies $(\varphi-1)b = x$ (resp. $(\varphi-1)b' = \frac{\gamma-1}{\gamma'-1}x$). Since $\frac{\log_p^0(\gamma)}{\gamma-1} - \frac{\log_p^0(\gamma')}{\gamma'-1} \in \mathbf{Z}_p[[\Gamma_K]]$, we can write the cocycle associates to $\log_p^0(\gamma')\iota_{\varphi,\gamma} \circ \iota_{\gamma,\gamma'}(x,y) - \log_p^0(\gamma)\iota_{\varphi,\gamma}(x,y)$ as $\sigma \mapsto (\sigma-1)c$, where

$$c = (\frac{\log_p^0(\gamma')}{\gamma'-1} - \frac{\log_p^0(\gamma)}{\gamma-1})y - (\log_p^0(\gamma')b' - \log_p^0(\gamma)b)$$

and the relation $(\varphi-1)y = (\gamma-1)x$ implies $(\varphi-1)c = 0$, hence $c \in V$ and the cocycle is indeed a coboundary, which leads to the conclusion. □

4.2. **The operator ψ.** To calculate $H^1(C_{\varphi,\gamma}(K,V))$ we have to understand the group $D(V)^{\varphi=1}$ and $\frac{D(V)}{\varphi-1}$. The problem is that the group $\frac{D(V)}{\varphi-1}$ is too complicated to write it down. To solve this difficulty, we introduce the left inverse of φ.

The field $\mathbf{B}$ is an extension of degree p of $\varphi(B)$, which allows up to define the operator $\psi : \mathbf{B} \to \mathbf{B}$ by the formula $\psi(x) = \frac{1}{p}\varphi^{-1}(\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x))$. More explicitly, one can verify that $\{1, [\varepsilon], ..., [\varepsilon]^{p-1}\}$ is a basis of $\mathbf{A}$ over $\varphi(\mathbf{A})$ (hence $\mathbf{B}$ over $\varphi(\mathbf{B})$) so we have

$$\psi(\sum_{i=0}^{p-1}[\varepsilon]^i\varphi(x_i)) = x_0 \quad x_i \in \mathbf{B} \quad \text{and} \quad \psi(\varphi(x)) = x \quad x \in \mathbf{B}.$$

The operator ψ commute with the action of G_K and $\psi(\mathbf{A}) \subset \mathbf{A}$.

Since ψ commutes with the action of G_K, if V is a $\mathbf{Z}_p$-representation or a p-adic representation of G_K, the module $D(V)$ inherit the action of ψ and commute with Γ_K. That is, the unique map $\psi : D(V) \to D(V)$ with

$$\psi(\varphi(a)x) = a\psi(x), \quad \psi(a\varphi(a)) = \psi(a)x$$

if $a \in \mathbf{A}_K$, $x \in D(V)$.

Proposition 4.3. *If V is a $\mathbf{Z}_p$-representation or a p-adic representation of G_K, then $\gamma - 1$ is invertible on $D(V)^{\psi=0}$.*

Proof. See [Her98]. □

Lemma 4.4. *We have a commutative diagram of complexes*

$$\begin{array}{ccccccccc}
C_{\varphi,\gamma}(K,V) : 0 & \longrightarrow & D(V) & \longrightarrow & D(V) \oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle (-\psi, id)} & & \downarrow{\scriptstyle -\psi} & & \\
C_{\psi,\gamma}(K,V) : 0 & \longrightarrow & D(V) & \longrightarrow & D(V) \oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0
\end{array}$$

which induces an isomorphism ι from $H^1(C_{\varphi,\gamma}(K,V))$ to $H^1(C_{\psi,\gamma}(K,V))$.

Proof. The commutativity of diagram follows from definition. Since ψ is surjective, the cokernel complex is 0. The kernel complex is

$$0 \longrightarrow 0 \longrightarrow D(V)^{\psi=0} \xrightarrow{\gamma-1} D(V)^{\psi=0} \longrightarrow 0,$$

which has no cohomology by proposition 4.3. □

Notation 4.5. We denote $\iota_{\psi,\gamma}$ the isomorphism of $H^1(C_{\psi,\gamma}(K,V))$ to $H^1(K,V)$ obtained by composite $\iota_{\varphi,\gamma}$ and ι^{-1}.

Remark 4.6. The same proof as lemma 4.2 shows that $\log_p^0(\gamma)\iota_{\psi,\gamma}$ does not depend on the generator γ of Γ_K.

Lemma 4.7. *The map which sends $(x,y) \in Z^1((C_{\varphi,\gamma}(K,V))$ to the image of x in $\frac{D(V)}{\psi-1}$ induces an exact sequence*

$$0 \longrightarrow D(V)^{\psi=1}_{\Gamma_K} \longrightarrow H^1(C_{\psi,\Gamma_K}(K,V)) \longrightarrow \left(\frac{D(V)}{\psi-1}\right)^{\Gamma_K} \longrightarrow 0$$

Proof. $\overline{x} \in \frac{D(V)}{\psi-1}$ is fixed by Γ_K if and only if there exists $(x,y) \in Z^1(C_{\varphi,\gamma}(K,V))$ whose image in $\frac{D(V)}{\psi-1}$ is equal to $\overline{x}$. The kernel of the map is the sum of $B^1(C_{\psi,\gamma}(K,V))$ and the set X of elements of the form $(0,y)$ where $y \in D(V)^{\psi=1}$. One observes that $X \cap B^1(C_{\varphi,\gamma}(K,V))$ is constituted by couples of the form $(0,y)$ where $y \in (\gamma-1)D(V)^{\psi=1}$. □

Remark 4.8. By [Her98], one can show that Herr complex indeed computes Galois cohomology group $H^i(K,V)$, hence we have

- $H^0(K,V) \simeq D(V)^{\psi=1,\gamma=1} \simeq D(V)^{\varphi=1,\gamma=1}$.
- $H^2(K,V) \simeq \frac{D(V)}{(\psi-1,\gamma-1)}$.
- $H^i(K,V) = 0$ if $i \geq 2$.

Similar to case of φ, the modules $D(V)^{\psi=1}$ and $\frac{D(V)}{\psi-1}$ can be interpreted naturally as Iwasawa algebra. Moreover, the module $\frac{D(V)}{\psi-1}$ is "small" compared to $\frac{D(V)}{\varphi-1}$, thus we can write $H^1(K,V)$ mainly as the submodule $D(V)^{\psi=1}$. More precisely, we have the following proposition whose proof would be in the following two subsections.

Proposition 4.9. *If V is a $\mathbf{Z}_p$-representation (resp. a p-adic representation) of G_K, then*

i) $D(V)^{\psi=1}$ is compact (resp. locally compact) and generates the $\mathbf{A}_K$-module ($\mathbf{B}_K$-vector space) $D(V)$.

ii) $\frac{D(V)}{\psi-1}$ is a free $\mathbf{Z}_p$-module of finite rank (resp. a finite dimensional $\mathbf{Q}_p$-vector space).

Remark 4.10. Since the p-adic representation case can be deduce from $\mathbf{Z}_p$-representation case by tensor $\mathbf{Q}_p$, we only need to treat the $\mathbf{Z}_p$-representation case.

4.3. **The compactness of $D(V)^{\psi=1}$.** The goal of this paragraph is to prove the following lemma. In particular, when $n=0$ and $N=+\infty$ is equivalent to the compactness of $D(V)^{\psi=1}$.

Lemma 4.11. *If V is a $\mathbf{Z}_p$-representation of G_K, $x \in D(V)$ and $N \in \mathbf{N}\cup\{+\infty\}$, the set of solutions $y \in D(V)/p^{N+1}D(V)$ of the equation $(\psi-1)y = x$ is compact.*

Let $\mathbf{A}^+_{Q_p}$ is the subring $\mathbf{Z}_p[[\pi]]$ of $\mathbf{A}_{Q_p}$, and let $A = \mathbf{A}^+_{Q_p}[[\frac{p}{\pi^{p-1}}]]$, then A is a compact subring of $\mathbf{A}_{Q_p}$ such that elements of A can be written as $x = \sum_{n\in\mathbf{Z}} x_n\pi^n$ where $(x_n)_{n\in\mathbf{Z}}$ is a sequence in $\mathbf{Z}_p$ such that we have $\nu_p(x_n) \geq -\frac{n}{p-1}$ if $n \leq 0$.

If $x \in \mathbf{A}_{Q_p}$, let $w_n(x) \in \mathbf{N}$ the smallest integer k such that x belongs to $\pi^{-k}A + p^{n+1}\mathbf{A}_{Q_p}$. If x is fixed, the sequence $\{w_n(x)\}_{n\in\mathbf{N}}$ is increasing and we have

$$w_n(x+y) \leq \sup(w_n(x), w_n(y))$$

$$w_n(xy) \leq \sup_{i+j=n}(w_i(x)+w_j(y)) \leq w_n(x)+w_n(y)$$

$$w_n(\varphi(x)) \leq pw_n(x)$$

the first two inequality follow from A is a ring and the third is because $\frac{\varphi(\pi)}{\pi^p}$ is an unit in A (This is the reason for working with A instead of $\mathbf{A}^+_{Q_p}$ by defining the map w_n) and such that $x \in \pi^{-k}A + p^{n+1}\mathbf{A}_{Q_p}$ implies $\varphi(x) \in \varphi(\pi)^{-k}A + p^{n+1}\mathbf{A}_{Q_p} = \pi^{-pk}A + p^{n+1}\mathbf{A}_{Q_p}$.

Lemma 4.12.

i) If $k \in \mathbf{N}$, then $\psi(\pi^k) \in \mathbf{A}^+_{Q_p}$ and $\psi(\pi^{-k}) \in \pi^{-k}\mathbf{A}^+_{Q_p}$

ii) $\psi(A) \subset A$.

Proof. ii) follows from i) and the definition of A. Since $\varphi(\pi) = (1+\pi)^p - 1$ is a monic polynomial of degree p in π and $[\varepsilon]^i = (1+\pi)^i$ is a monic polynomial of degree i in π, hence $\{[\varepsilon]^i\varphi(\pi)^j\}_{0\le i\le p-1, j\in\mathbf{N}}$ forms a basis of polynomials in π. Moreover, $\psi([\varepsilon]^i\varphi(\pi)^j) = \begin{cases} 0 & i \neq 0 \\ \pi^j & i = 0 \end{cases}$, we thus deduce that $\psi(\pi^k) \in \mathbf{A}^+_{Q_p}$ if $k \ge 0$. If $k \ge 1$, then

$$\mathrm{Tr}_{\mathbf{A}_{Q_p}/\varphi(\mathbf{A}_{Q_p})}(\pi^{-k}) = \sum_{\zeta^p=1}((1+\pi)\zeta - 1)^{-k},$$

which can be written as the form $\frac{P(\varphi(\pi))}{\varphi(\pi)^k}$, where P is a polynomial with coefficient in $\mathbf{Z}_p$. Thus the conclusion follows. □

Corollary 4.13. *If $x \in \mathbf{A}_{Q_p}$ and $n \in \mathbf{N}$, then $w_n(\psi(x)) \le 1 + [\frac{w_n(x)}{p}] \le 1 + \frac{w_n(x)}{p}$.*

Proof. Since $\frac{\varphi(\pi)}{\pi^p}$ is an unit in A and $\psi(\frac{x}{\varphi(\pi)^k}) = \frac{\psi(x)}{\pi^k}$, we have

$$\psi(\pi^{-kp}A + p^{n+1}\mathbf{A}_{Q_p}) = \psi(\varphi(\pi)^{-k}A + p^{n+1}\mathbf{A}_{Q_p}) \subset \pi^{-k}A + p^{n+1}\mathbf{A}_{Q_p},$$

the conclusion follows. □

If $U = (a_{i,j})_{1\le i,j\le d} \in M_d(\mathbf{A}_{Q_p})$ and $n \in \mathbf{N}$, we define $w_n(U)$ by $w_n(U) = \sup_{i,j} w_n(a_{i,j})$. Similarly if V is a $\mathbf{Z}_p$-representation of G_K and if $e_1, ..., e_d$ is a basis of $D(V)$ over $\mathbf{A}_{Q_p}$, we put $w_n(a) = \sup_i w_n(a_i)$ if $a = \sum_{i=1}^d a_i e_i \in D(V)$. Note that w_n depends on the choice of basis $e_1, ..., e_d$.

Lemma 4.14. *Let V be a $\mathbf{Z}_p$-representation of G_K, $e_1, ..., e_d$ is a basis of $D(V)$ over $\mathbf{A}_{Q_p}$ and $\Phi = (a_{i,j})$ the matrix defined by $e_j = \sum_{i=1}^d a_{i,j}\varphi(e_i)$. If $x, y \in \mathbf{A}_{Q_p}$ satisfy the equation $(\psi - 1)y = x$, then $w_n(y) \le sup\Big(w_n(x), \frac{p}{p-1}(w_n(\Phi)+1)\Big)$ for all $n \in \mathbf{N}$.*

Proof. Since $\varphi(e_1), ..., \varphi(e_d)$ is a basis of $D(V)$ over $\phi(D(V))$, we can write $x = \sum_{i=1}^d x_i\varphi(e_i)$ and $y = \sum_{i=1}^d y_i\varphi(e_i)$. We have $\psi(y) = \sum_{i=1}^d \psi(y_i)e_i$ and the equation $\psi(y) - y = x$ translate to system of equation

$$y_i = -x_i + \sum_{j=1}^d a_{i,j}\psi(y_j) \quad 1 \le j \le d.$$

One get the inequalities

$$w_n(y_i) \le \sup\left(w_n(x_i), \sup_{1\le j\le d}(w_n(a_{i,j}) + w_n(\psi(y_j)))\right) \le \sup\left(w_n(x), w_n(\Phi) + \frac{w_n(y)}{p} + 1\right)$$

for $1 \leq i \leq d$, which gives us the inequality

$$w_n(y) \leq \sup\left(w_n(x), w_n(\Phi) + \frac{w_n(y)}{p} + 1\right)$$

and the conclusion follows. $\square$

To deduce lemma 4.11. If $n \in \mathbf{N} \cup \{+\infty\}$, let X_n be the set of solutions of the equation $(\psi - 1)y = x$ in $D(V)/p^{n+1}D(V)$. We want to show that X_n is compact. If $n \in N$, let $r_n = \sup(w_n(x), \frac{p}{p-1}(w_n(\Phi) + 1))$. The set X_n is closed (since $\psi - 1$ is continuous). By the previous lemma, the image of $(\pi^{-r_n}A)^d$ is compact since A is. If N is finite, it suffices to take $n = N$ to conclude. If $N = +\infty$, the map from $x \in X_{+\infty}$ to the sequence of its images modulo p^{n+1} allows us to identify $X_{+\infty}$ with the closed subset of compact set $\prod_{n\in\mathbf{N}} X_n$, and the conclusion follows.

4.4. **The module $\frac{D(V)}{\psi-1}$.**

Lemma 4.15. *If V be a $\mathbf{Z}_p$-representation of G_K, the module $\frac{D(V)}{\psi-1}$ has no nonzero p-divisible element.*

Proof. Let x be a p-divisible element of $\frac{D(V)}{\psi-1}$. For each $n \in \mathbf{N}$, there exist elements y_n, z_n of $D(V)$ such that $x = p^n y_n + (\psi - 1)z_n$. If we fix $m \in \mathbf{N}$ and if $n \geq m + 1$, then z_n is a solution of equation $\psi(z) - z = x \mod p^{m+1}$. Since the set of solutions is compact due to lemma 4.11, there exists a subsequence of $\{z_n\}_{n\in\mathbf{N}}$ which converges modulo p^m for all m and we have a limit Z in $D(V)$. By passing to limit, we obtain $x = (\psi - 1)z$ and hence $x = 0$ in $\frac{D(V)}{\psi-1}$. $\square$

Lemma 4.16. *If V is a $\mathbf{F}_p$-representation of G_K and $x \in \mathfrak{m}_\mathbf{E} \otimes V$, then the series $\sum_{n=0}^{+\infty} \varphi^n(x)$ and $\sum_{n=1}^{+\infty} \varphi^n(x)$ converges in $\mathfrak{m}_\mathbf{E} \otimes V$ and we have*

$$(\psi - 1)\left(\sum_{n=0}^{+\infty} \varphi^n(x)\right) = \psi(x) \quad \textit{and} \quad (\psi - 1)\left(\sum_{n=1}^{+\infty} \varphi^n(x)\right) = x.$$

Proof. If $e_1, ..., e_d$ is a basis of V over F_p and $x = x_1e_1 + ... + x_ne_d \in \mathfrak{m}_\mathbf{E} \otimes V$, there exists $r \geq 0$ such that if $\nu_E(x_i) \geq r$ for $1 \leq i \leq d$ implies that $\nu_E(\varphi^n(x_i)) \geq p^n r$ tends to $+\infty$ and hence we have $\varphi^n(x)$ tends to 0 as n tends to $+\infty$. We thus deduce the convergence of the series. These formulas are consequence of the fact that ψ is a left inverse of φ. $\square$

Lemma 4.17.

i) If V be a $\mathbf{F}_p$-representation of G_K, then $\frac{D(V)}{\psi-1}$ is a finite dimensional $\mathbf{F}_p$-vector space.

ii) There exists a open subgroup of Γ_K which acts trivially on $\frac{D(V)}{\psi-1}$.

Proof. Let $M = (\mathfrak{m}_\mathbf{E} \otimes V)^{H_K}$, which is a lattice of $D(V)$ fixed by φ. If $x \in M$, the sereis $\sum_{n=1}^{+\infty} \varphi^n(x)$ converges in M, and by previous lemma, we have $x = (\psi - 1)(\sum_{n=1}^{+\infty} \varphi^n(x))$, which proves that $(\psi - 1)D(V)$ is contains M.

Since ψ is continuous, there exists $c \in \mathbf{N}$ such that $\psi(M) \subset \pi^{-c}M$ and since $\psi(\pi^{-pk}x) = \pi^{-k}x$, we have $\psi(\pi^{-pk}M) \subset \pi^{-k-c}M$. We deduce that if $n \geq b = [\frac{pc}{p-1}] + 1$, then $\psi = 0$ in $\frac{\pi^{-n+1}M}{\pi^{-n}M}$ and such that $\psi - 1$ is bijective on $\frac{\pi^{-n+1}M}{\pi^{-n}M}$. Since $D(V) = \bigcup_{n\in\mathbf{N}} \pi^{-n}M$, which implies the natural map from $\frac{\pi^{-b}M}{\psi-1}$ to $\frac{D(V)}{\psi-1}$ is an isomorphism.

To prove i), it suffices to note that $(\psi-1)M$ contained in M, which implies that $\frac{D(V)}{\psi-1}$ is a quotient of $\frac{\pi^{-b}M}{\psi-1}$. To prove ii), this is because that Γ_K fixes M and hence $\pi^k M$ for all $k\in\mathbf{Z}$ and the action of Γ_K is continuous on $D(V)$ and M is closed in $D(V)$, there exists an open subgroup of Γ_K acts trivially on $\frac{\pi^{-b}M}{\psi-1}$ since the module is endowed with discrete topology. □

Corollary 4.18. *If V be a $\mathbf{Z}_p$-representation of G_K, then $\frac{D(V)}{\psi-1}$ is a $\mathbf{Z}_p$-module of finite type.*

Proof. $\frac{D(V)}{\psi-1}/p\frac{D(V)}{\psi-1}=\frac{D(V)}{(p,\psi-1)}=\frac{D(V/p)}{\psi-1}$ is a $\mathbf{F}_p$-vector space of finite type by the preceding lemma, together with lemma 4.15, we get the conclusion. □

Hence we deduce ii) of proposition 4.9 and it remains to prove that $D(V)^{\psi=1}$ generate $D(V)$. We will need the following lemma.

Lemma 4.19. *If V be a $\mathbf{F}_p$-representation of G_K and X is a sub-$\mathbf{F}_p$-vector space of $D(V)^{\psi=1}$ of finite codimension, then X contains a basis of $D(V)$ over $\mathbf{E}_K$.*

Proof. Let $M=(\mathfrak{m}_{\mathbf{E}}\otimes V)^{H_K}$ as above. Note that by lemma 4.16, if $x\in M^{\psi=0}$, then the series $\sum_{n=0}^{+\infty}\varphi^n(x)$ converges in $D(V)$ to an element of $D(V)^{\psi=1}$. We denote it by $eul(x)$. Let $e_1,...,e_d$ be a basis of M over $\mathbf{E}_K^+$. Let r the codimension of X in $D(V)^{\psi=1}$. If $1\le i\le d$ and $j\ge 1$, let $z_{i,j}=eul(\varepsilon\varphi(\pi^j e_i))$. If i and $n\ge 1$ are fixed, the $\{z_{i,j}\}_{n\le j\le n+r}$ form a set of $r+1$ elements in $D(V)^{\psi=1}$ and since X is of codimension r in $D(V)^{\psi=1}$, we can find elements $\{a_{i,j}^{(n)}\}_{0\le j\le r}$ of $\mathbf{F}_p$ such that $f_{i,n}=\sum_{j=0}^r a_{i,j}^{(n)}z_{i,j+n}$ belongs to X. Let $\beta_{i,n}=\pi^n\sum_{j=0}^r a_{i,j}^{(n)}\pi^j$. We have $\lim_{n\to+\infty}(\varepsilon\varphi(\beta_{i,n}))^{-1}f_{i,n}=\varphi(e_i)$, which implies that the determinant of $f_{1,n},...,f_{d,n}$ in the basis $\varphi(e_1),...,\varphi(e_d)$ is nonzero if $n\gg 0$ and we have $f_{1,n},...,f_{d,n}$ form a basis of $D(V)$ over $\mathbf{E}_K$ if n is great enough. The lemma follows. □

Corollary 4.20. *If V is a $\mathbf{Z}_p$-representation of G_K, then $D(V)^{\psi=1}$ generates the $\mathbf{A}_K$-module $D(V)$.*

Proof. The snake lemma shows that the cokernel of the injective map $D(V)^{\psi=1}/pD(V)^{\psi=1}$ to $D(V/p)^{\psi=1}$ is identified with the p-torsion part of $D(V)/(\psi-1)$. In particular, it is of finite dimension over $\mathbf{F}_p$. By the preceeding lemma, we have $D(V)^{\psi=1}/pD(V)^{\psi=1}$ contains a basis of $D(V/p)$ over $\mathbf{E}_K$, which lifts to a basis in $D(V)^{\psi=1}$ that generates $D(V)$ over $\mathbf{A}_K$. □

5. Iwasawa theory and p-adic representations

5.1. **Iwasawa cohomology.** Recall that if $n\in\mathbf{N}$, we denote K_n the field $K(\varepsilon^{(n)})=K(\mu_{p^n})$. On the other hand, if $n\ge 1$ (resp. $n\ge 2$ if $p=2$), the group Γ_{K_n} is isomorphic to Z_p. We choose a generator γ_1 of Γ_{K_1} and put $\gamma_n=\gamma_1^{[K_N:K_1]}$ if $n\ge 1$ (if $p=2$, we can start from $n=2$), this makes γ_n a generator of Γ_{K_n}.

Let V be a p-adic representation of G_K. The Iwasawa cohomology groups $H^i_{\mathrm{Iw}}(K,V)$ are defined by $H^i_{\mathrm{Iw}}(K,V)=\mathbf{Q}_p\otimes_{\mathbf{Z}_p}H^i_{\mathrm{Iw}}(K,T)$ where T is any G_K-stable lattice of V and where

$$H^i_{\mathrm{Iw}}(K,T)=\varprojlim_{\mathrm{cor}_{K_{n+1}/K_n}}H^i(K_n,T)$$

Each of the $H^i(K,T)$ is a $\mathbf{Z}_p[\Gamma_K/\Gamma_{K_n}]$-module, and $H^i_{\mathrm{Iw}}(K,T)$ is then endowed with the structure of Λ_K-module. Roughly speaking, theses cohomology groups are where Euler system live (at least locally).

If V is a Z_p-representation or a p-adic representation of G_K, we endow $\Lambda_K \otimes_{\mathbf{Z}_p} V$ the natural diaganal action of G_K. If we consider $\Lambda_K \otimes_{\mathbf{Z}_p} V$ the space of measure of Γ_K with values in V, the measure $\sigma(\mu)$ is the map sends continuous map $f : \Gamma_K \mapsto V$ to the element

$$\int_{\Gamma_K} f(x)\sigma(\mu) = \sigma(\int_{\Gamma_K} f(\sigma x)\mu) \in V$$

If V is a $\mathbf{Z}_p$-representation or a p-adic representation of G_K and $k \in \mathbf{Z}$, we denote $V(k)$ the twist of V by the k-th power of the cyclotomic character and if $x \in V$, we denote $x(k)$ its image in $V(k)$.

If $\mu \in H^m(K, \Lambda_K \otimes_{\mathbf{Z}_p} V)$ and if $\tau \mapsto \mu_{\tau_1,\ldots,\tau_m}$ is a continuous m-cocycle represents μ, then $\tau \mapsto (\int_{\Gamma_{K_n}} \chi(x)^k \mu_{\tau_1,\ldots,\tau_m})(k)$ is a m-cycle of G_K with values in $V(k)$ whose class $(\int_{\Gamma_{K_n}} \chi(x)^k \mu)(k)$ in $H^m(K_n, V(k))$ does not depend on the choice of cocycle represent μ.

The Shapiro's lemma allows us to replace the projective limit in the definition of $H^m_{\mathrm{Iw}}(K, V)$ by a group cohomology.

Proposition 5.1. *Let V be a $\mathbf{Z}_p$-representation or a p-adic representation of G_K.If $m \in \mathbf{N}$ and $k \in \mathbf{Z}$, the map sends μ to $(\ldots, \int_{K_n} \chi(x)^k \mu(k), \ldots)$ is an isomorphism from $H^i(K, \Gamma_K \otimes_{\mathbf{Z}_p} V)$ to $H^i_{\mathrm{Iw}}(K, V(k))$. In particular, if $k \in \mathbf{Z}$, the cohomology group $H^m_{\mathrm{Iw}}(K, V)$ and $H^m_{\mathrm{Iw}}(K, V(k))$ are isomorphic.*

Proof. The case of $\mathbf{Q}_p$ follows from the case of $\mathbf{Z}_p$ by tensoring $\mathbf{Q}_p$. If M is a G_{K_n}-module, we denote $\mathrm{Ind}_{K_n}^K M$ the set of continuous maps from G_K to M satisfies $a(hx) = ha(x)$ if $h \in G_{K_n}$. The module $\mathrm{Ind}_{K_n}^K M$ is provided with a continuous action of G_K, the image ga of a by $g \in G_K$, is given by the formula $(ga)(x) = a(xg)$. If M is a G_K module, and $a \in \mathrm{Ind}_{K_n}^K M$, the map sends $x \in G_K$ to $x^{-1}(a(x))$ is constant modulo G_{K_n}, and the map of $\mathrm{Ind}_{K_n}^K M$ to $\mathbf{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M$ which sends a to $\sum_{x \in \mathrm{Gal}(K_n/K)} x^{-1}(ax)\delta_{x^{-1}}$ is an isomorphism of G_K-modules. By Shapiro's lemma, we have an canonical isomorphism from $H^i(K, \mathbf{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M)$ to $H^i(K_n, M)$. On the other hand, the corestriction map from $H^i(K_{n+1}, M)$ to $H^i(K_n, M)$ is derived from the previous isomorphosm and the natural map form $\mathbf{Z}_p[\mathrm{Gal}(K_{n+1}/K)]$ to $\mathbf{Z}_p[\mathrm{Gal}(K_n/K)]$. we thus deduce the natural map from $H^i(K, \Lambda_K \otimes M)$ to

$$\varprojlim H^i(K, (\Lambda/\omega_n) \otimes M) = \varprojlim H^i(K_n, M).$$

It remains to show that this map is an isomorphism.

Surjectivity is a obvious. To prove injectivity, it suffices to verify that the map from $H^i(K, \Lambda_K \otimes M)$ to $H^i(K, \Lambda/(\omega_n, p^n) \otimes M)$ is injective. Since $\Lambda_K = \varprojlim \Lambda_K/(\omega_n, p^n)$, it suffices to show that $H^i(K, (\Lambda/\omega_n, p^n) \otimes M)$ satifies the Mittag-Leffler condition (c.f. [NSK]), which is obvious since the group is finite. ∎

By lemma 4.7, the map ι_{ψ,γ_n} identifies $\frac{D(V)^{\psi=1}}{\gamma_n - 1}$ with a subgroup of $H^1(K_n, V)$ if Γ_{K_n} is torsion free, we thus obtained a map $h^1_{K_n,V} : D(V)^{\psi=1} \to H^1(K_n, V)$. Explicitly, if $y \in D(V)^{\psi=1}$, then $(\varphi - 1)y \in D(V)^{\psi=0}$ and since $\gamma_n - 1$ is invertible on $D(V)^{\psi=0}$, there exist $x_n \in D(V)^{\psi=0}$ satisfies $(\gamma_n - 1)x_n = (\varphi - 1)y$ (i.e. $(x_n, y) \in Z^1_{\varphi,\gamma_n}(K_n, V)$). On the other hand, lemma 4.2 implies that the image $\iota_{\psi,n}(y)$ and $\log^0_p(\gamma_n)\iota_{\varphi,\gamma_n}(x_n, y)$ in $H^1(K_n, V)$ does not depend on the choice of γ.

By lemma 5.3 below, we have $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$. On the other hand, if Γ_{K_n} is no longer torsion free, we define $h^1_{K_n,V}$ by the relation $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$. By this way, we associate every element in $D(V)^{\psi=1}$ to a collection of Galois cohomology class $h^1_{K_n,V}(y) \in H^1(K_n, V)$ for $n \geq 1$. The main result of this section is:

Theorem 5.2. *(Fontaine) Let V be a $\mathbf{Z}_p$-representation or a p-adic representation of G_K.*
i) If $y \in D(V)^{\psi=1}$, then $(..., h^1_{K_n,V}(y), ...) \in H^1_{\mathrm{Iw}}(K, V)$.
ii) The map $\mathrm{Log}^*_{V^*(1)} : D(V)^{\psi=1} \to H^1_{\mathrm{Iw}}(K, V)$ defined by above is an isomorphism.

5.2. **Corestriction and (φ, Γ)-modules.** i) of theorem 5.2 is a consequence of the following lemma.

Lemma 5.3. *If $n \geq 1$, let*

$$T_{\gamma,n} : H^1(C_{\varphi,\gamma_n}(K_n, V)) \to H^1(C_{\varphi,\gamma_{n-1}}(K_{n-1}, V))$$

the map induced by $(x, y) \in Z^1(C_{\varphi,\gamma_n}(K_n, V))$ to $(\frac{\gamma_n-1}{\gamma_{n-1}-1}x, y) \in Z^1(C_{\varphi,\gamma_{n-1}}(K_{n-1}, V))$. Then the diagram

$$\begin{array}{ccc} H^1(C_{\varphi,\gamma_n}(K_n, V)) & \xrightarrow{T_{\gamma,n}} & H^1(C_{\varphi,\gamma_{n-1}}(K_{n-1}, V)) \\ \downarrow{\scriptstyle \iota_{\varphi,\gamma_n}} & & \downarrow{\scriptstyle \iota_{\varphi,\gamma_{n-1}}} \\ H^1(K_n, V) & \xrightarrow{cor_{K_n/K_{n-1}}} & H^1(K_{n-1}, V) \end{array}$$

is commutative.

Proof. Recall that if G is a group, M is a G-module and H a subgroup of finite index of G, the corestriction map $\mathrm{cor} : H^1(H, N) \to H^1(G, M)$ can be written in the following way: let $X \subset G$ is a system of representatives of G/H and, if $g \in G$, let τ_g is the permutation of X defined by $\tau_g(x)H = gxH$ if $x \in X$. If $c \in H^1(H, M)$ and $h \mapsto c_h$ is a cocycle which represents c, then

$$g \to \sum_{x \in X} \tau_g(x)(c_{\tau_g(x)^{-1}gx})$$

is a cocycle of G with values in M whose class in $H^1(G, M)$ does not depend on the choice of X and is equal to $\mathrm{cor}(c)$.

If N is a G-submodule of M such that the image of c in $H^1(H, N)$ is trivial (i.e. there exists $b \in N$ such that we have $c_h = (h - 1)b$ for all $h \in H$), then $\mathrm{cor}(c)$ is the class of the cocycle $g \mapsto (g - 1)(\sum_{x \in X} xb)$.

In particular, we put $G = G_{K_{n-1}}$, $H = G_{K_n}$ and, if $\widetilde{\gamma}_{n-1}$ is a lift of γ_{n-1} in $G_{K_{n-1}}$, we take $X = \{1, \widetilde{\gamma}_{n-1}, ..., \widetilde{\gamma}^{p-1}_{n-1}\}$. Take $N = \mathrm{Frac}(\mathbf{Z}_p[[G_{K_{n-1}}]]) \otimes_{\mathbf{Z}_p[[G_{K_{n-1}}]]} (A \otimes_{\mathbf{Z}_p} V)$. If $(x, y) \in Z^1(C_{\varphi,\gamma}(K_n, V))$ and if $b \in \mathbf{A} \otimes T$, the cocycle $c_{x,y}$ is given by the formula $c_{x,y}(\tau) = (\tau - 1)c$, where $c = \frac{y}{\widetilde{\gamma}_n-1} - b \in N$. It follows that $\mathrm{cor}_{K_n/K_{n-1}}(\iota_{\varphi,\gamma_n}(x, y))$ is represented by the cocycle

$$\tau \to (\sigma - 1)(\sum_{i=0}^{p-1} \widetilde{\gamma}^i_{n-1}c) = (\sigma - 1)(\frac{y}{\widetilde{\gamma}_{n-1} - 1} - \sum_{i=0}^{p-1} \widetilde{\gamma}^i_{n-1}b)$$

and since

$$(\varphi - 1)(\sum_{i=0}^{p-1} \widetilde{\gamma}^i_{n-1}b) = \sum_{i=0}^{p-1} \widetilde{\gamma}^i_{n-1}((\varphi - 1)b) = \frac{\widetilde{\gamma}^p_{n-1} - 1}{\widetilde{\gamma}_{n-1} - 1}x = \frac{\gamma_n - 1}{\gamma_{n-1} - 1}x,$$

we see that this cocycle is just $\iota_{\varphi,\gamma_{n-1}}(T_{\gamma,n}(x, y))$, and the conclusion follows. $\square$

Remark 5.4. One can also hide the explicit calculation by noting that, if $n \geq 1$, the diagram

$$\begin{array}{cccccccccc}
C_{\varphi,\gamma_n}(K_n,V): 0 & \longrightarrow & D(V) & \longrightarrow & D(V)\oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \frac{\gamma_n-1}{\gamma_{n-1}-1}} & & \downarrow{\scriptstyle (\frac{\gamma_n-1}{\gamma_{n-1}-1}, id)} & & \downarrow{\scriptstyle id} & & \\
C_{\varphi,\gamma_{n-1}}(K_n,V): 0 & \longrightarrow & D(V) & \longrightarrow & D(V)\oplus D(V) & \longrightarrow & D(V) & \longrightarrow & 0
\end{array}$$

is commutative and functorial on V and induces a homomorphism of cohomology group from $H^*(K_n,\cdot)$ to $H^*(K_{n-1},\cdot)$ which coincides with with the corestriction map at $*=0$ and hence is corestriction map.

5.3. **Interpretation of $D(V)^{\psi=1}$ and $\frac{D(V)}{\psi-1}$ in Iwasawa theory.** We turn to prove ii) of theorem 5.2. The lemma 5.3 implies that the map $(\iota_{\psi,\gamma_n})_{n\in\mathbf{N}}$ induces an isomorphism from the projective limit of $H^1(C_{\psi,\gamma_n}(K_n,V))$ with respect to the map $T_{\gamma,n}$ to $H^1_{\mathrm{Iw}}(K,V)$. On the other hand, lemma 4.7, implies by passing to the projective limit, that we have an exact sequence:

$$0 \longrightarrow \varprojlim \frac{D(V)^{\psi=1}}{\gamma_n-1} \longrightarrow \varprojlim H^1(C_{\psi,\gamma_n}(K_n,V)) \longrightarrow \varprojlim (\frac{D(V)}{\psi-1})^{\Gamma_{K_n}}$$

The projective limit of $\frac{D(V)^{\psi=1}}{\gamma_n-1}$ is by the natural maps induced by the identity on $D(V)^{\psi-1}$ and that of $(\frac{D(V)}{\psi-1})^{\gamma_n=1}$ with respect to the map

$$\frac{\gamma_{n+1}-1}{\gamma_n-1} : (\frac{D(V)}{\psi-1})^{\gamma_n=1} \to (\frac{D(V)}{\psi-1})^{\gamma_{n-1}=1}.$$

Hence ii) of theorem 5.2 is followed by the following proposition:

Proposition 5.5. *If V is a $\mathbf{Z}_p$-representation of G_K, then*

i) The natural map from $D(V)^{\psi=1}$ to $\varprojlim \frac{D(V)^{\psi=1}}{\gamma_n-1}$ is an isomorphism.

ii) $\varprojlim(\frac{D(V)}{\psi-1})^{\gamma_n=1}=0$

Proof. i) Let $(x_n)_{n\in\mathbf{N}} \in \varprojlim \frac{D(V)^{\psi=1}}{\gamma_n-1}$. The compactness of $D(V)^{\psi=1}$ [c.f. proposition 4.9 i)] implies that the sequence x_n admits a accumulation points $x \in D(V)^{\psi=1}$ and the image of $x \in \varprojlim \frac{D(V)^{\psi=1}}{\gamma_n-1}$ is by construction $(x_n)_{n\in\mathbf{N}}$. The natural map from $D(V)^{\psi=1}$ to $\varprojlim \frac{D(V)^{\psi=1}}{\gamma_n-1}$ is hence surjective.

By the compactness of $D(V)^{\psi=1}$ and the fact that if $x\in D(V)$, then $(\gamma_n-1)x$ tend to 0 when n tend to $+\infty$ implies that if U is open in $D(V)$ fixed by Γ, then there exist $n_U\in\mathbf{N}$ such that $(\gamma_n-1)D(V)^{\psi=1}\subset U$ if $n\geq n_V$. This implies that $\bigcap_{n\in\mathbf{N}}(\gamma_n-1)(D(V)^{\psi=1})=\{0\}$ and we prove the injectivity.

ii) $\frac{D(V)}{\psi-1}$ is a free $\mathbf{Z}_p$-module of finite rank [c.f. proposition 4.9 ii)], the sequence $(\frac{D(V)}{\psi-1})^{\gamma_n=1}$ is stationary since it is increasing. One can deduce the fact that there exists $n_0\in\mathbf{N}$ such that $\frac{\gamma_n-1}{\gamma_{n-1}-1}$ is multiplication by p on $(\frac{D(V)}{\psi-1})^{\gamma_n=1}$ if $n\geq n_0$, which proves the statement since $\frac{D(V)}{\psi-1}$ has no p-divisible element [c.f. lemma 4.15]. □

Remark 5.6. We have $H^2(K_n,V)\cong H^2(C_{\psi,\gamma_n}(K_n,V))=\frac{D(V)}{(\psi-1,\gamma_n-1)}$. We deduce that if V is a $\mathbf{Z}_p$-representation, then $H^2_{\mathrm{Iw}}(K,V)$ is a projective limit of $\frac{D(V)}{(\psi-1,\gamma_n-1)}$ since $\frac{D(V)}{\psi-1}$ is a $\mathbf{Z}_p$-module of finite type which Γ_K acts continuously by ii) of lemma 4.17, the natural map from $\frac{D(V)}{\psi-1}$ to the projective limit of $\frac{D(V)}{(\psi-1,\gamma_n-1)}$ is an isomorphism, this proves that $\frac{D(V)}{\psi-1}$ is identified with $H^2_{\mathrm{Iw}}(K,V)$.

By above proposition, one can summarize the the above results as follows:

Corollary 5.7. *The complex of $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-modules*

$$0 \longrightarrow D(V) \xrightarrow{1-\psi} D(V) \longrightarrow 0$$

computes the Iwasawa cohomology of V.

There is a natural projection map $\mathrm{pr}_{K_n,V} : H^i_{\mathrm{Iw}}(K,V) \to H^i(K_n,V)$ and when $i=1$ it is of course equal to the composition of:

$$H^1_{\mathrm{Iw}}(K,V) \longrightarrow D(V)^{\psi=1} \xrightarrow{h^1_{K_n,V}} H^1(K_n,V).$$

The $H^i_{\mathrm{Iw}}(K,V)$ have been studied in detail by Perrin-Riou, who proved the following

Proposition 5.8. *If V is a p-adic representation of G_K, then*

- *i) The torsion submodule of $H^1_{\mathrm{Iw}}(K,V)$ is a $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-module isomorphic to V^{H_K} and $H^1_{\mathrm{Iw}}(K,V)/V^{H_K}$ is a free $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-module whose rank is $[K:\mathbf{Q}_p] dim_{\mathbf{Q}_p} V$.*
- *ii) $H^2_{\mathrm{Iw}}(K,V)$ is isomorphic to $V(-1)^{H_K}$ as $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-module. In particular, it is torsion.*
- *iii) $H^i_{\mathrm{Iw}}(K,V) = 0$ when $i \neq 1,2$.*

Proof. See [Per94, 3.2.1]. □

6. De Rham representations and overconvergent representations

6.1. **De Rham representations and crystalline representations.** Recall $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}^+$ and if $x \in \widetilde{\mathbf{E}}^+$. We define the homomorphism $\theta : \widetilde{\mathbf{A}}^+ \to \mathscr{O}_{\mathbf{C}_p}$ by

$$\theta(\sum_{k\geq 0} p^k [x_k]) = \sum_{k\geq 0} p^k x_k^{(0)}$$

One can show that this is a surjective map and $\ker(\theta : \widetilde{\mathbf{A}}^+ \to \mathscr{O}_{\mathbf{C}_p})$ is generated by $\omega = \pi/\varphi^{-1}(\pi)$.

We can extend θ to a homomorphism from $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[\frac{1}{p}]$ to $\mathbf{C}_p$, and we denote $\mathbf{B}^+_{\mathrm{dR}}$ the ring $\varprojlim \widetilde{\mathbf{B}}^+/(\ker\theta)^n$, thus θ can be extended by continuity to a homomorphism from $\mathbf{B}^+_{\mathrm{dR}}$ to $\mathbf{C}_p$. This makes $\mathbf{B}^+_{\mathrm{dR}}$ a discrete valuation ring with maximal ideal $\ker\theta$ and residue field $\mathbf{C}_p$. The action of $G_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{A}}^+$ extend by continuity to an action of $G_{\mathbf{Q}_p}$ on $\mathbf{B}^+_{\mathrm{dR}}$. The series $\log[\varepsilon] = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}\pi^n$ converges in $\mathbf{B}^+_{\mathrm{dR}}$ to an element which we denote it by t, which is a generator of $\ker\theta$ with an $G_{\mathbf{Q}_p}$ action defined by $\sigma(t) = \chi(\sigma)t$ where $\sigma \in G_{\mathbf{Q}_p}$. This element can be viewed as p-adic analogy of $2\pi i$.

We put $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}^+_{\mathrm{dR}}[t^{-1}]$, this makes $\mathbf{B}_{\mathrm{dR}}$ a field with filtration defined by $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} = t^i\mathbf{B}^+_{\mathrm{dR}}$. This filtration is stable by the action of G_K.

Let K is a finite extension of $\mathbf{Q}_p$ and V be a p-adic representation of G_K. We say V is de Rham if the $\mathbf{B}_{\mathrm{dR}}$-admissible, which is equivalent to the K-vector space $\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is of dimension $d = \dim_{\mathbf{Q}_p}(V)$. On the other hand, $\mathbf{D}_{\mathrm{dR}}(V)$ is endowed with a filtration induced by $\mathbf{B}_{\mathrm{dR}}$. We have $\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$ if $i \ll 0$ and $\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V) = \{0\}$ if $i \gg 0$.

The ring $\mathbf{B}^+_{\text{cris}}$ is defined by

$$\mathbf{B}^+_{\text{cris}} = \{\sum_{n\geq 0} a_n \frac{\omega^n}{n!} \mid a_n \in \widetilde{\mathbf{B}}^+ \text{ is a sequence converges to } 0\},$$

and $\mathbf{B}^+_{\text{cris}}[\frac{1}{t}]$. The ring $\mathbf{B}_{\text{cris}}$ is a subring of $\mathbf{B}_{\text{dR}}$ stable under $G_{\mathbf{Q}_p}$ containing t and the action of φ on $\widetilde{\mathbf{B}}^+$ is extended by continuity to an action of $\mathbf{B}^+_{\text{cris}}$. In particular, we have $\varphi(t) = pt$.

We say V is crystalline if it is $\mathbf{B}_{\text{cris}}$-admissible, which is equivalent to the $F = K \cap \mathbf{Q}_p^{ur}$-vector space $\mathbf{D}_{\text{cris}}(V) = (\mathbf{B}_{\text{cris}} \otimes V)^{G_K}$ is of dimension $d = \dim_{\mathbf{Q}_p}(V)$. The action of φ on $\mathbf{B}_{\text{cris}}$ commutes with the action of $G_{\mathbf{Q}_p}$, which endows $\mathbf{D}_{\text{cris}}$ a natural semi-linear action of φ.

We have $(\mathbf{B}_{\text{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K} = \mathbf{D}_{\text{dR}}(V) = K \otimes_F \mathbf{D}_{\text{cris}}(V)$, thus the crystalline representation is de Rham and $K \otimes_F \mathbf{D}_{\text{cris}}(V)$ is a filtered K-vector space. Hence if V is de Rham (resp. crystalline) and $k \in \mathbf{Z}$, so is $V(k)$, and we have $\mathbf{D}_{\text{dR}}(V(k)) = t^{-k}\mathbf{D}_{\text{dR}}(V)$ (resp. $\mathbf{D}_{\text{cris}}(V(k)) = t^{-k}\mathbf{D}_{\text{cris}}(V)$).

6.2. **Overconvergent elements.** Every element x of $\widetilde{\mathbf{B}}$ can be written uniquely as the form $\sum_{k\gg -\infty} p^k[x_k]$, where x_k is element of $\widetilde{\mathbf{E}}$ and the series converges in $\mathbf{B}^+_{\text{dR}}$ if and only if the series $\sum_{k\gg-\infty} p^k x_k^{(0)}$ converges in $\mathbf{C}_p$, which is equivalent to $k + \nu_E(x_k)$ tends to $+\infty$ as k tends to $+\infty$. More generally, if $n \in \mathbf{N}$, $\varphi^{-n}(x)$ converges if and only if $k + p^{-n}\nu_E(x_k)$ tends to $+\infty$ as k tends to $+\infty$.

For $r \geq 0$, we set

$$\widetilde{\mathbf{B}}^{\dagger,r} = \{x \in \widetilde{\mathbf{B}} \mid \lim_{k\to+\infty} \nu_E(x_k) + \frac{pr}{p-1}k = +\infty\}.$$

This makes $\widetilde{\mathbf{B}}^{\dagger,r}$ into an intermediate ring between $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$. We denote $\widetilde{\mathbf{B}}^\dagger = \cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}$, which is a subfield of $\widetilde{\mathbf{B}}$ with action of G_K and φ. On the other hand, we have a well-defined injection map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n} \to \mathbf{B}^+_{\text{dR}}$, where $r_n = p^{n-1}(p-1)$.

We denote $\widetilde{\mathbf{A}}^{\dagger,r} = \widetilde{\mathbf{B}}^{\dagger,r} \cap \widetilde{\mathbf{A}}$, that is, the subring of elements $x = \sum_{k=0}^{+\infty} p^k[x_k]$ of $\widetilde{\mathbf{A}}$ such that $\nu_E(x_k) + \frac{pr}{p-1}k$ tends to $+\infty$ as k tends to $+\infty$. We have $\widetilde{\mathbf{B}}^{\dagger,n} = \widetilde{\mathbf{A}}^{\dagger,n}[\frac{1}{p}]$.

By putting $\mathbf{B}^\dagger = \mathbf{B} \cap \widetilde{\mathbf{B}}^\dagger$, $\mathbf{A}^{\dagger,n} = \mathbf{A} \cap \widetilde{\mathbf{A}}^{\dagger,r}$ and $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,n}$, we define a subring $\mathbf{B}^\dagger$ of $\mathbf{B}$ fixed by φ and $G_{\mathbf{Q}_p}$, and if $n \in \mathbf{N}$, subrings $\mathbf{A}^{\dagger,r}$ and $\mathbf{B}^{\dagger,r}$ of $\mathbf{B}$ are fixed by $G_{\mathbf{Q}_p}$. By construction, $\varphi^{-n}(\mathbf{B}^{\dagger,r_n})$ is naturally identified with a subring of $\mathbf{B}^+_{\text{dR}}$. Finally, if K is a finite extension of $\mathbf{Q}_p$, we set $\mathbf{B}^\dagger_K = (\mathbf{B}^\dagger)^{H_K}$, $\mathbf{A}^{\dagger,r}_K = (\mathbf{A}^{\dagger,r})^{H_K}$ and $\mathbf{B}^{\dagger,r}_K = (\mathbf{B}^{\dagger,r})^{H_K}$.

Let e_K the ramification index of K_∞ over F_∞ and $F' \subset K_\infty$ be the maximal unramified extension of $\mathbf{Q}_p$ contained in K_∞. Let $\overline{\pi}_K$ be a uniformizer of $\mathbf{E}_K = k_{F'}((\overline{\pi}_K))$ and $\overline{P}_K \in E_{F'}$ be a minimal polynomial of $\overline{\pi}_K$ and $\delta = \nu_E(\overline{P}'(\overline{\pi}_K))$. Choose $P_K \in \mathbf{A}_{F'}$ such that it modulo p is $\overline{P}_K$. By Hensel's lemma, there exists a unique $\pi_K \in \mathbf{A}_K$ such that $P_K(\pi_K) = 0$ and $\pi_K = \overline{\pi}_K$ modulo p. In particular, if $K = F'$, one can take $\pi_K = \pi$.

The terminology "overconvergent" can be explained by the following proposition:

Proposition 6.1. *If $r \geq r(K)$, then the map $f \mapsto f(\pi_K)$ from $\mathcal{B}^{e_K r}_{F'}$ to $\mathbf{B}^{\dagger,r}_K$ is an isomorphism, where $\mathcal{B}^\alpha_{F'}$ is the set of power series $f(T) = \sum_{k\in\mathbf{Z}} a_k T^k$ such that a_k is a bounded sequence of elements of F', and such that $f(T)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha} \leq \|T\| < 1\}$.*

Proof. See lemma II.2.2 [CC98]. □

Proposition 6.2. *If K is a finite extension of $\mathbf{Q}_p$, then $\mathbf{B}^\dagger_K$ is an extension of $\mathbf{B}^\dagger_{\mathbf{Q}_p}$ of degree $[\mathbf{B}_K : \mathbf{B}_{Q_p}] = [K_\infty : \mathbf{Q}_p(\mu_{p^\infty})]$ and there exists $a(K) \in \mathbf{N}$ such that if $n \geq a(K)$, then $\varphi^{-n}(\mathbf{B}^{\dagger,r_n}_K) \subset K_n[[t]]$, where $r_n = p^{n-1}(p-1)$.*

Proof. In the case K is unramified over $\mathbf{Q}_p$, one can follow proposition 6.1 i) using the fact that $K_n[[t]]$ is closed in $\mathbf{B}^+_{\mathrm{dR}}$ and the formula

$$\varphi^{-n}(\pi) = \varphi^{-n}([\varepsilon]-1) = [\varepsilon^{p^{-n}}] - 1 = \varepsilon^{(n)} \exp(t/p^n) - 1 \in K_n[[t]].$$

For general case, by remark 3.2, there exists $\omega = (\omega^{(n)})_{n\in\mathbf{N}} \in \varprojlim \mathscr{O}_{K_n}$ such that $\omega^{(n)}$ is a uniformizer of $\mathscr{O}_{K_n}$ if n large enough and then $\overline{\pi}_K = \iota_K(\omega)$ is a uniformizer of $\mathbf{E}_K$ such that it is totally ramified of degree e_K over $\mathbf{E}_{F'}$. Let $\overline{P}(X) = X^{e_K} + \overline{a}_{e_K-1}X^{e_K-1} + ... + \overline{a}_0 \in \mathbf{E}_{F'}[X]$ be the minimal polynimial of $\overline{\pi}_K$ over $\mathbf{E}_{F'}$ and let $\delta = \nu_E(\overline{P}'(\overline{\pi}_K))$. If $0 \le i \le e_K - 1$, let $a_i \in \mathscr{O}_F[[\pi]] \subset \mathbf{A}_F$ whose reduction modulo p is $\overline{a}_i$ and let $P(X) = X^{e_K} + a_{e_K-1}X^{e_K-1} + ... + a_0 \in \mathbf{A}_F[X]$. By Hensel's lemma, the equation $P(X) = 0$ has a unique solution π_K in A_K whose reduction modulo p is $\overline{\pi}_K$ and we can write it in the form

$$\pi_K = [\overline{\pi}_K] + \sum_{i=1}^{+\infty} p^i[\alpha_i], \tag{1}$$

where α_i are elements of $\widetilde{\mathbf{E}}$ verify $\nu_E(\alpha_i) \ge -i\delta$. In particular, $\pi_K \in \mathbf{A}_K^{\dagger,r}$ if $\frac{p}{p-1}r \ge \delta$, hence we have $\mathbf{A}_K^{\dagger,r} = \mathbf{A}_F^{\dagger,r}[\pi_K]$ if $\frac{p}{p-1}r \ge \delta$. Thus it suffices to prove it when n large enough, then $\pi_{K,n} = \varphi^{-n}(\pi_K) \in K_n[[t]]$.

Let P_n (resp. Q_n) be polynomial obtained by the map $\theta \circ \varphi^{-n}$ (resp. φ^{-n}) apply on the coefficients of P, which is a polynomial with coefficients in $\mathscr{O}_{F_n}$ (resp. $F_n[[t]]$) with $\theta(\pi_{K,n})$ (resp. $\pi_{K,n}$) as a root. On the other hand, by definition of ι_K (c.f. 3.2), we have $\nu_p(\omega^{(n)} - \overline{\pi}_K^{(n)}) \ge \frac{1}{p}$ if n large enough and formula (1) shows that $\nu_p(\theta(\pi_{K,n}) - \overline{\pi}_K^{(n)}) \ge (1 - \frac{\delta}{p^n})$. Then we have $\nu_p(P_n(\omega^{(n)})) \ge \frac{1}{p}$ if n large enough and

$$\nu_p(P_n'(\omega^{(n)}) = \frac{1}{p^n}\nu_E(P'(\overline{\pi}_K)) = \frac{\delta}{p^n} < \frac{1}{2p}$$

if n large enough. By Hensel's lemma, the equation $P_n(X) = 0$ has a unique solution in $\mathbf{C}_p$ close to $\omega^{(n)}$ and hence belongs to $\mathscr{O}_{K_n}$ since $\omega^{(n)}$ and the coefficients of P_n do. We deduce that $\theta(\pi_{K,n})$ belongs to K_n. By using the Hensel's lemma again, one can show that Q_n has a unique solution in $\mathbf{B}^+_{\mathrm{dR}}$ whose image by θ is $\theta(\pi_{K,n})$ and thus belongs to $K_n[[t]]$. $\square$

We endow $\mathbf{B}_{Q_p}$ the differential operator ∂ defined by continuity and the derivation $\partial\pi = 1 + \pi$. We therefore have $\partial = [\varepsilon]\frac{d}{d\pi} = \frac{d}{dt}$ (Note that $t \notin \mathbf{B}_{Q_p}$). The derivation can be extended uniquely to a maximal unramified extension of $\mathbf{B}_{Q_p}$ in $\widetilde{\mathbf{B}}$, hence by continuity to a derivation ∂ from $\mathbf{B}$ to $\mathbf{B}$.

Lemma 6.3. *If K is a finite extension of $\mathbf{Q}_p$, there exists $m(K) \in \mathbf{Z}$ such that, if $n \ge m(K)$ and x in $\mathbf{B}_K^{\dagger,r_n}$, then*

i) $\partial x \in \mathbf{B}_K^{\dagger,r_n}$.
ii) $\varphi^{-n}(\partial x) = p^n \partial(\varphi^{-n}(x))$.

Proof. If $K = \mathbf{Q}_p$, explicit calculation using proposition 6.1 i), shows that we can take $m(K) = 1$. For the general case, let α be a generator of $\mathbf{B}_K^{\dagger}$ over $\mathbf{B}_{\mathbf{Q}_p}^{\dagger}$ and P be its minimal polynomial. The identity,

$$0 = \partial(P(\alpha)) = P'(\alpha)\partial\alpha + \partial P(\alpha),$$

where ∂P is the polynomial obtained by applying ∂ on the coefficients of P, shows that $\partial\alpha = -\frac{\partial P(\alpha)}{P'(\alpha)} \in \mathbf{B}_K^\dagger$. It is then possible to take $m(K)$ any integer such that $\mathbf{B}_K^{\dagger,m(K)}$ contains $\partial\alpha$ and α.

For ii), it suffices to note that $\varphi^{-n}\circ\partial$ is $p^n\partial\circ\varphi^{-n}$ are two derivations of $\mathbf{B}_K^{\dagger,r_n}$ coincides on $\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r_n}$ by

$$\varphi^{-n}\circ\partial([\varepsilon]) = \varphi^{-n}([\varepsilon]) = \varepsilon^{(n)}\exp(p^{-n}t)$$
$$p^n\partial\circ\varphi^{-n}([\varepsilon]) = p^n\frac{d}{dt}(\varepsilon^{(n)}\exp(p^{-n}t)) = \varepsilon^{(n)}\exp(p^{-n}t).$$

□

6.3. **Overconvergent representations.**

Definition 6.4. If V is a p-adic representation of G_K, we set

$$\mathbf{D}^\dagger(V) = (\mathbf{B}^\dagger\otimes_{\mathbf{Q}_p} V)^{H_K} \quad \textit{and} \quad \mathbf{D}^{\dagger,r}(V) = (\mathbf{B}^{\dagger,r}\otimes_{\mathbf{Q}_p} V)^{H_K}$$

We have $\dim_{\mathbf{B}_K^\dagger}\mathbf{D}^\dagger(V) \le \dim_{\mathbf{Q}_p}V$ and we say that V is overconvergent if the equality holds, which is equivalent to $D(V)$ has a basis over $\mathbf{B}_K$ made up of elements of $\mathbf{D}^\dagger(V)$.

Proposition 6.5.

i) Every p-adic representations of G_K is overconvergent.

ii) There exists $r(V)$ such that $D(V)^{\psi=1}\subset \mathbf{D}^{\dagger,r(V)}(V)$.

iii) If V is overconvergent and $n\in\mathbf{N}$, then γ_n-1 admits a an continuous inverse on $\mathbf{D}^\dagger(V)^{\psi=0}$. Moreover, there exists $n_2(V)$ such that if $n\ge n_2(V)$, then

$$(\gamma_n-1)^{-1}(\mathbf{D}^{\dagger,r_n}(V)^{\psi=0})\subset \mathbf{D}^{\dagger,r_{n+1}}(V)^{\psi=0}$$

Proof. i), iii) see [CC98]. ii) follows from lemma 4.14. □

7. Explicit reciprocity laws and de Rham Representation

7.1. **The Bloch-Kato exponential map and its dual.** Let K be a finite extension of $\mathbf{Q}_p$ and V a p-adic representation of G_K. We have fundamental exact sequence

$$0\longrightarrow \mathbf{Q}_p\longrightarrow \mathbf{B}_{\mathrm{cris}}^{\varphi=1}\longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+\longrightarrow 0$$

(c.f. [Col98, proposition III 3.5]). Tensoring this exact sequence with V and taking the invariant under the action of G_K, we obtain:

$$0\longrightarrow V^{G_K}\longrightarrow \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}\longrightarrow ((\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+)\otimes V)^{G_K}\longrightarrow H^1_e(K,V)\longrightarrow 0$$

where we denote $H^1_e(K,V)$ the kernel of the natural map from $H^1(K,V)$ to $H^1(K,\mathbf{B}_{\mathrm{cris}}^{\varphi=1}\otimes V)$. We denote the isomorphism induced by connecting homomorphism

$$\exp_{K,V}: \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)+\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}}\longrightarrow H^1_e(K,V)\subset H^1(K,V)$$

the Bloch-Kato exponential of V over K and we denote its inverse by

$$\log_{K,V}: H^1_e(K,V)\longrightarrow \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)+\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}}$$

the Bloch-Kato logarithm of V over K. Moreover, if V is de Rham and $k\gg 0$, then $\exp_{K,V(k)}$ is an isomorphism from $\mathbf{D}_{\mathrm{dR}}(V(k))$ to $H^1(K,V(k))$.

The choice of t gives an isomorphism from $\mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1)) = t^{-1}K$ to K. If V is a p-representation of G_K, the couple $[,]_{\mathbf{D}_{\mathrm{dR}}(V)}$ is defined by compositing the maps

$$\mathbf{D}_{\mathrm{dR}}(V) \otimes \mathbf{D}_{\mathrm{dR}}(V^*(1)) \cong \mathbf{D}_{\mathrm{dR}}(V \otimes V^*(1)) \longrightarrow \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong K \xrightarrow{\mathrm{Tr}_{K/Qp}} \mathbf{Q}_p$$

is non-degenerate, hence $\mathbf{D}_{\mathrm{dR}}(V^*(1))$ can be naturally identified with the dual of $\mathbf{D}_{\mathrm{dR}}(V)$. Similarly, via the cup product

$$H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbf{Q}_p(1)) = \mathbf{Q}_p,$$

$H^1(K,V^*(1))$ is naturally identified with the dual of $H^1(K,V)$. This allows us to view the map $\exp^*_{K,V^*(1)}$ as transpose of the map $\exp_{K,V^*(1)} : \mathbf{D}_{\mathrm{dR}}(V^*(1)) \to H^1(K,V^*(1))$ as a map from $H^1(K,V)$ to $\mathbf{D}_{\mathrm{dR}}(V)$, whose image is contained in $\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V))$. If V is de Rham and $k \gg 0$, the map $\exp^*_{K,V^*(1+k)}$ is an isomorphism from $H^1(K,V(-k))$ to $\mathbf{D}_{\mathrm{dR}}(V(-k))$.

If $x \in K_\infty$ and $n \in \mathbf{N}$, then $\frac{1}{p^m}\mathrm{Tr}_{K_m/K_n}(x)$ does not depend on the choice of integer $m \geq n+1$ such that x is belongs to K_m. We denote T_n the above $\mathbf{Q}_p$-linear map from K_∞ to K_n. If $n \geq 1$ and $x \in K_n$, then $\mathrm{T}_n(x) = p^{-n}x$. We have

$$\mathrm{T}_m = \mathrm{Tr}_{K_n/K_m} \circ \mathrm{T}_n \quad \text{if } n \geq m.$$

We also denote T_n the map from $K_\infty((t))$ to $K_n((t))$ defined by $\mathrm{T}_n(\sum_{k=0}^{+\infty} a_k t^k) = \sum_{k=0}^{+\infty} \mathrm{T}_n(a_k)t^k$.

Proposition 7.1.

i) $K_\infty((t))$ is dense in $\mathbf{B}_{\mathrm{dR}}^{H_K}$ and T_n can be extended to a $\mathbf{Q}_p$-linear map from $\mathbf{B}_{\mathrm{dR}}^{H_K}$ to $K_n((t))$.

ii) If $F \in \mathbf{B}_{\mathrm{dR}}^{H_K}$, then $\lim_{n\to+\infty} p^n\mathrm{T}_n(F) = F$.

Proof. See [Col98], proposition V.4.5. □

Let V be a de Rham representation of G_K, we have $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V \cong \mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$ and $H^1(K,\mathbf{B}_{\mathrm{dR}} \otimes V) = H^1(K,\mathbf{B}_{\mathrm{dR}} \otimes \mathbf{D}_{\mathrm{dR}}(V)) = H^1(K,\mathbf{B}_{\mathrm{dR}}) \otimes \mathbf{D}_{\mathrm{dR}}(V)$. Since $K \cong H^1(K,\mathbf{B}_{\mathrm{dR}})$ via $x \mapsto x \cup \log\chi$,we get an isomorphism

$$\mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,\mathbf{B}_{\mathrm{dR}} \otimes V); \quad x \mapsto x \cup \log\chi$$

Proposition 7.2. *If V is a de Rham representation, the map sends $x \in \mathbf{D}_{\mathrm{dR}}(V)$ to a cocycle $\tau \mapsto x\log\chi(\tau) \in \mathbf{D}_{\mathrm{dR}}(V) \subset \mathbf{B}_{\mathrm{dR}} \otimes V$ induces an isomorphism from $\mathbf{D}_{\mathrm{dR}}(V)$ to $H^1(K,\mathbf{B}_{\mathrm{dR}} \otimes V)$ and the map $\exp^*_{V^*(1)}$ is the composition of the inverse of the above isomorphism and the natural map from $H^1(K,V)$ to $H^1(K,\mathbf{B}_{\mathrm{dR}} \otimes V)$.*

Proof. See [Kato93] proposition 1.4. of chapter II. □

We define the map $\mathrm{pr}_{K_n} : \mathbf{B}_{\mathrm{dR}}^{H_K} \to K_n((t))$ by the formula $\mathrm{pr}_{K_n}(x) = \frac{1}{[K_m:K_n]}\mathrm{Tr}_{K_m/K_n}(x)$ if $x \in K_\infty$ and $m \geq n$ such that $x \in K_m$ and there exists $a'(K) \geq 1$ such that one has $p^n\mathrm{T}_n = \mathrm{pr}_{K_n}$ if $n \geq a'(K)$. From (ii) of proposition 7.1, we can shows that $\lim_{n\to+\infty} \mathrm{pr}_{K_n} x = x$ if $x \in \mathbf{B}_{\mathrm{dR}}^{H_K}$

If V is a de Rham representation, the natural map from $\mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$ to $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{H_K}$ is an isomorphism and we can extend the map T_n and pr_{K_n} for $n \in \mathbf{N}$ by linearity to $\mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$. On the other hand, if $F \in K_\infty((t)) \otimes \mathbf{D}_{\mathrm{dR}}(V)$, we can write F uniquely as the form $\sum_{k\gg-\infty} t^k d_k$, where $d_k \in K_\infty \otimes \mathbf{D}_{\mathrm{dR}}(V)$. We denote $\partial_{V(-k)}(F)$ the element $t^k d_k$ of $K_\infty \otimes \mathbf{D}_{\mathrm{dR}}(V(-k))$.

Proposition 7.3. *Let V is a p-adic representation of G_K and $n, m \in \mathbf{N}$ be two integers. If $c \in H^1(K_m, V(-k))$, there exists cocycle $\tau \mapsto c_\tau$ on Γ_{K_m} with values in $(\mathbf{B}_{\mathrm{dR}} \otimes V(-k))^{H_K}$ which has the same image as c in $H^1(K_m, \mathbf{B}_{\mathrm{dR}} \otimes V(-k))$. Moreover, if V is de Rham. then*

$$\exp^*_{V^*(1+k)}(c) = \partial_{V(-k)} \circ \mathrm{pr}_{K_m}(\frac{1}{\log_p(\chi(\gamma))} c_\gamma)$$

for all $\gamma \in \Gamma_{K_m}$ such that $\log_p(\chi(\gamma)) \neq 0$

Proof. Since $H^1(K_\infty, \mathbf{B}_{\mathrm{dR}} \otimes V)$ is zero (c.f. [Col98] theorem IV.3.1), the inflation map from $H^1(\Gamma_{K_m}, (\mathbf{B}_{\mathrm{dR}} \otimes V)^{H_K})$ to $H^1(K_m, \mathbf{B}_{\mathrm{dR}} \otimes V)$ is an isomorphism, hence we have the existance of coycle $\tau \mapsto c_\tau$. On the other hand, if V is de Rham, the map $\tau \mapsto \partial_{V(-k)} \circ \mathrm{pr}_{K_m}(c_\tau)$ is a cocycle on Γ_{K_m} with values in $\mathbf{D}_{\mathrm{dR}}(V(-k))$ which Γ_{K_m} acts trivially. It is of the form $\tau \mapsto d \log_p \chi(\tau)$, where $d \in \mathbf{D}_{\mathrm{dR}}(V(-k))$ and if c is zero, which implies $\tau \mapsto c_\tau$ is a coboundary, hence $d = 0$. One cae deduce that $\partial_{V(-k)} \circ \mathrm{pr}_{K_m}(\frac{1}{\log_p \chi(\gamma)} c_\gamma) \in \mathbf{D}_{\mathrm{dR}}(V(-k))$ does not depend on $\gamma \in \Gamma_{K_m}$ such that $\chi(\gamma) \neq 0$ and the choice of cocycle $\tau \mapsto c_\tau$ representing c, which provides us a natural map from $H^1(K, V(-k))$ to $\mathbf{D}_{\mathrm{dR}}(V(-k))$ coincides with $\exp^*_{V^*(1+k)}$ by proposition 7.2. □

7.2. **Explicit reciprocity law.** Let V be a de Rham representation of G_K and let $n(V) \geq n_1(V)$ smallest integer satisfies $r_{n(V)} \geq r_V$ (c.f. proposition 6.5). If $\mu \in H^1_{\mathrm{Iw}}(K, V)$, then $\mathrm{Exp}^*_{V^*(1)}(\mu) \in D(V)^{\psi=1}$. On the other hand, $D(V)^{\psi=1} \subset \mathbf{D}^{\dagger, r_V}(V)$. If $n \geq n(V)$, we can view $\varphi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu))$ as an element in $\mathbf{B}_{\mathrm{dR}} \otimes V$. Since $\varphi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu))$ is an element of $\mathbf{B}_{\mathrm{dR}} \otimes V$ fixed by H_K, we can consider its image under T_m.

Theorem 7.4. *Let V be a de Rham representation and $m \in \mathbf{N}$.*
- *i) If $n \geq \sup(m, n(V))$ and $\mu \in H^1_{\mathrm{Iw}}(K, V)$, then $\mathrm{T}_m(\varphi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu)))$ is an element in $K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$ independent of n, we denote it by $\mathrm{Exp}^*_{V^*(1), K_m}(\mu)$.*
- *ii) If $\mu \in H^1_{\mathrm{Iw}}(K, V)$, then*

$$\mathrm{Exp}^*_{V^*(1), K_m}(\mu) = \sum_{k \in \mathbf{Z}} \exp^*_{V^*(1+k)}(\int_{\Gamma_{K_m}} \chi(x)^{-k} \mu).$$

- *iii) There exists $m(V) \geq n(V)$ such that if $m \geq m(V)$ and $\mu \in H^1_{\mathrm{Iw}}(K, V)$, then*

$$\mathrm{Exp}^*_{V^*(1), K_m}(\mu) = p^{-m} \varphi^{-m}(\mathrm{Exp}^*_{V^*(1)}(\mu)).$$

Remark 7.5.
- i) The image of $H^1(K_m, V(-k))$ by $\exp^*_{V^*(1+k)}$ is contained in $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V(-k)) = \mathrm{Fil}^0(t^K \mathbf{D}_{\mathrm{dR}}(V))$ which is zero if $k \ll 0$. Hence the series in ii) converges in $\mathbf{B}_{\mathrm{dR}} \otimes \mathbf{D}_{\mathrm{dR}}(V)$.
- ii) We have a map $\mu \in H^1_{\mathrm{Iw}}(K, V) \mapsto \int_{\Gamma_{K_n}} \chi^k \mu \in H^1(G_{K_n}, V(k))$, thus $\exp^*_{V(1+k)}(\int_{\Gamma_{K_n}} \chi^{-k} \mu) \in t^k K_n \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$.
- iii) For $n \geq n(V)$, we have $\varphi^{-n}(\mathbf{D}^{\dagger, r_n}(V)) \subset K_n((t)) \otimes \mathbf{D}_{\mathrm{dR}}(V)$.

Proof. Given that $\mathrm{T}_r = \mathrm{Tr}_{K_m/K_r} \circ \mathrm{T}_m$ if $r \leq m$ and if $L_1 \subset L_2$ are two finite extension of K, then the diagram

$$\begin{array}{ccc} H^1(L_2, V) & \xrightarrow{\exp^*_{V^*(1)}} & L_2 \otimes \mathbf{D}_{\mathrm{dR}}(V) \\ \Big\downarrow{\scriptstyle \mathrm{cor}_{L_2/L_1}} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{L_2/L_1} \otimes id} \\ H^1(L_1, V) & \xrightarrow{\exp^*_{V^*(1)}} & L_1 \otimes \mathbf{D}_{\mathrm{dR}}(V) \end{array}$$

is commutative. Thus, to prove i) and ii), it suffices to prove them for m large enough. We can therefore suppose that $m \geq n(V)+1$, $\mathrm{pr}_{K_m} = p^m \mathrm{T}_m$ and $\log_p^0(\gamma_m) = \frac{\log_p(\chi(\gamma_m))}{p^m}$.

Denote y the element $\mathrm{Exp}^*_{V^*(1)}(\mu)$ in $D(V)^{\psi=1}$ and if $i \in \mathbf{Z}$, denote $y(i)$ the image of y in $D(V(i))^{\psi=1} = D(V)^{\psi=1}$ (same as set but differerent as Galois module by twist χ^i). By construction of $\mathrm{Exp}^*_{V^*(1)}$ (indeed its inverse), $\int_{\Gamma_{K_n}} \chi(x)^{-k}\mu$ is represented by the cocycle

$$\sigma \mapsto c'_\sigma = \log_p^0(\gamma_m)\Big(\frac{\sigma-1}{\gamma_m-1}y(-k) - (\sigma-1)b\Big),$$

where $b \in \mathbf{A} \otimes V$ is a solution of the equation $(\varphi-1)b = (\gamma_m-1)^{-1}((\varphi-1)y)(-k))$.

By definition of $n(V)$, we have $y \in \mathbf{D}^{\dagger,r_{n(V)}}(V) \subset \mathbf{D}^{\dagger,r_{m-1}}(V)$ and $(\varphi-1)y \in \mathbf{D}^{\dagger,r_m}(V)$, which implies that $(\gamma_m-1)^{-1}(\varphi-1)y(-k) \in \mathbf{D}^{\dagger,r_{m+1}}(V)$ by proposition 6.5, and the same argument as lemma 4.14 implies that $b \in \mathbf{A}^{\dagger,r_m} \otimes V$. Since we suppose that $n \geq \sup(m, n(V))$, we have $\varphi^{-n}(b)$ and $\varphi^{-n}(y)$ are both in $\mathbf{B}^+_{\mathrm{dR}} \otimes V$ and $c'_\sigma = \varphi^{-n}(c'_\sigma)$ is a cocycle with values in $\mathbf{B}_{\mathrm{dR}} \otimes V$ which differs from the coycle

$$\sigma \mapsto c_\sigma = \frac{\log_p \chi(\gamma_m)}{p^m}\frac{\sigma-1}{\gamma_m-1}\varphi^{-n}(y(-k))$$

by a coboundary $\sigma \mapsto \frac{\log_p \chi(\gamma_m)}{p^m}(\sigma-1)\varphi^{-n}(b)$. Since y is fixed by H_K, the cocycle $\sigma \mapsto c_\sigma$ has values in $(\mathbf{B}_{\mathrm{dR}} \otimes V)^{H_K}$ which allows us to use proposition 7.3 to calculate it and we obtain

$$\exp^*_{V^*(1+k)}\Big(\int_{\Gamma_{K_m}} \chi(x)^{-k}\mu\Big) = \frac{1}{\log_p \chi(\gamma_m)}\partial_{V(-k)}(\mathrm{pr}_{K_m}(c_{\gamma_m})) = \frac{1}{p^m}\partial_{V(-k)}(\mathrm{pr}_{K_m}(\varphi^{-n}(y)))$$

and since $\frac{1}{p^m}\mathrm{pr}_{K_m} = \mathrm{T}_m$ and $\mathrm{T}_m(x) = \sum_{k\in\mathbf{Z}} \partial_{V(-k)}(\mathrm{T}_m(x))$ if $x \in (\mathbf{B}_{\mathrm{dR}} \otimes V)^{H_K}$, we deduce i) and ii).

To prove iii), it suffices to show that if m is large enough, then $\varphi^{-m}(\mathrm{Exp}^*_{V^*(1)}(\mu)) \in K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$. We need the following lemma:

Lemma 7.6. *Let d be an integer ≥ 1. If $U \in GL_d(\mathbf{B}_{\mathrm{dR}}^{H_K})$ and there exists $n \in \mathbf{N}$ such that $U^{-1}\gamma(U) \in GL_d(K_n((t)))$, then there exists $m \in \mathbf{N}$ such that $U \in GL_d(K_m((t)))$.*

Proof. Let $A = U^{-1}\gamma(U)$. If $m \geq n$, let $U_m = \mathrm{pr}_{K_m}(U)$. Using the fact that pr_{K_m} is $K_n((t))$-linear if $m \geq n$, we obtain, by applying pr_{K_m} to the identity $UA = \gamma(U)$, the relation $U_m A = \gamma(U_m)$. On the other hand, since $\lim_{m\to+\infty} U_m = U$, there exists $m \geq n$ such that U_m is invertible. Subtract A by the above identity, We have UU_m^{-1} is fixed by γ and therefore belongs to $GL_d(K)$. We hence deduce that U belongs to $GL_d(K_m((t)))$. □

Let $e_1, ..., e_d$ be a basis of $\mathbf{D}^{\dagger,r_{n(V)}}(V)$ over $\mathbf{B}_K^{\dagger,r_{n(V)}}$ which contains $D(V)^{\psi=1}$ and $f_1, ..., f_d$ a basis of $\mathbf{D}_{\mathrm{dR}}(V)$ over K. Let $A = (a_{i,j}) \in GL_d(\mathbf{B}_K^\dagger)$, $B = (b_{i,j}) \in GL_d(\mathbf{B}_K^\dagger)$ and if $m \geq n(V)$, $C^{(m)} = (c_{i,j}^{(m)}) \in GL_d(\mathbf{B}_{\mathrm{dR}}^{H_K})$ the matrices defined by

$$\gamma(e_i) = \sum_{j=1}^d a_{i,j}e_j, \quad \varphi(e_i) = \sum_{j=1}^d b_{i,j}e_j \quad \text{and} \quad \varphi^{-m}(e_i) = \sum_{j=1}^d c_{i,j}^{(m)} f_j.$$

The relation $\gamma \circ \varphi^{-m} = \varphi^{-m} \circ \gamma$ and $\varphi^{-m} = \varphi^{-(m+1)} \circ \varphi$ is translated to

$$\gamma(C^{(m)}) = C^{(m)}\varphi^{-m}(A) \quad \textit{and} \quad C^{(m)} = C^{(m+1)}\varphi^{-(m+1)}(C^{-1})$$

since $f_1, ..., f_d$ is fixed by γ. There exists $n_0 \geq n(V)$ such that A and B belongs to $GL_d(\mathbf{B}_K^{\dagger, r_{n_0}})$. Since there exists $m_0 \in \mathbf{N}$ such that $\varphi^{-m}(\mathbf{B}_K^{\dagger, r_m}) \in K_m[[t]]$, if $m \geq m_0$. By above relations and lemma 7.6, there exists $m(V) \geq \sup(n_0, m_0) = m_1$ such that $C^{(m_1)} \in GL_d(K_{m(V)}((t)))$, which implies that $C^{(m)} \in GL_d(K_{m(V)}((t)))$ for $m \geq m(V)$ by second relation. Since $x \in D(V)^{\psi=1}$ is of the form $\sum_{i=1}^d x_i e_i$ where $x \in \mathbf{B}^{\dagger, r_{n(V)}}$ and $\varphi^{-m}(\mathbf{B}^{\dagger, r_{n(V)}}) \subset K_m[[t]]$ if $m \geq m(V)$ by the choice of $m(V)$, we have the inclusion $\varphi^{-m}(D(V)^{\psi=1}) \subset K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$ if $m \geq m(V)$. This proves iii). $\square$

7.3. **Connection with the Perrin-Riou's logarithm.** Our Goal in this paragraph is to compare $\mathrm{Exp}^*_{V^*(1)}$ and Perrin-Riou's logarithm constructed in [Col98]. Let's recall the construction of logarithm map.

Proposition 7.7. *Let V be a de Rham representation. Let W be the finite dimensional $\mathbf{Q}_p$-vector space $\cup_{n \in \mathbf{N}} (\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes V)^{G_{K_n}}$. Let $\mu \in H^1_{\mathrm{Iw}}(K_n, V)$ such that $\int_{\Gamma_{K_n}} \mu \in H^1_e(K, V)$ for all $n \in \mathbf{N}$ and $\tau \to \mu_\tau$ a continuous cocycle represent μ. Finally, if $n \gg 0$, let c_n be the unique element of $(\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes V)/W$ verifies $(1-\tau)c_n = \int_{K_n} \mu_\tau$ for all $\tau \in G_{K_n}$.*

i) The sequence $p^n c_n$ converges in $(\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes V)/W$ to an element of $(\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes V)^{G_K}/W$ denoted by $\mathrm{Log}_V(\mu)$.

ii) If $n \in \mathbf{N}$, then

$$t\frac{d}{dt} T_n(\mathrm{Log}_V(\mu)) = \sum_{k \in \mathbf{N}} \exp^*_{V^*(1+k)}(\int_{\Gamma_{K_n}} \chi(x)^{-k} \mu).$$

Proof. See [Col98, Theorem VI.3.1 and Theorem VII.1.1]. $\square$

Remark 7.8.

i) There exists $k_0 \in \mathbf{N}$ such that the condition $\int_{\Gamma_{K_n}} \mu \in H^1_e(K, V)$ for all $n \in \mathbf{N}$ holds automatically if we replace V by $V(k)$ for $k \geq k_0$.

ii) The operator $\frac{d}{dt}$ annihilates $K_\infty \otimes \mathbf{D}_{\mathrm{dR}}(V)$ and hence W, which explains why we don't need to pass to quotient W in formula (ii).

The connection between Log_V and $\mathrm{Exp}^*_{V^*(1)}$ in the case V is de Rham is by:

Theorem 7.9. *Let V be a de Rham representation of G_K. There exists $m(V) \geq n(V)$ such that if $m \geq m(V)$ and $\mu \in H^1_{\mathrm{Iw}}(K, V)$ such that $\int_{\Gamma_{K_n}} \mu \in H^1_e(K_n, V)$ for all $n \in \mathbf{N}$, then*

$$p^{-m}\varphi^{-m}(\mathrm{Exp}^*_{V^*(1)}(\mu)) = t\frac{d}{dt}(T_m(\mathrm{Log}_V(\mu)))$$

Proof. Given ii) of proposition 7.7, it is immediately followed by theorem 7.4. $\square$

Remark 7.10. It is possible that the theorem is empty, that is there exists no nonzero element in $H^1_{\mathrm{Iw}}(K, V)$ satisfies the assumptions in proposition 7.7, but as we note above, if we replace V by $V(k)$ for $k \gg 0$, then the assumptions of the theorem is verified for all elements of $H^1_{\mathrm{Iw}}(K, V)$.

8. The $\mathbf{Q}_p(1)$ representation and Coleman's power series

8.1. **The module $D(\mathbf{Z}_p(1))^{\psi=1}$.** The module $\mathbf{Z}_p(1)$ is just $\mathbf{Z}_p$ with the action of $G_{\mathbf{Q}_p}$ defined by $g \in G_{\mathbf{Q}_p}$, $x \in \mathbf{Z}_p(1)$, $g(x) = \chi(g)x$. We shall study the exponential map

$$\mathrm{Exp}^*_{\mathbf{Q}_p} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p, \mathbf{Z}_p(1)) \to D(\mathbf{Z}_p(1))^{\psi=1}.$$

Note that $D(\mathbf{Z}_p(1)) = (\mathbf{A}\otimes\mathbf{Z}_p(1))^{H_{\mathbf{Q}_p}} = \mathbf{A}_{Q_p}(1)$, with usual actions of φ and ψ, and for $\gamma\in\Gamma$, $\gamma(f(\pi)) = \chi(\gamma)f((1+\pi)^{\chi(\gamma)}-1)$, for all $f(\pi)\in\mathbf{A}_{Q_p}(1)$.

Proposition 8.1. $(\mathbf{A}_{Q_p})^{\psi=1} = \mathbf{Z}_p\cdot\frac{1}{\pi}\oplus(\mathbf{A}^+_{Q_p})^{\psi=1}$.

Proof. Note that we have $\psi(\mathbf{A}^+_{Q_p})\subset\mathbf{A}^+_{Q_p}$, $\psi(\frac{1}{\pi}) = \frac{1}{\pi}$ and $\nu_E(\psi(x)\geq[\frac{\nu_E(x)}{p}]$ if $x\in\mathbf{E}^+_{\mathbf{Q}_p}$. These facts imply that $\psi-1$ is bijective on $\mathbf{E}_{\mathbf{Q}_p}/\overline{\pi}^{-1}\mathbf{E}^+_{\mathbf{Q}_p}$ and hence it is also bijective on $\mathbf{A}_{Q_p}/\pi^{-1}\mathbf{A}^+_{Q_p}$. Thus $\psi(x)=x$ implies $x\in\pi^{-1}\mathbf{A}^+_{Q_p}$. □

8.2. **Kummer theory.** We define the Kummer map $\kappa : K^*\to H^1(K,\mathbf{Q}_p(1))$ as follows: For $a\in K^*$, we choose x any element in $\widetilde{\mathbf{E}}$ satisfying $x^{(0)}=a$, then $\tau\mapsto(1-\tau)(\frac{\log[x]}{t}(1))$ is a 1-cocycle on G_K with values in $\mathbf{Q}_p(1)$ whose image in $H^1(K,\mathbf{Q}_p(1))$ is defined to be $\kappa(a)$.

Recall that $\varepsilon=(1,\varepsilon^{(1)},\cdots)\in\mathbf{E}^+_{\mathbf{Q}_p}$, $\varepsilon^{(1)}\neq1$. Let $F_n=\mathbf{Q}_p(\varepsilon^{(n)})$ and $\kappa_n : F_n^*\to H^1(F_n,\mathbf{Q}_p(1))$ be the Kummer maps defined above. Since $\mathrm{cor}_{F_{n+1}/F_n}\circ\kappa_{n+1}=\kappa_n\circ\mathrm{N}_{F_{n+1}/F_n}$, which induces a map

$$\kappa : \varprojlim F_n^*\to H^1_{\mathrm{Iw}}(\mathbf{Q}_p,\mathbf{Q}_p(1)).$$

We have

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p,\mathbf{Z}_p(1)) = \mathbf{Z}_p\cdot\kappa(\pi)\oplus\kappa(\varprojlim\mathscr{O}^*_{F_n}).$$

8.3. **Multiplicative representatives.** Recall $\mathbf{B}$ is a extension of degree p of $\varphi(\mathbf{B})$ (totally ramified since residual extension is purely inseparable). Define the multiplicative map $\mathrm{N}:\mathbf{B}\to\mathbf{B}$ by the formula $\mathrm{N}(x)=\varphi^{-1}(\mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}(x))$. This is an multiplicative analogy of ψ.

Lemma 8.2. *If $x\in\mathbf{E}^*$ and U_x denote the set $y\in\mathbf{A}$ whose reduction modulo p is x, then N is a contractible map of U_x for the p-adic topology.*

Proof. Note that N induces the identity on $\mathbf{E}$ and thus the fixes U_x. On the other hand, if $y\equiv 1 \mod p^k$, we have

$$\mathrm{N}(y)\equiv 1+\varphi^{-1}\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(y-1) = 1+p\psi(y-1) \mod p^{2k},$$

which implies in particular $\mathrm{N}(y)-1\in p^{k+1}\mathbf{A}$. We deduce that if y_1,y_2 two elements of U_x verified $y_1-y_2\in p^k\mathbf{A}$, then $\mathrm{N}(y_1)-\mathrm{N}(y_2)=\mathrm{N}(y_2)(\mathrm{N}(y_2^{-1}y_1)-1)\in p^{k+1}\mathbf{A}$, which proves the lemma. □

Corollary 8.3.

i) If $x\in\mathbf{E}$, there exists an unique element $\hat{x}\in\mathbf{A}$ whose image modulo p is x and $\mathrm{N}(\hat{x})=\hat{x}$.
ii) If x and y are two elements in $\mathbf{E}$, the $\widehat{xy}=\hat{x}\hat{y}$.

Proof. i) follows from the above lemma if $x\neq0$ and completeness of U_x for the p-adic topology. On the other hand, $\mathrm{N}(p^k\mathbf{A})\subset p^{pk}\mathbf{A}$, this proves that 0 is the only element of y of $p\mathbf{A}$ verified $\mathrm{N}(y)=y$ and the uniqueness follows. ii) follows from the uniqueness of i). □

Remark 8.4. There are two multiplicative maps from $\mathbf{E}$ to $\widetilde{\mathbf{A}}$, namely the map $x\to\hat{x}$ and the Techmuller map $[x]$. We have $\hat{x}\neq[x]$ unless $x\in\overline{\mathbf{F}}_p$.

Lemma 8.5. *Let K be a finite extension of $\mathbf{Q}_p$ and $d=[\mathbf{B}_K:\mathbf{B}_{Q_p}]=[K_\infty:\mathbf{Q}_p(\mu_{p^\infty)})]$. If $n(K)$ is the smallest integer $n\geq2$ such that there exist $e_1,...,e_d\in\widetilde{\mathbf{A}}^{\dagger,r_n}_K$ such that $\varphi(e_1),...,\varphi(e_d)$ form a basis of $\widetilde{\mathbf{A}}^{\dagger,r_{n+1}}_K$ over $\widetilde{\mathbf{A}}^{\dagger,r_{n+1}}_{\mathbf{Q}_p}$ and if $n\geq n(K)$, then $\mathrm{N}(\widetilde{\mathbf{A}}^{\dagger,r_{n+1}}_K)\subset\widetilde{\mathbf{A}}^{\dagger,r_n}_K$.*

Proof. By definition of $n(K)$, if $n \geq n(K)$ and $x \in \mathbf{A}_K^{\dagger, r_{n+1}}$, we can write x as the form $x = \sum_{i=1}^d x_i \varphi(e_i)$ where $x_i \in \mathbf{A}_{\mathbf{Q}_p}^{\dagger, r_{n+1}}$. On the other hand, we can write x_i of the form $x_i = \sum_{j=0}^{p-1} x_{i,j}[\varepsilon]^j$ where $x_{i,j} = \varphi(\psi([\varepsilon]^{-j} x_i))$ and corollary 4.13 and proposition 6.1 show that we have $x_{i,j} \in \varphi(\mathbf{A}_{\mathbf{Q}_p}^{\dagger, r_n})$. We hence deduce the coordinate $y_j = \sum_{i=1}^d x_{i,j}\varphi(e_i)$ of x in basis $1, [\varepsilon], ..., [\varepsilon]^{p-1}$ of $\mathbf{B}$ over $\varphi(\mathbf{B})$ belongs to to $\mathbf{A}_K^{\dagger, r_{n+1}} \cap \varphi(\mathbf{B}) = \varphi(\mathbf{A}_K^{\dagger, r_n})$. On the other hand, $\mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}$ is the determinant of the multiplication by x in $\mathbf{B}$ considered as a vector space of dimension p over $\varphi(\mathbf{B})$, therefore the determinant of the matrix

$$\begin{pmatrix} y_0 & [\varepsilon]^p y_{p-1} & \cdots & [\varepsilon]^p y_1 \\ y_1 & y_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & [\varepsilon]^p y_{p-1} \\ y_{p-1} & y_{p-2} & \cdots & y_0 \end{pmatrix}.$$

We deduce that $\mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}$ belongs to $\varphi(\mathbf{A}_K^{\dagger, r_n})$ Together with the relation $\mathrm{N} = \varphi^{-1} \circ \mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}$, we complete the proof. □

Corollary 8.6. *If $x \in \mathbf{E}_K^+$, then $\hat{x} \in \mathbf{A}_K^{\dagger, r_{n(K)}}$. Moreover, if K is a unramified extension over $\mathbf{Q}_p$ and $x \in \mathbf{E}_K^+$, then $\hat{x} \in \mathbf{A}_K^+ = \mathbf{A}_K \cap \mathbf{A}_{Q_p}^+ = \mathscr{O}_K[[\pi]]$.*

Proof. Let $v \in \mathbf{A}_K^\dagger$ whose image in $\mathbf{E}_K$ is x and let $n \geq n(K)$ such that $v \in \mathbf{A}_K^{\dagger, r_n}$. Let $(v_k)_{k \in \mathbf{N}}$ the sequence of elements in $\mathbf{A}_K$ defined by $v_0 = v$ and $v_k = \mathrm{N}(v_{k-1})$ if $k \geq 1$. By lemma 8.2, the sequence tends to $\hat{x}$ in $\mathbf{A}_K$ as k tends to $+\infty$. On the other hand, lemma 8.5, implies that $v_k \in \mathbf{A}_K^{\dagger, r_n}$ for $k \in \mathbf{N}$ and since $\mathbf{A}_K^{\dagger, r_n}$ is relatively compact in $\mathbf{A}_K^{\dagger, r_{n+1}}$, which implies $\hat{x} \in \mathbf{A}^{\dagger, r_{k+1}}$ and the result follows by using the lemma 8.5 by descending $\mathbf{A}_K^{\dagger, r_{n+1}}$ to $\mathbf{A}_K^{\dagger, r_{n(K)}}$.

In the case where K is unramified over $\mathbf{Q}_p$, the reduction modulo p induces a surjection from $\mathbf{A}_K^+$ to $\mathbf{E}_K^+$ and since $\mathbf{A}_K^+$ is a closed subring of $\mathbf{A}_K$ fixed by N, similar proof shows that $v \in \mathbf{A}_K^+$ implies that $\hat{x} \in \mathbf{A}_K^+$. □

8.4. **Generalized Coleman's power series.** Let's recall the construction of Coleman's power series.

Proposition 8.7. *Let F be a finite unramified extension of $\mathbf{Q}_p$. If $u = (u^{(n)})_{n \in \mathbf{N}}$ is an element of the projective limit $\varprojlim \mathscr{O}_{F_n}^*$ of $\mathscr{O}_{F_n}^*$ with respect to the norm map, there exists a unique power series $\mathrm{Col}_u(T)$ in $\mathscr{O}_F[[T]]^*$ such that we have $\mathrm{Col}_u^{\varphi^{-n}}(\varepsilon^{(n)} - 1) = u^{(n)}$ for all $n \in \mathbf{N}$.*

Proof. See [Co79]. □

Lemma 8.8. *If K is a finite extension of $\mathbf{Q}_p$ and $n \geq n(K)$, then the diagram*

$$\begin{array}{ccc} \mathbf{A}_K^{\dagger, r_{n+1}} & \xrightarrow{\mathrm{N}} & \mathbf{A}_K^{\dagger, r_n} \\ \Big\downarrow{\scriptstyle \varphi^{-(n+1)}} & & \Big\downarrow{\scriptstyle \varphi^{-n}} \\ K_{n+1}[[t]] & \xrightarrow{\mathrm{N}_{K_{n+1}/K_n}} & K_n[[t]] \end{array}$$

is commutative.

Proof. By definition, $\mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}$ (resp. $\mathrm{N}_{K_{n+1}/K_n}(\varphi^{-(n+1)}(x))$) is the determinant of the multiplication by x (resp. $\varphi^{-(n+1)}(x)$) over $\mathbf{B}$ (resp. $K_{n+1}[[t]]$) considered as a $\varphi(\mathbf{B})$-vector space (resp. $K_n[[t]]$-module) and the commutativity of the diagram follows from the fact $\varphi^{(n+1)}$ is a ring homomorphism and $\varphi^{-n} \circ \mathrm{N} = \varphi^{-(n+1)} \circ \mathrm{N}_{\mathbf{B}/\varphi(\mathbf{B})}$. $\square$

Denote the map θ_n the homorphism $\theta \circ \varphi^{-n}$ from $\mathbf{B}^{\dagger, r_n}$ to $\mathbf{C}_p$.

Lemma 8.9. *If $u = (u^{(n)}) \in \varprojlim \mathscr{O}_{K_n}$ and $n \geq n(K)$, then $\theta_n(\widehat{\iota_K(u)}) = u^{(n)}$.*

Proof. By the preceeding lemma, $(\theta_n(\widehat{\iota_K(u)}))_{n \geq n(K)}$ belongs to $\varprojlim \mathscr{O}_{K_n}$. On the other hand, since $[\iota_K(u)] - \widehat{\iota_K(u)} \in \widetilde{\mathbf{A}}_K^{\dagger, r_{n(K)}} \cap p\widetilde{\mathbf{A}}$, which implies that if $n \geq n(K)$, then $\nu_p(\theta_n([\iota_K(u)] - \theta_n(\widehat{\iota_K(u)})) \geq 1 - \frac{1}{p^{n-n(K)}}$ and since $\nu_p(\theta_n([\iota_K(u)] - u^{(n)}) \geq \frac{1}{p}$ if n large enough, we show that $(\theta_n(\widehat{\iota_K(u)})_{n \geq n(K)}$ has same image as u in $\mathbf{E}_K^+$ (c.f. proposition 3.1), so it is equal. $\square$

Proposition 8.10. *Let K be a finite extension of $\mathbf{Q}_p$, $F = K_\infty \cap \mathbf{Q}_p^{ur}$ and $e_K = [K_\infty : F_\infty]$.*

i) If $e_K = 1$ and $u \in \varprojlim \mathscr{O}_{K_n}$, then $\widehat{\iota_K(u)} = \mathrm{Col}_u(\pi)$.

ii) When $e_K \geq 2$, there exist Laurent series $f_0, \cdots, f_{e_K-1} \in \mathscr{O}_F((T))$ converges in the annulus $0 < \nu_p(x) < \frac{1}{(p-1)p^{n(K)-1}}$ such that, if $n \geq n(K)$, then $(u^n)^{e_K} + f_{e_K-1}^{\varphi^{-n}}(\varepsilon^{(n)} - 1)(u^{(n)})^{e_K-1} + \cdots + f_0^{\varphi^{-n}}(\varepsilon^{(n)} - 1) = 0$.

Proof. By corollary 8.6, $\widehat{\iota_K(u)} \in \mathbf{A}_F^+$ if $u \in \varprojlim \mathscr{O}_{K_n}$. In particular, there exists $f \in \mathscr{O}_F[[T]]$ such that $\widehat{\iota_K(u)} = f(\pi)$. On the other hand, by applying lemma 8.9 to the map θ_n, we obtained $u^{(n)} = f^{\varphi^{-n}}(\varepsilon^{(n)} - 1)$, which shows $f = \mathrm{Col}_u$ by the characterization of Col_u.

ii) By corollary 8.6, $\widehat{\iota_K(u)} \in \mathbf{A}_K^{\dagger, r_{n(K)}}$. On the other hand, $\mathbf{A}_K^{\dagger, r_{n(K)}}$ is of dimension e_K over $\mathbf{A}_F^{\dagger, r_{n(K)}}$ (by the definition of $n(K)$); so we can find elements $\widetilde{f}_0, \cdots, \widetilde{f}_{e-1} \in \mathbf{A}_F^{\dagger, r_{n(K)}}$ such that we have $\widehat{\iota_K(u)}^{e_K} + \widetilde{f}_{e-1}\widehat{\iota_K(u)}^{e_K-1} + \cdots + \widetilde{f}_0 = 0$, by lemma 8.9, we obtain the result. $\square$

8.5. The map $\mathrm{Log}_{\mathbf{Q}_p(1)}$ and $\mathrm{Exp}^*_{\mathbf{Q}_p}$.

Lemma 8.11. *If $u \in \mathbf{E}_K$, the sequence $(\varphi^{-n}(\widehat{\iota_K(u)}))^{p^n}$ converge in to $[\iota_K(u)]$ in $\widetilde{\mathbf{A}}$ and $\mathbf{B}_{\mathrm{dR}}^+$.*

Proof. Since $\widehat{\iota_K(u)} \in \mathbf{A}^{\dagger, r_{n(K)}}$ with image $\iota_K(u)$ in $\mathbf{E}$, it can be written as the form $[\iota_K(u)] + \sum_{k=1}^{+\infty} p^k [x_k]$, where x_k are elements of $\widetilde{\mathbf{E}}$ satisfying $\nu_E(x_k) \geq -kp^{n(K)}$. We have the formula $v_n = \varphi^{-n}(\widehat{\iota_K(u)}) = [\iota_K(u)^{p^{-n}}] + \sum_{k=1}^{+\infty} p^k [x_k^{p^{-n}}]$ and the congruence $v_n^{p^n} \equiv [\iota_K(u)] \bmod p^{n+1}\widetilde{\mathbf{A}}$, thus it converges in $\widetilde{\mathbf{A}}$.

Let α an element in $\widetilde{\mathbf{E}}^+$ verified $\nu_E(\alpha) = \frac{p-1}{p}$, thus $(\frac{p}{[\alpha]})^i$ tends to 0 in $\mathbf{B}_{\mathrm{dR}}^+$ as i tends to $+\infty$. If $n \geq n(K) + 1$, the above formula shows that v_n belongs to the subring A (c.f. section 4.4) of $\mathbf{B}_{\mathrm{dR}}^+$ of elements of the form $y = \sum_{i=0}^{+\infty} y_i (\frac{p}{[\alpha]})^i$, where y_i are elements in $\mathbf{A}$ and we have $v_n - [\iota_K(u)^{p^{-n}}] \in \frac{p}{[\alpha]} A$. We deduce that $v_n^{p^n}$ tends to $[\iota_K(u)]$ in A and a fortiori in $\mathbf{B}_{\mathrm{dR}}^+$. $\square$

Proposition 8.12. *Let K be a finite extension of $\mathbf{Q}_p$ and $u \in \varprojlim \mathscr{O}_{K_n}^*$.*

i) $\mathrm{Log}_{\mathbf{Q}_p(1)}(\kappa(u)) = t^{-1} \log[\iota_K(u)]$

ii) If $n \geq n(K)$, then $T_n(\mathrm{Log}_{\mathbf{Q}_p(1)}(\kappa(u))) = t^{-1} \log \varphi^{-n}(\widehat{\iota_K(u)})$.

iii) $\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u)) = \widehat{\iota_K(u)}^{-1} \partial \widehat{\iota_K(u)}$, where ∂ is the derivation $(1+\pi)\frac{d}{d\pi}$ (see 6.2).

Proof. By construction of Kummer map, if u_n any element in $\widetilde{\mathbf{E}}^+$ satisfying $u_n^{(0)} = u^{(n)}$, then $\tau \mapsto (1-\tau)(\frac{\log[u_n]}{t}(1))$ is a 1-cocycle on G_{K_n} with values in $\mathbf{Q}_p(1)$ whose image in $H^1(K_n, \mathbf{Q}_p(1))$ is equal to $\kappa_n(u^{(n)})$. Since we suppose that $u^{(n)} \in \mathscr{O}^*_{K_n}$, we have $\log[u_n] \in \mathbf{B}_{\mathrm{cris}}$ and $\frac{\log[u_n]}{t}(1) \in \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes \mathbf{Q}_p(1)$, proving that $\kappa_n(u^{(n)}) \in H^1_e(K_n, \mathbf{Q}_p(1))$. Hence we deduce the formula

$$\mathrm{Log}_{\mathbf{Q}_p(1)}(\kappa(u)) = \lim_{n\to+\infty} p^n \frac{\log[u_n]}{t}(1) = t^{-1} \lim_{n\to+\infty} \log([u_n]^{p^n}).$$

Finally, we have $\nu_p(\theta([\iota_K(u)^{p^{-n}}]) - \theta([u_n])) \geq \frac{1}{p}$ if n large enough, therefore $[u_n]^{p^n}$ tends to $[\iota_K(u)]$ as n tends to $+\infty$. We complete i).

By i) and lemma 8.11, we have

$$\mathrm{T}_n(\mathrm{Log}_{\mathbf{Q}_p(1)}(\kappa(u))) = t^{-1} \lim_{m\to+\infty} \mathrm{T}_n(p^m \log(\varphi^{-m}(\widehat{\iota_K(u)}))).$$

On the other hand, if $m \geq n$, we have $\mathrm{T}_n = \mathrm{Tr}_{K_m[[t]]/K_n[[t]]} \circ T_m$ and sicne $\varphi^{-m}(\widehat{\iota_K(u)})) \in K_m[[t]]$ and the restriction of T_m on $K_m[[t]]$ is multiplication by p^{-m}, we obtain the formula

$$\begin{aligned}\mathrm{T}_n(\mathrm{Log}_{\mathbf{Q}_p(1)}(\kappa(u))) =& t^{-1} \lim_{m\to+\infty} \mathrm{Tr}_{K_m[[t]]/K_n[[t]]}(\log(\varphi^{-m}(\widehat{\iota_K(u)}))) \\ =& t^{-1} \lim_{m\to+\infty} \log(\mathrm{N}_{K_m[[t]]/K_n[[t]]}(\varphi^{-m}(\widehat{\iota_K(u)})))\end{aligned}$$

and this completes ii) by using lemma 8.8.

Note that t^{-1} is a generator of $\mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1))$. ii) and theorem 7.9 implies that if n is large enough, we have

$$\begin{aligned}\varphi^{-n}(\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u))) =& t^{-1}\left(t\frac{d}{dt}\big(p^n \log \varphi^{-n}((\widehat{\iota_K(u)}))\big)\right) \\ =& p^n \frac{\frac{d}{dt}(\varphi^{-n}(\widehat{\iota_K(u)}))}{\varphi^{-n}(\widehat{\iota_K(u)})} \\ =& \varphi^{-n}\big(\widehat{\iota_K(u)}^{-1} \partial \widehat{\iota_K(u)}\big) \quad \text{by lemma 6.3 ,}\end{aligned}$$

which complete iii). □

8.6. **Cyclotomic units and Coates-Wiles homomorphisms.**

Example 8.13. Let $K = \mathbf{Q}_p$, $V = \mathbf{Q}_p(1)$ and $u = (\frac{\zeta_{p^n}-1}{\zeta_{p^n}})_{n\geq 1} \in \varprojlim \mathscr{O}^*_{F_n}$. Then its Coleman's power series is $\mathrm{Col}_u(T) = \frac{1+T}{T}$. By iii) of proposition 8.12, we have $\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u)) = (\frac{\partial \mathrm{Col}_u(T)}{\mathrm{Col}_u(T)})(\pi) = \frac{1}{\pi}$. On the other hand, since

$$\varphi^{-1}(\pi)^{-1} = ([\varepsilon^{\frac{1}{p}}] - 1)^{-1} = \frac{1}{\varepsilon^{(1)}\exp(t/p) - 1},$$

by iii) of theorem 7.4, we have

$$\begin{aligned}\mathrm{Exp}^*_{\mathbf{Q}_p,F_1}(\kappa(u)) =&\frac{1}{p}\mathrm{Tr}_{F_1/\mathbf{Q}_p}\varphi^{-1}(\frac{1}{\pi})\\ =&\frac{1}{p}\sum_{\zeta^p=1,\zeta\neq 1}\frac{1}{\zeta\exp(t/p)}\\ =&\frac{-1}{t}(\frac{t}{1-\exp(t)}-\frac{t/p}{1-\exp(t/p)})\\ =&\sum_{k=1}^{+\infty}(1-p^{-k})\zeta(1-k)\frac{(-t)^{k-1}}{(k-1)!}.\end{aligned}$$

Thus by ii) of theorem 7.4,

$$\exp^*_{\mathbf{Q}_p(1+k)^*}(\int_{\Gamma_{F_1}}\chi^{-k}\kappa(\mu))=\begin{cases}0 & \text{if } k\leq 0\\ (1-p^{-k})\zeta(1-k)\frac{(-t)^{k-1}}{(k-1)!} & \text{if } k\geq 1\end{cases}.$$

Example 8.14. Let $K=\mathbf{Q}_p$, $V=\mathbf{Q}_p(1)$ and $u=(\frac{\zeta^a_{p^n}-1}{\zeta_{p^n}-1})_{n\geq 1}\in\varprojlim\mathscr{O}^*_{F_n}$, where $a\in\mathbf{Z}$. Then its Coleman's power series is $\mathrm{Col}_u(T)=\frac{(1+T)^a-1}{T}$. By iii) of proposition 8.12, we have $\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u))=(\frac{\partial\mathrm{Col}_u(T)}{\mathrm{Col}_u(T)})(\pi)=\frac{a(1+\pi)^a}{(1+T)^a-1}-\frac{1+T}{T}$. On the other hand, since

$$\varphi^{-1}(\pi)^{-1}=([\varepsilon^{\frac{1}{p}}]-1)^{-1}=\frac{1}{\varepsilon^{(1)}\exp(t/p)-1},$$

by iii) of theorem 7.4, we have

$$\begin{aligned}\mathrm{Exp}^*_{\mathbf{Q}_p,F_1}(\kappa(u)) =&\frac{1}{p}\mathrm{Tr}_{F_1/\mathbf{Q}_p}\varphi^{-1}(\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u)))\\ =&a-1+\frac{1}{p}\sum_{\zeta^p=1,\zeta\neq 1}\frac{a}{\zeta\exp at/p-1}-\frac{1}{\zeta\exp t/p}\\ =&a-1+\frac{-1}{t}(\frac{at}{1-\exp(at)}-\frac{at/p}{1-\exp(at/p)}-\frac{t}{1-\exp(t)}+\frac{t/p}{1-\exp(t/p)})\\ =&\sum_{k=1}^{+\infty}(1-p^{-k})(a^k-1)\zeta(1-k)\frac{(-t)^{k-1}}{(k-1)!}.\end{aligned}$$

Thus by ii) of theorem 7.4,

$$\exp^*_{\mathbf{Q}_p(1+k)^*}(\int_{\Gamma_{F_1}}\chi^{-k}\kappa(\mu))=\begin{cases}0 & \text{if } k\leq 0\\ (a^k-1)(1-p^{-k})\zeta(1-k)\frac{(-t)^{k-1}}{(k-1)!} & \text{if } k\geq 1\end{cases}.$$

Example 8.15. Let $K=\mathbf{Q}_p(\zeta_d)$, $V=\mathbf{Q}_p(1)$ and ε is a Dirichlet character of conductor $d\geq 1$ prime to p. Set $u=\big(\frac{-1}{G(\varepsilon^{-1})}\sum_{b \bmod d}\frac{\varepsilon^{-1}(b)}{\zeta^b_d\zeta_{p^n}-1}\big)_{n\geq 1}\in\varprojlim\mathscr{O}^*_{K_n}$, then we have

$$\mathrm{Col}_u(T)=\frac{-1}{G(\varepsilon^{-1})}\sum_{b \bmod d}\frac{\varepsilon^{-1}(b)}{\zeta^b_d(1+T)-1}$$

and thus

$$\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u)) = \frac{-1}{G(\varepsilon^{-1})}\sum_{b \bmod d}\frac{\varepsilon^{-1}(b)}{\zeta_d^b(1+\pi)-1}.$$

Hence we have,

$$\begin{aligned}
\mathrm{Exp}^*_{\mathbf{Q}_p,K_1}(\kappa(u)) =& p^{-1}\mathrm{Tr}_{K_1/K}\varphi^{-1}(\mathrm{Exp}^*_{\mathbf{Q}_p}(\kappa(u)))\\
=& p^{-1}\frac{-1}{G(\varepsilon^{-1})}\sum_{z^p=1,z\neq 1}\sum_{b \bmod d}\varepsilon^{-1}(b)\frac{1}{\zeta_d^{b/p}z\exp(t/p)-1}\\
=& \frac{-1}{G(\varepsilon^{-1})}\sum_{b \bmod d}\varepsilon^{-1}(b)\left(\frac{1}{1-\zeta_d^b\exp(t)}-p^{-1}\frac{1}{1-\zeta_d^{b/p}\exp(t/p)}\right)\\
=& \sum_{b \bmod d}\frac{\varepsilon(b)\exp(bt)}{1-\exp(dt)}-p^{-1}\varepsilon(p)\frac{\varepsilon(b)\exp(bt/p)}{1-\exp(dt/p)}\\
=& \sum_{k=1}^{+\infty}(1-\varepsilon(p)p^{-k})\mathrm{L}(1-k,\varepsilon)\frac{t^{k-1}}{(k-1)!}
\end{aligned}$$

and thus

$$\exp^*_{\mathbf{Q}_p^*(1+k)}(\int_{\Gamma_{K_1}}\chi^{-k}\kappa(\mu)) = \begin{cases}0 & \text{if } k\leq 0\\ (1-\varepsilon(p)p^{-k})\mathrm{L}(1-k,\varepsilon)\frac{t^{k-1}}{(k-1)!} & \text{if } k\geq 1\end{cases}.$$

These allow us to define a homomorphism $\mathrm{CW}_{k,n}$ from $H^1_{\mathrm{Iw}}(K,V)$ to $K_n\otimes\mathbf{D}_{\mathrm{dR}}(V)$ by putting

$$\mathrm{CW}_{k,n}(\mu) = \partial_k(\mathrm{T}_{K_n}(\mathrm{Log}_V(\mu))).$$

for each $n\in\mathbf{N}$ and $k\in\mathbf{Z}$, the homomorphism is a generalization of Coates-Wiles homomorphism and we have the following theorem by proposition 7.7.

Theorem 8.16. *If $\mu\in H^1_{\mathrm{Iw}}(K,V)$, if $n\in\mathbf{N}$ and $k\in\mathbf{Z}$, then*

$$\mathrm{CW}_{k,n}(\mu) = -\exp^*(\int_{\Gamma_{K_n}}\chi(x)^{-k}\mu).$$

Remark 8.17. The map

$$\varprojlim \mathscr{O}^*_{\mathbf{Q}_p(\mu_{p^n})}\to H^1_{\mathrm{Iw}}(\mathbf{Q}_p,\mathbf{Q}_p(1))\to\mathbf{Q}_p,\quad u\mapsto \exp^*_{\mathbf{Q}_p^*(1+k)}(\int_{\Gamma_{F_1}}\chi^{-k}\kappa(\mu))$$

is just the Coates-Wiles homomorphism.

9. (φ,Γ)-modules and differential equations

9.1. The rings $\mathbf{B}_{\max}$ and $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$.

The ring $\mathbf{B}^+_{\max}$ is defined by

$$\mathbf{B}^+_{\max} = \{\sum_{n\geq 0}a_n\frac{\omega^n}{p^n}\mid a_n\in\widetilde{\mathbf{B}}^+ \text{ is a sequence converges to } 0\},$$

and $\mathbf{B}_{\max}=\mathbf{B}^+_{\max}[\frac{1}{t}]$. It is closely related to $\mathbf{B}_{\mathrm{cris}}$ but tends to be more amendable. One could replace ω by any generator of $\ker(\theta)$ in $\widetilde{\mathbf{A}}^+$. The ring $\mathbf{B}_{\max}$ injects canonically into $\mathbf{B}_{\mathrm{dR}}$ and, in particular, it is endowed with the induced Galois action and filtration, as well as with a continuous Frobenius φ, extending the map $\varphi:\widetilde{\mathbf{B}}^+\to\widetilde{\mathbf{B}}^+$. Note that φ deos not extend continuously to $\mathbf{B}_{\mathrm{dR}}$. We sets $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}=\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\max})$.

We recall a representation V of G_K is crystalline if it is $\mathbf{B}_{\mathrm{cris}}$-admissible, which is equivalent to $\mathbf{B}_{\max}$-admissible or $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[\frac{1}{t}]$-admissible (because $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\max}) = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$ and the periods of crystalline representations live in finite dimensional F-vector subspaces of $\mathbf{B}_{\max}$, stable by φ and so in fact in $\cap_{n=0}^{\infty}\varphi^n(\mathbf{B}^+_{\max}[\frac{1}{t}])$; that is, the F-vector space

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K} = (\mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[\frac{1}{t}] \otimes_{\mathbf{Q}_p} V)^{G_K}$$

is of dimension $d = \dim_{\mathbf{Q}_p}(V)$. Then $\mathbf{D}_{\mathrm{cris}}(V)$ is endowed with a Frobenius φ induced by that of $\mathbf{B}_{\max}$ and $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K} = \mathbf{D}_{\mathrm{dR}}(V) = K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ so that a crystalline representation is also de Rham and $K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ is a filtered K-vector space.

If V is a p-adic representation, we say V is Hodge-Tate, with Hodge Tate weights $h_1, ..., h_d$, if we have a decomposition $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V \cong \oplus_{j=1}^d \mathbf{C}_p(h_j)$. We say that V is positive if its Hodge-Tate weights are all negative. By using the map $\theta : \mathbf{B}^+_{\mathrm{dR}} \to \mathbf{C}_p$, it is easy to see that a de Rham representation is Hodge-Tate and that the Hodge-Tate weights of V are those integers h such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-h+1}\mathbf{D}_{\mathrm{dR}}(V)$.

9.2. **The structure of $D(T)^{\psi=1}$.** Recall in section 3, we introduce (φ, Γ)-modules and their relation with Galois representation. Let us now set $K = F$ (i.e. we are working in an unramified extension of $\mathbf{Q}_p$). We say that a p-adic representation V of G_F is of finite height if $D(V)$ has a basis over $\mathbf{B}_F$ made up of elements of $D^+(V) = (\mathbf{B}^+ \otimes_{\mathbf{Q}_p} V)^{H_F}$. A result of [Col99, proposition III.2] shows that V is of finite height if and only if $D(V)$ has a sub-$\mathbf{B}^+_F$-module which is free of rank d, and stable by φ. Let us recall the main result of [Col99, theorem 1] regarding crystalline representation of G_F:

Theorem 9.1. *If V is a crystalline representation of G_F, then V is of finite height.*

Let V be a crystalline representation of G_F and let T denote a G_F stable lattice of V. The following proposition is proved in [Ber04, proposition II.1.1]

Proposition 9.2. *If T is a lattice in a positive crystalline representation V, then there exists a unique sub-$\mathbf{A}^+_F$-module $\mathbf{N}(T)$ of $D^+(T)$, which satisfies the following conditions:*

1. $\mathbf{N}(T)$ is an free $\mathbf{A}^+_F$-module of rank $d = \dim_{\mathbf{Q}_p} V$;
2. the action of Γ_F preserves $\mathbf{N}(T)$ and is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;
3. there exists an integer $r \geq 0$ such that $\pi^r D^+(T) \subset \mathbf{N}(T)$.

Moreover, $\mathbf{N}(T)$ is stable by φ, and the $\mathbf{B}^+_F$-module $\mathbf{N}(V) = \mathbf{B}^+_F \otimes_{\mathbf{A}^+_F} \mathbf{N}(T)$ is the unique sub-$\mathbf{B}^+_F$-module of $D^+(V)$ satisfying the corresponding conditions.

The $\mathbf{A}^+_F$-module $\mathbf{N}(T)$ is called the Wach module associated to T.

Notice that $\mathbf{N}(T(-1)) = \pi\mathbf{N}(T) \otimes e_{-1}$. When V is no longer positive, we can therefore defined $\mathbf{N}(T)$ as $\pi^{-h}\mathbf{N}(T(-h)) \otimes e_h$ for h large enough so that $V(-h)$ is positive. Using the results of [Ber04, III.4], one can show that:

Proposition 9.3. *If T is a lattice in a crystalline representation V of G_F, whose Hodge-Tate weights are in $[a; b]$, then $\mathbf{N}(T)$ is the unique sub-$\mathbf{A}^+_F$-module of $D^+(T)[1/\pi]$ which is free of rank d, stable by Γ_F with the action of Γ_F being trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$ and such that $\mathbf{N}(T)[1/\pi] = D^+(T)[1/\pi]$.*

Finally, we have $\varphi(\pi^b\mathbf{N}(R)) \subset \pi^b\mathbf{N}(T)$ and $\pi^b\mathbf{N}(T)/\varphi^*(\pi^b)$ is killed by q^{b-a}, where $q = \varphi(\omega)$. The construction $T \mapsto \mathbf{N}(T)$ gives a bijection between Wach modules over $\mathbf{A}^+_F$ which are lattices in $\mathbf{N}(V)$ and Galois lattices T in V.

Indeed $D(V)^{\psi=1}$ is not very far from being included in $\mathbf{N}(V)$:

Theorem 9.4. *If V is a crystalline representation of G_F, whose Hodge-Tate weights are in $[a;b]$, then $D(V)^{\psi=1} \subset \pi^{a-1}\mathbf{N}(V)$. In addition, if V has no quotient isomorphic to $\mathbf{Q}_p(a)$, then actually $D(V)^{\psi=1} \subset \pi^a\mathbf{N}(V)$.*

Proof. See [Ber03, Theorem A.3]. □

9.3. **p-adic representations and differential equations.** In this paragraph, we recall some of the results of [Ber02], which allow us to recover $\mathbf{D}_{\mathrm{cris}}(V)$ from the (φ,Γ)-module associated to V. Let $\mathcal{H}^{\alpha}_{F'}$ be the set of power series $f(T) = \sum_{k\in\mathbf{Z}} a_k T^k$ such that a_k is a sequence (not necessarily bounded) of elements of F', and such that $f(T)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha} \leq |T| < 1\}$.

For $r \geq r(K)$ (c.f. proposition 6.1), define $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$ as the set of $f(\pi_K)$ where $f(T) \in \mathcal{H}^{e_k r}_{F'}$. Obviously, $\mathbf{B}^{\dagger,r}_K \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$ and the second ring is the completion of the first one for the natural Fréchet topology. If V is a p-adic representation, let

$$\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_K} \mathbf{D}^{\dagger,r}(V) \quad \text{and} \quad \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{H_K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V).$$

One of the main technical tools of [Ber02] is the construction of a large ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, which contains $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ and $\widetilde{\mathbf{B}}^{\dagger}$. This ring is a bridge between p-adic Hodge theory and the thoery of (φ,Γ)-modules.

As a consequence of the two above inclusions, we have:

$$\mathbf{D}_{\mathrm{cris}}(V) \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[\frac{1}{t}] \otimes_{\mathbf{Q}_p} V)^{G_K} \quad \text{and} \quad \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[\frac{1}{t}] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[\frac{1}{t}] \otimes_{\mathbf{Q}_p} V)^{H_K}.$$

One of the main result of [Ber02] is:

Theorem 9.5. *If V is a p-adic representation of G_K then $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[\frac{1}{t}])^{\Gamma_K}$. If V is positive, then $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\Gamma_K}$.*

Proof. See [Ber02, theorem 3.6]. □

Note that $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$ is the completion of $\mathbf{B}^{\dagger,r}_K$ for the ring's natural Fréchet topology and that $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is the union of the $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$. Similarly, there is a natural Fréchet topology on $\widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ is the completion of $\widetilde{\mathbf{B}}^{\dagger,r}$ for that topology and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. Actually, one can show that $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ for any r and there is an exact sequence

$$0 \longrightarrow \widetilde{\mathbf{B}}^{+} \longrightarrow \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \oplus \widetilde{\mathbf{B}}^{\dagger,r} \longrightarrow \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \longrightarrow 0$$

which one can take as providing a definition of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$.

Recall that if $n \geq 0$ and $r_n = p^{n-1}(p-1)$, then there is a well-defined injective map φ^{-n} : $\widetilde{\mathbf{B}}^{\dagger,r_n} \to \mathbf{B}^{+}_{\mathrm{dR}}$ (c.f. section 6.2), and the map extends to an injective map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^{+}_{\mathrm{dR}}$ (see [Ber02, corollary 2.13]).

Let $\mathbf{B}^{+}_{\mathrm{rig},F}$ be the set of $f(\pi)$ where $f(T) = \sum_{k\geq 0} a_k T^k$ with $a_k \in F$, and such that $f(T)$ is holomorphic on the p-adic open unit disk. Set $\mathbf{D}^{+}_{\mathrm{rig}}(V) = \mathbf{B}^{+}_{\mathrm{rig},F} \otimes_{\mathbf{B}^{+}_F} D^{+}(V)$. One can show the following refinement of theorem 9.5:

Proposition 9.6. *We have $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{+}_{\mathrm{rig}}(V)[1/t])^{\Gamma_F}$ and if V is positive then $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^{+}_{\mathrm{rig}}(V)^{\Gamma_F}$.*

Indeed if $\mathbf{N}(V)$ is the Wach module associated to V, then $\mathbf{N}(V) \subset D^+(V)$ when V is positive and it is shown in [Ber03, II.2] that under that hypothesis, $\mathbf{D}_{\text{cris}}(V) = (\mathbf{B}^+_{\text{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\Gamma_F}$.

9.4. **The Fontaine isomorphism revisit.** The purpose of this section is to recall the constructions in section 4.2 and extend them a little bit. Let V be a p-adic representation of G_K. Recall in section 4.2, we construct a map $h^1_{K,V} : D(V)^{\psi=1} \to H^1(K,V)$, and when Γ_K is torsion free, it gives rise to an exact sequence:

$$0 \longrightarrow D(V)^{\psi=1}_{\Gamma_K} \xrightarrow{h^1_{K,V}} H^1(K,V) \longrightarrow (\tfrac{D(V)}{\psi-1})^{\Gamma_K} \longrightarrow 0$$

We shall extend $h^1_{K,V}$ to a map $h^1_{K,V} : \mathbf{D}^\dagger_{\text{rig}}(V)^{\psi=1} \to H^1(K,V)$.

Lemma 9.7. *If r is large enough and $\gamma \in \Gamma_K$ then*

$$1-\gamma : \mathbf{D}^{\dagger,r}_{\text{rig}}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}_{\text{rig}}(V)^{\psi=0}$$

is an isomorphism

Proof. We first show that $1-\gamma$ is injective. By theorem 9.5, an element in the kernel of $1-\gamma$ would be in $\mathbf{D}_{\text{cris}}(V)$ and therefore in $\mathbf{D}_{\text{cris}}(V)^{\psi=0}$, which is obviously 0.

To prove surjectivity. Recall that by iii) of proposition 6.5, if r is large enough and $\gamma \in \Gamma_K$ thern $1-\gamma : \mathbf{D}^{\dagger,r}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is an isomorphism whose inverse is uniformly continuous for the Fréchet topology of $\mathbf{D}^{\dagger,r}(V)$.

In order to show the surjectivity of $1-\gamma$ it is therefore enough to show that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\text{rig}}(V)^{\psi=0}$ for the Fréchet topology. For r large enough, $\mathbf{D}^{\dagger,r}(V)$ has a basis in $\varphi(\mathbf{D}^{\dagger,r/p}(V))$ so that

$$\begin{aligned}\mathbf{D}^{\dagger,r}(V)^{\psi=0} &= (\mathbf{B}^{\dagger,r}_K)^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}(V))\\ \mathbf{D}^{\dagger,r}_{\text{rig}}(V)^{\psi=0} &= (\widetilde{\mathbf{B}}^{\dagger,r}_{\text{rig},K})^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}_{\text{rig}}(V)).\end{aligned}$$

The fact that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\text{rig}}(V)^{\psi=0}$ for the Fréchet topology will therefore follow from the density of $(\mathbf{B}^{\dagger,r}_K)^{\psi=0}$ in $(\widetilde{\mathbf{B}}^{\dagger,r}_{\text{rig},K})^{\psi=0}$. The last statement follows from the facts that by definition $\mathbf{B}^{\dagger,r/p}_K$ is dense in $\widetilde{\mathbf{B}}^{\dagger,r/p}_{\text{rig},K}$ and that

$$(\mathbf{B}^{\dagger,r}_K)^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\mathbf{B}^{\dagger,r/p}_K) \quad \textit{and} \quad (\widetilde{\mathbf{B}}^{\dagger,r}_{\text{rig},K})^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\widetilde{\mathbf{B}}^{\dagger,r/p}_{\text{rig},K}).$$

□

Lemma 9.8. *The following maps are all surjective and the kernel is $\mathbf{Q}_p$*

$$1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger, \quad 1-\varphi : \widetilde{\mathbf{B}}^+_{\text{rig}} \to \widetilde{\mathbf{B}}^+_{\text{rig}} \quad \textit{and} \quad 1-\varphi : \widetilde{\mathbf{B}}^\dagger_{\text{rig}} \to \widetilde{\mathbf{B}}^\dagger_{\text{rig}}$$

Proof. Since $\widetilde{\mathbf{B}}^+_{\text{rig}} \subset \mathbf{B}^\dagger_{\text{rig}}$ and $\widetilde{\mathbf{B}}^\dagger \subset \widetilde{\mathbf{B}}^\dagger_{\text{rig}}$ it is enough to show that $(\widetilde{\mathbf{B}}^\dagger_{\text{rig}})^{\varphi=1} = \mathbf{Q}_p$. If $x \in (\widetilde{\mathbf{B}}^\dagger_{\text{rig}})^{\varphi=1}$, then [Ber02, proposition 3.2] shows that actually $x \in (\widetilde{\mathbf{B}}^+_{\text{rig}})^{\varphi=1}$, and therefore $x \in (\widetilde{\mathbf{B}}^+_{\text{rig}})^{\varphi=1} = (\mathbf{B}^+_{\max})^{\varphi=1} = \mathbf{Q}_p$ by [Col98, proposition III 3.5].

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^+_{\text{rig}} \to \widetilde{\mathbf{B}}^+_{\text{rig}}$ results from the surjectivity of $1-\varphi$ on the first two spaces since by [Ber02, lemma 2.18], one can write $\alpha \in \widetilde{\mathbf{B}}^+_{\text{rig}}$ as $\alpha = \alpha^+ + \alpha^-$ with $\alpha^+ \in \widetilde{\mathbf{B}}^+_{\text{rig}}$ and $\alpha^- \in \widetilde{\mathbf{B}}^\dagger$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ follows from the facts that $1-\varphi : \mathbf{B}^+_{\max} \to \mathbf{B}^+_{\max}$ is surjective ([Col98, proposition III 3.1]) and that $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} = \cap_{n=0}^{\infty}\varphi^n(\mathbf{B}^+_{\max})$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger$ follows from the facts that $1-\varphi : \widetilde{\mathbf{B}} \to \widetilde{\mathbf{B}}$ is surjective (it is surjective on $\widetilde{\mathbf{A}}$ as can be seen by reducing modulo p and using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed) and that if $\beta \in \widetilde{\mathbf{B}}$ is such that $(1-\varphi)\beta \in \widetilde{\mathbf{B}}^\dagger$, then $\beta \in \widetilde{\mathbf{B}}^\dagger$.

If $x = \sum_{i=0}^{+\infty} p^i[x_i] \in \widetilde{\mathbf{A}}$, let us set $w_k(x) = \inf_{i\leq k} \nu_E(x_i) \in \mathbb{R}\cup\{+\infty\}$. The definition of $\widetilde{\mathbf{B}}^{\dagger,r}$ shows that $x \in \widetilde{\mathbf{B}}^{\dagger,r}$ if and only if $\lim_{k\to+\infty} w_k(x) + \frac{pr}{p-1}k = +\infty$. A short computation shows that $w_k(\varphi(x)) = pw_k(x)$ and that $w_k(x+y) \geq \inf(w_k(x), w_k(y))$ with equality if $w_k(x) \neq w_k(y)$. It is then clear that

$$\lim_{k\to+\infty} w_k((1-\varphi)x) + \frac{pr}{p-1}k = +\infty \Longrightarrow \lim_{k\to+\infty} w_k(x) + \frac{p(r/p)}{p-1}k = +\infty$$

and so if $x \in \widetilde{\mathbf{A}}$ is such that $(1-\varphi)x \in \widetilde{\mathbf{B}}^{\dagger,r}$ then $x \in \widetilde{\mathbf{B}}^{\dagger,r/p}$ and likewise for $x \in \widetilde{\mathbf{B}}$ by multiplication by a suitable power of p. This shows the second fact. $\square$

Proposition 9.9. *If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$ and Γ_K is torsion free, there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\gamma-1)(\varphi-1)b = (\varphi-1)y$ and the formula*

$$h^1_{K,V}(y) = \log^0_p(\gamma)[\sigma \mapsto \frac{\sigma-1}{\gamma-1}y - (\sigma-1)b]$$

then defines a map $h^1_{K,V} : \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}_{\Gamma_K} \mapsto H^1(K,V)$ which does not depend either on the choice of generator γ of Γ_K or on the particular solution b, and if $y \in D(V)^{\psi=1} \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}(y)$ coincides with the cocycle constructed in section 4.2.

Proof. Our construction closely follows section 4.2; to simplify the notations, we may assume that $\log^0_p(\gamma) = 1$. The fact that $h^1_{K,V}$ is independent of the choice of γ is same as lemma 4.2.

Let us start by showing the existence of $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$. If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $(\varphi-1)y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$. By lemme 9.7, there exists $x \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$ such that $(\gamma-1)x = (\varphi-1)y$. By lemma 9.8, there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\varphi-1)b = x$.

Recall that we define $h^1_{K,V}(y) \in H^1(K,V)$ by the formula:

$$h^1_{K,V}(y)(\sigma) = \frac{\sigma-1}{\gamma-1}y - (\sigma-1)b.$$

Notice that, a priori, $h^1_{K,V}(y) \in H^1(K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)$, but

$$\begin{aligned}(\varphi-1)h^1_{K,V}(y)(\sigma) =& \frac{\sigma-1}{\gamma-1}(\varphi-1)y - (\sigma-1)(\varphi-1)b\\ =& \frac{\sigma-1}{\gamma-1}(\gamma-1)x - (\sigma-1)x\\ =& 0,\end{aligned}$$

so that $h^1_{K,V}(y)(\sigma) \in (\mathbf{B}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$. In addition, two different choices of b differ by an element of $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$, and therefore give rise to two cohomologous cocycles.

It is clear that if $y \in D(V)^{\psi=1} \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}$ coincide with the cocycle constructed in section 4.2, as can be seen by their identical construction, and it is immediate that if $y \in (\gamma-1)\mathbf{D}^\dagger_{\mathrm{rig}}(V)$, then $h^1_{K,V}(y) = 0$. $\square$

Lemma 9.10. *We have* $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$.

Proof. Same as lemma 5.3. □

9.5. **Iwasawa algebra and power series.** Given a finite unramified extension F of $\mathbf{Q}_p$, denote by $\Lambda(\Gamma_F)$ (resp. $\Lambda(\Gamma_F^1)$) the Iwasawa algebra $\mathbf{Z}_p[[\Gamma_F]]$ (resp. $\mathbf{Z}_p[[\Gamma_F^1]]$).

Let
$$\mathcal{H} = \{f \in \mathbf{Q}_p[\Delta][[X]] \mid f \text{ convergs on the open unit disk}\},$$
and define $\mathcal{H}(\Gamma_F)$ to be the set of $f(\gamma - 1)$ with $f(X) \in \mathcal{H}$ and γ a topological generator of Γ. We may identify $\Lambda(\Gamma_F) \otimes \mathbf{Q}_p$ with the subring of $\mathcal{H}(\Gamma_F)$ consisting of power series with bounded coefficients. Note that $\mathcal{H}(\Gamma)$ may be identified with the continuous dual of the space of locally analytic functions on Γ_F, with multiplication corresponding to convolution, implying that its definition is independent of the choice of generator γ_F (c.f. section 1.2).

The action of Γ_F on $\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p}$ gives an isomorphism of $\mathcal{H}(\Gamma_F)$ with $(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ via the Mellin transform [Per01, corollary B.2.8]
$$\begin{aligned} \mathfrak{M} : \mathcal{H}(\Gamma_F) &\to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0} \\ f(\gamma-1) &\mapsto f(\gamma-1)(\pi+1). \end{aligned}$$
In particular, $\Lambda(\Gamma_F)$ corresponds to $(\mathbf{A}^+_{\mathbf{Q}_p})^{\psi=0}$ under $\mathfrak{M}$. Similarly, we define $\mathcal{H}(\Gamma_F^1)$ as the subring of $\mathcal{H}(\Gamma_F)$ defined by power series over $\mathbf{Q}_p$, rather than $\mathbf{Q}_p[\Delta]$. Then, $\mathcal{H}(\Gamma_F^1)$ (resp. $\Lambda(\Gamma_F^1)$) corresponds to $(1+\pi)\varphi(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})$ (resp. $(1+\pi)\varphi(\mathbf{A}^+_{\mathbf{Q}_p})$) under $\mathfrak{M}$.

9.6. **Iwasawa algebras and differential equations.** By [Ber02, proposition 2.24], we have maps $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$ whose restriction to $\mathbf{B}^+_{\mathrm{rig},F}$ satisfied $\varphi^{-n}(\mathbf{B}^+_{\mathrm{rig},F}) \subset F_n[[t]]$ and which can be characterized by the fact that π maps to $\varepsilon^{(n)}\exp(t/p^n) - 1$.

Recall if $z \in F_n((t)) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, we denote the constant coefficient of z by $\partial_V(z) \in F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$.

Lemma 9.11. *If* $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, *then for any* $m \geq n \geq 0$, *the element*
$$p^{-m}\mathrm{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y)) \in F_n \otimes \mathbf{D}_{\mathrm{cris}}(V)$$
does not depend on m *and we have*
$$p^{-m}\mathrm{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0 \end{cases}$$

Proof. Recall that if $y = t^{-l}\sum_{k=0}^{+\infty} a_k\pi^k \in \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then
$$\varphi^{-m}(y) = p^{ml}t^{-l}\sum_{k=0}^{+\infty}\varphi^{-m}(a_k)(\varepsilon^{(m)}\exp(t/p^m) - 1)^k,$$
and that by the definition of ψ, $\psi(y) = y$ means that:
$$\varphi(y) = \frac{1}{p}\sum_{\zeta^p=1} y(\zeta(1+T) - 1).$$
The lemma then follows from the fact that if $m \geq 2$, then the conjugates of $\varepsilon^{(m)}$ under $\mathrm{Gal}(F_m/F_{m-1})$ are the $\zeta\varepsilon^{(m)}$, where $\zeta^p = 1$, while if $m = 1$, then the conjugates of $\varepsilon^{(1)}$ under $\mathrm{Gal}(F_1/F)$ are the ζ, where $\zeta^p = 1$ but $\zeta \neq 1$. □

Recall that since F is an unramified extension of $\mathbf{Q}_p$, $\Gamma_F \simeq \mathbf{Z}_p^*$ and that $\Gamma_{F_n} = \mathrm{Gal}(F_\infty/F_n)$ is the set of elements $\gamma \in \Gamma_F$ such that $\chi(\gamma) \in 1 + p^n\mathbf{Z}_p$.

The Iwasawa algebra of Γ_F is $\Lambda_{\mathscr{O}_F} = \mathbf{Z}_p[[\Gamma_F]] \cong \mathbf{Z}_p[\Delta_F] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma_{F_1}]]$, and we set $\mathcal{H}(\Gamma_F) = \mathbf{Q}_p[\Delta_F] \otimes_{\mathbf{Q}_p} \mathcal{H}(\Gamma_F^1)$ where $\mathcal{H}(\Gamma_F^1)$ is the set of $f(\gamma - 1)$ with $\gamma \in \Gamma_F^1$ and where $f(X) \in \mathbf{Q}_p[[X]]$ is convergent on the p-adic open unit disk. We define $\nabla_i \in \mathcal{H}(\Gamma_F)$ by

$$\nabla_i = \frac{\log(\gamma)}{\log_p(\chi(\gamma))} - i.$$

We will also use the operator $\nabla_0/(\gamma_n - 1)$, where γ_n is a topological generator of Γ_F^n. It is defined by the formula

$$\frac{\nabla_0}{\gamma_n - 1} = \frac{\log(\gamma_n)}{\log_p(\chi(\gamma_n))(\gamma_n - 1)} = \frac{1}{\log_p(\chi(\gamma_n))} \sum_{i \geq 1} \frac{(1-\gamma_n)^{i-1}}{i},$$

or equivalently by

$$\frac{\nabla_0}{\gamma_n - 1} = \lim_{\eta \in \Gamma_F^n, \eta \to 1} \frac{\eta - 1}{\gamma_n - 1} \frac{1}{\log_p(\chi(\eta))}.$$

It is easy to see that $\nabla_0/(\gamma_n - 1)$ acts on F_n by $1/\log_p(\chi(\gamma_n))$.

The algebra $\mathcal{H}(\Gamma_F)$ acts on $\mathbf{B}^+_{\mathrm{rig},F}$ and one can easily check that

$$\nabla_i = t\frac{d}{dt} - i = \log(1+\pi)\partial - i, \quad \text{where} \quad \partial = (1+\pi)\frac{d}{d\pi}.$$

In particular, $\nabla_0 \mathbf{B}^+_{\mathrm{rig},F} \subset t\mathbf{B}^+_{\mathrm{rig},F}$ and if $i \geq 1$, then

$$\nabla_{i-1} \circ \cdots \circ \nabla_0 \subset t^i \mathbf{B}^+_{\mathrm{rig},F}.$$

Lemma 9.12. *If $n \geq 1$, then $\nabla_0/(\gamma_n - 1)(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset (t/\varphi^n(\pi))(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$ so that if $i \geq 1$, then*

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1}(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset (\frac{t}{\varphi^n(\pi)})^i(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

Proof. Since $\nabla_i = t \cdot d/dt - i$, the second claim follows easily from the first one. By the standard properties of p-adic holomorphic functions, what we need to do is to show that if $x \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$, then

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = 0$$

for all $m \geq n+1$.

On the other hand, up to a scalar factor, one has for $m \geq n+1$:

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = \mathrm{Tr}_{F_m/F_n} x(\varepsilon^{(m)} - 1),$$

which can be seen from the fact that

$$\frac{\nabla_0}{\gamma_n - 1} = \lim_{\eta \in \Gamma_F^n, \eta \to 1} \frac{\eta - 1}{\gamma_n - 1} \cdot \frac{1}{\log_p(\chi(\eta))}.$$

On the other hand, the fact $\psi(x) = 0$ implies that for every $m \geq 2$, $\mathrm{Tr}_{F_m/F_{m-1}} x(\varepsilon^{(m)} - 1) = 0$. This completes the proof. $\square$

Finally, let us point out that the actions of any element of $\mathcal{H}(\Gamma_F)$ and φ commute. Since $\varphi(t) = pt$, we also see that $\partial \circ \varphi = p\varphi \circ \partial$.

We will henceforth assume that $\log_p(\chi(\gamma_n)) = p^n$, and in addition $\nabla_0/(\gamma_n - 1)$ acts on F_n by p^{-n}.

10. Bloch-Kato's exponential maps: Three explicit reciprocity formulas

In this section, we explain the results of Berger in [Ber03] on explicit reciprocity formulas when V is a crystalline representation of an unramified field.

Recall $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$, let Δ_K be the torsion subgroup of $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$ and let $\Gamma_K^1 = \mathrm{Gal}(K_\infty/K(\mu_p))$, so that $\Gamma_K \simeq \Delta_K \times \Gamma_K^1$. Let $\Gamma_K = \mathbf{Z}_p[[\Gamma_K]]$ and $\mathcal{H}(\Gamma_K) = \mathbf{Q}_p[\Delta_K] \otimes_{\mathbf{Q}_p} \mathcal{H}(\Gamma_K^1)$ where $\mathcal{H}(\Gamma_K^1)$ is the set of $f(\gamma_1 - 1)$ with $\gamma_1 \in \Gamma_{K_1}$ and where $f(T) \in \mathbf{Q}_p[[T]]$ is a power series which converges on the p-adic unit disk.

When F is an unramified extension of and V is a crystalline representation of G_F, Perrin-Riou has constructed in [Per94] a period map $\Omega_{V,h}$ which interpolates the $\exp_{F,V(k)}$ as k runs over the positive integers. It is crucial ingredient in the construction of p-adic L-funtions, and is a vast generalization of Coleman's isomorphism.

The main result of [Per94] is the construction, for a crystalline representation of V of G_F of a family of maps (parameterized by $h \in \mathbf{Z}$):

$$\Omega_{V,h} : (\mathcal{H}(\Gamma_F) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathcal{H}(\Gamma_F) \otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F, V)/V^{H_F},$$

whose main property is that they interpolate Bloch-Kato's exponential map. More precisely, if $h, j \gg 0$, then the diagram:

$$\begin{tikzcd}
(\mathcal{H}(\Gamma_F) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j)))^{\Delta=0} \arrow[r, "\Omega_{V(j),h}"] \arrow[d, "\Xi_{n,V(j)}"'] & \mathcal{H}(\Gamma_F) \otimes_{\Gamma_F} H^1_{\mathrm{Iw}}(F, V(j))/V(j)^{H_F} \arrow[d, "\mathrm{pr}_{F_n,V(j)}"] \\
F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \arrow[r, "(h+j-1)!\times\exp^*_{F_n,V(j)}"] & H^1(F_n, V(j)).
\end{tikzcd}$$

is commutative where Δ and Ξ are two maps whose definition is rather technical (see section 10.2 for a precisely definition).

Using the inverse of Perrin-Riou's map, one can then associate to an Euler system a p-adic L-function. For example, if one starts with $V = \mathbf{Q}_p(1)$, then Perrin-Riou's map is the inverse of the Coleman isomorphism and one recovers Kubota-Leopoldt p-adic L-functions (See section 11.2).

The goal of this section is to give formulas for $\exp_{K,V}$, $\exp^*_{K,V^*(1)}$ and $\Omega_{V,h}$ in terms of the (φ, Γ)-module associated to V.

10.1. **The Bloch-Kato's exponential map and its dual revisit.** Recall in section 7.1, we defined the Bloch-Kato's exponential map and its dual. The goal of theis paragraph is to compute Bloch-Kato's exponential map and its dual in terms of the (φ, Γ)-module of V. Let $h \geq 1$ be an integer such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$.

Recall that we have seen that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^+_{\mathrm{rig}}[1/t])^{\Gamma_F}$ and by [Ber04, II.3], there is an isomorphism

$$\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}^+_{\mathrm{rig}}(V).$$

If $y \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then the fact that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ implies by result of [Ber04, II.3] that $t^h y \in \mathbf{D}^+_{\mathrm{rig}}(V)$, so that if

$$y = \sum_{i=0}^{d} y_i \otimes d_i \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1},$$

then

$$\nabla_{h-1} \circ \cdots \nabla_0(y) = \sum_{i=0}^{d} t^h \partial^h y_i \otimes d_i \in \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}.$$

One can apply the operator $h^1_{F_n,V}$ to $\nabla_{h-1} \circ \cdots \nabla_0(y)$, then we have:

Theorem 10.1. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, then*

$$h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} \exp_{F_n,V}(p^{-n}\partial_V(\varphi^{-n}(y))) & \text{if } n \geq 1 \\ \exp_{F,V}((1-p^{-1}\varphi^{-1})\partial_V(y)) & \text{if } n = 0 \end{cases}$$

Proof. Because the diagram

$$\begin{array}{ccc} F_{n+1} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_{n+1},V}} & H^1(F_{n+1}, V) \\ \downarrow{\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n} \otimes id} & & \downarrow{\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \\ F_n \otimes \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_n,V}} & H^1(F_n, V) \end{array}$$

is commutative, it is enough to prove the theorem under the assumption that Γ^n_F is torsion free. Let us set $y_h = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$. Since we are assuming for simplicity that $\chi(\gamma_n) = p^n$, the cocycle $h^1_{F_n,V}(y_h)$ is defined by:

$$h^1_{F_n,V}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1) b_{n,h}$$

where $b_{n,h}$ is a solution of the equation $(\gamma_n - 1)(\varphi - 1)b_{n,h} = (\varphi - 1)y_h$. In lemma 9.12 above, we prove that

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1}(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset (\frac{t}{\varphi^n(\pi)})^i (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

It is then clear that if one sets

$$z_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1}(\varphi - 1)y,$$

then

$$z_{n,h} \in (\frac{t}{\varphi^n(\pi)})^h (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \subset \varphi^n(\pi^{-h})\mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=0} \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}.$$

Let $q = \varphi(\pi)/\pi$. By lemma 10.2 below, there exists an element $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that

$$(\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h)b_{n,h}) = \varphi^n(\pi^h)z_{n,h},$$

so that $(1-\varphi)b_{n,h} = z_{n,h}$ with $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes \mathbf{Q}_p$.

If we set $w_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1} y$, then $w_{n,h}$ and $b_{n,h} \in \mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V$ and the cocycle $h^1_{F_n,V}(y_h)$ is then given by the formula $h^1_{F_n,V}(y_h)(\sigma) = (\sigma - 1)(w_{n,h} - b_{n,h})$. Now $(\varphi - 1)b_{n,h} = z_{n,h}$ and $(\varphi - 1)w_{n,h} = z_{n,h}$ as well, so that $w_{n,h} - b_{n,h} \in \mathbf{B}^{\varphi=1}_{\max} \otimes_{\mathbf{Q}_p} V$.

We can also write

$$h^1_{F_n,V}(y_h)(\sigma) = (\sigma-1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})).$$

Since we know that $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\mathbf{B}^+_{\max} \otimes_{\mathbf{Q}_p} V$, we have $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

The definition of Bloch-Kato exponential gives rise to the following construction: if $x \in \mathbf{D}_{\mathrm{dR}}(V)$ and $\widetilde{x} \in \mathbf{B}^{\varphi=1}_{\max} \otimes_{\mathbf{Q}_p} V$ is such that $x - \widetilde{x} \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ then $\exp_{K,V}(x)$ is the class of the coclycle $g \mapsto g(\widetilde{x}) - \widetilde{x}$.

The theorem therefore follow from the fact that:

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\partial_V(\varphi^{-n}(y) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

since we already know that $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

In order to show this, first notice that

$$\varphi^{-n}(y) - \partial_V(\varphi^{-n}(y)) \in tF_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V).$$

We can therefore write

$$\frac{\nabla_0}{\gamma_n - 1}\varphi^{-n}(y) = p^{-n}\partial_V(\varphi^{-n}(y)) + tz_1$$

and a simple recurrence shows that

$$\nabla_{i-1} \circ \cdots \circ \frac{\nabla_0}{1-\gamma_n}\varphi^{-n}(y) = (-1)^{i-1}(i-1)!p^{-n}\partial_V(\varphi^{-n}(y)) + t^i z_i,$$

with $z_i \in F_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$. By taking $i = h$, we see that

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\partial_V(\varphi^{-n}(y)) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

Since we choose h such that $t^h\mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. □

Lemma 10.2. *If $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, the there exists $\beta \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that*

$$(\varphi - \varphi^{n-1}(q^h))\beta = \alpha.$$

Proof. By [Ber02, proposition 2.19], the ring $\widetilde{\mathbf{B}}^+$ is dense in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ for the Fréchet topology. Hence, if $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, then there exists $\alpha_0 \in \widetilde{\mathbf{B}}^+$ such that $\alpha - \alpha_0 = \varphi^n(\pi^h)\alpha_1$ with $\alpha_1 \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$.

The map $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$ is surjective because $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$ is surjective, as can be seen by reducing modulo p using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed and that $\widetilde{\mathbf{E}}^+$ is its ring of integers.

One can therefore write $\alpha_0 = (\varphi - \varphi^{n-1}(q^h))\beta_0$. Finally by lemma 9.8, there exists $\beta \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that $\alpha_1 = (\varphi - 1)\beta_1$, so that $\varphi^n(\pi^h)\alpha_1 = (\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h)\beta_1)$. □

Theorem 10.3. *If $y \in (\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\psi=1}$ and $y \in \mathbf{D}^+_{\mathrm{rig}}(V)[1/t]$ (so that in particular $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$), then*

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0 \end{cases}$$

Proof. Since the following diagram

$$\begin{array}{ccc} H^1(F_{n+1},V) & \xrightarrow{\exp^*_{F_{n+1},V^*(1)}} & F_{n+1}\otimes \mathbf{D}_{\mathrm{cris}}(V) \\ {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}\otimes id} \\ H^1(F_n,V) & \xrightarrow{\exp^*_{F_n,V^*(1)}} & F_n\otimes \mathbf{D}_{\mathrm{cris}}(V) \end{array}$$

is commutative, we only need to prove the theorem when Γ_F^n is torsion free by lemma 10.1. We then have (assuming that $\chi(\gamma_n)=p^n$ for simplicity) :

$$h^1_{F_n,V}(y)(\sigma)=\frac{\sigma-1}{\gamma_n-1}y-(\sigma-1)b,$$

where $(\gamma_n-1)(\varphi-1)b=(\varphi-1)y$. Recall that $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}=\cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. Since $b\in\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\otimes_{\mathbf{Q}_p}V$, there exists $m\gg 0$ such that $b\in\widetilde{\mathbf{B}}^{\dagger,r_m}_{\mathrm{rig}}\otimes_{\mathbf{Q}_p}V$ and that the map φ^{-m} embeds $\widetilde{\mathbf{B}}^{\dagger,r_m}_{\mathrm{rig}}$ into $\mathbf{B}^+_{\mathrm{dR}}$. we can then write

$$h^1(y)(\sigma)=\frac{\sigma-1}{\gamma_n-1}\varphi^{-m}(y)-(\sigma-1)\varphi^{-m}(b),$$

and $\varphi^{-m}(b)\in\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V$. In addition, $\varphi^{-m}(y)\in F_m((t))\otimes_F\mathbf{D}_{\mathrm{cris}}(V)$ and γ_n-1 is invertible on $t^kF_m\otimes_F\mathbf{D}_{\mathrm{cris}}(V)$ for every $k\neq 0$ This shows that the cocycle $h^1_{F_n,V}$ is cohomologous in $H^1(F_n,\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V)$ to

$$\sigma\mapsto\frac{\sigma-1}{\gamma_n-1}(\partial_V(\varphi^{-m}(y)))$$

which is itself cohomologous (since γ_n-1 is invertible on $F_m^{\mathrm{Tr}_{F_m/F_n}=0}$) to

$$\sigma\mapsto\frac{\sigma-1}{\gamma_n-1}\big(p^{n-m}\mathrm{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y))\big)=\sigma\mapsto p^{-n}\log_p(\chi(\overline{\sigma}))p^{n-m}\mathrm{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y)).$$

It follows from this and proposition 7.2 and lemma 9.11 that

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y))=p^{-m}\mathrm{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y))=\begin{cases}p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n\geq 1\\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n=0.\end{cases}$$

□

10.2. **Perrin-Riou's big exponential map.** By using the results of the previous paragraphs, we can give a uniform formula for the image of an element $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ in $H^1(F_n,V(j))$ under the composition of the following maps:

$$\big(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V)\big)^{\psi=1}\xrightarrow{\nabla_{h-1}\circ\cdots\circ\nabla_0}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}\xrightarrow{\otimes e_j}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V(j))^{\psi=1}\xrightarrow{h^1_{F_n,V(j)}}H^1(F_n,V(j))$$

Here e_j is a basis of $\mathbf{Q}_p(j)$ such that $e_{j+k}=e_j\otimes e_k$ so that if V is a p-adic representation, then we have compatible isomorphisms of $\mathbf{Q}_p$-vector spaces $V\to V(j)$ given by $v\mapsto v\otimes e_j$.

Theorem 10.4. *If $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, and $h\geq 1$ is an integer such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$, then for all j with $h+j\geq 1$, we have :*

$$\begin{aligned}h^1_{F_n,V(j)}(\nabla_{h-1}\circ\cdots\nabla_0(y)\otimes e_j)=&(-1)^{h+j-1}(h+j-1)!\times\\ &\begin{cases}\exp_{F_n,V(j)}(p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y\otimes t^{-j}e_j))) & \text{if } n\geq 1\\ \exp_{F,V(j)}((1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y\otimes t^{-j}e_j)) & \text{if } n=0\end{cases}\end{aligned}$$

while if $h+j \leq 0$, then we have:

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V}(\nabla_{h-1}\circ\cdots\nabla_0(y)\otimes e_j)) = \frac{1}{(-h-j)!}\begin{cases} p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^j y\otimes t^{-j}e_j)) & \text{if } n\geq 1\\ (1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y\otimes t^{-j}e_j) & \text{if } n=0\end{cases}$$

Proof. If $h+j\geq 1$, then we have the following commutative diagram:

$$\begin{tikzcd} \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} \arrow[r, "\otimes e_j"] & \mathbf{D}^+_{\mathrm{rig}}(V(j))^{\psi=1} \\ \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} \arrow[u, "\nabla_{h-1}\circ\cdots\circ\nabla_0"] \arrow[r, "\partial^{-j}\otimes t^{-j}e_j"] & \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V(j))\right)^{\psi=1}. \arrow[u, "\nabla_{h+j-1}\circ\cdots\circ\nabla_0"'] \end{tikzcd}$$

and the theorem is then a straightforward consequence of theorem 10.1 applied to $\partial^{-j}y\otimes t^{-j}e_j$, $h+j$ and $V(j)$.

On the other hand, if $h+j\leq 0$, and Γ^n_F is torsion free, then theorem 10.3 shows that

$$\exp^*_{F_n,V*(1-j)}(h^1_{F_n,V(j)}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j))$$

in $\mathbf{D}_{\mathrm{cris}}(V(j))$, and a short computation involving Taylor series shows that

$$p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = (-h-j)!^{-1}p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y\otimes t^{-j}e_j)).$$

Finally, to get the case $n=0$, one just needs to use the corresponding statement of theorem 10.3 or equivalently corestrict. □

Remark 10.5. The notation ∂^{-j} is not injective on $\mathbf{B}^+_{\mathrm{rig},F}$ (it is surjective by integration) but it can be checked that it leads to no ambiguity in the formulas above.

We will now use the above result to give a construction of Perrin-Riou's exponential map. If $f\in \mathbf{B}^+_{\mathrm{rig},F}\otimes\mathbf{D}_{\mathrm{cris}}(V)$, we define $\Delta(f)$ to be the image of $\oplus^h_{k=0}\partial^k(f)(0)$ in $\oplus^h_{k=0}(\mathbf{D}_{\mathrm{cris}}(V))/(1-p^k\varphi)(k)$. There is then an exact sequence of $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda_F$-modules (cf [Per94, section 2.2]):

$$0\longrightarrow \oplus^h_{k=0}t^k\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}}\longrightarrow (\mathbf{B}^+_{\mathrm{rig},F}\otimes\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}\xrightarrow{1-\varphi}$$
$$(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V)\xrightarrow{\Delta}\oplus^h_{k=0}\frac{\mathbf{D}_{\mathrm{cris}}(V)}{1-p^k\varphi}(k)\longrightarrow 0.$$

If $f\in((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$, then by the above exact sequence there exists

$$y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$$

such that $f=(1-\varphi)y$, and since $\nabla_{h-1}\circ\cdots\nabla_0$ kills $\oplus^{h-1}_{k=0}t^k\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}}$ we see that $\nabla_{h-1}\circ\cdots\nabla_0(y)$ does not depend upon the choice of such y unless $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}\neq 0$.

Definition 10.6. Let $h\geq 1$ be an integer such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$ and such that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}=0$. One deduce from the above construction a well-defined map

$$\Omega_{V,h}:\left((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V)\right)^{\Delta=0}\to\mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}$$

given by $\Omega_{V,h}(f)=\nabla_{h-1}\circ\cdots\nabla_0(y)$, where $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ is such that $f=(1-\varphi)y$.

If $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}\neq 0$ then we get a map

$$\Omega_{V,h}:\left((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V)\right)^{\Delta=0}\to\mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{G_F=\chi^h}.$$

Theorem 10.7. *If V is a crystalline representation and $h \geq 1$ is such that we have $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$, then the map*

$$\Omega_{V,h} : \left((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{H_F}$$

which takes $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1} \circ \cdots \nabla_0((1-\varphi)^{-1}f)$ is well defined and coincides with Perrin-Riou's exponential map.

Proof. The map $\Omega_{V,h}$ is well defined because as we seen above the kernel of $1-\varphi$ is killed by $\nabla_{h-1} \circ \cdots \circ \nabla_0$, except for $t^h\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}$, which is mapped to copies of $\mathbf{Q}_p(h) \in V^{H_F}$.

The fact that $\Omega_{V,h}$ coincides with Perrin-Riou's exponential map follows directly from theorem 10.4 above applied to those j's for which $h+j \geq 1$, and the fact that by [Per94, theorem 3.2.3], the $\Omega_{V,h}$ are uniquely determined by the requirement that they satisfy the following diagram for $h, j \gg 0$:

$$\begin{tikzcd}
(\mathcal{H}(\Gamma_F) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j))^{\Delta=0} \arrow[r, "\Omega_{V(j),h}"] \arrow[d, "\Xi_{n,V(j)}"'] & \mathcal{H}(\Gamma_F) \otimes_{\Gamma_F} (H^1_{\mathrm{Iw}}(F, V(j)/V(j)^{H_F}) \arrow[d, "\mathrm{pr}_{F_n,V(j)}"] \\
F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \arrow[r, "(h+j-1)!\exp_{F_n,V(j)}"] & H^1(F_n, V(j)).
\end{tikzcd}$$

Here $\Xi_{n,V(j)}(g) = p^{-n}(\varphi \otimes \varphi)^{-n}(f)(\varepsilon^{(n)} - 1)$ where f is such that

$$(1-\varphi)f = g(\gamma - 1)(1+\pi) \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

and the φ on the left of $\varphi \otimes \varphi$ is the Frobenius on $\mathbf{B}^+_{\mathrm{rig},F}$ while the φ on the right is the Frobenius on $\mathbf{D}_{\mathrm{cris}}(V)$.

Note that by theorem 5.2, we have an isomorphism $D(V)^{\psi=1} \simeq H^1_{\mathrm{Iw}}(F, V)$ and therefore we get a map $\mathcal{H}(\Gamma_F) \otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F, V) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$. On the other hand, there is a map

$$\mathcal{H}(\Gamma_F) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j)) \to (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

which sends $\sum f_i(\gamma - 1) \otimes d_i$ to $\sum f_i(\gamma-1)(1+\pi) \otimes d_i$. These two maps allow us to compare the diagram above with the formulas given by theorem 10.4. $\square$

Remark 10.8. By the above remarks, if V is a crystalline representation and $h \geq 1$ is such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{Q}_p(h) \not\subset V$, then the map

$$\Omega_{V,h} : \left((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}$$

which takes $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1} \circ \cdots \nabla_0((1-\varphi)^{-1}f)$ is well defined, without having to kill the Λ_F-torsion of $H^1_{\mathrm{Iw}}(F, V)$.

Remark 10.9. It is clear from theorem 10.4 that we have:

$$\Omega_{V,h}(x) \otimes e_j = \Omega_{V(j),h+j}(\partial^j x \otimes t^{-j}e_j) \quad \text{and} \quad \nabla_h \circ \Omega_{V,h}(x) = \Omega_{V,h+1}(x)$$

and following Perrin-Riou, one can use these formulas to extend the definition of $\Omega_{V,h}$ to all $h \in \mathbf{Z}$ by tensoring all $\mathcal{H}(\Gamma_F)$-modules with the field of fractions of $\mathcal{H}(\Gamma_F)$

10.3. **The explicit reciprocity formula.** Recall we have a map $\mathcal{H}(\Gamma_F) \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ which sends $f(\gamma-1)$ to $f(\gamma-1)(1+\pi)$, this map is a bijection and its inverse is the Mellin transform so that if $g(\pi) \in (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$, then $g(\pi) = \mathfrak{M}(g)(1+\pi)$. If $f, g \in (\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ then we define $f*g$ by the formula $\mathfrak{M}(f*g) = \mathfrak{M}(f)\mathfrak{M}(g)$. Let $[-1] \in \Gamma_F$ be the element such that $\chi([-1]) = -1$, and let ι be the involution of Γ_F which sends γ to γ^{-1}. The operator ∂^j on $(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ corresponds to Tw_j on Γ_F (Tw_j is defined by $\mathrm{Tw}_j(\gamma) = \chi(\gamma^j)\gamma$). We will make use of the facts that $\iota\circ\partial^j = \partial^{-j}\circ\iota$ and $[-1] \circ \partial^j = (-1)^j \partial^j \circ [-1]$.

If V is a crystalline representation, then the natural maps

$$\mathbf{D}_{\mathrm{cris}}(V) \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)) \longrightarrow \mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_{F/\mathbf{Q}_p}} \mathbf{Q}_p$$

allow us to define a perfect pairing $[\cdot,\cdot]_V : \mathbf{D}_{\mathrm{cris}}(V) \times \mathbf{D}_{\mathrm{cris}}(V^*(1))$ which we extend by linearity to

$$[\cdot,\cdot]_V : (\mathbf{B}^+_{\mathrm{rig},F} \otimes \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0} \times (\mathbf{B}^+_{\mathrm{rig},F} \otimes \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula $[f(\pi) \otimes d_1, g(\pi) \otimes d_2]_V = (f*g)(\pi)[d_1, d_2]_V$.

We can also define a semi-linear pairing (with respect to ι)

$$\langle\cdot,\cdot\rangle_V : \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} \times \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula

$$\langle\cdot,\cdot\rangle_V = \varprojlim \sum_{\tau \in \Gamma_F/\Gamma_F^n} \big\langle \tau^{-1}(h^1_{F_n,V}(y_1)), h^1_{F_n,V^*(1)}(y_2)\big\rangle_{F_n,V} \cdot \tau(1+\pi)$$

where the pairing $\langle\cdot,\cdot\rangle_{F_n,V}$ is given by the cup product:

$$\langle\cdot,\cdot\rangle_{F_n,V} : H^1(F_n,V) \times H^1(F_n, V^*(1)) \to H^2(F_n, \mathbf{Q}_p(1)) \cong \mathbf{Q}_p.$$

The pairing $\langle\cdot,\cdot\rangle_V$ satisfies the relation $\langle \gamma_1 x_1, \gamma_2 x_2\rangle_V = \gamma_1\iota(\gamma_2)\langle x_1, x_2\rangle_V$. Perrin-Riou's explicit reciprocity formula is then:

Theorem 10.10. *If $x_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$ and $x_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0}$, then for every h, we have*

$$(-1)^h \langle \Omega_{V,h}(x_1), [-1]\cdot\Omega_{V^*(1),1-h}(x_2)\rangle_V = -[x_1, \iota(x_2)]_V.$$

Proof. By the theory of p-adic interpolation, it is enough to prove that if $x_i = (1-\varphi)y_i$ with $y_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ and $y_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=1}$, then for all $j \gg 0$;

$$\big(\partial^{-j}(-1)^h \big\langle \Omega_{V,h}(x_1), [-1]\cdot\Omega_{V^*(1),1-h}(x_2)\big\rangle_V\big)(0) = -(\partial^{-j}[x_1,\iota(x_2)]_V)(0).$$

The above formula is equivalent to:

$$(1) \qquad (-1)^{h+j}\langle h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_{-j}), h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^j x_2 \otimes t^j e_{-j})\rangle_{F,V(j)}$$
$$= [\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j), \partial_{V^*(1-j)}(\partial^j x_2 \otimes t^j e_{-j}))]_{V(j)}.$$

By combining theorems 10.4 and 10.7 with remark 10.9, we see that for $j \gg 0$:

$$h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j) = (-1)^{h+j-1}\exp_{F,V(j)}((h+j-1)!(1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j)),$$

and that

$$h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^j x_2 \otimes t^j e_{-j})$$
$$= (\exp^*_{F,V^*(1-j)})^{-1}(h+j-1)!^{-1}((1-p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^j y_2 \otimes t^j e_{-j})).$$

Using the fact that by definition, if $x \in \mathbf{D}_{\mathrm{cris}}(V(j))$ and $y \in H^1(F, V(j))$ then

$$[x, \exp^*_{F,V^*(1-j)} y]_{V(j)} = \langle \exp_{F,V(j)} x, y \rangle_{F,V(j)},$$

we see that

$$(2) \qquad \langle h^1_{F,V(j)} \Omega_{V(j),h+j}(\partial^{-j} x_1 \otimes t^{-j} e_j), h^1_{F,V^*(1-j)} \Omega_{V^*(1-j),1-h-j}(\partial^j x_2 \otimes t^j e_{-j}) \rangle_{F,V(j)}$$
$$= (-1)^{h+j-1}[(1 - p^{-1}\varphi^{-1})\partial_{V(j)}(\varphi^{-j} y_1 \otimes t^{-j} e_j), (1 - p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^j y_2 \otimes t^j e_{-j})]_{V(j)}.$$

It is easy to see that under $[,]$, the adjoint of $(1 - p^{-1}\varphi^{-1})$ is $1 - \varphi$ and that if $x_i = (1 - \varphi) y_i$, then

$$\partial_{V(j)}(\partial^{-j} x_1 \otimes t^{-j} e_j) = (1 - \varphi)\partial_{V(j)}(\partial^{-j} y_1 \otimes t^{-j} e_j),$$
$$\partial_{V^*(1-j)}(\partial^j x_2 \otimes t^j e_{-j}) = (1 - \varphi)\partial_{V^*(1-j)}(\partial^j y_2 \otimes t^j e_{-j}),$$

So that (2) implies (1), and this proves the theorem. □

11. Perrin-Riou's big regulator map

Let F be an finite unramified extension over $\mathbf{Q}_p$ and V a continuous p-adic representation of G_F, which is crystalline with Hodge-Tate weights ≥ 0 and with no quotient isomorphic to the trivial representation. In [Per95], Perrin-Riou construct a big logarithm map

$$\mathcal{L}^{\Gamma_F}_{F,V} : H^1_{\mathrm{Iw}}(F, V) \longrightarrow \mathcal{H}(\Gamma_F) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V)$$

which interpolates the values of Bloch-Kato's dual exponential and logarithm maps for $V(j)$, $j \in \mathbf{Z}$, over each F_n.

In this section, we follow [LZ11, Appendix B] to adapt Berger's explicit formulas to construct Perrin-Riou's big logarithm and use it to calculate Kubota-Leopoldt p-adic L-function.

11.1. **Perrin-Riou's big logarithm map.** Let V be a positive crystalline representation of $\mathrm{Gal}(F_\infty/F)$ and $x \in \mathcal{H}(\Gamma_F) \otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F, V)$. We write x_j for the image of x in $H^1_{\mathrm{Iw}}(F, V(-j))$, and $x_{j,n}$ for the image of x_j in $H^1(F_n, V(-j))$. If we identify x with its image in $D(V)^{\psi=1}$, then x_j corresponds to the element $x \otimes e_{-j} \in D(V)^{\psi=1} \otimes e_{-j} = D(V(-j))^{\psi=1}$.

Since V is positive, we may interpret x as an element of the module $(\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$.

We shall assume:

$$(3) \qquad x \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\psi=1} \subset (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}.$$

The condition is satisfied if V has no quotient isomorphoic to $\mathbf{Q}_p$ (c.f. theorem 9.4).

Recall in section 6.2, we define ∂ denote the differential operator $(1 + \pi)\frac{d}{d\pi}$ (or $\frac{d}{dt}$) on $\mathbf{B}^+_{\mathrm{rig},F}$ and we have a map

$$\partial_V \circ \varphi^{-n} : \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \to F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$$

which sends $\pi^k \otimes d$ to the constant coefficient of $(\zeta_n \exp(t/p^n) - 1)^k \otimes \varphi^{-n}(d) \in F_n((t)) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$.

For $m \in \mathbf{Z}$, define $\Gamma^*(m)$ the leading term of Taylor series expansion of $\Gamma(x)$ at $x = m$; thus

$$\Gamma^*(m) = \begin{cases} j! & \text{if } n \geq 0 \\ \frac{(-1)^{-j-1}}{(-j-1)!} & \text{if } n \leq -1 \end{cases}$$

Proposition 11.1. *Define*

$$R_{j,n}(x) = \frac{1}{j!} \times \begin{cases} p^{-n}\partial_{V(-j)}(\varphi^{-n}(\partial^j x \otimes t^j e_{-j})) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_{V(-j)}(\partial^j x \otimes t^j e_{-j}) & \text{if } n = 0 \end{cases}$$

Then we have

$$R_{j,n}(x) = \begin{cases} \exp^*_{F_n, V^*(1-j)}(x_{j,n}) & \text{if } j \geq 0 \\ \log_{F_n, V(-j)}(x_{j,n}) & \text{if } j \leq -1 \end{cases}$$

Proof. This result is essentially a minor variation on thoerem 10.4. The case $j \geq 0$ is immediate from theorem 10.1 applied with V replaced by $V(-j)$ and x by $x \otimes e_{-j}$, using the formula

$$\partial_{V(-j)}(\varphi^{-n}(x \otimes e_{-j})) = \frac{1}{j!}\partial_{V(-j)}(\varphi^{-n}(\partial^j x \otimes t^j e_{-j})).$$

For the formula for $j \leq -1$, we choose h such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$. The element $\partial^j x \otimes t^j e_{-j}$ lies in $(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V(-j)))^{\psi=1}$. Applying theorem 10.1 with V,h and x replaced by $V(-j)$, $h-j$, and $\partial^j x \otimes t^{-j} e_j$, we see that

$$\Gamma^*(j+1)R_{j,n}(x) = \Gamma^*(j-h+1)\log_{F_n,V(-j)}[(\nabla_0 \circ \cdots \circ \nabla_{h-1}x)_{j,n}].$$

For $x \in \mathcal{H}(\Gamma_F) \otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F,V)$, we have

$$(\nabla_r x)_{j,n} = (j-r)x_{j,n},$$

so se have

$$(\nabla_0 \circ \cdots \circ \nabla_{h-1}x)_{j,n} = (j)(j-1)\cdots(j-h+1)x_{j,n}$$

as require. □

For ω a finite order character on Γ_F of conductor n, we denote

$$G(\omega) = \sum_{\sigma \in \Gamma_F/\Gamma_{F_n}} \omega(\sigma)\zeta_{p^n}^{\sigma}.$$

the Gauss sum of ω.

Proposition 11.2. *If x is as above, and $\mathcal{L}_V^{\Gamma_F}(x)$ is the unique element of $\mathcal{H}(\Gamma_F) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ such that $\mathcal{L}_V^{\Gamma_F}(x) \cdot (1+\pi) = (1-\varphi)x$, then for any $j \in \mathbf{Z}$ we have*

$$(1-\varphi)\partial_{V(-j)}(\varphi^{-n}(\partial^j x \otimes t^j e_{-j})) = \mathcal{L}_V^{\Gamma_F}(x)(\chi^j) \otimes t^j e_{-j},$$

while for any finite order character ω of Γ_F of conductor $n \geq 1$, we have

$$\left(\sum_{\sigma \in \Gamma_F/\Gamma_F^n} \omega(\sigma)^{-1}\sigma\right) \cdot \partial_{V(-j)}(\varphi^{-n}(\partial^j x \otimes t^j e_{-j}) = G(\omega)\varphi^{-n}(\mathcal{L}_V^{\Gamma_F}(x)(\chi^j\omega) \otimes t^j e_{-j}).$$

Proof. We note that

$$\mathcal{L}^{\Gamma_F}_{V(-j)}(\partial^j x \otimes t^j e_{-j}) = \mathrm{Tw}_j(\mathcal{L}_V^{\Gamma_F}(x)) \otimes t^j e_{-j},$$

so it suffices to prove the result for $j=0$. Suppose we have $x = \sum_{k \geq 0} v_k \pi^k$ where $v_k \in \mathbf{D}_{\mathrm{cris}}(V)$. Then

$$\partial_V(\varphi^{-n}(x)) = \sum_{k \geq 0} \varphi^{-n}(v_k)(\zeta_{p^n}-1)^k.$$

On the other hand,

$$\partial_V(\varphi^{-n}((1-\varphi)x)) = \sum_{k\geq 0}\varphi^{-n}(v_k)(\zeta_{p^n}-1)^k - \sum_{k\geq 0}\varphi^{1-n}(v_k)(\zeta_{p^{n-1}}-1)^k.$$

Applying the operator $e_\omega = \sum_{\sigma\in\Gamma_F/\Gamma_F^n}\omega(\sigma)\sigma$, we have for $n\geq 1$

$$e_\omega\cdot\partial_V(\varphi^{-n}(x)) = e_\omega\cdot\partial_V(\varphi^{-n}((1-\varphi)x)),$$

since e_ω is zero on $F_{n-1}((t))$.

However, since the map $\partial_V\circ\varphi^{-n}$ is a homomorphism of Γ_F-modules, we have

$$\begin{aligned} e_\omega\cdot\partial_V(\varphi^{-n}((1-\varphi)x)) =& e_\omega\cdot\partial_V(\varphi^{-n}(\mathcal{L}_V^{\Gamma_F}(x)\cdot(1+\pi)))\\ =&\varphi^{-n}(\mathcal{L}_V^{\Gamma_F}(x))\cdot e_\omega\partial_F(\varphi^{-n}(1+\pi))\\ =&G(\omega)\varphi^{-n}(\mathcal{L}_V^{\Gamma_F}(x)(\omega)).\end{aligned}$$

This completes the proof of the proposition for $j=0$. □

Definition 11.3. Let $x\in H^1_{\mathrm{Iw}}(F,V)$ If η is any continuous character of Γ_F, denote by x_η the image of x in $H^1_{\mathrm{Iw}}(F,V(\eta^{-1}))$. If $n\geq 0$, denote by $x_{\eta,n}$ the image of x_η in $H^1(F_n,V(\eta^{-1}))$.

Thus $x_{\chi^j,n}=x_{j,n}$ in the previous notation. The next lemma is valid for arbitrary de Rham representations of G_F (with no restriction on Hodge-Tate weights):

Lemma 11.4. *For any finite-order character ω factoring through Γ_F/Γ_F^n, with values in a finite extension E/F, we have*

$$\sum_{\sigma\in\Gamma_F/\Gamma_F^n}\omega(\sigma)^{-1}\exp^*_{F_n,V^*(1)}(x_{0,n})^\sigma = \exp^*_{F_n,V(\omega^{-1})^*(1)}(x_{\omega,0})$$

and

$$\sum_{\sigma\in\Gamma/\Gamma_n}\omega(\sigma)^{-1}\log_{F_n,V}(x_{0,n})^\sigma = \log_{F_n,V(\omega^{-1})}(x_{\omega,0})$$

where we identify $\mathbf{D}_{\mathrm{dR}}(V(\omega^{-1}))\cong(E\otimes_F F_n\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Gamma=\omega}$.

Proof. This follows from the compatibility of the maps $\exp^*$ and log with the corestriction maps (c.f. Theorem 10.1 and 10.3). □

Combining the three results above, we obtain:

Theorem 11.5. *Let $j\in\mathbf{Z}$ and let x satisfies (3). Let η be a continuous character of Γ_F of the form $\chi^j\omega$, where ω is a finite-order character of conductor n.*

i) If $j\geq 0$, we have

$$\mathcal{L}_V^{\Gamma_F}(x)(\eta) = j!\times\begin{cases}(1-p^j\varphi)(1-p^{-1-j}\varphi^{-1})^{-1}\left(\exp^*_{F,V(\eta^{-1})^*(1)}(x_{\eta,0})\otimes t^{-j}e_j\right) & \text{if } n=0\\ G(\omega)^{-1}p^{n(1+j)}\varphi^n\left(\exp^*_{F,V(\eta^{-1})^*(1)}(x_{\eta,0})\otimes t^{-j}e_j\right) & \text{if } n\geq 1.\end{cases}$$

ii) If $j\leq -1$, we have

$$\mathcal{L}_V^{\Gamma}(x)(\eta) = \frac{(-1)^{-j-1}}{(-j-1)!}\times\begin{cases}(1-p^j\varphi)(1-p^{-1-j}\varphi^{-1})^{-1}\left(\log_{F,V(\eta^{-1})}(x_{\eta,0})\otimes t^{-j}e_j\right) & \text{if } n=0\\ G(\omega)^{-1}p^{n(1+j)}\varphi^n\left(\log_{F,V(\eta^{-1})}(x_{\eta,0})\otimes t^{-j}e_j\right) & \text{if } n\geq 1.\end{cases}$$

In both case, we assume that $(1-p^{-1-j}\varphi^{-1})$ is invertible on $\mathbf{D}_{\mathrm{cris}}(V)$ when $\eta=\chi^j$.

11.2. **Cyclotomic units and Kubota-Leopoldt p-adic L-functions.** The relation between Coleman's power series and the Perrin-Riou's big logarithm map is given by the following diagram:

$$\begin{tikzcd} \varprojlim \mathscr{O}_{F_n}^* \arrow[rr, "\kappa"] \arrow[d, "\mathrm{Col}"'] & & H^1_{\mathrm{Iw}}(F,\mathbf{Z}_p(1)) \arrow[dd, "\mathscr{L}^{\Gamma}_{F,\mathbf{Q}_p(1)}"] \\ \mathscr{O}_F[[\pi]]^* \arrow[d, "(1-\frac{\varphi}{p})\log"'] & & \\ \mathscr{O}_F[[\pi]]^{\psi=0} \arrow[rr] & & \mathcal{H}(\Gamma)\otimes_{\mathbf{Q}_p}\mathbf{D}_{\mathrm{cris}}(F,\mathbf{Q}_p(1)) \end{tikzcd}$$

If we identify $\mathbf{D}_{\mathrm{cris}}(F,\mathbf{Q}_p(1))$ with F via the basis vector $t^{-1}\otimes e_1$, then the bottom map sends $f\in\mathscr{O}_F[[\pi]]^{\psi=0}$ to $\nabla_0\cdot\mathfrak{M}^{-1}(f)$, where $\nabla_0=\frac{\log\gamma}{\log\chi(\gamma)}$ for any non-identity element $\gamma\in\Gamma_1$ and $\mathfrak{M}$ is the Mellin transform defined in section 9.5. Thus the image of the bottom map is precisely $\nabla_0\cdot\Lambda_{\mathscr{O}_F}(\Gamma)\subset\mathcal{H}_F(\Gamma)$; and if we define

$$h_F(u)=\nabla_0^{-1}\cdot\mathscr{L}^{\Gamma}_{F,\mathbf{Q}_p(1)}(\kappa(u))\in\Lambda_{\mathscr{O}_F}(\Gamma),$$

then we have

$$\mathfrak{M}(h_F(u))=(1-\frac{\varphi}{p})\log\mathrm{Col}_u(u).$$

By calculation in section 8.6, we can use theorem 11.5 to calculate the Kubota-Leopoldt p-adic L-functions.

Example 11.6. (Kubota-Leopoldt p-adic zeta-function) Let $K=\mathbf{Q}_p$, $V=\mathbf{Q}_p(1)$ and

$$u=(\frac{\zeta_{p^n}-1}{\zeta_{p^n}})_{n\geq1}\in\varprojlim\mathscr{O}^*_{\mathbf{Q}_p(\mu_{p^n})}.$$

Then by calculation in section 8.6, we have

$$\begin{aligned} h_F(u)(\chi^k) =&\chi^k(\nabla_0^{-1})\cdot k!\cdot\frac{(1-p^k\varphi)}{(1-p^{-1-k}\varphi^{-1})})\Big(\exp^*_{\mathbf{Q}_p,V^*(1-j)}(u_{k,0})\otimes t^{-k}e_k\Big)\\ =&\frac{1}{k}k!\cdot\frac{(1-p^k\varphi)}{(1-p^{-1-k}\varphi^{-1})}\Big((1-p^{-k})\zeta(1-k)\frac{t^{k-1}}{(k-1)!}\otimes t^{-k}e_k\Big)\\ =&(1-p^{k-1})\zeta(1-k)t^{-1} \end{aligned}$$

and for ω a finite order character of Γ of conductor n, we have

$$\begin{aligned} h_F(u)(\chi^k\omega) =&\chi^k(\nabla_0^{-1})\cdot k!\cdot G(\omega)^{-1}p^{n(1+k)}\varphi^n\Big(\exp^*_{\mathbf{Q}_p,V(\eta^{-1})^*(1)}(u_{\eta,0})\otimes t^{-k}e_k\Big)\\ =&\frac{1}{k}k!\cdot G(\omega)^{-1}p^{n(1+k)}\varphi^n\Big(p^{-(n+1)k}G(\omega)\mathrm{L}(1-k,\omega)\frac{t^{k-1}}{(k-1)!}\otimes t^{-k}e_k\Big)\\ =&\mathrm{L}(1-k,\omega)t^{-1} \end{aligned}$$

Example 11.7. (Kubota-Leopoldt p-adic L-function) Let $K=\mathbf{Q}_p(\zeta_d)$, $V=\mathbf{Q}_p(1)$ and ε is a Dirichlet character of conductor $d\geq1$ prime to p. Set $u=\big(\frac{-1}{G(\varepsilon^{-1})}\sum_{0\leq a\leq d-1}\varepsilon(a)^{-1}\frac{\zeta_d^a\zeta_{p^n}}{\zeta_d^a\zeta_{p^n}-1}\big)_{n\geq1}$.

Then by calculation in section 8.6, we have

$$\begin{aligned}
h_F(u)(\chi^k) =&\chi^k(\nabla_0^{-1}) \cdot k! \cdot \frac{(1-p^k\varphi)}{(1-p^{-1-k}\varphi^{-1})^{-1}}\left(\exp^*_{K,V^*(1-j)}(u_{k,0}) \otimes t^{-k}e_k\right) \\
=&\frac{1}{k}k! \cdot \frac{(1-p^k\varphi)}{(1-p^{-1-k}\varphi^{-1})^{-1}}\left((1-\varepsilon(p)p^{-k})\mathrm{L}(1-k,\varepsilon)\frac{t^{k-1}}{(k-1)!} \otimes t^{-k}e_k\right) \\
=&(1-\varepsilon(p)p^{k-1})\mathrm{L}(1-k,\varepsilon)t^{-1}
\end{aligned}$$

and for ω a finite order character of Γ of conductor n, we have

$$\begin{aligned}
h_F(u)(\chi^k\omega) =&\chi^k(\nabla_0^{-1}) \cdot k! \cdot G(\omega)^{-1}p^{n(1+k)}\varphi^n\left(\exp^*_{\mathbf{Q}_p,V(\eta^{-1})^*(1)}(u_{\eta,0}) \otimes t^{-k}e_k\right) \\
=&\frac{1}{k}k! \cdot G(\omega)^{-1}p^{n(1+k)}\varphi^n\left(p^{-(n+1)k}G(\omega)\mathrm{L}(1-k,\omega\varepsilon)\frac{t^{k-1}}{(k-1)!} \otimes t^{-k}e_k\right) \\
=&\mathrm{L}(1-k,\omega\varepsilon)t^{-1}
\end{aligned}$$

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