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Harmonic Analysis on Exponential Solvable Lie Groups

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Preface

In this book we treat some topics in the theory of unitary representations of solvable Lie groups and we present through various examples the present situation, some open problems and possible directions of research. In the early 1970s Auslander and Kostant [3] succeeded in constructing the unitary dual of connected and simply connected type I solvable Lie groups by holomorphically induced representations and this result was extended by Pukanszky [65] to non-type I solvable Lie groups. These works are landmarks in the representation theory for solvable Lie groups. Nevertheless, it is still difficult even now to study in detail these representations and their applications.

For example, concerning the study of induced or restricted representations, we wish to decompose them into irreducibles, construct intertwining operators and understand the related algebras of invariant differential operators. However, we know little about these classical problems even in the case of exponential solvable Lie groups. It is only for nilpotent Lie groups that we have more tools in our hands. The representation theory of semisimple Lie groups differs very much from that of solvable Lie groups. Rich algebraic structures of semisimple Lie groups offer many research tools; the less rigid structure of solvable Lie groups allows only the induction procedure as the unique efficient method to obtain results. Fortunately, for solvable Lie groups, we have the orbit method to understand their unitary duals. The fundamental idea of Kirillov [48] to associate a coadjoint orbit to an irreducible unitary representation allows us to describe problems in representation theory in terms of data coming from the coadjoint orbits in the dual space of the Lie algebra of the group.

The authors have studied the orbit method since the 1970s. During this time it seems that some progress has been made for nilpotent Lie groups. On the other hand, almost all the problems one can ask for solvable non-nilpotent Lie groups remain unanswered or unexplored, for instance, questions surrounding holomorphically induced representation. It follows that the materials of this book are mainly the analysis for exponential solvable Lie groups, especially for nilpotent Lie groups. However, if we want to go beyond exponential solvable Lie groups, we necessarily must be confronted with the holomorphically induced representation. Therefore, the book introduces also the Auslander–Kostant theory, because the authors believe that this theory will be an important research subject for the future.

This book is an enlargement of the Japanese book [39] written by the first author.

The first chapter presents preliminaries to treat unitary representations of solvable Lie groups. Following Professor Sugiura's book [74], we describe the general theory of Lie groups and Lie algebras which will be needed from the second chapter downward.

In the second chapter, we give some generalities on harmonic analysis for locally compact topological groups.

In the third chapter, we study the Mackey theory for induced representations which is the background of the orbit method. A good reference is his lecture note [56].

In fourth chapter we study in detail four typical solvable Lie groups. These observations will offer to us a certain perspective on the orbit method.

The fifth chapter is somewhat central to this book and we explain in detail the orbit method for exponential solvable Lie groups. As a guideline we use Mackey theory for a minimal non-central normal subgroup.

We shall see in the sixth chapter some more details on Kirillov theory for nilpotent Lie groups. There we recognize that we can manipulate more tools in this case.

In the seventh chapter we study in detail holomorphically induced representations for exponential solvable Lie groups.

From the eighth chapter forward we shall explain some topics on which we have been working. Though the ordering of these topics is not necessarily chronological along the research life of the authors, all topics are closely related to each other. In the eighth chapter we first describe in terms of orbits the canonical central decomposition of monomial representations and restrictions onto subgroups of irreducible unitary representations in the case of exponential solvable Lie groups.

In the ninth chapter we examine e-central elements, which are very useful to study invariant differential operators and which were introduced for nilpotent Lie groups in Corwin and Geenleaf [17].

In the tenth chapter we consider the Frobenius reciprocity in the distribution version.

In the 11th chapter we explicitly describe by the orbit method the abstract Plancherel formula for monomial representations. Investigating many examples, we try to understand its mechanism.

In the 12th chapter we prove for monomial representations the so-called commutativity conjecture due to Duflo, and Corwin and Geenleaf. This conjecture will be translated in the final chapter into a claim concerning the restriction of representations.

All topics treated in the last five chapters still have to be studied in the exponential case. We believe that they all present interesting and important but difficult problems, which should be addressed in the near future.

The materials of this book are specialized, but they are accessible to researchers and students of graduate schools or universities who intend to be researchers in the future. The reader should have a basic background of the theory of Lie groups and Lie algebras, and some elementary knowledge on representation theory. It often happens that proofs by induction become long and tedious since often many cases have to be considered. So occasionally we sketch only a guideline of the proof. Moreover, we are obliged sometimes to omit proofs which need too much preparation, but which can be found in the classical literature.

Finally, among many texts concerning the theory of Lie group, Lie algebra and representation, we list some references relating to the materials of this book. On the theory of representation for general topological groups: [20, 49, 56], on representations of nilpotent Lie groups: [16, 64], on representations of solvable Lie groups: [4, 10, 51] (the principal part of [51] is devoted to a proof of Theorem 5.3.31 in the fifth chapter and addressed to experts). Further, on the enveloping algebra of Lie algebra: [21] and on the orbit method: [50].

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Contents

Chapter 1
Preliminaries: Lie Groups and Lie Algebras

1.1 Lie Groups and Lie Algebras

There are many texts on Lie groups and Lie algebras. Here following Sugiura [74] we gather general preliminaries. As other references we list [45, 62, 70, 71].

Definition 1.1.1. When the set G satisfies the following three conditions, G is called a **(real) Lie group**:

(1) G is a group.
(2) G is a real analytic manifold.
(3) The group operation $G \times G \ni (x, y) \mapsto xy^{-1} \in G$ is an analytic mapping.

Definition 1.1.2. Let $\mathfrak{g}$ be a vector space over the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. When a bilinear mapping $\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto [X, Y] \in \mathfrak{g}$ is defined and satisfies the following two conditions, $\mathfrak{g}$ is called a **(real) Lie algebra** or a **complex Lie algebra**:

(1) $[X, X] = 0$.
(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (**Jacobi identity**).

Let M be a real analytic manifold and $p \in M$. We consider a real-valued analytic function in a neighbourhood of p and denote its totality by $C^\omega(p)$. If a linear mapping $X_p : C^\omega(p) \to \mathbb{R}$ satisfies

$$X_p(\varphi\psi) = X_p(\varphi)\psi(p) + \varphi(p)X_p(\psi),$$

it is called a **tangent vector** at p. We designate its totality by M_p or $T_p(M)$ and call it the **tangent space** of M at p.

Definition 1.1.3. Let M, N be analytic manifolds, and suppose that the mapping $\lambda : M \to N$ is analytic at $p \in M$. Then, since $f \circ \lambda \in C^\omega(p)$ for $f \in C^\omega(\lambda(p))$, $Y_{\lambda(p)} \in T_{\lambda(p)}(N)$ is defined for $X_p \in T_p(M)$ by the formula

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$$Y_{\lambda(p)}(f) = X_p(f \circ \lambda),\ \forall\, f \in C^{\omega}(\lambda(p)).$$

The mapping $X_p \mapsto Y_{\lambda(p)}$ is linear from $T_p(M)$ to $T_{\lambda(p)}(N)$, called the differential of λ and denoted by $(d\lambda)_p$.

Definition 1.1.4. Let G be a Lie group. L_a denotes the **left translation** and R_a the **right translation** by $a \in G$. Namely,

$$L_a(g) = ag,\ R_a(g) = ga\ (g \in G).$$

For $a,\ b \in G$, we clearly have $L_a \circ R_b = R_b \circ L_a$. A **vector field** $X : G \ni g \mapsto X_g \in T_g(G)$ on G is said to be **left-invariant** if

$$\left(dL_g\right)_h (X_h) = X_{gh}$$

holds for any $g,\ h \in G$.

Now let $X : g \mapsto X_g$ and $Y : g \mapsto Y_g$ be two left-invariant vector fields on a Lie group G. Then, using the fact that the group operation on G is analytic mapping, we see that the vector fields $X,\ Y$ are analytic. It means that, if $f \in C^{\omega}(g)$ is analytic in an open neighbourhood U of g, we get analytic functions on U by $(Xf)(p) = X_p(f),\ (Yf)(p) = Y_p(f)$. So, putting

$$[X, Y]_g(f) = X_g(Yf) - Y_g(Xf),$$

we have $[X, Y]_g \in T_g(G)$. In fact, for $\varphi,\ \psi \in C^{\omega}(g)$ we simply calculate

$$\begin{aligned}
&[X, Y]_g(\varphi\psi) = X_g\left(Y(\varphi\psi)\right) - Y_g\left(X(\varphi\psi)\right) \\
&= X_g\left((Y\varphi)\psi + \varphi(Y\psi)\right) - Y_g\left((X\varphi)\psi + \varphi(X\psi)\right) \\
&= \big(X_g(Y\varphi) - Y_g(X\varphi)\big)\,\psi(g) + \varphi(g)\,\big(X_g(Y\psi) - Y_g(X\psi)\big) \\
&= [X, Y]_g(\varphi)\psi(g) + \varphi(g)[X, Y]_g(\psi).
\end{aligned}$$

Moreover, the vector field $g \mapsto [X, Y]_g$ on G is left-invariant. Indeed, for $a \in G$ and $f \in C^{\omega}(ag)$,

$$\begin{aligned}
&(dL_a)\big([X, Y]_g\big)(f) = \big([X, Y]_g\big)(f \circ L_a) \\
&= X_g\left(Y\left(f \circ L_a\right)\right) - Y_g\left(X\left(f \circ L_a\right)\right) \\
&= X_g\left((Yf) \circ L_a\right) - Y_g\left((Xf) \circ L_a\right) \\
&= (dL_a)(X_g)(Yf) - (dL_a)(Y_g)(Xf) \\
&= X_{ag}(Yf) - Y_{ag}(Xf) = [X, Y]_{ag}(f).
\end{aligned}$$

It is easily checked that the vector space $\mathfrak{g}$ of all the left-invariant vector fields on G becomes a Lie algebra with respect to the **bracket product** introduced in this way. Let e be the unit element of G. Through the linear mapping $X \mapsto X_e$, $\mathfrak{g}$ is isomorphic as vector space to $T_e(G)$ and dim $\mathfrak{g}$ = dim G. Of course we could consider the **right-invariant** vector fields instead of the left-invariant ones, but nothing new appears since the left and the right transform to each other by the diffeomorphism $g \mapsto g^{-1}$ of G.

Definition 1.1.5. $\mathfrak{g}$ is called the Lie algebra of the Lie group G.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. It is the exponential mapping that connects $\mathfrak{g}$ and G. We consider an C^r map $c(t)$ from an open interval I in $\mathbb{R}$ to G, namely an C^r curve in G. Let $r \geq 1$. The element $(dc)_t\left(\left(\frac{d}{dt}\right)_t\right)$ of $T_{c(t)}(G)$ is called the **tangent vector** of the curve c at the point $c(t)$ and designated by $\dot{c}(t)$. Furthermore, for a vector field X on an open set U in G, a C^r curve c $(r \geq 1)$ in U satisfying $\dot{c}(t) = X_{c(t)}$ $(\forall\, t \in I)$ is said to be an **integral curve** of X.

Definition 1.1.6. An analytic homomorphism a from the additive group $\mathbb{R}$ to Lie group G is called a **one-parameter subgroup** of G. In this case $a(s+t) = a(s)a(t)$ holds for arbitrary $s,\ t \in \mathbb{R}$. If this relation holds only locally in a neighbourhood of $0 \in \mathbb{R}$, that is to say, there exists a certain $\delta > 0$ such that the above relation holds when $s,\ t,\ s+t \in I(\delta)$, then the analytic mapping a from $I(\delta) = \{t \in \mathbb{R}; |t| < \delta\}$ to G is called a **local one-parameter subgroup** of G.

Proposition 1.1.7. *Let $a(t)$ be a local one-parameter subgroup of Lie group G. Clearly $a(0) = e$, $a(-t) = a(t)^{-1}$. There uniquely exists $X \in \mathfrak{g}$ such that $\dot{a}(0) = X_e$, and $a(t)$ is an integral curve, passing through e, of the left-invariant vector field X.*

Proof. For $f \in C^{\omega}(a(t))$, we see by the left invariance of X that

$$\begin{aligned}\dot{a}(t)(f) &= \left.\frac{d}{ds} f\left(a(t+s)\right)\right|_{s=0} = \left.\frac{d}{ds} f\left(a(t)a(s)\right)\right|_{s=0} \\ &= \left.\frac{d}{ds}\left(f \circ L_{a(t)}\right)(a(s))\right|_{s=0} = X_e\left(f \circ L_{a(t)}\right) = X_{a(t)}(f).\end{aligned}$$

Namely, $a(t)$ is an integral curve of X. ∎

Since an integral curve is given as a solution of a system of ordinary differential equations on a local coordinate system, using the theorem on the existence and the uniqueness of the solution, we get the following.

Proposition 1.1.8. *For any point g of Lie group G and any element X of the Lie algebra $\mathfrak{g}$ of G, there exists a certain $\delta > 0$ so that we have a unique integral curve $c : I(\delta) \to G$ of X verifying $c(0) = g$. If we write this c as $c(t; g)$, $c(t; g) = gc(t; e)$ holds for arbitrary $t \in I(\delta)$.*

Since two one-parameter subgroups coincide globally if they do locally, we have:

Lemma 1.1.9. *Two one-parameter subgroups $a(t),\ b(t)$ of Lie group G coincide if $\dot{a}(0) = \dot{b}(0)$.*

An integral curve of a left-invariant vector field turns out to be a local one-parameter subgroup, and extending it to a one-parameter subgroup we establish the following.

Theorem 1.1.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any element X of $\mathfrak{g}$, there exists a unique one-parameter subgroup $a(t)$ of G satisfying $\dot{a}(0) = X_e$.*

Definition 1.1.11. $a(t)$ in the theorem being written as $a_X(t)$, we define the **exponential map** $\exp : \mathfrak{g} \to G$ by $\exp X = a_X(1)$.

Then we easily see the following.

Proposition 1.1.12. *Let X be any element of the Lie algebra $\mathfrak{g}$ of Lie group G and let $s,\ t$ be real numbers.*

(1) $a_X(st) = a_{sX}(t)$;
(2) $a_X(t) = exp(tX)$;
(3) $exp((s+t)X) = exp(sX)exp(tX)$;
(4) $\dot{a}_X(0) = X_e$;
(5) $\frac{d}{dt} f(gexp(tX))\big|_{t=0} = X_g f,\ g \in G,\ f \in C^\omega(g)$;
(6) *for any one-parameter subgroup $c(t)$ of G, there exists $X \in \mathfrak{g}$ such that*

$$c(t) = exp(tX)\ (\forall\, t \in \mathbb{R}).$$

A **homomorphism** between Lie groups means an analytic mapping which is a group homomorphism. An **isomorphism** between Lie groups is defined likewise. The symbol $G \cong G'$ means that the Lie groups G and G' are **isomorphic**. We denote by $\mathrm{Aut}(G)$ the group formed by all the **automorphisms** of Lie group G, and call it the **automorphism group** of G. V being a real or complex vector space of finite dimension, the totality $GL(V)$ of regular endomorphisms of V is a Lie group. A homomorphism from a Lie group G to $GL(V)$ is called a **(finite-dimensional) representation** of G in V.

Let's make a similar consideration for Lie algebras. A **homomorphism** of Lie algebras means a linear mapping preserving bracket products. To be **isomorphic** is designated by the same notation $\cong$. We denote by $\mathrm{Aut}(\mathfrak{g})$ the **automorphism group** of the Lie algebra $\mathfrak{g}$. A vector subspace closed under the bracket product is called a **Lie subalgebra**. V being a real or complex vector space of finite dimension, the totality $\mathfrak{gl}(V)$ of endomorphisms of V becomes a Lie algebra by the bracket product $[X, Y] = XY - YX$. A homomorphism from the Lie algebra $\mathfrak{g}$ to $\mathfrak{gl}(V)$ is called a **(finite-dimensional) representation** of $\mathfrak{g}$ in V.

Proposition 1.1.13. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ respectively and $\varphi : G \to H$ a homomorphism. Then, for any $X \in \mathfrak{g}$, there exists a unique $Y = (d\varphi)(X) \in \mathfrak{h}$ verifying $Y_{\varphi(g)} = (d\varphi)_g(X_g)$ $(\forall\, g \in G)$. And $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism.*

Proof. As before, we denote by e the unit element of G and by e' that of H. For any $X \in \mathfrak{g}$, there uniquely exists $Y \in \mathfrak{h}$ such that $Y_{e'} = (d\varphi)_e(X_e)$. Then, we shall see that $Y_{\varphi(g)} = (d\varphi)_g(X_g)$ at any $g \in G$. In fact,

$$\begin{aligned} Y_{\varphi(g)} &= \left(dL_{\varphi(g)}\right)_{e'}(Y_{e'}) = \left(dL_{\varphi(g)}\right)_{e'}((d\varphi)_e(X_e)) \\ &= \left(d\left(L_{\varphi(g)}\circ\varphi\right)\right)_e(X_e) = \left(d\left(\varphi\circ L_g\right)\right)_e(X_e) \\ &= (d\varphi)_g\left(\left(dL_g\right)_e(X_e)\right) = (d\varphi)_g(X_g). \end{aligned}$$

Let's put $Y_j = (d\varphi)(X_j)$ $(j = 1, 2)$. For any $f \in C^{\omega}(\varphi(g))$,

$$\begin{aligned} ((d\varphi)_g[X_1, X_2])f &= [X_1, X_2]_g(f\circ\varphi) = (X_1)_g(X_2(f\circ\varphi)) - (X_2)_g(X_1(f\circ\varphi)) \\ &= (Y_1)_{\varphi(g)}(Y_2 f) - (Y_2)_{\varphi(g)}(Y_1 f) = [Y_1, Y_2]_{\varphi(g)} f. \end{aligned}$$

So $d\varphi$ is a Lie algebra homomorphism. ∎

Definition 1.1.14. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $a \in G$, and consider the **inner automorphism** $i_a : g \mapsto aga^{-1}$ of G by a. We denote by Ad a the differential $d(i_a)$ of i_a. Ad a is an element of Aut($\mathfrak{g}$) and Ad is a homomorphism from G to Aut($\mathfrak{g}$). The representation Ad of G on $\mathfrak{g}$ is called the **adjoint representation** of G.

We often make use of the following result.

Theorem 1.1.15. *Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras of Lie groups G, H respectively. Concerning a homomorphism φ from G to H, we have*

$$\varphi(exp\ X) = exp(d\varphi(X))\ (\forall\, X \in \mathfrak{g}).$$

Proof. We consider a one-parameter subgroup $b(t) = \varphi(\exp(tX))$ $(t \in \mathbb{R})$ of H. Let e' be the unit element of H. As we already saw, using $Y \in \mathfrak{h}$ such that $\dot{b}(0) = Y_{e'}$, we have $b(t) = \exp(tY)$, $t \in \mathbb{R}$. On the other hand, we put $a(t) = \exp(tX)$. Since $b = \varphi\circ a$,

$$\begin{aligned} Y_{e'} &= \dot{b}(0) = (db)_0\left(\left(\frac{d}{dt}\right)_0\right) \\ &= (d\varphi)_e\left((da)_0\left(\left(\frac{d}{dt}\right)_0\right)\right) = (d\varphi)_e(\dot{a}(0)) = (d\varphi)_e(X_e). \end{aligned}$$

Hence $Y = (d\varphi)(X)$. Therefore, $\varphi(\exp(tX)) = \exp(td\varphi(X))$ for any $t \in \mathbb{R}$. Putting $t = 1$, we obtain the desired result. ∎

Theorem 1.1.16. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *The exponential map exp is an analytic mapping from $\mathfrak{g}$ to G.*
(2) *The differential $(d\,exp)_0 : \mathfrak{g}_0 = T_0(\mathfrak{g}) \to T_e(G)$ at the point 0 is an isomorphism of vector spaces.*
(3) *The exponential map exp induces a diffeomorphism between an open neighbourhood V of 0 in $\mathfrak{g}$ and an open neighbourhood W of e in G.*

Proof. We show (1). Let $\dim G = n$. We take a local coordinate system (U, ψ), $\psi(e) = 0$, around the unit element e in G and denote by $x = (x_1, \ldots, x_n)$ the coordinate function in the open set $\psi(U)$ of $\mathbb{R}^n$. We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. If we write on U

$$X_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j} \ (1 \le i \le n),$$

a_{ij} are analytic functions on U. Using $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, any $X \in \mathfrak{g}$ is uniquely written as

$$X = X(u) = \sum_{k=1}^{n} u_k X_k = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} u_k a_{kj} \right) \frac{\partial}{\partial x_j}.$$

So, if we write

$$a_X(t) = a(t, u) = (a_1(t, u), \ \ldots, \ a_n(t, u)), \ c_j(x, u) = \sum_{k=1}^{n} u_k a_{kj}(x),$$

these functions $a_i(t, u)$ satisfy a system of ordinary differential equations

$$\frac{d}{dt} a_i(t, u) = c_i(a(t, u), u), \ 1 \le i \le n$$

and the initial condition

$$a_i(0, u) = 0.$$

Since $c_i(x, u)$ is an analytic function on $U \times \mathbb{R}^n$, the theorem on the existence and uniqueness of a solution for a system of ordinary differential equations and the theorem on the differentiability with respect to parameter u give us the following. For $u = (u_1, \ldots, u_n)$, put $\|u\| = \max_{1 \le i \le n} |u_i|$. Then, for a certain $\delta > 0$, there exists in $J(2\delta) = I(2\delta) \times \{u \in \mathbb{R}^n; \|u\| < 2\delta\}$ a unique solution $a(t, u)$ of this initial value problem and a is an analytic mapping from $J(2\delta)$ to U.

As $a(t, u) = a\left(\delta, \frac{tu}{\delta}\right)$, an analytic solution $a(t, u) \in U$ is finally defined for arbitrary $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ satisfying $\|tu\| < \delta^2$. In particular, if $\|u\| < \delta^2$,

$\exp(X(u)) = a(1, u)$ is analytic with respect to u and belongs to U. Thus if we put

$$U_0 = \{X(u) \in \mathfrak{g};\ \|u\| < \delta^2\},$$

U_0 is an open neighbourhood of 0 in $\mathfrak{g}$ and exp is an analytic mapping from U_0 to U. For any $X \in \mathfrak{g}$, $\frac{X}{m} \in U_0$ with a sufficiently large positive integer m. So, $\exp X = \left(\exp\left(\frac{X}{m}\right)\right)^m$ is analytic around X.

Since (3) is immediately obtained from (2) and the inverse function theorem, let's show (2). The tangent space $\mathfrak{g}_0$ at 0 of the vector space $\mathfrak{g}$ becomes $\mathfrak{g}_0 = \mathfrak{g}$, when we identify the differential operator in the direction of X at 0,

$$D(X)_0 f = \frac{d}{dt} f(tX)\bigg|_{t=0}, \quad f \in C^{\omega}(0),$$

with $X \in \mathfrak{g}$. Under this identification,

$$((d\exp)_0 X)\, f = D(X)_0(f \circ \exp) = \frac{d}{dt} f(\exp(tX))\bigg|_{t=0} = X_e f.$$

Hence $(d\exp)_0 X = 0 \Leftrightarrow X_e = 0 \Leftrightarrow X = 0$. ∎

Definition 1.1.17. We denote by $\log : W \to V$ the inverse mapping of the mapping $\exp : V \to W$ in (3) of the theorem. We take a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ and its dual basis $\{X_1^*, \ldots, X_n^*\}$. If we put $x_i = X_i^* \circ \log$, then $x = (x_1, \ldots, x_n)$ is a local coordinate of G defined on W. Moreover, if we consider the mapping

$$g = \exp(y_1 X_1) \cdots \exp(y_n X_n) \mapsto y = (y_1, \ldots, y_n) \in \mathbb{R}^n,$$

the value taken at the unit element e of G by the Jacobian of y with respect to x is equal to 1. Therefore, y is also a local coordinate around e. x, y are respectively called the **canonical coordinate of the first and the second kind** relative to the basis $\{X_i\}_{i=1}^n$.

Now let's see relations existing through the exponential map between the product in a Lie group and the bracket product in its Lie algebra. Let $f(t)$, $g(t)$ be numerical or vector-valued functions defined on a neighbourhood of 0 in $\mathbb{R}$ and let $|\cdot|$ denote the absolute value or the norm. If there exists such a constant $C > 0$ that $|f(t)| \leq C|g(t)|$ holds in some neighbourhood of 0, we write $f(t) = O(g(t))$ with Landau notation. Using the notations introduced until now:

Lemma 1.1.18. *Let $g \in G$, $f \in C^{\omega}(g)$, n be a positive integer, and $X, X_1, \ldots, X_n$ elements of* $\mathfrak{g}$.

(1) $(X^n f)(g) = \frac{d^n}{dt^n} f(g exp(tX))\big|_{t=0}$.

(2) $(X_1 \cdots X_n f)(g) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} f\left(g exp(t_1 X_1) \cdots exp(t_n X_n)\right)\big|_{t_1 = \cdots = t_n = 0}$.

(3) *In a neighbourhood of* $t = 0$,

$$f(g\mathit{exp}(tX)) = f(g) + t(Xf)(g) + \frac{t^2}{2}(X^2 f)(g) + O(t^3).$$

Proof. (1), (2) are clear. (3) is obtained by substituting (1) in the Maclaurin expansion

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2}t^2 + O(t^3)$$

of the function $F(t) = f(g\exp(tX))$. ■

Theorem 1.1.19. *Let* $X, Y \in \mathfrak{g}$ *and* $t \in \mathbb{R}$.

(1) $\mathit{exp}(tX)\mathit{exp}(tY) = \mathit{exp}\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$.
(2) $\mathit{exp}(tX)\mathit{exp}(tY)\mathit{exp}(-tX) = \mathit{exp}\left(tY + t^2[X,Y] + O(t^3)\right)$.
(3) $\mathit{exp}(tX)\mathit{exp}(tY)\mathit{exp}(-tX)\mathit{exp}(-tY) = \mathit{exp}\left(t^2[X,Y] + O(t^3)\right)$.

Proof. We show (1). By (2) of the preceding lemma,

$$(X^m Y^n f)(e) = \frac{\partial^{m+n}}{\partial t^m \partial s^n} f(\exp(tX)\exp(sY))\bigg|_{t=s=0}$$

for any $f \in C^\omega(e)$, $m, n \in \mathbb{N}$. Applying this formula to the Taylor expansion of the function

$$F(t,s) = f(\exp(tX)\exp(sY))$$

with two variables and letting $r = \sqrt{t^2+s^2}$, we get

$$\begin{aligned} f(\exp(tX)\exp(sY)) =& f(e) + ((tX+sY)f)(e) \\ &+ \frac{1}{2}((t^2X^2 + 2tsXY + s^2Y^2)f)(e) + O(r^3). \end{aligned}$$

In particular, letting $s = t$,

$$\begin{aligned} f(\exp(tX)\exp(tY)) =& f(e) + t((X+Y)f)(e) \\ &+ \frac{t^2}{2}((X^2 + 2XY + Y^2)f)(e) + O(t^3). \end{aligned}$$

From what we have seen until now, there exists a neighbourhood V of 0 in $\mathfrak{g}$ and for sufficiently small $|t|$ we can uniquely write as

$$\exp(tX)\exp(tY) = \exp(Z(t)),\ Z(t) \in V.$$

Since $Z(t) = \log(\exp(tX)\exp(tY))$ is analytic in a neighbourhood of 0 and since $Z(0) = 0$, we have the Maclaurin expansion

$$Z(t) = tA + t^2B + O(t^3),$$

here $A,\ B$ are elements of $\mathfrak{g}$ not depending on t.

Let's take a basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$ and introduce the canonical coordinate of the first kind $x = (x_1, \ldots, x_n)$ around the unit element e of G. By (3) of the preceding lemma,

$$\begin{aligned} x_i(\exp(Z(t))) &= x_i(e) + (Z(t)x_i)(e) + \frac{1}{2}\left(Z(t)^2x_i\right)(e) + O(t^3) \\ &= \left((tA + t^2B)x_i\right)(e) + \frac{1}{2}\left((tA + t^2B)^2x_i\right)(e) + O(t^3) \\ &= \left(\left(tA + t^2B + \frac{t^2}{2}A^2\right)x_i\right)(e) + O(t^3). \end{aligned}$$

Comparing this with the above equation, for any $1 \le i \le n$,

$$\begin{aligned} (Ax_i)(e) &= ((X + Y)x_i)(e) \\ \left((2B + A^2)x_i\right)(e) &= \left((X^2 + 2XY + Y^2)x_i\right)(e). \end{aligned}$$

As x is a local coordinate around e, we have $A_e = (X + Y)_e$ and $A = X + Y$ by the left invariance. Likewise,

$$2B + A^2 = X^2 + 2XY + Y^2$$

and

$$B = \frac{1}{2}\left((X^2 + 2XY + Y^2) - (X + Y)^2\right) = \frac{1}{2}(XY - YX) = \frac{1}{2}[X, Y].$$

We show (2). Just like (1), if we put

$$Z(t) = \log(\exp(tX)\exp(tY)\exp(-tX)) = tA + t^2B + O(t^3),$$

$$\begin{aligned} f(\exp(Z(t))) &= \sum_{m+n+p\le 2} \frac{t^{m+n+p}}{m!n!p!}\left(X^mY^n(-X)^pf\right)(e) + O(t^3) \\ &= f(e) + t((X + Y - X)f)(e) + \frac{t^2}{2}\left((X^2 + Y^2 + X^2 \right. \\ &\quad \left. + 2(XY - YX) - 2X^2)f\right)(e) + O(t^3) \\ &= f(e) + t(Yf)(e) + t^2\left(\left(\frac{1}{2}Y^2 + [X, Y]\right)f\right)(e) + O(t^3). \end{aligned}$$

Setting $f = x_i$ $(1 \leq i \leq n)$ as above, we get

$$A = Y, \ B + \frac{1}{2}A^2 = \frac{1}{2}Y^2 + [X, Y]$$

and $B = [X, Y]$.

We show (3). If we put

$$Z(t) = \log(\exp(tX)\exp(tY)\exp(-tX)\exp(-tY)) = tA + t^2B + O(t^3),$$

$$\begin{aligned} f(\exp(Z(t))) &= \sum_{m+n+p+q \leq 2} \frac{t^{m+n+p+q}}{m!n!p!q!} (X^m Y^n (-X)^p (-Y)^q f)(e) + O(t^3) \\ &= f(e) + t((X + Y - X - Y)f)(e) + \frac{t^2}{2}\big((X^2 + Y^2 + X^2 + Y^2 \\ &\quad + 2(XY - X^2 - XY - YX - Y^2 + XY)\big) f\big)(e) + O(t^3) \\ &= f(e) + t^2((XY - YX)f)(e) + O(t^3). \end{aligned}$$

Thus, $A = 0, \ B = [X, Y]$. ■

Definition 1.1.20. For each element X of a real or complex Lie algebra $\mathfrak{g}$ we define a linear transformation $\mathrm{ad}X$ on $\mathfrak{g}$ by $(\mathrm{ad}X)(Y) = [X, Y]$ $(Y \in \mathfrak{g})$. The Jacobi identity says that the mapping ad is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{g})$. We call this the **adjoint representation** of $\mathfrak{g}$ and its kernel the **centre** of $\mathfrak{g}$.

Theorem 1.1.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *For any $g \in G$, $X \in \mathfrak{g}$, $g(\exp X)g^{-1} = \exp((Adg)X)$.*
(2) *$Ad : G \to GL(\mathfrak{g})$ is an analytic mapping.*
(3) $d(Ad) = ad$ (cf. Proposition 1.1.13).
(4) *For any $X \in \mathfrak{g}$, $Ad(\exp X) = \exp(adX)$.*
(5) *If G is connected, the kernel of the adjoint representation of G coincides with the centre of G.*

Proof. The claim (1) comes from Definition 1.1.14 and Theorem 1.1.15.

We show (2). We take a basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$ and identify $GL(\mathfrak{g})$ with the set $GL(n, \mathbb{R})$ of all the real regular matrices of order n. Namely, if we write $(\mathrm{Ad}g)(X_j) = \sum_{i=1}^n a_{ij}(g)X_i$ for $g \in G$, the linear transformation $\mathrm{Ad}g$ is identified with the matrix $\big(a_{ij}(g)\big)$. Then by (1)

$$g(\exp(tX_j))g^{-1} = \exp(t(\mathrm{Ad}g)X_j) = \exp\left(t\sum_{i=1}^n a_{ij}(g)X_i\right)$$

and the left member of this equation is an analytic function of g. Relative to the basis $\{X_i\}_{i=1}^n$, let $(U, (x_1, \dots, x_n))$ be a canonical coordinate system of the first kind around the unit element e of G. g_0 being any element of G and taking $t > 0$ small enough, we have $g(\exp(tX_j))g^{-1} \in U$ as far as g remains in a certain open neighbourhood W of g_0. We get from the above equation that

$$x_i\left(g(\exp(tX_j))g^{-1}\right) = ta_{ij}(g).$$

Hence $a_{ij}(g)$ is analytic in the open neighbourhood W of g_0 and the mapping Ad is analytic on G.

We show (3). We put $\varphi = d(\mathrm{Ad})$ and let $X,\ Y \in \mathfrak{g},\ t \in \mathbb{R}$. With $g(t) = \exp(tX)$, Theorem 1.1.15 gives

$$\mathrm{Ad}(g(t)) = \exp(t\varphi(X)).$$

Together with (1) we get

$$\begin{aligned} i_{g(t)}(\exp(tY)) &= \exp(t(\mathrm{Ad}(g(t)))(Y)) = \exp(t\exp(t\varphi(X))Y) \\ &= \exp(tY + t^2\varphi(X)Y + O(t^3)). \end{aligned}$$

On the other hand, Theorem 1.1.19 (2) gives

$$i_{g(t)}(\exp(tY)) = \exp(tY + t^2[X, Y] + O(t^3)).$$

Comparing these two equations, we see for sufficiently small $|t|$ that

$$\varphi(X)Y = [X, Y] + O(t).$$

Setting $t \to 0$, $\varphi(X)Y = [X, Y] = (\mathrm{ad}X)(Y)$. Namely $\varphi = \mathrm{ad}$.

The claim (4) comes from (3) and Theorem 1.1.15.

We show (5). If $g \in G$ belongs to the kernel of the adjoint mapping, Definition 1.1.14 means that $d(i_g)$ becomes the identity mapping of $\mathfrak{g}$ and i_g is the identity mapping in some neighbourhood of e. As G is connected, i_g finally becomes the identity mapping on the whole of G. Thus, $i_g(a) = gag^{-1} = a$ for any $a \in G$. That is to say, $ga = ag$ and g belongs to the centre of G. The inverse inclusion is clear. ∎

We sketch the correspondence between Lie subgroups and Lie subalgebras. Please refer to Sugiura [74] for the details.

Definition 1.1.22. Let M be a subset of an analytic manifold N. We say that M is an **analytic submanifold** of N, if the following two statements hold.

(1) M is an analytic manifold.
(2) The inclusion map $i : M \to N$, $i(x) = x$ $(\forall\, x \in M)$, is an analytic map and its differential $(di)_p : M_p \to N_p$ is injective at each point $p \in M$.

Definition 1.1.23. Let G be a Lie group. When a subgroup H of G is an analytic submanifold of G and becomes a Lie group with these two structures, we say that H is a **Lie subgroup** of G. A connected Lie subgroup is called an **analytic subgroup**.

Theorem 1.1.24. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *The Lie algebra $\mathfrak{h}$ of any Lie subgroup H of G is isomorphic to the Lie subalgebra $(di)(\mathfrak{h})$ of $\mathfrak{g}$. Here $i : H \to G$ denotes the inclusion map. Let's identify $\mathfrak{h}$ with $(di)(\mathfrak{h})$.*
(2) *Conversely, for any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a connected Lie subgroup H of G the Lie subalgebra of which is $\mathfrak{h}$. H is nothing but the maximal connected integral manifold containing e of $\mathfrak{h}$.*
(3) *The correspondence $H \leftrightarrow \mathfrak{h}$ is a bijection between the totality of connected Lie subgroups of G and the totality of Lie subalgebras of $\mathfrak{g}$.*

Here we recall the definitions of solvable Lie algebras and nilpotent Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. When $[X, Y] = 0$ for any $X,\ Y \in \mathfrak{g}$, we say that $\mathfrak{g}$ is **commutative** or **abelian**. A linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an **ideal** if $[X, \mathfrak{a}] \subset \mathfrak{a}$ for any $X \in \mathfrak{g}$. Then, the quotient space $\mathfrak{g}/\mathfrak{a}$ becomes a Lie algebra by the operation induced from the bracket product of $\mathfrak{g}$. This is called the **quotient Lie algebra** of $\mathfrak{g}$ by $\mathfrak{a}$. Now, letting

$$D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \left\{ \sum_{i=1}^{m} [x_i, y_i];\ x_i,\ y_i \in \mathfrak{g},\ m \geq 1 \right\},$$

$D\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and the quotient Lie algebra $\mathfrak{g}/D\mathfrak{g}$ is commutative. If we set inductively

$$D^0\mathfrak{g} = \mathfrak{g},\ D^k\mathfrak{g} = D(D^{k-1}\mathfrak{g})\ \ (k = 1,\ 2,\ \cdots),$$

we get a sequence of ideals

$$\mathfrak{g} = D^0\mathfrak{g} \supset D^1\mathfrak{g} \supset D^2\mathfrak{g} \supset \cdots .$$

When there exists k such that $D^k\mathfrak{g} = \{0\}$, Lie algebra $\mathfrak{g}$ is said to be **solvable**. Changing slightly this procedure, if we set inductively

$$C^0\mathfrak{g} = \mathfrak{g},\ C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}]\ \ (k = 1,\ 2,\ \cdots),$$

we get a sequence of ideals

$$\mathfrak{g} = C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset C^2\mathfrak{g} \supset \cdots.$$

When there exists k such that $C^k\mathfrak{g} = \{0\}$, Lie algebra $\mathfrak{g}$ is said to be **nilpotent**. Besides, a Lie group is **solvable** when its Lie algebra is solvable and the same for the **nilpotent** case.

From the above, nilpotent Lie groups are solvable Lie groups, but the gap between these two categories is very big. Later we shall define the category of **exponential solvable Lie groups** which will be our object, and survey the **orbit method** for these groups. Here we collect fundamental properties of nilpotent and solvable Lie groups.

Proposition 1.1.25. *Let $\mathfrak{g} \neq \{0\}$ be a (real or complex) Lie algebra and $\mathfrak{z}$ its centre.*

(1) *If $\mathfrak{g}$ is nilpotent, any Lie subalgebra $\mathfrak{h}$ and the image of any homomorphism $\varphi(\mathfrak{g})$ are also nilpotent.*
(2) *If $\mathfrak{g}$ is nilpotent, $\mathfrak{z} \neq \{0\}$.*
(3) *If the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$ is nilpotent, $\mathfrak{g}$ is also nilpotent.*

Proof. For any $k \in \mathbb{N}$, $C^k\mathfrak{h} \subset C^k\mathfrak{g}$ and $C^k\varphi(\mathfrak{g}) = \varphi(C^k\mathfrak{g})$. So, we have (1). Besides, if $C^k\mathfrak{g}$ first becomes $\{0\}$, then $\{0\} \neq C^{k-1}\mathfrak{g}$ is contained in the centre $\mathfrak{z}$ and (2) follows. We finally show (3). If $C^k(\mathfrak{g}/\mathfrak{z}) = \{0\}$, $C^k\mathfrak{g} \subset \mathfrak{z}$ and hence $C^{k+1}\mathfrak{g} = \{0\}$. ■

Lemma 1.1.26. *Let V be a finite-dimensional vector space, and we consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(V)$. If $X \in \mathfrak{g}$ is a nilpotent endomorphism, then adX is a nilpotent linear transformation of $\mathfrak{g}$.*

Proof. The two linear transformations $L_X(Y) = XY$, $R_X(Y) = YX$ of $\mathfrak{g}$ are nilpotent and mutually commutative and $\mathrm{ad}X = L_X - R_X$. Thus, we can apply the binomial expansion formula to conclude that $\mathrm{ad}X$ is nilpotent. ■

Theorem 1.1.27 (Engel). *Let $V \neq \{0\}$ be a finite-dimensional vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$. If every element of $\mathfrak{g}$ is a nilpotent linear transformation, there exists $0 \neq v \in V$ such that $\mathfrak{g}{\cdot}v = \{0\}$.*

Proof. We proceed by induction on $n = \dim \mathfrak{g}$. The assertion is clear when $n = 1$. Let $n \geq 2$ and $\mathfrak{h}$ be a proper Lie subalgebra of $\mathfrak{g}$. The preceding proposition implies that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is a Lie algebra composed of nilpotent linear transformations of $\mathfrak{g}$, which keep $\mathfrak{h}$ stable. Therefore, $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}\mathfrak{h}$ is a Lie algebra composed of nilpotent linear transformations of $\mathfrak{g}/\mathfrak{h}$. Since $\dim \mathfrak{h} < \dim \mathfrak{g}$, the induction hypothesis implies the existence of $X \in (\mathfrak{g} \setminus \mathfrak{h})$ such that $[\mathfrak{h}, X] \subset \mathfrak{h}$. Hence, the **normalizer**

$$\mathfrak{n}(\mathfrak{h}) = \{A \in \mathfrak{g}; [A, \mathfrak{h}] \subset \mathfrak{h}\}$$

of $\mathfrak{h}$ in $\mathfrak{g}$ properly contains $\mathfrak{h}$. So, if we take $\mathfrak{h}$ as a maximal proper Lie subalgebra of $\mathfrak{g}$, then $\mathfrak{n}(\mathfrak{h}) = \mathfrak{g}$ and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Here, if $\dim(\mathfrak{g}/\mathfrak{h}) > 1$, taking the inverse image $\mathfrak{k}$ in $\mathfrak{g}$ of a one-dimensional Lie subalgebra of $\mathfrak{g}/\mathfrak{h}$, we have $\mathfrak{h} \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}$, which contradicts the maximality of $\mathfrak{h}$. Hence $\dim(\mathfrak{g}/\mathfrak{h}) = 1$. Let's utilize once again the above $X \in (\mathfrak{g} \setminus \mathfrak{h})$. If we put $W = \{w \in V; \mathfrak{h}\cdot w = \{0\}\}$, the induction hypothesis gives $W \neq \{0\}$. On the other hand, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and so W is even $\mathfrak{g}$-invariant. Now if we take an eigenvector $v \in W$ for the eigenvalue 0 of $X|_W$, v is the element of V we are looking for. ∎

Theorem 1.1.28. *For a Lie algebra* $\mathfrak{g}$, *the following are equivalent:*

(a) $\mathfrak{g}$ *is nilpotent.*
(b) *For any* $X \in \mathfrak{g}$, *ad*X *is a nilpotent linear transformation.*
(c) *There exists a decreasing sequence of ideals*

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{0\},$$

such that dim$(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$ *and such that* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *for* $i = 0, \cdots, n-1$.

Proof. (a) ⇒ (b): If we assume $C^n\mathfrak{g} = \{0\}$, then $(\mathrm{ad}X)^n = 0$.

(b) ⇒ (a): We proceed by induction on $n = \dim \mathfrak{g}$. The claim being clear when $n = 0$, let $n \geq 1$. The previous theorem says that there exists an element $X \neq 0$ of $\mathfrak{g}$ such that $[\mathfrak{g}, X] = \{0\}$. Thus the centre $\mathfrak{z}$ of $\mathfrak{g}$ is not $\{0\}$ and $\mathrm{ad}(\mathfrak{g}/\mathfrak{z})$ is formed only by nilpotent linear transformations. Hence the induction hypothesis implies that $\mathfrak{g}/\mathfrak{z}$ is nilpotent. Finally, $\mathfrak{g}$ is nilpotent by Proposition 1.1.25.

(a) ⇔ (c): This is trivial. ∎

Definition 1.1.29. We say that a sequence $\mathcal{M} = (\mathfrak{h}_i)_{i=0}^n$ of subalgebras of a Lie algebra $\mathfrak{g}$ is a **Malcev sequence** if

$$\mathfrak{g} = \mathfrak{h}_0 \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n = \{0\},$$

such that $\dim(\mathfrak{h}_i/\mathfrak{h}_{i+1}) = 1$ for $i = 0, \cdots, n-1$.

We say that the Malcev sequence $\mathcal{M}$ passes through a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, if $\mathfrak{k} = \mathfrak{h}_j$ for some $j \in \{1, \cdots, n\}$. A Malcev sequence $\mathcal{M}$ is called a **Jordan–Hölder sequence**, if $\mathfrak{h}_j$ is an ideal of $\mathfrak{g}$ for every $j \in \{1, \ldots, n\}$.

Let $\mathfrak{g}$ be a Lie algebra. A decreasing sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\}$$

of ideals of $\mathfrak{g}$ is called a **composition series**, if the $\mathfrak{g}$-module $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is **irreducible** for every $i \in \{0, \cdots, m-1\}$. A Malcev sequence $(\mathfrak{h}_j)_j$ of $\mathfrak{g}$ is called **strong**, if whenever $\mathfrak{h}_j$ is not an ideal of $\mathfrak{g}$, then $\mathfrak{h}_{j-1}$ and $\mathfrak{h}_{j+1}$ are ideals and the $\mathfrak{g}$-module $\mathfrak{h}_{j-1}/\mathfrak{h}_{j+1}$ is irreducible. Every composition sequence of $\mathfrak{g}$ can be refined to give a strong Malcev sequence.

Remark 1.1.30. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let G be a simply connected, connected Lie group with Lie algebra $\mathfrak{g}$. Then for any $X, Y \in \mathfrak{g}$, we have that

$$\exp X \cdot \exp Y = \exp(CB(X, Y)),$$

where $CB : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the **Campbell–Hausdorff product**

$$CB(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots,$$

which is a finite expression in the brackets of X and Y, since $\mathfrak{g}$ is nilpotent. Hence we can define on $\mathfrak{g}$ a group multiplication

$$(X, Y) \mapsto X \cdot_{CB} Y := CB(X, Y)$$

which is polynomial in X and Y. This new group $(\mathfrak{g}, \cdot_{CB})$ is evidently isomorphic to G, since the Lie algebra of $(\mathfrak{g}, \cdot_{CB})$ is equal to $\mathfrak{g}$.

Proposition 1.1.31. *Let $\mathfrak{g}$ be a Lie algebra.*

(1) *If $\mathfrak{g}$ is solvable, any Lie subalgebra $\mathfrak{h}$ and the image of any homomorphism $\varphi(\mathfrak{g})$ are also solvable.*
(2) *If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$ and the quotient algebra $\mathfrak{g}/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable.*
(3) *If $\mathfrak{a}$, $\mathfrak{b}$ are solvable ideals of $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ is also a solvable ideal.*

Proof. For any $k \in \mathbb{N}$, $D^k\mathfrak{h} \subset D^k\mathfrak{g}$ and $D^k\varphi(\mathfrak{g}) = \varphi(D^k\mathfrak{g})$. So, the assertion (1) follows.

We show (2). Letting $D^n(\mathfrak{g}/\mathfrak{a}) = \{0\}$, $D^n\mathfrak{g} \subset \mathfrak{a}$. Further supposing $D^m\mathfrak{a} = \{0\}$, $D^{n+m}\mathfrak{g} = \{0\}$.

The claim (3) is obtained by applying (1) and (2) to $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$. ■

Lemma 1.1.32. *Let V be a real or complex vector space of finite dimension, $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{h}$ an ideal of $\mathfrak{g}$. We assume that, for an element α of the dual vector space $\mathfrak{h}^*$, the common eigenspace $W = \{w \in V; Y \cdot w = \alpha(Y)w \ \forall\, Y \in \mathfrak{h}\}$ of $\mathfrak{h}$ is not $\{0\}$. Then:*

(1) $\alpha([\mathfrak{g}, \mathfrak{h}]) = \{0\}$;
(2) $\mathfrak{g} \cdot W \subset W$.

Proof. We fix $0 \neq w \in W$ and $X \in \mathfrak{g}$. For any $0 \neq i \in \mathbb{N}$ let W_i be the subspace of V generated by $\{w,\ X \cdot w,\ \ldots,\ X^{i-1} \cdot w\}$ and $W_0 = \{0\}$. If we take the minimal positive integer n satisfying $W_{n+1} = W_n$, then $\dim W_n = n$, $X \cdot W_n \subset W_n$ and $W_{n+j} = W_n$ for any $j \geq 0$.

Now, for any $i \in \mathbb{N}$, $Y \in \mathfrak{h}$,

$$YX^i \cdot w \equiv \alpha(Y) X^i \cdot w \mod W_i$$

holds. In fact, this is clear for $i = 0$. If we assume it for $i - 1$, then modulo W_i

$$\begin{aligned} YX^i \cdot w &= YXX^{i-1} \cdot w = XYX^{i-1} \cdot w + [Y, X]X^{i-1} \cdot w \\ &= \alpha(Y)X^i \cdot w + \alpha([Y, X])X^{i-1} \cdot w \equiv \alpha(Y)X^i \cdot w \end{aligned}$$

and it holds for i too. From this equation, we see $Y \cdot W_i \subset W_i$ for any $i \in \mathbb{N}$. Consequently, relative to the basis $\{w, X \cdot w, \ldots, X^{n-1} \cdot w\}$ of W_n, the restriction $Y|_W$ of Y to W is represented by an upper triangular matrix with diagonal entries $\alpha(Y)$. Therefore its trace is $\mathrm{Tr}\,(Y|_{W_n}) = n\alpha(Y)$. In particular, $\alpha([X, Y]) = 0$ because of $0 = \mathrm{Tr}\,([X, Y]|_{W_n}) = n\alpha([X, Y])$.

(2) The assertion (1) implies

$$\begin{aligned} YX \cdot w &= XY \cdot w + [Y, X] \cdot w = \alpha(Y) X \cdot w + \alpha([Y, X]) w \\ &= \alpha(Y) X \cdot w \end{aligned}$$

for any $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$, $w \in W$. So, $X \cdot w \in W$. ∎

Theorem 1.1.33 (Lie). *Let $V \neq \{0\}$ be a complex vector space of finite dimension and $\mathfrak{g}$ a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then, there exists a common eigenvector $w \neq 0$ for $\mathfrak{g}$. Namely, there exists $\alpha \in \mathfrak{g}^*$ such that $X \cdot w = \alpha(X) w$ ($\forall\, X \in \mathfrak{g}$).*

Proof. We proceed by induction on $n = \dim \mathfrak{g}$. The statement being clear when $n = 0$, we assume $n > 0$. As $\mathfrak{g} \neq \{0\}$ is solvable, $\bar{\mathfrak{g}} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. If we take a subspace of codimension 1 in $\bar{\mathfrak{g}}$ and write $\mathfrak{h}$ its inverse image in $\mathfrak{g}$, then $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$. From the induction hypothesis,

$$W = \{w \in V; Y \cdot w = \beta(Y) w, \ \forall\, Y \in \mathfrak{h}\}$$

is not $\{0\}$ for some $\beta \in \mathfrak{h}^*$. Besides, the previous proposition says that W is $\mathfrak{g}$-invariant. Now, if we write $\mathfrak{g} = \mathbb{C}T \oplus \mathfrak{h}$, $T|_W$ has an eigenvector $0 \neq w \in W$ and $T \cdot w = \lambda w$, $\lambda \in \mathbb{C}$. Clearly, w is a common eigenvector for $\mathfrak{g}$. ∎

Corollary 1.1.34. *Let $\mathfrak{g}$ be a solvable Lie algebra and (ρ, V) a complex finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$. Let*

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = \{0\}$$

be a composition series of V. Then $\dim(V_j / V_{j+1}) = 1$ for every $j = 0, \ldots, m-1$. In particular there exists a basis $\mathcal{B} = \{b_1, \ldots, b_m\}$ of V, such that the matrix $M(X)$ of $\rho(X)$ in this basis is upper triangular for every $X \in \mathfrak{g}$.

Proof. The simple modules V_j / V_{j+1} are necessarily one-dimensional by Theorem 1.1.33. Applying this theorem several times, we find a sequence $(V_j)_j$ of $\mathfrak{g}$-invariant subspaces of V such that $V_j = \mathbb{R} b_j \oplus V_{j+1}$ and such that $\rho(X) v_j = \lambda_j(X) v_j$ modulo V_{j+1} for every $X \in \mathfrak{g}$, for some linear functional $\lambda_j : \mathfrak{g} \to \mathbb{C}$, $j = 1, \ldots, m-1$. ∎

Corollary 1.1.35. *Let $\mathfrak{g}$ be a solvable Lie algebra. Then the ideal $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is nilpotent.*

Proof. Let $(\rho, V) = (\mathrm{ad}, \mathfrak{g}_{\mathbb{C}})$ be the adjoint representation of $\mathfrak{g}$. Then for every $X \in D\mathfrak{g}$ its matrix $M(X)$ has 0 diagonal entries. Hence $\mathrm{ad}(X)$ is nilpotent. This shows that $D\mathfrak{g}$ is nilpotent by Engel's theorem. ■

Definition 1.1.36. Let $\mathfrak{g}$ be a solvable Lie algebra. Let (π, V) be a finite-dimensional complex $\mathfrak{g}$-module. We put

$$S_V = \{\lambda \in \mathfrak{g}_{\mathbb{C}}^*; \exists v \in V \setminus \{0\}; \pi(X)v = \lambda(X)v,\ \forall X \in \mathfrak{g}_{\mathbb{C}}\}.$$

We call an element of S_V an *eigenvalue* of π. To an eigenvalue λ of π is associated the *eigenspace* of V, defined by

$$E_\lambda(V) = E_\lambda = \{v \in V; \pi(X)v = \lambda(X)v\ \forall X \in \mathfrak{g}_{\mathbb{C}}\}.$$

We also let

$$E(V) = E = \sum_{\lambda \in S_V} E_\lambda(V).$$

Since $\mathfrak{g}$ is solvable, S_V is nonempty by the theorem of Lie.

Definition 1.1.37. Let (π, V) be a finite-dimensional complex $\mathfrak{g}$-module. A linear form λ on $\mathfrak{g}_{\mathbb{C}}$, or its restriction on $\mathfrak{g}$, is called a **weight** of (π, V) if there exists a $\mathfrak{g}$-invariant subspace V_0 of V and an element v in $V \setminus V_0$ such that for all X in $\mathfrak{g}$ we have

$$\pi(X)v - \lambda(X)v \in V_0.$$

A weight of $(\mathrm{ad}, \mathfrak{g}_{\mathbb{C}})$ is called a **root** of $\mathfrak{g}_{\mathbb{C}}$ or $\mathfrak{g}$. We denote by R_V the set of the weights of (π, V) and by $\mathcal{R}$ the set of the roots of $\mathfrak{g}_{\mathbb{C}}$.

We fix finally an element $T \in \mathfrak{g}$ which is in general position with respect to π and ad, i.e.

$$T \in \mathfrak{g} \setminus \left(\bigcup_{\mu \neq \mu' \in R_V} \ker(\mu - \mu') \cup \bigcup_{\mu \neq \mu' \in \mathcal{R}} \ker(\mu - \mu') \right)$$

and let $\sigma_T(\pi)$, resp. $\sigma_T(\mathrm{ad})$, be the spectrum of $\pi(T)$, resp. of $\mathrm{ad}(T)$. The mappings

$$R_V \to \sigma_T(\pi), \mathrm{res}.\mathcal{R} \to \sigma_T(\mathrm{ad}),\ \lambda \mapsto \lambda(T), \tag{1.1.1}$$

are then automatically injective. Let

$$V_\lambda = \left\{ v \in V; (\pi(T) - \lambda(T) Id_V)^{\dim(V)}(v) = 0 \right\}, \qquad \lambda \in R_V,$$

and

$$(\mathfrak{g}_{\mathbb{C}})_\mu = \left\{ X \in \mathfrak{g}_{\mathbb{C}}; (\mathrm{ad}(T) - \mu(T) Id_{\mathfrak{g}_{\mathbb{C}}})^{\dim(\mathfrak{g})}(X) = 0 \right\}, \qquad \mu \in \mathcal{R}.$$

We then obtain the **Jordan decomposition** of V with respect to T:

$$V = \bigoplus_{\lambda \in \sigma_T(\pi)} V_\lambda \tag{1.1.2}$$

and

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\mu \in \sigma_T(\mathrm{ad})} (\mathfrak{g}_{\mathbb{C}})_\mu.$$

Definition 1.1.38. Let $\mathfrak{g}$ be a solvable Lie algebra and let T be an element of $\mathfrak{g}$ in general position with respect to ad. Let for $\lambda \in \sigma_T(\mathrm{ad})$,

$$\mathfrak{g}_\lambda := ((\mathfrak{g}_{\mathbb{C}})_\lambda + (\mathfrak{g}_{\mathbb{C}})_{\overline{\lambda}}) \cap \mathfrak{g} = ((\mathfrak{g}_{\mathbb{C}})_\lambda + \overline{(\mathfrak{g}_{\mathbb{C}})_\lambda}) \cap \mathfrak{g}.$$

Evidently $\mathfrak{g}_\lambda = \mathfrak{g}_{\overline{\lambda}}$ for every λ. Let

$$\mathfrak{g}_1 := \sum_{\lambda \in \sigma_T(\mathrm{ad}), \lambda \neq 0} \mathfrak{g}_\lambda.$$

Then we have that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1. \tag{1.1.3}$$

Proposition 1.1.39. *We have that* $\mathfrak{g}_\lambda \subset [\mathfrak{g}, \mathfrak{g}]$ *for every* $\lambda \neq 0, \lambda \in \sigma_T(ad)$.

Proof. By definition of $(\mathfrak{g}_{\mathbb{C}})_\lambda$, we see that $\mathrm{ad}(T) - \lambda Id_V$ is nilpotent on $(\mathfrak{g}_{\mathbb{C}})_\lambda$, so for $\lambda \neq 0$, $\mathrm{ad}(T)$ is invertible on $(\mathfrak{g}_{\mathbb{C}})_\lambda$. Hence

$$(\mathfrak{g}_{\mathbb{C}})_\lambda = [T, (\mathfrak{g}_{\mathbb{C}})_\lambda] \subset [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}].$$

∎

We have the following bracket relation.

Proposition 1.1.40. *Let* (π, V) *be a finite-dimensional* $\mathfrak{g}$*-module. For all* λ *in* $\sigma_T(\pi)$ *and all* μ *in* $\sigma_T(ad)$ *we have that*

$$\pi(\mathfrak{g}_\mu)(V_\lambda) \subset V_{\lambda+\mu}.$$

In particular $\mathfrak{g}_0 = (\mathfrak{g}_{\mathbb{C}})_0 \cap \mathfrak{g}$ *is a nilpotent subalgebra of* $\mathfrak{g}$ *and* $\pi(\mathfrak{g}_0) V_\lambda \subset V_\lambda$ *for all* $\lambda \in \sigma_T(\pi)$.

Proof. Let $v \in V_\lambda$ and $X \in (\mathfrak{g}_\mathbb{C})_\mu$. We choose a composition series $(V_i)_i$ of V and let j be the index for which $v \in V_j \backslash V_{j+1}$. Then we observe that $\lambda = \lambda_j(T)$ since $v \in V_j$ and so

$$\begin{aligned}\pi(T)(\pi(X)v) - (\lambda + \mu)\pi(X)v &= \pi([T, X])v - \mu\pi(X)v + \pi(X)(\pi(T)v - \lambda v)\\ &= \pi(\mathrm{ad}(T)X - \mu X)v + \pi(X)(\pi(T)v - \lambda v)\\ &= \pi(\mathrm{ad}(T)X - \mu X)v \bmod \mathfrak{g}_{j+1}.\end{aligned}$$

This shows that inductively on i,

$$(\pi(T) - (\lambda + \mu)Id_V)^i(\pi(X)v) = (\mathrm{ad}(T)X - \mu X)^i(v) \bmod \mathfrak{g}_{\mathrm{j}+1}.$$

Hence for i big enough, it follows that

$$(\pi(T) - (\lambda + \mu)Id_V)^i(\pi(X)v) \in \mathfrak{g}_{j+1}.$$

Repeating this game for $j + 1$, we finally see that

$$(\pi(T) - (\lambda + \mu)Id_V)^k(\pi(X)v) = 0$$

for some $k \in \mathbb{N}^*$.

If $\mathfrak{g}_0$ is not nilpotent, then there exists a root $\mu \neq 0$ of the Lie algebra $\mathfrak{g}_0$, which defines then also a root of $\mathfrak{g}$. But this root is 0 on T by definition of $\mathfrak{g}_0$. This contradiction tells us that 0 is the only root of $\mathfrak{g}_0$, i.e. $\mathfrak{g}_0$ is nilpotent. ∎

Remark 1.1.41. Let $\mathfrak{g}$ be a solvable Lie algebra. We can use the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of Definition 1.1.3 to describe the multiplication of the simply connected group G associated to $\mathfrak{g}$. Let $\mathfrak{w}$ be a subspace of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{w} \oplus (\mathfrak{g}_0 \cap [\mathfrak{g}, \mathfrak{g}])$. Then for $X, Y \in \mathfrak{w}$, we can write

$$X \cdot_{CB} Y = (X + Y, P(X, Y)),$$

where $P(X, Y)$ is a polynomial expression in X, Y, since $\mathfrak{g}_0$ is nilpotent. We now write $\mathfrak{g}$ as a semi-direct product of $\mathfrak{w}$ and $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{n}$ being a nilpotent ideal of $\mathfrak{g}$. This allows us to define a group multiplication $\cdot$ on $G := \mathfrak{w} \times \mathfrak{n}$ by letting

$$(w, n)\cdot(w', n') := (w + w', P(w, w') \cdot_{CB} (e^{-w'} n \cdot_{CB} n')). \tag{1.1.4}$$

Here the symbol e^w for $w \in \mathfrak{w}$ denotes the automorphism $e^w := \exp(\mathrm{ad}(w))$ of the nilpotent Lie algebra $\mathfrak{n}$. The group $(G, \cdot)$ admits as Lie algebra the semi-direct product $\mathfrak{w} \times \mathfrak{n}$ with the Lie bracket

$$[(w, X), (w', X')] = (0, [w, w'] + [X, X'] + [w, X'] + [X, w'])$$

which is isomorphic to $\mathfrak{g}$. Hence the Lie group G, which is simply connected, admits $\mathfrak{g}$ as its Lie algebra.

Proposition 1.1.42. *Let $\mathfrak{g}$ be a solvable Lie algebra, let $\mathfrak{b}$ be an ideal of $\mathfrak{g}$, such that $\mathfrak{g}/\mathfrak{b}$ is abelian. Let $\mathfrak{v} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{b}$. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then G contains a closed simply connected subgroup B with Lie algebra $\mathfrak{b}$ and the mapping $E : \mathfrak{v} \times B \to G; (v, b) \mapsto (exp\ v)b$ is a diffeomorphism.*

Proof. We proceed by induction on the dimension of $\mathfrak{g}$. If dim $\mathfrak{g} = 1$, there is nothing to prove. If dim $\mathfrak{v} = 1$, i.e. $\mathfrak{v} = \mathbb{R}X$, then we take a simply connected Lie group B' with Lie algebra $\mathfrak{b}$ and we consider the derivation $d(U) = \mathrm{ad}X(U), U \in \mathfrak{b}$, of $\mathfrak{b}$. This defines a one-parameter group of automorphisms $a(t),\ t \in \mathbb{R}$, of B', where $a(t)(\exp U) = \exp(e^{td}(U)),\ U \in \mathfrak{b}$. We can thus construct a new Lie group $G' = \mathbb{R} \times B'$ with the multiplication

$$(s, u)\cdot(s', u') := (s + s', a(-s')u\cdot u').$$

The Lie algebra of this group is isomorphic to $\mathfrak{g}$ and so there exists an isomorphism $\psi : G' \to G$. The subgroup $B := \psi(B')$ is closed and simply connected with Lie algebra $\mathfrak{b}$, since the same is true for $\{0\} \times B' \subset G'$. Furthermore, $\exp(sX) = \psi(s, 0)$ for every $s \in \mathbb{R}$. This shows that the mapping $E(s, b) = \exp(sX)\cdot b = \psi(s, \psi^{-1}(b))$ from $\mathbb{R} \times B$ onto G is a diffeomorphism.

Suppose now that dim $\mathfrak{v} > 1$. Choose a vector $X \in \mathfrak{v}$ and a subspace $\mathfrak{w}$ in $\mathfrak{v}$, such that $\mathfrak{v} = \mathbb{R}X \oplus \mathfrak{w}$ and let $\mathfrak{g}' = \mathfrak{w} + \mathfrak{b}$. Applying the induction hypothesis to $\mathfrak{g}'$, we see that there exists a closed normal simply connected subgroup B of G, such that the mapping

$$\mathbb{R} \times \mathfrak{w} \times B \to G;\ (s, U, b) \mapsto \exp(sX)(\exp U)b$$

is a diffeomorphism. Since G/B is abelian, we have that

$$\exp(sX)\exp U = \exp(sX + U)q(s, U)^{-1},$$

where $(s, U) \mapsto q(s, U) \in B$ is a C^∞ mapping. This shows that our mapping $(sX + U, b) \mapsto \exp(sX + U)b = \exp(sX)(\exp U)q(s, U)b$ is a diffeomorphism of $\mathfrak{v} \times B$ onto G. ∎

1.2 Enveloping Algebra

In this section, following Dixmier [21] we define the enveloping algebra of a Lie algebra and give the fundamental theorem of Poincaré–Birkhoff–Witt. Let $\mathfrak{g}$ be a real or complex Lie algebra. We designate the base field by k. Namely, k is $\mathbb{R}$ or $\mathbb{C}$. Let $T = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \cdots$ be the tensor product of $\mathfrak{g}$. Here T^n denotes an n tensor product of $\mathfrak{g}$, in particular $T^0 = k \cdot 1$ and $T^1 = \mathfrak{g}$. We denote by J the two-sided ideal of T generated by the tensors

$$X \otimes Y - Y \otimes X - [X, Y]\ (X, Y \in \mathfrak{g}).$$

The algebra T/J is called the **enveloping algebra** of $\mathfrak{g}$ and denoted by $\mathcal{U}(\mathfrak{g})$. The composite mapping $\sigma : \mathfrak{g} \to T \to \mathcal{U}(\mathfrak{g})$ is called the canonical mapping of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. Thus, for any $X,\ Y \in \mathfrak{g}$,

$$\sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) = \sigma([X, Y]).$$

When $\mathfrak{g}$ is a commutative Lie algebra, $\mathcal{U}(\mathfrak{g})$ is nothing but the symmetric algebra $S(\mathfrak{g})$ of the vector space $\mathfrak{g}$. We denote the canonical image in $\mathcal{U}(\mathfrak{g})$ of the two-sided ideal $T_+ = T^1 \oplus T^2 \oplus \cdots$ of T by $\mathcal{U}_+(\mathfrak{g})$. Since $T = T^0 \oplus T_+ = k \cdot 1 \oplus T_+$ and $J \subset T_+$, if we denote by $\mathcal{U}^0$ the image of T^0 in $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{g}) = \mathcal{U}^0 \oplus \mathcal{U}_+(\mathfrak{g})$ and $\mathcal{U}^0 = k \cdot 1$ is one-dimensional. Let's identify $\mathcal{U}^0$ with k. The associative algebra $\mathcal{U}(\mathfrak{g})$ is generated by 1 and the canonical image of $\mathfrak{g}$ in $\mathcal{U}(\mathfrak{g})$.

Lemma 1.2.1. *Let σ be the canonical mapping of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. Let A be an algebra with unity, τ a linear mapping of $\mathfrak{g}$ into A such that $\tau(X)\tau(Y) - \tau(Y)\tau(X) = \tau([X, Y])$ for any $X,\ Y \in \mathfrak{g}$. Then, there uniquely exists a homomorphism τ' of $\mathcal{U}(\mathfrak{g})$ into A satisfying $\tau'(1) = 1$ and $\tau' \circ \sigma = \tau$.*

Proof. $\mathcal{U}(\mathfrak{g})$ is generated by 1 and $\sigma(\mathfrak{g})$, τ' is unique if it exists. On the other hand, there uniquely exists a homomorphism φ of T into A, which extends τ and satisfies $\varphi(1) = 1$. As

$$\varphi(X \otimes Y - Y \otimes X - [X, Y]) = \tau(X)\tau(Y) - \tau(Y)\tau(X) - \tau([X, Y]) = 0$$

for all $X,\ Y \in \mathfrak{g}$, $\varphi(J) = \{0\}$ and passing to the quotient φ gives us the desired homomorphism τ'. ■

We fix a basis $\{X_1,\ X_2,\ \ldots,\ X_n\}$ of $\mathfrak{g}$ and put $Y_i = \sigma(X_i)$ $(1 \le i \le n)$. For a finite sequence $I = (i_1,\ \ldots,\ i_p)$ of positive integers from 1 to n, we set $Y_I = Y_{i_1} Y_{i_2} \cdots Y_{i_p}$. When $i \le i_1,\ \ldots,\ i \le i_p$ for an integer i, we write $i \le I$. Furthermore, we denote by $\mathcal{U}_q(\mathfrak{g})$ the image of $T^0 \oplus T^1 \oplus \cdots \oplus T^q$ in $\mathcal{U}(\mathfrak{g})$.

Lemma 1.2.2. *Let $Y_1,\ \ldots,\ Y_p$ be elements of $\mathfrak{g}$ and π a permutation of $(1,\ \ldots,\ p)$. Then,*

$$\sigma(Y_1) \cdots \sigma(Y_p) - \sigma\left(Y_{\pi(1)}\right) \cdots \sigma\left(Y_{\pi(p)}\right)$$

belongs to $\mathcal{U}_{p-1}(\mathfrak{g})$.

Proof. It is sufficient to prove the claim when π is the transposition of j and $j + 1$. In that case, the claim follows from the equality

$$\sigma(Y_j)\sigma(Y_{j+1}) - \sigma(Y_{j+1})\sigma(Y_j) = \sigma([Y_j, Y_{j+1}]).$$ ■

From this we immediately see the following.

Corollary 1.2.3. *The Y_I for all non-decreasing sequences I of length smaller than or equal to p generate the vector space $\mathcal{U}_p(\mathfrak{g})$.*

Let P be the polynomial ring of n variables $z_1, \ldots, z_n$. We denote by P_i for $i \geq 0$ the totality of polynomials the degree of which is smaller than or equal to i. Besides, we set $z_I = z_{i_1} z_{i_2} \cdots z_{i_p}$ for the finite sequence $I = (i_1, \ldots, i_p)$ of natural numbers between 1 and n.

Lemma 1.2.4. *For any integer $p \geq 0$, there uniquely exists a linear mapping f_p from the vector space $\mathfrak{g} \otimes P_p$ to P satisfying the following conditions. Moreover, the restriction of f_p on $\mathfrak{g} \otimes P_{p-1}$ is f_{p-1}.*

(A_p) *if* $i \leq I$, $z_I \in P_p$, $f_p(X_i \otimes z_I) = z_i z_I$;
(B_p) *if* $z_I \in P_q$, $q \leq p$, $f_p(X_i \otimes z_I) - z_i z_I \in P_q$;
(C_p) *if* $z_J \in P_{p-1}$ $(p \geq 1)$,

$$f_p(X_i \otimes f_p(X_j \otimes z_J)) = f_p(X_j \otimes f_p(X_i \otimes z_J)) + f_p([X_i, X_j] \otimes z_J).$$

Proof. For $p = 0$, the condition (A_0) means $f_0(X_i \otimes 1) = z_i$, and the condition (B_0) holds. Let us assume the existence and the uniqueness of the mapping f_{p-1}. If f_p exists, its restriction on $\mathfrak{g} \otimes P_{p-1}$ satisfies the conditions (A_{p-1}), (B_{p-1}), (C_{p-1}) and is equal to f_{p-1}. Therefore, we show that there uniquely exists a linear extension f_p of f_{p-1} to $\mathfrak{g} \otimes P_p$ satisfying the conditions (A_p), (B_p), (C_p). That is to say, we need to define $f_p(X_i \otimes z_I)$ for non-decreasing sequence I of p elements. When $i \leq I$, this is done by the condition (A_p). Otherwise, we can write $I = (j, J)$, $j < i$, $j \leq J$. Then, by (A_{p-1}) and (C_p),

$$\begin{aligned} f_p(X_i \otimes z_I) &= f_p(X_i \otimes f_{p-1}(X_j \otimes z_J)) \\ &= f_p(X_j \otimes f_{p-1}(X_i \otimes z_J)) + f_{p-1}([X_i, X_j] \otimes z_J). \end{aligned}$$

Now, (B_{p-1}) implies $f_{p-1}(X_i \otimes z_J) = z_i z_J + w$, $w \in P_{p-1}$ and by (A_p)

$$\begin{aligned} f_p(X_j \otimes f_{p-1}(X_i \otimes z_J)) &= z_j z_i z_J + f_{p-1}(X_j \otimes w) \\ &= z_i z_J + f_{p-1}(X_j \otimes w). \end{aligned}$$

Through these processes f_{p-1} is uniquely extended to $\mathfrak{g} \otimes P_p$ by linearity, and this extension f_p satisfies (A_p), (B_p). We verify that f_p satisfies (C_p) too.

From the above construction, when $j < i$ and $j \leq J$, the condition (C_p) is satisfied. As $[X_j, X_i] = -[X_i, X_j]$, (C_p) is also satisfied when $i < j$ and $i \leq J$, and clearly in the case $j = i$ as well. In consequence, if $i \leq J$ or $j \leq J$, then (C_p) is satisfied. In the other case, we can write $J = (k, K)$, $k \leq K$, $k < i$, $k < j$. To simplify the notations, we put from now on $f_p(X \otimes z) = Xz$ for $X \in \mathfrak{g}$, $z \in P_p$. From the induction hypothesis,

$$X_j z_J = X_j(X_k z_K) = X_k(X_j z_K) + [X_j, X_k] z_K. \tag{1.2.1}$$

Now we can write $X_j z_K = z_j z_K + w,\ w \in P_{p-2}$. Because of $k \leq K$ and $k < j$, the condition (C_p) is applied to $X_i(X_k(z_j z_K))$, to $X_i(X_k w)$ by the induction hypothesis and hence to $X_i(X_k(X_j z_K))$. Taking Eq. (1.2.1) into account,

$$\begin{aligned} X_i(X_j z_J) =& X_k(X_i(X_j z_K)) + [X_i, X_k](X_j z_K) \\ &+ [X_j, X_k](X_i z_K) + [X_i, [X_j, X_k]] z_K. \end{aligned}$$

Interchanging i and j then taking the difference of these two equations,

$$\begin{aligned} &X_i(X_j z_J) - X_j(X_i z_J) \\ &= X_k(X_i(X_j z_K) - X_j(X_i z_K)) + [X_i, [X_j, X_k]] z_K - [X_j, [X_i, X_k]] z_K \\ &= X_k([X_i, X_j] z_K) + [X_i, [X_j, X_k]] z_K + [X_j, [X_k, X_i]] z_K \\ &= [X_i, X_j] X_k z_K + [X_k, [X_i, X_j]] z_K + [X_i, [X_j, X_k]] z_K + [X_j, [X_k, X_i]] z_K \\ &= [X_i, X_j] X_k z_K = [X_i, X_j] z_J. \end{aligned}$$

■

Lemma 1.2.5. *The Y_I for all non-decreasing sequence I constitute a basis of the vector space $\mathcal{U}(\mathfrak{g})$.*

Proof. We use the notations of the preceding lemma. The lemma says that there exists a bilinear mapping f from $\mathfrak{g} \times P$ to P such that $f(X_i, z_I) = z_i z_I$ for $i \leq I$ and

$$f(X_i, f(X_j, z_J)) = f(X_j, f(X_i, z_J)) + f([X_i, X_j], z_J)$$

for any $i,\ j,\ J$. In other words, there exists a representation ρ of $\mathfrak{g}$ in P such that $\rho(X_i) z_I = z_i z_I$ when $i \leq I$. By Lemma 1.2.1, there exists a homomorphism φ of $\mathcal{U}(\mathfrak{g})$ into $\mathrm{End}(P)$ such that $\varphi(Y_i) z_I = z_i z_I$ when $i \leq I$. Therefore,

$$\varphi(Y_{i_1} Y_{i_2} \cdots Y_{i_p}) \cdot 1 = z_{i_1} z_{i_2} \cdots z_{i_p}$$

when $i_1 \leq i_2 \leq \cdots \leq i_p$. Hence, the Y_I for non-decreasing sequence I are linearly independent. Together with Corollary 1.2.3, we get the assertion. ■

Corollary 1.2.6. *The canonical map of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$ is injective.*

By this corollary we regard from now on that $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$. By what proceeds:

Theorem 1.2.7 (Poincaré–Birkhoff–Witt). *Let $(X_1, \ldots, X_n)$ be a basis of the vector space $\mathfrak{g}$. When $\nu = (\nu_1, \ldots, \nu_n)$ runs over $\mathbb{N}^n$, $X^\nu = X_1^{\nu_1} X_2^{\nu_2} \cdots X_n^{\nu_n}$ form a basis of $\mathcal{U}(\mathfrak{g})$.*

Corollary 1.2.8. *The algebra $\mathcal{U}(\mathfrak{g})$ has no zero divisor.*

Proof. For $\nu = (\nu_1, \dots, \nu_n)$, we set $|\nu| = \nu_1 + \cdots + \nu_n$. Now computing the product of two non zero elements

$$A = \sum_{|\alpha|=0}^{\ell} c_\alpha X^\alpha, \quad B = \sum_{|\beta|=0}^{m} d_\beta X^\beta$$

of $\mathcal{U}(\mathfrak{g})$,

$$\begin{aligned} AB &= \sum_{|\alpha|=\ell, |\beta|=m} c_\alpha d_\beta X^\alpha X^\beta + \sum_{|\gamma|<\ell+m} e_\gamma X^\gamma \\ &= \sum_{|\gamma|=\ell+m} \left(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \right) X^\gamma + \sum_{|\gamma|<\ell+m} e_\gamma X^\gamma. \end{aligned}$$

Thus, letting $AB = 0$, the product of two polynomials

$$p = \sum_{|\alpha|=\ell} c_\alpha x^\alpha, \quad q = \sum_{|\beta|=m} d_\beta x^\beta$$

in the polynomial ring $k[x_1, \dots, x_n]$ becomes 0, which is impossible. ■

Definition 1.2.9. We define a linear map ψ of the vector space $\mathfrak{g}$ by $\psi(X) = -X$ and extend it to an anti-isomorphism of the tensor product T. Since $\psi(J) \subset J$, ψ induces an anti-isomorphism $\diamondsuit$ on $\mathcal{U}(\mathfrak{g})$, called the **principal anti-isomorphism** of $\mathcal{U}(\mathfrak{g})$.

Definition 1.2.10. We denote by $S(\mathfrak{g})$ the symmetric algebra of the vector space $\mathfrak{g}$ and by S^n the set of its homogeneous elements of degree n. $\mathfrak{S}_n$ being the symmetric group of degree n, the mapping

$$x_1 x_2 \cdots x_n \mapsto \frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_n} x_{\alpha(1)} x_{\alpha(2)} \cdots x_{\alpha(n)} \ (x_j \in \mathfrak{g},\ 1 \le j \le n)$$

from S^n to $\mathcal{U}_n(\mathfrak{g})$ gives a bijection of $S(\mathfrak{g})$ onto $\mathcal{U}(\mathfrak{g})$, called the **symmetrization map**.

1.3 Unitary Representations

In this section we only touch upon unitary representations of groups; their detailed discussion will be given in the next chapter. In general a representation of a group G means a homomorphism of G to the group of linear automorphisms of a linear space. The first results were obtained by Frobenius at the end of the nineteenth century and were extended, making use of integrals on group manifolds, by Weyl in the 1920s to compact groups. Let M be a space where a group G acts. If we try to develop the analysis on the space M utilizing the action of G, we would be interested in the representations of G. When there exists on M an G-invariant measure μ, denoting by $L^2(M,\mu)$ the space of all complex-valued functions on M which are square integrable with respect to μ, we get a unitary representation of G on $L^2(M,\mu)$ by the formula

$$(\pi(g)\varphi)(m) = \varphi(g^{-1}\cdot m) \quad (g \in G,\ m \in M)$$

for any $\varphi \in L^2(M,\mu)$. This construction is in some sense fundamental. For example, letting $M = G$ be a separable locally compact topological group, μ a left Haar measure on G, we get the left regular representation of G. The disintegration of this representation would lead to a development from the commutative Fourier analysis to the non-commutative harmonic analysis. Furthermore, H being a closed subgroup of G, G acts on the homogeneous space G/H and by extending this manner of construction we obtain a method to induce a representation of G starting from that of H. It was in the 1940s that the study of these infinite-dimensional unitary representations really began.

We only consider locally compact topological groups G satisfying the second countability condition and complex Hilbert spaces $\mathcal{H}$ are always supposed to be separable. Let $\mathcal{H}$ be a (separable) Hilbert space. A **complete orthonormal system** $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}$ is called a basis of $\mathcal{H}$.

Lemma 1.3.1. *Let $\{x_j\}_{j=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ be two bases of a Hilbert space $\mathcal{H}$. If we set for a bounded linear operator A of $\mathcal{H}$*

$$T_x(A) = \sum_{j=1}^{\infty} \|Ax_j\|^2, \ T_y(A) = \sum_{k=1}^{\infty} \|Ay_k\|^2,$$

then $T_x(A) = T_x(A^*) = T_y(A)$.

Proof. In fact,

$$\begin{aligned} T_x(A) &= \sum_{j=1}^{\infty} \|Ax_j\|^2 = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |(Ax_j, y_k)|^2 \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} |(A^* y_k, x_j)|^2 \right) = \sum_{k=1}^{\infty} \|A^*(y_k)\|^2 = T_y(A^*) \end{aligned}$$

and in particular $T_x(A) = T_x(A^*)$. Replacing A by A^* in this relation, $T_x(A) = T_x(A^*) = T_y(A)$. ∎

Definition 1.3.2. Let $\{e_j\}_{j=1}^{\infty}$ be a basis of a Hilbert space $\mathcal{H}$. A bounded linear operator A of $\mathcal{H}$ is of **Hilbert–Schmidt class** if $\|A\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \|Ae_j\|^2 < +\infty$. The preceding lemma confirms that this definition does not depend on the choice of the basis $\{e_j\}_{j=1}^{\infty}$. The number $\|A\|_{\mathrm{HS}}$ is called the **Hilbert–Schmidt norm** of A.

The next lemma is evident from the definition.

Lemma 1.3.3. (1) *The sum of two operators of Hilbert–Schmidt class is of Hilbert–Schmidt class.*
(2) *The adjoint operator of a Hilbert–Schmidt class operator is of Hilbert–Schmidt class.*
(3) *If A is of Hilbert–Schmidt class and B is any bounded linear operator, $AB,\ BA$ are of Hilbert–Schmidt class and $\|AB\|_{\mathrm{HS}} \leq \|B\|_{\mathrm{op}}\|A\|_{\mathrm{HS}}$.*

Definition 1.3.4. An operator A is of **trace class** or a nuclear operator if it is written as a finite sum of operators of the form BC, where $B,\ C$ are of Hilbert–Schmidt class.

Lemma 1.3.5. *An operator A of trace class is written in the form $A = \sum_{j=1}^{n} c_j A_j^* A_j$ $(c_j \in \mathbb{C})$, where $\{A_j\}_{j=1}^{n}$ are Hilbert–Schmidt class operators.*

Proof. For any bounded linear operators $A,\ B$,

$$\begin{aligned} 4A^*B = {} & \{(A+B)^*(A+B) - (A-B)^*(A-B)\} \\ & -i\,\{(A+iB)^*(A+iB) - (A-iB)^*(A-iB)\}\,. \end{aligned}$$

Thus, the assertion is clear from the definition. ■

Definition 1.3.6. Let A be a trace class operator in a Hilbert space $\mathcal{H}$ and $\{e_j\}_{j=1}^{\infty}$ a basis of $\mathcal{H}$. From what we have seen above, the series $\sum_{j=1}^{\infty}(Ae_j, e_j)$ is absolutely convergent and its sum does not depend on the choice of the basis $\{e_j\}_{j=1}^{\infty}$. We call this sum the **trace** of the operator A and denote it by $\mathrm{Tr}(A)$.

Definition 1.3.7. A **unitary representation** of G in $\mathcal{H}$ means a homomorphism π of G into the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$ such that for any $x \in \mathcal{H}$ the mapping $G \ni g \mapsto \pi(g)(x) \in \mathcal{H}$ is continuous relative to the topology of $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is called the **representation space** of π, and we agree to denote it by $\mathcal{H}(\pi)$ or $\mathcal{H}_\pi$. According to the dimension of $\mathcal{H}(\pi)$, π is said to be finite- or infinite-dimensional.

Definition 1.3.8. An **invariant subspace** of π means a subspace $\mathcal{L}$ of $\mathcal{H}(\pi)$ satisfying $\pi(g)(\mathcal{L}) \subset \mathcal{L}$ for any $g \in G$. When there is no closed invariant subspace other than $\{0\}$ and $\mathcal{H}(\pi)$, π is said to be **irreducible**.

Definition 1.3.9. Let $\pi,\ \rho$ be two unitary representations of G. A bounded linear operator T from $\mathcal{H}(\pi)$ to $\mathcal{H}(\rho)$ is called a **intertwining operator** from π to ρ, if it satisfies $T \circ \pi(g) = \rho(g) \circ T$ for any $g \in G$. We denote by $R(\pi, \rho)$ the vector space of all such operators. When there is as such an intertwining operator which is unitary, we say that π and ρ are **equivalent**. To indicate the equivalence we use the notation $\pi \simeq \rho$. The set of all equivalence classes of irreducible unitary representations of G is called the **unitary dual** of G and denoted by $\hat{G}$.

An irreducible unitary representation π of G and its equivalence class are often identified and we write $\pi \in \hat{G}$ etc.

By the way, what are standard problems in the representation theory? We may list for example the problems of constructing and decomposing representations, checking the irreducibility and equivalence of representations, computing an explicit intertwining operator between two equivalent representations, analysing particular representation and so on. In order to treat these problems we sketch the direct integral of unitary representations. Two unitary representations π, ρ of G being given, let $\mathcal{H}$ be the direct sum Hilbert space of their spaces. The unitary representation $\pi \oplus \rho$ on $\mathcal{H}$ is defined by the formula

$$(\pi \oplus \rho)(g)(v, w) = (\pi(g)v, \rho(g)w),\ g \in G,\ v \in \mathcal{H}(\pi),\ w \in \mathcal{H}(\rho).$$

We call $\pi \oplus \rho$ the **direct sum** of π and ρ. More generally:

Definition 1.3.10. Let $\{\pi_i\}$ be a family of unitary representations of G in their Hilbert spaces $\{\mathcal{H}(\pi_i)\}$. The unitary representation $\pi = \sum_i \pi_i$ of G constructed as follows on the Hilbert space $\mathcal{H}$, which is the direct sum of the Hilbert spaces $\{\mathcal{H}(\pi_i)\}$, is called the direct sum of $\{\pi_i\}$. The vectors of $\mathcal{H} = \sum_i \mathcal{H}(\pi_i)$ are the sequences $v = (v_i; v_i \in \mathcal{H}(\pi_i))$ such that $\sum_i \|v_i\|^2 < \infty$ and $\pi(g)v = (\pi_i(g)v_i)$ $(g \in G)$. In particular, when all the π_i are equivalent to a same representation ρ, the direct sum $\sum_i \pi_i$ is also written as $n\rho$. Here, $n \in \{1, 2, 3, \dots, \infty\}$ denotes the number of indices.

Let us extend this sum to integrals. For the details on the Borel structure in $\hat{G}$, direct integrals of representations and groups of type I etc., the readers may refer to Mackey's lecture note [56].

A Borel structure on a set X means the existence of a σ-algebra defined on X, i.e. a family $\mathcal{B}$ of subsets of X, closed under taking the complement and countable unions. A Borel space is a set X equipped with a Borel structure $\mathcal{B}$. The members of $\mathcal{B}$ are called Borel sets. The Borel structures we mainly consider here are the σ-algebras generated by the closed subsets of separable topological spaces and in particular complete separable metric spaces.

Let μ be a Borel measure on a Borel space X. Let $\mathcal{H}$ be a separable Hilbert space and $L^2(X, \mu, \mathcal{H})$ the set of all functions f on X with values in $\mathcal{H}$, and satisfying:

(1) for any $\varphi \in \mathcal{H}$, $(f(x), \varphi)$ is a Borel function of $x \in X$,
(2) $\int_X \|f(x)\|^2 d\mu(x) < \infty$.

If we identify in $L^2(X, \mu, \mathcal{H})$ two functions which coincide almost everywhere with respect to μ and define the inner product by

$$(f, h) = \int_X (f(x), h(x))_{\mathcal{H}} \, d\mu(x),$$

we get a separable Hilbert space $\mathcal{L}$.

Now, let $\{\pi_x; x \in X\}$ be a field of unitary representations of G such that $\mathcal{H}(\pi_x) = \mathcal{H}, \ (\forall x \in X)$ and that

$$X \ni x \mapsto \langle \pi_x(g) (f(x)), h(x) \rangle$$

is a measurable function for any $g \in G$ and $f, h \in \mathcal{L}$. Then, we get a unitary representation π of G in $\mathcal{L}$ by the formula

$$(\pi(g) f)(x) = \pi_x(g) (f(x)), \ \ g \in G, \ f \in \mathcal{L}.$$

This representation π is called the **direct integral** of $\{\pi_x; x \in X\}$ and written as $\pi = \int_X^{\oplus} \pi_x d\mu(x)$.

The exponential solvable Lie groups G which we will treat later are groups of type I and its unitary representations on separable Hilbert spaces can be uniquely decomposed into a direct integral of irreducible unitary representations. On $\hat{G}$ there exists a canonical topology, called the Fell topology, which gives us a canonical Borel structure on $\hat{G}$. For any unitary representation ρ of G, there exists a Borel measure μ on $\hat{G}$ and a Borel function $m : \hat{G} \to \mathbb{N} \cup \{\infty\}$ such that

$$\rho \simeq \int_{\hat{G}}^{\oplus} m(\pi) \pi d\mu(\pi).$$

We call this the **canonical irreducible decomposition** of ρ and the function m the **multiplicity function** in this decomposition. If there is another canonical irreducible decomposition

$$\rho \simeq \int_{\hat{G}}^{\oplus} m'(\pi) \pi d\mu'(\pi)$$

of ρ, the measures $\mu, \ \mu'$ are mutually equivalent and the multiplicity functions $m, \ m'$ coincide almost everywhere with respect to μ.

Chapter 2
Haar Measure and Group Algebra

2.1 The Haar Measure of a Locally Compact Group

Definition 2.1.1. Let G be a locally compact topological group. We denote by $C_c(G)$ the vector space of all the continuous complex-valued functions on G with compact support. On this space we put the norm

$$\|f\|_\infty := \sup_{g\in G} |f(g)|, \quad g \in G.$$

The completion of $C_c(G)$ with respect to this norm is the function-space

$$C_0(G) := \{f : G \to \mathbb{C}; f \text{ continuous, tending to 0 at } \infty\}.$$

The **support** of a continuous function defined on a topological space X is by definition the closure $\mathrm{supp}(f)$ of the subset $\{x \in X;\ f(x) \neq 0\}$.

Definition 2.1.2. We say that a linear functional

$$\mu : C_c(G) \to \mathbb{C},$$

is a **Borel measure** on G, if for every compact subset K of G, there exists a constant $C_K > 0$, such that $|\mu(f)| \leq C_K\|f\|_\infty$ for all f with $\mathrm{supp}(f) \subset K$.

A linear functional μ on $C_c(G)$ is said to be positive if $\mu(f) \geq 0$ for every non-negative function $f \in C_c(G)$. First we remark that such a positive linear functional μ maps real-valued functions into real numbers. It is furthermore easy to see that μ is in fact a (positive) Borel measure. Indeed, given a compact subset K of G, we can find a non-negative function ϕ in $C_c(G)$, such that ϕ is equal to 1 on a neighbourhood of K. Hence for every real-valued $f \in C_c(G)$ with $\mathrm{supp}(f) \subset K$, we have that

$$f(g) \leq \|f\|_\infty\phi(g), \quad g \in G,$$

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and so $\mu(\|f\|_\infty\phi - f) \geq 0$, since μ is positive. Hence $\mu(f) \leq C_K\|f\|_\infty$, where $C_K = \mu(\phi)$. Replacing f by $-f$, we see that $|\mu(f)| \leq C_K\|f\|_\infty$. Now let f be any element of $C_c(G)$ supported by K. Multiplying f by complex number α of modulus one, we can suppose that $\mu(\alpha f)$ is a real number. Let $\alpha f = a + ib$ be the decomposition of αf into its real part a and its imaginary part b. Then $\mathbb{R} \ni \mu(\alpha f) = \mu(a) + i\mu(b)$ implies that $\mu(b) = 0$ and so

$$|\mu(f)| = |\mu(\alpha f)| = |\mu(a)| \leq C_K\|a\|_\infty \leq C_K\|f\|_\infty.$$

Hence μ is a (positive) Borel measure.

By the theorem of Stone (see [44]), there exists on the σ-algebra generated by the compact subsets of G a unique positive measure $d\mu$ such that

$$\mu(f) = \int_G f(g)d\mu(g), \quad f \in C_c(G).$$

Definition 2.1.3. Let X be a complex vector space. For a function or mapping $F : G \to X$, set

$$\lambda(s)F(t) := F(s^{-1}t), \quad s, t \in G.$$

We call this action of G on the function-space $\mathcal{F}(G, X) := \{F : G \to X\}$ the left translation. It is immediately checked that $\lambda(s)$ defines a linear bijection of the vector space $\mathcal{F}(G, X)$ and that

$$\lambda(st) = \lambda(s) \circ \lambda(t), \quad \lambda(s)^{-1} = \lambda(s^{-1}), \quad s, t \in G.$$

In particular $\lambda(s)$ leaves the spaces $C_c(G)$ and $C_0(G)$ invariant.

Definition 2.1.4. We call a Borel measure μ on a locally compact group G **left-invariant** if

$$\mu(\lambda(s)f) = \mu(f), \text{ for all } s \in G, f \in C_c(G).$$

Proposition 2.1.5. *Let $\mu \neq 0$ be a left-invariant positive measure on a locally compact group G. Then $\mu(f) > 0$ for any non-negative function $f \in C_c(G)$ different from 0.*

Proof. Let $0 \neq f \in C_c(G)$ be a non-negative function which is annihilated by μ. There exists then an element u in the support of f and an neighbourhood U of u, such that $f(s) \geq \varepsilon > 0$ for all $s \in U$. Let us show that $\mu(h) = 0$ for all non-negative $h \in C_c(G)$. Indeed, for such a function h and for every t in a compact neighbourhood K of the support of h, consider the compact neighbourhood $U_t := tU$ of t. Then the translated function $f_t := \lambda(t)f$ is $\geq \varepsilon$ on U_t, because for $s = tu', u' \in U$, we have that $f_t(s) = f(t^{-1}tu') = f(u') \geq \varepsilon$. Since K is compact, there exists a finite number of points $t_1, \dots, t_N$ in K such that $\bigcup_{j=1}^N U_{t_j} \supset K$. Then the function $a := \sum_{j=1}^N f_{t_j}$ has the following property. Take any $t \in K$. Since $K \subset \bigcup_{j=1}^N U_{t_j}$, there exists $l \in \{1, \dots, N\}$ such that $t \in U_{t_l}$ and so

$$a(t) = \sum_{j=1}^{N} f_{t_j}(t) \geq f_{t_l}(t) \geq \varepsilon.$$

Therefore we have for our t that

$$h(t) \leq \|h\|_\infty \leq \frac{\|h\|_\infty}{\varepsilon}\varepsilon \leq \frac{\|h\|_\infty}{\varepsilon} a(t).$$

This means that $a - \frac{\varepsilon}{\|h\|_\infty} h \geq 0$ and so $\mu(a) \geq \frac{\varepsilon}{\|h\|_\infty}\mu(h) \geq 0$, since μ is positive. But

$$\mu(a) = \mu\left(\sum_{j=1}^{N} f_{t_j}\right) = \sum_{j=1}^{N} \mu(\lambda(t_j)f) = \sum_{j=1}^{N} \mu(f) = 0,$$

because μ is left-invariant. Hence $\mu(h) = 0$. This implies that $\mu(h) = 0$ for every $h \in C_c(G)$. ■

Theorem 2.1.6. *Let G be a locally compact group. There exists on G a unique* (*up to multiplication by a positive constant*) *left-invariant positive Borel measure* (*called* **Haar measure**) *which we denote by*

$$\mu(f) := \int_G f(g)dt, \quad f \in C_c(G).$$

Proof. We shall show only the uniqueness of the Haar measure; a complete proof can be found for instance in [13]. Take two functions $f, g \in C_c(G)$. Then it is easy to check that the function

$$f*g(t) := \int_G f(s)g(s^{-1}t)ds,$$

is continuous with compact support. Indeed, for $t, t' \in G$, we have that

$$\begin{aligned}|f*g(t) - f*g(t')| &= \Big|\int_G f(s)(g(s^{-1}t) - g(s^{-1}t'))ds\Big| \\ &\leq \int_G |f(s)||\check{g}(t^{-1}s) - \check{g}(t'^{-1}s)|ds \\ &\leq \|\lambda(t)\check{g} - \lambda(t')\check{g}\|_\infty \int_G |f(s)|ds.\end{aligned}$$

(Here we used the notation

$$\check{F}(s) := F(s^{-1}), \quad F \in \mathcal{F}(G, X).)$$

Since the function $\check{g}$ is continuous with compact support, it follows that it is uniformly continuous, hence $\lim_{t' \to t} \|\lambda(t)g - \lambda(t')g\|_\infty = 0$. Therefore $f * g$ is continuous. Furthermore, if $f * g(t) \neq 0$ then necessarily $g(s^{-1}t) \neq 0$ for at least one $s \in \mathrm{supp}(f)$. Hence $s^{-1}t \in \mathrm{supp}(g)$ and so

$$t \in \mathrm{supp}(f)\mathrm{supp}(g).$$

This shows that

$$\mathrm{supp}(f * g) \subset \mathrm{supp}(f)\mathrm{supp}(g),$$

since the product of the two compact subsets $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ is compact and hence closed in G. Let us take two left-invariant Borel measures on G, denoted by $d_1 g$ and $d_2 g$. We choose an element $h \in C_c(G)$ such that $\int_G h(g) d_2 g = 1$. Then for any $f \in C_c(G)$, we have that

$$\begin{aligned}\int_G f(s) d_1 s &= \int_G f(s) d_1 s \int_G h(g) d_2 g = \int_G \left(\int_G f(g^{-1}s) d_1 s \right) h(g) d_2 g \\ &= \int_G \left(\int_G f(g^{-1}s) h(g) d_2 g \right) d_1 s = \int_G f(g^{-1}) \left(\int_G h(sg) d_1 s \right) d_2 g\end{aligned}$$

by the left invariance of $d_1 s,\ d_2 g$ and Fubini's theorem. Hence

$$\int_G f(s) d_1 s = \int_G f(s^{-1}) \varphi_h^1(s) d_2 s,$$

where $\varphi_h^1(s) := \int_G h(gs) d_1 g,\ s \in G$. Choosing a function $q \in C_c(G)$ such that

$$\int_G q(g) d_1 g = 1,$$

we get in a similar way that

$$\int_G f(s) d_2 s = \int_G f(s^{-1}) \varphi_q^2(s) d_1 s,$$

where $\varphi_q^2(s) := \int_G q(gs) d_2 g,\ s \in G$. Combining the two identities above, this shows that:

$$\int_G f(s) d_1 s = \int_G f(s) \varphi_h^1(s^{-1}) \varphi_q^2(s) d_1 s,$$

for all $f \in C_c(G)$. Therefore

$$\varphi_h^1(s^{-1})\varphi_q^2(s) = 1$$

for all $h, q \in C_c(G)$ such that

$$\int_G h(g)d_2g = 1, \int_G q(g)d_1g = 1.$$

Fixing $q \geq 0$ with $\int_G q(g)d_1g = 1$ and letting $s = e$, we see that, for every $h \in C_c(G)$ with $\int_G h(g)d_2(g) \neq 0$,

$$\int_G \frac{h(g)}{\int_G h(s)d_2s} d_1g = \frac{1}{\int_G q(g)d_2g}.$$

Hence

$$\int_G h(g)d_2g = c\int_G h(g)d_1g, \text{where } c := \int_G q(t)d_2t > 0,$$

by Proposition 2.1.5. This shows that $d_2g = cd_1g$. ■

Definition 2.1.7. Let G be a locally compact group. For every $x \in G$, the linear functional $\varphi \mapsto \int_G \varphi(gx^{-1})dg$ is positive and left translation invariant. Hence it defines a Haar measure and so by Theorem 2.1.6, there exists a positive number denoted by $\Delta_G(x)$ such that

$$\int_G \varphi(gx^{-1})dg = \Delta_G(x)\int_G \varphi(g)dg, \ \varphi \in C_c(G).$$

We call the positive function $x \mapsto \Delta_G(x)$ on G the **modular function**.

Proposition 2.1.8. *Let G be a locally compact group. The modular function is a continuous homomorphism of G into the multiplicative group of the strictly positive real numbers.*

Proof. We have for any non-negative nonzero element $\varphi \in C_c(G)$, that

$$\Delta_G(x) = \frac{\int_G \varphi(gx^{-1})dg}{\int_G \varphi(g)dg}, \ x \in G,$$

and this relation tells us that Δ_G is a continuous function. Furthermore, for $x, y \in G, \varphi \in C_c(G)$, we have that

$$\begin{aligned}\Delta_G(xy)\int_G \varphi(g)dg &= \int_G \varphi(g(xy)^{-1})dg = \int_G \varphi((gy^{-1})x^{-1})dg \\ &= \Delta_G(y)\int_G \varphi(gx^{-1})dg = \Delta_G(y)\Delta_G(x)\int_G \varphi(g)dg.\end{aligned}$$ ■

Proposition 2.1.9. *Let G be a locally compact group. Then for any $f \in C_c(G)$ we have that*

$$\int_G f(g^{-1})\Delta_G(g^{-1})dg = \int_G f(g)dg.$$

Proof. The integral $f \mapsto \int_G f(g^{-1})\Delta_G(g^{-1})dg$ is left-invariant since for any $t \in G$

$$\begin{aligned}\int_G \lambda(t)f(g^{-1})\Delta_G(g^{-1})dg &= \int_G f(t^{-1}g^{-1})\Delta_G(g^{-1})dg\\ &= \int_G f((gt)^{-1})\Delta_G((gt)^{-1})\Delta_G(t)dg\\ &= \Delta_G(t^{-1})\int_G f(g^{-1})\Delta_G(g^{-1})\Delta_G(t)dg\\ &= \int_G f(g^{-1})\Delta_G(g^{-1})dg.\end{aligned}$$

Hence, by the uniqueness of Haar measure, there exists $c > 0$, such that

$$\int_G f(g^{-1})\Delta_G(g^{-1})dg = c\int_G f(g)dg, \ f \in C_c(G).$$

Let us show that $c = 1$. If we take $f \in C_c(G)$, $f \geq 0$, such that $f(g) = f(g^{-1})$ for all $g \in G$ (for instance taking $h \geq 0$ in $C_c(G)$ and putting $f(g) = h(g) + h(g^{-1})$), then

$$\begin{aligned}\int_G f(g^{-1})\Delta_G(g^{-1})dg &= c\int_G f(g)dg\\ &= c\int_G f(g^{-1})dg = c\int_G f(g^{-1})\Delta_G(g)\Delta_G(g^{-1})dg\\ &= c^2\int_G f(g)\Delta_G(g^{-1})dg = c^2\int_G f(g^{-1})\Delta_G(g^{-1})dg.\end{aligned}$$

Therefore $c^2 = 1$. ■

Example 2.1.10. 1. Let G be any group. We give it the discrete topology. This means that every subset of G is open and closed. The Haar measure is the counting measure:

$$\int_G f(g)dg = \sum_{g\in G} f(g), \ f \in C_c(G).$$

2. Let $G = Gl_n(\mathbb{R}), n \in \mathbb{N}^*$. Then G is an open subset of the vector space $M_n(\mathbb{R})$. The Haar measure is given by:

$$\int_G f(g)dg = \int_{M_n(\mathbb{R})} f(v)|\det(v)|^{-n}dv.$$

Indeed, for any $u \in G$, we have that

$$\begin{aligned}\int_G f(ug)dg &= \int_{M_n(\mathbb{R})} f(uv)|\det(v)|^{-n}dv \\ &= \int_{M_n(\mathbb{R})} |\det(u)|^n f(uv)|\det(uv)|^{-n}dv \\ &= \int_{M_n(\mathbb{R})} f(v)|\det(v)|^{-n}dv = \int_G f(g)dg, f \in C_c(G).\end{aligned}$$

3. Let $G = \exp(\mathfrak{g})$ be a connected Lie group for which the exponential mapping is a diffeomorphism. We can then transfer the multiplication in the group G to the vector space $\mathfrak{g}$ and we obtain what we can call a **vector group**, i.e. a finite-dimensional real vector space V space endowed with an analytic group multiplication $\cdot$ such that

$$(sX) \cdot (tX) = (s+t)X, \forall s, t \in \mathbb{R}, X \in V.$$

In particular 0 is the identity element in V and for every $X \in V$ its inverse X^{-1} is the opposite vector $-X$.

Furthermore, we can identify the Lie algebra $\mathfrak{g}$ of V with the vector space V and then the exponential mapping is just the identity mapping from V to V, since for any $X \in V$ the curve $t \mapsto tX$ is a homomorphism from the group $(\mathbb{R}, +)$ to the group $(V, \cdot)$.

Of course in our case we can take $V = \mathfrak{g}$ and $X \cdot Y = \log(\exp X \cdot \exp Y), X, Y \in \mathfrak{g}$.

Let us denote as before by $L_X : V \to V$ the left multiplication in V. Then for any $X, Y \in V$ we have that $L_X \circ L_Y = L_{X \cdot Y}$ and so for the differentials dL_X we have that

$$dL_{X\cdot Y}(U) = dL_X(L_Y(U)) \circ dL_Y(U), U \in V. \tag{2.1.1}$$

Let for $X \in V$

$$j(X) := |\det(dL_X(0))|^{-1}$$

be the inverse of the absolute value of the determinant of the differential dL_X in 0.

The Haar measure on the group $(V, \cdot)$ is then given by the integral

$$\int_V f(X)j(X)dX, \ f \in C_c(V), \tag{2.1.2}$$

dX denoting Lebesgue measure on V.

Indeed for $f \in C_c(V)$, $X \in V$:

$$\int_V f(X \cdot Y)j(Y)du = \int_V f(X \cdot Y)j(X^{-1} \cdot (X \cdot Y))dY$$

$$= \int_V f(Y)j(X^{-1} \cdot Y)dY\,|\det(dL_{X^{-1}}(Y))|dY$$

(change of variables $Y \to L_{X^{-1}}Y$)

$$= \int_V f(Y)|det(dL_{X^{-1}Y}((0))|^{-1}|\det(dL_{X^{-1}}(Y))|dY$$

$$= \int_V f(Y)|det(dL_{X^{-1}}(Y))|^{-1}\,|\det(dL_Y(0))|^{-1}\,|\det(dL_{X^{-1}}(Y))|dY$$

(by(2.1.1))

$$= \int_V f(Y)j(Y)dY.$$

Remark 2.1.11. For a Lie group G we have that

$$\Delta_G(g) = |\det \mathrm{Ad}(g)|^{-1} \ (g \in G).$$

Let us show this identity for vector groups $(V, \cdot)$.

First we remark that the function $j : V \to \mathbb{R}_+^*$ is conjugation invariant. This is so because the integral

$$f \mapsto \int_V f(U)j(X \cdot U \cdot X^{-1})dU$$

$$= \int_V f(U)j(Ad(X)U)dU = |\det(Ad(X))|^{-1} \int_V f(Ad(X^{-1}(U)))j(U)dU$$

is also left-invariant. Hence there exists for $X \in V$ a positive number $c(X)$ such that

$$c(X) \int_V f(U)j(U)dU = \int_G f(X)j(X \cdot U \cdot X^{-1})dX, \forall f \in C_c(V).$$

Whence $c(X)j(U) = j(X \cdot U \cdot X^{-1})$ for all $U \in V$. In particular for $U = 0$ we have that $j(0) = 1$ and so we get

$$c(X) = c(X)j(0) = j(X \cdot 0 \cdot X^{-1}) = j(0).$$

This means that $c(X) = 1$ for all $X \in V$. For $f \in C_c(V)$ we then have that

$$\begin{aligned}\Delta_G(X)\int_V f(U)j(U)dU &= \int_V f(U\cdot X^{-1})j(U)du\\ &= \int_V f(X\cdot U\cdot X^{-1})j(U)du\\ &= \int_V f(Ad(X)(U))j(U)du\\ &= \int_V f(Ad(X)(U))j(Ad(X)U)du\\ &= |\det(Ad(X))|^{-1}\int_V f(U)j(U)dU,\end{aligned}$$

i.e. $\Delta_G(X) = |\det(Ad(X))|^{-1}$.

2.2 The Group Algebra

Definition 2.2.1. Let G be a locally compact group. For $1 \le p < \infty$, we denote by $L^p(G)$ the space of all measurable (with respect to Haar measure) complex-valued functions $f : G \to \mathbb{C}$ such that

$$\|f\|_p^p := \int_G |f(g)|^p dg < \infty.$$

Let $f, g \in L^1(G)$. Since

$$\begin{aligned}\int_G\left(\int_G |f(t)g(t^{-1}s)|ds\right)dt &= \int_{G\times G}|f(t)g(s)|dsdt\\ &= \int_G |f(t)|dt\int_G |g(s)|dt < \infty,\end{aligned}$$

Fubini's theorem implies that

$$\int_G\left(\int_G |f(t)g(t^{-1}s)|dt\right)ds = \int_G\left(\int_G |f(t)g(t^{-1}s)|ds\right)dt < \infty$$

and so again by Fubini's theorem, there exists a measurable subset $F(f, g)$ in G whose complement has Haar measure 0, such that the integral

$$\int_G f(t)g(t^{-1}s)dt$$

exists for every $s \in F(f, g)$ and the function $f*g$ is defined by this integral, namely

$$f*g(s) = \begin{cases} 0, & \text{if } s \text{ is not in } F(f,g); \\ \int_G f(t)g(t^{-1}s)ds, & \text{if } s \text{ is in } F(f,g) \end{cases} .$$

It follows from the above that

$$\|f*g\|_1 \leq \|f\|_1\|g\|_1.$$

It is easy to check that for $f, g, h \in C_c(G)$, we have that

$$f*(g*h) = (f*g)*h.$$

This implies that $(L^1(G), *)$ is an associative Banach algebra which has also an isometric involution $f \mapsto f^*$, namely

$$f^*(s) = \Delta_G(s^{-1})\overline{f(s^{-1})}, s \in G.$$

The mapping $f \mapsto f^*$ has the following properties: it is anti-linear, $(f^*)^* = f$ and $(f*g)^* = g^* * f^*$ for every $f, g \in L^1(G)$.

Definition 2.2.2. Let $s \in G$ and $f \in L^1(G)$. Then we define the right translation $\rho(s)$ on f by

$$\rho(s)f(t) := \Delta_G(s)f(ts),\ t \in G.$$

It follows from the definition that

$$\|\rho(s)f\|_1 = \|f\|_1.$$

Proposition 2.2.3. *Let $f, g \in L^1(G),\ s \in G$. We then have that*

$$f*(\lambda(s)g) = (\rho(s^{-1})f)*g,\ \lambda(s)(f*g) = (\lambda(s)f)*g.$$

Proof. Indeed, for $t \in G$, we have that

$$\begin{aligned} f*(\lambda(s)g)(t) &= \int_G f(u)\lambda(s)g(u^{-1}t)du = \int_G f(u)g(s^{-1}u^{-1}t)du \\ &= \Delta_G(s^{-1})\int_G f(us^{-1})g(u^{-1}t)du = (\rho(s^{-1})f)*g(t). \\ (\lambda(s)f)*g(t) &= \int_G f(s^{-1}u)g(u^{-1}t)du \\ &= \int_G f(u)g(u^{-1}s^{-1}t)du = \lambda(s)(f*g)(t). \end{aligned}$$ ∎

We denote by $\mathcal{V}$ the set of all compact neighbourhoods of the neutral element of G.

Proposition 2.2.4. *Let $f \in L^p(G), 1 \le p < \infty$. Then* $\lim_{x\to y} \lambda(x)f = \lambda(y)f$.

Proof. Let $\varepsilon > 0$ and let $f \in L^p(G)$, $1 \le p < \infty$. We choose $h \in C_c(G)$ such that $\|f - h\|_p < \varepsilon$. Since h is uniformly continuous, there exists a neighbourhood $V \in \mathcal{V}$ such that

$$\|\lambda(y)h - h\|_\infty < \frac{\varepsilon^p}{|V\operatorname{supp}(h)|}$$

for all $y \in V$. Here

$$|V\operatorname{supp}(h)| = \int_{V\operatorname{supp}(h)} dg.$$

Hence for $v \in V$

$$\begin{aligned}\|\lambda(yv)h - \lambda(y)h\|_p^p &= \int_G |\lambda(yv)h(t) - \lambda(y)h(t)|^p dt \\ &\le \|\lambda(yv)h - \lambda(y)h\|_\infty \int_{yV\operatorname{supp}(h)} dt \\ &= \|\lambda(v)h - h\|_\infty |V\operatorname{supp}(h)| \le \varepsilon^p.\end{aligned}$$

Therefore

$$\begin{aligned}\|\lambda(yv)f - \lambda(y)f\|_p &\le \|\lambda(yv)h - \lambda(y)h\|_p + \|\lambda(yv)f - \lambda(yv)h\|_p \\ &\quad + \|\lambda(y)f - \lambda(y)h\|_p \le 3\varepsilon, \quad v \in V.\end{aligned}$$

■

Corollary 2.2.5. *For every $f, g \in L^1(G)$, we have that*

$$f * g = \int_G f(t)\lambda(t)g\,dt$$

Proof. Indeed, if $f, g \in C_c(G)$ then the integral $\int_G f(t)\lambda(t)g\,dt$ converges in $C_c(G)$ and so if we evaluate this integral at a point $s \in G$, it follows that

$$\int_G f(t)\lambda(t)g\,dt(s) = \int_G f(t)\lambda(t)g(s)dt = \int_G f(t)g(t^{-1}s)dt = f * g(s).$$

Hence $\int_G f(t)\lambda(t)g\,dt = f * g$. Since $C_c(G)$ is dense in $L^1(G)$ and since translation is continuous in $L^1(G)$, it follows that $\int_G f(t)\lambda(t)g\,dt = f * g$ for every $f, g \in L^1(G)$. ■

Corollary 2.2.6. *Let $f, g \in L^1(G)$. Then $f * g$ can be approximated in $L^1(G)$ by sums of the form $\sum_{j=1}^m c_j\lambda(t_j)g$, where the constants c_j are in $\mathbb{C}$ and the t_j are certain elements in G.*

Definition 2.2.7. Let A be a Banach algebra. We say that A possesses a (left) **bounded approximate identity**, if there exists a net $(e_i)_{i\in I}$ in A such that $\lim_{i\to\infty} e_i a = a$ for every $a \in A$ and such that $\sup_{i\in I} \|e_i\| < \infty$.

Proposition 2.2.8. *Let G be a locally compact group. Then $L^1(G)$ has a bounded approximate identity.*

Proof. As before, let $\mathcal{V}$ be the set of all compact neighbourhoods of the neutral element. We order $\mathcal{V}$ by inclusion, i.e. $U \le V \Leftrightarrow V \subset U$ for $U, V \in \mathcal{V}$. Choose for $V \in \mathcal{V}$ a non-negative function $f_V \in C_c(G)$ with support in V, such that $\int_G f_V(t)dt = 1$. We obtain in this fashion a net $(f_V)_{V\in\mathcal{V}}$ which is bounded by 1. Now let $f \in L^1(G)$. Then for $V \in \mathcal{V}$ we have that

$$\begin{aligned}
\|f_V * f - f\|_1 &= \int_G \Big| \int_G f_V(s) f(s^{-1}t)ds - f(t)\Big| dt \\
&= \int_G \Big| \int_G f_V(s) f(s^{-1}t)ds - \int_G f_V(s) ds f(t)\Big| dt \\
&= \int_G \Big| \int_G f_V(s)(f(s^{-1}t) - f(t))ds\Big| dt \\
&\le \int_G \int_G f_V(s)|f(s^{-1}t) - f(t)|dtds \\
&= \int_G f_V(s)\|\lambda(s)f - f\|_1 ds
\end{aligned}$$

by Fubini's theorem. Choose now a neighbourhood V_0 of e such that $\|\lambda(s) f - f\|_1 \le \varepsilon$ for any s in V_0. Then, for $V \subset V_0$, $V \in \mathcal{V}$, we have that

$$\|f_V * f - f\|_1 \le \int_V f_V(s)\|\lambda(s)f - f\|_1 ds \le \varepsilon \int_V f_V(s)ds = \varepsilon. \qquad \blacksquare$$

2.3 Representations

Definition 2.3.1. Let G be a locally compact group. A **representation** of G on a Banach space X is a strongly continuous homomorphism (π, X) of the group G into the group $Gl(X)$ of the bounded invertible linear operators on X. Strongly continuous means that the mappings

$$G \to X, \quad s \mapsto \pi(s)x \ (\forall x \in X)$$

are continuous.

The representation (π, X) is called **bounded**, if $C_\pi := \sup_{x\in G} \|\pi(x)\|_{\mathrm{op}} < \infty$.

The representation $(\pi, \mathcal{H})$ of G is called **unitary**, if $\mathcal{H}$ is a Hilbert space and for if all $g \in G$, $\pi(g)$ is a unitary operator, i.e. $\pi(g)^{-1} = \pi(g)^*$. Here $\pi(g)^*$ denotes the adjoint of $\pi(g)$.

Similarly, let A be an involutive Banach algebra. A **representation** π of A on a Banach space X is a bounded algebra homomorphism of A into the algebra $B(X)$ of all bounded linear operators of X.

The representation (π, X) is called **non-degenerate** if the subspace generated by the subset $\{\pi(f)x; f \in L^1(G), x \in X\}$ is dense in X.

The representation $(\pi, \mathcal{H})$ is called **unitary**, if $\mathcal{H}$ is a Hilbert space and if $\pi(a)^* = \pi(a^*)$ for all $a \in A$.

Remark 2.3.2. Let (π, X) be a bounded representation of a locally compact group G. For $f \in C_c(G), x \in X$, the mapping $g \to f(g)\pi(g)x$ is continuous with compact support, and so the integral $\int_G f(g)\pi(g)xdg$ converges in X. Furthermore, the mapping $x \to \int_G f(g)\pi(g)xdg$ is obviously linear and

$$\begin{aligned}\left\| \int_G f(g)\pi(g)xdg \right\|_X &\leq \int_G |f(g)| \|\pi(g)x\|_X ds \\ &\leq C_\pi \|x\|_X \int_G |f(s)|ds = C_\pi \|x\|_X \|f\|_1.\end{aligned}$$

Hence the operator $\pi(f) := \int_G f(g)\pi(g)dg$ is bounded by $C_\pi \|f\|_1$.

Since $C_c(G)$ is dense in $L^1(G)$, it follows that the integral $\pi(f) := \int_G f(g)\pi(g)dg$ converges in the strong topology for every f in $L^1(G)$ and

$$\|\pi(f)\|_{\text{op}} \leq C_\pi \|f\|_1, \forall f \in L^1(G).$$

Hence, for $f \in L^1(G), s \in G$, we have that

$$\begin{aligned}\pi(\lambda(s)f) &= \int_G \lambda(s)f(t)\pi(t)dt = \int_G f(s^{-1}t)\pi(t)dt \\ &= \int_G f(t)\pi(st)dt = \int_G f(t)\pi(s) \circ \pi(t)dt = \pi(s) \circ \pi(f).\end{aligned}$$

Furthermore, for $f, g \in L^1(G)$, we have

$$\begin{aligned}\pi(f * g) &= \pi\left(\int_G f(s)\lambda(s)gds\right) = \int_G f(s)\pi(\lambda(s)g)ds \\ &= \int_G f(s)\pi(s) \circ \pi(g)ds = \pi(f) \circ \pi(g).\end{aligned}$$

Theorem 2.3.3. *Let G be a locally compact group.*

1. *Let (π, X) be a bounded representation of G. There exists a unique bounded and non-degenerate representation $(\overline{\pi}, X)$ of the algebra $L^1(G)$ such that $\overline{\pi}(\lambda(s)f) = \pi(s) \circ \overline{\pi}(f)$ for all $f \in L^1(G),\ s \in G$.*
2. *Let $(\overline{\pi}, X)$ be a non-degenerate representation of $L^1(G)$ and let C be the bound of $\overline{\pi}$. Then there exists a unique bounded representation (π, X) of G on X such that $\overline{\pi}(f) = \int_G f(s)\pi(s)ds$ for all $f \in L^1(G)$ and which is furthermore bounded by C.*

Proof. 1. The existence of the representation π of $L^1(G)$ has been shown in Remark 2.3.2. Suppose that we have a second non-degenerate representation (π', X) of $L^1(G)$, such that $\pi(s) \circ \pi'(f) = \pi'(\lambda(s)f)$ for all s and f. Then for a bounded approximate identity $(e_i)_i$ of $L^1(G)$, we have

$$\pi'(f * e_i) = \pi'\left(\int_G f(s)\lambda(s)e_i\,ds\right) = \int_G f(s)\pi'(\lambda(s)e_i)ds$$
$$= \int_G f(s)\pi(s) \circ \pi'(e_i)ds = \int_G f(s)\pi(s)ds \circ \pi'(e_i) = \overline{\pi}(f) \circ \pi'(e_i).$$

Hence, since π' is non-degenerate, $\lim_i \pi'(e_i) = \mathbb{I}_X$ strongly and so

$$\pi'(f) = \lim_i \pi'(f * e_i) = \lim_i \pi(f) \circ \pi'(e_i) = \pi(f) \circ \lim_i \pi'(e_i) = \pi(f).$$

2. Let $(\overline{\pi}, X)$ be a non-degenerate representation of the group algebra $L^1(G)$. Given $s \in G$ and a finite family $(\xi_i)_i$ of vectors in X resp. of elements $(f_i)_i$ of $L^1(G)$ such that

$$\sum_i \overline{\pi}(f_i)\xi_i = 0,$$

let us show that also

$$\sum_i \overline{\pi}(\lambda(s)f_i)\xi_i = 0.$$

Indeed, it follows from the above that for every $f \in L^1(G)$

$$0 = \overline{\pi}(f)\left(\sum_i \overline{\pi}(f_i)\xi_i\right) = \sum_i \overline{\pi}(f * f_i)\xi_i.$$

Hence for our approximate identity $(e_j)_j$, we obtain the relation

$$0 = \sum_i \overline{\pi}((\lambda(s)e_j) * f_i)\xi_i = \sum_i \overline{\pi}(\lambda(s)(e_j * f_i))\xi_i \overset{j\to\infty}{\to} \sum_i \overline{\pi}(\lambda(s)(f_i))\xi_i.$$

This identity allows us to define $\pi(s)$, $s \in G$, on the vector space $X^\infty := \pi(C_c(G))X$ by the formula

$$\pi(s)\Big(\sum_i \overline{\pi}(f_i)\xi_i\Big) := \sum_i \overline{\pi}((\lambda(s)f_i)\xi_i).$$

Let us show that $\|\pi(s)\xi\|_X \le C\|\xi\|_X$ for every $\xi \in X^\infty$. Let $\eta \in X^\infty$ and let $\varepsilon > 0$. We take an element f in our bounded approximate identity defined in Proposition 2.2.8 such that

$$\|\overline{\pi}(f)(\pi(s)\eta) - \pi(s)\eta\|_X < \varepsilon.$$

Then for $s \in G$, by Proposition 2.2.3

$$\begin{aligned}\|\pi(s)(\eta)\|_X &\le \|\overline{\pi}(f)(\pi(s)\eta)\|_X + \|\overline{\pi}(f)(\pi(s)\eta) - \pi(s)\eta\|_X\\ &= \|\overline{\pi}(\rho(s^{-1})f)(\eta)\|_X + \|\overline{\pi}(f)(\pi(s)\eta) - \pi(s)\eta\|_X\\ &\le C\|\rho(s^{-1})f\|_1\|\eta\|_X + \varepsilon \le C\|\eta\|_X + \varepsilon.\end{aligned}$$

Hence $\|\pi(s)\eta\|_X \le C\|\eta\|_X$. We can now extend the linear operator $\pi(s)$ to the whole space X, since $\overline{\pi}$ is non-degenerate. Since $\pi(s)(\overline{\pi}(f)\xi) = \overline{\pi}(\lambda(s)f)\xi$ for every $\xi \in X$, $s \in G$, the mappings $s \mapsto \pi(s)\xi$ ($\xi \in X$) are all continuous and we obtain in this way a representation (π, X) of G on the Banach space X which is bounded by the constant C. Furthermore, for $f, f' \in C_c(G)$, $\eta \in X$, we have that

$$\begin{aligned}\overline{\pi}(f)(\overline{\pi}(f')\eta) &= \overline{\pi}(f * f')\eta = \overline{\pi}\Big(\int_G f(u)\lambda(u)f'du\Big)\eta\\ &= \int_G f(u)\overline{\pi}(\lambda(u)f')\eta du = \int_G f(u)\pi(u)\pi(f')\eta du.\end{aligned}$$

This shows that the representation π of G is unique and that the representation $\overline{\pi}$ of $L^1(G)$ is the integrated version of the representation π of the group G. ■

Remark 2.3.4. For simplicity of notation, we shall always write in the following π instead of $\overline{\pi}$ for the integrated representation $\int_G f(x)\pi(x)dx$ of a bounded representation (π, X) of G.

Definition 2.3.5. Let (π, X) be a representation of a locally compact group G. Let $Y \subset X$ be a subspace of X. We say that Y is **invariant** or G**-invariant**, if

$$\pi(s)\xi \in Y, \text{ for all } s \in G \text{ and } \xi \in Y.$$

We say that a representation (π, X) of the group G on the Banach space is **irreducible**, if $\{0\}$ and X are the only closed invariant subspaces of X.

An element ξ of X is said to be **cyclic**, if the subspace X_ξ spanned by the vectors $\pi(g)\xi, g \in G$ is dense in X.

Corollary 2.3.6. *1. Let (π, X) be a representation of a locally compact group G. Then a closed subspace Y of X is invariant, if and only if Y is $L^1(G)$-invariant, i.e. if and only if for every $f \in L^1(G)$, $\xi \in Y$ we have that $\pi(f)\xi \in Y$.*
2. A representation (π, X) of G on a Banach space X is irreducible, if and only if it is irreducible for the representation π of the algebra $L^1(G)$, i.e. if and only if the only closed $L^1(G)$-invariant subspaces of X are the trivial ones, $\{0\}$ and X.

Proof. 1. Let Y be a closed G-invariant subspace of X and let $\varphi \in X'$ be a bounded linear functional which annihilates Y. Then, for every $\xi \in Y$, $f \in L^1(G)$

$$\langle \varphi, \pi(f)\xi\rangle = \left\langle \varphi, \int_G f(s)\pi(s)\xi ds\right\rangle = \int_G f(s)\langle \varphi, \pi(s)\xi\rangle ds = \int_G f(s)\cdot 0 ds = 0.$$

Hence by the Hahn–Banach theorem, $\pi(f)\xi \in Y$. Suppose that Y is $L^1(G)$-invariant. Then for every $\eta \in Y, f \in L^1(G)$ and $s \in G$, we have that $\pi(f)(\pi(s)\eta) = \pi(\rho(s^{-1})f)\eta \in Y$. Hence $\pi(s)\eta = \lim_{V\to\{e\}} \pi(f_V)(\pi(s)\eta) \in Y$, since Y is closed.
2. This is an immediate consequence of 1. ∎

Theorem 2.3.7 (Schur's Lemma). *Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. The following conditions are equivalent:*

1. *π is irreducible.*
2. *Every nonzero vector in $\mathcal{H}$ is cyclic.*
3. *Every bounded linear endomorphism of $\mathcal{H}$, which commutes with all the operators $\pi(g), g \in G$, is a scalar multiple of the identity.*

Proof. 1 $\Longleftrightarrow$ 2. Suppose that π is irreducible. Let $\xi \in \mathcal{H}$ be nonzero. Then the subspace $\mathcal{H}_\xi$ spanned by the vectors $\pi(g)\xi, g \in G$, is G-invariant and hence dense in $\mathcal{H}$, since π is irreducible. Hence ξ is a cyclic vector. If every nonzero vector in $\mathcal{H}$ is cyclic, then any nonzero G-invariant closed subspace F of $\mathcal{H}$ equals $\mathcal{H}$, since it contains cyclic vectors.

1 $\Longleftrightarrow$ 3 Suppose that π is irreducible. Let u be a bounded endomorphism which commutes with the operators $\pi(g), g \in G$. Then its adjoint u^* also commutes with them and so does $v := \frac{1}{2}(u+u^*)$ and $w := \frac{1}{2i}(u-u^*)$. Hence if we show that every self-adjoint linear operator commuting with all the $\pi(g)$'s is a scalar multiple of the identity, then so is our $u = v + iw$. We can now assume that u is self-adjoint. By the spectral theorem, we have the spectral decomposition of u:

$$u = \int_{\text{spectrum}(u)} \lambda dP_\lambda$$

and we know that all the projections P_λ commute with the operators $\pi(g), g \in G$. Hence the kernels and the images of the P_λ' s are closed G-invariant subspaces, hence either trivial or the whole space and so P_λ is either the identity operator or the zero operator. It then follows that $u = \lambda_0 Id$ for some real number λ_0. If every bounded endomorphism of $\mathcal{H}$, which commutes with the $\pi(g)$'s, is a multiple of the identity, then, given a closed G-invariant subspace F of $\mathcal{H}$, the orthogonal projection P onto F commutes with the operators $\pi(g), g \in G$, and so $P = \lambda Id$ for some $\lambda \in \mathbb{R}$. But then $\lambda = 1$ or 0, since $P^2 = P$ and so $F = \mathcal{H}$ or $F = \{0\}$. Hence π is irreducible. ∎

Definition 2.3.8. Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be two representations of a locally compact group G. We say that a bounded linear operator $U : \mathcal{H} \to \mathcal{H}'$ is an intertwining operator for π and π' or intertwines π and π', if $U \circ \pi(g) = \pi'(g) \circ U$ for all $g \in G$. We denote by $BL_G(\mathcal{H}, \mathcal{H}')$ the set of all intertwining operators for π and π'.

Definition 2.3.9. Two unitary representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are said to be equivalent if there exists in $BL_G(\mathcal{H}, \mathcal{H}')$ an invertible isometry.

Proposition 2.3.10. *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be two irreducible unitary representations of a locally compact group G. Then $BL_G(\mathcal{H}, \mathcal{H}')$ is one-dimensional if π and π' are equivalent and equal to $\{0\}$ if they are not.*

Proof. Let $a,\ b \in BL_G(\mathcal{H}, \mathcal{H}')$. Then $b^* \circ a$ commutes with π and so $b^* \circ a$ is a multiple of the identity by Schur's lemma (Theorem 2.3.7). In particular $a^* \circ a = c Id_{\mathcal{H}}$ for some $c > 0$ and similarly $a \circ a^* = d Id_{\mathcal{H}'}$ for some $d > 0$. This shows that a is a scalar multiple of a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$, and that every other $b \in BL_G(\mathcal{H}, \mathcal{H}')$ is then a scalar multiple of a if $a \neq 0$. ∎

Definition 2.3.11. Let G be a locally compact group. We define the **dual space** or **spectrum** $\hat{G}$ of G to be the space of equivalence classes of irreducible unitary representations of G.

Similarly for any involutive Banach algebra A we define the dual space or spectrum $\hat{A}$ to be the space of equivalence classes of irreducible unitary representations of A.

Definition 2.3.12. Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. Let $\xi \in \mathcal{H}$. The function

$$c^\pi_\xi(g) := \langle \pi(g)\xi, \xi \rangle,\ g \in G,$$

is called a **coefficient** of the representation π. We denote by $E(\pi)$ the set of the coefficients $\{c^\pi_\xi; \xi \in \mathcal{H}, \|\xi\| = 1\}$.

Proposition 2.3.13. *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be two irreducible unitary representations of a locally compact group G. Then the two representations π and π' are equivalent if and only if $E(\pi) \cap E(\pi') \neq \emptyset$.*

Proof. If π and π' are equivalent, let $U : \mathcal{H} \to \mathcal{H}'$ be a unitary intertwining operator. Choose $\xi \in \mathcal{H}$ of norm 1. Let $\xi' := U(\xi)$. Then

$$\begin{aligned} c^{\pi'}_{\xi'}(g) &= \langle \pi'(g)\xi', \xi'\rangle = \langle \pi'(g)U(\xi), U(\xi)\rangle \\ &= \langle U^* \circ \pi'(g)\circ U(\xi), \xi\rangle = \langle \pi(g)\xi, \xi\rangle = c^{\pi}_{\xi}(g),\ g \in G. \end{aligned} \tag{2.3.1}$$

If now $c^{\pi}_{\xi} = c^{\pi'}_{\xi'}$, then we have that

$$\begin{aligned} \|\pi(f)\xi\|^2_{\mathcal{H}} &= \langle \pi(f)\xi, \pi(f)\xi\rangle = \langle \pi(f)^*\circ\pi(f)\xi, \xi\rangle = \langle \pi(f^* * f)\xi, \xi\rangle \\ &= \int_G \int_G f^*(u) f(u^{-1}v)\langle \pi(v)\xi, \xi\rangle du dv \\ &= \int_G \int_G f^*(u) f(u^{-1}v) c^{\pi}_{\xi}(v) dv du \\ &= \int_G \int_G f^*(u) f(u^{-1}v) c^{\pi'}_{\xi'}(v) dv du = \|\pi'(f)\xi'\|^2_{\mathcal{H}'},\ f \in L^1(G). \end{aligned}$$

We can now obtain a linear operator $U : \pi(L^1(G)\xi) \to \mathcal{H}'$ by first defining $U(\pi(f))\xi := \pi'(f)\xi'$ for $f \in L^1(G)$, then using the preceding identity, we can extend this linear mapping to an isometry from $\mathcal{H}$ into $\mathcal{H}'$, since ξ is a cyclic vector. This linear mapping intertwines obviously the representations π and π' and therefore has dense image (as π' is irreducible), i.e. is unitary. ∎

Definition 2.3.14. Let A be an algebra. A two-sided ideal K of A is called **prime** if for any two-sided ideals I, J of A, for which $IJ \subset K$, we have that either $I \subset K$ or $J \subset K$.

Proposition 2.3.15. *Let A be a Banach algebra and (π, V) be a bounded topologically irreducible representation of A on a Banach space V. Then the kernel $K := \ker\pi$ of π is a closed prime ideal of A.*

Proof. Let I, J be two two-sided ideals of A such that $IJ \subset K$. Suppose that neither I nor J is contained in K. Then there exists a vector v in V, such that $W := \pi(J)v \neq \{0\}$. Since W is an A-invariant subspace of V (as J is a two-sided ideal) it follows that W is dense in V. Hence $\pi(I)W \neq \{0\}$, because I is also not in the kernel of π. But then $\pi(IJ)v = \pi(I)W \neq \{0\}$ and so IJ is not contained in the kernel of π. ∎

Definition 2.3.16. Let (π, X) be a bounded representation of an algebra A on a Banach space X. We denote by F_π the two-sided ideal of A consisting of all the elements $a \in A$, for which $\pi(a)$ is of finite rank. Let $X_{fin} = \sum_{a\in F_\pi} \mathrm{Im}(\pi(a))$. The elements ξ of X_{fin} are then of the form

$$\xi = \sum_{j=1}^{m} \pi(a_j)\xi_j,\ a_1, \dots, a_m \in F_\pi, \xi_1, \dots, \xi_m \in X.$$

Theorem 2.3.17. *Let (π, X_π) be an irreducible bounded representation of an algebra A on Banach space X. Suppose that the ideal F_π is different from $\{0\}$. Then the sub-module (π, X_{fin}) of A is simple.*

Proof. We must show that every element $\xi \in X_{fin}, \xi \neq 0$ is cyclic. Let $\eta = \sum_j \pi(a_j)\eta_j \in X_{fin}$. Since $\pi(a_j)X = \pi(a_j)\pi(A)\xi$ as $\pi(A)\xi$ is dense in $\mathcal{H}$ (π being irreducible) and as the image of $\pi(a_j)$ is finite-dimensional for every j, it follows that $\pi(a_j)\eta_j = \pi(a_j)\pi(b_j)\xi$ for some $b_j \in A$ and for every j . Hence $\eta = (\sum_j \pi(a_j b_j))\xi \in \pi(A)\xi$ and ξ is therefore a cyclic vector. ∎

Definition 2.3.18. We say that an involutive Banach algebra $(A, \|\cdot\|_A)$ is a C^*-algebra, if

$$\|u^* u\|_A = \|u\|_A^2, \ u \in A.$$

We can associate to every locally compact group its C^*-algebra $C^*(G)$ which is defined to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_{C^*} := \sup_{\pi \in \hat{G}} \|\pi(f)\|_{\text{op}}, \ f \in L^1(G).$$

We now define the Fourier transform on a C^*-algebra A in the following way.

We need first to define the algebra of bounded operator fields defined over the spectrum $\hat{A}$ of A. Choose for every $\gamma \in \hat{A}$ a representative $(\pi_\gamma, \mathcal{H}_\gamma)$ of the equivalence class γ and consider the algebra $l^\infty(\hat{A})$ of all bounded operator fields

$$(\phi(\gamma) \in B(\mathcal{H}_\gamma)_{\gamma \in \hat{A}}), \|\phi\| := \sup_{\gamma \in \hat{A}} \|\phi(\gamma)\|_{op} < \infty.$$

The Fourier transform $\mathcal{F} : A \to l^\infty(\hat{A})$ is now defined by

$$\mathcal{F}(a)(\gamma) = \hat{a}(\gamma) := \pi_\gamma(a), a \in A.$$

We obtain in this way for every $a \in A$ a field of operators $(\hat{a}(\gamma))_{\gamma \in \hat{A}}$. This field is bounded: $\sup_{\gamma \in \hat{A}} \|\hat{a}(\gamma)\|_{\text{op}} = \|a\|_A$.

Proposition 2.3.19. *Let A be a commutative C^*-algebra with unit. For every $a = a^* \in A$, the spectrum $\sigma(a)$ is contained in $\mathbb{R}$. In particular for every character $\chi : A \to \mathbb{C}$, we have that $\chi(a^*) = \overline{\chi(a)}, \ a \in A$.*

Proof. Since for every $t \in \mathbb{R}$ we have that

$$1 = \|1_A\|_A = \|e^{-ita} e^{ita}\|_A = \|e^{ita}\|_A^2,$$

we see that $\|e^{ita}\|_A = 1$ for all $t \in \mathbb{R}$. Let χ be a character of A. Then the relation

$$|e^{it\chi(a)}| = |\chi(e^{ita})| \leq \|e^{ita}\|_A = 1$$

tells us that $\chi(a)$ must be a real number. Hence for every $a \in A$, $\chi(a + a^*)$ is in $\mathbb{R}$ and also $\chi(i(a - a^*))$ is in $\mathbb{R}$. This tells us that $\chi(a^*) = \overline{\chi(a)}$. We know from the general theory of Banach algebras that the spectrum of an element $a \in A$ is also the set of the values of all the characters of A on a. Hence $\sigma(a) \subset \mathbb{R}$. ■

Proposition 2.3.20. *Let A be an abelian C^*-algebra.*

1. *A is isomorphic as C^*-algebra with the algebra $C_0(\hat{A})$ of the algebra of continuous functions vanishing at infinity defined on the dual space $\hat{A}$ of A.*
2. *Let $A = C_0(X)$ for some locally compact space X. Then $\hat{A}$ is homeomorphic with X, the homeomorphism being given by $X \ni x \mapsto \delta_x$, where δ_x denotes point evaluation at x.*
3. *Let I be a closed ideal of A. Let S be the hull of I in $\hat{A}$, i.e. $S = \{\chi \in \hat{A}; \chi(I) = \{0\}\}$. Then $I = \{a \in A;\ \hat{a} = 0 \text{ on } S\}$.*

Proof. 1. The dual space $\hat{A}$ of A consists of the unitary characters $\chi : A \to \mathbb{C}$ by Schur's lemma, i.e. $\hat{A}$ is the set of all characters of A by Proposition 2.3.19. Hence $\hat{A}$ is contained in the unit ball of the dual space A' of the bounded linear functionals on A. We equip $\hat{A}$ with the weak-star topology $\sigma(A', A)$. If A has a unit, then $\hat{A}$ is a compact topological space, if not $\hat{A} \cup \{0\}$ is a compact subset of A'. Hence $\hat{A}$ is locally compact. The mapping $a \to \hat{a}; \hat{a}(\chi) = \langle \chi, a \rangle,\ \chi \in \hat{A}$, is then a homomorphism of A into $C_0(\hat{A})$. The image of A in $C_0(\hat{A})$ is an involutive subalgebra which separates the points of $\hat{A}$. Hence this subalgebra is dense in $C_0(\hat{A})$ by the Stone–Weierstrass theorem. Let us show that $\|\hat{a}\|_\infty = \|a\|_A$ for every $a \in A$. For $a = a^* \in A$ we have by the spectral radius theorem, that

$$\|a\|_A = \|a^2\|_A^{1/2} = \|a^4\|_A^{1/4} = \cdots = \|a^{2^k}\|_A^{1/2^k} \to \rho(a) \text{ (as } k \to \infty),$$

where $\rho(a) = \sup_{\lambda \in spectrum(a)} |\lambda|$ denotes the spectral radius of a. Hence

$$\|a\|_A = \sup_{\chi \in \hat{A}} |\chi(a)| = \|\hat{a}\|_\infty.$$

If a is any element of A, we have that

$$\|a\|_A^2 = \|a^* a\|_A = \|\widehat{a^* a}\|_\infty = \|\hat{a}\|_\infty^2.$$

Hence the Fourier transform $a \mapsto \hat{a}$ is an isometry and so A is isometrically isomorphic to $C_0(\hat{A})$.

2. The mapping $X \ni x \to \delta_x \in \hat{A}$ is clearly injective. It is also continuous. If a net $(x_i)_i$ converges in X to some x_∞, we have for any $\varphi \in C_0(X)$ that

$$\lim_i \langle \delta_{x_i}, \varphi \rangle = \lim_i \varphi(x_i) = \varphi(x_\infty) = \langle \delta_{x_\infty}, \varphi \rangle.$$

Hence $\lim_i \delta_{x_i} = \delta_{x_\infty}$ in $\hat{A}$.

Let us show that this mapping is also surjective. Suppose that there exists a homomorphism $\chi : A \to \mathbb{C}$, which is not of the form $\delta_x,\ x \in X$. Then $\ker\chi \not\subset \ker\delta_x$ for any $x \in X$. Hence for every $x \in X$, there exists $\varphi_x \in \ker\chi$ such that $\varphi_x(x) \neq 0$. If we replace φ_x by $\varphi_x\overline{\varphi_x}$, then we can even suppose that φ_x is non-negative and multiplying it with a convenient scalar, we can even assume that $\varphi_x(x) = 1,\ x \in X$. Now let $\psi \in C_c(X)$ and let $K := \text{supp}(\psi)$. For every $x \in K$, there exists a neighbourhood U_x of x, such that $\varphi_x(y) > \frac{1}{2}$ for every $y \in U_x$. Since K is compact, we find a finite subset F of K such that $\bigcup_{x\in F} U_x \supset K$. Let $h := \sum_{x\in F} \varphi_x$. Then $h \in \ker\chi$ and $h(y) > \frac{1}{2}$ for every $y \in K$. Choose a continuous function q with compact support on X such that $q(y) = \frac{1}{h(y)},\ y \in K$. Then $qh = 1$ on K and so $\psi = \psi qh \in \ker\chi$. This tells us that χ annihilates every $\psi \in C_c(X)$. This contradiction implies that $\hat{A} = \{\delta_x, x \in X\}$.

To finish the proof, we have to show that our mapping is open. Let $(x_i)_i$ be a net which converges in $\hat{A}$ to some δ_{x_∞}. Let U be a compact neighbourhood of x_∞. Suppose that there exists a sub-net $(y_j := x_{i_j})$, such that $y_j \notin U$ for every j. Choose a function $\varphi \in C_c(X)$ such that $\varphi(x_\infty) = 1, \varphi = 0$ outside U. Then

$$\lim_i \langle \delta_{x_i}, \varphi\rangle = \langle \delta_{x_\infty}, \varphi\rangle = 1,$$

but $\lim_j \langle \delta_{y_j}, \varphi\rangle = \lim_j 0 = 0$. This contradiction tells us that eventually $x_i \in U$. Hence our mapping is a homeomorphism.

3. We can assume that $A = C_0(X)$ for some locally compact space X and that $S = \{x \in X; f(x) = 0, \text{for all } f \in I\}$. Let

$$j(S) = \{f \in A;\ \text{supp}(f) \text{ is compact and disjoint from } S\}.$$

Let us show that $j(S) \subset I$. Since $j(S)$ is dense in $\ker S := \{f \in A;\ f(S) = \{0\}\}$, by the Stone–Weierstrass theorem, it will follow that

$$\ker S = \overline{j(S)} \subset I \subset \ker S \Rightarrow I = \ker S.$$

Suppose that there exists $f \in j(S), f \notin I$. Then we can find an element $g \in \ker S$ such that $gf = f$. For that, it suffices to take $g \in \ker S$, such that $g = 1$ on $\text{supp}(f)$. Let

$$J := \{kg - k + i,\ i \in I, k \in A\}.$$

We see that J is a modular ideal of A with modular unit g. If $g \in J$, then $g = kg - k + i$ for some $k \in A$ and some $i \in I$. But then

$$f = gf = f(kg - k + i) = k(fg - f) + fi = fi \in I.$$

This contradiction tells us that J is a proper modular ideal and so there exists a maximal modular ideal M, which does not contain g (since g is the modular unit) and which contains J. There exists hence a character χ of A, i.e. a point $\chi \in X$, with $\chi(g) \neq 0$ and $\chi(M) = \{0\}$. Since $I \subset M$, we see that $\chi \in S$. But then $g(\chi) = 0$. This contradiction tells us that $j(S) \subset I$. ■

Definition 2.3.21. Let A be an abelian locally compact group. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. The subset of $\hat{A}$ consisting of all characters annihilating the kernel of π in $C^*(A)$, i.e. the hull of $\ker\pi$ in $\hat{A}$, is called the **support of** π **in** $\hat{A}$.

Let C be a closed normal subgroup of the locally compact group G. For every function f on C and $g \in G$, let ${}^g f$ be the function on C defined by: ${}^g f(c) := f(g^{-1}cg), c \in C$.

For a representation (ρ, X) of C and $g \in G$, let $(g \cdot \rho, X)$ be the representation of C defined by

$$g \cdot \rho(c) := \rho(g^{-1}cg), g \in G.$$

Proposition 2.3.22. *Let G be a locally compact group and let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G. Let C be a closed abelian normal subgroup of G. Suppose that there exists in $\hat{C}$ an open G-invariant subset U, whose complement consists of fixed points for the action of G and which has the following separation property: two distinct G-orbits in U have disjoint open G-invariant neighbourhoods. Then the support of the representation $\rho := \pi_{|C}$ is the closure of a G-orbit.*

Proof. The support S of ρ is of course not empty, since if it is empty the whole algebra $C^*(C)$ would be contained in $\ker\rho$. It is also clear that S is G-invariant, since $\ker\rho$ is G-invariant. Suppose that $S \cap U$ contains two G-orbits O and O'. Take an open G-invariant neighbourhood W of O and an open G-invariant neighbourhood W' of O' such that $W \cap W' = \emptyset$. We choose functions f, f' in $L^1(C)$ such that their Fourier transforms $\hat{f}$ resp. $\hat{f'}$ do not vanish identically on O, resp. on O' and such that the support of $\hat{f}$, resp. of $\widehat{f'}$ is contained in W, resp. in W'. This implies that $f, f' \notin \ker\rho$ and that

$$\hat{f} \cdot \widehat{({}^g f')} = 0 \Rightarrow f * ({}^g f') = 0 \text{ for all } g \in G. \tag{2.3.2}$$

Here ${}^g f$ is the function defined by

$${}^g f(u) := f(g^{-1}ug),\ u \in C, g \in G.$$

Consider the two ideals $J := L^1(G)*f*L^1(G)$, $J' = L^1(G)*f'*L^1(G)$ in $L^1(G)$. Then J, J' are not contained in $\ker\pi$ and $J*J' = \{0\} \subset \ker\pi$, since for $\varphi, \psi, \psi' \in L^1(G)$, we have by Corollary 2.2.6 and the implication (2.3.2) that

$$\begin{aligned}\varphi*f*\psi*f'*\psi' &= \varphi*f*(\lim \sum_j c_j\lambda(t_j)(f'*\psi'))\\ &= \lim \sum_j c_j\varphi*(f*({}^{t_j}f')\Delta_G(t_j))*(\lambda(t_j)\psi')\\ &= \lim \sum_j c_j\varphi*(0)*(\lambda(t_j)\psi') = 0.\end{aligned}$$

Hence $J*J' = \{0\} \subset \ker\pi$. Since $\ker\pi$ is a prime ideal, it follows that J or J' is contained in $\ker\pi$. This contradiction tells us that $S \cap U$ can contain only one G-orbit. Hence $S \cap U$ is a G-orbit. Denote the closure of $S \cap U$ by S_U. If now $S \neq S_U$ then there exists an open G-invariant neighbourhood V of a point $s_0 \in S$ such that $V \cap S \cap U = \emptyset$. Choose again as above f, f' in $L^1(C)$ such that their Fourier transforms $\hat{f}$ resp. $\widehat{f'}$ do not vanish identically in s_0, resp. on $S \cap U$ and such that the support of $\hat{f}$, resp. of $\widehat{f'}$, is contained in V, resp. in U. Then for every $g \in G$ and $s \in S \cap U$ we have that $\widehat{f * {}^g f'}(s) = \hat{f}(s)\widehat{f'}(g \cdot s) = 0$ since $\hat{f} = 0$ on $S \cap U$ and if $s \in S \setminus U$:

$$\widehat{f * {}^g f'}(s) = \hat{f}(s)\widehat{f'}(g \cdot s) = \hat{f}(s)\widehat{f'}(s) = 0, \text{ since } \widehat{f'} = 0 \text{ on } S \setminus U.$$

and so $f_2*{}^g f_2 \in \ker\rho$. But then we can conclude that the product $J*J'$ of the ideals $J := L^1(G)*f*L^1(G)$, $J' = L^1(G)*f'*L^1(G)$ is contained in $\ker\pi$, whence f or $f' \in \ker\rho$. This contradiction tells us that $S = S_U$ and so S is the closure of a G-orbit. ∎

Proposition 2.3.23. *Let A be a closed normal subgroup of a locally compact group G. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Let $\rho := \pi|_A$. If $I \subset L^1(A)$ is a G-invariant closed two-sided ideal, which is not contained in $\ker\rho$, then the subspace $\rho(I)\mathcal{H}$ is dense in $\mathcal{H}$.*

Proof. Let $\mathcal{K}$ be the closure of $\rho(I)\mathcal{H}$. Since $\rho(I) \neq \{0\}$, we have that $\mathcal{K} \neq \{0\}$. For every $g \in G$, $f \in I$ and $\xi \in \mathcal{H}$, we have that $\pi(g)\rho(f)\xi = c(g^{-1})\rho({}^g f)\pi(g)\xi$, where for the Haar measure da on A we write $d(g^{-1}ag) = c(g^{-1})da, g \in G$. This shows that $\mathcal{K}$ is a non-trivial closed G-invariant subspace of $\mathcal{H}$. Hence $\mathcal{K} = \mathcal{H}$, since π is irreducible. ∎

Chapter 3
Induced Representations

3.1 Measures on Quotient Spaces

Let G be a locally compact group and H a closed subgroup of G. We are going to show that there exists a G-invariant Borel measure on the quotient space G/H, if and only if for the modular functions Δ_H and Δ_G we have that

$$\Delta_H = \Delta_G|_H \tag{3.1.1}$$

If this condition is not satisfied, then there exists a unique G-invariant positive linear form on a certain left translation invariant space of continuous complex-valued functions denoted by $\mathcal{E}(G/H)$, which reduces to $C_c(G/H)$ if relation (3.1.1) is satisfied.

Definition 3.1.1. Let H be a closed subgroup of a locally compact group G. We define the functions $\Delta_{G,H}$ and $\Delta_{H,G}$ on H by

$$\Delta_{G,H}(h) := \frac{\Delta_G(h)}{\Delta_H(h)}, \quad \Delta_{H,G}(h) := \frac{\Delta_H(h)}{\Delta_G(h)}, \quad h \in H.$$

Remark 3.1.2. When G is a Lie group and H is a subgroup with Lie algebra $\mathfrak{h}$, then

$$\Delta_{H,G}(\exp X) = e^{\mathrm{Trad}_{\mathfrak{g}/\mathfrak{h}} X} (X \in \mathfrak{h}).$$

This follows from Remark 2.1.11.

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Definition 3.1.3. Let H be a closed subgroup of a locally compact group G. Let $\mathcal{E}(G/H)$ be defined by

$$\mathcal{E}(G/H) = \{\xi : G \to \mathbb{C}; \xi(gh) = \Delta_{H,G}(h)\xi(g), \quad \forall g \in G, h \in H,$$
$$\xi \text{ is continuous with compact support modulo } H\}$$

We remark that the space $\mathcal{E}(G/H)$ is left translation and complex conjugation invariant and that the non-negative functions contained in $\mathcal{E}(G/H)$ span $\mathcal{E}(G/H)$ as a vector space.

Proposition 3.1.4. *Let H be a closed subgroup of a locally compact group G and $p : G \to G/H$ the canonical projection. The linear mappings*

$$R_{G/H} : C_c(G) \to \mathcal{E}(G/H), f \mapsto \left\{G \ni x \mapsto \int_H f(xh)\Delta_{G,H}(h)dh\right\},$$

$$Q_{G/H} : C_c(G) \to C_c(G/H), f \mapsto \left\{p(x) \mapsto \int_H f(xh)dh,\ x \in G\right\} \tag{3.1.2}$$

are positive, G-equivariant and surjective.

Proof. Let us show first that the functions $r := R_{G/H} f$ and $q := Q_{G/H} f$ (which are well defined since $f \in C_c(G)$) are continuous with compact support modulo H. Indeed, let K be a compact neighbourhood of the support of f. Then for $x \in G$ the relation $r(x) \neq 0$ implies that $f(xh) \neq 0$ for at least one $h \in H$ and so $xh \in \text{supp}(f)$, hence $x \in \text{supp}(f)H$ and $h \in x^{-1}\text{supp}(f)$. Therefore the support of r is contained in the compact subset KH of G/H; similarly for the function q. Furthermore, since for y close enough to x (i.e. for y in a small neighbourhood Vx of x, for which $V^{-1}\text{supp}(f) \subset K$) we have that $H \cap y^{-1}\text{supp}(f) \subset H \cap x^{-1}K$ and so

$$R_{G/H} f(y) = \int_{H\cap y^{-1}\text{supp}(f)} f(yh)\Delta_{G,H}(h)dh = \int_{H\cap x^{-1}K} f(yh)\Delta_{G,H}(h)dh.$$

It follows that for these y's

$$\begin{aligned}|R_{G/H} f(y) - R_{G/H} f(x)| &\leq \int_{H\cap x^{-1}K} |f(yh) - f(xh)|\Delta_{G,H}(h)dh \\ &\leq \|\lambda(y^{-1})f - \lambda(x^{-1})f\|_\infty \int_{H\cap x^{-1}K} \Delta_{G,H}(h)dh.\end{aligned}$$

This shows that $R_{G/H} f$ is continuous; similarly for $Q_{G/H}$.

Let us now show that $Q_{G/H} f$ is surjective. Let $\psi \in C_c(G/H)$. Choose a compact subset $L \subset G$ such that $\text{supp}(\psi) \subset LH$. We take a non-negative $\varphi \in C_c(G)$ such that $\varphi(x) = 1$ for all $x \in L$. Then the function $\tilde{\varphi} := Q_{G/H}\varphi$

is strictly positive on the support of ψ and so the function $\sigma := \frac{\psi}{\tilde{\varphi}}$ is well-defined and continuous on G/H. Let $a = (\sigma \circ p)\varphi \in C_c(G)$, where $p : G \to G/H$ denotes the canonical projection. Then

$$\begin{aligned} Q_{G,H}a(p(x)) &= \int_H (\sigma \circ p)(xh)\varphi(xh)dh = \sigma(p(x)) \int_H \varphi(xh)dh \\ &= \sigma(p(x))\tilde{\varphi}(p(x)) = \frac{\psi(p(x))}{\tilde{\varphi}(p(x))}\tilde{\varphi}(p(x)) = \psi(p(x)), \quad x \in G. \end{aligned}$$

Hence $Q_{G/H}$ is surjective.

Now let $\psi \in \mathcal{E}(G/H)$. Then as before, we choose a compact subset $L \subset G$ such that $\operatorname{supp}(\psi) \subset LH$. We take a non-negative $\varphi \in C_c(G)$ such that $\varphi(x) = 1$ for all $x \in L$. Then the function $\tilde{\varphi} := Q_{G/H}\varphi$ is strictly positive on the support of ψ and so the function $\sigma := \frac{\psi}{\tilde{\varphi}}$ is well-defined and continuous on G/H. Let $a = \sigma\varphi \in C_c(G)$. Then

$$\begin{aligned} R_{G,H}a(x) &= \int_H \sigma(xh)\varphi(xh)\Delta_{G,H}(h)dh = \sigma(x) \int_H \varphi(xh)dh \\ &= \sigma(x)\tilde{\varphi}(x) = \frac{\psi(x)}{\tilde{\varphi}(x)}\tilde{\varphi}(x) = \psi(x), \quad x \in G. \end{aligned}$$

Hence $R_{G/H}$ is also surjective. ■

Corollary 3.1.5. *Let A be a closed normal subgroup of a locally compact group G. Then for every $\varphi \in C_c(G)$ we have that*

$$\int_G \varphi(g)dg = \int_{G/A} \int_A \varphi(ga)da\,d\dot{g}.$$

Proof. By the definition (3.1.2) the mapping

$$\varphi \mapsto \{\dot{g} \mapsto \tilde{\varphi}(\dot{g}) := \int_A \varphi(gb)db,\ g \in G\}$$

from $C_c(G)$ to $C_c(G/A)$ is surjective and G-equivariant and the linear functional $\varphi \mapsto \int_{G/A} \tilde{\varphi}(\dot{g})d\dot{g}$ on $C_c(G)$ is positive and left-invariant. Hence it describes a Haar measure $\int_G \varphi dg$ on G. ■

Proposition 3.1.6. *Let H be a closed subgroup of a locally compact group G. There exists a unique (up to multiplication by a positive constant) G-invariant positive linear functional, denoted by*

$$k \mapsto \mu_{G,H}(k) = \oint_{G/H} k(x)d\mu_{G,H}(x) = \oint_{G/H} k(x)d\dot{x},$$

on the space $\mathcal{E}(G, H)$. *We have that*

$$\int_G k(t)dt = \oint_{G/H} \left(\int_H k(th)\Delta_{G,H}(h)dh \right) d\dot{t}, \quad \forall k \in C_c(G). \tag{3.1.3}$$

Proof. (a) Let us first prove the following relation. Let $f \in C_c(G)$ such that

$$R_{G/H}(f)(x) = \int_H f(xh)\Delta_{G,H}(h)dh = 0$$

for all $x \in G$. Then

$$\int_G f(x)dx = 0.$$

Indeed, we have for any $\psi \in C_c(G)$, that

$$\int_G \psi(x) \left(\int_H \Delta_{G,H}(h) f(xh)dh \right) dx = 0.$$

Hence,

$$\begin{aligned}
0 &= \int_G \psi(x) \left(\int_H \Delta_{G,H}(h) f(xh)dh \right) dx \\
&= \int_H \left(\int_G \psi(x)\Delta_{G,H}(h) f(xh)dx \right) dh \quad \text{(by Fubini)} \\
&= \int_H \left(\int_G \Delta_G^{-1}(h)\psi(xh^{-1})\Delta_{G,H}(h) f(x)dx \right) dh \quad (x \mapsto xh^{-1}) \\
&= \int_G f(x) \left(\int_H \Delta_G^{-1}(h)\Delta_{G,H}(h)\psi(xh^{-1})dh \right) dx \quad \text{(by Fubini)} \\
&= \int_G f(x) \left(\int_H \Delta_G(h)\Delta_H^{-1}(h)\Delta_{G,H}(h^{-1})\psi(xh)dh \right) dx \quad (h \mapsto h^{-1}) \\
&= \int_G f(x) \left(\int_H \psi(xh)dh \right) dx = \int_G f(x) Q_{G/H}(\psi)(\dot{x})dx.
\end{aligned}$$

Since by Proposition 3.1.4 the mapping $Q_{G/H} : C_c(G) \to C_c(G/H)$ is surjective, we can choose a $\psi \in C_c(G)$, such that $Q_{G/H}(\psi)(\dot{x}) = 1$ for all $x \in \mathrm{supp}(f) \cdot H$. Hence

$$\int_G f(x)dx = 0.$$

(b) Let us construct now the positive definite linear functional $\oint_{G/H} k(x)d\dot{x}$. If $k \in \mathcal{E}(G/H)$ is of the form

$$R_{G/H}(f) = k = R_{G/H}(f')$$

for two elements $f, f' \in C_c(G)$, then by (a)

$$\int_G f(x)dx = \int_G f'(x)dx.$$

Hence the function

$$k \mapsto \int_G f(x)dx \quad \text{(whenever } Q_{G/H}(f) = k\text{)} \tag{3.1.4}$$

is well defined on $\mathcal{E}(G/H)$ and gives us a linear functional on this space which we shall denote by $\oint_{G/H} d\dot{x}$. Since $R_{G/H}$ is positive and commutes with left translation, it follows that $\oint_{G/H} d\dot{x}$ is itself positive. Furthermore, for $k = R_{G/H}(f) \in \mathcal{E}(G/H)$ and $s \in G$ we have that

$$\begin{aligned}\oint_{G/H} \lambda(s)k(x)d\dot{x} &= \oint_{G/H} \lambda(s)R_{G/H}(f)(x)d\dot{x} \\ &= \oint_{G/H} R_{G/H}(\lambda(s)f)(x)d\dot{x} \\ &= \int_G f(s^{-1}x)dx = \int_G f(x)dx = \oint_{G/H} k(x)d\dot{x}.\end{aligned}$$

Hence $\oint_{G/H} d\dot{x}$ is a positive, translation-invariant linear functional on $\mathcal{E}(G/H)$. The uniqueness follows from the fact that any positive, left-invariant linear functional μ on $\mathcal{E}(G/H)$ defines also a left Haar-measure $d_\mu x$ on G through the relation

$$\int_G f(x)d_\mu x := \mu(R_{G/H}(f)), \quad f \in C_c(G)$$

By the uniqueness of the Haar measure on G, it follows that

$$d_\mu x = c\,dx$$

for some $c > 0$. Therefore, since the mapping $R_{G/H}$ is surjective, this implies that for every $k = R_{G/H}(f) \in \mathcal{E}(G/H)$:

$$\mu(k) = \mu(R_{G/H}(f)) = \int_G f(x)d_\mu x = c\int_G f(x)dx = c\oint_{G/H} k(x)d\dot{x}.$$

Hence $\mu = c\oint_{G/H} d\dot{x}$. Finally, formula (3.1.3) is a consequence of the definition (3.1.4). ■

Here is a very useful property for us, the **transitivity** of the form $\mu_{G,H}$ (cf. [14, Chap. V]) . We consider a closed subgroup K of H. Let us show that

$$\oint_{G/H}\left(\oint_{H/K}\Delta_{G,H}(h)\varphi(gh)d\dot{h}\right)d\dot{g}=\oint_{G/K}\varphi(g)d\dot{g},\ \varphi\in\mathcal{E}(G/K). \tag{3.1.5}$$

The function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(g):=\oint_{H/K}\Delta_{G,H}(h)\varphi(gh)d\dot{h},\ g\in G\ (\varphi\in\mathcal{E}(G/K)),$$

is contained in $\mathcal{E}(G/H)$, since it is obviously continuous with compact support modulo H and since, for $h'\in H$,

$$\begin{aligned}\tilde{\varphi}(gh')&=\oint_{H/K}\Delta_{G,H}(h)\varphi(gh'h)d\dot{h}=\oint_{H/K}\Delta_{G,H}(h'^{-1}h)\varphi(gh)d\dot{h}\\&=\Delta_{H,G}(h')\oint_{H/K}\Delta_{G,H}(h)\varphi(gh)d\dot{h}=\Delta_{H,G}(h')\tilde{\varphi}(g).\end{aligned}$$

It follows that the linear functional $\varphi\mapsto\oint_{G/H}\tilde{\varphi}(g)d\dot{g},\ \varphi\in\mathcal{E}(G/K)$ is positive and obviously G-invariant and so it must be equal to the positive invariant linear functional $\varphi\mapsto\oint_{G/K}\varphi(g)d\dot{g}$.

Theorem 3.1.7. *Let H be closed subgroup of a locally compact group G. Then the quotient space G/H has a G-invariant Borel measure if and only if $\Delta_{G,H}=1$.*

Proof. Let $p: G\to G/H$ denote the canonical projection. If $\Delta_{G,H}=1$, then the space $\mathcal{E}(G/H)$ is just $C_c(G/H)$ and the positive linear functional $\oint_{G/H}d\dot{s}$ is a G-invariant measure on G/H.

Suppose now that there exists a G-invariant measure on G/H. Then the functional $f\mapsto\int_{G/H}Q_{G/H}(f)(\dot{x})d\dot{x}$ defines a G-invariant Borel measure on G. Hence

$$\int_G f(s)ds=\int_{G/H}\left(\int_H f(xh)dh\right)d\dot{x},\quad f\in C_c(G).$$

Now let $f,\psi\in C_c(G)$. Then we easily see that

$$Q_{G/H}(\psi)Q_{G/H}(f)=Q_{G/H}(\psi Q_{G/H}(f)\circ p). \tag{3.1.6}$$

Therefore

$$\int_G f(x)Q_{G/H}\psi(x)dx=\int_{G/H}Q_{G/H}(\psi)(\dot{x})Q_{G/H}(f)(\dot{x})d\dot{x}$$

$$= \int_G \psi(x) Q_{G/H}(f)(x)dx = \int_G \psi(x)\Big(\int_H f(xh)dh\Big)dx$$

$$= \int_H \left(\int_G f(x)\Delta_G^{-1}(h)\psi(xh^{-1})dx\right)dh \quad (x \mapsto xh^{-1})$$

$$= \int_G f(x)\left(\int_H \psi(xh)\Delta_H(h^{-1})\Delta_G(h)dh\right)dx \quad (h \to h^{-1})$$

$$= \int_G f(x)\left(\int_H \psi(xh)\Delta_{G,H}(h)dh\right)dx$$

$$= \int_G f(x)R_{G/H}(\psi)(x)dx.$$

Hence

$$\int_G f(x)\left(Q_{G/H}(\psi)(\dot{x})\right)dx = \int_G f(x)R_{G/H}(\psi)(x)dx, \quad f, \psi \in C_c(G),$$

and so

$$\int_H \psi(xh)dh = \int_H \psi(xh)\Delta_{G,H}(h)dh, \quad \psi \in C_c(G), x \in G.$$

This implies that $\Delta_{G,H}(h) = 1$ for all $h \in H$. ■

Proposition 3.1.8. *Let A be a closed normal subgroup of a locally compact group G. Then $\Delta_G|_A = \Delta_A$.*

Proof. We have by Proposition 3.1.5 that

$$\Delta_G(a)\int_G \varphi(y)dy = \int_G \varphi(ya^{-1})dy = \int_{G/A}\int_A \varphi(gba^{-1})db\,d\dot{g}$$

$$= \int_{G/A} \Delta_A(a)\int_A \varphi(gb)db\,d\dot{g} = \Delta_A(a)\int_G \varphi(y)dy,$$

for every $a \in A$ and $\varphi \in C_c(G)$. Hence $\Delta_A = \Delta_G|_A$. ■

3.2 Definition of an Induced Representation

Definition 3.2.1. Let H be a closed subgroup of a locally compact group G. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of H. Define the space of mappings

$$\mathcal{E}(G/H, \rho)$$

by

$$\mathcal{E}(G/H,\rho) := \{\xi : G \to \mathcal{H}_\rho;\ \xi(gh) = \Delta_{H,G}^{1/2}(h)\rho(h)^*(\xi(g)), \quad \forall g \in G, h \in H,$$
$$\xi \text{ is continuous with compact support modulo } H\}. \tag{3.2.1}$$

We remark that the space $\mathcal{E}(G/H,\rho)$ is left translation invariant and that for $\xi, \eta \in \mathcal{E}(G/H,\rho)$ the function

$$x \to \langle \xi(x), \eta(x)\rangle_\rho =: q_{\xi,\eta}(x), x \in G$$

is continuous with compact support modulo H and satisfies the relation

$$\begin{aligned} q_{\xi,\eta}(xh) &= \langle \xi(xh), \eta(xh)\rangle_\rho \\ &= \Delta_{H,G}(h)\langle \rho(h)^*(\xi(x)), \rho(h)^*(\eta(xh))\rangle_\rho \\ &= \Delta_{H,G}(h) q_{\xi,\eta}(x), x \in G, h \in H, \end{aligned}$$

and so $q_{\xi,\eta} \in \mathcal{E}(G/H)$. We can thus define a scalar product on $\mathcal{E}(G/H,\rho)$ by

$$\langle \xi, \eta\rangle_{\mathrm{ind}\rho} := \oint_{G/H} \langle \xi(x), \eta(x)\rangle d\dot{x}$$

and a norm

$$\|\xi\|_{\mathrm{ind}\rho} := \sqrt{\langle \xi, \xi\rangle_{\mathrm{ind}\rho}}.$$

The left invariance of the linear functional $\oint_{G/H} d\dot{x}$ tells us that

$$\langle \lambda(x)\xi, \lambda(x)\eta\rangle_{\mathrm{ind}\rho} = \langle \xi, \eta\rangle_{\mathrm{ind}\rho}, \text{ for all } x \in G, \quad \xi, \eta \in \mathcal{E}(G/H,\rho).$$

We can construct explicit elements in $\mathcal{E}(G/H,\rho)$ through the mapping

$$R_{G/H,\rho} : C_c(G) \times \mathcal{H}_\rho \to \mathcal{E}(G/H,\rho),$$

$$(f,u) \to \left\{x \to \int_H f(xh)\Delta_{G,H}^{1/2}(h)\rho(h)u dh,\ x \in G\right\}.$$

It is easy to verify that $R_{G/H,\rho}(f,u)$ is effectively in the space $\mathcal{E}(G/H,\rho)$ whenever $f \in C_c(G)$ and $u \in \mathcal{H}_\rho$ and that the mapping $G \ni t \to R_{G/H,\rho}(f,u)(t) \in \mathcal{H}_\rho$ is continuous. We set:

Definition 3.2.2.

$$L^2(G/H,\rho) := \overline{\mathcal{E}(G/H,\rho)}^{\|\cdot\|_{\mathrm{ind}\rho}}.$$

Since the left translation is isometric on $\mathcal{E}(G/H,\rho)$, we obtain an isometric action of G on the Hilbert space $L^2(G/H,\rho)$. We denote this action by $\tau_{\rho,H} := \tau_\rho = \mathrm{ind}_H^G \rho$, where

$$\tau_{\rho,H}(t)\xi(s) := \xi(t^{-1}s), \xi \in L^2(G/H,\rho), s,t \in G. \tag{3.2.2}$$

Since obviously for $\xi \in \mathcal{E}(G/H,\rho)$

$$\lim_{t\to e} \|\lambda(t)\xi - \xi\|_\infty = 0,$$

it follows also that

$$\lim_{t\to e} \|\lambda(t)\xi - \xi\|_{\mathrm{ind}\rho} = 0.$$

Hence the mapping $t \to \tau_\rho(t)\xi$ from G into $L^2(G/H,\rho)$ is continuous and so τ_ρ is a **unitary representation** of G. We say that τ_ρ is an **induced representation** of G.

In order to construct representations of a group from those of subgroups, this method is utilized very often. Especially representations induced from unitary characters, i.e. one-dimensional unitary representation, are called **monomial representations**. When every irreducible representation is equivalent to a monomial representation, the group G is said to be **monomial**. Exponential solvable Lie groups which we introduce later are monomial [10, 75], but in general solvable Lie groups are not necessarily monomial.

Proposition 3.2.3. *Let $(\rho,\mathcal{H})$ be a unitary representation of a closed subgroup H of a locally compact group G. Let $f \in C_c(G)$. Then $\tau_\rho(f)$ is a kernel operator with a kernel function $f_\rho : G \times G \to B(\mathcal{H})$ given by the formula*

$$f_\rho(s,t) = \Delta_G(t^{-1}) \int_H f(sht^{-1})\Delta_{G,H}^{1/2}(h)\rho(h)dh \in B(\mathcal{H}), s,t \in G.$$

Proof. Let $\xi \in L^2(G/H,\rho)$ such that the function $g \mapsto |f(g)\xi(g^{-1}x)|$ is in $L^1(G)$ for every $x \in G$ (for instance if $f \in C_c(G)$ and $\xi \in \mathcal{E}(G/H,\rho)$) and let $s \in G$. Then

$$\begin{aligned}
\tau_\rho(f)\xi(s) &= \int_G f(t)\xi(t^{-1}s)dt = \int_G \Delta_G(t^{-1})f(st^{-1})\xi(t)dt \\
&= \oint_{G/H}\int_H \Delta_G((th)^{-1})f(s(th)^{-1})\xi(th)\Delta_{G,H}(h)dhd\dot{t} \\
&\quad \text{(by Eq. (3.1.3))}
\end{aligned}$$

$$
\begin{aligned}
&= \oint_{G/H} \int_H \Delta_G(t^{-1}) f(s(th)^{-1}) \xi(th) \Delta_H(h^{-1}) dh d\dot{t} \\
&= \oint_{G/H} \int_H \Delta_G(t^{-1}) f(sh^{-1}t^{-1}) \rho(h^{-1}) \Delta_{H,G}^{1/2}(h) \xi(t) \Delta_H(h^{-1}) dh d\dot{t} \\
&= \oint_{G/H} \left(\int_H \Delta_G(t^{-1}) f(sht^{-1}) \Delta_{G,H}^{1/2}(h) \rho(h) dh \right) \xi(t) d\dot{t} \\
&= \oint_{G/H} f_\rho(s,t) \xi(t) d\dot{t}.
\end{aligned}
$$

Lebesgue's theorem of dominated convergence tells us that the mapping $(s,t) \to f_\rho(s,t)$ is continuous on $G \times G$ if $f \in C_c(G)$. ■

Definition 3.2.4. Let us denote by $CB(G/H)$ the space of all complex-valued continuous and bounded functions on G/H. We equip this space with the infinity norm $\|\cdot\|_\infty$ and we obtain a commutative C^*-algebra which contains the space $C_0(G/H)$ of the continuous functions on G/H which vanish at infinity as C^*-subalgebra. Let $\phi \in CB(G/H)$. Then for any $\xi \in \mathcal{E}(G/H, \rho)$ we have that the mapping $\phi\xi$, where

$$
\phi\xi(t) := \phi(t)\xi(t), \quad t \in G,
$$

is also an element of $\mathcal{E}(G/H, \rho)$ and so $\mathcal{E}(G/H, \rho)$ is an $CB(G/H)$ module. Furthermore we have the following estimate

$$
\|\phi\xi\|_{\mathrm{ind}\rho} \le \|\phi\|_\infty \|\xi\|_{\mathrm{ind}\rho}.
$$

Hence we can extend the module action of $CB(G/H)$ to $L^2(G/H, \rho)$. Let us put

$$
M_\rho(\phi)\xi := \phi\xi, \quad \phi \in CB(G/H), \xi \in L^2(G/H, \rho).
$$

We obtain in this way a $*$-representation M_ρ of $CB(G/H)$ on the Hilbert space $L^2(G/H, \rho)$.

Let us see how the representations M_ρ of $CB(G/H)$ and τ_ρ are linked together. For $\xi \in \mathcal{E}(G/H, \rho), \phi \in CB(G/H)$ and $s, t \in G$ we have that

$$
\begin{aligned}
\tau_\rho(s)(M_\rho(\phi)\xi)(t) &= \phi(s^{-1}t)\xi(s^{-1}t) \\
&= (\lambda(s)\phi)(t)(\tau_\rho(s)\xi)(t) = M_\rho(\lambda(s)\phi)(\tau_\rho(s)\xi)(t).
\end{aligned}
$$

This shows that

$$
M_\rho(\lambda(s)\phi) = \tau_\rho(s) \circ M_\rho(\phi) \circ \tau_\rho(s)^{-1}, \quad s \in G, \phi \in CB(G/H). \tag{3.2.3}
$$

This is **Mackey's imprimitivity relation**.

Proposition 3.2.5. *Let ρ be a unitary representation of a closed subgroup H of a locally compact group G. Let $f \in C_c(G), \eta \in L^2(G/H, \rho)$ and let $\xi := \tau_\rho(f)\eta$. Then the mapping $G \ni t \mapsto \xi(t) \in \mathcal{H}_\rho$ is continuous.*

Proof. We have for $s, t \in G$ that

$$\begin{aligned}\xi(s) - \xi(t) &= \int_G f(u)\eta(u^{-1}s)du - \int_G f(u)\eta(u^{-1}t)du \\ &= \int_G (f(su) - f(tu))\eta(u^{-1})du \\ &= \int_G \Delta_G(u^{-1})(f(su^{-1}) - f(tu^{-1}))\eta(u)du.\end{aligned}$$

Hence

$$\begin{aligned}&\|\xi(s) - \xi(t)\|_\rho^2 = \Big\| \int_G \Delta_G(u^{-1})(f(su^{-1}) - f(tu^{-1}))\eta(u)du\Big\|_\rho^2 \\ &\leq \left(\int_G \Delta_G(u^{-1})|f(su^{-1}) - f(tu^{-1})|\|\eta(u)\|_\rho du\right)^2 \\ &\leq \Big(\int_G \Delta_G^2(u^{-1})|f(su^{-1}) - f(tu^{-1})|du\Big) \\ &\quad \times \Big(\int_G |f(su^{-1}) - f(tu^{-1})|\|\eta(u)\|_\rho^2 du\Big) \text{ (by Cauchy–Schwartz)} \\ &\leq \int_G \Delta_G^2(u^{-1})|f(su^{-1}) - f(tu^{-1})|du \\ &\quad \times \oint_{G/H} \left(\int_H |f(s(uh)^{-1}) - f(t(uh)^{-1})|\Delta_{G,H}(h)\|\rho(h)^{-1}(\eta(u))\|_\rho^2 dh\right) d\dot{u} \\ &\quad \text{(by Eq. (3.1.3))} \\ &= \int_G \Delta_G^2(u^{-1})|f(su^{-1}) - f(tu^{-1})|du \\ &\quad \times \oint_{G/H} \|(\eta(u))\|_\rho^2 \left(\int_H |f(s(uh)^{-1}) - f(t(uh)^{-1})|\Delta_{G,H}(h)dh\right) d\dot{u}.\end{aligned}$$

Since the functions

$$(s, t, u) \to \int_H |f(s(uh)^{-1}) - f(t(uh)^{-1})|\Delta_{G,H}(h)dh$$

and

$$(s,t) \to \int_G \Delta_G^2(u^{-1})|f(su^{-1}) - f(tu^{-1})|du$$

are continuous, it follows that

$$\lim_{s\to t} \|\xi(s) - \xi(t)\|_\rho = 0. \qquad \blacksquare$$

Proposition 3.2.6. *Let ρ be a unitary representation of a closed subgroup H of a locally compact group G. Let K be a closed subgroup of G containing H. Let $f \in C_c(G)$, $\eta \in L^2(G/H,\rho)$ and let $\xi := \tau_\rho(f)\eta$. Then the mapping*

$$G \ni t \mapsto (K \ni k \mapsto \Delta_{G,K}^{1/2}(k)\xi(tk)) =: \tilde{\xi}(t) \in L^2(K/H,\rho)$$

is continuous.

Proof. Let $\sigma = \mathrm{ind}_H^K \rho$. We have for $s,t \in G, k \in K$ that

$$\begin{aligned}
\xi(sk) - \xi(tk) &= \int_G f(u)\eta(u^{-1}sk)du - \int_G f(u)\eta(u^{-1}tk)du \\
&= \int_G (f(su) - f(tu))\eta(u^{-1}k)du \\
&= \int_G \Delta_G^{-1}(u)(f(su^{-1}) - f(tu^{-1}))\eta(uk)du.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\tilde{\xi}(s) - \tilde{\xi}(t)\|_\sigma^2 = \oint_{K/H} \left\| \int_G \Delta_G^{-1}(u)(f(su^{-1}) - f(tu^{-1}))\Delta_{G,K}^{1/2}(k)\eta(uk)du\right\|_\rho^2 d\dot{k} \\
&\le \oint_{K/H} \left(\int_G \Delta_G^{-1}(u)|f(su^{-1}) - f(tu^{-1})|\Delta_{G,K}^{1/2}(k)\|\eta(uk)\|_\rho du\right)^2 d\dot{k} \\
&\le \oint_{K/H} \left(\int_G \Delta_G^{-1}(u)|f(su^{-1}) - f(tu^{-1})|du\right) \\
&\quad \times \int_G \Delta_G^{-1}(v)|f(sv^{-1}) - f(tv^{-1})|\Delta_{G,K}(k)\|\eta(vk)\|_\rho^2 dv d\dot{k} \text{ (by Cauchy-Schwartz)} \\
&\le \int_G |\Delta_G^{-1}(u)g(u)|du \\
&\quad \times \oint_{K/H}\oint_{G/H}\int_H \Delta_G^{-1}((uh))|g(uh)|\Delta_{G,H}(h)\Delta_{G,K}(k)\|\eta(uhk)\|_\rho^2 \Delta_{G,H}(h) dh d\dot{u} d\dot{k}
\end{aligned}$$

where $g(u) := g_{s,t}(u) := f(tu^{-1}) - f(su^{-1}), u, s, t \in G$ (by Eq. (3.1.3))

$$= \int_G |\Delta_G^{-1}(u)g(u)|du$$
$$\times \oint_{G/H} \oint_{K/H} \Delta_{G,K}(k)\|(\eta(uk))\|_\rho^2 \left(\int_H \Delta_G^{-1}((uh))\Delta_{K,H}(h)|g(uh)|dh\right) d\dot{u}d\dot{k}$$
$$\leq \int_G |g(u^{-1})|du \|\tilde{g}\|_\infty \|\eta\|_2^2,$$

where

$$\tilde{g}(u) := \int_K \Delta_{K,H}(h)\Delta_G(h^{-1}u^{-1})|g(uh)|dh, \ u \in G.$$

Since the function

$$(s,t) \to \int_G |f(su) - f(tu)|du$$

is continuous, f being in $C_c(G)$, it follows that

$$\lim_{s \to t} \|\tilde{\xi}(s) - \tilde{\xi}(t)\|_\sigma = 0.$$

∎

Proposition 3.2.7. *The image of the mapping $R_{G/H,\rho}$ is total in $L^2(G/H, \rho)$.*

Proof. Suppose that $\eta \in L^2(G/H, \rho)$ is orthogonal to the closed subspace L_0 generated by the image of the mapping $R_{G/H,\rho}$. Since this mapping is G-equivariant in the first variable, its image is G-invariant, hence for every $f \in L^1(G)$, the vector $\tau_\rho(f)\eta$ is also in $L_0^\perp$. Let $(h_i)_{i\in I}$ be a bounded approximate identity in $L^1(G)$ consisting of continuous functions with compact support. Let

$$\eta_i := \tau_\rho(h_i)\eta \in L_0^\perp, i \in I.$$

Then $\lim_i \tau_\rho(h_i)\eta = \eta$ and for every i, the mapping $G \ni t \mapsto \tau_\rho(t)\eta_i$ is continuous by Proposition 3.2.5. Therefore we can assume that η is continuous. If now $\phi : G/H \to \mathbb{C}$ is a continuous bounded function, we have by Definition 3.2.4 and the definition of $R_{G/H,\rho}$ that for every $\xi \in L_0$, the mapping $\phi\xi$ is also in L_0 and so

$$0 = \langle \eta, \phi\xi \rangle_{\mathrm{ind}\rho} = \oint_{G/H} \overline{\phi(t)} \langle \eta(t), \xi(t) \rangle_\rho d\dot{t}.$$

Hence we must have that

$$\langle \eta(t), \xi(t) \rangle_\rho = 0 \text{ for all } t \in G.$$

In particular, taking $\xi = R_{G/H,\rho}(f, u)$ (where $f \in C_c(G)$ and $u \in \mathcal{H}_\rho$), this gives us

$$0 = \int_H f(th)\Delta_{G,H}^{1/2}(h)\langle \eta(t), \rho(h)u\rangle dh, \forall t \in G.$$

Hence $\langle \eta(t), \rho(h)u\rangle = 0$ for all $u \in \mathcal{H}_\rho$ and all $t \in G$. Finally, $\eta(t) = 0$ for all $t \in G$. ■

Theorem 3.2.8 (Induction in Stages). *Let $K \subset H \subset G$ be two closed subgroups of a locally compact group G. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of K. Then the representations $ind_K^G \rho$ and $ind_H^G(ind_K^H \rho)$ are unitarily equivalent.*

Proof. Let $\mathcal{E}(G/H, \mathcal{E}(H/K, \rho))$ be the space

$$\begin{aligned}
&\mathcal{E}(G/H, \mathcal{E}(H/K, \rho)) \\
&= \{\eta : G \to \mathcal{E}(H/K, \rho), \eta \text{ is continuous with compact support modulo } H, \\
&\quad \eta(gh) = \Delta_{H,G}^{1/2}(h)\lambda(h^{-1})(\eta(g)), \forall g \in G, h \in H, \text{ and there exists} \\
&\quad \text{a compact subset } L_\eta \subset G, \text{ such that supp } \eta(g) \subset g^{-1}L_\eta K \cap H \ \forall g \in G\},
\end{aligned}$$

where $\lambda := \mathrm{ind}_K^H \rho$.

The mapping

$$U : \mathcal{E}(G/K, \rho) \to \mathcal{E}(G/H, \mathcal{E}(H/K, \rho)),$$

$$U(\varphi)(g)(h) := \Delta_{G,H}^{1/2}(h)\varphi(gh), \ g \in G, h \in H,$$

is then well defined, i.e. $U(\varphi)$ is in the space $\mathcal{E}(G/H, \mathcal{E}(H/K, \rho))$, if φ is in $\mathcal{E}(G/K, \rho)$, since φ is continuous with compact support modulo K, hence $U(\varphi)$ is continuous with compact support modulo H and $U(\varphi)(g)$ has its support in LH for some compact subset L of G. Furthermore

$$\begin{aligned}
U(\varphi)(g)(hk) &= \Delta_{G,H}^{1/2}(hk)\varphi(ghk) = \Delta_{G,H}^{1/2}(hk)\Delta_{K,G}^{1/2}(k)\rho(k^{-1})(\varphi(gh)) \\
&= \Delta_{K,H}^{1/2}(k)\rho(k^{-1})U(\varphi)(g)(h), \ \forall g \in G, h \in H, k \in K.
\end{aligned}$$

Hence $U(\varphi)(g) \in \mathcal{E}(H/K, \rho)$ for all $g \in G$. Also, for $h, h' \in H, g \in G$, we have that

$$\begin{aligned}
U(\varphi)(gh)(h') &= \Delta_{G,H}^{1/2}(h')\varphi(ghh') = \Delta_{G,H}^{1/2}(hh')\Delta_{H,G}^{1/2}(h)\varphi(ghh') \\
&= \Delta_{H,G}^{1/2}(h)U(\varphi)(g)(hh') = \Delta_{H,G}^{1/2}(h)\lambda(h^{-1})U(\varphi)(g)(h').
\end{aligned}$$

Hence $U(\varphi)$ is contained in $\mathcal{E}(G/H, \mathcal{E}(H/K, \rho))$.

The inverse mapping

$$V : \mathcal{E}(G/H, \mathcal{E}(H/K, \rho)) \to \mathcal{E}(G/K, \rho)$$

is given by

$$V(\eta)(g) := \eta(g)(e), \ g \in G.$$

Indeed, for $g \in G, k \in K$, we have that

$$\begin{aligned} V\eta(gk) &= \eta(gk)(e) = \Delta_{H,G}^{1/2}(k)\lambda(k^{-1})\eta(g)(e) \\ &= \Delta_{H,G}^{1/2}(k)\eta(g)(k) = \Delta_{K,G}^{1/2}(k)\rho(k^{-1})\eta(g) \end{aligned}$$

and $V(\eta)$ is obviously continuous and its support is contained in the set $L_\eta K$. This shows that $V(\eta)$ is indeed in $\mathcal{E}(G/K, \rho)$. Furthermore, for $\eta \in \mathcal{E}(G/H, \mathcal{E}(H/K, \rho)), g \in G, h \in H$, we have that

$$\begin{aligned} U \circ V(\eta)(g)(h) &= \Delta_{G,H}^{1/2}(h)V(\eta)(gh) = \Delta_{G,H}^{1/2}(h)\eta(gh)(e) \\ &= \Delta_{G,H}^{1/2}(h)\Delta_{H,G}^{1/2}(h)\eta(g)(h) = \eta(g)(h). \end{aligned}$$

Similarly for $\varphi \in \mathcal{E}(G/K, \rho), \ g \in G$, we have that

$$V \circ U(\varphi)(g) = U(\varphi)(g)(e) = \Delta_{G,H}^{1/2}(e)\varphi(ge) = \varphi(g).$$

Hence $U \circ V = Id$ and $V \circ U = Id$.

The mapping U is obviously G-equivariant and by Eq. (3.1.5) it is also isometric, since

$$\begin{aligned} \|U(\varphi)\|^2_{\text{ind}(\text{ind}\rho)} &= \oint_{G/H} \|U(\varphi)(g)\|^2_{\text{ind}\rho} d\dot{g} \\ &= \oint_{G/H} \left(\oint_{H/K} \Delta_{G,H}(h)\|\varphi(gh)\|^2_{\mathcal{H}_\rho} d\dot{h} \right) d\dot{g} \\ &= \oint_{G/K} \|\varphi(g)\|^2_{\mathcal{H}_\rho} d\dot{g} = \|\varphi\|^2_{\text{ind}\rho}. \end{aligned}$$

Therefore U is a unitary intertwining operator. ■

Proposition 3.2.9. *Let G be a locally compact group, let H be a closed subgroup of G and let A be a closed normal subgroup of G contained in H. Let $\tilde{G} = G/A$, $\tilde{H} = H/A$ and let (ρ, V) be a unitary representation of H, which is trivial on A.*

Let $\Theta : G \to G/A$ be the canonical projection. Denote by $\tilde{\rho}$ the representation of $\tilde{H}$ which is defined through the relation $\tilde{\rho}\circ\Theta = \rho$. Then the unitary representations $\tau := ind_H^G \rho$ and $\tilde{\tau} := ind_{\tilde{H}}^{\tilde{G}} \tilde{\rho}\circ\Theta$ of G are equivalent.

Proof. The two spaces $\mathcal{E}(G/H, \rho)$ and $\mathcal{E}(\tilde{G}/\tilde{H}, \tilde{\rho})$ are isomorphic, an isomorphism $U : \mathcal{E}(G/H, \rho) \to \mathcal{E}(\tilde{G}/\tilde{H}, \tilde{\rho})$ being given by

$$U\xi(gA) := \xi(g),\ g \in G, \xi \in \mathcal{E}(G/H, \rho), \tag{3.2.4}$$

since $\Delta_{G,H}(a) = \Delta_{G,H}(a) = \Delta_{A,H}(a) = 1$, by Proposition 3.1.8 and since A is a normal subgroup of G. By the covariance relation we have that

$$\xi(ga) = \Delta_{H,G}^{1/2}(a)\rho(a^{-1})\xi(g) = \xi(g),\ g \in G, a \in H.$$

The spaces $\mathcal{E}(G/H)$ and $\mathcal{E}((G/A)/(H/A))$ are for the same reason G-isomorphic too through a map denoted also by U and therefore

$$\oint_{G/H} \varphi(g)d\dot{g} = \oint_{(G/A)/(H/A)} U\varphi(\tilde{g})d\dot{\tilde{g}}, \varphi \in \mathcal{E}(G/H).$$

This shows that the left representation of G on the spaces $L^2(G/H, \rho)$ and $L^2(\tilde{G}/\tilde{A}, \tilde{\rho})$ i.e. $\mathrm{ind}_{\tilde{H}}^{\tilde{G}} \tilde{\rho}\circ\Theta$ and $\mathrm{ind}_H^G \rho$ are equivalent. ■

Proposition 3.2.10. *Let $\tau_\rho = ind_H^G \rho$ be an induced representation of a locally compact group G. If τ_ρ is irreducible, then ρ must be irreducible too.*

Proof. If ρ is not irreducible, then the Hilbert space $\mathcal{H}_\rho$ of ρ can be written as the direct orthogonal sum of two non-trivial closed H-invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}_\rho$ and $\rho = \rho_1 \oplus \rho_2$, where $\rho_i = \rho|_{\mathcal{H}_i}$, $i = 1, 2$. It follows that $\mathcal{E}(G/H, \rho)$ is the orthogonal direct sum of the G-invariant subspaces $\mathcal{E}_i = \mathcal{E}(G/H, \rho_i)$, $i = 1, 2$. This implies that τ_ρ is not irreducible. ■

3.3 Conjugation of Induced Representations

Definition 3.3.1. Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. For an automorphism a of G, we write $(\pi^a, \mathcal{H})$ for the representation

$$\pi^a(s) := \pi(a^{-1}(s)),\ s \in G.$$

If $a = I_t$ is the inner automorphism $a(s) = tst^{-1}$, $t \in G$, then we write

$$\pi^t := \pi^{I_t}.$$

Let H be a closed subgroup of the locally compact group G and let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of H. Then $(\rho^a, \mathcal{H}_\rho)$ is the representation of the group $H^a := a(H)$ defined by

$$\rho^a(h) := \rho(a^{-1}(h)),\ h \in H^a.$$

Proposition 3.3.2. *Let H be a closed subgroup of a locally compact group G and let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of H. For every automorphism a of G we have that $(\tau_{\rho,H})^a$ is equivalent to τ_{ρ^a,H^a}.*

Proof. The Haar measure of the group H^a can be written in the following way:

$$\int_{H^a} \varphi(h)dh = \int_H \varphi(a(h'))dh',\ \varphi \in C_c(H^a),$$

since the mapping $C_c(H^a) \to C_c(H)$, $\varphi \mapsto \varphi \circ a =: U(\varphi)$ is a positive linear bijection and since $U(\lambda(s)\varphi) = \lambda(a^{-1}(s))U(\varphi)$ for every $s \in H^a$ and so the linear functional $\varphi \mapsto \int_H \varphi(a(h))dh$ is positive and $\mathcal{H}^a$-invariant, hence is a Haar measure on H^a.

In particular, we have that

$$\Delta_{H^a}(a(h)) = \Delta_H(h) \text{ for every } h \in H, \tag{3.3.1}$$

because

$$\begin{aligned}\int_{H^a} \varphi(hs^{-1})dh &= \int_H \varphi(a(h')s^{-1})dh' \\ &= \int_H \varphi(a(h')a(a^{-1}(s^{-1})))dh' = \int_H \varphi(a(h'a^{-1}(s^{-1})))dh' \\ &= \Delta_H(a^{-1}(s)) \int_H \varphi(a(h'))dh' = \Delta_H(a^{-1}(s)) \int_{H^a} \varphi(h)dh.\end{aligned}$$

In a similar way, we have that the invariant "measure" on $G/(H^a)$ is given by

$$\oint_{G/H^a} \varphi(g)d\dot{g} = \oint_{G/H} \varphi(a(g))d\dot{g}. \tag{3.3.2}$$

This follows from the fact that the mapping $\varphi \to \varphi \circ a$ is a positive linear bijection of $\mathcal{E}(G/H^a)$ onto $\mathcal{E}(G/H)$. Let us show that the mapping

$$U : \mathcal{E}(G/H^a, \rho^a) \to \mathcal{E}(G/H, \rho);\ \varphi \mapsto \varphi \circ a$$

defines a unitary operator which intertwines both representations. First we observe that $U|_{\mathcal{E}(G/H^a,\rho^a)}$ is a linear bijection onto $\mathcal{E}(G/H, \rho)$, since for $\varphi \in \mathcal{E}(G/H^a, \rho^a)$, we have that

$$
\begin{aligned}
U\varphi(gh) = \varphi(a(g)a(h)) &= \Delta_{H^a,G}^{1/2}(a(h))\rho^a(a(h))^{-1}\varphi(a(s)) \\
&= \Delta_{H,G}^{1/2}(h)\rho(h)^{-1}\varphi(a(s)) \\
&\quad \text{(by the definition of } \rho^a \text{ and (3.3.1))} \\
&= \Delta_{H,G}^{1/2}(h)\rho(h)^{-1}U\varphi(g).
\end{aligned}
$$

This shows that $U(\varphi)$ satisfies the covariance condition of the definition (3.2.1). Furthermore, since obviously $U\varphi$ is continuous and its support is compact modulo H, hence $U\varphi$ is an element of $\mathcal{E}(G/H, \rho)$. Furthermore, for $\varphi \in \mathcal{E}(G/H^a, \rho^a)$, we have by (3.3.2) that

$$
\|U(\varphi)\|_{\mathrm{ind}\rho}^2 = \oint_{G/H} \|\varphi(a(t))\|_{\mathcal{H}_\rho}^2 d\dot{t} = \oint_{G/H^a} \|\varphi(t)\|_{\mathcal{H}_{\rho^a}}^2 d\dot{t} = \|\varphi\|_{\mathrm{ind}\rho^a}^2
$$

and for $t, s \in G, \varphi \in \mathcal{E}(G/H^a, \rho^a)$:

$$
\begin{aligned}
U(\tau_{\rho^a}(t)\varphi)(s) &= \tau_{\rho^a}(t)\varphi(a(s)) = \varphi(t^{-1}a(s)) \\
&= \varphi(a((a^{-1}(t))^{-1}s)) = (\tau_\rho)^a(t)U\varphi(s).
\end{aligned}
$$

Hence τ_{ρ^a} is equivalent to the representation $(\tau_\rho)^a$. ∎

Let us give a criterion for the irreducibility of an induced representation. This criterion will be used in Sect. 5.3.

Theorem 3.3.3. *Let H be a closed subgroup of a connected locally compact group G. Let $(\tau, \mathcal{H}_\tau)$ be an irreducible representation of H. Let C be a closed normal subgroup of G contained in the centre of H. Let $\chi : C \to \mathbb{T}$ be the unitary character of C defined through τ, i.e. $\tau(c) = \chi(c)Id_{\mathcal{H}_\tau}$, $c \in C$. Suppose that the G-orbit of χ in $\hat{C}$ is locally closed and that the stabilizer $G_\chi = \{g \in G;\ \chi^g = \chi\}$ of χ is equal to H. Then the unitary representation $\pi := ind_H^G \tau$ is irreducible.*

Proof. Since the G-orbit O of χ is locally closed, we can write $O = A \cap U$ for some open subset U and some closed subset A of $\hat{C}$. Hence $\overline{O} \subset A$ and so

$$
O \subset \overline{O} \cap U \subset A \cap U = O \Rightarrow O = \overline{O} \cap U.
$$

Since G is connected, it is the union of an increasing sequence of compact sets and so we know from the general theory of locally compact groups, since O is locally closed, hence locally compact, that the quotient space $G/G_\chi = G/H$ is homeomorphic to O. Such a homeomorphism is given by the mapping

$$
G/H \ni gH \mapsto \chi^g,\ g \in G.
$$

Now let $\psi \in L^1(C)$ and let us compute $\pi|_C(\psi)$. Let $\eta \in \mathcal{E}(G/H, \tau)$ and $g \in G$. We then have that

$$\pi|_C(\psi)\eta(g) = \int_C \psi(c)\eta(c^{-1}g)dc = \int_C \psi(c)\eta(g(g^{-1}c^{-1}g))dc$$
$$= \int_C \psi(c)\chi(g^{-1}cg)\eta(g)dc = \hat{\psi}(\chi^g)\eta(g).$$

Hence $\pi|_C(\psi)$ is the multiplication operator with the continuous bounded function $m_\psi(gH) := \hat{\psi}(\chi^g), gH \in G/H$. If we choose ψ such that $\hat{\psi}$ has a compact support which is contained in U, then the function m_ψ has compact support in G/H and the collection I of all these functions m_ψ is a subalgebra of $C_0(G/H)$ which separates the points in G/H and which is complex conjugation invariant. Hence I is dense in $C_0(G/H)$ in the uniform norm.

Let us show now that π is irreducible. Let $\eta' \neq 0$ be an element in $L^2(G/H, \tau)$ and let $\mathcal{H}_0 := \pi(C_c(G))\eta'$. We must show that $\mathcal{H}_0^\perp = \{0\}$. Let $\varphi' \in \mathcal{H}_0^\perp$. Let $f \in C_c(G)$ run through an approximate identity and let $\varphi := \pi(f)\varphi', \eta := \pi(f)\eta'$. The elements η and φ of $\mathcal{H}_\pi = L^2(G/H, \tau)$ have the property that the mappings $G \to \mathcal{H}_\tau; g \mapsto \eta(g)$, $g \to \varphi(g)$ are continuous (see Proposition 3.2.6). We shall show that $\varphi = 0$. This will imply then that $\varphi' = 0$.

We have for all $u, v \in G$, $\psi \in L^1(C)$ that

$$0 = \langle \pi|_C(\psi)\pi(u)\eta, \pi(v)\varphi\rangle = \oint_{G/H} m_\psi(g)\langle \eta(u^{-1}g), \varphi(v^{-1}g)\rangle_{\mathcal{H}_\tau} d\dot{g}.$$

It follows that

$$0 = \langle \eta(u^{-1}g), \varphi(v^{-1}g)\rangle_{\mathcal{H}_\tau} \ \forall u, v, g \in G.$$

Hence, in particular, if we replace u by gu^{-1} and v by gv^{-1}, then we obtain the relation

$$0 = \langle \tau(h)(\eta(u)), \varphi(v)\rangle_{\mathcal{H}_\tau} \ \forall u, v \in G, h \in H.$$

This shows that, using the fact that τ is irreducible, that $\varphi(v) = 0$ for all $v \in G$, since the element $\eta(u) \in \mathcal{H}_\tau$ is not 0 for all $u \in G$. ■

Proposition 3.3.4. *Let H be a closed subgroup of a connected locally compact group G. Let $(\tau, \mathcal{H}_\tau)$ and $(\tau', \mathcal{H}_{\tau'})$ be two unitary representations of H and let $\pi := ind_H^G \tau,\ \pi' := ind_H^G \tau'$.*

1. *Let $u : \mathcal{H}_\tau \to \mathcal{H}_{\tau'}$ be a bounded intertwining operator for τ and τ'. Then the mapping $U_u : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ defined by*

$$U_u(\eta)(g) := u(\eta(g)),\ g \in G, \eta \in \mathcal{E}(G/H, \tau),$$

is a bounded intertwining operator for π and π'.

2. *Let $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ be a bounded intertwining operator for π and π'. Suppose that U intertwines also the representations M_τ resp. $M_{\tau'}$ of $C_0(G/H)$ on $\mathcal{H}_\pi$ resp. on $\mathcal{H}_{\pi'}$. Then there exists a bounded intertwining operator u for τ and τ' such that*

$$U = U_u.$$

Proof. 1. Take $\xi \in \mathcal{E}(G/H, \tau)$. Then the mapping $g \mapsto u(\xi(g)) \in \mathcal{H}_{\tau'}$ is continuous with compact support modulo H and for $g \in G, h \in H$, we have that

$$u(\xi(gh)) = u(\Delta_{H,G}^{1/2}(h)\tau(h^{-1})(\xi(g))) = \Delta_{H,G}^{1/2}(h)\tau'(h^{-1})u(\xi(g)),$$

since $u \in BL_H(\mathcal{H}_\tau, \mathcal{H}_{\tau'})$. Hence $U_u(\xi)$ is an element of $\mathcal{H}_{\pi'}$. Furthermore

$$\|U_u(\xi)\|_{\pi'}^2 = \oint_{G/H} \|U_u(\xi)(g)\|_{\tau'}^2 d\dot{g} = \oint_{G/H} \|u(\xi(g))\|_{\tau'}^2 d\dot{g}$$
$$\leq \|u\|_{\text{op}}^2 \oint_{G/H} \|\xi(g))\|_\tau^2 d\dot{g} = \|u\|_{\text{op}}^2 \|\xi\|_\pi^2.$$

This shows that U_u is bounded by $\|u\|_{\text{op}}$.

For $t, g \in G$ we have that

$$U_u(\pi(t)\xi)(g) = u(\pi(t)\xi(g)) = u(\xi(t^{-1}g)) = \pi'(t)u \circ \xi(g) = \pi'(t)(U_u\xi)(g).$$

Hence U_u is an element of $BL_G(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$.

2. In order to find the operator $u \in BL_H(\mathcal{H}_\tau, \mathcal{H}_{\tau'})$, let $\mathcal{H}_\pi^0$, resp. $\mathcal{H}_{\pi'0}$ be the span of the vectors $\pi(f)\xi,\ \xi \in \mathcal{H}_\pi,\ f \in C_c(G)$, resp. $\pi'(f)\xi,\ \xi \in \mathcal{H}_{\pi'},\ f \in C_c(G)$. Then every element η of $\mathcal{H}_{\pi'}^0$ has the property that the mapping $G \ni g \mapsto \eta(g) \in \mathcal{H}_{\tau'}$ is continuous. Since U maps $\mathcal{H}_\pi^0$ into $\mathcal{H}_{\pi'}^0$, we can evaluate $U(\eta)$, $\eta \in \mathcal{H}_\pi^0$, at every $g \in G$. We observe first that $\|U(\xi)(g)\|_{\tau'} \leq \|U\|_{\text{op}} \|\xi(g)\|_\tau$ for every $\xi \in \mathcal{H}_\pi^0, g \in G$. Indeed, we have that for every $\varphi \in C_c(G/H)$

$$\oint_{G/H} |\varphi(t)|^2 \|U(\xi)(t)\|_{\tau'}^2 d\dot{t} = \|M_\tau(\varphi)(U(\xi))\|_{\pi'}^2 = \|U(M_\tau(\varphi)\xi)\|_{\pi'}^2$$
$$\leq \|U\|_{\text{op}}^2 \|M_\tau(\varphi)(\xi)\|_\pi^2 = \|U\|_{\text{op}}^2 \oint_{G/H} |\varphi(t)\|^2 \|\xi(t)\|_\tau^2 d\dot{t}.$$

Hence, for every $g \in G$, we must have that

$$\|U(\xi)(g)\|_{\tau'} \leq \|U\|_{\text{op}} \|\xi(g)\|_\tau. \tag{3.3.3}$$

In particular, whenever for two elements η and η' in $\mathcal{H}_\pi^0$, we have that $\eta(g) = \eta'(g)$, then necessarily $U(\eta)(g) = U(\eta')(g)$. This allows us to define a linear operator u from the subspace $\mathcal{H}_\tau^0 = \{\xi(e); \xi \in \mathcal{H}_\pi^0\}$ into $\mathcal{H}_{\tau'}$ by letting

$$u(v) := U(\xi)(e), \text{ whenever } v = \xi(e) \text{ for some } \xi \in \mathcal{H}_\pi^0. \tag{3.3.4}$$

It follows from (3.3.3) that u is bounded by $\|U\|_{\mathrm{op}}$. We extend u to a bounded operator on the whole space $\mathcal{H}_\pi$. It follows from its definition that

$$U(\xi)(g) = \pi'(g^{-1})U(\xi)(e) = U(\pi(g^{-1})\xi)(e) = u(\pi(g^{-1})\xi(e)) = u(\xi(g)),$$

for every $\xi \in \mathcal{H}_\pi^0$ and $g \in G$. Hence $U = U_u$.

We still have to show that u intertwines τ and τ'. For $v = \xi(e) \in \mathcal{H}_\tau^0$, $h \in H$, we have that

$$\begin{aligned} u(\tau(h)v) &= u(\tau(h)\xi(e)) = u(\Delta_{H,G}^{1/2}(h)\xi(h^{-1})) \\ &= \Delta_{H,G}^{1/2}(h)U\xi(h^{-1}) \\ &= \tau'(h)(U\xi(e)) = \tau'(h)(u(v)). \end{aligned}$$

This shows that $u \in BL_H(\mathcal{H}_\tau, \mathcal{H}_{\tau'})$. ■

3.4 The Imprimitivity Theorem

Definition 3.4.1. Let H be a closed subgroup of a locally compact group G. Let $(\tau, \mathcal{H}_\tau)$ be a unitary representation of G and let $(M, \mathcal{H}_\tau)$ be a **non-degenerate** unitary representation of $C_0(G/H)$ on the same Hilbert space $\mathcal{H}_\tau$. We say that the pair (τ, M) is a **system of imprimitivity**, if the following relation holds:

$$M(\lambda(s)\phi) = \tau(s)\circ M(\phi) \circ \tau(s)^{-1}, \quad s \in G, \phi \in C_0(G/H).$$

Relation (3.2.3) tells us that for every unitary representation ρ of H, the pair (τ_ρ, M_ρ) is a system of imprimitivity.

Theorem 3.4.2 (Mackey's Imprimitivity Theorem). *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of a locally compact group G. If there exists a closed subgroup H of G and a non-degenerate unitary representation M of $C_0(G/H)$ on the space $\mathcal{H}_\pi$ of π such that the pair (π, M) is a system of imprimitivity, then π is unitarily equivalent to a representation which is induced from a unitary representation ρ of H and the representation M is equivalent to M_ρ under the same equivalence.*

Proof. Let $\mathcal{H}_\pi^\infty$ be the Garding space of π, i.e. the span of all the vectors of the form $(\pi(f)\xi)$ with $f \in C_c(G), \xi \in \mathcal{H}_\pi$. It is immediate that $\mathcal{H}_\pi^\infty$ is invariant under G and under $C_c(G)$.

For $f \in C_c(G)$, write also $\tilde{f} = Q_{G/H} f$. For $\xi, \eta \in \mathcal{H}_\pi^\infty$, let $\nu_{\xi,\eta}$ be the linear functional on $C_c(G)$ defined by

$$\nu_{\xi,\eta}(f) = \langle M(\tilde{f})\xi, \eta\rangle_\pi, \quad f \in C_c(G).$$

We now make the following assumption:

(A) The subset $\mathcal{H}_c^2$ of $\mathcal{H}_\pi \times \mathcal{H}_\pi$ consisting of all the ξ's and η's for which there exists a continuous function $q_{\xi,\eta} : G \to \mathbb{C}$, such that

$$\nu_{\xi,\eta}(f) = \int_G f(x) q_{\xi,\eta}(x) dx, \qquad \forall f \in C_c(G)$$

contains $\mathcal{H}_\pi^\infty \times \mathcal{H}_\pi^\infty$. Furthermore, for $(\xi,\eta) \in \mathcal{H}_\pi^\infty \times \mathcal{H}_\pi^\infty$ the function $G \times G \times G \ni (z, y, t) \to q_{\pi(z)\xi,\pi(y)\eta}(t)$ is continuous.

We shall prove assertion (A) at the end of the present proof.

The following properties of the function $q_{\xi,\eta}$, $(\xi,\eta) \in \mathcal{H}_c^2$, follow immediately from its definition.

1. $q_{\xi,\eta}$ is linear in ξ and anti-linear in η and $q_{\xi,\eta} = \overline{q_{\eta,\xi}}$.
2. $q_{\xi,\xi}(t) \geq 0$ for every t in G, since for $C_c(G) \ni f \geq 0$ we have that $\tilde{f} \geq 0$ and so $M(\tilde{f})$ is a non-negative operator, whence $\int_G f(t) q_{\xi,\xi}(t) dt = \langle M(\tilde{f})\xi, \xi\rangle \geq 0$.
3. $q_{\xi,\eta}(th) = \Delta_{H,G}(h) q_{\xi,\eta}(t)$ for all $t \in G$ and $h \in H$.

   Indeed, for any $f \in C_c(G)$, we have for the right translated function $\rho(h^{-1})f,\ h \in H$, that

$$(\rho(h^{-1})f)\tilde{}(\dot{t}) = \int_H f(tkh^{-1}) dk = \Delta_H(h) \int_H f(tk) dk = \Delta_H(h) \tilde{f}(\dot{t}), \ \ t \in G.$$

   Hence

$$\begin{aligned}\int_G f(t) q_{\xi,\eta}(th) dt &= \Delta_G^{-1}(h) \int_G f(th^{-1}) q_{\xi,\eta}(t) dt \\ &= \Delta_G^{-1}(h) \langle M(Q_{G/H}(\rho(h^{-1})f))\xi, \eta\rangle \\ &= \Delta_H(h) \Delta_G^{-1}(h) \langle M(Q_{G/H}(f))\xi, \eta\rangle \\ &= \int_G f(t) \Delta_{H,G}(h) q_{\xi,\eta}(t) dt.\end{aligned}$$

4. $q_{M(\phi)\xi,\eta}(t) = \phi(\dot{t}) q_{\xi,\eta}(t), t \in G, \phi \in C_c(G/H)$.

   Indeed, for $f \in C_c(G)$, we have

$$\begin{aligned}\int_G f(t)\phi(\dot{t}) q_{\xi,\eta}(t) dt &= \langle M((f\phi)\tilde{})\xi, \eta\rangle_\pi = \langle M(\tilde{f}\phi)\xi, \eta\rangle_\pi \\ &= \langle M(\tilde{f}) M(\phi)\xi, \eta\rangle_\pi = \int_G f(t) q_{M(\phi)\xi,\eta}(t) dt.\end{aligned}$$

5. $q_{\xi,\eta}(z^{-1}t) = q_{\pi(z)\xi,\pi(z)\eta}(t), \forall z, t \in G$.

Indeed, for $f \in C_c(G)$ we have

$$\begin{aligned}\int_G f(t)q_{\xi,\eta}(z^{-1}t)dt &= \int_G f(zt)q_{\xi,\eta}(t)dt = \langle M((\lambda(z^{-1})f)\tilde{)}\xi, \eta\rangle \\ &= \langle \pi(z^{-1})(M(\tilde{f}))\pi(z)(\xi), \eta\rangle = \int_G f(t)q_{\pi(z)\xi,\pi(z)\eta}(t)dt. \\ &\quad \text{(by the imprimitivity relation)}\end{aligned}$$

We need now to define a unitary representation of the group H. Let us therefore consider on the vector space $\mathcal{H}_\pi^\infty$ the bilinear form

$$\langle \xi, \eta\rangle_\rho := q_{\xi,\eta}(e), \quad \xi, \eta \in \mathcal{H}_\pi^\infty.$$

It follows from properties 1 and 2 of the functions $q_{\xi,\eta}$ that $\langle\cdot,\cdot\rangle_\rho$ is a scalar product. We denote by $\mathcal{H}_0$ the subspace of $\mathcal{H}_\pi^\infty$ of all ξ for which $\langle\xi,\xi\rangle_\rho = 0$. By Property 3, the subspace $\mathcal{H}_0$ is H-invariant. Let

$$\mathcal{H}_\rho := \overline{(\mathcal{H}_\pi^\infty/\mathcal{H}_0)}^{\langle\cdot,\cdot\rangle_\rho}.$$

be the closure of $\mathcal{H}_\pi^\infty/\mathcal{H}_0$ with respect to the norm obtained from $\langle\cdot,\cdot\rangle_\rho$.

We let the group H act on $\mathcal{H}_\pi^\infty/\mathcal{H}_0$ in the following way:

$$\rho(h)(\xi + \mathcal{H}_0) := \Delta_{H,G}^{1/2}(h)\pi(h)\xi + \mathcal{H}_0,\ h \in H, \xi \in \mathcal{H}_\pi^\infty.$$

We see that

$$\begin{aligned}\|\rho(h)(\xi + \mathcal{H}_0)\|_\rho^2 &= \Delta_{H,G}(h)q_{\pi(h)\xi,\pi(h)\xi}(e) \\ &= \Delta_{H,G}(h)q_{\xi,\xi}(h^{-1}) \text{ (by property 5)} \\ &= \Delta_{H,G}(h)\Delta_{G,H}(h)q_{\xi,\xi}(e) = \|\xi\|_\rho^2 \text{ (by property 3).}\end{aligned}$$

Hence this action is isometric. Furthermore, by the continuity of the functions $q_{\xi,\eta}$ we have that

$$\begin{aligned}&\lim_{h\to h_0} \|\rho(h)(\xi + \mathcal{H}_0) - \rho(h_0)(\xi + \mathcal{H}_0)\|_\rho^2 \\ &= \lim_{h\to h_0} q_{(\Delta_{H,G}^{1/2}(h)\pi(h)-\Delta_{H,G}^{1/2}(h_0)\pi(h_0))\xi,(\Delta_{H,G}^{1/2}(h)\pi(h)-\Delta_{H,G}^{1/2}(h_0)\pi(h_0))\xi}(e) \\ &= \lim_{h\to h_0} (2\|\xi + \mathcal{H}_0\|_\rho^2 - \Delta_{H,G}^{1/2}(h)\Delta_{H,G}^{1/2}(h_0)(q_{\pi(h)\xi,\pi(h_0)\xi}(e) + q_{\pi(h_0)\xi,\pi(h)\xi}(e))) \\ &= 2\|\xi + \mathcal{H}_0\|_\rho^2 - 2\Delta_{H,G}(h_0)(q_{\pi(h_0)\xi,\pi(h_0)\xi}(e)) = 0.\end{aligned}$$

This shows that $(\rho, \mathcal{H}_\rho)$ is a unitary representation of H.

Let us show that π is unitarily equivalent to the induced representation τ_ρ. We define a mapping U from $\mathcal{H}_\pi^\infty$ to the space $\mathcal{H}_{\tau_\rho}$ in the following way:

$$U(\xi)(t) := \pi(t^{-1})\xi + \mathcal{H}_0,\ t \in G, \xi \in \mathcal{H}_\pi^\infty.$$

We must of course verify that the mapping

$$U\xi : G \ni t \to \pi(t^{-1})\xi + \mathcal{H}_0$$

is really an element of $\mathcal{H}_{\tau_\rho}$. Since $\pi(t^{-1})\xi \in \mathcal{H}_\pi^\infty$ for any $t \in G$ and $\xi \in \mathcal{H}_\pi^\infty$ we have that $U\xi(t) \in \mathcal{H}_\rho$ for every $t \in G$ and $\xi \in \mathcal{H}_\pi^\infty$. Furthermore $U\xi$ is continuous. Indeed, for $s, t \in G$, we have that

$$\begin{aligned}\|U\xi(t) - U\xi(s)\|_\rho^2 &= \|\pi(t^{-1})\xi - \pi(s^{-1})\xi + \mathcal{H}_0\|_\rho^2 \\ &= q_{\pi(t^{-1})\xi-\pi(s^{-1})\xi,\pi(t^{-1})\xi-\pi(s^{-1})\xi}(e) \to 0,\end{aligned}$$

if $t \to s$, by the bilinearity in ξ, η and the continuity of the function $G \times G \times G \ni (z, y, t) \mapsto q_{\pi(z)\xi,\pi(y)\eta}(t)$.

Let us show that $U\xi$ satisfies the covariance condition. Let $t \in G$ and $h \in H$. Then

$$\begin{aligned}U\xi(th) &= \pi((th)^{-1})\xi + \mathcal{H}_0 = \pi(h^{-1})(\pi(t^{-1})\xi) + \mathcal{H}_0 \\ &= \pi(h^{-1})(U\xi(t)) = \Delta_{H,G}^{1/2}(h)\rho(h)^{-1}(U\xi(t)).\end{aligned}$$

The last line follows from the definition of the representation ρ.

Let us show that U is an isometry. Choose $0 \le f \in C_c(G)$ and $\xi \in \mathcal{H}_\pi^\infty$. Then we have

$$\begin{aligned}&\oint_{G/H} \tilde{f}(\dot{t})\|U\xi(t)\|_\rho^2 d\dot{t} = \oint_{G/H} \tilde{f}(\dot{t}) q_{\pi(t^{-1})\xi,\pi(t^{-1})\xi}(e) d\dot{t} \\ &= \oint_{G/H} \tilde{f}(\dot{t}) q_{\xi,\xi}(t) d\dot{t} \text{ (by property 5)} \\ &= \oint_{G/H} \left(\int_H f(th) q_{\xi,\xi}(th)\Delta_{G,H}(h) dh\right) d\dot{t} \text{ (by property 3)} \\ &= \int_G f(t) q_{\xi,\xi}(t) dt = \langle M(\tilde{f})\xi, \xi\rangle_\pi\end{aligned}$$

by Eq. (3.1.3).

We now take functions $f \in C_c(G)$, such that the $\tilde{f}$'s form a bounded approximate identity in $C_0(G/H)$. It then follows that $M(\tilde{f})\xi$ tends to ξ in $\mathcal{H}_\pi$, since the representation M is not degenerate. Therefore also $\oint_{G/H} \tilde{f}(t)\|U\xi(t)\|_\rho^2 d\dot{t}$ goes to $\oint_{G/H} \|U\xi(t)\|_\rho^2 d\dot{t}$; this yields that

$$\oint_{G/H} \|U\xi(t)\|_\rho^2 d\dot{t} = \|\xi\|_\pi^2$$

and so U is a linear isometry, which can be extended to an isometry of $\mathcal{H}_\pi$ into $\mathcal{H}_{\tau_\rho}$.

Let us show that U intertwines π and τ_ρ. For ξ in $\mathcal{H}_\pi^\infty$, $s, t \in G$ we have that

$$\begin{aligned}\tau_\rho(s)(U(\xi))(t) &= U\xi(s^{-1}t) = \pi(t^{-1}s)\xi + \mathcal{H}_0 \\ &= \pi(t^{-1})(\pi(s)\xi) + \mathcal{H}_0 = U(\pi(s)\xi)(t).\end{aligned}$$

Hence

$$U \circ \pi(s) = \tau_\rho(s) \circ U, \text{ for all } s \in G.$$

Let us now show that U intertwines also the representations M and M_ρ. Indeed for $\phi \in C_c(G/H), \xi \in \mathcal{H}_\pi^\infty$ and $t \in G$, we have that

$$\begin{aligned}U(M(\phi)\xi)(t) &= \pi(t^{-1})(M(\phi)\xi) + \mathcal{H}_0 \\ &= \pi(t^{-1}) \circ M(\phi) \circ \pi(t) \circ \pi(t^{-1})\xi + \mathcal{H}_0 \\ &= M(\lambda(t^{-1})\phi) \circ \pi(t^{-1})\xi + \mathcal{H}_0.\end{aligned}$$

Now by property 4 for every $\xi, \eta \in \mathcal{H}_\pi^\infty$ we have that

$$q_{M(\lambda(t^{-1})\phi)\xi,\eta}(e) = \phi(\dot{t})q_{\xi,\eta}(e) = q_{\phi(\dot{t})\xi,\eta}(e),$$

which shows that

$$M(\lambda(t^{-1})\phi)\pi(t^{-1})\xi + \mathcal{H}_0 = \phi(\dot{t})\pi(t^{-1})\xi + \mathcal{H}_0,$$

namely

$$U \circ M(\phi) = M_\rho(\phi) \circ U, \quad \phi \in C_c(G/H). \tag{3.4.1}$$

Let us show that the image of U is dense, which tells us then that U is a unitary operator. Let $\eta \in \mathcal{H}_{\tau_\rho}$, such that η is orthogonal to the image of U. By Proposition 3.2.5, we can assume that the mapping $G \ni t \to \eta(t) \in \mathcal{H}_\rho$ is continuous. Then we have by (3.4.1) for any $\xi \in \mathcal{H}_\pi^\infty$ and $\phi \in C_0(G/H)$ that

$$0 = \langle \eta, U(M(\phi)\xi)\rangle_{\mathrm{ind}\rho} = \oint_{G/H} \phi(\dot{t})\langle \eta(t), \pi(t)^{-1}(\xi) + \mathcal{H}_0\rangle_\rho d\dot{t},$$

whence we have that

$$0 = \langle \eta(t), \pi(t)^{-1}(\xi) + \mathcal{H}_0\rangle_\rho \text{ for all } t \in G \text{ and } \xi \in \mathcal{H}_\pi^\infty.$$

Replacing ξ by $\pi(t)\xi$ we get

$$0 = \langle \eta(t), \xi + \mathcal{H}_0 \rangle_\rho \text{ for all } t \in G \text{ and } \xi \in \mathcal{H}_\pi^\infty.$$

Since $\mathcal{H}_\pi^\infty$ mod $\mathcal{H}_0$ is dense in $\mathcal{H}_\rho$ it follows that $\eta(t) = 0$ for all t and so $\eta = 0$. This shows that $U : \mathcal{H}_\pi \to \mathcal{H}_{\tau_\rho}$ is a unitary intertwining operator for π and τ_ρ. In order to finish the proof of the imprimitivity theorem we must give the following.

Proof of Assumption (A):

Let $f \in C_c(G)$ and $a, b \in C_c(G), \xi_0, \eta_0 \in \mathcal{H}_\pi$ and $\xi = \pi(a)\xi_0, \eta = \pi(b)\eta_0 \in \mathcal{H}_\pi^\infty$. Then we have

$$\begin{aligned}
\langle M(\tilde{f})\xi, \eta \rangle_\pi &= \int_G \int_G a(t)\overline{b(s)} \langle M(\tilde{f})\pi(t)\xi_0, \pi(s)\eta_0 \rangle_\pi \, ds dt \\
&= \int_G \int_G a(t)\overline{b(s)} \langle \pi(t) M(\lambda(t^{-1})\tilde{f})\xi_0, \pi(s)\eta_0 \rangle_\pi \, ds dt \\
&\quad \text{(by the imprimitivity relation)} \\
&= \int_G \int_G \overline{b(s)} \langle \pi(s^{-1}t) M(a(t)\lambda(t^{-1})\tilde{f})\xi_0, \eta_0 \rangle_\pi \, ds dt \\
&= \int_G \int_G \langle \pi(s^{-1}) M(\overline{b(ts)} a(t)\lambda(t^{-1})\tilde{f})\xi_0, \eta_0 \rangle_\pi \, ds dt \\
&= \int_G \left\langle \left(\int_G M(\overline{b(ts)} a(t)\lambda(t^{-1})\tilde{f}) dt \right) \xi_0, \pi(s)\eta_0 \right\rangle_\pi ds.
\end{aligned}$$

For $s \in G$, the function

$$\phi_s(u) := \int_G \overline{b(ts)} a(t)\lambda(t^{-1})\tilde{f}(\dot{u}) dt = \int_G \int_H \overline{b(ts)} a(t) f(tuh) dh dt, \quad u \in G,$$

can be rewritten as

$$\begin{aligned}
\phi_s(u) &= \int_G \int_H \Delta_G^{-1}(uh)\overline{b(t(uh)^{-1}s)} a(t(uh)^{-1}) f(t) dh dt \\
&= \int_G f(t) \left(\int_H \Delta_G^{-1}(st)\overline{(\Delta_G b)(t(uh)^{-1}s)} a(t(uh)^{-1}) dh \right) dt.
\end{aligned}$$

Let us put

$$\begin{aligned}
c_{s,t}(u) : &= \Delta_G^{-1}(st)\overline{(\Delta_G b)(tu^{-1}s)} a(tu^{-1}) \\
&= \Delta_G^{-1}(st)\overline{(\Delta_G b)((ut^{-1})^{-1}s)} a((ut^{-1})^{-1}), \; s, t, u \in G.
\end{aligned} \tag{3.4.2}$$

Then the functions $c_{s,t}$ $(s, t \in \mathbb{R})$ are in $C_c(G)$ and the mapping $G \times G \ni (s, t) \mapsto c_{s,t} \in (C_c(G), \|\cdot\|_\infty)$ is continuous. We have still to verify that the mapping $G \times G \ni$

$(s,t) \mapsto \tilde{c_{s,t}} \in C_c(G/H)$ is continuous with compact support in the variables s and u for fixed t. We can pick a compact subset K of G such that the support of the function $G \times G \ni (s,u) \mapsto b(u^{-1}s)a(u^{-1})$ is contained in $K \times K$. Then it follows from (3.4.2) that for fixed t, the function $(s,u) \mapsto c_{s,t}(u)$ has its support in $K \times Kt$. Hence, for $s,u,t \in G$ the function $G \times H \ni (s,h) \mapsto c_{s,t}(uh)$ is supported by $K \times (u^{-1}Kt \cap H)$ and so

$$\tilde{c_{s,t}}(u) = \int_{H \cap u^{-1}Kt} \Delta_G^{-1}(st)\overline{(\Delta_G b)(t(uh)^{-1}s)}a(t(uh)^{-1})dh,$$

which shows that the function $G \times G \times G \ni (s,t,u) \mapsto \tilde{c_{s,t}}(u)$ is continuous and its support in (s,u) (for fixed t) is contained in $K \times (KtH)$. Hence the integral

$$\phi_s = \int_G f(t)\tilde{c_{s,t}}dt$$

converges in $C_c(G/H)$ and so

$$M(\phi_s) = \int_G f(t)M(\tilde{c_{s,t}})dt,$$

which yields that

$$\langle M(\tilde{f})\xi,\eta\rangle_\pi = \int_G f(t)\langle M(\tilde{c_{s,t}})\xi_0,\pi(s)\eta_0\rangle_\pi dsdt = \int_G f(t)q_{\xi,\eta}(t)dt,$$

where

$$q_{\xi,\eta}(t) := \int_G \langle M(\tilde{c_{s,t}})\xi_0,\pi(s)\eta_0\rangle_\pi ds,\ t \in G,$$

is continuous in t.

A similar argument works for the continuity of the mapping $G \times G \times G \ni (z,y,t) \mapsto q_{\pi(z)\xi,\pi(y)\eta}(t)$. Indeed, since

$$\pi(z)\pi(a)\xi_0 = \int_G a(s)\pi(z)\pi(s)\xi_0 ds = \int_G a(s)\pi(zs)\xi_0 ds = \int_G a(z^{-1}s)\pi(s)\xi_0 ds,$$

we compute as above that for $z, y \in G$, $f, a, b \in C_c(G)$, $\xi_0, \eta_0 \in \mathcal{H}_\pi$ and $\xi = \pi(a)$ ξ_0, $\eta = \pi(b)\eta_0$ we have as above that

$$\langle M(\tilde{f})\pi(z)(\xi),\pi(y)\eta\rangle_\pi = \int_G\int_G a(z^{-1}t)\overline{b(y^{-1}s)}\langle M(\tilde{f})\pi(t)\xi_0,\pi(s)\eta_0\rangle_\pi dsdt$$

$$= \int_G \left\langle\left(\int_G M(\overline{b(y^{-1}ts)}a(z^{-1}t)\lambda(t^{-1})\tilde{f})dt\right)\xi_0,\pi(s)\eta_0\right\rangle_\pi ds.$$

Hence, proceeding as before, we see that

$$q_{\pi(z)\xi,\pi(y)\eta}(t) := \int_G \langle M(\widetilde{c_{z,y,s,t}})\xi_0, \pi(s)\eta_0\rangle_\pi ds,\ t \in G,$$

where

$$\widetilde{c_{z,y,s,t}}(u) = \int_{H\cap u^{-1}Kz^{-1}t} \Delta_G^{-1}(st)\overline{(\Delta_G b)(y^{-1}t(uh)^{-1}s)}a(z^{-1}t(uh)^{-1})dh. \quad \blacksquare$$

Corollary 3.4.3. *Let G be a connected locally compact group and let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G. Let C be a closed abelian normal subgroup of G. Suppose that the spectrum $\hat{C}$ of the group C contains an open G-invariant subset U of $\hat{C}$, whose complement in $\hat{C}$ consists of G-fixed points, which has the following separation property. Two distinct G-orbits in U have disjoint open G-invariant neighbourhoods. Then there exist a character $\chi \in S$ and an irreducible representation π_0 of the stabilizer G_0 of χ in G such that $\pi = ind_H^G \pi_0$ and such that $\pi_0|_C = \chi Id_{\mathcal{H}_{\pi_0}}$.*

Proof. We know from Proposition 2.3.22 that the support S of the representation $\rho := \pi_{|C}$ is the closure of a G-orbit $O := G \cdot \chi$ for a certain unitary character χ of C. In particular $S \cap U = O$ and so the G-orbit O is locally closed. Let G_0 be the stabilizer of χ in G. In order to construct a system of imprimitivity, we use the identification of G/G_0 with $G \cdot \chi$. Let $C_{0,0}(S)$ be the subspace of $C_0(S)$, consisting of all functions which vanish on the boundary $S \setminus G \cdot \chi$ of S. The mapping

$$C_0(G/G_0) \to C_0(S);\ \varphi \mapsto \tilde{\varphi}, \tag{3.4.3}$$

where $\tilde{\varphi}(g \cdot \chi) := \varphi(g), g \in G$, is an isomorphism of C^* algebras. Since the kernel of $\pi|_C$ in $C^*(C)$ is $\{f \in C^*(C); \hat{f} \equiv 0 \text{ on } S\}$, we can define a representation ρ of $C_0(S)$ on $\mathcal{H}_\pi$ by letting

$$\rho(\psi) := \pi|_C(f), (\hat{f})|_S = \psi. \tag{3.4.4}$$

This allows us to define the representation M of $C_0(G/G_0)$ on $\mathcal{H}_\pi$ through

$$M(\varphi) := \rho(\tilde{\varphi}),\ \varphi \in C_c(G/G_0). \tag{3.4.5}$$

Since $C_{0,0}(S)$ is not contained in the kernel of ρ, it follows from Proposition 2.3.23 that M is not degenerate. The imprimitivity relation also holds. Let $u \in G, \varphi \in C_c(G/G_0)$. Then there exists $f \in C^*(C)$, such that $\hat{f} = \tilde{\varphi}$ on S and so, letting $d(u^{-1}xu) = \delta_C(u)dx$ $(x \in C,\ u \in G)$,

$$\begin{aligned}\pi(u)\circ M(\varphi)\pi(u^{-1}) &= \pi(u) \circ \pi|_C(f)\pi(u^{-1}) \\ &= \delta_C(u)\pi|_C({}^u f) = \rho({}^u \hat{f}|_S)\end{aligned}$$

$$(\text{where } {}^u\psi(r) := \psi({}^{u^{-1}}r), r \in \hat{C}, \psi \in C_0(\hat{C}))$$
$$= \rho({}^u\tilde{\varphi}) = \rho((\lambda(u)\varphi)\tilde{}) = M(\lambda(u)\varphi).$$

The imprimitivity theorem tells us that $\pi \simeq \mathrm{ind}_{G_0}^G \rho$, for some irreducible representation ρ of G_0. Let $U : \mathcal{H}_\pi \to L^2(G/G_0, \rho)$ be an intertwining operator for the two imprimitivity systems. Then for $f \in C_c(C)$, we have that $\pi|_C(f) = M(\varphi)$, where as above $\tilde{\varphi} = \hat{f}|_S$. Then for $\xi \in \mathcal{H}_\pi$ we get

$$\hat{f}(\chi)U(\xi)(e) = \varphi(e)U(\xi)(e) = U(M(\varphi)\xi)(e) = U(\pi|_C(f)\xi)(e)$$
$$= (\mathrm{ind}_{G_0}^G \rho)(f)(U\xi)(e) = \rho|_C(f)(U\xi)(e).$$

We deduce from this that $\rho|_C(f) = \hat{f}(\chi)Id_{\mathcal{H}_\rho}$. Hence C is in the projective kernel of ρ and $\rho(c) = \chi(c)Id_{\mathcal{H}_\rho}, c \in C$. ∎

Mackey Theory: résumé

It is well known that a connected and simply connected solvable Lie group G is obtained starting from $\mathbb{R}$ and repeating a semi-direct product by $\mathbb{R}$. Therefore, if we would like to study the unitary representation of G, the Mackey theory we have seen is fundamental. We resume it in such a way that we shall need it later and apply it to typical exponential solvable Lie groups in the next chapter. It will be the objective of Chap. 5 to generalize these examples. Readers are recommended to refer to the lecture note [56].

Let G be a locally compact group satisfying the second countability condition and A a closed commutative normal subgroup of G on which G acts smoothly (for instance if $A = C$ with the properties of Theorem 3.3.3 and Corollary 3.4.3). Then G acts on the space $\hat{A}$ of all unitary characters of A: for $g \in G, \chi \in \hat{A}$, $(g \cdot \chi)(a) = \chi(g^{-1}ag)$ $(a \in A)$ in a nice way and we have the following

Theorem 3.4.4 (1). *Let $G(\chi)$ be the stabilizer of $\chi \in \hat{A}$ in G.*

1. *Let ρ be an irreducible unitary representation of $G(\chi)$ such that its restriction $\rho|_A$ to A is a multiple of χ, i.e. there exists a certain $m \in \mathbb{N} \cup \{\infty\}$ such that $\rho|_A = m\chi$. Then the induced representation $\mathrm{ind}_{G(\chi)}^G \rho$ is irreducible.*
2. *Let ρ_1, ρ_2 be two irreducible representations of $G(\chi)$ such that $\rho_1|_A, \rho_2|_A$ are multiples of χ. Then $\mathrm{ind}_{G(\chi)}^G \rho_1 \simeq \mathrm{ind}_{G(\chi)}^G \rho_2$ if and only if $\rho_1 \simeq \rho_2$.*

(2) *Suppose that the G-orbit $G \cdot \chi$ is for every $\chi \in \hat{A}$ a locally closed set in $\hat{A}$. Then any irreducible unitary representation π of G, which is not trivial on A, is obtained as $\pi \simeq \mathrm{ind}_{G(\chi)}^G \rho$ with a certain $\chi \in \hat{A}$ and an irreducible unitary representation ρ of $G(\chi)$ such that $\rho|_A$ is a multiple of χ.*

(See Proposition 3.3.4, Theorem 3.3.3 and Corollary 3.4.3.)

Chapter 4
Four Exponential Solvable Lie Groups

4.1 The Group $\mathbb{R}^n$

The group $(\mathbb{R}^n, +), n \in \mathbb{N}^*$, is the only connected and simply connected abelian Lie group of dimension n. Its irreducible unitary representations are one-dimensional by Schur's lemma (2.3.7), and define in this way unitary characters $\chi : \mathbb{R}^n \to \mathbb{C}$ of $\mathbb{R}^n$. Such a character χ is of the form χ_ℓ for some real-valued linear functional ℓ on $\mathbb{R}^n$, i.e.

$$\chi(x) = \chi_\ell(x) = e^{-2\pi i \ell(x)}, \ x \in \mathbb{R}^n.$$

For $\varphi \in L^1(\mathbb{R}^n)$, we obtain the Fourier transform $\hat{\varphi}$ of φ, where

$$\hat{\varphi}(\ell) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i \ell(x)} dx, \ \ell \in (\mathbb{R}^n)^*.$$

The functions $\hat{\varphi}$ are continuous, tend to 0 at infinity and $\|\hat{\varphi}\|_\infty \le \|\varphi\|_1$. Furthermore for $\varphi, \psi \in L^1(\mathbb{R}^n)$, we have that

$$\widehat{\varphi * \psi} = \hat{\varphi} \cdot \hat{\psi}.$$

Also,

$$\widehat{f^*} = \overline{\hat{f}}, f \in L^1(\mathbb{R}^n).$$

H. Fujiwara, J. Ludwig, *Harmonic Analysis on Exponential Solvable Lie Groups*, Springer Monographs in Mathematics, DOI 10.1007/978-4-431-55288-8_4

If $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then $\hat{f}$ is in $L^2(\mathbb{R}^n)$ too and we have Plancherel's formula

$$\|f\|_2 = \|\hat{f}\|_2.$$

This allows us to define the Fourier transform also for general L^2-functions and this generalized Fourier transform becomes a unitary operator from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. Fourier's inversion formula tells us that for any $\varphi \in L^1(\mathbb{R}^n)$, such that $\hat{\varphi}$ is also integrable, we have that

$$\varphi(x) = \int_{\mathbb{R}^n} \hat{\varphi}(\ell) e^{2\pi i \ell(x)} d\ell, \ x \in \mathbb{R}^n, \text{ i.e. } \varphi = \check{\hat{\hat{\varphi}}},$$

where $\check{\varphi}$ is defined by $\check{\varphi}(x) = \varphi(-x), \ x \in \mathbb{R}^n$.

A particularly interesting subalgebra of $L^1(\mathbb{R}^n)$ is the Schwartz algebra $\mathcal{S}_n$ consisting of the rapidly decreasing C^∞-functions on $\mathbb{R}^n$, i.e.

$$\mathcal{S}_n := \{\varphi : \mathbb{R}^n \to \mathbb{C}, \ P\partial^\alpha \varphi \in L^2(\mathbb{R}^n) \\ \text{for every polynomial function } P \text{ and } \alpha \in \mathbb{N}^n\}.$$

Here ∂^α is the partial differential operator

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

for $\alpha = (\alpha_1 \cdots \alpha_n) \in \mathbb{N}^n$. The space $\mathcal{S}_n$ is mapped under the Fourier transform onto itself and the Fourier transform is thus an isomorphism of Fréchet spaces.

Let us remark that the C^*-algebra of $L^1(\mathbb{R}^n)$ is the algebra $C_0(\mathbb{R}^n)$ of the continuous complex-valued functions on $\mathbb{R}^n$ which go to 0 at infinity. The Fourier transform maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$ and the C^*-norm $\|f\|_{C^*}$ of $f \in L^1(\mathbb{R}^n)$ is given by

$$\|f\|_{C^*} = \sup_{y \in \mathbb{R}^n} |\hat{f}(y)| = \|\hat{f}\|_\infty.$$

If now π is a unitary representation of $\mathbb{R}^n$, let S be the support of π. Then S is a closed subset of $\mathbb{R}^n$ and $\ker\pi = \{f \in C^*(\mathbb{R}^n); \hat{f} = 0 \text{ on } S\}$. In particular $C^*(\mathbb{R}^n)/\ker\pi$ is isomorphic with $C_0(S)$, an isomorphism being given by $f \mapsto \hat{f}|_S$. This tells us that for $f \in C^*(\mathbb{R}^n)$

$$\|\pi(f)\|_{\text{op}} = \|\hat{f}_{|S}\|_\infty \leq \|\hat{f}\|_\infty. \tag{4.1.1}$$

4.2 The Heisenberg Group

Let $H_1 = \exp \mathfrak{g}$, where $\mathfrak{g} = \langle X, Y, Z\rangle_{\mathbb{R}}$; $[X, Y] = Z$. The Heisenberg group H_1 of dimension 3 is a typical nilpotent Lie group. In matrix form,

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_1 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\}.$$

More generally, let

$$H_n := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}.$$

We equip H_n with the following multiplication:

$$(x, y, t)(x', y', t') := \left(x + x', y + y', t + t' + \frac{1}{2} \left(xy' - x'y\right) \right),$$

where for two elements $x = (x_1, \ldots, x_n)$, $y' = (y'_1, \ldots, y'_n) \in \mathcal{R}^n$ the symbol xy' means

$$xy' := \sum_{j=1}^{n} x_j y'_j.$$

We verify that this multiplication is associative, that the element $0 = (0_n, 0_n, 0)$ of H_n is the neutral element and that the inverse of $h = (x, y, t)$ of H_n is the vector $h^{-1} = (-x, -y, -t) = -h$. The centre C_n of the group H_n is the subgroup

$$C_n = \{(0, 0, t);\ t \in \mathbb{R}\} = \{0\} \times \{0\} \times \mathbb{R}.$$

We see that for every (x, y, t) and (u, v, s) in H_n, we have that

$$\begin{aligned}[] [(x, y, t), (u, v, s)] &= (x, y, t)(u, v, s)(-x, -y, -t)(-u, -v, -s) \\ &= (0_n, 0_n, x{\cdot}v - y{\cdot}u). \end{aligned}$$

Hence

$$[H_n, H_n] = C_n.$$

The Lie algebra $\mathfrak{h}_n$ of H_n is the real vector space $\mathfrak{h}_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the bracket

$$[(x, y, t), (x', y', t')] := (0_n, 0_n, xy' - x'y).$$

This follows easily from the fact that

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{d}{dt}(tx, ty, tz)\left(sx', sy', sz'\right)\Big|_{t=0}\right)\Big|_{s=0} \\
&= \frac{d}{ds}\left(\frac{d}{dt}\left(\left(tx + sx', ty + sy', tz + sz' + \frac{st}{2}\left(xy' - x'y\right)\right)\Big|_{t=0}\right)\Big|_{s=0}\right. \\
&= (0_n, 0_n, \frac{1}{2}(xy' - x'y)).
\end{aligned}$$

Hence we can also write the multiplication in H_n as

$$UV = U + V + \frac{1}{2}[U, V],\ U, V \in H_n.$$

It follows that

$$UVU^{-1} = V + [U, V],\ U, V \in H_n.$$

The mapping $\mathbb{R} \ni t \to tU$ for a fixed element U of H_n is then a one-parameter subgroup of H_n and every such a one-parameter subgroup is of this form. Indeed, for $s, t \in \mathbb{R}$, $U \in H_n$, we have that

$$(sU)(tU) = sU + tU + \frac{st}{2}[U, U] = sU + tU = (s + t)U.$$

Hence the exponential mapping $\exp : \mathfrak{h}_n \to H_n$ is the identity.

We take the following basis $\mathcal{Z}$ of the Lie algebra $\mathfrak{h}_n$:

$$\mathcal{Z} := \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\},$$

where

$$X_j = ((\delta_{i,j})_i, 0_n, 0), Y_j = (0_n, (\delta_{i,j})_i, 0), Z = (0_n, 0_n, 1).$$

This gives us the non-trivial brackets

$$[X_i, Y_j] = \delta_{i,j} Z.$$

We see from the definition of the multiplication in H_n that Lebesgue measure on H_n is left- and right-invariant, hence it is our Haar measure, and H_n is unimodular.

4.2.1 Induced Representations

Let $P := \{0\} \times \mathbb{R}^n \times \mathbb{R} \subset H_n$. Obviously P is a closed, connected and abelian subgroup of H_n. We can consider P also as an abelian subalgebra $\mathfrak{p}$ of $\mathfrak{h}_n$ and write formally $P = \exp \mathfrak{p}$. Since P is abelian, every linear functional f of $\mathfrak{h}_n$ defines a unitary character χ_f of P through the rule

$$\chi_f(\exp U) := e^{-2\pi i f(U)},\ U \in \mathfrak{p}.$$

We can now define the induced representation $\tau_f := \tau_{\chi_f}$ of G as in the definition (3.2.2). Let us show that the Hilbert space $\mathcal{H}_f$ of τ_f is naturally isomorphic to the Hilbert space $L^2(\mathbb{R}^n)$. Indeed, since H_n and P are unimodular, the homogeneous space H_n/P has a (unique) left invariant Borel measure and it is easy to check that this measure $\int_{H_n/P} d\dot{x}$ is given by

$$\varphi \mapsto \int_{\mathbb{R}^n} \varphi(U, 0, 0) dU, \varphi \in C_c(H_n/P).$$

Indeed, for $\varphi \in C_c(H_n/P)$, $g = (x, y, t) \in H_n$ we have that

$$\int_{\mathbb{R}^n} \varphi(g^{-1}(u, 0, 0)) du = \int_{\mathbb{R}^n} \varphi\left(\left(u - x, -y, -t + \frac{1}{2} yu\right)\right) du$$
$$= \int_{\mathbb{R}^n} \varphi\left((u - x, 0_n, 0)\left(0_n, -y, -t + \frac{1}{2} yu + \frac{1}{2}(u - x)y\right)\right) du$$
$$= \int_{\mathbb{R}^n} \varphi(u - x, 0_n, 0) du = \int_{\mathbb{R}^n} \varphi(u, 0_n, 0) du.$$

In particular the norm of φ is given by

$$\|\varphi\|_{\tau_f} = \left(\int_{\mathbb{R}^n} |\varphi(u, 0_n, 0)|^2 du\right)^{1/2}.$$

For $\xi \in \mathcal{E}(H_n/P, \chi_f)$, the function $R\xi$ defined on $\mathbb{R}^n$ by

$$R\xi(U) := \xi(U, 0_n, 0),\quad U \in \mathbb{R}^n,$$

is in $C_c(\mathbb{R}^n)$ and

$$\|R\xi\|_2^2 = \int_{\mathbb{R}^n} |\xi(U, 0_n, 0)|^2 dU = \int_{H_n/P} |\xi(x)|^2 dx = \|\xi\|^2_{\mathrm{ind}\chi_f}.$$

Furthermore, for every $\psi \in C_c(\mathbb{R}^n)$, if we define the function ξ on G by

$$\xi((U, 0_n, 0)(0, V, t)) := e^{-2\pi i f(0,V,t)} \psi(U);\ U, V \in \mathbb{R}^n, t \in \mathbb{R},$$

then ξ is an element of $\mathcal{E}(H_n/P, \chi_f)$ and $R\xi = \psi$. This shows that the mapping R is a linear bijection between $\mathcal{E}(H_n/P, \chi_f)$ and $C_c(\mathbb{R}^n)$ and extends to a unitary operator, which we shall also denote by R, between $\mathcal{H}_f$ and $L^2(\mathbb{R}^n)$.

By transferring the representation τ_f to $L^2(\mathbb{R}^n)$, we obtain a representation $\pi_{f,P} = \pi_f$ of H_n on $L^2(\mathbb{R}^n)$. For $\psi = R(\xi) \in L^2(\mathbb{R}^n)$, $g = (X, Y, t) \in H_n$ and $U \in \mathbb{R}^n$ we have

$$
\begin{aligned}
\pi_f(g)\psi(U) &= R(\tau_f(g)\xi)(U) = \xi((-X, -Y, -t)(U, 0_n, 0)) \\
&= \xi\left(U - X, -Y, -t + \frac{1}{2}Y \cdot U\right) \\
&= \xi\left((U - X, 0_n, 0)\left(0_n, -Y, -t + \frac{1}{2}Y \cdot U + \frac{1}{2}(U - X)Y\right)\right) \\
&= \xi\left((U - X, 0_n, 0)\left(0_n, -Y, -t + Y \cdot U - \frac{1}{2}X \cdot Y\right)\right) \\
&= e^{-2\pi i f(t + \frac{1}{2}X\cdot Y - Y\cdot U)}\psi(U - X). \qquad (4.2.1)
\end{aligned}
$$

Let us now write $\mu := \langle f, Z\rangle$. We write f_μ for the linear functional which is zero on all the X_i's and all the Y_j's and which takes the value μ on Z. We remark that for

$$(0_n, 0_n, t) \in C_n, \pi_f(0_n, 0_n, t) = e^{-2\pi i\mu t} Id_{L^2(\mathbb{R}^n)}. \qquad (4.2.2)$$

Remark 4.2.1. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of H_n. By Schur's lemma (2.3.7), there exists a unitary character χ of the centre C_n, such that $\pi(z) = \chi_{\mu_\pi}(z)\mathbb{I}_{\mathcal{H}},\ z \in C_n$ for some $\mu_\pi \in \mathbb{R}$. Since C_n is isomorphic to the real line, it follows that

$$\chi(tZ) = e^{-2\pi i\mu_\pi t},\ t \in \mathbb{R},$$

for some $\mu_\pi \in \mathbb{R}$. We shall write χ_π for this character of C_n.

Theorem 4.2.2. *Let $\mu \in \mathbb{R}^*$. The representation $\pi_\mu := \pi_{f_\mu,P}$ is irreducible. Every irreducible unitary representation $(\pi, \mathcal{H})$ of H_n, which restricts to the centre C_n of H_n to a multiple of the character χ_μ is equivalent to π_μ.*

Proof. We shall use Theorem 3.3.3 and Proposition 3.4.3.

We describe first the H_n-orbits of a character $\chi_f, f \in \mathfrak{h}_n^*$, of the normal abelian closed and connected subgroup P of H_n. For $g = (u, v, 0) \in H_n$ and $p = (0, y, t) \in P, \mu = f(Z_n) \in \mathbb{R}$, we have that

$$
\begin{aligned}
\chi_f^g(p) = g \cdot \chi_f(p) = \chi_f(g^{-1}pg) &= e^{-2\pi i f((-u,-v,0)(0,y,t)(u,v,0))} \\
&= e^{2\pi i\mu u\cdot y}e^{-2\pi i\mu t} = \chi_{Ad^*(g)f}(p),
\end{aligned}
$$

where $Ad^*(g)f,\ f \in \mathfrak{h}_n^*$ is the linear form defined on $\mathfrak{h}_n$ by

$$Ad^*(g)f(x,y,t) = f(g^{-1}(x,y,t)g) = f(x,y,t - u{\cdot}y + v{\cdot}x).$$

Hence identifying the spectrum $\hat{P}$ of the abelian group P with the linear dual space of $\mathfrak{p}$ and then with $\mathbb{R}^n \times \mathbb{R}$, we see that

$$\begin{aligned}&\chi_f^g \simeq (\mu u, \mu) \in \mathbb{R}^n \times \mathbb{R} \text{ and}\\ &G \cdot \chi_f \simeq \mathbb{R}^n \times \{\mu\} =: O_\mu \text{ if } \mu \neq 0, G \cdot \chi_f = \{f\}, \text{ if } \mu = 0.\end{aligned}$$

Now let $U := \{f \in \mathfrak{p}^*, f(Z_n) \neq 0\}$. Then U is an open H_n-invariant subset of $\hat{P}$ and its complement is the H_n-invariant subset $Z_n^\perp$. Two distinct H_n-orbits O_μ and $O_{\mu'}$ have disjoint open neighbourhoods $V = \{f \in \mathfrak{p}^*, |f(Z_n) - \mu| < \varepsilon\}$ resp. $V = \{f \in \mathfrak{p}^*, |f(Z_n) - \mu'| < \varepsilon\}$, where $2\varepsilon := |\mu - \mu'|$.

Furthermore, the stabilizer in H_n of a character χ_f with $\mu = f(Z_n) \neq 0$ is obviously the group P itself.

Hence every induced representation $\pi_f := \operatorname{ind}_P^{H_n} \chi_f$ with $f(Z_n) \neq 0$ is irreducible.

Furthermore if $(\pi, \mathcal{H})$ is an irreducible unitary representation, then by Schur's lemma it restricts on the centre to a multiple of a character χ_μ, where $\chi_\mu(0_n, 0_n, t) = e^{-2\pi i \mu t}, t \in \mathbb{R}$. By Proposition 3.4.3, π restricts to P to the H_n-orbit O_μ and π is equivalent to any representation $\pi_f, f \in \mathfrak{p}^*$, with $f(Z_n) = \mu$. ■

Definition 4.2.3. Let $SB(L^2(\mathbb{R}^n))$ be the space of bounded linear operators on the Hilbert space $L^2(\mathbb{R}^n)$ with Schwartz kernels, which means that every element b of $SB(L^2(\mathbb{R}^n))$ is a kernel operator with a kernel function F_b which is a Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$.

We denote by $\mathcal{S}(H_n)$ the space of the rapidly decreasing C^∞-functions on H_n, i.e.

$$\begin{aligned}\mathcal{S}(H_n) := \{f : H_n \to \mathbb{C}; f \in C^\infty(H_n),\\ p\partial^\alpha f \in L^2(H_n), p \text{ is a polynomial function}, \alpha \in \mathbb{N}^n\}.\end{aligned}$$

(It is easy to see that $\mathcal{S}(H_n)$ is an involutive Fréchet subalgebra of $L^1(H_n)$.)

Proposition 4.2.4. *Let $\mu \in \mathbb{R}^*$. For every $f \in \mathcal{S}(H_n)$ the operator $\pi_\mu(f)$ is a smooth bounded linear operator on $L^2(\mathbb{R}^n)$ and the mapping $\mathcal{S}(H_n) \mapsto SB(L^2(\mathbb{R}^n)) : f \to \pi_\mu(f)$ is surjective.*

Proof. For $f \in \mathcal{S}(H_n)$, the operator $\pi_\mu(f)$ is a Hilbert–Schmidt operator, whose kernel function f_μ is a Schwartz function. Indeed, for $\varphi \in \mathcal{S}(\mathbb{R}^n), u \in \mathbb{R}^n$, we have by (4.2.1) that

$$\begin{aligned}\pi_\mu(f)\varphi(u) &= \int_{H_n} f(x,y,t)e^{-2\pi i(\mu t+\frac{1}{2}\mu xy-\mu yu)}\varphi(u-x)dxdydt \\ &= \int_{H_n} f(u-x,y,t)e^{-2\pi i(\mu t+\frac{1}{2}\mu(-u-x)y)}\varphi(x)dxdydt \\ &= \int_{H_n} \hat{f}^{2,3}\left(u-x,-\mu\frac{1}{2}(u+x),\mu\right)\varphi(x)dx.\end{aligned}$$

where

$$\hat{f}^{2,3}(u,\nu,\mu) := \int_{\mathbb{R}^n\times\mathbb{R}} f(u,y,t)e^{-2\pi i(\nu y+\mu t)}dydt,\ (\nu,t)\in\mathbb{R}^n\times\mathbb{R},$$

is the partial Fourier transform in the variables y and t. This tells us that the operator $\pi_\mu(f)$ is a kernel operator with kernel function f_μ given by

$$f_\mu(u,x) = \hat{f}^{2,3}\left(u-x,-\frac{\mu}{2}(u+x),\mu\right),\ x,u\in\mathbb{R}^n, \tag{4.2.3}$$

which is a Schwartz function on $\mathbb{R}^n\times\mathbb{R}^n$. Hence $\pi_\mu(f)\in B(L^2(\mathbb{R}^n))^\infty$.

Now let $b\in B(L^2(\mathbb{R}^n))^\infty$ and let $F_b\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ be its kernel function.

Define the Schwartz function $\psi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ by

$$\psi(u,v) := F_b\left(-\frac{-u}{2}-\frac{v}{\mu},\frac{u}{2}-\frac{v}{\mu}\right), u,v\in\mathbb{R}^n.$$

Then

$$\psi(s-t,-\frac{\mu}{2}(s+t)) = F_b(s,t), s,t\in\mathbb{R}^n.$$

Take also a Schwartz function $\beta:\mathbb{R}\to\mathbb{C}$, such that $\hat{\beta}(\mu)=1$. We can now define the Schwartz function f on H_n by

$$f(x,y,t) := \hat{\psi}^{2,3}(x,-y)\hat{\beta}(t), (x,y,t)\in H_n.$$

Since

$$\hat{f}^{2,3}(s-t,-\frac{\mu}{2}(s+t),\mu)\hat{\beta}(\mu) = \psi(s-t,-\frac{\mu}{2}(s+t)) = F_b(s,t), s,t\in\mathbb{R}^n*,$$

it follows that $\pi_\mu(f)=b$. ■

We now give another proof of the imprimitivity theorem for the Heisenberg groups H_n. This result was first obtained by Stone and Von Neumann independently in 1929.

Theorem 4.2.5. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of H_n, such that $\mu = -\frac{1}{2\pi i} d\pi(Z) \neq 0$. Then π is equivalent to the representation π_μ of Proposition 4.2.4.*

Proof. Let us show first that the kernel of π in $L^1(H_n)$ is equal to the kernel of π_μ. By Formula (4.2.3) it follows that

$$\begin{aligned}\ker\pi_\mu &= \{f \in L^1(H_n), \hat{f}^{2,3}(s,u,\mu) = 0 \text{ for all } u, s \in \mathbb{R}^n\}\\ &= \{f \in L^1(H_n); \int_{\mathbb{R}} f(a\cdot(tZ))e^{-2\pi i\mu t}dt = 0, a \in H_n \text{ almost everywhere}\}.\end{aligned}$$

If now $f \in \ker\pi_\mu$, then

$$\begin{aligned}\pi(f) &= \int_{H_n} f(g)\pi(g)dg\\ &= \int_{G/C_n}\int_{\mathbb{R}} \pi(g\cdot tZ)f(a\cdot tZ)dt\,d\dot{g}\\ &= \int_{G/C_n} \pi(g)\left(\int_{\mathbb{R}} e^{-2\pi i\mu t} f(a\cdot tZ)dt\right)d\dot{g}\\ &= \int_{G/C_n} \pi(g)(0)d\dot{g}\\ &= 0.\end{aligned}$$

Hence $\ker\pi_\mu \subset \ker\pi$. Since $\pi(l(a)\circ r(a))f \in \ker\pi$ for every $f \in \ker\pi, a = u+v, u \in X, v \in Y$, it follows that

$$\begin{aligned}0 &= \pi(l(a)\circ r(a)(f))\\ &= \int_{H_n} f(a^{-1}ga)\pi(g)dg\\ &= \int_{H_n} f(g)\pi(aga^{-1})dg\\ &= \int_{H_n} f(g)\pi(g+[a,g])dg\\ &= \int_{H_n} f(g)\pi(g)e^{-2\pi i\mu(u\cdot y - v\cdot x)}dg\\ &= \int_{X+Y}\left(\int_{\mathbb{R}} f(x+y+tZ)e^{-2\pi i\mu t}dt\right)\pi(x+y)e^{-2\pi i\mu(u\cdot y-v\cdot x)}dx\,dy.\end{aligned}$$

for all $a \in H_n$. Choosing any $\xi, \eta \in \mathcal{H}_\pi$ it then follows that

$$0 = \int_{X+Y} \left(\int_{\mathbb{R}} f(x + y + tZ) e^{-2\pi i \mu t} dt \right) c^\pi_{\xi,\eta}(x + y) e^{-2\pi i \mu (u \cdot y - v \cdot x)} dx dy$$

for $u \in X, v \in Y$ (where $c^\pi_{\xi,\eta}(x + y) := \langle \pi(x + y)\xi, \eta \rangle$). Hence the functions

$$(x, y) \mapsto c^\pi_{\xi,\eta}(x + y) \left(\int_{\mathbb{R}} f(x + y + tZ) e^{-2\pi i \mu t} dt \right)$$

are 0 for any $\xi, \eta \in \mathcal{H}_\pi$. This means that $\int_{\mathbb{R}} f(x + y + tZ) e^{-2\pi i \mu t} dt = 0$ for every $x \in X, y \in Y$. Hence $\ker\pi \subset \ker\pi_\mu$.

We take now a Schwartz function $\xi \in \mathcal{S}((\mathbb{R}^n))$. There exists a self-adjoint $f_\xi \in \mathcal{S}(H_n)$, such that $\pi_\mu(f_\xi)$ is the self-adjoint projection P_ξ onto the one-dimensional subspace $\mathbb{C}\xi$ of $L^2(\mathbb{R}^n)$. But then $\pi(f_\xi)$ is also a self-adjoint projection in $B(\mathcal{H}_\pi)$ different from 0, since $\ker\pi = \ker\pi_\mu$ and $\pi(f_\xi * f_\xi - f_\xi) = 0$. Take now η in the image of $\pi(f_\xi)$ of length one. Then for any $f \in L^1(H_n)$ we have that

$$\begin{aligned} \pi_\mu(f_\xi * f * f_\xi) &= P_\xi \circ \pi_\mu(f) \circ P_\xi \\ &= \langle \pi_\mu(f)\xi, \xi \rangle P_\xi \\ &= \pi_\mu(\langle \pi_\mu(f)\xi, \xi \rangle f_\xi). \end{aligned}$$

Hence $f_\xi * f * f_\xi = \langle \pi_\mu(f)\xi, \xi \rangle f_\xi$ modulo $\ker\pi_\mu$ and so finally

$$\pi(f_\xi * f * f_\xi) = \langle \pi_\mu(f)\xi, \xi \rangle \pi(f_\xi).$$

Therefore

$$\begin{aligned} \langle \pi(f)\eta, \eta \rangle &= \langle \pi(f)(\pi(f_\xi)\eta), \pi(f_\xi)(\eta) \rangle \\ &= \langle \pi(f_\xi * f * f_\xi)(\eta), \eta \rangle \\ &= \langle \pi_\mu(f)\xi, \xi \rangle \langle \pi(f_\xi(\eta)), \eta \rangle \\ &= \langle \pi_\mu(f)\xi, \xi \rangle. \end{aligned}$$

Hence $E(\pi) \cap E(\pi_\mu) \neq \emptyset$ and therefore π and π_μ are equivalent. ■

Theorem 4.2.6. *The unitary dual $\widehat{H_n}$ is parametrized by $\mathbb{R}^* \bigcup \mathbb{R}^{2n}$. For every $\mu \in \mathbb{R}^*$, the representation π_μ acts on $L^2(\mathbb{R}^n)$ and for every $h \in \mathbb{R}^{2n}$, we have the unitary character χ_h of H_n defined by $\chi_h(x, y, t) := e^{-2\pi i h \cdot (x,y)}$, $(x, y, t) \in H_n$.*

Proof. We have seen that every irreducible representation of H_n defines a character χ_π on the centre C_n. If χ_π is not trivial then π is equivalent to π_{μ_π} for some $\mu_\pi \neq 0$ by Proposition 4.2.5. If χ_π is trivial, then π is also trivial on $C_n = [H_n, H_n]$, hence $\pi(g)$ commutes with $\pi(g')$ for every $g, g' \in H_n$. This shows that π must be a character by Schur's lemma. But every character of H_n is of the form $\chi_h, h \in \mathbb{R}^{2n}$.

■

Definition 4.2.7. Let

$$L^2(\widehat{H_n}) := \{\alpha : \mathbb{R}^* \to SB(L^2(\mathbb{R}^n)); \alpha \text{ measurable},$$
$$\|\alpha\|_2^2 := \int_{\mathbb{R}^*} \|a(\mu)\|_{\mathrm{HS}}^2 |\mu|^n d\mu < \infty\}.$$

Here $\|a\|_{\mathrm{HS}}^2 = \mathrm{tr}(a^* a)$ denotes the square of the Hilbert–Schmidt norm of a bounded operator a. We define a representation $(\nu, L^2(\widehat{H_n}))$ of H_n by

$$(\nu(g)\alpha)(\mu) := \pi_\mu(g) \circ \alpha(\mu), \mu \in \mathbb{R}^*.$$

One has to check that the mappings $g \mapsto \nu(g)a$ are continuous. This is easy for a finite rank operator b and a mapping $\alpha(\mu) := \varphi(\mu)b, \mu \in \mathbb{R}^*$, for some $\varphi \in C_c(\mathbb{R}^*)$. But these mappings are total in the Hilbert space $L^2(\widehat{H_n})$.

Proposition 4.2.8. *For every $F \in \mathcal{S}(H_n)$ and $\mu \in \mathbb{R}^*$, the operator $\pi_\mu(F)$ is Hilbert–Schmidt and*

$$\|\pi_\mu(F)\|_{\mathrm{HS}}^2 = \frac{1}{|\mu|^n} \int_{Z^\perp} |\hat{F}(l)|^2 dl.$$

Proof. Since the kernel function of $\pi_\mu(F)$ is the Schwartz function $F_\mu(s, x) = \hat{F}^{2,3}(s - x, -\frac{\mu}{2}(s + x), \mu)$ it is clear that $\pi_\mu(F)$ is Hilbert–Schmidt.

$$\begin{aligned}
\|\pi_\mu(F)\|_{\mathrm{HS}}^2 &= \int_{X \times Y^*} |F_\mu(x, u)|^2 dx du \\
&= \int_{X \times Y^*} |\hat{F}^{2,3}(x - u, -\frac{1}{2}\mu(x + u), \mu)|^2 dx du \\
&= \int_{X \times Y^*} |\hat{F}^{2,3}(x, \mu u, \mu)|^2 dx du \\
&= \int_{X \times Y^*} \left| \int_{Y + \mathbb{R}Z} F(x + y + tZ) e^{-2\pi i(\mu t + y \cdot \mu u)} dt dy \right|^2 du dx \\
&= \frac{1}{|\mu|^n} \int_{X^* \times Y^*} |\hat{F}(v + u + \mu Z^*)|^2 du dv \\
&= \frac{1}{|\mu|^n} \int_{Z^\perp} |\hat{F}(l)|^2 dl.
\end{aligned}$$ ∎

Theorem 4.2.9. *The mapping*

$$\mathcal{F}(F)(\mu) := \pi_\mu(F), \mu \in \mathbb{R}^*, F \in \mathcal{S}(H_n),$$

sends $\mathcal{S}(H_n)$ into $L^2(\widehat{H_n})$ and extends to an isometry from $L^2(H_n)$ onto the Hilbert space $L^2(\widehat{H_n})$ and intertwines the left regular representation l and the representation ν.

Proof. For $F \in \mathcal{S}(H_n)$ we have that

$$\begin{aligned}\|F\|_2^2 &= \int_{H_n^*} |F(l)|^2 dl \\ &= \int_{\mathbb{R}^*} \frac{1}{|\mu|^n} \left(\int_{Z^\perp} |\hat{F}(l+\mu Z^*)|^2 dl \right) |\mu|^n d\mu \\ &= \int_{\mathbb{R}^*} \|\pi_\mu(F)\|_{\mathrm{HS}}^2 |\mu|^n d\mu.\end{aligned}$$

This shows that $\mathcal{F}$ is an isometry. Let us show that $\mathcal{F}$ is onto. Let α be in the orthogonal of the image of $\mathcal{F}$. If we take two C^∞ mappings $\mathbb{R} \to \mathcal{S}(\mathbb{R}^n)$, ξ and η, and a function $\varphi \in C_c^\infty(\mathbb{R}^*)$, and if we write

$$a_{\varphi,\xi,\eta}(\mu) := \varphi(\mu) P_{\xi(\mu),\eta(\mu)}, \mu \in \mathbb{R}^*,$$

where for two elements $a, b \in L^2(\mathbb{R}^n)$ we have the linear operator $P_{a,b}(\delta) := \langle \delta, b\rangle a, \delta \in L^2(\mathbb{R}^n)$, then the function $F_{\varphi,\xi,\eta} : G \to \mathbb{C}$ defined by

$$\begin{aligned}F_{\varphi,\xi,\eta}(x+y+tZ) &= \int_{\mathbb{R}^*} \mathrm{tr}(\pi_\mu(-x-y-z) \circ a(\mu)) |\mu|^n d\mu \\ = \int_{\mathbb{R}^*} \int_{\mathbb{R}^n} &\xi(s)\overline{\eta(s-x)} e^{2\pi i t\mu} e^{-2\pi i \mu y\cdot s} ds e^{\pi i \mu y\cdot x} |\mu|^n d\mu, x \in X, y \in Y, t \in \mathbb{R},\end{aligned}$$

is contained in $\mathcal{S}(H_n)$ and $\mathcal{F}(F_{\varphi,\xi,\eta}) = a_{\varphi,\xi,\eta}$, as can be easily verified. Hence

$$\begin{aligned}0 &= \langle \alpha, a_{\varphi,\xi,\eta}\rangle \\ &= \int_{\mathbb{R}^*} \overline{\varphi}(\mu) \mathrm{tr}(\alpha(\mu) \circ P_{\eta,\xi}) |\mu|^n d\mu, \varphi \in C_c^\infty(\mathbb{R}^*), \xi, \eta \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

Hence

$$0 = \mathrm{tr}(\alpha(\mu) \circ P_{\eta,\xi}), \xi, \eta \in \mathcal{S}(\mathbb{R}^n), \mu \in \mathbb{R}^*$$

and finally $\alpha = 0$. This shows that the mapping $\mathcal{F}$ is onto. The intertwining property is easy. ∎

4.3 The “$ax + b$” Group

Let $G = \mathbb{R}^2$ be equipped with the multiplication

$$(a,b)\cdot(a',b') = (a+a', e^{-a'}b + b'),\ a,a',b,b' \in \mathbb{R}.$$

It is easy to verify that $(G,\cdot)$ is a Lie group, that $(0,0) =: e$ is the neutral element in G and that

$$(a,b)^{-1} = (-a, -e^{a}b),\ (a,b) \in G.$$

The Lie algebra $\mathfrak{g}$ of G is the vector space $\mathfrak{g} = \mathbb{R}^2$ with the basis $\{A = (1,0), B = (0,1)\}$ and the bracket

$$[A,B] = B.$$

Indeed, it suffices to observe that for $f \in C^{\infty}(G)$ we have that

$$\begin{aligned} A{*}f(a,b) &= \frac{d}{dt} f((-t,0)(a,b))|_{t=0} \\ &= \frac{d}{dt} f((a-t,b))|_{t=0} = -\frac{\partial}{\partial a}(f)(a,b). \end{aligned}$$

and

$$\begin{aligned} B{*}f(a,b) &= \frac{d}{dt} f((0,-t)(a,b))|_{t=0} \\ &= \frac{d}{dt} f((a,-e^{-a}t + b))|_{t=0} \\ &= -e^{-a}\frac{\partial}{\partial b}(f)(a,b). \end{aligned}$$

Hence

$$[A,B] = \left[-\frac{\partial}{\partial a}, -e^{-a}\frac{\partial}{\partial b}\right] = -e^{-a}\frac{\partial}{\partial b} = B.$$

As we shall see later, G is a typical completely solvable Lie group. In matrix form,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; a > 0, b \in \mathbb{R} \right\}.$$

The exponential mapping $\exp : \mathfrak{g} \to G$ is given by

$$\exp(s,t) = \left(s, \frac{e^{-s}-1}{-s}t\right) = E(s,t),\ s,t \in \mathbb{R}.$$

Indeed, we have for $u, v \in \mathbb{R},\ (s,t) \in G$ that

$$\begin{aligned}
E(u(s,t))E(v(s,t)) &= E(us,ut)E(vs,vt)\\
&= \left(us, \frac{e^{-us}-1}{-us}ut\right)\left(vs, \frac{e^{-vs}-1}{-vs}vt\right)\\
&= \left(us+vs, e^{-vs}\frac{(e^{-us}-1)}{-us}ut + \frac{e^{-vs}-1}{-vs}vt\right)\\
&= \left((u+v)s, \frac{e^{-(u+v)s}-1}{-s}t\right)\\
&= \left((u+v)s, \frac{e^{-(u+v)s}-1}{-(u+v)s}(u+v)t\right)\\
&= E((u+v)(s,t)).
\end{aligned}$$

This shows that the mapping $u \mapsto E(u(s,t))$ is a homomorphism. Furthermore

$$\frac{d}{du}E(u(s,t))|_{u=0} = (s,t).$$

Hence E is indeed the exponential mapping.

The left Haar measure dx on G is given by

$$\int_G \varphi(x)dx = \int_{\mathbb{R}^2} \varphi(E(sA)E(tB))dsdt = \int_{\mathbb{R}^2} \varphi(s,t)dsdt,\ \varphi \in C_c(G),$$

a since for any $(a,b) \in G$ and $\varphi \in C_c(G)$ we have that

$$\begin{aligned}
\int_G \lambda(a,b)\varphi(x)dx &= \int_G \varphi((-a,-e^a b)(s,0)(0,t))dsdt\\
&= \int_G \varphi(-a+s, -e^{a-s}b+t)dsdt\\
&= \int_{\mathbb{R}^2} \varphi(s,t)dsdt = \int_G \varphi(x)dx.
\end{aligned}$$

We see immediately that G is not unimodular. For $(a, b) \in G$ and $\varphi \in C_c(G)$ we have that

$$\int_G \varphi(g(a,b)^{-1})dg = \int_{\mathbb{R}^2} \varphi(s-a, e^a t - e^a b)dsdt$$
$$= e^{-a} \int_{\mathbb{R}^2} \varphi(s,t)dsdt = e^{-a} \int_G \varphi(g)dg.$$

Hence

$$\Delta_G(a,b) = e^{-a},\ (a,b) \in G.$$

4.3.1 Induced Representations

Let $P = \{0\} \times \mathbb{R}$. It is easy to see that P is a closed abelian and normal subgroup of G. Choose any $\mu \neq 0 \in \mathbb{R}$ and let χ_μ be the character of P defined by

$$\chi_\mu(0, p) := e^{-i\mu p}, p \in \mathbb{R}.$$

Define the representation $(\tau_\mu, \mathcal{H}_\mu)$ of G by

$$\tau_\mu := \mathrm{ind}_P^G \chi_\mu.$$

As can easily be checked, the invariant measure on G/P is given by

$$\int_{G/P} \varphi(g)d\dot{g} = \int_{\mathbb{R}} \varphi(E(s,0))ds,\ \varphi \in C_c(G/P). \tag{4.3.1}$$

Let us realize the representation τ_μ on $L^2(\mathbb{R})$. The mapping

$$U : L^2(\mathbb{R}) \to \mathcal{H}_\mu; U(\xi)(s,t) = \xi(s)\chi_\mu(0,t)^{-1} = e^{i\mu t}\xi(s), (s,t) \in G,$$

is an isometry by (4.3.1) and so for the representation π_μ obtained in this way on $L^2(\mathbb{R})$ we have that

$$\pi_\mu(s,u)\xi(a) = U^{-1} \circ \tau_\mu(s,u) \circ U(\xi)(a)$$
$$= U(\xi)(a-s, e^{-a+s}(-u))$$
$$= e^{-i\mu(e^{-a+s}u)}\xi(a-s), \tag{4.3.2}$$

for $(s, u) \in G$ and $a \in \mathbb{R}$.

Let us compute $\pi_\mu(f)$ for $f \in L^1(G)$. We have

$$\begin{aligned}\pi_\mu(f)\xi(u) &= \int_{\mathcal{R}^2} f(s,t)\pi_\mu(s,t)\xi(u)dsdt \\ &= \int_{\mathbb{R}^2} f(s,t)e^{-i\mu te^{s-u}}\xi(u-s)dsdt \\ &= \int_{\mathbb{R}} \hat{f}^2(s,\mu e^{s-u})\xi(u-s)ds \\ &= \int_{\mathbb{R}} \hat{f}^2(u-s,\mu e^{-s})\xi(s)ds.\end{aligned}$$

Hence the operator $\pi_\mu(f)$ is a kernel operator with kernel function

$$f_\mu(u,s) = \hat{f}^2(u-s,\mu e^{-s}),\ s,u \in \mathbb{R}, \tag{4.3.3}$$

where

$$\hat{f}^2(s,u) = \int_{\mathbb{R}} f(s,t)e^{-itu}dt,\ f \in L^1(G), s,u \in \mathbb{R}.$$

Hence the kernel K_μ of the representation π_μ in the algebra $L^1(G)$ is given by the functions

$$K_\mu = \{f \in L^1(G); \hat{f}^2(\mathbb{R} \times \{(\text{sign}\ \mu)\mathbb{R}_+\}) = \{0\}\}.$$

Definition 4.3.1. Denote by $\mathfrak{p} := \mathbb{R}B$ the Lie algebra of the group P. We identify the dual space $\mathfrak{p}^*$ with $\mathbb{R} \simeq \mathbb{R}B^*$.

Proposition 4.3.2. *The subset $\mathbb{R}^* \simeq \mathbb{R}^*B^*$ of $\mathfrak{p}^*$ is open and G-invariant, consists of two open orbits and its complement $\{0\}$ is the G-fixed point set of $\mathfrak{p}^*$.*

Proof. For $\lambda \in \mathbb{R}^*B^*$ and $g = (a,b) \in G$, we have that

$$\mathrm{A}d^*(g)(\lambda B^*)(B) = \lambda e^a$$

and so we have the two open G-orbits $O_+ := \mathbb{R}^*_+B^*$ and $O_- := \mathbb{R}^*_-B^*$. The complement $\{0\}$ is obviously made of G-fixed points. ∎

Theorem 4.3.3. *For every $\mu \in \mathbb{R}^*$ the representation π_μ is irreducible. If $(\pi,\mathcal{H})$ is an irreducible representation of G, which is not trivial on the subgroup $P = [G,G]$, then π is equivalent to $\pi_+ := \pi_1$ or to $\pi_- := \pi_{-1}$. The representations π_+ and π_- are not equivalent.*

Proof. Since the stabilizer of any $\mu B^*, \mu \in \mathbb{R}^*$, is the group $P = \exp(\mathbb{R}B)$, it follows from Theorem 3.3.3 and Proposition 3.4.3 that for every $\mu \in \mathbb{R}^*$

the representation π_μ is irreducible. Furthermore, if $(\pi, \mathcal{H})$ is an irreducible representation of G, which is not trivial on the subgroup $P = [G, G]$, then the support of restriction to P of π is the closure of a non-trivial G-orbit, i.e. is either $\mathbb{R}_+ B^*$ or $\mathbb{R}_- B^*$ and so π is either equivalent to π_+ or to π_-.

In order to show that π_+ and π_- are not equivalent, choose two functions f, g in $L^1(G)$, such that

$$\hat{f}^2(s,u) = 0 \text{ for all } u > 0, \hat{f}^2(s,-1) \neq 0, s \in \mathbb{R},$$

and

$$\hat{g}^2(s,u) = 0 \text{ for all } u < 0, \hat{g}^2(s,1) \neq 0, s \in \mathbb{R}.$$

Then relation (4.3.3) implies that

$$\begin{aligned} \pi_-(f) \neq 0,\ \pi_+(f) = 0 \\ \pi_-(g) = 0,\ \pi_+(g) \neq 0. \end{aligned}$$

Hence π_+ and π_- cannot be equivalent, since otherwise their kernels in $L^1(G)$ would be the same. ∎

Remark 4.3.4. Let us give a direct proof that the representations $\pi_\mu, \mu \in \mathbb{R}^*$, are irreducible.

Let V be a closed G-invariant subspace of $L^2(\mathbb{R})$. Then its orthogonal complement $V^\perp$ is also G-invariant. Let $\xi' \in V$ and $\eta' \in V^\perp$. We take any $\varphi \in C_c(\mathbb{R})$ and we let

$$\xi = \int_{\mathbb{R}} \varphi(s)\pi_\mu(s)\xi' ds \in V,\ \eta = \int_{\mathbb{R}} \varphi(s)\pi_\mu(s)\eta' ds \in V^\perp.$$

Since

$$\xi(a) = \int_{\mathbb{R}} \varphi(s)\xi'(a-s)ds, a \in \mathbb{R},$$

it follows that ξ is a continuous function on $\mathbb{R}$ and so is the function η. We have that

$$\langle \pi_\mu(0,b)\xi, \eta\rangle = 0,\ b \in \mathbb{R}.$$

Hence

$$0 = \int_{\mathbb{R}} e^{-i\mu b e^{-a}} \xi(a)\overline{\eta(a)} da = \int_{\mathbb{R}_+} e^{-i\mu b t} \xi(-\log(t))\overline{\eta(-\log(t))}\frac{1}{t} dt,\ b \in \mathbb{R}.$$

This means that the Fourier transform of the L^1-function

$$t \mapsto 1_{]0,\infty[}\xi(-\log(t))\overline{\eta(-\log(t))}\frac{1}{t}$$

vanishes for every $b \in \mathbb{R}$. Hence this continuous function is identically zero on $]0, \infty[$ and so is the function $\mathbb{R} \ni a \mapsto \xi(a)\eta(a)$. The same argument works with the function $\pi_\mu(s, 0)\xi,\ s \in \mathbb{R}$ and therefore we have that

$$\xi(a-s)\overline{\eta(a)} = 0,\ s, a \in \mathbb{R}.$$

Hence, if ξ is not identically zero, we must have that $\eta(a) = 0$ for all $a \in \mathbb{R}$. Finally, the function η' itself must be 0, because η' can be approximated in $L^2(\mathbb{R})$ by functions η of the type above. This shows that if $V \neq \{0\}$ then $V^\perp$ must be $\{0\}$, hence $V = L^2(\mathbb{R})$. ■

Proposition 4.3.5. *The representations τ_μ are equivalent to $\tau_+ := \tau_1$, if $\mu > 0$ (resp. to $\tau_- := \tau_{-1}$ if $\mu < 0$).*

Proof. Let $a \in \mathbb{R}$ and $U_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$U_a(\xi)(u) := \xi(u-a), \xi \in L^2(\mathbb{R}), u \in \mathbb{R}.$$

Let for $\mu \in \mathbb{R}^*$

$$\pi_\mu^a = U_a \circ \pi_\mu \circ U_a^*.$$

For $\xi \in L^2(\mathbb{R})$ we have by (4.3.2) that

$$\begin{aligned}\pi_\mu^a(s,t)U_a^*\xi(u) &= \pi_\mu(s,t)U_a^*\xi(u-a) = e^{-i\mu(e^{a-u+s}t)}U_a^*\xi(u-a-s)\\ &= e^{-i\mu e^a e^{-u+s}t}\xi(u-s) = \pi_{\mu e^a}(s,t)\xi(u).\end{aligned}$$

Hence again by (4.3.2)

$$\pi_\mu^a = \pi_{\mu e^a}.$$

■

4.3.2 The Plancherel Theorem

Let δ be the unbounded self-adjoint operator defined on $L^2(\mathbb{R})$ by

$$\delta\xi(u) = e^{-\frac{1}{2}u}\xi(u),\ u \in \mathbb{R}, \xi \in L^2(\mathbb{R}).$$

For every $f \in L^2(G) \cap L^1(G)$ we have that

$$\pi_\mu(f) \circ \delta\xi(u) = \int_{\mathbb{R}} \hat{f}^2(u-s, \mu e^{-s})e^{-\frac{1}{2}s}\xi(s)ds, \xi \in L^2(G),\ u \in \mathbb{R},$$

and so the kernel function f'_μ of the operator $\pi_\mu(f) \circ \delta$ is given by

$$f'_\mu(u,s) = \hat{f}^2(u-s, \mu e^{-s})e^{-\frac{1}{2}s},\ s, u \in \mathbb{R}.$$

We compute the Hilbert–Schmidt norm $\|\pi_\mu(f)\|_{\mathrm{HS}}$ of the operator $\pi_\mu(f)$:

$$\begin{aligned}\|\pi_\mu(f)\|^2_{\mathrm{HS}} &= \int_{\mathbb{R}^2} |f'_\mu(u,s)|^2 duds \\ &= \int_{\mathbb{R}^2} |\hat{f}^2(u-s, \mu e^{-s})|^2 e^{-s} duds \\ &= \int_{\mathbb{R}} \int_0^\infty |\hat{f}^2(s, \mu t)|^2 dudt.\end{aligned}$$

Hence for $f \in L^1(G) \cap L^2(G)$ we have that

$$\begin{aligned}\|\pi_+(f)\|^2_{\mathrm{HS}} + \|\pi_-(f)\|^2_{\mathrm{HS}} &= \int_{\mathbb{R}} \int_0^\infty |\hat{f}^2(u,t)|^2 dudt + \int_{\mathbb{R}} \int_0^\infty |\hat{f}^2(u,-t)|^2 dudt \\ &= \int_{\mathbb{R}^2} |\hat{f}^2(s,t)|^2 dtds \\ &= 2\pi \int_{\mathbb{R}^2} |f(s,u)|^2 dsdu = 2\pi \|f\|^2_2.\end{aligned}$$

This is Plancherel's formula for the group G.

4.3.3 $d\pi$ for the Enveloping Algebra

Let us compute $d\pi_\mu$ for the elements A, B of $\mathfrak{g}$. We have for any C^∞-function ξ with compact support that

$$d\pi_\mu(A)\xi(u) = \frac{d}{dt}\xi(u-t)|_{t=0} = -\xi'(u).$$

and

$$d\pi_\mu(B)\xi(u) = \frac{d}{dt}e^{-i\mu(e^{-u}t)}\xi(u)|_{t=0} = -i\mu e^{-u}\xi(u),\ u \in \mathbb{R},$$

This shows that the C^∞-vectors of the representations $\pi_\mu,\ \mu \neq 0$, are not the Schwartz functions, but the C^∞ functions f such that

$$e^{-mu} f^{(k)}(u) \in L^2(G)$$

for all $k, m \in \mathbb{N}$.

Remark 4.3.6. Let us give a direct proof of the imprimitivity theorem for the $ax+b$ group.

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Let us construct an imprimitivity system for π.

Proposition 4.3.7. *Suppose that ker(π) contains ker(π_-) + ker(π_+). Then π is one-dimensional.*

Proof. Indeed, let $f \in L^1(\mathbb{R})$, such that $\hat{f}(0) = 0$. We show that

$$\pi^P(f) := \int_{\mathbb{R}} f(t)\pi(0,t)dt$$

is zero. It suffices to choose a sequence of elements (f_k) in $C_c^\infty(\mathbb{R})$ such that $\hat{f_k}$ is zero in a neighbourhood of 0 and such that $\hat{f_k}$ converges in the uniform norm to $\hat{f}$. Then $\pi^P(f_k)$ converges in operator norm to $\pi^P(f)$. But for any $\mu \neq 0$ and for $\xi \in L^2(\mathbb{R}), u \in \mathbb{R}$, we have that

$$\pi_\mu^P(f_k)\xi(u) = \int_{\mathbb{R}} f_k(t)e^{-i\mu e^{-u}t}\xi(u)dt = \hat{f_k}(\mu e^{-u})\xi(u).$$

Since we can write $f_k = h_+ + h_-$, where $h_\pm$ is in $C_0(\mathbb{R})$, where $\hat{h}_+$ vanishes identically on $\mathbb{R}_+$ and $\hat{h}_-$ on $\mathbb{R}_-$, it follows that for every $\varphi \in L^1(G)$,

$$\pi_\mu(\varphi * h_+) = \pi_\mu(\varphi) \circ \pi_\mu^P(h_+) = 0,$$

if $\mu < 0$ and in the same way $\pi_\mu(\varphi * h_-) = 0$ if $\mu > 0$. This tells us that $\varphi * f_k \in \ker(\pi_+) + \ker(\pi_-) \subset \ker(\pi)$. Finally, $f \in \ker(\pi^P)$, i.e. $0 = \pi^P(f) = \int_{\mathbb{R}} f(t)\pi(0,t)dt$. But then for any $f \in L^1(\mathbb{R})$, we have that for any $s \in \mathbb{R}$,

$$\pi^P(f) - \pi^P(0,s) \circ \pi^P(f) = \pi^P(f - \lambda(s)f) = 0,$$

since $(f - \lambda(s)f)\hat{}(0) = 0$. Hence $(\pi(0,s) - Id_{\mathcal{H}}) \circ \pi^P(f) = 0$ and so for every $s \in \mathbb{R}$ we have that $\pi(0,s) = Id_{\mathcal{H}}$. But then π is trivial on $P = [G,G]$ and so by Schur's lemma, π is one-dimensional. ■

Theorem 4.3.8. *Let $(\pi, \mathcal{H})$ be an infinite-dimensional representation. Then π is equivalent to π_+ or to π_-.*

Proof. We have that

$$\ker(\pi_+) * \ker(\pi_-) \subset \ker(\pi_+) \cap \ker(\pi_-) = \{0\},$$

since any $f \in \ker(\pi_+) \cap \ker(\pi_-)$ has the property that

$$\hat{f}^2(s,u) = 0 \text{ for all } s,u \in \mathbb{R}$$

and so $f = 0$. Since $\ker(\pi)$ is a prime ideal in the algebra $L^1(G)$, it follows from Proposition 2.3.15 that either $\ker(\pi_+)$ or $\ker(\pi_-)$ is contained in $\ker(\pi)$. Suppose now that $\ker(\pi_+)$ is contained in $\ker(\pi)$. Then by Proposition 4.3.7, $\pi(\ker(\pi_-)) \neq \{0\}$.

We identify G/P with $\mathbb{R}$ via the mapping $\mathbb{R} \ni t \to \exp(tA)P$. For $\varphi \in C_c^\infty(\mathbb{R})$, let

$$\tilde{\varphi}(t) := \begin{cases} \varphi(-\log(t)) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then the mapping $\varphi \to \tilde{\varphi}$ is a $*$-isomorphism from the algebra $C_c^\infty(G/P)$ to the algebra $C_c^\infty(\mathbb{R})$ and we define the operator $\Delta(\varphi)$ for $\varphi \in C_c(G/P)$ on the Hilbert space $\mathcal{H}$ by

$$\Delta(\varphi) = \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(-b)\pi(0,b)db.$$

We then have that

$$\|\Delta(\varphi)\|_{\mathrm{op}} \leq \|\tilde{\varphi}\|_\infty \leq \|\varphi\|_\infty$$

and that $\Delta(C_c^\infty(G/P))\mathcal{H}$ is dense in $\mathcal{H}$, since the functions $\tilde{\varphi}$ are not 0 on $\mathbb{R}_+$ if $\varphi \neq 0$ and so $\pi_+^P(\check{\widehat{\tilde{\varphi}}}) \neq 0$. Furthermore for $\varphi, \phi \in C_c(G/P)$, we have that

$$\begin{aligned}\Delta(\varphi\phi) = \pi^P\left({}^\vee((\widehat{\tilde{\varphi}\tilde{\phi}}))\right) &= (2\pi)\pi^P\left({}^\vee((\widehat{\tilde{\varphi}})) *{}^\vee ((\widehat{\tilde{\phi}}))\right) \\ &= (2\pi)\pi^P\left({}^\vee((\widehat{\tilde{\varphi}}))\right) \circ \pi^P\left({}^\vee((\widehat{\tilde{\varphi}}))\right) = (2\pi)\Delta(\varphi) \circ \Delta(\phi).\end{aligned}$$

Hence Δ can be extended as a $*$-homomorphism to the whole C^*-algebra $C_0(G/P)$. Let us verify the imprimitivity relation. For any $a \in \mathbb{R}$, $\varphi \in C_c^\infty(G/P)$, we have that

$$\begin{aligned}\pi(a,0) \circ \Delta(\varphi) \circ \pi(-a,0) &= \pi(a,0) \circ \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(0,-b)db \circ \pi(-a,0) \\ &= \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(a,0) \circ \pi(0,-b) \circ \pi(-a,0)db \\ &= \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(0,-e^a b)db \\ &= e^{-a} \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(e^{-a}b)\pi(0,-b)db.\end{aligned}$$

Since

$$\widehat{\tilde{\varphi}}(e^{-a}b) = \int_{\mathbb{R}} \varphi(-\log(t))e^{-ie^{-a}bt}dt = e^a \int_{\mathbb{R}} \varphi(-\log(e^a t))e^{-ibt}dt$$
$$= e^a \int_{\mathbb{R}} \varphi(-a - \log(t))e^{-ibt}dt = e^a \int_{\mathbb{R}} \int_{\mathbb{R}} (\lambda(a)\varphi)(-\log(t))e^{-ibt}dt$$
$$= e^a \widehat{\widetilde{(\lambda(a,0)\varphi)}}(b),$$

it follows that

$$\pi(a,0) \circ \Delta(\varphi) \circ \pi(-a,0) = \Delta(\lambda(a,0)\varphi).$$

Similarly, for $c \in \mathbb{R}$,

$$\pi(0,c) \circ \Delta(\varphi) \circ \pi(0,-c) = \pi(0,c) \circ \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(0,-b)db \circ \pi(0,-c)$$
$$= \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(0,c) \circ \pi(0,-b) \circ \pi(0,-c)db$$
$$= \int_{\mathbb{R}} \widehat{\tilde{\varphi}}(b)\pi(0,-b)db = \Delta(\varphi) = \Delta(\lambda(0,c)\varphi),$$

because $\lambda(0,c)\varphi = \varphi$, P being a normal subgroup of G.

This shows that (π, Δ) is a system of imprimitivity. By the imprimitivity theorem, π is equivalent to a representation which is induced from an irreducible representation, hence from a unitary character χ_μ of P. Since π is of infinite dimension, μ cannot be 0. Hence π is equivalent to π_+. The arguments are similar if $\ker(\pi)$ contains $\ker(\pi_-)$. ∎

We have thus found the irreducible unitary representations of "$ax + b$" group.

Theorem 4.3.9. *The unitary dual of the group $ax + b$ contains two infinite dimensional irreducible unitary representations π_+ and π_- and the unitary characters $\chi_\lambda((a,b)) := e^{-i\lambda a}, (a,b) \in G$ $(\lambda \in \mathbb{R})$.*

Proof. Every character must vanish on $P = [G,G]$. Hence it is of the form χ_λ for some $\lambda \in \mathbb{R}$. ∎

4.4 Grélaud's Group

Definition 4.4.1. Let $\theta \in \mathbb{R}^*$ and let

$$A = A_\theta = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} = I_2 + J \in M_2(\mathbb{R}) \text{ where } J = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

We equip $\mathbb{R} \times \mathbb{R}^2$ with the multiplication

$$(s,u) \cdot (s',u') := (s,u) \cdot_\theta (s',u') := (s+s', e^{-s'A}u + u'),\ s,s' \in \mathbb{R},\ u,u' \in \mathbb{R}^2,$$

and we write G_θ for the group $(\mathbb{R} \times \mathbb{R}^2, \cdot_\theta)$. Here

$$e^{tA} = \begin{pmatrix} e^t \cos(\theta t) & e^t \sin(\theta t) \\ -e^t \sin(\theta t) & e^t \cos(\theta t) \end{pmatrix} = e^t \begin{pmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{pmatrix} = e^t e^{tJ}$$

and for $u = (u_1, u_2) \in \mathbb{R}^2$

$$e^{tA}u = (e^t \cos(\theta t) u_1 + e^t \sin(\theta t) u_2, -e^t \sin(\theta t) u_1 + e^t \cos(\theta t) u_2).$$

It is easy to verify that G_θ is a Lie group, that $(0,0) =: e$ is the neutral element in G_θ and that

$$(s,u)^{-1} = (-s, -e^{sA}u),\ (s,u) \in G_\theta.$$

The Lie algebra $\mathfrak{g}_\theta$ of G_θ is the vector space $\mathfrak{g}_\theta = \mathbb{R} \times \mathbb{R}^2$ with the basis $\{T = (1,0,0), X = (0,1,0), Y = (0,0,1)\}$ and the brackets

$$[T,X] = X - \theta Y,\ [T,Y] = Y + \theta X.$$

Indeed, it suffices to observe that for $f \in C^\infty(G_\theta)$ we have that

$$\begin{aligned} T * f(s,u) &= \frac{d}{dt} f((-t,0,0)(s,u))|_{t=0} \\ &= \frac{d}{dt} f((s-t,u))|_{t=0} = -\frac{\partial}{\partial s}(f)(s,u), \end{aligned}$$

that

$$\begin{aligned} X * f(s,u) &= \frac{d}{dt} f((0,-t,0)(s,u))|_{t=0} \\ &= \frac{d}{dt} f((s, -e^{-sA}(t,0) + u))|_{t=0} \\ &= \frac{d}{dt} f\left((s, -(e^{-s}\cos(\theta s)t, e^{-s}\sin(\theta s)t) + u)\right)|_{t=0} \\ &= (-e^{-s}\cos(\theta s)\partial_2 - e^{-s}\sin(\theta s)\partial_3) f(s,u) \end{aligned}$$

and that

$$
\begin{aligned}
Y * f(s,u) &= \frac{d}{dt} f((0,0,-t)(s,u))|_{t=0} \\
&= \frac{d}{dt} f((s, -e^{-sA}((0,t)) + u))|_{t=0} \\
&= \frac{d}{dt} f\ (((s, -(e^{-s}\sin(-\theta s)t, e^{-s}\cos(\theta s)t) + u)\ |_{t=0} \\
&= (e^{-s}\sin(\theta s)\partial_2 - e^{-s}\cos(\theta s)\partial_3) f(s,u).
\end{aligned}
$$

Hence

$$
\begin{aligned}
[T,X] &= \left[-\frac{\partial}{\partial s}, -e^{-s}\cos(\theta s)\partial_2 - e^{-s}\sin(\theta s)\partial_3 \right] \\
&= -e^{-s}\cos(\theta s)\partial_2 - \theta e^{-s}\sin(\theta s)\partial_2 - e^{-s}\sin(\theta s)\partial_3 + \theta e^{-s}\cos(\theta s)\partial_3 \\
&= X - \theta Y.
\end{aligned}
$$

and

$$
\begin{aligned}
[T,Y] &= \left[-\frac{\partial}{\partial s}, e^{-s}\sin(\theta s)\partial_2 - e^{-s}\cos(\theta s)\partial_3 \right] \\
&= e^{-s}\sin(\theta s)\partial_2 - \theta e^{-s}\cos(\theta s)\partial_2 - e^{-s}\cos(\theta s)\partial_3 - \theta e^{-s}\sin(\theta s)\partial_3 \\
&= Y + \theta X.
\end{aligned}
$$

As we shall see later, G_θ is a typical exponential solvable Lie group. In matrix form,

$$
T = \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
G_\theta = \left\{ \begin{pmatrix} e^t\cos(\theta t) & e^t\sin(\theta t) & x \\ e^t\sin(-\theta t) & e^t\cos(\theta t) & y \\ 0 & 0 & 1 \end{pmatrix} ; t, x, y \in \mathbb{R} \right\}.
$$

The exponential mapping $\exp : \mathfrak{g}_\theta \to G_\theta$ is given by

$$
E(s,t) := \exp(s,t) = \left(s, \frac{e^{-sA} - 1}{-sA} t \right),\ (s,t) \in G_\theta.
$$

Indeed, we have for $u, v \in \mathbb{R}$, $(s,t) \in G_\theta$ that

$$\begin{aligned}
E(u(s,t))E(v(s,t)) &= E(us,ut)E(vs,vt) \\
&= \left(us, \frac{e^{-usA}-1}{-usA}ut\right)\left(vs, \frac{e^{-vsA}-1}{-vsA}vt\right) \\
&= \left(us+vs, e^{-vsA}\frac{(e^{-usA}-1)}{-usA}ut + \frac{e^{-vsA}-1}{-vsA}vt\right) \\
&= \left((u+v)s, \frac{e^{-(u+v)sA}-1}{-sA}t\right) \\
&= \left((u+v)s, \frac{e^{-(u+v)sA}-1}{-(u+v)sA}(u+v)t\right) \\
&= E((u+v)(s,t)).
\end{aligned}$$

This shows that the mapping $u \mapsto E(u(s,t))$ is a homomorphism. Furthermore

$$\frac{d}{du}E(u(s,t))|_{u=0} = (s,t).$$

Hence E is indeed the exponential mapping.

The left Haar measure dx on $G = G_\theta$ is given by

$$\int_G \varphi(x)dx = \int_{\mathbb{R}\times\mathbb{R}^2} \varphi(E(s,0)E(0,u))dsdu = \int_{\mathbb{R}\times\mathbb{R}^2} \varphi(s,u)dsdu,\ \varphi \in C_c(G),$$

since for any $(a,b) \in G$ and $\varphi \in C_c(G)$ we have that

$$\begin{aligned}
\int_G \lambda(a,b)\varphi(x)dx &= \int_G \varphi((-a,-e^{aA}b)(s,0)(0,u))dsdu \\
&= \int_G \varphi((-a+s,-e^{(a-s)A}b+u)dsdu \\
&= \int_{\mathbb{R}\times\mathbb{R}^2} \varphi(s,u)dsdu = \int_G \varphi(x)dx.
\end{aligned}$$

We see immediately that G is not unimodular. For $(a,b) \in G$ and $\varphi \in C_c(G)$ we have that

$$\begin{aligned}
\int_G \varphi(g(a,b)^{-1})dg &= \int_{\mathbb{R}\times\mathbb{R}^2} \varphi(s-a, e^{aA}u - e^{aA}b)dsdu \\
&= e^{-2a}\int_{\mathbb{R}\times\mathbb{R}^2} \varphi(s,u)dsdu = e^{-2a}\int_G \varphi(g)dg.
\end{aligned}$$

Hence

$$\Delta_G(a,b) = e^{-2a},\ (a,b) \in G.$$

4.4.1 Induced Representations

Let $P = \{0\} \times \mathbb{R}^2$ and $\mathfrak{p} = \mathbb{R}^2$. It is easy to see that P is a closed abelian and normal subgroup of G. Choose any $\mu \neq 0 \in \mathbb{R}^2$ and let χ_μ be the character of P defined by

$$\chi_\mu(0, p) := e^{-i\mu p}, p \in \mathbb{R}^2.$$

Define the representation $(\tau_\mu, \mathcal{H}_\mu)$ of G by

$$\tau_\mu := \mathrm{ind}_P^G \chi_\mu.$$

As can easily be checked, the invariant measure on G/P is given by

$$\int_{G/P} \varphi(g) d\dot{g} = \int_{\mathbb{R}} \varphi(E(s,0)) ds,\ \varphi \in C_c(G/P). \tag{4.4.1}$$

Let us realize the representation τ_μ on $L^2(\mathbb{R})$. The mapping

$$U : L^2(\mathbb{R}) \to \mathcal{H}_\mu; U(\xi)(s,t) = \xi(s)\chi_\mu(0,t)^{-1} = e^{i\mu t}\xi(s), (s,t) \in G_\theta,$$

is an isometry by (4.4.1) and so for the representation π_μ, we obtain in this way on $L^2(\mathbb{R})$:

$$\begin{aligned} \pi_\mu(s,u)\xi(a) &= U^{-1} \circ \tau_\mu(s,u) \circ U(\xi)(a) \\ &= U(\xi)(a-s, e^{(-a+s)A}(-u)) \\ &= e^{-i\mu(e^{(-a+s)A}u)}\xi(a-s), \end{aligned} \tag{4.4.2}$$

for $(s,u) \in G_\theta$ and $a \in \mathbb{R}$.

Definition 4.4.2. Let $S = \{\mu \in \mathbb{R}^2,\ \|\mu\|_2 = 1\}$ be the unit circle in $\mathbb{R}^2$. We denote by $d\mu$ the Haar measure on S with total measure 1, i.e.

$$\int_S f(\mu) d\mu = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt,\ f \in C_c(S).$$

Proposition 4.4.3. *Let $\mu, \nu \in \mathfrak{p}^* \backslash \{0\}$. The representations τ_μ and τ_ν are equivalent if and only if $\nu = e^{sA^t}\mu$ for some $s \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ and $U_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$U_a(\xi)(u) := \xi(u-a), \xi \in L^2(\mathbb{R}), u \in \mathbb{R}.$$

Let for $\mu \in \mathfrak{p}^*$

$$\pi_\mu^a = U_a \circ \pi_\mu \circ U_a^*.$$

For $\xi \in L^2(\mathbb{R})$ we have by (4.4.2) that

$$\begin{aligned}\pi_\mu^a(s,u)\xi(r) &= \pi_\mu(s,u)U_a^*\xi(r-a) = e^{-i\mu(e^{(a-r+s)A}u)}U_a^*\xi(r-a-s)\\ &= e^{-i(e^{aA^t}\mu)(e^{(-r+s)A}u)}\xi(r-s) = \pi_{e^{aA^t}\mu}(s,u)\xi(r).\end{aligned}$$

Hence if $\nu = e^{aA^t}\mu$, then π_ν is equivalent to π_μ.

Suppose now that $e^{\mathbb{R}A^t}\mu \cap e^{\mathbb{R}A^t}\nu = \emptyset$. By Sect. 4.4.3 we can find two Schwartz functions φ and ψ in $\mathcal{S}(P)$, such that their Fourier transforms have compact support, such that $\hat{\varphi}$ vanishes on $e^{\mathbb{R}A^t}\mu$ but not on $e^{\mathbb{R}A^t}\nu$ and such that $\hat{\psi}$ vanishes on $e^{\mathbb{R}A^t}\nu$ but not on $e^{\mathbb{R}A^t}\mu$. We then have that

$$\pi_\mu(\varphi)\xi(s) = \int_{\mathbb{R}} \varphi(0,u)e^{-i\mu(e^{-sA}u)}\xi(s)du = \hat{\varphi}(e^{-sA^t}\mu)\xi(s)$$

for $\xi \in L^2(\mathbb{R})$ and $s \in \mathbb{R}$. Hence $\pi_\mu(\varphi) = 0$. Similarly, $\pi_\nu(\psi) = 0$, $\pi_\mu(\psi) \neq 0$ and $\pi_\nu(\varphi) \neq 0$. This shows that π_μ and π_ν are not equivalent. ∎

4.4.2 $d\pi_\mu$ for the Enveloping Algebra

Let us compute $d\pi_\mu$ for the elements T, X, Y of $\mathfrak{g}$. We have for any C^∞-function ξ with compact support that

$$d\pi_\mu(T)\xi(r) = \frac{d}{dt}\xi(r-t)|_{t=0} = -\xi'(r).$$

and for $u \in \mathfrak{p}$,

$$\begin{aligned}d\pi_\mu(0,u)\xi(r) &= \frac{d}{dt}e^{-i\mu(e^{-rA})tu}\xi(r)|_{t=0}\\ &= -i\mu(e^{-rA}(u))\xi(r),\ r \in \mathbb{R},\end{aligned}$$

Since

$$A(X+iY)=\begin{pmatrix}1 & \theta\\ -\theta & 1\end{pmatrix}(1,i)=(1+i\theta)(1,i),$$

it follows that

$$d\pi_\mu(X+iY)\xi(r)=-ie^{-r(1+i\theta)}\mu(1,i)\xi(r), r\in\mathbb{R}, \xi\in L^2(\mathbb{R}).$$

This shows that the C^∞-vectors of the representations $\pi_\mu,\ \mu\neq 0$ are not the Schwartz functions, but the C^∞ functions f such that

$$e^{-mr}f^{(k)}(r)\in L^2(\mathbb{R}) \text{ for all } k,m\in\mathbb{N}.$$

4.4.3 The Imprimitivity Theorem for Grélaud's Group

We need to apply Theorem 3.3.3 and Proposition 3.4.3.

Proposition 4.4.4. *Let S be the unit circle in $\mathbb{R}^2$. The mapping*

$$\Phi:\mathbb{R}\times S\to\mathbb{R}^2\setminus\{0\}, \Phi(r,s)=e^{rA^t}s,$$

is a diffeomorphism.

Proof. It suffices to show that Φ has an inverse of class C^∞, since Φ is obviously smooth itself. For $u\in\mathbb{R}^2, u\neq 0$, we write

$$\Psi(u):=(\log(\|u\|_2), e^{-\log(\|u\|_2)A^t}(u))\in\mathbb{R}\times S.$$

Since

$$\|e^{rA^t}s\|_2=e^r,$$

it follows that

$$\Psi(\Phi(r,s))=(r,e^{-rA^t}(e^{rA^t}s))=(r,s)$$

and

$$\Phi(\Psi(u))=\Phi(\log(\|u\|_2), e^{-\log(\|u\|_2)A^t}(u))=e^{\log(\|u\|_2)A^t}e^{-\log(\|u\|_2)A^t}(u)=u.$$

Hence Ψ is the inverse mapping of Φ. ∎

Proposition 4.4.5. *Identifying the character space $\hat{P}$ of the normal closed subgroup P of G with $\mathbb{R}^2$, we have that the open subset $U := \mathbb{R}^2 \setminus \{0\}$ has the property that two distinct G-orbits in U are separated by two open G-invariant subsets in U and that the complement $\{0\}$ of U in $\hat{P}$ is made out of a single fixed point.*

Proof. Indeed any G-orbit O in U is determined by its intersection with the unit circle, i.e. there exists a unique $s_O \in S$, such that $O = G \cdot s_O$. So for two distinct orbits O, O' we take the subsets $\mathcal{U} := G \cdot I$, resp. $\mathcal{U}' := G \cdot I'$, where $I = \{t \in S, |t - s_O| < \varepsilon\}$, resp. $I' = \{t \in S, |t - s_{O'}| < \varepsilon\}$, where $2\varepsilon = |s_0 - s_{O'}|$ ($|t - t'|$ denoting the Euclidean distance of the two elements $t, t' \in S$.) By Proposition 4.4.4, the two sets $\mathcal{U}, \mathcal{U}'$ are open and G-invariant and their intersections are empty. ∎

Proposition 4.4.6. *The representation π_μ is irreducible for every $\mu \in \mathbb{R}^*$. Every unitary irreducible representation of infinite dimension of Grélaud's group is equivalent to π_μ for some $\mu \in \mathbb{R}^*$.*

Proof. This follows again immediately from Theorem 3.3.3 and Proposition 3.4.3. ∎

Let us give concrete proofs of these results in the following propositions.

Proposition 4.4.7. *The representation π_μ is irreducible for every $\mu \in \mathbb{R}^*$.*

Proof. Let V be a closed G_θ-invariant subspace of $L^2(\mathbb{R})$. Then its orthogonal complement $V^\perp$ is also G_θ-invariant. Let $\xi' \in V$ and $\eta' \in V^\perp$. We take any $\varphi \in C_c(\mathbb{R})$ and we let

$$\xi = \int_{\mathbb{R}} \varphi(s)\pi_\mu(s,0)\xi' ds \in V, \ \eta = \int_{\mathbb{R}} \varphi(s)\pi_\mu(s,0)\eta' ds \in V^\perp.$$

Since

$$\xi(a) = \int_{\mathbb{R}} \varphi(s)\xi'(a-s)ds, a \in \mathbb{R},$$

it follows that ξ is a continuous function on $\mathbb{R}$ and so is the function η. We have that

$$\langle \pi_\mu(0,b)\xi, \eta\rangle = 0, \ b \in \mathbb{R}^2.$$

Hence

$$0 = \int_{\mathbb{R}} e^{-i\mu(e^{-aA}b)}\xi(a)\overline{\eta(a)}da, \ b \in \mathbb{R}. \tag{4.4.3}$$

Let f be any radial function in the Schwartz space $\mathcal{S}(P)$. It follows from (4.4.3) that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} f(b) \int_{\mathbb{R}} e^{-i\mu(e^{-aA}b)}\xi(a)\overline{\eta(a)}da db \\ &= \int_{\mathbb{R}} \xi(a)\overline{\eta(a)} \left(\int_{\mathbb{R}} f(b)e^{-i\mu(e^{-aA}b)}db \right) da \end{aligned}$$

and so

$$0 = \int_{\mathbb{R}} \hat{f}(e^{-aA^t}\mu)\xi(a)\overline{\eta(a)}da$$

for all these functions f. Since the functions $t \mapsto \hat{f}(e^{-t}\mu),\ f \in \mathcal{S}(P)$, give us among others all the C^∞-functions with compact support on $\mathbb{R}$, it follows that

$$\xi(a)\overline{\eta(a)} = 0 \text{ for all } a \in \mathbb{R}.$$

Replacing ξ by $\lambda(s,0)\xi,\ s \in \mathbb{R}$, we see that

$$\xi(a-s)\overline{\eta(a)} = 0 \text{ for all } s, a \in \mathbb{R}$$

and so $\eta = 0$, if $\xi \neq 0$. This shows that the orthogonal complement of V is reduced to $\{0\}$ if $V \neq \{0\}$. ■

Proposition 4.4.8. *Let $(\pi, \mathcal{H})$ be an infinite-dimensional irreducible representation of G_θ and let $\rho := \pi|_P$. Then $\rho(f)$ is not 0 for all the functions $f \in C_c(P)$, for which $\hat{f}(0) = 0$*

Proof. Indeed, suppose the contrary. Then for any $u \in P$ and $f \in C_c(P)$, we have that

$$(\lambda(u)f - f)\widehat{\ }(0) = \hat{f}(0) - \hat{f}(0) = 0$$

and so $\rho(f) = \rho(\lambda(u)(f)) = \rho(u) \circ \rho(f)$ which implies that $\pi((0,u) = Id_{\mathcal{H}}$ for all $u \in \mathbb{R}^2$. Since $P = [G_\theta, G_\theta]$, this implies that π is one-dimensional. ■

Theorem 4.4.9. *Let $(\pi, \mathcal{H})$ be an infinite-dimensional irreducible representation of the group G_θ. Then π is equivalent to π_μ for some $\mu \in S$.*

Proof. As before, let $\rho = \pi|_P$. Let N be the hull in $\mathbb{R}^2$ of the ideal $I := \ker(\rho)_{C^*(P)}$. We have seen in Proposition 4.4.8 that N is not equal to $\{0\}$. Denote for any $f \in C_c(P)$ and $s \in \mathbb{R}$ the function

$$\mathbb{R}^2 \ni (0,u) \mapsto f((-s,0)(0,u)(s,0))$$

by $i(s)f$. Then

$$\pi(s,0) \circ \rho(f) \circ \pi(-s,0) = \int_{\mathbb{R}^2} e^{-2s} f(e^{-sA}u)\pi(0,u)du$$

and so for $f \in C^*(P)$ and $s \in \mathbb{R}$, we have that

$$\begin{aligned}
\hat{f}(e^{sA^t}\mu) = 0 \text{ for all } \mu \in N &\iff \pi(s,0) \circ \rho(f) \circ \pi(-s,0) = 0 \\
&\iff \rho(f) = 0 \\
&\iff \hat{f}(\mu) = 0 \text{ for all } \mu \in N.
\end{aligned}$$

This shows that N is invariant under the $\mathbb{R}$-action defined by the matrices $e^{sA^t}, s \in \mathbb{R}$. Suppose that N contains more than two $\mathbb{R}$-orbits. This means that N contains two distinct points μ and ν of S. If we take two open disjoint $\mathbb{R}$-invariant subsets U and V of $\mathbb{R}^2$, U containing μ and V containing ν, then the two ideals I_U and I_V of $C^*(P)$ defined by

$$I_U = \{f \in C^*(P); \hat{f} \text{ has a compact support contained in } U\}$$

and similarly for I_V, then $I_U * I_V = \{0\}$ and both ideals are invariant under conjugation by elements of $\mathbb{R} \times \{0\}$. Hence $\rho(I_U * I_V) = \{0\}$ and so, since π is irreducible, either I_U or I_V is contained in $\ker(\rho)$. But there exist elements $f \in I_U$, for which $\hat{f}(\nu) \neq 0$ and so this f is not contained in $\ker(\rho)$; similarly for I_V. Hence the hull of the ideal $\ker(\rho)_{C^*(P)}$ is the closure $N_\mu := \{e^{sA^t}\mu,\ s \in \mathbb{R}\} \cup \{0\}$ of a unique $\mathbb{R}$-orbit. We can define a representation $(\sigma, \mathcal{H})$ of $C_1(N) := \{\tilde{\varphi} \in C_0(N), \tilde{\varphi}(0) = 0\}$ by letting

$$\sigma(\varphi) := \rho(f), f \in C^*(P) \text{ with } \hat{f} = \varphi \text{ on } N.$$

It is easy to check that σ is a bounded $*$-representation of the C^*-algebra $C_1(N_\mu)$, since this algebra is isomorphic to $C^*(P)/\ker\rho_{C^*(P)}$. We now identify G_θ/P with $N_\mu^0 := N_\mu \backslash \{0\}$ via the mapping $\mathbb{R} \ni s \mapsto \exp(sA^t)\mu$ for $\varphi \in C_0(\mathbb{R})$, and let

$$\tilde{\varphi}(e^{sA^t}\mu) := \varphi(-s),\ s \in \mathbb{R}.$$

Then the mapping $\varphi \to \tilde{\varphi}$ is a $*$-isomorphism from the algebra $C_0(G_\theta/P)$ onto the algebra $C_0(N_\mu^0)$. We obtain a representation $(\Delta, \mathcal{H})$ of $C_0(G_\theta/P)$ on the Hilbert space $\mathcal{H}$ by

$$\Delta(\varphi) = \sigma(\tilde{\varphi}),\ \varphi \in C_c(G_\theta/P).$$

We then have for any $s \in \mathbb{R}$ and $\tilde{\varphi} \in C_0(N_\mu^0)$, for which there exists $f \in \mathcal{S}(P)$ with $\hat{f}|_{N_\mu^0} = \tilde{\varphi}$, that

$$\begin{aligned}
\pi(s,0) \circ \Delta(\varphi) \circ \pi(-s,0) &= \pi(s,0) \circ \left(\int_{\mathbb{R}^2} f(b)\pi(0,b)db \right) \circ \pi(-s,0) \\
&= \int_{\mathbb{R}^2} f(b)\pi(s,0) \circ \pi(0,b) \circ \pi(-s,0)db \\
&= \int_{\mathbb{R}^2} f(b)\pi(0, e^{sA}b)db \\
&= e^{-2s} \int_{\mathbb{R}^2} f(e^{-sA}b)\pi(0,b)db.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^2} e^{-2s} f(e^{-sA}b)e^{-i\nu(b)}db = \hat{f}(e^{sA^t}\nu), \nu \in \mathfrak{p}^*,$$

and since for $s, r \in \mathbb{R}$,

$$\hat{f}(e^{(s+r)A^t}\mu) = \tilde{\varphi}(e^{(s+r)A^t}\mu) = (\widetilde{\lambda(s)\varphi})(e^{rA^t}\mu),$$

we have that

$$\pi(s,0) \circ \Delta(\varphi) \circ \pi(-s,0) = \Delta(\lambda(s,0)\varphi)$$

for all $s \in \mathbb{R}$ and $\varphi \in C_0(G_\theta/P)$.

Similarly, for $(0,c) \in P$ and $\tilde{\varphi}$ as above,

$$\begin{aligned}\pi(0,c) \circ \Delta(\varphi) \circ \pi(0,-c) &= \pi(0,c) \circ \int_{\mathbb{R}^2} f(b)\pi(0,b)db \circ \pi(0,-c)\\ &= \int_{\mathbb{R}^2} f(b)\pi(0,b)db = \Delta(\varphi) = \Delta(\lambda(0,c)\varphi),\end{aligned}$$

because $\lambda(0,c)\varphi = \varphi$, P being a normal subgroup of G_θ.

This shows that (π, Δ) is a system of imprimitivity. By the imprimitivity theorem, π is equivalent to a representation which is induced from an irreducible representation, hence from a unitary character χ_μ of P. Since π is of infinite dimension, μ cannot be 0. Therefore $\pi \simeq \pi_\mu$. ■

We have thus found the irreducible unitary representations of Grélaud's group.

Theorem 4.4.10. *Let G_θ be Grélaud's group. The unitary dual of G_θ consists of the infinite-dimensional representations π_μ, $\mu \in \mathfrak{p}^*$ of length 1, and the unitary characters $\chi_\lambda((a,u)) := e^{-i\lambda a}, (a,u) \in G_\theta$ $(\lambda \in \mathbb{R})$.*

Proof. Every character must vanish on $P = [G_\theta, G_\theta]$. Hence it is of the form χ_λ for some $\lambda \in \mathbb{R}^*$. ■

4.4.4 The Plancherel Theorem

Let us compute $\pi_\mu(f)$ for $f \in L^1(G_\theta)$. We have

$$\begin{aligned}\pi_\mu(f)\xi(r) &= \int_{\mathbb{R}\times\mathbb{R}^2} f(s,u)\pi_\mu(s,u)\xi(r)dsdu\\ &= \int_{\mathbb{R}\times\mathbb{R}^2} f(s,u)e^{-i\mu e^{(s-r)A}u}\xi(r-s)dsdt\\ &= \int_{\mathbb{R}} \hat{f}^2(s, e^{(s-r)A^t}\mu)\xi(r-s)ds\\ &= \int_{\mathbb{R}} \hat{f}^2(r-s, e^{-sA^t}\mu)\xi(s)ds.\end{aligned}$$

Hence the operator $\pi_\mu(f)$ is a kernel operator with kernel function

$$f_\mu(r,s) := \hat{f}^2(r-s, e^{-sA^t}\mu),\ r,s \in \mathbb{R},$$

where

$$\hat{f}^2(s,u) = \int_{\mathbb{R}} f(s,t)e^{-itu}dt,\ f \in L^1(G_\theta), (s,u) \in G_\theta.$$

Therefore the kernel K_μ of the representation π_μ in the algebra $L^1(G_\theta)$ is given by the functions

$$K_\mu = \{f \in L^1(G_\theta); \hat{f}^2(r, e^{-sA^t}\mu) = 0 \text{ for all } r,s \in \mathbb{R}\}.$$

Let δ be the unbounded self-adjoint operator defined on $L^2(\mathbb{R})$ by

$$\delta\xi(u) = e^{-u}\xi(u),\ u \in \mathbb{R}, \xi \in L^2(\mathbb{R}).$$

For every $f \in L^2(G_\theta) \cap L^1(G_\theta)$ we have that

$$\pi_\mu(f) \circ \delta\xi(u) = \int_{\mathbb{R}} \hat{f}^2(u-s, e^{-sA^t}\mu)e^{-s}\xi(s)ds,\ \xi \in L^2(\mathbb{R}),\ u \in \mathbb{R},$$

and so the kernel function f'_μ of the operator $\pi_\mu(f) \circ \delta$ is given by

$$f'_\mu(u,s) = \hat{f}^2(u-s, e^{-sA^t}\mu)e^{-s},\ s,u \in \mathbb{R}.$$

We compute the Hilbert–Schmidt norm $\|\pi_\mu(f)\circ\delta\|_{\mathrm{HS}}$ of the operator $\pi_\mu(f)\circ\delta$:

$$\begin{aligned}
\|\pi_\mu(f) \circ \delta\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^2} |f'_\mu(u,s)|^2 duds \\
&= \int_{\mathbb{R}^2} |\hat{f}^2(u-s, e^{-sA^t}\mu)|^2 e^{-2s} duds.
\end{aligned}$$

Hence for $f \in L^1(G_\theta) \cap L^2(G_\theta)$ we have that

$$\begin{aligned}
\int_S \|\pi_\mu(f) \circ \delta\|_{\mathrm{HS}}^2 d\mu &= \int_S \int_{\mathbb{R}^2} |\hat{f}^2(r-s, e^{-sA^t}\mu)|^2 e^{-2s} drdsd\mu \\
&= \int_{\mathbb{R}\times\mathbb{R}^2} |\hat{f}^2(s,u)|^2 duds \\
&= (2\pi)^2 \int_{\mathbb{R}\times\mathbb{R}^2} |f(s,u)|^2 dsdu = (2\pi)^2 \|f\|_2^2.
\end{aligned}$$

This is Plancherel's formula for the group G_θ.

Chapter 5
Orbit Method

5.1 Auslander–Kostant Theory

Auslander and Kostant [3] extended the orbit method to solvable Lie groups of type I. To explain their theory, we first prepare the ingredients. Let G be a solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. G acts on $\mathfrak{g}$ by the adjoint action and on $\mathfrak{g}^*$ by its contragradient representation:

$$(g\cdot f)(X) = \left(\mathrm{Ad}^*(g)\cdot f\right)(X) = f(\mathrm{Ad}(g^{-1})X) \ (g \in G, f \in \mathfrak{g}^*, X \in \mathfrak{g}).$$

The action of G obtained in this way is called the **coadjoint representation** of G. Let $G(f)$ be the stabilizer of $f \in \mathfrak{g}^*$ in G. Hence the Lie algebra of $G(f)$ is

$$\mathfrak{g}(f) = \{X \in \mathfrak{g}; f([X, Y]) = 0, \forall\, Y \in \mathfrak{g}\}.$$

We define the anti-symmetric bilinear form B_f on $\mathfrak{g} \times \mathfrak{g}$ by $B_f(X, Y) = f([X, Y])$. For a vector subspace $\mathfrak{a}$ of $\mathfrak{g}$, we denote by $f|_{\mathfrak{a}}$ the restriction of f to $\mathfrak{a}$ and put

$$\mathfrak{a}^{\perp,\mathfrak{g}^*} = \{f \in \mathfrak{g}^*; f|_{\mathfrak{a}} = 0\},$$

$$\mathfrak{a}^f = \{X \in \mathfrak{g}; B_f(X, Y) = 0, \forall\, Y \in \mathfrak{a}\}.$$

When there is no risk of confusion, we simply write $\mathfrak{a}^{\perp}$ instead of $\mathfrak{a}^{\perp,\mathfrak{g}^*}$. When $\mathfrak{a} \subset \mathfrak{a}^f$, we say that $\mathfrak{a}$ is **isotropic** (with respect to B_f). Then,

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$$\begin{aligned}
&\mathfrak{a} \text{ is a maximal isotropic subspace}\\
&\Leftrightarrow \mathfrak{a} = \mathfrak{a}^f\\
&\Leftrightarrow \mathfrak{a} \subset \mathfrak{a}^f \text{ and } \dim \mathfrak{a} = \frac{1}{2}(\dim \mathfrak{g} + \dim(\mathfrak{g}(f))).
\end{aligned}$$

We designate the set of Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ satisfying $\mathfrak{h} \subset \mathfrak{h}^f$ (resp. $\mathfrak{h} = \mathfrak{h}^f$) by $S(f, \mathfrak{g})$ (resp. $M(f, \mathfrak{g})$).

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and we linearly extend f, B_f to $\mathfrak{g}_{\mathbb{C}}$.

Definition 5.1.1. Let $\mathfrak{p}$ be a complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. When $\mathfrak{p}$ satisfies the following conditions, it is called a **polarization** of G at $f \in \mathfrak{g}^*$:

(1) $\mathfrak{p}$ is a maximal isotropic subspace with respect to B_f;
(2) $\mathfrak{p} + \overline{\mathfrak{p}}$ is a complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$;
(3) $\mathfrak{p}$ is stable by $\mathrm{Ad}(G(f))$.

$\mathfrak{p}$ is a polarization of $\mathfrak{g}$, which means that $\mathfrak{p}$ is a polarization of the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. We denote by $P(f, G)$ the set of polarizations of G at $f \in \mathfrak{g}^*$.

Definition 5.1.2. $\mathfrak{p} \in P(f, G)$ being given, we define real Lie subalgebras $\mathfrak{d}, \mathfrak{e}$ of $\mathfrak{g}$ by

$$\mathfrak{d} = \mathfrak{p} \cap \mathfrak{g}, \ \mathfrak{e} = (\mathfrak{p} + \bar{\mathfrak{p}}) \cap \mathfrak{g}.$$

As is easily seen, $\mathfrak{d}_{\mathbb{C}} = \mathfrak{p} \cap \bar{\mathfrak{p}}$, $\mathfrak{e}_{\mathbb{C}} = \mathfrak{p} + \bar{\mathfrak{p}}$ and $\mathfrak{d} = \mathfrak{e}^f$. Hence B_f induces a non-degenerate bilinear form $\hat{B}_f$ on the quotient vector space $\mathfrak{e}/\mathfrak{d}$. Moreover,

$$(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}} \simeq \mathfrak{e}_{\mathbb{C}}/\mathfrak{d}_{\mathbb{C}} = (\mathfrak{p} + \bar{\mathfrak{p}})\,/\mathfrak{p} \cap \bar{\mathfrak{p}} = \mathfrak{p}/\mathfrak{d}_{\mathbb{C}} \oplus \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}.$$

Definition 5.1.3. We define a linear operator J on $(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}}$ by the following: $J(X) = -iX, i = \sqrt{-1}$ for $X \in \mathfrak{p}/\mathfrak{d}_{\mathbb{C}}$, $J(X) = iX$ for $X \in \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}$.

Then J defines a real linear transformation on $\mathfrak{e}/\mathfrak{d}$ with square equal to -1, namely a natural complex structure, and for $u \in \mathfrak{e}/\mathfrak{d}$,

$$u + iJu \in \mathfrak{p}/\mathfrak{d}_{\mathbb{C}}, \ u - iJu \in \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}.$$

Definition 5.1.4. We define a bilinear form S_f on $\mathfrak{e}/\mathfrak{d}$ by

$$S_f(u, v) = \hat{B}_f(u, Jv), \ u, v \in \mathfrak{e}/\mathfrak{d}.$$

Proposition 5.1.5. *S_f is a non-degenerate symmetric bilinear form on $\mathfrak{e}/\mathfrak{d}$ and $\hat{B}_f, S_f$ are J-invariant: i.e.*

$$\hat{B}_f(Ju, Jv) = \hat{B}_f(u, v), \ S_f(Ju, Jv) = S_f(u, v).$$

Proof. $\mathfrak{p}/\mathfrak{d}_{\mathbb{C}}$ is clearly isotropic relative to $\hat{B}_f$ and, for $u, v \in \mathfrak{e}/\mathfrak{d}$,

$$\begin{aligned} 0 &= \hat{B}_f(u + iJu, v + iJv) \\ &= \hat{B}_f(u, v) - \hat{B}_f(Ju, Jv) + i\left(\hat{B}_f(Ju, v) + \hat{B}_f(u, Jv)\right). \end{aligned}$$

Taking the imaginary part of this equality,

$$\hat{B}_f(u, Jv) = -\hat{B}_f(Ju, v) = \hat{B}_f(v, Ju). \tag{5.1.1}$$

Hence $S_f(u, v) = S_f(v, u)$, i.e. S_f is symmetric. Since $\hat{B}_f$, J are non-degenerate, S_f too. From relation (5.1.1) and $J^2 = -1$,

$$\begin{aligned} \hat{B}_f(Ju, Jv) &= -\hat{B}_f(u, J(Jv)) = \hat{B}_f(u, v), \\ S_f(Ju, Jv) &= \hat{B}_f(Ju, J(Jv)) = -\hat{B}_f(Ju, v) \\ &= \hat{B}_f(v, Ju) = S_f(v, u) = S_f(u, v). \end{aligned}$$

■

Definition 5.1.6. When the symmetric form S_f is positive definite or $\mathfrak{e}/\mathfrak{d} = \{0\}$, we say that $\mathfrak{p} \in P(f, G)$ is **positive**. In particular, when $\mathfrak{p} = \bar{\mathfrak{p}}$, we say that $\mathfrak{p}$ is **real**.

On real polarizations of connected simply connected type I solvable Lie groups, please refer to [29], for example.

We denote by $P^+(f, G)$ the set of all positive polarizations of G at f. Taking $\mathfrak{p} \in P(f, G)$, we define the Lie subalgebras $\mathfrak{d}$, $\mathfrak{e}$ as before and denote by D_0 (resp. E_0) the connected Lie subgroup of G corresponding to $\mathfrak{d}$ (resp. $\mathfrak{e}$). Since $\mathfrak{p}$ is stable by the action of $\mathrm{Ad}(G(f))$,

$$D = G(f)D_0, \ E = G(f)E_0$$

are two subgroups of G.

Proposition 5.1.7. *D, D_0 are closed subgroups of G. Moreover D_0 is the connected component of the unit in D and $\mathfrak{d}$ is the Lie algebra of D.*

Proof. As $\mathfrak{d}$ and $\mathfrak{e}$ are mutually the orthogonal complement of each other relative to B_f, for $X \in \mathfrak{g}$:

$$X \in \mathfrak{d} \Leftrightarrow X{\cdot}f(Y) = B_f(Y, X) = 0 \ (\forall\, Y \in \mathfrak{e}).$$

Taking the image of the exponential map, for $a \in D_0$:

$$(a{\cdot}f - f)(Y) = 0 \ (\forall\, Y \in \mathfrak{e}). \tag{5.1.2}$$

Thus Eq. (5.1.2) holds for any $a \in \overline{D_0}$ and any element X of the Lie algebra of $\overline{D_0}$ satisfies

$$X\cdot f(Y) = 0\ (\forall\, Y \in \mathfrak{e}).$$

Hence $X \in \mathfrak{d}$ and $D_0 = \overline{D_0}$.

Let us repeat a similar argument to prove the rest of the proposition. Let D_1 denote the connected component of the unit in $\overline{D} = \overline{G(f)D_0}$. For $a \in D_1$,

$$(a\cdot f - f)(Y) = 0\ (\forall\, Y \in \mathfrak{e}).$$

We know from this that the Lie algebra $\mathfrak{d}_1$ of D_1 is contained in $\mathfrak{d}$. On the other hand, $D_0 \subset D$ implies $\mathfrak{d} \subset \mathfrak{d}_1$. Hence $\mathfrak{d} = \mathfrak{d}_1$ and $D_0 = D_1$, while $D_0 \subset D \subset \overline{D}$. In this way D is a closed subgroup and D_0 is the connected component of the unit in D. ■

Here we consider the D-orbit $D\cdot f$ in $\mathfrak{g}^*$.

Proposition 5.1.8. *$D\cdot f$ is an open set of the affine space $f + \mathfrak{e}^{\perp}$. Clearly $D\cdot f = D_0\cdot f$.*

Proof. First let us see that $f + \mathfrak{e}^{\perp}$ is stable by the action of D. Since $\mathfrak{e}$ is D-stable, $\mathfrak{e}^{\perp}$ is too. As $D = D_0G(f)$, $D\cdot f = D_0G(f)\cdot f = D_0\cdot f$. Therefore, for $d \in D$ and $\ell \in \mathfrak{e}^{\perp}$, there exists $a \in D_0$ such that

$$d\cdot(f + \ell) - f = a\cdot f - f + d\cdot\ell.$$

Then, taking into account relation (5.1.2), $d\cdot(f + \ell) - f \in \mathfrak{e}^{\perp}$. Thus $f + \mathfrak{e}^{\perp}$ is D-stable and $\mathfrak{d}\cdot f \subset \mathfrak{e}^{\perp}$. On the other hand, $\mathfrak{d}\cdot f \cong \mathfrak{d}/\mathfrak{g}(f)$ and $\mathfrak{e} = \mathfrak{d}^f$ give

$$\dim \mathfrak{d} + \dim \mathfrak{e} = \dim \mathfrak{g} + \dim(\mathfrak{g}(f)).$$

Therefore

$$\begin{aligned}\dim(\mathfrak{d}\cdot f) &= \dim(\mathfrak{d}/\mathfrak{g}(f)) = \dim \mathfrak{d} - \dim(\mathfrak{g}(f))\\ &= \dim \mathfrak{g} - \dim \mathfrak{e} = \dim(\mathfrak{e}^{\perp}).\end{aligned}$$

Consequently $\mathfrak{d}\cdot f = \mathfrak{e}^{\perp}$ and, $\mathfrak{d}\cdot f$ being the tangent space of $D_0\cdot f \subset f + \mathfrak{e}^{\perp}$ at f, the implicit function theorem concludes that $D_0\cdot f = D\cdot f$ is an open set of $f + \mathfrak{e}^{\perp}$. ■

Now we investigate E-orbit $E\cdot f = E_0\cdot f$ in $\mathfrak{g}^*$.

Definition 5.1.9. When $E\cdot f$ is a closed set in $\mathfrak{g}^*$, we say that $\mathfrak{p} \in P(f, G)$ satisfies the **Pukanszky condition**.

Remark 5.1.10. If $\mathfrak{p}$ is real, $D = E$ and Proposition 5.1.8 means:

$$\mathfrak{p} \text{ satisfies the Pukanszky condition } \Leftrightarrow D{\cdot}f = f + \mathfrak{e}^{\perp}.$$

Lemma 5.1.11. *If $\mathfrak{p} \in P(f, G)$ satisfies the Pukanszky condition, E_0, E are closed subgroups of G and E_0 is the connected component of the unit in E.*

Proof. Let $\psi : G \to \mathfrak{g}^*$ be the mapping defined by $\psi(g) = g{\cdot}f$. Evidently $E = \psi^{-1}(E{\cdot}f) = \psi^{-1}(E_0{\cdot}f)$ and consequently E is a closed set in G. Let E_1 be the connected component of the unit in E. Clearly $E_0 \subset E_1$. On the other hand, $\psi(E_0) = \psi(E_1)$ and, $G(f)_0$ denoting the connected component of the unit in $G(f)$, $G(f)_0 \subset E_0$. Comparing their dimension $E_0 = E_1$. ∎

Proposition 5.1.12. *If $\mathfrak{p} \in P(f, G)$ satisfies the Pukanszky condition, then $D{\cdot}f = f + \mathfrak{e}^{\perp}$.*

Proof. We put $K = \{a \in E_0; a{\cdot}f \in f + \mathfrak{e}^{\perp}\}$. Evidently K is a closed set in E_0 and consequently a closed set in G by Lemma 5.1.11. Furthermore, $\mathfrak{e}^{\perp}$ being stable by E_0, K is a subgroup of E_0 and $f + \mathfrak{e}^{\perp}$ is K-invariant. As we immediately see, $D_0 \subset K$ and Proposition 5.1.8 means that $D_0{\cdot}f$ is an open set in $f + \mathfrak{e}^{\perp}$. Dividing K into cosets by D_0, $K{\cdot}f$ turns out to be an open set in $f + \mathfrak{e}^{\perp}$. On the other hand, the Pukanszky condition implies that $K{\cdot}f = (E_0){\cdot}f \cap (f + \mathfrak{e}^{\perp})$ is a closed set in $f + \mathfrak{e}^{\perp}$. Thus $K{\cdot}f = f + \mathfrak{e}^{\perp}$.

Next let $\mathfrak{k}$ be the Lie algebra of K. From what we have seen by now, $B_f(\mathfrak{k}, \mathfrak{e}) = \{0\}$. Therefore $\mathfrak{k} \subset \mathfrak{d}$. Since the inclusion $\mathfrak{d} \subset \mathfrak{k}$ is trivial, $\mathfrak{k} = \mathfrak{d}$ and D_0 is nothing but the connected component of the unit in K. In particular, D_0 is a normal subgroup of K. For $\ell \in \mathfrak{e}^{\perp}$, we write $f + \ell = k{\cdot}f$ with $k \in K$. Then,

$$D_0{\cdot}(f + \ell) = D_0{\cdot}(k{\cdot}f) = k{\cdot}(D_0{\cdot}f).$$

Proposition 5.1.8 means that $D_0{\cdot}f$ is an open set in $f + \mathfrak{e}^{\perp}$ and that each orbit $D_0{\cdot}(f + \ell)$ is an open set in $f + \mathfrak{e}^{\perp}$. Hence $D_0{\cdot}f = f + \mathfrak{e}^{\perp}$, and we finally conclude by $D = D_0G(f)$ that $D{\cdot}f = f + \mathfrak{e}^{\perp}$. ∎

Lemma 5.1.13. *We assume that $\mathfrak{p} \in P(f, G)$ satisfies the Pukanszky condition. If we denote by $G(f)_0$ the connected component of the unit in $G(f)$, $D_0 \cap G(f) = G(f)_0$. Let D_1 be the simply connected covering group of D_0 and $\tau : D_1 \to D_0$ the canonical projection. Then $\tau^{-1}(G(f)_0) = G(f)_1$ is connected.*

Proof. As $D{\cdot}f = D_0{\cdot}f$, $D{\cdot}f \simeq D_0/(D_0 \cap G(f))$. Since $G(f)_0 \subset D_0$, $G(f)_0$ is the connected component of the unit in $D_0 \cap G(f)$. From the proof of Proposition 5.1.12 $D_0{\cdot}f$ is simply connected and $D_0 \cap G(f)$ turns out to be connected. Hence $D_0 \cap G(f) = G(f)_0$ and

$$D_1/G(f)_1 \simeq D_0/G(f)_0 = D_0/D_0 \cap G(f) = D{\cdot}f = D_0{\cdot}f.$$

$D_0{\cdot}f$ being simply connected, $G(f)_1 = \tau^{-1}(G(f)_0)$ is connected. ∎

By the definition of $\mathfrak{g}(f)$, the restriction $f|_{\mathfrak{g}(f)}$ of f to $\mathfrak{g}(f)$ gives a homomorphism of the Lie algebra $\mathfrak{g}(f)$ into $\mathbb{R}$.

Definition 5.1.14. We say that $f \in \mathfrak{g}^*$ is **integral** if there is a homomorphism $\eta_f : G(f) \to \mathbb{T}$ satisfying $d\eta_f = if|_{\mathfrak{g}(f)}$.

Hereafter we assume that $f \in \mathfrak{g}^*$ is integral and denote by η_f the corresponding character of $G(f)$. From the relation $f([\mathfrak{d}, \mathfrak{e}]) = \{0\}$, $f|_{\mathfrak{d}}$ gives a homomorphism $\mathfrak{d} \to \mathbb{R}$.

Proposition 5.1.15. *When $\mathfrak{p} \in P(f, G)$ satisfies the Pukanszky condition, η_f is uniquely extended to a character $\chi_f : D \to \mathbb{T}$ satisfying $d\chi_f = if|_{\mathfrak{d}}$.*

Proof. Let us adopt the notations of Lemma 5.1.13. By $f([\mathfrak{d}, \mathfrak{d}]) = \{0\}$ and D_1 being simply connected, there uniquely exists a character $\chi_f^1 : D_1 \to \mathbb{T}$ satisfying $d\chi_f^1 = if|_{\mathfrak{d}}$. When $\mathfrak{p} \in P(f, G)$ satisfies the Pukanszky condition, $G(f)_1$ is connected from Lemma 5.1.13 and

$$\chi_f^1|_{G(f)_1} = \left(\eta_f|_{G(f)_0}\right) \circ \tau.$$

The kernel K of the homomorphism $\tau : D_1 \to D_0$ is contained in $G(f)_1 = \tau^{-1}(G(f)_0)$ and $\chi_f^1|_K$ is trivial. Therefore there is a homomorphism $\chi_f^0 : D_0 \to \mathbb{T}$ such that $\chi_f^1 = \chi_f^0 \circ \tau$. Evidently, $d\chi_f^0 = if|_{\mathfrak{d}}$. Now, $G(f)$ acts on D_0 and so on the group of its unitary characters. By the way, the unitary character of a connected Lie group is determined by its differential. On account of $G(f)\cdot f = f$,

$$\chi_f^0(udu^{-1}) = \chi_f^0(d) \ (u \in G(f), d \in D_0).$$

Now letting $A = D_0 \rtimes G(f)$ be the semi-direct product of D_0 by $G(f)$, we define the mapping $\mu_f : A \to \mathbb{T}$ by

$$\mu_f(d, u) = \chi_f^0(d)\eta_f(u) \ (d \in D_0, u \in G(f)).$$

Then μ_f is a unitary character of A. Next we consider a homomorphism σ of A onto D defined by $\sigma(d, u) = du$. By Lemma 5.1.13,

$$\ker\sigma = \{(u, u^{-1}); u \in G(f) \cap D_0 = G(f)_0\}.$$

Since χ_f^0 coincides on $G(f)_0$ with η_f, the homomorphism μ_f becomes trivial on $\ker\sigma$ and induces a unitary character χ_f of D. χ_f is clearly provided with the required property. Since χ_f coincides on $G(f)$ with η_f and is determined on D_0 by its differential, its uniqueness is evident by $D = D_0 G(f)$. ■

We start from $\mathfrak{p} \in P(f, G)$ satisfying the Pukanszky condition and construct a unitary representation of G. From $E = E_0 D$, $X = E/D$ is connected. On the other hand, the alternating bilinear form $\hat{B}_f$ on $\mathfrak{e}/\mathfrak{d}$ is non-degenerate and D-invariant.

Let $\dim(\mathfrak{e}/\mathfrak{d}) = 2m$. Extending E-invariantly $\frac{1}{m!}\wedge^m \hat{B}_f$, we induce a measure μ_X on X which is invariant by the action of E. We denote by $M(E, \chi_f)$ the set of all measurable functions ϕ on E which satisfy the covariance relation

$$\phi(ab) = \chi_f(b)^{-1}\phi(a) \ (a \in E, b \in D),$$

consider the space of $\phi \in M(E, \chi_f)$ such that

$$\int_X |\phi|^2 d\mu_X < \infty$$

and identify functions which coincide almost everywhere. In this fashion, we get a Hilbert space $\hat{\mathcal{H}}(E, \chi_f)$. Namely, $\hat{\mathcal{H}}(E, \chi_f)$ is the Hilbert space of the induced representation $\mathrm{ind}_D^E \chi_f$.

Let $C^\infty(E)$ be the space of all C^∞ functions on E. For $z = x + iy$ $(x, y \in \mathfrak{e})$ and $\psi \in C^\infty(E)$, we put $\psi{\cdot}z = \psi{\cdot}x + i\psi{\cdot}y$. Here

$$(\psi{\cdot}x)(a) = \frac{d}{dt}\psi(a\exp(tx))|_{t=0} \ (a \in E).$$

Moreover we set

$$C^\infty(E, f, \mathfrak{p}) = \{\psi \in C^\infty(E); \psi{\cdot}z = -if(z)\psi,\ z \in \mathfrak{p}\},$$

$$\mathcal{L} = C^\infty(E, f, \mathfrak{p}) \cap M(E, \chi_f),\ \ \mathcal{H}(f, \eta_f, \mathfrak{p}, E) = \mathcal{L} \cap \hat{\mathcal{H}}(E, \chi_f).$$

Proposition 5.1.16 ([3])). *The space $\mathcal{H}(f, \eta_f, \mathfrak{p}, E)$ is a closed subspace of the Hilbert space $\hat{\mathcal{H}}(E, \chi_f)$.*

The space $\mathcal{H}(f, \eta_f, \mathfrak{p}, E)$ is regarded as the space of all complex holomorphic functions belonging to the L^2-space with a certain weight function, stable by the action of $\mathrm{ind}_D^E \chi_f$ and gives its subrepresentation. We designate this by $\mathrm{ind}_D^E(\eta_f, \mathfrak{p})$. Finally, setting

$$\rho(f, \eta_f, \mathfrak{p}, G) = \mathrm{ind}_E^G\left(\mathrm{ind}_D^E(\eta_f, \mathfrak{p})\right),$$

we get a subrepresentation of $\mathrm{ind}_D^G \chi_f$ and call it a **holomorphically induced representation**. $\hat{\mathcal{H}}(f, \mathfrak{p}, G)$ denotes the Hilbert space of $\mathrm{ind}_D^G \chi_f$ and $\mathcal{H}(f, \eta_f, \mathfrak{p}, G)$ its closed subspace corresponding to $\rho(f, \eta_f, \mathfrak{p}, G)$.

Let us consider an exact sequence of Lie groups

$$1 \to N \to G \xrightarrow{p} \tilde{G} \to 1.$$

Let $\mathfrak{n}$ (resp. $\mathfrak{g}$, $\tilde{\mathfrak{g}}$) be the Lie algebra of N (resp. G, $\tilde{G}$) and $dp : \mathfrak{g} \to \tilde{\mathfrak{g}}$ the differential mapping of p. We use the same notation for the linear extension of dp to $\mathfrak{g}_{\mathbb{C}}$.

Proposition 5.1.17. *We assume that* $\tilde{f} \in \tilde{\mathfrak{g}}^*$ *is integral and denote by* $\eta_{\tilde{f}}$ *the corresponding unitary character of* $\tilde{G}(\tilde{f})$. *We suppose that* $\tilde{\mathfrak{p}} \in P(\tilde{f}, \tilde{G})$ *satisfies the Pukanszky condition and put* $f = \tilde{f} \circ dp \in \mathfrak{g}^*, \mathfrak{p} = (dp)^{-1}(\tilde{\mathfrak{p}})$. *Then* f *is integral,* $G(f) = p^{-1}\left(\tilde{G}(\tilde{f})\right)$ *and the character* η_f *of* $G(f)$ *defined by* $\eta_f = \eta_{\tilde{f}} \circ p$ *is what corresponds to* f. *Furthermore,* $\mathfrak{p}$ *is a polarization of* G *at* f *satisfying the Pukanszky condition. In addition,*

$$\rho(f, \eta_f, \mathfrak{p}, G) \simeq \rho(\tilde{f}, \eta_{\tilde{f}}, \tilde{\mathfrak{p}}, \tilde{G}) \circ p.$$

Proof. It suffices to check the various definitions, paying attention to the following points. First, $\tilde{\mathfrak{g}}^*$ is identified with $\mathfrak{n}^{\perp} \subset \mathfrak{g}^*$ and $\mathfrak{p}$ satisfies the Pukanszky condition. Next, $D = p^{-1}(\tilde{D}), \chi_f = \chi_{\tilde{f}} \circ p, E = p^{-1}(\tilde{E})$ and p induces an isomorphism between E/D (resp. $G/D, G/E$) and $\tilde{E}/\tilde{D}$ (resp. $\tilde{G}/\tilde{D}, \tilde{G}/\tilde{E}$). Finally,

$$\mathcal{H}(\tilde{f}, \eta_{\tilde{f}}, \tilde{\mathfrak{p}}, \tilde{G}) \circ p = \mathcal{H}(f, \eta_f, \mathfrak{p}, G). \qquad \blacksquare$$

Let G be a connected and simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ and N the corresponding analytic subgroup of G. We take $f \in \mathfrak{g}^*$ and put $f_0 = f|_{\mathfrak{n}}$. Since $\mathfrak{n}$ is stable by $\mathrm{Ad}(G)$, G acts on $\mathfrak{n}^*$. We write as $G(f_0)$ the stabilizer of f_0 in G.

Definition 5.1.18. Let $\mathfrak{p} \in P(f, G)$. We say that $\mathfrak{p}$ is **$\mathfrak{n}$-admissible**, if $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}} \in P(f_0, N)$. In addition, if $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}}$ is stable by $\mathrm{Ad}\,(G(f_0))$, we say that $\mathfrak{p}$ is **strongly $\mathfrak{n}$-admissible**.

Remark 5.1.19. If $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}}$ is a maximal isotropic subspace relative to B_{f_0}, $\mathfrak{p}$ is $\mathfrak{n}$-admissible.

In these circumstances we write down two important theorems obtained in Auslander and Kostant [3]. Though we cannot go into details, the proof of the second theorem is based on the Mackey theory. Being a connected and simply connected nilpotent Lie group, $N = \exp \mathfrak{n}$ is of type I and its unitary dual $\hat{N}$ is realized by the space $\mathfrak{n}^*/N$ of coadjoint orbits of N, as we shall see later. Keeping this fact in mind, they consider the G-action on $\hat{N}$ and apply the Mackey theory.

Theorem 5.1.20. *At any* $f \in \mathfrak{g}^*$, *there exists an* $\mathfrak{n}$*-admissible* $\mathfrak{p} \in P^+(f, G)$. *Moreover, if* $\mathfrak{p} \in P(f, G)$ *is* $\mathfrak{n}$*-admissible,* $\mathfrak{p}$ *satisfies the Pukanszky condition.*

Theorem 5.1.21. *We assume that* $f \in \mathfrak{g}^*$ *is integral and that* $\mathfrak{p} \in P^+(f, G)$ *is strongly* $\mathfrak{n}$*-admissible. Then*

$$\mathcal{H}(f, \eta_f, \mathfrak{p}, G) \neq \{0\},$$

$\rho(f, \eta_f, \mathfrak{p}, G)$ *gives an irreducible unitary representation of* G *and its equivalence class does not depend on either* $\mathfrak{p}$ *or* $\mathfrak{n}$.

At the end of this section we note one result which we shall need later.

Definition 5.1.22. Let $\mathfrak{g}$ be a real Lie algebra. A complex Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}_{\mathbb{C}}$ is said to be **totally complex**, if $\mathfrak{m} + \bar{\mathfrak{m}} = \mathfrak{g}_{\mathbb{C}}$ holds.

Theorem 5.1.23. *([11]) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $f \in \mathfrak{g}^*$. We suppose that $\mathfrak{p} \in P(f, G)$ is totally complex and satisfies the Pukanszky condition. Then, if $\mathcal{H}(f, \eta_f, \mathfrak{p}, G) \neq \{0\}$, $\rho(f, \eta_f, \mathfrak{p}, G)$ is irreducible.*

5.2 Exponential Solvable Lie Groups

In this section we give the definition of the exponential solvable Lie groups and explain the orbit method for these groups.

5.2.1 Co-exponential Sequence

Definition 5.2.1. Let $\mathfrak{g}$ be a real solvable Lie algebra. We consider a series of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\}$$

of $\mathfrak{g}$ such that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian for every $0 \leq i \leq m-1$. Take for every j a subspace $\mathfrak{v}_j$ of $\mathfrak{g}_j$ such that $\mathfrak{g}_j = \mathfrak{v}_j \oplus \mathfrak{g}_{j+1}$. Hence

$$\mathfrak{g} = \oplus_{j=0}^{m-1} \mathfrak{v}_j.$$

We say that $(\mathfrak{v}_j)_j$ is a **co-exponential sequence**.

As an example we can take a composition series $(\mathfrak{g}_i)_i$, since for any i, the algebra $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ must be abelian, since otherwise its commutator is a non-trivial $\mathfrak{g}$-submodule.

Definition 5.2.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We write e^X for the element $\exp X$, $X \in \mathfrak{g}$, of G.

Applying $(m-1)$ times Proposition 1.1.42 we obtain:

Proposition 5.2.3. *Let $\mathfrak{g}$ be a solvable Lie algebra and let G be a connected and simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{v} = (\mathfrak{v}_0, \cdots, \mathfrak{v}_{m-1})$ be a co-exponential sequence. Then the mapping*

$$E_{\mathfrak{v}} : \mathfrak{v} \to G; E_{\mathfrak{v}}(v_1, \cdots, v_{m-1}) = e^{v_1} \cdots e^{v_{m-1}}$$

is a diffeomorphism.

Proposition 5.2.4. *Let G be a simply connected solvable Lie group and let*

$$\mathcal{B} = (\mathfrak{v}_0, \cdots, \mathfrak{v}_{m-1})$$

be a co-exponential sequence. Let $\mathfrak{g} := \oplus_{i=0}^{m-1} \mathfrak{v}_i$ be the Lie algebra of G. We define on $\mathfrak{g}$ a group multiplication

$$v \cdot_{\mathcal{B}} v' = (\psi_0(v, v'), \cdots, \psi_{m-1}(v, v')) := E_{\mathcal{B}}^{-1} \left(E_{\mathcal{B}}(v) \cdot E_{\mathcal{B}}(v') \right), \ v, v' \in \mathfrak{g}.$$

Then the functions $\psi_i : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{v}_i$, $i = 0, \cdots, m-1$ are smooth and $\psi_i(v, v')$ depends only on the coordinates $v_0, \cdots, v_i$ of v and on the coordinates $v'_0, \cdots, v'_i$ of v'. In particular $\psi_j((0, \cdots, 0, v_j), (0, \cdots, 0, v'_j)) = v_j + v'_j$ for every j.

Proof. It is clear that the coordinate functions ψ_i of the multiplication $\cdot_{\mathcal{B}}$ are smooth, since the mapping $E_{\mathcal{B}}$ is smooth. Let $\mathcal{B}_i := (\mathfrak{v}_j)_{j=0}^{i}$. Then $\mathcal{B}_i$ is a co-exponential sequence for the algebra $\mathfrak{g}/\mathfrak{g}_{i+1}$ and the group G/G_{i+1}, where $G_{i+1} = \exp(\mathfrak{g}_{i+1})$. This shows that ψ_i does not depend on the coordinates $v_k, k > i$ of v and also not on the coordinates $v'_k, k > i$ of v', since e^{v_k} and $e^{v'_k}$ are contained in G_{i+1} if $k > i$. ■

Definition 5.2.5. Again let $\mathfrak{g}$ be a solvable Lie algebra, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We take a sequence of co-abelian ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\}$$

and the corresponding co-exponential sequence $(\mathfrak{v}_i)_i$.

We consider the index set $I^{\mathfrak{g}/\mathfrak{h}} \subset \{0, \cdots, m-1\}$ consisting of all the i's for which

$$\mathfrak{h} + \mathfrak{g}_i \not\supset \mathfrak{h} + \mathfrak{g}_{i+1}.$$

For every $i \in I^{\mathfrak{g}/\mathfrak{h}}$, we choose a subset $\mathfrak{w}_i \subset \mathfrak{v}_i$ such that

$$\mathfrak{g}_i + \mathfrak{h} = \mathfrak{w}_i \oplus \mathfrak{g}_{i+1} + \mathfrak{h}.$$

Let $\mathfrak{w} := (\mathfrak{w}_j)_{j \in I^{\mathfrak{g}/\mathfrak{h}}}$.

We say that the sequence $\mathfrak{w} := (\mathfrak{w}_j)_{j \in I^{\mathfrak{g}/\mathfrak{h}}}$ is a co-exponential sequence with respect to $\mathfrak{h}$. In the following, if we use a co-exponential sequence $\mathcal{B} = (\mathfrak{w}_j)_j$ relatively to a subalgebra $\mathfrak{h}$, we always are given also the co-abelian sequence $(\mathfrak{g}_i)_i$ of ideals in $\mathfrak{g}$, which passes through the nilradical, the sequence of subspaces $(\mathfrak{v}_i)_i$, such that $\mathfrak{g}_i = \mathfrak{g}_{i+1} \oplus \mathfrak{v}_i$ for every $0 \le i \le m-1$ and the index set $I^{\mathfrak{g}/\mathfrak{h}}$.

Proposition 5.2.6. *Let G be a connected and simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then there exists a closed connected subgroup H of G with Lie algebra $\mathfrak{h}$. Let $\mathfrak{w} = (\mathfrak{w}_j)_{j \in I^{\mathfrak{g}/\mathfrak{h}}}$ be a co-exponential sequence with respect to $\mathfrak{h}$. Then the mapping*

$$\mathfrak{w} \times H \to G; ((w_j), h) \mapsto E_{\mathcal{B}}((w_j))h = \prod_{j \in I^{\mathfrak{g}/\mathfrak{h}}} e^{w_j} h,$$

is a diffeomorphism.

Proof. We proceed by induction on dim $\mathfrak{g}$. If $\mathfrak{g}$ is abelian, then everything is clear. Suppose first that $\mathfrak{h} \subset \mathfrak{g}_1 = \sum_{j=1}^{m-1} \mathfrak{v}_j$. Then 0 is contained in $I^{\mathfrak{g}/\mathfrak{h}}$ and $\mathfrak{w}_0 = \mathfrak{v}_0$. By Proposition 5.2.3, there exists a closed normal subgroup G_1 of G with Lie algebra $\mathfrak{g}_1$, such that the mapping $\mathfrak{w}_0 \times \mathfrak{g}_1 \to G; (w, u) \mapsto e^w u$ is a diffeomorphism. Hence, using the induction hypothesis for $\mathfrak{g}_1$ and $\mathfrak{h}$, we obtain the result for G and $\mathfrak{h}$.

Suppose now that $\mathfrak{h}$ is not contained in $\mathfrak{g}_1$. Choose a vector $X \in \mathfrak{h} \setminus \mathfrak{g}_1$ and a subspace $\mathfrak{v}^0$ of $\mathfrak{v}_0$, such that

$$\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{v}^0 \oplus \mathfrak{g}_1.$$

Let $\mathfrak{g}^1 := \mathfrak{g}_1 + \mathfrak{v}^0, \mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}^1$ and apply the induction hypothesis to $\mathfrak{g}^1, \mathfrak{h}_1$ and to the closed connected normal subgroup G^1 of G with Lie algebra $\mathfrak{g}^1$, which exists by Proposition 1.1.42. We then know that there exists a closed connected subgroup H_1 in G^1, such that the mapping $E'_{\mathfrak{w}}$ from $\mathfrak{w} \times H_1$ onto G^1 is a diffeomorphism. Applying again Proposition 1.1.42, we can conclude that the mapping

$$E : \mathbb{R} \times \mathfrak{w} \times H_1 \to G;\ (s, (V_j), h_1) \mapsto e^{sX} \prod_j e^{V_j} h_1,$$

is a diffeomorphism. The subset $H = \{e^{sX} h_1,\ s \in \mathbb{R}, h_1 \in H_1\}$ is then closed in G and it is also a subgroup, since $\mathfrak{h}_1$ is an ideal in $\mathfrak{h}$. The Lie algebra of H is evidently isomorphic to $\mathfrak{h}$. Since the mapping

$$\mathbb{R} \times G^1 \to \mathbb{R} \times G^1; (s, u) \mapsto (s, e^{-sX} u e^{sX})$$

is a diffeomorphism, it follows that the mapping

$$\mathbb{R} \times \mathfrak{w} \times H_1 \to G;\ (s, (V_j), h_1) \mapsto \prod_j e^{V_j} h_1 e^{sX},$$

is a diffeomorphism too. The mapping $\mathbb{R} \times H_1 \to H;\ (s, h_1) \mapsto h_1 e^{sX}$ being also a diffeomorphism, we see that the proposition is true. ■

Corollary 5.2.7. *Let $G = exp\ \mathfrak{g}$ be a simply connected solvable Lie group, let $H = exp\ \mathfrak{h}$ be a closed subgroup of G and let $A = exp\ \mathfrak{a}$ be a closed normal subgroup of G. Then HA is a closed subgroup too.*

Proof. We take a composition sequence $(\mathfrak{g}_j)$ of $\mathfrak{g}$, which passes through $\mathfrak{a}$, i.e. such that $\mathfrak{g}_j = \mathfrak{a}$ for some j. Then if we put

$$\mathfrak{w}^j := \oplus_{k \in I^{\mathfrak{g}/\mathfrak{h}}, k \geq j} \mathfrak{w}_k,$$

we have that

$$AH = E_{\mathcal{B}}(\mathfrak{w}^j)H$$

and so AH is a closed subgroup of G. ■

Proposition 5.2.8. *Let $G = exp\ \mathfrak{g}$ be a simply connected solvable Lie group and let $H = exp\ \mathfrak{h}$ be a closed subgroup of G. Choose a co-exponential sequence $\mathcal{B} = (\mathfrak{w}_j)_j$ of $\mathfrak{g}$ relatively to $\mathfrak{h}$ associated to a co-abelian sequence $(\mathfrak{g}_i)_i$ such that $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}]$ for some k. For any $X \in \mathfrak{h}$ and $w = (w_j)_j \in \mathfrak{w} = \oplus_j \mathfrak{w}_j$, we have that*

$$e^X E_{\mathcal{B}}(w)e^{-X} = \prod_j e^{\exp(\lambda_j(X))w_j + p_j(X, w_0, \cdots, w_{j-1})} q_j(X, w_0, \cdots, w_j) \ mod\ H, \tag{5.2.1}$$

where $q_j(X, w_0, \cdots, w_j) \in G_{j+1}$ and $p_j(X, w_0, \cdots, w_{j-1})$ is in $\mathfrak{w}_j$ if $j \geq k$, and $p_j = 0, q_j \in \mathfrak{g}_k$ if $j < k$ and where $\lambda_j(X)$ is the endomorphism defined by $ad(X)$ on the space $\mathfrak{w}_j \cong \mathfrak{g}_j + \mathfrak{h}/\mathfrak{g}_{j+1} + \mathfrak{h}$.

Proof. We have for $g = \exp X \in G$, that $g e^{w_j} g^{-1} = e^{\exp(\lambda_j(X))w_j}$ modulo $G_{j+1} \cap [G, G]$. Hence, using Proposition 5.2.4, we see that

$$g e^{w_j} g^{-1} \mathrm{mod}(G_{j+1}H) = e^{\exp(\lambda_j(X))w_j \ \mathrm{mod}(\mathfrak{h}+\mathfrak{g}_{j+1})} \mathrm{mod}(G_{j+1}H).$$

■

Let H be a closed connected subgroup of the simply connected solvable Lie group G. We can now explicitly describe the invariant positive linear functional $\oint_{G/H}$ which exists on the functional space $\mathcal{E}(G/H)$ by Proposition 3.1.6.

Proposition 5.2.9. *Let $G = exp\ \mathfrak{g}$ be a simply connected solvable Lie group and let $H = exp\ \mathfrak{h}$ be a closed connected subgroup of G. Let $\mathcal{B} = (\mathfrak{w}_j)_j$ be a co-exponential sequence relative to $\mathfrak{h}$ and $\mathfrak{w} = \oplus_j \mathfrak{w}_j$. Then the positive linear functional defined on $\mathcal{E}(G/H)$ by*

$$\varphi \mapsto \int_{\mathfrak{w}} \varphi(E_{\mathcal{B}}(w))dw =: \oint_{G/H} \varphi(t)d\dot{t},\ \varphi \in \mathcal{E}(G/H),$$

is G-invariant.

Proof. We proceed by induction on dim $\mathfrak{g}$. If G is abelian, the proposition is evident. Suppose first that $\mathfrak{h} \subset \mathfrak{g}_1$. Then $0 \in I^{\mathfrak{g}/\mathfrak{h}}$. Let

$$\mathfrak{w}^1 := \sum_{j \in I^{\mathfrak{g}/\mathfrak{h}}, j>0} \mathfrak{w}_j.$$

We apply the induction hypothesis to $G_1 = \exp(\mathfrak{g}_1)$, to H and to $\mathcal{B}_1 := (\mathfrak{w}_j)_{j=1}^{m-1}$. Since G_1 is a normal closed subgroup, for every $g \in G$, we have that $\Delta_{G_1,H} = \Delta_{G,H}|_{G_1}$ and so the function $\varphi_g(u) := \varphi(gu), u \in G_1$, is in $\mathcal{E}(G_1/H)$. Therefore for $g \in G_1$, by the induction hypothesis,

$$\begin{aligned}\int_{\mathfrak{w}} \varphi(gE_{\mathcal{B}}(w))dw &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(ge^{w_0}E_{\mathcal{B}_1}(w^1))dw_0dw^1 \\ &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(e^{w_0}(e^{-w_0}ge^{w_0})E_{\mathcal{B}_1}(w^1))dw^1dw_0 \\ &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(e^{w_0}E_{\mathcal{B}_1}(w^1))dw^1dw_0 = \int_{\mathfrak{w}} \varphi(E_{\mathcal{B}}(w))dw.\end{aligned}$$

For $g = e^{w_0'}$ with $w_0' \in \mathfrak{w}_0 = \mathfrak{v}_0$, we have that $e^{w_0'}e^{w_0} = e^{w_0+w_0'}q(w_0, w_0')$ where $q(w_0, w_0')$ is an element of G_1. Hence,

$$\begin{aligned}\int_{\mathfrak{w}} \varphi(gE_{\mathcal{B}}(w))dw &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(ge^{w_0}E_{\mathcal{B}_1}(w^1))dw_0dw^1 \\ &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(e^{w_0'+w_0}(q(w_0, w_0'))E_{\mathcal{B}_1}(w^1))dw_0dw^1 \\ &= \int_{\mathfrak{w}_0}\int_{\mathfrak{w}^1} \varphi(e^{w_0}E_{\mathcal{B}_1}(w^1))dw_0dw^1 = \int_{\mathfrak{w}} \varphi(E_{\mathcal{B}}(w))dw.\end{aligned}$$

Suppose now that $\mathfrak{h} \not\subset \mathfrak{g}_1$. Choose a vector $X \in \mathfrak{h} \setminus \mathfrak{g}_1$ and a subspace $\mathfrak{v}^0$ of $\mathfrak{v}_0$, such that $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{v}^0 \oplus \mathfrak{g}_1$. Let

$$\mathfrak{g}^1 := \mathfrak{g}_1 + \mathfrak{v}^0, \ \mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}^1$$

and apply the induction hypothesis to $\mathfrak{g}^1, \mathfrak{h}_1$ and to the closed connected normal subgroup $G^1 = \exp(\mathfrak{g}^1)$ of G. We have now that $\mathcal{E}(G/H)$ is isomorphic to $\mathcal{E}(G^1/H_1)$ with $H_1 = \exp(\mathfrak{h}_1)$, an example of an isomorphism being the restriction mapping $\varphi \mapsto \varphi|_{G^1}$. Hence, it suffices to see that our integral is invariant under $\exp(sX)$ (recall that $G = \exp(\mathbb{R}X)G^1$). Now by (5.2.1)

$$\begin{aligned}&\int_{\mathfrak{w}} \varphi(e^X E_{\mathcal{B}}(w))dw \\ &= \int_{\mathfrak{w}} \varphi\Big(\big(\prod_j e^{\exp(\lambda_j(X))w_j + p_j(X, w_0, \cdots, w_{j-1})} q_j(X, w_1, \cdots, w_j)\big)e^X\Big)dw \\ &= \Delta_{H,G}(e^X) \int_{\mathfrak{w}} \varphi\big(\prod_j e^{\exp(\lambda_j(X))w_j}\big)dw\end{aligned}$$

$$= e^{\mathrm{Tr}(\mathrm{ad}(X)_{\mathfrak{g}/\mathfrak{h}})} \int_{\mathfrak{w}} e^{-\sum_j \mathrm{Tr}(\lambda_j(X)_{(\mathfrak{v}_j/\mathfrak{v}_j\cap\mathfrak{h})})} \varphi\Big(\prod_j e^{w_j}\Big) dw$$

$$= \int_{\mathfrak{w}} \varphi(E_{\mathcal{B}}(w)) dw,$$

since $\sum_j \mathrm{Tr}(\lambda_j(X)_{(\mathfrak{v}_j/\mathfrak{v}_j\cap\mathfrak{h})}) = \mathrm{Tr}(\mathrm{ad}(X)_{\mathfrak{g}/\mathfrak{h}})$. ■

Corollary 5.2.10. *Let G be a simply connected solvable Lie Group with Lie algebra $\mathfrak{g}$. Let $\mathcal{B} = (\mathfrak{v}_j)_j$ be an co-exponential sequence for $\mathfrak{g}$. Then the linear functional*

$$\varphi \mapsto \int_{\mathfrak{g}} \varphi(E_{\mathcal{B}}(v)) dv$$

is a left Haar measure on G.

5.2.2 *The Exponential Mapping for Solvable Lie Groups*

Definition 5.2.11. Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{B} = (\mathfrak{v}_i)_{i=0}^{m-1}$ be a co-exponential sequence of $\mathfrak{g}$ associated to a composition series $(\mathfrak{g}_j)_j$, such that $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}]$ for some k. We identify $\mathfrak{g}$ with $\mathfrak{v} = \oplus_{i=0}^{m-1} \mathfrak{v}_i$ and we write

$$\exp v = (\alpha_0(v), \cdots, \alpha_{m-1}(v)),\ v \in \mathfrak{v}.$$

Then for any $X \in \mathfrak{v}_j, t \in \mathbb{R}$, we have that

$$\exp((0, \cdots, 0, tX, 0, \cdots, 0)) = (0, \cdots, 0, tX, 0, \cdots, 0).$$

We say that a solvable Lie group G with Lie algebra $\mathfrak{g}$ is **exponential**, if the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism. We remark that an exponential group is automatically connected and simply connected.

We say that a solvable Lie algebra is **exponential** if for every $X \in \mathfrak{g}$, the spectrum of the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}_{\mathbb{C}}$ does not contain a complex number of the form $ib, b \in \mathbb{R}^*$.

Example 5.2.12. 1. A connected and simply connected nilpotent Lie group is an exponential solvable Lie group.
2. Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathfrak{g} = n$. When there is a sequence of ideals

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_j) = j\ (0 \le j \le n),$$

we say that $\mathfrak{g}$ is **completely solvable**. A completely solvable Lie algebra is an exponential solvable Lie algebra.

3. Any subalgebra and any quotient algebra of an exponential solvable Lie algebra is exponential.

Proposition 5.2.13. *A solvable Lie algebra* $\mathfrak{g}$ *is exponential if and only if every root* λ *of the* $\mathfrak{g}$*-module* $\mathfrak{g}_{\mathbb{C}}$ *is of the form* $\lambda = (1 + i\theta)\mu$*, where* $\mu : \mathfrak{g} \to \mathbb{R}$ *is real character of* $\mathfrak{g}$ *and* θ *is some real number.*

Proof. Every composition series $((\mathfrak{g}_{\mathbb{C}})_j)_j$ of the $\mathfrak{g}$-module $\mathfrak{g}_{\mathbb{C}}$ has the property that the simple $\mathfrak{g}$-module $(\mathfrak{g}_{\mathbb{C}})_j/(\mathfrak{g}_{\mathbb{C}})_{j+1}$ is one-dimensional and gives us a complex character λ_j of $\mathfrak{g}$ for $j = 0, \cdots, \dim(\mathfrak{g}_{\mathbb{C}}) - 1$. Choosing a Jordan–Hölder basis

$$\mathcal{Z} = (Z_j \in (\mathfrak{g}_{\mathbb{C}})_j \setminus (\mathfrak{g}_{\mathbb{C}})_{j+1}),$$

we have that for every $X \in \mathfrak{g}$, the matrix of the endomorphism $\mathrm{ad}(X)$ is upper triangular and has as diagonal entries the numbers $\lambda_j(X)$. Hence the spectrum of $\mathrm{ad}(X)$ is the set of numbers $\{\lambda_j(X); j = 0, \cdots, \dim \mathfrak{g} - 1\}$. This proves the assertion of the proposition. ∎

Example 5.2.14. The group $E(2)$. Let

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and let $\mathfrak{g}(2)$ be the Lie algebra, whose underlying vector space is $\mathbb{R} \times \mathbb{R}^2$ with the bracket relation

$$[(s, (a, b)), (s', (a', b'))] = (0, s'(a, b)\cdot J^t - s(a', b')\cdot J^t) = (0, (s'b - sb', sa' - s'a)).$$

Namely, $\mathfrak{g}(2) = \langle X, P, Q \rangle_{\mathbb{R}}; [X, P] = -Q, [X, Q] = P$ with

$$X = (1, (0, 0)), P = (0, (0, 1)), Q = (0, (1, 0)).$$

In matrix form,

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

The corresponding simply connected Lie group $G(2)$ is the space $\mathbb{R} \times \mathbb{R}^2$ equipped with the multiplication

$$\begin{aligned}(s, (a, b))(s', (a', b')) &= (s + s', (a, b)\cdot e^{s'J} + (a', b')) \\ &= (s + s', (a \cos s' - b \sin s' + a', a \sin s' + b \cos s' + b')).\end{aligned}$$

The exponential mapping can be explicitly computed. We have for $s \in \mathbb{R}$ and $u \in \mathbb{R}^2$ that

$$\exp(s,u) = \left(s, u \cdot \frac{e^{sJ} - I_2}{sJ}\right).$$

Indeed, for $t, t' \in \mathbb{R}$, we have that

$$\exp((t+t')(s,u)) = \left((t+t')s, u \cdot \frac{e^{s(t+t')J} - I_2}{sJ}\right)$$

and

$$\begin{aligned}\exp(ts,tu)\exp(t's,t'u) &= \left(ts, u \cdot \frac{e^{tsJ} - I_2}{sJ}\right)\left(t's, u \cdot \frac{e^{t'sJ} - I_2}{sJ}\right)\\ &= \left((t+t')s, u \cdot (\frac{e^{tsJ} - I_2}{sJ})e^{t'sJ} + u \cdot \frac{e^{t'sJ} - I_2}{sJ}\right)\\ &= \left((t+t')s, u \cdot \frac{e^{s(t+t')J} - e^{t'sJ} + e^{t'sJ} - I_2}{sJ}\right)\\ &= \left((t+t')s, u \cdot \frac{e^{s(t+t')J} - I_2}{sJ}\right).\end{aligned}$$

Hence the mapping $t \mapsto \psi(t) := \exp(ts, tu)$ is a homomorphism of $\mathbb{R}$ into $G(2)$ and if we differentiate in t we see that $d\psi = (s,u)$. This shows that we have found the exponential mapping. But if $s = 2\pi k$ with $k \in \mathbb{Z}^*$, then we see that

$$\exp(2\pi k, u) = \left(2\pi k, u \cdot \frac{e^{2\pi kJ} - I_2}{2\pi kJ}\right) = \left(2\pi k, u \cdot \frac{I_2 - I_2}{2\pi kJ}\right) = (2\pi k, 0)$$

for every $u \in \mathbb{R}^2$. Hence the exponential mapping is not injective and not surjective. Furthermore the Lie algebra $\mathfrak{g}(2)$ is not exponential since the spectrum of the vector $T = (1, 0_{\mathbb{R}^2})$ is the set $\{0, i, -i\}$.

Remark 5.2.15. Let G be a Lie group, $A = \exp \mathfrak{a}$ be a closed normal subgroup of G. Then $\exp(t(X+Y)) \in \exp(tX)A$ for every $t \in \mathbb{R}$, $Y \in \mathfrak{a}$. Indeed, the mappings $t \mapsto \exp(tX)A$ and $t \mapsto \exp(t(X+Y))A$ are continuous homomorphisms from $\mathbb{R}$ into G/A, whose differentials are equal to $X \bmod \mathfrak{a}$ resp. to $X + Y \bmod \mathfrak{a}$. Hence these two homomorphisms coincide.

Theorem 5.2.16 (cf. [18]). *A connected and simply connected solvable Lie group G with Lie algebra $\mathfrak{g}$ is exponential $\Leftrightarrow$ exp is injective $\Leftrightarrow$ exp is surjective $\Leftrightarrow$ the Lie algebra $\mathfrak{g}$ is exponential.*

Proof. We proceed by induction on the dimension of G. Every simply connected abelian Lie group and every abelian Lie algebra is exponential. As before, let $(\mathfrak{g}_j)_{j=0}^m$ be a composition series of $\mathfrak{g}$. Let $\mathfrak{a} := \mathfrak{g}_{m-1} = \mathfrak{v}_{m-1}$ and consider $\tilde{\mathfrak{g}} = \oplus_{j=0}^{m-2}\mathfrak{v}_j \simeq \mathfrak{g}/\mathfrak{a}$, $\tilde{G} = \oplus_{j=0}^{m-2}\mathfrak{v}_j \simeq G/A$, where $A = \exp \mathfrak{a}$. If we write the group G as $G = \tilde{G} \times \mathfrak{a}$, then the multiplication in G has the following form. For $u, u' \in \tilde{G}, a, a' \in \mathfrak{a}$, we have that

$$(u, a)\cdot_G(u', a') = (u\cdot_{\tilde{G}}u', e^{-\lambda(u')}a + a' + p(u, u')),$$

where $p : \tilde{G}\times\tilde{G} \to \mathfrak{a}$ is a smooth function. We can express the exponential mapping exp in terms of the root $\lambda = \lambda_{m-1}$ associated to $\mathfrak{v}_{m-1}$ and of the exponential mapping $\widetilde{\exp} : \tilde{\mathfrak{g}} \to \tilde{G}$. For $u \in \tilde{\mathfrak{g}}$, we have

$$\exp(u, 0_{\mathfrak{a}}) = \Big(\widetilde{\exp}(u), q(u)\Big),$$

for some smooth mapping $q : \tilde{\mathfrak{g}} \to \mathfrak{a}$, by Remark 5.2.15, and we obtain the following expression for $u \in \tilde{\mathfrak{g}}, a \in \mathfrak{a}$

$$\exp(u, a) = \left(\widetilde{\exp}(u), \frac{e^{-\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a) + q(u)\right), \tag{5.2.2}$$

for some C^∞ mapping $q : \tilde{\mathfrak{g}} \to \mathfrak{a}$. It follows from the property of the exponential mapping that

$$q((t+t')u) = e^{-t'\lambda(u)}(q(tu))+q(t'u)+p(\widetilde{\exp}(tu), \widetilde{\exp}(t'u)), t, t' \in \mathbb{R}, \tag{5.2.3}$$

and that $\frac{d}{dt}q(tu)|_{t=0} = 0$. Indeed, if we check the properties of these mappings, we see that for $t, t' \in \mathbb{R}$,

$$\begin{aligned}
&\exp(tu, ta)\exp(t'u, t'a)\\
=&\left(\widetilde{\exp}(tu), \frac{e^{-t\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a) + q(tu)\right)\left(\widetilde{\exp}(t'u), \frac{e^{-t'\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a) + q(t'u)\right)\\
=&\Bigg(\widetilde{\exp}((t + t')u), e^{-t'\lambda(u)} \cdot \frac{e^{-t\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a)\\
&+ e^{-t'\lambda(u)}q(tu) + \frac{e^{-t'\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a) + q(t'u) + p(\widetilde{\exp}(tu), \widetilde{\exp}(t'u))\Bigg)\\
=&\Bigg(\widetilde{\exp}((t + t')u), \frac{e^{-(t+t')\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a)\\
&+ e^{-t'\lambda(u)}q(tu) + q(t'u) + p(\widetilde{\exp}(tu), \widetilde{\exp}(t'u))\Bigg)
\end{aligned}$$

$$= \left(\widetilde{\exp}((t+t')u), \frac{e^{-(t+t')\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)}(a) + q((t+t')u) \right).$$

Together with (5.2.3), this shows that (5.2.2) describes the exponential mapping of G.

Suppose first that the spectrum of λ does not contain a number of the form ia with $a \in \mathbb{R}^*$. Then the linear mapping

$$\frac{e^{-\lambda(u)} - Id_{\mathfrak{a}}}{-\lambda(u)} : \mathfrak{a} \to \mathfrak{a}$$

is invertible for every $u \in \tilde{\mathfrak{g}}$. This implies that exp is injective resp. surjective resp. a diffeomorphism, if and only if $\tilde{\exp}$ is injective, resp. surjective, resp. a diffeomorphism. Furthermore, if $\tilde{\exp}$ is not injective, resp. not surjective, then so is exp.

Suppose now that the spectrum of λ contains a number of the form ia with $a \in \mathbb{R}^*$, which means that the mapping $e^{-\lambda(u)}w$ is a rotation for every $u \in \mathfrak{g}$. This gives us an element $u_0 \in \mathfrak{g}$ such that the linear map $\frac{e^{-\lambda(2\pi k u_0)} - Id_{\mathfrak{a}}}{-\lambda(2\pi k u_0)}$ is 0 for every $k \in \mathbb{Z}^*$. Hence

$$\exp(2\pi k u_0, a) = \exp(2\pi k u_0, 0) = (\widetilde{\exp}(2\pi k u_0), q(2\pi k u_0))$$

for every $k \in \mathbb{Z}^*$ and $(\widetilde{\exp}(2\pi k u_0), a + q(2\pi k u_0))$ is not contained in the image of exp for every $a \in \mathfrak{a}, a \neq 0$. ■

Remark 5.2.17. Let G be an exponential group. Then we can define a group multiplication on the Lie algebra $\mathfrak{g}$ of G by using the exponential mapping. For $X, Y \in \mathfrak{g}$, let

$$X \cdot_{\mathfrak{g}} Y := \exp^{-1}(\exp X \cdot_G \exp Y) = X + Y + \frac{1}{2}[X, Y] + \cdots$$

according to the Campbell–Baker–Hausdorff formula. It follows that the exponential mapping $(\mathfrak{g}, [,]_{\mathfrak{g}}) \to (\mathfrak{g}, \cdot_{\mathfrak{g}})$ is the identity and so every subalgebra of $\mathfrak{g}$ is a closed connected subgroup of $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ and every closed connected subgroup of $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ is also a subalgebra of $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$ we have that

$$X \cdot_{\mathfrak{g}} Y \cdot_{\mathfrak{g}} (-X) = e^{\mathrm{ad}(X)}(Y).$$

Lemma 5.2.18. *Let H be a closed proper subgroup of the group $(\mathbb{R}, +)$. Then there exists a $0 < h_0 \in H$ such that $H = \mathbb{Z}h_0$.*

Proof. Let $h_0 := \inf\{0 < h \in H\}$. If $h_0 = 0$, then for every $\varepsilon > 0$, there exists $h_\varepsilon \in H$ such that $0 < h_\varepsilon \leq \varepsilon$. Let $x \in \mathbb{R}$. There exists $k \in \mathbb{Z}$, such that $x \in [kh_\varepsilon, (k+1)h_\varepsilon[$. Since $kh_\varepsilon \in H$, we have that the distance of x to H is less or equal to ε. Hence $H = \mathbb{R}$. Since H is proper, h_0 must be strictly positive. Let $x \in H$, such that $x \notin \mathbb{Z}h_0$. Then there exists $k \in \mathbb{Z}$, such that $x \in]kh_0, (k+1)h_0[$ and then $0 < x - kh_0 < h_0$ and $x - kh_0 \in H$. This relation contradicts the minimality of h_0. Hence $H = \mathbb{Z}h_0$. ■

5.3 Kirillov–Bernat Mapping

From Sect. 5.2.1 an exponential solvable Lie group $G = \exp \mathfrak{g}$ is equipped with the following useful property: being given a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is a basis $\{X_1, \ldots, X_p\}$ of a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$ such that the mapping

$$(t_1, \ldots, t_p, X) \mapsto g_1(t_1) \cdots g_p(t_p) \exp X$$

with $g_j(t) = \exp(tX_j)$ $(t \in \mathbb{R})$ is a diffeomorphism from $\mathbb{R}^p \times \mathfrak{h}$ onto G. Such a basis is said to be **coexponential** in $\mathfrak{g}$ to $\mathfrak{h}$ and is constructed as follows.

First of all, $\mathfrak{k}$ being a Lie subalgebra containing $\mathfrak{h}$, we may put together a coexponential basis in $\mathfrak{g}$ to $\mathfrak{k}$ and that in $\mathfrak{k}$ to $\mathfrak{h}$ to make a coexponential basis in $\mathfrak{g}$ to $\mathfrak{h}$. Thus it suffices to consider the following cases:

(1) $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$;
(2) $\mathfrak{h}$ is not an ideal and $\mathfrak{g}/\mathfrak{h}$ is an irreducible $\mathfrak{h}$-module.

In case (1), any element of $\mathfrak{g}$ not belonging to $\mathfrak{h}$ gives a coexponential basis. Hence, if $\mathfrak{g}$ is nilpotent, every Lie subalgebra of codimension 1 is an ideal so that we can construct by iteration of case (1) a coexponential basis to any Lie subalgebra.

Next we consider case (2). Let $\mathfrak{n}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Because of $\mathfrak{n} \not\subset \mathfrak{h}$, $\mathfrak{n}/(\mathfrak{h} \cap \mathfrak{n})$ is identified with a non-trivial $\mathfrak{h}$-submodule of $\mathfrak{g}/\mathfrak{h}$, hence with $\mathfrak{g}/\mathfrak{h}$. Since every subspace containing $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, we can construct by the iteration of case (1) a coexponential basis in $\mathfrak{g}$ to $\mathfrak{n}$, all vectors of which are supposed by hypothesis taken in $\mathfrak{h}$. Applying case (1), we can construct a coexponential basis $\{X\}$ or $\{X_1, X_2\}$, depending on the dimension, in $\mathfrak{n}$ to $\mathfrak{h} \cap \mathfrak{n}$. It is obvious that this is also coexponential in $\mathfrak{g}$ to $\mathfrak{h}$.

Definition 5.3.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and V an G-module or $\mathfrak{g}$-module. We say that V is of exponential type if every weight of $\mathfrak{g}$ in V is written

$$X \mapsto \lambda(X)(1 + i\alpha)$$

with $\alpha \in \mathbb{R}, \lambda \in \mathfrak{g}^*$.

Theorem 5.3.2. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group (with Lie algebra $\mathfrak{g}$) and V an G-module of exponential type. Then the stabilizer in G of any point of V is connected.*

Proof. We denote by ρ the action of G on V. For $X \in \mathfrak{g}, v \in V$ suppose $\rho(\exp X)v = v$. The set of $t \in \mathbb{R}$ satisfying $\rho(\exp(tX))v = v$ is a closed subgroup of $\mathbb{R}$. If this is discrete ($\neq \{0\}$), we take its minimal positive element t_0. Then:

$$\rho\left(\exp\left(\frac{t_0}{2}X\right)\right)\left(\rho\left(\exp\left(\frac{t_0}{2}X\right)\right)v - v\right) = -\left(\rho\left(\exp\left(\frac{t_0}{2}X\right)\right)v - v\right) \neq 0.$$

From this $d\rho\left(\frac{t_0}{2}X\right)$ has an eigenvalue $in\pi$ for some nonzero integer n. ■

Remark 5.3.3. This theorem implies that the stabilizer $G(\ell)$ in G under the coadjoint representation of an element $\ell \in \mathfrak{g}^*$ is connected and hence $G(\ell) = \exp(\mathfrak{g}_{(\ell)})$.

We can parametrize G-orbits in this situation and get the following lemma.

Lemma 5.3.4. *Let G be an exponential solvable Lie group and V an G-module of exponential type. We denote by $G(v)$ the stabilizer in G of a point $v \in V$ and introduce on the orbit $G{\cdot}v$ the relative topology from V. Then $G{\cdot}v$ is homeomorphic to the homogeneous space $G/G(v)$. These spaces are homeomorphic to $\mathbb{R}^d$ with some non-negative integer d.*

Hereafter in this section $G = \exp \mathfrak{g}$ denotes an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$. We define an alternating bilinear form B_f on $\mathfrak{g} \times \mathfrak{g}$ by the formula $B_f(X, Y) = f([X, Y])$. We denote by $S(f, \mathfrak{g})$ (resp. $M(f, \mathfrak{g})$) the set of Lie subalgebras which are isotropic, i.e. self-orthogonal (resp. maximal isotropic) subspaces for B_f. We call elements of $M(f, \mathfrak{g})$ (real) polarizations of $\mathfrak{g}$ at f.

Now $\mathfrak{h} \in S(f, \mathfrak{g})$ being given, we define a unitary character χ_f of $H = \exp \mathfrak{h}$ by the formula

$$\chi_f(\exp X) = e^{if(X)}\ (\forall\, X \in \mathfrak{h}),$$

and construct the induced representation

$$\hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f.$$

We denote by $\hat{\mathcal{H}}(f, \mathfrak{h}, G)$ the Hilbert space of $\hat{\rho}(f, \mathfrak{h}, G)$. Finally, we denote by $I(f, \mathfrak{g})$ the set of $\mathfrak{h} \in S(f, \mathfrak{g})$ such that $\hat{\rho}(f, \mathfrak{h}, G)$ is irreducible.

Definition 5.3.5. Let G be a locally compact group and let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. The **projective kernel** of π, denoted by the symbol $\mathrm{pker}(\pi)$ is the subgroup

$$\mathrm{pker}(\pi) := \{s \in G;\ \pi(s) = \lambda Id_{\mathcal{H}}, \text{for some } \lambda \in \mathbb{C}\}.$$

Proposition 5.3.6. *Let G be a locally compact group and let $(\pi, \mathcal{H}_\pi)$ be an irreducible representation of G. Then $K := pker(\pi)$ is a closed normal subgroup of G and there exists a unitary character χ of K, such that $\pi(k) = \chi(k) Id_{\mathcal{H}_\pi}$ for* *any $k \in K$. If G is a Lie group, then there exists a G-invariant linear functional ℓ on the Lie algebra $\mathfrak{k}$ of K, such that $\pi(e^U) = e^{i\ell(U)} Id_{\mathcal{H}_\pi}$ for every $U \in \mathfrak{k}$.*

Proof. It is clear that K is closed, since π is strongly continuous. If k is in K and $g \in G$, then $\pi(k) = \lambda(k) Id_{\mathcal{H}_\pi}$ for some $\lambda(k) \in \mathbb{T}$ and so

$$\pi(gkg^{-1}) = \pi(g) \circ \pi(k) \circ \pi(g)^{-1} = \pi(g) \circ \lambda(k) Id_{\mathcal{H}_\pi} \circ \pi(g)^{-1} = \lambda(k) Id_{\mathcal{H}_\pi}.$$

This shows that K is normal. Since for $k, k' \in K$,

$$\lambda(k)\lambda(k') Id_{\mathcal{H}_\pi} = \pi(k) \circ \pi(k') = \pi(kk') = \lambda(kk') Id_{\mathcal{H}_\pi},$$

we see that the function $k \mapsto \lambda(k)$ is a continuous homomorphism and hence a continuous unitary character of K. From what we saw above this character is G-invariant. If G is a Lie group, then the restriction of the G-invariant unitary character λ of K to its connected component defines the G-invariant linear functional ℓ on $\mathfrak{k}$. ■

Lemma 5.3.7. *The projective kernel of $\hat{\rho}(\ell, \mathfrak{h}, G)$, $\mathfrak{h} \in I(\ell, \mathfrak{g})$, is the largest closed normal subgroup A_ℓ of G, which is contained in the stabilizer $G(\ell)$ of ℓ under the coadjoint representation of G. Its Lie algebra is the largest ideal $\mathfrak{a}_\ell$ contained in $\mathfrak{g}(\ell)$ and for every $s \in pker(\hat{\rho}(\ell, \mathfrak{h}, G))$, $\hat{\rho}(\ell, \mathfrak{h}, G)(s) = \chi_\ell(s) Id_{\hat{\mathcal{H}}(\ell,\mathfrak{h},G)}$.*

Proof. Let $s \in A_\ell$. Then for $\varphi \in \mathcal{E}(G/H, \chi_\ell)$ and $g \in G$, we have that

$$\hat{\rho}(\ell, \mathfrak{h}, G)(s)\varphi(g) = \varphi(s^{-1}g) = \varphi(g(g^{-1}s^{-1}g)) = \chi_\ell(g^{-1}sg)\varphi(g),$$

since A_ℓ is a normal subgroup contained in $G(\ell) \subset H$ by (2) of the next theorem. Let us write $s = e^S$, with $S \in \mathfrak{a}_\ell$. Since $\mathfrak{a}_\ell$ is an ideal of $\mathfrak{g}$, we see that

$$\langle \ell, \mathrm{ad}(-X)^k(S) \rangle = \langle \ell, [-X, \mathrm{ad}(-X)^{k-1}(S)] \rangle = 0 \ (X \in \mathfrak{g})$$

for all $\mathbb{Z} \ni k > 0$, since then $\mathrm{ad}(-X)^k(S) \in \mathfrak{g}_{(\ell)}$. Hence for $g = e^X \in G$, we see that

$$\chi_\ell(g^{-1}sg) = e^{i\langle \ell, \mathrm{Ad}(g^{-1})S \rangle} = e^{i \sum_{j=0}^{\infty} \frac{1}{j!} \langle \ell, \mathrm{ad}^j(-X)S \rangle} = e^{i\ell(S)} = \chi_\ell(s).$$

This shows that $\hat{\rho}(\ell, \mathfrak{h}, G)(s) = \chi_\ell(s) Id_{\hat{\mathcal{H}}(\ell,\mathfrak{h},G)}$ and so $A_\ell \subset \mathrm{pker}(\hat{\rho}(\ell, \mathfrak{h}, G))$.

If conversely $s = e^S \in \mathrm{pker}(\hat{\rho}(\ell, \mathfrak{h}, G))$, then

$$\varphi(s^{-1}g) = \lambda(s)\varphi(g)$$

for all $\varphi \in \mathcal{E}(G/H, \chi_\ell)$ and all $g \in G$ and so if $S \notin \mathfrak{h}$, we can use Proposition 5.2.6 and choose φ in $\mathcal{E}(G/H, \chi_\ell)$ with $\varphi(e) = 1$ and we see that for $k \in \mathbb{N}$ big enough,

$$0 = \varphi(e^{kS}) = \varphi(s^k) = \lambda(s)^{-k}\varphi(e) = \lambda(s^{-k}).$$

This contradiction implies that $S \in \mathfrak{h}$. Since for any $t \in \mathrm{pker}(\hat{\rho}(\ell, \mathfrak{h}, G))$

$$\hat{\rho}(\ell, \mathfrak{h}, G)(g) \circ \hat{\rho}(\ell, \mathfrak{h}, G)(t) \circ \hat{\rho}(\ell, \mathfrak{h}, G)(g)^{-1}$$
$$= \hat{\rho}(\ell, \mathfrak{h}, G)(g) \circ \lambda(t) Id_{\hat{\mathcal{H}}(\ell,\mathfrak{h},G)} \circ \hat{\rho}(\ell, \mathfrak{h}, G)(g)^{-1} = \lambda(t) Id_{\hat{\mathcal{H}}(\ell,\mathfrak{h},G)},$$

we have that $\mathrm{pker}(\hat{\rho}(\ell, \mathfrak{h}, G))$ is a closed normal subgroup, hence $g^{-1}sg \in H$ for any $g \in G$ and therefore

$$\hat{\rho}(\ell, \mathfrak{h}, G)(s)\varphi(g) = \varphi(g(g^{-1}s^{-1}g)) = \chi_\ell(g^{-1}sg)\varphi(g) = \lambda(s)\varphi(g),$$

for $\varphi \in \mathcal{E}(G/H, \chi_\ell)$, $g \in G$. Hence

$$e^{i\langle \ell, \mathrm{Ad}(\exp(tX))S\rangle} = \chi_\ell(e^{tX} s e^{-tX}) = \lambda(s)$$

for any $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Hence differentiating this equation at $t = 0$, we find that

$$i\langle \ell, [X, S]\rangle e^{i\ell(S)} = 0, \;\; X \in \mathfrak{g}.$$

This relation tells us that $S \in \mathfrak{g}(\ell)$. Hence

$$A_\ell = \mathrm{pker}(\hat{\rho}(\ell, \mathfrak{h}, G)),$$

since $A_\ell = \bigcap_{g \in G} g G(\ell) g^{-1}$. ■

Theorem 5.3.8 ([10], Chap.VI). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and let $f \in \mathfrak{g}^*$. Then:*

(1) *$I(f, \mathfrak{g}) \neq \emptyset$;*
(2) *$I(f, \mathfrak{g}) \subset M(f, \mathfrak{g})$;*
(3) *for $\mathfrak{h}_1, \mathfrak{h}_2 \in I(f, \mathfrak{g})$, $\hat{\rho}(f, \mathfrak{h}_1, G) \simeq \hat{\rho}(f, \mathfrak{h}_2, G)$.*

From now on we aim to prove this theorem following [10]. When $\mathfrak{m}$ is a linear subspace of $\mathfrak{g}$, we put

$$\mathfrak{m}^f = \{X \in \mathfrak{g}; f([X, \mathfrak{m}]) = \{0\}\}.$$

For $X \in \mathfrak{m}$,

$$X \cdot f(\mathfrak{m}^f) = f([\mathfrak{m}^f, X]) = \{0\}$$

so that $\mathfrak{m} \cdot f \subset (\mathfrak{m}^f)^\perp$. Further,

$$\dim(\mathfrak{m} \cdot f) = \dim \mathfrak{m} - \dim(\mathfrak{m} \cap \mathfrak{g}(f)) = \dim \mathfrak{g} - \dim(\mathfrak{m}^f)$$

so that $\mathfrak{m} \cdot f = (\mathfrak{m}^f)^\perp$. We denote by $\mathfrak{z}$ the centre of $\mathfrak{g}$.

Lemma 5.3.9. *Let $\mathfrak{g}$ be a solvable Lie algebra, (V, ρ) an irreducible $\mathfrak{g}$-module of exponential type and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Then dim $V \leq 2$. The representation $\rho|_{\mathfrak{h}}$ is either trivial or irreducible and the codimension of its kernel in $\mathfrak{h}$ is smaller than or equal to* 1.

Proof. By Lie's theorem $d = \dim V \leq 2$. If $d = 1$, the lemma is clear. If $d = 2$, ρ is defined by two weights of the form $\lambda(1 \pm i\alpha), \lambda \in \mathfrak{g}^*, 0 \neq \alpha \in \mathbb{R}$. If $\lambda|_{\mathfrak{h}} = 0$, $\rho|_{\mathfrak{h}}$ is trivial. If not, $\rho|_{\mathfrak{h}}$ is irreducible and its kernel is $\mathfrak{h} \cap \ker\lambda$. ∎

Lemma 5.3.10. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{a}$ its* ***minimal non-central ideal****, i.e. minimal among non-central ideals. Then $\mathfrak{a}$ is commutative and*

$$dim(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})) \leq 2.$$

Proof. By definition the $\mathfrak{g}$-module $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})$ is irreducible. By the previous lemma $d = \dim(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})) \leq 2$ and $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a} \cap \mathfrak{z}$. If $d = 1$, the lemma is clear. Assume $d = 2$ in what follows. $\mathfrak{a}_{\mathbb{C}}$ is spanned by two vectors $A, \bar{A}$ linearly independent over $(\mathfrak{a} \cap \mathfrak{z})_{\mathbb{C}}$, and we may assume that these vectors are eigenvectors in $\mathfrak{a}_{\mathbb{C}}$ of the $\mathfrak{g}$-module $\mathfrak{a}_{\mathbb{C}}/(\mathfrak{a} \cap \mathfrak{z})_{\mathbb{C}}$. Then there exist $X \in \mathfrak{g}$ and $0 \neq \alpha \in \mathbb{R}$ such that

$$[X, A] = (1 + i\alpha)A,\ [X, \bar{A}] = (1 - i\alpha)\bar{A}$$

modulo $(\mathfrak{a} \cap \mathfrak{z})_{\mathbb{C}}$. Hence,

$$0 = [X, [A, \bar{A}]] = (1 + i\alpha)[A, \bar{A}] + (1 - i\alpha)[A, \bar{A}] = 2[A, \bar{A}]. \qquad ∎$$

We denote by $m(f, \mathfrak{g})$ the dimension of the elements of $M(f, \mathfrak{g})$, namely $\frac{1}{2}(\dim \mathfrak{g} + \dim(\mathfrak{g}(f)))$.

Lemma 5.3.11. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{a}$ its minimal non-central ideal. Then $m(f, \mathfrak{g}) = m(f, \mathfrak{a}^f)$ and $f + (\mathfrak{a}^f)^{\perp} \subset G{\cdot}f$.*

Proof. We set $f' = f|_{\mathfrak{a}^f}$ and let $\mathfrak{p} \subset \mathfrak{a}^f$ be a maximal isotropic subspace with respect to $B_{f'}$. Then

$$B_{f'}(\mathfrak{p} + \mathfrak{a}, \mathfrak{p} + \mathfrak{a}) = f([\mathfrak{p} + \mathfrak{a}, \mathfrak{p} + \mathfrak{a}]) \subseteq f([\mathfrak{a}^f, \mathfrak{a}]) = \{0\}.$$

Hence $\mathfrak{p} + \mathfrak{a} = \mathfrak{p} \supset \mathfrak{a}$ and $\mathfrak{p}^f \subset \mathfrak{a}^f$. Therefore $\mathfrak{p} = \mathfrak{p}^{f'} = \mathfrak{p}^f$, and we get the first assertion. Since $\mathfrak{a}$ is commutative, $(\exp X){\cdot}f = f + X{\cdot}f$ for any $X \in \mathfrak{a}$. Setting $A = \exp \mathfrak{a}$,

$$f + (\mathfrak{a}^f)^{\perp} = f + \mathfrak{a}{\cdot}f = A{\cdot}f \subset G{\cdot}f. \qquad ∎$$

Lemma 5.3.12. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra, $\mathfrak{a}$ a minimal non-central ideal of $\mathfrak{g}$ and $f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$. We set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{a}^f, \mathfrak{h}' = \mathfrak{h}_0 + \mathfrak{a}, \mathfrak{k} = \mathfrak{h} + \mathfrak{a}$.*

(1) *dim*$(\mathfrak{h}/\mathfrak{h}_0) \leq$ *dim*$(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$.
(2) $\mathfrak{h}' \in S(f, \mathfrak{g})$ *and dim*$(\mathfrak{h}') \geq$ *dim* $\mathfrak{h}$. *Hence, if* $\mathfrak{h} \in M(f, \mathfrak{g})$*, then the equality holds.*
(3) *dim* $\mathfrak{h} =$ *dim*$(\mathfrak{h}')$ *if and only if dim*$(\mathfrak{h}/\mathfrak{h}_0) =$ *dim*$(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$.
(4) *If dim* $\mathfrak{h} =$ *dim*$(\mathfrak{h}')$*, passing to the quotient spaces,* B_f *induces a bilinear form* $B : \mathfrak{h}/\mathfrak{h}_0 \times \mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}) \to \mathbb{R}$. B *gives the duality between* $\mathfrak{h}/\mathfrak{h}_0$ *and* $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ *and induces an isomorphism as* $\mathfrak{h}_0$*-modules from* $\mathfrak{h}/\mathfrak{h}_0$ *onto* $(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))^*$.
(5) *If dim* $\mathfrak{h} =$ *dim*$(\mathfrak{h}')$*,*

$$Tr\ ad_{\mathfrak{h}'/\mathfrak{h}_0} X = \frac{1}{2}\left(Tr\ ad_{\mathfrak{k}/\mathfrak{h}} X - Tr\ ad_{\mathfrak{k}/\mathfrak{h}'} X\right)$$

for all $X \in \mathfrak{h}_0$.

Proof. As $f([\mathfrak{h}, \mathfrak{h} \cap \mathfrak{a}]) = \{0\} = f([\mathfrak{h}_0, \mathfrak{a}])$, B_f defines by passing to the quotient spaces the bilinear form $B : \mathfrak{h}/\mathfrak{h}_0 \times \mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}) \to \mathbb{R}$ and hence a linear mapping $\tilde{B}$ from $\mathfrak{h}/\mathfrak{h}_0$ to $(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))^*$. $\tilde{X} \in \mathfrak{h}/\mathfrak{h}_0$ being the image of $X \in \mathfrak{h}$,

$$\tilde{B}(\tilde{X}) = 0 \iff f([X, \mathfrak{a}]) = \{0\} \iff X \in \mathfrak{h} \cap \mathfrak{a}^f = \mathfrak{h}_0 \iff \tilde{X} = 0.$$

Thus $\tilde{B}$ is injective and the assertion (1) follows.

Since $\mathfrak{a} \subset \mathfrak{a}^f$, $\mathfrak{h}_0 \cap \mathfrak{a} = \mathfrak{h} \cap \mathfrak{a}^f \cap \mathfrak{a} = \mathfrak{h} \cap \mathfrak{a}$. Hence

$$\dim(\mathfrak{h}') = \dim(\mathfrak{h}_0) + \dim(\mathfrak{a}/(\mathfrak{h}_0 \cap \mathfrak{a})) \geq \dim(\mathfrak{h}_0) + \dim(\mathfrak{h}/\mathfrak{h}_0) = \dim \mathfrak{h}.$$

Since $[\mathfrak{h}', \mathfrak{h}'] \subset [\mathfrak{h}, \mathfrak{h}] + [\mathfrak{a}^f, \mathfrak{a}]$, $\mathfrak{h}' \in S(f, \mathfrak{g})$. In this fashion we get the claims (2) and (3).

In what follows we suppose that $\dim \mathfrak{h} = \dim(\mathfrak{h}')$. As we have seen above, $\tilde{B}$ becomes bijective and B furnishes a duality between $\mathfrak{h}/\mathfrak{h}_0$ and $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$. Since $[\mathfrak{h}_0, \mathfrak{h} \cap \mathfrak{a}] \subset \mathfrak{h} \cap \mathfrak{a}$ and $[\mathfrak{h}_0, \mathfrak{h}_0] \subset \mathfrak{h}_0$, $\mathfrak{h}/\mathfrak{h}_0$ and $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ are $\mathfrak{h}_0$-modules. Let $X_0 \in \mathfrak{h}_0, X \in \mathfrak{h}, Y \in \mathfrak{a}$ and let $\tilde{X} \in \mathfrak{h}/\mathfrak{h}_0, \tilde{Y} \in \mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ be respectively the images of X, Y. Then

$$\begin{aligned} B(X_0 \cdot \tilde{X}, \tilde{Y}) + B(\tilde{X}, X_0 \cdot \tilde{Y}) &= f([[X_0, X], Y]) + f([X, [X_0, Y]]) \\ &= f([X_0, [X, Y]]) \in f([\mathfrak{a}^f, \mathfrak{a}]) = \{0\} \end{aligned}$$

and the assertion (4) follows. Furthermore, since

$$\mathfrak{h}'/\mathfrak{h}_0 = (\mathfrak{h}_0 + \mathfrak{a})/\mathfrak{h}_0 \cong \mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}) = \mathfrak{a}/(\mathfrak{h}_0 \cap \mathfrak{a})$$

as $\mathfrak{h}_0$-modules, taking the assertion (4) into account

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}'/\mathfrak{h}_0} X_0 = \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})} X_0 = -\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}/\mathfrak{h}_0} X_0.$$

By the way, since

$$\begin{aligned}\operatorname{Tr} \operatorname{ad}_{\mathfrak{k}/\mathfrak{h}_0} X_0 &= \operatorname{Tr} \operatorname{ad}_{\mathfrak{k}/\mathfrak{h}} X_0 + \operatorname{Tr} \operatorname{ad}_{\mathfrak{h}/\mathfrak{h}_0} X_0 \\ &= \operatorname{Tr} \operatorname{ad}_{\mathfrak{k}/\mathfrak{h}'} X_0 + \operatorname{Tr} \operatorname{ad}_{\mathfrak{h}'/\mathfrak{h}_0} X_0,\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}\left(\operatorname{Tr} \operatorname{ad}_{\mathfrak{k}/\mathfrak{h}} X_0 - \operatorname{Tr} \operatorname{ad}_{\mathfrak{k}/\mathfrak{h}'} X_0\right) \\ =&\frac{1}{2}\left(\operatorname{Tr} \operatorname{ad}_{\mathfrak{h}'/\mathfrak{h}_0} X_0 - \operatorname{Tr} \operatorname{ad}_{\mathfrak{h}/\mathfrak{h}_0} X_0\right) = \operatorname{Tr} \operatorname{ad}_{\mathfrak{h}'/\mathfrak{h}_0} X_0.\end{aligned}$$ ■

Lemma 5.3.13. *Let* $\mathfrak{a}$ *be a minimal non-central ideal and* $\mathfrak{h}$ *a Lie subalgebra containing* $\mathfrak{z}$. *Then dim*$(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) \leq 2$. *If dim*$(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) = 2$, $\mathfrak{h} \cap \mathfrak{a} = \mathfrak{z} \cap \mathfrak{a}$ *and the action of* $\mathfrak{g}$ *on* $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ *is irreducible.*

Proof. By hypothesis $\mathfrak{h} \supset \mathfrak{h} \cap \mathfrak{a} \supset \mathfrak{z} \cap \mathfrak{a}$. By the way, since the action of $\mathfrak{g}$ on $\mathfrak{a}/(\mathfrak{z} \cap \mathfrak{a})$ is irreducible, $\dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) \leq 2$ and the equality holds only when $\mathfrak{h} \cap \mathfrak{a} = \mathfrak{z} \cap \mathfrak{a}$. ■

Lemma 5.3.14. *We keep the notations in Lemma 5.3.12. Let* $\mathfrak{a}$ *be a minimal non-central ideal and suppose* $\mathfrak{h} \supset \mathfrak{z}$. *Let* σ *be the action of* $\mathfrak{h}$ *on* $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ *and denote its kernel by* $\mathfrak{j}$.

(1) *The action* σ *is either trivial or irreducible.*
(2) $\mathfrak{j}$ *and* $\mathfrak{j} + \mathfrak{h}_0$ *are ideals of* $\mathfrak{h}$.
(3) $\mathfrak{j} + \mathfrak{h}_0$ *coincides with either* $\mathfrak{h}$ *or* $\mathfrak{h}_0$ *and the first possibility occurs if dim* $(\mathfrak{h}/\mathfrak{h}_0) = 2$.
(4) $[\mathfrak{j}, \mathfrak{j}] \subset \mathfrak{h}_0 \cap \mathfrak{j}$.

Proof. By the previous lemma $d = \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) \leq 2$. Concerning the assertion (1), it is clear if $d = 1$ and comes from the previous lemma if $d = 2$. In any way the codimension of $\mathfrak{j}$ in $\mathfrak{h}$ is smaller than or equal to 1, $\mathfrak{j} \supset [\mathfrak{h}, \mathfrak{h}]$ and $\sigma(\mathfrak{h})$ is commutative. These give (2).

We show (3). If $\mathfrak{h}_0 \not\subset \mathfrak{j}$, $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{j}$. If $\mathfrak{h}_0 \subset \mathfrak{j}$ and if $\dim(\mathfrak{h}/\mathfrak{h}_0)$ is either 0 or 1, $\mathfrak{j}$ coincides with either $\mathfrak{h}_0$ or $\mathfrak{h}$. It remains the case where $\mathfrak{h}_0 \subset \mathfrak{j}$ and $\dim(\mathfrak{h}/\mathfrak{h}_0) = 2$. In this case $\mathfrak{j} \neq \mathfrak{h}_0$, so it suffices to show $\mathfrak{j} = \mathfrak{h}$.

Now suppose $\mathfrak{j} \neq \mathfrak{h}$. By (1) σ is irreducible. By Lemma 5.3.12 $2 = \dim(\mathfrak{h}/\mathfrak{h}_0) \leq \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) \leq 2$ and B provides a duality between $\mathfrak{h}/\mathfrak{h}_0$ and $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$. The orthogonal space of $\mathfrak{j}/\mathfrak{h}_0$ with respect to B is the image in $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ of the subspace $\mathfrak{a}(\mathfrak{j}) = \{Y \in \mathfrak{a}; f([Y, \mathfrak{j}]) = \{0\}\}$ of $\mathfrak{a}$. Hence by duality $\mathfrak{h}_0 \subsetneq \mathfrak{j} \subsetneq \mathfrak{h}$ implies $\mathfrak{h} \cap \mathfrak{a} \subsetneq \mathfrak{a}(\mathfrak{j}) \subsetneq \mathfrak{a}$. Now, in order to arrive at a contradiction, it is enough to see that $\mathfrak{a}(\mathfrak{j})$ is an $\mathfrak{h}$-module. Indeed, since $[\mathfrak{j}, \mathfrak{h}] \subset \mathfrak{j}$ and $[\mathfrak{j}, \mathfrak{a}] \subset \mathfrak{h} \cap \mathfrak{a}$,

$$\begin{aligned}f([\mathfrak{j}, [\mathfrak{h}, \mathfrak{a}(\mathfrak{j})]]) &= f([[\mathfrak{j}, \mathfrak{h}], \mathfrak{a}(\mathfrak{j})]) + f([\mathfrak{h}, [\mathfrak{j}, \mathfrak{a}(\mathfrak{j})]]) \\ &\subset f([\mathfrak{j}, \mathfrak{a}(\mathfrak{j})]) + f([\mathfrak{h}, \mathfrak{h} \cap \mathfrak{a}]) = \{0\}.\end{aligned}$$

Thus $[\mathfrak{h}, \mathfrak{a}(\mathfrak{j})] \subset \mathfrak{a}(\mathfrak{j})$. ■

Lemma 5.3.15. *Let $G = exp\ \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. For $X_1, X_2 \in \mathfrak{g}$,*

$$exp(X_1)exp(X_2) = exp(X_1 + X_2)exp\ Y = exp(Y')exp(X_1 + X_2)$$

with some $Y, Y' \in [\mathfrak{g}, \mathfrak{g}]$.

Proof. Since the group G is exponential, the commutator subgroup $[G, G]$ coincides with $\exp([\mathfrak{g}, \mathfrak{g}])$. Let $p : G \to G/[G, G]$, $q = dp : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ be canonical projections. Since these projections commute with the exponential maps and the group $G/[G, G]$ is commutative,

$$\begin{aligned} p(\exp(X_1 + X_2)) &= \exp(q(X_1) + q(X_2)) \\ &= \exp(q(X_1))\exp(q(X_2)) = p(\exp(X_1)\exp(X_2)). \end{aligned}$$

That is, $\exp(X_1)\exp(X_2)\exp(-(X_1 + X_2)) \in [G, G] = \exp([\mathfrak{g}, \mathfrak{g}])$. ■

Lemma 5.3.16. *We use the notations of Lemma 5.3.12. Let $\mathfrak{w}$ be a linear subspace of $\mathfrak{a}$ complementary to $\mathfrak{h} \cap \mathfrak{a}$. Then, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{w}$ and the mapping $(X, Y) \mapsto (exp\ X)(exp\ Y)$ is a diffeomorphism from $\mathfrak{h} \times \mathfrak{w}$ onto K.*

Proof. Put $H = \exp \mathfrak{h}$, $K = \exp \mathfrak{k}$. If we write as φ the mapping in question, φ is analytic. Let us see that φ is bijective. Since $\mathfrak{a}$ is a commutative ideal of $\mathfrak{k}$, $K = HA$ and $A = \exp(\mathfrak{h} \cap \mathfrak{a})\exp \mathfrak{w}$. Hence $K = H\exp \mathfrak{w}$ so that φ is surjective. Next if

$$\exp(X_1)\exp(Y_1) = \exp(X_2)\exp(Y_2)$$

for $(X_j, Y_j) \in \mathfrak{h} \times \mathfrak{w}$ $(j = 1, 2)$, since $\mathfrak{a}$ is commutative and $\mathfrak{h}$ is exponential, there is $X \in \mathfrak{h}$ such that

$$\exp(Y_1 - Y_2) = \exp(Y_1)\exp(-Y_2) = \exp(-X_1)\exp(X_2) = \exp X.$$

Hence $Y_1 - Y_2 \in \mathfrak{h} \cap \mathfrak{w} = \{0\}$ and $(X_1, Y_1) = (X_2, Y_2)$.

Finally, we show that φ^{-1} is analytic. For $k \in K$ we put $\varphi^{-1}(k) = (\psi(k), \zeta(k)), \Psi = \psi \circ \exp : \mathfrak{k} \to \mathfrak{h}, \Upsilon = \zeta \circ \exp : \mathfrak{k} \to \mathfrak{w}$. By definition $\exp X = \exp(\Psi(X))\exp(\Upsilon(X))$ for $X \in \mathfrak{k}$. Now, H being a closed subgroup of K, the canonical projection $p : K \to H\backslash K$ is analytic and, if we identify $H\backslash K$ with $\exp\mathfrak{w}$, $p(\exp X) = \exp(\Upsilon(X))$. Thus, since the exponential map is a diffeomorphism, the mapping $\exp \circ \Psi : X \to (\exp X)p(\exp X)^{-1}$ and $\psi, \zeta, \varphi^{-1}$ are analytic. ■

Lemma 5.3.17. *We use the notations of Lemma 5.3.14.*

(1) *Suppose that $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{j}$. Let $\mathfrak{t}$ be a linear subspace of $\mathfrak{h}$ complementary to $\mathfrak{h}_0$ and contained in $\mathfrak{j}$. Then the mapping $(T, X) \to (exp\ T)(exp\ X)$ is a diffeomorphism from $\mathfrak{t} \times \mathfrak{h}_0$ onto $H = exp\ \mathfrak{h}$.*

(2) *Suppose that* $\mathfrak{j} = \mathfrak{h}_0 \neq \mathfrak{h}$, *and let*

$$X_1 \in \mathfrak{h} \backslash \mathfrak{h}_0.$$

Then the mapping

$$(t, X_0) \mapsto exp(tX_1)exp(X_0)$$

is a diffeomorphism from $\mathbb{R} \times \mathfrak{h}_0$ *onto* H.

Proof. If $\mathfrak{j} = \mathfrak{h}_0 \neq \mathfrak{h}$, $\mathfrak{j}$ is an ideal of codimension 1 and the assertion (2) is well known. So, assume $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{j}$ and denote by φ the mapping in the statement of (1). φ is analytic and, if it is shown to be bijective, φ^{-1} turns out to be analytic as seen in the latter part of the proof of the previous lemma.

Let

$$\exp(T_1)\exp(X_1) = \exp(T_2)\exp(X_2)$$

for $(T_j, X_j) \in \mathfrak{t} \times \mathfrak{h}_0$ $(j = 1, 2)$. From Lemma 5.3.14(4) and Lemma 5.3.15 there is $X_0 \in \mathfrak{h}_0$ so that

$$\exp(X_2)\exp(-X_1) = \exp(-T_2)\exp(T_1) = \exp(T_1 - T_2)\exp(X_0).$$

Hence $T_1 - T_2 \in \mathfrak{h}_0 \cap \mathfrak{t} = \{0\}$ and φ is injective.

In $\mathfrak{h}$, $\mathfrak{h}_0$ is a Lie subalgebra, $\mathfrak{j}$ is an ideal and $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{j}$, hence $H = (\exp \mathfrak{j})\exp(\mathfrak{h}_0)$. To show from this that φ is surjective, it suffices to show that $(\exp \mathfrak{t})\exp(\mathfrak{h}_0 \cap \mathfrak{j})$, which generates $\exp \mathfrak{j}$, is a subgroup. In fact

$$H = (\exp \mathfrak{t})\exp(\mathfrak{h}_0 \cap \mathfrak{j})\exp(\mathfrak{h}_0) = (\exp \mathfrak{t})\exp(\mathfrak{h}_0).$$

Now we take $(T_j, X_j) \in \mathfrak{t} \times (\mathfrak{h}_0 \cap \mathfrak{j})$ $(j = 1, 2)$. Again from Lemma 5.3.14(4) and Lemma 5.3.15

$$\begin{aligned}\exp(T_1)\exp(X_1)\exp(-X_2)\exp(-T_2) &= \exp(T_1)\exp(X_3)\exp(-T_2)\\ &= \exp(T_1)\exp(-T_2)\exp\left(\exp(T_2)\cdot X_3\right)\\ &= \exp(T_1 - T_2)\exp(X_4)\exp(X_5) = \exp(T_1 - T_2)\exp(X_6),\end{aligned}$$

here $X_j \in \mathfrak{h}_0 \cap \mathfrak{j}$ $(3 \leq j \leq 6)$. ∎

Proposition 5.3.18. *Let* $\mathfrak{a}$ *be a minimal non-central ideal and* $f \in \mathfrak{g}^*$, $\mathfrak{h} \in S(f, \mathfrak{g})$, *and assume* $\mathfrak{h} \supset \mathfrak{z}$. *Set*

$$\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{a}^f, \ \mathfrak{h}' = \mathfrak{h}_0 + \mathfrak{a}, \ \mathfrak{k} = \mathfrak{h} + \mathfrak{a}.$$

Further, we denote by $\mathfrak{j}$ *the kernel of the action of* $\mathfrak{h}$ *on* $\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})$. *We suppose that*

$$dim(\mathfrak{h}/\mathfrak{h}_0) = dim(\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a}))$$

and also $\mathfrak{h} = \mathfrak{j} + \mathfrak{h}_0$. *We denote by* $\mathfrak{t}$ *a linear subspace of* $\mathfrak{h}$ *complementary to* $\mathfrak{h}_0$ *and contained in* $\mathfrak{j}$, *and by* $\mathfrak{w}$ *a linear subspace of* $\mathfrak{a}$ *complementary to* $\mathfrak{h}\cap\mathfrak{a}$. *Then,*

(1) *the restriction of* B_f *to* $\mathfrak{t}\times\mathfrak{w}$ *is non-degenerate,*
(2) *the mapping* $(T, X, Y) \mapsto (exp\ T)(exp\ X)(exp\ Y)$ *is a diffeomorphism from* $\mathfrak{t}\times\mathfrak{h}_0\times\mathfrak{w}$ *onto* $K = exp\ \mathfrak{k}$,
(3) $\hat{\rho}(f, \mathfrak{h}, K) \simeq \hat{\rho}(f, \mathfrak{h}', K)$.

Proof. The subspaces $\mathfrak{t}$, $\mathfrak{w}$ are respectively identified with

$$\mathfrak{h}/\mathfrak{h}_0,\ \mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})$$

and the claim (1) follows from Lemma 5.3.12. Besides, the claim (2) follows from the last two lemmas. Also from this the mapping $(X, Y) \mapsto (\exp X)(\exp Y)$ is a diffeomorphism from $\mathfrak{h}_0\times\mathfrak{w}$ onto $H' = \exp(\mathfrak{h}')$.

To simplify the notations, we set

$$\rho = \hat{\rho}(f, \mathfrak{h}, K),\ \rho' = \hat{\rho}(f, \mathfrak{h}', K),$$
$$\mathcal{H} = \hat{\mathcal{H}}(f, \mathfrak{h}, K),\ \mathcal{H}' = \hat{\mathcal{H}}(f, \mathfrak{h}', K),$$
$$\delta = \Delta_{H,K},\ \delta' = \Delta_{H',K}.$$

We construct an intertwining operator $R : \mathcal{H} \to \mathcal{H}'$ between ρ and ρ'. Let us first consider it on the space $\mathcal{C}$ of continuous functions $\psi \in \mathcal{H}$ with compact support modulo H. Fixing $\psi \in \mathcal{C}$ and $k \in K$, we define the complex-valued function $\Phi_{k,\psi}$ on H' by

$$h' \mapsto \psi(kh')\chi_f(h')(\delta')^{-1/2}(h'). \tag{5.3.1}$$

Then, let us see that $\Phi_{k,\psi}$ belongs to the space of functions $\mathcal{E}(H'/H_0)$, $H_0 = \exp(\mathfrak{h}_0)$ defined in Sect. 2.1. In fact, if we put $\delta_0 = \Delta_{H_0,H'}$,

$$\delta_0(h_0) = \big(\delta(h_0)/\delta'(h_0)\big)^{1/2} \quad (h_0 \in H_0)$$

by Lemma 5.3.12(5). Hence, with $h' \in H'$,

$$\begin{aligned}\Phi_{k,\psi}(h'h_0) &= \psi(kh'h_0)\chi_f(h'h_0)(\delta')^{-1/2}(h'h_0)\\ &= \psi(kh')\delta^{1/2}(h_0)\chi_f(h_0)^{-1}\chi_f(h')\chi_f(h_0)(\delta')^{-1/2}(h')(\delta')^{-1/2}(h_0)\\ &= \Phi_{k,\psi}(h')\delta_0(h_0).\end{aligned}$$

On the other hand, by $K = AH = H'H$, the canonical projection $K \to K/H$ induces a surjection $H' \to K/H$ and K/H becomes a homogeneous space of H' so that we get a diffeomorphism $\lambda : H'/H_0 \to K/H$. Let S be the support of $\Phi_{k,\psi}$ modulo H_0. Since the image of S by λ is compact, S itself is compact.

So, as in Sect. 2.1, making use of the H'-invariant positive linear form μ_{H',H_0} on $\mathcal{E}(H'/H_0)$, we define the complex-valued function $R(\psi)$ on K by the formula

$$R(\psi)(k) = \mu_{H',H_0}\left(\Phi_{k,\psi}\right) \ \ (k \in K).$$

The operator R defined in this manner on $\mathcal{C}$ obviously intertwines the actions $\rho,\ \rho'$, that is to say, is commutative with the left translation of K.

Let us verify that R extends to an isometry from $\mathcal{H}$ onto $\mathcal{H}'$. From what we have seen, K/H and H'/H_0 are identified with $\mathfrak{w}$, besides K/H' with $\mathfrak{t}$. Namely, $\mathcal{H}$ is identified with $L^2(\mathfrak{w})$ by the mapping $\psi \to \tilde{\psi},\ \tilde{\psi}(Y) = \psi(\exp Y)$, $\mathcal{C}$ and $\mathcal{E}(H'/H_0)$ are with $C_c(\mathfrak{w})$ by $\Phi \to \tilde{\Phi},\ (\tilde{\Phi})(Y) = \Phi(\exp Y)$, $\mathcal{H}'$ is with $L^2(\mathfrak{t})$ by $\varphi \to \tilde{\varphi},\ \tilde{\varphi}(T) = \varphi(\exp T)$. Besides, μ_{H',H_0} is identified with the Lebesgue measure dY on $\mathfrak{w}$. For $\Phi \in \mathcal{E}(H'/H_0),\ h_0 \in H_0,\ Y_0 \in \mathfrak{w}$,

$$\int_{\mathfrak{w}} \Phi(\exp(Y_0)h_0 \exp Y)dY = \int_{\mathfrak{w}} \Phi(\exp(Y_0{+}h_0{\cdot}Y))\delta_0(h_0)dY = \int_{\mathfrak{w}} \Phi(\exp Y)dY.$$

Moreover, since $\mathfrak{w} \subset \mathfrak{a}$, $\delta'|_W \equiv 1,\ W = \exp \mathfrak{w}$. Therefore, for $Y \in \mathfrak{w}$

$$\widetilde{\Phi_{k,\psi}}(Y) = \Phi_{k,\psi}(\exp Y) = \psi(k\exp Y)e^{if(Y)}$$

and

$$R(\psi)(k) = \int_{\mathfrak{w}} \psi(k\exp Y)e^{if(Y)}dY.$$

Now by Definition 5.3.1 and the left invariance of μ_{H',H_0} we have

$$R(\psi)(kh') = \chi_f(h')^{-1}\delta'(h')^{1/2}R(\psi)(k) \ \ (k \in K,\ h' \in H')$$

so that $R(\psi)$ is identified with the function $\widetilde{R(\psi)}$ on $\mathfrak{t}$,

$$\widetilde{R(\psi)}(T) = R(\psi)(\exp T) = \int_{\mathfrak{w}} \psi((\exp T)\exp Y)e^{if(Y)}dY$$

$$= \int_{\mathfrak{w}} \psi(\exp((\exp T){\cdot}Y)\exp T)e^{if(Y)}dY \ \ (T \in \mathfrak{t}).$$

By the way, $\mathfrak{t} \subset \mathfrak{j}$ and, since $[\mathfrak{j}, \mathfrak{w}] \subset \mathfrak{h} \cap \mathfrak{a}$, $\delta(\exp T) = 1$ and

$$(\exp T){\cdot}Y = Y + [T, Y] + Y',\ \ [T, Y] \in \mathfrak{h} \cap \mathfrak{a},\ \ Y' \in [\mathfrak{h}, \mathfrak{h}] \cap \mathfrak{a}.$$

Hence

$$\begin{aligned}\psi((\exp T)\exp Y) &= \psi(\exp((\exp T)\cdot Y))\delta(\exp T)^{1/2}e^{-if(T)}\\ &= \psi((\exp Y)\exp([T,Y])\exp(Y'))e^{-if(T)}\\ &= \psi(\exp Y)e^{-if([T,Y])}e^{-if(T)}\end{aligned}$$

and consequently

$$\widetilde{R(\psi)}(T) = e^{-if(T)}\int_{\mathfrak{w}} \tilde{\psi}(Y)e^{iB_f(Y,T)}e^{if(Y)}dY.$$

Since the restriction of B_f to $\mathfrak{t}\times\mathfrak{w}$ is non-degenerate, B_f defines a Fourier transform $\mathcal{F}_B : L^2(\mathfrak{w}) \to L^2(\mathfrak{t})$ and

$$\widetilde{R(\psi)}(T) = e^{-if(T)}\left(\mathcal{F}_B\hat{\psi}\right)(T),\ \ \hat{\psi}(Y) = \tilde{\psi}(Y)e^{if(Y)}.$$

In this way $\widetilde{R(\psi)} \in \mathcal{H}' = L^2(\mathfrak{t})$ is shown, and we get, the Fourier transform being isometric, the desired isometry by multiplying R with some positive constant. ■

Proposition 5.3.19. *As before, we define* $\mathfrak{h},\ \mathfrak{h}_0,\ \mathfrak{h}',\ \mathfrak{k},\ \mathfrak{j}$ *and assume* $\mathfrak{h} \supset \mathfrak{z}$, *dim*$(\mathfrak{h}/\mathfrak{h}_0) =$ *dim*$(\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a}))$, $\mathfrak{h} \neq \mathfrak{h}_0 + \mathfrak{j}$.

(1) $\mathfrak{h}_0 = \mathfrak{j}$, *dim*$(\mathfrak{h}/\mathfrak{h}_0) = 1$ *and there are* $X \in \mathfrak{h}\setminus\mathfrak{h}_0$, $Y \in \mathfrak{a}\setminus(\mathfrak{h}\cap\mathfrak{a})$ *such that*

$$\mathfrak{k} = \mathbb{R}X \oplus \mathfrak{h}_0 \oplus \mathbb{R}Y,\ \ [X,Y] = Y,\ \ f(Y) = 1$$

and the mapping

$$(\alpha, X_0, \beta) \mapsto exp(\alpha X)exp(X_0)exp(\beta Y)$$

provides a diffeomorphism from $\mathbb{R}\times\mathfrak{h}_0\times\mathbb{R} \simeq \mathfrak{k}$ *onto* K.

(2) $\mathfrak{h}_0,\ \mathfrak{h}'$ *are ideals of* $\mathfrak{k}$ *and* $[\mathfrak{h}_0,\mathfrak{a}] \subset \mathfrak{h}\cap\mathfrak{a} = \mathfrak{z}\cap\mathfrak{a}$.

(3) *For* $v \in \mathbb{R}$ *we give* $f_v \in \mathfrak{k}^*$ *by* $f_v \in f + \mathfrak{h}^{\perp}$, $f_v(Y) = v$. *Then* $exp(\alpha X)\cdot f_v = f_{ve^{-\alpha}}$ *for any* $\alpha \in \mathbb{R}$. *Moreover, if* $v \neq 0$, $\mathfrak{h}' \in M(f_v, \mathfrak{k})$.

(4) *If* v_+ *and* v_- *denote respectively a positive and negative real number,*

$$\hat{\rho}(f,\mathfrak{h},K) \simeq \hat{\rho}(f_{v_+},\mathfrak{h}',K) \oplus \hat{\rho}(f_{v_-},\mathfrak{h}',K).$$

Proof. Since $\mathfrak{h}_0 + \mathfrak{j} \neq \mathfrak{h}$, from Lemma 5.3.14 and another one just before it, $\mathfrak{h}_0 = \mathfrak{j}$ and $\dim(\mathfrak{h}/\mathfrak{h}_0) = 1$. Because $\mathfrak{h} \neq \mathfrak{j}$, the action of $\mathfrak{h}$ on $\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})$, and hence of course on $\mathfrak{a}/(\mathfrak{z}\cap\mathfrak{a})$, is not trivial, so the latter is irreducible. Since $\mathfrak{h}\cap\mathfrak{a}$ is a submodule of the $\mathfrak{h}$-module $\mathfrak{a}$ different from $\mathfrak{a}$, $\mathfrak{h}\cap\mathfrak{a} = \mathfrak{z}\cap\mathfrak{a}$.

While, by Lemma 5.3.14, $\mathfrak{j} = \mathfrak{h}_0$ is an ideal of $\mathfrak{h}$, by definition $[\mathfrak{j},\mathfrak{a}] \subset \mathfrak{h}\cap\mathfrak{a} \subset \mathfrak{h}_0$ and $\mathfrak{h}_0$ is an ideal of $\mathfrak{k}$, likewise for $\mathfrak{h}' = \mathfrak{h}_0 + \mathfrak{a}$. Now the last claim of (1) follows.

Next let $X_1 \in \mathfrak{h} \setminus \mathfrak{h}_0,\ Y_1 \in \mathfrak{a} \setminus (\mathfrak{a} \cap \mathfrak{h})$. Then there are $0 \neq \lambda \in \mathbb{R}$ and $Y_2 \in \mathfrak{h} \cap \mathfrak{a}$ such that $[X_1, Y_1] = \lambda Y_1 + Y_2$. So, if we put $X = \lambda^{-1} X_1,\ Y = \gamma(\lambda Y_1 + Y_2)$ $(0 \neq \gamma \in \mathbb{R})$, we get $[X, Y] = Y$. As $f([X, Y]) \neq 0$, we can choose γ in such a fashion that $f(Y) = 1$. Finally, since $[\mathfrak{h}_0, \mathfrak{a}] = [\mathfrak{j}, \mathfrak{a}] \subset \mathfrak{h} \cap \mathfrak{a}$, the statements (1), (2) follow.

We show the assertion (3). If $X' \in \mathfrak{h}$, $\exp(\alpha X)\cdot X' \in X' + [\mathfrak{h}, \mathfrak{h}]$. Since $\mathfrak{h} \in S(f, \mathfrak{g})$, $\exp(\alpha X)\cdot f_v \in f + \mathfrak{h}^{\perp}$. Hence we can write $\exp(\alpha X)\cdot f_v = f_{v'}$ with

$$v' = (\exp(\alpha X)\cdot f_v)\,(Y) = f_v(\exp(-\alpha X)\cdot Y) = e^{-\alpha} v.$$

Since $[\mathfrak{h}', \mathfrak{h}'] = [\mathfrak{h}_0, \mathfrak{h}_0] + [\mathfrak{h}_0, \mathfrak{a}] \subset \mathfrak{h}$, Lemma 5.3.12 gives $f_v([\mathfrak{h}', \mathfrak{h}']) = f([\mathfrak{h}', \mathfrak{h}']) = \{0\}$ so that $\mathfrak{h}' \in S(f_v, \mathfrak{k})$, while, as $f_v([X, Y]) = f(Y) = v$, $\mathfrak{k} \notin S(f_v, \mathfrak{k})$ if $v \neq 0$ so that $\mathfrak{h}' \in M(f_v, \mathfrak{k})$.

The representation $\rho = \hat{\rho}(f, \mathfrak{h}, K)$ is realized on the space $\mathcal{H} = \hat{\mathcal{H}}(f, \mathfrak{h}, K)$ of all complex-valued functions ψ on K satisfying the relation

$$\psi(kh) = \chi_f(h)^{-1}\delta^{1/2}(h)\psi(k),\ k \in K,\ h \in H,\ \delta = \Delta_{H,K}.$$

Here note that $\delta|_{H_0} = 1$ by $[\mathfrak{h}_0, \mathfrak{a}] \subset \mathfrak{h}$ and $\delta(\exp X) = e$ by $[X, Y] = Y$. Likewise, sine $\mathfrak{h}'$ is an ideal of $\mathfrak{k}$, the representation $\rho'_v = \hat{\rho}(f_v, \mathfrak{h}', K)$ is realized on the space $\mathcal{H}'_v = \hat{\mathcal{H}}(f_v, \mathfrak{h}', K)$ of all complex-valued functions φ on K satisfying the relation

$$\varphi(kh') = \chi_{f_v}(h')^{-1}\varphi(k),\ k \in K,\ h' \in H'. \tag{5.3.2}$$

As in the proof of the previous proposition, we argue on the space $\mathcal{C}$ of continuous functions $\psi \in \mathcal{H}$ with compact support modulo H. Fixing $\psi \in \mathcal{C}$ and $k \in K$, we consider the function

$$\Phi_{k,\psi} : h' \mapsto \psi(kh')\chi_{f_v}(h'),\ h' \in H' \tag{5.3.3}$$

on H'. The spaces K/H and H'/H_0 being identified, $\Phi_{k,\psi}$ has a compact support modulo H_0. Moreover, since $\delta|_{H_0} = 1$ and $f|_{\mathfrak{h}_0} = f_v|_{\mathfrak{h}_0}$,

$$\Phi_{k,\psi}(h'h_0) = \Phi_{k,\psi}(h'),\ \ h' \in H',\ h_0 \in H_0,$$

namely $\Phi_{k,\psi} \in \mathcal{E}(H'/H_0)$. Now we set

$$(R_v\psi)(k) = \mu_{H',H_0}(\Phi_{k,\psi}).$$

By Definition 5.3.3 of $\Phi_{k,\psi}$,

$$\Phi_{kh',\psi}(h'') = \chi_{f_v}(h')^{-1}\Phi_{k,\psi}(h'h''),\ k \in K,\ h', h'' \in H'$$

and, by the invariance of μ_{H',H_0}, $R_v\psi$ satisfies relation (5.3.2).

Let us see $R_\nu\psi \in \mathcal{H}'_\nu$ and that the operator R_ν defined on $\mathcal{C}$ extends to a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'_\nu$. If this is done, since R_ν is by definition commutative with the left translation by the elements of K, R_ν gives an intertwining operator between ρ and ρ'_ν.

Since $(\beta, X_0) \mapsto \exp(\beta Y)\exp(X_0)$ is a homeomorphism from $\mathbb{R} \times \mathfrak{h}_0$ onto H', under the mapping $\psi \mapsto \tilde{\psi}, \tilde{\psi}(\beta) = \psi(\exp(\beta Y))$ we identify $\mathcal{E}(H'/H_0)$ with $C_c(\mathbb{R})$ and the measure μ_{H',H_0} with the Lebesgue measure $d\beta$. Thus

$$\widetilde{\Phi_{k,\psi}}(\beta) = \Phi_{k,\psi}(\exp(\beta Y)) = \psi(k\exp(\beta Y))e^{i\beta\nu}$$

and

$$(R_\nu\psi)(k) = \int_{\mathbb{R}} \psi(k\exp(\beta Y))e^{i\nu\beta}d\beta.$$

Similarly, since $(\beta, X_1) \mapsto \exp(\beta Y)\exp(X_1)$ is a homeomorphism from $\mathbb{R} \times \mathfrak{h}$ onto K, using the mapping $\psi \mapsto \tilde{\psi}, \tilde{\psi}(\beta) = \psi(\exp(\beta Y))$ we can identify $\mathcal{H}$ with $L^2(\mathbb{R})$. Likewise, by the mapping $\varphi \mapsto \tilde{\varphi}, \tilde{\varphi}(\alpha) = \varphi(\exp(\alpha X))$, $\mathcal{H}'_\nu$ is identified with $L^2(\mathbb{R})$ and $R_\nu\psi$ with the function $\widetilde{R_\nu\psi} : \alpha \mapsto R_\nu\psi(\exp(\alpha X))$. Under these identifications, $\mathcal{F}$ denoting the Fourier transformation in $L^2(\mathbb{R})$,

$$\begin{aligned}
\left(\widetilde{R_\nu\psi}\right)(\alpha) &= (R_\nu\psi)(\exp(\alpha X)) = \int_{\mathbb{R}} \psi(\exp(\alpha X)\exp(\beta Y))e^{i\nu\beta}d\beta \\
&= \int_{\mathbb{R}} \psi\left(\exp\left(e^{\alpha}\beta Y\right)\right)\delta(\exp(\alpha X))^{1/2}e^{-i\alpha f(X)}e^{i\nu\beta}d\beta \\
&= e^{-i\alpha f(X)}\int_{\mathbb{R}} e^{\alpha/2}\tilde{\psi}\left(e^{\alpha}\beta\right)e^{i\nu\beta}d\beta \\
&= e^{-i\alpha f(X)}\int_{\mathbb{R}} e^{-\alpha/2}\tilde{\psi}(\beta)e^{i\nu\beta e^{-\alpha}}d\beta \\
&= \sqrt{2\pi}e^{-\alpha(if(X)+1/2)}\left(\mathcal{F}\tilde{\psi}\right)\left(\nu e^{-\alpha}\right).
\end{aligned}$$

In this way, if $\nu \neq 0$, by $d(\nu e^{-\alpha}) = -\nu e^{-\alpha}d\alpha$, $\widetilde{R_\nu\psi} \in L^2(\mathbb{R})$, $R_\nu\psi \in \mathcal{H}'_\nu$ and $\|R_\nu\psi\|^2 \leq \frac{2\pi}{|\nu|}\|\psi\|^2$. Hence $\frac{|\nu|^{1/2}}{\sqrt{2\pi}}R_\nu$ is extended to an intertwining operator between ρ and ρ'_ν. Let $L^2_+(\mathbb{R})$ and $L^2_-(\mathbb{R})$ be the space of all functions in $L^2(\mathbb{R})$ whose Fourier transforms vanish almost everywhere in the outside of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. From the above computations, if $\nu > 0$ (resp. $\nu < 0$), $\nu e^{-\alpha} \in \mathbb{R}_\pm$ and $\frac{|\nu|^{1/2}}{\sqrt{2\pi}}R_\nu$ provides an isometry from $L^2_+(\mathbb{R})$ (resp. $L^2_-(\mathbb{R})$) onto $L^2(\mathbb{R})$ and the assertion (4) follows since $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$. ■

Corollary 5.3.20. *We use the notations in the previous two propositions. Suppose* $\mathfrak{h} \in I(f, \mathfrak{g})$, $dim(\mathfrak{h}/\mathfrak{h}_0) = dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$. *Then* $\hat{\rho}(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}', G)$.

Let us do concrete computations for groups of low dimension.

Lemma 5.3.21. (1) *Let $G_2 = exp(\mathfrak{g}_2)$, $\mathfrak{g}_2 = \langle X, Y\rangle_{\mathbb{R}}; [X, Y] = Y$. Namely, G_2 is the "$ax + b$" group of Chap. 4, the unique connected and simply connected non-commutative Lie group of dimension 2 and a typical completely solvable Lie group. For any $f \in \mathfrak{g}^*$, the monomial representation $\hat{\rho}(f, \mathbb{R}X, G_2)$ is reducible.*

(2) *Let $G_3(\alpha) = exp(\mathfrak{g}_3(\alpha))$, $\mathfrak{g}_3(\alpha) = \langle T, X, Y\rangle_{\mathbb{R}}$ $(0 \neq \alpha \in \mathbb{R}); [T, X] = X - \alpha Y, [T, Y] = \alpha X + Y$. Namely, $G_3(\alpha)$ is the Grélaud group G_α of Chap. 4, a typical exponential solvable Lie group which is not completely solvable. For any $f \in \mathfrak{g}^*$, the monomial representation $\hat{\rho}(f, \mathbb{R}T, G_3(\alpha))$ is reducible.*

Proof. The reducibility of $\hat{\rho}(f, \mathbb{R}X, G_2)$ follows from the preceding proposition. We show the reducibility of $\rho = \hat{\rho}(f, \mathbb{R}T, G_3(\alpha))$. Put $\mathfrak{a} = \mathbb{R}X + \mathbb{R}Y$. If we identify $\mathfrak{a}$ with $\mathbb{C}$ by the mapping $\beta X + \gamma Y \mapsto \beta + i\gamma$, we have $[T, z] = (1 - i\alpha)z$ for any $z \in \mathbb{C}$. The group $G_3(\alpha)$ is the semi-direct product of $H = \exp(\mathbb{R}T)$ by $A = \exp \mathbb{C}$ and

$$\exp(\kappa T)\exp z = \exp\left(e^{\kappa(1-i\alpha)}z\right)\exp(\kappa T),\ \kappa \in \mathbb{R},\ z \in \mathbb{C}.$$

Let $f \in \mathfrak{g}^*$, $\lambda = f(T)$. Since $\Delta_{H,G_3(\alpha)}(\exp(\kappa T)) = e^{2\kappa}$, $\chi_f(\exp(\kappa T)) = e^{i\lambda\kappa}$, $\mathcal{H} = \hat{\mathcal{H}}(f, \mathbb{R}T, G_3(\alpha))$ is the space of all complex-valued functions φ on $G_3(\alpha)$ satisfying the relation

$$\varphi(g\exp(\kappa T)) = e^{\kappa(1-i\lambda)}\varphi(g),\ (\kappa, g) \in \mathbb{R} \times G_3(\alpha).$$

When we identify $\mathcal{H}$ with $L^2(\mathbb{C}) = L^2(\mathbb{R}^2)$ through the correspondence $\varphi \mapsto \tilde{\varphi}, \tilde{\varphi}(z) = \varphi(\exp z)$, with $\kappa \in \mathbb{R},\ z_0,\ z \in \mathbb{C},\ \varphi \in \mathcal{H}$,

$$\begin{aligned}(\rho(\exp(\kappa T)\exp(z_0))\tilde{\varphi})(z) &= \varphi(\exp(-z_0)\exp(-\kappa T)\exp z)\\ &= \varphi\left(\exp\left(ze^{-\kappa(1-i\alpha)} - z_0\right)\exp(-\kappa T)\right) = e^{\kappa(i\lambda-1)}\tilde{\varphi}\left(ze^{-\kappa(1-i\alpha)} - z_0\right).\end{aligned}$$

We denote by $\mathcal{F}$ the Fourier transformation in $L^2(\mathbb{C})$ and put $\rho_1 = \mathcal{F} \circ \rho \circ \mathcal{F}^{-1}$. We denote by $(\cdot|\cdot)$ the inner product in $\mathbb{R}^2 = \mathbb{C}$ and by dz the Lebesgue measure on $\mathbb{R}^2$. With $g = \exp(\kappa T)\exp(z_0) \in G_3(\alpha)$, $\psi \in C_c(\mathbb{C})$, $v \in \mathbb{C}$,

$$\begin{aligned}2\pi\,(\mathcal{F}(\rho(g))\psi)(v) &= \int_{\mathbb{C}}(\rho(g)\psi)(z)e^{i(v|z)}dz\\ &= \int_{\mathbb{C}} e^{\kappa(i\lambda-1)}\psi\left(ze^{-\kappa(1-i\alpha)} - z_0\right)e^{i(v|z)}dz\\ &= \int_{\mathbb{C}} e^{\kappa(i\lambda+1)}\psi(z')e^{i\left(v|e^{\kappa(1-i\alpha)}(z'+z_0)\right)}dz'\\ &= 2\pi e^{\kappa(i\lambda+1)+i\left(v|e^{\kappa(1-i\alpha)}z_0\right)}(\mathcal{F}\psi)\left(e^{\kappa(1+i\alpha)}v\right).\end{aligned}$$

We deduce from this the reducibility of ρ_1 and hence of ρ. Because $\mathbb{R}$ acts on $z \in \mathbb{C}$ by the formula $\kappa \cdot z = e^{\kappa(1+i\alpha)}z$ and there exists a non-negligible measurable set M of $\mathbb{C}$, which is invariant by the action of $\mathbb{R}$ and whose complement is also non-negligible, the closed subspace formed by all $\Phi \in L^2(\mathbb{C})$ which vanish almost everywhere in the outside of M is invariant by ρ_1. ■

Henceforth we fix an exponential solvable Lie group $G = \exp \mathfrak{g}$ and $f \in \mathfrak{g}^*$. The sum of all minimal ideals of $\mathfrak{g}$ is called the **pedestal** of $\mathfrak{g}$, which we denote by $\gamma(\mathfrak{g})$.

Proposition 5.3.22. *Every* $\mathfrak{h} \in I(f, \mathfrak{g})$ *contains* $\gamma(\mathfrak{g})$.

Proof. Let $\mathfrak{a}$ be a minimal ideal of $\mathfrak{g}$. Then $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \in I(f, \mathfrak{k})$. If $[\mathfrak{h}, \mathfrak{a}] = \{0\}$, $\mathfrak{h} \supset \mathfrak{a}$. In fact, if this is not true, $K = \exp \mathfrak{k}$ is a direct product of $H = \exp \mathfrak{h}$ with a non-trivial subgroup, and $\rho_1 = \hat{\rho}(f, \mathfrak{h}, K)$ is reducible. So, we suppose $[\mathfrak{h}, \mathfrak{a}] \neq \{0\}$ in what follows. Then the action of $\mathfrak{h}$ on $\mathfrak{a}$ is irreducible so that $\mathfrak{h} \cap \mathfrak{a}$ is equal to either $\{0\}$ or $\mathfrak{a}$. Therefore, it is enough to show that ρ_1 is reducible if $[\mathfrak{h}, \mathfrak{a}] \neq \{0\}$ and if $\mathfrak{h} \cap \mathfrak{a} = \{0\}$. In this case, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{a}$ and K is a semi-direct product of H by $A = \exp \mathfrak{a}$. Now $\mathcal{H}_1 = \hat{\mathcal{H}}(f, \mathfrak{h}, K)$ is identified with $L^2(\mathfrak{a})$, a continuous function $\varphi \in \mathcal{H}_1$ corresponding to $\tilde{\varphi} : Y \mapsto \varphi(\exp Y)$. With $X \in \mathfrak{h},\ Y \in \mathfrak{a}$,

$$
\begin{aligned}
(\rho_1(\exp X)\tilde{\varphi})\,(Y) &= \varphi(\exp(-X)\exp Y) = \varphi(\exp(\exp(-X)\cdot Y)\exp(-X)) \\
&= \tilde{\varphi}(\exp(-X)\cdot Y)\Delta_{H,K}^{-1/2}(\exp X)e^{if(X)}.
\end{aligned}
$$

If $\mathfrak{j}$ denotes the kernel of the action of $\mathfrak{h}$ on $\mathfrak{a}$, as we saw before, $\mathfrak{j}$ is a hyperplane of $\mathfrak{h}$. Let us write $\mathfrak{h} = \mathfrak{j} \oplus \mathbb{R}X_1$. Since $\exp(-X)\cdot Y = Y$, $\Delta_{H,K}(\exp X) = 1$ for $X \in \mathfrak{j}$, the above computations say that $\rho_1(\exp X)$ is a scalar operator $e^{if(X)}$. Hence, putting $\mathfrak{k}_1 = \mathbb{R}X_1 \oplus \mathfrak{a},\ K_1 = \exp(\mathfrak{k}_1)$, it suffices to see that $\rho_2 = \rho_1|_{K_1}$ is reducible, while $\rho_2 = \hat{\rho}(f, \mathbb{R}X_1, K_1)$ and $\mathfrak{k}_1$ is isomorphic to either $\mathfrak{g}_2$ or $\mathfrak{g}_3(\alpha)$ according to $\dim \mathfrak{a} = 1,\ 2$. Now the desired result comes from the previous lemma. ■

Proposition 5.3.23. $I(f, \mathfrak{g}) \subset M(f, \mathfrak{g})$.

Proof. Take $\mathfrak{h} \in I(f, \mathfrak{g})$. We show $\dim \mathfrak{h} = m(f, \mathfrak{g})$ by induction on $\dim \mathfrak{g}$. Since $\mathfrak{h} \supset \gamma(\mathfrak{g})$ by the last proposition, $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$.

We first assume that $\ker f$ contains a non-trivial ideal of $\mathfrak{g}$, and let $\mathfrak{a}$ be a minimal ideal such that $f|_{\mathfrak{a}} = 0$. The last proposition assures that $\mathfrak{h} \supset \mathfrak{a}$. We set $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$. Let $\tilde{\mathfrak{h}}$ (resp. $\tilde{f}$) be the canonical image of $\mathfrak{h}$ (resp. f) in $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{g}}^*$) and $p : G \to \tilde{G} = G/A,\ A = \exp \mathfrak{a}$, the canonical projection. Then, since $\hat{\rho}(f, \mathfrak{h}, G) = \hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \circ p$, $\tilde{\mathfrak{h}} \in I(\tilde{f}, \tilde{\mathfrak{g}})$. Thus, $\tilde{\mathfrak{h}} \in M(\tilde{f}, \tilde{\mathfrak{g}})$ by the induction hypothesis. The bilinear form $B_{\tilde{f}}$ being obtained from B_f by passing to the quotient space, $m(\tilde{f}, \tilde{\mathfrak{g}}) + \dim \mathfrak{a} = m(f, \mathfrak{g})$. Therefore $\mathfrak{h} \in M(f, \mathfrak{g})$.

In what follows suppose that $\ker f$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then obviously $\dim \mathfrak{z} \leq 1$. Take a minimal non-central ideal $\mathfrak{a}$ of $\mathfrak{g}$. Then $\mathfrak{a}^f \neq \mathfrak{g}$. Indeed, if $\mathfrak{a}^f = \mathfrak{g}$, f vanishes on the ideal $[\mathfrak{g}, \mathfrak{a}] \neq \{0\}$, contradictory to the hypothesis.

We set $G_0 = \exp(\mathfrak{a}^f)$. If $\mathfrak{h} \subset \mathfrak{a}^f$, because of $\hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_{G_0}^G \hat{\rho}(f, \mathfrak{h}, G_0)$, $\mathfrak{h} \in I(f, \mathfrak{a}^f)$. Hence, by the induction hypothesis and Lemma 5.3.11, we get $\mathfrak{h} \in M(f, \mathfrak{g})$.

Suppose hereafter that $\mathfrak{h} \not\subset \mathfrak{a}^f$. Let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{a}^f$, $\mathfrak{h}' = \mathfrak{h}_0 + \mathfrak{a}$, $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$. By Lemma 5.3.12,

$$1 \leq \dim(\mathfrak{h}/\mathfrak{h}_0) \leq \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) \leq 2.$$

If $\dim(\mathfrak{h}/\mathfrak{h}_0) = \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$, $\dim \mathfrak{h} = \dim(\mathfrak{h}')$ by Lemma 5.3.12 and $\hat{\rho}(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}', G)$ by Corollary 5.3.20. Since $\mathfrak{h}' \subset \mathfrak{a}^f$, $\mathfrak{h}' \in M(f, \mathfrak{g})$ as we have seen above. Thus, $\mathfrak{h} \in M(f, \mathfrak{g})$.

Now remains the case where $\mathfrak{h} \not\subset \mathfrak{a}^f$ and $\dim(\mathfrak{h}/\mathfrak{h}_0) < \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$. We will see there is no such possibility. In fact, if we assume such a situation, $\dim(\mathfrak{h}/\mathfrak{h}_0) = 1$, $\dim(\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})) = 2$ and $\mathfrak{h}\cap\mathfrak{a} = \mathfrak{z}\cap\mathfrak{a}$ by Lemma 5.3.13. Furthermore, $\mathfrak{z}\cap\mathfrak{a} \neq \{0\}$. If not, $\mathfrak{a}$ turns out to be a minimal ideal and $\mathfrak{h} \supset \mathfrak{a}$ by the last proposition. Therefore $\mathfrak{h} \subset \mathfrak{h}^f \subset \mathfrak{a}^f$, which is contradictory to the hypothesis. As $\dim \mathfrak{z} \leq 1$,

$$\mathfrak{h} \cap \mathfrak{a} = \mathfrak{z} \cap \mathfrak{a} = \mathfrak{z}, \ \dim \mathfrak{z} = 1.$$

If we write $\mathfrak{z} = \mathbb{R}Z$, we may take $f(Z) = 1$ since $f|_{\mathfrak{z}} \neq 0$.

Lemma 5.3.24. *We denote by* $\mathfrak{j}$ *the kernel of the action of* $\mathfrak{h}$ *on* $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ *and by* $\mathfrak{c}(\mathfrak{a})$ *the centralizer of* $\mathfrak{a}$ *in* $\mathfrak{g}$*, namely*

$$\mathfrak{c}(\mathfrak{a}) = \{X \in \mathfrak{g}; [X, Y] = 0, \forall Y \in \mathfrak{a}\}.$$

Then $\mathfrak{h} \cap \mathfrak{c}(\mathfrak{a}) = \mathfrak{h}_0 \subset \mathfrak{j}$.

Proof. If we set $\tilde{B} = B_f|_{\mathfrak{a}\times\mathfrak{j}}$, $\mathfrak{j}\cap\mathfrak{h}_0$ is the orthogonal space of $\mathfrak{a}$ relative to $\tilde{B}$. Since $[X, \mathfrak{a}] \subset \mathfrak{h} \cap \mathfrak{a} = \mathbb{R}Z$ for any $X \in \mathfrak{j}$, there is $\lambda_X \in \mathfrak{a}^*$ such that $[X, Y] = \lambda_X(Y)Z$ for all $Y \in \mathfrak{a}$ and so $f([X, Y]) = \lambda_X(Y)$. Here the subspace $\mathfrak{a}(\mathfrak{j}) = \{Y \in \mathfrak{a}; [Y, \mathfrak{j}] = \{0\}\}$ is the orthogonal space of $\mathfrak{j}$ relative to $\tilde{B}$. Since $\dim(\mathfrak{h}/\mathfrak{h}_0) = 1$,

$$\dim(\mathfrak{a}/\mathfrak{a}(\mathfrak{j})) = \dim(\mathfrak{j}/(\mathfrak{j} \cap \mathfrak{h}_0)) \leq 1.$$

Consequently $\mathfrak{z} \cap \mathfrak{a} \subsetneq \mathfrak{a}(\mathfrak{j}) \subset \mathfrak{a}$. By Lemma 5.3.14, $\mathfrak{j}$ being an ideal of $\mathfrak{h}$ and $\mathfrak{a}(\mathfrak{j})$ becomes an $\mathfrak{h}$-module by Jacobi identity.

If $\mathfrak{h} = \mathfrak{j}$, of course $\mathfrak{h}_0 \subset \mathfrak{j}$. Otherwise, the action of $\mathfrak{h}$ on $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ is irreducible. Hence $\mathfrak{a}(\mathfrak{j}) = \mathfrak{a}$, $\mathfrak{j} = \mathfrak{j} \cap \mathfrak{h}_0$, $\mathfrak{j} \subset \mathfrak{h}_0 \subsetneq \mathfrak{h}$. Thus $\mathfrak{j} = \mathfrak{h}_0$ since $\dim(\mathfrak{h}/\mathfrak{j}) \leq 1$. From what we have seen, $\mathfrak{h}_0 \subset \mathfrak{j}$ in any case. So, if $X \in \mathfrak{h}_0 \subset \mathfrak{j} \cap \mathfrak{a}^f$, $\lambda_X = 0$ and $X \in \mathfrak{h} \cap \mathfrak{c}(\mathfrak{a})$. Thus

$$\mathfrak{h}_0 \subset \mathfrak{h} \cap \mathfrak{c}(\mathfrak{a}) \subset \mathfrak{h} \cap \mathfrak{a}^f = \mathfrak{h}_0. \qquad \blacksquare$$

We continue the proof of the proposition. Take $X \in \mathfrak{h} \setminus \mathfrak{h}_0$ so that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{R}X$ as vector space. First suppose $\mathfrak{h} = \mathfrak{j}$. Since $\dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) > \dim(\mathfrak{h}/\mathfrak{h}_0)$ and B_f

induces the bilinear mapping $\mathfrak{a}/(\mathfrak{z}\cap\mathfrak{a})\times\mathfrak{h}/\mathfrak{h}_0 \to \mathbb{R}$, there is $Y \in (\mathfrak{a}\setminus(\mathfrak{h}\cap\mathfrak{a}))\cap\mathfrak{h}^f$ such that $[X,Y] \in [\mathfrak{j},\mathfrak{a}] \subset \mathfrak{z}$, $f([X,Y]) = 0$. Hence $[X,Y] = 0$. As we saw in the last lemma, $\mathfrak{h}_0 \subset \mathfrak{c}(\mathfrak{a})$ so that $[\mathfrak{h},Y] = \{0\}$. In consequence, Y belongs to the centre of $\mathfrak{k}$, but this contradicts $\mathfrak{h} \in I(f,\mathfrak{k})$.

We next suppose $\mathfrak{h} \neq \mathfrak{j}$ so that the action of $\mathfrak{h}$ on $\mathfrak{a}/(\mathfrak{z}\cap\mathfrak{a})$ is irreducible. Take $Y = Y_1 + iY_2 \in \mathfrak{a}_{\mathbb{C}}$ which represents an eigenvector of the action of $\mathfrak{h}$ on $(\mathfrak{a}/(\mathfrak{z}\cap\mathfrak{a}))_{\mathbb{C}}$. So there are $\alpha,\ \lambda \in \mathbb{R}\setminus\{0\}$ and $\gamma \in \mathbb{C}$ such that $[X,Y] = \lambda(1+i\alpha)Y + \gamma Z$. Replacing if necessary Y by $[X,Y]$ and X by $\lambda^{-1}X$, we may assume $[X,Y] = (1+i\alpha)Y$. Since $\mathfrak{h}_0 \subset \mathfrak{c}(\mathfrak{a})$ by the last lemma, $\mathfrak{a}' = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ is a minimal ideal of $\mathfrak{k}$, and we are led to a contradiction as in the case described above. ■

Lemma 5.3.25. *Let, as before, $f \in \mathfrak{g}^*$ and let $\mathfrak{a}$ be a commutative ideal of $\mathfrak{g}$. Then $I(f,\mathfrak{a}^f) \subset I(f,\mathfrak{g})$. We set $G_0 = exp(\mathfrak{a}^f)$, $A = exp\ \mathfrak{a}$. If $\mathfrak{h} \in I(f,\mathfrak{a}^f)$, $\mathfrak{h} \supset \mathfrak{a}$ and the restriction of $\hat{\rho}(f,\mathfrak{h},G_0)$ to A is $\chi_f Id$.*

Proof. The stabilizer of χ_f with respect to the action of G on $\hat{A}$ is nothing but G_0. Taking Theorem 3.4.4 into account, it is enough to verify that the restriction of $\rho = \hat{\rho}(f,\mathfrak{h},G_0)$ to A is $\chi_f Id$. By Proposition 5.3.23 $\mathfrak{h} \in M(f,\mathfrak{a}^f)$ and, since $\mathfrak{h}+\mathfrak{a} \in S(f,\mathfrak{a}^f)$, $\mathfrak{h} \supset \mathfrak{a}$, while ρ is realized by the left translation in the space $\hat{\mathcal{H}}(f,\mathfrak{h},G_0)$ of complex-valued functions φ on G_0 satisfying the relation

$$\varphi(gh) = \chi_f(h)^{-1}\Delta_{H,G_0}^{1/2}(h)\varphi(g)\ \ g \in G_0,\ h \in H,$$

and $\Delta_{H,G_0}|_A = 1$. Since $\mathfrak{a}$ is commutative, with $X \in \mathfrak{a}^f,\ Y \in \mathfrak{a}$,

$$\begin{aligned}(\rho(\exp Y)\varphi)(\exp X) &= \varphi(\exp(-Y)\exp X)\\ &= \varphi\left((\exp X)\exp(\exp(-X)\cdot(-Y))\right).\end{aligned}$$

Here we can write $\exp(-X)\cdot(-Y) = -Y + Y',\ Y' \in [\mathfrak{a}^f,\mathfrak{a}] \subset \mathfrak{a}$. As $f(Y') = 0$,

$$(\rho(\exp Y)\varphi)(\exp X) = \varphi((\exp X)\exp(-Y)\exp(Y')) = \chi_f(\exp Y)\varphi(\exp X). \ ■$$

Proposition 5.3.26. *At any $f \in \mathfrak{g}^*$, $I(f,\mathfrak{g}) \neq \emptyset$.*

Proof. We proceed by induction on $\mathfrak{g}$. If there is an ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{g}$ such that $f|_{\mathfrak{a}} = 0$, let $A = \exp\mathfrak{a}$, $\tilde{G} = G/A$, $p : G \to \tilde{G}$ the canonical projection, $dp : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ its differential and $\tilde{f} \in \tilde{\mathfrak{g}}^*$ the image of f. By the induction hypothesis there exists $\tilde{\mathfrak{h}} \in I(\tilde{f},\tilde{\mathfrak{g}})$. So, if we put $\mathfrak{h} = (dp)^{-1}(\tilde{\mathfrak{h}})$, $\mathfrak{h} \in S(f,\mathfrak{g})$ and $\hat{\rho}(f,\mathfrak{h},G) = \hat{\rho}(\tilde{f},\tilde{\mathfrak{h}},\tilde{G})\circ p$. Hence $\mathfrak{h} \in I(f,\mathfrak{g})$.

In the case where there is no such ideal as above, if $\mathfrak{a}$ is a minimal non-central ideal of $\mathfrak{g}$, $\mathfrak{a}^f \subsetneq \mathfrak{g}$ and $I(f,\mathfrak{a}^f) \neq \emptyset$ by the induction hypothesis. Thus the result follows from the last proposition. ■

Proposition 5.3.27. *Let $f \in \mathfrak{g}^*$ and $\mathfrak{h}_1,\ \mathfrak{h}_2$ be two elements of $I(f, \mathfrak{g})$. Then the representations $\hat{\rho}(f, \mathfrak{h}_1, G)$ and $\hat{\rho}(f, \mathfrak{h}_2, G)$ are equivalent.*

Proof. We reason in the same scheme as in the proof of the previous proposition. If there is a non-trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$ where f vanishes, we may assume that $\mathfrak{a}$ is a minimal ideal. Then Proposition 5.3.22 assures that $\mathfrak{h}_1,\ \mathfrak{h}_2$ contain $\mathfrak{a}$. Using the notations introduced in the proof of the previous proposition, we put $\tilde{\mathfrak{h}}_j = dp(\mathfrak{h}_j)$ for $j = 1,\ 2$. Then $\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_j, \tilde{G})\circ p = \hat{\rho}(f, \mathfrak{h}_j, G)$, while $\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_1, \tilde{G})$ and $\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_2, \tilde{G})$ are equivalent by the induction hypothesis. So we have the conclusion.

In the case where there is no such ideal as the above, if $\mathfrak{a}$ is a minimal non-central ideal of $\mathfrak{g}$, $\mathfrak{a}^f \subsetneq \mathfrak{g}$. When $\mathfrak{h}_1,\ \mathfrak{h}_2$ are both contained in $\mathfrak{a}^f$, the induction hypothesis says that $\rho_j = \hat{\rho}(f, \mathfrak{h}_j, G_0)$ $(j = 1,\ 2)$ with $G_0 = \exp(\mathfrak{a}^f)$ are mutually equivalent, and likewise for representations $\hat{\rho}(f, \mathfrak{h}_j, G) = \mathrm{ind}_{G_0}^G \rho_j$ $(j = 1,\ 2)$.

After these considerations, the proof will be complete if we show the following. For $\mathfrak{h} \in I(f, \mathfrak{g})$ not contained in $\mathfrak{a}^f$, there is $\mathfrak{h}' \in S(f, \mathfrak{g})$ contained in $\mathfrak{a}^f$ so that $\hat{\rho}(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}', G)$. Indeed, if we set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{a}^f$, $\mathfrak{h}' = \mathfrak{h}_0 + \mathfrak{a}$ by modifying $\mathfrak{h}$, Proposition 5.3.23 and Lemma 5.3.12(2) imply $\dim \mathfrak{h} = \dim(\mathfrak{h}')$. Similarly Lemma 5.3.12(3) tells us that $\dim(\mathfrak{h}/\mathfrak{h}_0) = \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a}))$. Now we are ready to apply Corollary 5.3.20 to get $\hat{\rho}(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}', G)$. ∎

Theorem 5.3.8 is now proved. If we agree to identify unitary representations with their equivalence classes, this theorem allows us to simplify the notation $\hat{\rho}(f, \mathfrak{h}, G)$ as $\hat{\rho}(f)$ and $f \mapsto \hat{\rho}(f)$ provides a mapping from $\mathfrak{g}^*$ to the unitary dual $\hat{G}$ of G. Moreover, for $\mathfrak{h} \in S(f, \mathfrak{g}), g \in G$, the simple transfers of structures give $g{\cdot}\mathfrak{h} \in S(g{\cdot}f, \mathfrak{g})$ and $\hat{\rho}(g{\cdot}f, g{\cdot}\mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}, G)$. Therefore, the above mapping induces a mapping from the space $\mathfrak{g}^*/G$ of all **coadjoint orbits** of G to $\hat{G}$, denoted again by $\hat{\rho} = \hat{\rho}_G$.

Theorem 5.3.28 (Bernat [10]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. The mapping $\hat{\rho}$ is a bijection of $\mathfrak{g}^*/G$ onto $\hat{G}$.*

Proof. To begin with, we show following Takenouchi [75] the surjectivity of the mapping $\hat{\rho}$. Take $\pi \in \hat{G}$. We have to see the existence of a closed connected subgroup H of G and its unitary character χ so that $\pi \simeq \mathrm{ind}_H^G \chi$, because, if this is checked, then $\mathfrak{h}$ being the Lie algebra of H, there exists some $f_1 \in \mathfrak{h}^*$ such that $\chi(\exp X) = e^{if_1(X)},\ \forall X \in \mathfrak{h}$. So, choosing $f \in \mathfrak{g}^*$ in such a manner that $f|_{\mathfrak{h}} = f_1$, we get $\pi \simeq \hat{\rho}(f, \mathfrak{h}, G)$.

As usual we show the claim by induction on the dimension of G. Let $\mathfrak{a}$ be a minimal non-central ideal of $\mathfrak{g}$ and $\mathfrak{b} = [\mathfrak{g}, \mathfrak{a}] \neq \{0\}$, $B = \exp \mathfrak{b}$. If $\pi|_B = 1$, we can write $\pi = \tilde{\pi}\circ p$ with $p : G \to \tilde{G} = G/B$, where $\tilde{\pi}$ is an irreducible unitary representation of $\tilde{G}$. By the induction hypothesis there are a connected closed subgroup $\tilde{H}$ of $\tilde{G}$ and its unitary character $\tilde{\chi}$ so that $\tilde{\pi} \simeq \mathrm{ind}_{\tilde{H}}^{\tilde{G}} \tilde{\chi}$. Hence putting $H = p^{-1}(\tilde{H}),\ \chi = \tilde{\chi}\circ p, \pi \simeq \mathrm{ind}_H^G \chi$.

Assume in what follows that $\pi|_B \neq 1$. We denote as before by $\mathfrak{z}$ the centre of $\mathfrak{g}$ and put $C = \exp \mathfrak{z}$. Schur's Lemma means that $\pi|_C$ is a unitary character of C and our hypothesis implies $\dim \mathfrak{z} \leq 1$. In order to apply Theorem 3.4.4 of

Chap. 3, we verify that the commutative closed normal subgroup $A = \exp \mathfrak{a}$ is regularly embedded in G. From now on we identify $\hat{A}$ with $\mathfrak{a}^*$ and calculate case by case the action of G on $\hat{A}$. $\dim(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z}))$ being 1 or 2, accordingly to these possibilities we write the root coming from the action of G on $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})$ as λ or $\lambda(1 + i\alpha)$ $(0 \neq \alpha \in \mathbb{R})$ with $\lambda \in \mathfrak{g}^*$.

(1) $\mathfrak{a} \cap \mathfrak{z} = \{0\}$, $\dim \mathfrak{a} = 1$.

If we put $\mathfrak{a} = \mathbb{R}Y$, $[X, Y] = \lambda(X)Y$ for all $X \in \mathfrak{g}$. Hence $(\exp X) \cdot \ell = e^{-\lambda(X)}\ell$ $(\ell \in \mathfrak{a}^*)$ and the G-orbit in $\mathfrak{a}^*$ is either one point or a half line.

(2) $\mathfrak{a} \cap \mathfrak{z} = \{0\}$, $\dim \mathfrak{a} = 2$.

There exist $Y_1,\ Y_2$ in $\mathfrak{a}$ so that

$$[X, Y_1] = \lambda(X)(Y_1 - \alpha Y_2),\ [X, Y_2] = \lambda(X)(\alpha Y_1 + Y_2)$$

for any $X \in \mathfrak{g}$. Let $\{Y_1^*,\ Y_2^*\}$ be the dual basis of $\{Y_1,\ Y_2\}$, and we calculate $\exp X \cdot \ell = a'Y_1^* + b'Y_2^*$ with $\ell = aY_1^* + bY_2^* \in \mathfrak{a}^*$ $(a,\ b \in \mathbb{R})$:

$$a' = e^{-\lambda(X)}\left(a \cos(\alpha\lambda(X)) + b \sin(\alpha\lambda(X))\right)$$

$$b' = e^{-\lambda(X)}\left(-a \sin(\alpha\lambda(X)) + b \cos(\alpha\lambda(X))\right).$$

Thus the G-orbit in $\mathfrak{a}^*$ is either one point or a spiral.

(3) $\mathfrak{a} \cap \mathfrak{z} = \mathfrak{z} \neq \{0\}$, $\dim \mathfrak{a} = 2$.

Let $\mathfrak{z} = \mathbb{R}Z$ and $\mathfrak{a} = \mathbb{R}Y \oplus \mathfrak{z}$, and take the dual basis $Y^*,\ Z^*$ of $Y,\ Z$. Now there is $\gamma \in \mathfrak{g}^*$ so that we can write

$$[X, Y] = \lambda(X)Y + \gamma(X)Z$$

for any $X \in \mathfrak{g}$. Separating again various cases, we compute the G-orbit of $\ell = aY^* + bZ^* \in \mathfrak{a}^*$ $(a,\ b \in \mathbb{R})$.

(i) $\lambda = 0$.

If we put $(\exp X) \cdot \ell = a'Y^* + b'Z^*$,

$$a' = a - \gamma(X)b,\ b' = b.$$

Hence the G-orbit in $\mathfrak{a}^*$ is either a point or a line.

(ii) The case where $\lambda \neq 0$ and $\lambda,\ \gamma \in \mathfrak{g}^*$ are linearly dependent.

Assume $\gamma = c\lambda$ $(c \in \mathbb{R})$ and put $Y' = Y + cZ$. Then $[X, Y'] = \lambda(X)Y'$ for any $X \in \mathfrak{g}$ and we are reduced to case (1).

(iii) The case where $\lambda,\ \gamma \in \mathfrak{g}^*$ are linearly independent.

In this case, we can take $X_1,\ X_2 \in \mathfrak{g}$ so that

$$\lambda(X_1) = 1, \quad \gamma(X_1) = 0$$

$$\lambda(X_2) = 0, \quad \gamma(X_2) = 1.$$

If we write $\exp(tX_1) \cdot \ell$ ($t \in \mathbb{R}$) as $a'Y^* + b'Z^*$,

$$a' = e^{-t}a,\ b' = b.$$

Likewise for $\exp(tX_2) \cdot \ell$,

$$a' = a - tb,\ b' = b.$$

Hence the G-orbit in $\mathfrak{a}^*$ is either one point, a half-line or a line.

(4) $\mathfrak{a} \cap \mathfrak{z} = \mathfrak{z} \neq \{0\}$, $\dim \mathfrak{a} = 3$.
Let $\mathfrak{z} = \mathbb{R}Z$. If we choose conveniently a basis $\{Y_1,\ Y_2\}$ of $\mathfrak{a}$ modulo $\mathfrak{z}$, there are $\gamma,\ \delta \in \mathfrak{g}^*$ so that

$$[X, Y_1] = \lambda(X)(Y_1 - \alpha Y_2) + \gamma(X)Z,$$
$$[X, Y_2] = \lambda(X)(\alpha Y_1 + Y_2) + \delta(X)Z$$

for any $X \in \mathfrak{g}$. Besides $\lambda \neq 0$ by hypothesis. Here once again we examine various cases separately. Replacing $Y_1,\ Y_2$ and trying a modification similar to (3)(ii), it is enough to treat the following three possibilities.

(i) $\gamma = \delta = 0$.
This case is clearly reduced to case (2).

(ii) The case where $\lambda,\ \gamma,\ \delta$ are linear independent.
There are $X_1,\ X_2,\ X_3$ in $\mathfrak{g}$ so that

$$\lambda(X_1) = 1,\quad \lambda(X_2) = 0,\quad \lambda(X_3) = 0,$$
$$\gamma(X_1) = 0,\quad \gamma(X_2) = 1,\quad \gamma(X_3) = 0,$$
$$\delta(X_1) = 0,\quad \delta(X_2) = 0,\quad \delta(X_3) = 1.$$

Using the dual basis $\{Y_1^*,\ Y_2^*,\ Z^*\}$ of $\{Y_1,\ Y_2,\ Z\}$, we compute the G-orbit passing $\ell = aY_1^* + bY_2^* + cZ^* \in \mathfrak{a}^*$ ($a,\ b,\ c \in \mathbb{R}$). If we write $\exp(tX_1) \cdot \ell$ ($t \in \mathbb{R}$) as $a'Y_1^* + b'Y_2^* + c'Z^*$,

$$a' = e^{-t}\left(a\cos(\alpha t) + b\sin(\alpha t)\right),\ b' = e^{-t}\left(-a\sin(\alpha t) + b\cos(\alpha t)\right),\ c' = c.$$

Likewise for $\exp(tX_2) \cdot \ell$,

$$a' = a - tc,\ b' = b,\ c' = c.$$

For $\exp(tX_3) \cdot \ell$,

$$a' = a,\ b' = b - tc,\ c' = c.$$

Consequently, the G-orbit $G \cdot \ell$ is one point if $a = b = c = 0$, a spiral if $|a| + |b| \neq 0,\ c = 0$ and a plane if $c \neq 0$.

(iii) The case where $\lambda,\ \gamma$ are linearly independent and $\delta = d\gamma$ $(d \in \mathbb{R})$.

Let us see that in reality there is no such a possibility like this. At first, if we put $Y_1' = Y_1 + dY_2,\ Y_2' = -dY_1 + Y_2$,

$$[X, Y_1'] = \lambda(X)(Y_1' - \alpha Y_2') + (1 + d^2)\gamma(X)Z,$$
$$[X, Y_2'] = \lambda(X)(\alpha Y_1' + Y_2')$$

for any $X \in \mathfrak{g}$. Hence we may assume $d = 0$ from the beginning. Now if we take $X_1,\ X_2$ in $\mathfrak{g}$ so that

$$\lambda(X_1) = \gamma(X_2) = 1,\ \lambda(X_2) = \gamma(X_1) = 0,$$

namely that

$$[X_1, Y_1] = Y_1 - \alpha Y_2, \quad [X_2, Y_1] = Z,$$
$$[X_1, Y_2] = \alpha Y_1 + Y_2, \quad [X_2, Y_2] = 0,$$

on the one hand,

$$[X_2, [X_1, Y_2]] = [X_2, \alpha Y_1 + Y_2] = \alpha Z.$$

On the other hand,

$$[X_2, [X_1, Y_2]] = [[X_2, X_1], Y_2] + [X_1, [X_2, Y_2]] = [[X_2, X_1], Y_2]$$

from Jacobi identity. But this last term never coincides with αZ.

Neither does it for the possibility where $\lambda,\ \delta$ are linearly independent and $\gamma = d\delta$ $(d \in \mathbb{R})$.

In each of the above cases, it is easy to confirm that the G-orbits in $\hat{A}$ are countably separated for the Borel structure of $\hat{A}$ and hence the commutative closed normal subgroup A is regularly embedded in G.

Theorem 3.4.4(2) tells us that there are some $\kappa \in \hat{A}$ and an irreducible unitary representation σ satisfying $\sigma|_A = \kappa Id$ of the stabilizer $G(\kappa)$ of κ in G so that $\pi \simeq \mathrm{ind}_{G(\kappa)}^G \sigma$. Besides, Theorem 5.3.2 says that $G(\kappa)$ is a connected closed subgroup of G. Now, if we take $f \in \mathfrak{g}^*$ such that $\kappa = \chi_f$, $G(\kappa) = \exp(\mathfrak{a}^f)$. Hence if $G(\kappa) = G$, $\mathfrak{a}^f = \mathfrak{g}$, namely $f([\mathfrak{g}, \mathfrak{a}]) = f(\mathfrak{b}) = \{0\}$. So, it follows that $\pi|_B = \kappa|_B = 1$. But this is contradictory to the assumption. Therefore $G(\kappa) \subsetneq G$ and by the induction hypothesis there are a connected closed subgroup H of $G(\kappa)$ and its unitary character χ so that $\sigma \simeq \mathrm{ind}_H^{G(\kappa)} \chi$. In this way, $\pi = \mathrm{ind}_H^G \chi$.

Next we show the mapping $\hat{\rho}$ is injective.

Lemma 5.3.29. *Let* $\mathfrak{a}$ *be a minimal non-central ideal of* $\mathfrak{g}$, $\mathfrak{b} = [\mathfrak{g}, \mathfrak{a}]$, $B = exp\ \mathfrak{b}$ *and* $f \in \mathfrak{g}^*$. *Then* $f \in \mathfrak{b}^\perp$ *if and only if* $\hat{\rho}(f)|_B = 1$.

Proof. Using Lemma 5.3.25 we take $\mathfrak{h} \in I(f, \mathfrak{a}^f) \subset I(f, \mathfrak{g})$. Since $\mathfrak{h} + \mathfrak{a} \in S(f, \mathfrak{g})$, Proposition 5.3.23 gives $\mathfrak{h} \supset \mathfrak{a}$. Now, if we put $\rho = \hat{\rho}(f, \mathfrak{h}, G)$, for $X \in \mathfrak{g}$, $Y \in \mathfrak{b}$ and a continuous function $\varphi \in \hat{\mathcal{H}}(f, \mathfrak{h}, G)$,

$$(\rho(\exp Y)\varphi)\,(\exp X) = \varphi(\exp(-Y)\exp X)$$
$$=\varphi\,((\exp X)\exp(-\exp(-X)\cdot Y)) = \varphi(\exp X)e^{if(\exp(-X)\cdot Y)}.$$

Hence if $f|_\mathfrak{b} = 0$, we have $\rho|_B = 1$. Conversely, if $\rho|_B = 1$, $f(G\cdot\mathfrak{b}) = f(\mathfrak{b}) = 0$ follows. ∎

Let $\mathfrak{a}$ be a minimal non-central ideal of $\mathfrak{g}$, $\mathfrak{b} = [\mathfrak{g}, \mathfrak{a}] \neq \{0\}$ and $B = \exp \mathfrak{b}$. If we designate the images in $\mathfrak{g}^*/G$ of $\mathfrak{b}^\perp$ and of $\mathfrak{g}^* \setminus \mathfrak{b}^\perp$ by $\mathcal{C}$ and by $\mathcal{D}$, the space of orbits $\mathfrak{g}^*/G$ is decomposed into the disjoint union of $\mathcal{C}$ and $\mathcal{D}$. By Lemma 5.3.29, $\hat{\rho}(\mathcal{C}) \cap \hat{\rho}(\mathcal{D}) = \emptyset$. Therefore if we see that $\rho_1 = \hat{\rho}|_\mathcal{C}$ and $\rho_2 = \hat{\rho}|_\mathcal{D}$ are both injective, then the mapping $\hat{\rho}$ turns out to be injective.

We suppose that $\hat{\rho}(f_1) = \hat{\rho}(f_2)$ for $f_1,\ f_2 \in \mathfrak{b}^\perp$. Put $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{b}$, and denote by $\tilde{f}_1,\ \tilde{f}_2$ the canonical images in $\tilde{\mathfrak{g}}^*$ of $f_1,\ f_2$ respectively. Let $\tilde{\mathfrak{h}}_j \in I(\tilde{f}_j, \tilde{\mathfrak{g}})$ $(j = 1,\ 2)$, be p the canonical projection from G onto $\tilde{G} = G/B$ and its differential dp, then for $j = 1,\ 2$

$$\mathfrak{h}_j = (dp)^{-1}(\tilde{\mathfrak{h}}_j),\ \hat{\rho}(f_j, \mathfrak{h}_j, G) = \hat{\rho}(\tilde{f}_j, \tilde{\mathfrak{h}}_j, \tilde{G})\circ p.$$

Hence $\hat{\rho}(\tilde{f}_1) \simeq \hat{\rho}(\tilde{f}_2)$ and by the induction hypothesis there is $g \in G$ so that $\tilde{f}_2 = p(g)\cdot\tilde{f}_1$, namely $f_2 = g\cdot f_1$. This means that the mapping ρ_1 is injective.

We next assume that $\hat{\rho}(f_1) \simeq \hat{\rho}(f_2)$ for $f_1,\ f_2 \in \mathfrak{g}^* \setminus \mathfrak{b}^\perp$. Since $f_j([\mathfrak{g}, \mathfrak{a}]) \neq \{0\}$ with $j = 1,\ 2$, $\mathfrak{a}^{f_j} \subsetneq \mathfrak{g}$. If we set $G_j = \exp(\mathfrak{a}^{f_j})$, there exist $\mathfrak{h}_j \in I(f_j, \mathfrak{a}^{f_j})$ so that

$$\hat{\rho}(f_j) = \hat{\rho}(f_j, \mathfrak{h}_j, G) \simeq \mathrm{ind}_{G_j}^G \hat{\rho}(f_j, \mathfrak{h}_j, G_j).$$

As $\mathfrak{h}_j \supset \mathfrak{a}$, the restriction of $\hat{\rho}(f_j, \mathfrak{h}_j, G_j)$ to A is a scalar operator corresponding to the unitary character χ_{f_j} of A. Hence, Mackey's theory asserts that $\hat{\rho}(f_j)|_A$ is defined by a measure μ_j supported on the G-orbit $\Omega_j = G \cdot \chi_{f_j} \subset \hat{A}$. The representations $\hat{\rho}(f_1)$ and $\hat{\rho}(f_2)$ being equivalent, the measures μ_1 and μ_2 are also equivalent. Consequently $\Omega_1 = \Omega_2$. Therefore there is $g \in G$ so that $\chi_{f_2} = g \cdot \chi_{f_1} = \chi_{g\cdot f_1}$. Hence replacing f_1 by $g\cdot f_1$, we may assume $\chi_{f_1} = \chi_{f_2}$. Then, $f_2 - f_1 \in \mathfrak{a}^\perp$ and $\mathfrak{a}^{f_1} = \mathfrak{a}^{f_2}$. We denote this common Lie subalgebra by $\mathfrak{g}_0$ and let $G_0 = \exp(\mathfrak{g}_0),\ f_0 = f_j|_\mathfrak{a}$.

The representations $\hat{\rho}(f_j, \mathfrak{h}_j, G_0)$ $(j = 1, 2)$ are irreducible representations of G_0, whose restrictions to A are the scalar operators corresponding to the same unitary character $\chi_{f_0} : \exp Y \mapsto e^{if_0(Y)}$. Since these two representations induce equivalent representations of G, they are mutually equivalent by Theorem 3.4.4(1). Therefore by the induction hypothesis there is $g \in G$ such that $g \cdot \left(f_1|_{\mathfrak{g}_0}\right) = f_2|_{\mathfrak{g}_0}$. That is, $g \cdot f_1 \in f_2 + {\mathfrak{g}_0}^{\perp}$. However, Lemma 5.3.11 asserts that

$$f_2 + \mathfrak{g}_0^{\perp} = f_2 + (\mathfrak{a}^{f_2})^{\perp} \subset G \cdot f_2.$$

In consequence, $G \cdot f_1 = G \cdot f_2$. In this way it has been shown that the mapping $\hat{\rho}$ is injective too. ■

Definition 5.3.30. The mapping $\hat{\rho}$ from $\mathfrak{g}^*$ or $\mathfrak{g}^*/G$ to $\hat{G}$ is called a **Kirillov–Bernat mapping**.

We give a more precise statement of this result. We provide $\hat{G}$ with **Fell topology** [24, 26, 41]. Let $\pi \in \hat{G}$ with its Hilbert space $\mathcal{H}_\pi$. We define a neighbourhood of π in $\hat{G}$. For finite vectors $v_1, \ldots, v_k$ of $\mathcal{H}_\pi$, a compact set C of G and a positive number $\epsilon > 0$, the neighbourhood $\mathcal{U}(v_1, \ldots, v_k; C; \epsilon)$ of π is constituted by such $\rho \in \hat{G}$ that there exist vectors $w_1, \ldots, w_k$ in $\mathcal{H}_\rho$ satisfying

$$|(\pi(g)v_i, v_j) - (\rho(g)w_i, w_j)| < \epsilon, \ g \in C, \ 1 \leq i, j \leq k.$$

Then, the unitary dual of an exponential solvable Lie group is realized including its topology by the space of the coadjoint orbits. This fact has been open for a long time under the name of Kirillov's conjecture before being proved by the following theorem. However, its proof is too long to be included here.

Theorem 5.3.31 ([51]). *We equip the orbit space $\mathfrak{g}^*/G$ with the quotient topology and the unitary dual $\hat{G}$ with Fell topology. Then the Kirillov–Bernat mapping $\hat{\rho}$ is a homeomorphism.*

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. We denote simply by dg a left Haar measure on G and consider the space $L^1(G)$ with respect to this measure. With $\varphi \in L^1(G)$ we define an operator $\pi(\varphi)$ in $\mathcal{H}_\pi$ by the formula

$$\pi(\varphi) = \int_G \varphi(g)\pi(g)dg.$$

When $\pi(\varphi)$ is a compact operator for arbitrary $\varphi \in L^1(G)$, π is called **CCR**. G is called **CCR** when its all irreducible unitary representation are CCR [4, 20]. It is known that $\pi \in \hat{G}$ is CCR if and only if $\{\pi\} \subset \hat{G}$ is a closed set for Fell topology [2, 25]. Then, all coadjoint orbits of a connected and simply connected nilpotent Lie group are closed sets so that these groups are CCR by Theorem 5.3.31. In the exponential solvable case, the coadjoint orbits are locally closed sets but not necessarily closed. Theorem 5.3.31 establishes also Moore's conjecture [57] which asserts the correspondence between closed coadjoint orbits and CCR representations. Likewise, Theorem 5.3.31 which translates various problems concerning Fell topology to those concerning the quotient topology is a significant result. What information on Fell topology offers the construction of the unitary dual due to Auslander and Kostant of connected and simply connected type I solvable Lie groups?

5.4 Pukanszky Condition

Different from the nilpotent case, it is possible that $I(f, \mathfrak{g}) \neq M(f, \mathfrak{g})$ in the case of exponential solvable Lie groups. The set $I(f, \mathfrak{g})$ is characterized by the following theorem.

Theorem 5.4.1. *([10]) Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$ and $H = \exp \mathfrak{h}$. Then the following statements are equivalent:*

(1) $H{\cdot}f = f + \mathfrak{h}^{\perp}$;
(2) $f + \mathfrak{h}^{\perp} \subset G{\cdot}f$ *and* $\mathfrak{h} \in M(f, \mathfrak{g})$;
(3) *for any* $\lambda \in \mathfrak{h}^{\perp}$, $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$;
(4) $\mathfrak{h} \in I(f, \mathfrak{g})$.

Proof. (1) $\Longrightarrow$ (2): By assumption the mapping $h \mapsto h{\cdot}f$ is a surjection from H onto $f + \mathfrak{h}^{\perp}$. Hence Sard's theorem tells us that there is at least one point $h_0 \in H$ at which the differential of this mapping becomes a surjection. Thus the mapping $X \mapsto X{\cdot}(h_0{\cdot}f)$ is a surjection from $\mathfrak{h}$ onto $\mathfrak{h}^{\perp}$. But $\mathfrak{h}{\cdot}(h_0{\cdot}f) = \left(\mathfrak{h}^{h_0 \cdot f}\right)^{\perp}$. Therefore $\mathfrak{h}^{h_0 \cdot f} = \mathfrak{h}$ and $\mathfrak{h}^f = \mathfrak{h}$.
(2) $\Longrightarrow$ (3): If $\lambda \in \mathfrak{h}^{\perp}$, then clearly $\mathfrak{h} \in S(f + \lambda, \mathfrak{g})$. So, it suffices to see $\dim(\mathfrak{g}(f + \lambda)) = \dim(\mathfrak{g}(f))$. From the assumption there exists $g \in G$ such that $f + \lambda = g{\cdot}f$. Hence $\mathfrak{g}(f + \lambda) = g{\cdot}(\mathfrak{g}(f))$ and $\dim(\mathfrak{g}(f + \lambda)) = \dim(\mathfrak{g}(f))$.
(3) $\Longrightarrow$ (1): The image of $\mathfrak{h}$ by the differential map $X \mapsto X{\cdot}f$ at the origin of the mapping $h \mapsto h{\cdot}f$ from H to $f + \mathfrak{h}^{\perp}$ is $\mathfrak{h}{\cdot}f = \left(\mathfrak{h}^f\right)^{\perp} = \mathfrak{h}^{\perp}$. Hence $H{\cdot}f$ contains a neighbourhood of f in $f + \mathfrak{h}^{\perp}$. This argument may be applied to each point of $f + \mathfrak{h}^{\perp}$ and we get (1).
(4) $\Longrightarrow$ (3): For any $\lambda \in \mathfrak{h}^{\perp}$, $\hat{\rho}(f, \mathfrak{h}, G) = \hat{\rho}(f + \lambda, \mathfrak{h}, G)$. Hence Proposition 5.3.23 implies $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$.
(2) $\Longrightarrow$ (4): This follows from Theorem 11.2.1 described later in Chap. 11. ∎

Definition 5.4.2. When $\mathfrak{h} \in S(f, \mathfrak{g})$ satisfies the assertion (1) of Theorem 5.4.1, $\mathfrak{h}$ is said to satisfy the **Pukanszky condition**.

Remark 5.4.3. $\mathfrak{h}$ satisfies the Pukanszky condition if and only if $\mathfrak{h}_{\mathbb{C}} \in P^+(f, G)$ satisfies the Pukanszky condition.

Concerning the construction of the elements of $I(f, \mathfrak{g})$, a standard method due to M. Vergne is well known. Let us consider a strong Malcev sequence of Lie subalgebras of $\mathfrak{g}$. Remember that this means a sequence of Lie subalgebras

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1\ (1 \le j \le n)$$

with the following properties: if $\mathfrak{g}_j$ is not ideal of $\mathfrak{g}$, $\mathfrak{g}_{j-1}$ and $\mathfrak{g}_{j+1}$ are both ideals of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is irreducible. By interpolating if necessary a composition series of ideals of $\mathfrak{g}$, we recognize that such a sequence of Lie subalgebras always exists. Put $f_j = f|_{\mathfrak{g}_j}$ for $1 \le j \le n$.

Theorem 5.4.4 ([10], Chap. IV). $\mathfrak{h} = \sum_{j=1}^{n} \mathfrak{g}_j(f_j)$ *is an element of* $I(f, \mathfrak{g})$.

Proof. We first see that $\mathfrak{g}_{n-1}$ is an ideal of $\mathfrak{g}$. Otherwise, the action of $\mathfrak{g}$ on $\mathfrak{g}/\mathfrak{g}_{n-2}$ being irreducible, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_{n-2}$ and $\mathfrak{g}_{n-1}$ becomes an ideal, contrary to the assumption.

We next notice that, for $0 \le p \le n$, $(\mathfrak{g}_j)_{0 \le j \le p}$ is a strong Malcev sequence of $\mathfrak{g}_p$. Indeed, if $\mathfrak{g}_k$ $(1 \le k \le p-1)$ is not an ideal of $\mathfrak{g}_p$, of course it is not an ideal of $\mathfrak{g}$. Consequently $\mathfrak{g}_{k\pm1}$ are ideals of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\mathfrak{g}_{k+1}/\mathfrak{g}_{k-1}$ is irreducible with root $\mathfrak{g} \to \lambda(X)(1 \pm i\alpha)$ $(\lambda \in \mathfrak{g}^*,\ \alpha \in \mathbb{R}\setminus\{0\})$. Hence by assumption $\lambda|_{\mathfrak{g}_p} \ne 0$ and the action of $\mathfrak{g}_p$ on $\mathfrak{g}_{k+1}/\mathfrak{g}_{k-1}$ is irreducible.

Let us verify by induction on $n = \dim \mathfrak{g}$ that $\mathfrak{h}$ is a maximal isotropic subspace regarding B_f. If $\mathfrak{g}(f) \subset \mathfrak{g}_{n-1}$, $\mathfrak{g}(f) \subset \mathfrak{g}_{n-1}(f_{n-1})$ and $\dim(\mathfrak{g}(f)) = \dim(\mathfrak{g}_{n-1}(f_{n-1}))-1$. So, by the induction hypothesis, $\mathfrak{h} = \sum_{j=1}^{n-1} \mathfrak{g}_j(f_j)$ is isotropic relative to B_f and

$$\dim \mathfrak{h} = \frac{1}{2}(n - 1 + \dim(\mathfrak{g}_{n-1}(f_{n-1}))) = \frac{1}{2}(n + \dim(\mathfrak{g}(f))).$$

If $\mathfrak{g}(f) \not\subset \mathfrak{g}_{n-1}$, $\mathfrak{g}_{n-1}(f_{n-1}) \subset \mathfrak{g}(f)$ and $\dim(\mathfrak{g}(f)) = \dim(\mathfrak{g}_{n-1}(f_{n-1})) + 1$. By the induction hypothesis, $\mathfrak{h} = \left(\sum_{j=1}^{n-1} \mathfrak{g}_j(f_j)\right) + \mathfrak{g}(f)$ is isotropic and

$$\begin{aligned}\dim \mathfrak{h} &= \dim \left(\sum_{j=0}^{n-1} \mathfrak{g}_j(f_j)\right) + 1 \\ &= \frac{1}{2}(n - 1 + \dim(\mathfrak{g}(f)) - 1) + 1 = \frac{1}{2}(n + \dim(\mathfrak{g}(f))).\end{aligned}$$

In any case

$$\mathfrak{h} \cap \mathfrak{g}_{n-1} = \sum_{j=1}^{n-1} \mathfrak{g}_j(f_j).$$

Let us show that $\mathfrak{h}$ is a Lie subalgebra. Suppose that there exists a positive integer p such that $\sum_{j=1}^{p-1} \mathfrak{g}_j(f_j)$ is a Lie subalgebra but that $\sum_{j=1}^{p} \mathfrak{g}_j(f_j)$ is no longer a Lie subalgebra. Thus there are $X \in \mathfrak{g}_p(f_p)$ and $Y_k \in \mathfrak{g}_k(f_k),\ k \le p-1$ so that

$$Y_{k+1} = [X, Y_k] \notin \sum_{j=1}^{p} \mathfrak{g}_j(f_j).$$

Then, in the first place $\mathfrak{g}_k$ is not an ideal of $\mathfrak{g}$. Otherwise, $Y_{k+1} \in \mathfrak{g}_k$ and

$$f([Y_{k+1}, Y]) = f([[X, Y_k], Y]) = f([[X, Y], Y_k]) - f([X, [Y, Y_k]]) = 0$$

for any $Y \in \mathfrak{g}_k$. Hence $Y_{k+1} \in \mathfrak{g}_k(f_k)$ follows, which contradicts the assumption.

From the preceding, $\mathfrak{g}_{k\pm1}$ are ideals of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\mathfrak{g}_{k+1}/\mathfrak{g}_{k-1}$ is irreducible. In particular, $[\mathfrak{g}_{k+1}, \mathfrak{g}_{k+1}] \subset \mathfrak{g}_{k-1}$. Now, we can compute as above to get

$$f([Y_{k+1}, Y]) = f([[X, Y_k], Y]) = 0$$

for any $Y \in \mathfrak{g}_{k-1}$. Therefore $Y_k \notin \mathfrak{g}_{k-1}$ and $\mathfrak{g}_k = \mathfrak{g}_{k-1} \oplus \mathbb{R}Y_k$, because, if $Y_k \in \mathfrak{g}_{k-1}$, we would have $Y_{k+1} \in \mathfrak{g}_{k-1}(f_{k-1})$ from the above equation. But this is absurd.

Moreover, $Y_{k+1} \notin \mathfrak{g}_k$. Otherwise, $f([Y_{k+1}, Y_k]) = 0$ since $Y_k \in \mathfrak{g}_k(f_k)$. Together with the above result of computations $Y_{k+1} \in \mathfrak{g}_k(f_k)$. Finally,

$$\mathfrak{g}_{k+1} = \mathfrak{g}_{k-1} \oplus \mathbb{R}Y_k \oplus \mathbb{R}Y_{k+1},$$

while $f([Y_{k+1}, Y_k]) \neq 0$ follows from $Y_{k+1} \notin \mathfrak{g}_{k+1}(f_{k+1})$.

Next, since $[Y_k, Y_{k+1}] \in \mathfrak{g}_{k-1}$ and $f([[Y_k, Y_{k+1}], Y]) = 0, \ \forall\, Y \in \mathfrak{g}_{k-1}$, we have

$$[Y_k, Y_{k+1}] \in \mathfrak{g}_{k-1}(f_{k-1}).$$

Here we put

$$[X, Y_{k+1}] = \beta Y_k + \gamma Y_{k+1} + Y_{k-1},\ \beta,\ \gamma \in \mathbb{R},\ Y_{k-1} \in \mathfrak{g}_{k-1}.$$

Then, from what we have seen until now, $Y_{k-1} \in \mathfrak{g}_{k-1}(f_{k-1})$. On the other hand, since $X \in \mathfrak{g}_p(f_p)$,

$$f([[X, Y_{k+1}], Y_k]) = f([X, [Y_{k+1}, Y_k]]) + f([Y_{k+1}, Y_{k+1}]) = 0.$$

From this and taking $f([Y_{k+1}, Y_k]) \neq 0$ into account, $\gamma = 0$. Hence the eigenvalue of the action of $\mathrm{ad}X$ on $\mathfrak{g}_{k+1}/\mathfrak{g}_{k-1}$ is a root of the equation $x^2 - \beta = 0$. Consequently

$$\lambda(X)(1 + i\alpha) = -\lambda(X)(1 - i\alpha)$$

and $\lambda(X) = 0$. But this is absurd and thus $\mathfrak{h}$ is a Lie subalgebra. In this manner, we have arrived at $\mathfrak{h} \in M(f, \mathfrak{g})$.

Finally, we show that $\mathfrak{h}$ satisfies the Pukanszky condition. Again we use the induction on $n = \dim \mathfrak{g}$. Let us see $f + \lambda \in G{\cdot}f$ for any $\lambda \in \mathfrak{h}^\perp$. If we put $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}_{n-1}$ and $H' = \exp(\mathfrak{h}')$, the induction hypothesis means

$$f_{n-1} + (\mathfrak{h}')^\perp = H'{\cdot}(f_{n-1})$$

in $\mathfrak{g}_{n-1}^*$, while, since $\mathfrak{g}_{n-1}$ is an ideal of $\mathfrak{g}$ and the group H' leaves $f+\mathfrak{h}^\perp$ stable, we may assume that λ vanishes on $\mathfrak{g}_{n-1}+\mathfrak{h}$, by moving $f+\lambda$ by an element of H'. By the way, $\mathfrak{g}_{n-1}$ is an ideal of codimension 1 in $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}_{n-1}$ and $\mathfrak{g}(f)\subset\mathfrak{h}$. In consequence, putting $\mathfrak{m}=[\mathfrak{g},\mathfrak{g}]+\mathfrak{g}(f)$, $\lambda\in\mathfrak{m}^\perp$. Then

$$G(f|_\mathfrak{m})\cdot f=f+\mathfrak{m}^\perp.$$

Indeed, $G(f|_\mathfrak{m})\cdot f\subset f+\mathfrak{m}^\perp$ by definition. On the other hand, if $X\in\mathfrak{g}(f|_\mathfrak{m})$, $X\cdot f\in\mathfrak{m}^\perp$ and, since $\mathfrak{m}$ contains $[\mathfrak{g},\mathfrak{g}]$, $(\exp X)\cdot f=f+X\cdot f$. Hence

$$f+\mathfrak{g}(f|_\mathfrak{m})\cdot f\subset G(f|_\mathfrak{m})\cdot f$$

and $f+\mathfrak{m}^\perp\subset G(f|_\mathfrak{m})\cdot f$ since

$$\mathfrak{g}(f|_\mathfrak{m})\cdot f=\left(\mathfrak{g}(f|_\mathfrak{m})^f\right)^\perp=\mathfrak{m}^\perp. \qquad \blacksquare$$

An element of $I(f,\mathfrak{g})$ constructed by this method is called a **Vergne polarization**.

Concerning Theorem 5.3.8(3), for $\mathfrak{h}_1,\mathfrak{h}_2\in I(f,\mathfrak{g})$, it is natural to ask how to construct explicitly an intertwining operator between two monomial representations $\hat\rho(f,\mathfrak{h}_1,G)$ and $\hat\rho(f,\mathfrak{h}_2,G)$. First of all, we can prove

$$\mathrm{Tr}\,\mathrm{ad}_{\mathfrak{h}_1/(\mathfrak{h}_1\cap\mathfrak{h}_2)}X+\mathrm{Tr}\,\mathrm{ad}_{\mathfrak{h}_2/(\mathfrak{h}_1\cap\mathfrak{h}_2)}X=0$$

for all $X\in\mathfrak{h}_1\cap\mathfrak{h}_2$ and consequently

$$\Delta_{H_1,G}(h)=\Delta_{H_2,G}(h)\Delta^2_{H_1\cap H_2,H_2}(h)\ (h\in H_1\cap H_2).$$

Hence if we define, $\phi\in\hat{\mathcal{H}}(f,\mathfrak{h}_1,G)$ and $g\in G$ being given, the function Φ_g on H_2 by

$$\Phi_g(h)=\phi(gh)\chi_f(h)\Delta^{-1/2}_{H_2,G}(h),$$

the relation

$$\Phi_g(hx)=\Delta_{H_1\cap H_2,H_2}(x)\Phi_g(h)\ (h\in H_2,x\in H_1\cap H_2)$$

holds. Therefore, with $\nu=\mu_{H_2,H_1\cap H_2}$, we can formally consider the integral

$$(T_{\mathfrak{h}_2\mathfrak{h}_1}\phi)(g)=\oint_{H_2/H_1\cap H_2}\phi(gh)\chi_f(h)\Delta^{-1/2}_{H_2,G}(h)d\nu(h)\ (g\in G). \tag{5.4.1}$$

At least at formal level, the function $T_{\mathfrak{h}_2\mathfrak{h}_1}\phi$ satisfies the covariance relation required to the elements of the space $\hat{\mathcal{H}}(f,\mathfrak{h}_2,G)$ and the operator $T_{\mathfrak{h}_2\mathfrak{h}_1}$ obviously commutes with the left translation of G. Besides, if the integral (5.4.1) is convergent on the space of C^∞-vectors, we could prove that $T_{\mathfrak{h}_2\mathfrak{h}_1}$ gives a true intertwining operator. Therefore, it is no exaggeration to say that the main problem would be the convergence of (5.4.1). The simple product set H_2H_1 turns out locally closed in G so that the homogeneous space $H_2/H_2\cap H_1$ is homeomorphic to the quotient space H_2H_1/H_1. Thus, if H_2H_1 is closed in G, which is true for example when G is nilpotent, the integral (5.4.1) is convergent for continuous functions ϕ with compact support modulo H_1. To our regret, we do not know whether H_2H_1 is always closed or not.

Question. Let G be an exponential solvable Lie group, $f \in \mathfrak{g}^*$, $\mathfrak{h}_j \in I(f, \mathfrak{g})$ and $H_j = \exp(\mathfrak{h}_j)$ $(j = 1, 2)$. Is the product set H_2H_1 a closed set in G?

If $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is a Vergne polarization, we can show by choosing appropriately a coexponential basis to $\mathfrak{h}_1$ in $\mathfrak{g}$ that H_2H_1 is closed in G, and by utilizing the transitivity of the form $\mu_{\cdot,\cdot}$ that the operator $T_{\mathfrak{h}_2\mathfrak{h}_1}$ furnishes a true intertwining operator.

Now as always let $f \in \mathfrak{g}^*$ and consider three maximal isotropic subspaces $W_j (1 \leq j \leq 3)$ of $\mathfrak{g}$ for the bilinear form B_f. Following M. Kashiwara we define a quadratic form Q on $W_1 \oplus W_2 \oplus W_3$ by the formula

$$Q(x_1, x_2, x_3) = f([x_1, x_2]) + f([x_2, x_3]) + f([x_3, x_1]).$$

We define the **Maslov index** of the triplet (W_1, W_2, W_3) as the index of the quadratic form Q and denote it by $\tau(W_1, W_2, W_3)$. That is, regarding a proper basis of the direct sum space $W_1 \oplus W_2 \oplus W_3$ the matrix of Q is diagonalized with diagonal entries $\pm 1,\ 0$. Let p be the number of $+1$ and q the number of -1. Then $\tau(W_1, W_2, W_3) = p - q$. We list the main properties of this index.

Lemma 5.4.5 ([52, 53]). *We write τ_{ijk} instead of $\tau(W_i, W_j, W_k)$.*

(a) $\tau_{123} = -\tau_{213} = -\tau_{132}$.
(b) $\tau_{234} - \tau_{134} + \tau_{124} - \tau_{123} = 0$.
(c) *If $\mathfrak{p}$ is an isotropic subspace of $\mathfrak{g}$ for B_f containing $\mathfrak{g}(f)$ and if W is a maximal isotropic subspace of $\mathfrak{g}$, $W^{\mathfrak{p}} = (W \cap \mathfrak{p}^f) + \mathfrak{p}$ is also a maximal isotropic subspace. Besides, if $\mathfrak{p}$ is contained in $(W_1 \cap W_2) + (W_2 \cap W_3) + (W_3 \cap W_1)$, $\tau_{123} = \tau(W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}, W_3^{\mathfrak{p}})$.*

Proof. Put $V = \mathfrak{g}/\mathfrak{g}(f)$ and denote by B the image of B_f to V. Then, it is enough to discuss the symplectic space (V, B). The property (a) is clear from the definition. Let us show the property (b). Let $L,\ M$ be two transversal, i.e. $L \cap M = \{0\}$, maximal isotropic subspaces for B. We write $V = L \oplus M$ and denote by $p_{L,M}$ the projection from V to L along this decomposition. Then, clearly

$$B\left(p_{L,M}(x), y\right) = B\left(x, p_{M,L}(y)\right) = B\left(p_{L,M}(x), p_{M,L}(y)\right),\ \forall\,(x, y) \in V \times V.$$

In the case where $W_i,\ W_j$ are transversal, the projection p_{W_i,W_j} will be simply written as $p_{i,j}$.

Lemma 5.4.6. *If $W_1,\ W_3$ are transversal, τ_{123} is equal to the index of the quadratic form*

$$W_2 \ni x \to B\left(p_{1,3}(x), p_{3,1}(x)\right)$$

on W_2.

Proof. Since we can deform

$$\begin{aligned} Q(x_1, x_2, x_3) &= B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1) \\ &= B\left(x_1, p_{3,1}(x_2)\right) + B\left(p_{1,3}(x_2), x_3\right) - B(x_1, x_3) \\ &= B\left(x_1 - p_{1,3}(x_2), -x_3 + p_{3,1}(x_2)\right) + B\left(p_{1,3}(x_2), p_{3,1}(x_2)\right), \end{aligned}$$

the change of variables

$$y_1 = x_1 - p_{1,3}(x_2),\ y_3 = -x_3 + p_{3,1}(x_2)$$

gives

$$Q(x_1, x_2, x_3) = B(y_1, y_3) + B\left(p_{1,3}(x_2), p_{3,1}(x_2)\right).$$

So, indicating the index by ι,

$$\tau_{123} = \iota(Q) = \iota\left(B(y_1, y_3)\right) + \iota\left(B\left(p_{1,3}(x_2), p_{3,1}(x_2)\right)\right).$$

Here $\iota\left(B(y_1, y_3)\right) = 0$ and the claim follows. ■

Lemma 5.4.7. *If M is a fourth maximal isotropic subspace transversal to W_j $(1 \leq j \leq 3)$,*

$$\tau_{123} = \tau(W_1, W_2, M) + \tau(W_2, W_3, M) + \tau(W_3, W_1, M).$$

Proof. By the preceding lemma, the right member of the desired equality is the index of the quadratic form

$$\begin{aligned} Q'(y_1, y_2, y_3) =& B\left(p_{1,M}(y_2), p_{M,1}(y_2)\right) + B\left(p_{2,M}(y_3), p_{M,2}(y_3)\right) \\ &+ B\left(p_{3,M}(y_1), p_{M,3}(y_1)\right) \\ =& B\left(p_{1,M}(y_2), y_2\right) + B\left(p_{2,M}(y_3), y_3\right) + B\left(p_{3,M}(y_1), y_1\right) \end{aligned}$$

on $W_1 \oplus W_2 \oplus W_3$. Now, let us prove that the changes of variables, which are reciprocal to each other,

$$x_1 = y_1 + p_{1,M}(y_2),\ x_2 = y_2 + p_{2,M}(y_3),\ x_3 = y_3 + p_{3,M}(y_1)$$

and

$$\begin{aligned} y_1 &= \frac{1}{2}\left(x_1 - p_{1,M}(x_2) + p_{1,M}(x_3)\right), \\ y_2 &= \frac{1}{2}\left(x_2 - p_{2,M}(x_3) + p_{2,M}(x_1)\right), \\ y_3 &= \frac{1}{2}\left(x_3 - p_{3,M}(x_1) + p_{3,M}(x_2)\right) \end{aligned}$$

make $Q(x_1, x_2, x_3)$ and $Q'(y_1, y_2, y_3)$ equivalent. In fact,

$$\begin{aligned} B(x_1, x_2) =& B(p_{1,M}(y_2), y_2) + B(y_1, y_2) \\ & + B(y_1, p_{2,M}(y_3)) + B(p_{1,M}(y_2), p_{2,M}(y_3)). \end{aligned}$$

Interchanging the variables by cyclic permutations, it is enough to show

$$B(y_1, y_2) + B(y_2, p_{3,M}(y_1)) + B(p_{3,M}(y_1), p_{1,M}(y_2)) = 0.$$

Now, writing $y_1 = p_{M,3}(y_1) + p_{3,M}(y_1)$,

$$\begin{aligned} & B(y_1, y_2) + B(y_2, p_{3,M}(y_1)) + B(p_{3,M}(y_1), p_{1,M}(y_2)) \\ &= B(p_{M,3}(y_1), y_2) + B(p_{3,M}(y_1), p_{1,M}(y_2)). \end{aligned}$$

However, as $B(y_1, p_{1,M}(y_2)) = 0$, the above value is equal to

$$B(p_{M,3}(y_1), y_2) - B(p_{M,3}(y_1), y_2 - p_{M,1}(y_2)) = 0.$$

In this way $Q(x_1, x_2, x_3) = Q'(y_1, y_2, y_3)$. ∎

In order to deduce Lemma 5.4.5(b), take a maximal isotropic subspace M transversal to W_j $(1 \le j \le 4)$, describe each term $\tau(W_i, W_j, W_k)$ by $\tau(W_p, W_q, M)$ and use the property (a), then the desired property is immediately found.

Let us finally show the property (c).

Lemma 5.4.8. *Let L, M be two maximal isotropic subspaces of V, $\mathfrak{p}$ a isotropic subspace of V and $\mathfrak{p}^\perp$ be the orthogonal subspace of $\mathfrak{p}$ with respect to B. Then, if*

$$\mathfrak{p} \subset \left(L \cap \mathfrak{p}^\perp\right) + \left(M \cap \mathfrak{p}^\perp\right),$$

$\tau(L, L^{\mathfrak{p}}, M) = 0$.

Proof. Since $L^{\mathfrak{p}} = \left(L \cap \mathfrak{p}^\perp\right) + \mathfrak{p}$, the assumption implies $L^{\mathfrak{p}} \subset \left(L \cap \mathfrak{p}^\perp\right) + \left(M \cap \mathfrak{p}^\perp\right)$. If $\mathfrak{p} \subset L$, the assertion is clear. Now assume $\mathfrak{p} \not\subset L$ and take a linear complement $\mathfrak{m}$ of $L \cap \mathfrak{p}^\perp$ in $L^{\mathfrak{p}}$ contained in $M \cap \mathfrak{p}^\perp$. For an element $y = (y_1, u+v, y_2)$ of the direct sum space $L \oplus \left((L \cap \mathfrak{p}^\perp) \oplus \mathfrak{m}\right) \oplus M$, the quadratic form Q whose index must be computed becomes

$$Q(y) = B(y_1, v) + B(u, y_2) + B(y_2, y_1) = B(y_2 - v, y_1 - u)$$

because $B(u, v) = 0$. Hence the index of Q is equal to the index of the quadratic form $(w_1, w_2) \mapsto B(w_1, w_2)$ on $M \oplus L$, and evidently to 0. ∎

Let us consider the situation of Lemma 5.4.5(c). Since $\mathfrak{p}$ is contained in the isotropic subspace $(W_1 \cap W_2) + (W_2 \cap W_3) + (W_3 \cap W_1)$, $\mathfrak{p}^\perp$ contains $W_1 \cap W_2$, $W_2 \cap W_3$ and $W_3 \cap W_1$. Since $(W_1 \cap W_2) + (W_3 \cap W_1) \subset W_1 \cap \mathfrak{p}^\perp$ and $W_2 \cap W_3 \subset W_2 \cap \mathfrak{p}^\perp$, we get in particular that $\mathfrak{p} \subset (W_1 \cap \mathfrak{p}^\perp) + (W_2 \cap \mathfrak{p}^\perp)$. Applying the last lemma to the pairs (W_1, W_2) and $(W_1, W_2^{\mathfrak{p}})$,

$$\tau(W_1, W_1^{\mathfrak{p}}, W_2) = \tau(W_1, W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}) = 0.$$

If we establish similar equalities for all indices (i, j), $1 \leq i, j \leq 3$ and apply repeatedly the property (b), we obtain

$$\tau(W_1, W_2, W_3) = \tau(W_2, W_3, W_1^{\mathfrak{p}}) = \tau(W_3, W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}) = \tau(W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}, W_3^{\mathfrak{p}}).$$

∎

Through a Vergne polarization $\mathfrak{h}_0$ at $f \in \mathfrak{g}^*$, we set

$$T'_{\mathfrak{h}_2\mathfrak{h}_1} = e^{\frac{i\pi}{4}\tau(\mathfrak{h}_1,\mathfrak{h}_0,\mathfrak{h}_2)} T_{\mathfrak{h}_2\mathfrak{h}_0} \circ T_{\mathfrak{h}_0\mathfrak{h}_1}.$$

We mention the following result without proof.

Theorem 5.4.9 ([1], Theorem 5.1). *The intertwining operator $T'_{\mathfrak{h}_2\mathfrak{h}_1}$ does not depend on the choice of $\mathfrak{h}_0$ and satisfies the **composition formula***

$$T'_{\mathfrak{h}_1\mathfrak{h}_3} \circ T'_{\mathfrak{h}_3\mathfrak{h}_2} \circ T'_{\mathfrak{h}_2\mathfrak{h}_1} = e^{\frac{i\pi}{4}\tau(\mathfrak{h}_3,\mathfrak{h}_2,\mathfrak{h}_1)}.$$

Furthermore, if at least one of $\mathfrak{h}_1$, $\mathfrak{h}_2$ is of Vergne, $T'_{\mathfrak{h}_2\mathfrak{h}_1}$ coincides with $T_{\mathfrak{h}_2\mathfrak{h}_1}$.

Chapter 6
Kirillov Theory for Nilpotent Lie Groups

6.1 Jordan–Hölder and Malcev Sequences

In this chapter, we examine in detail the Kirillov theory for nilpotent Lie groups, which are always assumed to be connected and simply connected.

6.1.1 Unipotent Representations

Definition 6.1.1. Let G be a locally compact group and let (T, V) be a finite-dimensional representation on a real vector space V. We say that the representation T is **unipotent** if for every $x \in G$, we have that

$$T(x) = Id_V + n(x),$$

where $n(x)$ is a nilpotent endomorphism of V.

Proposition 6.1.2. *Let G be a nilpotent Lie group and let (T, V) be a unipotent representation of G. Then there exists for every G-invariant subspace W of V a sequence*

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = \{0\}$$

of G-invariant subspaces, such that $dim(V_i / V_{i+1}) = 1$ for every $i = 0, \dots, m-1$ and such that $W = V_k$ for some $k \in \{0, \dots, m\}$.

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Proof. Let $X \in \mathfrak{g}$. Since $T(e^X) = e^{dT(X)} = Id_V + n(e^X)$, it follows that

$$dT(X) = \log(I + n(e^X)) = \sum_{j=1}^{\dim V} \frac{1}{j} n(e^X)^j$$

is itself nilpotent. Hence we can apply the theorem of Engel to the modules V/W and W. ■

Definition 6.1.3. We say that the sequence $(V_j)_{j=0}^m$ is a Jordan–Hölder sequence which **passes through** W.

Corollary 6.1.4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. There exists for every ideal $\mathfrak{a}$ of $\mathfrak{g}$ a Jordan–Hölder sequence which passes through $\mathfrak{a}$.*

Definition 6.1.5. We say that a basis $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ of the nilpotent Lie algebra $\mathfrak{g}$ is a **Jordan–Hölder basis**, if the subspaces

$$\mathfrak{g}_j := \operatorname{span}\{Z_j, \dots, Z_n\}, \; j = 1, \dots, n,$$

form a Jordan–Hölder sequence in $\mathfrak{g}$.

We say that a basis $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ of the nilpotent Lie algebra is a **Malcev basis**, if the subspaces $\mathfrak{g}_j := \operatorname{span}\{Y_j, \dots, Y_n\}, j = 1, \dots, n$, form a Malcev sequence in $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of $\mathfrak{g}$. A family of vectors $\mathcal{Y} = \{Y_1, \dots, Y_d\}$ is called a **Malcev basis of $\mathfrak{g}$ relative to** $\mathfrak{h}$, if $\mathfrak{g} = \oplus_{j=1}^d \mathbb{R} Y_j \oplus \mathfrak{h}$ and if for every $j = 1, \dots, d$, the subspace $\mathfrak{h}_j := \sum_{i=j}^d \mathbb{R} Y_i + \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

Proposition 6.1.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. There exists a Malcev sequence*

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{n+1} = \{0\}$$

such that $\mathfrak{g}_j = \mathfrak{h}$ for some $j \in \{1, \dots, n+1\}$.

Proof. We consider the restriction of the representation $\operatorname{ad}\mathfrak{h}$ to $\mathfrak{g}/\mathfrak{h}$. There exists by Engel's theorem a vector $\xi = X_1 \bmod \mathfrak{h}$, such that $[X_1, \mathfrak{h}] \subset \mathfrak{h}$. Let $\mathfrak{h}_1' = \mathbb{R}X_1 + \mathfrak{h}$. Then $\mathfrak{h}_1'$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and $\dim(\mathfrak{h}_1'/\mathfrak{h}) = 1$. We continue in this fashion and we find a sequence of subalgebras

$$\mathfrak{g} = \mathfrak{h}_k' \supset \cdots \supset \mathfrak{h}_1' \supset \mathfrak{h}$$

such that $\dim(\mathfrak{h}_i'/\mathfrak{h}_{i-1}') = 1$ for all i. We choose also a Jordan–Hölder sequence in $\mathfrak{h}$. This gives us our Malcev sequence in $\mathfrak{g}$ which passes through $\mathfrak{h}$. ■

Remark 6.1.7. Construction of a Malcev basis relative to a subalgebra.

Let $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ be a Jordan–Hölder basis of $\mathfrak{g}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We denote by $I^{\mathfrak{h}}$ the index set

$$I^{\mathfrak{h}} := \{i; \mathfrak{h} + \mathfrak{g}_i = \mathfrak{h} + \mathfrak{g}_{i+1}\},$$

where $\mathfrak{g}_i := \operatorname{span}\{Z_i, \dots, Z_n\}$. For $i \in I^{\mathfrak{h}}$, there exists an element $H_i \in \mathfrak{h} \cap (\mathfrak{g}_i \setminus \mathfrak{g}_{i+1})$. Hence we can assume that $Z_i = H_i \in \mathfrak{h}$ for those indices i.

Let $I^{\mathfrak{g}/\mathfrak{h}}$ be the index set

$$I^{\mathfrak{g}/\mathfrak{h}} := \{i; \mathfrak{g}_i + \mathfrak{h} \supsetneq \mathfrak{g}_{i+1} + \mathfrak{h}\}.$$

Let us write $I^{\mathfrak{g}/\mathfrak{h}} = \{i_1, \dots, i_d\}$. If now $\sum_{i \in I^{\mathfrak{g}/\mathfrak{h}}} x_i Z_i + H = 0$ for some scalars x_i and some vector $H \in \mathfrak{h}$, then for the smallest index $i = i_j \in I^{\mathfrak{g}/\mathfrak{h}}$, such that $x_i \neq 0$, we have that $Z_i \in \mathfrak{g}_{i_j+1} + \mathfrak{h}$. This contradiction tells us that the sum $\sum_{i \in I^{\mathfrak{g}/\mathfrak{h}}} \mathbb{R} Z_i + \mathfrak{h}$ is direct. Furthermore

$$\mathfrak{g} = \sum_i \mathbb{R} Z_i = \sum_{i \in I^{\mathfrak{g}/\mathfrak{h}}} \mathbb{R} Z_i + \sum_{i \in I^{\mathfrak{h}}} \mathbb{R} Z_i = \sum_{i \in I^{\mathfrak{g}/\mathfrak{h}}} \mathbb{R} Z_i + \mathfrak{h}.$$

Finally, the subspace

$$\mathfrak{h}_j := \sum_{i \in I^{\mathfrak{g}/\mathfrak{h}}, i \geq j} \mathbb{R} Z_i + \mathfrak{h}$$

is a subalgebra since it is equal to $\mathfrak{h} + \mathfrak{g}_j$.

Definition 6.1.8. Let $\mathfrak{h}$ be a Lie subalgebra of a nilpotent Lie algebra $\mathfrak{g}$. Let $\mathcal{Y} = \{Y_1, \dots, Y_d\}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{h}$. We denote by $E_{\mathcal{Y}}$ the mapping

$$E_{\mathcal{Y}} : \mathbb{R}^d \to G;\ E_{\mathcal{Y}}(t) := \exp(t_1 Y_1) \cdots \exp(t_d Y_d).$$

Proposition 6.1.9. *The mapping*

$$E_{\mathcal{Y},H} : \mathbb{R}^d \times H \to G = exp\, \mathfrak{g};\ E_{\mathcal{Y},H}(t, h) := E_{\mathcal{Y}}(t) h$$

is a bi-polynomial diffeomorphism.

Proof. We proceed by induction on the dimension of G. If G is abelian, there is nothing to prove. Let $\mathfrak{g}_1 = (\sum_{j=2}^d \mathbb{R} Y_j) + \mathfrak{h}$ and $G_1 = \exp(\mathfrak{g}_1)$. Then $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$, since it is a subalgebra of codimension 1. The mapping

$$E : \mathbb{R} \times G_1 \to G;\ (t, g_1) \to \exp(t Y_1) g_1$$

is a bi-polynomial diffeomorphism by Proposition 1.1.42. Hence by the induction hypothesis applied to $E_{\mathcal{Y}_1,H}$ (where $\mathcal{Y}_1 = \{Y_2, \dots, Y_d\}$), since

$$E_{\mathcal{Y},H}(t_1, t_2, \dots, t_d, h) = E(t_1, E_{\mathcal{Y}_1,H}(t_2, \dots, t_d, h)),$$

we have that $E_{\mathcal{Y},H}$ is a bi-polynomial diffeomorphism. ■

Remark 6.1.10. We can use the mapping

$$E_{G/H}: \mathbb{R}^d \to G/H; E_{G/H}(t) := E_{\mathcal{Y}}(t)H,\ t \in \mathbb{R}^d,$$

to describe the invariant measure on G/H. Indeed, by Proposition 5.2.9 the measure

$$C_c(G/H) \ni \varphi \mapsto \int_{\mathbb{R}^d} \varphi(E_{G/H}(t))dt$$

is G invariant and therefore it is our G-invariant measure on G/H.

6.1.2 Polynomial Vector Groups

Definition 6.1.11. A group $(V, \cdot)$ is called a polynomial vector group, if V is a real finite-dimensional vector space, if the multiplication $\cdot$ is polynomial, which means that for every basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ of V, there exists polynomial functions $p_j : \mathbb{R}^n \to \mathbb{R}$, such that

$$X \cdot Y = \sum_{j=1}^{n} p_j((x_1, \ldots, x_n), (y_1, \ldots, y_n))X_j,$$

$$X = \sum_{j=1}^{n} x_j X_j, Y = \sum_{k=1}^{n} y_k X_k \in V,$$

and if for every $X \in V, r, s \in \mathbb{R}$, we have that

$$(sX) \cdot (tX) = (s+t)X. \tag{6.1.1}$$

Remark 6.1.12. It follows from the definition that 0 is the identity in V and that for any $X \in V$, its inverse X^{-1} is the vector $-X$.

Furthermore it is clear that a polynomial vector group is a Lie group and we can identify the Lie algebra $\mathfrak{v}$ of V with the vector space V. Since for every $X \in V$, the mapping $\mathbb{R} \mapsto V, t \to tX$ is a group homomorphism, it follows that the exponential mapping $\exp : V \to V$ is the identity.

Let us give now a direct proof of the existence of the Campbell–Baker–Hausdorf multiplication on a nilpotent Lie algebra.

Theorem 6.1.13. *(a) Every polynomial vector group is a (simply connected) nilpotent Lie group.*

(b) Let $\mathfrak{g}$ be a real nilpotent Lie algebra. Then there exists on $\mathfrak{g}$ a unique group multiplication $\cdot_{\mathfrak{g}}$ such that the group $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ is a polynomial vector group, which admits the Lie algebra $\mathfrak{g}$ as its Lie algebra.

Furthermore, for every Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ of $\mathfrak{g}$, there exist polynomial functions q_j, $j = 2, \dots, n$, defined on $\mathbb{R}^{j-1} \times \mathbb{R}^{j-1}$, such that for any $t, t' \in \mathbb{R}^n$, for $g = \sum_{i=1}^n t_i Z_i$, $g' = \sum_{i=1}^n t_i Z_i$, we have that

$$g \cdot_{\mathfrak{g}} g' = \sum_{j=1}^n (t_j + t'_j + q_j(t_1, \dots, t_{j-1}, t'_1, \dots, t'_{j-1})) Z_j \qquad (6.1.2)$$

with $q_1 \equiv 0$.

Proof. (a) Since the exponential mapping is the identity, we have for every $X, Y \in V$, $s \in \mathbb{R}$, that

$$(sX) \cdot Y \cdot (-sX) = Ad(sX)(Y).$$

Hence the mapping $s \to Ad(sX)(Y) = \sum_{k=0} \frac{s^k}{k!} ad(X)^k(Y)$ is polynomial in s and therefore $ad(X)^k = 0$ for k large enough. This shows that the Lie algebra $(V, [,])$ is nilpotent.

(b) We proceed by induction on $\dim \mathfrak{g}$. The result is trivial for an abelian algebra.

Choose any Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ of $\mathfrak{g}$. Let $\mathfrak{g}_1 := \sum_{j=2}^n \mathbb{R} Z_j$. We apply the induction hypothesis to $\mathfrak{g}_1$.

Let us define a group multiplication $\cdot_s$ on $\mathfrak{g}$ by

$$(t_1 Z_1 + U) \cdot_s (t'_1 Z_1 + U') := (t_1 + t'_1) Z_1 + (e^{-t'_1 \mathrm{ad}(Z_1)})(U) \cdot_{\mathfrak{g}_1} U',$$

for $U, U' \in \mathfrak{g}_1, t_1, t'_1 \in \mathbb{R}$. Then $(\mathfrak{g}, \cdot_s)$ is a Lie group, whose Lie algebra is equal to $(\mathfrak{g}, [,])$, since by induction the Lie algebra of the normal subgroup $\mathfrak{g}_1$ of $(\mathfrak{g}, \cdot_s)$ is isomorphic to $(\mathfrak{g}_1, [,])$ and furthermore for any $U \in \mathfrak{g}_1$ and $t \in \mathbb{R}$, we have that

$$(tZ_1) \cdot_s U \cdot_s (-tZ_1) = e^{t \mathrm{ad} Z_1} U$$

and so $\mathrm{ad}_s(Z_1) = \mathrm{ad}(Z_1)$. Here $\mathrm{ad}_s(Z_1)$ denotes the adjoint action coming from the new multiplication $\cdot_s$. It follows from the induction hypothesis that the multiplication $\cdot_s$ has property (6.1.2) of the theorem. Let us show that the exponential mapping $\exp_s : \mathfrak{g} \to (\mathfrak{g}, \cdot_s)$ is a bipolynomial diffeomorphism of the form

$$\exp_s\left(\sum_{j=1}^n t_j Z_j\right) = \sum_{j=1}^n (t_j + r_j(t_1, \dots, t_{j-1})) Z_j,$$

where the functions r_j are polynomial. Indeed, it suffices to remark that for $X = \sum_{j=1}^n x_j Z_j \in \mathfrak{g}$, the left-invariant vector field D_X determined by X is given at the point $Y = \sum_{j=1}^n y_j Y_j$ by

$$D_X(Y) = \frac{d}{dt} Y \cdot_s (tX)|_{t=0} = \sum_{j=2}^{n} (x_j + \frac{\partial}{\partial t} p_j(y_1, \ldots, y_{j-1}, tx_1, \ldots, tx_{j-1})|_{t=0}) Z_j,$$

for some polynomial functions p_j, $j = 2, \ldots, n$. Hence the integral curve

$$A(t) = \sum_{j=1}^{n} a_j(t) Z_j$$

determined by X satisfies the equations

$$\begin{aligned} \dot{a}_1(t) &= x_1 \\ \dot{a}_2(t) &= x_2 + p_2(a_1(t), x_1) \\ \vdots &= \vdots \\ \dot{a}_n(t) &= x_n + p_n(a_1(t), \ldots, a_{n-1}(t), x_1, \ldots, x_{n-1}). \end{aligned}$$

Hence

$$a_1(t) = tx_1, \ldots, a_n(t) = tx_n + r_n'(t, x_1, \ldots, x_{n-1}),$$

where the r_j's are polynomial functions. Therefore the exponential mapping $\exp_s : \mathfrak{g} \to (\mathfrak{g}, \cdot_s)$, given by $\exp_s X = (x_1, \ldots, x_n + r_n'(1, x_1, \ldots, x_{n-1}))$ is a bi-polynomial diffeomorphism. We define now a new product $\cdot_\mathfrak{g}$ on $\mathfrak{g}$ in the following way:

$$X \cdot_\mathfrak{g} Y := \log_s(\exp X \cdot_s \exp Y),\ X, Y \in \mathfrak{g}.$$

Here $\log_s$ denotes the inverse of the mapping $\exp_s$. Then the new Lie group $(\mathfrak{g}, \cdot_\mathfrak{g})$ is isomorphic to $(\mathfrak{g}, \cdot_s)$ and it satisfies the condition (6.1.2), since $\log_s$ has the same property. Furthermore, for $s, t \in \mathbb{R}$, $X \in \mathfrak{g}$, we have that

$$(sX) \cdot_\mathfrak{g} (tX) = \log_s(\exp(sX) \cdot_s \exp(tX)) = \log_s(\exp_s(s+t)X) = (s+t)X.$$

Hence $\exp_\mathfrak{g} : \mathfrak{g} \to (\mathfrak{g}, \cdot_\mathfrak{g})$ is the identity of $\mathfrak{g}$. The uniqueness follows from the following theorem. ∎

Theorem 6.1.14. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $K = exp_\mathfrak{k} \mathfrak{k}$ be a Lie group. Let φ be a homomorphism of $\mathfrak{g}$ into a Lie algebra $\mathfrak{k}$. Then there exists a unique homomorphism Φ of $(\mathfrak{g}, \cdot_\mathfrak{g})$ into the group K associated to $\mathfrak{k}$ such that $\Phi(X) = exp_\mathfrak{k}(\varphi(X))$, $X \in \mathfrak{g}$.*

Proof. We proceed by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ is abelian, then it suffices to write $\Phi(X) := \exp(\varphi(X))$, $X \in \mathfrak{g}$. Then Φ is a homomorphism. We use now the notations and procedure of the proof of Theorem 6.1.13. We define Φ in the following way. By the induction hypothesis $\Phi_1(U) = \exp(\varphi(U)),\ U \in \mathfrak{g}_1$, defines a homomorphism of $(\mathfrak{g}_1, \cdot_{\mathfrak{g}_1})$. We remark that $(tZ_1)\cdot_s U = tZ_1 + U, U \in \mathfrak{g}_1, t \in \mathbb{R}$, and use the fact that the mapping $h_s := \exp_s : \mathfrak{g} \mapsto \mathfrak{g}$ is an isomorphism of the group $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ onto the group $(\mathfrak{g}, \cdot_s)$. We define:

$$\Phi_s(tZ_1 + U_1) := \exp_{\mathfrak{k}}(t\varphi(Z_1)) \cdot \Phi_1(U),\ t \in \mathbb{R}, U_1 \in \mathfrak{g}_1.$$

Then Φ_s is a group homomorphism: for $t, t' \in \mathbb{R}, U, U' \in \mathfrak{g}_1$ we have that

$$\begin{aligned}
&\Phi(tZ + U)\Phi(t'Z + U') \\
&= \exp_{\mathfrak{k}}(t\varphi(Z_1)) \cdot \Phi_1(U)\exp_{\mathfrak{k}}(t'\varphi(Z_1)) \cdot \Phi_1(U') \\
&= \exp_{\mathfrak{k}}((t + t')\varphi(Z_1))\exp_{\mathfrak{k}}(-t'\varphi(Z_1))\exp_{\mathfrak{k}}(\varphi(U))\exp_{\mathfrak{k}}(t'\varphi(Z_1))\exp_{\mathfrak{k}}(\varphi(U')) \\
&= \exp_{\mathfrak{k}}((t + t')\varphi(Z_1))\exp_{\mathfrak{k}}(e^{-t'\mathrm{ad}(\varphi(Z_1))}\varphi(U))\exp_{\mathfrak{k}}(\varphi(U')) \\
&= \exp_{\mathfrak{k}}((t + t')\varphi(Z_1))\exp_{\mathfrak{k}}(\varphi(e^{-t'\mathrm{ad}(Z_1)}(U)))\exp_{\mathfrak{k}}(\varphi(U')) \\
&= \exp_{\mathfrak{k}}((t + t')\varphi(Z_1))\Phi_1(e^{-t'\mathrm{ad}(Z_1)}(U)\cdot_{\mathfrak{g}_1} U') \\
&= \Phi_s((tZ_1 + U)\cdot_{\mathfrak{g}}(t'Z_1 + U')).
\end{aligned}$$

Hence the mapping $\Phi : (\mathfrak{g}, \cdot_{\mathfrak{g}}) \mapsto K, \Phi(X) := \Phi_s(h_s(A)), X \in \mathfrak{g}$, is also a homomorphism. Now by Theorem 1.1.15:

$$\Phi(X) = \Phi(\exp_{\mathfrak{g}} X) = \Phi_s(\exp_s X) = \exp_{\mathfrak{k}}(\varphi(X)), X \in \mathfrak{g}. \quad \blacksquare$$

6.1.3 Unipotent Orbits

Following Pukanszky [64] we describe now an orbit of a unipotent representation. Let for a while $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let ρ be a unipotent representation of G on a finite-dimensional real vector space V. Applying as above Engel's theorem to the nilpotent Lie algebra $\{d\rho(X) : X \in \mathfrak{g}\}$, there exists in V a Jordan–Hölder basis $\mathcal{E} = \{e_j : 1 \leq j \leq n\}$, $n = \dim V$ with respect to ρ. Namely,

$$\rho(g)(e_n) = e_n,\ \rho(g)(e_j) \equiv e_j \bmod \sum_{k=j+1}^{n} \mathbb{R}e_k\ (1 \leq k \leq n - 1)$$

for any $g \in G$. Now we fix an G-orbit Ω in V. Then Ω is parametrized as follows. In particular, Ω is an algebraic subset of V.

Theorem 6.1.15. *Let (ρ, V) be a unipotent representation of the nilpotent Lie group $G = exp\ \mathfrak{g}$. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a Jordan–Hölder basis of V. There exist*

n polynomials $\{P_j : 1 \le j \le n\}$ of $d = \dim \Omega$, real variables $\{x_k : 1 \le k \le d\}$ and d indices $1 \le j_1 < \cdots < j_d < n$ such that:

1. *$\Omega = \{\sum_{j=1}^n P_j(x)e_j, (x_1, \ldots, x_d) \in \mathbb{R}^n\}$;*
2. *$P_{j_k}(x) = x_k$ $(1 \le k \le d)$;*
3. *P_j $(1 \le j \le n)$ depends only on $\{x_1, \ldots, x_k\}$, where k is the maximal natural number such that $j_k \le j$.*

Proof. To avoid the complexity of notations, we simply write $g{\cdot}v$, $X{\cdot}v$ instead of $\rho(g)(v)$, $d\rho(X)(v)$ $(g \in G,\ X \in \mathfrak{g},\ v \in V)$. Setting $V_{n+1} = \{0\}$, $V_j = \sum_{k=j}^n \mathbb{R}e_k$ $(1 \le j \le n)$, we obtain a composition series

$$\{0\} = V_{n+1} \subset V_n \subset \cdots \subset V_1 = V$$

of the G-module V. For $j = n, \ldots, 1, v \in V$ let

$$\mathfrak{g}_j(v) := \{X \in \mathfrak{g}, X \cdot v \in V_j\}.$$

Then $\mathfrak{g}_j(v) \subset \mathfrak{g}_{j-1}(v), j = 1, \ldots, n$ and the stabilizer $\mathfrak{g}(v)$ of v in $\mathfrak{g}$ is the subalgebra $\mathfrak{g}_{n+1}(v)$.

Let $I^v = \{1 < j_1 < j_2 < \cdots < j_d \le n\}$ be the index set of v defined to be the collection of all $j \in \{1, \ldots, n\}$ such that $\mathfrak{g}_j(v) \ne \mathfrak{g}_{j+1}(v)$. If $j \in I^v$, then there exists $X_j \in \mathfrak{g}_j(v)$ such that $X_j \cdot v \in V_j \setminus V_{j+1}$. We can then replace X_j by a scalar multiple and assume that $X_j \cdot v = e_j \bmod V_{j+1}$. We see that $\dim(\mathfrak{g}_j(v)/\mathfrak{g}_{j+1}(v)) = 1$ and so the family of vectors $\mathcal{B}_j(v) = \{X_j, j \in I^v\}$ is a Malcev basis relative to $\mathfrak{g}(v)$. Now, we put $g_k(t) = \exp(tX_{j_k})(t \in \mathbb{R}, j_k \in I^v)$ and

$$g(T) = g_1(t_1)g_2(t_2)\cdots g_d(t_d)$$

for $T = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d$.

Then by Proposition 6.1.9, we have that

$$\begin{aligned}\Omega = \rho(G)(v) &= \{(\rho(\Pi_{k=1}^d \exp(\mathbb{R}X_j))v\} \\ &= \{\rho(g(T))v, T \in \mathbb{R}^d\}.\end{aligned}$$

We prove the other statements by induction on $\dim V$. Let us first show that

$$g(T) \cdot v = \sum_{j=1}^n Q_j(t_1, \ldots, t_d)e_j, T \in \mathbb{R}^d,$$

where the Q_j's are polynomial functions such that

2. $Q_{j_k}((t_1, \ldots, t_d)) = x_k + Q'_j(t_1, \ldots, t_{j-1})$ $(1 \le k \le d)$;

3. Q_j $(1 \le j \le n)$ depends only on $\{t_1, \dots, t_k\}$,

   where k is the maximal natural number such that $j_k \le j$.

If dim $V = 1$, then $\rho(g) = \mathbb{I}_V$ and the conclusion is obvious. Suppose that everything is clear for dimension $n-1$ and take V of dimension 1 with its Jordan–Hölder basis $\mathcal{B} = \{e_1, \dots, e_n\}$. Then $V/\mathbb{R}e_n$ defines a unipotent G module of dimension $n-1$ and according to the induction hypothesis we have the polynomial functions $Q_1, \dots, Q_{n-1}$ with the properties 2 and 3 above.

If now n is not an index, then $\mathfrak{g}_n(v) = \mathfrak{g}_{n+1}(v) = \mathfrak{g}(v)$ and writing $g(T)v = \sum_{j=1}^{n} Q_j\ (t_1, \dots, t_d)e_j, T \in \mathbb{R}^d$, we obtain the polynomial function Q_n in the variables $T = (t_1, \dots, t_d)$.

Suppose now that $n = j_d$ is an index for v. Then there exists $X_d \in \mathfrak{g}$ such that $X_d \cdot v = e_n$.

Therefore

$$\begin{aligned} g(t_1, \dots, t_d)v &= \exp(t_1X_1) \cdots \exp(t_{d-1}X_{d-1})(v + t_d e_n) \\ &= \exp(t_1X_1) \cdots \exp(t_{d-1}X_{d-1})(v) + t_d e_n \\ &= \sum_{j=1}^{n-1} Q_j(t_1, \dots, t_{d-1})e_j + (t_d + Q'_n(t_1, \dots, t_{d-1}))e_n, \end{aligned}$$

for some polynomial function Q'_n in $d-1$ variables. Hence putting

$$Q_n(t_1, \dots, t_d) := Q'_n(t_1, \dots, t_{d-1}) + t_d$$

and using the induction hypothesis for the polynomials $Q_j, j = 1, \dots, n-1$ we can conclude.

It suffices now to let $x_i = t_i + Q'_i(t_1, \dots, t_{i-1}), i = 1, \dots, d$ to finish the proof of the theorem. ■

Corollary 6.1.16. *Every orbit of a unipotent representation of a connected and simply connected nilpotent Lie group is a Zariski closed subset of V.*

Corollary 6.1.17. *If $G = \exp \mathfrak{g}$ is nilpotent, then $I(f, \mathfrak{g}) = M(f, \mathfrak{g})$ at any $f \in \mathfrak{g}^*$.*

Proof. Let $\mathfrak{h} \in M(f, \mathfrak{g})$ and $H = \exp \mathfrak{h}$. As we have already seen, $H \cdot f$ is an open set of $f + \mathfrak{h}^{\perp}$ and at the same time a closed set by Corollary 6.1.16. Thus $H \cdot f = f + \mathfrak{h}^{\perp}$, since it is connected. ■

Remark 6.1.18. Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group. We fix a Jordan–Hölder basis $\mathcal{Z} = \{Z_n, \dots, Z_1\}$ of $\mathfrak{g}$. i.e. $\mathcal{Z}$ is a basis of $\mathfrak{g}$ and the sequence $\mathfrak{g}_j := \text{span}\{Z_1, \dots, Z_j\}$ is an ideal of $\mathfrak{g}$ for every $j = 1, \dots, n$. Its dual basis $\mathcal{Z}^* = \{Z_1^*, \dots, Z_n^*\}$ is then a Jordan–Hölder basis of the G-module $\mathfrak{g}^*$ and the subspaces $\mathfrak{g}_j^* := \text{span}\{Z_j^*, \dots, Z_n^*\}$ are then G-invariant for every $j = 1, \dots, n$. Let $\ell \in \mathfrak{g}^*$ and $O = O(\ell)$ its coadjoint orbit. The subalgebra

$$\begin{aligned}\mathfrak{g}_j(\ell) &:= \{U \in \mathfrak{g}; \mathrm{ad}^*(U)\ell \in \mathfrak{g}_j^*\}\\ &= \{U \in \mathfrak{g}; \mathrm{ad}^*(U)\ell \in \mathrm{span}\{Z_j^*, \ldots, Z_n^*\}\}\\ &= \{U \in \mathfrak{g}; \langle \ell, [U, \mathfrak{g}_{j-1}]\rangle = \{0\}\}.\end{aligned}$$

is then just the stabilizer in $\mathfrak{g}$ of the restriction of ℓ to the ideal $\mathfrak{g}_{j-1}$. Hence

$$\mathfrak{g}_j(\ell) = \mathfrak{g}_{j-1}^{B_\ell}.$$

Here B_ℓ denotes the skew symmetric bilinear form

$$B_\ell(U, V) := \langle \ell, [U, V]\rangle,\ U, V \in \mathfrak{g},$$

of $\mathfrak{g}$ and for a subspace $\mathfrak{v}$ of $\mathfrak{g}$, $\mathfrak{v}^{B_\ell} := \{U \in \mathfrak{g}; B_\ell(U, \mathfrak{v}) = \{0\}\}$ denotes as before the orthogonal of $\mathfrak{v}$ with respect to B_ℓ. It is easy to see that

$$(\mathfrak{v}^{B_\ell})^{B_\ell} = \mathfrak{v} + \mathfrak{g}(\ell).$$

Therefore, since for any j in $\{1, \ldots, n\}$, j is in the index set

$$I^\ell = I^\ell_{\mathcal{Z}^*} := \{1 \le j \le n; \mathfrak{g}_j(\ell) \neq \mathfrak{g}_{j-1}(\ell)\}$$

of ℓ if and only if

$$\mathfrak{g}_{j-1}^{B_\ell} \neq \mathfrak{g}_j^{B_\ell}$$

or equivalently

$$\mathfrak{g}_{j-1} + \mathfrak{g}(\ell) \neq \mathfrak{g}_j + \mathfrak{g}(\ell).$$

Hence

$$I^\ell_{\mathcal{Z}^*} = I^\ell = \{j \in \{1, \ldots, n\}; \mathfrak{g}_j + \mathfrak{g}(\ell) \neq \mathfrak{g}_{j-1} + \mathfrak{g}(\ell)\}.$$

In particular, we see that an index j is not contained in I^ℓ, if and only if there exists an element of $\mathfrak{g}$ of the form

$$T'_j(\ell) = T'_j = Z_j + \sum_{1 \le i < j} a_i(\ell) Z_i \in \mathfrak{g}(\ell).$$

If we replace the vectors $Z_i,\ i \notin I^\ell, 1 \le i \le j-1$, by T'_i, then we get a vector

$$T_j(\ell) = T_j = Z_j + \sum_{1 \le i < j, i \in I^\ell} a_i(\ell) Z_i \in \mathfrak{g}(\ell). \tag{6.1.3}$$

Proposition 6.1.19. *Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group, let $\ell \in \mathfrak{g}^*$ and let as in Theorem 6.1.15:*

$$O = O(\ell) = \left\{ \sum_{j=1}^{n} p_j^{\mathcal{Z}}(z) Z_j^*; z \in \mathbb{R}^d \right\}$$

be its coadjoint orbit. There exists, up to a scalar multiplication, a unique G-invariant Borel measure $d\mu_O$ on O. This measure can be described in the following way. Let $\mathcal{Z}^*$ be a Jordan–Hölder basis of the G-module $\mathfrak{g}^*$. Then we have that

$$\int_O \varphi(q) d\mu_O(q) = \int_{\mathbb{R}^d} \varphi\left(\sum_{i=1}^{n} p_i^{\mathcal{Z}}(z) Z_i^*\right) dz, \ \varphi \in C_c(\mathfrak{g}^*). \tag{6.1.4}$$

Proof. Let $d := \dim O = \dim(\mathfrak{g}/\mathfrak{g}(\ell))$. As in the proof of the Theorem 6.1.15, we choose for every $j \in I^\ell = \{j_1, \dots, j_d\}$ an element $X_j \in \mathfrak{g}$, such that $\mathrm{ad}^*(X_j)\ell = Z_j^*$ modulo $\mathfrak{g}_{j-1}^*$. This gives us a Malcev basis $\mathcal{X} := \{X_{j_1}, \dots, X_{j_d}\}$ of $\mathfrak{g}$ relative to $\mathfrak{g}(\ell)$. Since O is closed in $\mathfrak{g}^*$, we see that the mapping

$$\tilde{E}_{\mathcal{X}} : \mathbb{R}^d \to O; \tilde{E}_{\mathcal{X}}(t_1, \dots, t_d) = Ad^*\left(\prod_{i=1}^{d} \exp(t_i X_{j_i})\right)\ell$$

is a homeomorphism and so we can define a measure $d\mu_O$ by

$$\int_O \varphi(q) d\mu_O(q) := \int_{\mathbb{R}^d} \varphi(\tilde{E}_{\mathcal{X}}(t)) dt, \ \varphi \in C_c(\mathfrak{g}^*).$$

To see the invariance of this measure, let $g \in G$ and $\varphi \in C_c(\mathfrak{g}^*)$. Then

$$\begin{aligned}
\int_O \varphi(\mathrm{Ad}^*(g)q) d\mu_O(q) &= \int_{\mathbb{R}^d} \varphi(\mathrm{Ad}^*(g)(\tilde{E}_{\mathcal{X}}(t))) dt \\
= \int_{\mathbb{R}^d} \varphi\left(\mathrm{Ad}^*\big(g \prod_{i=1}^{d} (\exp(t_i X_{j_i}))\big)\ell\right) dt &= \int_{G/G(\ell)} \varphi(\mathrm{Ad}^*(gh)\ell) d\dot{h} \\
= \int_{G/G(\ell)} \varphi(\mathrm{Ad}^*(h)\ell) d\dot{h} = \int_{\mathbb{R}^d} \varphi(\tilde{E}_{\mathcal{X}}(t)) dt &= \int_O \varphi(q) d\mu_O(q),
\end{aligned}$$

where $h = \prod_{i=1}^{d} \exp(t_i X_{j_i})$. In order to prove Eq. (6.1.2), it suffices to observe that by Theorem 6.1.15 and the notations of its proof:

$$\tilde{E}_{\mathcal{X}}(t) = \sum_{j=1}^{n} (q_j(t_1, \dots, t_d)) Z_j^* \tag{6.1.5}$$

and so, writing $q_j(t_1, \ldots, t_d) = p_j^{\mathcal{Z}}(z_1, \ldots, z_d)$,

$$\int_O \varphi(q) d\mu_O(q) = \int_{\mathbb{R}^d} \varphi\Big(\sum_{j=1}^n q_j(t) Z_j^*\Big) dt$$
$$= \int_{\mathbb{R}^d} \varphi\Big(\sum_{j=1}^n p_j^{\mathcal{Z}}(z) Z_j^*\Big) dz$$

(by the change of variables $z_i := t_i + q'_{j_i}(t_1, \ldots, t_{i-1})$). ■

Proposition 6.1.20. *Let $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ be a Jordan–Hölder basis of the nilpotent Lie algebra $\mathfrak{g}$. Then there exist polynomial functions $p_j : \mathbb{R}^{j-1} \times \mathfrak{g} \to \mathbb{R}$, $j = 2, \ldots, n$, which are linear in $X \in \mathfrak{g}$, such that the vector field X at the point $Y = \sum_{i=1}^n y_i Z_i \in \mathfrak{g}$ is given by*

$$X(Y) = \sum_{j=1}^n (x_j + p_j(y_1, \ldots, y_{j-1}, X)) Z_j.$$

Proof. Indeed, we know from (6.1.4), that for $X, Y \in \mathfrak{g}$,

$$Y \cdot_{\mathfrak{g}} X = \sum_{j=1}^n (y_j + x_j + q_j(y_1, \ldots, y_{j-1}, x_1, \ldots, x_{j-1})) Z_j.$$

Hence

$$X(Y) = \frac{\partial}{\partial t} Y \cdot_{\mathfrak{g}} (tX)|_{t=0}$$
$$= \sum_{j=1}^n \Big(x_j + \frac{\partial}{\partial t} q_j(y_1, \ldots, y_{j-1}, tx_1, \ldots, tx_{j-1})|_{t=0}\Big) Z_j$$
$$= \sum_{j=1}^n (x_j + p_j(y_1, \ldots, y_{j-1}, X)) Z_j,$$

where

$$p_j(y_1, \ldots, y_{j-1}, X) = \frac{\partial}{\partial t} q_j(y_1, \ldots, y_{j-1}, tx_1, \ldots, tx_{j-1})|_{t=0}.$$ ■

6.2 Schwartz Spaces

Definition 6.2.1. We denote for a finite-dimensional real vector space V the algebra of all partial differential operators with complex polynomial coefficients on V by $\mathcal{PD}(V)$.

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a basis of V. We define for $\alpha \in \mathbb{N}^n$ the monomial function

$$X = \sum_{j=1}^{n} x_j B_j \mapsto X^\alpha := x_1^{\alpha_1} \cdots x_n{}^{\alpha_n}$$

and the partial differential operator

$$D^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}.$$

The **Schwartz space** $\mathcal{S}(V)$ is by definition the space of all smooth functions $f : V \to \mathbb{C}$, such that for every $\Lambda \in \mathcal{PD}(V)$ the function Λf is in $L^2(V)$ with respect to Lebesgue measure on V. We put on $\mathcal{S}(V)$ the family of norms

$$\|f\|_k^2 := \sum_{|\alpha| \le k} \int_V (1 + \|X\|_2)^k \|D^\alpha f(t)\|^2 dt, \; f \in \mathcal{S}(V).$$

(Here $\|X\|_2$ denotes a Euclidean norm on V.) The space $\mathcal{S}(V)$ becomes in this way a Fréchet space.

Definition 6.2.2. Let $G = (\mathfrak{g}, \cdot_{\mathfrak{g}})$ a nilpotent Lie group. We choose again a Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ of $\mathfrak{g}$ and we define for $\alpha \in \mathbb{N}^n$, the element $Z^\alpha := Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. These elements $Z^\alpha, \alpha \in \mathbb{N}^n$, form a basis of $\mathcal{U}(\mathfrak{g})$, by the Birkhoff–Poincare–Witt theorem. Let $\mathcal{PU}(\mathfrak{g})$ be the algebra of partial differential operators generated by the elements of $\mathcal{U}(\mathfrak{g})$ and the algebra $\mathcal{P}(\mathfrak{g})$ of complex-valued polynomial functions on $\mathfrak{g}$. It follows from Proposition 6.1.20 that

$$\mathcal{PD}(\mathfrak{g}) = \mathcal{PU}(\mathfrak{g}).$$

An element X of $\mathfrak{g}$ acts on the left on a smooth function $\varphi : G \to \mathbb{C}$ by

$$X * \varphi(g) := \frac{d}{dt} \varphi(\exp(-tX)g)|_{t=0}, \; g \in G.$$

We extend this action to an action of $\mathcal{U}(\mathfrak{g})$ on the space $C^\infty(G)$.

Definition 6.2.3. Let G be a connected locally compact group and let $U = U^{-1}$ be a compact symmetric neighbourhood of the identity. Then we know that

$G = \bigcup_{n\in\mathbb{N}} U^n$, where U^n denotes the compact subset $U^n := \{u_1\cdots u_n; u_i \in U, i = 1,\dots,n\}$ and $U^0 = \emptyset$. We define a function $w_U = w : G \to \mathbb{N}$ by

$$w(x) = n, \text{ if } x \in U^n \backslash U^{n-1}.$$

In particular $w(e) = 1$. Let $\omega(x) = 1 + w(x),\ x \in G$.

Proposition 6.2.4. *For $x, y \in G$, we have that $w(xy) \leq w(x)+w(y)$. Furthermore $\omega(xy) \leq \omega(x)\omega(y)$.*

Proof. Let $n = w(x)$, $m = w(y)$. Then $xy \in U^{n+m}$ and so $w(xy) \leq n + m = w(x) + w(y)$. Therefore

$$\omega(xy) = 1 + w(xy) \leq 1 + w(x) + w(y) \leq (1 + w(x))(1 + w(y)) = \omega(x)\omega(y).$$

■

Proposition 6.2.5. *Let $\|\cdot\|$ be a Euclidean norm on the nilpotent Lie algebra $\mathfrak{g}$ and let $G = (\mathfrak{g}, \cdot_{\mathfrak{g}})$. Let U be the unit ball of centre 0 in $\mathfrak{g}$. There exists a constant $C > 0$ and an integer d such that*

$$w_U(X) \leq 1 + \|X\| \leq C(1 + w_U(X))^d,\ X \in \mathfrak{g}.$$

Proof. Let $X \in \mathfrak{g}$ and choose $n \in \mathbb{N}$, such that $n \leq \|X\| < n + 1$. Let $Y := \frac{1}{n+1}X$. Then $Y \in U$ and $X = (n + 1)Y = Y^{n+1} \in U^{n+1}$. This shows that $w(X) \leq n + 1 \leq \|X\| + 1$.

In order to find the constant C and the integer d, we show by induction on the dimension of $\mathfrak{g}$ that there exists a polynomial function with non-negative coefficients p, such that for any $m \in \mathbb{N}$, $X \in U^m$, we have that

$$\|X\| \leq p(m).$$

If $\mathfrak{g}$ is abelian, then $U^m = mU$ and we can take $p(s) = s,\ s \in \mathbb{R}$. Let $\mathfrak{g}_1$ be an ideal of $\mathfrak{g}$ of codimension 1. We write $\mathfrak{g} = \mathbb{R}Y \oplus \mathfrak{g}_1$ for some $Y \in \mathfrak{g} \setminus \mathfrak{g}_1$. We take a neighbourhood V of 0 of the form $V = \exp(]-1, 1[Y)W$, where W is the exponential of the unit ball of centre 0 in $\mathfrak{g}_1$. Any element X of V^m can be written as

$$\begin{aligned} X &= \prod_{j=1}^{m} \exp(s_j Y)\exp(U_j) \\ &= \exp((s_1 + \cdots + s_m)Y) \prod_{j=1}^{m} \exp(e^{-(s_j+1+\cdots+s_m)\mathrm{ad}Y} U_j)\exp(U_m), \end{aligned}$$

where $s_1, \dots, s_m \in]-1, 1[$ and $U_j \in W$ for $j = 1, \dots, m$.

Since $\mathrm{ad}(Y)$ is a nilpotent endomorphism of $\mathfrak{g}$, it follows that

$$\|e^{-s\mathrm{ad}Y}(U)\| \leq q(|s|),\ s \in \mathbb{R}, U \in W,$$

for some polynomial q with integer coefficients and therefore $\exp(e^{-s\mathrm{ad}Y})U \subset W^{q([s])}$, taking now $q(s+1)$ as the new $q(s)$. Hence

$$X \in \exp(]-m, m[Y) W^{q(m-1)+\cdots+q(1)+1} \subset \exp(]-m, m[Y) W^{mq(m-1)}$$

and so $\|X\| \leq p'(m)$ for some polynomial p' with non-negative coefficients, since there exists by the induction hypothesis a polynomial p_1 with non-negative coefficients such that for any $U \in W^k$, $\|U\| \leq p_1(k)$, $k \in \mathbb{N}$. Since the unit ball U in $\mathfrak{g}$ is compact, there exists an integer r, such that $U \subset V^r$, since $G = \bigcup_{r=1}^{\infty} V^r$. Hence, $U^m \subset V^{rm}$ and we see that we can take as p the polynomial $p(s) := p'(rs)$, $s \in \mathbb{R}$. It follows that for $X \in \mathfrak{g}$ with $w_U(X) = m$, X is contained in U^m, therefore $\|X\| \leq p(m) = p(w_U(X)) \leq C(1 + \|X\|)^d$ for some constant $C > 0$ and $d \in \mathbb{N}$ being the degree of the polynomial p. ■

Definition 6.2.6. Let $\mathfrak{g}$ be a nilpotent Lie algebra and let G be its connected and simply connected Lie group. We denote by $\mathcal{S}(G)$ the Fréchet space of all smooth functions $\varphi : G \to \mathbb{C}$, such that

$$\|\varphi\|_N := \sum_{|\alpha| \leq N} \int_G \omega(x)^N |Z^\alpha \varphi(x)| dx < \infty,\ \forall N \in \mathbb{N}.$$

It follows from Proposition 6.2.5 and from Proposition 6.1.20 that the Fréchet space $\mathcal{S}(G)$ is equal to the Fréchet space $\mathcal{S}(\mathfrak{g})$. It is easy to see that $\mathcal{S}(G)$ is also a Fréchet algebra. Indeed, for $f, g \in \mathcal{S}(G)$ we have that

$$\begin{aligned}
\|f * g\|_N &= \sum_{|\alpha| \leq N} \int_G \omega(x)^N |Z^\alpha f * g(x)| dx \\
&= \sum_{|\alpha| \leq N} \int_G \omega(x)^N |(Z^\alpha f) * g(x)| dx \\
&= \sum_{|\alpha| \leq N} \int_G \omega(x)^N \left| \int_G (Z^\alpha f)(y) g(y^{-1}x) dy \right| dx \\
&\leq \sum_{|\alpha| \leq N} \int_{G \times G} \omega(yx)^N |(Z^\alpha f)(y)||g(x)| dy dx \\
&\leq \sum_{|\alpha| \leq N} \int_{G \times G} \omega(y)^N \omega(x)^N |(Z^\alpha f)(y)||g(x)| dy dx
\end{aligned}$$

$$\leq \left(\sum_{|\alpha| \leq N} \int_G \omega(x)^N |Z^\alpha f| dx \right) \int_G \omega(y)^N |g(y)| dy$$

$$\leq \|f\|_N \|g\|_N.$$

Here we have used the fact that ω^N is submultiplicative.

Definition 6.2.7. Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of the connected and simply connected nilpotent Lie group $G = \exp \mathfrak{g}$. Let $\chi : H \to \mathbb{T}$ be a unitary character of H. Let $\mathcal{Y} = \{Y_1, \ldots, Y_d\}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{h}$. We define the Schwartz space $\mathcal{S}(G/H, \chi)$ of the representation $\tau := \mathrm{ind}_H^G \chi$ as the subspace of all smooth functions $\varphi : G \to \mathbb{C}$ contained in the Hilbert space $\mathcal{H}_\tau = L^2(G/H, \chi)$ of τ for which the function $\varphi \circ E_{\mathcal{Y}} : \mathbb{R}^d \to \mathbb{C}$, is in $\mathcal{S}(\mathbb{R}^d)$. It follows from Proposition 6.1.9, that the definition of the space $\mathcal{S}(G/H, \chi)$ does not depend on the choice of the Malcev basis $\mathcal{Y}$.

Definition 6.2.8. Let $(\pi, \mathcal{H})$ be a unitary representation of the Lie group G. We denote by $\mathcal{H}^\infty$ the space of the C^∞-vectors of $\mathcal{H}$. This means that a vector ξ is in $\mathcal{H}^\infty$, if and only if the mapping $G \to \mathcal{H};\ g \mapsto \pi(g)\xi$ is C^∞. We can give the vector space $\mathcal{H}^\infty$ the structure of a Fréchet space. It suffices to take on $\mathcal{H}^\infty$ the family of norms

$$\|\xi\|_N := \sup_{|\alpha| \leq N} \|d\pi(Z^\alpha)\xi\|_{\mathcal{H}},\ \xi \in \mathcal{H}^\infty (N \in \mathbb{N}). \qquad \blacksquare$$

We denote by $\mathcal{K}(\mathcal{H})$ (resp. by $\mathcal{B}(\mathcal{H})$), the space of compact linear operators (resp. of bounded linear mappings), on $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^\infty$ be the space of all smooth linear operators on $\mathcal{H}$. Here a linear operator u on $\mathcal{H}$ is called smooth or C^∞, if the mapping $G \times G \to \mathcal{K}(\mathcal{H});\ (g, g') \mapsto \pi(g) \circ u \circ \pi(g')^{-1}$ is smooth in the strong operator topology. This means that we use the representation R_π of $G \times G$ on Banach space $\mathcal{K}(\mathcal{H})$ defined by

$$R_\pi(g, g')u := \pi(g) \circ u \circ \pi(g')^{-1},\ g, g' \in G, u \in \mathcal{B}(\mathcal{H}).$$

If we take the subspace $L^2(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ consisting of all Hilbert–Schmidt operators on $\mathcal{H}$, the representation R_π of G restricts to a unitary representation, since $L^2(\mathcal{H})$ is a Hilbert space and the mapping $(g, g') \to R_\pi(g, g')u, u \in L^2(\mathcal{H})$, is continuous for the Hilbert–Schmidt norm.

Definition 6.2.9. Let G be a Lie group. We denote by $L^1(G)^\infty$ the smooth vectors in $L^1(G)$ for the regular representation κ of $G \times G$ on $L^1(G)$:

$$\kappa(g, g')f(x) = f(g^{-1} x g'),\ g, g', x \in G,\ f \in L^1(G).$$

If $(\pi, \mathcal{V})$ is a bounded representation of G on a Banach space $\mathcal{V}$, then $\pi(f)\mathcal{V} \subset \mathcal{V}^\infty$ for any $f \in L^1(G)^\infty$. If G is a connected and simply connected nilpotent Lie group, then the Schwartz algebra $\mathcal{S}(G)$ is contained in $L^1(G)^\infty$.

We want now to study the image of the enveloping algebra under an irreducible representation of a nilpotent Lie group. Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We study the differential representation $d\pi$ of $\pi \in \hat{G}$. We know that

$$\pi(\exp(tX)) = \exp(t(d\pi)(X)) \quad (t \in \mathbb{R})$$

for any $X \in \mathfrak{g}$. In this situation, we can do the following.

Remark 6.2.10. Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group. Let $\ell \in \mathfrak{g}^*$. Choose a polarization $\mathfrak{p}$ at ℓ let $\pi_{\ell,\mathfrak{p}} = \mathrm{ind}_P^G \chi_\ell$ be the corresponding irreducible representation. We choose a Malcev basis $\mathcal{X} = \{X_1, \ldots, X_d\}$ relative to $\mathfrak{p}$, for instance as in Remark 6.1.7. We can then identify as in Definition 6.2.7 the Hilbert space $L^2(G/P, \chi_\ell)$ with the Hilbert space $L^2(\mathbb{R}^k)$, using the unitary operator $U_{\mathcal{X}}$:

$$U_{\mathcal{X}}(\xi)(x) := \xi(E_{\mathcal{X}}(x)), \ x \in \mathbb{R}^k.$$

We denote by $(\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}, L^2(\mathbb{R}^k))$ the representation $\pi^{\mathcal{X}}_{\ell,\mathfrak{p}} := U_{\mathcal{X}} \circ \pi_{\ell,\mathfrak{p}} \circ U^*_{\mathcal{X}}$.

Definition 6.2.11. We denote as before by $\mathcal{PD}(\mathbb{R}^n)$ the algebra of differential operators with polynomial coefficients on $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$.

Lemma 6.2.12. *Let $\mathfrak{g}$ be a real nilpotent Lie algebra and let $0 \neq \ell \in \mathfrak{g}^*$. Let (as in Lemma 5.3.7) $\mathfrak{a}_\ell$ be the largest ideal of $\mathfrak{g}$ contained in the subalgebra $\mathfrak{g}(\ell)$. Then $\ell_{|\mathfrak{a}_\ell} \neq 0$. Let $Y \in \mathfrak{g} \setminus \mathfrak{a}_\ell$, such $[\mathfrak{g}, Y] \subset \mathfrak{a}_\ell$. Then the subspace*

$$\mathfrak{g}_1 := \{U \in \mathfrak{g}; \langle \ell, [U, Y]\rangle = 0\} = ker(ad^*(Y)\ell)$$

of $\mathfrak{g}$ is an ideal of codimension 1. For any $g \in G$ and $a \in A_\ell := exp(\mathfrak{a}_\ell)$ we have that $\langle \ell, \log(gag^{-1})\rangle = \langle \ell, \log a\rangle$.

Proof. Suppose that ℓ is zero on $\mathfrak{a}_\ell$. Then for our Y we have that $\langle \ell, [\mathfrak{g}, Y]\rangle = \{0\}$ and so $Y \in \mathfrak{g}(\ell)$ and the subspace $\mathbb{R}Y + \mathfrak{a}_\ell$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{g}(\ell)$, a contradiction. The same argument tells us that there must exist $X \in \mathfrak{g}$, such that $\langle \ell, [X, Y]\rangle = 1$. Since for any $U, V \in \mathfrak{g}$ we have that

$$\begin{aligned}\langle \ell, [[U, V], Y]\rangle &= \langle \ell, [[U, [V, Y]]\rangle - \langle \ell, [[V, [U, Y]]\rangle \\ &= 0 + 0 = 0,\end{aligned}$$

(as $[\mathfrak{g}, Y] \subset \mathfrak{g}(\ell)$) it follows that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_1$. So we have that $\mathfrak{g}_1 = \ker(ad^*(Y)\ell)$ is an ideal of $\mathfrak{g}$ of co-dimension 1. Furthermore, by the definition of $\mathfrak{a}_\ell$, we know that for any $S \in \mathfrak{a}_\ell$

$$ad^k(\mathfrak{g})(Y) \subset \mathfrak{g}(\ell)$$

for any $k \in \mathbb{N}^*$. ∎

Theorem 6.2.13 (Kirillov). *Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group. Let $\pi = \pi_{\ell,\mathfrak{p}}$ be an irreducible unitary representation of G. Let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{p}$. Identify the Hilbert space $L^2(G/P, \chi_\ell)$ with $L^2(\mathbb{R}^d)$ via the unitary operator*

$$U_{\mathcal{X}} : L^2(G/P, \chi_\ell) \to L^2(\mathbb{R}^k) : \xi \mapsto \xi \circ E_{\mathcal{X}}.$$

Then the image under $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}$ of $\mathcal{U}(\mathfrak{g})$ is equal to the algebra $\mathcal{PD}(\mathbb{R}^n)$.

Proof. It is easy to see that if we have proved the theorem for one Malcev basis $\mathcal{X}$ of $\mathfrak{g}$ relative to $\mathfrak{p}$, the theorem is also proved for every other Malcev basis $\mathcal{X}'$ relative to $\mathfrak{p}$, since the passage from $\mathcal{X}$ to $\mathcal{X}'$ is given by bi-polynomial bijections, which preserve the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{PD}(\mathbb{R}^d)$.

Let us first show that $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(\mathcal{U}(\mathfrak{g})) \subset \mathcal{PD}(\mathbb{R}^d)$.

Choose a Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ of $\mathfrak{g}$ to introduce coordinates of the second kind on G. For $g \in G$ and $t \in \mathbb{R}^d$, we write the product $g^{-1}E_{\mathcal{X}}(t)$ in the coordinates of $\mathbb{R}^d$ and $\mathfrak{p}$. We have a polynomial mapping $Q : G \times \mathbb{R}^d \to \mathbb{R}^d$ and a polynomial mapping $R : G \times \mathbb{R}^d \to \mathfrak{p}$, such that

$$g^{-1}E_{\mathcal{X}}(t) = E_{\mathcal{X}}(Q(g,t))\exp(R(g,t)),\ t \in \mathbb{R}^d, g \in G.$$

Hence

$$\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(g)\xi(E_{\mathcal{X}}(t)) = e^{i\langle \ell, R(g,t)\rangle}\xi(E_{\mathcal{X}}(Q(g,t))), t \in \mathbb{R}^d, g \in G, \xi \in L^2(G/P, \chi_\ell).$$

This implies that

$$\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(g)\eta(t) = e^{i\langle \ell, R(g,t)\rangle}\eta(Q(g,t)), t \in \mathbb{R}^d, g \in G, \eta \in L^2(\mathbb{R}^d).$$

Hence by differentiation, we see that for every $X \in \mathfrak{g}$, $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(X) \in \mathcal{PD}(\mathbb{R}^d)$ and therefore $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(\mathcal{U}(\mathfrak{g})) \subset \mathcal{PD}(\mathbb{R}^d)$.

In order to show that $\mathcal{PD}(\mathbb{R}^d) \subset d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(\mathcal{U}(\mathfrak{g}))$, we proceed by induction on the dimension of $\mathfrak{g}$.

If G is abelian, then $d = 0$ and there is nothing to prove.

We can assume now that ℓ does not vanish on $\mathfrak{a}_\ell$ and so we have $X \in \mathfrak{g}, Y \in \mathfrak{g} \setminus \mathfrak{a}_\ell$, such that $[\mathfrak{g}, Y] \subset \mathfrak{g}_\ell$, such that $\langle \ell, [X, Y]\rangle = 1$ and such that $\ker(\mathrm{a}d^*(Y)\ell) = \mathfrak{g}_1$ is a co-one-dimensional ideal in $\mathfrak{g}$. Let $G_1 = \exp(\mathfrak{g}_1)$. Then G_1 is a closed normal subgroup of G.

Suppose first that $\mathfrak{p} \subset \mathfrak{g}_1$. Choose our Malcev basis $\mathcal{X}$ relative to $\mathfrak{p}$, such that $\mathcal{X} = \{X\} \cup \mathcal{X}_1$, where $\mathcal{X}_1$ a Malvev basis of $\mathfrak{g}_1$ relative to $\mathfrak{p}$. Then $\mathfrak{p}$ is a polarization at $\ell_1 = \ell_{|\mathfrak{g}_1}$ and we have that $\pi_{\ell,\mathfrak{p}} \simeq \mathrm{ind}_{G_1}^G \pi^{\mathcal{X}_1}_{\ell_1,\mathfrak{p}}$. To simplify notations, let $\pi_1 := \pi^{\mathcal{X}_1}_{\ell_1,\mathfrak{p}}$.

We can identify the Hilbert space $L^2(\mathbb{R}^d)$ of the representation $\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}$ with

$$L^2(\mathbb{R}, L^2(\mathbb{R}^{d-1}))$$

through the mapping

$$U(\xi)(t)(T_1) := \xi(t, T_1), t \in \mathbb{R}, T_1 \in \mathbb{R}^{d-1}.$$

We define a new representation π of G on the space $\mathcal{H}_\pi = L^2(\mathbb{R}, \mathcal{H}_{\pi_1})$ by

$$\pi(g) := U \circ \pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(g) \circ U^*, \ g \in G. \tag{6.2.1}$$

Then of course π is equivalent $\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}$.

Let us compute the operators $\pi(g)$ for $g \in G$.

Let $\eta \in L^2(\mathbb{R}, \mathcal{H}_{\pi_1}), \ t \in \mathbb{R}$. Then we have

$$\begin{aligned} \pi(\exp(xX))\eta(t) &= \eta(t-x) \qquad (6.2.2)\\ \pi(\exp(yY))\eta(t) &= e^{iyt}\eta(t) \\ \pi(g_1)\eta(t) &= \pi_1(g_1^t)(\eta(t)), \end{aligned}$$

where $g_1 \in G_1, t, x, y \in \mathbb{R}$ and where $g_1^t := \exp(-tX)g_1\exp(tX)$. This gives us for the elements X, Y of $\mathfrak{g}$ and $d_1 \in \mathcal{U}(\mathfrak{g}_1)$, for a smooth vector η that

$$\begin{aligned} d\pi(X)\eta(t) &= -\frac{d}{dt}\eta(t) \\ d\pi(Y)\eta(t) &= it\eta(t) \\ d\pi(d_1)\eta(t) &= d\pi_1(d_1^t)(\eta(t)), \qquad (6.2.3) \end{aligned}$$

where $d_1^t = \sum_{j=0}^{\infty} \frac{1}{j!}(-t)^j \operatorname{ad}^j(X)d_1$. The mapping $d_1 \mapsto d_1^t$ is an automorphism of $\mathcal{U}(\mathfrak{g})$ for every $t \in \mathbb{R}$ and we have that $(\operatorname{ad}^k(A)(d_1))^t = \operatorname{ad}^k(A^t)d_1^t$ for any $A \in \mathfrak{g}$.

Since g is nilpotent, we see that the sum above is finite and so d^t is contained in $\mathcal{U}(\mathfrak{g}_1)$.

Let for $d \in \mathcal{U}(\mathfrak{g}_1)$

$$\tilde{d} = \sum_{j=0}^{\infty} \frac{1}{j!(-i)^j} Y^j \operatorname{ad}^j(X)d.$$

Again the sum above is finite and so $\tilde{d}$ is contained in $\mathcal{U}(\mathfrak{g})$.

Let us compute $d\pi(\tilde{d}_1)$ where d_1 is contained in $\mathcal{U}(\mathfrak{g}_1)$. We have for a smooth vector $\eta \in L^2(\mathbb{R}, \mathcal{H}_{\pi_1})$ and $t \in \mathbb{R}$, that

$$\begin{aligned} d\pi(\tilde{d}_1)\eta(t) &= d\pi_1((\tilde{d}_1)^t)(\eta(t)) \\ &= d\pi_1\Bigg(\sum_{j=0}^{\infty} \frac{1}{j!(-i)^j}(Y^j \operatorname{ad}^j(X)d_1)^t\Bigg)(\eta(t)) \end{aligned}$$

$$= d\pi_1\bigg(\sum_{j=0}^{\infty}\frac{1}{j!(-i)^j}(Y^j)^t(\mathrm{ad}^j(X)d_1)^t\bigg)(\eta(t))$$

$$= d\pi_1\bigg(\sum_{j=0}^{\infty}\frac{1}{j!(-i)^j}(Y^j)^t(\mathrm{ad}^j(X)(d_1^t))\bigg)(\eta(t)).$$

Since $d\pi_1(Y^t) = d\pi_1(Y - tZ) = -it\mathbb{I}_{\mathcal{H}_{\pi_1}}$, we get therefore

$$d\pi(\tilde{d}_1)(\eta(t)) = d\pi_1\bigg(\sum_{j=0}^{\infty}\frac{t^j}{j!}(\mathrm{ad}^j(X)(d_1^t)))(\eta(t))$$

$$= d\pi_1((d_1^t)^{-t}\bigg)(\eta(t)) = d\pi_1(d_1)(\eta(t)).$$

Using the induction hypothesis, this computation shows that the algebra of $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(\mathcal{U}(\mathfrak{g}))$ contains the operators $\frac{\partial}{\partial t_1}$, M_{t_1} (i.e. the multiplication operator with the variable t_1) and every partial differential operator with polynomial coefficients in the variables $t_2, \ldots, t_d$. Hence $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(\mathcal{U}(\mathfrak{g})) = \mathcal{PD}(\mathbb{R}^d)$.

If $\mathfrak{p} \not\subset \mathfrak{g}_1$, then we can take our X in $\mathfrak{p}$, and even assume that $\langle \ell, X\rangle = 0$, and we consider the polarization $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}_1 + \mathbb{R}Y$. The intertwining operator U for $\pi = \pi_{\ell,\mathfrak{p}}$ and $\pi' = \pi_{\ell,\mathfrak{p}'}$ from $L^2(G/P, \chi_\ell)$ to $L^2(G/P', \chi_\ell)$ is given by

$$U(\xi)(g_1\exp(xX)) = \int_{\mathbb{R}}\xi(g_1\exp(xX)\exp(yY))dy$$

$$= \int_{\mathbb{R}}\xi(g_1\exp(yY)\exp(yxZ)\exp(xX))dy$$

$$= \int_{\mathbb{R}}e^{-iyx}\xi(g_1\exp(yY))dy,$$

for $x \in \mathbb{R}$ and $g_1 \in G_1$. This shows that if we take a Malcev basis $\mathcal{X} = \{X_1, \ldots, X_{d-1}, Y\}$ of $\mathfrak{g}$ relative to $\mathfrak{p}$ with $\{X_1, \ldots, X_{d-1}\} \subset \mathfrak{g}_1$, which gives us a Malcev basis $\mathcal{X}' = \{X_1, \ldots, X_{d-1}, X\}$ of $\mathfrak{g}$ relative to $\mathfrak{p}'$, then the intertwining operator for $\pi^{\mathcal{X}'}_{\ell,\mathfrak{p}'}$ and $\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}$ is just a partial Fourier transform in the variable y_1. This intertwining operator then maps $\mathcal{PD}(\mathbb{R}^d)$ into itself. ■

Corollary 6.2.14. *Let $\pi_{\ell,\mathfrak{p}}$ be an irreducible representation of the simply connected nilpotent Lie group G. Then*

$$\mathcal{H}^{\infty}_{\pi^{\mathcal{X}}_{\ell,\pi}} = \mathcal{S}(\mathbb{R}^{dim(G/P)}).$$

Proof. It is well known that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is also the space of all L^2 functions ξ on $\mathbb{R}^n$, such that for every $D \in \mathcal{PD}(\mathbb{R}^n)$, the distribution $D\xi$ is also in $L^2(\mathbb{R}^n)$. The corollary then follows from Theorem 6.2.13. ■

Proposition 6.2.15. *Let $\pi_{\ell,\mathfrak{p}}$ be an irreducible unitary representation of the nilpotent Lie group $G = \exp \mathfrak{g}$. Let $u \in B(L^2(\mathbb{R}^d))$ ($d = dim(G/P)$) be a smooth operator for the representation $\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}$. Then u is trace class and its kernel function $k_u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is contained in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$.*

Proof. For any $D \in \mathcal{PD}(\mathbb{R}^n)$ the operator $u \circ D$, which is a priory defined on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, extends to a bounded operator on $L^2(\mathbb{R}^d)$. There exist elements $D \in \mathcal{PD}(\mathbb{R}^d)$, which can be inverted on $\mathcal{S}(\mathbb{R}^d)$, and whose inverse D^{-1} extends to a trace class operator T on $L^2(\mathbb{R}^d)$. Hence $u = (u \circ D^{-1}) \circ T$ is a trace class operator. Let us construct such an operator D, using Hermite functions. Let $\xi_m(s) := e^{s^2/2}(\frac{d}{ds})^m(e^{-s^2}), s \in \mathbb{R}$. Then $\xi_m(s) = e^{-s^2/2}Q_m(s)$, where Q_m is a polynomial of degree $m,\ m \in \mathbb{N}$). Let

$$h_m = \frac{1}{\|\xi_m\|_2}\xi_m,\ m \in \mathbb{N}.$$

Then the functions $(h_m)_{m\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R})$. Let $D_1 := -\frac{d^2}{ds^2} + s^2$. It follows inductively on m, that

$$D_1 h_m = (2m+1)h_m,\ m \in \mathbb{N}. \tag{6.2.4}$$

We define now the element $D \in \mathcal{PD}(\mathbb{R}^d)$ by

$$D := \underbrace{D_1^2 \otimes \cdots \otimes D_1^2}_{d-\text{times}}$$

and for $\mu = (m_1, \ldots, m_d) \in \mathbb{N}^d$, let

$$h_\mu := h_{m_1} \otimes \cdots \otimes h_{m_d}.$$

Then the family $(h_\mu)_{\mu\in\mathbb{N}^d}$ constitutes an orthonormal basis of $L^2(\mathbb{R}^d)$. Furthermore, by (6.2.1), we have that

$$Dh_\mu = (2m_1+1)^2 \cdots (2m_d+1)^2 h_\mu,\ \mu \in \mathbb{N}^d.$$

Hence D is invertible on $\mathcal{S}(\mathbb{R}^d)$ and its inverse T acts on the vectors h_μ by

$$Th_\mu = \frac{1}{(2m_1+1)^2 \cdots (2m_d+1)^2} h_\mu,\ \mu \in \mathbb{N}^d.$$

This shows that T is a positive trace class operator, since

$$\mathrm{Tr}(T) = \sum_{\mu\in\mathbb{N}^d} \frac{1}{(2m_1+1)^2 \cdots (2m_d+1)^2} = \left(\sum_{m\in\mathbb{N}} (\frac{1}{2m+1})^2\right)^d < \infty.$$

Since now u is a trace-class and hence a Hilbert–Schmidt operator, its kernel function is in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. But u is a smooth operator, hence for every $D = d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(A), D' = d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}((\check{A}'))\in \mathcal{PD}(\mathbb{R}^n)$, the derivative in the distribution sense of the kernel function k_u by D in the first variable and D' in the second variable gives us the kernel function of the operator $d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(A)\circ u\circ d\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}((\check{A}'))$, which is again an L^2-function. This shows that k_u is a smooth function and also an element of $S(\mathbb{R}^n \times \mathbb{R}^n)$. ■

Corollary 6.2.16. *Let G be a connected and simply connected nilpotent Lie group and let $\pi_{\ell,\mathfrak{p}}$ be an irreducible unitary representation of G. Then for every element $f \in L^1(G)$, the operator $\pi_{\ell,\mathfrak{p}}(f)$ is compact.*

Proof. The Schwartz algebra $\mathcal{S}(G)$ is dense in $L^1(G)$ and for every $f \in \mathcal{S}(G)$ the operator $\pi_{\ell,\mathfrak{p}}(f)$ is trace class, hence compact. ■

Remark 6.2.17. Let us compute for $f \in \mathcal{S}(G)$ the kernel function of the operator $\pi_{\ell,\mathfrak{p}}$ for an element $\ell \in \mathfrak{g}^*$ and a polarization $\mathfrak{p}$ at ℓ. Let $g \in G$ and $\xi \in L^2(G/P,\chi_\ell)$. Then

$$\begin{aligned}
\pi_{\ell,\mathfrak{p}}(f)\xi(g) &= \int_G f(t)\pi_{\ell,\mathfrak{p}}(t)\xi(g)dt\\
&= \int_G f(t)\xi(t^{-1}g)dt = \int_G f(gt^{-1})\xi(t)dt\\
&= \int_{G/P}\int_P f(gp^{-1}t^{-1})\chi_\ell(p^{-1})dp\xi(t)dt\\
&= \int_{G/P} f_{\ell,\mathfrak{p}}(g,t)\xi(t)dt,
\end{aligned}$$

where

$$f_{\ell,\mathfrak{p}}(g,t) := \int_P f(gpt^{-1})\chi_\ell(p)dp,\ g,t\in G.$$

This function $f_{\ell,\mathfrak{p}}$ is smooth, and it satisfies the covariance condition

$$f_{\ell,\mathfrak{p}}(gp,tp') = \chi_\ell(p^{-1})\chi_\ell(p')f_{\ell,\mathfrak{p}}(g,t),\ p,p'\in P, g,t\in G.$$

Let $\mathcal{X}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{p}$. Then we know from Proposition 6.2.15, that the function $f_{\ell,\mathfrak{p}}\circ(E_{\mathcal{Y}}\otimes E_{\mathcal{Y}})$ is contained in $\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$, where $d=\dim(G/P)$, since it is the kernel function of the operator $\rho_{\ell,\mathfrak{p}}(f)$.

This allows us to define the space of the **Schwartz kernels**.

Definition 6.2.18.

$$\begin{aligned}\mathcal{SK}(G/P,\chi_\ell) := \big\{K : G\times G \to \mathbb{C};\\ &K(gp,g'p') = \chi_\ell(p^{-1})\chi_\ell(p')K(g,g'), g,g' \in G, p,p' \in P\\ &\text{and } K\circ(E_{\mathcal{Y}}\otimes E_{\mathcal{Y}}) \in \mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)\big\}.\end{aligned}$$

These Schwartz kernels are just the kernel functions of smooth operators on $L^2(G/P,\chi_\ell)$.

We can also define the Schwartz space of functions with covariance conditions Let $H = \exp\mathfrak{h}$ be a closed connected subgroup of the nilpotent Lie group $G = \exp\mathfrak{g}$ and let χ be a unitary character of H. Let $\mathcal{Y}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then

$$\mathcal{S}(G/H,\chi) := \{\xi \in L^2(G/H,\chi),\ \xi\circ E_{\mathcal{Y}} \in \mathcal{S}(\mathbb{R}^{\dim(G/H)})\}.$$

Theorem 6.2.19. *Let $G = \exp\mathfrak{g}$ be a simply connected nilpotent Lie group. Let $\ell \in \mathfrak{g}^*$ and let $\mathfrak{p}$ be a polarization at ℓ. Let $d := \dim(\mathfrak{g}/\mathfrak{p})$. Let $\mathcal{X}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{p}$. There exists for every $N \in \mathbb{N}$ a continuous linear mapping, which we call a retract,*

$$R = R_{\ell,\mathfrak{p},\mathcal{X}} : \mathcal{S}(\mathbb{R}^N,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)) \mapsto \mathcal{S}(\mathbb{R}^N,\mathcal{S}(G))$$

such that for every $s \in \mathbb{R}^N$ and $F \in \mathcal{S}(\mathbb{R}^N,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$ the (smooth operator)

$$\pi^{\mathcal{X}}_{\ell,\mathfrak{p}}(R(F(s)))$$

admits the function $F(s)$ as kernel function.

Proof. It is easy to see that if the theorem is true for one Malcev basis relative to $\mathfrak{p}$, it is also true for any such Malcev basis. For simplicity of notation, let $\mathcal{S}(N,d) := \mathcal{S}(\mathbb{R}^N,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$.

We proceed by induction on the dimension of G. If G is abelian, then $d = 0$. We choose a Schwartz function $\varphi \in \mathcal{S}(G)$, such that $\hat{\varphi}(\ell) = 1$ and it suffices to put $R(F(s)) := F(s)\varphi,\ s \in \mathbb{R}^d$.

We can assume now that ℓ does not vanish on $\mathfrak{a}_\ell$ and that $\mathfrak{a}_\ell \neq \mathfrak{g}$. So we have $X \in \mathfrak{g}, Y \in \mathfrak{g}\setminus\mathfrak{a}_\ell$, such that $[\mathfrak{g},Y] \subset \mathfrak{g}_\ell$, such that $\langle\ell,[X,Y]\rangle = 1$, $\langle\ell,Y\rangle = \langle\ell,X\rangle = 0$ and such that $\ker(ad^*(Y)\ell) = \mathfrak{g}_1$ is a co-one-dimensional ideal in $\mathfrak{g}$. Let $G_1 = \exp(\mathfrak{g}_1)$. Then G_1 is a closed normal subgroup of G. Let $\ell_1 := \ell|_{\mathfrak{g}_1}$. We take first $\mathfrak{p}$ inside $\mathfrak{g}_1$ and $\mathfrak{p}$ is then a polarization at ℓ_1. Choose our Malcev basis $\mathcal{X}$ relative to $\mathfrak{p}$, such that $\mathcal{X} = \{X\}\cup\mathcal{X}_1$, where $\mathcal{X}_1$ a Malvev basis of $\mathfrak{g}_1$ relative to $\mathfrak{p}$. Then we have that $\pi_{\ell,\mathfrak{p}} \simeq \mathrm{ind}_{G_1}^G \pi^{\mathcal{X}_1}_{\ell_1,\mathfrak{p}}$. To simplify notation, let $\pi_1 := \pi^{\mathcal{X}_1}_{\ell_1,\mathfrak{p}}$.

We can identify the Hilbert space $L^2(\mathbb{R}^d)$ of the representation $\pi_{\ell,\mathfrak{p}}^{\mathcal{X}}$ with

$$L^2(\mathbb{R}, L^2(\mathbb{R}^{d-1}))$$

through the mapping

$$U(\xi)(t)(T_1) := \xi(t, T_1), t \in \mathbb{R}, T_1 \in \mathbb{R}^{d-1}.$$

We define a new representation π of G on the space $\mathcal{H}_\pi = L^2(\mathbb{R}, \mathcal{H}_{\pi_1})$ by

$$\pi(g) := U \circ \pi_{\ell,\mathfrak{p}}^{\mathcal{X}}(g) \circ U^*, \ g \in G. \tag{6.2.5}$$

Then of course π is equivalent $\pi_{\ell,\mathfrak{p}}^{\mathcal{X}}$.

As before, we compute $\pi(f)$ on a vector $\xi \in \mathcal{H}_\pi$. Let $t \in \mathbb{R}$. Then

$$\begin{aligned}
\pi(f)\xi(t) &= \int_{\mathbb{R}} \int_{G_1} f(\exp(xX)g_1)\pi(\exp(xX)g_1)\xi(t)dxdg_1 \\
&= \int_{\mathbb{R}} \int_{G_1} f(\exp(xX)g_1)\pi_1(g_1^{t-x})\xi(t-x)dxdg_1 \\
&= \int_{\mathbb{R}} \int_{G_1} f(\exp((t-x)X)g_1^{-x})\pi_1(g_1)\xi(x)dxdg_1 \\
&= \int_{\mathbb{R}} \pi_1(f^x(t-x))(\xi(x))dx,
\end{aligned} \tag{6.2.6}$$

where $f^x(u) \in \mathcal{S}(G_1)$ denotes the Schwartz function

$$\begin{aligned}
f^x(u)(g_1) &:= f(\exp(uX)g_1^{-x}) \\
&= f(\exp(uX)\exp(xX)g_1\exp(-xX)), x, u \in \mathbb{R}, \ g_1 \in G_1.
\end{aligned}$$

Hence the operator $\pi(f)$ is given by the operator-valued kernel function

$$k(t, x) = \pi_1(f^x(t-x)) \in B(\mathcal{H}_{\pi_1}), \ t, x \in \mathbb{R}. \tag{6.2.7}$$

Define the (normal) subgroup A of G by $A := \exp \mathfrak{a}$ where $\mathfrak{a} := \mathbb{R}Y + \mathfrak{a}_\ell$.

Now let $S_1 : \mathcal{S}(\mathbb{R}^N, \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)) \to \mathcal{S}(\mathbb{R}^{N+2}, \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})), F \to F_1$ be defined by the rule:

$$F_1((s, t, x), \overline{t}, \overline{x}) := F(s, (t + x, \overline{t}), (x, \overline{x})), \text{ where } \overline{t}, \overline{x} \in \mathbb{R}^{d-1}, s \in \mathbb{R}^N, t, x \in \mathbb{R}.$$

The function F_1 is obviously in $\mathcal{S}(\mathbb{R}^{N+2}, \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}))$ and the mapping $F \mapsto F_1, \mathcal{S}(N, d) \to \mathcal{S}(N + 2, d - 1)$, is continuous.

We apply the induction hypothesis and we get the retract $R_1 : \mathcal{S}(N+2, d-1) \to \mathcal{S}(\mathbb{R}^{N+2}, \mathcal{S}(G_1))$.

We define a new function $h_F : \mathbb{R}^{N+2} \times G_1 \to \mathbb{C}$ by

$$h_F((s,t,x), g_1) := \int_A R_1 F_1((s,t,x))(g_1 a)\chi_\ell(a)da.$$

By definition, the function h_F satisfies the following covariance relation:

$$h_F((s,t,x), g_1 a) = \chi_\ell(a)^{-1} h_F((s,t,x), g_1), s \in \mathbb{R}^N, t, x \in \mathbb{R}, g_1 \in G_1, a \in \mathfrak{a}.$$

Choose a Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ of $\mathfrak{g}$ relative to $\mathfrak{a}_\ell$, such that

$$Z_1 = X, Z_m = Y, \mathfrak{g}_1 = \mathrm{span}\{Z_2, \ldots, Z_M, \mathfrak{a}_\ell\}.$$

Let $\mathfrak{v} := \mathrm{span}\{Z_2, \ldots, Z_{m-1}\}$. Then we have $\mathfrak{g}_1 = \mathfrak{v} \oplus \mathbb{R}Y \oplus a_\ell$. Let $V := \exp \mathfrak{v}$. We obtain the decomposition

$$V \times \mathbb{R}^2 \times A_\ell \to G_1, (v, x, y, a) \mapsto v\exp(yY)\exp(zZ)\exp a$$

of G_1. It follows again from the definition of h_F that the function

$$\mathbb{R}^N \times \mathbb{R}^2 \times V \ni (s,t,x,v) \mapsto h_F((s,t,x), v)$$

is a Schwartz function and that the linear mapping $\mathcal{S}(N, d) \to \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^2 \times V)$ is continuous.

Let $\varphi \in \mathcal{S}(A_\ell)$ such that $\hat{\varphi}(0) = 1$. We obtain now our retract R in the following way: for $F \in \mathcal{S}(N, d)$ let

$$RF(s)(\exp(xX)g_1 a) := \varphi(a)\int_{\mathbb{R}} h_F((s,x,u), g_1^{-u})du,$$

for $s \in \mathbb{R}^N, x \in \mathbb{R}, g_1 \in V\exp(\mathbb{R}Y), a \in A_\ell$. Here, as before, for $g_1 \in G_1$ we use the notation: $g_1^u := \exp(-uX)g_1\exp(uX), u \in \mathbb{R}$.

We observe that

$$\begin{aligned} & RF(s)(\exp(xX)v\exp(yY)a) \\ &= \varphi(a)\int_{\mathbb{R}} h_F((s,x,u), (v\exp(yY)a)^u)du \\ &= \varphi(a)\overline{\chi_\ell(a)}\int_{\mathbb{R}} e^{iuy} h_F((s,x,u), v^u)du. \end{aligned}$$

for $s \in \mathbb{R}^N, x, y, z \in \mathbb{R}, v \in V$. From this formula, we easily deduce that the function RF is in $\mathcal{S}(\mathbb{R}^N \times G)$ and that the linear mapping $F \mapsto RF$ is continuous. We have to check that $F(s) = k_{\pi(RF(s))}$ for every $F \in \mathcal{S}(N,d)$. Let $f(s) := RF(s) \in \mathcal{S}(G), s \in \mathbb{R}^N$. We use formula (6.2.6): let $t, x \in \mathbb{R}$. Then

$$\begin{aligned}
\pi_1(f(s)^x(t-x)) &= \int_{G_1} \pi(g_1) f(s)(\exp((t-x)X)(g_1^{-x}))dg_1 \\
&= \int_V \pi_1(v) \left(\int_{\mathbb{R}} e^{-iyx} \int_A \chi_\ell(a)\overline{\chi_\ell(a)}\varphi(a)\pi_1(v) \right. \\
&\quad \times \left. \left(\int_{\mathbb{R}} e^{iyu} h_F((s,t-x,u),(v^{-x})^u)du \right) dady \right) dv \\
&= \hat{\varphi}(0) \int_V \pi_1(v) h_F((s,t-x,x),v)dv \\
&= \int_V \int_A \pi_1(v\exp(yY)a) R_1 F_1(s,t-x,x)(v\exp(yY)a)dydadv \\
&= \pi_1(R_1 F_1(s,t-x,x)),
\end{aligned}$$

by the property of the retract R_1. Hence the kernel of the operator $\pi_{\ell,\mathfrak{p}}^{\mathcal{X}}(RF(s))$ is the function

$$F(s,t-x+x,\overline{t},x,\overline{x}) = F(s,t,\overline{t},x,\overline{x}).$$

If now $\mathfrak{p} \not\subset \mathfrak{g}_1$, then we replace $\mathfrak{p}$ by the polarization $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{g}_1 + \mathbb{R}Y$. Since the spaces of the Schwartz kernels $\mathcal{SK}(G/P,\chi_\ell)$ and $\mathcal{SK}(G/P',\chi_\ell)$ are isomorphic and since this isomorphism is given by the composition with the intertwining operator U between $\pi_{\ell,\mathfrak{p}}$ and $\pi_{\ell,\mathfrak{p}'}$, i.e.

$$UF(g,g') = \int_{\mathbb{R}^2} F(g\exp(yY), g'\exp(y'Y))dydy',$$

it suffices to use the retract R' constructed for $\mathfrak{p}'$ to compose R with U. This finishes the proof of the theorem. ∎

Corollary 6.2.20. *Let G be a connected and simply connected nilpotent Lie group. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Then the ideal*

$$F_\pi = \{f \in L^1(G); rank\ (\pi(f)) < \infty\}$$

of $L^1(G)$ is different from $\{0\}$ and the sub-module $\mathcal{H}_0 = \sum_{f \in F_\pi} Im(\pi(f))$ of $\pi(L^1(G))$ is simple. The same statement is true if we replace $L^1(G)$ by the involutive algebra $\mathcal{S}(G)$.

Proof. Let $\pi = \pi_{\ell,\mathfrak{p}}^{\mathcal{X}}$ for some $\ell \in \mathfrak{g}^*$, some polarization $\mathfrak{p}$ at ℓ and some Malcev basis relative to $\mathfrak{p}$. Let $d := \dim(\mathfrak{g}/\mathfrak{p})$. We choose any Schwartz function $\xi \in \mathcal{S}(\mathbb{R}^d)$ of L^2 norm 1 and let $F := \xi \otimes \overline{\xi} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. By Theorem 6.2.19, there exists a Schwartz function $f \in \mathcal{S}(G)$, such that $\rho_{\ell,\mathfrak{p}}(f)$ admits F as kernel function. Hence $\pi(f)$ is the orthogonal projection onto $\mathbb{C}\xi$ and $f \in F_\pi$. It suffices then to use Theorem 2.3.17. ■

6.3 Intertwining Operator for Irreducible Representations

Let $G = \exp\mathfrak{g}$ be a nilpotent Lie group and let $\ell \in \mathfrak{g}^*$. Choose two polarizations $\mathfrak{p}, \mathfrak{p}'$ at ℓ. We know from Proposition 5.3.27 that there exists a unitary intertwining operator $U : L^2(G/P, \chi_\ell) \to L^2(G/P', \chi_\ell)$, which is unique up to a constant of modulus 1. We want to give now an explicit expression for this operator U. We remark first that U maps the C^∞-vectors for $\pi_{\ell,\mathfrak{p}}$ onto the C^∞-vectors for $\pi_{\ell,\mathfrak{p}'}$. This means by Corollary 6.2.14 that U maps $\mathcal{S}(G/P, \chi_\ell)$ onto $\mathcal{S}(G/P', \chi_\ell)$. We have another linear operator T, which could be an intertwining operator. Namely

$$T\xi(g) := \int_{P'/P\cap P'} \xi(gp')\chi_\ell(p')d\dot{p}', \ \xi \in \mathcal{S}(G/P, \chi_\ell), g \in G. \tag{6.3.1}$$

In order to show that the operators U and T coincide on Schwartz functions, we need some preparations.

Lemma 6.3.1. *Let $H = \exp\mathfrak{h}$ and $H' = \exp(\mathfrak{h}')$ be two closed connected subgroups of the nilpotent Lie group G. Then the product $H'H$ is closed in G.*

Proof. If G is abelian, then $H'H = \exp(\mathfrak{h}' + \mathfrak{h})$ is the exponential of the sum of the two subspaces $\mathfrak{h}'$ and $\mathfrak{h}$. We can also assume that both subgroups are proper. There then exists an ideal $\mathfrak{g}_0$ of codimension 1, which contains the subalgebra $\mathfrak{h}$. If $\mathfrak{h}'$ is also contained in $\mathfrak{g}_0$, then we can apply the induction hypothesis and $H'H$ is closed in $G_0 = \exp(\mathfrak{g}_0)$ and hence in G. If $\mathfrak{h}' \not\subset \mathfrak{g}_0$, we take a vector $X \in \mathfrak{h}' \setminus \mathfrak{g}_0$. Then $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$, $\mathfrak{h}' = \mathbb{R}X \oplus \mathfrak{h}'_0$ where $\mathfrak{h}'_0 := \mathfrak{h}' \cap \mathfrak{g}_0$ and $H' = \exp(\mathbb{R}X)H'_0$ with $H'_0 = \exp(\mathfrak{h}'_0)$. Take a sequence $g_n = \exp(t_nX)h'_nh_n \in H'H, h'_n \in H'_0, h_n \in H$, $t_n \in \mathbb{R}, n \in \mathbb{N}$, which converges to some $u = \exp(tX)b \in G = \exp(\mathbb{R}X)G_0$. Since the function $G \ni g = \exp(tX)a \mapsto t$ is continuous, it follows that $\lim_{n\to\infty} t_n = t$ and so h'_nh_n converges to b, which by the induction hypothesis is contained in $H'_0H \subset G_0$. Hence $u \in H'H$. ■

Now let $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ be a Jordan–Hölder basis of $\mathfrak{g}$ and as before, we let $\mathfrak{g}_j := \operatorname{span}\{Z_i; i \geq j\}$ for $j = 1, \ldots, n$. For a subspace $\mathfrak{q}$ of $\mathfrak{g}$ let $I^{\mathfrak{q}}$ be the index set

$$I^{\mathfrak{q}} := \{1 \leq j \leq n; \mathfrak{g}_j + \mathfrak{q} = \mathfrak{g}_{j+1} + \mathfrak{q}\} = \{1 \leq j \leq n; \mathfrak{g}_j \cap \mathfrak{q} \neq \mathfrak{g}_{j+1} \cap \mathfrak{q}\}.$$

With this definition it is obvious that $I^{\mathfrak{p}\cap\mathfrak{p}'} \subset I^{\mathfrak{p}'} \subset I^{\mathfrak{p}'+\mathfrak{p}}$.

Now let $i \in I^{\mathfrak{p}'} \cap I^{\mathfrak{p}} \setminus I^{\mathfrak{p}'\cap\mathfrak{p}}$. Then given $X_i^1 \in \mathfrak{g}_i \cap \mathfrak{p} \setminus \mathfrak{g}_{i+1}$, there exists $X_i^2 \in \mathfrak{g}_i \cap \mathfrak{p}' \setminus \mathfrak{g}_{i+1}$, such that $Z_j := X_i^1 + X_i^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$ for some $j > i$. We choose now X_i^1 and X_i^2, such that j is maximal. This index j is then unique and we put $j(i) := j$. Then it follows that $j \in I^{\mathfrak{p}'+\mathfrak{p}}$ and the maximality of j implies that $j \notin I^{\mathfrak{p}'} \cup I^{\mathfrak{p}}$.

Let us verify that the mapping

$$I^{\mathfrak{p}'} \cap I^{\mathfrak{p}} \setminus I^{\mathfrak{p}\cap\mathfrak{p}'} \to I^{\mathfrak{p}'+\mathfrak{p}} \setminus (I^{\mathfrak{p}} \cup I^{\mathfrak{p}'}); i \mapsto j = j(i),$$

which had been defined above, is a bijection.

If for $i < i'$ we have that $j(i) = j(i')$, then there exists $X_i^1 \in \mathfrak{p} \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_{i'}^1 \in \mathfrak{p} \cap \mathfrak{g}_{i'} \setminus \mathfrak{g}_{i'+1}, X_i^2 \in \mathfrak{p}' \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_{i'}^2 \in \mathfrak{p}' \cap \mathfrak{g}_{i'} \setminus \mathfrak{g}_{i'+1}$, such that $Z_j := X_i^1 + X_i^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$ and $Z'_j := X_{i'}^1 + X_{i'}^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$. For a certain scalar $\lambda \in \mathbb{R}^*$, we then have that $Z_j - \lambda Z'_j \in \mathfrak{g}_{j+1}$ and so $X_i^1 - \lambda X_{i'}^1 \in \mathfrak{g}_i \cap \mathfrak{p} \setminus \mathfrak{g}_{i+1}, X_i^2 - \lambda X_{i'}^2 \in \mathfrak{g}_{i'} \cap \mathfrak{p}' \setminus \mathfrak{g}_{i'+1}$ and $X_i^1 - \lambda X_{i'}^1 + X_i^2 - \lambda X_{i'}^2 \in \mathfrak{g}_{j+1}$, contradicting the maximality of j.

For the indices $j \in I^{\mathfrak{p}+\mathfrak{p}'} \setminus (I^{\mathfrak{p}} \cup I^{\mathfrak{p}'})$, we have an index $i \leq j$, and elements $X_i^1 \in \mathfrak{p} \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_i^2 \in \mathfrak{p}' \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$, such that $X_i^1 + X_i^2 =: Z_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$. Since $j \notin I^{\mathfrak{p}} \cap I^{\mathfrak{p}'}$, the index i must be strictly smaller than the index j. We choose i maximal with these properties. Then $i \in I^{\mathfrak{p}} \cap I^{\mathfrak{p}'}$. Furthermore $j = j(i)$, since letting $j' := j(i)$, we have that $j' \geq j$ and in the case where $j' > j$, there exists $U_i^1 \in \mathfrak{g}_i \cap \mathfrak{p} \setminus \mathfrak{g}_{i+1}, U_i^2 \in \mathfrak{g}_i \cap \mathfrak{p}' \setminus \mathfrak{g}_{i+1}$, such that $Z_{j'} := U_i^1 + U_i^2 \in \mathfrak{g}_{j'} \setminus \mathfrak{g}_{j'+1}$. But we can then take a scalar $\lambda \in \mathbb{R}^*$, such that $X_i^1 - \lambda U_i^1 \in \mathfrak{g}_{i+1}$. This implies that $X_i^2 - \lambda U_i^2 \in \mathfrak{g}_{i+1}$ too and so $(X_i^1 - \lambda U_i^1) + (X_i^2 - \lambda U_i^2) = Z_j - \lambda Z_{j'} \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$, contradicting the maximality of i. This shows that our mapping is also surjective.

We now order the indices $i(j)$ and we write $J := \{i_1 < \cdots < i_r\} := \{i(j), j \in I^{\mathfrak{p}'} \cap I^{\mathfrak{p}} \setminus I^{\mathfrak{p}'\cap\mathfrak{p}}\}$.

We can write for $j \in I^{\mathfrak{p}'} \cap I^{\mathfrak{p}} \setminus I^{\mathfrak{p}\cap\mathfrak{p}'}$

$$Z_j = B_j + c_j Z_{i(j)} + Q_{i(j)+1},$$

where $c_j \in \mathbb{R}$ is not zero, $Z_j \in \mathfrak{p}'$, $B_j \in \mathfrak{p}$ and where $Q_{i(j)+1} \in \mathfrak{g}_{i(j)+1}$. Then for $t_j \in \mathbb{R}$, we have

$$\begin{aligned} \exp(t_j Z_j) &= \exp(t_j c_j Z_{i(j)} + t_j(B_j + Q_{i(j)+1})) \\ &= \exp(t_j c_j Z_{i(j)})\exp(R_j(t_j)) \bmod P, \end{aligned} \tag{6.3.2}$$

for some element $R_j(t_j) \in \mathfrak{g}_{i(j)+1}$, which varies polynomially in t_j.

Furthermore, we always can write for two indices $j < j'$

$$\exp(sZ_{j'})\exp(tZ_j) = \exp(tZ_j)\exp(sZ_{j'}) \bmod \mathfrak{g}_{j'+1},\ s, t \in \mathbb{R}, \tag{6.3.3}$$

Write $I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'} = \{j_1 < \cdots, j_s\}$ and define for $\xi \in \mathcal{S}(G/P, \chi_\ell), g \in G$, the function $\xi(g) : \mathbb{R}^s \mapsto \mathbb{C}$ by

$$\xi(g)(t_1, \ldots, t_s) := \xi(g\exp(t_1 Z_{j_1}) \cdots \exp(t_s Z_{j_s})).$$

We observe that the function $\xi(g)$ is contained in $\mathcal{S}(\mathbb{R}^s)$ for every $g \in G$ and that its Schwartz norms are bounded uniformly in g. Indeed, let us take our Jordan–Hölder basis $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ of G, such that the vectors Z_j are in $\mathfrak{p}$ for $j \in I^{\mathfrak{p}}$ and such that $Z_j \in \mathfrak{p}'$, for $j \in I^{\mathfrak{p}'} \setminus I^{\mathfrak{p}}$. Let $\mathcal{Y}' = \{Z_j; j \in I^{\mathfrak{p}'} \setminus I^{\mathfrak{p}\cap\mathfrak{p}'}\}$. Then for $g = E_{\mathcal{Z}}(x_1, \ldots, x_n)$ and $p' = E_{\mathcal{Y}'}((t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}})$ it follows from relations (6.3.2) and (6.3.3) that

$$gp' = \prod_{k\in I^{\mathfrak{g}/\mathfrak{p}}} \exp(q_k((x_i)_{i=1}^n, (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}})Z_k),$$

where for $k \notin I^{\mathfrak{p}+\mathfrak{p}'}$

$$q_k((x_i)_{i=1}^n, (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}}) = x_k + q_k'(((x_i)_{i=1}^n, i < k), (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}, j<k}),$$

where for $k \in I^{\mathfrak{p}'} \setminus I^{\mathfrak{p}}$

$$q_k((x_i)_{i=1}^n, (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}}) = x_k + t_k + q_k'(((x_i)_{i=1}^n, i < k), (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}, j<k}),$$

and where for $k = i(j') \in J$

$$q_k((x_i)_{i=1}^n, (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}}) = x_k + t_{j'} + q_k'(((x_i)_{i=1}^n, i < k), (t_j)_{j\in I^{\mathfrak{p}'/\mathfrak{p}\cap\mathfrak{p}'}, j<j'}),$$

for some new polynomial functions q_k'.

Proposition 6.3.2. *For every $\xi \in \mathcal{S}(G/P, \chi_\ell)$, the function*

$$T\xi(g) := \int_{P'/P\cap P'} \xi(gp')\chi_\ell(p')d\dot{p}', \ \xi \in \mathcal{S}(G/P, \chi_\ell), g \in G,$$

is contained in $\mathcal{S}(G/P', \chi_\ell)$ and

$$T(\pi_{\ell,\mathfrak{p}}(f)\xi) = \pi_{\ell,\mathfrak{p}'}(f)(T\xi), \ f \in S(G).$$

Proof. We remark first that the function $P' \ni p' \mapsto \xi(gp')\chi_\ell(p')$ is contained in $\mathcal{S}(P'/P'\cap P)$, since first we observe that for $p' \in P, p_0 \in P \cap P'$,

$$\xi(gp'p_0)\chi_\ell(p'p_0) = \xi(gp')\chi_\ell(p')\chi_\ell(p_0^{-1})\chi_\ell(p_0) = \xi(gp')\chi_\ell(p').$$

Hence it is constant modulo $P \cap P'$ and so the integral $\int_{P'/P\cap P'} \xi(gp')\chi_\ell(p')d\dot{p}'$ converges for ξ and all its derivatives $D * \xi, D \in \mathcal{U}(\mathfrak{g})$. Furthermore $T(\xi)$ is a Schwartz function in the variable $p' \in P'/(P\cap P')$ and uniformly bounded in g by the observation above. It follows from the definition of the operator T that

$$T\xi(gp') = \chi_\ell(p'^{-1})T\xi(g), \ g \in G, p' \in P', \xi \in \mathcal{S}(G/P, \chi_\ell).$$

It is clear from the formula for the operator T that T commutes with left translation and also with derivations on the left and so for any element $D \in \mathcal{U}(\mathfrak{g})$ we have the relation

$$D{*}T(\xi) = T(D{*}\xi), D \in \mathcal{U}(\mathfrak{g}).$$

Since $T(\xi)$ satisfies the covariance condition with respect to P' and χ_ℓ, it follows that

$$d\pi_{\ell,\mathfrak{p}'}(D)T(\xi) = T(d\pi_{\ell,\mathfrak{p}}(D)\xi), D \in \mathcal{U}(\mathfrak{g}).$$

This shows that $d\pi_{\ell,\mathfrak{p}'}(D)(T\xi)$ is a bounded function for every $D \in \mathcal{U}(\mathfrak{g})$ and hence it must be in $\mathcal{S}(G/P', \chi_\ell)$ by Theorem 6.2.13. The last relation follows immediately from Fubini's theorem, since

$$\begin{aligned} T(\pi_{\ell,\mathfrak{p}}(f)\xi)(g) &= \int_{P'/(P'\cap P)} \int_G f(u)\xi(u^{-1}gp')\chi_\ell(p')du\,d\dot{p}' \\ &= \int_G f(u) \int_{P'/(Pv\cap P)} \xi(u^{-1}gp')\chi_\ell(p')d\dot{p}'du \\ &= \pi_{\ell,\mathfrak{p}'}(f)(T\xi)(g), g \in G, \xi \in \mathcal{S}(G/P, \chi_\ell). \end{aligned}$$ ■

Theorem 6.3.3 (cf. [52]). *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group and let $\ell \in \mathfrak{g}^*$. Choose two polarizations $\mathfrak{p}$ and $\mathfrak{p}'$ at ℓ. The operator $T : \mathcal{S}(G/P, \chi_\ell) \to \mathcal{S}(G/P', \chi_\ell)$ defined in (6.3.1) is isometric and defines a unitary intertwining operator for the representations $\pi_{\ell,\mathfrak{p}}$ and $\pi_{\ell,\mathfrak{p}'}$.*

Proof. We know that the algebra $\mathcal{S}(G)$ acts irreducibly and simply on the space of the C^∞-vectors for the representation $\pi_{\ell,\mathfrak{p}}$ (see Corollary 6.2.20). Furthermore, if we take our official unitary intertwining operator

$$U : L^2(G/P, \chi_\ell) \to L^2(G/P', \chi_\ell),$$

then this operator also maps the C^∞-vectors onto C^∞-vectors. Hence if we compose T with U^{-1}, then we get an intertwining operator $S = U^{-1} \circ T : \mathcal{S}(G/P, \chi_\ell) \to \mathcal{S}(G/P, \chi_\ell)$ for $\pi_{\ell,\mathfrak{p}}$. But such an operator must be a multiple c of the identity by Schur's lemma for simple modules. Hence $T = cU$. ■

6.4 Traces and Plancherel Theorem

Theorem 6.4.1 (Kirillov's Character Formula). *Let $(\pi_{\ell,\mathfrak{p}}, L^2(G/P, \chi_\ell))$ be an irreducible representation of a nilpotent Lie group $G = \exp \mathfrak{g}$. There exists a G-invariant measure $d\nu_O$ on the coadjoint orbit O of ℓ, such that*

$$Tr(\pi_{\ell,\mathfrak{p}}(f)) = \int_O \widehat{f \circ exp}(q) d\nu_O(q), \;\; f \in \mathcal{S}(G).$$

Proof. We know from Proposition 6.2.15 that for $f \in \mathcal{S}(G)$ the operator $\pi_{\ell,\mathfrak{p}}(f)$ is trace class and that its kernel function is Schwartz class. Therefore

$$\begin{aligned}
\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) &= \int_{\mathbb{R}^d} k_{\rho_{\ell,\mathfrak{p}}(f)}(t,t)dt = \int_{G/P} k_{\pi_{\ell,\mathfrak{p}}(f)}(g,g)d\dot{g} \\
&= \int_{G/P}\int_P f(gpg^{-1})\chi_\ell(p)dpd\dot{g} \\
&= \int_{G/P}\int_{\mathfrak{p}} f \circ \exp(\mathrm{Ad}(g)U)e^{-i\ell(U)}dUd\dot{g}.
\end{aligned}$$

Using the Fourier inversion formula, we see that

$$\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) = \frac{1}{(2\pi)^d}\int_{G/P}\int_{\mathfrak{p}^\perp} \widehat{f \circ \exp}(\mathrm{Ad}^*(g)(\ell+q))dqd\dot{g},$$

where $d = \dim(G/P)$. If we choose a Jordan–Hölder basis $\{X_1^*,\dots,X_n^*\}$ for the action of $\mathfrak{p}$ on $\mathfrak{g}^*$, such that $\mathcal{X} := \{X_1^*,\dots,X_d^*\}$ is a Jordan–Hölder basis of $\mathfrak{g}^*$ relative to $\mathfrak{p}^*$, then we see that the index set I^ℓ for the action of P must be the set $I^\ell = \{1,\dots,d\}$, since $Ad^*(P)\ell = \ell + \mathfrak{p}^\perp$, and therefore by formula (6.1.2)

$$\int_{\mathfrak{p}^\perp} \varphi(\ell+q)dq = \int_{\mathbb{R}^d} \varphi(\ell + \sum_{j=1}^d z_j X_j^*)dz = \int_{P/G(\ell)} \varphi(\mathrm{Ad}^*(p)\ell)d\dot{p}, \; \varphi \in \mathcal{S}(\mathfrak{g}^*).$$

Hence

$$\begin{aligned}
\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) &= \frac{1}{(2\pi)^d}\int_{G/P}\int_{P/G(\ell)} \widehat{f \circ \exp}(\mathrm{Ad}^*(g)\mathrm{Ad}^*(p)\ell)d\dot{p}d\dot{g} \\
&= \frac{1}{(2\pi)^d}\int_O \widehat{f \circ \exp}(q)d\nu_O(q).
\end{aligned}$$

■

Remark 6.4.2. We can use Theorem 6.4.1 to have another proof of the fact that the Kirillov mapping for nilpotent Lie groups is injective. Suppose that we have two different orbits O and O'. Choose a nonnegative Schwartz function ψ on $\mathfrak{g}^*$, such that ψ is nonzero on O, but vanishes on O'. Let $f \in \mathcal{S}(G)$ be defined by $\widehat{f \circ \exp} = \psi$. Then by Theorem 6.4.1 $\mathrm{Tr}(\hat{\rho}(O)(f)) = \int_O \psi(q)d\nu_O(q) > 0$ and $\mathrm{Tr}(\hat{\rho}(O')(f)) = \int_{O'} \psi(q)d\nu_{O'}(q) = 0$. Hence $\hat{\rho}(O)$ and $\hat{\rho}(O')$ cannot be equivalent.

6.5 Parametrization of All Orbits

Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group. Let $\mathcal{Z} = \{Z_n, \dots, Z_1\}$ be a Jordan–Hölder basis of $\mathfrak{g}$ and for $j \in \{1, \dots, n\}$ let, as before, $\mathfrak{g}_j = \text{span}\{Z_1, \dots, Z_j\}$, $\mathfrak{g}_0 := \{0\}$. We had defined for $\ell \in \mathfrak{g}^*$:

$$I_{\mathcal{Z}}^{\ell} = \{1 \leq i \leq n; \mathfrak{g}(\ell) + \mathfrak{g}_i \neq \mathfrak{g}(\ell) + \mathfrak{g}_{i-1}\}.$$

Definition 6.5.1. For a subset I of $\{1, \dots, n\}$, let

$$\mathfrak{g}_I^* = \{\ell \in \mathfrak{g}^*; I_{\mathcal{Z}}^{\ell} = I\}.$$

We denote by $\mathcal{I}_{\mathcal{Z}} = \mathcal{I}$ the family of the subsets I of $\{1, \dots, n\}$, for which $\mathfrak{g}_I^* \neq \emptyset$ and we define the polynomial function Q_I on $\mathfrak{g}^*$ by

$$Q_I(\ell) = \det(\langle \ell, [Z_i, Z_j] \rangle_{i,j \in I}), I \in \mathcal{I}_{\mathcal{Z}}.$$

Theorem 6.5.2. *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group. There exists an ordering on the family $\mathcal{I}_{\mathcal{Z}}$ such that for every $I \in \mathcal{I}_{\mathcal{Z}}$:*

1. The algebraic subset

$$F_I := \{\ell \in \mathfrak{g}^*; Q_J(\ell) = 0, Q_I(\ell) \neq 0, J < I\}$$

is G-invariant and $F_I = \mathfrak{g}_I^*$.
2. $\mathfrak{g}^* = \bigcup_{I \in \mathcal{I}_{\mathcal{Z}}} \mathfrak{g}_I^*$.
3. The largest $I = \emptyset$ corresponds to the set of all characters of $\mathfrak{g}$.

Proof. We define an ordering on $\mathcal{I}$ in the following way. Let $I = \{i_d > \cdots > i_1\}$, $J = \{j_e > \cdots > j_1\} \in \mathcal{I}$. We say that $I < J$
if either $|I| > |J|$
or otherwise (i.e. $|I| = d = |J|$), if $i_1 = j_1, \dots, i_{r-1} = j_{r-1}$ but $i_r < j_r$.
Let us show that

$$\mathfrak{g}_I^* = \{\ell \in \mathfrak{g}^*; Q_I(\ell) \neq 0, Q_J(\ell) = 0, J < I\} =: F_I.$$

We use the following criterion for linear independence.

Lemma 6.5.3. *Let ω be a non-degenerate skew symmetric bilinear form on a finite-dimensional vector space V. Then a family of vectors $\mathcal{B} = \{B_1, \dots, B_d\}$ is linearly independent, if and only if the determinant of the matrix*

$$M = \big(\omega(B_i, B_j)\big)_{1 \leq i,j \leq d}$$

is different from 0.

Let $I = \{i_d > \cdots > i_1\}$ and now take $\ell \in \mathfrak{g}_I^*$. Let $J = \{j_e > \cdots > j_1\} < I$. If $|J| > |I|$, then the family of vectors Z_j, $j \in J$, cannot be linearly independent modulo $\mathfrak{g}(\ell)$, since $\dim(\mathfrak{g}/\mathfrak{g}(\ell)) = d = |I|$. Hence $Q_J(\ell) = 0$ by the above lemma. If $d = |J|$, then there exists an index $j_r = j \in J$ which is not in I, but the indices $j_s, s < r$, are in $I \cap J$. Hence there exists a vector

$$T_r = Z_{j_r} + \sum_{\substack{j_r - 1 \geq k \geq 1 \\ k \in I}} a_k(\ell) Z_k \in \mathfrak{g}(\ell).$$

But then the vectors $\{Z_j; j \in J\}$ cannot be linearly independent modulo $\mathfrak{g}(\ell)$ and therefore $Q_J(\ell) = 0$. Of course, by definition $Q_I(\ell) \neq 0$. Conversely, let $\ell \in \mathfrak{g}^*$ such that $Q_I(\ell) \neq 0$, but $Q_J(\ell) = 0$ for all $J < I$. The fact that $Q_I(\ell) \neq 0$ tells us that $I \subset I^\ell$, since otherwise there exists a smallest index i which is contained in I but not in I^ℓ, which implies again that $Z_i \in \mathfrak{g}(\ell)$ modulo $\mathfrak{g}_{i-1}$ so that $Q_I(\ell) = 0$. Suppose that $I^\ell \neq I$. Then $|I| < |I^\ell|$ and so $I^\ell < I$, whence $Q_{I^\ell}(\ell) = 0$, which is a contradiction. Hence $I = I^\ell$ and so $\mathfrak{g}_I^* = F_I$. ■

Theorem 6.5.4. *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group, $\mathcal{Z} = \{Z_n, \ldots, Z_1\}$ be a Jordan–Hölder basis of $\mathfrak{g}$ and let $\mathcal{Z}^*$ be its dual basis. For every index set $I = \{j_d > \cdots > j_1\} \in \mathcal{I}_{\mathcal{Z}}$ there exist rational functions $R^I_{j,\mathcal{Z}} = R^I_j : \mathbb{R}^d \times \mathfrak{g}^* \to \mathbb{R}$, $j = 1, \ldots, n$, which have no singularities on $\mathfrak{g}_I^*$, such that:*

1. *$R^I_j(z, \ell)$ is polynomial in $z \in \mathbb{R}^d$ for $j = 1, \ldots, n$.*
2. *$R^I_{j_i}(z, \ell) = z_i, z \in \mathbb{R}^d, \ell \in \mathfrak{g}_I^*$, for every $j_i \in I$.*
3. *For $j \notin I$, $j_{i+1} > j > j_i$, the function $R^I_j(z, \ell)$ does only depend on $(z_1, \ldots, z_i, \ell)$.*
4. *For all $\ell \in \mathfrak{g}_I^*$*

$$O(\ell) := \mathrm{Ad}^*(G)\ell = \left\{ \sum_{j=1}^n R^I_j(z, \ell) Z_j^*; z \in \mathbb{R}^d \right\}.$$

5. *$R^I_j(z, \mathrm{Ad}^*(g)\ell) = R^I_j(z, \ell)$ for all $z \in \mathbb{R}^d, \ell \in \mathfrak{g}_I^*$.*

Proof. We know that the matrix

$$M_I(\ell) = (m_{ij})_{i,j \in I} := (\langle \ell, [Z_i, Z_j] \rangle)_{i,j \in I}$$

is invertible for every $\ell \in \mathfrak{g}_I^*$, since its determinant $Q_I(\ell) \neq 0$. Let $N_I(\ell) = (n_{i,j}(\ell))$ be its inverse. Since the coefficients of $M_I(\ell)$ are linear functions in ℓ, it follows that the coefficients $n_{i,j}(\ell)$ of its inverse $N_I(\ell)$ are rational functions without any singularities in $\ell \in \mathfrak{g}_I^*$ and the denominator is a power of the polynomial Q_I. Let for any $j \in I$

$$X_j(\ell) := -\sum_{i \in J} n_{j,i}(\ell) Z_i.$$

Then we have for $k \in I$

$$\begin{aligned}\langle \mathrm{ad}^*(X_j(\ell))\ell, Z_k\rangle &= \langle \ell, -[X_j(\ell), Z_k]\rangle \\ &= \sum_{i\in I} n_{j,i}(\ell) m_{i,k}(\ell) = \delta_{j,k}.\end{aligned}$$

For the indices $1 \le k \le j, k \notin I$, we have that $Z_k \in \mathrm{span}\{Z_i; 1 \le i < k, i \in I\}$ modulo $\mathfrak{g}(\ell)$ and so $\langle \mathrm{ad}^*(X_j(\ell))\ell, Z_k\rangle = 0$ too. This means that $X_j(\ell) \in \mathfrak{g}_{j-1}(\ell) \setminus \mathfrak{g}_j(\ell)$ and $\mathrm{ad}^*(X_j(\ell))(\ell) = Z_j^*$ modulo $\mathfrak{g}_{j-1}^{\perp}$ in the notations of Remark 6.1.18. Hence

$$\begin{aligned}O(\ell) &= \left\{\prod_{j\in I} \mathrm{Ad}^*(\exp(t_j X_j(\ell))(\ell), t \in \mathbb{R}^d\right\} \\ &= \sum_{i=1}^{n} (l_i + q_i(t_1, \ldots, t_d, \ell)) Z_i^*, \qquad (6.5.1)\end{aligned}$$

where for $j = j_i \in I$, we have that

$$q_{j_i}(t_1, \ldots, t_d, \ell) = t_i + q'_{j_i,\ell}(t_1, \ldots, t_{i-1})$$

for some polynomial function $q'_{j_i,\ell}$ in $t \in \mathbb{R}^d$, which varies rationally in $\ell \in \mathfrak{g}_I^*$. If $j \notin I$, then $j_i > j > j_{i+1}$ for some $1 \le i \le d$ and therefore the polynomial function $q_{j,\ell}$ depends only on $t_1, \ldots, t_i$. Now let inductively

$$z_1 = l_{j_1} + t_1, \ldots, z_i = l_{j_i} + t_i + q'_{j_i,\ell}(t_1, \ldots, t_{i-1}),\ i = 2, \ldots d.$$

It then follows that conversely $t_i = z_i + r_i(z_1, \ldots, z_{i-1}, \ell)$, for some polynomial function r_i, for $i = 1, \ldots, d$. Hence, replacing the t_i's by the z_i's in (6.5.1), we obtain a canonical description of the orbit of ℓ (once the basis $\mathcal{Z}$ is fixed). There exist polynomial functions $R_j^I(z_1, \ldots, z_d, \ell), j = 1, \ldots, n$, on $\mathbb{R}^d$ which vary rationally in $\ell \in \mathfrak{g}_I^*$, such that $R_{j_i}^I(z_1, \ldots, z_d, \ell) = z_i$ for all $j_i \in I$, such that for $j \notin I$ and $j_{i+1} > j > j_i$ for some i, $R_j^I(z_1, \ldots, z_d, \ell) = R_j(z_1, \ldots, z_i, \ell)$ and such that

$$O(\ell) = \left\{\sum_{j=1}^{n} R_j^I(z, \ell) Z_j^*;\ z \in \mathbb{R}^d\right\}.$$

We must prove the invariance in ℓ of the functions $R_j^I,\ j \notin I$. For any $g \in G$ and any $q = \sum_j q_j Z_j^*$ in the orbit $O = O(\ell) = O(\mathrm{Ad}^*(g)\ell)$, we have for $g \in G, j \notin I, j_i > j > j_{i+1}$,

$$R_j^I(q_{j_1}, \ldots, q_{j_i}, \mathrm{Ad}^*(g)\ell) = q_j = R_j^I(q_{j_1}, \ldots, q_{j_i}, \ell).$$

This shows that

$$R^I_j(z_1,\dots,z_d,\mathrm{Ad}^*(g)\ell) = R^I_j(z_1,\dots,z_d,\ell),\ z=(z_1,\dots,z_d)\in\mathbb{R}^d. \qquad \blacksquare$$

Definition 6.5.5. Let $G = \exp\mathfrak{g}$ be a nilpotent Lie group and let $\mathcal{Z}$ be a Jordan–Hölder basis of $\mathfrak{g}$. We denote by $I_{\min}$ the smallest element in $\mathcal{I}_{\mathcal{Z}}$ and by $\mathfrak{g}^*_{\mathrm{gen}}$ the Zariski open subset $\mathfrak{g}^*_{I_{\min}}$ of $\mathfrak{g}^*$. We say that the elements of $\mathfrak{g}^*_{\mathrm{gen}}$ are in **general position**. We consider the rational functions $\rho_j,\ j\notin I_{\min}$, defined by

$$\rho_j(\ell) := R^{I_{\min}}_j(0,\ell),\ \ell\in\mathfrak{g}^*_{\mathrm{gen}}.$$

These rational functions ρ_j are regular in the points $Q_{I_{\min}}$, since $\rho_j = \frac{P_j}{Q^k_{I_{\min}}}$ for some $k\in\mathbb{N}^*$ by the proof of Theorem 6.5.4 and they are also G-invariant by the same theorem, hence $\rho_j\in R(\mathfrak{g}^*)^G$, the algebra of the G-invariant rational functions on $\mathfrak{g}^*$, for every $j\notin I_{\min}$. Furthermore, since for $\ell=\sum_j l_jZ^*_j$ we have by (6.1.3) that $\rho_j(\ell) = l_j + \rho'_j(l_1,\dots,l_{j-1})$, we see that the functions $\rho_j,\ j\notin I_{\min}$, are algebraically independent. Let $d_{\max}$ be the maximum of the dimensions of all coadjoint orbits. Then $d_{\max} = |I_{\min}|$.

We shall now parametrize the orbits in general position in the following way. For every orbit $O = G\cdot\ell$ in $\mathfrak{g}^*_{\mathrm{gen}}$, there exists exactly one element ℓ_O, which is contained in $\mathfrak{v}^*$ introduced below. Indeed, if $\ell_O = \sum_{j=1}^n R^{I_{\min}}_j(0,\ell)Z^*_j$, ℓ_O is the only element of O, for which the components $l_j,\ j\in I_{\min}$, are all 0.

Definition 6.5.6. Let

$$\mathfrak{v}^* = \mathfrak{v}^*_{\mathcal{Z}} := \left\{\ell = \sum_{j=1}^n l_jZ^*_j\in\mathfrak{g}^*;\ l_j = 0,\ \text{for all } j\in I_{\min}\right\},$$

and we let

$$\mathfrak{v}^*_{\mathrm{gen}} = \mathfrak{g}^*_{\mathrm{gen}}\cap\mathfrak{v}^* = \{\ell\in\mathfrak{v}^*;\ Q_{I_{\min}}(\ell)\neq 0\}.$$

Then $\mathfrak{v}^*$ is a $(\dim\mathfrak{g} - d_{\max})$-dimensional subspace of $\mathfrak{g}^*$ and $\mathfrak{v}^*_{\mathrm{gen}}$ is Zariski open in $\mathfrak{v}^*$.

Definition 6.5.7. Let $G = \exp\mathfrak{g}$ be a nilpotent Lie group. Let $f = \frac{P}{Q} : \mathfrak{g}^*\to\mathbb{C}$ be a rational function on $\mathfrak{g}^*$. Let us write $g\cdot f$ for the function

$$g\cdot f(\ell) = f(\mathrm{Ad}^*(g^{-1})\ell) = \frac{P(\mathrm{Ad}^*(g^{-1})\ell)}{Q(\mathrm{Ad}^*(g^{-1})\ell)},\ell\in\mathfrak{g}^*.$$

We say that f is G-invariant if $g\cdot f = f$ for all $g\in G$. This is equivalent to saying that

$$X\cdot f := \frac{d}{dt}\exp(tX)\cdot f|_{t=0} = \frac{(X\cdot P)Q - P(X\cdot Q)}{Q^2} = 0 \text{ for all } X \in \mathfrak{g}.$$

Let $R(\mathfrak{g}^*)^G$ be the algebra of all G-invariant rational functions on $\mathfrak{g}^*$.

Proposition 6.5.8. *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group.*

1. *For any $f \in R(\mathfrak{g}^*)^G$ there exist two G-invariant polynomial functions P, Q on $\mathfrak{g}^*$ such that $f = \frac{P}{Q}$.*
2. *Any polynomial $P \in R(\mathfrak{g}^*)^G$ is a polynomial expression in the functions $\rho_j,\ j \notin I_{\min}$.*

Proof. Let

$$\mathfrak{g} = \mathfrak{g}_n \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

be a composition series of $\mathfrak{g}$. Let us show by induction on j that there exist two polynomial functions P_j, Q_j, which are $\mathfrak{g}_j$ invariant, such that $f = \frac{P_j}{Q_j}$. Suppose that we have found $\mathfrak{g}_j$-invariant polynomial functions P_j and Q_j, such that $f = \frac{P_j}{Q_j}$, then we are looking for such $\mathfrak{g}_{j+1}$-invariant ones. Choose $X \in \mathfrak{g}_{j+1} \setminus \mathfrak{g}_j$. If a polynomial function R is $\mathfrak{g}_j$-invariant, then the function $X\cdot R$ is also $\mathfrak{g}_j$-invariant since

$$U \cdot (X \cdot R) = X \cdot (U \cdot R) + [U, X] \cdot R = 0 + 0 = 0.$$

Now $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by derivations. Hence there exists $m \in \mathbb{N}^*$, such that $X^m\cdot Q_j = 0$, $X^{m-1}\cdot Q_j \neq 0$. If $m > 1$, then the equation $X\cdot f = 0$ tells us that $(X\cdot P_j)$ $Q_j - P_j(X\cdot Q_j) = 0$, i.e. $\frac{X\cdot P_j}{X\cdot Q_j} = \frac{P_j}{Q_j} = f$. Continuing in this way we see that $\frac{X^{m-1}\cdot P_j}{X^{m-1}\cdot Q_j} = f$. Let $P_{j+1} := X^{m-1}\cdot P_j, Q_{j+1} := X^{m-1}\cdot Q_j$. Then both are $\mathfrak{g}_{j+1}$-invariant and $f = \frac{P_{j+1}}{Q_{j+1}}$.

Suppose now that P is a G-invariant polynomial function on $\mathfrak{g}^*$. Then for $\ell \in \mathfrak{g}^*_{\text{gen}}$, there exists an element $g \in G$, such that $\mathrm{Ad}^*(g)(\ell) = \sum_j R_j^{I_{\min}}(0, \ell)Z_j^*$. Hence

$$P(\ell) = P(\mathrm{Ad}^*(g)\ell) = P\Big(\sum_{j\notin I_{\min}} \rho_j(\ell)Z_j^*\Big).$$

Hence P is a polynomial in the $\rho_j, j \notin I_{\min}$. ∎

Remark 6.5.9. Let us recall that we have two precise measures on a coadjoint orbit O, the first one given by the trace formula (see Theorem 6.4.1)

$$\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) = \int_O \widehat{f\circ\exp}(q)d\nu_O(q),\ f \in \mathcal{S}(G), \ell \in O, \tag{6.5.2}$$

the second being given by the canonical description of the coadjoint orbit O using a Jordan–Hölder basis $\mathcal{Z}$ of $\mathfrak{g}$. Let $I = I_{\mathcal{Z}}$ the index set of O. Then we have the second invariant measure

$$\int_O \varphi(q)d\mu_O(q) := \int_{\mathbb{R}^d} \varphi\Big(\sum_j R_j^I(t,O)\Big)dt, \varphi \in C_c(O).$$

Theorem 6.5.10. *We have that* $d\nu_O = \frac{1}{(2\pi)^{d/2}Q_I(O)^{1/2}}d\mu_O$, *where* $O \in \mathfrak{g}_I^*$ *and* d *denotes the dimension of* O.

Proof. We recall the computations of Theorem 6.4.1. Let $\ell \in O$ and let $\mathfrak{p}$ be a polarization at ℓ. Let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a Malcev basis of $\mathfrak{g}(\ell)$, $\mathcal{C} = \{C_1, \dots, C_m\}$ be a Malcev basis of $\mathfrak{p}$ relative to $\mathfrak{g}(\ell)$ and let $\mathcal{D} = \{D_1, \dots, D_m\}$ be a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{p}$. We denote by

$$\mathcal{Y} := \{D_1, \dots, D_m, C_1, \dots, C_m, E_1, \dots, E_k\} =: \{Y_1, \dots, Y_n\}$$

the corresponding Malcev basis of $\mathfrak{g}$ and we use this basis to describe the Haar measure of G. We have seen in the proof of Theorem 6.4.1 that

$$\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) = \frac{1}{(2\pi)^m} \int_{G/P} \int_{\mathfrak{p}^\perp} \widehat{f \circ \exp}(\mathrm{Ad}^*(g)(\ell + q))dq d\dot{g}, \ f \in \mathcal{S}(G). \tag{6.5.3}$$

This formula can be given the form

$$\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) = \frac{\alpha}{(2\pi)^m} \int_{G/P} \int_{P/G(\ell)} \widehat{f \circ \exp}(\mathrm{Ad}^*(g)\mathrm{Ad}^*(p)\cdot\ell)d\dot{p}d\dot{g}, \ f \in \mathcal{S}(G).$$

Here we use the invariant measure on $P/G(\ell)$ given by the basis $\mathcal{C}$. We obtain the constant α, which relates the invariant measures on $P/G(\ell)$ and on $\mathfrak{p}^\perp$. In order to compute this constant, it suffices to write

$$\mathrm{Ad}^*(E_{\mathcal{C}}(t))\ell = \ell + \sum_{j=1}^m Q_j(t)D_j^*, \ t \in \mathbb{R}^m,$$

and to make in the integral (6.5.3) the change of variables $\mathfrak{p}^\perp \ni q = \sum_{j=1}^m Q_j(t)D_j^*$. The Jacobian of this change of variables is this constant α. We compute the Jacobian for $t = 0$. This gives us

$$\alpha = \left| \det(\frac{\partial Q_j(t)}{\partial t_i}) \right|_{t=0}.$$

Since

$$\frac{\partial Q_j(t)}{\partial t_i}\Big|_{t=0} = \frac{\partial\Big(\mathrm{Ad}^*(\exp(t_i C_i)\ell)(D_j)\Big)}{\partial t_i}\Bigg|_{t=0} = \langle \ell, [D_j, C_i]\rangle,$$

we see that

$$\alpha = \Big|\det\Big(\langle \ell, [D_j, C_i]\rangle\Big)_{i,j}\Big|. \tag{6.5.4}$$

Hence for $f \in \mathcal{S}(G)$

$$\mathrm{Tr}(\pi_{\ell,\mathfrak{p}}(f)) = \frac{\alpha}{(2\pi)^m}$$
$$\times \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widehat{f \circ \exp}\left(\prod_{i=1}^m \mathrm{Ad}^*(\exp(t_i D_i))\prod_{i=1}^m \mathrm{Ad}^*(\exp(s_i C_i))\ell\right)\prod_{i=1}^m ds_i \prod_{i=1}^m dt_i.$$

We now use a new change of variables to pass from the basis $\mathcal{Y}$ to the basis $\mathcal{Z}$. We write

$$\sum_{j=1}^n Q_j(t_1,\ldots,t_m,s_1,\ldots,s_m)Z_j^* = \prod_{i=1}^m \mathrm{Ad}^*(\exp(t_i D_i))\prod_{i=1}^m \mathrm{Ad}^*(\exp(s_i C_i))\ell$$

and we have the change of variables

$$z_i = Q_{j_i}(t_1,\ldots,t_m,s_1,\ldots,s_m),\, j_1,\ldots,j_{2m} \in I^\ell.$$

We obtain in this way a constant β such that for $\varphi \in C_c(\mathfrak{g}^*)$

$$\int_{\mathbb{R}^{2m}} \varphi\left(\sum_{j=1}^n R_j^I(z,\ell)Z_j^*\right)dz = \beta\int_{G/G(\ell)} \varphi(\mathrm{Ad}^*(g)\ell)d\dot{g}. \tag{6.5.5}$$

Again the change of variables above gives us the value of β as

$$\beta = \left|\det\begin{pmatrix}\frac{\partial Q_{j_i}(t,s)}{\partial t_p}\\ \frac{\partial Q_{j_i}(t,s)}{\partial s_q}\end{pmatrix}\right|_{t=s=0}.$$

Since

$$\frac{\partial Q_{j_i}(t,s)}{\partial t_p}\Big|_{t=s=0} = \frac{\partial(\mathrm{Ad}^*(\exp(t_p D_p)\ell)(Z_{j_i}))}{\partial t_p}\Big|_{t=s=0} = -\langle \ell, [D_p, Z_{j_i}]\rangle,$$

and

$$\frac{\partial Q_{j_i}(t,s)}{\partial s_q}\Big|_{t=s=0} = \frac{\partial(\mathrm{Ad}^*(\exp(s_q C_q)\ell)(Z_{j_i}))}{\partial s_q}\Big|_{t=s=0} = -\langle \ell, [C_q, Z_{j_i}]\rangle,$$

we see that

$$\beta = \left| \det \begin{pmatrix} \langle \ell, [D_p, Z_{j_i}]\rangle \\ \langle \ell, [C_q, Z_{j_i}]\rangle \end{pmatrix} \right|.$$

We develop now the vectors D_p, C_q in the basis $\mathcal{Z}$ modulo $\mathfrak{g}(\ell)$, i.e.

$$D_p = \sum_{i=1}^{2m} a_{i,p} Z_{j_i},\ C_q = \sum_{i=1}^{2m} a_{i,q} Z_{j_i} \bmod \mathfrak{g}(\ell),\, p = 1, \ldots, m, q = 1, \ldots, m.$$

This change of basis gives us the $2m$ by $2m$ matrix

$$A = \begin{pmatrix} a_{i,p} \\ a_{i,q} \end{pmatrix}.$$

We then see that

$$\beta = |\det A||Q_I(\ell)|. \tag{6.5.6}$$

Furthermore, since $\langle \ell, [C_i, C_{i'}]\rangle = 0$ for all i, i', we observe that

$$\begin{aligned} \alpha &= \left|\det(\langle \ell, [D_j, C_i]\rangle_{i,j})\right| = \left|\det\left(\begin{pmatrix} \langle \ell, [D_j, C_i]\rangle & \langle \ell, [D_j, D_{j'}]\rangle \\ \langle \ell, [C_i, C_{i'}]\rangle & \langle \ell, [C_i, D_j]\rangle \end{pmatrix}\right)\right|^{1/2} \\ &= |\det A||Q_I(\ell)|^{1/2}. \end{aligned} \tag{6.5.7}$$

Therefore, by Eqs. (6.5.4)–(6.5.7)

$$\frac{\alpha}{\beta} = \frac{|\det A||Q_I(\ell)|^{1/2}}{|\det A||Q_I(\ell)|} = \frac{1}{|Q_I(\ell)|^{1/2}}.$$ ■

Definition 6.5.11. We define the function $\delta = \delta_{\mathcal{Z}}$ on $\mathfrak{g}^*$ by

$$\delta(\ell) := (2\pi)^{d_{\max}/2}|Q_{I_{\min}}(\ell)|^{1/2},\ \ell \in \mathfrak{g}^*.$$

The function δ is of course G-invariant, since so is $Q_{I_{\min}}$, it is homogeneous of degree $\frac{d_{\max}}{2}$, since $Q_{I_{\min}}$ is homogeneous of degree $d_{\max}$ and δ is also a polynomial. In order to see that, let $A_{2m}(\mathbb{R})$ denote the space of all skew symmetric real matrices of order $2m$.

Lemma 6.5.12. *Let $m \in \mathbb{N}$. There exists a polynomial mapping $Q : A_{2m} \to \mathbb{R}$ in the variables $(a_{i,j})_{i,j}$ (called the Pfaffian of A) such that for every element $A = (a_{i,j}) \in A_{2m}(\mathbb{R})$ we have*

$$\det(a_{i,j}) = (Q(a_{i,j}))^2.$$

Proof. We define for $A \in A_{2m}(\mathbb{R})$ the skew symmetric bilinear form

$$w_A(x, y) := \sum_{i,j} a_{i,j} x_i y_j.$$

Let $\mathcal{X} = \{X_1, \ldots, X_{2m}\}$ be the canonical basis of $\mathbb{R}^{2m}$. Starting with

$$X_1(A) := \frac{1}{a_{1,2}} X_1,\ X_2(A) = X_2$$

and considering the subspace $V_1(A) := \{U \in \mathbb{R}^{2m},\ w_A(U, X_j) = 0, j = 1, 2\}$ and the basis

$$\begin{aligned}\mathcal{X}_1(A) = \{&X_3 + w_A(X_3, X_1(A))X_2(A) - w_A(X_3, X_2(A))X_1(A), \ldots,\\ &X_{2m} + w_A(X_{2m}, X_1(A))X_2(A) - w_A(X_{2m}, X_2(A))X_1(A)\}\end{aligned}$$

we construct inductively a symplectic basis $\mathcal{X}(A) = \{X_1(A), \ldots, X_{2m}(A)\}$ for the skew symmetric bilinear form w_A for all A in a Zariski open subset of $A_{2m}(\mathbb{R})$. The vectors $X_j(A)$ of $\mathcal{X}(A)$ vary rationally in the coefficients of A. Let $U(A)$ be the matrix which gives the change of the basis $\mathcal{X}$ to the basis $\mathcal{X}(A)$. Then

$$\begin{aligned}\det A &= \det\left(U(A)^2\right) \det\left(\begin{pmatrix} 0_m & \mathbb{I}_m \\ -\mathbb{I}_m & 0_m \end{pmatrix}\right)\\ &= \frac{P_1(A)^2}{P_2(A)^2} \det\left(\begin{pmatrix} 0_m & \mathbb{I}_m \\ -\mathbb{I}_m & 0_m \end{pmatrix}\right) = \frac{P_1(A)^2}{P_2(A)^2}\end{aligned}$$

for two polynomial functions P_1, P_2 in the coefficients $a_{i,j}$ of A. Hence $P_2(A)^2 \det(A) = P_1(A)^2$ and therefore P_2 divides P_1. This shows that $\det(A) = Q(A)^2$ for some polynomial function $Q(A)$ in the variables $a_{i,j}$. ■

We define now a measure $d\lambda$ on the subspace $\mathfrak{v}^*$ in the following way. Since $\mathfrak{v}^*$ is a real finite-dimensional vector space we can define the measure $d\lambda$ by

$$\int_{\mathfrak{v}^*} \varphi(q)\delta(q)dq,\ \varphi \in C_c(\mathfrak{g}^*).$$

Here $dq = \prod_{j \notin I_{\min}} dl_j$ denotes the Lebesgue measure on $\mathfrak{v}^*$.

Theorem 6.5.13. *Let $G = exp\,\mathfrak{g}$ be a nilpotent Lie group. Let $n_G = dimG - d_{\max}$. For every Schwartz function $f \in \mathcal{S}(G)$, we have that*

$$f(g) = \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} Tr(\pi_v(f) \circ \pi_v(g^{-1}))d\lambda(v),\ g \in G.$$

Proof. By the ordinary Fourier inversion formula we have for $f \in \mathcal{S}(G)$, that

$$f(e) = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} \widehat{f \circ \exp}(\ell)d\ell. \tag{6.5.8}$$

Choose again a Jordan–Hölder basis $\mathcal{Z} = \{Z_n, \dots, Z_1\}$ of $\mathfrak{g}$, which gives us the Haar measure of the group G and the functions $R_j(z, \ell) := R_j^{I_{\min}}(z, \ell), \ell \in \mathfrak{g}^*_{\text{gen}}$, $z \in \mathbb{R}^{d_{\max}}$. Let us recall that

$$R_j(z, \ell) = l_j + R'_j(l_1, \dots, l_{j-1}),\ \ell = \sum_{k=1}^{n} l_k Z_k^*,\ j \notin I_{\min},$$

where R'_j is a rational function in the variables $l_1, \dots, l_{j-1}$ which is regular on $\mathfrak{g}^*_{\text{gen}}$ (see Theorem 6.1.15). Hence we can transform Eq. (6.5.8) in the following way:

$$\begin{aligned}
f(e) &= \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} \widehat{f \circ \exp}\Big(\sum_j l_j Z_j^*\Big)d\ell \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{d_{\max}} \times \mathfrak{v}^*} \widehat{f \circ \exp}\Big(\sum_j R_j(z, v)Z_j^*\Big)dzdv \\
&= \frac{1}{(2\pi)^n} \int_{\mathfrak{v}^*} \int_{O(v)} \widehat{f \circ \exp}(q)d\mu_O(q)dv \\
&= \frac{(2\pi)^{d_{\max}/2}}{(2\pi)^n} \int_{\mathfrak{v}^*} Q_{I_{\min}}(v)^{1/2} \int_{O(v)} \widehat{f \circ \exp}(q)d\nu_{O(v)}(q)dv \text{ (by Theorem 6.5.10)} \\
&= \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \mathrm{Tr}(\pi_\ell(f))\delta(v)dv \text{ (by formula (6.5.2))}.
\end{aligned}$$

Now for $g \in G$ we get

$$\begin{aligned}
f(g) &= \lambda(g^{-1})f(e) \\
&= \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \mathrm{Tr}(\pi_\ell(\lambda(g^{-1})f))\delta(v)dv \\
&= \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \mathrm{Tr}(\pi_v(g^{-1}) \circ \pi_v(f))d\lambda(v).
\end{aligned}$$

■

Corollary 6.5.14 (Plancherel Formula). *Let* $f \in \mathcal{S}(G)$. *Then*

$$\|f\|_2^2 = \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \|\pi_v(f))\|_{\mathrm{HS}}^2 d\lambda(v).$$

Proof. We have

$$\begin{aligned}
\|f\|_2^2 &= \int_G f(t)\overline{f(t)}dt = f * f^*(e) \\
&= \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \mathrm{Tr}(\pi_v(f) \circ \pi_v(f)^*) d\lambda(v) \\
&= \frac{1}{(2\pi)^{n_G}} \int_{\mathfrak{v}^*} \|\pi_v(f)\|_{\mathrm{HS}}^2 d\lambda(v).
\end{aligned}$$

■

Chapter 7
Holomorphically Induced Representations $\rho(f, \mathfrak{h}, G)$ for Exponential Solvable Lie Groups

7.1 First Trial

Let us consider holomorphically induced representations for an exponential solvable Lie group $G = \exp \mathfrak{g}$. Since the stabilizer $G(f)$ in G of any $f \in \mathfrak{g}^*$ is connected (Theorem 5.3.2), there uniquely exists the homomorphism $\eta_f : G(f) \to \mathbb{T}$ such that $d\eta_f = if|_{\mathfrak{g}(f)}$. So, we write the holomorphically induced representation $\rho(f, \eta_f, \mathfrak{h}, G)$ constructed from a polarization $\mathfrak{h} \in P(f, G)$ satisfying the Pukanszky condition and its representation space $\mathcal{H}(f, \eta_f, \mathfrak{h}, G)$ simply as $\rho(f, \mathfrak{h}, G)$ and $\mathcal{H}(f, \mathfrak{h}, G)$. We show the following theorem.

Theorem 7.1.1 ([27]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $f \in \mathfrak{g}^*$. Suppose that $\mathfrak{h} \in P^+(f, G)$ satisfies the Pukanszky condition. Provided that $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$, $\rho(f, \mathfrak{h}, G)$ is irreducible and equivalent to the irreducible unitary representation of G corresponding to the coadjoint orbit $G{\cdot}f$.*

Proof. When $\dim G = 1$, the theorem is clear. Now we show the theorem by induction on $\dim G$. Suppose that $\dim G = n$ and that the theorem holds for exponential solvable Lie groups whose dimension is smaller than n.

Case 1. There is an ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{g}$ such that $f|_{\mathfrak{a}} = 0$.

Let $A = \exp \mathfrak{a}, \tilde{G} = G/A = \exp(\tilde{\mathfrak{g}}), \tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ and $p : G \to \tilde{G}$ the canonical projection. We denote by dp the differential of p and extend it linearly on $\mathfrak{g}_{\mathbb{C}}$. Then we consider the exact sequence

$$1 \to A \to G \xrightarrow{p} \tilde{G} \to 1$$

of exponential solvable Lie groups and take $\tilde{f} \in \tilde{\mathfrak{g}}^*$ such that $\tilde{f} \circ dp = f$.

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Since $\mathfrak{a} \subset \mathfrak{g}(f) \subset \mathfrak{h}$, $\tilde{\mathfrak{h}} = dp(\mathfrak{h})$ is an element of $P^+(\tilde{f}, \tilde{G})$. As in Proposition 5.1.17, $\tilde{\mathfrak{g}}^*$ is naturally identified with $\mathfrak{a}^\perp \subset \mathfrak{g}^*$ and $\tilde{\mathfrak{h}}$ satisfies the Pukanszky condition. By Proposition 5.1.17

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \circ p \simeq \rho(f, \mathfrak{h}, G). \tag{7.1.1}$$

It follows from the assumption that $\mathcal{H}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \neq \{0\}$. Since $\dim \tilde{G} < \dim G$, there exists $\tilde{\mathfrak{h}}_0 \in I(\tilde{f}, \tilde{\mathfrak{g}})$ so that

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \simeq \hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}). \tag{7.1.2}$$

Putting $\mathfrak{h}_0 = (dp)^{-1}(\tilde{\mathfrak{h}}_0)$,

$$\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}) \circ p \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \tag{7.1.3}$$

and $\mathfrak{h}_0 \in I(f, \mathfrak{g})$. Now Eqs. (7.1.1)–(7.1.3) give the desired result.

Case 2. There is no ideal $\mathfrak{a} \neq \{0\}$ such that $f|_{\mathfrak{a}} = 0$.

(i) $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g} \neq \mathfrak{g}$:

We choose and fix a complementary subspace $\mathfrak{m}$ of $\mathfrak{e}$ in $\mathfrak{g}$. If we identify $\mathfrak{e}^*$ with $\mathfrak{m}^\perp$, $\mathfrak{e}^* \subset \mathfrak{g}^*$. Letting $p : \mathfrak{g}^* \to \mathfrak{e}^*$ be the restriction mapping, $p|_{\mathfrak{e}^*}$ is the identity map. For $\ell \in \mathfrak{g}^*$, we put $\ell' = p(\ell) = \ell|_{\mathfrak{e}}$.

As is known immediately, $\mathfrak{h} \in P^+(f', E)$. Let us verify that $\mathfrak{h}$ satisfies the Pukanszky condition as polarization of $\mathfrak{e}$. First, since $\mathfrak{e}$ is a Lie subalgebra of $\mathfrak{g}$,

$$p(E{\cdot}f) = (E{\cdot}f)' = E{\cdot}f' \subset \mathfrak{e}^*. \tag{7.1.4}$$

Next, let us see $p^{-1}(E{\cdot}f') = E{\cdot}f$. Evidently $a{\cdot}\ell \in \mathfrak{e}^\perp$ for all $a \in E, \ell \in \mathfrak{e}^\perp$. Let us decompose $a{\cdot}f \ (a \in E)$ as

$$a{\cdot}f = (a{\cdot}f)' + \hat{f}, \ \hat{f} \in \mathfrak{e}^\perp.$$

Thus, for any $\ell \in \mathfrak{e}^\perp$,

$$a{\cdot}\left(f - a^{-1}{\cdot}(\hat{f} - \ell)\right) = (a{\cdot}f)' + \ell,$$

where $a^{-1}{\cdot}(\hat{f} - \ell) \in \mathfrak{e}^\perp$. Since $\mathfrak{h}$ satisfies the Pukanszky condition as polarization of $\mathfrak{g}$ at $f \in \mathfrak{g}^*$, there is by Proposition 5.1.12 $b \in D \subset E$ such that $b{\cdot}f = f - a^{-1}{\cdot}(f - \ell)$. Hence

$$(ab){\cdot}f = (a{\cdot}f)' + \ell. \tag{7.1.5}$$

Since $a \in E,\ \ell \in \mathfrak{e}^{\perp}$ are arbitrary, the relations (7.1.4), (7.1.5) indicate

$$p^{-1}(E{\cdot}f') = E{\cdot}f. \tag{7.1.6}$$

Therefore $E{\cdot}f' = E{\cdot}f \cap \mathfrak{e}^*$ is a closed set of $\mathfrak{e}^*$ and $\mathfrak{h}$ satisfies the Pukanszky condition as polarization of $\mathfrak{e}$ at f'.

Meanwhile we have

$$\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f', \mathfrak{h}, E).$$

Hence, provided $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$,

$$\mathcal{H}(f', \mathfrak{h}, E) \neq \{0\}.$$

As $\dim E < \dim G$, there is by the induction hypothesis $\mathfrak{h}_0 \in I(f', \mathfrak{e})$ so that

$$\rho(f', \mathfrak{h}, E) \simeq \hat{\rho}(f', \mathfrak{h}_0, E). \tag{7.1.7}$$

Theorem 5.3.8 means $\mathfrak{h}_0 \in M(f', \mathfrak{e})$. As $\mathfrak{h} \in P(f', E)$,

$$\dim_{\mathbb{R}}(\mathfrak{h}_0) = \dim_{\mathbb{C}}\mathfrak{h} = \frac{1}{2}(\dim_{\mathbb{R}}\mathfrak{g} + \dim_{\mathbb{R}}(\mathfrak{g}(f))).$$

Consequently $\mathfrak{h}_0 \in M(f, \mathfrak{g})$. Let us see that $\mathfrak{h}_0$ satisfies the Pukanszky condition. To begin with,

$$f + {\mathfrak{h}_0}^{\perp, \mathfrak{g}^*} = f' + {\mathfrak{h}_0}^{\perp, \mathfrak{e}^*} + \mathfrak{e}^{\perp, \mathfrak{g}^*}. \tag{7.1.8}$$

As $\mathfrak{h}_0 \in I(f', \mathfrak{e})$, Theorem 5.4.1 assures that $\mathfrak{h}_0$ satisfies the Pukanszky condition at the stage of E, i.e.

$$f' + {\mathfrak{h}_0}^{\perp, \mathfrak{e}^*} \subset E{\cdot}f'. \tag{7.1.9}$$

From properties (7.1.6), (7.1.8), (7.1.9),

$$f + {\mathfrak{h}_0}^{\perp, \mathfrak{g}^*} \subset G{\cdot}f,$$

That is, $\mathfrak{h}_0 \in M(f, \mathfrak{g})$ satisfies the Pukanszky condition and $\mathfrak{h}_0 \in I(f, \mathfrak{g})$. Consequently $\mathrm{ind}_E^G \hat{\rho}(f', \mathfrak{h}_0, E) = \hat{\rho}(f, \mathfrak{h}_0, G)$ is irreducible. Hence by (7.1.7),

$$\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f', \mathfrak{h}, E) \simeq \mathrm{ind}_E^G \hat{\rho}(f', \mathfrak{h}_0, E) = \hat{\rho}_G(f).$$

(ii) $\mathfrak{e} = \mathfrak{g}$ (i.e. $\mathfrak{h}$ is totally complex).

In this case, Theorem 5.1.23 tells us that $\rho(f, \mathfrak{h}, G)$ is irreducible. Then we assume $\rho(f, \mathfrak{h}, G) = \hat{\rho}_G(f_0)$ and show $f_0 \in G{\cdot}f$.

Lemma 7.1.2. *When G is an exponential solvable Lie group, $\mathfrak{d}$ is an ideal of $\mathfrak{e}$.*

Proof. Since $\mathfrak{d}$ and $\mathfrak{e}$ are mutually orthogonal regarding B_f,

$$0 = f([x,[y,z]]) = f([[x,y],z]) + f([y,[x,z]]) \; (x \in \mathfrak{d}, \; y,z \in \mathfrak{e}).$$

Therefore,

$$B_f(\mathrm{ad}(x)y,z) = -B_f(y,\mathrm{ad}(x)z). \tag{7.1.10}$$

We designate by $A(x)$ the linear transformation on $\mathfrak{e}/\mathfrak{d}$ induced by $\mathrm{ad}(x)$. As $x \in \mathfrak{h} \cap \bar{\mathfrak{h}}$, the linear transformation $\mathrm{ad}(x)$ extended on $\mathfrak{e}_{\mathbb{C}}$ keeps $\mathfrak{h}, \bar{\mathfrak{h}}$ stable. Hence $A(x)$ keeps $\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}, \bar{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}}$ stable and consequently is commutative with J. Then, as $y,z \in \mathfrak{e}/\mathfrak{d}$, Eq. (7.1.10) gives

$$\hat{B}_f(A(x)y, Jz) = -\hat{B}_f(y, A(x)Jz) = -\hat{B}_f(y, JA(x)z),$$

in other words,

$$S_f(A(x)y,z) = -S_f(y,A(x)z).$$

Namely, $A(x)$ is anti-symmetric with respect to the positive definite symmetric bilinear form S_f and its eigenvalues are purely imaginary, whereas, as $\mathfrak{d}$-module through A, $\mathfrak{e}/\mathfrak{d}$ is of exponential type. Finally, $A(x) = 0$ and $[\mathfrak{d},\mathfrak{e}] \subset \mathfrak{d}$. ■

Now let us return to the sequel of the proof of the theorem.

Corollary 7.1.3. *$\mathfrak{d}$ is an ideal of $\mathfrak{g}$ and $\dim \mathfrak{d} \leq 1$. Besides, let $\mathfrak{z}$ be the centre of $\mathfrak{g}$. Then, $\mathfrak{d} = \mathfrak{g}(f) = \mathfrak{z}$.*

Proof. As $\mathfrak{e} = \mathfrak{g}$, Lemma 7.1.2 means that $\mathfrak{d}$ is an ideal of $\mathfrak{g}$. We put $\mathfrak{b} = \mathfrak{d} \cap \ker f$. Since $[\mathfrak{g},\mathfrak{d}] \subset \mathfrak{d}$ and $f([\mathfrak{g},\mathfrak{d}]) = f([\mathfrak{e},\mathfrak{d}]) = \{0\}$, $[\mathfrak{g},\mathfrak{d}] \subset \mathfrak{b}$. Hence $\mathfrak{b}$ is an ideal of $\mathfrak{g}$ and $f(\mathfrak{b}) = \{0\}$. From the assumption of our case, $\mathfrak{b} = \{0\}$. In the sequel, $\dim \mathfrak{d} \leq 1$ and $\mathfrak{d} \subset \mathfrak{z}$. On the other hand, $\mathfrak{z} \subset \mathfrak{g}(f) \subset \mathfrak{d}$ is obvious and finally $\mathfrak{d} = \mathfrak{g}(f) = \mathfrak{z}$. ■

To continue the proof of the theorem, we need the following fact.

Theorem 7.1.4 (Invariance of Domain [72]). *Let U_1, U_2 be the subsets of $\mathbb{R}^m$ homeomorphic to each other. Provided that U_1 is an open set of $\mathbb{R}^m$, so is U_2.*

Let us continue the proof of the theorem. If $\dim \mathfrak{d} = 0$, $\mathfrak{g}(f) = \{0\}$ and $\dim(G\cdot f) = \dim(\mathfrak{g}^*) = n$. Taking Lemma 5.3.4 into account, the orbit $G\cdot f$ is homeomorphic to $\mathbb{R}^n$ and hence is an open set of $\mathfrak{g}^*$ by Theorem 7.1.4, whereas, because $\mathfrak{h}$ satisfies the Pukanszky condition, $G\cdot f = E\cdot f$ is a closed set of $\mathfrak{g}^*$. Thus $G\cdot f = \mathfrak{g}^*$ but this is impossible. Hence $\dim \mathfrak{d} = 1$.

From the preceding, $\mathfrak{d} = \mathfrak{g}(f) = \mathfrak{z} = \mathbb{R}Z$, $f(Z) \neq 0$ and

$$\dim(G\cdot f) = \dim \mathfrak{g} - \dim(\mathfrak{g}(f)) = n - 1. \tag{7.1.11}$$

Put $V = \{\ell \in \mathfrak{g}^*; \ell(Z) = f(Z)\}$. Since $Z \in \mathfrak{z}$, $(a\cdot f)(Z) = f(Z)$ for any $a \in G$, i.e. $G\cdot f \subset V$. By Lemma 5.3.4 and Eq. (7.1.11), the orbit $G\cdot f$ is homeomorphic to $\mathbb{R}^{n-1}$ and by Theorem 7.1.4 $G\cdot f$ is an open set of V. However, since $\mathfrak{h}$ satisfies the Pukanszky condition, $G\cdot f$ is a closed set of V. Finally,

$$G\cdot f = V. \tag{7.1.12}$$

Now if we take $\mathfrak{h}_0 \in I(f_0, \mathfrak{g})$, $\mathfrak{z} \subset \mathfrak{h}_0$. Let

$$R : \hat{\mathcal{H}}(f_0, \mathfrak{h}_0, G) \to \mathcal{H}(f, \mathfrak{h}, G)$$

be the intertwining operator between $\hat{L} = \hat{\rho}(f_0, \mathfrak{h}_0, G) = \hat{\rho}_G(f_0)$ and $L = \rho(f, \mathfrak{h}, G)$. Thus, for $\phi \in \hat{\mathcal{H}}(f_0, \mathfrak{h}_0, G)$ and $g \in G$,

$$\big(R\circ\hat{L}(g)\big)(\phi) = \big(L(g)\circ R\big)(\phi).$$

Fix $t_0 \in \mathbb{R}$ and put $g_0 = \exp(t_0 Z)$. Since g_0 is an central element of G,

$$\big(\hat{L}(g_0)\phi\big)(g) = \phi(\exp(-t_0 Z)g) = \phi(g\exp(-t_0 Z)) = e^{it_0 f_0(Z)}\phi(g)$$

for $g \in G$. Hence,

$$\big(R\circ\hat{L}(g_0)\big)(\phi) = e^{it_0 f_0(Z)} R(\phi).$$

On the other hand,

$$\begin{aligned}((L(g_0)\circ R)(\phi))\,(g) &= (R\phi)(\exp(-t_0 Z)g) \\ &= (R\phi)(g\exp(-t_0 Z)) = e^{it_0 f(Z)}(R\phi)(g).\end{aligned}$$

That is,

$$(L(g_0)\circ R)(\phi) = e^{it_0 f(Z)} R(\phi).$$

Taking all into consideration, $e^{it_0 f_0(Z)} = e^{it_0 f(Z)}$ for any $t_0 \in \mathbb{R}$. From this,

$$f_0(Z) = f(Z). \tag{7.1.13}$$

Finally, (7.1.12) and (7.1.13) conclude $f_0 \in G\cdot f$. ■

Now, following [28], we proceed to more detailed study of holomorphically induced representations for exponential solvable Lie groups.

7.2 Exponential j-Algebras

Definition 7.2.1. The triplet $(\mathfrak{g}, j, \beta)$ consisting of an exponential solvable Lie algebra $\mathfrak{g}$, a linear operator j and an alternating bilinear form β on $\mathfrak{g}$ is called an **exponential Kähler algebra** if it has the following properties:

(1) $j^2 = -1$,
(2) $[jX, jY] = j[jX, Y] + j[X, jY] + [X, Y]$,
(3) $\beta(jX, jY) = \beta(X, Y)$,
(4) $\beta(jX, X) > 0$ for $X \neq 0$,
(5) $\beta([X, Y], Z) + \beta([Y, Z], X) + \beta([Z, X], Y) = 0$.

If, in addition to these properties, there is a linear form $\omega \in \mathfrak{g}^*$ such that

$$\beta(X, Y) = \omega([X, Y]) \quad \text{for any } X, Y \in \mathfrak{g}$$

the triplet $(\mathfrak{g}, j, \omega)$ is called an exponential j-algebra. By abuse of language, we shall often say that $\mathfrak{g}$ is an **exponential j-algebra** or Kähler algebra.

Let $\mathcal{J}$ be a hermitian vector space of finite dimension, j the complex structure in $\mathcal{J}$ and β the imaginary part of the hermitian scalar product on $\mathcal{J}$.

Definition 7.2.2. A representation ρ of an exponential Kähler algebra $\mathfrak{g}$ by real linear transformations of a hermitian vector space $\mathcal{J}$ is called **symplectic** if it satisfies the following two conditions:

$$[\rho(jX) - j\rho(X), j] = 0, \tag{7.2.1}$$

$$\beta(\rho(X)x, y) + \beta(x, \rho(X)y) = 0 \tag{7.2.2}$$

for all $X \in \mathfrak{g},\ x, y \in \mathcal{J}$.

For the next lemma, the proof of Lemma 3 in Part II of Gindikin, Pjatetskii-Shapiro and Vinberg [40] remains valid.

Lemma 7.2.3. *Every symplectic representation ρ of exponential type of a commutative Kähler algebra $\mathfrak{g}$ is trivial, i.e. $\rho(X) = 0$ for $X \in \mathfrak{g}$.*

Lemma 7.2.4. *Let $\mathfrak{g}$ be a two-dimensional exponential Kähler algebra with basis $\{js, s\}$ satisfying the commutation law $[js, s] = s$. If ρ is a symplectic representation of exponential type of $\mathfrak{g}$ on a hermitian space $\mathcal{J}$, then the operator $L = \rho(js)$ is semi-simple and there is a direct sum decomposition*

$$\mathcal{J} = \mathcal{J}_- \oplus \mathcal{J}_0 \oplus \mathcal{J}_+$$

which has the following properties:

(1) *The subspaces $\mathcal{J}_-, \mathcal{J}_0$ and $\mathcal{J}_+$ are stable under L.*

(2) *The real parts of the eigenvalues of* L *on* $\mathcal{J}_-, \mathcal{J}_0$ *and* $\mathcal{J}_+$ *are equal to* $-\frac{1}{2}, 0$ *and* $\frac{1}{2}$ *respectively.*
(3) $j(\mathcal{J}_-) = \mathcal{J}_+,\ j(\mathcal{J}_0) = \mathcal{J}_0.$
(4) $L|_{\mathcal{J}_0} = 0$ *and* $\rho(s) = \begin{cases} j \text{ on } \mathcal{J}_-, \\ 0 \text{ on } \mathcal{J}_0 + \mathcal{J}_+. \end{cases}$

For later use we prove this by the same process employed in Proposition 5.14 of Rossi [67] for a normal symplectic representation.

Proof of Lemma 7.2.4. We proceed by induction on dim $\mathcal{J}$. Let $M = \rho(s)$. Since $[L, M] = LM - ML = M$, we conclude that M is singular, and L leaves ker$M$ invariant. The condition (7.2.1) implies that

$$M + jMj + jL - Lj = 0. \tag{7.2.3}$$

Suppose first that dim $\mathcal{J} = 2$.

Case 1. Assume that L has a real eigenvalue. Let e be a common eigenvector for L and M such that $Le = \alpha e,\ Me = 0,\ \alpha \in \mathbb{R}$. Now relative to the basis $\{e, je\}$, we can write

$$L = \begin{pmatrix} \alpha & a \\ 0 & a' \end{pmatrix},\quad M = \begin{pmatrix} 0 & b \\ 0 & b' \end{pmatrix}\quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

but $\beta(Le, je) + \beta(e, Lje) = 0$, so $a' = -\alpha$ and similarly $b' = 0$. By (7.2.3),

$$\begin{pmatrix} -a & 2\alpha + b \\ 2\alpha + b & a \end{pmatrix} = 0.$$

Thus $a = 0,\ b = -2\alpha$ and $[L, M] = M$ implies that $2\alpha b = b$. Therefore if $b = 0$, then $\alpha = 0$ and we have $\mathcal{J} = \mathcal{J}_0$. If $b \neq 0$, then $\alpha = \frac{1}{2}$, so $b = -1$, and we have $\mathcal{J} = \mathbb{R}je \oplus \mathbb{R}e = \mathcal{J}_- \oplus \mathcal{J}_+$.

Case 2. Assume that L has no real eigenvalue. In this case, $M = 0$ and $Lj = jL$. Let e be a nonzero element. Then relative to the basis $\{e, je\}$,

$$L = \lambda \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \text{ and } j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\alpha, \lambda \in \mathbb{R},\ \alpha\lambda \neq 0$. Since $\beta(Le, je) + \beta(e, Lje) = 0$, we have $2\lambda\beta(e, je) = 0$ which is a contradiction.

Next we assume that Lemma 7.2.4 is valid for symplectic representations of exponential type of $\mathfrak{g}$ on hermitian spaces of lower dimension than $\mathcal{J}$.

Case 1. Suppose that $L|_{\ker M}$ has a real eigenvalue.
Choose an eigenvector e of $L|_{\ker M}$ such that $Le = \alpha e$ ($\alpha \in \mathbb{R}$) and $\beta(je, e) = 1$. Let W be the annihilator of $\mathbb{R}je \oplus \mathbb{R}e$ with respect to β. Then W is invariant under j and, since

$$\beta(Lx, e) = -\beta(x, Le) = 0 = -\beta(x, Me) = \beta(Mx, e)$$

for $x \in W$, $W \oplus \mathbb{R}e$ is invariant under both L and M. For $x \in W$, write

$$Lx = L_W x + l(x)e \text{ and } Mx = M_W x + m(x)e$$

with $L_W x, M_W x \in W$ and $l(x), m(x) \in \mathbb{R}$. Then relation (7.2.3) becomes

$$M_W + jM_W j + jL_W - L_W j = 0,$$

and $[L, M] = M$ implies $[L_W, M_W] = M_W$. Further, since

$$L|_{W \oplus \mathbb{R}e} = \begin{pmatrix} L_W & 0 \\ l & \alpha \end{pmatrix} \text{ and } M|_{W \oplus \mathbb{R}e} = \begin{pmatrix} M_W & 0 \\ m & 0 \end{pmatrix},$$

both L_W and M_W have no nonzero purely imaginary eigenvalue. So the representation ρ_W of $\mathfrak{g}$ on W defined by $\rho_W(js) = L_W$ and $\rho_W(s) = M_W$ is of exponential type. Furthermore, ρ_W is clearly symplectic. Thus the induction hypothesis gives the splitting

$$W = W_- \oplus W_0 \oplus W_+.$$

Therefore, if one can show that L and M leave $\mathbb{R}je \oplus \mathbb{R}e$ invariant, we shall have one of the two possibilities:

$$\mathcal{J}_- = W_-, \ \ \mathcal{J}_0 = W_0 \oplus \mathbb{R}je \oplus \mathbb{R}e, \ \ \mathcal{J}_+ = W_+$$

or

$$\mathcal{J}_- = W_- \oplus \mathbb{R}je, \ \ \mathcal{J}_0 = W_0, \ \ \mathcal{J}_+ = W_+ \oplus \mathbb{R}e.$$

Since $\beta(e, Lje) = -\beta(Le, je) = \alpha$, we can write

$$Lje = w - \alpha je + ae \text{ and } Mje = w' + a'e$$

with $w, w' \in W,\ a, a' \in \mathbb{R}$. Now

$$0 = (M + jMj + jL - Lj)e = jw' + a'je + \alpha je + \alpha je - w - ae,$$

so $a = 0, a' = -2\alpha$ and $w = jw'$. Since $\beta(Lx, je) + \beta(x, Lje) = 0$ for $x \in W$, $l(x) = \beta(x, w)$ and similarly $m(x) = \beta(x, w')$. From the equality $[L, M]je = Mje$, we deduce that

$$2\alpha(2\alpha - 1) = \beta(w', w) - \beta(w, w') = 2\beta(w', jw') \leq 0,$$

i.e. $0 \leq \alpha \leq 1/2$ and $(1 - \alpha)w' = L_W w' - M_W w$. But we can write $w' = w'_- + w'_0 + w'_+$, so $jw' = jw'_- + jw'_0 + jw'_+$, and thus we get

$$(1 - \alpha)w'_- = L_W w'_-, \ \ (1 - \alpha)w'_0 = 0, \ \ -\alpha w'_+ = L_W w'_+.$$

These equalities together with the inequality $0 \leq \alpha \leq 1/2$ imply that $w'_- = w'_0 = w'_+ = 0$, i.e. $w = w' = 0$.

Case 2. Suppose that $L|_{\ker M}$ has no real eigenvalue.

Let e_1, e_2 be two linearly independent vectors in $\ker M$ such that the subspace $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ is L-invariant, $\beta(je_1, e_1) = 1$ and that

$$L|_V = \lambda \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}$$

with $\lambda, \alpha \in \mathbb{R}$ and $\lambda\alpha \neq 0$, relative to the basis $\{e_1, e_2\}$. Since $\beta(Le_1, e_2) + \beta(e_1, Le_2) = 0, \beta(e_1, e_2) = \beta(je_1, je_2) = 0$. We set $\beta(je_2, e_2) = k(> 0)$ and $\beta(e_1, je_2) = \beta(e_2, je_1) = \mu$, then

$$0 \leq \mu^2 < k. \tag{7.2.4}$$

Since $\beta(e_1, e_2) = 0$, $je_1 \notin V$. Let W be the annihilator of $\mathbb{R}je_1 \oplus \mathbb{R}je_2 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with respect to β. Then W is invariant under j and, since

$$\beta(Lx, e_1) = \beta(Lx, e_2) = \beta(Mx, e_1) = \beta(Mx, e_2) = 0$$

for $x \in W$, $W \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2$ is invariant under both L and M. For $x \in W$, we write

$$Lx = L_W x + l_1(x)e_1 + l_2(x)e_2 \text{ and } Mx = M_W x + m_1(x)e_1 + m_2(x)e_2$$

with $L_W x, M_W x \in W$ and $l_i(x), m_i(x) \in \mathbb{R} (i = 1, 2)$. By (7.2.3), we have

$$M_W + jM_W j + jL_W - L_W j = 0,$$

and $[L, M] = M$ implies $[L_W, M_W] = M_W$. Further, since

$$L|_{W \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2} = \begin{pmatrix} L_W & 0 \\ * & \lambda \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \end{pmatrix}$$

and

$$M|_{W\oplus\mathbb{R}e_1\oplus\mathbb{R}e_2} = \begin{pmatrix} M_W & 0 \\ * & 0 \end{pmatrix},$$

we can define a symplectic representation ρ_W of exponential type of $\mathfrak{g}$ just as in Case 1 above. Thus the induction hypothesis gives the splitting $W = W_- \oplus W_0 \oplus W_+$ and L_W is semi-simple. Write

$$Lje_1 = w_1 + aje_1 + bje_2 + pe_1 + qe_2,\ Lje_2 = w_2 + cje_1 + dje_2 + re_1 + se_2$$

and

$$Mje_1 = w_1' + \xi e_1 + \gamma e_2,\ Mje_2 = w_2' + \delta e_1 + \zeta e_2$$

with $w_i, w_i' \in W$ $(i = 1, 2)$ and $a, b, c, d, p, q, r, s, \xi, \gamma, \delta, \zeta \in \mathbb{R}$. Now, $\beta(e_1, Lje_1) = -\beta(Le_1, je_1)$ implies $-a + b\mu = \lambda(1 + \alpha\mu)$. Similarly, we have

$$a\mu - kb = \lambda(\alpha - \mu),\ -c + d\mu = -\lambda(\mu + \alpha k), c\mu - kd = \lambda(k - \alpha\mu).$$

Thus,

$$a = \lambda\left(\frac{\alpha\mu(1+k)}{\mu^2 - k} - 1\right), b = \lambda\alpha\left(1 + \frac{1+k}{\mu^2 - k}\right),$$
$$c = -\alpha\lambda\left(1 + \frac{k(1+k)}{\mu^2 - k}\right), d = -\lambda\left(1 + \frac{\alpha\mu(1+k)}{\mu^2 - k}\right).$$

From the relation $0 = (M + jMj + jL - Lj)e_1$, we deduce that

$$jw_1' = w_1, \xi + \lambda = a, \gamma - \alpha\lambda = b, p = q = 0$$

and

$$jw_2' = w_2, \delta + \lambda\alpha = c, \zeta + \lambda = d, r = s = 0.$$

Therefore,

$$\xi = \lambda\left(\frac{\alpha\mu(1+k)}{\mu^2 - k} - 2\right), \gamma = \alpha\lambda\left(2 + \frac{1+k}{\mu^2 - k}\right),$$
$$\delta = -\alpha\lambda\left(2 + \frac{k(1+k)}{\mu^2 - k}\right), \zeta = -\lambda\left(2 + \frac{\alpha\mu(1+k)}{\mu^2 - k}\right).$$

Since $\beta(Lje_1, x) + \beta(je_1, Lx) = 0$ ($x \in W$), we have

$$\beta(x, w_1) - l_1(x) + \mu l_2(x) = 0.$$

Similarly,

$$\beta(x, w_2) + \mu l_1(x) - k l_2(x) = 0,$$
$$\beta(x, w_1') - m_1(x) + \mu m_2(x) = 0,$$
$$\beta(x, w_2') + \mu m_1(x) - k m_2(x) = 0.$$

From the equation $[L, M]je_1 = Mje_1$,

$$L_W w_1' - M_W w_1 = (1+\alpha)w_1' + b w_2', \tag{7.2.5}$$

$$\begin{aligned} l_1(w_1') - m_1(w_1) &= \xi(1 + a - \lambda) + b\delta - \gamma\lambda\alpha, \\ l_2(w_1') - m_2(w_1) &= \gamma(1 + a - \lambda) + b\zeta + \xi\lambda\alpha. \end{aligned} \tag{7.2.6}$$

Likewise,

$$L_W w_2' - M_W w_2 = c w_1' + (1+d)w_2', \tag{7.2.7}$$

$$\begin{aligned} l_1(w_2') - m_1(w_2) &= (1 + d - \lambda)\delta + c\xi - \zeta\lambda\alpha, \\ l_2(w_2') - m_2(w_2) &= (1 + d - \lambda)\zeta + c\gamma + \delta\lambda\alpha. \end{aligned} \tag{7.2.8}$$

From these relations, we obtain

$$(\xi - \mu\gamma)(1 + a - \lambda) + b(\delta - \mu\zeta) - \lambda\alpha(\gamma + \mu\xi) = 2\beta(w_1', jw_1') \le 0,$$
$$(k\zeta - \mu\delta)(1 + d - \lambda) + c(k\gamma - \mu\xi) + \lambda\alpha(k\delta + \mu\zeta) = 2\beta(w_2', jw_2') \le 0.$$

Therefore,

$$2\lambda(2\lambda - 1) + 2\alpha\lambda(4\mu\lambda - 2\alpha\lambda - \mu) - 2\alpha^2\lambda^2(1+k) + \frac{2\alpha^2\lambda^2(1+k)^2}{k - \mu^2} \le 0, \tag{7.2.9}$$

$$2k\lambda(2\lambda - 1) + 2\alpha\lambda(4\mu\lambda + 2k\alpha\lambda - \mu) - 2\alpha^2\lambda^2(1+k) + \frac{2\alpha^2\lambda^2 k(1+k)^2}{k - \mu^2} \le 0. \tag{7.2.10}$$

Summing up these inequalities, we get

$$2\lambda(2\lambda - 1)(1+k) - 8\alpha^2\lambda^2(1+k) + \frac{2\alpha^2\lambda^2(1+k)^2}{k - \mu^2} \le 0,$$

so

$$\lambda(2\lambda - 1) + \frac{\alpha^2\lambda^2\left((1-k)^2 + 4\mu^2\right)}{k - \mu^2} \leq 0. \tag{7.2.11}$$

On the other hand, we can write

$$w_1' = w_1^- + w_1^0 + w_1^+,\ \ w_2' = w_2^- + w_2^0 + w_2^+,$$

so the relations (7.2.5), (7.2.7) and $jw_1' = w_1,\ jw_2' = w_2$ imply that

$$L_W w_1^- = (1+a)w_1^- + bw_2^-, L_W w_1^+ = aw_1^+ + bw_2^+$$
$$L_W w_2^- = cw_1^- + (1+d)w_2^-, L_W w_2^+ = cw_1^+ + dw_2^+.$$

We set $P^- = \mathbb{R}w_1^- + \mathbb{R}w_2^-$ and $P^+ = \mathbb{R}w_1^+ + \mathbb{R}w_2^+$, then P^- and P^+ are both invariant under L_W.

We first suppose that $\dim(P^-) = 2$ (resp. $\dim(P^+) = 2$) and let x be an eigenvalue of $L_W|_{P^-}$ (resp. $L_W|_{P^+}$). Then x satisfies the equation

$$\begin{vmatrix} 1-x+a & c \\ b & 1-x+d \end{vmatrix} = (1-x-\lambda)^2 + \lambda^2\alpha^2 = 0$$
$$\left(\text{resp.} \begin{vmatrix} -x+a & c \\ b & -x+d \end{vmatrix} = (x+\lambda)^2 + \lambda^2\alpha^2 = 0\right).$$

Thus $x = 1 - \lambda \pm i\lambda\alpha$ (resp. $x = -\lambda \pm i\lambda\alpha$). This equality together with the inequality $0 \leq \lambda \leq 1/2$, which comes from the inequalities (7.2.4) and (7.2.11), implies that the real part of x is not equal to $-1/2$ (resp. $1/2$). Thus we have a contradiction.

Next we suppose that $w_2^- = tw_1^-$ or $w_2^+ = tw_1^+$ with $0 \neq t \in \mathbb{R}$. Then,

$$(1+\mu^2)t^2 + 2(1+k)\mu t + k^2 + \mu^2 = 0.$$

Since

$$\mu^2(1+k)^2 - (1+\mu^2)(k^2+\mu^2) = 2k\mu^2 - k^2 - \mu^4 = -(k-\mu^2)^2 < 0,$$

we conclude that $t \notin \mathbb{R}$. This is a contradiction.

Therefore $P^- = P^+ = \{0\}$. Since

$$\begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1-\lambda)^2 + \lambda^2\alpha^2,$$

we have $w_1^0 = w_2^0 = 0$ by (7.2.4), (7.2.11) and $\lambda\alpha \neq 0$. In consequence, $w_1' = w_2' = 0$ so $w_1 = jw_1' = 0$ and $w_2 = jw_2' = 0$. Therefore the inequalities (7.2.9)–(7.2.11) are replaced by corresponding equalities. By (7.2.9) and (7.2.10),

$$2\alpha\lambda\,(4\mu\lambda(k+1)-\mu(k+1))-2\alpha^2\lambda^2(1+k)(k-1)=0,$$

hence we have

$$k-1=\frac{\mu}{\alpha\lambda}(4\lambda-1). \tag{7.2.12}$$

On the other hand, by (7.2.6),

$$\frac{(1+k)(1-4\lambda+2\lambda\alpha\mu)}{\mu^2-k}+2-8\lambda=0.$$

By (7.2.8),

$$\frac{(1+k)(4\lambda k-k+2\lambda\alpha\mu)}{\mu^2-k}+8\lambda-2=0,$$

which is added to the above equality to get

$$(4\lambda-1)(k-1)+4\lambda\alpha\mu=0. \tag{7.2.13}$$

By (7.2.12) and (7.2.13), $\mu\left((4\lambda-1)^2+4\alpha^2\lambda^2\right)=0$. Thus $\mu=0$. By (7.2.11) and (7.2.12), $k=1$ and $\lambda=1/2$. Therefore $l_i(x)=m_i(x)=0$ $(i=1,2)$ for all $x\in W$, and W is invariant under both L and M. Further,

$$Lje_1=-\frac{1}{2}(je_1+\alpha je_2),\quad Lje_2=-\frac{1}{2}(je_2-\alpha je_1)$$

and $Mje_1=-e_1,\ Mje_2=-e_2$. Thus we have the splitting

$$\mathcal{J}=\mathcal{J}_-\oplus\mathcal{J}_0\oplus\mathcal{J}_+,$$

where

$$\mathcal{J}_-=W_-\oplus jV=W\oplus\mathbb{R}je_1\oplus\mathbb{R}je_2,\mathcal{J}_0=W_0,\mathcal{J}_+=W_+\oplus V=W_+\oplus\mathbb{R}e_1\oplus\mathbb{R}e_2.$$

∎

We generalize I. I. Pjatetskii-Shapiro's structure theorem for a normal j-algebra (cf. Theorem 2 at Section 3 in Chapter 2 of [60], Theorem 5.13 of [67]) to an exponential j-algebra.

Theorem 7.2.5. *Let $(\mathfrak{g},j,\omega)$ be an exponential j-algebra. We define an inner product S on $\mathfrak{g}$ by $S(X,Y)=\omega([jX,Y])$ for $X,Y\in\mathfrak{g}$. Let $\mathfrak{a}$ be the orthogonal complement of $\eta=[\mathfrak{g},\mathfrak{g}]$ with respect to the form S. Then $\mathfrak{a}$ is a commutative Lie subalgebra of $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{a}+\eta$, and the adjoint representation of $\mathfrak{a}$ on η is complex diagonalizable. For $\alpha\in\mathfrak{a}^*$, we set*

$$\eta^{\alpha} = \{X \in \eta; [A, X] = \alpha(A)X \ \forall\, A \in \mathfrak{a}\}$$

and let η^{α_i}, $1 \le i \le r$, *be those root spaces* η^{α} *for which* $j(\eta^{\alpha}) \subset \mathfrak{a}$. *Then* $dim(\eta^{\alpha_i}) = 1$ $(1 \le i \le r)$ *and* $r = dim\, \mathfrak{a}$ (r *is called the rank of* $\mathfrak{g}$). *If we order* $\alpha_1, \ldots, \alpha_r$ *in an appropriate way, then all other roots are of the form*

$$\frac{1}{2}(\alpha_m - \alpha_k),\ \ \frac{1}{2}(\alpha_m + \alpha_k) \qquad\qquad (1 \le k < m \le r),$$
$$\frac{1}{2}\alpha_k\ \ (1 \le k \le r)$$

(*not all possibilities need occur*), *and* η *can be decomposed as follows:*

$$\eta = \sum_{k<m} \eta^{\frac{1}{2}(\alpha_m - \alpha_k)} \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \sum_{k \le m} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)},$$

where $\mathfrak{g}_{\frac{1}{2}} = \sum_k \tilde{\eta}^{\frac{1}{2}\alpha_k}$ *and* $\tilde{\eta}^{\frac{1}{2}\alpha_k}$ *is an* $(ad\, \mathfrak{a})$-*invariant subspace, the complexification of which is the sum of the root spaces of* $ad\mathfrak{a}$ *with the roots of the form*

$$A \mapsto \frac{1}{2}\alpha_k(A)(1 + i\beta_{k,p})\ (A \in \mathfrak{a})$$

with $\beta_{k,p} \in \mathbb{R}$. *Let*

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \sum_{k<m} \eta^{\frac{1}{2}(\alpha_m - \alpha_k)} \text{ and } \mathfrak{g}_1 = \sum_{k \le m} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}.$$

Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1$$

and $[\mathfrak{g}_i, \mathfrak{g}_k] \subset \mathfrak{g}_{i+k}$ *with the convention that* $\mathfrak{g}_{i+k} = \{0\}$ *if* $i + k \ne 0,\ 1/2$ *or* 1. *We have*

$$j\left(\eta^{\frac{1}{2}(\alpha_m - \alpha_k)}\right) = \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\ (1 \le k < m \le r),\ j\left(\tilde{\eta}^{\frac{1}{2}\alpha_k}\right) = \tilde{\eta}^{\frac{1}{2}\alpha_k}\ (1 \le k \le r).$$

Let U_i *be the nonzero element of* η^{α_i} *such that* $[jU_i, U_i] = U_i$ *and let* $s = \sum_{i=1}^{r} U_i$. *Then*

$$\alpha_k(jU_i) = \delta_{ik},\ adjs|_{\mathfrak{g}_0} = 0,\ adjs|_{\mathfrak{g}_1} = Id$$

and $adjs|_{\mathfrak{g}_{\frac{1}{2}}}$ *is semi-simple and its eigenvalues have the real part* $1/2$. *Finally,* $jX = [s, X]$ *for* $X \in \mathfrak{g}_0$.

Proof. We proceed by induction on dim $\mathfrak{g}$. Let $\mathbb{R}U$ be an ideal of dimension 1, which certainly exists by Lemma 1 in Sect. 2 of [61]. Then, since $\omega([jU, U]) > 0$, we have $[jU, U] \neq 0$, so we can choose U in such a way that $[jU, U] = U$. This completes the initial step of our induction, where dim $\mathfrak{g} = 2$.

Let dim $\mathfrak{g} > 2$ and W the orthogonal complement of $\mathbb{R}jU \oplus \mathbb{R}U$ with respect to S. Then W is invariant under both j and $\mathrm{ad}\,jU$ and $L = \mathrm{ad}_W\,jU = (\mathrm{ad}\,jU)|_W$ commutes with j. We regard L as a complex linear transformation on the complex space W. Then the transformation L is semi-simple and the eigenvalues have the form either $\frac{1}{2} + i\lambda_1$ or $i\lambda_2$, where $\lambda_1,\ \lambda_2$ are real numbers. We denote by $W_{\frac{1}{2}}$ the subspace spanned by the eigenvectors for the eigenvalues of the form $\frac{1}{2} + i\lambda_1$ ($\lambda_1 \in \mathbb{R}$) and by W_0 for the eigenvalues of the form $i\lambda_2$ ($\lambda_2 \in \mathbb{R}$). Then W_0, $W_{\frac{1}{2}}$ are clearly j-invariant and

$$W = W_0 \oplus W_{\frac{1}{2}},\ [W_0, W_0] \subset W_0,\ [W_0, W_{\frac{1}{2}}] \subset W_{\frac{1}{2}},\ [W_{\frac{1}{2}}, W_{\frac{1}{2}}] \subset \mathbb{R}U$$

(cf. Theorem 1 of [61]). In our case, $L|_{W_0} = 0$ by the assumption that $\mathfrak{g}$ is an exponential algebra. Since $(W_0, j|_{W_0}, \omega|_{W_0})$ is an exponential j-algebra, the induction hypothesis allows us to write W_0 as

$$W_0 = \mathfrak{a}_{W_0} \oplus \sum_{1\le k<m\le r-1} {\eta_0}^{\frac{1}{2}(\tilde{\alpha}_m - \tilde{\alpha}_k)} \oplus \sum_{1\le k\le r-1} \tilde{\eta}_0^{\frac{1}{2}\tilde{\alpha}_k} \oplus \sum_{1\le k\le m\le r-1} {\eta_0}^{\frac{1}{2}(\tilde{\alpha}_m \oplus \tilde{\alpha}_k)}$$

with $\tilde{\alpha}_k \in \mathfrak{a}^*_{W_0}$ for $1 \le k \le r-1$ and $\eta_0 = [W_0, W_0]$. Let $X \in W_0$ and let $Y \in W_{\frac{1}{2}}$ be an eigenvector of L corresponding to the eigenvalue $\frac{1}{2} + i\lambda$ ($\lambda \in \mathbb{R}$). Then

$$[jU, Y] = \left(\frac{1}{2} + i\lambda\right) Y,\ \text{i.e.}\ [jU, Y] = \frac{1}{2}Y + \lambda jY.$$

Hence $[jU, jY] = \frac{1}{2}jY - \lambda Y$ and

$$\frac{1}{2}[jU, Y] - \lambda[jU, jY] = \left(\frac{1}{4} + \lambda^2\right) Y.$$

Thus $Y = [jU, Y_0]$ with

$$Y_0 = \frac{4}{(1 + 4\lambda^2)} \left(\frac{1}{2}Y - \lambda jY\right) \in W_{\frac{1}{2}},$$

so

$$\begin{aligned} S(X, Y) = \omega([jX, Y]) &= \omega([jX, [jU, Y_0]]) \\ &= \omega([jU, [jX, Y_0]]) = S(U, [jX, Y_0]). \end{aligned}$$

Since $[jX, Y_0] \in W_{\frac{1}{2}}$, $S(X, Y) = 0$. Thus, the orthogonal complement $\mathfrak{a}$ of

$$\eta = [\mathfrak{g}, \mathfrak{g}] = \eta_0 \oplus W_{\frac{1}{2}} \oplus \mathbb{R}U$$

with respect to S is $\mathfrak{a}_{W_0} \oplus \mathbb{R}jU$, which clearly is a commutative subalgebra of $\mathfrak{g}$. Define $\alpha_k \in \mathfrak{a}^*$ by $\alpha_k|_{\mathfrak{a}_{W_0}} = \tilde{\alpha}_k$ and $\alpha_k(jU) = 0$ for $1 \leq k \leq r-1$, and let $\alpha_r \in \mathfrak{a}^*$ be such that $\alpha_r|_{\mathfrak{a}_{W_0}} = 0$ and $\alpha_r(jU) = 1$. Then it is easy to see that the spaces

$${\eta_0}^{\frac{1}{2}(\tilde{\alpha}_m - \tilde{\alpha}_k)},\ \tilde{\eta}_0^{\frac{1}{2}\tilde{\alpha}_k},\ {\eta_0}^{\tilde{\alpha}_k},\ {\eta_0}^{\frac{1}{2}(\tilde{\alpha}_m + \tilde{\alpha}_k)}$$

are the same as

$$\eta^{\frac{1}{2}(\alpha_m - \alpha_k)},\ \tilde{\eta}^{\frac{1}{2}\alpha_k},\ \eta^{\alpha_k},\ \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}$$

respectively, so

$$\begin{aligned} \mathfrak{g} = \mathfrak{a} \oplus & \sum_{1 \leq k < m \leq r-1} \eta^{\frac{1}{2}(\alpha_m - \alpha_k)} \oplus \sum_{1 \leq k \leq r-1} \tilde{\eta}^{\frac{1}{2}\alpha_k} \\ & \oplus \sum_{1 \leq k \leq m \leq r-1} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)} \oplus W_{\frac{1}{2}} \oplus \mathbb{R}U. \end{aligned}$$

Now on the finite-dimensional vector space $W_{\frac{1}{2}}$ over $\mathbb{R}$ with the complex structure $j|_{W_{\frac{1}{2}}}$, the bilinear form β is determined by $\beta(X, Y) = \omega([X, Y])$ $(X, Y \in W_{\frac{1}{2}})$, and the symplectic representation ρ_k $(1 \leq k \leq r-1)$ of the exponential Kähler algebra $\mathfrak{g}_k = \mathbb{R}jU_k + \mathbb{R}U_k$ defined by $\rho_k(jU_k) = L_k = \mathrm{ad}_{W_{\frac{1}{2}}} jU_k$ and $\rho_k(U_k) = M_k = \mathrm{ad}_{W_{\frac{1}{2}}} U_k$ is of exponential type. Therefore we can write

$$W_{\frac{1}{2}} = W_-^k \oplus W_0^k \oplus W_+^k$$

with the properties stated in Lemma 7.2.4. For $k \neq l$, we have

$$L_k|_{W_-^l \oplus W_+^l} = 0 = M_k|_{W_-^l \oplus W_+^l}.$$

In fact, since L_k and L_l are both semi-simple and commute mutually, W_-^l is invariant under L_k, and the transformations $L_k|_{(W_-^l)_{\mathbb{C}}}$ and $L_l|_{(W_-^l)_{\mathbb{C}}}$ are simultaneously diagonalizable. Thus, if we suppose $L_k|_{W_-^l} \neq 0$, we shall have one of the following two possibilities:

(1) There is a nonzero vector $X \in W_-^l$ such that $L_l X = -\frac{1}{2}X$ and $L_k X = \pm\frac{1}{2}X$. But we can apply Lemma 7.2.4 to the symplectic representation $\rho_{k,l}$ of the exponential Kähler algebra

$$\mathfrak{g}_{k,l} = \mathbb{R}(jU_k + jU_l) + \mathbb{R}(U_k + U_l)$$

defined by

$$\rho_{k,l}(jU_k + jU_l) = L_k + L_l \ \text{ and } \ \rho_{k,l}(U_k + U_l) = M_k + M_l.$$

Hence if $L_k X = -\frac{1}{2}X$, we should have -1 as an eigenvalue for $L_k + L_l$, which is impossible. If $L_k X = \frac{1}{2}X$, then $M_k X = 0$ so that $(L_k + L_l)X = 0$ with $(M_k + M_l)X = jX$, which is also impossible.

(2) There are two nonzero vectors $X_1, X_2 \in W_-^l$ such that

$$L_l X_1 = -\frac{1}{2}(X_1 - \alpha X_2),\ L_l X_2 = -\frac{1}{2}(X_2 + \alpha X_1)$$

and

$$L_k X_1 = \lambda(X_1 - \alpha X_2),\ L_k X_2 = \lambda(X_2 + \alpha X_1)$$

with $\lambda = \pm\frac{1}{2},\ 0 \neq \alpha \in \mathbb{R}$. As in case (1), we have a contradiction.

Similarly we can show $L_k|_{W_+^l} = 0$. Thus $M_k|_{W_-^l \oplus W_+^l} = 0$. Next, the spaces $W_\pm^k$ are L-invariant for $1 \leq k \leq r-1$. We show

$$L|_{W_-^k \oplus W_+^k} = \frac{1}{2}Id,\ L_k|W_-^k = -\frac{1}{2}Id,\ L_k|_{W_+^k} = \frac{1}{2}Id.$$

It suffices to show that $L|_{W_+^k} = \frac{1}{2}Id$ and $L_k|_{W_+^k} = \frac{1}{2}Id$. In fact, that $j \circ L = L \circ j$ on W implies $L|_{W_-^k} = \frac{1}{2}Id$, and that $[L_k, M_k]X = M_k X$ for $X \in W_-^k$ implies $jL_k X = -\frac{1}{2}jX$, so $L_k|_{W_-^k} = -\frac{1}{2}Id$. We suppose that $L|_{W_+^k} \neq \frac{1}{2}Id$ or $L_k|_{W_+^k} \neq \frac{1}{2}Id$. Then there are nonzero vectors $X_1, X_2 \in W_+^k$ such that

$$LX_1 = L_k X_1 = \frac{1}{2}(X_1 - \alpha X_2),\ LX_2 = L_k X_2 = \frac{1}{2}(X_2 + \alpha X_1)$$

with $0 \neq \alpha \in \mathbb{R}$. Then, since $L \circ j = j \circ L$, on W,

$$LjX_1 = \frac{1}{2}(jX_1 - \alpha jX_2),\ LjX_2 = \frac{1}{2}(jX_2 + \alpha jX_1).$$

But by the proof of Lemma 7.2.4,

$$L_k jX_1 = -\frac{1}{2}(jX_1 + \alpha jX_2),\ L_k jX_2 = -\frac{1}{2}(jX_2 - \alpha jX_1).$$

If we set $[jX_1, jX_2] = tU$ with $t \in \mathbb{R}$, then $[jU_k, [jX_1, jX_2]] = t[jU_k, U]$ and hence $[jX_1, jX_2] = 0$. Furthermore, we have

$$[j(U + U_k), jX_1] = -\alpha jX_2,\ [j(U + U_k), jX_2] = \alpha jX_1,$$

which contradicts the assumption that $\mathfrak{g}$ is an exponential algebra (cf. Theorem 1 of [69], I). Therefore it is clear that $W_-^k = W_{\frac{1}{2}}^{-\frac{1}{2}\tilde{\alpha}_k}$ is the $-\frac{1}{2}\tilde{\alpha}_k$-weight space of $W_{\frac{1}{2}}$ with respect to $\mathrm{ad}\mathfrak{a}_{W_0}$. Also $W_+^k = W_{\frac{1}{2}}^{\frac{1}{2}\tilde{\alpha}_k}$, so that

$$W_{\frac{1}{2}} = \sum_{1 \le k \le r-1} W_{\frac{1}{2}}^{-\frac{1}{2}\tilde{\alpha}_k} \oplus \sum_{1 \le k \le r-1} W_{\frac{1}{2}}^{\frac{1}{2}\tilde{\alpha}_k} \oplus W_{\frac{1}{2}}^0,$$

where

$$W_{\frac{1}{2}}^0 = \{X \in W_{\frac{1}{2}};\, [A, X] = 0\ \forall\ A \in \mathfrak{a}_{W_0}\}.$$

Taking $L|_{W_-^k \oplus W_+^k} = \frac{1}{2}Id$ into account, we have, relative to the new system $\{\alpha_1, \ldots, \alpha_r\}$ of roots,

$$W_{\frac{1}{2}} = \sum_{1 \le k \le r-1} \eta^{\frac{1}{2}(\alpha_r - \alpha_k)} \oplus \sum_{1 \le k \le r-1} \eta^{\frac{1}{2}(\alpha_r + \alpha_k)} + \tilde{\eta}^{\frac{1}{2}\alpha_r}.$$

These facts with the observation that $j(W_+^k) = W_-^k,\ j(W_0^k) = W_0^k$ by Lemma 7.2.4 prove Theorem 7.2.5 except for the last assertion with $\mathbb{R}U = \eta^{\alpha_r},\ U = U_r$. Since $\eta^{\frac{1}{2}(\alpha_r - \alpha_k)} \subset W$, the relation $X_{r,k} \in \eta^{\frac{1}{2}(\alpha_r - \alpha_k)}$ implies

$$[U_r, X_{r,k}] = 0,\ [U_k, X_{r,k}] = [M_k, X_{r,k}] = jX_{r,k}.$$

So, if $X \in \mathfrak{g}_0$ is written in the form

$$X = X' + \lambda jU_r + \sum_{1 \le k \le r-1} X_{r,k}$$

with $X' \in W_0,\ \lambda \in \mathbb{R}$ and $X_{r,k} \in \eta^{\frac{1}{2}(\alpha_r - \alpha_k)}$, then

$$\begin{aligned} [s, X] &= \sum_{k=1}^{r-1} \left[U_k + U_r, X' + \lambda jU_r + \sum_{1 \le k \le r-1} X_{r,k} \right] \\ &= jX' + \left[\sum_{k=1}^{r-1} U_k, \sum_{1 \le k \le r-1} X_{r,k} \right] - \lambda U_r = jX. \end{aligned}$$

∎

7.3 Exponential Kähler Algebras

We generalize the fundamental theorem for a normal Kähler algebra in Part III of [40]. Let $(\mathfrak{g}, j, \beta)$ be an exponential Kähler algebra. A j-invariant Lie subalgebra (resp. ideal) of $\mathfrak{g}$ is called a Kähler subalgebra (resp. ideal).

Lemma 7.3.1. *Let $\mathfrak{g}$ be an exponential Kähler algebra and let $\mathfrak{a}$ be a minimal ideal of $\mathfrak{g}$ such that dim $\mathfrak{a} = 2$. Then $\mathfrak{k} = j\mathfrak{a} + \mathfrak{a}$ is a commutative Kähler algebra and dim $\mathfrak{k} = 4$.*

Proof. Since $\mathfrak{g}$ is an exponential solvable Lie algebra, there is a basis $\{e_1, e_2\}$ of $\mathfrak{a}$ such that

$$[x, e_1] = \lambda(x)(e_1 - \alpha e_2),\ [x, e_2] = \lambda(x)(e_2 + \alpha e_1)\ (\forall\, x \in \mathfrak{g})$$

with $\lambda \in \mathfrak{g}^*$ and $0 \neq \alpha \in \mathbb{R}$. This implies that $[e_1, e_2] = 0$. Since $\mathfrak{a}$ is a minimal ideal, there is an element $x_0 \in \mathfrak{g}$ such that $\lambda(x_0) = 1$. Then,

$$0 = \beta(x_0, [e_1, e_2]) = \beta([x_0, e_1], e_2) + \beta(e_1, [x_0, e_2]) = 2\beta(e_1, e_2).$$

It follows that dim $\mathfrak{k} = 4$. Let $\lambda_1 = \lambda(je_1)$ and $\lambda_2 = \lambda(je_2)$. Then

$$[je_1, je_2] = j[je_1, e_2] + j[e_1, je_2] = (\lambda_1\alpha - \lambda_2)je_1 + (\lambda_1 + \alpha\lambda_2)je_2.$$

We apply ade_1 to both sides of this equation to get

$$0 = \alpha(\lambda_1^2 + \lambda_2^2)(e_1 - \alpha e_2).$$

Therefore $\lambda_1 = \lambda_2 = 0$, $[je_1, je_2] = 0$, which implies that $\mathfrak{k}$ is a commutative Kähler algebra. ∎

Theorem 7.3.2. *Let $\mathfrak{g}$ be an exponential Kähler algebra, then $\mathfrak{g}$ can be decomposed into a semi-direct sum*

$$\mathfrak{g} = \mathcal{J} + \mathcal{H},$$

where $\mathcal{J}$ is a commutative Kähler ideal and $\mathcal{H}$ is an exponential j-subalgebra.

This section will be fully devoted to the proof of this theorem. The proof will follow one given in [40] for the case of a normal Kähler algebra with slight modifications. As it is rather complicated, we sketch here the outline of its logical structure. The proof will proceed by induction on dim $\mathfrak{g}$. We first present some consequences of the theorem as Lemmas 7.3.3–7.3.8, which will be included in the induction hypothesis. The theorem for dim $\mathfrak{g} = 2$ is trivial. The essential part of the proof is that of key Proposition 7.3.10, which will be divided into two cases and each case will be proved in Proposition 7.3.13 and Proposition 7.3.14 respectively. (The induction hypothesis will be used only in the proof of Proposition 7.3.14.)

Assume that the theorem is true and let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1$ be the decomposition of $\mathcal{H}$ given in Theorem 7.2.5 and let $s \in \mathcal{H}$ be the element used there. The element s is called the principal idempotent of the Kähler algebra $\mathfrak{g}$.

Lemma 7.3.3. *The transformation $ad_{\mathcal{J}}\, js$ is semi-simple and its eigenvalue is either 0 or $\pm\frac{1}{2} + i\mu$ $(\mu \in \mathbb{R})$. We denote by $\mathcal{J}_0$ the eigenspace corresponding to the eigenvalue 0 and by $\mathcal{J}^{\sim}_{\frac{1}{2}}$ (resp. $\mathcal{J}^{\sim}_{-\frac{1}{2}}$) the sum of the eigenspaces of $\mathcal{J}_{\mathbb{C}}$ corresponding to the eigenvalues $\frac{1}{2} + i\mu$ (resp. $-\frac{1}{2} + i\mu$). Then clearly $\overline{\mathcal{J}^{\sim}_{\frac{1}{2}}} = \mathcal{J}^{\sim}_{\frac{1}{2}}$, $\overline{\mathcal{J}^{\sim}_{-\frac{1}{2}}} = \mathcal{J}^{\sim}_{-\frac{1}{2}}$. We set $\mathcal{J}_{\frac{1}{2}} = \mathcal{J}^{\sim}_{\frac{1}{2}} \cap \mathcal{J}$ (resp. $\mathcal{J}_{-\frac{1}{2}} = \mathcal{J}^{\sim}_{-\frac{1}{2}} \cap \mathcal{J}$), then*

$$\mathcal{J} = \mathcal{J}_{-\frac{1}{2}} \oplus \mathcal{J}_0 \oplus \mathcal{J}_{\frac{1}{2}},\ j\mathcal{J}_{\lambda} = \mathcal{J}_{-\lambda}\ \big(\lambda = \pm\frac{1}{2},\ 0\big)$$

and $[s, x] = jx$ for $x \in \mathcal{J}_{-\frac{1}{2}}$.

Proof. We consider the representation of the two-dimensional Kähler algebra generated by s and js on the ideal $\mathcal{J}$ induced by the adjoint representation of $\mathfrak{g}$. This is easily seen to be a symplectic representation of exponential type. Lemma 7.3.3 is an immediate consequence of Lemma 7.2.4. ∎

We define an inner product S on $\mathfrak{g}$ by $S(x, y) = \beta(jx, y)$ for $x, y \in \mathfrak{g}$.

Lemma 7.3.4. *The subspaces $\mathcal{H}_\lambda$ and $\mathcal{J}_\mu$ are mutually orthogonal with respect to S.*

Proof. We extend β to $\mathfrak{g}_{\mathbb{C}}$ by linearity and denote it by the same letter β. Let $x, y \in \mathfrak{g}_{\mathbb{C}}$ be eigenvectors of $\mathrm{ad}js$ corresponding to eigenvalues $\alpha, \gamma \in \mathbb{C}$ respectively. Then $\beta(js, [x, y]) = (\alpha+\gamma)\beta(x, y)$ and $\bar{x}$ (resp. $\bar{y}$) is an eigenvector corresponding to the eigenvalue $\bar{\alpha}$ (resp. $\bar{\gamma}$). Thus, if we denote by V_λ the space $\mathcal{J}_\lambda$ or the space $\mathcal{H}_\lambda$, then $\beta(js, [V_\lambda, V_\mu]) = 0$ and $\lambda + \mu \neq 0$ will imply $\beta(V_\lambda, V_\mu) = 0$.

(a) $S(\mathcal{H}_\lambda, \mathcal{H}_\mu) = \{0\}$ for $\lambda < \mu$. In fact, $j\mathcal{H}_\lambda = \mathcal{H}_{1-\lambda}$ and, since $1 - \lambda + \mu > 1$, $[\mathcal{H}_{1-\lambda}, \mathcal{H}_\mu] = \{0\}$. Therefore, $S(\mathcal{H}_\lambda, \mathcal{H}_\mu) = \beta(\mathcal{H}_{1-\lambda}, \mathcal{H}_\mu) = \{0\}$.
(b) $S(\mathcal{J}_\lambda, \mathcal{J}_\mu) = \{0\}$ for $\lambda < \mu$. In fact, $j\mathcal{J}_\lambda = \mathcal{J}_{-\lambda}$, $-\lambda+\mu > 0$ and $[\mathcal{J}_{-\lambda}, \mathcal{J}_\mu] = \{0\}$ since $\mathcal{J}$ is commutative. Thus, $S(\mathcal{J}_\lambda, \mathcal{J}_\mu) = \beta(\mathcal{J}_{-\lambda}, \mathcal{J}_\mu) = \{0\}$.
(c) It remains to show that $S(\mathcal{J}, \mathcal{H}) = \{0\}$. We first prove that $\beta(js, \mathcal{J}_{\frac{1}{2}}) = \{0\}$. We can write each element of $\mathcal{J}_{\frac{1}{2}}$ in the form jw, $w \in \mathcal{J}_{-\frac{1}{2}}$,

$$\begin{aligned}\beta(s, w) &= \beta(js, jw) = \beta(js, [s, w]) \\ &= \beta([js, s], w) + \beta(s, [js, w]) = \beta(s, w) + \beta(s, [js, w]).\end{aligned}$$

This shows that $\beta(s, \mathcal{J}_{-\frac{1}{2}}) = \beta(s, [js, \mathcal{J}_{-\frac{1}{2}}]) = \{0\}$, so $\beta(js, \mathcal{J}_{\frac{1}{2}}) = \{0\}$. We show that $S(\mathcal{J}_\lambda, \mathcal{H}_\mu) = \{0\}$ for $\lambda < \mu$. In fact, $j\mathcal{J}_\lambda = \mathcal{J}_{-\lambda}$ and $-\lambda + \mu > 0$, thus

$[\mathcal{J}_{-\lambda}, \mathcal{H}_\mu] \subset \mathcal{J}_{\frac{1}{2}}$ and $\beta(js, [\mathcal{J}_{-\lambda}, \mathcal{H}_\mu]) \subset \beta(js, \mathcal{J}_{\frac{1}{2}}) = \{0\}$. Hence $S(\mathcal{J}_\lambda, \mathcal{H}_\mu) = \beta(\mathcal{J}_{-\lambda}, \mathcal{H}_\mu) = \{0\}$. If $\lambda \geq \mu$, then $-\lambda < 1 - \mu$ and

$$S(\mathcal{J}_\lambda, \mathcal{H}_\mu) = S(j\mathcal{J}_\lambda, j\mathcal{H}_\mu) = S(\mathcal{J}_{-\lambda}, \mathcal{H}_{1-\mu}) = \{0\}. \qquad \blacksquare$$

Lemma 7.3.5. $[\mathcal{H}_0, \mathcal{J}_0] = \{0\}$.

Proof. We consider the operator $A = \mathrm{ad}_{\mathcal{J}_0} h$ with $h \in \mathcal{H}_0$. Since $[\mathcal{H}_1, \mathcal{J}_0] = \{0\}$, using the integrability condition, it follows that A commutes with j. Since β is closed and since $\mathcal{J}$ is commutative,

$$\beta(Ax, y) + \beta(x, Ay) = 0 \; (x, y \in \mathcal{J}_0).$$

This shows that the operator A is skew-symmetric with respect to the scalar product S. Since its eigenvalues must be purely imaginary, we have $A = 0$. ■

Lemma 7.3.6. *Assume that $\lambda + \mu + \gamma = 0$. If an element $h \in \mathcal{H}_\lambda$ commutes with $\mathcal{J}_\mu$, then it commutes with $\mathcal{J}_\gamma$.*

Proof. Let $a \in \mathcal{J}_\gamma$. The form β being closed, the assumption implies $\beta([h, a], x) = 0$ for all $x \in \mathcal{J}_\mu$. Since $j[h, a] \in \mathcal{J}_\mu$ by Lemma 7.3.3, we have $\beta([h, a], j[h, a]) = 0$ and therefore $[h, a] = 0$. ■

Lemma 7.3.7. $\mathcal{J} = \{a \in \mathfrak{g}; [ja, a] = 0\}$.

Proof. It is sufficient to prove that $a \in \mathcal{J}$ if $[ja, a] = 0$. Write $a = x + h$ with $x \in \mathcal{J},\ h \in \mathcal{H}$. Then

$$0 = [ja, a] = [jh, h] \; (\mathrm{mod}\, \mathcal{J}).$$

Since $\mathcal{H}$ is an exponential j-algebra, we have $h = 0$ as desired. ■

Lemma 7.3.8. *The centre $\mathfrak{z}$ of $\mathfrak{g}$ is contained in $\mathcal{J}_0$ and j-invariant.*

Proof. The last lemma shows that $\mathfrak{z} \subset \mathcal{J}$. Moreover, it is clear that $\mathfrak{z} \subset \mathcal{J}_0$. From Lemma 7.3.3 and the fact that $[\mathcal{H}_1, \mathcal{J}_0] = \{0\}$, it follows that $\mathfrak{z}$ coincides with the centralizer of $\mathcal{H}_{\frac{1}{2}}$ in $\mathcal{J}_0$. Let $a \in \mathfrak{z}$. From the integrability condition, we find that $[ja, jh] = j[ja, h]$ for all $h \in \mathcal{H}_{\frac{1}{2}}$ so that

$$\beta(j[ja, h], [ja, h]) = \beta([ja, jh], [ja, h]).$$

The form β being closed, the equalities $[ja, [ja, h]] = 0$ and $[jh, [ja, h]] = 0$ imply $\beta(j[ja, h], [ja, h]) = 0$. This proves that $[ja, h] = 0$ and $ja \in \mathfrak{z}$. ■

We now begin the proof of Theorem 7.3.2 by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 2$, the theorem is trivially valid. We assume $\dim \mathfrak{g} = n$ and that Theorem 7.3.2 and therefore Lemmas 7.3.3–7.3.8 are valid for exponential Kähler algebras of dimension less than n.

Definition 7.3.9. An exponential j-algebra of rank 1 is said to be elementary.

Proposition 7.3.10. *In $\mathfrak{g}$, there is either an elementary or commutative nonzero Kähler algebra.*

In what follows, we aim at the proof of Proposition 7.3.10. We choose and fix a minimal ideal $\mathfrak{a}$ of $\mathfrak{g}$. Then one of the following three cases will occur (cf. Lemma 7.3.1).

(i) $\dim \mathfrak{a} = 1,\ [j\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$. Let $r \in \mathfrak{a}$ be the nonzero element such that $[jr, r] = r$.
(ii) $\dim \mathfrak{a} = 1,\ [j\mathfrak{a}, \mathfrak{a}] = \{0\}$. Let $r \in \mathfrak{a}$ be a nonzero element.
(iii) $\dim \mathfrak{a} = 2,\ \mathfrak{k} = j\mathfrak{a} \oplus \mathfrak{a}$ is a four-dimensional commutative Kähler subalgebra. Let $\{r_1, r_2\}$ be a basis of $\mathfrak{a}$ such that

$$[a, r_1] = l(a)(r_1 - \zeta r_2),\ [a, r_2] = l(a)(r_2 + \zeta r_1)$$

for all $a \in \mathfrak{g}$ with $l \in \mathfrak{g}^*,\ \zeta \in \mathbb{R}$.

In each case, we set $\mathfrak{k} = j\mathfrak{a} \oplus \mathfrak{a}$.

Lemma 7.3.11. *We put* $P = \{a \in \mathfrak{g}; [a, \mathfrak{a}] = [ja, \mathfrak{a}] = \{0\}\}$. *Then* P *is invariant under both* j *and* $ad\, jx$ $(x \in \mathfrak{a})$. *Moreover, the operator* $ad_P\, jx$ $(x \in \mathfrak{a})$ *commutes with* j.

Proof. The invariance of P under j is immediate from the definitions. From the integrability condition, we have

$$[jx, jp] = j[jx, p] \tag{7.3.1}$$

for all $p \in P$. Using the Jacobi identity and (7.3.1), we find $[[jx, p], y] = 0$ and $[j[jx, p], y] = [[jx, jp], y] = 0$ for $y \in \mathfrak{a}$, so that $[jx, p] \in P$. The commutativity of j and $\mathrm{ad}_P\, jx$ follows from (7.3.1). ■

Lemma 7.3.12. *For all $u, v \in \mathfrak{g}$ and $x \in \mathfrak{a}$,*

$$\frac{d}{dt}\beta\left(e^{t(adjx)}u, e^{t(adjx)}v\right) = \beta\left(jx, e^{t(adjx)}[u, v]\right). \tag{7.3.2}$$

Proof.

$$\begin{aligned}
&\frac{d}{dt}\beta\left(e^{t(\mathrm{ad}jx)}u, e^{t(\mathrm{ad}jx)}v\right)\\
=&\beta\left([jx, e^{t(\mathrm{ad}jx)}u], e^{t(\mathrm{ad}jx)}v\right) + \beta\left(e^{t(\mathrm{ad}jx)}u, [jx, e^{t(\mathrm{ad}jx)}v]\right)\\
=&\beta\left(jx, [e^{t(\mathrm{ad}jx)}u, e^{t(\mathrm{ad}jx)}v]\right) = \beta\left(jx, e^{t(\mathrm{ad}jx)}[u, v]\right).
\end{aligned}$$
■

Now, we prove Proposition 7.3.13, from which Proposition 7.3.10 will follow in case (i). Note that we shall not use the induction hypothesis for the proof of Proposition 7.3.13.

Proposition 7.3.13. *If the minimal ideal* $\mathfrak{a}$ *is of type* (i)*, then* $\mathfrak{g}$ *can be decomposed into a semi-direct sum* $\mathfrak{g} = \mathcal{N} + \mathfrak{g}'$ *which satisfies the following conditions.*

(1) $\mathcal{N}$ *is a Kähler ideal and is an elementary* j*-algebra, and is decomposed into a direct sum of subspaces:* $\mathcal{N} = \mathbb{R}jr + \mathbb{R}r + U$*, where* $U = \mathcal{N}_{\frac{1}{2}}$ *in conformity with the notation of Theorem 7.2.5.*
(2) $\mathfrak{g}'$ *is a Kähler subalgebra orthogonal to* $\mathcal{N}$ *and* $[\mathfrak{k}, \mathfrak{g}'] = \{0\}$.

Proof. Let P be the subspace constructed in Lemma 7.3.11. By the relation $[jr, r] = r$ one can find in a unique way, for any $u \in \mathfrak{g}$, numbers λ and μ so that $u - \lambda jr - \mu r \in P$. This means that $\mathfrak{g}$ decomposes into the direct sum of subspaces

$$\mathfrak{g} = \mathbb{R}jr \oplus \mathbb{R}r \oplus P. \tag{7.3.3}$$

Using the last lemma, we study the eigenvalues of $\operatorname{ad} jr$ on P.

(a) If in (7.3.2) we set $u = r$ and $v \in P$, then the right-hand side is 0. Using the equality

$$e^{t(\operatorname{ad} jr)} r = e^t r, \tag{7.3.4}$$

we obtain for $v \in P$

$$\beta\left(r, e^{t(\operatorname{ad} jr)} v\right) = \alpha e^{-t}, \tag{7.3.5}$$

where α is some real number. According to Lemma 7.3.11, the operator $\operatorname{ad} jr$ commutes with j on P. Therefore, for $v \in P$,

$$\beta\left(jr, e^{t(\operatorname{ad} jr)} v\right) = \alpha e^{-t}. \tag{7.3.6}$$

By (7.3.3) and (7.3.4), we obtain, for $x \in \mathfrak{g}$,

$$\beta\left(jr, e^{t(\operatorname{ad} jr)} x\right) = \alpha e^{-t} + \gamma e^{t} \tag{7.3.7}$$

with some real numbers α, γ. Now for $u, v \in \mathfrak{g}$, formula (7.3.2) gives

$$\beta\left(e^{t(\operatorname{ad} jr)} u, e^{t(\operatorname{ad} jr)} v\right) = \alpha e^{-t} + \gamma e^{t} + \delta \tag{7.3.8}$$

with some real numbers α, γ, δ. Since $\operatorname{ad}_P jr \circ j = j \circ \operatorname{ad}_P jr$, formula (7.3.8) implies, for $u \in P$, $v \in \mathfrak{g}$,

$$S\left(e^{t(\operatorname{ad} jr)} u, e^{t(\operatorname{ad} jr)} v\right) = \alpha e^{-t} + \gamma e^{t} + \delta \tag{7.3.9}$$

with some real numbers α, γ, δ.
(b) We regard the operator $\operatorname{ad} jr|_P$ as a complex linear transformation on P. Let $p \in P$ be its eigenvector corresponding to an eigenvalue $\lambda + i\mu$ ($\lambda, \mu \in \mathbb{R}$), then $e^{t(\operatorname{ad} jr)} p = e^{(\lambda + i\mu)t} p$. So by (7.3.9),

$$e^{2\lambda t}S((\cos\mu t + i\sin\mu t)p, (\cos\mu t + i\sin\mu t)p) = \alpha e^{-t} + \gamma e^{t} + \delta.$$

The left-hand side equals

$$\begin{aligned} &e^{2\lambda t}S(p\cos\mu t + jp\sin\mu t, p\cos\mu t + jp\sin\mu t) \\ &= e^{2\lambda t}(\cos^2\mu t + \sin^2\mu t)S(p,p) = e^{2\lambda t}S(p,p). \end{aligned}$$

Thus we have

$$e^{2\lambda t}S(p,p) = \alpha e^{-t} + \gamma e^{t} + \delta.$$

Since $S(p,p) \neq 0$, λ can have only the values $0,\ \pm\frac{1}{2}$.

We now check that the operator $\mathrm{ad}_P\, jr$ is semi-simple (and therefore $\mathrm{ad}\, jr$ is also semi-simple on $\mathfrak{g}$). Let $p,\ q$ be vectors in P such that

$$[jr,p] = (\lambda + i\mu)p,\ [jr,q] = (\lambda + i\mu)q + p\ (\lambda,\mu \in \mathbb{R}).$$

Then

$$e^{t(\mathrm{ad}\, jr)}q = e^{(\lambda+i\mu)t}q + te^{(\lambda+i\mu)t}p$$

and by formula (7.3.9)

$$\begin{aligned} S\left(e^{t(\mathrm{ad}\, jr)}p, e^{t(\mathrm{ad}\, jr)}q\right) &= e^{2\lambda t}S((\cos\mu t + i\sin\mu t)p, (\cos\mu t + i\sin\mu t)q) \\ &\quad + te^{2\lambda t}S((\cos\mu t + i\sin\mu t)p, (\cos\mu t + i\sin\mu t)p) \\ &= e^{2\lambda t}S(p,q) + te^{2\lambda t}S(p,p) \\ &= \alpha e^{-t} + \gamma e^{t} + \delta. \end{aligned}$$

$S(p,p)$ being nonzero, this equality is impossible.

(c) Since the operator $\mathrm{ad}_P\, jr$ has no eigenvalue with real part -1, we have $\alpha = 0$ in (7.3.5). Looking at the formulas deduced from (7.3.5), we see that in each of them $\alpha = 0$, and therefore $-\frac{1}{2} + i\mu$ $(\mu \in \mathbb{R})$ is not an eigenvalue of $\mathrm{ad}_P\, jr$. We denote by P_0 the eigenspace of $\mathrm{ad}_P\, jr$ corresponding to the eigenvalue 0, and by $P_{\frac{1}{2}}$ the sum of its eigenspaces corresponding to the eigenvalues of the form $\frac{1}{2} + i\mu$ $(\mu \in \mathbb{R})$. The algebra $\mathfrak{g}$ decomposes into a direct sum of subspaces:

$$\mathfrak{g} = \mathbb{R}jr + \mathbb{R}r + P_{\frac{1}{2}} + P_0.$$

The subspaces $P_{\frac{1}{2}},\ P_0$ are clearly j-invariant. The formulas (7.3.5), (7.3.6) and (7.3.9) show that P_0 is orthogonal to $\mathcal{N} = \mathbb{R}jr + \mathbb{R}r + P_{\frac{1}{2}}$. As in Sect. 2 of [61], we can show that $\mathcal{N}$ is an ideal of $\mathfrak{g}$. Now the relations (7.3.5) and (7.3.6) show that $P_{\frac{1}{2}}$ and $\mathfrak{k} = \mathbb{R}jr \oplus \mathbb{R}r$ are mutually orthogonal with respect to S. Let u (resp. v) be an eigenvector of $\mathrm{ad}_P\, jr$ corresponding to an eigenvalue $\frac{1}{2} + i\mu$ (resp. $\frac{1}{2} + i\nu,\ \nu \in \mathbb{R}$). By (7.3.9),

$$e^t\, (S(u, v)\cos((\mu - \nu)t) + S(ju, v)\sin((\mu - \nu)t)) = \gamma e^t + \delta.$$

Hence if $\mu \neq \nu$, it follows that $S(u, v) = S(ju, v) = 0$. Therefore $\beta(u, v) = 0$ and $\beta(jr, [u, v]) = 0$. If $\mu = \nu$,

$$\beta(jr, [u, v]) = \frac{1}{2}\beta(u, v) + \mu\beta(ju, v) + \frac{1}{2}\beta(u, v) + \mu\beta(u, jv) = \beta(u, v).$$

In either case we have

$$\beta(jr, [u, v]) = \beta(u, v). \tag{7.3.10}$$

Let $\omega \in \mathcal{N}^*$ be defined by $\omega(x) = \beta(jr, x)$ $(x \in \mathcal{N})$. Then it can be shown that

$$\omega([x, y]) = \beta(x, y)$$

for all $x, y \in \mathcal{N}$. In fact, let

$$x = ajr + u + br,\ \ y = cjr + v + dr\ \ (a, b, c, d \in \mathbb{R})$$

with $u, v \in P_{\frac{1}{2}}$. Then

$$[x, y] = (ad - bc)r + w + [u, v]$$

with $w \in P_{\frac{1}{2}}$. Thus

$$\omega([x, y]) = \beta(jr, (ad - bc)r + w + [u, v]) = (ad - bc)\beta(jr, r) + \beta(jr, [u, v]).$$

On the other hand,

$$\beta(x, y) = (ad - bc)\beta(jr, r) + \beta(u, v)$$

and Eq. (7.3.10) means $\omega([x, y]) = \beta(x, y)$.

In consequence, $\mathcal{N}$ is an exponential j-algebra. If we set $\mathfrak{g}' = P_0$, then $\mathfrak{g}'$ is the orthogonal complement of the Kähler ideal $\mathcal{N}$ so that $\mathfrak{g}'$ is a Kähler subalgebra of $\mathfrak{g}$ (cf. Lemma 2 at Sect. 6 in Part II of [40]). ■

From now on, we always assume that the minimal ideal $\mathfrak{a}$ is of type (ii) or (iii).

Proposition 7.3.14. *There is a commutative Kähler ideal $\mathcal{N}$ which contains $\mathfrak{a}$.*

We prepare several lemmas for the proof of this proposition. We denote by P the space constructed in Lemma 7.3.11.

Lemma 7.3.15. $[j\mathfrak{a}, \mathfrak{g}] \subset P$.

Proof. The Jacobi identity and the integrability condition imply that

$$[[j\mathfrak{a}, \mathfrak{g}], \mathfrak{a}] = [j\mathfrak{a}, [\mathfrak{g}, \mathfrak{a}]] = \{0\}, [j[j\mathfrak{a}, \mathfrak{g}], \mathfrak{a}] = [[j\mathfrak{a}, \mathfrak{g}], \mathfrak{a}] = \{0\}.$$

That is, $[j\mathfrak{a}, \mathfrak{g}] \subset P$. ■

Lemma 7.3.16. *$ad(jx) \circ ad(jy) = 0$ for any $x, y \in \mathfrak{a}$.*

Proof. Since $\mathrm{ad}(jx) \circ \mathrm{ad}(jy) = \mathrm{ad}(jy) \circ \mathrm{ad}(jx)$, it suffices to show that $(\mathrm{ad}(jx))^2 = 0$ for all $x \in \mathfrak{a}$.

(a) We show that the set $[j\mathfrak{a}, P]$ is orthogonal to $\mathfrak{k}$. Let $p \in P$ and $x, y \in \mathfrak{a}$. Since β is closed, we have $\beta(y, [jx, p]) = \beta(jx, [y, p]) = 0$. Using Lemma 7.3.11, we also obtain

$$\beta(jy, [jx, p]) = -\beta(y, j[jx, p]) = -\beta(y, [jx, jp]) = 0.$$

(b) From Lemma 7.3.15 and (a), it follows that $\beta(jy, (\mathrm{ad}\, jx)^2\mathfrak{g}) = \{0\}$. Then, formula (7.3.2) shows that, for $u, v \in \mathfrak{g}$,

$$\begin{aligned}\frac{d^3}{dt^3}\beta\left(e^{t(\mathrm{ad}\, jx)}u, e^{t(\mathrm{ad}\, jx)}v\right) &= \frac{d^2}{dt^2}\beta\left(jx, e^{t(\mathrm{ad}\, jx)}[u, v]\right) \\ &= \beta\left(jx, [jx, [jx, e^{t(\mathrm{ad}\, jx)}[u, v]]]\right) = 0,\end{aligned}$$

i.e.

$$\beta\left(e^{t(\mathrm{ad}\, jx)}u, e^{t(\mathrm{ad}\, jx)}v\right) = \alpha t^2 + \gamma t + \delta \ (\alpha, \gamma, \delta \in \mathbb{R}).$$

Whence for $p \in P,\ q \in \mathfrak{g}$,

$$S\left(e^{t(\mathrm{ad}\, jx)}p, e^{t(\mathrm{ad}\, jx)}q\right) = \alpha t^2 + \gamma t + \delta. \tag{7.3.11}$$

Setting $q = p$, we obtain that the operator $\mathrm{ad}\, jx$ has no nonzero eigenvalue on P (and therefore on $\mathfrak{g}$).

(c) We assume that $(\mathrm{ad}\, jx)^2 \neq 0$. Then, there exist elements $u, v, w \in \mathfrak{g}$ such that

$$[jx, u] = 0,\ [jx, v] = u,\ [jx, w] = v.$$

It is clear that

$$e^{t(\mathrm{ad}\, jx)}u = u,\ e^{t(\mathrm{ad}\, jx)}v = v + tu,\ e^{t(\mathrm{ad}\, jx)}w = w + tv + \frac{t^2}{2}u.$$

We have $v \in P$ by Lemma 7.3.15 and we can put $p = v,\ q = w$ in (7.3.11) to get

$$S\left(v + tu, w + tv + \frac{t^2}{2}u\right) = \alpha t^2 + \gamma t + \delta.$$

Since $S(u, u) \neq 0$, this is impossible. ∎

Let L be a Lie algebra over $\mathbb{R}$. A decreasing sequence of Lie subalgebras $\{L^{(k)}\}_{k\in\mathbb{Z}}$,

$$\cdots \supset L^{(-2)} \supset L^{(-1)} \supset L^{(0)} \supset L^{(1)} \supset L^2 \supset \cdots$$

with the following properties is called a filtration of L:

(1) $\cup_k L^{(k)} = L$;
(2) $\cap_k L^{(k)} = \{0\}$;
(3) $[L^{(k)}, L^{(l)}] \subset L^{(k+l)}$.

With the aid of the operator $\mathrm{ad}\, jx$ $(x \in \mathfrak{a})$, we construct a j-invariant filtration of $\mathfrak{g}$. Let $U = \mathfrak{g}/\mathfrak{k}$ and π the natural projection of $\mathfrak{g}$ onto U.

Case 1. $\mathfrak{a}$ is of type (ii). Let A_0 be the operator on U induced by $\mathrm{ad}\, jr$. We set

$$\mathfrak{g}^{(k)} = \mathfrak{g}(k < 0),\ \mathfrak{g}^{(0)} = \pi^{-1}(\ker A_0),\ \mathfrak{g}^{(1)} = \pi^{-1}(\mathrm{Im}\, A_0),$$
$$\mathfrak{g}^{(2)} = \mathfrak{k},\ \mathfrak{g}^{(k)} = \{0\}(k > 2).$$

Case 2. $\mathfrak{a}$ is of type (iii). Let $A_i\, (i = 1, 2)$ be the operators on U induced by $\mathrm{ad}\, jr_i$. We set

$$\mathfrak{g}^{(k)} = \mathfrak{g}(k < 0),\ \mathfrak{g}^{(0)} = \pi^{-1}(\ker A_1 \cap \ker A_2),$$
$$\mathfrak{g}^{(1)} = \pi^{-1}(\mathrm{Im}\, A_1 + \mathrm{Im}\, A_2),\ \mathfrak{g}^{(2)} = \mathfrak{k},\ \mathfrak{g}^{(k)} = \{0\}(k > 2)$$

From the integrability condition, it follows that $A_i\, (i = 0, 1, 2)$ commute with j on U.

Lemma 7.3.17. *The subspaces $\mathfrak{g}^{(k)}$ form a j-invariant filtration of the Lie algebra $\mathfrak{g}$. Furthermore*

$$[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \{0\}. \tag{7.3.12}$$

Proof. (a) It follows from the preceding lemma that

$$A_i^2 = A_1 \cdot A_2 = A_2 \cdot A_1 = 0\ (i = 0, 1, 2).$$

Therefore

$$\ker A_0 \supset \operatorname{Im} A_0,\ \ker A_1 \cap \ker A_2 \supset \operatorname{Im} A_1 + \operatorname{Im} A_2,$$

so we have the inclusions

$$\mathfrak{g}^{(-1)} \supset \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)}.$$

The invariance of the subspaces $\mathfrak{g}^{(k)}$ with respect to j follows from the commutativity of A_i with j.

(b) We prove that $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \{0\}$. In Case 1, let $g_1, g_2 \in \mathfrak{g}$ and put $u_1 = [jr, g_1],\ u_2 = [jr, g_2]$. Since $(\mathrm{ad}\, jr)^2 = 0$, $0 = (\mathrm{ad}\, jr)^2([g_1, g_2]) = 2[u_1, u_2]$. Therefore $[[jr, \mathfrak{g}], [jr, \mathfrak{g}]] = \{0\}$. Since $\mathfrak{g}^{(1)} = [jr, \mathfrak{g}] + \mathfrak{k}$, it remains to prove that $[\mathfrak{k}, [jr, \mathfrak{g}]] = \{0\}$, which follows from the last two lemmas.

In Case 2, we first show that $\mathfrak{g}^{(1)}$ is a Lie subalgebra. In fact, we have

$$[[jr_1, \mathfrak{g}], [jr_1, \mathfrak{g}]] = [[jr_2, \mathfrak{g}], [jr_2, \mathfrak{g}]] = \{0\}$$

as above and

$$[[jr_1, \mathfrak{g}], [jr_2, \mathfrak{g}]] \subset [jr_1, \mathfrak{g}] \cap [jr_2, \mathfrak{g}].$$

Since

$$\mathfrak{g}^{(1)} = \mathfrak{k} + [jr_1, \mathfrak{g}] + [jr_2, \mathfrak{g}],$$

it follows that $\mathfrak{g}^{(1)}$ is a Lie subalgebra. Moreover $[\mathfrak{g}^{(1)}, [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]] = \{0\}$. Thus $\mathfrak{g}^{(1)}$ is a nilpotent Kähler algebra and the minimality of the ideal $\mathfrak{a}$ proves that the dimension of $\mathfrak{g}^{(1)}$ is less than n. Hence, by the induction hypothesis concerning Theorem 7.3.2, $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \{0\}$.

(c) It follows immediately from the definitions that

$$[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(1)},\ [\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}.$$

(d) We show that $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}$. Let $l \in \mathfrak{g}^{(0)},\ u \in \mathfrak{g}^{(1)}$. Since $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(1)}$, it is sufficient to consider the case where $u = [jx, g]$ with $x \in \mathfrak{a}$ and $g \in \mathfrak{g}$. We have $[jx, [l, g]] = [l, u] + [[jx, l], g]$. Since $[jx, [l, g]], [[jx, l], g] \in \mathfrak{g}^{(1)}$, we also have $[l, u] \in \mathfrak{g}^{(1)}$.

(e) We prove that $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(0)}$. Let $g \in \mathfrak{g}$ and $u \in \mathfrak{g}^{(1)}$. Since $[jx, u] = 0$ and $[jx, g] \in \mathfrak{g}^{(1)}$ for $x \in \mathfrak{a}$, the Jacobi identity and the already proved commutativity of $\mathfrak{g}^{(1)}$ show that $[jx, [g, u]] = 0$. Therefore $[g, u] \in \mathfrak{g}^{(0)}$.

(f) It remains to prove that $\mathfrak{g}^{(0)}$ is a Lie subalgebra. Let $g_1, g_2 \in \mathfrak{g}^{(0)}$ and let $x \in \mathfrak{a}$. Since $[\mathfrak{g}^{(2)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(2)}$, we have

$$[jx, [g_1, g_2]] = [[jx, g_1], g_2] + [g_1, [jx, g_2]] \in \mathfrak{g}^{(2)}.$$

Consequently, $[g_1, g_2] \in \mathfrak{g}^{(0)}$. ■

If $\mathfrak{g}^{(0)} = \mathfrak{g}$ in the filtration constructed above, then $\mathcal{N} = \mathfrak{g}^{(2)}$ is a commutative Kähler ideal in $\mathfrak{g}$ and Proposition 7.3.14 is proved. Therefore in the following, we shall assume that $\mathfrak{g}^{(0)} \neq \mathfrak{g}$.

The induction hypothesis may be applied to the Kähler algebra $\mathfrak{g}^{(0)}$. Let

$$\mathfrak{g}^{(0)} = \mathcal{J} + \mathcal{H} \tag{7.3.13}$$

be the decomposition corresponding to Theorem 7.3.2.

Lemma 7.3.18. $\mathfrak{g}^{(1)} \subset \mathcal{J}$.

Proof. Let $g \in \mathfrak{g}^{(1)}$. Then $jg \in \mathfrak{g}^{(1)}$. It follows from (7.3.12) that $[jg, g] = 0$. By Lemma 7.3.7, this means that $g \in \mathcal{J}$. ■

Lemma 7.3.19. $[\mathfrak{g}, \mathcal{J}] \subset \mathfrak{g}^{(0)}$.

Proof. Let $g \in \mathfrak{g}$ and $u \in \mathcal{J}$. Then $[jx, g] \in \mathfrak{g}^{(1)}$ for $x \in \mathfrak{a}$ and $[[jx, \mathfrak{g}], u] = \{0\}$ by the previous lemma. Therefore $[jx, [g, u]] = [g, [jx, u]] = 0$, which proves the lemma. ■

From this lemma, it follows in particular that if the subalgebra $\mathfrak{g}^{(0)}$ is commutative, then it is an ideal of $\mathfrak{g}$, so that Proposition 7.3.14 is proved in this case.

In what follows, we assume that $\mathfrak{g}^{(0)}$ is not commutative, i.e. $\mathcal{H} \neq \{0\}$. We denote by s the principal idempotent of the Kähler algebra $\mathfrak{g}^{(0)}$. For $\alpha \in \mathbb{C},\ \lambda \in \mathbb{R}$, we set

$$(\mathfrak{g}_{\mathbb{C}})_\alpha = \{x \in \mathfrak{g}_{\mathbb{C}};\, (\mathrm{ad}\, js - \alpha)^m x = 0, 0 < \exists m \in \mathbb{N}\},$$

$\mathfrak{g}_{\tilde{\lambda}} = \sum_{\mu \in \mathbb{R}} (\mathfrak{g}_{\mathbb{C}})_{\lambda + i\mu}$ and set $\mathfrak{g}_\lambda = \mathfrak{g}_{\tilde{\lambda}} \cap \mathfrak{g}$. Then it is clear that

$$\overline{\mathfrak{g}_{\tilde{\lambda}}} = \mathfrak{g}_{\tilde{\lambda}},\ \mathfrak{g} = \sum_\lambda \mathfrak{g}_\lambda,\ [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu},\ \mathfrak{g}_\lambda \cap \mathfrak{g}^{(0)} = \mathfrak{g}_\lambda^{(0)}.$$

Lemma 7.3.20. *If* $\lambda + \mu > 0$ *or* $\lambda = \mu = 0$, *then* $[[\mathfrak{g}, \mathcal{J}_\lambda], \mathcal{J}_\mu] = \{0\}$.

Proof. Let $g \in \mathfrak{g}_\gamma$ and $x \in \mathcal{J}_\lambda$. Then by the last lemma

$$[g, x] \in \mathfrak{g}^{(0)}_{\lambda+\gamma} \subset \mathcal{H}_{\lambda+\gamma} + \mathcal{J}.$$

According to Lemma 7.3.6 it suffices to prove that $[[g, x], \mathcal{J}_{-(\lambda+\mu+\gamma)}] = \{0\}$. Let $y \in \mathcal{J}_{-(\lambda+\mu+\gamma)}$. The commutativity of $\mathcal{J}$ implies $[[g, x], y] = [[g, y], x]$. We have

$$[g, y] \in \mathfrak{g}^{(0)}_{-(\lambda+\mu)} \subset \mathcal{H}_{-(\lambda+\mu)} + \mathcal{J}.$$

If $\lambda + \mu > 0$, then $[g, y] \in \mathcal{J}$ since the operator $\mathrm{ad}_{\mathcal{H}}\, js$ has only eigenvalues with non-negative real part. Consequently, $[[g, y], x] = 0$ in this case. If $\lambda = \mu = 0$, then $[g, y] \in \mathcal{H}_0 + \mathcal{J}$, $x \in \mathcal{J}_0$ and $[[g, y], x] = 0$ by Lemma 7.3.5. ■

We consider the graded Lie algebra

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)} + \bar{\mathfrak{g}}^{(0)} + \bar{\mathfrak{g}}^{(1)} + \bar{\mathfrak{g}}^{(2)},$$

which is associated to the filtered Lie algebra $\mathfrak{g}$. For every element $g \in \mathfrak{g}^{(k)}$, we denote by $\bar{g}$ the corresponding element of $\bar{\mathfrak{g}}^{(k)}$. If we regard the same element $g \in \mathfrak{g}^{(k)}$ as an element of $\mathfrak{g}^{(k-1)}$, then $\bar{g} = 0$. In the following, however, it will always be clear which of the subspaces $\mathfrak{g}^{(k)}$ we have in mind.

It follows from (7.3.12) that

$$[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \{0\}. \tag{7.3.14}$$

For a nonzero element $x \in \mathfrak{a}$, we define on $\bar{\mathfrak{g}}^{(-1)}$ a trilinear operation $(a, b, c) \to (abc)_x$ by

$$(abc)_x = [[[\overline{jx}, a], b], c]. \tag{7.3.15}$$

We establish some properties of this operation.

Lemma 7.3.21. (1) *The operation* (7.3.15) *is commutative.*
(2) *For* $y \in \mathfrak{a}$, $[\overline{jy}, (abc)_x] = [[[\overline{jx}, a], b], [\overline{jy}, c]]$.

Proof. (1) By the Jacobi identity

$$[[[\overline{jx}, a], b], c] - [[[\overline{jx}, b], a], c] = [[\overline{jx}, [a, b]], c] = 0$$

since $[\bar{\mathfrak{g}}^{(-1)}, \bar{\mathfrak{g}}^{(-1)}] = \{0\}$. This shows that $(abc)_x = (bac)_x$. It is proved analogously that $(abc)_x = (acb)_x$.

(2) We take the commutator of both sides of Eq. (7.3.15) with $\overline{jy}$ and use the Jacobi identity. From the properties of the graduation and relation (7.3.14), it follows that $[\overline{jy}, [\overline{jx}, a]] = 0$ and $[[\overline{jx}, a], [\overline{jy}, b]] = 0$, so that there remains only $[[[\overline{jx}, a], b], [\overline{jy}, c]]$ of the three terms on the right-hand side. ■

Lemma 7.3.22. $(abc)_x = 0$ *for all* $a, b, c \in \mathfrak{g}^{(-1)}$ *and* $x \in \mathfrak{a}$.

Proof. To the decomposition (7.3.13) of the algebra $\mathfrak{g}^{(0)}$, there corresponds the decomposition

$$\bar{\mathfrak{g}}^{(0)} = \overline{\mathcal{J}} + \overline{\mathcal{H}} \tag{7.3.16}$$

of the algebra $\bar{\mathfrak{g}}^{(0)}$. By Lemma 7.3.18,

$$[\bar{\mathfrak{g}}^{(1)} + \bar{\mathfrak{g}}^{(2)}, \overline{\mathcal{J}}] = \{0\} \tag{7.3.17}$$

and from Lemma 7.3.19 it follows that

$$[\bar{\mathfrak{g}}^{(-1)}, \overline{\mathcal{J}}] = \{0\}. \tag{7.3.18}$$

Lemma 7.3.18 implies that $x \in \mathcal{J}$.

In Case 1, $[js, r] = \alpha r$ ($\alpha \in \mathbb{R}$) since $\mathfrak{a} = \mathbb{R}r$ is an ideal. By Lemma 7.3.3, $\alpha = 0$ or $\pm\frac{1}{2}$, and if $\alpha = -\frac{1}{2}$, then $[s, r] = jr \notin \mathfrak{a}$, so that this case is impossible. Furthermore, if $r \in \mathcal{J}_\alpha$, then $jr \in \mathcal{J}_{-\alpha}$ and $[js, jr] = -\alpha jr$ (cf. the proof of Lemma 7.2.4). Going over to the algebra $\bar{\mathfrak{g}}$, we obtain the following relation

$$[\overline{js}, \overline{jr}] = -\alpha \overline{jr} \; (\alpha = 0 \text{ or } \alpha = \frac{1}{2}). \tag{7.3.19}$$

By the definition of the subspaces $\bar{\mathfrak{g}}^{(k)}$, it is clear that the operator $\mathrm{ad}\overline{jr}$ maps $\bar{\mathfrak{g}}^{(-1)}$ isomorphically onto $\bar{\mathfrak{g}}^{(1)}$. The operator $\mathrm{ad}\overline{js}$ is semi-simple on $\bar{\mathfrak{g}}^{(1)}$ and has eigenvalues $0, \; \pm\frac{1}{2} + i\gamma$ ($\gamma \in \mathbb{R}$) on it. Relation (7.3.19) shows that $\mathrm{ad}\overline{js}$ is semi-simple also on $\bar{\mathfrak{g}}^{(-1)}$ and has eigenvalues $\alpha, \; \pm\frac{1}{2} + i\gamma + \alpha$ there. We define the space $\bar{\mathfrak{g}}^{(-1)}_\lambda$ ($\lambda \in \mathbb{R}$) just as the space $\mathfrak{g}_\lambda$ by making use of the operator $\mathrm{ad}\overline{js}$. Let $a \in \bar{\mathfrak{g}}^{(-1)}_\lambda$ $b \in \bar{\mathfrak{g}}^{(-1)}_\mu$, $c \in \bar{\mathfrak{g}}^{(-1)}_\gamma$ such that

$$(abc)_r = [[[\overline{jr}, a], b], c] \neq 0. \tag{7.3.20}$$

Then also

$$[\overline{jr}, (abc)_r] = [[[\overline{jr}, a], b], [\overline{jr}, c]] \neq 0. \tag{7.3.21}$$

From what has been mentioned above, it is clear that

$$\lambda, \; \mu, \; \gamma = \alpha \text{ or } \pm\frac{1}{2} + \alpha. \tag{7.3.22}$$

Moreover,

$$[[\overline{jr}, a], b] \in \bar{\mathfrak{g}}^{(0)}_{\lambda+\mu-\alpha} = \overline{\mathcal{H}}_{\lambda+\mu-\alpha} + \overline{\mathcal{J}}_{\lambda+\mu-\alpha}.$$

If $\lambda + \mu - \alpha \neq 0, \; 1$ or $\frac{1}{2}$, then $[[\overline{jr}, a], b] \in \overline{\mathcal{J}}$, which is impossible in view of (7.3.18) and (7.3.20). Using the symmetry of $(abc)_r$, we know that the possible values of $\lambda + \mu, \; \mu + \gamma, \; \gamma + \lambda$ are one of $\alpha, \; 1 + \alpha, \; \frac{1}{2} + \alpha$:

$$\lambda + \mu, \; \mu + \gamma, \; \gamma + \lambda = \alpha, \; 1 + \alpha, \text{ or } \frac{1}{2} + \alpha. \tag{7.3.23}$$

Lemma 7.3.20 shows that if $\zeta + \delta > 0$ or $\zeta = \delta = 0$, then $[[\bar{\mathfrak{g}}^{(-1)}, \bar{\mathfrak{g}}^{(1)}_\zeta], \bar{\mathfrak{g}}^{(1)}_\delta] = \{0\}$. Since $[\overline{jr}, a] \in \bar{\mathfrak{g}}^{(1)}_{\lambda-\alpha}$ and $[\overline{jr}, c] \in \bar{\mathfrak{g}}^{(1)}_{\gamma-\alpha}$, the condition (7.3.21) can be satisfied only in the case where $\lambda+\gamma-2\alpha\le 0$ and $\lambda-\alpha,\ \gamma-\alpha$ are not simultaneously zero. Using the symmetry of $(abc)_r$, we obtain

$$\lambda+\mu,\ \mu+\gamma,\ \gamma+\lambda\le 2\alpha, \tag{7.3.24}$$

where at most one of the three numbers λ,μ,γ is equal to α. This condition and (7.3.22)–(7.3.24) can be simultaneously satisfied when neither $\alpha = 0$ nor $\alpha=\frac{1}{2}$.

In Case 2, we have

$$[js,r_1]=\alpha(r_1-\zeta r_2),\ [js,r_2]=\alpha(r_2+\zeta r_1)\ (\zeta\in\mathbb{R})$$

with $\alpha=0$ or $\alpha=\pm\frac{1}{2}$ (cf. Lemma 7.3.3). If $\alpha=-\frac{1}{2}$, then $[s,r_1]=jr_1\notin\mathfrak{a}$, which is impossible. Furthermore, the proof of Lemma 7.2.4 shows that

$$[\overline{js},\overline{jr_1}]=-\alpha(\overline{jr_1}+\zeta\overline{jr_2}),\ [\overline{js},\overline{jr_2}]=-\alpha(\overline{jr_2}-\zeta\overline{jr_1}).$$

Let $\bar{u}_1,\bar{u}_2$ be two elements of $\bar{\mathfrak{g}}^{(-1)}$ not simultaneously zero such that

$$[\overline{js},\bar{u}_1]=\lambda(\bar{u}_1-\gamma\bar{u}_2),\ [\overline{js},\bar{u}_2]=\lambda(\bar{u}_2+\gamma\bar{u}_1)\ (\lambda,\gamma\in\mathbb{R}).$$

We set

$$x_1=[\overline{jr_1},\bar{u}_1]-[\overline{jr}_2,\bar{u}_2],\ y_1=[\overline{jr_1},\bar{u}_2]+[\overline{jr}_2,\bar{u}_1],$$
$$x_2=[\overline{jr_1},\bar{u}_1]+[\overline{jr}_2,\bar{u}_2],\ y_2=[\overline{jr_1},\bar{u}_2]-[\overline{jr}_2,\bar{u}_1].$$

Then,

$$[\overline{js},x_1+iy_1]=\{(\lambda-\alpha)+i(\lambda\gamma+\alpha\zeta)\}(x_1+iy_1),$$
$$[\overline{js},x_2+iy_2]=\{(\lambda-\alpha)+i(\lambda\gamma-\alpha\zeta)\}(x_2+iy_2).$$

By definition of the subspaces $\mathfrak{g}^{(k)}$, at least one of the elements x_1+iy_1, x_2+iy_2 is different from zero. The real part of the eigenvalues of the operator $\mathrm{ad}\,\overline{js}$ on $\bar{\mathfrak{g}}^{(1)}$ is 0 or $\pm\frac{1}{2}$. Thus $\lambda=\alpha$ or $\lambda=\pm\frac{1}{2}+\alpha$. Let $a\in\bar{\mathfrak{g}}^{(-1)}_\lambda,\ b\in\bar{\mathfrak{g}}^{(-1)}_\mu,\ c\in\bar{\mathfrak{g}}^{(-1)}_\gamma$ be such that

$$(abc)_x=[[[\overline{jx},a],b],c]\neq 0. \tag{7.3.25}$$

Then there is a nonzero element $y\in\mathfrak{a}$ such that

$$[\overline{jy},(abc)_x]=[[[\overline{jx},a],b],[\overline{jy},c]]\neq 0. \tag{7.3.26}$$

From what has been mentioned above, the conditions (7.3.25), (7.3.26) give rise to the same restrictions on the numbers λ, μ, γ as in Case 1. ■

The statement of this last lemma can be rephrased as follows:

$$[[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}] \subset \mathfrak{g}^{(0)}. \tag{7.3.27}$$

From the Jacobi identity and the commutativity of $\mathfrak{g}^{(1)}$, it follows that

$$[\mathfrak{g}^{(2)}, [\mathfrak{g}^{(1)}, \mathfrak{g}]] = \{0\}. \tag{7.3.28}$$

The condition (7.3.27) means $[jx, [[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}]] \subset \mathfrak{g}^{(2)}$ for $x \in \mathfrak{a}$. By (7.3.28) this is equivalent to

$$[[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(2)}. \tag{7.3.29}$$

The form β being closed, (7.3.28) implies

$$\beta([[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}^{(1)}], \mathfrak{g}^{(2)}) = \{0\}.$$

Comparing this with (7.3.29), we finally obtain the relation

$$[[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}^{(1)}] = \{0\}. \tag{7.3.30}$$

Lemma 7.3.23. *The centralizer $\mathfrak{z}(\mathfrak{g}^{(1)})$ of the subalgebra $\mathfrak{g}^{(1)}$ in $\mathfrak{g}$ is a Kähler ideal.*

Proof. Note that

$$\mathfrak{z}(\mathfrak{g}^{(1)}) \subset \mathfrak{g}^{(0)} \tag{7.3.31}$$

since $j\mathfrak{a} \subset \mathfrak{g}^{(1)}$ and $\mathfrak{z}(j\mathfrak{a}) \subset \mathfrak{g}^{(0)}$. Equation (7.3.30) means

$$[\mathfrak{g}^{(1)}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g}^{(1)}). \tag{7.3.32}$$

From the Jacobi identity, we have

$$[jx, [\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}]] \subset [\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}^{(1)}] = \{0\}$$

for any $x \in \mathfrak{a}$. Consequently $[\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}] \subset \mathfrak{g}^{(0)}$ and

$$[[\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}], \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}. \tag{7.3.33}$$

Furthermore from (7.3.32), it follows that

$$[[\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}], \mathfrak{g}^{(1)}] \subset [\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{z}(\mathfrak{g}^{(1)})]$$

and

$$\beta([[\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}], \mathfrak{g}^{(1)}], \mathfrak{g}^{(1)}) \subset \beta([\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{z}(\mathfrak{g}^{(1)})], \mathfrak{g}^{(1)}) = \{0\}. \tag{7.3.34}$$

Combining (7.3.33) with (7.3.34), we find

$$[[\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}], \mathfrak{g}^{(1)}] = \{0\}, i.e. [\mathfrak{z}(\mathfrak{g}^{(1)}), \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g}^{(1)}).$$

Hence $\mathfrak{z}(\mathfrak{g}^{(1)})$ is an ideal of $\mathfrak{g}$. Now we show that the ideal $\mathfrak{z}(\mathfrak{g}^{(1)})$ is invariant under j. If $z \in \mathfrak{z}(\mathfrak{g}^{(1)})$, then $jz \in \mathfrak{g}^{(0)}$. From the integrability condition, it follows that the operator $A = \mathrm{ad}_{\mathfrak{g}^{(1)}} jz$ commutes with j. Since β is closed and that $\mathfrak{g}^{(1)}$ is commutative, it follows that A is skew-symmetric with respect to β. Therefore the operator A is skew-symmetric with respect to the scalar product S and since it has no nonzero purely imaginary eigenvalue, we have $A = 0$. This means that $jz \in \mathfrak{z}(\mathfrak{g}^{(1)})$, and the lemma is proved. ■

Now we can prove Proposition 7.3.14. We denote by $\mathcal{N}$ the centre of $\mathfrak{z}(\mathfrak{g}^{(1)})$. This is a commutative ideal of $\mathfrak{g}$ and is j-invariant by Lemma 7.3.8 applied to the Kähler algebra $\mathfrak{z}(\mathfrak{g}^{(1)})$. Since $\mathcal{N} \supset \mathfrak{g}^{(1)}$, Proposition 7.3.14 has been proved under the induction hypothesis of Theorem 7.3.2.

Combining Propositions 7.3.13 and 7.3.14, Proposition 7.3.10 has been proved.

Proof of Theorem 7.3.2. Let $\mathcal{N}$ be a Kähler ideal of $\mathfrak{g}$ satisfying the conditions of either Proposition 7.3.13 or Proposition 7.3.14. Let $\mathfrak{g}'$ be the orthogonal complement of $\mathcal{N}$. Then $\mathfrak{g}'$ is a Kähler subalgebra of $\mathfrak{g}$. By the induction hypothesis, $\mathfrak{g}'$ can be decomposed into a semi-direct sum

$$\mathfrak{g}' = \mathcal{J}' + \mathcal{H}', \tag{7.3.35}$$

where $\mathcal{J}'$ is a commutative Kähler ideal and $\mathcal{H}'$ is an exponential j-subalgebra. Applying Lemma 7.2.3 on symplectic representations to the Kähler algebra $\mathcal{N} + \mathcal{J}'$, we see that

$$[\mathcal{N}, \mathcal{J}'] = \{0\}. \tag{7.3.36}$$

We consider separately the two cases corresponding the two possible types of the ideal $\mathcal{N}$.

(a) $\mathcal{N}$ is an elementary Kähler ideal.

We set $\mathcal{J} = \mathcal{J}', \mathcal{H} = \mathcal{N} + \mathcal{H}'$ and show that $\mathcal{H}$ is an exponential j-algebra. Since $\mathcal{N}$ and $\mathcal{H}'$ are j-algebras, there are $\omega_1 \in \mathcal{N}^*$ and $\omega_2 \in (\mathcal{H}')^*$ such that

$$\beta(x, y) = \omega_1([x, y]) \ \text{ for all } x, y \in \mathcal{N}, \tag{7.3.37}$$

$$\beta(u, v) = \omega_2([u, v]) \text{ for all } u, v \in \mathcal{H}'. \tag{7.3.38}$$

If we set $\omega = \omega_1 + \omega_2 \in \mathcal{H}^*$, then

$$\beta(p,q) = \omega([p,q]) \text{ for all } p,q \in \mathcal{H}.$$

In fact, we write $p = x + u,\ q = y + v$ with $x, y \in \mathcal{N}$ and $u, v \in \mathcal{H}'$. Note that

$$\mathcal{N} = \mathbb{R}jr + \mathbb{R}r + U,\ [\mathbb{R}jr + \mathbb{R}r, \mathfrak{g}'] = \{0\},\ [U, \mathfrak{g}'] \subset U$$

(cf. Proposition 7.3.13). Moreover, we have $\omega_1|_U = 0$ by the proof of Proposition 7.3.13. Thus

$$\beta(p,q) = \beta(x,y) + \beta(u,v)$$

and

$$\begin{aligned}\omega([p,q]) &= \omega([x+u, y+v]) = \omega_1([x,y]) + \omega_1([x,v] + [u,y]) + \omega_2([u,v]) \\ &= \omega_1([x,y]) + \omega_2([u,v]).\end{aligned}$$

These equalities together with the formulas (7.3.37), (7.3.38) show that $\beta(p,q) = \omega([p,q])$. From (7.3.35) and (7.3.36), it follows that $\mathcal{J}$ is an ideal of $\mathfrak{g}$. This finishes the proof in case (a).

(b) $\mathcal{N}$ is a commutative Kähler ideal.

We set $\mathcal{J} = \mathcal{N} + \mathcal{J}'$ and $\mathcal{H} = \mathcal{H}'$. From (7.3.35) and (7.3.36), it follows that the ideal $\mathcal{J}$ is commutative, so that the decomposition $\mathfrak{g} = \mathcal{J} + \mathcal{H}$ satisfies the requirements of the theorem. ■

7.4 Structure of Positive Polarizations

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in P^+(f,G)$. Further let $\mathfrak{d}$, $\mathfrak{e}$ be Lie subalgebras of $\mathfrak{g}$ defined in Definition 5.1.2 and let $\mathfrak{b} = \mathfrak{d} \cap \ker f$. Then $\mathfrak{d}$, $\mathfrak{b}$ are ideals of $\mathfrak{e}$ by Lemma 7.1.2 and $\mathfrak{z} = \mathfrak{d}/\mathfrak{b}$ is the centre of $\tilde{\mathfrak{e}} = \mathfrak{e}/\mathfrak{b}$ of dimension at most 1. Let $\pi : \mathfrak{e} \to \tilde{\mathfrak{e}}$ be the natural projection, $f_0 = f|_{\mathfrak{e}} \in \mathfrak{e}^*$, $\tilde{\mathfrak{h}} = \pi(\mathfrak{h})$ and let $\tilde{f} \in \tilde{\mathfrak{e}}^*$ such that $\tilde{f} \circ \pi = f_0$.

Let $\mathfrak{k}$ be a Lie algebra over $\mathbb{R}$. Recall that $\mathfrak{k}$ is called a Heisenberg algebra if its centre $\mathfrak{c}$ is of dimension 1 and $\mathfrak{k}/\mathfrak{c}$ is commutative.

Theorem 7.4.1. *$\tilde{\mathfrak{e}}$ can be decomposed into a semi-direct sum $\tilde{\mathfrak{e}} = \mathfrak{n} + \mathfrak{m}$, $\mathfrak{m}$ being a Lie subalgebra and $\mathfrak{n}$ being an ideal, satisfying the following conditions. Note that $\mathfrak{n}$ or $\mathfrak{m}$ may be $\{0\}$. Let $\mathfrak{h}_1 = \tilde{\mathfrak{h}} \cap \mathfrak{n}_{\mathbb{C}}$, $\mathfrak{h}_2 = \tilde{\mathfrak{h}} \cap \mathfrak{m}_{\mathbb{C}}$, $\tilde{f}_1 = \tilde{f}|_{\mathfrak{n}} \in \mathfrak{n}^*$ and $\tilde{f}_2 = \tilde{f}|_{\mathfrak{m}} \in \mathfrak{m}^*$.*

(a) $\mathfrak{n}$ *is a Heisenberg algebra with centre* $\mathfrak{z}$ *and* $\mathfrak{h}_1 \in P^+(\tilde{f}_1, N)$ *where* $N = exp\, \mathfrak{n}$.
(b) $\mathfrak{h}_2 \in P^+(\tilde{f}_2, M)$ *with* $M = exp\, \mathfrak{m}$, $\mathfrak{h}_2 + \overline{\mathfrak{h}_2} = \mathfrak{m}_{\mathbb{C}}$ *and* $\mathfrak{h}_2 \cap \mathfrak{m} = \{0\}$. *We define the linear operator* j *on* $\mathfrak{m}$ *by* $j(x) = -ix$ *if* $x \in \mathfrak{h}_2$ *and* $j(x) = ix$ *if* $x \in \overline{\mathfrak{h}_2}$. *Then* $(\mathfrak{m}, j, -\tilde{f}_2)$ *is an exponential* j*-algebra.*
(c) $\mathfrak{m} \cdot \tilde{f}_1 = \{0\}$, *where the operation is the coadjoint action.*

Proof. We know that $\mathfrak{d}$ is the orthogonal subspace to $\mathfrak{e}$ relative to B_f, so that B_f induces a non-degenerate alternating form $\hat{B}_f$ on $\hat{\mathfrak{e}} = \mathfrak{e}/\mathfrak{d}$. $\hat{\mathfrak{e}}_{\mathbb{C}}$ may be identified with $\mathfrak{e}_{\mathbb{C}}/\mathfrak{d}_{\mathbb{C}}$ and $\hat{\mathfrak{e}}_{\mathbb{C}} = \mathfrak{h}/\mathfrak{d}_{\mathbb{C}} + \overline{\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}}$ where the sum is direct. If we consider $\hat{\mathfrak{e}}$ as the real form of $\hat{\mathfrak{e}}_{\mathbb{C}}$, then $\bar{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}} = \overline{\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}}$. We define $j \in \mathrm{End}\hat{\mathfrak{e}}_{\mathbb{C}}$ by $j = -i$ on $\mathfrak{h}/\mathfrak{d}_{\mathbb{C}}$ and $j = i$ on $\bar{\mathfrak{h}}/\mathfrak{d}_{\mathbb{C}}$. Then, j maps $\hat{\mathfrak{e}}$ to itself. If we define an alternating form β by $\beta(X,Y) = -\hat{B}_f(X,Y)$ for $X, Y \in \hat{\mathfrak{e}}$, $(\hat{\mathfrak{e}}, j, \beta)$ is an exponential Kähler algebra (cf. Section 4 in Chapter I of [3]).

Case 1. We assume $f|_{\mathfrak{d}} = 0$. Since $\tilde{\mathfrak{e}} = \mathfrak{e}/\mathfrak{b} = \mathfrak{e}/\mathfrak{d} = \hat{\mathfrak{e}}$, $\tilde{\mathfrak{h}} \in P^+(\tilde{f}, \tilde{E})$ with $\tilde{E} = \exp(\tilde{\mathfrak{e}})$. $\tilde{\mathfrak{h}} + \overline{\tilde{\mathfrak{h}}} = \tilde{\mathfrak{e}}_{\mathbb{C}}$ and $\tilde{\mathfrak{h}} \cap \tilde{\mathfrak{e}} = \{0\}$, it is easy to see that $(\tilde{\mathfrak{e}}, j, -\tilde{f})$ is an exponential j-algebra.

Case 2. Assume $f|_{\mathfrak{d}} \neq 0$. Let $Z \in \mathfrak{z}$ be such that $\tilde{f}(Z) = 1$ and let $\hat{\pi} : \tilde{\mathfrak{e}} \to \hat{\mathfrak{e}}$ be the natural projection. By Theorem 7.3.2, $\hat{\mathfrak{e}}$ can be decomposed into a semi-direct sum $\hat{\mathfrak{e}} = \mathcal{J} + \mathcal{H}$, where $\mathcal{J}$ is a commutative Kähler ideal and $\mathcal{H}$ is an exponential j-algebra. For $X \in \hat{\mathfrak{e}}$, there uniquely exists an element $X' \in \tilde{\mathfrak{e}} \cap \ker \tilde{f}$ such that $\hat{\pi}(X') = X$. We identify $\hat{\mathfrak{e}}$ with $\ker \tilde{f}$ as vector space by the correspondence $X \leftrightarrow X'$. Now by Theorem 7.2.5,

$$
\begin{aligned}
&\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1, \ \mathcal{H}_0 = \mathfrak{a} \oplus \sum_{m>k} \eta^{\frac{1}{2}(\alpha_m - \alpha_k)}, \\
&\mathcal{H}_{\frac{1}{2}} = \sum_k \tilde{\eta}^{\frac{1}{2}\alpha_k}, \ \mathcal{H}_1 = \sum_{m \geq k} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)} \\
&\dim(\eta^{\alpha_k}) = 1 \ (1 \leq k, m \leq r),
\end{aligned}
$$

where r denotes the rank of $\mathcal{H}$. We take the nonzero element $U_k \in \eta^{\alpha_k}$ introduced in Theorem 7.2.5. Then,

$$
\mathfrak{a} = \sum_{k=1}^{r} \mathbb{R} jU_k, \ [jU_k, U_l] = \delta_{kl} U_l, \ \text{i.e. } \alpha_l(jU_k) = \delta_{kl}.
$$

Since $\mathcal{H}$ is an exponential j-algebra, there exists $\omega \in \mathcal{H}^*$ such that

$$
\beta(X,Y) = -\tilde{f}([X', Y']) = \omega([X,Y]) \tag{7.4.1}
$$

for all $X, Y \in \mathcal{H}$. This shows that

$$
\omega(U_k) = \omega([jU_k, U_k]) = \tilde{f}([U_k', (jU_k)']) > 0.
$$

Since $\tilde{f}([\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}]) = \{0\}$, we have

$$\tilde{f}([A' + i(jA)', X' + i(jX)']) = 0$$

for $A \in \mathfrak{a}$ and $X \in \mathcal{H}_0 + \mathcal{H}_{\frac{1}{2}}$. Hence by (7.4.1),

$$\omega([A, X]) = 0 = \omega([jA, X] + [A, jX]) = \omega(j[A, X])$$

and ω vanishes on

$$\sum_{m>k} \left(\eta^{\frac{1}{2}(\alpha_m - \alpha_k)} + \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\right) \oplus \mathcal{H}_{\frac{1}{2}}.$$

Thus, if we set $\xi_0 = \omega|_{\mathcal{H}_1}$, ξ_0 can be written in the form $\xi_0 = \sum_{k=1}^{r} a_k U_k^*$ with $a_k > 0$ $(1 \leq k \leq r)$. Let $[(jU_k)', U_l'] = \delta_{kl} U_l' + b_{k,l} Z$. Then

$$b_{k,l} = f([(jU_k)', U_l']) = -\omega([jU_k, U_l]) = -\delta_{kl} a_k \ (1 \leq k, l \leq r). \tag{7.4.2}$$

We set

$$\mathfrak{m} = (\mathcal{H}_0)' \oplus \left(\mathcal{H}_{\frac{1}{2}}\right)' \oplus \sum_{k=1}^{r} \mathbb{R}(U_k' - a_k Z) \oplus \left(\sum_{m>k} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\right)',$$

where $V' = \{X'; X \in V\}$ for subspaces V of $\hat{\mathfrak{e}}$. We show that $\mathfrak{m}$ is a Lie subalgebra of $\tilde{\mathfrak{e}}$. First, the subspaces $\mathcal{H}_\lambda$ $(\lambda = 0, 1/2, 1)$ are mutually orthogonal with respect to the scalar product S, which is defined on $\hat{\mathfrak{e}}$ by $S(X, Y) = \beta(jX, Y)$ for $X, Y \in \hat{\mathfrak{e}}$ (cf. Lemma 7.3.4). So, if we set

$$\mathcal{H}_1'' = \sum_{k=1}^{r} \mathbb{R}(U_k' - a_k Z) \oplus \left(\sum_{m>k} \eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\right)',$$

then $(\mathcal{H}_0)'$, $\mathcal{H}_1''$ are Lie subalgebras of $\tilde{\mathfrak{e}}$ and

$$\left[(\mathcal{H}_0)', \left(\mathcal{H}_{\frac{1}{2}}\right)'\right] \subset \left(\mathcal{H}_{\frac{1}{2}}\right)', \ \left[\left(\mathcal{H}_{\frac{1}{2}}\right)', \mathcal{H}_1''\right] = \{0\}.$$

Since ω vanishes on $\eta^{\frac{1}{2}(\alpha_m + \alpha_k)}$ $(m > k)$,

$$\left[\mathfrak{a}', \left(\eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\right)'\right] \subset \left(\eta^{\frac{1}{2}(\alpha_m + \alpha_k)}\right)',$$
$$\left[\left(\eta^{\frac{1}{2}(\alpha_m - \alpha_k)}\right)', \left(\eta^{\frac{1}{2}(\alpha_p + \alpha_k)}\right)'\right] \subset \left(\eta^{\frac{1}{2}(\alpha_m + \alpha_p)}\right)'$$

for $m > k, p > k, m \neq p$ and

$$\left[\left(\eta^{\frac{1}{2}(\alpha_m-\alpha_k)}\right)', \left(\eta^{\frac{1}{2}(\alpha_k+\alpha_p)}\right)'\right] \subset \left(\eta^{\frac{1}{2}(\alpha_m+\alpha_p)}\right)'$$

for $m > k > p$. For $X \in \eta^{\frac{1}{2}(\alpha_m-\alpha_k)}$ and $Y \in \eta^{\frac{1}{2}(\alpha_m+\alpha_k)}$ $(m > k)$, we write $[X, Y] = \gamma U'_m + \delta Z$. Similarly to (7.4.2), we get $\delta = -a_m\gamma$. Thus $[X', Y'] = \gamma(U'_m - a_m Z)$. These considerations and relation (7.4.2) show that $[(\mathcal{H}_0)', \mathcal{H}''_1] \subset \mathcal{H}''_1$. Finally, we have $\left[\left(\mathcal{H}_{\frac{1}{2}}\right)', \left(\mathcal{H}_{\frac{1}{2}}\right)'\right] \subset \mathcal{H}''_1$. In fact, we see

$$\left[\left(\tilde{\eta}^{\frac{1}{2}\alpha_m}\right)', \left(\tilde{\eta}^{\frac{1}{2}\alpha_k}\right)'\right] \subset \left(\eta^{\frac{1}{2}(\alpha_m+\alpha_k)}\right)' \ (m \neq k)$$

as above. For $X, Y \in \tilde{\eta}^{\frac{1}{2}\alpha_k}$, we write $[X', Y'] = \alpha U'_k + \zeta Z$. Then just like (7.4.2), it follows that $\zeta = -a_k\alpha$, which proves $\left[\left(\mathcal{H}_{\frac{1}{2}}\right)', \left(\mathcal{H}_{\frac{1}{2}}\right)'\right] \subset \mathcal{H}''_1$. This finishes the verification that $\mathfrak{m}$ is a Lie subalgebra. Next we set $\mathfrak{n} = \mathcal{J}' \oplus \mathfrak{z}$. It is clear that $\mathfrak{n}$ is an ideal of $\tilde{\mathfrak{e}}$. Since $\beta(jx, x) > 0$ for any nonzero element $x \in \mathcal{J}$, $\mathfrak{n}$ has the one-dimensional centre $\mathfrak{z}$. Since $\mathfrak{n}/\mathfrak{z}$ is commutative, $\mathfrak{n}$ is a Heisenberg algebra. Finally, the orthogonality of the spaces $\mathcal{J}$ and $\mathcal{H}$ with respect to the scalar product S (cf. Lemma 7.3.4) means that $\mathfrak{m}\cdot f_1 = \{0\}$. ■

7.5 Non-vanishing of $\mathcal{H}(f, \mathfrak{h}, G)$

A. Siegel domains of type II.

Definition 7.5.1. Let Ω be a convex cone in $\mathbb{R}^n$. Its **dual cone** Ω^* is defined by

$$\Omega^* = \{\xi \in \mathbb{R}^n; \langle \xi, y\rangle > 0, \ \forall\, y \in \overline{\Omega}\backslash\{0\}\},$$

where $\langle \xi, y\rangle = \sum_{i=1}^n \xi_i y_i$ for $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. A convex cone Ω in $\mathbb{R}^n$ such that $\Omega^* \neq \emptyset$ is called **proper**.

Definition 7.5.2. Let Ω be an open proper convex cone in $\mathbb{R}^n$. A real-bilinear form $Q : \mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}^n$ is called a **Ω-hermitian form** if it has the following properties:

(1) Q is complex-linear in the first variable.
(2) $Q(u, u') = \overline{Q(u', u)}$.
(3) $Q(u, u) \in \overline{\Omega}$ for all $u \in \mathbb{C}^q$.
(4) $Q(u, u) = 0$ implies $u = 0$.

Definition 7.5.3. Let Ω be an open proper convex cone in $\mathbb{R}^n$ and Q an Ω-hermitian form. The domain

$$D(\Omega, Q) = \{(x + iy, u) \in \mathbb{C}^n \times \mathbb{C}^q; y - Q(u, u) \in \Omega\}$$

is called the **Siegel domain of type II** associated to (Ω, Q). We shall often write simply D for $D(\Omega, Q)$.

Let $\mathcal{O}(D)$ represent the space of holomorphic functions on the Siegel domain $D(\Omega, Q)$. Let Ψ (resp. Φ) be a positive continuous function on Ω (resp. $\mathbb{C}^q$). We set

$$H(D, \Psi, \Phi) = \big\{F \in \mathcal{O}(D);\\ \|F\|^2_{\Psi,\Phi} = \int_D |F(z,u)|^2 \Psi(y - Q(u,u))\Phi(u)dxdydu < \infty\big\}.$$

The following two lemmas can be proved just as in Chapter 2 of [68].

Lemma 7.5.4. *$H(D, \Psi, \Phi)$ is a Hilbert space.*

Lemma 7.5.5. *For $F \in H(D, \Psi, \Phi)$, let*

$$N_F^2(y,u) = \int_{\mathbb{R}^n} |F(x + iy, u)|^2 dx.$$

For almost all u, $N_F^2(y,u)$ is finite for all y. Further, for such u,

$$\hat{F}(\xi, u) = \int_{\mathbb{R}^n} F(x + iy, u) e^{-i\langle \xi, z\rangle} dx$$

is independent of y and

$$F(z,u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{F}(\xi, u) e^{i\langle \xi, z\rangle} d\xi,$$

$$N_F^2(y,u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{F}(\xi, u)|^2 e^{-2\langle \xi, y\rangle} d\xi,$$

$$\|F\|^2_{\Psi,\Phi} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{C}^q} |\hat{F}(\xi, u)|^2 e^{-2\langle \xi, Q(u,u)\rangle} \Phi(u) du \right) I_\Psi(\xi) d\xi,$$

where

$$I_\Psi(\xi) = \int_\Omega e^{-2\langle \xi, t\rangle} \Psi(t) dt.$$

Let $\Phi_m(u) = e^{m|u|^2} = e^{m\langle u,u\rangle}$ $(u \in \mathbb{C}^q)$, where m is some positive real number. Let $\hat{\Omega}_\Psi$ be the set of $\xi \in \mathbb{R}^n$ such that

$$I_\Psi(\xi) = \int_\Omega e^{-2\langle \xi, t\rangle} \Psi(t) dt < \infty,$$

and let $\hat{\Omega}^{Q}_{\Phi_m}$ be the set of $\xi \in \mathbb{R}^n$ such that $2\langle \xi, Q(u,u)\rangle - m|u|^2$ is a positive definite form on $\mathbb{C}^q$.

Lemma 7.5.6. *If $\hat{\Omega}_\Psi \cap \hat{\Omega}^{Q}_{\Phi_m}$ has interior points, then $H(D, \Psi, \Phi_m) \neq \{0\}$.*

Proof. Let B be a closed ball contained in $\hat{\Omega}_\Psi \cap \hat{\Omega}^{Q}_{\Phi_m}$, and let $K = \max\{I_\Psi(\xi); \xi \in B\}$ whose existence follows from the continuity of I_Ψ on $\hat{\Omega}_\Psi$ (cf. Chapter 2 of [68]). For $\xi \in B$, $2\langle \xi, Q(u,u)\rangle - m|u|^2$ is a positive definite form on $\mathbb{C}^q$ and the mapping $\xi \mapsto 2\langle \xi, Q(u,u)\rangle - m|u|^2$ is continuous. Thus, there is a positive number k such that $2\langle \xi, Q(u,u)\rangle - m|u|^2 \geq k|u|^2$ for all $u \in \mathbb{C}^q$ and $\xi \in B$. Now

$$F(z,u) = \frac{1}{(2\pi)^{n/2}} \int_B e^{i\langle \xi, z\rangle} d\xi$$

is an entire function on $\mathbb{C}^n \times \mathbb{C}^q$ and

$$\begin{aligned} \|F\|^2_{\Psi,\Phi_m} &= \int_B \left\{ \int_{\mathbb{C}^q} e^{-2\langle \xi, Q(u,u)\rangle + m|u|^2} du \right\} I_\Psi(\xi) d\xi \\ &\leq K \mathrm{vol}(B) \int_{\mathbb{C}^q} e^{-k|u|^2} du < \infty. \end{aligned}$$ ∎

B. The non-vanishing of $\mathcal{H}(f, \mathfrak{h}, G)$.

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in P^+(f, G)$. We keep the notations used in the previous sections. Let $E = \exp \mathfrak{e}$, $\tilde{E} = \exp(\tilde{\mathfrak{e}})$ and $\pi' : E \to \tilde{E}$ the natural projection. Then

$$\mathfrak{h} \in P^+(f_0, E),\ \tilde{\mathfrak{h}} \in P^+(\tilde{f}, \tilde{E}),\ \mathcal{H}(f_0, \mathfrak{h}, E) = \mathcal{H}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{E}) \circ \pi'$$

and $\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f_0, \mathfrak{h}, E)$, so that $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$ if and only if $\mathcal{H}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{E}) \neq \{0\}$. Therefore, as for the non-vanishing of $\mathcal{H}(f, \mathfrak{h}, G)$, we can confine ourselves to the consideration of the following two cases by Theorem 7.4.1. The triplet $(\mathfrak{g}, \mathfrak{h}, f)$ satisfies:

Case I. $\mathfrak{e} = \mathfrak{g}$ and $\mathfrak{d} = \{0\}$.
Case II. $\mathfrak{e} = \mathfrak{g}$, $\mathfrak{d}$ is equal to the one-dimensional centre $\mathfrak{z}$ of $\mathfrak{g}$ and $f|_{\mathfrak{z}} \neq 0$.

We separately examine these two cases.

B_1 : Case I. Theorem 7.2.5 enables us to generalize to exponential groups, without any modification, the method and the results of [68] for completely solvable Lie groups. For details of the following observations, see Section 5 in Chapter 2 of [60] and Chapter 4 of [68].

We employ the notations of Theorem 7.2.5 by which we can realize G as a transitive group of affine automorphisms on a Siegel domain of type II as follows. Let $G_k = \exp(\mathfrak{g}_k)$ $(k = 0, 1/2, 1)$ and Ω the orbit of s in $\mathfrak{g}_1$ under the adjoint representation of G_0. Then Ω is an open proper convex cone in $\mathfrak{g}_1$. Next, we consider the space $\mathfrak{g}_{\frac{1}{2}}$ with complex structure $j|_{\mathfrak{g}_{\frac{1}{2}}}$ and define the form $Q : \mathfrak{g}_{\frac{1}{2}} \times \mathfrak{g}_{\frac{1}{2}} \to \mathfrak{g}_1 + i\mathfrak{g}_1$ by

$$Q(U, V) = \frac{1}{4}([jV, U] - i[V, U]).$$

Then Q is an Ω-hermitian form. Thus we can construct the Siegel domain of type II:

$$D(\Omega, Q) = \{(X + iY, U); X, Y \in \mathfrak{g}_1,\ U \in \mathfrak{g}_{\frac{1}{2}},\ Y - Q(U, U) \in \Omega\}.$$

The group $G = G_0 \cdot G_{\frac{1}{2}} \cdot G_1$ acts on the domain $D(\Omega, Q)$ by the following rules:

(i) $R(h_0) : (z, u) \mapsto (h_0 \cdot z, h_0 \cdot u),\ h_0 \in G_0$,
(ii) $R(\exp(u_0)) : (z, u) \mapsto (z + 2iQ(u, u_0) + iQ(u_0, u_0), u + u_0),\ u_0 \in \mathfrak{g}_{\frac{1}{2}}$,
(iii) $R(\exp(x_1)) : (z, u) \mapsto (z + x_1, u), x_1 \in \mathfrak{g}_1$.

The point $(is, 0)$ is in $D(\Omega, Q)$ and the map α from G to $D(\Omega, Q)$ defined by

$$\alpha : h_0 \cdot \exp u \cdot \exp x \mapsto (h_0 \cdot (x + is + iQ(u, u)), h_0 \cdot u)$$

is a bijection of G onto $D(\Omega, Q)$. A function $\phi \in C^\infty(G)$ is holomorphic on $D(\Omega, Q)$ if and only if $\phi \cdot (X + ijX) = 0$ for all $X \in \mathfrak{g}$. These facts imply that the non-vanishing of the space $\mathcal{H}(f, \mathfrak{h}, G)$ is equivalent to the existence of a nonzero holomorphic function on $D(\Omega, Q)$ which belongs to the L^2-space with respect to some Radon measure.

Definition 7.5.7. Let α_k, $\eta^{\frac{1}{2}(\alpha_m - \alpha_k)}$ and $\tilde{\eta}^{\frac{1}{2}\alpha_k}$ be as defined in Theorem 7.2.5. Let

$$\mathcal{L}_k = \sum_{m>k} \eta^{\frac{1}{2}(\alpha_m - \alpha_k)},\ \mathcal{L}'_k = \sum_{k>i} \eta^{\frac{1}{2}(\alpha_k - \alpha_i)},$$

$$p_k = \dim(\mathcal{L}'_k),\ q_k = \dim(\mathcal{L}_k), r_k = \dim(\tilde{\eta}^{\frac{1}{2}\alpha_k}).$$

The following two theorems can be proved in a way similar to that used in [68].

Theorem 7.5.8. *$\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$ if and only if*

$$-2f_k - \left(p_k + 1 + \frac{1}{2}(q_k + r_k)\right) > 0\ (1 \leq k \leq r),$$

where $f_k = f(U_k)$ and r denotes the rank of $\mathfrak{g}$.

Theorem 7.5.9. *Suppose $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$. Then $\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(G \cdot f)$, the irreducible unitary representation of G corresponding to the coadjoint orbit $G \cdot f$.*

B_2 : Case II. In this case, we have from Theorem 7.4.1 the following situation. The Lie algebra $\mathfrak{g}$ is decomposed into a semi-direct sum $\mathfrak{g} = \mathfrak{n} + \mathfrak{m}$, where $\mathfrak{n}$ is an ideal and $\mathfrak{m}$ is a Lie subalgebra. We set $N = \exp \mathfrak{n}$ and $M = \exp \mathfrak{m}$. Thus $G = N \rtimes M$ (semi-direct product). Let

$$\tilde{f}_1 = f|_{\mathfrak{n}} \in \mathfrak{n}^*, \ \tilde{f}_2 = f|_{\mathfrak{m}} \in \mathfrak{m}^*, \ \mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{n}_{\mathbb{C}}, \ \mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{m}_{\mathbb{C}}$$

and let $\mathfrak{d}_1, \ \mathfrak{e}_1$ (resp. $\mathfrak{d}_2, \ \mathfrak{e}_2$) be the Lie subalgebras of $\mathfrak{n}$ (resp. $\mathfrak{m}$) defined in Definition 5.1.2. Then,

$$\mathfrak{h}_1 \in P^+(\tilde{f}_1, N), \ \mathfrak{h}_2 \in P^+(\tilde{f}_2, M),$$
$$\mathfrak{d}_1 = \mathfrak{z}, \ \mathfrak{e}_1 = \mathfrak{n}, \ \mathfrak{d}_2 = \{0\}, \ \mathfrak{e}_2 = \mathfrak{m}, \ \mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2.$$

Furthermore, $\mathfrak{n}$ is a Heisenberg algebra with centre $\mathfrak{z}$ and $(\mathfrak{m}, j, -\tilde{f}_2)$ is an exponential j-algebra with j defined in Theorem 7.4.1 and the triplet $(\mathfrak{m}, \mathfrak{h}_2, \tilde{f}_2)$ satisfies the condition of Case I. So the notions, the notations and the results in B_1 can be transferred to $\mathfrak{m}$, which we shall do without explicit mentioning. For example, the numbers $p_k, \ q_k$ and r_k are defined as in Definition 7.5.7.

Let $W = \ker \tilde{f}_1 \subset \mathfrak{n}$. Since $\mathfrak{m} \cdot \tilde{f}_1 = 0$, W is invariant under $\mathrm{ad}_{\mathfrak{n}}\mathfrak{m}$. If we regard W as commutative Lie algebra, then the semi-direct sum $W + \mathfrak{m}$ has the structure of an exponential Kähler algebra.

Lemma 7.5.10. *The adjoint representation $\mathrm{ad}_{W_{\mathbb{C}}}\mathfrak{a}$ of the commutative subalgebra $\mathfrak{a} \subset \mathfrak{m}$ on $W_{\mathbb{C}}$ is diagonalizable and $W_{\mathbb{C}}$ can be decomposed into root spaces $(W_{\mathbb{C}})^{\gamma}$ with roots of the form $\gamma(A) = \pm\frac{1}{2}\alpha_k(A)(1 + i\gamma_{k,l})$ $(A \in \mathfrak{a})$ with $\gamma_{k,l} \in \mathbb{R}$ or $\gamma = 0$ (not all possibilities need occur). We put, for $1 \le k \le r$,*

$$\tilde{W}_{\mathbb{C}}^{\frac{1}{2}\alpha_k} = \sum_{\gamma=\frac{1}{2}(1+i\gamma_{k,l})\alpha_k} (W_{\mathbb{C}})^{\gamma} \ \left(resp. \ \tilde{W}_{\mathbb{C}}^{-\frac{1}{2}\alpha_k} = \sum_{\gamma=-\frac{1}{2}(1+i\gamma_{k,l})\alpha_k} (W_{\mathbb{C}})^{\gamma} \right),$$
$$\tilde{W}^{\frac{1}{2}\alpha_k} = \tilde{W}_{\mathbb{C}}^{\frac{1}{2}\alpha_k} \cap W \ \left(resp. \ \tilde{W}^{-\frac{1}{2}\alpha_k} = \tilde{W}_{\mathbb{C}}^{-\frac{1}{2}\alpha_k} \cap W \right).$$

Then W can be decomposed into the direct sum

$$W = \sum_k \tilde{W}^{-\frac{1}{2}\alpha_k} + W_0 + \sum_k \tilde{W}^{\frac{1}{2}\alpha_k},$$

where $W_0 = \{X \in W; [A, X] = 0, \ \forall \ A \in \mathfrak{a}\}$.

Note that $j\tilde{W}^{-\frac{1}{2}\alpha_k} = \tilde{W}^{\frac{1}{2}\alpha_k}$ and that $W = W_{-\frac{1}{2}} \oplus W_0 \oplus W_{\frac{1}{2}}$ gives the decomposition in Lemma 7.3.3 with $W_{-\frac{1}{2}} = \sum_k \tilde{W}^{-\frac{1}{2}\alpha_k}, \ W_{\frac{1}{2}} = \sum_k \tilde{W}^{\frac{1}{2}\alpha_k}$.

Proof. We assume that there exist two elements $X, \ Y$ in W not simultaneously zero and two numbers k, l $(1 \le k, l \le r)$ such that

$$[jU_k, X] = -\frac{1}{2}(X - \mu Y), \ \ [jU_k, Y] = -\frac{1}{2}(Y + \mu X)$$

$$[jU_l, X] = \frac{1}{2}(X - \mu Y), \quad [jU_l, Y] = \frac{1}{2}(Y + \mu X),$$

with $\mu \in \mathbb{R}$. To prove the lemma it is enough to show that this assumption gives rise to a contradiction. In fact, we have $[U_k, X] = jX$, $[U_l, X] = 0$ by Lemma 7.2.4. We consider the adjoint representation of the elementary Kähler algebra $\mathbb{R}j(U_k + U_l) \oplus \mathbb{R}(U_k + U_l)$ of dimension 2. Then $[j(U_k + U_l), X] = 0$ so that $[U_k + U_l, X] = 0$ since the representation is also symplectic. Thus we have $jX = 0$ and $X = 0$. Similarly $Y = 0$, which is impossible. ■

Definition 7.5.11. We set

$$t_k = \dim(\tilde{W}^{-\frac{1}{2}\alpha_k}) = \dim(\tilde{W}^{\frac{1}{2}\alpha_k}) \ (1 \leq k \leq r),$$

$$m = \dim(W_{-\frac{1}{2}}) = \dim(W_{\frac{1}{2}}) = \sum_{k=1}^{r} t_k$$

and $n = \frac{1}{2}\dim(W_0)$.

The space W possesses the inner product S: $S(X, Y) = f([X, jY])$ for $X, Y \in W$. As in Lemma 7.3.4 it is found that the subspaces W_0, $\tilde{W}^{\pm\frac{1}{2}\alpha_k}$ $(1 \leq k \leq r)$ are mutually orthogonal with respect to S. This fact together with the proof of Theorem 7.2.4 enables us to take a basis

$$\{P_1, P_2, \ldots, P_{m+n}, Q_1, Q_2, \ldots, Q_{m+n}\}$$

with the following properties. Let C be the nonzero element of $\mathfrak{z}$ such that $f(C) = 1$. Then,

$$jP_k = Q_k, \ [P_k, Q_l] = \delta_{kl}C, \ [P_k, P_l] = [Q_k, Q_l] = 0$$

for $1 \leq k, l \leq m + n$.

$$\tilde{W}^{-\frac{1}{2}\alpha_k} = \langle P_{i_{k-1}+1}, P_{i_{k-1}+2}, \ldots, P_{i_k} \rangle_{\mathbb{R}}$$

with $i_0 = 0$, $i_r = m$, $i_{k-1} \leq i_k$ $(1 \leq k \leq r)$ and

$$W_0 = \langle P_{m+1}, P_{m+2}, \ldots, P_{m+n}, Q_{m+1}, Q_{m+2}, \ldots Q_{m+n} \rangle_{\mathbb{R}}.$$

So

$$\tilde{W}^{\frac{1}{2}\alpha_k} = \langle Q_{i_{k-1}+1}, Q_{i_{k-1}+2}, \ldots, Q_{i_k} \rangle_{\mathbb{R}}$$

and $t_k = i_k - i_{k-1}$ $(1 \leq k \leq r)$. Moreover,

$$[jU_k, P_\nu] = -\frac{1}{2}P_\nu,\ [jU_k, Q_\nu] = \frac{1}{2}Q_\nu\ (i_{k-1}+1 \le \nu \le i'_k),$$

$$[jU_k, P_\nu] = -\frac{1}{2}(P_\nu - \gamma_\nu P_{\nu+i''_k}),\ [jU_k, P_{\nu+i''_k}] = -\frac{1}{2}(P_{\nu+i''_k} + \gamma_\nu P_\nu),$$

$$[jU_k, Q_\nu] = \frac{1}{2}(Q_\nu + \gamma_\nu Q_{\nu+i''_k}),$$

$$[jU_k, Q_{\nu+i''_k}] = \frac{1}{2}(Q_{\nu+i''_k} - \gamma_\nu Q_\nu)\ (i'_k + 1 \le \nu \le i'_k + i''_k)$$

with $i_{k-1} \le i'_k \le i'_k + i''_k$ so that

$$i_k = i'_k + 2i''_k,\ t_k = i'_k - i_{k-1} + 2i''_k\ (1 \le k \le r).$$

Finally,

$$\mathfrak{h} = \sum_{k=1}^{m+n} \mathbb{C}(P_k + iQ_k) + \mathbb{C}C + \mathfrak{h}_2.$$

We set $Z = \exp \mathfrak{z}$.

Proposition 7.5.12. *Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in P^+(f, G)$. We set*

$$\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g},\ \mathfrak{b} = \mathfrak{d} \cap ker f,\ \mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$$

and $E = exp\ \mathfrak{e}$. If $E{\cdot}f$ is closed in $\mathfrak{g}^*$, $\tilde{\mathfrak{e}} = \mathfrak{e}/\mathfrak{b}$ is a Heisenberg algebra with centre $\mathfrak{z} = \mathfrak{d}/\mathfrak{b}$ so that $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$. In this case, $\rho(f, \mathfrak{h}, G)$ is irreducible and $\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(G{\cdot}f)$. In particular, The equivalence class of $\rho(f, \mathfrak{h}, G)$ is independent of such $\mathfrak{h}$.

Proof. By Theorem 7.1.1, it suffices to show that $\tilde{\mathfrak{e}}$ is a Heisenberg algebra. Let $B = \exp \mathfrak{b}$, $\tilde{E} = E/B$, $\tilde{e} = \mathfrak{n} + \mathfrak{m}$ be the semi-direct decomposition given in Theorem 7.4.1, $N = \exp \mathfrak{n}$ and $M = \exp \mathfrak{m}$. Then, $\tilde{E}$ is decomposed into the semi-direct product $\tilde{E} = N \rtimes M$. Let $\tilde{f} \in \tilde{\mathfrak{e}}^*$ be the linear form induced by f and $\mathcal{O}(\tilde{f})$ the coadjoint orbit of $\tilde{E}$ through $\tilde{f}$. Then we have

$$\mathcal{O}(\tilde{f}) = \{\ell \in \tilde{\mathfrak{e}}^*; \ell|_{\mathfrak{z}} = f|_{\mathfrak{z}}\}. \tag{7.5.1}$$

Suppose that $\mathfrak{m} \neq \{0\}$ and denote by s the principal idempotent in $\mathfrak{m}$. Then, $(h{\cdot}\tilde{f})(s) < 0$ and $(h{\cdot}\tilde{f})|_{\mathfrak{n}} = \tilde{f}|_{\mathfrak{n}}$ for $h \in M$. For any $a \in N$, we write

$$a^{-1} = \exp X,\ X = X_1 + X_2 + X_3 + X_4$$

with

$$X_1 \in W_{-\frac{1}{2}},\ X_2 \in W_0,\ X_3 \in W_{\frac{1}{2}},\ X_4 \in \mathfrak{z}.$$

Then,

$$(\mathrm{ad}X)(s) = [X_1 + X_2 + X_3 + X_4, s] = -jX_1,$$
$$(\mathrm{ad}X)^2(s) = [X_1, -jX_1] \in \mathfrak{z},\ (\mathrm{ad}X)^k(s) = 0\ (k \geq 3).$$

It follows that

$$((a{\cdot}h) \cdot \tilde{f})(s) = (h \cdot \tilde{f})(a^{-1}{\cdot}s) = (h \cdot \tilde{f})(s - jX_1 - \frac{1}{2}[X_1, jX_1])$$
$$= (h \cdot \tilde{f})(s) - \frac{1}{2}f([X_1, jX_1]) < 0.$$

Since $h \in M,\ a \in N$ are arbitrary, this contradicts (7.5.1). ■

By the integrability of j,

$$j[X, Y] - j[jX, Y] = [jX, Y] + [X, jY]$$

for any $X \in \mathfrak{m}, Y \in W$. In particular,

$$j[X, Y] = [jX, Y] \tag{7.5.2}$$

for $X \in \eta^{\frac{1}{2}(\alpha_k - \alpha_l)},\ Y \in \tilde{W}^{-\frac{1}{2}\alpha_k}$ and for $X \in \tilde{\eta}^{\frac{1}{2}\alpha_k},\ Y \in \tilde{W}^{-\frac{1}{2}\alpha_k}$. Also we have

$$[jX, jY] = [X, Y] \tag{7.5.3}$$

for $X \in \eta^{\frac{1}{2}(\alpha_k + \alpha_l)}(k > l),\ Y \in \tilde{W}^{-\frac{1}{2}\alpha_l}$ and for $X \in \tilde{\eta}^{\frac{1}{2}\alpha_k},\ Y \in W_0$. Let V be an element in $\eta^{\frac{1}{2}(\alpha_k - \alpha_l)}$ or in $\tilde{\eta}^{\frac{1}{2}\alpha_k}$ such that $[jV, V] = 2U_k$. Then by (7.5.2) and (7.5.3), we have for $P \in \tilde{W}^{-\frac{1}{2}\alpha_k}$

$$[V, [jV, P]] = [V, j[V, P]] = -[jV, [V, P]]$$

and

$$2jP = 2[U_k, P] = [[jV, V], P] = [jV, [V, P]] - [V, [jV, P]].$$

Thus

$$[jV, [V, P]] = -[V, [jV, P]] = jP$$

and

$$\begin{aligned}B_f([V,P],j[V,P]) &= B_f([V,P],[jV,P]) = B_f([[V,P],jV],P)\\ &= B_f([[V,jV],P],P) + B_f([V,[P,jV]],P)\\ &= -2B_f(jP,P) - B_f(P,jP) = B_f(P,jP). \qquad (7.5.4)\end{aligned}$$

The spaces $\tilde{\eta}^{\frac{1}{2}\alpha_k}$ are mutually orthogonal with respect to S. By considering an orthogonal basis in each space $\tilde{\eta}^{\frac{1}{2}\alpha_k}$, we can take a basis

$$\left\{V_1^k, V_2^k, \ldots, V_{r'_k}^k, jV_1^k, jV_2^k, \ldots, jV_{r'_k}^k\right\}\ \left(r'_k = \frac{1}{2}r_k\right)$$

in $\tilde{\eta}^{\frac{1}{2}\alpha_k}$ such that

$$[jV_l^k, V_l^k] = -[V_l^k, jV_l^k] = 2U_k\ (1 \le l \le r'_k)$$

and all the other brackets are equal to zero. Let $\{E_{k,l}^p\}$ be the basis of $\eta^{\frac{1}{2}(\alpha_k-\alpha_l)}$ introduced in [68]: that is, we consider the scalar product Q' on $\eta^{\frac{1}{2}(\alpha_k-\alpha_l)}$ defined by $Q'(L,L')U_k = [L,[L',U_l]]$ and choose $\{E_{k,l}^p\}$ as an orthonormal basis relative to $\frac{1}{2}Q'$. We can write

$$[E_{k,l}^p, P_\nu] = \sum_{\tau=i_{l-1}+1}^{i_l} c_{k,l,\nu}^{p,\tau} P_\tau\ \ (i_{k-1}+1 \le \nu \le i_k), \qquad (7.5.5)$$

$$[V_\mu^k, P_\nu] = \sum_{l=m+1}^{m+n} (d_{\mu,\nu}^{k,l} P_l + e_{\mu,\nu}^{k,l} Q_l)\ \ (i_{k-1}+1 \le \nu \le i_k). \qquad (7.5.6)$$

Since $B_f([E_{k,l}^p, P_\nu], Q_\tau) = B_f([E_{k,l}^p, Q_\tau], P_\nu)$ for $i_{k-1}+1 \le \nu \le i_k$ and $i_{l-1}+1 \le \tau \le i_l$, we have

$$[E_{k,l}^p, Q_\tau] = -\sum_{\nu=i_{k-1}+1}^{i_k} c_{k,l,\nu}^{p,\tau} Q_\nu.$$

By (7.5.2) and (7.5.3),

$$[jE_{k,l}^p, P_\nu] = \sum_{\tau=i_{l-1}+1}^{i_l} c_{k,l,\nu}^{p,\tau} Q_\tau\ (i_{k-1}+1 \le \nu \le i_k),$$

$$[jE^p_{k,l}, P_\tau] = \sum_{\nu=i_{k-1}+1}^{i_k} c^{p,\tau}_{k,l,\nu} Q_\nu \ (i_{l-1}+1 \leq \tau \leq i_l),$$

$$[jV^k_\mu, P_\nu] = \sum_{l=m+1}^{m+n} (-e^{k,l}_{\mu,\nu} P_l + d^{k,l}_{\mu,\nu} Q_l) \ (i_{k-1}+1 \leq \nu \leq i_k).$$

For $i_{k-1}+1 \leq \nu \leq i_k$ and $m+1 \leq l \leq m+n$, we have

$$B_f(P_\nu, [V^k_\mu, P_l]) = B_f([P_\nu, V^k_\mu], P_l) = e^{k,l}_{\mu,\nu},$$
$$B_f(P_\nu, [V^k_\mu, Q_l]) = B_f([P_\nu, V^k_\mu], Q_l) = -d^{k,l}_{\mu,\nu}.$$

Hence

$$[V^k_\mu, P_l] = \sum_{\nu=i_{k-1}+1}^{i_k} e^{k,l}_{\mu,\nu} Q_\nu, \ [V^k_\mu, Q_l] = -\sum_{\nu=i_{k-1}+1}^{i_k} d^{k,l}_{\mu,\nu} Q_\nu.$$

By (7.5.3),

$$[jV^k_\mu, P_l] = \sum_{\nu=i_{k-1}+1}^{i_k} d^{k,l}_{\mu,\nu} Q_\nu, \ [jV^k_\mu, Q_l] = \sum_{\nu=i_{k-1}+1}^{i_k} e^{k,l}_{\mu,\nu} Q_\nu.$$

Recall that the space $\mathcal{H}(f, \mathfrak{h}, G)$ consists of all C^∞-functions ϕ on G which satisfy the following two conditions:

$$\phi \cdot X = -if(X)\phi \ \ (\forall X \in \mathfrak{h}), \tag{7.5.7}$$

$$\|\phi\|^2 = \int_{G/Z} |\phi(g)| d\dot{g} < +\infty, \tag{7.5.8}$$

where $d\dot{g}$ denotes an invariant measure on G/Z. Each element $g \in G = M \cdot N$ can be uniquely written in the form $g = ha$ with $h \in M$, $a \in N$ and a can be uniquely written in the form

$$a = \exp \sum_{i=1}^{m+n} (x_i P_i + y_i Q_i) \exp(wC).$$

Through this expression, we regard $\phi \in C^\infty(G)$ as a C^∞-function relative to variables

$$(h, x_1, y_1, x_2, y_2, \ldots, x_{m+n}, y_{m+n}, w) \in M \times \mathbb{R}^{2(m+n)+1}.$$

We calculate the condition (7.5.7) for $X \in \mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{m}_{\mathbb{C}}$. For $X, Y \in \mathfrak{m}$ and $\phi \in C^\infty(G)$, we set

$$\begin{aligned}
&(\phi \underset{M}{\cdot} X)(h, x_1, y_1, \ldots, x_{m+n}, y_{m+n}, w) \\
&= \frac{d}{dt}\phi(h\exp(tX), x_1, y_1, \ldots, x_{m+n}, y_{m+n}, w)\bigg|_{t=0}, \\
&\phi \underset{M}{\cdot} (X + iY) = \phi \underset{M}{\cdot} X + i\phi \underset{M}{\cdot} Y.
\end{aligned}$$

We have the following relations:

$$\begin{aligned}
&\phi \cdot (U_k + ijU_k) = \phi \underset{M}{\cdot} (U_k + ijU_k) \\
&- \sum_{\nu=i_{k-1}+1}^{i'_k} \left\{ x_\nu \frac{\partial\phi}{\partial y_\nu} - \frac{i}{2}\left(x_\nu \frac{\partial\phi}{\partial x_\nu} - y_\nu \frac{\partial\phi}{\partial y_\nu} \right) \right\} \\
&- \sum_{\nu=i'_k+1}^{i'_k+i''_k} \left[x_\nu \frac{\partial\phi}{\partial y_\nu} + x_{\nu+i''_k} \frac{\partial\phi}{\partial y_{\nu+i''_k}} - \frac{i}{2}\left\{ (x_\nu + \gamma_\nu x_{\nu+i''_k}) \frac{\partial\phi}{\partial x_\nu} \right.\right. \\
&+ (x_{\nu+i''_k} - \gamma_\nu x_\nu) \frac{\partial\phi}{\partial x_{\nu+i''_k}} \\
&\left.\left. -(y_\nu - \gamma_\nu y_{\nu+i''_k}) \frac{\partial\phi}{\partial y_\nu} - (y_{\nu+i''_k} + \gamma_\nu y_\nu) \frac{\partial\phi}{\partial y_{\nu+i''_k}} \right\} \right].
\end{aligned} \tag{7.5.9}$$

$$\begin{aligned}
&\phi\cdot(E^p_{k,l} + ijE^p_{k,l}) = \phi \underset{M}{\cdot} (E^p_{k,l} + ijE^p_{k,l}) \\
&- \sum_{\tau=i_{l-1}+1}^{i_l} \left(\sum_{\nu=i_{k-1}+1}^{i_k} c^{p,\tau}_{k,l,\nu} x_\nu \right) \left(\frac{\partial\phi}{\partial x_\tau} + i\frac{\partial\phi}{\partial y_\tau} \right) \\
&- i \sum_{\nu=i_{k-1}+!}^{i_k} \left\{ \sum_{\tau=i_{l-1}+1}^{i_l} c^{p,\tau}_{k,l,\nu}(x_\tau + iy_\tau) \right\} \frac{\partial\phi}{\partial y_\nu}.
\end{aligned} \tag{7.5.10}$$

$$\begin{aligned}
&\phi \cdot (V^k_\mu + ijV^k_\mu) = \phi \underset{M}{\cdot} (V^k_\mu + ijV^k_\mu) \\
&- \sum_{l=m+1}^{m+n} \sum_{\nu=i_{k-1}+1}^{i_k} (e^{k,l}_{\mu,\nu} + i d^{k,l}_{\mu,\nu}) \left\{ (x_l + iy_l)\frac{\partial\phi}{\partial y_\nu} - ix_\nu \left(\frac{\partial\phi}{\partial x_l} + i\frac{\partial\phi}{\partial y_l} \right) \right\}.
\end{aligned} \tag{7.5.11}$$

We define $\phi_0 \in C^\infty(G)$ by

$$\phi_0(g) = e^{-iw} e^{-\frac{1}{4}\sum_{i=1}^{m+n}(x_i^2+y_i^2)}$$

when $g = h\exp\sum_{i=1}^{m+n}(x_i P_i + y_i Q_i)\exp(wC)$. Then ϕ_0 satisfies the infinitesimal condition (7.5.7) and the following relations:

$$\phi_0 \cdot (U_k + ijU_k) = \left\{ -\frac{i}{4} \sum_{\nu=i_{k-1}+1}^{i_k} (x_\nu + iy_\nu)^2 \right\} \phi_0, \tag{7.5.12}$$

$$\phi_0 \cdot (E_{k,l}^p + ijE_{k,l}^p) = \left\{ \frac{1}{2} \sum_{\nu=i_{k-1}+1}^{i_k} \sum_{\tau=i_{l-1}}^{i_l} c_{k,l,\nu}^{p,\tau} (x_\nu + iy_\nu)(x_\tau + iy_\tau) \right\} \phi_0, \tag{7.5.13}$$

$$\phi_0 \cdot (V_\mu^k + ijV_\mu^k) = \left\{ -\frac{i}{2} \sum_{l=m+1}^{m+n} \sum_{\nu=i_{k-1}+1}^{i_k} (e_{\mu,\nu}^{k,l} + id_{\mu,\nu}^{k,l})(x_\nu + iy_\nu)(x_l + iy_l) \right\} \phi_0. \tag{7.5.14}$$

Therefore, if we set $\hat{\phi} = \phi\cdot\phi_0^{-1}$ for any $\phi \in \mathcal{H}(f, \mathfrak{h}, G)$, then $\hat{\phi}$ is independent of w and is a C^∞-function on $M \times \mathbb{C}^{m+n}$ with $z_k = x_k + iy_k$ $(1 \leq k \leq m+n)$. Furthermore, $\hat{\phi}$ is holomorphic relative to the variables z_k $(1 \leq k \leq m+n)$ and the above relations (7.5.9)–(7.5.11) imply the following relations:

$$\begin{aligned}
&\hat{\phi} \underset{M}{\cdot} (U_k + ijU_k) - \frac{i}{2} \sum_{\nu=i_{k-1}+1}^{i'_k} z_\nu \frac{\partial\hat{\phi}}{\partial x_\nu} \\
&- \frac{i}{2} \sum_{\nu=i'_k+1}^{i'_k+i''_k} \left\{ (z_\nu - \gamma_\nu z_{\nu+i''_k}) \frac{\partial\hat{\phi}}{\partial x_\nu} + (z_{\nu+i''_k} + \gamma_\nu z_\nu) \frac{\partial\hat{\phi}}{\partial x_{\nu+i''_k}} \right\} \\
&= -i\tilde{f}_2(U_k + ijU_k)\hat{\phi} + \left(\frac{i}{4} \sum_{\nu=i_{k-1}+1}^{i_k} z_\nu^2 \right) \hat{\phi},
\end{aligned} \tag{7.5.15}$$

$$\begin{aligned}
&\hat{\phi} \underset{M}{\cdot} (E_{k,l}^p + ijE_{k,l}^p) + \sum_{\nu=i_{k-1}+1}^{i_k} \sum_{\tau=i_{l-1}+1}^{i_l} c_{k,l,\nu}^{p,\tau} z_\tau \frac{\partial\hat{\phi}}{\partial x_\nu} \\
&= -\frac{1}{2} \left(\sum_{\nu=i_{k-1}+1}^{i_k} \sum_{\tau=i_{l-1}+1}^{i_l} c_{k,l,\nu}^{p,\tau} z_\nu z_\tau \right) \hat{\phi},
\end{aligned} \tag{7.5.16}$$

$$\hat{\phi} \underset{M}{\cdot} (V_\mu^k + ijV_\mu^k) - i \sum_{l=m+1}^{m+n} \sum_{\nu=i_{k-1}+1}^{i_k} (e_{\mu,\nu}^{k,l} + i d_{\mu,\nu}^{k,l}) z_l \frac{\partial \hat{\phi}}{\partial x_\nu}$$
$$= \frac{i}{2} \left\{ \sum_{l=m+1}^{m+n} \sum_{\nu=i_{k-1}+1}^{i_k} (e_{\mu,\nu}^{k,l} + i d_{\mu,\nu}^{k,l}) z_\nu z_l \right\} \hat{\phi}. \tag{7.5.17}$$

As for the infinitesimal conditions on the space $\mathcal{H}(\tilde{f}_2, \mathfrak{h}_2, M)$, Rossi and Vergne [68] gave a particular solution: let $\kappa : \mathfrak{m} \to \mathfrak{h}_2$ be the linear map defined by

$$\kappa(X) = \begin{cases} X + ijX, & X \in \mathfrak{m}_0, \\ \frac{1}{2}(X + ijX), & X \in \mathfrak{m}_{\frac{1}{2}}, \\ 0 & X \in \mathfrak{m}_1, \end{cases}$$

and let $\lambda_{\tilde{f}_2} \in C^\infty(M)$ be defined by

$$\lambda_{\tilde{f}_2}(\exp X) = e^{i \tilde{f}_2(\kappa(X))} \ \ (X \in \mathfrak{m}).$$

Then $\lambda_{\tilde{f}_2}^{-1}$ satisfies the infinitesimal conditions which the space $\mathcal{H}(\tilde{f}_2, \mathfrak{h}_2, M)$ requires.

Lemma 7.5.13. *Let $g = ha$ $(h \in M,\ a \in N)$ with $a = exp \sum_{i=1}^{m+n}(x_i P_i + y_i Q_i) exp(wC)$.*

(1) *The C^∞-function*

$$\phi_1(g) = \lambda_{\tilde{f}_2}^{-1}(h) e^{-iw} e^{\left(-\frac{1}{2} \sum_{j=1}^{m+n} (x_j^2 + i x_j y_j)\right)}$$

satisfies the infinitesimal condition (7.5.7).

(2) *If $\phi_M \in C^\infty(M)$ satisfies the infinitesimal conditions imposed on the space $\mathcal{H}(\tilde{f}_2, \mathfrak{h}_2, M)$, the function*

$$\phi(g) = \frac{1}{(2\pi)^{(m+n)/2}} \phi_M(h) e^{-iw} e^{\left(-\frac{1}{2} \sum_{j=1}^{m+n} (x_j^2 + i x_j y_j)\right)}$$
$$\times \int_B e^{B_f\left(h^{-1} \cdot \left(\sum_{j=1}^{m+n} \xi_j P_j\right), \sum_{j=1}^{m+n} z_j Q_j\right)} d\xi \tag{7.5.18}$$

satisfies the infinitesimal condition (7.5.7)*, where $d\xi$ denotes a Lebesgue measure on $\mathbb{R}^{m+n}$, B a compact set in $\mathbb{R}^{m+n}$ and $z_j = x_j + iy_j$ $(1 \leq j \leq m + n)$.*

Proof. (1) Simple calculations show that the function

$$\phi_2(g) = \lambda_{\tilde{f}_2}^{-1}(h) e^{\left(-\frac{1}{4}\sum_{j=1}^{m+n} z_j^2\right)},$$

where $z_j = x_j + iy_j$ $(1 \leq j \leq m+n)$, satisfies the conditions (7.5.15)–(7.5.17). It follows that the function $\phi_1 = \phi_0 \cdot \phi_2$ satisfies the infinitesimal condition (7.5.7).

(2) The function

$$\phi_3(g) = \int_B e^{B_f\left(h^{-1}\cdot\left(\sum_{j=1}^{m+n}\xi_j P_j\right), \sum_{j=1}^{m+n} z_j Q_j\right)} d\xi$$

is clearly holomorphic in z_j $(1 \leq j \leq m+n)$. Thus, it suffices to show that the function $\phi_\xi \in C^\infty(G)$ given by

$$\phi_\xi(g) = B_f\left(h^{-1}\cdot\left(\sum_{j=1}^{m+n}\xi_j P_j\right), \sum_{j=1}^{m+n} z_j Q_j\right)$$

satisfies the condition

$$\phi_\xi \cdot X = 0 \;\; (\forall\, X \in \mathfrak{h}_2). \tag{7.5.19}$$

In fact, since ϕ_ξ is holomorphic in z_j $(1 \leq j \leq m+n)$, the relations (7.5.9)–(7.5.11) imply at first that $\phi_\xi \cdot X$ $(X \in \mathfrak{h}_2)$ is a homogeneous polynomial in $z_1, z_2, \ldots, z_{m+n}$ of degree at most 1 and then give us the desired equation (7.5.19) by direct computations. ■

Now we estimate $\|\phi\|^2$ for ϕ given by (7.5.18). We denote by dh a left-invariant measure on M, then $d\dot{g} = dhdxdy$ where $dx,\ dy$ denote Lebesgue measures on $\mathbb{R}^m,\ \mathbb{R}^n$. We set $\mathfrak{q} = \langle Q_1, Q_2, \ldots, Q_{m+n}\rangle_{\mathbb{R}}$, which is invariant under the adjoint representation $\mathrm{Ad}_W M$ of M on W. For any $h \in M$, we set $\mathrm{Ad}_{W/\mathfrak{q}} h = \hat{h}$ and

$$h^{-1}\cdot\left(\sum_{j=1}^{m+n}\xi_j P_j\right) \equiv \sum_{j=1}^{m+n}\xi'_j P_j \mod \mathfrak{q}.$$

By Plancherel formula

$$\begin{aligned}\|\phi\|^2 = \int_{G/Z}|\phi(g)|^2 d\dot{g} = \int_M |\phi_M(h)|^2 \int_{\mathbb{R}^{m+n}} e^{\left(-\sum_{j=1}^{m+n} x_j^2\right)} \\ \times \int_{\mathbb{R}^{m+n}} \left|\frac{1}{(2\pi)^{(m+n)/2}}\int_B e^{\left(\sum_{j=1}^{m+n}\xi'_j z_j\right)} d\xi\right|^2 dhdxdy\end{aligned}$$

$$\begin{aligned}
&= \int_M |\phi_M(h)|^2 \left(\det \hat{h}\right)^2 \int_{\mathbb{R}^{m+n}} e^{\left(-\sum_{j=1}^{m+n} x_j^2\right)} \\
&\quad \times \int_{\mathbb{R}^{m+n}} \left| \frac{1}{(2\pi)^{(m+n)/2}} \int_{\hat{h}^{-1}\cdot B} e^{\left(\sum_{j=1}^{m+n} \xi_j z_j\right)} d\xi \right|^2 dy\,dx\,dh \\
&= \int_M |\phi_M(h)|^2 \left(\det \hat{h}\right)^2 \int_{\mathbb{R}^{m+n}} e^{\left(-\sum_{j=1}^{m+n} x_j^2\right)} \int_{\hat{h}^{-1}\cdot B} e^{\left(2\sum_{j=1}^{m+n} x_j y_j\right)} dy\,dx\,dh \\
&= \pi^{(m+n)/2} \int_M |\phi_M(h)|^2 \left(\det \hat{h}\right)^2 \int_{\hat{h}^{-1}\cdot B} e^{\left(\sum_{j=1}^{m+n} y_j^2\right)} dy\,dh \\
&= \pi^{(m+n)/2} \int_M |\phi_M(h)|^2 \left(\det \hat{h}\right) \int_B e^{\left(\sum_{j=1}^{m+n} (y_j')^2\right)} dy\,dh. \qquad (7.5.20)
\end{aligned}$$

If we write $h = h_0 \cdot \exp U \cdot h_1$ with $h_0 \in M_0,\ U \in \mathfrak{m}_{\frac{1}{2}},\ h_1 \in M_1$, then $\det \hat{h} = \det_{W_{-\frac{1}{2}}} h_0$ and

$$\begin{aligned}
h^{-1} \cdot \left(\sum_{j=1}^{m+n} y_j P_j \right) &\equiv \exp(-U) \cdot h_0^{-1} \cdot \left(\sum_{j=1}^{m+n} y_j P_j \right) \\
&\equiv h_0^{-1} \cdot \exp(-h_0 \cdot U) \cdot \left(\sum_{j=1}^{m+n} y_j P_j \right) \mod \mathfrak{q}.
\end{aligned}$$

Using the expressions

$$\begin{aligned}
h_0 &= \left(\prod_{i=1}^{r} \exp(a_i j U_i) \right) \exp(L_1) \exp(L_2) \cdots \exp(L_{r-1}),\ L_i = \sum_{k>i} \sum_{p} x_{k,i}^p E_{k,i}^p, \\
h_0 \cdot U &= \sum_k \sum_\mu \left(u_\mu^k V_\mu^k + v_\mu^k j V_\mu^k \right), \qquad (7.5.21)
\end{aligned}$$

we calculate y_j' and estimate $\sum_{j=1}^{m+n} (y_j')^2$. For fixed k $(1 \le k \le r)$, we have

$$y_\tau' = e^{a_k/2} y_\tau - I_{k,\tau}$$

when $i_{k-1} \le \tau \le i_k'$, and

$$\begin{aligned}
y_\tau' &= e^{a_k/2} \left(y_\tau \cos\left(\frac{\gamma_\tau a_k}{2}\right) + y_{\tau+i_k''} \sin\left(\frac{\gamma_\tau a_k}{2}\right)\right) - I_{k,\tau} \\
y_{\tau+i_k''}' &= e^{a_k/2} \left(y_{\tau+i_k''} \cos\left(\frac{\gamma_\tau a_k}{2}\right) - y_\tau \sin\left(\frac{\gamma_\tau a_k}{2}\right)\right) - I_{k,\tau+i_k''}
\end{aligned}$$

when $i'_k + 1 \le \tau \le i'_k + i''_k$. Here,

$$I_{k,\tau} = \sum_{j>k} \sum_{p} \Bigg[\sum_{\nu=i_{j-1}+1}^{i'_j} c^{p,\tau}_{j,k,\nu} y_\nu \\ + \sum_{\nu=i'_j+1}^{i'_j+i''_j} \Big\{ c^{p,\tau}_{j,k,\nu} \Big(y_\nu \cos\Big(\frac{\gamma_\nu a_j}{2}\Big) + y_{\nu+i''_j} \sin\Big(\frac{\gamma_\nu a_j}{2}\Big)\Big) \\ + c^{p,\tau}_{j,k,\nu+i''_j} \Big(y_{\nu+i''_j} \cos\Big(\frac{\gamma_\nu a_j}{2}\Big) - y_\nu \sin\Big(\frac{\gamma_\nu a_j}{2}\Big)\Big)\Big\}\Bigg] e^{a_j/2} x^p_{j,k}.$$

Further

$$y'_l = y_l - \sum_{k=1}^{r} \sum_{\mu=1}^{r'_k} \sum_{\nu=i_{k-1}+1}^{i_k} y_\nu \left(u^k_\nu d^{k,l}_{\mu,\nu} - v^k_\mu e^{k,l}_{\mu,\nu} \right)$$

when $m+1 \le l \le m+n$. Since $E^p_{k,l} \in \eta^{\frac{1}{2}(\alpha_k - \alpha_l)}$ satisfies the relation $[jE^p_{k,l}, E^p_{k,l}] = 2U_k$, we have

$$\sum_{\tau=i_{k-1}+1}^{i_k} (c^{p,\tau}_{j,k,\nu})^2 = 1, \quad \sum_{l=m+1}^{m+n} \left\{ (d^{k,l}_{\mu,\nu})^2 + (e^{k,l}_{\mu,\nu})^2 \right\} = 1$$

by (7.5.4)–(7.5.6). For fixed k $(1 \le k \le r)$ and τ $(i_{k-1} + 1 \le \tau \le i_k)$, we set

$$\hat{I}_{k,\tau} = \sum_{j>k} \sum_{p} \Bigg[\sum_{\nu=i_{j-1}+1}^{i'_j} (c^{p,\tau}_{j,k,\nu})^2 (y_\nu)^2 \\ + \sum_{\nu=i'_j+1}^{i'_j+i''_j} \Bigg((c^{p,\tau}_{j,k,\nu})^2 \Big(y_\nu \cos\Big(\frac{\gamma_\nu a_j}{2}\Big) + y_{\nu+i''_j} \sin\Big(\frac{\gamma_\nu a_j}{2}\Big)\Big)^2 \\ + (c^{p,\tau}_{j,k,\nu+i''_j})^2 \Big(y_{\nu+i''_j} \cos\Big(\frac{\gamma_\nu a_j}{2}\Big) - y_\nu \sin\Big(\frac{\gamma_\nu a_j}{2}\Big)\Big)^2\Bigg)\Bigg] e^{a_j} (x^p_{j,k})^2.$$

Then,

$$\sum_{j=1}^{m+n} (y'_j)^2 \le c \left[\sum_{k=1}^{r} \left[\sum_{\tau=i_{k-1}+1}^{i'_k} \left(e^{a_k} (y_\tau)^2 + \hat{I}_{k,\tau} \right) \right.\right.$$

$$+ \sum_{\tau=i'_k+1}^{i'_k+i''_k} \left\{ e^{a_k} \left\{ \left(y_\tau \cos\left(\frac{\gamma_\tau a_k}{2}\right) + y_{\tau+i''_k} \sin\left(\frac{\gamma_\tau a_k}{2}\right)\right)^2 \right.\right.$$

$$\left.\left. + \left(y_{\tau+i''_k} \cos\left(\frac{\gamma_\tau a_k}{2}\right) - y_\tau \sin\left(\frac{\gamma_\tau a_k}{2}\right)\right)^2 \right\} + \hat{I}_{k,\tau} + \hat{I}_{k,\tau+i''_k} \right\} \Bigg]$$

$$+ \sum_{l=m+1}^{m+n} \left[(y_l)^2 + \sum_{k=1}^{r} \sum_{\mu=1}^{r'_k} \sum_{\nu=i_{k-1}+1}^{i_k} (y_\nu)^2 \left((u^k_\mu)^2 (d^{k,l}_{\mu,\nu})^2 + (v^k_\mu)^2 (e^{k,l}_{\mu,\nu})^2 \right) \right] \Bigg]$$

$$\leq c \left[\sum_{k=1}^{r} \left[e^{a_k} \left(\sum_{\tau=i_{k-1}+1}^{i_k} (y_\tau)^2 \right) + \sum_{j>k} \sum_{p} \left[\sum_{\nu=i_{j-1}+1}^{i'_j} (y_\nu)^2 \right.\right.\right.$$

$$+ \sum_{\nu=i'_j+1}^{i'_j+i''_j} \left\{ \left(y_\nu \cos\left(\frac{\gamma_\nu a_j}{2}\right) + y_{\nu+i''_j} \sin\left(\frac{\gamma_\nu a_j}{2}\right)\right)^2 \right.$$

$$\left.\left. + \left(y_{\nu+i''_j} \cos\left(\frac{\gamma_\nu a_j}{2}\right) - y_\nu \sin\left(\frac{\gamma_\nu a_j}{2}\right)\right)^2 \right\} \right] e^{a_j} (x^p_{j,k})^2 \Bigg]$$

$$\left. + \sum_{l=m+1}^{m+n} (y_l)^2 + \sum_{k=1}^{r} \sum_{\mu=1}^{r'_k} \left(\sum_{\nu=i_{k-1}+1}^{i_k} (y_\nu)^2 \right) \left((u^k_\mu)^2 + (v^k_\mu)^2 \right) \right]$$

$$= c \left[\sum_{k=1}^{r} \left\{ e^{a_k} \left(\sum_{\tau=i_{k-1}+1}^{i_k} (y_\tau)^2 \right) + \sum_{j>k} \sum_{p} \left(\sum_{\nu=i_{j-1}+1}^{i_j} (y_\nu)^2 \right) e^{a_j} (x^p_{j,k})^2 \right\} \right.$$

$$\left. + \sum_{l=m+1}^{m+n} (y_l)^2 + \sum_{k=1}^{r} \sum_{\mu=1}^{r'_k} \left(\sum_{\nu=i_{k-1}+1}^{i_k} (y_\nu)^2 \right) \left((u^k_\mu)^2 + (v^k_\mu)^2 \right) \right],$$

where c denotes some positive constant.

Therefore, if we take

$$B = \{\xi \in \mathbb{R}^{m+n}; \sum_{j=1}^{m+n} (\xi_j)^2 \leq c^{-1}\},$$

we obtain by (7.5.20)

$$\|\phi\|^2 \leq \pi^{(m+n)/2} \int_M |\phi_M(h)|^2 \left(\det_{W_{-\frac{1}{2}}} h_0\right) e^{\sum_{k=1}^{r} e^{a_k} \left\{1+\sum_{j<k,\, p} (x^p_{k,j})^2\right\}}$$

$$\times\ e^{\sum_{k=1}^{r}\sum_{\mu=1}^{r'_k}\left((u_\mu^k)^2+(v_\mu^k)^2\right)}dh\left(\int_B e\,dy\right)$$

$$= e\pi^{(m+n)/2}\mathrm{vol}(B)\int_M |\phi_M(h)|^2\left(\det{}_{W_{-\frac{1}{2}}} h_0\right)$$

$$\times\, e^{\sum_{k=1}^{r} e^{a_k}\left\{1+\sum_{j\underset{p}{<}k}(x_{k,j}^p)^2\right\}} e^{\sum_{k=1}^{r}\sum_{\mu=1}^{r'_k}\left((u_\mu^k)^2+(v_\mu^k)^2\right)}dh. \tag{7.5.22}$$

As stated in subsection A, the subgroup $M = \exp \mathfrak{m}$ can be identified with the Siegel domain $D(\Omega, Q)$ which was constructed in A under the mapping $\alpha : h \mapsto h{\cdot}(is, 0)$. Then ϕ_M can be written in the form

$$\phi_M(h) = \lambda_{\tilde{f}_2}^{-1}(h)F(\alpha(h)), \tag{7.5.23}$$

where F is a holomorphic function on $D(\Omega, Q)$. Under the identification α, a Haar measure on M is given by

$$dh = \det{}_{\mathfrak{m}_{\frac{1}{2}}}\left(\alpha^{-1}(y - Q(u,u))\right)^{-1}\det{}_{\mathfrak{m}_1}\left(\alpha^{-1}(y - Q(u,u))\right)^{-2}dxdydu,$$

where $\det_{\mathfrak{m}_i} h$ ($h \in M_0$, $i = 1/2, 1$) denotes the determinant of the adjoint representation. Let $d = \dim(\mathfrak{m}_1)$, $q = \frac{1}{2}\dim(\mathfrak{m}_{\frac{1}{2}})$. We define a positive continuous function Ψ_0 on $\Omega = M_0{\cdot}s$ by

$$\Psi_0(h_0{\cdot}s) = e^{\sum_{k=1}^{r} e^{a_k}\left\{1+\sum_{j\underset{p}{<}k}(x_{k,j}^p)^2\right\}}|\lambda_{\tilde{f}_2}(h_0)|^{-2}$$

$$\times\left(\det{}_{W_{-\frac{1}{2}}} h_0\right)\left(\det{}_{\mathfrak{m}_{\frac{1}{2}}} h_0\right)^{-1}(\det{}_{\mathfrak{m}_1} h_0)^{-2}$$

for $h_0 \in M_0$. Further we set

$$\Phi_1(u) = e^{|u|^2}$$

for $u \in \mathbb{C}^q$. Then by (7.5.22)

$$\|\phi\|^2 \le \int_D |F(z,u)|^2\Psi_0(y - Q(u,u))\Phi_1(u)dxdydu. \tag{7.5.24}$$

Therefore, if there exists a nonzero holomorphic function F_0 which makes the right-hand side of (7.5.24) finite, we can construct a nonzero element $\phi \in \mathcal{H}(f, \mathfrak{h}, G)$ by means of (7.5.18) and (7.5.23). For the existence of such F_0, it suffices by Lemma 7.5.6 that $\hat{\Omega}_{\Psi_0} \cap \hat{\Omega}^Q_{\Phi_1}$ has interior points. Let $\xi_0 = \sum_{k=1}^{r} U_k^* \in \Omega^*$ and let Ψ_1 be the positive continuous function on Ω given by

$$\Psi_1(h_0{\cdot}s) = |\lambda_{\tilde{f}_2}(h_0)|^{-2}\left(\det_{W_{-\frac{1}{2}}} h_0\right)\left(\det_{\mathfrak{m}_{\frac{1}{2}}} h_0\right)^{-1}(\det_{\mathfrak{m}_1} h_0)^{-2}. \tag{7.5.25}$$

Then, under expression (7.5.21) of $h_0 \in M_0$,

$$\Psi_0(h_0{\cdot}s) = e^{\left[\sum_{k=1}^{r} e^{a_k}\left\{1+\sum_{j \underset{p}{<} k}(x_{k,j}^p)^2\right\}\right]}\Psi_1(h_0{\cdot}s) = e^{\xi_0(h_0\cdot s)}\Psi_1(h_0{\cdot}s)$$

(cf. Theorem 4.10 in [68]). Hence

$$I_{\Psi_0}(\xi) = \int_\Omega e^{-2\xi(t)}\Psi_0(t)dt = \int_\Omega e^{-2(\xi-\frac{1}{2}\xi_0)(t)}\Psi_1(t)dt = I_{\Psi_1}\left(\xi - \frac{1}{2}\xi_0\right). \tag{7.5.26}$$

The calculations similar to those of [68] show that the integral $I_{\Psi_1}(\xi_0)$ is, up to a constant factor, the product of the r-integrals

$$\int_0^{+\infty} e^{-2t}t^{\left[-2f_i-\left\{p_i+1+\frac{1}{2}(q_i+r_i+t_i)\right\}-1\right]}dt,$$

where $f_i = \tilde{f}_2(U_i)$ $(1 \le i \le r)$. Therefore, if

$$-2f_i - \left\{p_i + 1 + \frac{1}{2}(q_i + r_i + t_i)\right\} > 0 \; (1 \le i \le r), \tag{7.5.27}$$

then $I_{\Psi_1}(\xi_0) < +\infty$ so that $I_{\Psi_1}(\xi) < +\infty$ throughout Ω^*, since Ψ_1 is homogeneous in the sense that $\Psi_1(py) = p^\lambda y$ $(0 < p \in \mathbb{R},\ y \in \Omega)$ for some real number λ (cf. Theorem 2.20 in [68]). Thus by (7.5.26),

$$\frac{1}{2}\xi_0 + \Omega^* \subset \hat{\Omega}_{\Psi_0},\ \left(\frac{1}{2}\xi_0 + \Omega^*\right) \cap \hat{\Omega}_{\Phi_1}^Q \subset \hat{\Omega}_{\Psi_0} \cap \hat{\Omega}_{\Phi_1}^Q.$$

Since the left-hand side of the last inclusion relation is an open set which contains $\xi = \alpha\xi_0$ with $\alpha > 1$, it follows that $\hat{\Omega}_{\Psi_0} \cap \hat{\Omega}_{\Phi_1}^Q$ has interior points.

In conclusion, the inequalities (7.5.27) give a sufficient condition for the non-vanishing of the space $\mathcal{H}(f, \mathfrak{h}, G)$.

Now we prove that the inequalities (7.5.27) are also necessary for the non-vanishing of $\mathcal{H}(f, \mathfrak{h}, G)$. By Theorem 7.4.1 and our assumption, $f \in \mathfrak{g}^*$ has the form

$$f = C^* + \sum_{k=1}^{r} f_k U_k^*$$

with $f_k < 0$ $(1 \le k \le r)$. Let

$$\mathfrak{m}' = \mathfrak{m}_0 + \mathfrak{m}_1,\ \mathfrak{g}' = \mathfrak{n} + \mathfrak{m}',$$

where the sums are semi-direct. We set $f' = f|_{\mathfrak{g}'} \in (\mathfrak{g}')^*$ and $\mathfrak{h}' = \mathfrak{h} \cap (\mathfrak{g}')_{\mathbb{C}}$. Then $\mathfrak{m}'$ (resp. $\mathfrak{g}'$) is a Lie subalgebra of $\mathfrak{m}$ (resp. $\mathfrak{g}$) and $\mathfrak{h}'$ is a positive polarization of $\mathfrak{g}'$ at f', i.e. $\mathfrak{h}' \in P^+(f', G')$ with $G' = \exp(\mathfrak{g}')$. We set $M' = \exp(\mathfrak{m}')$ and let $\pi : G \to G/Z$ be the natural projection. If we write $\dot{g} = \pi(g) \in G/Z$ in the form $\dot{g} = \pi(h\exp Y)$ with $h \in M$ and $Y \in W$, then $d\dot{g} = dh dY$ with a Haar measure dh on M and a Lebesgue measure dY on W. Moreover, h can be uniquely written in the form $h = \exp u{\cdot}h'$ with $u \in \mathfrak{m}_{\frac{1}{2}}$ and $h' \in M'$ so that $dh = \left(\det_{\mathfrak{m}_{\frac{1}{2}}} h'\right)^{-1} du dh'$ with a Lebesgue measure du on $\mathfrak{m}_{\frac{1}{2}}$ and a Haar measure dh' on M'. Thus, we have

$$d\dot{g} = \left(\det_{\mathfrak{m}_{\frac{1}{2}}} h'\right)^{-1} du dh' dY,$$

when we write $g \in G$ uniquely in the form

$$g = \exp u{\cdot}g',\ g' = h'\exp Y \exp(wC)$$

with $u \in \mathfrak{m}_{\frac{1}{2}},\ h' \in M',\ Y \in W,\ w \in \mathbb{R}$.

We suppose $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$. Then there is a nonzero C^∞-function ϕ on G which satisfies the infinitesimal condition (7.5.7) and

$$\int_{G/Z} |\phi(\exp u{\cdot}g')|^2 \left(\det_{\mathfrak{m}_{\frac{1}{2}}} h'\right)^{-1} du d\dot{g}' < +\infty \tag{7.5.28}$$

where $d\dot{g}' = dh' dY$. By (7.5.28), we have

$$\int_{G'/Z} |\phi(\exp u{\cdot}g')|^2 \left(\det_{\mathfrak{m}_{\frac{1}{2}}} h'\right)^{-1} d\dot{g}' < +\infty \tag{7.5.29}$$

for almost all $u \in \mathfrak{m}_{\frac{1}{2}}$. We take and fix such an element $u_0 \in \mathfrak{m}_{\frac{1}{2}}$ and define a nonzero C^∞-function ϕ_{u_0} on G' by $\phi_{u_0}(g') = \phi(\exp(u_0){\cdot}g')$. Then ϕ_{u_0} satisfies the infinitesimal condition on the space $\mathcal{H}(f', \mathfrak{h}', G')$. We write

$$g' = h'\exp \sum_{j=1}^{m+n} (x_j P_j + y_j Q_j) \exp(wC)$$

and define the function ψ on G' by

$$\psi(g') = \lambda_{\tilde{f}_2}^{-1}(h') e^{-iw} e^{\left(-\frac{1}{2}\sum_{j=1}^{m+n}(x_j^2 + i x_j y_j)\right)}.$$

Then by virtue of Lemma 7.5.13 ψ satisfies the infinitesimal condition on the space $\mathcal{H}(f', \mathfrak{h}', G')$. Thus $\hat{\phi}_{u_0} = \phi_{u_0}{\cdot}\psi^{-1}$ is independent of w and is a C^∞-function on

$M' \times \mathbb{C}^{m+n}$, and holomorphic in $z_j = x_j + iy_j$ $(1 \leq j \leq m+n)$. Furthermore, by (7.5.15) and (7.5.16)

$$(\hat{\phi}_{u_0}) \underset{M}{\cdot} (U_k + ijU_k) - \frac{i}{2} \sum_{\nu=i_{k-1}+1}^{i'_k} z_\nu \frac{\partial \hat{\phi}_{u_0}}{\partial x_\nu}$$

$$- \frac{i}{2} \sum_{\nu=i'_k+1}^{i'_k+i''_k} \left\{ (z_\nu - \gamma_\nu z_{\nu+i''_k}) \frac{\partial \hat{\phi}_{u_0}}{\partial x_\nu} + (z_{\nu+i''_k} + \gamma_\nu z_\nu) \frac{\partial \hat{\phi}_{u_0}}{\partial x_{\nu+i''_k}} \right\} = 0, \tag{7.5.30}$$

$$(\hat{\phi}_{u_0}) \underset{M}{\cdot} (E^p_{k,l} + ijE^p_{k,l}) + \sum_{\nu=i_{k-1}+1}^{i_k} \sum_{\tau=i_{l-1}+1}^{i_l} c^{p,\tau}_{k,l,\nu} \frac{\partial \hat{\phi}_{u_0}}{\partial x_\nu} = 0, \tag{7.5.31}$$

and

$$\int_{G'/Z} |\hat{\phi}_{u_0}|^2 |\psi|^2 \left(\det_{\mathfrak{m}_{\frac{1}{2}}} h' \right)^{-1} d\dot{g}' < +\infty \tag{7.5.32}$$

by (7.5.29). We set $z = (z_1, z_2, \ldots, z_{m+n}) \in \mathbb{C}^{m+n}$ and define the function $\tilde{\phi}_{u_0}$ on $M' \times \mathbb{C}^{m+n}$ by

$$\tilde{\phi}_{u_0}(h', z) = \hat{\phi}_{u_0}(h', \hat{z}),$$

where $\hat{z} = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{m+n})$ given by $\sum_{j=1}^{m+n} \hat{z}_j Q_j = h_0^{-1} \cdot (\sum_{j=1}^{m+n} z_j Q_j)$ for $h' = h_0 \cdot h_1$ with $h_0 \in M_0,\ h_1 \in M_1$. Then by (7.5.30) and (7.5.31), it follows that

$$(\tilde{\phi}_{u_0}) \underset{M}{\cdot} (U_k + ijU_k) = 0,\ (\tilde{\phi}_{u_0}) \underset{M}{\cdot} (E^p_{k,l} + ijE^p_{k,l}) = 0. \tag{7.5.33}$$

Moreover, it is obvious that $\tilde{\phi}_{u_0}$ is holomorphic in z_j $(1 \leq j \leq m+n)$. Let $\Omega = M_0 \cdot s$. Then the Siegel domain corresponding to M' is the tube domain $D' = \mathfrak{m}_1 + i\Omega$. We denote again by α the identification between M' and D'. By (7.5.33), the function F_{u_0} on $D' \times \mathbb{C}^{m+n}$ defined by

$$F_{u_0}(\alpha(h'), z) = \tilde{\phi}_{u_0}(h', z)$$

is a nonzero holomorphic function. We denote the coordinates in D' by

$$z' = x' + iy',\ x' = (x'_1, x'_2, \ldots, x'_d),\ y' = (y'_1, y'_2, \ldots, y'_d)$$

with $d = \dim(\mathfrak{m}_1)$. If we introduce the positive continuous function Ψ_2 on $\Omega = M_0 \cdot s$ by

$$\Psi_2(h_0{\cdot}s) = |\lambda_{\tilde{f}_2}(h_0)|^{-2} \left(\det\nolimits_{\mathfrak{m}_{\frac{1}{2}}} h_0\right)^{-1} (\det\nolimits_{\mathfrak{m}_1} h_0)^{-2} \left(\det\nolimits_{W_{-\frac{1}{2}}} h_0\right)^{-2} \ (h_0 \in M_0),$$

then we obtain by (7.5.32)

$$\int_{D' \times \mathbb{C}^{m+n}} |F_{u_0}(z', z)|^2 e^{\left(-\sum_{j=1}^{m+n} (\hat{x}_j)^2\right)} \Psi_2(y') dx' dy' dx dy < +\infty. \tag{7.5.34}$$

By Theorem 2.8 in [68], a nonzero holomorphic function F_{u_0} which satisfies (7.5.34) exists if and only if the set of $\xi \in \mathbb{R}^{m+n}$ such that

$$I(\xi) = \int_{\Omega \times \mathbb{R}^{m+n}} e^{-2\xi(t)} e^{\left(-\sum_{j=1}^{m+n} (\hat{x}_j)^2\right)} \Psi_2(y'') dt < +\infty$$

has interior points, where $t = (y'', x)$, $y'' = h_0{\cdot}s$ and $\sum_{j=1}^{m+n} \hat{x}_j Q_j = h_0^{-1}{\cdot}(\sum_{j=1}^{m+n} x_j Q_j)$. We write $\xi = (\xi_1, \xi_2)$, $\xi_1 \in \mathbb{R}^d$, $\xi_2 \in \mathbb{R}^{m+n}$ and transform the integral into an integral over M_0:

$$\begin{aligned} I(\xi) = &\int_{M_0} e^{-2\xi_1(h_0\cdot s)} \Psi_2(h_0{\cdot}s)\, (\det\nolimits_{\mathfrak{m}_1} h_0) \\ &\times \int_{\mathbb{R}^{m+n}} e^{-2\xi_2(x)} e^{\left(-\sum_{j=1}^{m+n} (\hat{x}_j)^2\right)} \Psi_2(y'') dx dh_0. \end{aligned}$$

And

$$\begin{aligned} &\int_{\mathbb{R}^{m+n}} e^{-2\xi_2(x)} e^{\left(-\sum_{j=1}^{m+n} (\hat{x}_j)^2\right)} dx \\ &= \left(\det\nolimits_{W_{-\frac{1}{2}}} h_0\right) \int_{\mathbb{R}^{m+n}} e^{\left(-\sum_{j=1}^{m+n} (x_j)^2 - 2\xi_2(\tilde{x})\right)} dx, \end{aligned} \tag{7.5.35}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{m+n})$ with $\sum_{j=1}^{m+n} \tilde{x}_j Q_j = h_0{\cdot}(\sum_{j=1}^{m+n} x_j Q_j)$, and the integral in the right-hand side of (7.5.35) has, up to a constant factor, the form $e^{\Lambda(h_0, \xi_2)}$ with $\Lambda(h_0, \xi_2) \geq 0$ for all $h_0 \in M_0$, $\xi_2 \in \mathbb{R}^{m+n}$. Therefore it follows that if the set $\{\xi \in \mathbb{R}^{d+m+n}; I(\xi) < +\infty\}$ has interior points, then the set of $\xi \in \mathbb{R}^d$ such that

$$\begin{aligned} I_1(\xi) &= \int_{M_0} e^{-2\xi(h_0\cdot s)} \Psi_2(h_0{\cdot}s)\, (\det\nolimits_{\mathfrak{m}_1} h_0) \left(\det\nolimits_{W_{-\frac{1}{2}}} h_0\right) dh_0 \\ &= \int_{\Omega} e^{-2\xi(t)} \Psi_1(t) dt = I_{\Psi_1}(\xi) < +\infty \end{aligned}$$

has interior points, where Ψ_1 is the function on Ω defined by (7.5.25). Again by Theorem 2.20 in [68], $\hat{\Omega}_{\Psi_1}$ has interior points if and only if $I_{\Psi_1}(\xi_0) < +\infty$ for

$\xi_0 = \sum_{k=1}^{r} U_k^* \in \Omega^*$, since Ψ_1 is homogeneous. Therefore if $\hat{\Omega}_{\Psi_1}$ has interior points, then the condition (7.5.29) is satisfied.

B_3 : General case. Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in P^+(f, G)$. Using the notations introduced at the beginning of Sect. 7.4, we decompose $\tilde{\mathfrak{e}}$ into a semi-direct sum $\tilde{\mathfrak{e}} = \mathfrak{n} + \mathfrak{m}$ by Theorem 7.4.1. As the triplet $(\mathfrak{m}, \mathfrak{h}_2, \tilde{f_2})$ satisfies the condition of Case I, Definition 7.5.7 applied to $\mathfrak{m}$ gives us the numbers p_i, q_i, r_i for $\mathfrak{m}$. Similarly, the number f_i is defined as in Theorem 7.5.8.

Define the numbers t_i $(1 \le i \le r = \operatorname{rank} \mathfrak{m})$ as follows. If $\mathfrak{n} = \{0\}$, put $t_i = 0$ $(1 \le i \le r)$. If $\mathfrak{n} \neq \{0\}$, the triplet $(\tilde{\mathfrak{e}}, \tilde{\mathfrak{h}}, \tilde{f})$ satisfies the condition of Case II, so we apply Definition 7.5.11 of t_i to $(\tilde{\mathfrak{e}}, \tilde{\mathfrak{h}}, \tilde{f})$.

Then the considerations in B_1 and B_2 give us the next theorem which generalizes Theorem 7.5.8.

Theorem 7.5.14. *$\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$ if and only if*

$$-2f_i - \left(p_i + 1 + \frac{1}{2}(q_i + r_i + t_i)\right) > 0 \; (1 \le i \le r).$$

7.6 Irreducibility and Equivalence of $\rho(f, \mathfrak{h}, G)$

We generalize the notion of the Pukanszky condition (cf. Definition 5.1.9).

Definition 7.6.1 (cf. Proposition 5.1.12). We say that $\mathfrak{h} \in P^+(f, G)$ satisfies the Pukanszky condition if $D{\cdot}f = f + \mathfrak{e}^{\perp}$, where $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ and $D = \exp \mathfrak{d}$ with $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ as before.

Note that, if $\mathfrak{h}$ is totally complex, i.e. $\mathfrak{e} = \mathfrak{g}$, then $\mathfrak{h}$ satisfies the Pukanszky condition. We denote as before by $\hat{\rho}(\sigma)$ the irreducible unitary representation of G associated with the coadjoint orbit $\sigma \in \mathfrak{g}^*/G$ in the Kirillov–Bernat sense (cf. Definition 5.3.30). Now we prove the following theorem which generalizes Theorems 7.1.1 and 7.5.9.

Theorem 7.6.2. *Let $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in P^+(f, G)$. Suppose $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$. Then $\rho(f, \mathfrak{h}, G)$ is irreducible if and only if $\mathfrak{h}$ satisfies the Pukanszky condition. In this case, $\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(G{\cdot}f)$. In particular, (the equivalence class of) $\rho(f, \mathfrak{h}, G)$ is independent of $\mathfrak{h}$.*

Proof (cf. Proof of Theorem 7.1.1). (A) First we prove that $\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(G{\cdot}f)$ for a positive polarization $\mathfrak{h}$ satisfying the Pukanszky condition. This claim is trivial when $\dim G = 1$, so we prove it by induction on $\dim G$ and assume $\dim G = n$.

Case 1. There is an ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{g}$ such that $f|_{\mathfrak{a}} = 0$.

Let $A = \exp \mathfrak{a}$, $\tilde{G} = G/A$ and $\pi : G \to \tilde{G}$ the canonical projection. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{G}$ and $d\pi : \mathfrak{g}_{\mathbb{C}} \to \tilde{\mathfrak{g}}_{\mathbb{C}}$ the differential of π. Now we consider the exact sequence of exponential solvable Lie groups:

$$1 \to A \to G \xrightarrow{\pi} \tilde{G} \to 1.$$

Let $\tilde{f} \in \tilde{\mathfrak{g}}^*$ be such that $\tilde{f} \circ d\pi = f$ and let $\tilde{\mathfrak{h}} = d\pi(\mathfrak{h})$. Since $\mathfrak{h} \supset \mathfrak{g}(f) \supset \mathfrak{a}$, it is clear that $\tilde{\mathfrak{h}} \in P^+(\tilde{f}, \tilde{G})$ and $\tilde{\mathfrak{g}}^*$ is naturally isomorphic to $\mathfrak{a}^{\perp, \mathfrak{g}^*}$, so that $\tilde{\mathfrak{h}}$ satisfies the Pukanszky condition. So by Proposition I.5.13 in [3],

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \circ \pi \simeq \rho(f, \mathfrak{h}, G). \tag{7.6.1}$$

Hence by our assumption, $\mathcal{H}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \neq \{0\}$. The induction hypothesis implies that $\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \simeq \hat{\rho}(\tilde{G} \cdot \tilde{f})$. That is, there exists $\tilde{\mathfrak{h}}_0 \in I(\tilde{f}, \tilde{\mathfrak{g}})$ such that

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \simeq \hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}). \tag{7.6.2}$$

Set $\mathfrak{h}_0 = (d\pi)^{-1}(\tilde{\mathfrak{h}}_0)$. Then it is obvious that

$$\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}) \circ \pi \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \tag{7.6.3}$$

and $\mathfrak{h}_0 \in I(f, \mathfrak{g})$. From (7.6.1)–(7.6.3),

$$\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \simeq \hat{\rho}(G \cdot f).$$

Case 2. There is no ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{g}$ such that $f|_{\mathfrak{a}} = 0$. This case is divided into two subcases.

(1) Suppose $\mathfrak{e} \neq \mathfrak{g}$. We choose and fix one complementary linear subspace $\mathfrak{c}$ of $\mathfrak{e}$ in $\mathfrak{g}$, and let $\kappa : \mathfrak{e}^* \to \mathfrak{g}^*$ be the injection such that $\kappa(l)(x) = l(y)$ for $l \in \mathfrak{e}^*$ and $x = y + z$ with $y \in \mathfrak{e}$, $z \in \mathfrak{c}$. From now on, we identify $\mathfrak{e}^*$ with its image $\kappa(\mathfrak{e}^*)$, so that $\mathfrak{e}^* \subset \mathfrak{g}^*$. Let $\pi : \mathfrak{g}^* \to \mathfrak{e}^*$ be the restriction mapping such that $\pi(l) = l_0 = l|_{\mathfrak{e}}$ for $l \in \mathfrak{g}^*$. Then $\pi|_{\mathfrak{e}^*} = I$ (the identity mapping of $\mathfrak{e}^*$) under the above identification. Now $\mathfrak{h}$ is clearly a positive polarization of $E = \exp \mathfrak{e}$ at $\pi(f) = f_0 = f|_{\mathfrak{e}}$ and satisfies the Pukanszky condition as a polarization of E at f_0 since $\mathfrak{h} \in P^+(f_0, E)$ is totally complex.

Since $\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f_0, \mathfrak{h}, E)$, $\mathcal{H}(f_0, \mathfrak{h}, E) \neq \{0\}$ provided $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$. We can apply the induction hypothesis to have $\rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(O(f_0))$, where $O(f_0)$ is the coadjoint orbit of E in $\mathfrak{e}^*$ through f_0. That is to say, there exists $\mathfrak{h}_0 \in I(f_0, \mathfrak{e}) \subset M(f_0, \mathfrak{e})$ such that

$$\rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(f_0, \mathfrak{h}_0, E) \tag{7.6.4}$$

Since $\mathfrak{h} \in P^+(f_0, E)$,

$$\dim_{\mathbb{R}}(\mathfrak{h}_0) = \dim_{\mathbb{C}} \mathfrak{h} = \frac{1}{2}(\dim_{\mathbb{R}} \mathfrak{g} + \dim_{\mathbb{R}}(\mathfrak{g}(f))).$$

Thus $\mathfrak{h}_0 \in M(f, \mathfrak{g})$.

We show that $\mathfrak{h}_0 \in I(f, \mathfrak{g})$, i.e. $\mathfrak{h}_0$ satisfies the Pukanszky condition. For the proof of the next lemma, see Proposition 3.1.5 in [10].

Lemma 7.6.3 (cf. Lemma 5.1.11). *Let $l \in \mathfrak{g}^*$ and $\mathfrak{k} \in S(l, \mathfrak{g}_{\mathbb{C}})$ such that $\mathfrak{g}(l)_{\mathbb{C}} \subset \mathfrak{k}$. We put $\mathfrak{p} = \mathfrak{k} \cap \mathfrak{g}$, $\mathfrak{q} = (\mathfrak{k} + \bar{\mathfrak{k}}) \cap \mathfrak{g}$ and set $P = exp\ \mathfrak{p}$. Then, the following conditions are equivalent:*

(1) $P{\cdot}l = l + \mathfrak{q}^{\perp}$;
(2) $l + \mathfrak{q}^{\perp} \subset G{\cdot}l$ *and* $\mathfrak{k} \in M(l, \mathfrak{g}_{\mathbb{C}})$;
(3) $\mathfrak{k} \in M(l + \varphi, \mathfrak{g}_{\mathbb{C}})$ *for any* $\varphi \in \mathfrak{q}^{\perp}$.

Now we continue the proof of the theorem. Since $\mathfrak{e}$ is a Lie subalgebra of $\mathfrak{g}$,

$$\pi(E{\cdot}f) = (E{\cdot}f)_0 = O(f_0). \tag{7.6.5}$$

Let us see that $\pi^{-1}(O(f_0)) = E{\cdot}f$. If $a \in E$ and $l \in \mathfrak{e}^{\perp}$, we have $a{\cdot}l \in \mathfrak{e}^{\perp}$. For $a \in E$, we write

$$a{\cdot}f = (a{\cdot}f)_0 + \hat{f}$$

with $\hat{f} \in \mathfrak{e}^{\perp}$. Then we have, for any $\ell \in \mathfrak{e}^{\perp}$, $a^{-1}{\cdot}(\hat{f} - \ell) \in \mathfrak{e}^{\perp}$ and

$$a{\cdot}(f - a^{-1}{\cdot}(\hat{f} - \ell)) = (a{\cdot}f)_0 + \ell.$$

Since $\mathfrak{h}$ satisfies the Pukanszky condition as a polarization of $\mathfrak{g}$ at f,

$$f - a^{-1}{\cdot}(\hat{f} - \ell) = b{\cdot}f$$

for some $b \in D \subset E$. Hence

$$(ab){\cdot}f = (a{\cdot}f)_0 + \ell. \tag{7.6.6}$$

Since $a \in E$ and $\ell \in \mathfrak{e}^{\perp}$ are arbitrary, (7.6.5) and (7.6.6) imply that

$$\pi^{-1}(O(f_0)) = E{\cdot}f. \tag{7.6.7}$$

Now we have

$$f + \mathfrak{h}_0^{\perp,\mathfrak{g}^*} = f_0 + \mathfrak{h}_0^{\perp,\mathfrak{e}^*} + \mathfrak{e}^{\perp}. \tag{7.6.8}$$

But since $\mathfrak{h}_0 \in I(f_0, \mathfrak{e})$, Theorem 5.4.1 asserts that $\mathfrak{h}_0$ satisfies the Pukanszky condition as an element of $S(f_0, \mathfrak{e})$ so that

$$f_0 + \mathfrak{h}_0^{\perp,\mathfrak{e}^*} \subset O(f_0). \tag{7.6.9}$$

One knows from (7.6.7)–(7.6.9) that

$$f + \mathfrak{h}_0^{\perp, \mathfrak{g}^*} \subset G{\cdot}f.$$

Thus $\mathfrak{h}_0$ satisfies the Pukanszky condition as an element of $S(f, \mathfrak{g})$. Hence Theorem 5.4.1 implies that $\mathfrak{h}_0 \in I(f, \mathfrak{g})$. That is, $\hat{\rho}(f, \mathfrak{h}_0, G) = \mathrm{ind}_E^G \hat{\rho}(f_0, \mathfrak{h}_0, E)$ is irreducible. So by (7.6.4),

$$\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f_0, \mathfrak{h}, E) \simeq \mathrm{ind}_E^G \hat{\rho}(f_0, \mathfrak{h}_0, E) \simeq \hat{\rho}(G{\cdot}f).$$

(ii) Suppose $\mathfrak{e} = \mathfrak{g}$ (i.e. $\mathfrak{h}$ is totally complex). This case is divided into two subcases, that is, Case I and Case II in B of the previous section. Thus by Theorem 7.5.9 we can again assume Case II and consider the situation B_2 of the previous section. Put

$$f' = f + \sum_{i=1}^{r} \frac{t_i}{4} U_i^*, \quad \tilde{f}_2' = f'|_{\mathfrak{m}} = \tilde{f}_2 + \sum_{i=1}^{r} \frac{t_i}{4} U_i^*$$

and

$$\mathfrak{h}' = \sum_{k=1}^{m+n} \mathbb{C} Q_k \oplus \mathbb{C} C \oplus \mathfrak{h}_2$$

with $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{m}_{\mathbb{C}}$ as before. Then

$$\mathfrak{e}' = (\mathfrak{h}' + \overline{\mathfrak{h}'}) \cap \mathfrak{g} = \sum_{k=1}^{m+n} \mathbb{R} Q_k \oplus \mathbb{R} C \oplus \mathfrak{m} \neq \mathfrak{g}$$

and

$$\mathfrak{d}' = \mathfrak{h}' \cap \mathfrak{g} = \sum_{k=1}^{m+n} \mathbb{R} Q_k \oplus \mathbb{R} C.$$

Lemma 7.6.4. *$f' \in G{\cdot}f$, $\mathfrak{h}' \in P^+(f', G)$ and $\mathfrak{h}'$ also satisfies the Pukanszky condition.*

Proof. Note that f has the form

$$f = C^* + \sum_{k=1}^{r} f_k U_k^* \ (f_k < 0, 1 \le k \le r)$$

with respect to the dual basis

$$\{P_k^*, Q_k^*, C^*, (jU_k)^*, (E_{k,l}^p)^*, (V_\mu^k)^*, (jV_\mu^k)^*, (jE_{k,l}^p)^*, (U_k)^*\}$$

and that $\mathfrak{m}\cdot\tilde{f}_1 = 0$. By these facts, it is easy to see that $f' \in G\cdot f$ since

$$f_k + \frac{t_k}{4} < 0 \ (1 \leq k \leq r)$$

by our assumption $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$ (cf. Theorem 7.5.14). Since $\mathfrak{h}_2$ is a Lie subalgebra of $\mathfrak{m}_{\mathbb{C}}$,

$$[\mathfrak{h}_2, \mathfrak{h}_2] \subset \mathfrak{h}_2 \cap [\mathfrak{m}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}}] \subset \sum_{k>l} \left(\eta^{\frac{1}{2}(\alpha_k - \alpha_l)} + \eta^{\frac{1}{2}(\alpha_k + \alpha_l)}\right)_{\mathbb{C}} \oplus (\mathfrak{m}_{\frac{1}{2}})_{\mathbb{C}}$$

and $f'([\mathfrak{h}', \mathfrak{h}']) = 0$ by $\mathfrak{m}\cdot\tilde{f}_1 = 0$. Further, $\dim(\mathfrak{h}') = \dim \mathfrak{h}$ and $f'(U_k) < 0$ $(1 \leq k \leq r)$. These facts imply that $\mathfrak{h}' \in P^+(f', G)$. Finally, $\mathfrak{h}'$ satisfies the Pukanszky condition since

$$\exp(tQ_k)\cdot f' = f' + tP_k^* \ (t \in \mathbb{R})$$

$1 \leq k \leq m+n$. ∎

By this lemma together with the fact $\mathfrak{e}' \neq \mathfrak{g}$, it suffices to show that $\rho(f, \mathfrak{h}, G) \simeq \rho(f', \mathfrak{h}', G)$. We write $g \in G = N\cdot M$ in the form

$$g = \exp\left(\sum_{k=1}^{m+n} x_k P_k\right)\cdot\exp\left(\sum_{k=1}^{m+n} y_k Q_k\right)\cdot\exp(wC)\cdot h$$

with $h \in M$. Let $E' = \exp(\mathfrak{e}')$ and let $\pi : G \to G/E'$ be the canonical projection. Under the correspondence

$$\pi(g) \leftrightarrow x = (x_1, x_2, \ldots, x_{m+n})$$

between G/E' and $\mathbb{R}^{m+n}$, the Lebesgue measure dx on $\mathbb{R}^{m+n}$ gives a quasi-invariant measure on G/E'. A function Φ of the form

$$\Phi(g) = e^{-iw}\phi(x)\psi(h) \ \left(\phi \in L^2(\mathbb{R}^{m+n}),\ \psi \in \mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M)\right)$$

is clearly contained in $\mathcal{H}(f', \mathfrak{h}', G)$. We denote as before by $C_c(\mathbb{R}^{m+n})$ the space of complex-valued continuous functions on $\mathbb{R}^{m+n}$ with compact support and set

$$\mathcal{S} = \left\{ \sum_{k:\text{finite}} c_k \Phi_k ; c_k \in \mathbb{C},\ \Phi_k(g) = e^{-iw}\phi_k(x)\psi_k(h), \right.$$
$$\left. \phi_k \in C_c(\mathbb{R}^{m+n}),\ \psi_k \in \mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M) \right\}.$$

The space $\mathcal{S}$ is invariant under $\rho(f', \mathfrak{h}', G)$. Now we show that $\mathcal{S}$ is dense in $\mathcal{H}(f', \mathfrak{h}', G)$. For $\Psi \in \mathcal{H}(f', \mathfrak{h}', G)$, we set

$$\hat{\Psi}(x, h) = \Psi\left(\exp\left(\sum_{k=1}^{m+n} x_k P_k\right) \cdot h\right)$$

with $x = (x_1, x_2, \ldots, x_{m+n}) \in \mathbb{R}^{m+n},\ h \in M$. Suppose that Ψ is orthogonal to $\mathcal{S}$. Then

$$\int \phi(x)\psi(h)\overline{\hat{\Psi}(x, h)}dxdh = \int \phi(x) \int \psi(h)\overline{\hat{\Psi}(x, h)}dhdx = 0 \tag{7.6.10}$$

for all $\phi \in C_c(\mathbb{R}^{m+n})$ and $\psi \in \mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M)$. Here

$$\begin{aligned}\int \left|\int \psi(h)\overline{\hat{\Psi}(x, h)}dh\right|^2 dx &\leq \int \left(\int |\psi(h)|^2 dh \int |\hat{\Psi}(x, h)|^2 dh\right) dx \\ &= \|\psi\|^2_{\mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M)} \cdot \|\hat{\Psi}\|^2 < +\infty.\end{aligned}$$

By letting ϕ run over $C_c(\mathbb{R}^{m+n})$, this implies that

$$\int \psi(h)\overline{\hat{\Psi}(x, h)}dh = 0 \tag{7.6.11}$$

for almost all $x \in \mathbb{R}^{m+n}$. But, if the function Ψ_x on M is given by $\Psi_x(h) = \hat{\Psi}(x, h)$ $(h \in M)$, then we have $\Psi_x \in \mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M)$ for almost all $x \in \mathbb{R}^{m+n}$. Since ψ is arbitrary, this implies that $\Psi_x = 0$ in $\mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M)$ for almost all $x \in \mathbb{R}^{m+n}$ and hence that $\Psi = 0$. Next we write $g \in G = M \cdot N$ in the form

$$g = h\exp\left(\sum_{k=1}^{m+n} x_k P_k\right) \exp\left(\sum_{k=1}^{m+n} y_k Q_k\right) \exp(wC)\ (h \in M).$$

Then, $\mathcal{S}$ is the set of all functions

$$g \mapsto \sum_{k:\text{finite}} c_k e^{\left(-iw - \frac{i}{2}\sum_{\nu=1}^{m+n} x'_\nu x''_\nu\right)} \phi'_k(x)\psi_k(h),$$

where

$$c_k \in \mathbb{C},\ \psi_k \in \mathcal{H}(\tilde{f}_2', \mathfrak{h}_2, M),\ \sum_{\nu=1}^{m+n} x'_\nu P_\nu + \sum_{\nu=1}^{m+n} x''_\nu Q_\nu = h \cdot \left(\sum_{\nu=1}^{m+n} x_\nu P_\nu\right)$$

and $\phi'_k(x) = \phi_k(x'_1, x'_2, \ldots, x'_{m+n})$ with $\phi_k \in C_c(\mathbb{R}^{m+n})$.

For

$$\Phi(g) = \sum_{k:\text{finite}} c_k e^{\left(-iw-\frac{i}{2}\sum_{\nu=1}^{m+n} x'_\nu x''_\nu\right)} \phi'_k(x)\psi_k(h),$$

we set

$$\begin{aligned}
&(R\Phi)\left(h\exp\sum_\nu (x_\nu P_\nu + y_\nu Q_\nu)\exp(wC)\right)\\
=&e^{\left(-iw-\frac{1}{2}\sum_\nu x_\nu^2-\frac{i}{2}\sum_\nu x_\nu y_\nu\right)}\left(\det{}_{W_{-\frac{1}{2}}} h_0\right)^{\frac{1}{2}}\\
&\times\int e^{\left(\sum_\nu \xi_\nu z_\nu-\frac{1}{2}\sum_\nu \xi_\nu^2\right)}\Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)d\xi\\
=&\sum_{k:\text{finite}} c_k e^{\left(-iw-\frac{1}{2}\sum_\nu x_\nu^2-\frac{i}{2}\sum_\nu x_\nu y_\nu\right)}\left(\det{}_{W_{-\frac{1}{2}}} h_0\right)^{\frac{1}{2}}\psi_k(h)\\
&\times\int e^{\left(\sum_\nu \xi_\nu z_\nu\right)}e^{\left(-\frac{1}{2}\sum_\nu \xi_\nu^2+\frac{i}{2}\sum_\nu \xi'_\nu\xi''_\nu\right)}\phi'_k(\xi)d\xi\\
=&\sum_{k:\text{finite}} c_k e^{\left(-iw-\frac{1}{2}\sum_\nu x_\nu^2-\frac{i}{2}\sum_\nu x_\nu y_\nu\right)}\left(\det{}_{W_{-\frac{1}{2}}} h_0\right)^{-\frac{1}{2}}\psi_k(h)\\
&\times\int e^{\left(\sum_\nu \hat{\xi}_\nu z_\nu-\frac{1}{2}\sum_\nu (\hat{\xi}_\nu)^2+\frac{i}{2}\sum_\nu \xi_\nu\hat{\hat{\xi}}_\nu\right)}\phi_k(\xi)d\xi,
\end{aligned}\tag{7.6.12}$$

where

$$h = h_0\cdot h', h_0 \in M_0,\ h' \in M_{\frac{1}{2}}\cdot M_1,\ z_\nu = x_\nu + iy_\nu\ (1 \le \nu \le m+n),$$

and

$$h\cdot\left(\sum_{k=1}^{m+n}\hat{\xi}_k P_k\right) = \sum_{k=1}^{m+n}\xi_k P_k + \sum_{k=1}^{m+n}\hat{\hat{\xi}}_k Q_k. \tag{7.6.13}$$

Lemma 7.6.5. *The operator*

$$2^{-(m+n)/2}\pi^{-3(m+n)/4}R$$

gives an isometry from $\mathcal{S}$ into $\mathcal{H}(f,\mathfrak{h},G)$.

Proof. Note that $R\Phi$ is a C^∞-function on G. Let Φ_0 be the C^∞-function on G given by

$$\Phi_0(g) = \left(\det_{W_{-\frac{1}{2}}} h\right)^{-\frac{1}{2}}$$

for

$$g = h\exp\sum_{\nu=1}^{m+n}(x_\nu P_\nu + y_\nu Q_\nu)\exp(wC) \ \ (h \in M), \tag{7.6.14}$$

where $h = h_0 \cdot h',\ h_0 \in M_0,\ h' \in M_{\frac{1}{2}} \cdot M_1$. It is easy to see that

$$\Phi_0 \cdot (U_k + ijU_k) = \frac{it_k}{4}\Phi_0 \ \ (1 \le k \le r),$$

$$\Phi_0 \cdot X = 0 \ \ (X \in \mathfrak{h}_1 \oplus (\mathfrak{h}_2 \cap [\mathfrak{m}, \mathfrak{m}]_{\mathbb{C}}))\,.$$

This fact together with Lemma 7.5.13 implies that if the function Φ_1 given by

$$\Phi_1(g) = e^{-\frac{1}{2}\sum_\nu \left(\hat{\xi}_\nu\right)^2 + \frac{i}{2}\sum_\nu \left(\xi_\nu \hat{\hat{\xi}}_\nu\right)}$$

satisfies

$$\Phi_1 \cdot X = 0 \ \ (X \in \mathfrak{h}_2), \tag{7.6.15}$$

then $R\Phi$ $(\Phi \in \mathcal{S})$ satisfies the infinitesimal condition on the space $\mathcal{H}(f, \mathfrak{h}, G)$ by virtue of formula (7.6.12). By (7.6.13),

$$\hat{\xi}_k = \hat{\xi}_k(h, \xi) = \sum_\nu \xi_\nu B_f(h^{-1} \cdot P_\nu, Q_k), \tag{7.6.16}$$

$$\begin{aligned}\hat{\hat{\xi}}_k = \hat{\hat{\xi}}_k(h, \xi) &= \sum_{\nu=1}^{m+n} \hat{\xi}_\nu B_f(h^{-1} \cdot P_k, P_\nu) \\ &= \sum_{\nu,\tau=1}^{m+n} \xi_\tau B_f(h^{-1} \cdot P_\tau, Q_\nu) B_f(h^{-1} \cdot P_k, P_\nu)\end{aligned} \tag{7.6.17}$$

for $1 \le k \le m+n$. For $X \in \mathfrak{m}$ and $t \in \mathbb{R}$, we set

$$\left(\hat{\xi}_k\right)_{tX} = \hat{\xi}_k(h\exp(tX), \xi),\ \left(\hat{\hat{\xi}}_k\right)_{tX} = \hat{\hat{\xi}}_k(h\exp(tX), \xi).$$

In order to show (7.6.15), it is sufficient to verify that

$$\sum_{k=1}^{m+n}\left(\hat{\xi}_k \frac{d\left(\hat{\xi}_k\right)_{tX}}{dt}\bigg|_{t=0} - i\,\frac{\xi_k}{2}\frac{d\left(\hat{\hat{\xi}}_k\right)_{tX}}{dt}\bigg|_{t=0} + i\,\hat{\xi}_k\frac{d\left(\hat{\xi}_k\right)_{tjX}}{dt}\bigg|_{t=0} + \frac{\xi_k}{2}\frac{d\left(\hat{\hat{\xi}}_k\right)_{tjX}}{dt}\bigg|_{t=0}\right) \tag{7.6.18}$$

is equal to zero for any $X \in \mathfrak{m}$. Using (7.6.16) and (7.6.17), we make the following calculations:

$$\left(\hat{\xi}_k\right)_{tU_l} = \sum_{\nu}\xi_\nu B_f(h^{-1}\cdot P_\nu, \exp(tU_l)\cdot Q_k) = \sum_{\nu}\xi_\nu B_f(h^{-1}\cdot P_\nu, Q_k) = \hat{\xi}_k,$$

$$\left(\hat{\hat{\xi}}_k\right)_{tU_l} = \sum_{\nu}\hat{\xi}_\nu B_f(h^{-1}\cdot P_k, \exp(tU_l)\cdot P_\nu) = \hat{\hat{\xi}}_k + t\sum_{\nu=i_{l-1}+1}^{i_l}\hat{\xi}_\nu B_f(h^{-1}\cdot P_k, Q_\nu).$$

$$\left(\hat{\xi}_k\right)_{tjU_l} = \sum_{\nu}\xi_\nu B_f(h^{-1}\cdot P_\nu, \exp(tjU_l)\cdot Q_k)$$

$$= \begin{cases} e^{t/2}\hat{\xi}_k & (i_{l-1}+1 \le k \le i_l'), \\ e^{t/2}\left(\hat{\xi}_k \cos\left(\frac{\gamma_k t}{2}\right) + \hat{\xi}_{k+i_l''}\sin\left(\frac{\gamma_k t}{2}\right)\right) & (i_l'+1 \le k \le i_l'+i_l''), \\ e^{t/2}\left(\hat{\xi}_k \cos\left(\frac{\gamma_{k-i_l''} t}{2}\right) - \hat{\xi}_{k-i_l''}\sin\left(\frac{\gamma_{k-i_l''} t}{2}\right)\right) & (i_l'+i_l''+1 \le k \le i_l), \\ \hat{\xi}_k & (\text{otherwise}) \end{cases}$$

$$\begin{aligned}\left(\hat{\hat{\xi}}_k\right)_{tjU_l} = &\sum_{\substack{1\le\nu\le i_{l-1} \\ i_\ell+1\le\nu\le m+n}}\hat{\xi}_\nu B_f(h^{-1}\cdot P_k, \exp(tjU_l)\cdot P_\nu) \\ &+ e^{t/2}\sum_{\nu=i_{l-1}+1}^{i_l''}\hat{\xi}_\nu B_f(h^{-1}\cdot P_k, \exp(tjU_l)\cdot P_\nu) \\ &+ e^{t/2}\sum_{\nu=i_l'+1}^{i_l'+i_l''}\left(\hat{\xi}_\nu\cos\left(\frac{\gamma_\nu t}{2}\right) + \hat{\xi}_{\nu+i_l''}\sin\left(\frac{\gamma_\nu t}{2}\right)\right) \\ &\times B_f(h^{-1}\cdot P_k, \exp(tjU_l)\cdot P_\nu)\end{aligned}$$

$$+ e^{t/2} \sum_{\nu=i_l'+i_l''+1}^{i_l} \left(\hat{\xi}_\nu \cos\left(\frac{\gamma_{\nu-i_l''}t}{2}\right) - \hat{\xi}_{\nu-i_l''} \sin\left(\frac{\gamma_{\nu-i_l''}t}{2}\right)\right)$$

$$\times B_f(h^{-1}\cdot P_k, \exp(tjU_l)\cdot P_\nu).$$

But the sum of the last two terms is equal to

$$\sum_{\nu=i_l'+1}^{i_l'+i_l''} \left(\hat{\xi}_\nu \cos\left(\frac{\gamma_\nu t}{2}\right) + \hat{\xi}_{\nu+i_l''} \sin\left(\frac{\gamma_\nu t}{2}\right)\right)$$

$$\times \left(B_f(h^{-1}\cdot P_k, P_\nu) \cos\left(\frac{\gamma_\nu t}{2}\right) + B_f(h^{-1}\cdot P_k, P_{\nu+i_l''}) \sin\left(\frac{\gamma_\nu t}{2}\right)\right)$$

$$+ \sum_{\nu=i_l'+i_l''+1}^{i_l} \left(\hat{\xi}_\nu \cos\left(\frac{\gamma_{\nu-i_l''}t}{2}\right) - \hat{\xi}_{\nu-i_l''} \sin\left(\frac{\gamma_{\nu-i_l''}t}{2}\right)\right)$$

$$\times \left(B_f(h^{-1}\cdot P_k, P_\nu) \cos\left(\frac{\gamma_{\nu-i_l''}t}{2}\right) - B_f(h^{-1}\cdot P_k, P_{\nu-i_l''}) \sin\left(\frac{\gamma_{\nu-i_l''}t}{2}\right)\right)$$

$$= \sum_{\nu=i_l'+1}^{i_l'+i_l''} \hat{\xi}_\nu B_f(h^{-1}\cdot P_k, P_\nu) + \sum_{\nu=i_l'+i_l''+1}^{i_l} \hat{\xi}_\nu B_f(h^{-1}\cdot P_k, P_\nu).$$

Thus we have

$$\left(\hat{\hat{\xi}}_k\right)_{tjU_l} = \hat{\hat{\xi}}_k.$$

Hence for $X = U_l$, the real part of (7.6.18) is equal to zero and its imaginary part is equal to

$$\frac{1}{2} \sum_{k=i_{l-1}+1}^{i_l'} \left(\hat{\xi}_k\right)^2 - \frac{1}{2} \sum_{k=i_l'+1}^{i_l'+i_l''} \hat{\xi}_k \left(\frac{1}{2}\hat{\xi}_k + \frac{\gamma_k}{2}\hat{\xi}_{k+i_l''}\right)$$

$$+ \sum_{k=i_l'+i_l''+1}^{i_l} \hat{\xi}_k \left(\frac{1}{2}\hat{\xi}_k - \frac{\gamma_{k-i_l''}}{2}\hat{\xi}_{k-i_l''}\right) - \frac{1}{2}\sum_{k=1}^{m+n} \xi_k \sum_{\nu=i_{l-1}+1}^{i_l} \hat{\xi}_\nu B_f(h^{-1}\cdot P_k, Q_\nu)$$

$$= \frac{1}{2} \sum_{\nu=i_{l-1}+1}^{i_l} \left(\hat{\xi}_k\right)^2 - \frac{1}{2} \sum_{\nu=i_{l-1}+1}^{i_l} \left(\hat{\xi}_k\right)^2 = 0$$

by (7.6.16). Using the same notations for the structure constants as in Sect. 7.5, B_2,

$$\left(\hat{\xi}_\nu\right)_{tE_{k,l}^p} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1}\cdot P_i, \exp(tE_{k,l}^p)\cdot Q_\nu)$$

$$= \sum_{i=1}^{m+n} \xi_i \left(B_f(h^{-1}\cdot P_i, Q_\nu) - t \sum_{\tau=i_{k-1}+1}^{i_k} c_{k,l,\tau}^{p,\nu} B_f(h^{-1}\cdot P_i, Q_\tau) \right)$$

$$= \hat{\xi}_\nu - t \sum_{\tau=i_{k-1}+1}^{i_k} \hat{\xi}_\tau$$

for $i_{l-1} + 1 \leq \nu \leq i_l$. Otherwise,

$$\left(\hat{\xi}_\nu\right)_{tE_{k,l}^p} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1}\cdot P_i, \exp(tE_{k,l}^p)\cdot Q_\nu) = \hat{\xi}_\nu.$$

Again,

$$\begin{aligned}
\left(\hat{\hat{\xi}}_\nu\right)_{tE_{k,l}^p} &= \sum_{\substack{1\leq i\leq i_{l-1} \\ i_l+1\leq i\leq m+n}} \hat{\xi}_i B_f(h^{-1}\cdot P_\nu, \exp(tE_{k,l}^p)\cdot P_i) \\
&\quad + \sum_{i=i_{l-1}+1}^{i_l} \left(\hat{\xi}_i - t \sum_{\tau=i_{k-1}+1}^{i_k} c_{k,l,\tau}^{p,i} \hat{\xi}_\tau \right) B_f(h^{-1}\cdot P_\nu, \exp(tE_{k,l}^p)\cdot P_i) \\
&= \sum_{\substack{1\leq i\leq i_{l-1} \\ i_l+1\leq i\leq m+n}} \hat{\xi}_i B_f(h^{-1}\cdot P_\nu, P_i) + t \sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{\tau=i_{l-1}}^{i_l} c_{k,l,i}^{p,\tau} B_f(h^{-1}\cdot P_\nu, P_\tau) \\
&\quad + \sum_{i=i_{l-1}+1}^{i_l} \left(\hat{\xi}_i - t \sum_{\tau=i_{k-1}+1}^{i_k} c_{k,l,\tau}^{p,i} \hat{\xi}_\tau \right) B_f(h^{-1}\cdot P_\nu, P_i) = \hat{\hat{\xi}}_\nu, \\
\left(\hat{\xi}_\nu\right)_{tjE_{k,l}^p} &= \sum_{i=1}^{m+n} \xi_i B_f(h^{-1}\cdot P_i, \exp(tjE_{k,l}^p)\cdot Q_\nu) = \hat{\xi}_\nu, \\
\left(\hat{\hat{\xi}}_\nu\right)_{tjE_{k,l}^p} &= \sum_{i=1}^{m+n} \hat{\xi}_i B_f(h^{-1}\cdot P_\nu, \exp(tjE_{k,l}^p)\cdot P_i) \\
&= \hat{\hat{\xi}}_\nu + t \left\{ \sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{\tau=i_{l-1}+1}^{i_l} c_{k,l,i}^{p,\tau} B_f(h^{-1}\cdot P_\nu, Q_\tau) \right. \\
&\quad \left. + \sum_{i=i_{l-1}+1}^{i_l} \hat{\xi}_i \sum_{\tau=i_{k-1}+1}^{i_k} c_{k,l,\tau}^{p,i} B_f(h^{-1}\cdot P_\nu, Q_\tau) \right\}
\end{aligned}$$

$$=\hat{\hat{\xi}}_\nu + t \sum_{\substack{i_{k-1}+1 \le i \le i_k \\ i_{l-1}+1 \le \tau \le i_l}} c^{p,\tau}_{k,l,i} (\hat{\xi}_i B_f(h^{-1} \cdot P_\nu, Q_\tau) + \hat{\xi}_\tau B_f(h^{-1} \cdot P_\nu, Q_i)).$$

Therefore the imaginary part of (7.6.18) is equal to zero for $X = E^p_{k,l}$. The real part of (7.6.18) is equal to

$$- \sum_{\substack{i_{l-1}+1 \le \nu \le i_l \\ i_{k-1}+1 \le \tau \le i_k}} c^{p,\nu}_{k,l,\tau} \hat{\xi}_\nu \hat{\xi}_\tau + \frac{1}{2} \sum_{\nu=1}^{m+n} \xi_\nu \sum_{\substack{i_{k-1}+1 \le i \le i_k \\ i_{l-1}+1 \le \tau \le i_l}} c^{p,\tau}_{k,l,i} \Big(\hat{\xi}_i B_f(h^{-1} \cdot P_\nu, Q_\tau)$$
$$+ \hat{\xi}_\tau B_f(h^{-1} \cdot P_\nu, Q_i)\Big) = - \sum_{\substack{i_{l-1}+1 \le \nu \le i_l \\ i_{k-1}+1 \le \tau \le i_k}} c^{p,\nu}_{k,l,\tau} \hat{\xi}_\nu \hat{\xi}_\tau + \sum_{\substack{i_{k-1}+1 \le i \le i_k \\ i_{l-1}+1 \le \tau \le i_l}} c^{p,\tau}_{k,l,i} \hat{\xi}_i \hat{\xi}_\tau = 0$$

by (7.6.16). Further,

$$\Big(\hat{\xi}_\nu\Big)_{tV^k_\mu} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1} \cdot P_i, \exp(tV^k_\mu) \cdot Q_\nu)$$
$$= \hat{\xi}_\nu - \sum_{i=1}^{m+n} \xi_i \sum_{\tau=i_{k-1}+1}^{i_k} d^{k,\nu}_{\mu,\tau} B_f(h^{-1} \cdot P_i, Q_\tau) = \hat{\xi}_\nu - t \sum_{\tau=i_{k-1}+1}^{i_k} d^{k,\nu}_{\mu,\tau} \hat{\xi}_\tau$$

for $m+1 \le \nu \le m+n$. When $1 \le \nu \le m$,

$$\Big(\hat{\xi}_\nu\Big)_{tV^k_\mu} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1} \cdot P_i, \exp(tV^k_\mu) \cdot Q_\nu) = \hat{\xi}_\nu.$$

Again,

$$\Big(\hat{\hat{\xi}}_\nu\Big)_{tV^k_\mu} = \sum_{i=1}^{m} \hat{\xi}_i B_f(h^{-1} \cdot P_\nu, \exp(tV^k_\mu) \cdot P_i)$$
$$+ \sum_{i=m+1}^{m+n} \left(\hat{\xi}_i - t \sum_{\tau=i_{k-1}+1}^{i_k} d^{k,i}_{\mu,\tau} \hat{\xi}_\tau \right) B_f(h^{-1} \cdot P_\nu, \exp(tV^k_\mu) \cdot P_i)$$
$$= \sum_{i=1}^{m} \hat{\xi}_i B_f(h^{-1} \cdot P_\nu, P_i) + t \sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} \Big(d^{k,l}_{\mu,i} B_f(h^{-1} \cdot P_\nu, P_l)$$
$$+ e^{k,l}_{\mu,i} B_f(h^{-1} \cdot P_\nu, Q_l) \Big) + \sum_{i=m+1}^{m+n} \left(\hat{\xi}_i - t \sum_{\tau=i_{k-1}+1}^{i_k} d^{k,i}_{\mu,\tau} \hat{\xi}_\tau \right)$$

$$\times \left(B_f(h^{-1}\cdot P_\nu, P_i) + t \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,i} B_f(h^{-1}\cdot P_\nu, Q_\tau) \right)$$

$$= \hat{\hat{\xi}}_\nu + t \left(\sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} e_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_l) \right.$$

$$\left. + \sum_{i=m+1}^{m+n} \hat{\xi}_i \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_\tau) \right) + \text{terms of degree} \geq 2 \text{ in } t.$$

Again,

$$\left(\hat{\xi}_\nu\right)_{tjV_\mu^k} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1}\cdot P_i, \exp(tjV_\mu^k)\cdot Q_\nu)$$

$$= \hat{\xi}_\nu + t \sum_{i=1}^{m+n} \xi_i \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,\nu} B_f(h^{-1}\cdot P_i, Q_\tau) = \hat{\xi}_\nu + t \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,\nu} \hat{\xi}_\tau$$

for $m + 1 \leq \nu \leq m + n$. When $1 \leq \nu \leq m$,

$$\left(\hat{\xi}_\nu\right)_{tjV_\mu^k} = \sum_{i=1}^{m+n} \xi_i B_f(h^{-1}\cdot P_i, \exp(tjV_\mu^k)\cdot Q_\nu) = \hat{\xi}_\nu.$$

Further,

$$\left(\hat{\hat{\xi}}_\nu\right)_{tjV_\mu^k} = \sum_{i=1}^{m} \hat{\xi}_i B_f(h^{-1}\cdot P_\nu, \exp(tjV_\mu^k)\cdot P_i)$$

$$+ \sum_{i=1}^{m+n} \left(\hat{\xi}_i + t \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,i} \hat{\xi}_\tau \right) B_f(h^{-1}\cdot P_\nu, \exp(tjV_\mu^k\cdot P_i))$$

$$= \sum_{i=1}^{m} \hat{\xi}_i B_f(h^{-1}\cdot P_\nu, P_i) - t \sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} \Big(e_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, P_l)$$

$$- d_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_l) \Big) + \sum_{i=m+1}^{m+n} \left(\hat{\xi}_i + t \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,i} \hat{\xi}_\tau \right)$$

$$\times \left(B_f(h^{-1}\cdot P_\nu, P_i) + t \sum_{\tau=i_{k-1}+1}^{i_k} d_{\mu,\tau}^{k,i} B_f(h^{-1}\cdot P_\nu, Q_\tau) \right)$$

$$=\hat{\hat{\xi}}_\nu + t\left(\sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} d_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_l)\right.$$

$$\left. + \sum_{i=m+1}^{m+n} \hat{\xi}_i \sum_{\tau=i_{k-1}+1}^{i_k} d_{\mu,\tau}^{k,i} B_f(h^{-1}\cdot P_\nu, Q_\tau)\right) + \text{terms of degree} \geq 2 \text{ in } t.$$

Thus by (7.6.16), the real part of (7.6.18) is equal to

$$-\sum_{\nu=m+1}^{m+n} \hat{\xi}_\nu \sum_{\tau=i_{k-1}+1}^{i_k} d_{\mu,\tau}^{k,\nu} \hat{\xi}_\tau + \frac{1}{2}\sum_{\nu=1}^{m+n} \xi_\nu \left(\sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} d_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_l)\right.$$

$$\left. + \sum_{i=m+1}^{m+n} \hat{\xi}_i \sum_{\tau=i_{k-1}+1}^{i_k} d_{\mu,\tau}^{k,i} B_f(h^{-1}\cdot P_\nu, Q_\tau)\right) = 0,$$

and the imaginary part of (7.6.18) is equal to

$$\sum_{\nu=m+1}^{m+n} \hat{\xi}_\nu \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,\nu} \hat{\xi}_\tau - \frac{1}{2}\sum_{\nu=1}^{m+n} \xi_\nu \left(\sum_{i=i_{k-1}+1}^{i_k} \hat{\xi}_i \sum_{l=m+1}^{m+n} e_{\mu,i}^{k,l} B_f(h^{-1}\cdot P_\nu, Q_l)\right.$$

$$\left. + \sum_{i=m+1}^{m+n} \hat{\xi}_i \sum_{\tau=i_{k-1}+1}^{i_k} e_{\mu,\tau}^{k,i} B_f(h^{-1}\cdot P_\nu, Q_\tau)\right) = 0.$$

Through these calculations we can conclude that, if $\Phi \in \mathcal{S}$, $R\Phi$ satisfies the infinitesimal condition imposed on the elements of $\mathcal{H}(f, \mathfrak{h}, G)$.

Furthermore, by the Plancherel formula,

$$\|R\Phi\|^2_{\mathcal{H}(f,\mathfrak{h},G)} = \int e^{-\sum_\nu (x_\nu)^2} \left(\det_{W_{-\frac{1}{2}}} h_0\right)$$

$$\times \left| \int e^{\sum_\nu \xi_\nu z_\nu} e^{-\frac{1}{2}\sum_\nu (\xi_\nu)^2} \Phi\left(h \exp\left(\sum_\nu \xi_\nu P_\nu\right)\right) d\xi \right|^2 dx\,dy\,dh$$

$$=(2\pi)^{m+n} \int e^{-\sum_\nu (x_\nu)^2} \left(\det_{W_{-\frac{1}{2}}} h_0\right)$$

$$\times \left(\int e^{2\sum_\nu x_\nu y_\nu - \sum_\nu (y_\nu)^2} \left|\Phi\left(h \exp\left(\sum_\nu y_\nu P_\nu\right)\right)\right|^2 dy\right) dx\,dh$$

$$=(2\pi)^{m+n}\int e^{-\sum_\nu (x_\nu-y_\nu)^2}dx\int\left(\det_{W_{-\frac{1}{2}}} h_0\right)$$
$$\times\left|\Phi\left(h\exp\left(\sum_\nu y_\nu P_\nu\right)\right)\right|^2 dy\,dh = 2^{m+n}\pi^{3(m+n)/2}\|\Phi\|^2_{\mathcal{H}(f',\mathfrak{h}',G)}. \qquad\blacksquare$$

We put $\rho_1=\rho(f,\mathfrak{h},G)$ and $\rho_2=\rho(f',\mathfrak{h}',G)$.

Lemma 7.6.6. *We have*

$$(\rho_1(g_0)\circ R)(\Phi)=(R\circ\rho_2(g_0))(\Phi) \tag{7.6.19}$$

for all $g_0\in G,\ \Phi\in\mathcal{S}$.

Proof. We again use expression (7.6.14).

(a) Let $g_0=\tilde{h}=\tilde{h}_0\cdot\tilde{h}'\in M,\ \tilde{h}_0\in M_0,\ \tilde{h}'\in M_{\frac{1}{2}}\cdot M_1$.

$$((\rho_1(g_0)\circ R)\Phi)(g)=(R\Phi)(g_0^{-1}g)$$
$$=e^{-iw-\frac{1}{2}\sum_\nu(x_\nu)^2-\frac{i}{2}\sum_\nu x_\nu y_\nu}\left(\det_{W_{-\frac{1}{2}}}\tilde{h}_0^{-1}h_0\right)^{\frac{1}{2}}$$
$$\times\int e^{\sum_\nu \xi_\nu z_\nu}e^{-\frac{1}{2}\sum_\nu(\xi_\nu)^2}\Phi\left(\tilde{h}^{-1}h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)d\xi$$
$$=e^{-iw-\frac{1}{2}\sum_\nu(x_\nu)^2-\frac{i}{2}\sum_\nu x_\nu y_\nu}\left(\det_{W_{-\frac{1}{2}}}\tilde{h}_0\right)^{\frac{1}{2}}$$
$$\times\int e^{\sum_\nu \xi_\nu z_\nu}e^{-\frac{1}{2}\sum_\nu(\xi_\nu)^2}(\rho_2(\tilde{h})\Phi)\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)d\xi$$
$$=((R\circ\rho_2(g_0))\Phi)(g).$$

(b) Let $g_0=\exp(w_0C)$. Since $\Phi\in\mathcal{H}(f',\mathfrak{h}',G)$ and $f'(C)=1$, (7.6.19) is trivially satisfied.

(c) Let $g_0=\exp\left(\sum_\nu y_\nu^0 Q_\nu\right)$ and let $h^{-1}\cdot\left(\sum_\nu y_\nu^0 Q_\nu\right)=\sum_\nu y'_\nu Q_\nu$. Then

$$((\rho_1(g_0)\circ R)\Phi)(g)=e^{-iw-\frac{1}{2}\sum_\nu(x_\nu)^2-\frac{i}{2}\sum_\nu x_\nu y_\nu}\left(\det_{W_{-\frac{1}{2}}}h_0\right)^{\frac{1}{2}}$$
$$\times\int e^{\sum_\nu \xi_\nu(x_\nu+i(y_\nu-y'_\nu))}e^{-\frac{1}{2}\sum_\nu(\xi_\nu)^2}\Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)$$
$$=((R\circ\rho_2(g_0))\Phi)(g)$$

since

$$
\begin{aligned}
&(\rho_2(g_0)\Phi)\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)\\
&= \Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\exp\left(-\sum_\nu y'_\nu Q_\nu\right)\exp\left(\sum_\nu \xi_\nu y'_\nu C\right)\right)\\
&= e^{-i\sum_\nu \xi_\nu y'_\nu}\Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right).
\end{aligned}
$$

(d) Let $g_0 = \exp\left(\sum_\nu x^0_\nu P_\nu\right)$ and let

$$
h^{-1}\cdot\left(\sum_\nu x^0_\nu P_\nu\right) = \sum_\nu x'_\nu P_\nu + \sum_\nu y'_\nu Q_\nu.
$$

Then, we have:

$$
\begin{aligned}
&((\rho_1(g_0)\circ R)\Phi)(g) = (R\Phi)\left(h\exp\sum_\nu((x_\nu - x'_\nu)P_\nu + (y_\nu - y'_\nu)Q_\nu)\right.\\
&\quad\times\left.\exp\left(\left(w + \frac{1}{2}\sum_\nu(x_\nu y'_\nu - x'_\nu y_\nu)\right)C\right)\right)\\
&= e^{-iw-\frac{1}{2}\sum_\nu(x_\nu)^2-\frac{i}{2}\sum_\nu x_\nu y_\nu-\frac{1}{2}\sum_\nu(x'_\nu)^2+\sum_\nu x_\nu x'_\nu+i\sum_\nu x'_\nu y_\nu-\frac{i}{2}\sum_\nu x'_\nu y'_\nu}\\
&\quad\times\left(\det_{W_{-\frac{1}{2}}} h_0\right)^{\frac{1}{2}}\int e^{\sum_\nu \xi_\nu((x_\nu - x'_\nu)+i(y_\nu - y'_\nu))}e^{-\frac{1}{2}\sum_\nu(\xi_\nu)^2}\\
&\quad\times\Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)d\xi.
\end{aligned}
\tag{7.6.20}
$$

On the other hand,

$$
\begin{aligned}
&(\rho_2(g_0)\Phi)\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)\\
&=\Phi\left(h\exp\left(-\sum_\nu(x'_\nu P_\nu + y'_\nu Q_\nu)\right)\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right)\\
&=e^{-i\sum_\nu\left(\xi_\nu y'_\nu - \frac{1}{2}x'_\nu y'_\nu\right)}\Phi\left(h\exp\left(\sum_\nu(\xi_\nu - x'_\nu)P_\nu\right)\right).
\end{aligned}
$$

Thus

$$((R\circ\rho_2(g_0))\Phi)(g) = e^{-iw-\frac{1}{2}\sum_\nu (x_\nu)^2 - \frac{i}{2}\sum_\nu x_\nu y_\nu}\left(\det_{W_{-\frac{1}{2}}} h_0\right)^{\frac{1}{2}} \int e^{\sum_\nu \xi_\nu z_\nu}$$

$$\times\, e^{-\frac{1}{2}\sum_\nu (\xi_\nu)^2} e^{-i\sum_\nu \left(\xi_\nu y'_\nu - \frac{1}{2}\sum_\nu x'_\nu y'_\nu\right)} \Phi\left(h\exp\left(\sum_\nu (\xi_\nu - x'_\nu) P_\nu\right) d\xi\right.$$

$$= e^{-iw-\frac{1}{2}\sum_\nu (x_\nu)^2 - \frac{i}{2}\sum_\nu x_\nu y_\nu}\left(\det_{W_{-\frac{1}{2}}} h_0\right)^{\frac{1}{2}} \int e^{\sum_\nu (\xi_\nu + x'_\nu) z_\nu}$$

$$\times\, e^{-\frac{1}{2}\sum_\nu (\xi_\nu + x'_\nu)^2 - i\sum_\nu \left((\xi_\nu + x'_\nu) y'_\nu - \frac{1}{2} x'_\nu y'_\nu\right)} \Phi\left(h\exp\left(\sum_\nu \xi_\nu P_\nu\right)\right) d\xi$$

$$= ((\rho_1(g_0)\circ R)\Phi)(g)$$

by (7.6.20). ∎

In view of Lemmas 7.6.5, 7.6.6, the linear operator

$$2^{-(m+n)/2}\pi^{-3(m+n)/4} R$$

defined on $\mathcal{S}$ can be extended to an isometry $\hat{R}$ between $\mathcal{H}(f', \mathfrak{h}', G)$ and some closed subspace $\mathcal{H}_0$ of $\mathcal{H}(f, \mathfrak{h}, G)$. It is obvious that $\hat{R}$ satisfies the relation $\rho_1(g)\circ\hat{R} = \hat{R}\circ\rho_2(g)$ for any $g \in G$, and hence the space $\mathcal{H}_0$ is invariant under $\rho_1 = \rho(f, \mathfrak{h}, G)$. But $\rho(f, \mathfrak{h}, G)$ is irreducible since $\mathfrak{h}$ is totally complex (cf. Theorem 5.1.23). Therefore $\mathcal{H}_0 = \mathcal{H}(f, \mathfrak{h}, G)$, which finishes the proof of our assertion (A).

(B) We prove the converse, namely, if $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$ and if $\rho(f, \mathfrak{h}, G)$ is irreducible, then $\mathfrak{h}$ satisfies the Pukanszky condition. Assume the hypotheses. Let $E = \exp\mathfrak{e}$ and $f_0 = f|_{\mathfrak{e}} \in \mathfrak{e}^*$. Then $\mathfrak{h} \in P^+(f_0, E)$ and $\mathfrak{h}$ satisfies the Pukanszky condition as an element of $P^+(f_0, E)$. Since $\mathcal{H}(f_0, \mathfrak{h}, E) \neq \{0\}$ by our assumption, we can deduce from (A) that $\rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(E\cdot f_0)$, i.e. $\rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(f_0, \mathfrak{h}_0, E)$ for any $\mathfrak{h}_0 \in I(f_0, \mathfrak{e}) \subset S(f, \mathfrak{g})$. Then

$$\hat{\rho}(f, \mathfrak{h}_0, G) \simeq \mathrm{ind}_E^G \hat{\rho}(f_0, \mathfrak{h}_0, E) \simeq \mathrm{ind}_E^G \rho(f_0, \mathfrak{h}, E) = \rho(f, \mathfrak{h}, G)$$

and hence $\hat{\rho}(f, \mathfrak{h}_0, G)$ is irreducible. Thus we obtain that $\mathfrak{h}_0 \in I(f, \mathfrak{g})$. By Theorem 5.4.1 and Lemma 5.1.11, it follows that $f + \mathfrak{h}_0^{\perp} \subset G\cdot f$ and hence $f + \mathfrak{e}^{\perp} \subset G\cdot f$. This implies by Lemma 5.1.11 again that $\mathfrak{h}$ satisfies the Pukanszky condition as an element of $P^+(f, G)$. The proof of Theorem 7.6.2 is complete. ∎

7.7 Decomposition of $\rho(f, \mathfrak{h}, G)$

In this last section, G stands for an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ as before. Let $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in M(f, \mathfrak{g}_{\mathbb{C}})$. Further, let $\mathfrak{e}$ be the subspace of $\mathfrak{g}$ defined by $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$, $\mathfrak{d}$ a Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ and $D = \exp \mathfrak{d}$. We denote by $U(f, \mathfrak{h})$ the set of all orbits $\sigma \in \mathfrak{g}^*/G$ such that $\sigma \cap (f + \mathfrak{e}^{\perp})$ is a non-empty open set in $f + \mathfrak{e}^{\perp}$. For $\sigma \in \mathfrak{g}^*/G$, we denote by $c(\sigma, f, \mathfrak{h})$ the number of the connected components in $\sigma \cap (f + \mathfrak{e}^{\perp})$.

Lemma 7.7.1 (cf. Lemma 11.2.2 in Chap. 11). *Let $\mathfrak{h} \in M(f, \mathfrak{g}_{\mathbb{C}})$. We denote by $V(f, \mathfrak{h})$ the set of all orbits $\sigma \in \mathfrak{g}^*/G$ of the form $\sigma = G\cdot(f + \phi)$, where $\phi \in \mathfrak{e}^{\perp}$ and $\mathfrak{h} \in M(f + \phi, \mathfrak{g}_{\mathbb{C}})$. Then $U(f, \mathfrak{h}) = V(f, \mathfrak{h})$. Moreover, let $\sigma \in U(f, \mathfrak{h})$ and $f + \phi$ $(\phi \in \mathfrak{e}^{\perp})$ an element of $\sigma \cap (f + \mathfrak{e}^{\perp})$. Then the connected component of $\sigma \cap (f + \mathfrak{e}^{\perp})$ which contains $f + \phi$ is equal to $D\cdot(f + \phi)$.*

Proof. (a) We prove that $V(f, \mathfrak{h}) \subset U(f, \mathfrak{h})$. Let $\phi \in \mathfrak{e}^{\perp}$ such that $\mathfrak{h} \in M(f + \phi, \mathfrak{g}_{\mathbb{C}})$. Then

$$f + \phi \in G\cdot(f + \phi) \cap (f + \mathfrak{e}^{\perp})$$

and hence $W = G\cdot(f + \phi) \cap (f + \mathfrak{e}^{\perp})$ is non-empty. We show that W is an open set in $f + \mathfrak{e}^{\perp}$. Let $f + \phi' \in W$, i.e. $\phi' \in \mathfrak{e}^{\perp}$ and $f + \phi' \in G\cdot(f + \phi)$. It follows that $\mathfrak{h} \in S(f + \phi', \mathfrak{g}_{\mathbb{C}})$ and that $\dim(\mathfrak{g}(f + \phi')) = \dim(\mathfrak{g}(f + \phi))$. Thus $\mathfrak{h} \in M(f + \phi', \mathfrak{g}_{\mathbb{C}})$ and hence $D\cdot(f + \phi') \subset W$ is an open set in $f + \mathfrak{e}^{\perp}$ (cf. Proposition 3.1.1 in Chapter IV of [10]).

(b) We prove the inverse inclusion $U(f, \mathfrak{h}) \subset V(f, \mathfrak{h})$. Let $\sigma \in U(f, \mathfrak{h})$ and

$$U_1 = \{\phi \in \mathfrak{e}^{\perp}; \dim(\mathfrak{g}(f + \phi)) = \dim(\mathfrak{g}(f))\}.$$

Since $\mathfrak{h} \in S(f + \phi, \mathfrak{g}_{\mathbb{C}})$ for any $\phi \in \mathfrak{e}^{\perp}$,

$$\dim(\mathfrak{g}(f)) = \min_{\phi \in \mathfrak{e}^{\perp}} \dim(\mathfrak{g}(f + \phi)).$$

It follows that U_1 is a non-empty Zariski open set in $\mathfrak{e}^{\perp}$ and thus dense in $\mathfrak{e}^{\perp}$. Consequently $\sigma \cap (f + \mathfrak{e}^{\perp})$ contains at least one point $f + \phi_1$ of $f + U_1$ and therefore $\sigma = G\cdot(f + \phi_1) \in V(f, \mathfrak{h})$.

(c) We prove the latter part of the lemma. Let $\sigma \in U(f, \mathfrak{h})$, $f + \phi$ $(\phi \in \mathfrak{e}^{\perp})$ an element of $\sigma \cap (f + \mathfrak{e}^{\perp})$ and let C denote the connected component of $\sigma \cap (f + \mathfrak{e}^{\perp})$ which contains $f + \phi$. Then C is stable under the coadjoint action of D on $\mathfrak{g}^*$. For any $f + \phi' \in C$, $D\cdot(f + \phi')$ is a connected open set in $\sigma \cap (f + \mathfrak{e}^{\perp})$ and hence in C. Thus $D\cdot(f + \phi)$ is an open set in C and its complement in C is a union of open D-orbits. Consequently $D\cdot(f + \phi)$ is not only open but also closed in C, and thus $C = D\cdot(f + \phi)$. ∎

The following theorem generalizes the result of M. Vergne for real polarizations (cf. Theorem 11.2.1).

Theorem 7.7.2. *Let* $f \in \mathfrak{g}^*$ *and* $\mathfrak{h} \in P^+(f, G)$. *If* $\mathcal{H}(f, \mathfrak{h}, G) \neq \{0\}$, *then:*

(a) $U(f, \mathfrak{h})$ *is a finite set,*
(b) $c(\sigma, f, \mathfrak{h}) < +\infty$ *for* $\sigma \in U(f, \mathfrak{h})$,
(c) $\rho(f, \mathfrak{h}, G) \simeq \sum_{\sigma \in U(f,\mathfrak{h})} c(\sigma, f, \mathfrak{h})\hat{\rho}(\sigma)$.

Proof. Let $f_0 = f|_{\mathfrak{e}} \in \mathfrak{e}^*$ and $E = \exp \mathfrak{e}$. Then $\mathfrak{h} \in P^+(f_0, E)$ and

$$\rho(f, \mathfrak{h}, G) = \mathrm{ind}_E^G \rho(f_0, \mathfrak{h}, E).$$

By our assumption, $\mathcal{H}(f_0, \mathfrak{h}, E) \neq \{0\}$ and hence, by Theorem 7.6.2,

$$\rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(E \cdot f_0), i.e. \rho(f_0, \mathfrak{h}, E) \simeq \hat{\rho}(f_0, \mathfrak{h}_0, E)$$

for any $\mathfrak{h}_0 \in I(f_0, \mathfrak{e})$. Since $\dim_{\mathbb{R}}(\mathfrak{h}_0) = \dim_{\mathbb{C}}\mathfrak{h}$, we have $\mathfrak{h}_0 \in M(f, \mathfrak{g})$. Therefore by Theorem 11.2.1, $\rho(f, \mathfrak{h}, G)$ can be decomposed into irreducible components as follows:

$$\begin{aligned} \rho(f, \mathfrak{h}, G) &\simeq \mathrm{ind}_E^G \hat{\rho}(f_0, \mathfrak{h}_0, E) \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \\ &\simeq \sum_{\sigma \in U(f_0, (\mathfrak{h}_0)_{\mathbb{C}})} c(\sigma, f, (\mathfrak{h}_0)_{\mathbb{C}})\hat{\rho}(\sigma). \end{aligned}$$

Thus it suffices to establish the following two assertions:

(A) $U(f, (\mathfrak{h}_0)_{\mathbb{C}}) = U(f, \mathfrak{h})$;
(B) $c(\sigma, f, (\mathfrak{h}_0)_{\mathbb{C}}) = c(\sigma, f, \mathfrak{h})$ for $\sigma \in U(f, (\mathfrak{h}_0)_{\mathbb{C}})$.

First we prove (A). Let $\sigma \in U(f, \mathfrak{h}) = V(f, \mathfrak{h})$, i.e. let σ be given by $\sigma = G \cdot (f + \phi)$ where $\phi \in \mathfrak{e}^{\perp}$ and $\mathfrak{h} \in M(f + \phi, \mathfrak{g}_{\mathbb{C}})$. Clearly $\phi \in (\mathfrak{h}_0)^{\perp}$, $\mathfrak{h}_0 \in M(f + \phi, \mathfrak{g})$, and hence $\sigma \in U(f, (\mathfrak{h}_0)_{\mathbb{C}})$. Conversely let $\sigma \in U(f, (\mathfrak{h}_0)_{\mathbb{C}})$ be given by $\sigma = G \cdot (f + \phi)$ where $\phi \in (\mathfrak{h}_0)^{\perp}$ and $\mathfrak{h}_0 \in M(f + \phi, \mathfrak{g})$. We write $f + \phi$ in the form

$$f + \phi = f_0 + l_1 + l_2,\ l_1 \in (\mathfrak{h}_0)^{\perp, \mathfrak{e}^*},\ l_2 \in \mathfrak{e}^{\perp}. \tag{7.7.1}$$

Let $\underset{E}{\cdot}$ denote the coadjoint representation of E. Since $\mathfrak{h}_0 \in I(f_0, \mathfrak{e})$, Theorem 5.4.1 asserts that there is some $a \in H_0 = \exp(\mathfrak{h}_0)$ such that

$$a \underset{E}{\cdot} f_0 = f_0 + l_1. \tag{7.7.2}$$

If we set

$$a^{-1} \cdot (f_0 + l_1) = a^{-1} \underset{E}{\cdot} (f_0 + l_1) + l_3\ (l_3 \in \mathfrak{e}^{\perp}),$$

then, since $\mathfrak{e}$ is a Lie subalgebra, $l_4 = l_3 + a^{-1}{\cdot}l_2 \in \mathfrak{e}^{\perp}$ and we have

$$a^{-1} \cdot (f + \phi) = f + l_4 \in f + \mathfrak{e}^{\perp}$$

by (7.7.1), (7.7.2). Thus $\dim(\mathfrak{g}(f + l_4)) = \dim(\mathfrak{g}(f))$, $\mathfrak{h} \in M(f + l_4, \mathfrak{g}_{\mathbb{C}})$ and $\sigma = G{\cdot}(f + l_4) \in U(f, \mathfrak{h})$.

Next we verify (B). Let $\sigma \in U(f, (\mathfrak{h}_0)_{\mathbb{C}})$ and $f + \phi \in f + (\mathfrak{h}_0)^{\perp}$ such that $\mathfrak{h} \in M(f + \phi, \mathfrak{g}_{\mathbb{C}})$. We showed in the proof of (A) that there exists some $a \in H_0 = \exp(\mathfrak{h}_0)$ such that $a{\cdot}(f + \phi) \in f + \mathfrak{e}^{\perp}$. This fact together with Lemma 7.7.1 implies that any connected component of $\sigma \cap (f + (\mathfrak{h}_0)^{\perp})$ is of the form $C = H_0{\cdot}(f + \phi')$ where $\phi' \in \mathfrak{e}^{\perp}$ and $f + \phi' \in \sigma \cap (f + \mathfrak{e}^{\perp})$. We prove that

$$C \cap (f + \mathfrak{e}^{\perp}) = D{\cdot}(f + \phi').$$

In fact, since $\mathfrak{d} = \mathfrak{e}^{f_0}$, $\mathfrak{d} \subset \mathfrak{h}_0$ and hence

$$D{\cdot}(f + \phi') \subset C \cap (f + \mathfrak{e}^{\perp}).$$

Conversely let $f + \psi \in C \cap (f + \mathfrak{e}^{\perp})$ with $\psi \in \mathfrak{e}^{\perp}$. Then there is $a' \in H_0$ such that $a'{\cdot}(f + \phi') = f + \psi$. But clearly $a' \underset{E}{\cdot} f_0 = f_0$ and hence $a' \in D$. Consequently $f + \psi = a'{\cdot}(f + \phi')$ with $a' \in D$ so that

$$C \cap (f + \mathfrak{e}^{\perp}) \subset D{\cdot}(f + \phi').$$

Hence by Lemma 7.7.1, the map

$$C = H_0{\cdot}(f + \phi') \mapsto C \cap (f + \mathfrak{e}^{\perp}) = D{\cdot}(f + \phi')$$

gives a one-to-one correspondence between the connected components of $\sigma \cap (f + (\mathfrak{h}_0)^{\perp})$ and those of $\sigma \cap (f + \mathfrak{e}^{\perp})$. ■

Chapter 8
Irreducible Decomposition

As usual let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. It is well known that there is a strong duality between the induction and the restriction of representations. In this chapter, we study the irreducible decomposition of the representation induced from a subgroup or the representation restricted to a subgroup.

8.1 Monomial Representations

We begin with a simple lemma.

Lemma 8.1.1. *Let V be a finite-dimensional real vector space where G acts by a representation of exponential type. Let v be a nonzero G-invariant vector of V: i.e. $g{\cdot}v = v$ for all $g \in G$. We arbitrarily fix $x \in V$ and consider the line $L_x = x + \mathbb{R}v$. We then have the following two possibilities:*

$$L_x \cap G{\cdot}x = x \text{ or } L_x \cap G{\cdot}x = L_x.$$

In other words, the line passing x and having the direction of the invariant vector v either meets the orbit $G{\cdot}x$ at only one point or is completely contained there.

Proof. Let V_0 be the subspace of V spanned by v, $\bar{V}$ the quotient space V/V_0 and $p : V \to \bar{V}$ the canonical projection. The representation of G obtained on $\bar{V}$ by passing to the quotient is obviously of exponential type. Then, in order that $g{\cdot}x \in L_x$ $(g \in G)$ it is necessary and sufficient that g belongs to $G(p(x))$. Whereas, if we put $g{\cdot}x = x + \lambda(g)v$, it is immediately known that λ gives a homomorphism of $G(p(x))$ to $\mathbb{R}$. Theorem 5.3.2 assures that $G(p(x))$ is connected so that either $\lambda \equiv 0$ or the image of λ coincides with the whole $\mathbb{R}$. Now it is enough to remark that the intersection of L_x and $G{\cdot}x$ is nothing but $\{g{\cdot}x; g \in G(p(x))\}$. ∎

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Definition 8.1.2. In the situation of the above lemma, provided $L_x \subset G{\cdot}x$, the orbit $G{\cdot}x$ is said to be **saturated** in the v-direction and otherwise $G{\cdot}x$ is said to be **non-saturated**.

When there is an ideal $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\dim(\mathfrak{g}/\mathfrak{g}_0) = 1$, a linear form $\ell \in \mathfrak{g}^*$ satisfying $\ell|_{\mathfrak{g}_0} = 0$ is an invariant vector relative to the coadjoint representation of G. We write $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}_0$. Let p be the projection from $\mathfrak{g}^*$ onto $\mathfrak{g}_0^*$ and $G_0 = \exp(\mathfrak{g}_0)$. The next lemma is immediately derived from the definition of the radical of an alternating bilinear form.

Lemma 8.1.3. *Let $\ell \in \mathfrak{g}^*$ and put $\ell_0 = p(\ell)$. If $\mathfrak{g}(\ell) \subset \mathfrak{g}_0$, then*

$$\mathfrak{g}(\ell) \subset \mathfrak{g}_0(\ell_0) \textit{ and } dim(\mathfrak{g}_0(\ell_0)) = dim(\mathfrak{g}(\ell)) + 1.$$

If $\mathfrak{g}(\ell) \not\subset \mathfrak{g}_0$, then

$$\mathfrak{g}_0(\ell_0) \subset \mathfrak{g}(\ell) \textit{ and } dim(\mathfrak{g}(\ell)) = dim(\mathfrak{g}_0(\ell_0)) + 1.$$

Lemma 8.1.4. *Let $\ell \in \mathfrak{g}^*, \ell_0 = p(\ell)$ and $\Omega = G{\cdot}\ell$.*

(1) *If the orbit Ω is saturated in the direction of $\mathfrak{g}_0^\perp$, there is a family $\{\omega_s\}_{s\in\mathbb{R}}$ of G_0-orbits in $\mathfrak{g}_0^*$ such that $p(\Omega) = \cup_{s\in\mathbb{R}}\omega_s$ and $\exp(tX)\cdot\omega_s = \omega_{s+t}$. Moreover, $G(\ell_0) \subset G_0$.*
(2) *If the orbit Ω is non-saturated in the direction of $\mathfrak{g}_0^\perp$, then $p(\Omega) = G_0{\cdot}\ell_0$.*

Proof. (1) First $G = \exp(\mathbb{R}X){\cdot}G_0 = G_0{\cdot}\exp(\mathbb{R}X)$. Now if we set $\omega_0 = G_0{\cdot}\ell_0$, $\omega_0 \subset p(\Omega)$ and, for any $t \in \mathbb{R}$, $\exp(tX){\cdot}\omega_0$ is an G_0-orbit contained in $p(\Omega)$. Put $\omega_t = \exp(tX){\cdot}\omega_0$. Since $p(\Omega) = p(G{\cdot}\ell) = G{\cdot}\ell_0 = \exp(\mathbb{R}X){\cdot}\omega_0$, The union of $\{\omega_t\}_{t\in\mathbb{R}}$ coincides with $p(\Omega)$. From the definition of ω_s, $\exp(tX){\cdot}\omega_s = \omega_{s+t}$. Furthermore, $\omega_s = \omega_t$ when and only when $s = t$. In fact, if $\exp(sX){\cdot}\ell_0 \in \omega_t$, $\exp(sX){\cdot}\ell_0 = \exp(tX){\cdot}g_0{\cdot}\ell_0$ with $g_0 \in G_0$. Hence $(\exp((t-s)X))\cdot g_0 \in G(\ell_0)$. So, if we show $G(\ell_0) \subset G_0$, then $\exp((t-s)X) \in G_0$ and consequently $s = t$. For this aim it is enough to show $\mathfrak{g}(\ell_0) \subset \mathfrak{g}_0$. By Lemma 8.1.3 there is $X_1 \in \mathfrak{g}_0(\ell_0)\backslash\mathfrak{g}(\ell)$ so that $\lambda = X_1{\cdot}\ell \neq 0$ belongs to $\mathfrak{g}_0^\perp$. Y being an arbitrary element of $\mathfrak{g}(\ell_0)$, $Y{\cdot}\ell \in \mathfrak{g}_0^\perp$ so that $Y{\cdot}\ell = (tX_1){\cdot}\ell$ with some $t \in \mathbb{R}$. From this $Y - tX_1 \in \mathfrak{g}(\ell) \subset \mathfrak{g}_0$ and $Y \in \mathfrak{g}_0$ since $X_1 \in \mathfrak{g}_0$.
(2) Since $\mathfrak{g}(\ell) \not\subset \mathfrak{g}_0$, $G = G_0{\cdot}G(\ell)$. Hence the orbit $G{\cdot}\ell$ is equal to $G_0{\cdot}\ell$. From this we immediately see $p(G{\cdot}\ell) = p(G_0{\cdot}\ell) = G_0{\cdot}\ell_0$. ∎

We simply write $\hat{\rho}_0$ instead of $\hat{\rho}_{G_0}$.

Proposition 8.1.5. *Let $\pi_0 \in \hat{G}_0$. We suppose $\pi_0 \simeq \hat{\rho}_0(\ell_0)$ with $\ell_0 \in \mathfrak{g}_0^*$. Let ℓ be an extension of ℓ_0 to $\mathfrak{g}$, and we put $\Omega = G{\cdot}\ell$.*

(1) *If Ω is saturated in the direction of $\mathfrak{g}_0^\perp$, $ind_{G_0}^G\pi_0 \simeq \hat{\rho}(\ell)$.*
(2) *If Ω is non-saturated in the direction of $\mathfrak{g}_0^\perp$, $ind_{G_0}^G\pi_0 \simeq \int_{\mathbb{R}}^{\oplus}\hat{\rho}(\ell^\nu)d\nu$. Here, X being a fixed element of $\mathfrak{g}(\ell)\backslash\,(\mathfrak{g}(\ell)\cap\mathfrak{g}_0)$, $\ell^\nu \in \mathfrak{g}^*$ is defined by $\ell^\nu|_{\mathfrak{g}_0} = \ell_0$ and $\ell^\nu(X) = -2\pi\nu$.*

Proof. (1) Let $\mathfrak{h}$ be the Vergne polarization at ℓ of $\mathfrak{g}$ constructed from a strong Malcev sequence of Lie subalgebras of $\mathfrak{g}$ passing $\mathfrak{g}_0$, $\mathfrak{h} \in I(\ell, \mathfrak{g})$ and $\mathfrak{h} \subset \mathfrak{g}_0$. Hence $\mathfrak{h} \in I(\ell_0, \mathfrak{g}_0)$. Therefore

$$\mathrm{ind}_{G_0}^{G} \pi_0 \simeq \mathrm{ind}_{G_0}^{G} \left(\mathrm{ind}_{H}^{G_0} \chi_{\ell_0} \right) \simeq \hat{\rho}(\ell)$$

with $H = \exp \mathfrak{h}$. The assertion follows from this.

(2) Each G-orbit which meets $\ell + \mathfrak{g}_0^{\perp}$ is non-saturated in the direction of $\mathfrak{g}_0^{\perp}$. Let X be a fixed element of $\mathfrak{g}(\ell) \backslash (\mathfrak{g}(\ell) \cap \mathfrak{g}_0)$, and define $\ell^{\nu} \in \mathfrak{g}^*$ ($\nu \in \mathbb{R}$) by

$$\ell^{\nu}|_{\mathfrak{g}_0} = \ell_0, \ \ell^{\nu}(X) = -2\pi\nu.$$

If we construct as in (1) $\mathfrak{h} \in I(\ell, \mathfrak{g})$, $\mathfrak{h} \in I(\ell^{\nu}, \mathfrak{g})$. We set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $H_0 = \exp(\mathfrak{h}_0)$.

We show $\mathrm{ind}_{H_0}^{H} \chi_{\ell_0} \simeq \int_{\mathbb{R}}^{\oplus} \chi_{\ell^{\nu}} d\nu$. We denote by $\mathcal{C}$ the space of all complex-valued continuous functions ϕ on H which satisfy

$$\phi(hh_0) = \chi_{\ell}(h_0)^{-1} \phi(h)$$

for any $h \in H$ and $h_0 \in H_0$, with compact support modulo H_0. Since $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$, there exists on H/H_0 an H-invariant measure γ. We denote by ρ the representation $\mathrm{ind}_{H_0}^{H} \chi_{\ell_0}$. Hence its representation space $\mathcal{H}_{\rho}$ is the completion of $\mathcal{C}$ regarding the norm

$$\|\phi\| = \left(\int_{H/H_0} |\phi(h)|^2 d\gamma(hH_0) \right)^{1/2}.$$

The space $\mathcal{C}$ is identified with $C_c(\mathbb{R})$ by the mapping $\phi \mapsto \tilde{\phi}$, $\tilde{\phi}(t) = \phi(\exp(tX))$. Then, γ is identified with a Lebesgue measure and $\mathcal{H}_{\rho}$ with $L^2(\mathbb{R})$. Thus $\|\phi\|^2 = \int_{\mathbb{R}} |\phi(\exp(tX))|^2 dt$.

The space of the representation $\sigma = \int_{\mathbb{R}}^{\oplus} \chi_{\ell^{\nu}} d\nu$ becomes $L^2(\mathbb{R})$ and, for $h \in H$ and $\phi \in L^2(\mathbb{R})$, $(\sigma(h)\phi)(\nu) = \chi_{\ell^{\nu}}(h)\phi(\nu)$.

$\mathcal{F}$ denotes the Fourier transformation of $L^2(\mathbb{R})$. We will see that the operator U from $\mathcal{H}_{\rho}$ to $L^2(\mathbb{R})$ defined by $U\phi = \mathcal{F}\tilde{\phi}$ furnishes an equivalence from ρ to σ. Since $\mathcal{F}$ and the mapping $\phi \mapsto \tilde{\phi}$ are both unitary bijection, the same for U. Hence it is enough to show that U intertwines ρ and σ. For $h \in H, \nu \in \mathbb{R}$ and $\phi \in \mathcal{H}_{\rho}$,

$$((U \circ \rho(h))\phi)(\nu) = \int_{-\infty}^{\infty} e^{-2i\pi\nu t} \phi(h^{-1}\exp(tX)) dt,$$

$$((\sigma(h) \circ U)(\phi))(\nu) = \chi_{\ell^{\nu}}(h) \int_{-\infty}^{\infty} e^{-2i\pi\nu t} \phi(\exp(tX)) dt.$$

Since $H = \exp(\mathbb{R}X)\cdot H_0 = H_0\cdot\exp(\mathbb{R}X)$, it suffices to show the equality of these two expressions for $h = \exp(sX)$ $(s \in \mathbb{R})$ and $h \in H_0$. By the way, with $X_0 \in \mathfrak{h}_0$,

$$\begin{aligned}((U \circ \rho(\exp(X_0))(\phi))\,(\nu) &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\phi(\exp(-X_0)\exp(tX))dt\\ &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\phi\left(\exp(tX)\exp\left(e^{\mathrm{ad}(-tX)}(-X_0)\right)\right)dt.\end{aligned}$$

But with $Y \in [\mathfrak{h}, \mathfrak{h}] \subset \ker\ell \cap \mathfrak{h}_0$, $\exp\left(e^{\mathrm{ad}(-tX)}(-X_0)\right) = \exp(-X_0 + Y)$. Therefore

$$((U \circ \rho(\exp(X_0)))(\phi))\,(\nu) = \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\chi_{\ell_0}\left(\exp(-X_0 + Y)\right)^{-1}\phi(\exp(tX))dt.$$

Further, $\chi_{\ell_0}(\exp(-X_0 + Y)) = e^{i\ell_0(-X_0+Y)} = e^{-i\ell_0(X_0)} = \chi_{\ell_0}(\exp(X_0))^{-1}$. Hence,

$$\begin{aligned}((U \circ \rho(\exp(X_0)))(\phi))\,(\nu) &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\chi_{\ell_0}(\exp(X_0))\phi(\exp(tX))dt\\ &= ((\sigma(\exp(X_0))\circ U)(\phi))\,(\nu).\end{aligned}$$

On the other hand,

$$\begin{aligned}((U \circ \rho(\exp(\kappa X)))(\phi))\,(\nu) &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\phi(\exp((t-\kappa)X)dt\\ &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t}e^{-2i\pi\nu\kappa}\phi(\exp(tX))dt\\ &= \chi_{\ell^\nu}(\exp(\kappa X))\int_{-\infty}^{\infty} e^{-2i\pi\nu t}\phi(\exp(tX))dt\\ &= ((\sigma(\exp(\kappa X))\circ U)\phi)\,(\nu)\end{aligned}$$

with $\kappa \in \mathbb{R}$. Thus $\mathrm{ind}_{H_0}^{H}\chi_\ell \simeq \int_{\mathbb{R}}^{\oplus}\chi_{\ell^\nu}d\nu$.

From this,

$$\mathrm{ind}_{G_0}^{G}\pi_0 \simeq \mathrm{ind}_{G_0}^{G}\left(\mathrm{ind}_{H_0}^{G_0}\chi_{\ell_0}\right) \simeq \mathrm{ind}_{H}^{G}\left(\mathrm{ind}_{H_0}^{H}\chi_{\ell_0}\right) \simeq \mathrm{ind}_{H}^{G}\left(\int_{\mathbb{R}}^{\oplus}\chi_{\ell^\nu}d\nu\right)$$

by use of Theorem 3.2.8 in the third chapter. Since $\mathrm{ind}_{H}^{G}\left(\int_{\mathbb{R}}^{\oplus}\chi_{\ell^\nu}d\nu\right) \simeq \int_{\mathbb{R}}^{\oplus}\left(\mathrm{ind}_{H}^{G}\chi_{\ell^\nu}\right)d\nu$ and $\mathfrak{h} \in I(\ell^\nu, \mathfrak{g})$ for any $\nu \in \mathbb{R}$,

$$\mathrm{ind}_{G_0}^G \pi_0 \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}(\ell^\nu) d\nu$$

is obtained. ■

Now $\mathfrak{h} \in S(f, \mathfrak{g})$ being given, let us examine the monomial representation $\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$. In this examination of τ, our stage turns out to be the affine subspace $\Gamma_\tau = f + \mathfrak{h}^\perp$ of $\mathfrak{g}^*$. We begin to examine the intersections of this affine space with G-orbits.

Theorem 8.1.6. *Let $\mathfrak{k}$ be an ideal of $\mathfrak{g}$, $f \in \mathfrak{k}^*$ and $\mathfrak{h} \in S(f, \mathfrak{k})$. We denote by $\mathfrak{h}^{\perp,\mathfrak{k}^*}$ the annihilator of $\mathfrak{h}$ in $\mathfrak{k}^*$, by μ a positive finite measure on $\mathfrak{k}^*$ equivalent to the Lebesgue measure in $f + \mathfrak{h}^{\perp,\mathfrak{k}^*}$ and by ν the image of μ by the canonical projection from $\mathfrak{k}^*$ onto the space $\mathfrak{k}^*/G$ of G-orbits. Then, the following holds for ν-almost all orbits $\Omega \in \mathfrak{k}^*/G$.*

(1) *Each connected component C of $\left(f + \mathfrak{h}^{\perp,\mathfrak{k}^*}\right) \cap \Omega$ is a manifold.*
(2) *The tangent space of C at a point $\ell \in C$ is equal to $\mathfrak{g} \cdot \ell \cap \mathfrak{h}^{\perp,\mathfrak{k}^*}$.*
(3) *When we decompose the measure μ with respect to ν as $\mu = \int \mu_\Omega d\nu(\Omega)$, the restriction of the fibre measure μ_Ω to any coordinate neighbourhood $(U; x_1, \ldots, x_m)$ is equivalent to $dx_1 \cdots dx_m$.*

Proof. We proceed by induction on $\dim \mathfrak{k}$. Let $\mathfrak{k}_0$ be an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ such that $\mathfrak{k}/\mathfrak{k}_0$ is irreducible as $\mathfrak{g}$-module. We identify $\mathfrak{k}_0^*$ with a subspace of $\mathfrak{k}^*$. If $\dim(\mathfrak{k}/\mathfrak{k}_0) = 1$, for $\lambda \in \mathfrak{g}^*$,

$$\mathfrak{k}^* = \mathbb{R}\xi \oplus \mathfrak{k}_0^*,\ \xi|_{\mathfrak{k}_0} = 0,\ X \cdot \xi = \lambda(X)\xi\ (X \in \mathfrak{g}).$$

And if $\dim(\mathfrak{k}/\mathfrak{k}_0) = 2$, using $\alpha \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathfrak{g}^*$,

$$\mathfrak{k}^* = \mathbb{R}\xi_1 \oplus \mathbb{R}\xi_2 \oplus \mathfrak{k}_0^*,\ \xi_1|_{\mathfrak{k}_0} = \xi_2|_{\mathfrak{k}_0} = 0,$$
$$X\cdot\xi_1 = \lambda(X)(\xi_1 - \alpha\xi_2),\ X\cdot\xi_2 = \lambda(X)(\xi_2 + \alpha\xi_1)\ (X \in \mathfrak{g}).$$

Then, the restriction map $\mathfrak{k}^* \ni \ell \mapsto \ell_0 = \ell|_{\mathfrak{k}_0} \in \mathfrak{k}_0^*$ is nothing but the projection $p : \mathfrak{k}^* \to \mathfrak{k}_0^*$ along this decomposition.

For $\ell \in \mathfrak{k}^*$, if we denote by $G(\ell_0) = \exp(\mathfrak{g}(\ell_0))$ the stabilizer of ℓ_0 in G, $p^{-1}(\ell_0) \cap G\cdot\ell = G(\ell_0)\cdot\ell$. Provided $\mathfrak{g}(\ell_0)$ is contained in $\ker\lambda$, $G(\ell_0)$ keeps ξ (or $\xi_1,\ \xi_2$) invariant. When $\dim(\mathfrak{k}/\mathfrak{k}_0) = 1$, for $g \in G(\ell_0)$, if we write $g\cdot\ell = \ell + a(g)\xi$ using $a(g) \in \mathbb{R}$, the function $a : G(\ell_0) \to \mathbb{R}$ satisfies $a(g_1g_2) = a(g_1) + a(g_2)$. Since $G(\ell_0)$ is connected, the image of the homomorphism a either degenerates to $\{0\}$ or is the whole $\mathbb{R}$. When $\dim(\mathfrak{k}/\mathfrak{k}_0) = 2$, setting likewise

$$g\cdot\ell = \ell + a_1(g)\xi_1 + a_2(g)\xi_2$$

for $g \in G(\ell_0)$, we get the two homomorphisms $a_1,\ a_2 : G(\ell_0) \to \mathbb{R}$ and we recognize that $p^{-1}(\ell_0) \cap G{\cdot}\ell$ has three possibilities: the plane $p^{-1}(\ell_0)$, a line or one point $\{\ell\}$. Besides, these possibilities depend on ℓ_0, not exactly on ℓ itself.

We next suppose that $\mathfrak{g}(\ell_0)$ is not contained in $\ker\lambda$ and take $X \in \mathfrak{g}(\ell_0)$ such that $\lambda(X) = 1$. If $\dim(\mathfrak{k}/\mathfrak{k}_0) = 1$. The relation

$$\exp(tX){\cdot}\ell = \ell + a(t)\xi \ (t \in \mathbb{R})$$

tells us that

$$a(t)(e^s - 1) = a(s)(e^t - 1)$$

holds for all $s,\ t \in \mathbb{R}$. From this,

$$a(t) = c(e^t - 1).$$

with some constant $c \in \mathbb{R}$. Hence $p^{-1}(\ell_0) \cap G{\cdot}\ell$ is either one point $\{\ell\}$, a half-line or the line $p^{-1}(\ell_0)$. If $\dim(\mathfrak{k}/\mathfrak{k}_0) = 2$, Writing

$$\exp(tX) \cdot \ell = \ell + a_1(t)\xi_1 + a_2(t)\xi_2 \ (t \in \mathbb{R})$$

and computing

$$\exp(tX)\exp(\pi X/\alpha) \cdot \ell = \exp(\pi X/\alpha)\exp(tX) \cdot \ell,$$

we have

$$a_1(t) + a_2(\pi/\alpha)\, e^t \sin(\alpha t) + a_1(\pi/\alpha)\, e^t \cos(\alpha t) = a_1(\pi/\alpha) - e^{\pi/\alpha} a_1(t),$$
$$a_2(t) + a_2(\pi/\alpha)\, e^t \cos(\alpha t) - a_1(\pi/\alpha)\, e^t \sin(\alpha t) = a_2(\pi/\alpha) - e^{\pi/\alpha} a_2(t).$$

From this, with certain constants $c_1,\ c_2 \in \mathbb{R}$,

$$a_1(t) = c_1 - e^t\left(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)\right),$$
$$a_2(t) = c_2 - e^t\left(c_2 \cos(\alpha t) - c_1 \sin(\alpha t)\right)$$

hold for all $t \in \mathbb{R}$. Now, $p^{-1}(\ell_0) \cap G{\cdot}\ell$ is either one point $\{\ell\}$, a spiral or the plane $p^{-1}(\ell_0)$.

To begin with, assume $\mathfrak{h} \subset \mathfrak{k}_0$. We denote by Θ the set of $\ell \in \Xi$ such that the G-orbit $G \cdot \ell \subset \mathfrak{k}^*$ has the maximal dimension among the G-orbits meeting $\Xi = f + \mathfrak{h}^{\perp,\mathfrak{k}^*}$. We set $G' = \exp(\mathfrak{g}'),\ \mathfrak{g}' = \ker\lambda$, and, replacing G by its connected normal subgroup G', introduce likewise the subset Θ' of Ξ. Next, using $\Xi_0 = f_0 + \mathfrak{h}^{\perp,\mathfrak{k}_0^*},\ f_0 = f|_{\mathfrak{k}_0}$, instead of Ξ we define the subsets $\Theta_0,\ \Theta_0'$ of Ξ_0. Now if we put $E = \Theta \cap \Theta' \cap p^{-1}(\Theta_0) \cap p^{-1}(\Theta_0')$, then $\Xi \cap G{\cdot}E \subset E$ and, since the μ-measure of $\Xi\backslash E$ is 0, we may restrict our reasoning on E, where the same eventuality of the possibilities of $G(\ell_0)\cdot\ell$ examined above occurs for all $\ell \in E$. Thus we obtain the claims (1)–(3) by applying the induction hypothesis to Ξ_0.

Hereafter we assume $\mathfrak{h} \not\subset \mathfrak{k}_0$ and put $\mathfrak{h}_0 = \mathfrak{h}\cap\mathfrak{k}_0$. Suppose first that $\dim(\mathfrak{k}/\mathfrak{k}_0) = 1$ and set $\mathfrak{h} = \mathbb{R}X \oplus \mathfrak{h}_0$, $\mathfrak{k} = \mathbb{R}X \oplus \mathfrak{k}_0$. In this case, we identify $\mathfrak{k}_0^*$ with the hyperplane $\{\ell \in \mathfrak{k}^*; \ell(X) = f(X)\}$ and apply the induction hypothesis to $\Xi_0 = f_0 + \mathfrak{h}_0^{\perp,\mathfrak{k}_0^*}$. As before, in order to construct a non-empty Zariski open set E of $f + \mathfrak{h}_0^{\perp,\mathfrak{k}^*}$, we utilize $\mathfrak{h}_0$ and Ξ_0. When $G(\ell_0)\cdot\ell$ is a line for $\ell\in E$, or when $G(\ell_0)\cdot\ell$ is a half-line for $\ell \in E$ and one point for $\ell\in\Xi$, there is nothing to prove. Besides, if $G(\ell_0)\cdot\ell$, $\ell\in E$, is a half-line and $\Xi\cap E \neq \emptyset$, it suffices to remark that each connected component of $\Xi\cap G\cdot\ell$ is an open submanifold of the connected component of $\Xi_0\cap G\cdot\ell_0$ containing it, which is a manifold by the induction hypothesis.

Let us examine the essential case where $G(\ell_0)\cdot\ell$ is one point for any $\ell\in E$. We fix $\ell\in\Xi$ such that each connected component of $M = \Xi_0\cap G\cdot\ell_0 \subset \Xi$ is a manifold. Here, ℓ_0 signifies ℓ regarded as an element of $\mathfrak{k}_0^*$. Now we define a function ϕ on M as follows. Provided $\eta = g\cdot\ell_0$ in $\mathfrak{k}_0^*$, namely if $\eta_0 = g\cdot\ell_0$,

$$\phi(\eta) = (g^{-1}\cdot\eta)(X),$$

here the action of G is taken in $\mathfrak{k}^*$. Notice that

$$\Xi\cap G\cdot\ell = \phi^{-1}(\ell(X)).$$

By the way, ϕ being analytic, the set N of the critical points of ϕ in a connected component of M is a null set with respect to the fibre measure induced on M as a result of the decomposition of the measure μ, namely a measure equivalent to the Lebesgue measure in a coordinate neighbourhood. Otherwise, ϕ becomes a constant map on this connected component.

Sard's theorem [73] tells us that the set W of the critical values of ϕ is a null set for the Lebesgue measure and each connected component of $\Xi\cap G\cdot\ell = \phi^{-1}(\ell(X))$ is a submanifold if $\ell(X)\notin W$. In a neighbourhood of a regular point of ϕ, we may take ϕ as one of local coordinates so that (1) and (3) follows.

Concerning (2), the function ϕ considered above is defined on the manifold $p^{-1}(G\cdot\ell_0)\subset\mathfrak{k}^*$ and separates there the orbits. By restricting this to Ξ, it turns out that the tangent space of $\Xi\cap G\cdot\ell = \Xi\cap\phi^{-1}(\ell(X))$ at a regular point ℓ is $\mathfrak{h}^{\perp,\mathfrak{k}^*}\cap\mathfrak{g}\cdot\ell$.

Relating to the above things we evoke the following computations. For arbitrary $Y\in\mathfrak{g}$,

$$\begin{aligned} Y\cdot\phi(\eta) &= Y\cdot(g\cdot\ell_0(g{\cdot}X)) = (Y\cdot\eta_0)(g{\cdot}X) + \eta([Y, g{\cdot}X]) \\ &= -\,\eta_0([Y,(g{\cdot}X)_0]) + \eta([Y, g{\cdot}X]) = c\eta([Y,X])\ (0\neq c\in\mathbb{R}), \end{aligned}$$

Here, $(g{\cdot}X)_0$ denotes the $\mathfrak{k}_0$-component of $g{\cdot}X\in\mathfrak{k} = \mathbb{R}X\oplus\mathfrak{k}_0$.

This reasoning is applied to other cases. For example, Assume $\dim(\mathfrak{k}/\mathfrak{k}_0) = 2$. We apply the induction hypothesis to the pair $(f_0, \mathfrak{h}_0)$ and as in the previous case fix $\ell \in \Xi$ in a general position. Provided $G(\ell_0)\cdot\ell$ coincides with the plane $p^{-1}(\ell_0)$, there is nothing left to prove. Assume $\dim(\mathfrak{h}/\mathfrak{h}_0) = 1$. By induction hypothesis $M = \Xi \cap p^{-1}(G\cdot\ell_0)$ is a manifold. In the case where $G(\ell_0)\cdot\ell$ is a line, as $\ell + \mathbb{R}\xi_1^*$ for instance. Then, for any $\eta \in M$, $G\cdot\eta$ meets the line $\ell + \xi_1^* + \mathbb{R}\xi_2^*$ at only one point $\ell + \xi_1^* + \phi(\eta)\xi_2^*$, and we get the function $\phi : M \to \mathbb{R}$. In the case where $G(\ell_0)\cdot\ell$ is a spiral, its gushing point is in the plane $p^{-1}(\ell_0)$, that is to say, the unique point $\{\ell_1\}$ such that $G(\ell_0)\cdot\ell_1 = \{\ell_1\}$. In the plane $p^{-1}(\ell_0)$ we draw at the centre ℓ_1 a circle S with a positive radius arbitrarily chosen. Then for any $\eta \in M$, $G\cdot\eta$ intersects S at only one point $\phi(\eta)$ and we obtain the mapping $\phi : M \to S$. In the case where $G(\ell_0)\cdot\ell = \{\ell\}$, our map $\phi : M \to p^{-1}(\ell_0)$ associates $\eta \in M$ with the common point $p^{-1}(\ell_0) \cap G\cdot\ell$.

Assume finally $\dim(\mathfrak{h}/\mathfrak{h}_0) = 2$. We identify $f_0 + \mathfrak{h}_0^{\perp,\mathfrak{k}_0^*} \subset \mathfrak{k}_0^*$ with $\Xi = f + \mathfrak{h}^{\perp,\mathfrak{k}^*} \subset \mathfrak{k}^*$, and fix $\ell \in \Xi$ such that $M = \Xi \cap G\cdot\ell_0$ is a manifold by the induction hypothesis. Next, completely as above we construct the function ϕ according to the cases which occur. Anyhow, we may apply Sard's theorem cited above to the mapping ϕ obtained in this manner. ■

If we take $\mathfrak{g}$ as $\mathfrak{k}$ in this theorem and continue the use of notations of the theorem:

Corollary 8.1.7. *Let $f \in \mathfrak{g}^*$, $\mathfrak{h} \in S(f, \mathfrak{g})$ and $\Gamma_\tau = \Xi = f + \mathfrak{h}^\perp$.*

(1) *For almost all coadjoint orbits $\Omega \in \mathfrak{g}^*/G$ with respect to the measure ν, the support of the fibre measure μ_Ω is the whole $\Gamma_\tau \cap \Omega$.*
(2) *For almost all coadjoint orbits $\Omega \in \mathfrak{g}^*/G$ with respect to the measure ν, each connected component C of $\Gamma_\tau \cap \Omega$ is a manifold, whose dimension is larger than or equal to $\frac{1}{2}\dim \Omega$.*
(3) *We have $\dim C = \frac{1}{2}\dim \Omega$ in the statement (2), if and only if $H\cdot\ell = C$ for every $\ell \in C$. Besides, in this case $\mathfrak{h} + \mathfrak{g}(\ell)$ is a maximal isotropic subspace for the bilinear form B_ℓ.*

Proof. The statement (1) is clear. Take a generic connected component C in question and let $\ell \in C$. By easy calculations:

$$\begin{aligned}\dim C &= \dim\left(\mathfrak{g}\cdot\ell \cap \mathfrak{h}^\perp\right) = \dim\left(\mathfrak{h}^\ell/\mathfrak{g}(\ell)\right)\\ &= \dim\mathfrak{g} - \dim\mathfrak{h} + \dim(\mathfrak{g}(\ell)\cap\mathfrak{h}) - \dim(\mathfrak{g}(\ell))\\ &= \dim\mathfrak{g} - \dim(\mathfrak{h} + \mathfrak{g}(\ell)) \geq \dim\mathfrak{g} - \frac{1}{2}(\dim\mathfrak{g} + \dim(\mathfrak{g}(\ell)))\\ &= \frac{1}{2}\dim\Omega.\end{aligned}$$

Here the equality holds if and only if $\mathfrak{h} + \mathfrak{g}(\ell)$ is a maximal isotropic subspace for B_ℓ. In such a case,

$$\dim(H \cdot \ell) = \dim(\mathfrak{h}/(\mathfrak{g}(\ell) \cap \mathfrak{h})) = \dim \mathfrak{h} - \dim(\mathfrak{g}(\ell) \cap \mathfrak{h})$$
$$= \dim(\mathfrak{h} + \mathfrak{g}(\ell)) - \dim(\mathfrak{g}(\ell)) = \frac{1}{2}\dim \Omega.$$

From this, we get the statements (2) and (3). ■

Now, let us obtain the canonical irreducible decomposition of τ. We denote by $\mathfrak{z}$ the centre of $\mathfrak{g}$ and let $\mathfrak{a}$ be a minimal non-central ideal of $\mathfrak{g}$. Of course, $\dim(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})) \leq 2$. Everybody has probably noticed that, when it is a matter of exponential solvable Lie groups, the main tool of proofs is the induction. We use induction on $\dim \mathfrak{g}, \dim \mathfrak{h}, \dim(\mathfrak{g}/\mathfrak{h})$ and $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{h})$. In many cases we are able to pass from $\mathfrak{h}$ to $\mathfrak{h} + \mathfrak{z}$ without problem and consequently we may consider only the case where $\mathfrak{h}$ contains $\mathfrak{z}$. If $f \in \mathfrak{g}^*$ vanishes on a non-trivial ideal of $\mathfrak{g}$, we could go down to the quotient space by this ideal. Through this routine we arrive at the case where $\dim \mathfrak{z} \leq 1, \dim \mathfrak{a} \leq 3$. Provided $\mathfrak{g} \neq \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$, there is an ideal $\mathfrak{g}_0$ of $\mathfrak{g}$ containing $\mathfrak{h}$ and such that $\dim(\mathfrak{g}/\mathfrak{g}_0) = 1$. In this case, we may apply the induction hypothesis to $G_0 = \exp(\mathfrak{g}_0)$ and combine with Proposition 8.1.5. If $\mathfrak{g} = \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$, set $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$, $K = \exp \mathfrak{k}$. Taking in mind the Theorem 3.2.8 in the third chapter concerning the induction by stage, what we have to do first is to analyse the monomial representation $\mathrm{ind}_H^K \chi_f$.

Lemma 8.1.8. (1) *Let $G = exp(\mathfrak{g}_2)$, $\mathfrak{g}_2 = \langle X, Y\rangle_{\mathbb{R}} = \mathbb{R}X + \mathbb{R}Y : [X, Y] = Y$. Take $f \in \mathfrak{g}_2^*$ and put $\mathfrak{h} = \mathbb{R}X$, $H = exp\ \mathfrak{h}$. Further, if we set $H' = exp(\mathfrak{h}'), \mathfrak{h}' = \mathbb{R}Y$,*

$$ind_H^G \chi_f \simeq ind_{H'}^G \chi_{Y^*} \oplus ind_{H'}^G \chi_{-Y^*}.$$

(2) *Take $G = exp(\mathfrak{g}_3(\alpha))$, $\mathfrak{g}_3(\alpha) = \langle T, Y_1, Y_2\rangle_{\mathbb{R}} : [T, Y_1] = Y_1 - \alpha Y_2, [T, Y_2] = Y_2 + \alpha Y_1$ $(0 \neq \alpha \in \mathbb{R})$, and let $f \in \mathfrak{g}_3(\alpha)^*, \mathfrak{h} = \mathbb{R}T$, $H = exp\ \mathfrak{h}$. Then, using $H' = exp(\mathbb{R}Y_1 + \mathbb{R}Y_2), \hat{\theta} = (\cos\theta)Y_1^* + (\sin\theta)Y_2^* \in \mathfrak{g}_3(\alpha)^*$,*

$$ind_H^G \chi_f \simeq \int_{[0,2\pi)}^{\oplus} ind_{H'}^G \chi_{\hat{\theta}} d\theta.$$

(3) *We take $G = exp(\mathfrak{g}_4)$, $\mathfrak{g}_4 = \langle T, X, Y, Z\rangle_{\mathbb{R}} : [T, X] = -X, [T, Y] = Y, [X, Y] = Z$. Let $f = \alpha T^* + \beta Z^* \in \mathfrak{g}_4^*$ $(\beta \neq 0), \mathfrak{h} = \langle T, X, Z\rangle_{\mathbb{R}}$ and $H = exp\ \mathfrak{h}$. Then, using $H' = exp(\mathfrak{h}'), \mathfrak{h}' = \langle T, Y, Z\rangle_{\mathbb{R}}$,*

$$ind_H^G \chi_f \simeq ind_{H'}^G \chi_f.$$

(4) *Take $G = exp(\mathfrak{g}_6)$, $\mathfrak{g}_6 = \langle T, X_1, X_2, Y_1, Y_2, Z\rangle_{\mathbb{R}} : [T, X_1] = -X_1 - \alpha X_2, [T, X_2] = -X_2 + \alpha X_1, [T, Y_1] = Y_1 - \alpha Y_2, [T, Y_2] = Y_2 + \alpha Y_1, [X_i, Y_j] = \delta_{ij} Z$ $(0 \neq \alpha \in \mathbb{R})$. Let $f = \beta T^* + \gamma Z^*$ $(\gamma \neq 0), \mathfrak{h} = \langle T, X_1, X_2, Z\rangle_{\mathbb{R}}$ and $H = exp\ \mathfrak{h}$. Then, using $H' = exp(\mathfrak{h}'), \mathfrak{h}' = \langle T, Y_1, Y_2, Z\rangle_{\mathbb{R}}$,*

$$ind_H^G \chi_f \simeq ind_{H'}^G \chi_f.$$

Proof. Probably these facts are well known except (2) [10, 77]. Here we show (2). For this aim, it is enough to continue a little more the observations on page 135 of [10]. We borrow the notations used there. We identify $\mathfrak{a} = \mathbb{R}Y_1 + \mathbb{R}Y_2$ with the complex plane $\mathbb{C}$ by the bijection $\xi Y_1 + \eta Y_2 \mapsto \xi + i\eta, i = \sqrt{-1}$. Put $\lambda = f(T)$. The monomial representation $\tau = \mathrm{ind}_H^G \chi_f$ is realized in the space $L^2(\mathbb{C}) = L^2(\mathbb{R}^2)$ by the following formula: for $\xi_0 \in \mathbb{R}, z_0, z \in \mathbb{C}$ and $\phi \in L^2(\mathbb{C})$,

$$\tau(\exp(\xi_0 T)\exp(z_0))\phi(z) = e^{\xi_0(i\lambda-1)}\phi\left(ze^{-\xi_0(1-i\alpha)} - z_0\right).$$

Let $\mathcal{F}$ be the Fourier transformation of $L^2(\mathbb{C})$, $(\cdot|\cdot)$ be the inner product in $\mathbb{R}^2 = \mathbb{C}$ and dz a Lebesgue measure of $\mathbb{R}^2$. A direct computation gives the following: for $\hat{g} = \exp(\xi_0 T)\exp(z_0) \in G$, $\psi \in L^2(\mathbb{C})$ and $v \in \mathbb{C}$,

$$\mathcal{F}\left(\tau(\hat{g})\psi\right)(v) = e^{\xi_0(i\lambda+1)+i(v|e^{\xi_0(1-i\alpha)}z_0)}\mathcal{F}\psi\left(e^{\xi_0(1+i\alpha)}v\right),$$

while, using the dual basis $\{Y_1^*, Y_2^*\}$ of $\mathfrak{a}^*$ and the bijection $\xi Y_1^* + \eta Y_2^* \mapsto \xi + i\eta$, $\mathfrak{a}^*$ is identified with $\mathbb{C}$. Then, if we compute $\xi' + i\eta' = \exp(tT)\cdot(\xi + i\eta)$,

$$\xi' = e^{-t}\left(\xi\cos(\alpha t) + \eta\sin(\alpha t)\right),\ \eta' = e^{-t}\left(\xi\sin(-\alpha t) + \eta\cos(\alpha t)\right).$$

From this the coadjoint orbit Ω_θ passing the point $e^{i\theta}$ is given by $e^{-t+i(\theta-\alpha t)}$ with t running over $\mathbb{R}$, and the Jacobian of the change of variables $(\xi', \eta') \mapsto (t, \theta)$ becomes e^{-2t}.

We set $\tau_1 = \mathcal{F}\circ\tau\circ\mathcal{F}^{-1}$. For $\hat{g} = \exp(\xi_0 T)\exp(z_0) \in G$,

$$\tau_1(\hat{g})\phi\left(e^{-t+i(\theta-\alpha t)}\right) = e^{\xi_0(1+i\lambda)+i(e^{-t+i(\theta-\alpha t)}|e^{\xi_0(\theta-\alpha t)}z_0)}\phi\left(e^{(-t+\xi_0)(1+i\alpha)+i\theta}\right).$$

Hence τ_1 keeps $L^2(\Omega_\theta)$ stable and the formula

$$\hat{\tau}(\hat{g})\Psi(t) = e^{\xi_0(1+i\lambda)+i(e^{i\theta}|e^{(\xi_0-t)(1-i\alpha)}z_0)}\Psi(t-\xi_0),$$

gives a representation $\hat{\tau}$ of G on the space of all functions Ψ satisfying

$$\int_{\mathbb{R}} e^{-2t}|\Psi(t)|^2 dt < +\infty.$$

However, $\hat{\tau}$ is nothing but the representation $\pi_\theta = \mathrm{ind}_{H'}^G \chi_{\hat{\theta}}$. Indeed, as before for $\hat{g} \in G$ and $\phi \in L^2(\mathbb{R})$,

$$\pi_\theta(\hat{g})\phi(x) = e^{i(e^{i\theta}|e^{(\xi_0-x)(1-i\alpha)}z_0)}\phi(x-\xi_0)\ (x \in \mathbb{R}).$$

Finally, by carrying π_θ by the mapping which sends $\phi \in L^2(\mathbb{R})$ to

$$\Phi \in L^2(\mathbb{R}; e^{-2x}dx),\ \Phi(x) = e^{(1+i\lambda)x}\phi(x),$$

we get $\hat{\tau}$ and finish the proof. ■

The reasoning the way of which we mentioned above leads us to the following result. Take a finite measure $\tilde{\mu}$ on Γ_τ equivalent to the Lebesgue measure and regard it a measure on $\mathfrak{g}^*$. Let $\mu = \hat{\rho}_*(\tilde{\mu})$ be the image of $\tilde{\mu}$ by the Kirillov–Bernat mapping $\hat{\rho} : \mathfrak{g}^* \to \hat{G}$. For $\pi \in \hat{G}$, we denote by $\Omega(\pi) = \Omega_G(\pi)$ the coadjoint orbit of G corresponding to π and by $m(\pi)$ the number of the H-orbits contained in $\Gamma_\tau \cap \Omega(\pi)$. By Corollary 8.1.7, this $m(\pi)$ is also given as follows. When each connected component of $\Gamma_\tau \cap \Omega(\pi)$ is a manifold of dimension $\frac{1}{2}\dim\, \Omega(\pi)$, it is the number of the connected components. Otherwise, it is $+\infty$.

Theorem 8.1.9 ([14, 32]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$, $\mathfrak{h} \in S(f, \mathfrak{g})$ and $H = \exp \mathfrak{h}$. Then, the monomial representation $\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$ is decomposed as*

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi). \tag{8.1.1}$$

Proof. Let us proceed by induction on dim G. We first suppose $\mathfrak{g} \neq \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$. Then there is an ideal $\mathfrak{g}_0$ of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{h}$ and we can apply the induction hypothesis to $G_0 = \exp(\mathfrak{g}_0)$. The coadjoint orbit Ω of G is either saturated or non-saturated regarding the restriction mapping $p : \mathfrak{g}^* \to \mathfrak{g}_0^*$ and the same eventuality occurs almost everywhere for $\tilde{\mu}$ [42]. Now, by Lemma 8.1.4 and Proposition 8.1.5, we know the following facts. If Ω is saturated, $p(\Omega)$ consists of a one-parameter family $\{\omega_t\}_{t\in\mathbb{R}}$ of G_0-orbits and

$$\dim\, \Omega = \dim(\omega_t) + 2,\ \hat{\rho}_G(\Omega) \simeq \mathrm{ind}_{G_0}^G \hat{\rho}_{G_0}(\omega_t)$$

for all $t \in \mathbb{R}$. On the contrary, if Ω is non-saturated, $\omega = p(\Omega)$ is a G_0-orbit, p gives a diffeomorphism from Ω onto ω, $p^{-1}(\omega)$ consists of a one-parameter family $\{\Omega_t\}_{t\in\mathbb{R}}$ of G-orbits and

$$\mathrm{ind}_{G_0}^G \hat{\rho}_{G_0}(\omega) \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_G(\Omega_t) dt.$$

Combining this with the induction hypothesis, we immediately obtain the assertion on the measure μ. Let us examine the multiplicity. In the case where almost all coadjoint G-orbits are non-saturated, this claim also would follow immediately from the above fact combined with the induction hypothesis. In the case where almost all coadjoint G-orbits are saturated, we set $f_0 = p(f) \in \mathfrak{g}_0^*$, $\tau_0 = \mathrm{ind}_H^{G_0} \chi_{f_0}$, $\Gamma_{\tau_0} = f_0 + \mathfrak{h}^{\perp, \mathfrak{g}_0^*}$, and write the canonical irreducible decomposition of τ_0 as

$$\tau_0 \simeq \int_{\widehat{G_0}}^{\oplus} m_0(\pi_0)\pi_0 d\mu_0(\pi_0).$$

Since G_0 is a normal subgroup of G, G acts as usual on $\widehat{G_0}$ and $\hat{G}$ is except for a null set for μ identified with the quotient set $\widehat{G_0}/G$. If we decompose the measure μ_0 with respect to μ, which is nothing but its image by the projection $q : \widehat{G_0} \to \widehat{G_0}/G$,

$$\begin{aligned}\tau \simeq \mathrm{ind}_{G_0}^G \tau_0 &\simeq \int_{\widehat{G_0}}^{\oplus} m_0(\pi_0)\left(\mathrm{ind}_{G_0}^G \pi_0\right) d\mu_0(\pi_0)\\ &\simeq \int_{\hat{G}}^{\oplus} d\mu(\pi) \int_{\widehat{G_0}}^{\oplus} m_0(\pi_0)\left(\mathrm{ind}_{G_0}^G \pi_0\right) d\mu_0^{\pi}(\pi_0),\end{aligned}$$

where μ_0^{π} is a measure on the fibre $q^{-1}(\pi)$. As we have seen until now, the irreducible representation which appears in the representation

$$\int_{\widehat{G_0}}^{\oplus} m_0(\pi_0)\left(\mathrm{ind}_{G_0}^G \pi_0\right) d\mu_0^{\pi}(\pi_0)$$

is only π and

$$m(\pi)\pi \simeq \int_{\widehat{G_0}}^{\oplus} m_0(\pi_0)\left(\mathrm{ind}_{G_0}^G \pi_0\right) d\mu_0^{\pi}(\pi_0)$$

for almost all π regarding μ. Therefore, $m(\pi)$ becomes finite for such π when and only when the measure μ_0^{π} is supported by finite π_0 such that $m_0(\pi_0)$ is finite. And in that case, if we write the G_0-orbit $\hat{\rho}_{G_0}^{-1}(\pi_0)$ as $\omega(\pi_0)$, the multiplicity $m_0(\pi_0)$ is given by the number of the connected components of $\Gamma_{\tau_0} \cap \omega(\pi_0)$.

Combining these observations with Corollary 8.1.7, we know that $q^{-1}(\pi)$ is a finite set and

$$m(\pi) = \sum_{\pi_0 \in q^{-1}(\pi)} m_0(\pi_0).$$

For each connected component C of $\Gamma_\tau \cap \Omega(\pi)$, $p(C)$ coincides with a connected component of $\Gamma_{\tau_0} \cap \omega(\pi_0)$ corresponding to a certain $\pi_0 \in q^{-1}(\pi)$ and, since the latter is a manifold of dimension $\frac{1}{2}\dim(\omega(\pi_0))$, C itself turns out to be a manifold of required dimension. Besides, $m(\pi)$ becomes equal to the number of the connected components of $\Gamma_\tau \cap \Omega(\pi)$.

From now on we suppose $\mathfrak{g} = \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$. Recall that we may assume that $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. Moreover, if $\mathfrak{h} \cap \ker f$ contains a non-trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$, we may apply the induction hypothesis to the quotient Lie group $G/\exp \mathfrak{a}$. Thus, we may assume $\dim \mathfrak{z} \leq 1$. Let $\mathfrak{a}$ be a minimal non-central ideal of $\mathfrak{g}$ and $A = \exp \mathfrak{a}$. By Lemma 5.3.10 of Chap. 5, $\mathfrak{a}$ is commutative and $\dim \mathfrak{a} \leq 3$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z})$ produces a root $\lambda \in \mathfrak{g}^*$ and there is by our assumption $T \in \mathfrak{h}$ verifying $\lambda(T) = 1$, while $\mathfrak{g}_0 = \ker\lambda$ is an ideal of $\mathfrak{g}$ so that $\mathfrak{g} = \mathbb{R}T \oplus \mathfrak{g}_0$. Let us examine the situations case by case.

(i) Suppose $\mathfrak{a} \cap \mathfrak{z} = \{0\}$. Then $\mathfrak{h} \cap \mathfrak{a} = \{0\}$. We set $\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{g}_0) + \mathfrak{a}$ and $H' = \exp(\mathfrak{h}')$. As far as we reason at the stage of the subgroup $K = \exp \mathfrak{k}$, $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$, the situation is totally the same as (1) or (2) of Lemma 8.1.8 according to the dimension of $\mathfrak{a}$.

Provided $\dim \mathfrak{a} = 1$, we write $\mathfrak{a} = \mathbb{R}Y$ and take $f_\pm \in \Gamma_\tau$ such that $f_\pm(Y) = \pm 1$. Clearly $\mathfrak{h}' \in S(f_\pm, \mathfrak{g})$ and the theorem is already established for the pair $(f_\pm, \mathfrak{h}')$ because $\mathfrak{g} \neq \mathfrak{h}' + [\mathfrak{g}, \mathfrak{g}]$. So, on the understanding of our notations concerning the multiplicity and the measure,

$$\begin{aligned}\tau &\simeq \mathrm{ind}_{H'}^G \chi_{f_+} \oplus \mathrm{ind}_{H'}^G \chi_{f_-} \\ &\simeq \int_{\hat{G}}^{\oplus} m_+(\pi)\pi d\mu_+(\pi) + \int_{\hat{G}}^{\oplus} m_-(\pi)\pi d\mu_-(\pi). \end{aligned} \tag{8.1.2}$$

Now put $P_\pm = \{\ell \in \mathfrak{g}^*; \ell(Y) \gtrless 0\}$, $\Gamma'_\pm = f_\pm + \mathfrak{h}'^\perp$, $\Gamma^0_\pm = f_\pm + \mathfrak{k}^\perp$ and $\Gamma_\pm = \Gamma_\tau \cap P_\pm$. Then

$$\Gamma_\pm = \exp(\mathbb{R}T) \cdot \Gamma^0_\pm, \ \Gamma'_\pm = A \cdot \Gamma^0_\pm$$

and $P_\pm$ are G-stable. In this way formula (8.1.2) furnishes the desired result.

When $\dim \mathfrak{a} = 2$, we choose $Y_1, \ Y_2 \in \mathfrak{a}$ so that the three elements $T, \ Y_1, \ Y_2$ satisfy the commutation relation of (2) in Lemma 8.1.8. Take $\hat{\theta} \in \Gamma_\tau$ such that

$$\hat{\theta}|_{\mathfrak{a}} = (\cos\theta)Y_1^* + (\sin\theta)Y_2^*.$$

Evidently $\mathfrak{h}' \in S(\hat{\theta}, \mathfrak{g})$. Similarly to (2) in Lemma 8.1.8,

$$\tau \simeq \int_{[0,2\pi)}^{\oplus} \mathrm{ind}_{H'}^G \chi_{\hat{\theta}} d\theta \simeq \int_{[0,2\pi)}^{\oplus} d\theta \int_{\hat{G}}^{\oplus} m_\theta(\pi)\pi d\mu_\theta(\pi). \tag{8.1.3}$$

We set $\Gamma'_\theta = \hat{\theta} + \mathfrak{h}'^\perp$ and $\Gamma^0_\theta = \hat{\theta} + \mathfrak{k}^\perp$. Now $\ell|_{\mathfrak{a}} \neq 0$ $(\ell \in \Omega)$, hence we take $\Omega \in \mathfrak{g}^*/G$ such that $\dim(\mathfrak{g}(\ell) \cap \mathfrak{a}) = 1$. For Ω like this, there uniquely exists $\theta \in [0, 2\pi)$ so that

$$\Gamma_\tau \cap \Omega = \exp(\mathbb{R}T) \cdot \left(\Gamma^0_\theta \cap \Omega\right), \ \Gamma'_\theta \cap \Omega = A \cdot \left(\Gamma^0_\theta \cap \Omega\right).$$

Taking these observations into account, formula (8.1.3) gives exactly the desired result.

(ii) Let us examine the case where $\mathfrak{a} \cap \mathfrak{z} = \mathfrak{z} \neq \{0\}$. Suppose first that $\mathfrak{a} \subset \mathfrak{h}$. If we introduce the proper Lie subalgebra $\mathfrak{g}_1 = \mathfrak{a}^f$ of $\mathfrak{g}$, then $\mathfrak{h} \subset \mathfrak{g}_1$. Applying the induction hypothesis to the subgroup $G_1 = \exp(\mathfrak{g}_1)$

$$\tau_1 = \mathrm{ind}_H^{G_1} \chi_f \simeq \int_{\widehat{G_1}}^{\oplus} m_1(\rho)\rho d\mu_1(\rho),$$

and so

$$\tau \simeq \mathrm{ind}_{G_1}^{G} \tau_1 \simeq \int_{\widehat{G_1}}^{\oplus} m_1(\rho) \left(\mathrm{ind}_{G_1}^{G} \rho\right) d\mu_1(\rho). \tag{8.1.4}$$

Here, since $\rho|A$ is a multiple of $\chi_f|_A$, Theorem 3.4.4 of the third chapter assures that these $\mathrm{ind}_{G_1}^{G} \rho$ are irreducible and non-equivalent to each other, whereas, $A \cdot \ell = \ell + (\mathfrak{a}^f)^{\perp}$ for any $\ell \in \Gamma_\tau$. Finally, formula (8.1.4) provides exactly the desired result.

Next we treat the case where $\mathfrak{h} \cap \mathfrak{a} = \mathfrak{z}$. Suppose first that $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \subset \mathfrak{g}_1$. Since $[\mathfrak{h}_0, \mathfrak{a}] = \{0\}$, as we described before $\mathfrak{h}' \in S(f_\pm, \mathfrak{g})$ or $\mathfrak{h}' \in S(\hat{\theta}, \mathfrak{g})$ according to the dimension of $\mathfrak{a}$. Further by the above arguments, we are led to the irreducible decomposition (8.1.2) if $\dim \mathfrak{a} = 2$ and (8.1.3) if $\dim \mathfrak{a} = 3$. However, this time a more complicated situation occurs. Assume $\dim \mathfrak{a} = 2$ and take $Y \in \mathfrak{a}$ such that $[T, Y] = Y$. With the previous notations the two sets $P_\pm$ are no longer G-invariant and the two direct integrals

$$\int_{\hat{G}}^{\oplus} m_+(\pi)\pi d\mu_+(\pi) \text{ and } \int_{\hat{G}}^{\oplus} m_-(\pi)\pi d\mu_-(\pi)$$

in formula (8.1.2) could have common irreducible components. Anyhow, passing to the quotient space regarding the action of G, we get the assertion on the measure.

Now concerning the multiplicity, Lemma 8.1.8(1) gives

$$m(\pi) = m_+(\pi) + m_-(\pi).$$

Let $\Omega \in \mathfrak{g}^*/G$ be such that each connected component C' of $\Gamma'_\pm \cap \Omega$ is an H'-orbit and that each connected component of $\Gamma_\tau \cap \Omega$ is a manifold, then

$$\exp(\mathbb{R}T) \cdot \left(\Gamma_\pm^0 \cap C'\right) = H \cdot \ell,\ \ell \in \Gamma_\pm^0 \cap C',$$

is the connected component of $\Gamma_\tau \cap \Omega$ containing $\Gamma_\pm^0 \cap C'$. This means it is impossible that such an $H \cdot \ell$ contained in $\Gamma_+ \cap \Omega$ and another $H \cdot \ell'$ contained in $\Gamma_- \cap \Omega$ could be included in the same connected component of $\Gamma_\tau \cap \Omega$. Indeed, this last connected component is by assumption a manifold and its dimension is necessarily $\frac{1}{2}\dim \Omega$. By Corollary 8.1.7(3), we obtain the result almost everywhere with respect to μ. This finally implies that for such an Ω the connected components of $\Gamma_\tau \cap \Omega$ contained in the hyperplane $\{\ell \in \mathfrak{g}^*; \ell(Y) = 0\}$ do not contribute by Corollary 8.1.7(1). We have finished the proof in this case.

Next assume $\dim \mathfrak{a} = 3$ and take $Y_1,\ Y_2 \in \mathfrak{a}$ so that the three elements $\{T,\ Y_1,\ Y_2\}$ satisfy the commutation relations in Lemma 8.1.8(2). We already know that

$$\tau \simeq \int_{[0,2\pi)}^{\oplus} d\theta \int_{\hat{G}}^{\oplus} m_\theta(\pi)\pi d\mu_\theta(\pi)$$

but, for $\theta_1 \neq \theta_2$, the elements of Γ'_{θ_1} and those of Γ'_{θ_2} are mixed by the action of G. Nevertheless, the statement concerning the measure is established by passing to the quotient space by the equivalence relation.

Regarding the multiplicity, if we denote by $\Theta(\pi)$ the set of $\theta \in [0, 2\pi)$ such that $\Gamma'_\theta \cap \Omega(\pi) \neq \emptyset$, Corollary 8.1.7(1) means

$$m(\pi) = \sum_{\theta \in \Theta(\pi)} m_\theta(\pi)$$

when $\Theta(\pi)$ is a countable set, otherwise $m(\pi) = +\infty$. Let us examine in more detail the case where $\Theta(\pi)$ is a countable set. Let C be a connected component of $\Gamma_\tau \cap \Omega(\pi)$. For all $\ell \in C$, the restriction of ℓ to $V = \mathbb{R}Y_1 + \mathbb{R}Y_2$ belongs to

$$\bigcup_{\theta \in \Theta(\pi)} \exp(\mathbb{R}T) \cdot (\theta|_V)$$

unless $\ell|_V = 0$. Furthermore, we suppose that each connected component C' of $\Gamma'_\theta \cap \Omega$, $\theta \in \Theta(\pi)$, is one H'-orbit, i.e. $C' = H' \cdot \ell'$ with $\ell' \in C'$ satisfying $\ell'(T) = f(T)$, and that each connected component of $\Gamma_\tau \cap \Omega(\pi)$ is a manifold. Exactly as in the preceding case, we understand that $H \cdot \ell'$ is the connected component of $\Gamma_\tau \cap \Omega(\pi)$ containing ℓ' and that we may neglect the connected components contained in the subspace $\{\ell \in \mathfrak{g}^*; \ell|_V = 0\}$. In this way we obtain the statement on the multiplicity.

Let us finally consider the case where $\mathfrak{h}_0 \not\subset \mathfrak{g}_1$. The situation is essentially similar to that in Lemma 8.1.8(3) or (4). If $\dim \mathfrak{a} = 2$, take $Z \in \mathfrak{z}$, $Y \in \mathfrak{a} \setminus \mathfrak{z}$ and $X \in \mathfrak{h}_0$ in such a fashion that the four elements $\{T, X, Y, Z\}$ satisfy the commutation relations similar to those in Lemma 8.1.8(3), while we replace $[T, X] = -X$ by the relation $[T, X] \equiv -X$ modulo $\mathfrak{g}_0 \cap \mathfrak{g}_1$. Likewise, if $\dim \mathfrak{a} = 3$, we choose $Z \in \mathfrak{z}$, $Y_1, Y_2 \in \mathfrak{a} \setminus \mathfrak{z}$ and $X_1, X_2 \in \mathfrak{h}_0$ so that these elements together with T satisfy the commutation relations similar to those in Lemma 8.1.8(4), while modulo $\mathfrak{g}_0 \cap \mathfrak{g}_1$, $[T, X_1] \equiv -X_1 - \alpha X_2$, $[T, X_2] \equiv -X_2 + \alpha X_1$ $(0 \neq \alpha \in \mathbb{R})$.

If we set $\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{g}_1) + \mathfrak{a}$, $H' = \exp(\mathfrak{h}')$, clearly $\mathfrak{h}' \in S(f, \mathfrak{g})$. Arguing at the stage of $K = \exp \mathfrak{k}$, $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$, Proposition 5.3.18 in Chap. 5 gives

$$\tau \simeq \operatorname{ind}_{H'}^G \chi_f.$$

Thus, this is reduced to the case already treated and, for $(f, \mathfrak{h}')$,

$$\tau \simeq \operatorname{ind}_{H'}^G \chi_f \simeq \int_{\hat{G}}^{\oplus} m'(\pi)\pi d\mu'(\pi)$$

holds with the measure μ' and the multiplicity $m'(\pi)$ given by the theorem.

To finish the proof of the theorem it is enough to notice the following. $\Omega \in \mathfrak{g}^*/G$ being arbitrarily given, each connected component C of $\Gamma_\tau \cap \Omega$ is written as $C = (\exp U)\cdot C_0$ with a connected component C_0 of $\Gamma_0 \cap \Omega$, $\Gamma_0 = f + \mathfrak{k}^\perp$, where $U = \mathbb{R}X$ or $U = \mathbb{R}X_1 + \mathbb{R}X_2$. Similarly, a connected component C' of $\Gamma' \cap \Omega$, $\Gamma' = f + \mathfrak{h}'^\perp$ is written as $C' = (\exp V)\cdot C_0$ with $V = \mathbb{R}Y$ or $V = \mathbb{R}Y_1 + \mathbb{R}Y_2$. ■

Let us generalize this result a little bit. Starting from a subgroup $K = \exp \mathfrak{k}$ and $\sigma \in \hat{K}$ and examine the induced representation $\mathrm{ind}_K^G \sigma$. We denote by $\omega(\sigma)$ the coadjoint orbit $\hat{\rho}_K^{-1}(\sigma) \subset \mathfrak{k}^*$ of K corresponding to σ and by $p : \mathfrak{g}^* \to \mathfrak{k}^*$ the restriction mapping. A K-invariant measure on $\omega(\sigma)$ and a Lebesgue measure on $\mathfrak{k}^\perp$ furnish the measure $\hat{\mu}$ on the submanifold $p^{-1}(\omega(\sigma))$ of $\mathfrak{g}^*$. Let $\tilde{\mu}$ be a finite measure on $\mathfrak{g}^*$ equivalent to $\hat{\mu}$, and put $\mu = \mu_G^\sigma = (\hat{\rho}_G)_*(\tilde{\mu})$. Moreover, for $\pi \in \hat{G}$, we denote by $n_\pi(\sigma)$ the number of the K-orbits contained in $\Gamma(\pi, \sigma) = \Omega(\pi) \cap p^{-1}(\omega(\sigma))$. Using the fact that σ is a monomial representation of K, we generalize Theorem 8.1.9:

Theorem 8.1.10 ([14, 34]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $K = \exp \mathfrak{k}$ an analytic subgroup of G and σ an irreducible unitary representation of K. Then,*

$$ind_K^G \sigma \simeq \int_{\hat{G}}^{\oplus} n_\pi(\sigma)\pi d\mu(\pi).$$

8.2 Restriction of Unitary Representations to Subgroups

To begin, we prepare Mackey's subgroup theorem [55]. However, most of the situations where we shall employ this theorem are very simple and we will not really need this theorem. Let H_1, H_2 be closed subgroups of a locally compact topological group G.

Definition 8.2.1. We denote by $H_2\backslash G/H_1$ the space of the (H_2, H_1)-double cosets and by $p : G \to H_2\backslash G/H_1$ the canonical projection. We introduce in $H_2\backslash G/H_1$ the Borel structure induced by the map p from that of G. We take on G a finite Borel measure μ equivalent to the Haar measure and define the Borel measure $\bar{\mu}$ on $H_2\backslash G/H_1$ by $\bar{\mu}(E) = \mu(p^{-1}(E))$. When $\bar{\mu}$ is a standard Borel measure, i.e. up to a $\bar{\mu}$-null set $H_2\backslash G/H_1$ is a standard Borel space, H_1 and H_2 are said to be regularly related.

Theorem 8.2.2 (Subgroup Theorem). *Suppose that H_1 and H_2 are regularly related. Let σ be a unitary representation of H_1 and $\pi = ind_{H_1}^G \sigma$. Besides, $\sigma_g (g \in G)$ denotes the unitary representation of the closed subgroup gH_1g^{-1} defined by the formula $\sigma_g(h) = \sigma(g^{-1}hg)$. The unitary representation ρ_g of H_2 induced from the restriction of σ_g to $H_2 \cap gH_1g^{-1}$ is up to equivalence determined* *by the double coset* $[g] = p(g)$ *of* g*, and we have*

$$\pi|_{H_2} \simeq \int_{H_2\backslash G/H_1}^{\oplus} \rho_g d\bar{\mu}([g]).$$

We only sketch a proof of the theorem. For a Borel set E of $H_2\backslash G/H_1$, we denote by φ_E the characteristic function of the set $p^{-1}(E)$ and consider the linear transformation of $\mathcal{H}_\pi$ defined by $(P_E\psi)(g) = \varphi_E(g)\psi(g)$ $(g \in G)$. Evidently, P_E is a projection in $\mathcal{H}_\pi$ and its range is stable by the actions of the representation $\pi|_{H_2}$. Next let us compute the actions of $\pi|_{H_2}$ in this range. Taking a Borel cross-section $s : E \to p^{-1}(E)$, we choose a representative element $g \in G$ of a double coset C contained in E in such a way that $g \in s(C)$. Hence $C = H_2gH_1$. If we set $\tilde{\varphi}(h) = \varphi(hg)$ $(h \in H_2)$ for φ belonging to the range of P_E, then $\tilde{\varphi}$ is a mapping from H_2 to $\mathcal{H}_\sigma$. As is easily seen, with $h \in H_2, x \in H_2 \cap gH_1g^{-1}$,

$$\tilde{\varphi}(hx) = \varphi(hxg) = \varphi(hgg^{-1}xg) = \sigma(g^{-1}xg)^{-1}\varphi(hg) = \sigma(g^{-1}xg)^{-1}\tilde{\varphi}(h).$$

Conversely, let ψ be a mapping from H_2 to $\mathcal{H}_\sigma$ which satisfy this relation on each double coset C contained in E. Then, putting

$$\varphi(h_2gh_1) = \sigma(h_1)^{-1}\psi(h_2) \ \ (h_1 \in H_1, h_2 \in H_2)$$

on $p^{-1}(C)$ and $\varphi(x) = 0$ $(x \notin p^{-1}(E))$, we get φ belonging to the range of P_E such that $\tilde{\varphi} = \psi$. The map $\varphi \to \tilde{\varphi}$ is linear and if, for φ belonging to the range of P_E, $\tilde{\varphi} = 0$ holds on each double coset contained in E, then $\varphi = 0$. Namely, the linear mapping $\varphi \to \tilde{\varphi}$ gives a bijection from the range of P_E to the representation space of the field of unitary representations $[g] \to \rho_g$ on E. This gives us the desired equivalence relation.

Returning to the situation mentioned at the beginning of the previous section and using the notations employed in Proposition 8.1.5, we would like to examine the restriction $\pi|_{G_0}$ of $\pi \in \hat{G}$ to G_0.

Proposition 8.2.3 ([42]). *Let* $G = exp\,\mathfrak{g}$ *be an exponential solvable Lie group with Lie algebra* $\mathfrak{g}$ *and* $G_0 = exp(\mathfrak{g}_0)$ *an analytic normal subgroup of codimension* 1 *in* G*. We denote by* $\hat{\rho}$, $\hat{\rho}_0$ *the Kirillov–Bernat mappings for* G, G_0*. Let* $\pi = \hat{\rho}(\ell)$ *with* $\ell \in \mathfrak{g}^*$ *and put* $\ell_0 = \ell|_{\mathfrak{g}_0}$*.*

(1) *When the orbit* $G{\cdot}\ell$ *is saturated in the direction of* $\mathfrak{g}_0^\perp$*,*

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_s)ds.$$

Here $\ell_s = exp(sX){\cdot}\ell_0$ *with* $X \in \mathfrak{g}\backslash\mathfrak{g}_0$*.*

(2) *When the orbit* $G{\cdot}\ell$ *is non-saturated in the direction of* $\mathfrak{g}_0^\perp$*,* $\pi|_{G_0} \simeq \hat{\rho}_0(\ell_0)$*.*

Proof. (1) Let us use Mackey's subgroup theorem although we do not necessarily need it. Take a Vergne polarization $\mathfrak{h}$ at ℓ of $\mathfrak{g}$ constructed from a strong Malcev sequence of Lie subalgebras passing $\mathfrak{g}_0$ and put $H = \exp \mathfrak{h}$. Then $\mathfrak{h} \in I(\ell, \mathfrak{g})$ and $\mathfrak{h} \subset \mathfrak{g}_0$. We see that H and G_0 are regularly related. Since $H \subset G_0$, the double coset $HgG_0 = HG_0g = G_0g$ is nothing but the coset modulo G_0. Therefore, the space of the double cosets becomes the group G/G_0 and there exists a countable separating family of Borel sets. Thus the subgroups H and G_0 are regularly related. The group G/G_0 is identified with $\mathbb{R}$ by the mapping $s \mapsto \exp(sX)G_0$ and we can take a Haar measure on G/G_0, hence a Lebesgue measure on $\mathbb{R}$, as a measure used in the direct integral decomposition [54] so that

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} V_t dt.$$

Here V_t is the representation of G_0 induced from the representation

$$\sigma_t : g_0 \mapsto \chi_\ell(\exp(tX)g_0\exp(-tX))$$

of the subgroup

$$H_t = G_0 \cap (\exp(-tX)H\exp(tX)) = \exp(-tX)H\exp(tX)$$

of G_0. However, σ_t is nothing but $\exp(-tX)\cdot\chi_\ell = \chi_{\exp(-tX)\cdot\ell}$. Hence,

$$V_t \simeq \mathrm{ind}_{H_t}^{G_0} \chi_{\exp(-tX)\cdot\ell} \simeq \exp(-tX) \cdot \left(\mathrm{ind}_H^{G_0} \chi_\ell\right) \simeq \exp(-tX) \cdot \hat{\rho}_0(\ell_0).$$

In this way V_t is irreducible and the Lie algebra of H_t belongs to $I(\exp(-tX) \cdot \ell_0, \mathfrak{g}_0)$. Thus, $V_t \simeq \hat{\rho}_0(\exp(-tX) \cdot \ell_0)$. Putting $\ell_s = \exp(sX) \cdot \ell_0$, we obtain the desired decomposition:

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_{-s})ds \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_s)ds.$$

(2) If we construct a Vergne polarization $\mathfrak{h} \in I(\ell, \mathfrak{g})$ just as above, $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \in I(\ell_0, \mathfrak{g}_0)$ and $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{g}(\ell)$. Since $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$, $H = H_0\exp(\mathbb{R}X)$ with $H_0 = \exp(\mathfrak{h}_0)$. We again apply (although not strictly necessary) Mackey's subgroup theorem to the pair (H, G_0). Here, since $HG_0 = H_0(\exp(\mathbb{R}X)G_0) = G$, there is only one double coset so that H and G_0 are regularly related. Thus

$$\pi|_{G_0} \simeq \mathrm{ind}_{G_0 \cap H}^{G_0} \chi_{\ell_0},$$

while $G_0 \cap H = H_0$ and $\mathfrak{h}_0 \in I(\ell_0, \mathfrak{g}_0)$, so

$$\mathrm{ind}_{G_0\cap H}^{G_0}\chi_{\ell_0} \simeq \hat{\rho}_0(\ell_0).$$

Hence $\pi|_{G_0} \simeq \hat{\rho}_0(\ell_0)$ and the assertion follows. ■

Now let $K = \exp \mathfrak{k}$ be a subgroup of G and $p : \mathfrak{g}^* \to \mathfrak{k}^*$ the restriction map. Let $\pi \in \hat{G}$ be given. We take in $\mathfrak{g}^*$ a finite measure $\tilde{\nu} = \tilde{\nu}_\pi$ equivalent to the G-invariant measure on the orbit $\Omega(\pi) = \Omega_G(\pi) = \hat{\rho}^{-1}(\pi)$ and set $\nu = \nu_K^\pi = (\hat{\rho}_K \circ p)_*(\tilde{\nu})$. Making use of the measure ν obtained in this manner on $\hat{K}$ and the same multiplicity $n_\pi(\sigma)$ as in Theorem 8.1.10, we procure the canonical irreducible decomposition of the restriction $\pi|_K$ of π to K.

Theorem 8.2.4 ([15, 34]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $K = \exp \mathfrak{k}$ an analytic subgroup of G. We take an irreducible unitary representation π of G. The restriction $\pi|_K$ of π to K is decomposed as*

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} n_\pi(\sigma)\sigma d\nu(\sigma).$$

Proof. As usual we employ the induction on the dimension of G. First of all, Proposition 8.2.3 offers the expected result when $\mathfrak{k}$ is an ideal of codimension 1. Suppose that $\mathfrak{k}$ is contained in an ideal $\mathfrak{g}_0$ of codimension 1, and put $G_0 = \exp(\mathfrak{g}_0)$. By Proposition 8.2.3 and the induction hypothesis applied to G_0, $\pi|_K = (\pi|_{G_0})|_K$ is equivalent to either

$$\left(\int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_s)ds\right)|_K \simeq \int_{\mathbb{R}}^{\oplus} (\pi_s|_K)\, ds \simeq \int_{\mathbb{R}}^{\oplus} ds \int_{\hat{K}}^{\oplus} n_{\pi_s}(\sigma)\sigma d\nu_K^{\pi_s}(\sigma) \tag{8.2.1}$$

with $\pi_s = \hat{\rho}_0(\ell_s)$ or

$$\int_{\hat{K}}^{\oplus} n_\rho(\sigma)\sigma d\nu_K^\rho(\sigma)$$

with $\rho = \pi|_{G_0} \in \widehat{G_0}$.

From this we get the statement about the measure ν. Besides, the claim concerning the multiplicity also follows immediately in the second case. Regarding the multiplicity in the first case, it is clear if the K-orbits are saturated in the direction of $\mathfrak{g}_0^\perp$ almost everywhere on $\Omega(\pi)$. Assume that the K-orbits are non-saturated almost everywhere. Then for almost all $s \in \mathbb{R}$ there exists $X(s) \in \mathfrak{g} \setminus \mathfrak{g}_0$ so that $X(s)\cdot\ell_s \in \mathfrak{k}^\perp$ in $\mathfrak{g}_0^*$. Hence, for $\sigma = \hat{\rho}_K(\ell|_\mathfrak{k}) \in \hat{K}$, we know that the number $n_\pi(\sigma)$ of the K-orbits contained in $\Omega(\pi) \cap p^{-1}(\omega(\sigma))$ and the multiplicity of σ in expression (8.2.1) are both $+\infty$.

Provided $\mathfrak{k}$ is contained in a proper ideal $\mathfrak{m}$ of $\mathfrak{g}$, $\tilde{p} : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{m}$ being the canonical map, we take an ideal $\tilde{\mathfrak{g}}_0$ of $\tilde{\mathfrak{g}}$ having the codimension 1 and containing $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. If we put $\mathfrak{g}_0 = \tilde{p}^{-1}(\tilde{\mathfrak{g}}_0)$, then $\mathfrak{g}_0$ is an ideal of codimension 1 in $\mathfrak{g}$ which contains $\mathfrak{k}$.

From now on we suppose that K is not contained in any proper normal subgroup of G. Besides, it is easily seen that we may assume $\mathfrak{k}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. We assume this in what follows. Using a polarization $\mathfrak{b}$ at $f \in \Omega(\pi)$, we realize π as $\pi \simeq \mathrm{ind}_B^G \chi_f$, $B = \exp \mathfrak{b}$. We further assume that $\ker f$ does not contain any non-trivial ideal of $\mathfrak{g}$, otherwise we may pass to the quotient Lie algebra by such an ideal and apply the induction hypothesis. Consequently, $\dim \mathfrak{z} \leq 1$ and $f|_{\mathfrak{z}} \neq 0$ if $\dim \mathfrak{z} = 1$. Take a minimal non-central ideal $\mathfrak{a}$ of $\mathfrak{g}$ and put $A = \exp \mathfrak{a}$, $\mathfrak{g}_1 = \mathfrak{a}^f$, $G_1 = \exp(\mathfrak{g}_1)$. Provided $\mathfrak{g}_1 = \mathfrak{g}$, f would vanish on the ideal $[\mathfrak{g}, \mathfrak{a}] \neq \{0\}$ of $\mathfrak{g}$, which is a contradiction. Hence $\mathfrak{g}_1$ is a proper Lie subalgebra of $\mathfrak{g}$. By Proposition 5.3.18 in Chap. 5, $\mathfrak{b}$ can be taken in $\mathfrak{g}_1$ and $\pi \simeq \mathrm{ind}_{G_1}^G \pi_1$ with $\pi_1 = \mathrm{ind}_B^{G_1} \chi_f$. Taking into account the existence of a coexponential basis and the proof of Proposition 5.3.18(2), we know that it is possible to choose a linear subspace $\mathfrak{t}$ complementary to $\mathfrak{g}_1$ in $\mathfrak{g}$ in such a fashion that $G = TG_1, T = \exp \mathfrak{t}$. Hereafter, we will do case-by-case studies.

(1) First of all we treat the case where $\mathfrak{k}$ is coirreducible, i.e. $\mathfrak{g}/\mathfrak{k}$ is irreducible as an $\mathfrak{k}$-module. Under this condition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$ or $\mathfrak{a} \subset \mathfrak{k}$.

 (a) Assume $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$. Clearly $G = KG_1$ and by Mackey's subgroup theorem [55]

$$\pi|_K \simeq \mathrm{ind}_{K_1}^K (\pi_1|_{K_1})$$

 with $K_1 = K \cap G_1$. We write the Lie algebra of K_1 as $\mathfrak{k}_1$. In this case $\pi_1|_A$ is a multiple of $\chi_f|_A$ and $G_1 = K_1 A$ so that $\pi_0 = \pi_1|_{K_1}$ is irreducible and corresponds to the coadjoint orbit of K_1 passing $f|_{\mathfrak{k}_1} \in \mathfrak{k}_1^*$. From the decomposition theorem 8.1.10 of induced representations,

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m(\sigma)\sigma d\mu_K^{\pi_0}(\sigma) \tag{8.2.2}$$

 holds. Here we recall the formula which gives the multiplicity $m(\sigma)$. If we denote by $p(\mathfrak{k}, \mathfrak{k}_1)$ the restriction map from $\mathfrak{k}^*$ to $\mathfrak{k}_1^*$ and put $\Xi = p(\mathfrak{k}, \mathfrak{k}_1)^{-1}(\Omega_{K_1}(\pi_0)) \subset \mathfrak{k}^*$, $m(\sigma)$ is obtained as the number $n_\sigma(\pi_0)$ of the K_1-orbits contained in $\Omega_K(\sigma) \cap \Xi$.

 Let us interpret formula (8.2.2). Taking $\mathfrak{t}$ in $\mathfrak{k}$, $\mathfrak{k} = \mathfrak{t} + \mathfrak{k}_1$ and $K = TK_1$. Then

$$\Omega(\pi) = T\cdot\left(K_1\cdot f + \mathfrak{g}_1^{\perp}\right),$$

$$p(\mathfrak{g}, \mathfrak{k})\,(\Omega(\pi)) = T\cdot\left(K_1\cdot(f|_{\mathfrak{k}_1}) + \mathfrak{k}_1^{\perp}\right) = T\cdot\Xi.$$

 Hence the quotient space $p(\mathfrak{g}, \mathfrak{k})\,(\Omega(\pi))\,/K$ is identified with Ξ/K, the measure $\mu_K^{\pi_0}$ is qualified as ν_K^{π}. As for the multiplicity, fixing arbitrarily the value on $\mathfrak{a}$, we know $n_\sigma(\pi_0) = n_\pi(\sigma)$ for σ belonging to the support of $\mu_K^{\pi_0}$.

(b) Next assume $\mathfrak{a} \subset \mathfrak{k}$ and divide this case into two subcases.

(i) If $\mathfrak{k} \not\subset \mathfrak{g}_1 = \mathfrak{a}^f$ for any choice of $f \in \Omega(\pi)$, the situation is similar to that examined above. As before, we can take $\mathfrak{t}$ in $\mathfrak{k}$ so that $G = KG_1$. Using Mackey's subgroup theorem

$$\pi|_K \simeq \mathrm{ind}_{K_1}^K (\pi_1|_{K_1})$$

with $K_1 = \exp(\mathfrak{k}_1) = K \cap G_1$. In general $\pi_1|_{K_1}$ is no longer irreducible. By the induction hypothesis

$$\pi_1|_{K_1} \simeq \int_{\widehat{K_1}}^{\oplus} n_{\pi_1}(\rho)\rho d\nu_{K_1}^{\pi_1}(\rho).$$

Therefore

$$\pi|_K \simeq \mathrm{ind}_{K_1}^K \int_{\widehat{K_1}}^{\oplus} n_{\pi_1}(\rho)\rho d\nu_{K_1}^{\pi_1}(\rho) \simeq \int_{\widehat{K_1}}^{\oplus} n_{\pi_1}(\rho) \left(\mathrm{ind}_{K_1}^K \rho\right) d\nu_{K_1}^{\pi_1}(\rho). \tag{8.2.3}$$

Let us examine formula (8.2.3) in more detail. Since $\rho|_A$ is a multiple of $\chi_f|_A$ and K_1 coincides with the stabilizer of $\chi_f|_A$ in K, Theorem 3.4.4 in the third chapter tells us that all $\sigma = \mathrm{ind}_{K_1}^K \rho$ are irreducible and non-equivalent to each other. Putting $f_1 = f|_{\mathfrak{g}_1} \in \mathfrak{g}_1^*$ we embed $\mathfrak{g}_1^*$ as the subspace $\{\lambda \in \mathfrak{g}^*; \lambda|_{\mathfrak{t}} = 0\}$ of $\mathfrak{g}^*$. Let us designate the coadjoint action of G (resp. G_1) by $\underset{G}{\cdot}$ (resp. $\underset{G_1}{\cdot}$). Then

$$\Omega(\pi) = G \cdot f = T \cdot \left(G_1 \underset{G_1}{\cdot} f_1 + \mathfrak{g}_1^{\perp} \right) = T \cdot \left(A \underset{G}{\cdot} G_1 \underset{G_1}{\cdot} f_1 \right)$$

and the quotient space

$$p(\mathfrak{g}, \mathfrak{k}) \left(\Omega_G(\pi) \right) / K$$

is identified with

$$p(\mathfrak{g}_1, \mathfrak{k}_1) \left(\Omega_{G_1}(\pi_1) \right) / K_1.$$

In these circumstances the action of T varies the value on $\mathfrak{a}$ and we find the measure $\nu_{K_1}^{\pi_1}$ qualified as ν_K^{π}. Moreover, paying attention to the intersection of each orbit with the affine subspace

$$p(\mathfrak{k}, \mathfrak{a})^{-1} (f|_{\mathfrak{a}}) = (f|_{\mathfrak{k}}) + \mathfrak{a}^{\perp} \subset \mathfrak{k}^*,$$

we recognize that the multiplicity $n_{\pi_1}(\rho)$ is equal to $n_{\pi}(\sigma)$.

(ii) Let us pass to the case where $\mathfrak{k} \subset \mathfrak{g}_1 = \mathfrak{a}^f$ for some $f \in \Omega(\pi)$. We denote by $\mathfrak{g}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and put $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$. By our assumption $f|_{\mathfrak{a}} \neq 0$ and $\mathfrak{k}$ is not contained in the ideal $\mathfrak{g}_0$. So, referring to the structural analysis concerning the minimal non-central ideal on the way of the proof of Theorem 5.3.8 in Chap. 5, this case occurs only when $\mathfrak{g}_0$ is a proper Lie subalgebra of $\mathfrak{g}_1$ and

$$\dim(\mathfrak{k}/\mathfrak{k}_0) = \dim(\mathfrak{a} \cap \mathfrak{z}) = \dim \mathfrak{z} = 1.$$

Put $K_0 = \exp(\mathfrak{k}_0)$ and notice that the theorem holds for $\pi|_{K_0}$ because $\mathfrak{k}_0$ is contained in a proper ideal of $\mathfrak{g}$.

Now set the desired irreducible decomposition as

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m'(\sigma)\sigma d\nu'(\sigma). \tag{8.2.4}$$

Restricting to K_0 the both sides of this formula

$$\pi|_{K_0} \simeq \int_{\hat{K}}^{\oplus} m'(\sigma)\,(\sigma|_{K_0})\,d\nu'(\sigma). \tag{8.2.5}$$

But this ought to give the expected formula for $\pi|_{K_0}$. Applying Proposition 8.2.3 to the pair $(\mathfrak{k}, \mathfrak{k}_0)$, we see that the measure ν' is equivalent to ν_K^{π}. If an element ξ of $\Lambda = p(\mathfrak{g}, \mathfrak{k})\,(\Omega(\pi)) \subset \mathfrak{k}^*$ satisfies $\xi|_{\mathfrak{a}} \neq f|_{\mathfrak{a}}$, then $A \cdot \xi = \xi + \mathfrak{k}_0^{\perp}$. So, case (2) of Proposition 8.2.3 occurs only when $\Omega_K(\sigma)$ is contained in the affine subspace $(f|_{\mathfrak{a}}) + \mathfrak{a}^{\perp} \subset \mathfrak{k}^*$. Considering

$$p(\mathfrak{k}, \mathfrak{a})(\Lambda) = \{\zeta \in \mathfrak{a}^*; \zeta|_{\mathfrak{z}} = f|_{\mathfrak{z}}\},$$

case (1) of Proposition 8.2.3 occurs almost everywhere in formula (8.2.5).

Setting $\sigma|_{K_0} \simeq \int_{\mathbb{R}}^{\oplus} \sigma_t dt$, formula (8.2.5) becomes

$$\int_{\widehat{K_0}}^{\oplus} n_{\pi_0}(\rho)\rho d\nu_{K_0}^{\pi_0}(\rho) \simeq \int_{\hat{K}}^{\oplus} m'(\sigma)d\nu'(\sigma)\int_{\mathbb{R}}^{\oplus} \sigma_t dt.$$

All these σ_t being non-equivalent to each other, we get $m'(\sigma) = n_{\pi_0}(\sigma_t)$ and this last quantity is equal to $n_\pi(\sigma)$. In this way we establish the claim on the multiplicity $m'(\sigma)$. Finally we understand that formula (8.2.4) has the desired form.

Now let us leave the assumption that $\mathfrak{k}$ is coirreducible.

(2) Here we prove the assertion on the measure. Let $\mathfrak{h}$ be a coirreducible proper Lie subalgebra containing $\mathfrak{k}$. Making an intervention of $H = \exp \mathfrak{h}$, we have from the arguments of (1) mentioned above and the induction hypothesis

$$\pi|_K \simeq (\pi|_H)|_K \simeq \left(\int_{\hat{H}}^{\oplus} n_\pi(\rho)\rho d\nu_H^\pi(\rho)\right)|_K \simeq \int_{\hat{H}}^{\oplus} n_\pi(\rho)(\rho|_K)d\nu_H^\pi(\rho)$$
$$\simeq \int_{\hat{H}}^{\oplus} n_\pi(\rho)d\nu_H^\pi(\rho)\int_{\hat{K}}^{\oplus} n_\rho(\sigma)\sigma d\nu_K^\rho(\sigma).$$

Decomposing the measure $p(\mathfrak{g},\mathfrak{h})_*(\tilde{\nu}_\pi)$ with respect to ν_H^π,

$$p(\mathfrak{g},\mathfrak{h})_*(\tilde{\nu}_\pi) = \int_{\hat{H}} \tilde{\nu}_H^\rho d\nu_H^\pi(\rho).$$

Since the equivalence class of the measure $\tilde{\nu}_\pi$ is invariant by the action of H, the fibre measures $\tilde{\nu}_H^\rho$ are almost everywhere equivalent to the H-invariant measure on $\Omega_H(\rho)$. By extending $\tilde{\nu}_H^\rho$ to $\mathfrak{h}^*$, we get the measure $\tilde{\nu}_\rho$.

For a Borel set $E \subset \hat{K}$,

$$\begin{aligned}
\nu_K^\pi(E) &= \tilde{\nu}_\pi\left(p(\mathfrak{g},\mathfrak{k})^{-1}\left(\hat{\rho}_K^{-1}(E)\right)\right)\\
&=\tilde{\nu}_\pi\left(p(\mathfrak{g},\mathfrak{h})^{-1}\left(p(\mathfrak{h},\mathfrak{k})^{-1}\left(\hat{\rho}_K^{-1}(E)\right)\right)\right)\\
&=\left(p(\mathfrak{g},\mathfrak{h})_*(\tilde{\nu}_\pi)\right)\left(p(\mathfrak{h},\mathfrak{k})^{-1}\left(\hat{\rho}_K^{-1}(E)\right)\right)\\
&=\int_{\hat{H}} \tilde{\nu}_H^\rho\left(\Omega_H(\rho)\cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\hat{\rho}_K^{-1}(E)\right)\right)d\nu_H^\pi(\rho)\\
&=\int_{\hat{H}}\left(p(\mathfrak{h},\mathfrak{k})_*(\tilde{\nu}_\rho)\right)\left(\hat{\rho}_K^{-1}(E)\right)d\nu_H^\pi(\rho) = \int_{\hat{H}} \nu_K^\rho(E)d\nu_H^\pi(\rho).
\end{aligned}$$

So, we obtain the result concerning the measure.

(3) Let us search for the multiplicity formula when $\Omega(\pi)$ is saturated with respect to a certain ideal $\mathfrak{g}_0$ of codimension 1 in $\mathfrak{g}$. Let $f \in \Omega(\pi)$, $f_0 = p(f)$, $p = p(\mathfrak{g},\mathfrak{g}_0)$, $G_0 = \exp(\mathfrak{g}_0)$ and π_0 the irreducible unitary representation of G_0 corresponding to the orbit $G_0{\cdot}f_0$. Thus $\pi \simeq \mathrm{ind}_{G_0}^G\pi_0$. Because we assume $\mathfrak{k} \not\subset \mathfrak{g}_0$, we confirm that $G = KG_0$. Putting $K_0 = K\cap G_0 = \exp(\mathfrak{k}_0)$, we utilize once again Mackey's subgroup theorem and apply the induction hypothesis to find

$$\pi|_K \simeq \mathrm{ind}_{K_0}^K(\pi_0|_{K_0}) \simeq \mathrm{ind}_{K_0}^K\int_{\widehat{K_0}}^{\oplus} n_{\pi_0}(\rho)\rho d\nu_{K_0}^{\pi_0}(\rho)$$
$$\simeq \int_{\widehat{K_0}}^{\oplus} n_{\pi_0}(\rho)\mathrm{ind}_{K_0}^K\rho d\nu_{K_0}^{\pi_0}(\rho). \tag{8.2.6}$$

By Proposition 8.1.5, either $\sigma = \mathrm{ind}_{K_0}^K\rho$ is irreducible or $\mathrm{ind}_{K_0}^K\rho \simeq \int_{\mathbb{R}}^{\oplus}\sigma_t dt$. In the latter case, we easily verify that the multiplicity of σ_t in formula (8.2.6) coincides with $n_\pi(\sigma_t)$. Let us compute the multiplicity of $\sigma \in \hat{K}$ in the former case. For $\sigma \in \hat{K}$ contained in the support of ν_K^π, we denote by $m(\sigma)$ the number of the K_0-orbits contained in $G_0{\cdot}f \cap p(\mathfrak{g},\mathfrak{k})^{-1}(\Omega_K(\sigma))$. As $G{\cdot}f = K{\cdot}G_0{\cdot}f$, $n_\pi(\sigma)$ is nothing but $m(\sigma)$. Replacing f if necessary, we consider the situation where $f|_{\mathfrak{k}} \in \Omega_K(\sigma)$. For all $\lambda \in \mathfrak{g}^*$ such that

$$p(\lambda) \in \Gamma(\pi_0,\sigma) = \Omega_{G_0}(\pi_0) \cap p(\mathfrak{g}_0,\mathfrak{k})^{-1}(\Omega_K(\sigma)),$$

if $p^{-1}p(\lambda) \subset K{\cdot}\lambda$, $m(\sigma)$ is equal to the sum of $n_{\pi_0}(\rho)$ in formula (8.2.6) regarding ρ such that $\sigma \simeq \operatorname{ind}_{K_0}^K \rho$. Provided $p^{-1}(f_0) \not\subset K{\cdot}f$, clearly $n_\pi(\sigma) = +\infty$. Now we assume $\sigma \simeq \operatorname{ind}_{K_0}^K \rho$. We have

$$\dim\left(G{\cdot}f_0 \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(\Omega_{K_0}(\rho))\right) = \dim(G{\cdot}f_0) - \dim(K_0{\cdot}f) + \dim(\Omega_{K_0}(\rho)),$$
$$\dim(\Gamma(\pi_0,\rho)) = \dim(G_0{\cdot}f_0) - \dim(K_0{\cdot}f_0) + \dim(\Omega_{K_0}(\rho)),$$

while

$$\dim(K_0{\cdot}f) = \dim(K_0{\cdot}f_0),\ \dim(G{\cdot}f_0) = \dim(G_0{\cdot}f_0) + 1$$

by our assumption, and

$$\dim(\Gamma(\pi_0,\rho)) < \dim\left(G{\cdot}f_0 \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(\Omega_{K_0}(\rho))\right). \tag{8.2.7}$$

Set $\mathfrak{k} = \mathbb{R}X + \mathfrak{k}_0$ and $k(s) = \exp(sX)$ for any $s \in \mathbb{R}$. Even if we replace f by $k(s){\cdot}f$, the inequality (8.2.7) remains still valid. On the other hand,

$$G{\cdot}f_0 \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(\Omega_{K_0}(\rho)) = \bigcup_{s\in\mathbb{R}} \left((k(s){\cdot}\Omega_{G_0}(\pi_0)) \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(\Omega_{K_0}(\rho))\right).$$

In the sequel,

$$(k(s){\cdot}\Omega_{G_0}(\pi_0)) \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(\Omega_{K_0}(\rho)) \neq \emptyset,$$

That is, the set of all $s \in \mathbb{R}$ satisfying

$$\Omega_{G_0}(\pi_0) \cap p(\mathfrak{g}_0,\mathfrak{k}_0)^{-1}(k(s){\cdot}\Omega_{K_0}(\rho)) \neq \emptyset$$

is not a null set for the Lebesgue measure. This means that σ appears with multiplicity $+\infty$ in the irreducible decomposition of $\pi|_K$. Thus we procure the desired result in any case.

(4) We finally suppose that $\Omega(\pi)$ is non-saturated with respect to any ideal of codimension 1. Let us utilize again a coirreducible proper ideal $\mathfrak{h}$ containing $\mathfrak{k}$. As usual, put $p = p(\mathfrak{g},\mathfrak{h})$ and $H = \exp \mathfrak{h}$. Let us take up the former formula

$$\begin{aligned}\pi|_K = (\pi|_H)|_K &\simeq \int_{\hat{H}}^{\oplus} n_\pi(\rho)(\rho|_K)d\nu_H^\pi(\rho)\\ &\simeq \int_{\hat{H}}^{\oplus} n_\pi(\rho)d\nu_H^\pi(\rho)\int_{\hat{K}}^{\oplus} n_\rho(\sigma)\sigma d\nu_K^\rho(\sigma).\end{aligned} \tag{8.2.8}$$

If we have

$$H\left(f|_{\mathfrak{h}}\right)\cdot f \subset K\cdot f$$

for all $f \in \Omega(\pi)$ such that $f|_{\mathfrak{k}} \in \Omega_K(\sigma)$, we get the multiplicity formula in (8.2.8) by counting the K-orbits in $\Gamma(\pi, \sigma)$.

It remains only for us to check the following. Provided

$$\dim(K\cdot f) < \dim(\Gamma(\pi, \sigma))$$

for $f \in \Gamma(\pi, \sigma)$, the multiplicity $m(\sigma)$ of σ in formula (8.2.8) is $+\infty$. Here we examine the case where $\dim(\mathfrak{g}/\mathfrak{h}) = 2$. The case where $\dim(\mathfrak{g}/\mathfrak{h}) = 1$ can be treated in exactly the same way.

Choose a coexponential basis $\{X_1,\ X_2\}$ to $\mathfrak{h}$ in $\mathfrak{g}$. Namely, the mapping

$$\mathbb{R}^2 \times H \ni ((s,t),h) \mapsto g(s,t)h,\ g(s,t) = \exp(sX_1)\exp(tX_2),$$

is a diffeomorphism onto G. Put $f \in \Gamma(\pi, \sigma),\ f_{s,t} = g(s,t)\cdot f$ and $f^0_{s,t} = p(f_{s,t})$, then

$$p\left(\Omega(\pi)\right) = \bigcup_{(s,t)\in\mathbb{R}^2} H\cdot f^0_{s,t}.$$

Since $\mathfrak{g}_0 = \{X \in \mathfrak{g}; [X, \mathfrak{g}] \subset \mathfrak{h}\}$ is an ideal of codimension 1 and by assumption $\Omega(\pi)$ is non-saturated with respect to $\mathfrak{g}_0$, there are two H-invariant possibilities for $\lambda \in p\left(\Omega(\pi)\right)$:

$$p^{-1}(\lambda) \cap \Omega(\pi) = \{\lambda\} \text{ or } p^{-1}(\lambda) \cap \Omega(\pi) = p^{-1}(\lambda).$$

We designate by E (resp. F) the set of all $(s,t) \in \mathbb{R}^2$ such that $f^0_{s,t}$ verifies the first (resp. second) eventuality. Remark that E is an open set of $\mathbb{R}^2$.

Regarding $(s,t) \in E$, the condition $H\cdot f^0_{s,t} \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right) \neq \emptyset$ implies

$$\dim\left(K\cdot f^0_{s,t}\right) < \dim\left(\left(\bigcup_{(s,t)\in E} H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right). \tag{8.2.9}$$

If there exists $(s,t) \in E$ satisfying

$$\dim\left(K\cdot f^0_{s,t}\right) < \dim\left(\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}(\Omega_K(\sigma))\right),$$

it turns out that $n_\rho(\sigma)$ is infinite for $\rho \in \hat{H}$ corresponding to $H\cdot f^0_{s,t}$. Otherwise, by relation (8.2.9), the set of $(s,t) \in E$ satisfying

$$\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right) \neq \emptyset$$

is not a null set, and the desired result follows.

Therefore let us assume

$$\left(\bigcup_{(s,t)\in E} H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right) = \emptyset.$$

Then we know immediately that

$$\begin{aligned}\dim\left(p\left(\Omega(\pi)\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right) &= \dim(\Gamma(\pi,\sigma)) - 2\\ &= \dim(\Omega(\pi)) - \dim(K\cdot f_{s,t}) + \dim(\Omega_K(\sigma)) - 2,\end{aligned} \tag{8.2.10}$$

while

$$\dim\left(\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right) = \dim\left(H\cdot f^0_{s,t}\right) - \dim\left(K\cdot f^0_{s,t}\right) + \dim(\Omega_K(\sigma)) \tag{8.2.11}$$

and

$$\dim\left(K\cdot f_{s,t}\right) \leq \dim\left(K\cdot f^0_{s,t}\right) + 2. \tag{8.2.12}$$

Since $\dim\left(H\cdot f^0_{s,t}\right) = \dim(\Omega(\pi)) - 4$, the formulas (8.2.10)–(8.2.12) give:

$$\begin{aligned}&\dim\left(\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right)\\ &\leq \dim(\Omega(\pi)) - 4 - \left(\dim\left(K\cdot f_{s,t}\right) - 2\right) + \dim(\Omega_K(\sigma))\\ &= \dim(\Omega(\pi)) - \dim\left(K\cdot f_{s,t}\right) + \dim(\Omega_K(\sigma)) - 2\\ &= \dim\left(p\left(\Omega(\pi)\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right).\end{aligned} \tag{8.2.13}$$

If the equality holds in the inequality (8.2.13), it does in the inequality (8.2.12) too. In such a case,

$$\dim\left(K\cdot f^0_{s,t}\right) = \dim\left(K\cdot f_{s,t}\right) - 2 < \dim\left(\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)\right)$$

and from this $n_\rho(\sigma) = +\infty$ for ρ corresponding to the orbit $H\cdot f^0_{s,t}$. If the equality in the inequality (8.2.13) holds nowhere, the relation

$$p\left(\Omega(\pi)\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right) = \bigcup_{(s,t)\in F}\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right)$$

assures that the set of all $(s,t) \in \mathbb{R}^2$ verifying

$$\left(H\cdot f^0_{s,t}\right) \cap p(\mathfrak{h},\mathfrak{k})^{-1}\left(\Omega_K(\sigma)\right) \neq \emptyset$$

is not a null set for the Lebesgue measure. From this we immediately get $m(\sigma) = +\infty$ and finish the proof of the theorem. ■

We generally consider a closed subgroup K of a Lie group G. Let σ be an irreducible unitary representation of K and π that of G. We call the Frobenius reciprocity the phenomenon that the multiplicity of π in the irreducible decomposition of the induced representation $\mathrm{ind}_K^G\sigma$ is equal to the multiplicity of σ in the irreducible decomposition of the restriction $\pi|_K$. Of course, when these irreducible decompositions are given by direct integrals, there is no meaning in talking about the multiplicity of a separate irreducible representation. Therefore in such a case we say that the **Frobenius reciprocity** holds when the multiplicity functions of $\mathrm{ind}_K^G\sigma$ and $\pi|_K$, the former being a function of π with parameter σ and the latter being a function of σ with parameter π, have the same expression.

Corollary 8.2.5. *The Frobenius reciprocity holds in these circumstances.*

Let π_j $(j = 1, 2)$ be two irreducible unitary representations of G. The outer Kronecker product $\pi_1 \times \pi_2$ of π_1 and π_2 corresponds to the orbit

$$\Omega_{G_1\times G_2}(\pi_1 \times \pi_2) = (\Omega(\pi_1), \Omega(\pi_2)) \subset \mathfrak{g}^* \oplus \mathfrak{g}^*.$$

We identify G with the subgroup of $G\times G$ consisting of all diagonal elements. Since the **(inner) Kronecker product** $\pi_1 \otimes \pi_2$ of π_1 and π_2 is by definition the restriction of $\pi_1 \times \pi_2$ to G:

Corollary 8.2.6 ([34]). *Let $p : \mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathfrak{g}^*$ be the restriction map. Then*

$$\pi_1 \otimes \pi_2 \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\nu(\pi),$$

Here $\nu = (\hat{\rho}_G \circ p)_*(\tilde{\nu}_{\pi_1\times\pi_2})$ and the multiplicity $m(\pi)$ is obtained as the number of all G-orbits contained in $(\Omega(\pi_1), \Omega(\pi_2)) \cap p^{-1}\left(\Omega_G(\pi)\right)$.

Chapter 9
e-Central Elements

9.1 Fundamental Result of Corwin and Greenleaf

In order to practise a more detailed analysis of monomial representations, we suppose in this chapter that $G = \exp \mathfrak{g}$ is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let us introduce e-central elements due to Corwin and Greenleaf [17]. Let

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_k) = k\ (0 \leq k \leq n) \tag{9.1.1}$$

be a composition series of ideals of $\mathfrak{g}$. Let $\{X_j\}_{1\leq j\leq n}$ be a Malcev basis of $\mathfrak{g}$ according to this composition series, i.e. $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ $(1 \leq j \leq n)$ and $\{X_j^*\}_{1\leq j\leq n}$ its dual basis in $\mathfrak{g}^*$. We denote the coordinates of $\ell \in \mathfrak{g}^*$ by $(\ell_1, \ldots, \ell_n), \ell_j = \ell(X_j)$. Then $\mathfrak{g}_j^\perp = \langle X_{j+1}^*, \ldots, X_n^* \rangle_{\mathbb{R}} \subset \mathfrak{g}^*, \mathfrak{g}_j^* \cong \mathfrak{g}^*/\mathfrak{g}_j^\perp$ and the projection $p_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ intertwines the actions of G on $\mathfrak{g}^*$ and $\mathfrak{g}_j^*$. For $\ell \in \mathfrak{g}^*$, we define $e_j(\ell) = \dim\big(G\cdot p_j(\ell)\big), e(\ell) = (e_1(\ell), \ldots, e_n(\ell))$ and set $\mathcal{E} = \{e(\ell); \ell \in \mathfrak{g}^*\}$. We also recognize $e_j(\ell) = \dim(G_j\cdot\ell)$ with $G_j = \exp(\mathfrak{g}_j)$. In fact,

$$\begin{aligned}\dim\big(G\cdot p_j(\ell)\big) &= \dim\big(\mathfrak{g}/\mathfrak{g}_j^\ell\big) = \dim(\mathfrak{g}/\mathfrak{g}(\ell)) - \dim\big(\mathfrak{g}_j^\ell/\mathfrak{g}(\ell)\big)\\ &= \dim(\mathfrak{g}/\mathfrak{g}(\ell)) - \big(\dim(\mathfrak{g}/\mathfrak{g}(\ell)) - \dim\big(\mathfrak{g}_j/\mathfrak{g}_j(\ell)\big)\big) = \dim(G_j\cdot\ell).\end{aligned}$$

For $e \in \mathcal{E}$ we define the G-invariant layer $U_e = \{\ell \in \mathfrak{g}^*; e(\ell) = e\}$. With $e_0 = 0$ we define the set of jump indices $S(e) = \{1 \leq j \leq n; e_j = e_{j-1} + 1\}$ and that of non-jump indices $T(e) = \{1 \leq j \leq n; e_j = e_{j-1}\}$. $\mathcal{U}(\mathfrak{g})$ being the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, $A \in \mathcal{U}(\mathfrak{g})$ is called an e**-central element** if, with $\pi_\ell = \hat{\rho}_G(\ell)$, $\pi_\ell(A) = d\pi_\ell(A)$ is a scalar operator for any $\ell \in U_e$.

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Now we mention a fundamental result of Corwin and Greenleaf. There exists a Zariski open set $\mathcal{Z}$ of $\mathfrak{g}^*$ which satisfies the following. $\mathcal{Z} \cap U_e$ is a non-empty G-invariant set and for each $j \in T(e)$ there is $A_j \in \mathcal{U}(\mathfrak{g}_j)$ with the following properties.

(a) Each A_j is e-central on $\mathcal{Z} \cap U_e$, i.e. $\pi_\ell(A_j)$ is a scalar operator for $\ell \in \mathcal{Z} \cap U_e$. Furthermore, A_j has a form $A_j = P_j X_j + Q_j$ such that:

1. P_j is a polynomial of A_k $(k \in T(e), k < j)$, in particular $P_j \in \mathcal{U}(\mathfrak{g}_{j-1})$;
2. P_j is e-central on $\mathcal{Z} \cap U_e$;
3. $Q_j \in \mathcal{U}(\mathfrak{g}_{j-1})$, in particular $P_1, Q_1 \in \mathbb{C}1$.

(b) $\pi_\ell(P_j) \neq 0$ at $\forall\, \ell \in \mathcal{Z} \cap U_e$.
(c) $\pi_\ell(A_j) = \varphi_j(\ell) Id$, where we can write $\varphi_j(\ell) = \tilde{p_j}(\ell')\ell_j + \tilde{q_j}(\ell')$ with two rational functions $\tilde{p_j}, \tilde{q_j}$ on $\mathcal{Z} \cap U_e$, which depend only on $\ell' = (\ell_1, \ldots, \ell_{j-1})$.
(d) $\tilde{p_j}(\ell')$ is G-invariant and $\tilde{p_j}(\ell') \neq 0$ for $\forall\, \ell \in \mathcal{Z} \cap U_e$.

Remark 9.1.1. We can repeat again the construction of these elements A_j on $U_e \backslash (\mathcal{Z} \cap U_e)$.

Let us return to a monomial representation $\tau = \mathrm{ind}_H^G \chi_f$, where $H = \exp \mathfrak{h}$ with $\mathfrak{h} \in S(f, \mathfrak{g})$, and consider the algebra $D_\tau(G/H)$ of G-invariant differential operators on the line bundle with base space G/H associated with data (H, χ_f). We fix a basis $\{Y_s\}_{1 \le s \le d}$ of $\mathfrak{h}$ and define the vector subspace

$$\mathfrak{a}_\tau = \sum_{s=1}^{d} \mathbb{C}\,(Y_s + i f(Y_s))$$

of $\mathcal{U}(\mathfrak{g})$. Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ spanned by $\mathfrak{a}_\tau$ and set

$$\mathcal{U}(\mathfrak{g}, \tau) = \{A \in \mathcal{U}(\mathfrak{g}); [A, Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau,\ \forall\, Y \in \mathfrak{h}\}.$$

The elements of $\mathcal{U}(\mathfrak{g})$ act as left G-invariant differential operators, i.e.

$$(R(X)\psi)\,(g) = \frac{d}{dt}\psi(g\exp(tX))|_{t=0}\ (\forall\, g \in G)$$

for $X \in \mathfrak{g}$ and $\psi \in C^\infty(G)$. Now the algebra $D_\tau(G/H)$ is the image of the map $R : \mathcal{U}(\mathfrak{g}, \tau) \ni A \mapsto R(A)$ and the kernel of this map is $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Thus $D_\tau(G/H)$ is isomorphic to

$$\mathcal{U}(\mathfrak{g}, \tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \cong (\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau)^H,$$

where the last expression represents the set of all H-invariant elements.

Corwin and Greenleaf [17] presented two conjectures about $D_\tau(G/H)$.

Commutativity Conjecture. The algebra $D_\tau(G/H)$ is commutative if and only if τ has finite multiplicities, i.e. $m(\pi) < \infty$ almost everywhere for μ in Theorem 8.1.9.

Polynomial Conjecture. When τ has finite multiplicities, the algebra $D_\tau(G/H)$ is isomorphic to the algebra $\mathbb{C}[\Gamma_\tau]^H$ of all H-invariant polynomial functions on Γ_τ.

Remark 9.1.2. The commutativity conjecture has been formerly conjectured by Duflo [22] in a more general context.

9.2 Monomial Representations and *e*-Central Elements

There uniquely exists $e \in \mathcal{E}$ such that $\Gamma_\tau \cap U_e$ is a non-empty Zariski open set of Γ_τ.

Theorem 9.2.1 ([36]). *Let $j \in T(e)$ and take the e-central element A_j. Then there exists a polynomial function $\varphi_j(\ell)$ on Γ_τ such that $\pi_\ell(A_j) = \varphi_j(\ell) Id$ at all $\ell \in \Gamma_\tau$.*

Indeed, following [36] let us prove the next theorem. To simplify the notations we merely write Γ instead of Γ_τ.

Theorem 9.2.2. *For an element σ of $\mathcal{U}(\mathfrak{g})$, the following statements are mutually equivalent:*

(1) *Regarding the ordinary topology there is a dense subset $\mathcal{M}$ of a certain non-empty open set in Γ such that $\pi_\ell(\sigma)$ is a scalar operator on $\mathcal{M}$. That is, there exists a complex-valued function $\varphi_0 : \ell \mapsto \varphi_0(\ell)$ on $\mathcal{M}$ so that $\pi_\ell(\sigma) = \varphi_0(\ell) Id$ holds at any ℓ in $\mathcal{M}$.*
(2) *$\pi_\ell(\sigma)$ is a scalar operator everywhere on $G{\cdot}\Gamma$. That is, there exists a complex-valued function $\varphi : \ell \mapsto \varphi(\ell)$ on $G{\cdot}\Gamma$ so that $\pi_\ell(\sigma) = \varphi(\ell) Id$ holds at any ℓ in $G{\cdot}\Gamma$.*

This function is a polynomial function on Γ and G-invariant on $G \cdot \Gamma$.

To prove the theorem let us utilize the theory [21, Chap. 6] on primitive ideals, i.e. kernels of algebraically irreducible representations, of $\mathcal{U}(\mathfrak{g})$ and elementary results (for instance, [47, Chap. 1]) of the algebraic geometry. A guideline of the proof is as follows. First we assume (1), then φ_0 will be extended to whole $\Gamma_{\mathbb{C}}$ as rational function. Since its graph is a connected closed set with Zariski topology, we shall conclude that it is a polynomial function. Some assertions in the following also hold for general Lie algebras or solvable Lie algebras, but we always assume that $\mathfrak{g}$ is nilpotent.

Let $\lambda \in \mathfrak{g}_{\mathbb{C}}^*$ and $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$. Namely, $\mathfrak{b}$ is a complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$, isotropic with respect to the bilinear form B_λ. Then, we denote by $\mathfrak{b}_\lambda$ the complex linear subspace of $\mathcal{U}(\mathfrak{g})$ generated by $\{Y - \lambda(Y)\}_{Y \in \mathfrak{b}}$. We set

$$M(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \lambda) = \mathcal{U}(\mathfrak{g}) / \left(\mathcal{U}(\mathfrak{g})\mathfrak{b}_\lambda\right)$$

and designate by $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})$ the representation of $\mathcal{U}(\mathfrak{g})$ induced naturally on the $\mathcal{U}(\mathfrak{g})$-module $M(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \lambda)$. Towards a proof of Theorem 9.2.2, we first prepare the necessary materials.

We give the natural structure of affine varieties to $\mathfrak{g}_{\mathbb{C}}^*$ and $\mathbb{C}$. For an integer r verifying $1 \leq r \leq n = \dim \mathfrak{g}$, we denote by $\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}, r)$ the Grassmann manifold formed by the r-dimensional complex linear subspaces of $\mathfrak{g}_{\mathbb{C}}$ and equip it with the structure of a natural projective algebraic variety. We designate by $\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$ their disjoint union $\sqcup_{1\leq r\leq n}\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}, r)$ regarded as an algebraic variety. We provide the direct product set $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \times \mathbb{C}$, $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$, $\mathfrak{g}_{\mathbb{C}}^* \times \mathbb{C}$ with the natural structure of the product algebraic variety and q_1, q_2, q_3 denote the following natural projections:

$$q_1 : \mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \to \mathfrak{g}_{\mathbb{C}}^*,$$
$$q_2 : \mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \times \mathbb{C} \to \mathfrak{g}_{\mathbb{C}}^* \times \mathbb{C},$$
$$q_3 : \mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \times \mathbb{C} \to \mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}).$$

Let $W \in \mathcal{U}(\mathfrak{g})$. We denote by $\tilde{F}_{W,r}$ the subset of $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}, r)$ composed of the pair $(\lambda, \mathfrak{b})$, where $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$ such that $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W)$ is a scalar operator. We write as $\tilde{F}_W$ the disjoint union $\sqcup_{1\leq r\leq n}\tilde{F}_{W,r}$ in $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$. Next we set $F_W = q_1(\tilde{F}_W)$ and $F_{W,r} = q_1(\tilde{F}_{W,r})$. These are subsets of $\mathfrak{g}_{\mathbb{C}}^*$, composed of λ where exists $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$, and of dimension r, such that $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W)$ is a scalar operator. Let us give the function $\tilde{\zeta}_W$ on $\tilde{F}_W$ by

$$\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W) = \tilde{\zeta}_W(\lambda, \mathfrak{b})Id.$$

Proposition 9.2.3. (1) *The subset $\tilde{F}_{W,r}$, $\tilde{F}_W$ of $\mathfrak{g}_{\mathbb{C}}^* \times Gr(\mathfrak{g}_{\mathbb{C}})$ is a Zariski closed set. Besides, the function $\tilde{\zeta}_W : \tilde{F}_W \to \mathbb{C}$ is a rational function.*
(2) *$F_{W,r}$ and $F_W = \sqcup_{1\leq r\leq n}F_{W,r}$ are Zariski closed sets of $\mathfrak{g}_{\mathbb{C}}^*$.*

Proof. (1) By the result of Conze and Duflo [21, Lemma 6.4.1], $\tilde{F}_{W,r}(1 \leq r \leq n)$ is a Zariski closed set of $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$ and the restriction of the function $\tilde{\zeta}_W$ to $\tilde{F}_{W,r}$ is a rational function.
(2) $\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}, r)$ is a projective variety and hence complete [47, Section 6.2]. Thus the projection q_1 is a closed mapping, and the assertion follows. ■

For $E \subset F_W \subset \mathfrak{g}_{\mathbb{C}}^*$, we set $\tilde{E} = q_1^{-1}(E) \cap \tilde{F}_W$. We indicate by $\tilde{E}^+$ the subset of $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \times \mathbb{C}$ composed of the triplet $(\lambda, \mathfrak{b}, \tilde{\zeta})$ satisfying the following three conditions:

(a) $\lambda \in E$;
(b) $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$;
(c) $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W) = \tilde{\zeta} Id$.

Set $E^+ = q_2(\tilde{E}^+)$. This is the subset of $\mathfrak{g}_{\mathbb{C}}^* \times \mathbb{C}$ composed of the pairs (λ, ζ) such that there exists $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$ verifying $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W) = \zeta Id$.

Proposition 9.2.4. (1) *If E is a Zariski closed set of $F_W \subset \mathfrak{g}_{\mathbb{C}}^*$, E^+ is also a Zariski closed set of $\mathfrak{g}_{\mathbb{C}}^* \times \mathbb{C}$.*
(2) *Suppose that E is an irreducible closed set of $\mathfrak{g}_{\mathbb{C}}^*$. Let r be the minimal integer such that $E \cap F_{W,r} \neq \emptyset$, then $E \subset F_{W,r}$.*
(3) *Suppose that E is an irreducible closed set of $\mathfrak{g}_{\mathbb{C}}^*$ and that the closed set E^+ of $\mathfrak{g}_{\mathbb{C}}^* \times \mathbb{C}$ is a graph of some complex-valued function on E. Then, E^+ is an irreducible closed set.*

Proof. (1) As the inverse image of some closed set, $\tilde{E}$ is a closed set of $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$ and in particular an algebraic subvariety. Namely, it satisfies the separation condition [47, Section 2.5, Examples (2), (3)] as $\mathfrak{g}_{\mathbb{C}}^*$ and $\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$, while, from (1) of the preceding proposition, the function $\tilde{\zeta}_W|_{\tilde{E}}$ is a morphism of varieties and $\tilde{E}^+$ is its graph in $\tilde{E} \times \mathbb{C}$. Since the varieties $\tilde{E}$, $\mathbb{C}$ satisfy the separation condition, $\tilde{E}^+$ is a closed set of $\tilde{E} \times \mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}}^* \times \mathrm{Gr}(\mathfrak{g}_{\mathbb{C}}) \times \mathbb{C}$ [47, Section 2.5, Proposition]. Since the variety $\mathrm{Gr}(\mathfrak{g}_{\mathbb{C}})$ is complete, the projection q_2 is a closed map and E^+ is a closed set.
(2) By the previous proposition each $F_{W,r}$ is a closed set and E is irreducible, so the assertion follows immediately.
(3) Let $\tilde{E} = \cup_{1 \le i \le q} \tilde{E}_i$ be the decomposition of $\tilde{E}$ into the irreducible components. Then, $E = \cup_{1 \le i \le q} q_1(\tilde{E}_i)$ and $\{q_1(\tilde{E}_i)\}_{1 \le i \le q}$ gives a closed covering of E. Since E is irreducible, one of the irreducible components $\tilde{E}_i$, for instance $\tilde{E}_1$, satisfies $q_1(\tilde{E}_1) = E$. Set $\tilde{E}_1^+ = q_3^{-1}(\tilde{E}_1) \cap \tilde{E}^+$. Since $\tilde{E}_1^+$ is the graph of the morphism $\tilde{\zeta}_W|_{\tilde{E}_1}$ from the variety $\tilde{E}_1$ to the variety $\mathbb{C}$, both satisfying the separation condition, the projection q_3 induces an isomorphism from $\tilde{E}_1^+$ onto $\tilde{E}_1$ [47, Section 2.5, Exercise 9]. Especially, as $\tilde{E}_1$, $\tilde{E}_1^+$ is an irreducible closed set. Since $q_1\left(q_3(\tilde{E}_1^+)\right) = E$, we get $q_2(\tilde{E}_1^+) = E^+$ using the fact that E^+ is a graph. In the sequel, E^+ is irreducible like $\tilde{E}_1^+$. ■

We denote by $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$ the set of all **primitive ideals** of $\mathcal{U}(\mathfrak{g})$. We mention the well-known orbit method only in the nilpotent case. Concerning the proofs, please refer to [21, Theorem 6.5.12 and Lemma 6.4.3]. Put $G_{\mathbb{C}} = \exp(\mathfrak{g}_{\mathbb{C}})$.

Theorem 9.2.5. *There exists a mapping $\lambda \mapsto I(\lambda)$ from $\mathfrak{g}_{\mathbb{C}}^*$ to $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$ which induces a bijection from $\mathfrak{g}_{\mathbb{C}}^*/Ad(G_{\mathbb{C}})$ to $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$, and the following hold:*

(1) *For any polarization $\mathfrak{p} \in M(\lambda, \mathfrak{g}_{\mathbb{C}}) \neq \emptyset$, $ker\rho(\lambda, \mathfrak{p}, \mathfrak{g}_{\mathbb{C}}) = I(\lambda)$.*
(2) *Provided $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$, $ker\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}}) \subset I(\lambda)$.*

We will show that the graph hypothesis in Proposition 9.2.4(3) is automatically satisfied. Let us begin with a simple lemma.

Lemma 9.2.6. *We fix $W \in \mathcal{U}(\mathfrak{g})$.*

(1) *When (ρ, V) is a representation of $\mathcal{U}(\mathfrak{g})$, the following two properties are equivalent:*

 (i) *$\rho(W)$ is a scalar operator ζId, $\zeta \in \mathbb{C}$;*
 (ii) *$W \in ker\rho + \mathbb{C}$. In this case, $W \equiv \zeta$ modulo $ker\rho$.*

(2) *Let (ρ', V') be another representation of $\mathcal{U}(\mathfrak{g})$ such that ker$\rho' \subset$ kerρ. Then, if $\rho'(W)$ is a scalar operator, $\rho(W)$ is also the scalar operator ζId.*

Proof. (1) This is clear from the following changes of expression:

$$\rho(W) = \zeta Id \iff \rho(W - \zeta) = 0 \iff W \equiv \zeta \bmod \ker\rho.$$

(2) Since $\ker\rho' \subset \ker\rho$ and $\mathbb{C} \cap \ker\rho = \{0\}$, $W \equiv \zeta \bmod \ker\rho'$ implies $W \equiv \zeta \bmod \ker\rho$. ■

Proposition 9.2.7. *We fix $W \in \mathcal{U}(\mathfrak{g})$. For $\lambda \in \mathfrak{g}_{\mathbb{C}}^*$, the next two properties are equivalent:*

(i) $W \in I(\lambda) \oplus \mathbb{C}$;
(ii) $\lambda \in F_W$. *In other words, there exists $\mathfrak{b} \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$ so that $\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})(W)$ is a scalar operator $\tilde{\zeta}_W(\lambda, \mathfrak{b})Id$.*

Then, the scalar value $\tilde{\zeta}_W(\lambda, \mathfrak{b})$ does not depend on the choice of $\mathfrak{b}$. Therefore, there is the function $\zeta_W : F_W \to \mathbb{C}$ such that $\tilde{\zeta}_W(\lambda, \mathfrak{b}) = \zeta_W(\lambda)$ and

$$W \equiv \zeta_W(\lambda) \ mod \ I(\lambda)$$

on F_W. Namely, for $E \subset F_W$, E^+ is the graph of $\zeta_W|_E$ in $E \times \mathbb{C}$.

Proof. We first assume (ii). From (1) of the last lemma, $W \in \ker\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}}) \oplus \mathbb{C}$. From Theorem 9.2.5(2), $\ker\rho(\lambda, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}}) \subset I(\lambda)$. So, using (2) of the last lemma, $W \equiv \tilde{\zeta}_W(\lambda, \mathfrak{b})$ modulo $I(\lambda)$ and $W \in I(\lambda) \oplus \mathbb{C}$.

Next, if we consider another $\mathfrak{b}' \in S(\lambda, \mathfrak{g}_{\mathbb{C}})$ such that $\rho(\lambda, \mathfrak{b}', \mathfrak{g}_{\mathbb{C}})(W)$ is a scalar operator $\tilde{\zeta}_W(\lambda, \mathfrak{b}')Id$,

$$W \equiv \tilde{\zeta}_W(\lambda, \mathfrak{b}) \bmod I(\lambda) \text{ and } W \equiv \tilde{\zeta}_W(\lambda, \mathfrak{b}') \bmod I(\lambda).$$

Since $\mathbb{C} \cap I(\lambda) = \{0\}$, this means $\tilde{\zeta}_W(\lambda, \mathfrak{b}) = \tilde{\zeta}_W(\lambda, \mathfrak{b}')$.

Conversely we assume (i). By Theorem 9.2.5(1), $\ker\rho(\lambda, \mathfrak{p}, \mathfrak{g}_{\mathbb{C}}) = I(\lambda)$ for any polarization $\mathfrak{p} \in M(\lambda, \mathfrak{g}_{\mathbb{C}})$ at λ. Hence, $\rho(\lambda, \mathfrak{p}, \mathfrak{g}_{\mathbb{C}})(W)$ is a scalar operator. ■

When $G = \exp \mathfrak{g}$ is nilpotent, let us clarify the relation between the kernels of irreducible unitary representations of G and the primitive ideals of $\mathcal{U}(\mathfrak{g})$.

Theorem 9.2.8 ([19]). *At every $\ell \in \mathfrak{g}^*$, ker$\pi_\ell = I(i\ell_{\mathbb{C}})$. Here, $\ell_{\mathbb{C}}$ denotes the element of $\mathfrak{g}_{\mathbb{C}}^*$ which extends ℓ complex linearly.*

Proof. This result itself is referred to in the preface of [21] and is well known. However, it seems rather difficult to read the reference [19] which preserves the proof. It is why we prove it anew. We choose a real polarization $\mathfrak{p}$ of $\mathfrak{g}$ at ℓ and, using a coexponential basis $\{X_k\}_{1 \le k \le p}$ to $\mathfrak{p}$ in $\mathfrak{g}$, realize π_ℓ in $L^2(\mathbb{R}^p)$. Then, as we saw in Corollary 6.2.14 in Chap. 6, the space of C^∞-vectors is just the space $\mathcal{S}(\mathbb{R}^p)$ of all rapidly decreasing functions of Schwartz and so its anti-dual space is the space $\mathcal{S}^*(\mathbb{R}^p)$ of the tempered distributions. We denote also by π_ℓ the actions of G and $\mathcal{U}(\mathfrak{g})$ extended to $\mathcal{S}^*(\mathbb{R}^p)$.

For $Y \in \mathfrak{p}$, let us consider the element $\tilde{Y} = Y - i\ell(Y)$ of $\mathcal{U}(\mathfrak{p})$. Let $\delta \in \mathcal{S}^*(\mathbb{R}^p)$ be the Dirac measure at the origin of $\mathbb{R}^p$. We first denote by $I(\delta)$ the left ideal of $\mathcal{U}(\mathfrak{g})$ composed by the annihilators of δ:

$$I(\delta) = \{U \in \mathcal{U}(\mathfrak{g}); \pi_\ell(U) \cdot \delta = 0\}.$$

Obviously, $\ker\pi_\ell$ is a two-sided ideal of $\mathcal{U}(\mathfrak{g})$ and contained in $I(\delta)$.

Put $\mathfrak{b} = \mathfrak{p}_{\mathbb{C}}$. By Theorem 9.2.5, the $\mathcal{U}(\mathfrak{g})$-module

$$M = M\,(i\ell_{\mathbb{C}}, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{g})/\,(\mathcal{U}(\mathfrak{g})\mathfrak{b}_{i\ell_{\mathbb{C}}})$$

is a simple module. Consider the representation $\rho = \rho(i\ell_{\mathbb{C}}, \mathfrak{b}, \mathfrak{g}_{\mathbb{C}})$ of $\mathcal{U}(\mathfrak{g})$ on M. When we denote by $I(1)$ the annihilator of 1_M in $\mathcal{U}(\mathfrak{g})$, $I(1)$ is a left ideal of $\mathcal{U}(\mathfrak{g})$ spanned by $\{\tilde{Y}\}_{Y \in \mathfrak{p}}$ and $I(i\ell_{\mathbb{C}})$ is a two-sided ideal of $\mathcal{U}(\mathfrak{g})$, which is maximal among those contained in $I(1)$.

From now on, we show the equality $I(\delta) = I(1)$. If this is shown, since $\ker\pi_\ell$ is a two-sided ideal, we have $\ker\pi_\ell \subset I(i\ell_{\mathbb{C}})$ at once. For this aim we take a basis $\{Y_1, \ldots, Y_{n-p}\}$ of $\mathfrak{p}$. Hence $\{Y_1, \ldots, Y_{n-p}, X_1, \ldots, X_p\}$ is a basis of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. By the Poincaré–Birkhoff–Witt theorem in the first chapter:

(a) $\left\{\tilde{Y}_{n-p}^{\alpha_{n-p}} \cdots \tilde{Y}_1^{\alpha_1}; (\alpha_1, \ldots, \alpha_{n-p}) \in \mathbb{N}^{n-p}\right\}$ is a basis of $\mathcal{U}(\mathfrak{p})$;
(b) $\left\{X_p^{\beta_p} \cdots X_1^{\beta_1} \tilde{Y}_{n-p}^{\alpha_{n-p}} \cdots \tilde{Y}_1^{\alpha_1}; (\alpha_1, \ldots, \alpha_{n-p}, \beta_1, \ldots, \beta_p) \in \mathbb{N}^n\right\}$ is a basis of $\mathcal{U}(\mathfrak{g})$.

Finally, from (a) and (b) the family

$$X_p^{\beta_p} \cdots X_1^{\beta_1} \tilde{Y}_{n-p}^{\alpha_{n-p}} \cdots \tilde{Y}_1^{\alpha_1}, \tag{9.2.1}$$

where

$$(\alpha_1, \ldots, \alpha_{n-p}, \beta_1, \ldots, \beta_p) \in \mathbb{N}^n, \alpha_1 + \cdots + \alpha_{n-p} \neq 0,$$

turns out to be a basis of $I(1)$.

We immediately get

$$\pi_\ell(Y)\delta = i\ell(Y)\delta, \quad Y \in \mathfrak{p},$$

$$\pi_\ell(X_k)\delta = -\frac{\partial}{\partial x_k}\delta, \quad 1 \le k \le p,$$

and

$$\pi_\ell\left(X_p^{\beta_p} \cdots X_1^{\beta_1} \tilde{Y}_{n-p}^{\alpha_{n-p}} \cdots \tilde{Y}_1^{\alpha_1}\right)\delta$$

$$= \begin{cases} 0, & \alpha_1 + \cdots + \alpha_{n-p} \neq 0 \\ \left(-\frac{\partial}{\partial x_p}\right)^{\beta_p} \cdots \left(-\frac{\partial}{\partial x_1}\right)^{\beta_1}\delta, & \alpha_1 = \cdots = \alpha_{n-p} = 0. \end{cases}$$

Now the family $\left\{\left(-\frac{\partial}{\partial x_p}\right)^{\beta_p}\cdots\left(-\frac{\partial}{\partial x_1}\right)^{\beta_1}\delta; (\beta_1,\cdots,\beta_p)\in\mathbb{N}^p\right\}$ forms in $\mathbb{R}^p$ a basis of the space of the distributions supported at the origin, so the family (9.2.1) forms a basis of $I(\delta)$. Thus $I(\delta) = I(1)$ and $\ker\pi_\ell \subset I(i\ell_{\mathbb{C}})$.

In order to show the inverse inclusion $I(i\ell_{\mathbb{C}}) \subset \ker\pi_\ell$, let us make use of the next two classical facts:

(a) since $I(i\ell_{\mathbb{C}})$ is a two-sided ideal, it is invariant by the adjoint representation of G;
(b) the linear subspace spanned by $\pi_\ell(G)\cdot\delta$ is dense in $\mathcal{S}^*(\mathbb{R}^p)$. In this case, δ is called a cyclic vector of the representation π_ℓ.

In particular from (a),

$$\pi_\ell(I(i\ell_{\mathbb{C}}))\cdot\pi_\ell(G)\cdot\delta = \{0\}.$$

Since the action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{S}^*(\mathbb{R}^p)$ by the representation π_ℓ is continuous,

$$\pi_\ell(I(i\ell_{\mathbb{C}}))\cdot\mathcal{S}^*(\mathbb{R}^p) = \{0\}$$

by (b). Thus $I(i\ell_{\mathbb{C}}) \subset \ker\pi_\ell$, and the theorem is proved. ■

Proposition 9.2.9. *Let $W \in \mathcal{U}(\mathfrak{g})$ and $\ell \in \mathfrak{g}^*$. In order that $\pi_\ell(W)$ is a scalar operator, it is necessary and sufficient that $i\ell_{\mathbb{C}} \in F_W$. In this case, $\pi_\ell(W) = \zeta_W(i\ell_{\mathbb{C}})Id$.*

Proof. From Proposition 9.2.7 and Theorem 9.2.8, $W \equiv \zeta_W(i\ell_{\mathbb{C}})$ modulo $\ker\pi_\ell$. Next by Lemma 9.2.6(1), $\pi_\ell(W) = \zeta_W(i\ell_{\mathbb{C}})Id$. ■

Proof of Theorem 9.2.2. Now we are ready to prove Theorem 9.2.2. It is enough to see that in the theorem assumption (1) leads to assumption (2). We set

$$\Gamma_{\mathbb{C}} = \{\lambda \in \mathfrak{g}^*_{\mathbb{C}}; (\lambda - if)|_{\mathfrak{h}} = 0\},$$

choose $W = \sigma$, $E = \Gamma_{\mathbb{C}}$ and put $\tilde{F} = \tilde{F}_\sigma$, $F = F_\sigma$, $\tilde{\zeta} = \tilde{\zeta}_\sigma$, $\zeta = \zeta_\sigma$. First we show the following.

Lemma 9.2.10. $\Gamma_{\mathbb{C}} \subset F$.

Proof. As we saw already, $F \cap \Gamma_{\mathbb{C}}$ is a Zariski closed set of $\Gamma_{\mathbb{C}}$ and consists of the common zero points of an ideal of polynomial functions on $\Gamma_{\mathbb{C}}$. Therefore, it suffices to verify that all elements P of this ideal vanish on whole $\Gamma_{\mathbb{C}}$. From the assumption and Proposition 9.2.9,

$$\ell \in \mathcal{M} \Rightarrow i\ell_{\mathbb{C}} \in F \cap \Gamma_{\mathbb{C}} \Rightarrow P(i\ell_{\mathbb{C}}) = 0.$$

Hence by continuity $P(i\ell_{\mathbb{C}}) = 0$ for all elements ℓ in the closure $\bar{\mathcal{M}}$ of $\mathcal{M}$. Since the interior of $\bar{\mathcal{M}}$ is not an empty set, $P(\ell) = 0$ at all points ℓ of some open set of $i\Gamma$ for the ordinary topology. Hence $P(\Gamma_{\mathbb{C}}) = \{0\}$. ∎

Let us show that $\zeta|_{\Gamma_{\mathbb{C}}}$ is a polynomial function on $\Gamma_{\mathbb{C}}$. Replacing E at Proposition 9.2.4(1), (3) by the irreducible closed set $\Gamma_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}^*$:

Lemma 9.2.11. *The graph $\Gamma_{\mathbb{C}}^+$ of $\zeta|_{\Gamma_{\mathbb{C}}}$ is an irreducible Zariski closed set of $\Gamma_{\mathbb{C}} \times \mathbb{C}$.*

The fact that $\zeta|_{\Gamma_{\mathbb{C}}}$ is a polynomial function is derived from the following lemma.

Lemma 9.2.12. *Let α be a function from $\mathbb{C}^p$ to $\mathbb{C}$ such that its graph Y in $\mathbb{C}^p \times \mathbb{C}$ is a connected closed set with respect to the Zariski topology. Then, α is a polynomial function on $\mathbb{C}^p$.*

Remark 9.2.13. We need here the assumption of the connectedness. In fact, let $p = 1$ for instance and consider the non-connected Zariski closed set

$$\{(z_1, z); z_1(z_1 z - 1) = 0 \text{ and } z(z_1 z - 1) = 0\}$$

of $\mathbb{C} \times \mathbb{C}$, this is a graph of the function α from $\mathbb{C}$ to $\mathbb{C}$ given by

$$\alpha(z_1) = \frac{1}{z_1} \ (z_1 \neq 0) \text{ and } \alpha(0) = 0.$$

Nevertheless, α is not a polynomial function.

Proof of the Lemma. To simplify the notations, the polynomial ring

$$\mathbb{C}[X_1, \dots, X_p, X] \ \ (\text{resp. } \mathbb{C}[X_1, \dots, X_p])$$

is denoted by $\mathcal{A}$ (resp. $\mathcal{A}_0$). We denote by

$$Z = (z_1, \dots, z_p, z) \ \ (\text{resp. } Z_0 = (z_1, \dots, z_p))$$

a variable of $\mathbb{C}^p \times \mathbb{C}$ (resp. $\mathbb{C}^p$) and, for $P \in \mathcal{A}$, by $d_X^\circ P$ the degree of P regarding the last variable X.

For $P \in \mathcal{A}$ (resp. $P \in \mathcal{A}_0$), we designate by $\mathcal{Z}(P)$ (resp. $\mathcal{Z}_0(P)$) the variety formed by all zero points of P in $\mathbb{C}^p \times \mathbb{C}$ (resp. $\mathbb{C}^p$). Let $\mathbb{C}(X_1, \dots, X_p)$ be the field of the rational functions of $\mathbb{C}[X_1, \dots, X_p]$ and $\mathcal{P} = \mathbb{C}(X_1, \dots, X_p)[X]$ the polynomial ring of one variable X with coefficients in $\mathbb{C}(X_1, \dots, X_p)$.

Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ composed of all polynomials which vanish on Y. Then let us see that all elements of $\mathcal{I}$ are divisible by a certain polynomial T_1 of $\mathcal{A}$. Let $\mathcal{I}'$ be the ideal generated by $\mathcal{I}$ in $\mathcal{P}$. The elements of $\mathcal{I}'$ are those written in the form $\frac{P}{A}$ with relatively prime $P \in \mathcal{I}$ and $A \in \mathcal{A}_0$. Since Y is a graph, $d_X^\circ P \geq 1$ if $P \in \mathcal{I} \setminus \{0\}$. Since $\mathcal{P}$ is a principal ideal ring, there exists a certain element of this ideal so that every element of $\mathcal{I}'$ is a multiple of this element. This element is written in the form $\frac{AT_1}{B}$ with $A, B \in \mathcal{A}_0$ and $T_1 \in \mathcal{A}$ such that $d_X^\circ T_1 \geq 1$. Here, we may assume that T_1 is not divisible in $\mathcal{A}_0$ and A, B are relatively prime. Besides, with $q > 0$, using $a_q^1 \in \mathcal{A}_0 \setminus \{0\}$ and $a_{q-1}^1, \ldots, a_0^1 \in \mathcal{A}_0$, we can write

$$T_1 = a_q^1 X^q + \cdots + a_0^1.$$

In particular, any element P of $\mathcal{I}$ is written in the form $P = \frac{P_1}{C} \cdot \frac{AT_1}{B}$. Here, P_1 and C are respectively in $\mathcal{A}$ and $\mathcal{A}_0$. Since $\mathcal{A}$ and $\mathcal{A}_0$ are unique factorization rings, B divides P_1 and C divides A. Finally, every element of $\mathcal{I}$ is a multiple of T_1. That is, for arbitrary $P \in \mathcal{I}$, there is $Q \in \mathcal{A}$ so that $P = QT_1$.

Therefore, $\mathcal{Z}(T_1) \subset Y$ and at arbitrary $Z_0 \in \mathbb{C}^p$ the zero point of the function

$$z \mapsto a_q^1(Z_0)z^q + \cdots + a_0^1(Z_0)$$

on $\mathbb{C}$ is at most one. Moreover, it has just one zero point at $Z_0 \notin \mathcal{Z}_0(a_q^1)$. This zero point is also the zero point of the function

$$z \mapsto qa_q^1(Z_0)z + a_{q-1}^1(Z_0)$$

which is obtained by differentiating T_1 for $q - 1$ times. Put $T_0 = qa_q^1 X + a_{q-1}^1$ and use the fact that $\mathcal{A}$ is a unique factorization ring to write down

$$T_0 = bT,\ b \in \mathcal{A}_0,\ T = a_1 X + a_0 \in \mathcal{A} \setminus \mathcal{A}_0$$

with an irreducible T. Since $\mathcal{Z}(b) \subset \mathcal{Z}(a_q^1)$, $\mathcal{Z}(T) \setminus \mathcal{Z}(a_q^1) \subset Y$. Besides, a_q^1 is not divisible by T. By Hilbert's zero point theorem, every element of $\mathcal{I}$ is a multiple of T which is a polynomial of X with degree one.

Next let us show by the absurdity $\mathcal{Z}_0(a_1) \cap \mathcal{Z}_0(a_0) = \emptyset$. Indeed, if there exists $Z_0^* \in \mathcal{Z}_0(a_1) \cap \mathcal{Z}_0(a_0)$, $(Z_0^*, \mathbb{C})$ is contained in the set of zero points of T and hence of all elements of $\mathcal{I}$, what contradicts the fact that Y is a graph. Hence $\mathcal{Z}(T) \subset (\mathbb{C}^p \times \mathbb{C}) \setminus \mathcal{Z}(a_1)$. From $\mathcal{Z}(T) \subset Y$, $\mathcal{Z}(T) = Y \setminus (Y \cap \mathcal{Z}(a_1))$ and since

$$Y = (Y \cap \mathcal{Z}(a_1)) \cup (Y \setminus (Y \cap \mathcal{Z}(a_1)),$$

we get

$$\mathcal{Z}(T) \cup (Y \cap \mathcal{Z}(a_1)) = Y,\ \ \mathcal{Z}(T) \cap (Y \cap \mathcal{Z}(a_1)) = \emptyset. \tag{9.2.2}$$

This result is obtained whenever the graph Y is any Zariski closed set. Here we use the assumption that Y is connected and let us see by absurdity that the element a_1 of $\mathcal{A}_0 \setminus \{0\}$ is in fact an element of $\mathbb{C}^*$. Provided $\mathcal{Z}(a_1) \neq \emptyset$, $Y \cap \mathcal{Z}(a_1) \neq \emptyset$ and from Eq. (9.2.2), Y becomes a disjoint union of two closed sets, which is absurd.

Summing up the above, $\mathcal{I}$ is the set of all multiples of the polynomial T with the form $X - a_0,\ a_0 \in \mathcal{A}_0$ and the lemma follows. ∎

Returning to the proof of the theorem, since $\varphi(\ell) = \zeta(i\ell_{\mathbb{C}})$ on Γ, φ becomes a polynomial function by the lemma. Thus the proof of Theorem 9.2.2 is complete. ■

Recall the sequence of ideals (9.1.1) of $\mathfrak{g}$. Set

$$\mathcal{I} = \{i_1 < i_2 < \cdots < i_d\} = \{1 \leq i \leq n; \mathfrak{h} \cap \mathfrak{g}_i \neq \mathfrak{h} \cap \mathfrak{g}_{i-1}\}$$

and $\mathcal{J} = \{j_1 < j_2 < \cdots < j_q\} = \{1, 2, \ldots, n\} \setminus \mathcal{I}$. Here $d = \dim \mathfrak{h}$ and $q = \dim(\mathfrak{g}/\mathfrak{h})$. On the one hand, setting $\mathfrak{k}_0 = \mathfrak{h}$, $\mathfrak{k}_r = \mathfrak{h} + \mathfrak{g}_{j_r}$ $(1 \leq r \leq q)$, we get the sequence of Lie subalgebras

$$\mathfrak{h} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{q-1} \subset \mathfrak{k}_q = \mathfrak{g}, \ \dim(\mathfrak{k}_r) = d + r. \tag{9.2.3}$$

On the other hand, setting $\mathfrak{h}_0 = \{0\}, \mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{g}_{i_s}$ $(1 \leq s \leq d)$, we procure a sequence of ideals of $\mathfrak{h}$:

$$\{0\} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_{d-1} \subset \mathfrak{h}_d = \mathfrak{h}, \ \dim(\mathfrak{h}_s) = s. \tag{9.2.4}$$

We choose a basis $\{Y_s\}_{1 \leq s \leq d}$ of $\mathfrak{h}$ in such a fashion that $Y_s \in \mathfrak{h}_s \setminus \mathfrak{h}_{s-1}$ $(1 \leq s \leq d)$. Next, we set

$$\mathfrak{a}_s = \sum_{j=1}^{s} \mathbb{C}(Y_j + if(Y_j))$$

with $1 \leq s \leq d$. We denote by $T(e_H)$ the set of indices $i_s \in \mathcal{I}$ such that $\mathfrak{h}_s \subset \mathfrak{h}_{s-1} + \mathfrak{g}(\ell)$ for almost all $\ell \in \Gamma_\tau$ with respect to $\tilde{\mu}$. As $T(e_H) \subset T(e)$, put $U(e) = T(e) \setminus T(e_H)$. By $\diamondsuit$ we indicate the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$. Let $i_s \in T(e_H)$ and $T(e) \cap \{1, 2, \ldots, i_s\} = \{m_1 < m_2 < \cdots < m_k = i_s\}$. For brevity, we write the Corwin–Greenleaf e-central element A_{m_j} $(1 \leq j \leq k)$ as σ_j. The next lemma will be a key lemma in Chap. 12 in order to prove the commutativity conjecture. To get the lemma we need much on e-central elements and this topic will be treated in Chap. 12.

Lemma 9.2.14. *$\diamondsuit(\sigma_k)$ is algebraic on $\{\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_{k-1})\}$ modulo $\mathcal{U}(\mathfrak{g}_{i_s})\mathfrak{a}_s$.*

Let us calculate in typical cases the Corwin–Greenleaf e-central elements and see what the lemma means. In these cases, those elements are nothing but elements in the centre of $\mathcal{U}(\mathfrak{g})$.

Example 9.2.15. We take up $\mathfrak{g} = \langle X_1, \ldots, X_n \rangle_{\mathbb{R}}$: $[X_n, X_k] = X_{k-1}$ $(2 \leq k \leq n-1)$. The centre of $\mathfrak{g}$ is $\mathfrak{z} = \mathbb{R}X_1$ and $\mathfrak{b} = \langle X_1, \ldots, X_{n-1} \rangle_{\mathbb{R}}$ is a commutative ideal of codimension 1 in $\mathfrak{g}$. When $n = 4$, we have in matrix form

$$X_1 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}.$$

Thus,

$$G = \left\{ \begin{pmatrix} 1 & w & w^2 & z \\ 0 & 1 & w & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \, w, x, y, z \in \mathbb{R} \right\}.$$

The generic coadjoint orbits of G are two-dimensional and their elements admit $\mathfrak{b}$ as a common polarization. Then, except for the case where $\mathfrak{h} = \mathfrak{z}$ and $f|\mathfrak{z} \neq 0$, our monomial representation $\tau = \mathrm{ind}_H^G \chi_f$ has finite multiplicities even if $\dim \mathfrak{h} = 1$. Setting $\mathfrak{g}_j = \langle X_1, \ldots, X_j \rangle_{\mathbb{R}}$ for $1 \leq j \leq n$, we get a Jordan–Hölder sequence of $\mathfrak{g}$:

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}.$$

Very simply, let us seek elements belonging to the centre of $\mathcal{U}(\mathfrak{g})$. For example, assuming $n \geq 5$,

$$\sigma_1 = X_1, \sigma_2 = 2X_1X_3 - X_2{}^2, \sigma_3 = X_1{}^2X_4 - X_1X_2X_3 + \frac{1}{3}X_2{}^3, \text{etc.}$$

First let $n = 5, \mathfrak{h} = \langle X_3, X_4 \rangle_{\mathbb{R}}$ and $f(X_3) = \lambda, f(X_4) = \kappa$. Then, τ has finite multiplicities and the algebra $D_\tau(G/H)$ coincides with the polynomial ring of X_1, X_2. Hence σ_3 does not contribute to the formation of $D_\tau(G/H)$. We set

$$\mathfrak{a}_\tau = \mathbb{C}(X_3 + if(X_3)) + \mathbb{C}(X_4 + if(X_4)) = \mathbb{C}(X_3 + i\lambda) + \mathbb{C}(X_4 + i\kappa)$$

and consider $\mathfrak{a}_1 = \mathbb{C}(X_3 + if(X_3)) = \mathbb{C}(X_3 + i\lambda)$. We have

$$\sigma_2 \equiv -2i\lambda\sigma_1 - X_2^2$$

modulo $\mathcal{U}(\mathfrak{g}_1)\mathfrak{a}_1$ and

$$\sigma_3 \equiv \sigma_1{}^2X_4 + \left(i\lambda\sigma_1 + \frac{1}{3}X_2{}^2\right) X_2$$

$$\equiv {\sigma_1}^2 X_4 + \left\{ i\lambda\sigma_1 - \frac{1}{3}(\sigma_2 + 2i\lambda\sigma_1) \right\} X_2$$
$$= {\sigma_1}^2 X_4 - \frac{1}{3}(\sigma_2 - i\lambda\sigma_1) X_2 \bmod \mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau$$

From this, if we take $\mathfrak{h}' = \mathbb{R}X_3$ and consider the induced representation $\tau' = \mathrm{ind}_{H'}^G \chi_f$ with $H' = \exp(\mathfrak{h}')$, σ_3 contributes substantially to form $D_{\tau'}(G/H')$. However,

$$9(\sigma_3 - \sigma_1^2 X_4)^2 + (\sigma_2 + 2i\lambda\sigma_1)(\sigma_2 - i\lambda\sigma_1)^2 \equiv 0 \bmod \mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau$$

and especially

$$9(\sigma_3 + i\kappa{\sigma_1}^2)^2 + (\sigma_2 + 2i\lambda\sigma_1)(\sigma_2 - i\lambda\sigma_1)^2 \equiv 0 \bmod \mathcal{U}(\mathfrak{g}_4)\mathfrak{a}_\tau.$$

We can continue likewise. Indeed, let $n \geq 6$. $\sigma_4 = 2X_1X_5 - 2X_2X_4 + {X_3}^2$ belongs to the centre of $\mathcal{U}(\mathfrak{g})$ and if we start from $\mathfrak{h} = \langle X_3, X_4, X_5 \rangle_{\mathbb{R}}$ with $f(X_5) = \zeta$,

$$\sigma_4 \equiv 2\sigma_1(-i\zeta) - 2X_2(-i\kappa) + (-i\lambda)^2 = -2i\zeta\sigma_1 + 2i\kappa X_2 - \lambda^2$$

modulo $\mathcal{U}(\mathfrak{g}_3)\mathfrak{a}_\tau$. Hence

$$(\sigma_4 + 2i\zeta\sigma_1 + \lambda^2)^2 \equiv -4\kappa^2 {X_2}^2 \equiv 4\kappa^2(\sigma_2 + 2i\lambda\sigma_1)$$

modulo $\mathcal{U}(\mathfrak{g}_5)\mathfrak{a}_\tau$.

Next let $n \geq 7$. If we take

$$\sigma_5 = {X_1}^4 X_6 - {X_1}^3 X_2 X_5 + \frac{1}{2}{X_1}^2{X_2}^2 X_4 - \frac{1}{6}X_1{X_2}^3 X_3 + \frac{1}{30}{X_2}^5,$$

this again belongs to the centre of $\mathcal{U}(\mathfrak{g})$. At generic $\ell \in \Gamma_\tau$, we take $\mathfrak{h} = \langle X_3, X_4, X_5, X_6 \rangle_{\mathbb{R}}$ in order that $\mathfrak{h} \cap \mathfrak{g}(\ell)$ becomes sufficiently large. Put $f(X_6) = \gamma$. Then,

$$\sigma_5 \equiv {\sigma_1}^4(-i\gamma) - {\sigma_1}^3 X_2(-i\zeta) + \frac{1}{2}{\sigma_1}^2{X_2}^2(-i\kappa) - \frac{1}{6}\sigma_1{X_2}^3(-i\lambda) + \frac{1}{30}{X_2}^5$$
$$= -i\gamma{\sigma_1}^4 + i\zeta{\sigma_1}^3 X_2 - \frac{i\kappa}{2}{\sigma_1}^2{X_2}^2 + \frac{i\lambda}{6}\sigma_1{X_2}^3 + \frac{1}{30}{X_2}^5$$

modulo $\mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau$. Hence

$$\sigma_5 \equiv -i\gamma{\sigma_1}^4 + i\zeta{\sigma_1}^3 X_2 + \frac{i\kappa}{2}{\sigma_1}^2(\sigma_2 + 2i\lambda\sigma_1)$$
$$- \frac{i\lambda}{6}\sigma_1(\sigma_2 + 2i\lambda\sigma_1)X_2 + \frac{1}{30}(\sigma_2 + 2i\lambda\sigma_1)^2 X_2$$

$$
\begin{aligned}
&= -i\gamma\sigma_1{}^4 - \lambda\kappa\sigma_1{}^3 + \frac{i\kappa}{2}\sigma_1{}^2\sigma_2 \\
&\quad + \frac{1}{30}(\sigma_2{}^2 + 4i\lambda\sigma_1\sigma_2 - 4\lambda^2\sigma_1{}^2 - 5i\lambda\sigma_1\sigma_2 + 10\lambda^2\sigma_1{}^2 + 30i\zeta\sigma_1{}^3)X_2 \\
&= -\frac{1}{2}(2i\gamma\sigma_1{}^4 + 2\lambda\kappa\sigma_1{}^3 - i\kappa\sigma_1{}^2\sigma_2) \\
&\quad + \frac{1}{30}(6\lambda^2\sigma_1{}^2 - i\lambda\sigma_1\sigma_2 + \sigma_2{}^2 + 30i\zeta\sigma_1{}^3)X_2
\end{aligned}
$$

modulo $\mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau$. In consequence,

$$
\begin{aligned}
&\left\{30\sigma_5 + 15(2i\gamma\sigma_1{}^4 + 2\lambda\kappa\sigma_1{}^3 - i\kappa\sigma_1{}^2\sigma_2)\right\}^2 \\
&\qquad + (6\lambda^2\sigma_1{}^2 - i\lambda\sigma_1\sigma_2 + \sigma_2{}^2 + 30i\zeta\sigma_1{}^3)^2(\sigma_2 + 2i\lambda\sigma_1) \equiv 0
\end{aligned}
$$

modulo $\mathcal{U}(\mathfrak{g}_6)\mathfrak{a}_\tau$. These all show that σ_1, σ_2, for example, generate algebraically the algebra of the Γ_τ-central elements.

Now take $\mathfrak{h} = \langle X_3, X_5\rangle_{\mathbb{R}}$ and put $f(X_3) = \lambda$, $f(X_5) = \zeta$ as before. Then,

$$
\sigma_4 \equiv 2\sigma_1(-i\zeta) - 2X_2X_4 + (-i\lambda)^2 = -2i\zeta\sigma_1 - 2X_2X_4 - \lambda^2
$$

modulo $\mathcal{U}(\mathfrak{g}_3)\mathfrak{a}_\tau$. Hence

$$
\begin{aligned}
\sigma_1{}^2\sigma_4 &\equiv -2i\zeta\sigma_1{}^3 - 2X_2\left\{\sigma_3 + \frac{1}{3}(\sigma_2 - i\lambda\sigma_1)X_2\right\} - \lambda^2\sigma_1{}^2 \\
&\equiv -2i\zeta\sigma_1{}^3 - 2X_2\sigma_3 + \frac{2}{3}(\sigma_2 - i\lambda\sigma_1)(\sigma_2 + 2i\lambda\sigma_1) - \lambda^2\sigma_1{}^2 \\
&= -2i\zeta\sigma_1{}^3 + \frac{1}{3}\lambda^2\sigma_1{}^2 + \frac{2i\lambda}{3}\sigma_1\sigma_2 + \frac{2}{3}\sigma_2{}^2 - 2X_2\sigma_3
\end{aligned}
$$

modulo $\mathcal{U}(\mathfrak{g}_3)\mathfrak{a}_\tau$. After all,

$$
(3\sigma_1{}^2\sigma_4 - 2\sigma_2{}^2 - 2i\lambda\sigma_1\sigma_2 + 6i\zeta\sigma_1{}^3 - \lambda^2\sigma_1{}^2)^2 + 36(\sigma_2 + 2i\lambda\sigma_1)\sigma_3{}^2 \equiv 0
$$

modulo $\mathcal{U}(\mathfrak{g}_5)\mathfrak{a}_\tau$. So, if we look at the case where $n = 6$, $D_\tau(G/H)$ is identified with the polynomial ring of three variables X_1, X_2, X_4.

Finally, let $\mathfrak{h} = \langle X_4, X_5\rangle_{\mathbb{R}}$ and put $f(X_4) = \kappa$, $f(X_5) = \zeta$ as before. This time

$$
\sigma_4 \equiv 2\sigma_1(-i\zeta) - 2X_2(-i\kappa) + X_3{}^2 = -2i\zeta\sigma_1 + 2i\kappa X_2 + X_3{}^2
$$

and

$$\begin{aligned} 4{\sigma_1}^2\sigma_4 &\equiv -8i\zeta{\sigma_1}^3 + 8i\kappa{\sigma_1}^2 X_2 + (\sigma_2 + {X_2}^2)^2 \\ &= -8i\zeta{\sigma_1}^3 + 8i\kappa{\sigma_1}^2 X_2 + {\sigma_2}^2 + 2\sigma_2{X_2}^2 + {X_2}^4 \end{aligned}$$

modulo $\mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau$, while

$$\sigma_3 \equiv {\sigma_1}^2(-i\kappa) - \frac{1}{3}\sigma_2 X_2 - \frac{\sigma_1}{6}(\sigma_2 + X_2^2) \bmod \mathcal{U}(\mathfrak{g}_2)\mathfrak{a}_\tau.$$

Combining these two relations together, we get modulo $\mathcal{U}(\mathfrak{g}_5)\mathfrak{a}_\tau$ an algebraic relation among $\sigma_1, \dots, \sigma_4$. In this case also, if $n = 6$, the algebra $D_\tau(G/H)$ becomes the polynomial ring of the three variables X_1, X_2, X_3.

We denote by $Z(\mathfrak{g}, \Gamma_\tau)$ the algebra composed by the function ζ on $G\cdot\Gamma_\tau$ such that there exists $W \in \mathcal{U}(\mathfrak{g})$ which satisfies $\pi_\ell(W) = \zeta(\ell)Id$ at any $\ell \in \Gamma_\tau$. Relating to Lemma 9.2.14, we will show the following theorem in Chap. 12:

Theorem 9.2.16 ([36]). $\{\varphi_j;\ j \in U(e)\}$ *forms a transcendental basis of* $Z(\mathfrak{g}, \Gamma_\tau)$.

Chapter 10
Frobenius Reciprocity

10.1 Frobenius Reciprocity in Distribution Version

Let G be a σ-compact Lie group with Lie algebra $\mathfrak{g}$, and we only consider unitary representations π whose Hilbert space $\mathcal{H}_\pi$ are separable. First we remember the C^∞-vectors. Let $v \in \mathcal{H}_\pi$. When the function $G \ni g \mapsto \pi(g)v \in \mathcal{H}_\pi$ is C^∞, v is called a C^∞-vector. We denote by $\mathcal{H}_\pi^\infty$ the space of the C^∞-vectors of π. $\{\psi_n\}_{n=1}^\infty$ being the approximate identity of $L^1(G)$ introduced in Proposition 2.2.8 and chosen in $\mathcal{D}(G)$, we see that $\|\pi(\psi_n)w - w\| \to 0$ $(n \to \infty)$ for any $w \in \mathcal{H}_\pi$. As $\pi(\psi_n)w \in \mathcal{H}_\pi^\infty$, $\mathcal{H}_\pi^\infty$ is a dense subspace of $\mathcal{H}_\pi$ and $\mathfrak{g}$ acts there by the differential representation $d\pi$ of π:

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} \ (X \in \mathfrak{g}, v \in \mathcal{H}_\pi^\infty).$$

The differential representation $d\pi$ is uniquely extended to a representation of $\mathcal{U}(\mathfrak{g})$. $\{X_1, \ldots, X_n\}$ being a basis of $\mathfrak{g}$, $\mathcal{H}_\pi^\infty$ becomes a Fréchet space with semi-norms

$$\rho_d(v) = \sum_{1 \le i_k \le n} \|d\pi(X_{i_1} \cdots X_{i_d})v\| \ (d \in \mathbb{N}).$$

We designate by $\mathcal{H}_\pi^{-\infty}$ the anti-dual space of $\mathcal{H}_\pi^\infty$, i.e. the space of the continuous anti-linear forms on $\mathcal{H}_\pi^\infty$ with values in $\mathbb{C}$. We call elements of $\mathcal{H}_\pi^{-\infty}$ generalized vectors of π. We equip $\mathcal{H}_\pi^{-\infty}$ with the strong dual topology of $\mathcal{H}_\pi^\infty$. Then, the anti-dual space of $\mathcal{H}_\pi^{-\infty}$ is identified with $\mathcal{H}_\pi^\infty$. For $a \in \mathcal{H}_\pi^{\pm\infty}$ and $b \in \mathcal{H}_\pi^{\mp\infty}$, we write $\langle a, b\rangle$ for the image of b by a. Hence $\langle a, b\rangle = \overline{\langle b, a\rangle}$. The actions of G and $\mathfrak{g}$ are continuously extended on $\mathcal{H}_\pi^{-\infty}$ by duality. Notice that

$$\pi(\varphi)\left(\mathcal{H}_\pi^{-\infty}\right) \subset \mathcal{H}_\pi^\infty$$

H. Fujiwara, J. Ludwig, *Harmonic Analysis on Exponential Solvable Lie Groups*, Springer Monographs in Mathematics, DOI 10.1007/978-4-431-55288-8_10

for $\varphi \in \mathcal{D}(G)$. When a closed subgroup K and its character $\chi : K \to \mathbb{C}^*$ are given, set

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{K,\chi} = \{a \in \mathcal{H}_\pi^{-\infty}; \pi(k)a = \chi(k)a,\ \forall\, k \in K\}.$$

Theorem 10.1.1 ([31, 46]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in I(f, \mathfrak{g})$. As before, we define the character χ_f of $H = \exp \mathfrak{h}$ by $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h})$ and consider $\tau = \mathrm{ind}_H^G \chi_f \in \hat{G}$. Then, for $\pi \in \hat{G}$,*

$$\dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = \begin{cases} 1, & \pi \simeq \tau, \\ 0, & \pi \not\simeq \tau. \end{cases}$$

Though we confirmed in Corollary 8.2.5 that Frobenius reciprocity holds, Theorem 10.1.1 also announces in a very special situation a kind of Frobenius reciprocity. Does this kind of reciprocity hold in the general situation?

Question. Does the following hold in the irreducible decomposition (8.1.1) of monomial representations:

$$m(\pi) = \dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

for μ-almost all $\pi \in \hat{G}$?

10.2 Realization of the Reciprocity in Nilpotent Case

Let us examine this problem in more detail for nilpotent Lie groups. Hereafter in this chapter, let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Penney [59] showed the inequality

$$m(\pi) \leq \dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

for μ-almost all $\pi \in \hat{G}$. So, we are interested in the inverse inequality.

Here we write down some comments on formula (8.1.1).

(1) The following alternative occurs: μ-almost everywhere the multiplicities $m(\pi)$ are uniformly bounded or μ-almost everywhere $m(\pi) = +\infty$. According to these two eventualities, we say that τ has finite or infinite multiplicities.
(2) τ has finite multiplicities if and only if $\tilde{\mu}$-almost everywhere $\mathfrak{h} + \mathfrak{g}(\ell)$ is a maximal isotropic subspace for the bilinear form B_ℓ.
(3) When τ has finite multiplicities, for μ-almost all $\pi \in \hat{G}$, each connected component of $\hat{\rho}^{-1}(\pi) \cap \Gamma_\tau$ consists of only one H-orbit and its dimension

is equal to $\frac{1}{2}\dim(\hat{\rho}^{-1}(\pi))$. Therefore, the multiplicity $m(\pi)$ is given by the number of the connected components of $\hat{\rho}^{-1}(\pi) \cap \Gamma_\tau$.

Suppose that $\tau = \mathrm{ind}_H^G \chi_f$ has finite multiplicities. For $\pi \in \hat{G}$, we write $\Omega(\pi)$ instead of $\hat{\rho}^{-1}(\pi)$. In Γ_τ, up to a null set for $\tilde{\mu}$, every connected component C_k $(1 \leq k \leq m(\pi))$ of $\Omega(\pi) \cap \Gamma_\tau$ turns out to be a single H-orbit. We fix $\ell \in \Omega(\pi)$ and $\mathfrak{b} \in M(\ell, \mathfrak{g})$. In other words, we realize $\pi = \mathrm{ind}_B^G \chi_\ell$ by means of $B = \exp \mathfrak{b}$. For $1 \leq k \leq m(\pi)$, we take $g_k \in G$ in such a manner that $g_k \cdot \ell \in C_k$ and take an invariant measure $d\dot{h}$ on the homogeneous space $H/(H \cap g_k B g_k^{-1})$.

Proposition 10.2.1 ([30]). *We can construct linearly independent elements a_π^k $(1 \leq k \leq m(\pi))$ in $\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f}$ by the following formula: for all $\phi \in \mathcal{H}_\pi^\infty$,*

$$\langle a_\pi^k, \phi \rangle = \int_{H/(H \cap g_k B g_k^{-1})} \overline{\phi(hg_k)\chi_f(h)} d\dot{h}. \tag{10.2.1}$$

Proof. First of all let us confirm that the integral in the right member is well-defined. In fact, with $h' \in H \cap g_k B g_k^{-1}$,

$$\begin{aligned}
\phi(hh'g_k)\chi_f(hh') &= \phi(hg_k g_k^{-1} h' g_k)\chi_f(h)\chi_f(h') \\
&= \chi_\ell(g_k^{-1} h'^{-1} g_k)\phi(hg_k)\chi_f(h)\chi_f(h') \\
&= \chi_{g_k \cdot \ell}(h'^{-1})\chi_f(h')\phi(hg_k)\chi_f(h) = \phi(hg_k)\chi_f(h).
\end{aligned}$$

Secondly, there exists in $\mathfrak{h}$ a coexponential basis to $\mathfrak{h} \cap g_k \cdot \mathfrak{b}$ which constitutes a part of a coexponential basis to $g_k \cdot \mathfrak{b}$ in $\mathfrak{g}$. Taking Corollary 6.2.14 into account, the space obtained from $\mathcal{H}_\pi^\infty$ through right translation by g_k is identified by means of this basis with the space $\mathcal{S}(\mathbb{R}^m)$, $m = \dim(G/B)$ of the rapidly decreasing functions and the measure $d\dot{h}$ is identified with the Lebesgue measure on $\mathbb{R}^p \subset \mathbb{R}^m$, $p = \dim(H/(H \cap g_k B g_k^{-1}))$, We derive from this the continuity of a_π^k. Actually, this right translation by g_k gives an intertwining operator between the two realizations, at ℓ and $g_k \cdot \ell$, of π. The semi-invariance required on a_π^k is easily verified by a direct computation.

Finally, we choose a Haar measure db on B and define, for $\psi \in \mathcal{D}(G)$, a function $\tilde{\psi}$ on G by

$$\tilde{\psi}(g) = \int_B \psi(gb)\chi_\ell(b)db.$$

Clearly $\tilde{\psi} \in \mathcal{H}_\pi^\infty$ and the generalized vector a_π^k supplies a distribution $\widetilde{a_\pi^k}$ on G by the formula $\widetilde{a_\pi^k}(\psi) = \langle \tilde{\psi}, a_\pi^k \rangle$. Then the support of $\widetilde{a_\pi^k}$ coincides with the double coset $Hg_k B$ which is closed. Hence to show that $\{a_\pi^k\}_{1 \leq k \leq m(\pi)}$ are linearly independent it is enough to check $Hg_j B \neq Hg_k B$ provided $j \neq k$. Otherwise, the connected set $g_j \cdot (\ell + \mathfrak{b}^\perp) \cap \Gamma_\tau$ of $\Omega(\pi) \cap \Gamma_\tau$ would intersect C_j and C_k at the same time, which is absurd. ∎

Remark 10.2.2. Up to a scalar multiplication the generalized vector a_π^k does not depend on the choice of either $g_k \in G$ or $\mathfrak{b} \in M(\ell, \mathfrak{g})$.

When G is nilpotent, we can answer affirmatively to the question of the previous section.

Theorem 10.2.3 ([38]). *Let $G = exp\, \mathfrak{g}$ be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$ and $\tau = ind_H^G \chi_f$. We consider the canonical irreducible decomposition*

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi)$$

of τ described in Theorem 8.1.9 of Chap. 8. In this situation, a kind of Frobenius reciprocity holds: for μ-almost all $\pi \in \hat{G}$,

$$m(\pi) = dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f}.$$

In particular, if τ has finite multiplicities,

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f} = \sum_{k=1}^{m(\pi)} \mathbb{C} a_\pi^k$$

for μ-almost all $\pi \in \hat{G}$.

Proof. Let us give only an outline of a proof. We employ the induction on $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{h})$. First we can assume that $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$ and that f does not vanish on any non-trivial ideal. Thus we arrive at the situation where $\dim \mathfrak{z} = 1, f|_{\mathfrak{z}} \neq 0$. As usual we take a Heisenberg triplet $\{X, Y, Z\}$ such that $\mathfrak{z} = \mathbb{R}Z, f(Z) = 1, [X, Y] = Z, \mathfrak{g} = \mathfrak{g}_0 + \mathbb{R}X$. Here, $\mathfrak{g}_0$ denotes the centralizer of Y in $\mathfrak{g}$. Let $\ell \in \Omega(\pi)$ and realize π by means of a polarization $\mathfrak{b}$ at ℓ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. According to the decomposition $G = \exp(\mathbb{R}X)G_0$, $G_0 = \exp(\mathfrak{g}_0)$, the space $\mathcal{H}_\pi^\infty$ becomes $\mathcal{S}(\mathbb{R}^m) \cong \mathcal{S}(\mathbb{R})\hat{\otimes}\mathcal{S}(\mathbb{R}^{m-1})$. Here $\mathcal{S}(\mathbb{R}^{m-1})$ represents the space $\mathcal{H}_{\pi_0}^\infty$ with $\pi_0 = \mathrm{ind}_B^{G_0} \chi_\ell \in \widehat{G_0}$. Now every $g \in G$ is uniquely written as $g = \exp(xX)g_0$ with $x \in \mathbb{R}, g_0 \in G_0$. We would like to go down to the subgroup G_0 by eliminating the first coordinate x.

Take $a \in \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f}$. If $\mathfrak{h} \not\subset \mathfrak{g}_0$, choose X in $\mathfrak{h} \cap \ker f$. Then from the semi-invariance of a, there is $a_0 \in \left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{H,\chi_f}$ so that

$$\langle a, \phi(x)\psi(g_0)\rangle = \left(\int_{\mathbb{R}} \overline{\phi(x)} dx\right) \langle a_0, \psi(g_0)\rangle.$$

Here $\phi \in \mathcal{S}(\mathbb{R}), x \in \mathbb{R}, \psi \in \mathcal{H}_{\pi_0}^\infty, g_0 \in G_0$. In this way we may descend to the subgroup G_0.

Next assume $\mathfrak{h} \subset \mathfrak{g}_0$. It is sufficient for us to consider the case where τ has finite multiplicities. So we assume that, by Lemma 9.2.14, there are Corwin-Greenleaf e-central elements $\{\sigma_1, \dots, \sigma_\kappa\}$ in such a way that we have a polynomial relation

$$P\left(\overline{(\Diamond(\sigma_1))}, \dots, \overline{(\Diamond(\sigma_\kappa))}, Y\right) \equiv 0$$

modulo $\mathcal{U}(\mathfrak{g})\overline{\mathfrak{a}_\tau}$. Here P is a polynomial of $\kappa+1$ variables and Y appears essentially. By applying this relation to our generalized vector a, we have $F(x)a = 0$ with some non-constant polynomial $F(x)$. Hence, $\{\alpha_j\}_{1\leq j\leq r}$ being real roots of the algebraic equation $F(x) = 0$, the support of the distribution $\tilde{a}$ is contained in the disjoint union of the submanifolds $M_j = \exp(\alpha_j X)G_0$ $(1 \leq j \leq r)$ of G. Thus, a is written as

$$a = \sum_{k=0}^{u} \frac{\partial^k}{\partial x^k} D_k$$

in a neighbourhood of M_j. Here, D_k $(1 \leq k \leq u)$ are distributions on G_0. It remains for us to show $u = 0$. This is done by choosing appropriately the polynomial P used above and by using the fact that τ has finite multiplicities. ■

Remark 10.2.4. When $G = \exp \mathfrak{g}$ is an exponential solvable Lie group and the monomial representation τ has finite multiplicities, is it possible to construct an element of

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

by a formula similar to (10.2.1): for $\phi \in \mathcal{H}_\pi^\infty$,

$$\langle a_\pi^k, \phi\rangle = \oint_{H/H\cap g_k B g_k^{-1}} \overline{\phi(hg_k)\chi_f(h)} \Delta_{H,G}^{-1/2}(h) d\nu(h) \ (g \in G) \tag{10.2.2}$$

with $\nu = \mu_{H,H\cap g_k B g_k^{-1}}$? In this case, already at the step of taking this value we encounter two problems: is this integral well defined? If so, as in the observation of intertwining operators, what can we say about its convergence? Finally, do the elements a_π^k $(1 \leq k \leq m(\pi))$ give generalized vectors of π? In the case of exponential solvable Lie groups, it is a difficult problem to determine the space $\mathcal{H}_\pi^\infty$; maybe we could say something by starting from a Vergne polarization $\mathfrak{b}$.

10.3 Examples of Semi-invariant Distributions

Example 10.3.1. Let $\mathfrak{g}$ be the completely solvable Lie algebra of dimension 4 with a basis (T, P, Q, E) satisfying the commutation relations

$$[T, P] = P/2, [T, Q] = Q/2, [T, E] = E, [P, Q] = E.$$

In matrix form

$$T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$G = \exp \mathfrak{g} = \left\{ \begin{pmatrix} w & x & z \\ 0 & 1 & y \\ 0 & 0 & w^{-1} \end{pmatrix}; w > 0, x, y, z \in \mathbb{R} \right\}.$$

Let $f = E^* \in \mathfrak{g}^*$ and $\mathfrak{h} = \mathbb{R}T + \mathbb{R}Q \in M(f, \mathfrak{g})$. We put as usual $H = \exp \mathfrak{h}$, $\tau = \mathrm{ind}_H^G \chi_f$. Here again $\tau \simeq \pi_+ \oplus \pi_-$ with square integrable irreducible unitary representations $\pi_\pm \in \hat{G}$ corresponding to the two open orbits $\pm G{\cdot}E^*$.

Let $a \in \left(\mathcal{H}_{\pi_\pm}^{-\infty}\right)^{H, \chi_f \Delta_{H,G}^{1/2}}$. When we realize $\pi_\pm$ by means of $\mathfrak{b} = \mathbb{R}Q + \mathbb{R}E \in I(\pm E^*, \mathfrak{g})$, their representation spaces $\mathcal{H}_{\pi_\pm}$ are identified with $L^2(\mathbb{R}^2)$ by the mapping $\psi \mapsto \tilde{\psi}(s,t) = \psi(\exp(sT)\exp(tP))$ $((s,t) \in \mathbb{R}^2)$. In this situation, the semi-invariance of a requires

$$\begin{cases} \langle a, \tilde{\psi}(s + x, t)\rangle = e^{x/4}\langle a, \tilde{\psi}\rangle, \\ \langle a, (1 - e^{\pm ixte^{-s/2}})\tilde{\psi}\rangle = 0 \end{cases}$$

for any $\psi \in \mathcal{H}_{\pi_\pm}^\infty$ and $x \in \mathbb{R}$. From the second condition, the support of a is contained in $\{(s,t) \in \mathbb{R}^2; t = 0\}$, then from the first condition

$$\langle a, \tilde{\psi}\rangle = \int_{\mathbb{R}} \overline{\tilde{\psi}(s, 0)} e^{-s/4} ds$$

up to a constant multiplication. Thus here also

$$\dim \left(\mathcal{H}_{\pi_\pm}^{-\infty}\right)^{H, \chi_f \Delta_{H,G}^{1/2}} = 1,$$

but of course this is no more than a very simple and special case obtained from the results of [9, 59] combined.

Next, for the point (α, β) of $\mathbb{R}^2$ not equal to the origin, we consider $\pi_{\alpha,\beta} \in \hat{G}$ corresponding to the coadjoint orbit through $\ell_{\alpha,\beta} = \alpha P^* + \beta Q^* \in \mathfrak{g}^*$. Let $a \in \left(\mathcal{H}_{\pi_{\alpha,\beta}}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$. Using $\mathfrak{b} = \langle P, Q, E\rangle_{\mathbb{R}} \in I(\ell_{\alpha,\beta}, \mathfrak{g})$, we identify the space $\mathcal{H}_{\pi_{\alpha,\beta}}$ with $L^2(\mathbb{R})$ by the mapping $\psi \mapsto \tilde{\psi}(t) = \psi(\exp(tT))$ $(t \in \mathbb{R})$. The semi-invariance of a requires

$$\begin{cases} \langle a, \tilde{\psi}(t+x)\rangle = e^{3x/4}\langle a, \tilde{\psi}\rangle, \\ \langle a, (1 - e^{-i\beta x e^{-t/2}})\tilde{\psi}(t)\rangle = 0 \end{cases}$$

for any $\psi \in \mathcal{H}_{\pi_{\alpha,\beta}}^{\infty}$ and $x \in \mathbb{R}$. From the second condition, $\beta \neq 0$ means $a = 0$. When $\beta = 0$, the first condition means

$$\langle a, \tilde{\psi}\rangle = \int_{\mathbb{R}} \overline{\tilde{\psi}(s)} e^{-3s/4} ds$$

up to a constant multiplication. Indeed, we know as in the last example that this formula furnishes a nonzero element of $\left(\mathcal{H}_{\pi_{\alpha,0}}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$. In consequence, though $\pi_{\alpha,0}$ does not appear in the irreducible decomposition of τ,

$$\dim \left(\mathcal{H}_{\pi_{\alpha,0}}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = 1.$$

Finally, if we denote by c_α the unitary character of G corresponding to αT^* $(\alpha \in \mathbb{R})$,

$$\left(\mathcal{H}_{c_\alpha}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = \{0\}.$$

As we have seen until now, we can say at least the following: regarding $\pi \in \hat{G}$ whose orbit does not encounter Γ_τ, $\left(\mathcal{H}_{\pi}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = \{0\}$.

Let us add a simple example of non-exponential solvable Lie group.

Example 10.3.2. Let G be the universal covering group of the motion group of a plane. Its Lie algebra $\mathfrak{g}$ is given by a basis (T, X, Y) with the commutation relations $[T, X] = Y, [T, Y] = -X$. An element of G is expressed by a triplet of real numbers (θ, a, b) and the multiplication among them is described by

$$(\theta, a, b)(\theta', a', b') = (\theta + \theta', a + a' \cos\theta - b' \sin\theta, b + a' \sin\theta + b' \cos\theta).$$

At a point $\ell_s = -sY^* \in \mathfrak{g}^*$ $(0 \neq s \in \mathbb{R})$, take a real polarization $\mathfrak{b} = \mathbb{R}X + \mathbb{R}Y$. The stabilizer $G(\ell_s)$ is nothing but the extension of $\exp(\mathbb{R}Y)$ by the centre $Z = \exp(2\pi\mathbb{Z}T)$ of G and its unitary character $\chi_\alpha, \alpha \in [0, 1)$, is given by

$\chi_\alpha(\exp(2m\pi T)) = e^{2\pi i m\alpha}$ $(m \in \mathbb{Z})$. Let $B_0 = \exp\mathfrak{b}$ and $B = ZB_0$. We extend χ_α to a unitary character $\chi_{s,\alpha}$ of B by the formula

$$\chi_{s,\alpha}(zb) = \chi_\alpha(z)\chi_{\ell_s}(b) \ (z \in Z, b \in B_0).$$

When we construct $\pi_{s,\alpha} = \mathrm{ind}_B^G \chi_{s,\alpha}$ by right translation, it is well known that $\pi_{s,\alpha}, s \in \mathbb{R}_+ = (0, +\infty), \alpha \in [0,1)$, are all irreducible and non-equivalent to each other. The action of $\pi_{s,\alpha}$ is described by the formula

$$(\pi_{s,\alpha}((\theta, a, b))\phi)(t) = e^{is(a\sin t + b\cos t)} e^{i\alpha[t+\theta]}\phi(\widehat{t+\theta})$$

in the space $L^2(\mathbb{T}), \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ endowed the measure $dt/2\pi$. Here $g = (\theta, a, b) \in G$ and we use the writing $t + \theta = [t+\theta] + \widehat{t+\theta}$ with $[t+\theta] \in 2\pi\mathbb{Z}$ and $\widehat{t+\theta} \in [0, 2\pi)$.

Now, let $\mathfrak{h} = \mathbb{R}X$ and $H_0 = \exp\mathfrak{h}$. Starting from the trivial representation 1_{H_0}, we construct the monomial representation $\tau = \mathrm{ind}_{H_0}^G 1_{H_0}$ of G. Benoist [8] asserts

$$\tau \simeq 2\int_{\mathbb{R}_+\times[0,1)}^{\oplus} \pi_{s,\alpha}\, ds d\alpha.$$

Besides, let δ_x be the Dirac measure at $x \in \mathbb{T}$. Then $\left(\mathcal{H}_{\pi_{s,\alpha}}^{-\infty}\right)^{H_0, 1_{H_0}} = \mathbb{C}\delta_0 \oplus \mathbb{C}\delta_\pi$, while the intersection $G\cdot\ell_s \cap \mathfrak{h}^\perp$ consists of two connected components, each of which is a H_0-orbit of dimension $\frac{1}{2}\dim(G\cdot\ell_s)$.

If we compute the subset $S = \{g \in G; g^{-1}\cdot(\ell_s + \mathfrak{b}^\perp) \cap \mathfrak{h}^\perp \neq \emptyset\}$ of G, it is easily seen that $S = \{g = (\theta, a, b) \in G; \theta \in \mathbb{Z}\pi\}$, and this coincides with the union of the supports as distribution of the elements of $\left(\mathcal{H}_{\pi_{s,\alpha}}^{-\infty}\right)^{H_0, 1_{H_0}}$.

Henceforth, we consider the extension $H = \exp(\mathbb{Z}\pi T)\cdot H_0$ and define its character $\rho_\gamma, \gamma \in [0,1)$, by

$$\rho_\gamma(\exp(m\pi T)\cdot h_0) = e^{2\pi i m\gamma} \ (m \in \mathbb{Z},\ h_0 \in H_0).$$

Now, $\tau_\gamma = \mathrm{ind}_H^G \rho_\gamma$ realized by right translation is given by

$$(\tau_\gamma(g)\phi)(t, \eta) = e^{2\pi i\gamma[[t+\theta]]}\phi(\widetilde{t+\theta}, (-1)^{[[t+\theta]]}(\eta + a\sin t + b\cos t)).$$

Here $\phi \in L^2([0,\pi) \times \mathbb{R}), g = (\theta, a, b) \in G$ and we use the writing $t + \theta = [[t+\theta]]\pi + \widetilde{t+\theta}$ with $[[t+\theta]] \in \mathbb{Z}$ and $\widetilde{t+\theta} \in [0, \pi)$.

By the Fourier transformation with respect to the second variable

$$\hat{\phi}(t,\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta y} \phi(t,y)dy,$$

another realization of τ_γ is written as

$$(\tau_\gamma(g)\hat{\phi})(t,\eta) = e^{2\pi i\gamma[[t+\theta]]} e^{-i\eta(a\sin t + b\cos t)} \hat{\phi}(\widetilde{t+\theta}, (-1)^{[[t+\theta]]}\eta).$$

Next by the mapping which sends $\hat{\phi} \in L^2([0,\pi)\times\mathbb{R})$ to $\tilde{\phi} \in L^2([0,2\pi)\times\mathbb{R}_+)$ given by

$$\tilde{\phi}(t,\eta) = \begin{cases} \hat{\phi}(t,\eta) & t\in[0,\pi), \\ e^{2\pi i\gamma}\hat{\phi}(t-\pi,-\eta) & t\in[\pi,2\pi), \end{cases}$$

we obtain the third realization in $L^2([0,2\pi)\times\mathbb{R}_+)$ written as

$$\left(\tau_\gamma(g)\tilde{\phi}\right)(t,\eta) = e^{2i\gamma[t+\theta]} e^{-i\eta(a\sin t + b\cos t)} \tilde{\phi}(t+\theta,\eta).$$

Thus,

$$\tau_\gamma \simeq \int_{\mathbb{R}_+}^{\oplus} \pi_{s,2\gamma(\mathrm{mod}\ \mathbb{Z})} ds$$

with uniform multiplicity 1 as a result of [9] asserts.

From a point of view of the orbit method, this phenomenon seems to result from the fact that the two H_0-orbits contained in $G\cdot\ell_s \cap \mathfrak{h}^\perp$ are linked together by the action of $\exp(\pi T) \in H$ so that $G\cdot\ell_s \cap \mathfrak{h}^\perp$ becomes only one H-orbit in spite of being non-connected.

Finally, in order that $a = \xi\delta_0 + \eta\delta_\pi \in \left(\mathcal{H}^{-\infty}_{\pi_{s,\alpha}}\right)^{H_0,1_{H_0}}$ ($\xi,\eta\in\mathbb{C}$) belongs to $\left(\mathcal{H}^{-\infty}_{\pi_{s,\alpha}}\right)^{H,\rho_\gamma}$, it is necessary and sufficient that $\langle \pi_{s,\alpha}(\exp(\pi T))a, \psi\rangle = \langle e^{2\pi i\gamma}a,\psi\rangle$ for all $\psi \in \mathcal{H}^{\infty}_{\pi_{s,\alpha}}$. Hence,

$$\left(\mathcal{H}^{-\infty}_{\pi_{s,\alpha}}\right)^{H,\rho_\gamma} = \begin{cases} \mathbb{C}(\delta_0 + e^{2\pi i\gamma}\delta_\pi) & \alpha = 2\gamma, \\ 0 & \alpha \neq 2\gamma. \end{cases}$$

So, we encounter again the above set S as the union of the supports as distribution of its elements.

Chapter 11
Plancherel Formula

11.1 Penney's Plancherel Formula

As before, let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$, $\mathfrak{h} \in S(f, \mathfrak{g})$, $H = \exp \mathfrak{h}$, $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h})$ and consider $\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$. We denote by e the unit element of G. Since every $\phi \in \mathcal{H}_\tau^\infty$ is an C^∞-function [63] on G, $\delta_\tau \in \left(\mathcal{H}_\tau^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$ is defined by $\delta_\tau(\phi) = \langle \delta_\tau, \phi \rangle = \overline{\phi(e)}$. In this chapter, we shall be interested in the **abstract Plancherel formula due to Penney** [59] and Bonnet [12] applied to the cyclic representation (τ, δ_τ).

Provided $\mathfrak{h} \in I(f, \mathfrak{g})$, τ is irreducible and, by Theorem 10.1.1 in Chap. 10, the space $\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$, $\pi \in \hat{G}$ is equal to $\mathbb{C}\delta_\pi$ when $\pi \simeq \tau$ and $\{0\}$ when $\pi \not\simeq \tau$. Provided $\mathfrak{h} \notin I(f, \mathfrak{g})$, let us decompose (τ, δ_τ) into irreducible components:

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi),\ \delta_\tau \simeq \int_{\hat{G}}^{\oplus} a_\pi d\mu(\pi).$$

Then, μ-almost everywhere a_π belongs to $\left(m(\pi)\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$ and at least when τ has finite multiplicities

$$a_\pi = (a_\pi^k)_{1 \le k \le m(\pi)},\ a_\pi^k \in \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

from the uniqueness of the irreducible decomposition [59]. So,

$$\langle \tau(\phi)\delta_\tau, \delta_\tau \rangle = \int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi)a_\pi^k, a_\pi^k \rangle d\mu(\pi)$$

for $\phi \in \mathcal{D}(G)$.

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Choosing a left Haar measure dh on H, we construct an element ϕ_H^f of $\mathcal{H}_\tau^\infty$ by the formula

$$\phi_H^f(g) = \int_H \phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh \ (g \in G)$$

for $\phi \in \mathcal{D}(G)$. With $\psi \in \mathcal{H}_\tau^\infty$,

$$\begin{aligned}\langle \tau(\phi)\delta_\tau, \psi\rangle &= \Big\langle \int_G \phi(g)\tau(g)\delta_\tau dg, \psi\Big\rangle = \Big\langle \delta_\tau, \int_G \overline{\phi(g)}\tau(g^{-1})\psi dg\Big\rangle \\ &= \int_G \phi(g)\overline{\psi(g)}dg = \oint_{G/H} d\mu_{G,H}(g)\int_H \phi(gh)\Delta_{H,G}^{-1/2}(h)\overline{\psi(gh)}dh \\ &= \oint_{G/H} \overline{\psi(g)}d\mu_{G,H}(g)\int_H \phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh = \langle \phi_H^f, \psi\rangle.\end{aligned}$$

Thus $\tau(\phi)\delta_\tau = \phi_H^f \in \mathcal{H}_\tau^\infty$ and

$$\langle \tau(\phi)\delta_\tau, \delta_\tau\rangle = \int_H \phi(h)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh = \phi_H^f(e)$$

for arbitrary $\phi \in \mathcal{D}(G)$.

In consequence, the abstract Plancherel formula for the cyclic monomial representation (τ, δ_τ) is described as follows.

Theorem 11.1.1 ([31, 59]). *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in S(f, \mathfrak{g})$. Let*

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi)$$

be the canonical irreducible decomposition of $\tau = \hat{\rho}(f, \mathfrak{h}, G)$. Assume that τ has finite multiplicities. There exist, for μ-almost all $\pi \in \hat{G}$, elements a_π^k, $1 \le k \le m(\pi)$, of $\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f\Delta_{G,H}^{1/2}}$ with which the formula

$$\phi_H^f(e) = \int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi)a_\pi^k, a_\pi^k\rangle d\mu(\pi) \tag{11.1.1}$$

holds for any $\phi \in \mathcal{D}(G)$.

Like the symmetric space of Benoist [8], we will practise the calculation of

$$(a_\pi^k)_{\pi \in \hat{G}, 1 \le k \le m(\pi)}$$

to procure concrete Plancherel formulas in certain cases.

Theorem 11.1.2 ([30]). *When $G = \exp \mathfrak{g}$ is nilpotent and τ has finite multiplicities, the generalized vectors constructed by formula (10.2.1) satisfy the abstract Plancherel formula (11.1.1), if we appropriately normalize the measures $d\dot{h}$ utilized in their construction.*

Proof. We begin by preparing a lemma. We choose invariant measures $dg, dh, d\dot{g}$ and $d\dot{h} = d_k\dot{h}$ respectively on $G, H, G/B$ and $H/(H \cap g_k B g_k^{-1})$. Next, we induce on G/H the quotient measure $d\ddot{g}$, determine by transitivity the invariant measure $d_k\dot{g}$ on $G/(H \cap g_k B g_k^{-1})$, take its image by the canonical map on $G/(g_k^{-1} H g_k \cap B)$ and finally induce the invariant measure $d_k\dot{b}$ on $B/(B \cap g_k^{-1} H g_k)$.

Lemma 11.1.3. *With $\phi \in \mathcal{D}(G)$,*

$$\left(\pi(\phi)a_\pi^k\right)(g) = \int_{B/(B\cap g_k^{-1}Hg_k)} \phi_H^f(gbg_k^{-1})\chi_\ell(b)d\dot{b} \ (g \in G),$$

Hence

$$\langle \pi(\phi)a_\pi^k, a_\pi^k\rangle = \int_{H/(H\cap g_k B g_k^{-1})} \chi_f(h)d\dot{h} \int_{B/(B\cap g_k^{-1}Hg_k)} \phi_H^f(hg_k b g_k^{-1})\chi_\ell(b)d\dot{b}. \tag{11.1.2}$$

Proof. With $\psi \in \mathcal{H}_\pi^\infty$ let us compute

$$\begin{aligned}
\langle \pi(\phi)a_\pi^k, \psi\rangle &= \int_G \phi(g)\langle a_\pi^k, \pi(g^{-1})\psi\rangle dg \\
&= \int_G \phi(g)dg \int_{H/(H\cap g_k B g_k^{-1})} \overline{\psi(ghg_k)\chi_f(h)}d_k\dot{h} \\
&= \int_{G/H} \phi_H^f(g)d\ddot{g} \int_{H/(H\cap g_k B g_k^{-1})} \overline{\psi(ghg_k)\chi_f(h)}d_k\dot{h} \\
&= \int_{G/H} d\ddot{g} \int_{H/(H\cap g_k B g_k^{-1})} \phi_H^f(gh)\overline{\psi(ghg_k)}d_k\dot{h} \\
&= \int_{G/(H\cap g_k B g_k^{-1})} \phi_H^f(g)\overline{\psi(gg_k)}d_k\dot{g} \\
&= \int_{G/B}\int_{B/(B\cap g_k^{-1}Hg_k)} \phi_H^f(gbg_k^{-1})\overline{\psi(gb)}d_k\dot{b} \\
&= \Big\langle \int_{B/(B\cap g_k^{-1}Hg_k)} \phi_H^f(gbg_k^{-1})\chi_\ell(b)d_k\dot{b}, \psi\Big\rangle.
\end{aligned}$$

From this we derive the conclusion. ∎

Let us show the theorem rather by normalizing the measure μ which appears in formula (8.1.1). Let $\mathfrak{g}_0$ be an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{h}$. Assume first that almost all orbits are non-saturated in the direction $\mathfrak{g}_0^{\perp}$. For μ-almost all $\pi \in \hat{G}$, the restriction π_0 of π to $G_0 = \exp(\mathfrak{g}_0)$ is irreducible, $m(\pi) = m(\pi_0)$ and, except for a set of measure 0, the Borel space $\hat{G}$ is identified with $\widehat{G_0} \times \mathbb{R}$. By the induction hypothesis, there exists a measure μ_0 on $\widehat{G_0}$ so that formula (11.1.1) holds for G_0. Under our identification, putting $a_\pi^k = a_{\pi_0}^k$ for $1 \leq k \leq m(\pi)$, the measure $\mu = \mu_0 \times dx$ satisfies our request, here dx denotes a Lebesgue measure on $\mathbb{R}$.

In fact, let $p : \mathfrak{g}^* \to \mathfrak{g}_0^*$ the restriction map. In order that the measure μ_0 exists, we can choose $d\dot{g}_0, d_k\dot{h}_0$ and hence $d_k\dot{b}_0$. Here the index 0 indicates corresponding objects at the stage of G_0. Let $\mathfrak{b}_0 \in M(\ell_0, \mathfrak{g}_0)$ be a real polarization appropriately chosen to give $\pi_0 = \hat{\rho}_{G_0}(\ell_0) \in \widehat{G_0}$. Let the mapping $\widehat{G_0} \ni \pi_0 \mapsto \ell_0 \in \mathfrak{g}_0^*$ be a Borel cross-section. For μ_0-almost all $\ell_0 \in \mathfrak{g}_0^*$, let $\ell \in \mathfrak{g}^*$ be such that $p(\ell) = \ell_0$. We can take an element $T = T(\ell_0)$ of $\mathfrak{g}$ depending on ℓ_0 so that $\mathfrak{g}(\ell) = \mathbb{R}T + \mathfrak{g}_0(\ell_0)$ and that except for an μ-null set the Borel map $\pi \mapsto (\pi|_{G_0}, \ell(T))$ would supply an identification between $\hat{G}$ and $\widehat{G_0} \times \mathbb{R}$. Since the generalized vector a_π^k does not depend up to a constant multiplication on the choice of the polarization, we can choose $\mathfrak{b}_0$ so that the polarization $\mathfrak{b} \in M(\ell, \mathfrak{g})$ would be obtained as $\mathbb{R}T + \mathfrak{b}_0$. Then if we express $b \in B = \exp \mathfrak{b}$ as $b = \exp(tT) \cdot b_0$ with $t \in \mathbb{R}$ and $b_0 \in B_0 = \exp(\mathfrak{b}_0)$, we know that $B/(B \cap g_k^{-1} H g_k)$ is isomorphic to $\mathbb{R} \times B_0/(B_0 \cap g_k^{-1} H g_k)$. We choose $d_k\dot{b}$ as $d_k\dot{b} = dt \cdot d_k\dot{b}_0$.

In this situation, if we put $\lambda = \ell(T)$, the measure $\mu = \mu_0 \times \frac{d\lambda}{2\pi}$ meets the request: for any $\phi \in \mathcal{D}(G)$, formula (8.1.2), the induction hypothesis on μ_0 and the Plancherel formula for $\mathbb{R}$ give

$$\begin{aligned}
&\int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi) a_\pi^k, a_\pi^k \rangle d\mu(\pi) \\
&\quad = \int_{\widehat{G_0}} d\mu_0(\pi_0) \int_{\mathbb{R}} \frac{d\lambda}{2\pi} \sum_{k=1}^{m(\pi_0)} \int_{H/(H \cap g_k B g_k^{-1})} \chi_f(h) d_k\dot{h} \\
&\quad\quad \times \int_{\mathbb{R}} e^{i\lambda t} dt \int_{B_0/(B_0 \cap g_k H g_k^{-1})} \phi_H^f(h g_k \exp(tT) b_0 g_k^{-1}) \chi_{\ell_0}(b_0) d_k\dot{b}_0 \\
&\quad = \int_{\widehat{G_0}} d\mu_0(\pi_0) \sum_{k=1}^{m(\pi_0)} \int_{H/(H \cap g_k B g_k^{-1})} \chi_f(h) d_k\dot{h} \\
&\quad\quad \times \int_{B_0/(B_0 \cap g_k H g_k^{-1})} \phi_H^f(h g_k b_0 g_k^{-1}) \chi_{\ell_0}(b_0) d_k\dot{b}_0 \\
&\quad = \int_{\widehat{G_0}} \sum_{k=1}^{m(\pi_0)} \langle \pi_0(\phi) a_{\pi_0}^k, a_{\pi_0}^k \rangle d\mu_0(\pi_0) = \phi_H^f(e).
\end{aligned}$$

Next, let us consider the case where μ-almost all orbits are saturated in the direction $\mathfrak{g}_0^{\perp}$. Let $\pi \in \hat{G}$ be such that each connected component of $\Omega(\pi) \cap \Gamma_\tau$ is only one H-orbit of dimension $\frac{1}{2}\dim(\Omega(\pi))$. There exist finite irreducible unitary representations $\pi_0^1, \ldots, \pi_0^s$ of G_0 such that $p\,(\Omega(\pi) \cap \Gamma_\tau)$ is a disjoint union of $\Omega_0(\pi_0^j) \cap \left(f_0 + \mathfrak{h}^{\perp, \mathfrak{g}_0^*}\right), 1 \leq j \leq s$ whose connected components are presented as $C_{0,1}^j, \ldots, C_{0,i_j}^j$ with $i_j = m(\pi_0^j)$. The connected components of $\Omega(\pi) \cap \Gamma_\tau$ are $C_k^j = p^{-1}(C_{0,k}^j)$ with $1 \leq k \leq m(\pi_0^j)$, $1 \leq j \leq s$. $C_{0,k}^j$ associates with $a_{\pi_0^j}^k \in \left(\mathcal{H}_{\pi_0^j}^{-\infty}\right)^{H,\chi_f}$ and C_k^j associates with $a_\pi^r \in \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f}$, with $r = \sum_{i=1}^{j-1} m(\pi_0^i) + k$.

By the induction hypothesis, we can appropriately choose $d\dot{g}_0, d_k\dot{h}_0, 1 \leq k \leq m(\pi_0)$ in order to get the measure μ_0 on $\widehat{G_0}$ which satisfies the theorem at the stage of G_0. Then, it is enough to take the image measure μ of μ_0 by the induction of representations. In fact, if we consider the decomposition

$$\mu_0 = \int_{\hat{G}} \mu_\pi \, d\mu(\pi)$$

of μ_0 with respect to μ, the fibre over π consists of π_0^j with $1 \leq j \leq s$, and it is easily seen that $\mu_\pi = \sum_{j=1}^s c_j \delta_{\pi_0^j}$ with $c_j > 0$. Here $\delta_{\pi_0^j}$ denotes the Dirac measure at the point $\pi_0^j \in \widehat{G_0}$. Let $\phi \in \mathcal{D}(G)$. The expressions (11.1.2) applied to $\langle \pi_0^j(\phi) a_{\pi_0^j}^k, a_{\pi_0^j}^k \rangle$ and $\langle \pi(\phi) a_\pi^r, a_\pi^r \rangle$ with $r = \sum_{i=1}^{j-1} m(\pi_0^i) + k$, and the transfers by intertwining operators permit us to choose $d\dot{g}, d_k\dot{h}$ so that

$$\langle \pi(\phi) a_\pi^r, a_\pi^r \rangle = c_j \langle \pi_0^j(\phi) a_{\pi_0^j}^k, a_{\pi_0^j}^k \rangle$$

for all indices j, k and any $\phi \in \mathcal{D}(G)$. With these choices we compute: for any $\phi \in \mathcal{D}(G)$,

$$\begin{aligned}
\phi_H^f(e) &= \int_{\widehat{G_0}} \sum_{k=1}^{m(\pi_0)} \langle \pi_0(\phi) a_{\pi_0}^k, a_{\pi_0}^k \rangle \, d\mu_0(\pi_0) \\
&= \int_{\hat{G}} d\mu(\pi) \int_{\widehat{G_0}} \sum_{k=1}^{m(\pi_0)} \langle \pi_0(\phi) a_{\pi_0}^k, a_{\pi_0}^k \rangle \, d\mu_\pi(\pi_0) \\
&= \int_{\hat{G}} \sum_{j=1}^{s} \sum_{k=1}^{m(\pi_0^j)} c_j \langle \pi_0^j(\phi) a_{\pi_0^j}^k, a_{\pi_0^j}^k \rangle \, d\mu(\pi) \\
&= \int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi) a_\pi^k, a_\pi^k \rangle \, d\mu(\pi).
\end{aligned}$$

Thus, the theorem has been proved. ■

Example 11.1.4. Let $\mathfrak{g} = \langle e_1, e_2, e_3, e_4\rangle_{\mathbb{R}} : [e_1, e_2] = e_3, [e_1, e_3] = e_4$. Put $f = e_4^*$ relative to the dual basis of $\mathfrak{g}^*$ and $\mathfrak{h} = \mathbb{R}e_2 + \mathbb{R}e_4 \in S(f, \mathfrak{g})$. Considering $f_\lambda = \lambda e_3^* + e_4^*$ ($\lambda \in \mathbb{R}$) in $\Gamma_\tau = f + \mathfrak{h}^\perp$, $\mathfrak{g}(f_\lambda) = \mathbb{R}(e_2 - \lambda e_3) \oplus \mathbb{R}e_4$. Hence, $\mathfrak{h} + \mathfrak{g}(f_\lambda)$ is a maximal isotropic subspace for B_{f_λ} when $\lambda \neq 0$. Put $G = \exp \mathfrak{g}$. For $\ell \in \Gamma_\tau$ such that $\ell(e_3) \neq 0$, the intersection $G \cdot \ell \cap \Gamma_\tau$ is composed of two lines each of which is a H-orbit with $H = \exp \mathfrak{h}$, while, concerning $\ell \in \Gamma_\tau$ which vanishes at e_3, this intersection is a line which contains infinite H-orbits. Let p be the canonical map of $\mathfrak{g}^*$ onto the orbit space $\mathfrak{g}^*/G$. Then, $p(\Gamma_\tau) = \{G\cdot f_\lambda; \lambda \geq 0\}$ and if we set $\pi_\lambda = \hat{\rho}(G\cdot f_\lambda)$ using the Kirillov map $\hat{\rho} : \mathfrak{g}^*/G \to \hat{G}$,

$$\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f \simeq 2\int_0^\infty \pi_\lambda d\lambda.$$

If we take a real polarization $\mathfrak{b} = \langle e_2, e_3, e_4\rangle$ at the point f_λ and realize $\pi_\lambda \in \hat{G}$ in $L^2(\mathbb{R})$ through the identification $\mathcal{H}_{\pi_\lambda} \ni \phi \leftrightarrow \Phi \in L^2(\mathbb{R})$ with $\Phi(t) = \phi(\exp(te_1))$ ($t \in \mathbb{R}$), the space $\mathcal{H}_{\pi_\lambda}^\infty$ is realized as the space $\mathcal{S}(\mathbb{R})$ of the rapidly decreasing functions and

$$\left(\mathcal{H}_{\pi_\lambda}^{-\infty}\right)^{H,\chi_f} = \mathbb{C}a_\lambda^1 \oplus \mathbb{C}a_\lambda^2,$$

Here $a_\lambda^1 : \Phi \mapsto \overline{\Phi(0)}$ and a_λ^2 is given by $\Phi \mapsto \overline{\Phi(2\lambda)}$ when $\lambda \neq 0$ and by $\Phi \mapsto \overline{\frac{d\Phi}{dt}(0)}$ when $\lambda = 0$. In this way,

$$\dim \left(\mathcal{H}_{\pi_\lambda}^{-\infty}\right)^{H,\chi_f} = 2,$$

which is equal to the multiplicity of π_λ in the irreducible decomposition of τ. Likewise, if $\pi \in \hat{G}$ is not equivalent to any π_λ,

$$\left(\mathcal{H}_{\pi}^{-\infty}\right)^{H,\chi_f} = \{0\}.$$

To write down the concrete Plancherel formula, set

$$\psi_H^f(g) = \int_H \psi(gh)\chi_f(h)dh \ (g \in G)$$

for $\psi \in \mathcal{D}(G)$. Here $dh = dx_2dx_4$ when $h = \exp(x_2e_2)\exp(x_4e_4)$. Now we put $dg = \Pi_{j=1}^4 dx_j$ for $g = \Pi_{j=1}^4 \exp(x_je_j)$. In this situation, the formula

$$\psi_H^f(e) = \frac{1}{2\pi}\int_0^\infty \{\langle \pi_\lambda(\psi)a_\lambda^1, a_\lambda^1\rangle + \langle \pi_\lambda(\psi)a_\lambda^2, a_\lambda^2\rangle\}d\lambda$$

holds. Indeed, put $B = \exp \mathfrak{b}$ and $db = \Pi_{j=2}^4 dx_j$ for $b = \Pi_{j=2}^4 \exp(x_je_j)$. Then, $\Psi_\lambda^1 = \pi_\lambda(\psi)a_\lambda^1 \in \mathcal{H}_{\pi_\lambda}^\infty = \mathcal{S}(\mathbb{R})$ is obtained by

$$\Psi_\lambda^1(t) = \int_B \psi(\exp(te_1)\cdot b)\chi_{f_\lambda}(b)db \ (t \in \mathbb{R}).$$

Therefore,

$$\langle \pi_\lambda(\psi)a_\lambda^1, a_\lambda^1\rangle = \int_B \psi(b)\chi_{f_\lambda}(b)db.$$

Similarly, $\Psi_\lambda^2 = \pi_\lambda(\psi)a_\lambda^2 \in \mathcal{S}(\mathbb{R})$ is obtained by

$$\Psi_\lambda^2(t) = \int_B \psi(\exp((t-2\lambda)e_1)\cdot b)\chi_{f_{-\lambda}}(b)db \ (t \in \mathbb{R})$$

and so

$$\langle \pi_\lambda(\psi)a_\lambda^2, a_\lambda^2\rangle = \int_B \psi(b)\chi_{f_{-\lambda}}(b)db.$$

Thus, using the Plancherel formula for $\mathbb{R}$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \{\langle \pi_\lambda(\psi)a_\lambda^1, a_\lambda^1\rangle + \langle \pi_\lambda(\psi)a_\lambda^2, a_\lambda^2\rangle\}d\lambda \\
&\quad = \frac{1}{2\pi}\int_0^\infty d\lambda \int_{\mathbb{R}} dx_3 \int_H \left(e^{i\lambda x_3} + e^{-i\lambda x_3}\right)\psi(h\exp(x_3e_3))\chi_{f_\lambda}(h)dh \\
&\quad = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\lambda x_3} d\lambda dx_3 \int_H \psi(h\exp(x_3e_3))\chi_f(h)dh \\
&\quad = \int_H \psi(h)\chi_f(h)dh = \psi_H^f(e).
\end{aligned}$$

11.2 Vergne's Decomposition Theorem

If we go beyond the nilpotent case, we know few. We describe here a particular case. Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group, $f \in \mathfrak{g}^*, \mathfrak{h} \in M(f, \mathfrak{g})$ and $H = \exp \mathfrak{h}$. The irreducible decomposition of $\tau = \mathrm{ind}_H^G \chi_f$ was obtained by Vergne [77] and it was our starting point toward Theorem 8.1.9 in Chap. 8. We denote by $U(f, \mathfrak{h})$ the set of the orbits $\Omega \in \mathfrak{g}^*/G$ whose intersections with Γ_τ are non-empty open subsets of Γ_τ.

Theorem 11.2.1. *When* $\mathfrak{h} \in M(f, \mathfrak{g})$:

(1) $U(f, \mathfrak{h})$ *is a finite set*;
(2) *if* $\Omega \in U(f, \mathfrak{h})$, *the number* $c(\Omega)$ *of the connected components of* $\Omega \cap \Gamma_\tau$ *is finite*;
(3) $\tau \simeq \sum_{\Omega \in U(f,\mathfrak{h})} c(\Omega)\hat{\rho}(\Omega)$.

Proof. First of all we prepare a lemma.

Lemma 11.2.2. *Let $\mathfrak{h} \in M(f, \mathfrak{g})$. If we denote by $V(f, \mathfrak{h})$ the set of all G-orbits $\Omega = G{\cdot}(f + \lambda)$ such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$, $\lambda \in \mathfrak{h}^{\perp}$, then $U(f, \mathfrak{h}) = V(f, \mathfrak{h})$. Besides, for $\Omega \in U(f, \mathfrak{h})$, the connected component containing a point $f + \lambda$ ($\lambda \in \mathfrak{h}^{\perp}$) of $\Omega \cap (f + \mathfrak{h}^{\perp})$ is equal to $H{\cdot}(f + \lambda)$.*

Proof. We repeat the same reasoning as in the complex case treated by Lemma 7.7.1 of Chap. 7. Let us show $V(f, \mathfrak{h}) \subset U(f, \mathfrak{h})$. Take $\lambda \in \mathfrak{h}^{\perp}$ such that $\mathfrak{h} \in M(f+\lambda, \mathfrak{g})$. Hence $E = G{\cdot}(f+\lambda) \cap (f+\mathfrak{h}^{\perp})$ is not empty. The question is to show that E is an open set of $f + \mathfrak{h}^{\perp}$. Provided $f + \gamma \in E$, $\gamma \in \mathfrak{h}^{\perp}$ and $f + \gamma \in G{\cdot}(f + \lambda)$. Hence $\mathfrak{h} \in S(f+\gamma, \mathfrak{g})$ and $\dim(\mathfrak{g}(f+\gamma)) = \dim(\mathfrak{g}(f+\lambda))$ so that $\mathfrak{h} \in M(f+\gamma, \mathfrak{g})$. Thus, as we saw in the proof of (3) $\Longrightarrow$ (1) of Theorem 5.4.1 in Chap. 5, $H{\cdot}(f + \gamma) \subset E$ becomes an open set of $f + \mathfrak{h}^{\perp}$ and the claim follows.

Now let us show the inverse inclusion $U(f, \mathfrak{h}) \subset V(f, \mathfrak{h})$. Take $\Omega \in U(f, \mathfrak{h})$ and set

$$U_1 = \left\{\lambda \in \mathfrak{h}^{\perp} : \dim(\mathfrak{g}(f + \lambda)) = \dim(\mathfrak{g}(f))\right\}.$$

Because $\mathfrak{h} \in S(f + \lambda, \mathfrak{g})$ for arbitrary $\lambda \in \mathfrak{h}^{\perp}$, the dimension of $\mathfrak{g}(f)$ is equal to the minimum of the dimensions of $\mathfrak{g}(f + \lambda)$ when λ moves through $\mathfrak{h}^{\perp}$. Hence U_1 is a non-empty Zariski open set of $\mathfrak{h}^{\perp}$ and dense in $\mathfrak{h}^{\perp}$. Therefore, $f + U_1$ meets $\Omega \cap (f+\mathfrak{h}^{\perp})$ at least at one point $f+\lambda_1$ and $\Omega = G{\cdot}(f+\lambda_1)$ belongs to $V(f, \mathfrak{h})$.

Lastly, take $\Omega \in U(f, \mathfrak{h})$. Let $f + \lambda$ ($\lambda \in \mathfrak{h}^{\perp}$) be a point of $\Omega \cap (f + \mathfrak{h}^{\perp})$ and C the connected component of $\Omega \cap (f + \mathfrak{h}^{\perp})$, which is an open set of $f + \mathfrak{h}^{\perp}$, containing $f + \lambda$. C is stable under the action of H and, for every point $f + \lambda'$ of C, $H{\cdot}(f+\lambda')$ is a connected open set of $\Omega \cap (f+\mathfrak{h}^{\perp})$ and hence of C. Moreover, its complement set in C is a union of open H-orbits by the same reasoning. In consequence, $H{\cdot}(f + \lambda)$ turns out to be open and closed in C and so coincides with C. ∎

Let us proceed to prove Theorem 11.2.1 by induction on dim $\mathfrak{g}$ as usual. Taking the last lemma into account, it is enough to show the following.

(A) $\hat{\rho}(f, \mathfrak{h}, G)$ is a finite sum of irreducible representations.
(B) Every irreducible component of $\hat{\rho}(f, \mathfrak{h}, G)$ is $\hat{\rho}(f+\lambda)$ corresponding to $f+\lambda$ such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ ($\lambda \in \mathfrak{h}^{\perp}$).
(C) Every irreducible representation corresponding to $f + \lambda$ such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ ($\lambda \in \mathfrak{h}^{\perp}$) appears as an irreducible component of $\hat{\rho}(f, \mathfrak{h}, G)$ with multiplicity equal to the number $c(\Omega, f, \mathfrak{h}, \mathfrak{g})$ of connected components of $\Omega \cap (f + \mathfrak{h}^{\perp})$ where $\Omega = G{\cdot}(f + \lambda)$.

(I) Suppose that there is a commutative ideal $\mathfrak{a} \neq \{0\}$ such that $f|_{\mathfrak{a}} = 0$. Then, since $\mathfrak{h} + \mathfrak{a} \in S(f, \mathfrak{g})$, $\mathfrak{h} \supset \mathfrak{a}$. Put $A = \exp \mathfrak{a}$, $\tilde{G} = G/A$. Let $p : G \to \tilde{G}$ be the canonical projection, $dp : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ its differential, $\tilde{f}$ the image of f in $\tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{h}} = dp(\mathfrak{h})$. Then, $\tilde{\mathfrak{h}} \in M(\tilde{f}, \tilde{\mathfrak{g}})$ and $\hat{\rho}(f, \mathfrak{h}, G) = \hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \circ p$. By the induction hypothesis,

$$\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \simeq \sum_{\tilde{\Omega} \in V(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{g}})} c(\tilde{\Omega}, \tilde{f}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{g}}) \hat{\rho}(\tilde{\Omega})$$

and we can write

$$\hat{\rho}(f, \mathfrak{h}, G) \simeq \sum_{\tilde{\Omega} \in V(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{g}})} c(\tilde{\Omega}, \tilde{f}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{g}}) \hat{\rho}(\tilde{\Omega}) \circ p.$$

Hence (A) is settled. Now let $\tilde{\Omega}$ be the $\tilde{G}$-orbit of $\tilde{f} + \tilde{\lambda}$ ($\tilde{\lambda} \in \tilde{\mathfrak{h}}^{\perp}$) and put $\lambda = \tilde{\lambda} \circ dp$. Then, $\hat{\rho}(\tilde{\Omega}) \circ p = \hat{\rho}(G \cdot (f + \lambda))$ and $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ if $\tilde{\mathfrak{h}} \in M(\tilde{f} + \tilde{\lambda}, \tilde{\mathfrak{g}})$. Hence (B) is settled. Finally, if $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ ($\lambda \in \mathfrak{h}^{\perp}$), $\lambda|_{\mathfrak{a}} = 0$ by $\mathfrak{h} \supset \mathfrak{a}$. So, let $\tilde{\lambda} \in \tilde{\mathfrak{g}}^*$ be such that $\lambda = \tilde{\lambda} \circ p$, then $\tilde{\mathfrak{h}} \in M(\tilde{f} + \tilde{\lambda}, \tilde{\mathfrak{g}})$. Furthermore, the canonical bijection $q : \mathfrak{a}^{\perp} \to \tilde{\mathfrak{g}}^*$ supplies a diffeomorphism between $G \cdot (f + \lambda) \cap (f + \mathfrak{h}^{\perp})$ and $\tilde{G} \cdot (\tilde{f} + \tilde{\lambda}) \cap (\tilde{f} + \tilde{\mathfrak{h}}^{\perp})$. Hence (C) is settled.

(II) Suppose that there does not exist such an ideal $\mathfrak{a}$ used in (I). If we take a minimal non-central ideal $\mathfrak{a}$ of $\mathfrak{g}$, $\mathfrak{a}^f \subsetneq \mathfrak{g}$ because $f([\mathfrak{g}, \mathfrak{a}]) \neq \{0\}$. First assume $\mathfrak{h} \supset \mathfrak{a}$. Hence $\mathfrak{a} \subset \mathfrak{h} \subset \mathfrak{a}^f$. We denote by $\ell \mapsto \bar{\ell}$ the restriction map from $\mathfrak{g}^*$ to $(\mathfrak{a}^f)^*$ and put $G_0 = \exp\left(\mathfrak{a}^f\right)$. Clearly $\mathfrak{h} \in M(\bar{f}, \mathfrak{a}^f)$ and

$$\hat{\rho}(f, \mathfrak{h}, G) \simeq \mathrm{ind}_{G_0}^{G} \hat{\rho}(\bar{f}, \mathfrak{h}, G_0).$$

By the induction hypothesis,

$$\hat{\rho}(\bar{f}, \mathfrak{h}, G_0) \simeq \sum_{\bar{\Omega} \in V(\bar{f}, \mathfrak{h}, \mathfrak{a}^f)} c(\bar{\Omega}, \bar{f}, \mathfrak{h}, \mathfrak{a}^f) \hat{\rho}(\bar{\Omega})$$

and we can write

$$\hat{\rho}(f, \mathfrak{h}, G) \simeq \sum_{\bar{\Omega} \in V(\bar{f}, \mathfrak{h}, \mathfrak{a}^f)} c(\bar{\Omega}, \bar{f}, \mathfrak{h}, \mathfrak{a}^f) \mathrm{ind}_{G_0}^{G} \hat{\rho}(\bar{\Omega}).$$

Now let $\bar{\Omega}$ be the G_0-orbit of $\bar{f} + \bar{\lambda}$ with $\bar{\lambda} \in \mathfrak{h}^{\perp}$ in $(\mathfrak{a}^f)^*$. Let $\lambda \in \mathfrak{g}^*$ be an extension of $\bar{\lambda}$. From $\bar{\lambda}|_{\mathfrak{a}} = 0$, $\mathfrak{a}^{f+\lambda} = \mathfrak{a}^f$ and by Lemma 5.3.25 in Chap. 5

$$\mathrm{ind}_{G_0}^{G} \hat{\rho}(\bar{\Omega}) = \hat{\rho}(G \cdot (f + \lambda)).$$

Hence claim (A) follows, while, by Lemma 5.3.11, $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ if $\mathfrak{h} \in M(\bar{f} + \bar{\lambda}, \mathfrak{a}^f)$. Claim (B) follows.

Next, if we take $\lambda \in \mathfrak{h}^{\perp}$ such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$, then $\bar{\lambda} \in \mathfrak{h}^{\perp}$ and $\mathfrak{h} \in M(\bar{f} + \bar{\lambda}, \mathfrak{a}^f)$. Hence, with $\bar{\Omega} = G_0 \cdot (\bar{f} + \bar{\lambda})$, $\hat{\rho}(G \cdot (f + \lambda)) = \mathrm{ind}_{G_0}^{G} \hat{\rho}(\bar{\Omega})$ appears as irreducible component of $\hat{\rho}(f, \mathfrak{h}, G)$. By Theorem 3.4.4 in the third chapter, the representations $\mathrm{ind}_{G_0}^{G} \hat{\rho}(\bar{\Omega})$ are all mutually non-equivalent. Therefore, the multiplicity of the representation $\hat{\rho}(G\cdot(f+\lambda))$ in the irreducible decomposition of $\hat{\rho}(f,\mathfrak{h},G)$ is equal to $c(\bar{\Omega},\bar{f},\mathfrak{h},\mathfrak{a}^f)$. Now, $r\ :\ \mathfrak{g}^*\ \to\ (\mathfrak{a}^f)^*$ denoting the restriction map,

$$r\left(G\cdot(f+\lambda)\cap(f+\mathfrak{h}^\perp)\right)=\bar{\Omega}\cap(\bar{f}+\mathfrak{h}^\perp)$$

and $G\cdot(f+\lambda)\cap(f+\mathfrak{h}^\perp)$ is equal to the whole inverse image of its projection. In fact, let

$$f+\lambda_1\in G\cdot(f+\lambda)\cap(f+\mathfrak{h}^\perp)$$

and denote by $\overline{\Omega_1}$ the G_0-orbit through $\bar{f}+\overline{\lambda_1}$. Then

$$\hat{\rho}(G\cdot(f+\lambda_1))\simeq\hat{\rho}(G\cdot(f+\lambda))\simeq\mathrm{ind}_{G_0}^G\hat{\rho}(\overline{\Omega_1})\simeq\mathrm{ind}_{G_0}^G\hat{\rho}(\bar{\Omega}).$$

Hence, by Theorem 3.4.4 in the third chapter, $\bar{f}+\overline{\lambda_1}\ \in\ \bar{\Omega}$. Conversely, assume $\bar{f}+\overline{\lambda_1}\in\bar{\Omega}\cap(\bar{f}+\mathfrak{h}^\perp)$ and let $\lambda_1\in\mathfrak{h}^\perp$ be an arbitrary extension of $\overline{\lambda_1}$. Then,

$$\mathrm{ind}_{G_0}^G\hat{\rho}(\bar{\Omega})\simeq\hat{\rho}(G\cdot(f+\lambda_1))\simeq\hat{\rho}(G\cdot(f+\lambda)).$$

So, $f+\lambda_1\in G\cdot(f+\lambda)\cap(f+\mathfrak{h}^\perp)$. Thus, for any $\lambda\in\mathfrak{h}^\perp$ such that $\mathfrak{h}\in M(f+\lambda,\mathfrak{g})$,

$$c(G\cdot(f+\lambda),f,\mathfrak{h},\mathfrak{g})=c(\bar{\Omega},\bar{f},\mathfrak{h},\mathfrak{a}^f)$$

and claim (C) follows.

Hereafter, we assume $\mathfrak{a}\not\subset\mathfrak{h}$, put $\mathfrak{h}_0=\mathfrak{h}\cap\mathfrak{a}^f$, $\mathfrak{h}'=\mathfrak{h}_0+\mathfrak{a}$, $\mathfrak{k}=\mathfrak{h}+\mathfrak{a}$ and denote by $\mathfrak{j}$ the kernel of the action of $\mathfrak{h}$ on $\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})$. We separate the cases where $\mathfrak{h}_0+\mathfrak{j}$ is equal to $\mathfrak{h}$ or not.

Suppose that $\mathfrak{h}_0+\mathfrak{j}=\mathfrak{h}$. Since $\mathfrak{h}\in M(f,\mathfrak{g})$, $\dim(\mathfrak{h}/\mathfrak{h}_0)=\dim(\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a}))$ by Lemma 5.3.12 in Chap. 5. Hence, if we define $\mathfrak{t}$, $\mathfrak{w}$ as in Lemma 5.3.18, this lemma means

$$\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{h}_0,\ \mathfrak{h}'=\mathfrak{h}_0\oplus\mathfrak{w},\ \mathfrak{k}=\mathfrak{t}\oplus\mathfrak{h}_0\oplus\mathfrak{w},\ [\mathfrak{t},\mathfrak{w}]\subset\mathfrak{h}\cap\mathfrak{a},$$

B_f gives a duality between $\mathfrak{t}$ and $\mathfrak{w}$, and the representations $\hat{\rho}(f,\mathfrak{h},G)$ and $\hat{\rho}(f,\mathfrak{h}',G)$ are equivalent. Since $\mathfrak{h}'\supset\mathfrak{a}$, if we set $\Omega'=G\cdot(f+\lambda')$ $(\lambda'\in(\mathfrak{h}')^\perp)$, the preceding arguments imply

$$\hat{\rho}(f,\mathfrak{h},G)\simeq\sum_{\Omega'\in V(f,\mathfrak{h}')}c(\Omega',f,\mathfrak{h}',\mathfrak{g})\hat{\rho}(\Omega')$$

and (A) is derived.

To show assertion (B), it suffices to show that, for any $\lambda'\in(\mathfrak{h}')^\perp$, there exists $\lambda\in\mathfrak{h}^\perp$ such that $f+\lambda\in G\cdot(f+\lambda')$. In fact, since then $\dim(\mathfrak{g}(f+\lambda))=\dim(\mathfrak{g}(f+\lambda'))$, we have $\mathfrak{h}\in M(f+\lambda,\mathfrak{g})$ if $\mathfrak{h}'\in M(f+\lambda',\mathfrak{g})$.

Since B_f gives a duality between $\mathfrak{t}$ and $\mathfrak{w}$, there uniquely exists $V(\lambda') \in \mathfrak{w}$ so that $\lambda'(T) = f([V(\lambda'), T])$ for any $T \in \mathfrak{t}$. Let us see that $(\exp(V(\lambda')))\cdot(f + \lambda')$ belongs to $f + \mathfrak{k}^{\perp}$. Since $V(\lambda') \in \mathfrak{h}'$, $(\exp(V(\lambda')))\cdot(f + \lambda')$ belongs to $f + (\mathfrak{h}')^{\perp}$. Hence it suffices to see that $(\exp(V(\lambda')))\cdot(f + \lambda')$ belongs to $f + \mathfrak{t}^{\perp}$. Because $\mathfrak{a} \supset \mathfrak{w}$ is commutative, $(\exp(-V(\lambda')))\cdot T = T - [V(\lambda'), T]$. Hence

$$\big(\exp(V(\lambda'))\big)\cdot(f + \lambda')(T) = f(T) + \lambda'(T) - f\big([V(\lambda'), T]\big) = f(T).$$

To show assertion (C), it suffices to show that, for $\lambda \in \mathfrak{h}^{\perp}$ such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$, there exists $\lambda' \in (\mathfrak{h}')^{\perp}$ so that $f + \lambda' \in G\cdot(f + \lambda)$ and that

$$c(G\cdot(f + \lambda'), f, \mathfrak{h}', \mathfrak{g}) = c(G\cdot(f + \lambda), f, \mathfrak{h}, \mathfrak{g}).$$

Let us proceed as in the above arguments. For $\lambda \in \mathfrak{h}^{\perp}$, there uniquely exists an element $T(\lambda) \in \mathfrak{t}$ so that $\lambda(Y) = f([T(\lambda), Y])$ for any $Y \in \mathfrak{w}$. Then, $(\exp(T(\lambda)))\cdot(f + \lambda)$ belongs to $f + \mathfrak{k}^{\perp}$. Indeed, as $T(\lambda) \in \mathfrak{h}$, $(\exp(T(\lambda)))\cdot(f + \lambda)$ belongs to $f + \mathfrak{h}^{\perp}$, while, because $[\mathfrak{t}, \mathfrak{w}] \subset \mathfrak{h}$, we can write

$$(\exp(-T(\lambda)))\cdot(Y) = Y - [T(\lambda), Y] + Y'$$

for any $Y \in \mathfrak{w}$. Here $Y' \in [\mathfrak{h}, \mathfrak{h}]$. Therefore,

$$(\exp(T(\lambda)))\cdot(f + \lambda)(Y) = f(Y) + \lambda(Y) - f([T(\lambda), Y]) = f(Y).$$

Now we define the mapping R from $f + \mathfrak{h}^{\perp}$ to $f + \mathfrak{k}^{\perp}$ by

$$R(f + \lambda) = (\exp(T(\lambda)))\cdot(f + \lambda)$$

and the mapping R' from $f + (\mathfrak{h}')^{\perp}$ to $f + \mathfrak{k}^{\perp}$ by

$$R'(f + \lambda') = \big(\exp(V(\lambda'))\big)\cdot(f + \lambda').$$

Now, if we show the following two claims (a), (b), then the desired assertion (C) will be settled.

(a) Let $\lambda \in \mathfrak{h}^{\perp}$ be such that $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ and C a connected component of $G\cdot(f + \lambda) \cap (f + \mathfrak{h}^{\perp})$, then $R(C)$ is a connected component of $G\cdot(f + \lambda) \cap (f + \mathfrak{k}^{\perp})$ and the mapping $C \mapsto R(C)$ supplies a bijection between the connected components of $G\cdot(f + \lambda) \cap (f + \mathfrak{h}^{\perp})$ and those of $G\cdot(f + \lambda) \cap (f + \mathfrak{k}^{\perp})$.
(b) Let $\lambda' \in (\mathfrak{h}')^{\perp}$ be such that $\mathfrak{h}' \in M(f + \lambda', \mathfrak{g})$ and C' a connected component of $G\cdot(f + \lambda') \cap (f + (\mathfrak{h}')^{\perp})$, then $R'(C')$ is a connected component of $G\cdot(f + \lambda') \cap (f + \mathfrak{k}^{\perp})$ and the mapping $C' \mapsto R'(C')$ supplies a bijection between the connected components of $G\cdot(f + \lambda') \cap (f + (\mathfrak{h}')^{\perp})$ and those of $G\cdot(f + \lambda') \cap (f + \mathfrak{k}^{\perp})$.

In fact, let $\lambda \in \mathfrak{h}^{\perp}$ be such that $\mathfrak{h} \in M(f+\lambda,\mathfrak{g})$. Then $R(f+\lambda) \in G{\cdot}(f+\lambda) \cap (f+(\mathfrak{h}')^{\perp})$ and $\mathfrak{h}' \in M(R(f+\lambda),\mathfrak{g})$. Hence the representation $\hat{\rho}(G{\cdot}(f+\lambda))$ appears as an irreducible component of $\hat{\rho}(f,\mathfrak{h},\mathfrak{g})$ with multiplicity $c(G{\cdot}(f+\lambda), f, \mathfrak{h}', \mathfrak{g})$, while, from (b) this value is equal to the number of connected components of $G{\cdot}(f+\lambda) \cap (f+\mathfrak{k}^{\perp})$, and further from (a) also to $c(G{\cdot}(f+\lambda), f, \mathfrak{h}, \mathfrak{g})$. Thus we obtain the desired equality

$$c(G{\cdot}(f+\lambda), f, \mathfrak{h}, \mathfrak{g}) = c(G{\cdot}(f+\lambda), f, \mathfrak{h}', \mathfrak{g}).$$

Because assertions (a), (b) are proved in the same way, let us show (a) for instance. Since $T(\lambda) = 0$ if $\lambda \in \mathfrak{k}^{\perp}$, the mapping R is the identity mapping on $f+\mathfrak{k}^{\perp}$. Now let C be a connected component of $G{\cdot}(f+\lambda) \cap (f+\mathfrak{h}^{\perp})$, C is invariant by the group H. Thus $R(C) \subset C \cap (f+\mathfrak{k}^{\perp})$ and $R(C) = C \cap (f+\mathfrak{k}^{\perp})$, while, $G{\cdot}(f+\lambda) \cap (f+\mathfrak{k}^{\perp})$ is the union of $C \cap (f+\mathfrak{k}^{\perp})$, where C runs through the connected components of $G{\cdot}(f+\lambda) \cap (f+\mathfrak{h}^{\perp})$. Hence (a) follows.

Finally, assume $\mathfrak{h}_0 + \mathfrak{j} \neq \mathfrak{h}$. By Lemma 5.3.12(2) in Chap. 5, $\dim(\mathfrak{h}/\mathfrak{h}_0) = \dim(\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a}))$ and the conditions of Proposition 5.3.19 in Chap. 5 are fulfilled. With the notations used there, $\mathfrak{j} = \mathfrak{h}_0$, $\mathfrak{k} = \mathbb{R}X \oplus \mathfrak{h}_0 \oplus \mathbb{R}Y$, Here $[X,Y] = Y$, $f(Y) = 1$. Besides, ν_+ and ν_- denoting respectively a positive and a negative real number, we had

$$\hat{\rho}(f,\mathfrak{h},K) \simeq \hat{\rho}(f_{\nu_+},\mathfrak{h}',K) \oplus \hat{\rho}(f_{\nu_-},\mathfrak{h}',K).$$

Therefore, $\tilde{f}_{\nu_+}$ and $\tilde{f}_{\nu_-}$ denoting respectively an extension of f_{ν_+} and f_{ν_-} to $\mathfrak{g}$,

$$\hat{\rho}(f,\mathfrak{h},G) \simeq \hat{\rho}(\tilde{f}_{\nu_+},\mathfrak{h}',G) \oplus \hat{\rho}(\tilde{f}_{\nu_-},\mathfrak{h}',G).$$

$\mathfrak{h}' \supset \mathfrak{a}$ and, if $\nu \neq 0$, the orthogonal space of $\mathfrak{a}$ with respect to $B_{\tilde{f}_\nu}$ is a proper Lie subalgebra of $\mathfrak{g}$. So, if we can choose $\nu_+, \nu_-, \tilde{f}_{\nu_+}, \tilde{f}_{\nu_-}$ so that $\mathfrak{h}' \in M(\tilde{f}_{\nu_+},\mathfrak{g}) \cap M(\tilde{f}_{\nu_-},\mathfrak{g})$, the problem will be reduced to the case already treated. Let us see such a choice is indeed possible. Let $\mathfrak{p}$ be a linear complement of $\mathfrak{k}$ in $\mathfrak{g}$ and we identify $\mathfrak{g}^*$ with $\mathfrak{k}^* \oplus \mathfrak{p}^*$. Then the linear variety $f+\mathfrak{h}^{\perp}$ is identified with $\mathbb{R}\times\mathfrak{p}^*$ by the mapping $\tilde{f}_\nu \mapsto (\nu, \tilde{f}_\nu|\mathfrak{p})$. Since $\mathfrak{h}' \in S(\tilde{f}_\nu,\mathfrak{g})$ for any ν, $\tilde{f}_\nu$, $\mathfrak{h}' \in M(f,\mathfrak{g})$ implies that the set of the pairs $(\nu,\lambda) \in \mathbb{R}\times\mathfrak{p}^*$ such that $\mathfrak{h}' \in M(f_\nu+\lambda,\mathfrak{g})$ is a non-empty Zariski open set and hence dense.

So, assume $\mathfrak{h}' \in M(\tilde{f}_{\nu_+},\mathfrak{g}) \cap M(\tilde{f}_{\nu_-},\mathfrak{g})$. Then, from what we saw already, putting $\Omega_+ = G{\cdot}(\tilde{f}_{\nu_+}+\lambda')$ and $\Omega_- = G{\cdot}(\tilde{f}_{\nu_-}+\lambda')$ with $\lambda' \in (\mathfrak{h}')^{\perp}$,

$$\hat{\rho}(f,\mathfrak{h},G) \simeq \sum_{\Omega_+ \in V(\tilde{f}_{\nu_+}+\lambda',\mathfrak{h}')} c(\Omega_+,\tilde{f}_{\nu_+},\mathfrak{h}',\mathfrak{g})\hat{\rho}(\Omega_+) \oplus \sum_{\Omega_- \in V(\tilde{f}_{\nu_-}+\lambda',\mathfrak{h}')} c(\Omega_-,\tilde{f}_{\nu_-},\mathfrak{h}',\mathfrak{g})\hat{\rho}(\Omega_-)$$

and assertion (A) holds.

To show assertion (B), it suffices to see that, for any $\lambda' \in (\mathfrak{h}')^{\perp}$ and $v \neq 0$, there exists $\lambda \in \mathfrak{h}^{\perp}$ so that $f + \lambda \in G{\cdot}(\tilde{f}_v + \lambda')$. Then, $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ if $\mathfrak{h}' \in M(\tilde{f}_v + \lambda', \mathfrak{g})$.

Indeed, we choose a real number $a(\lambda')$ so that $\tilde{f}_v([a(\lambda')Y, X]) = \lambda'(X)$, i.e. $a(\lambda') = -\frac{\lambda'(X)}{v}$. Then,

$$\exp(-a(\lambda')Y){\cdot}X = X - [a(\lambda')Y, X]$$

and

$$\exp(a(\lambda')Y){\cdot}(\tilde{f}_v + \lambda') \in \tilde{f}_v + \mathfrak{h}'^{\perp}$$

since $Y \in \mathfrak{h}'$. Hence

$$\exp(a(\lambda')Y){\cdot}(\tilde{f}_v + \lambda')(X) = f(X) + \lambda'(X) - \tilde{f}_v\left([a(\lambda')Y, X]\right) = f(X)$$

and

$$\exp(a(\lambda')Y){\cdot}(\tilde{f}_v + \lambda') \in \tilde{f}_v + \mathfrak{k}^{\perp} \subset f + \mathfrak{h}^{\perp}.$$

We define the mapping R'_v from $\tilde{f}_v + (\mathfrak{h}')^{\perp}$ to $\tilde{f}_v + \mathfrak{k}^{\perp}$ by

$$R'_v(\tilde{f}_v + \lambda') = \exp(a(\lambda')Y){\cdot}(\tilde{f}_v + \lambda').$$

Then as before, let $\lambda' \in (\mathfrak{h}')^{\perp}$ be such that $\mathfrak{h}' \in M(\tilde{f}_v + \lambda')$ and C' a connected component of $G{\cdot}(\tilde{f}_v + \lambda') \cap (\tilde{f}_v + (\mathfrak{h}')^{\perp})$, then $R'_v(C') = C' \cap (\tilde{f}_v + \mathfrak{k}^{\perp})$ and the mapping R'_v gives a bijection between the connected components of $G{\cdot}(\tilde{f}_v + \lambda') \cap (\tilde{f}_v + (\mathfrak{h}')^{\perp})$ and those of $G{\cdot}(\tilde{f}_v + \lambda') \cap (\tilde{f}_v + \mathfrak{k}^{\perp})$. Therefore, we have

$$c(G{\cdot}(\tilde{f}_v + \lambda'), \tilde{f}_v, \mathfrak{h}', \mathfrak{g}) = c(G{\cdot}(\tilde{f}_v + \lambda'), \tilde{f}_v, \mathfrak{k}, \mathfrak{g}).$$

Let us prove assertion (C). Let $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ and $\lambda \in \mathfrak{h}^{\perp}$. Then from the decomposition formula of $\hat{\rho}(f, \mathfrak{h}, G)$, the multiplicity of the representation $\hat{\rho}(G{\cdot}(f + \lambda))$ is equal to

$$c(G{\cdot}(f + \lambda), \tilde{f}_{v_+}, \mathfrak{h}', \mathfrak{g}) + c(G{\cdot}(f + \lambda), \tilde{f}_{v_-}, \mathfrak{h}', \mathfrak{g}).$$

Next, if we put

$$(f + \mathfrak{h}^{\perp})^+ = \left\{f + \ell \in f + \mathfrak{h}^{\perp} : (f + \ell)(Y) > 0\right\},$$
$$(f + \mathfrak{h}^{\perp})^- = \left\{f + \ell \in f + \mathfrak{h}^{\perp} : (f + \ell)(Y) < 0\right\},$$

$G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})$ is the union of $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})^{+}$ and $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})^{-}$. In fact, let $f+\ell\in G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})$, then $\mathfrak{h}\in M(f+\ell,\mathfrak{g})$ and Proposition 5.3.19(3) in Chap. 5 gives $(f+\ell)(Y)\neq 0$. Hence each connected component of $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})$ is contained in either $(f+\mathfrak{h}^{\perp})^{+}$ or $(f+\mathfrak{h}^{\perp})^{-}$. We denote by $c^{+}(G{\cdot}(f+\lambda),f,\mathfrak{h})$ and $c^{-}(G{\cdot}(f+\lambda),f,\mathfrak{h})$ the number of connected components of $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})^{+}$ and $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})^{-}$ respectively. From what we have seen until now, it is enough to show the equalities

$$c^{+}(G{\cdot}(f+\lambda),f,\mathfrak{h})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{+}},\mathfrak{h}',\mathfrak{g})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{+}},\mathfrak{k},\mathfrak{g})$$

and

$$c^{-}(G{\cdot}(f+\lambda),f,\mathfrak{h})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{-}},\mathfrak{h}',\mathfrak{g})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{-}},\mathfrak{k},\mathfrak{g}).$$

Because it is the same thing, let us prove

$$c^{+}(G{\cdot}(f+\lambda),f,\mathfrak{h})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{+}},\mathfrak{k},\mathfrak{g}).$$

Let $f+\ell\in(f+\mathfrak{h}^{\perp})^{+}$, then there uniquely exists a real number $x(\ell)$ satisfying

$$\exp(x(\ell)X){\cdot}(f+\ell)\in\tilde{f}_{v_{+}}+\mathfrak{k}^{\perp}.$$

In fact, since $\exp(x(\ell)X){\cdot}(f+\ell)\in\tilde{f}_{v_{+}}+\mathfrak{h}^{\perp}$, it suffices that $\exp(x(\ell)X){\cdot}(f+\ell)(Y)=v_{+}$. Namely,

$$e^{-x(\ell)}=\frac{v_{+}}{(f+\ell)(Y)}.$$

We define the mapping

$$R^{+}:(f+\mathfrak{h}^{\perp})^{+}\to\tilde{f}_{v_{+}}+\mathfrak{k}^{\perp}\subset f+\mathfrak{h}^{\perp}$$

by $R^{+}(f+\ell)=\exp(x(\ell)X){\cdot}(f+\ell)$. Then, if C^{+} is a connected component of $G{\cdot}(f+\lambda)\cap(f+\mathfrak{h}^{\perp})^{+}$, it turns out that $R^{+}(C^{+})=C^{+}\cap(\tilde{f}_{v_{+}}+\mathfrak{k}^{\perp})$. Thus, we find

$$c^{+}(G{\cdot}(f+\lambda),f,\mathfrak{h})=c(G{\cdot}(f+\lambda),\tilde{f}_{v_{+}},\mathfrak{k},\mathfrak{g})$$

and assertion (C) holds. ∎

To examine the concrete Plancherel formula, we begin with a lemma:

Lemma 11.2.3. *Take* $\mathfrak{h}\in M(f,\mathfrak{g})$*. Then there exists* $\mathfrak{b}\in I(f,\mathfrak{g})$ *provided with the following properties. We put* $B=exp\,\mathfrak{b},\pi=ind_{B}^{G}\chi_{f}$ *and denote by* $\left(\mathcal{H}_{\pi}^{\infty}\right)_{0}$ *the subspace of* $\mathcal{H}_{\pi}^{\infty}$ *composed of all functions with compact supports modulo* B*. Then*:

(1) $\Delta_{H\cap B,H}(h)\Delta_{H\cap B,B}(h) = 1$ *for any* $h \in H \cap B$;
(2) HB *is a closed set of* G;
(3) *From* (1) *and* (2)*, we are able to consider the anti-linear form*

$$a : \left(\mathcal{H}_\pi^\infty\right)_0 \ni \phi \mapsto \oint_{H/(H\cap B)} \overline{\phi(h)\chi_f(h)} \Delta_{H,G}^{-1/2}(h) d\mu_{H,H\cap B}(h) \in \mathbb{C}$$

and this form is extended to an element of $\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$.

Proof. If there is a non-trivial ideal of $\mathfrak{g}$ where f vanishes, we may immediately pass to the quotient by this ideal and apply the induction hypothesis. From now on we suppose that there is no such ideal. Let $\mathfrak{a}$ be a minimal non-central ideal, $\mathfrak{g}_0 = \mathfrak{a}^f \neq \mathfrak{g}, G_0 = \exp(\mathfrak{g}_0), \mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}_0 + \mathfrak{a} \in M(f, \mathfrak{g})$ and $H' = \exp(\mathfrak{h}')$. By the induction hypothesis, at the stage of the subgroup G_0 there is $\mathfrak{b} \in I(f_0, \mathfrak{g}_0)$ with three required properties. Here $f_0 = f|_{\mathfrak{g}_0} \in \mathfrak{g}_0^*$. It is a matter of the case where $\mathfrak{a} \not\subset \mathfrak{h}$.

(1) Put $\mathfrak{h}_0 = \mathfrak{h}\cap\mathfrak{g}_0$ and $H_0 = \exp(\mathfrak{h}_0)$. For all $h = \exp X \in H\cap B = H_0\cap B$ ($X \in \mathfrak{h}_0 \cap \mathfrak{b}$),

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}/\mathfrak{h}_0} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{a}/(\mathfrak{h}\cap\mathfrak{a})} X = 0. \tag{11.2.1}$$

However, $\mathfrak{b}$ is chosen so that

$$\Delta_{H'\cap B,H'}(h')\Delta_{H'\cap B,B}(h') = 1 \ (h' \in H' \cap B)$$

might hold, and this means

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}'/(\mathfrak{h}'\cap\mathfrak{b})} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{b}/(\mathfrak{h}'\cap\mathfrak{b})} X = 0$$

for all $X \in \mathfrak{h}' \cap \mathfrak{b} = (\mathfrak{h} \cap \mathfrak{b}) + \mathfrak{a}$. Again, this is the same as

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_0/(\mathfrak{h}_0\cap\mathfrak{b})} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{b}/(\mathfrak{h}_0\cap\mathfrak{b})} X - \mathrm{Tr}\ \mathrm{ad}_{(\mathfrak{h}'\cap\mathfrak{b})/(\mathfrak{h}_0\cap\mathfrak{b})} X = 0 \tag{11.2.2}$$

holds. From (11.2.1) and (11.2.2), identifying $(\mathfrak{h}' \cap \mathfrak{b})/(\mathfrak{h}_0 \cap \mathfrak{b})$ with $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$,

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_0/(\mathfrak{h}_0\cap\mathfrak{b})} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{b}/(\mathfrak{h}_0\cap\mathfrak{b})} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}/\mathfrak{h}_0} X = 0$$

for any $X \in \mathfrak{h} \cap \mathfrak{b}$. Besides,

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b})} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{b}/(\mathfrak{h}\cap\mathfrak{b})} X = 0.$$

So, we obtain the desired equality.

(2) We denote by $\mathfrak{z}$ the centre of $\mathfrak{g}$ and by $\mathfrak{j}$ (resp. $\mathfrak{g}_1$) the kernel of the adjoint representation of $\mathfrak{h}$ (resp. $\mathfrak{g}$) on $\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})$ (resp. $\mathfrak{a}/(\mathfrak{z} \cap \mathfrak{a})$). Then, there occurs

two possibilities: either $\mathfrak{j} + \mathfrak{h}_0 = \mathfrak{h}$ or $\mathfrak{j} + \mathfrak{h}_0 = \mathfrak{h}_0$. In the first case, let $\mathfrak{m}$ be a linear complement to $\mathfrak{h}_0$ in $\mathfrak{h}$ contained in $\mathfrak{j}$. Then a basis of $\mathfrak{m}$ constitutes a part of a coexponential basis to $\mathfrak{g}_0$ in $\mathfrak{g}$ and so HB is a closed set of G.

If $\mathfrak{j} + \mathfrak{h}_0 = \mathfrak{h}_0$, then $\mathfrak{j} = \mathfrak{h}_0, \mathfrak{h} \cap \mathfrak{a} = \mathfrak{z} \cap \mathfrak{a}$ and $\dim(\mathfrak{h}/\mathfrak{h}_0) = \dim(\mathfrak{a}/(\mathfrak{h} \cap \mathfrak{a})) = 1$. Hence $\mathfrak{g}(f) \subset \mathfrak{g}_1$ and by the induction hypothesis we can choose our polarization $\mathfrak{b}$ in $\mathfrak{g}_1$. If we take the coexponential basis $X \in \mathfrak{h}, X \notin \mathfrak{j}$ to $\mathfrak{g}_1$ in $\mathfrak{g}$, $HB = \exp(\mathbb{R}X)\cdot H'B$ is a closed set of G because $H'B$ is a closed set of $G_1 = \exp(\mathfrak{g}_1)$.

(3) Taking (1) into account, with $\phi \in \mathcal{H}_\pi$, we can apply the linear functional $\mu_{H,H\cap B}$ to the function

$$\phi_H : H \ni h \mapsto \overline{\phi(h)\chi_f(h)}\Delta_{H,G}^{-1/2}(h).$$

Now, when ϕ moves in $\left(\mathcal{H}_\pi^\infty\right)_0$, we have by (2) that

$$\mu_{H,H\cap B}(\phi_H) = \oint_{H/(H\cap B)} \overline{\phi(h)\chi_f(h)}\Delta_{H,G}^{-1/2}(h)d\mu_{H,H\cap B}(h) < \infty.$$

Then the anti-linear form $\left(\mathcal{H}_\pi^\infty\right)_0 \ni \phi \mapsto \mu_{H,H\cap B}(\phi_H)$ could be uniquely extended to a nonzero element a of $\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f\Delta_{H,G}^{1/2}}$. This is easily derived from what we have seen until now, but we add some comments.

If we can choose $\mathfrak{b} \in I(f, \mathfrak{g})$ so that an intertwining operator R between $\pi = \mathrm{ind}_B^G \chi_f$ and $\tau = \mathrm{ind}_H^G \chi_f$ is given by the formula

$$(R\phi)(g) = \oint_{H/H\cap B} \phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)d\mu_{H,H\cap B}(h)\ (g \in G),$$

the generalized vector $a = \delta_\tau \circ R$ is what we are looking for. Now we think about R. Put $\tau' = \mathrm{ind}_{H'}^G \chi_f$. If an intertwining operator T between τ' and τ is constructed by giving a sense to the formal operator defined by

$$(T\psi)(g) = \oint_{H/H_0} \psi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)d\mu_{H,H_0}(h)\ (g \in G)$$

for $\psi \in \mathcal{H}_{\tau'}$, the desired R is obtained from the induction hypothesis applied to G_0 and the transitivity of $\mu_{\cdot,\cdot}$.

As for T, it is enough to consider the Lie subalgebra $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$. So, we assume $\mathfrak{g} = \mathfrak{k}$. For instance, let us examine the case where τ splits into two parts by the modification $\mathfrak{h} \to \mathfrak{h}'$. Let $\mathfrak{a} = \mathbb{R}Y$ (or $\mathfrak{a} = \mathbb{R}Y + \mathfrak{z}$), $f(Y) = 1, \mathfrak{h} = \mathbb{R}X + \mathfrak{h}_0$ and $[X, Y] = Y$. Using the one-parameter subgroup $\exp(\mathbb{R}Y)$ (resp. $\exp(\mathbb{R}X)$), the space $\mathcal{H}_\tau$ (resp. $\mathcal{H}_{\tau'}$) is identified with $L^2(\mathbb{R})$ and

$$
\begin{aligned}
(T\psi)(s) = (T\psi)(\exp(sY)) &= \int_{\mathbb{R}} \psi(\exp(sY)\exp(tX))e^{-t/2}dt \\
&= \int_{\mathbb{R}} \psi(\exp(tX)\exp(se^{-t}Y))e^{-t/2}dt = \int_{\mathbb{R}} \psi(\exp(tX))e^{-ise^{-t}}e^{-t/2}dt
\end{aligned}
$$

for $\psi \in \mathcal{H}_{\tau'}$. Hence, by change of variables,

$$
\begin{aligned}
\|T\psi\|^2 &= \int_{\mathbb{R}} ds \Big| \int_{\mathbb{R}} \psi(\exp(tX))e^{-ise^{-t}}e^{-t/2}dt \Big|^2 \\
&= \int_{\mathbb{R}} ds \Big| \int_0^{+\infty} \psi(\exp((-\log u)X))e^{-isu}\frac{du}{\sqrt{u}} \Big|^2 \\
&= 2\pi \int_0^{+\infty} |\psi(\exp((-\log u)X))|^2 \frac{du}{u} \\
&= 2\pi \int_{\mathbb{R}} |\psi(\exp(-vX))|^2 dv = 2\pi \|\psi\|^2.
\end{aligned}
$$

When τ does not split, T is similarly interpreted as a Fourier transformation. ■

In the above situation we have the following:

Theorem 11.2.4 ([31]). *When Ω runs over $U(f, \mathfrak{h})$, take ℓ_Ω^k ($1 \le k \le c(\Omega)$) arbitrarily in each connected component C_Ω^k of $\Omega \cap \Gamma_\tau$. At these points $\ell_\Omega^k \in \mathfrak{g}^*$ ($\Omega \in U(f, \mathfrak{h})$, $1 \le k \le c(\Omega)$), we choose polarizations $\mathfrak{b}_\Omega^k \in I(\ell_\Omega^k, \mathfrak{g})$ of Lemma 11.2.3 and construct $a_\Omega^k \in \left(\mathcal{H}_{\hat{\rho}(\Omega)}^{-\infty}\right)^{H, \chi_f \Delta_{H,G}^{1/2}}$. Then by means of matrix elements concerning these a_Ω^k appropriately normalized, the concrete* Plancherel *formula for τ is written down: for any $\phi \in \mathcal{D}(G)$,*

$$
\phi_H^f(e) = \sum_{\Omega \in U(f, \mathfrak{h})} \sum_{k=1}^{c(\Omega)} \langle \hat{\rho}(\Omega) a_\Omega^k, a_\Omega^k \rangle.
$$

Examining various cases, we can prove this theorem by induction. However, we here abstain from mentioning its proof and give in what follows certain examples.

11.3 Examples of Penney's Plancherel Formula

Example 11.3.1. Let $\mathfrak{g}$ be the $ax + b$ Lie algebra taken up in Chap. 4: $\mathfrak{g} = \mathbb{R}e_1 + \mathbb{R}e_2, [e_1, e_2] = e_2$. $f = e_2^*$ and $\mathfrak{h} = \mathbb{R}e_1 \in M(f, \mathfrak{g})$. The irreducible unitary representations of the completely solvable Lie group $G = \exp \mathfrak{g}$, which are of infinite dimension, are only $\pi_\pm$ corresponding to two open orbits $\pm G\cdot e_2^*$. Now the monomial representation $\tau = \operatorname{ind}_H^G \chi_f$, $H = \exp \mathfrak{h}$, is equivalent to the direct sum $\pi_+ \oplus \pi_-$.

We realize $\pi_\pm$ by means of $B = \exp(\mathbb{R}e_2)$ as $\operatorname{ind}_B^G \chi_{\pm e_2^*}$ and identify $\mathcal{H}_{\pi_\pm}$ with $L^2(\mathbb{R})$ by the mapping $\psi \mapsto \tilde{\psi}(t) = \psi(\exp(te_1))$ $(t \in \mathbb{R})$. For $\psi \in \mathcal{H}_{\pi_\pm}^\infty$,

$$((\pi_\pm(e_2))^m\psi)^\sim(t) = (\pm i)^m e^{-mt}\psi(t) \in L^2(\mathbb{R})$$

with any $m \in \mathbb{N}$. Thus,

$$\begin{aligned}
\Big|\int_{\mathbb{R}} e^{-t/2}\tilde{\psi}(t)dt\Big| &\leq \int_{-\infty}^{0} |e^{-t/2}\tilde{\psi}(t)|dt + \int_{0}^{+\infty} |e^{-t/2}\tilde{\psi}(t)|dt \\
&\leq \left(\int_{-\infty}^{0} e^t dt\right)^{1/2} \left(\int_{-\infty}^{0} |e^{-t}\tilde{\psi}(t)|^2 dt\right)^{1/2} \\
&\quad + \left(\int_{0}^{+\infty} e^{-t} dt\right)^{1/2} \left(\int_{0}^{+\infty} |\tilde{\psi}(t)|^2 dt\right)^{1/2} \\
&= \left(\int_{0}^{+\infty} |\tilde{\psi}(t)|^2 dt\right)^{1/2} + \left(\int_{-\infty}^{0} |e^{-t}\tilde{\psi}(t)|^2 dt\right)^{1/2} \leq \|\psi\| + \|\pi_\pm(e_2)\psi\|.
\end{aligned}$$

From this, if we set

$$a_\pm(\psi) = \int_{\mathbb{R}} \overline{\psi(\exp(te_1))} e^{-t/2} dt$$

for $\psi \in \mathcal{H}_{\pi_\pm}^\infty$, then $a_\pm \in \left(\mathcal{H}_{\pi_\pm}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$. Take $\phi \in \mathcal{D}(G)$. Choosing dg on G, we compute

$$(\pi_+(\phi)a_+)^\sim(t) = \int_{\mathbb{R}^2} \phi(\exp(se_1)\exp(ue_2)) e^{iue^{s-t}} e^{(s-t)/2} ds du;$$

further,

$$\langle \pi_+(\phi)a_+, a_+\rangle = \int_0^{+\infty} dv \int_{\mathbb{R}^2} \phi(\exp(se_1)\exp(ue_2)) e^{iuv} e^{-s/2} ds du.$$

Likewise,

$$\langle \pi_-(\phi)a_-, a_-\rangle = \int_{-\infty}^0 dv \int_{\mathbb{R}^2} \phi(\exp(se_1)\exp(ue_2)) e^{iuv} e^{-s/2} ds du.$$

In this way our concrete Plancherel formula for τ reduces, normalizing appropriately various measures, to the obvious formula

$$\begin{aligned}&\langle \pi_+(\phi)a_+, a_+\rangle + \langle \pi_-(\phi)a_-, a_-\rangle \\ =&2\pi \int_{\mathbb{R}} \phi(\exp(se_1))e^{-s/2}ds = \phi_H^f(e) = \langle \tau(\phi)\delta_\tau, \delta_\tau\rangle.\end{aligned}$$

If we consider the trivial representation π_0 of G, clearly $\Omega(\pi_0) \cap \Gamma_\tau = \Omega(\pi_0) = \{0\}$, which is connected and one H-orbit. Besides, if we regard π_0 as constructed by means of $\mathfrak{g} \in I(0, \mathfrak{g})$, the closure of the set $\{g \in G; g\cdot(0 + \mathfrak{g}^\perp) \cap \Gamma_\tau \neq \emptyset\}$ is the whole G. In spite of these facts, $\left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = \{0\}$.

Example 11.3.2. Let $G = G_3(\alpha) = \exp(\mathfrak{g}_3(\alpha)), \mathfrak{g}_3(\alpha) = \langle T, X, Y\rangle_{\mathbb{R}}$ taken up in Chap. 4: $[T, X] = X - \alpha Y, [T, Y] = Y + \alpha X$. Take $f = X^* \in \mathfrak{g}_3(\alpha)^*$ and $\mathfrak{h} = \mathbb{R}X$. Here we may assume that α is negative. Let us parameterize using (x, y)-coordinates the orbit passing $\ell = (1, \lambda) \in \mathfrak{a}^* = \mathbb{R}X^* + \mathbb{R}Y^*$. Here,

$$x(t) = e^t(\cos(\alpha t) - \lambda \sin(\alpha t)), \tag{11.3.1}$$

$$y(t) = e^t(\sin(\alpha t) + \lambda \cos(\alpha t)). \tag{11.3.2}$$

If the line $x = 1$ is tangent to the orbit of ℓ, then $(dx/dt)_{t=0} = 1 - \alpha\lambda = 0$ and $\lambda = 1/\alpha$, We indicate this value by λ_0. We denote by t^* the first positive number t satisfying

$$e^t(\cos(\alpha t) - (1/\alpha)\sin(\alpha t)) = 1$$

by λ_1 the y-coordinate of the crossing point and choose $\ell = (1, \lambda) \in \mathfrak{a}^*, \lambda_0 \leq \lambda < \lambda_1$ as the representatives of the orbits which encounter the affine space Γ_τ. All these orbits are saturated in the direction of $\mathbb{R}T^*$.

Using the parametrization of the orbits and imposing

$$e^t(\cos(\alpha t) - \lambda \sin(\alpha t)) = 1,$$

let us look for the crossing point with Γ_τ. With θ satisfying

$$\sin\theta = \lambda(1 + \lambda)^{-1/2}, \ \cos\theta = (1 + \lambda^2)^{-1/2}, \ e^t(1 + \lambda^2)^{1/2}\cos(\alpha t + \theta) = 1.$$

Hence,

$$e^t(\sin(\alpha t) + \lambda\cos(\alpha t)) = (1 + \lambda^2)^{1/2}e^t\sin(\alpha t + \theta) = \tan(\alpha t + \theta).$$

Thus $\ell_n = (1, \tan(\alpha t_n + \theta)), \ell_0 = \ell$. Choosing arbitrarily $g_n \in G$ such that $g_n : \ell \mapsto \ell_n$, we construct the map

$$a_n : \phi \mapsto \oint_{H/(H\cap g_n B g_n^{-1})} \overline{\phi(hg_n)\chi_f(h)}\Delta_{H,G}^{-1/2}(h)d\nu(h) \ (\phi \in \mathcal{H}_\pi^\infty). \tag{11.3.3}$$

This last value is nothing but $\overline{\phi(g_n)}$. We put $B = \exp \mathfrak{b}$ according to the polarization $\mathfrak{b} = \mathbb{R}X + \mathbb{R}Y$ and denote by $\mathcal{H}_\pi$ the representation space of $\pi = \pi_\ell = \mathrm{ind}_B^G \chi_\ell$. As is easily seen, $a_n \in \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$. So, if we compute for $\psi \in \mathcal{D}(G)$,

$$(\pi(\psi)a_n)(g) = \Delta_G^{-1}(g_n) \int_B \psi(gbg_n^{-1})\chi_\ell(b)db \ (g \in G).$$

Moreover,

$$\langle \pi(\psi)a_n, a_n \rangle = \int_{\mathbb{R}} \psi_H^f(\exp(sY))\exp(is\tan(\alpha t_n + \theta))ds.$$

Writing simply y instead of $y(t)$, the formulas (11.3.1), (11.3.2) give

$$(1 - \alpha y)(dt/d\lambda) = e^t \sin(\alpha t), \tag{11.3.4}$$

$$(dy/d\lambda) - (y + \alpha)(dt/d\lambda) = e^t \cos(\alpha t). \tag{11.3.5}$$

On the other hand, Taking $e^{2t} = (1 + y^2)/(1 + \lambda^2)$ into account,

$$\{(1 + \lambda^2)(dt/d\lambda) + \lambda\}(1 + y^2)/(1 + \lambda^2) = y(dy/d\lambda). \tag{11.3.6}$$

Equations (11.3.4), (11.3.5) imply

$$\lambda(dy/d\lambda) + (1 - \alpha y - \lambda y - \alpha\lambda)(dt/d\lambda) = y,$$

$$(dy/d\lambda) + (\lambda\alpha y - \lambda - y - \alpha)(dt/d\lambda) = 1.$$

Therefore,

$$(\lambda - y)(dy/d\lambda) + (1 - \alpha\lambda)(1 + y^2)(dt/d\lambda) = 0.$$

Substituting (11.3.6),

$$(\lambda - y)(dy/d\lambda) + (1 - \alpha\lambda)\{y(dy/d\lambda) - \lambda(1 + y^2)/(1 + \lambda^2)\} = 0,$$

Namely,

$$\lambda(1 - \alpha y)(dy/d\lambda) = \lambda(1 + \alpha\lambda)(1 + y^2)/(1 + \lambda^2).$$

Finally, for λ not equal to 0,

$$(dy/d\lambda) = (1 - \alpha\lambda)(1 + y^2)/(1 - \alpha y)(1 + \lambda^2).$$

Now the formula which we must show becomes the following: because we may properly multiply a_n by a constant if necessary, $y_n = y_n(\lambda)$ denoting $\tan(\alpha t_n + \theta)$,

$$\psi_H^f(e) = \int_{\lambda_0}^{\lambda_1} \xi(\lambda)d\lambda \left(\sum_{n=0}^{\infty} \kappa(n) \int_{\mathbb{R}} \psi_H^f(\exp(sY))\exp(is\tan(\alpha t_n + \theta)) \right) ds$$

$$= (2\pi)^{-1/2} \int_{\lambda_0}^{\lambda_1} \xi(\lambda) \sum_{n=0}^{\infty} \kappa(n)(\psi_H^f)\widehat{\ }(y_n)d\lambda,$$

here $\xi(\lambda)$ denotes a certain measurable function, $\kappa(n) \geq 0$ and $(\psi_H^f)\widehat{\ }$ the Fourier transform of $\psi_H^f \circ \exp$.

In fact, set $\xi(\lambda) = (1 - \alpha\lambda)/2\pi(1 + \lambda^2)$ and $\kappa(n) = (1 + y_n^2)|1 - \alpha y_n|$, that is, take $\{(1 + y_n^2)/|1 - \alpha y_n|\}^{1/2} a_n$. Then,

$$(2\pi)^{-1/2} \int_{\lambda_0}^{\lambda_1} \left(\sum_{n=0}^{\infty} (\psi_H^f)\widehat{\ }(y_n)\kappa(n) \right) \xi(\lambda)d\lambda$$

$$= (2\pi)^{-1/2} \sum_{n=0}^{\infty} \int_{\lambda_0}^{\lambda_1} (\psi_H^f)\widehat{\ }(y_n)(1 - \alpha\lambda)(1 + y_n^2)/|1 - \alpha y_n|(1 + \lambda^2)d\lambda$$

$$= (2\pi)^{-1/2} \int_{\mathbb{R}} (\psi_H^f)\widehat{\ }(s)ds = (\psi_H^f \circ \exp)(0) = \psi_H^f(e).$$

Example 11.3.3. Like the last example let $G = G_3(\alpha) = \exp(\mathfrak{g}_3(\alpha))$. Here take $\mathfrak{h} = \mathbb{R}T$ and let $f \in \mathfrak{g}^*$ be arbitrary. Then,

$$\Gamma_\tau = f + \mathfrak{h}^\perp = f(T)T^* + \mathbb{R}X^* + \mathbb{R}Y^*.$$

A generic orbit has its representative $\hat{\theta} = (\cos\theta)X^* + (\sin\theta)Y^*$ and corresponds to the irreducible representation $\pi_\theta = \mathrm{ind}_B^G \chi_{\hat{\theta}}$ of G. Here B is the analytic subgroup having the polarization $\mathfrak{b} = \mathbb{R}X + \mathbb{R}Y$ as Lie algebra.

In this case the usual formula (11.3.3) to obtain an H-semi-invariant generalized vector gives

$$a_\theta : \phi \mapsto \int_H \overline{\phi(h)\chi_f(h)} \Delta_{H,G}^{-1/2}(h)dh.$$

We know by similar reasoning as in the case of $ax + b$ that a_θ fulfils our request. For $\psi \in \mathcal{D}(G)$,

$$(\pi_\theta(\psi)a_\theta)(g) = \int_B \psi_H^f(gb)\chi_{\hat{\theta}}(b)db \ (g \in G),$$

$$\langle \pi_\theta(\psi)a_\theta, a_\theta\rangle = \int_{\mathbb{R}} e^{2t}dt \int_B \psi_H^f(b)\chi_\ell(b)db,$$

where $\ell = h\cdot\hat{\theta} = e^t \cos(\alpha t + \theta)X^* + e^t \sin(\alpha t + \theta)Y^*,\ h = \exp(tT)$.

In this fashion,

$$\begin{aligned}(2\pi)^{-2}&\int_0^{2\pi} \langle \pi_\theta(\psi)a_\theta, a_\theta\rangle d\theta \\ &= (2\pi)^{-2}\int_0^{2\pi} d\theta \int_{\mathbb{R}} e^{2t}dt \int_{\mathbb{R}^2} \psi_H^f(\exp(b_1X + b_2Y)) \\ &\quad \times \exp\big(ie^t(b_1\cos(\alpha t + \theta) + b_2\sin(\alpha t + \theta))\big)\, db_1db_2.\end{aligned}$$

If we perform the change of variables

$$x = e^t\cos(\alpha t + \theta),\ y = e^t\sin(\alpha t + \theta),$$

its Jacobian is

$$\partial(x, y)/\partial(t, \theta) = e^{2t};\ dxdy = e^{2t}dtd\theta$$

and so

$$\begin{aligned}(2\pi)^{-2}&\int_0^{2\pi} \langle \pi_\theta(\psi)a_\theta, a_\theta\rangle d\theta \\ &= (2\pi)^{-2}\int_{\mathbb{R}^2} dxdy \int_{\mathbb{R}^2} (\psi_H^f)^*(b_1, b_2)\exp(i(xb_1 + yb_2))db_1db_2 \\ &= (\psi_H^f)^*(0,0) = \psi_H^f(e),\end{aligned}$$

where $(\psi_H^f)^*(b_1, b_2) = \psi_H^f(\exp(b_1X + b_2Y))$.

Example 11.3.4. Let $G = \exp(\mathfrak{g}_4), \mathfrak{g}_4 = \langle T, X, Y, Z\rangle_{\mathbb{R}}; [T, X] = X, [T, Y] = -Y, [X, Y] = Z$ (split oscillator group). In matrix form

$$T = \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix},\ \ X = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$Y = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\ \ Z = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

$$G = \exp \mathfrak{g} = \left\{ \begin{pmatrix} w & x & z \\ 0 & 1 & y \\ 0 & 0 & w \end{pmatrix}; w > 0, x, y, z \in \mathbb{R} \right\}.$$

Set $f(\alpha, \beta) = \alpha T^* + \beta Z^* \in \mathfrak{g}_4^*$ with $\beta \neq 0$ and $\Omega(\alpha, \beta) = G \cdot f(\alpha, \beta)$. Now take $f = f(\alpha_0, \beta_0)$.

(i) First let $\mathfrak{h} = \mathbb{R}T + \mathbb{R}X$. Then,

$$\tau = \mathrm{ind}_H^G \chi_f \simeq \int_{\mathbb{R}} \hat{\rho}(\Omega(\alpha_0, \beta)) d\beta.$$

Using $\ell = f(\alpha_0, \beta) \in \Gamma_\tau \cap \Omega(\alpha_0, \beta)$ and a polarization $\mathfrak{b} = \langle T, X, Z \rangle_{\mathbb{R}}$ at ℓ, we construct $\pi(\alpha_0, \beta) = \mathrm{ind}_B^G \chi_\ell = \hat{\rho}(\Omega(\alpha_0, \beta))$ with $B = \exp \mathfrak{b}$. In this situation, a_β given as usual by $\langle a_\beta, \phi \rangle = \overline{\phi(e)}$ clearly fulfils our request. $\mathfrak{b} = \mathfrak{h} + \mathbb{R}Z$ and Z is a central element. So,

$$\left(\mathcal{H}_{\pi(\alpha_0, \beta)}^{-\infty} \right)^{H, \chi_f \Delta_{G,H}^{1/2}} = \mathbb{C} a_\beta.$$

Now, for $\psi \in \mathcal{D}(G)$, $\pi(\alpha_0, \beta)(\psi) a_\beta = \psi_B^\ell$, i.e.

$$\left(\pi(\alpha_0, \beta)(\psi) a_\beta \right)(g) = \int_B \psi(gb) \chi_\ell(b) \Delta_{B,G}^{-1/2}(b) db \ (g \in G),$$

then

$$\langle \pi(\alpha_0, \beta)(\psi) a_\beta, a_\beta \rangle = \int_B \psi(b) \chi_\ell(b) \Delta_{H,G}^{-1/2}(b) db.$$

Therefore,

$$\begin{aligned} &(2\pi)^{-1} \int_{\mathbb{R}} \langle \pi(\alpha_0, \beta)(\psi) a_\beta, a_\beta \rangle d\beta \\ &\qquad = (2\pi)^{-1} \int_{\mathbb{R}} d\beta \int_{\mathbb{R}} \psi_H^f(\exp(wZ)) \exp(i\beta w) dw = \psi_H^f(e). \end{aligned}$$

(ii) Secondly, let $\mathfrak{h} = \mathbb{R}T + \mathbb{R}Z$. If we take $\ell = f(\alpha, \beta_0) = \alpha T^* + \beta_0 Z^*$, then

$$\Gamma_\tau \cap \Omega(\alpha, \beta_0) = \alpha_0 T^* + xX^* + yY^* + \beta_0 Z^*;\ xy = \beta_0(\alpha - \alpha_0).$$

If the value

$$\begin{aligned} (\exp(aX)\exp(bY) \cdot \ell)(T) &= (\exp(bY) \cdot \ell)(T + aX) \\ &= \ell(T + aX - bY + abZ) = \alpha + ab\beta_0 \end{aligned}$$

is equal to $f(T), \alpha + ab\beta_0 = \alpha_0$, i.e. $ab\beta_0 = \alpha_0 - \alpha$. Modifying the elements of the basis by appropriate constants, we may assume $\beta_0 = 1$. Then, the above equality becomes $ab = \alpha_0 - \alpha$ and, for $f(\alpha, \beta_0)$ such that $\alpha \neq \alpha_0$, $g_j \cdot f(\alpha, \beta_0) \in \Gamma_\tau$ $(j = 1, 2)$ with $g_1 = \exp X \exp((\alpha_0 - \alpha)Y), g_2 = \exp(-X)\exp((\alpha - \alpha_0)Y)$.

As in case (i), we realize the representation $\pi(\alpha, \beta_0) = \mathrm{ind}_B^G \chi(\alpha, \beta_0) = \hat{\rho}(\Omega(\alpha, \beta_0))$. Take $\phi \in \mathcal{H}^\infty_{\pi(\alpha,\beta_0)}$ and recall the usual formula:

$$\begin{aligned}\langle a^1_\alpha, \phi\rangle &= \oint_{H/(H\cap g_1 B g_1^{-1})} \overline{\phi(hg_1)\chi_f(h)}\Delta_{H,G}^{-1/2} d\nu(h)\\ &= \int_{\mathbb{R}} \overline{\phi(\exp(tT)\exp X)} e^{-it\alpha_0} dt = \int_{\mathbb{R}} \overline{\phi(\exp(e^t X))} e^{it(\alpha-\alpha_0)+t/2} dt\\ &= \int_0^\infty \overline{\phi(\exp(sX))} s^{i(\alpha-\alpha_0)-1/2} ds.\end{aligned}$$

This last integrand is integrable and

$$a^1_\alpha \in \left(\mathcal{H}^{-\infty}_{\pi(\alpha,\beta_0)}\right)^{H,\chi_f\Delta_{H,G}^{1/2}}.$$

Similarly the formula

$$\begin{aligned}\langle a^2_\alpha, \phi\rangle &= \oint_{H/(H\cap g_2 B g_2^{-1})} \overline{\phi(hg_2)\chi_f(h)}\Delta_{H,G}^{-1/2}(h) d\nu(h)\\ &= \int_0^\infty \overline{\phi(\exp(-sX))} s^{i(\alpha-\alpha_0)-1/2} ds\end{aligned}$$

defines the nonzero element

$$a^2_\alpha \in \left(\mathcal{H}^{-\infty}_{\pi(\alpha,\beta_0)}\right)^{H,\chi_f\Delta_{H,G}^{1/2}}.$$

Let a be any element of this space. By the semi-invariance of a relative to $h = \exp(tT), t \in \mathbb{R}$,,

$$\langle e^{it(\alpha_0-\alpha)-t/2} a, \phi(\exp(xX))\rangle = \langle a, \phi(\exp(e^t xX))\rangle.$$

Thus, if the support of ϕ is contained in $\mathbb{R}_+ = \{s \in \mathbb{R}; s > 0\}$,

$$\langle a, \phi\rangle = c_1 \int_{\mathbb{R}_+} \overline{\phi(\exp(xX))} x^{i(\alpha-\alpha_0)-1/2} dx \ (c_1 : \text{constant}),$$

that is, $a = c_1 a_\alpha^1$ (c_1: constant) on $\mathbb{R}_+$. Likewise, $a = c_2 a_\alpha^2$ (c_2: constant) on $\mathbb{R}_- = \{s \in \mathbb{R}; s < 0\}$.

Now assuming that the support of a is contained in B, we write

$$a = \sum_{j=0}^{m} \lambda_j D_j,\ \langle D_j, \phi \rangle = \overline{(d^j \hat{\phi}/dx^j)(0)}.$$

Here $\hat{\phi}(x) = \phi(\exp(xX))$. By the semi-invariance of a relative to $h = \exp(tT)$,

$$e^{i\alpha_0 t} \sum_{j=0}^{m} \overline{(d^j \hat{\phi}/dx^j)(0)} = \sum_{j=0}^{m} \lambda_j \overline{(d^j \hat{\phi}/dx^j)(0)} e^{(j+1/2)t} e^{i\alpha t}.$$

Here if we take ϕ satisfying $(d^j \hat{\phi}/dx^j)(0) = \delta_{jm}$,

$$\lambda_m e^{i\alpha_0 t} = \lambda_m e^{(m+1/2)t} e^{i\alpha t} \ (t \in \mathbb{R}).$$

Thus $\lambda_m = 0$ and $a = 0$. In consequence,

$$\left(\mathcal{H}^{-\infty}_{\pi(\alpha,\beta_0)}\right)^{H,\chi_f \Delta^{1/2}_{H,G}} = \mathbb{C}a_\alpha^1 \oplus \mathbb{C}a_\alpha^2.$$

Let us consider the concrete Plancherel formula for the monomial representation $\tau = \mathrm{ind}_H^G \chi_f,\ f = f(\alpha_0, \beta_0)$. To simplify the writing, we write π in place of $\pi(\alpha, \beta_0)$ below. Let $\psi \in \mathcal{D}(G)$ and assume that $\phi \in \mathcal{H}_\pi^\infty$ has compact support modulo B. Writing simply a^j instead of a_α^j ($j = 1, 2$), we compute

$$\begin{aligned}\langle \pi(\psi)a^1, \phi \rangle = &\oint_{H/(H\cap g_1 B g_1^{-1})} d\nu(h) \oint_{G/B} d\mu_{G,B}(g) \int_B \psi(gbg_1^{-1}h^{-1}) \\ &\times \overline{\phi(g)} \Delta_{B,G}^{-1/2}(h) \chi_\ell(b) \overline{\chi_f(h)} \Delta_G^{-1}(h) \Delta_{H,G}^{-1/2}(h) db.\end{aligned}$$

As we will see in what follows, we can change the order of the first two integrals in the right member. First $\Delta_G(h) = \Delta_{H,G}(h) = 1$. In the above expression, we write g as $g = \exp(xX)$ and make x move in some finite interval J. We denote the integrand by $\Xi(h, g, b)$ and write $h = \exp(tT), b = \exp(sT)\exp(yY)\exp(wZ)$. Then,

$$\begin{aligned}\int_B \Xi(h, g, b) db = \overline{\hat{\phi}(x)} \int_{\mathbb{R}^3} &\psi(\exp(xX)\exp(yY)\exp(wZ)\exp((\alpha - \alpha_0)Y) \\ &\times \exp(-X)\exp(-tT)) e^{-s/2} e^{i(w+\alpha s-\alpha_0)} ds dy dw.\end{aligned}$$

Here,

$$\begin{aligned}
&\exp(xX)\exp(sT)\exp(yY)\exp((\alpha-\alpha_0)Y)\exp(-X)\exp(-tT)\\
=&\exp\big((w+e^{-s}x(y+\alpha-\alpha_0))Z\big)\\
&\times\exp\big(e^{-s}(y+\alpha-\alpha_0)Y\big)\exp\big((x-e^{s})X\big)\exp((s-t)T).
\end{aligned}$$

Taking this into account,

$$\begin{aligned}
&\Big|\int_B \Xi(h,g,b)db\Big|\le\big|\hat{\phi}(x)\big|\int_{\mathbb{R}^3}|\psi(\exp(wZ)\exp\big(e^{-s}yY\big)\\
&\times\exp\big((x-e^{s})X\big)\exp((s-t)T)|e^{-s/2}dsdydw\\
&=|\hat{\phi}(x)|\int_{\mathbb{R}^3}|\psi\ (\exp(wZ)\exp(yY)\\
&\times\exp\big((x-e^{s+t})X\big)\exp(sT)\big)\big|e^{(s+t)/2}dsdydw.
\end{aligned}$$

Since x moves in the finite interval J, this expression is integrable with respect to $dtdx$ and we can change the order of the first two integrals in the equality concerning $\langle\pi(\psi)a^1,\phi\rangle$.

Thus, we get

$$\begin{aligned}
\langle\pi(\psi)a^1,\phi\rangle=\oint_{G/B}d\mu_{G,B}(g)\oint_{H/(H\cap g_1Bg_1^{-1})}d\nu(h)\int_B\psi(gbg_1^{-1}h^{-1})\\
\times\overline{\phi(g)}\Delta_{B,G}^{-1/2}(b)\chi_\ell(b)\overline{\chi_f(h)}\Delta_G^{-1}(h)\Delta_{H,G}^{-1/2}(h)db.
\end{aligned}$$

Hence, for $g\in G$,

$$\begin{aligned}
(\pi(\psi)a^1)(g)&=\oint_{H/(H\cap g_1Bg_1^{-1})}d\nu(h)\int_B\psi(gbg_1^{-1}h^{-1})\\
&\quad\times\Delta_{B,G}^{-1/2}(b)\chi_\ell(b)\overline{\chi_f(h)}\Delta_G^{-1}(h)\Delta_{H,G}^{-1/2}(h)db\\
&=\oint_{H/(H\cap g_1Bg_1^{-1})}d\nu(h)\oint_{B/(B\cap g_1^{-1}Hg_1)}\chi_\ell(b)\overline{\chi_f(h)}\Delta_G^{-1}(h)\Delta_{H,G}^{-1/2}(h)d\hat{\nu}(b)\\
&\quad\times\int_{B\cap g_1^{-1}Hg_1}\psi(gbb_0^{-1}g_1^{-1}h^{-1})\\
&\quad\times\Delta_{B,G}^{1/2}(b_0)\overline{\chi_\ell(b_0)}\Delta_{B\cap g_1^{-1}Hg_1,B}(b_0)\Delta_{B\cap g_1^{-1}Hg_1}(b_0)db_0,
\end{aligned}$$

where $\hat{\nu}=\mu_{B,B\cap g_1^{-1}Hg_1}$. Exactly the same arguments as above permit us to change the order of the first two integrals. In consequence,

$$\begin{aligned}(\pi(\psi)a^1)(g) = \oint_{B/(B\cap g_1^{-1}Hg_1)} & \chi_\ell(b)\Delta_{B,G}^{-1/2}(b)d\hat{\nu}(b) \\ \times \oint_{H/(H\cap g_1 B g_1^{-1})} & \overline{\chi_f(h)}\Delta_G^{-1}(h)\Delta_{H,G}^{-1/2}(h)d\nu(h) \\ \times \int_{B\cap g_1^{-1}Hg_1} & \psi(gbb_0^{-1}g_1^{-1}h^{-1}) \\ \times\, & \overline{\chi_\ell(b_0)}\Delta_{B,G}^{1/2}(b_0)\Delta_{B\cap g_1^{-1}Hg_1,B}(b_0)\Delta^{-1}_{B\cap g_1^{-1}Hg_1}(b_0)db_0.\end{aligned}$$

This last integral is equal to

$$\begin{aligned}\int_{H\cap g_1 B g_1^{-1}} & \psi(gbg_1^{-1}b'^{-1}h^{-1})\chi_{g_1\cdot\ell}(b') \\ & \times \Delta^{1/2}_{g_1Bg_1^{-1},G}(b')\Delta_{g_1Bg_1^{-1}\cap H,g_1Bg_1^{-1}}(b')\Delta_{g_1Bg_1^{-1}\cap H}(b')db'.\end{aligned}$$

Although it is easily verified in our case, if the equality

$$\Delta_H^{1/2}(b')\Delta_{g_1Bg_1^{-1}\cap H}(b') = \Delta_{g_1^{-1}Bg_1}(b')\ (b'\in g_1Bg_1^{-1}\cap H)$$

holds, we get

$$\begin{aligned}& \Delta^{1/2}_{g_1Bg_1^{-1},G}(b')\Delta_{g_1Bg_1^{-1}\cap H,g_1Bg_1^{-1}}(b')\Delta_{g_1Bg_1^{-1}\cap H}(b') \\ &= \Delta_H^{-1}(b')\Delta_{H,G}^{1/2}(b')\Delta_{g_1Bg_1^{-1}\cap H,H}(b')\end{aligned}$$

and so

$$\begin{aligned}(\pi(\psi)a^1)(g) &= \oint_{B/(B\cap g_1^{-1}Hg_1)} \Delta_{B,G}^{-1/2}(b)\chi_\ell(b)d\hat{\nu}(b) \\ &\quad\times \int_H \psi(gbg_1^{-1}h)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh \\ &= \oint_{B/(B\cap g_1^{-1}Hg_1)} \psi_H^f(gbg_1^{-1})\chi_\ell(b)\Delta_{B,G}^{-1/2}(b)d\hat{\nu}(b).\end{aligned}$$

Likewise,

$$(\pi(\psi)a^2)(g) = \oint_{B/(B\cap g_2^{-1}Hg_2)} \psi_H^f(gbg_2^{-1})\chi_\ell(b)d\tilde{\nu}(b),$$

where $\tilde{\nu} = \mu_{B,B\cap g_2^{-1}Hg_2}$. From what we have seen,

$$\begin{aligned}\langle\pi(\psi)a^1, a^1\rangle &= \oint_{H/(H\cap g_1Bg_1^{-1})} \chi_\ell(h)\Delta_{H,G}^{-1/2}(h)d\nu(h)\\ &\quad\times\oint_{B/(B\cap g_1^{-1}Hg_1)} \psi_H^f(hg_1bg_1^{-1})\chi_\ell(b)\Delta_{B,G}^{-1/2}(b)d\hat{\nu}(b)\\ &= \int_{\mathbb{R}} e^{it(\alpha_0-\alpha)+t/2}dt\\ &\quad\times\int_{\mathbb{R}} \psi_H^f\big(\exp\big(e^tX\big)\exp(xT)\exp(yY)\exp(-X)\big)e^{i\alpha x-x/2}dxdy\\ &= \int_{\mathbb{R}} e^{i(t-x)(\alpha_0-\alpha)+(t+x)/2}dt\int_{\mathbb{R}^2} \psi_H^f\big(\exp\big((e^t-e^x)X\big)\exp(yY)\big)e^{-iye^x}dxdy.\end{aligned}$$

Performing the change of variables $t\to t+x, e^x = s$, we get

$$\begin{aligned}\langle\pi(\psi)a^1, a^1\rangle &= \int_{\mathbb{R}} e^{it(\alpha_0-\alpha)+t/2}dt\\ &\quad\times\int_{\mathbb{R}\times\mathbb{R}_+} \psi_H^f\big(\exp\big(s(e^t-1)X\big)\exp(yY)\big)e^{-iys}dyds.1\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\pi(\psi)a^2, a^2\rangle &= \oint_{H/(H\cap g_2Bg_2^{-1})} \chi_f(h)\Delta_{H,G}^{-1/2}(h)d\nu(h)\\ &\quad\times\oint_{B/(B\cap g_2^{-1}Hg_2)} \psi_H^f(hg_2bg_2^{-1})\chi_\ell(b)\Delta_{B,G}^{-1/2}(b)d\tilde{\nu}(b)\\ &= \int_{\mathbb{R}} e^{it(\alpha_0-\alpha)+t/2}dt\int_{\mathbb{R}\times\mathbb{R}_+} \psi_H^f(\exp\big(s\big(1-e^t\big)X\big)\exp(yY))e^{iys}dyds.\end{aligned}$$

Finally,

$$\begin{aligned}&\langle\pi(\psi)a^1, a^1\rangle + \langle\pi(\psi)a^2, a^2\rangle\\ &\qquad= \int_{\mathbb{R}} e^{it(\alpha_0-\alpha)+t/2}dt\int_{\mathbb{R}^2} \psi_H^f(\exp(s(1-e^t)X)\exp(yY))e^{iys}dyds.\end{aligned}$$

Now, if we set

$$\Psi(t) = \int_{\mathbb{R}^2} \psi_H^f\big(\exp\big(s(1-e^t)X\big)\exp(yY)\big)e^{iys}dyds,$$

this function is infinitely differentiable at $t \neq 0$. In fact, in this case the integral is taken on a compact set. Further, the function $\mathbb{R}^2 \ni (s, y) \to \psi_H^f(\exp(sX)\exp(yY))$ belongs to $\mathcal{D}(\mathbb{R}^2)$. If we make an L^1-approximation with a function of the form

$$\sum_j \xi_j(s)\eta_j(y) \ (\xi_j, \eta_j \in \mathcal{D}(\mathbb{R})),$$

for any positive number ϵ, we can choose ξ_j, η_j in such a fashion that

$$|\Psi(t) - (2\pi)^{1/2} \sum_j \int_{\mathbb{R}} \xi_j \left(s(1-e^t)\right) \widehat{\eta_j}(s) ds| < \epsilon$$

holds for arbitrary $t \in \mathbb{R}$. Here $\widehat{\eta_j}$ denotes the Fourier inverse transform of η_j. When t tends to 0, the limit of

$$\int_{\mathbb{R}} \xi_j \left(s(1-e^t)\right) \widehat{\eta_j}(s) ds$$

is equal to $(2\pi)^{1/2}\xi_j(0)\eta_j(0)$. If we take this into consideration, $\Psi(t)$ is continuous even at $t = 0$.

Lastly, let us consider the function $e^{t/2}\Psi(t)$. When $t \to -\infty$, this function is rapidly decreasing because of $e^{t/2}$. Let us examine its movement when t tends to $+\infty$. By a change of variables, we have

$$e^{t/2}\Psi = e^{t/2}(1-e^t)^{-1} \int_{\mathbb{R}^2} \psi_H^f(\exp(sX)\exp(yY)) e^{iys(1-e^t)^{-1}} ds dy$$

and

$$|e^{t/2}\Psi(t)| \leq e^{t/2}|1-e^t|^{-1} \int_{\mathbb{R}^2} |\psi_H^f(\exp(sX)\exp(yY))| ds dy.$$

Hence, $e^{t/2}\Psi(t)$ is rapidly decreasing when t tends to $+\infty$.

Thus, the Fourier inversion formula supplies the concrete Plancherel formula for the monomial representation $\tau = \mathrm{ind}_H^G \chi_f$:

$$(2\pi)^{-2} \int_{\mathbb{R}} \left(\langle \pi(\psi) a_\alpha^1, a_\alpha^1 \rangle + \langle \pi(\psi) a_\alpha^2, a_\alpha^2 \rangle\right) d\alpha$$

$$= (2\pi)^{-1} \int_{\mathbb{R}} \psi_H^f(\exp(yY)) e^{iys} ds dy = \psi_H^f(e).$$

11.4 Bonnet's Plancherel Formula

Until now we have tried to write down explicitly Penney's abstract Plancherel formula for monomial representations $\tau = \mathrm{ind}_H^G \chi_f$ with finite multiplicities. At the end of this chapter, we will treat **Bonnet's Plancherel formula** in the case of nilpotent Lie groups. These two Plancherel formulas seem to be essentially the same, however the latter does not put in front the canonical irreducible decomposition of τ. Let dg, dh be Haar measures on $G = \exp \mathfrak{g}$, $H = \exp \mathfrak{h}$ respectively and χ be a unitary character of H. According to Bonnet [12], as a distribution of positive type on G, χ has its Fourier transform (μ, U). Namely, there exist a positive measure μ on $\hat{G}$ and a field of nuclear operators U, $U_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty} (\pi \in \hat{G})$ so that the abstract Plancherel formula

$$\int_H \phi(h)\chi(h)dh = \int_{\hat{G}} \mathrm{Tr}\left(\pi(\phi)U_\pi\right) d\mu(\pi)$$

holds for all $\phi \in \mathcal{D}(G)$. Here Tr denotes the trace of the operator. If we identify (μ, U) and (μ', U') when $\mu' = \lambda\mu$ $(\lambda > 0)$ and $U' = \lambda^{-1}U$, the pair (μ, U) satisfying the formula is unique.

In order to describe this formula explicitly, we write $\chi = \chi_f$ using $f \in \mathfrak{g}^*$. Then the measure μ in question is noting but the measure which appears in the canonical decomposition (8.1.1) of the monomial representation $\tau = \mathrm{ind}_H^G \chi_f$. Employing the notations used before, let us first look at the case where H is a normal subgroup. G acts on $\mathfrak{h}^*$ and it turns out that $G{\cdot}\ell \cap \Gamma_\tau = P{\cdot}\ell$, $P = G(f|_{\mathfrak{h}})$, for all $\ell \in \Gamma_\tau$. Moreover, it is well known [14] that the multiplicities $m(\pi)$ in formula (8.1.1) are uniformly equal to either 1 or ∞. In this situation, the arguments developed in Grélaud [43] work well. We realize $\pi \in \hat{G}$ belonging to the support of μ by means of a polarization $\mathfrak{b}$ at $\ell \in \hat{\rho}^{-1}(\pi) \cap \Gamma_\tau$: $\pi = \mathrm{ind}_B^G \chi_\ell$ with $B = \exp \mathfrak{b}$. What simplifies the state of affairs in this case is that we can choose $\mathfrak{b}$ in such a way that it contains $\mathfrak{h}$. Hence $H \subset B \subset P$. For $\alpha, \beta \in \mathcal{H}_\pi^\infty$, the function $\alpha\bar{\beta}$ is B-invariant from the right and integrable with respect to P-invariant measure $d\dot{g}$ on P/B. We define the nuclear operator U_π from $\mathcal{H}_\pi^\infty$ to $\mathcal{H}_\pi^{-\infty}$ by the formula

$$\langle U_\pi\alpha, \beta\rangle = \int_{P/B} \alpha(g)\overline{\beta(g)}d\dot{g}.$$

This operator is a positive self-adjoint operator, while, for $\phi \in \mathcal{D}(G)$, $\pi(\phi)$ is a operator with integral kernel which is expressed by the formula

$$K_\phi(x, y) = \int_B \phi(xby^{-1})\chi_\ell(b)db$$

with $(x, y) \in G \times G$. Here db is a Haar measure on B. Exactly as in [43],

$$\begin{aligned}\mathrm{Tr}\,(\pi(\phi)U_\pi) &= \int_{P/B} K_\phi(g,g)d\dot{g} = \int_{P/B} d\dot{g}\int_B \phi(gbg^{-1})\chi_\ell(b)db\\ &= \int_{P/B} d\dot{g}\int_{\mathfrak{b}} \phi^g(\exp X)e^{i\ell(X)}dX,\end{aligned}$$

Here $\phi^g(x) = \phi(gxg^{-1})$ $(x \in G)$. As $\ell|_{\mathfrak{b}} \neq 0$ if $\ell \neq 0$, we take a linear complement $\mathfrak{t}$ to $\mathfrak{b}$ in $\mathfrak{g}$ in such a manner that $\ell|_{\mathfrak{t}} = 0$. Taking $B{\cdot}\ell = \ell + \mathfrak{b}^\perp$ into account and denoting by $\mathcal{F}(\psi)$ the Fourier transform of $\psi \in \mathcal{D}(G)$, We derive from the Fourier inversion formula that

$$\begin{aligned}\mathrm{Tr}\,(\pi(\phi)U_\pi) &= \int_{P/B} d\dot{g}\int_{\mathfrak{b}^\perp} d\xi\int_{\mathfrak{t}} dY\int_{\mathfrak{b}} \phi^g(\exp(X+Y))e^{i\ell(X)+i\xi(Y)}dX\\ &= \int_{P/B} d\dot{g}\int_{\mathfrak{b}^\perp} d\xi\int_{\mathfrak{g}} \phi^g(\exp X)e^{i(\ell+\xi)(X)}dX\\ &= \int_{P/B} d\dot{g}\int_{B/G(\ell)} d\dot{b}\int_{\mathfrak{g}} \phi(\exp X)e^{ib\cdot\ell(g^{-1}\cdot X)}dX\\ &= \int_{P/G(\ell)} \mathcal{F}(\phi\circ\exp)(g\cdot\ell)d\dot{g}\\ &= \int_{G\cdot\ell\cap\Gamma_\tau} \mathcal{F}(\phi\circ\exp)(\lambda)d\lambda.\end{aligned}$$

In this way, $\Omega(\pi)$ denoting the orbit $\hat{\rho}^{-1}(\pi)$,

$$\mathrm{Tr}\,(\pi(\phi)U_\pi) = \int_{\Omega(\pi)\cap\Gamma_\tau} \mathcal{F}(\phi\circ\exp)(\lambda)d\lambda.$$

When H is trivial, this formula is evidently reduced to Kirillov's character formula.

From this immediately results the Plancherel formula: regarding all $\phi \in \mathcal{D}(G)$,

$$\begin{aligned}\int_H \phi(h)\chi_f(h)dh &= \int_{\mathfrak{h}} \phi(\exp X)e^{if(X)}dX = \int_{\Gamma_\tau} \mathcal{F}(\phi\circ\exp)(\xi)d\xi\\ &= \int_{\hat{G}} d\mu(\pi)\int_{\Omega(\pi)\cap\Gamma_\tau} \mathcal{F}(\phi\circ\exp)(\lambda)d\lambda = \int_{\hat{G}} \mathrm{Tr}\,(\pi(\phi)U_\pi)\,d\mu(\pi).\end{aligned}$$

Now we abandon the assumption that H is a normal subgroup, and show at first a lemma. Abusing a little the notation, we write Γ_τ/G instead of $(G{\cdot}\Gamma_\tau)/G$.

Lemma 11.4.1. *The Lebesgue measure ν on Γ_τ is decomposed relative to the base measure μ on Γ_τ/G as*

$$\nu = \int_{\Gamma_\tau/G} \nu_\Omega d\mu(\Omega),$$

where ν_Ω is a measure supported on the G-orbit Ω.

Proof. The proof is reduced to the case where $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$, and again by a classic reasoning to the case where $\dim \mathfrak{z} = 1$, $f|_\mathfrak{z} \neq 0$. We take in $\mathfrak{g}$ a Heisenberg triplet $\{X, Y, Z\}$, $[X, Y] = Z$, such that $\mathfrak{z} = \mathbb{R}Z$, $f(Y) = 0$, $f(Z) = 1$ and that $\mathfrak{a} = \mathbb{R}Y + \mathbb{R}Z$ is a commutative ideal of $\mathfrak{g}$. Then the centralizer $\mathfrak{g}_0$ of $\mathfrak{a}$ in $\mathfrak{g}$ is an ideal of codimension 1. By the induction hypothesis, we may assume that the lemma holds for $G_0 = \exp(\mathfrak{g}_0)$. We indicate the objects regarding G_0 by the index 0.

Let $p : \mathfrak{g}^* \to \mathfrak{g}_0^*$ be the restriction map and put $\Gamma_0 = p(\Gamma_\tau)$. For $\Omega \in \mathfrak{g}^*/G$ verifying $\Omega \cap \Gamma_\tau \neq \emptyset$, $\Omega = p^{-1}(p(\Omega))$ and, by Lemma 8.1.4 in Chap. 8, $p(\Omega)$ is decomposed into one-parameter family ω_t, $t \in \mathbb{R}$, of G_0-orbits where $\omega_t = \exp(tX) \cdot \omega_0$. To say more exactly, ω_t is, for instance P_t being the hyperplane $\{\ell \in \mathfrak{g}_0^*; \ell(Y) = t\}$ in $\mathfrak{g}_0^*$, given by $\omega_t = p(\Omega) \cap P_t$. Let ν_0 the Lebesgue measure on the affine space Γ_0.

(1) First assume $\mathfrak{h} \not\subset \mathfrak{g}_0$. In this case, ν is identified with ν_0. Besides, choosing X in $\mathfrak{h}$, $\exp(tX) \cdot (\omega_0 \cap \Gamma_0) = \omega_t \cap \Gamma_0$ for arbitrary $t \in \mathbb{R}$. Set $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_0 + \mathbb{R}Y$, $f_0 = p(f) \in \mathfrak{g}_0^*$ and $\Gamma_1 = \{\ell_0 \in \mathfrak{g}_0^*; \ell_0|_{\mathfrak{h}_1} = f_0|_{\mathfrak{h}_1}\}$. Then, the Borel space Γ_τ/G is identified with Γ_1/G_0.

The induction hypothesis decomposes according to the action of G_0 the Lebesgue measure ν_1 on Γ_1:

$$\nu_1 = \int_{\Gamma_1/G_0} \nu_\omega d\mu_0(\omega),$$

where μ_0 is the base measure and ν_ω is a measure on $\omega \cap \Gamma_1$. Hence,

$$\nu \cong \nu_0 = \int_{\Gamma_1/G_0} d\mu_0(\omega) \int_\mathbb{R} \exp(tX) \cdot \nu_\omega dt = \int_{\Gamma/G} \nu_\Omega d\mu(\Omega),$$

where $\mu(\Omega) = \mu_0(\omega_0) = \mu_0(p(\Omega) \cap P_0)$ and $\nu_\Omega = \int_\mathbb{R} \exp(tX) \cdot \nu_\omega dt$.

(2) Secondly, assume $\mathfrak{h} \subset \mathfrak{g}_0$. In this case, using the Lebesgue measure dx on $\mathfrak{g}_0^\perp \subset \mathfrak{g}^*$, $\nu = \nu_0 \times dx$. According to [32], except for a null set $\Omega \cap \Gamma_\tau$ (resp. $\omega \cap \Gamma_0$) is a differential manifold and its dimension r (resp. r_0) does not depend on $\Omega \in \mathfrak{g}^*/G$ (resp. $\omega \in \mathfrak{g}_0^*/G_0$). There occur two possibilities: either $r = r_0 + 2$ or $r = r_0 + 1$.

(a) Provided $r = r_0 + 2$, let us argue as in the case (1). Let $\mathfrak{h}_1 = \mathfrak{h} + \mathbb{R}Y$, $f_t \in \Gamma_\tau$ be such that $f_t(Y) = t$ and put $\Gamma_t = \{\ell_0 \in \mathfrak{g}_0^*; \ell_0|_{\mathfrak{h}_1} = f_t|_{\mathfrak{h}_1}\}$ ($t \in \mathbb{R}$). Then,

$$\nu_0 = \int_\mathbb{R} \nu^t dt,$$

where ν^t is the Lebesgue measure on the affine space Γ_t.

By the induction hypothesis,

$$\nu^t = \int_{\Gamma_t/G_0} \nu^t_\omega d\mu_t(\omega) \ (t \in \mathbb{R})$$

with agreement on the notations. All the spaces Γ_t/G_0 $(t \in \mathbb{R})$ are identified with Γ_τ/G, μ_t are mutually equivalent. Retaking ν^t_ω if necessary, we may think that all μ_t coincide with the measure μ on Γ_τ/G. Consequently,

$$\nu = \int_{\Gamma_\tau/G} \nu_\Omega d\mu(\Omega),$$

where $\nu_\Omega = \left(\int_{\mathbb{R}} \nu^t_{\omega_t} dt\right) \times dx$ with $p(\Omega) = \sqcup_{t\in\mathbb{R}}\omega_t$.

(b) Provided $r = r_0 + 1$, the induction hypothesis supplies the decomposition of ν_0:

$$\nu_0 = \int_{\Gamma_0/G_0} \nu_\omega d\mu_0(\omega).$$

The action of G on $\widehat{G_0}$ introduces an equivalence relation in Γ_0/G_0 and by it we get the projection $q : \Gamma_0/G_0 \to \Gamma_\tau/G$. As is immediately seen, there exists a constant $k > 0$ so that $\sharp\{q^{-1}(\Omega)\} < k$ for almost all $\Omega \in \Gamma_\tau/G$. Here $\sharp(A)$ denotes the cardinal of A.

From this (cf. [13]), μ_0 is decomposed with respect to q:

$$\mu_0 = \int_{\Gamma_\tau/G} \nu^0_\Omega d\mu(\Omega),$$

where ν^0_Ω is the finite sum of ν_ω regarding ω such that $q(\omega) = \Omega$. Finally,

$$\nu = \int_{\Gamma_\tau/G} \nu_\Omega d\mu(\Omega)$$

with $\nu_\Omega = \nu^0_\Omega \times dx$. ∎

Remark 11.4.2. By a standard reasoning, the measure class of the measure μ is the image of that of ν by the canonical projection from Γ_τ to Γ_τ/G and this decomposition of ν is essentially unique: if

$$\nu = \int_{\Gamma_\tau/G} \nu'_\Omega d\mu'(\Omega)$$

with other choices of the measures μ' and $(\nu'_\Omega)_{\Omega\in\Gamma_\tau/G}$, there exists a measurable function F with positive values so that $\mu' = F\mu$ and $\nu_\Omega = F\nu'_\Omega$ for μ-almost all Ω. Besides, these ν_Ω are of course invariant relative to the action of H.

Consider a coadjoint orbit Ω of G. Let $\ell \in \Omega$. Let ϵ denote the mapping $G \ni g \to g\cdot\ell \in \Omega$ and put $G_\ell = \epsilon^{-1}(\Gamma_\tau)$. Let $\mathfrak{b} \in M(\ell, \mathfrak{g})$ and $B = \exp\mathfrak{b}$. The set of all double classes $H\backslash G/B$ is a standard Borel space [23]. We write the canonical projection of G to $H\backslash G/B$ as p and a Borel section from $H\backslash G/B$ to G as σ. Put $\Xi = p(G_\ell)$ and $\Theta = \sigma(\Xi)$. For $x \in \Theta$, let $d\dot{h}$ be an invariant measure on $H/(H\cap xBx^{-1})$, $d\xi$ a Lebesgue measure on the affine space $\mathfrak{b}[x] = (\ell+\mathfrak{b}^\perp)\cap x^{-1}\cdot\Gamma_\tau$ and $B_x = \epsilon^{-1}(\mathfrak{b}[x])$. We designate by H_x the image of the section constructed by means of a coexponential basis to $\mathfrak{h}\cap x\cdot\mathfrak{b}$ in $\mathfrak{h}$. Then, each element $g \in G_\ell$ is uniquely written as

$$g = hxb,\ h \in H_x,\ x \in \Theta,\ b \in B_x.$$

Let us transfer the measure ν_Ω on $\widetilde{G_\ell} = G_\ell/G(\ell)$. In the proof of Lemma 11.4.1, we know the following by induction. ν_Ω decomposes with respect to a measure λ on Ξ:

$$\nu_\Omega = \int_\Xi \nu_x d\lambda(\dot{x})\ (\dot{x} = p(x)),$$

where the measure ν_x on the fibre $H_x \times B_x$ is obtained by $d\dot{h} \times \epsilon_*^{-1}(d\xi)$.

We realize π by means of $\mathfrak{b}$: $\pi = \mathrm{ind}_B^G \chi_\ell$. We define a nuclear operator U_π from $\mathcal{H}_\pi^\infty$ to $\mathcal{H}_\pi^{-\infty}$ by the following formula:

$$\langle\phi, U_\pi\psi\rangle = \int_\Xi d\lambda(\dot{x}) \int_{H/(H\cap xBx^{-1})} \phi(hx)\chi_f(h)d\dot{h} \times \int_{H/(H\cap xBx^{-1})} \overline{\psi(h'x)\chi_f(h')}d\dot{h}'$$

for arbitrary $\phi,\ \psi \in \mathcal{H}_\pi^\infty$. With the help of a coexponential basis to $\mathfrak{b}$ in $\mathfrak{g}$, $\mathcal{H}_\pi^\infty$ is identified with the space $\mathcal{S}(\mathbb{R}^m)$, $m = \dim(\mathfrak{g}/\mathfrak{b})$, of the Schwartz's rapidly decreasing functions and hence $\mathcal{H}_\pi^{-\infty}$ is identified with the space $\mathcal{S}'(\mathbb{R}^m)$ of all tempered distributions. In this situation, the operator U_π is defined by the integral kernel

$$K_\pi(hx, h'x) = \left(\chi_f(h)d_{H/(H\cap xBx^{-1})}\dot{h} \times \overline{\chi_f(h')}d_{H/(H\cap xBx^{-1})}\dot{h}'\right) d\lambda(\dot{x})$$

and we know by induction (cf. the proof of Lemma 11.4.1) that this kernel belongs to $\mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m)$. We derive from this (cf. [76, p. 531]) that U_π is a nuclear operator and can compute $\mathrm{Tr}\,(\pi(\phi)U_\pi)$ for $\phi \in \mathcal{D}(G)$. Set

$$\phi_H^f(g) = \int_H \phi(gh)\chi_f(h)dh \ (g \in G).$$

as before.

In this situation we can compute $\mathrm{Tr}\,(\pi(\phi)U_\pi)$.

Theorem 11.4.3 (Character Formula [33]). *For μ-almost all $\pi \in \hat{G}$, we have with an invariant measure $d\dot{b}$*

$$Tr\,(\pi(\phi)U_\pi) = \int_\Xi d\lambda(\dot{x}) \int_{H/(H\cap xBx^{-1})} d\dot{h} \times \int_{B/(B\cap x^{-1}Hx)} \phi_H^f(hxbx^{-1}h^{-1})\chi_\ell(b)d\dot{b}$$

for all $\phi \in \mathcal{D}(G)$. Further, up to a normalization of measures, the value of the right-hand side does not depend on the choice of either $\ell \in \Omega$ or $\mathfrak{b} \in M(\ell, \mathfrak{g})$.

Proof. The kernel of the operator $\pi(\phi)$ is given by the formula [66]:

$$K_\phi(g, g') = \int_B \phi(gbg'^{-1})\chi_\ell(b)db, \ (g, g') \in G \times G,$$

and by [43]

$$\begin{aligned}
&\mathrm{Tr}\,(\pi(\phi)U_\pi) = \langle K_\phi(g', g), K_\pi\rangle \\
&= \int_\Xi d\lambda(\dot{x}) \int_{x^{-1}Hx/(x^{-1}Hx\cap B)} \chi_{x\cdot f}(h)d\dot{h} \\
&\quad \times \int_{x^{-1}Hx/(x^{-1}Hx\cap B)} \overline{\chi_{x\cdot f}(h')}d\dot{h}' \int_B \phi(xhbh'^{-1}x^{-1})\chi_\ell(b)db \\
&= \int_\Xi d\lambda(\dot{x}) \int_{H/(H\cap xBx^{-1})} d\dot{h} \int_{B/(B\cap x^{-1}Hx)} \phi_H^f(hxbx^{-1}h^{-1})\chi_\ell(b)d\dot{b}.
\end{aligned}$$

It is clear that the mapping $\mathcal{D}(G) \ni \phi \mapsto \mathrm{Tr}\,(\pi(\phi)U_\pi)$ is, up to a normalization, independent of the choice of $\ell \in \Omega$. Let us see that it does not depend on the choice of $\mathfrak{b} \in M(\ell, \mathfrak{g})$. Remark first we can assume that $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. Next, if $\mathfrak{p} = \mathfrak{z} \cap \ker f$ is not trivial, we can pass to the quotient algebra $\mathfrak{g}/\mathfrak{p}$ to which the induction hypothesis is applied. Thus, we assume hereafter $\dim \mathfrak{z} = 1$ and $f|_\mathfrak{z} \neq 0$.

Let $\{X, Y, Z\}$ be a Heisenberg triplet such that

$$\mathfrak{z} = \mathbb{R}Z, \ f(Z) = 1, \ [X, Y] = Z$$

and that $\mathfrak{a} = \mathbb{R}Y \oplus \mathbb{R}Z$ is a minimal non-central ideal of $\mathfrak{g}$. Then the centralizer $\mathfrak{g}_0$ of $\mathfrak{a}$ in $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}_0$. Suppose that $\mathfrak{b} \not\subset \mathfrak{g}_0$. Knowing that

$\mathfrak{b}' = \mathfrak{b}_0 + \mathfrak{a}, \mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}_0$, is an element of $M(\ell, \mathfrak{g})$ contained in $\mathfrak{g}_0$, we show that this change from $\mathfrak{b}$ to $\mathfrak{b}'$ does not influence the mapping $\mathcal{D}(G) \ni \phi \mapsto \mathrm{Tr}\,(\pi(\phi)U_\pi)$. Set $G_0 = \exp(\mathfrak{g}_0)$.

First let $\mathfrak{h} \subset \mathfrak{g}_0$. We set

$$E = \{x \in G_\ell; \mathfrak{h} \cap x{\cdot}\mathfrak{b} = \mathfrak{h} \cap x{\cdot}\mathfrak{b}'\},\ F = \{x \in G_\ell; \mathfrak{h} \cap x{\cdot}\mathfrak{b} \neq \mathfrak{h} \cap x{\cdot}\mathfrak{b}'\}.$$

If $x \in E$, the elements $x\exp(tY), t \in \mathbb{R}$, belong to different double classes $H\backslash G/B$. In fact, $hxb = x\exp(tY)$ $(h \in H, b \in B)$ means $b \in B_0 = \exp(\mathfrak{b}_0)$ and $x\exp(tY)b^{-1}x^{-1} \in H \cap xB'x^{-1} = H \cap xBx^{-1}$, where $B' = \exp(\mathfrak{b}')$. This necessitates $t = 0$. Hence both sets $H\backslash E/B$ and $H\backslash E/B'$ are identified with $\mathbb{R} \times (H\backslash E_0/B_0), E_0 = E \cap G_0$; the first with $\exp(\mathbb{R}Y) \times (H\backslash E_0/B)$ and the second with $(\cup_{t\in\mathbb{R}} b_t) \times (H\backslash E_0/B_0)$, $b_t \in xBx^{-1}$ not belonging to xB_0x^{-1} except for only one value of $t \in \mathbb{R}$. This last affirmation is seen as follows. Because of $x \in E$, $Y \notin \mathfrak{h} + x{\cdot}\mathfrak{b}$. For any $t \in \mathbb{R}$, we consider $\zeta_t \in (\mathfrak{h} + x{\cdot}\mathfrak{b})^\perp \subset \mathfrak{g}^*$ satisfying $\zeta_t(Y) = t$. The Pukanszky condition says that there exists b_t satisfying $b_t x{\cdot}\ell = \zeta_t$, namely $b_t x \in G_\ell$.

Now let $x \in F$, which means $Y = V + x{\cdot}W$ with $V \in \mathfrak{h}, W \in \mathfrak{b}_0$. Then, taking X in $\mathfrak{b}$,

$$H/(H \cap xBx^{-1}) \cong \exp(\mathbb{R}V) \times (H/(H \cap xB'x^{-1})),$$
$$B/(B \cap x^{-1}Hx) \cong \exp(\mathbb{R}X) \times (B_0/(B_0 \cap x^{-1}Hx)),$$
$$B'/(B' \cap x^{-1}Hx) \cong B_0/(B_0 \cap x^{-1}Hx),$$

while, if $hxb \in G_\ell$ with $h \in H$ and $b \in B$, then

$$h{\cdot}Y = h{\cdot}V + hxbb^{-1}{\cdot}W$$

and

$$f(V) = hxb{\cdot}\ell(hV) = \ell(b^{-1}x^{-1}Y - b^{-1}W) = \ell(b^{-1}x^{-1}Y - W).$$

Therefore $b \in B_0$ and $H\backslash F/B \cong H\backslash F/B_0$. But for $x \in F$, we have

$$HxB' = Hx\exp(x^{-1}{\cdot}\mathbb{R}V)B_0 = HxB_0,$$

which allows us to identify $H\backslash F/B_0$ with $H\backslash F/B'$.

From these observations we immediately conclude that the formula of $\mathrm{Tr}\,(\pi(\phi)U_\pi)$ does not change under the modification from $\mathfrak{b}$ to $\mathfrak{b}'$.

Secondly, let $\mathfrak{h} \not\subset \mathfrak{g}_0$. X being taken in $\mathfrak{h}$, we set $\mathfrak{h} = \mathbb{R}X + \mathfrak{h}_0$ with $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $H_0 = \exp(\mathfrak{h}_0)$. Let

$$E = \{x \in G_\ell; \mathfrak{h} \cap x{\cdot}\mathfrak{b} \neq \mathfrak{h}_0 \cap x{\cdot}\mathfrak{b}\},\ F = G_\ell\backslash E.$$

For $x \in E$, we have $\mathfrak{h} \cap x{\cdot}\mathfrak{b}' = \mathfrak{h} \cap x{\cdot}\mathfrak{b}_0$. Therefore,

$$\begin{aligned}
B'/(B' \cap x^{-1}Hx) &\cong \exp(\mathbb{R}Y) \times (B_0/(B_0 \cap x^{-1}Hx)),\\
B/(B \cap x^{-1}Hx) &\cong B_0/(B_0 \cap x^{-1}Hx),\\
H/(H \cap xBx^{-1}) &\cong H_0/(H_0 \cap xBx^{-1}),\\
H/(H \cap xB'x^{-1}) &\cong \exp(\mathbb{R}X) \times (H_0/(H_0 \cap xBx^{-1})).
\end{aligned}$$

On the other hand, for $x \in E_0 = E \cap G_0$, the relation $y \in (HxB) \cap G_0$ leads to $y \in Hx\exp(\mathbb{R}\tilde{X})x^{-1}xB_0$ with $\tilde{X} \in \mathfrak{b}\backslash\mathfrak{b}_0$ such that $x{\cdot}\tilde{X} \in \mathfrak{h}$ and $y \in HxB_0$. This says that $H\backslash E/B \cong H\backslash E/B'$. Hence on the set E it suffices for us to apply the inversion formula of Fourier in the expression of $\mathrm{Tr}\,(\pi(\phi)U_\pi)$.

It remains for us to examine the set F, which is divided into

$$F_1 = \{x \in F; \mathfrak{h} \cap x{\cdot}\mathfrak{b} \neq \mathfrak{h} \cap x{\cdot}\mathfrak{b}'\} \text{ and } F_2 = \{x \in F; \mathfrak{h} \cap x{\cdot}\mathfrak{b} = \mathfrak{h} \cap x{\cdot}\mathfrak{b}'\}.$$

For $x \in F_1$, Y is written as $Y = x^{-1}{\cdot}V + W$ with certain $V \in \mathfrak{h}$ and $W \in \mathfrak{b}_0$. Hence

$$\begin{aligned}
H/(H \cap xBx^{-1}) &\cong \exp(\mathbb{R}V) \times (H/(H \cap xB'x^{-1})),\\
B/(B \cap x^{-1}Hx) &\cong \exp(\mathbb{R}X) \times (B_0/(B_0 \cap x^{-1}Hx)),\\
B'/(B' \cap x^{-1}Hx) &\cong B_0/(B_0 \cap x^{-1}Hx).
\end{aligned}$$

Moreover, if $y \in (HxB) \cap G_\ell$, then $xb = hy$ with certain $b \in B$ and $h \in H$. It follows that

$$xb{\cdot}\ell(V) = hy{\cdot}\ell(V) = f(h^{-1}{\cdot}V) = f(V)$$

on one side and

$$xb{\cdot}\ell(V) = b{\cdot}\ell(x^{-1}{\cdot}V) = b{\cdot}\ell(Y - W)$$

on the other. It results that $b \in B_0$ and $H\backslash F_1/B = H\backslash F_1/B'$. We conclude in this way that also on F_1 our claim comes from the inversion formula of Fourier.

Finally, we divide F_2 into two parts,

$$F_2^1 = \{x \in F_2; Y \in \mathfrak{h} + x{\cdot}\mathfrak{b}\} \text{ and } F_2^2 = \{x \in F_2; Y \notin \mathfrak{h} + x{\cdot}\mathfrak{b}\}.$$

For $x \in F_2^1$, $Y = V + x{\cdot}W$ with certain $V \in \mathfrak{h} \setminus \mathfrak{h}_0$ and $W \in \mathfrak{b} \setminus \mathfrak{b}_0$. Here,

$$\begin{aligned}
B/(B \cap x^{-1}Hx) &\cong \exp(\mathbb{R}W) \times (B_0/(B_0 \cap x^{-1}Hx)),\\
B'/(B' \cap x^{-1}Hx) &\cong \exp(\mathbb{R}Y) \times (B_0/(B_0 \cap x^{-1}Hx))
\end{aligned}$$

and

$$HxB' = \cup_{t\in\mathbb{R}} Hx\exp(tx^{-1}\cdot V)\exp(tW)B_0 = HxB.$$

It follows that

$$\int_{p(F_2^1)} d\lambda(\dot{x}) \int_{H/(H\cap xBx^{-1})} d\dot{h} \int_{B/(B\cap x^{-1}Hx)} \phi_H^f(hxbx^{-1}h^{-1})\chi_\ell(b)d\dot{b}$$

$$= \int_{p(F_2^1)} d\lambda(\dot{x}) \int_{\mathbb{R}} ds \int_{H_0/(H_0\cap xBx^{-1})} d\dot{h} \int_{B_0/(B_0\cap x^{-1}Hx)} d\dot{b}$$

$$\times \int_{\mathbb{R}} \phi_H^f(h\exp(sV)x\exp(tW)x^{-1}xb\exp(-sx^{-1}V)x^{-1}h^{-1})\chi_\ell(\exp(tW)b)dt$$

$$= \int_{p(F_2^1)} d\lambda(\dot{x}) \int_{\mathbb{R}} ds \int_{H_0/(H_0\cap xBx^{-1})} d\dot{h} \int_{B_0/(B_0\cap x^{-1}Hx)} d\dot{b}$$

$$\times \int_{\mathbb{R}} \phi_H^f(h\exp(sV)\exp(t(Y-V))\exp(-sV)xbx^{-1}h^{-1})\chi_\ell(\exp(tW)b)dt$$

$$= \int_{p(F_2^1)} d\lambda(\dot{x}) \int_{\mathbb{R}} ds \int_{H_0/(H_0\cap xBx^{-1})} d\dot{h} \int_{B_0/(B_0\cap x^{-1}Hx)} d\dot{b}$$

$$\times \int_{\mathbb{R}} \phi_H^f(h\exp(t(Y-V))xbx^{-1}h^{-1})\chi_\ell(\exp(tW)b)e^{ist}dt$$

$$= \int_{p(F_2^1)} d\lambda(\dot{x}) \int_{H_0/(H_0\cap xBx^{-1})} d\dot{h} \int_{B_0/(B_0\cap x^{-1}Hx)} \phi_H^f(hxbx^{-1}h^{-1})\chi_\ell(b)d\dot{b}.$$

In the same way this last member is equal to

$$\int_{p(F_2^1)} d\lambda(\dot{x}) \int_{H/(H\cap xB'x^{-1})} d\dot{h} \int_{B'/(B'\cap x^{-1}Hx)} \phi_H^f(hxbx^{-1}h^{-1})\chi_\ell(b)d\dot{b},$$

what finishes the computations on F_2^1.

For $x \in F_2^2$,

$$B/(B\cap x^{-1}Hx) \cong \exp(\mathbb{R}X) \times (B_0/(B_0\cap x^{-1}Hx))$$

with X chosen in $\mathfrak{b} \setminus \mathfrak{b}_0$ and

$$B'/(B'\cap x^{-1}Hx) \cong \exp(\mathbb{R}Y) \times (B_0/(B_0\cap x^{-1}Hx)).$$

Besides, as $Y \notin \mathfrak{h} + x\cdot\mathfrak{b}$, we recall the reasoning already done once before. For any $t \in \mathbb{R}$, we consider $\zeta_t \in (\mathfrak{h} + x\cdot\mathfrak{b})^\perp$ satisfying $\zeta_t(Y) = t$. By the Pukanszky condition, there exists $b_t \in xBx^{-1}$ outside of xB_0x^{-1} such that $b_tx\cdot\ell = \zeta_t$, namely $b_tx \in G_\ell$. We get from this $HxB = \cup_{t\in\mathbb{R}} Hb_txB_0$ and hence

$$H\backslash F_2^2/B_0 \cong \mathbb{R} \times (H\backslash F_2^2/B).$$

On the other hand, as $X \in x^{-1}\cdot\mathfrak{h} + \mathfrak{b}'$, for any $t \in \mathbb{R}$ we similarly define $\xi_t \in \mathfrak{g}^*$ such that $\xi_t \in (\mathfrak{h} + x\cdot\mathfrak{b}')^\perp, \xi_t(x\cdot X) = t$. Hence there exists $b'_t \in xB'x^{-1}$ not belonging to xB_0x^{-1} and satisfying $b'_t x \in G_\ell$. We now conclude that $HxB' = \cup_{t\in\mathbb{R}} Hb_t xB_0$ and

$$H\backslash F_2^2/B_0 \cong \mathbb{R} \times (H\backslash F_2^2/B').$$

These observations find that there exists a subset F_3 of F_2^2 such that

$$H\backslash F_2^2/B \cong \mathbb{R} \times (H\backslash F_3/B_0)$$

under the correspondence

$$\mathbb{R} \ni t \mapsto b'_t \in xB'x^{-1},\ b'_t \notin xB_0x^{-1}$$

and

$$H\backslash F_2^2/B' \cong \mathbb{R} \times (H\backslash F_3/B_0)$$

under

$$\mathbb{R} \ni t \mapsto b_t \in xBx^{-1},\ b_t \notin xB_0x^{-1}.$$

Finally, a simple calculation confirms that the formula of $\mathrm{Tr}\,(\pi(\phi)U_\pi)$ remains unchanged.

Until now we proved that the formula in question does not change under the modification from $\mathfrak{b}$ to $\mathfrak{b}'$. In order to see its independence of the choice of polarizations it suffices therefore to confirm the claim when polarizations are contained in $\mathfrak{g}_0$. By the induction hypothesis we assume the result at the level of G_0.

First, suppose $\mathfrak{h} \subset \mathfrak{g}_0$. We consider the slices $\Omega_t, t \in \mathbb{R}$, of $\Gamma_\tau \cap \Omega$ by hyperplane $\{\xi \in \mathfrak{g}^*; \xi(Y) = t\}$. On each Ω_t, the induction hypothesis assures the invariance of the formula. Integrating them with respect to the variable t, we arrive at the awaited result.

Secondly, suppose $\mathfrak{h} \not\subset \mathfrak{g}_0$. If we fix the element X in $\mathfrak{h}$ and if we look at the independence of the formula at the level of G_0 for $\phi_0 \in \mathcal{D}(G_0)$ constructed from $\phi \in \mathcal{D}(G)$ by

$$\phi_0(g_0) = \int_{\mathbb{R}} \phi(g_0\exp(tX))dt\ (g_0 \in G_0)$$

we notice immediately that there remains nothing to prove. ■

Now, the value of $\mathrm{Tr}\,(\pi(\phi)U_\pi)$ being independent of the choice of polarizations, the next theorem can be proved just like the last part of the proof of the above theorem.

Theorem 11.4.4 (Plancherel Formula [33]). *Normalizing appropriately the measures,*

$$\phi_H^f(e) = \int_{\Gamma_\tau/G} Tr\,(\pi_\Omega(\phi) U_{\pi_\Omega})\, d\mu(\Omega)$$

for all $\phi \in \mathcal{D}(G)$. *Here* π_Ω *is the irreducible unitary representation of* G *corresponding to the coadjoint orbit* $\Omega \in \mathfrak{g}^*/G$.

Chapter 12
Commutativity Conjecture: Induction Case

12.1 Toward the Commutativity Conjecture

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $H = \exp \mathfrak{h}$ an analytic subgroup of G with Lie algebra $\mathfrak{h}$. Given a unitary character χ of H, we construct the monomial representation $\tau = \mathrm{ind}_H^G \chi$ and examine the algebra $D_\tau(G/H)$ of the G-invariant differential operators on the line bundle $G \times_H \mathbb{C}$ associated with χ. Our target is the commutativity conjecture due to Duflo [22] and Corwin and Greenleaf [17]. The latter proved one direction of implication: if τ has finite multiplicities, then $D_\tau(G/H)$ is commutative. Therefore, we are interested in the inverse direction of implication.

We take an $f \in \mathfrak{g}^*$ such that $d\chi = if|_{\mathfrak{h}}, i = \sqrt{-1}$ and set $\Gamma_\tau = f + \mathfrak{h}^\perp$ as before. Our first step is:

Theorem 12.1.1 ([35]). *Assume that τ has infinite multiplicities. Let $\mathfrak{g}_0$ be a Lie subalgebra of codimension 1 containing $\mathfrak{h}$ and $G_0 = exp(\mathfrak{g}_0)$. Suppose that $\tau_0 = ind_H^{G_0} \chi$ has finite multiplicities. If there exists $W \in \mathcal{U}(\mathfrak{g}, \tau)$ such that $W \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, then there is $T \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$ such that $[W, T] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.*

Proof. Let us employ the induction on $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{h})$. Notice first that we may assume that $\mathfrak{h}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. In fact, provided $\mathfrak{z} \not\subset \mathfrak{h}$, we take $0 \neq Z \in \mathfrak{z}$ in the outside of $\mathfrak{h}$, and describe the representatives in $\mathcal{U}(\mathfrak{g}_0, \tau_0)$ by means of an adapted basis $\{Z, X_1, X_2, \ldots, X_p\}$ to $\mathfrak{h}$ in $\mathfrak{g}_0$. Here $\{X_j\}_{j=1}^p$ is an adapted basis to $\mathfrak{h}' = \mathfrak{h} + \mathbb{R}Z$ in $\mathfrak{g}_0$. At a generic $\ell \in \Gamma_\tau$, we put $\alpha = \ell(Z)$. Then, if we denote by χ_ℓ the character of $H' = \exp(\mathfrak{h}')$ defined by $d\chi_\ell = i\ell|_{\mathfrak{h}'}$, $\tau^\alpha = \mathrm{ind}_H^{G_0} \chi_\ell$ has finite multiplicities.

A subset $\mathcal{S}$ of an algebra $\mathcal{A}$ is said to be a system of rational generators if for arbitrary $a \in \mathcal{A}$ there exist two polynomials P, Q of the elements of $\mathcal{S}$ such that $Pa = Q$. Using $\{X_j\}_{j=1}^p$ and a generic α, there is a system of rational generators of the algebra $D_{\tau^\alpha}(G_0/H)$ and its element in $\mathcal{U}(\mathfrak{g}_0, \tau^\alpha)/\mathcal{U}(\mathfrak{g}_0)\mathfrak{a}_{\tau^\alpha}$ is expressed, by means of an element

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$$T = \sum_{k,J} c_{k,J} Z^k X_1^{j_1} X_2^{j_2} \cdots X_p^{j_p}$$

of $\mathcal{U}(\mathfrak{g}_0, \tau_0)$, as

$$T(\alpha) = \sum_{k,J} c_{k,J} (-i\alpha)^k X_1^{j_1} X_2^{j_2} \cdots X_p^{j_p}$$

with the notation $J = (j_1, j_2, \ldots, j_p)$ for p-tuple of non-negative integers.

We write $\mathfrak{g} = \mathfrak{g}_0 + \mathbb{R}X$ and describe a representative of W by means of the basis $\{Z, X_1, \ldots, X_p, X\}$ to construct $W(\alpha) \in \mathcal{U}(\mathfrak{g}, \tau_\alpha)$. Here $\tau_\alpha = \mathrm{ind}_{G_0}^G \tau^\alpha$. From the above, if $[W, T] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ for arbitrary $T \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$,

$$[W, T(\alpha)] = [W(\alpha) + \tilde{W}(Z + i\alpha), T(\alpha)] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau_\alpha}$$

with a certain $\tilde{W} \in \mathcal{U}(\mathfrak{g})$. Then, if we take α so that $W(\alpha) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau_\alpha}$, this contradicts the induction hypothesis applied to the pair $(\mathfrak{h}', \chi_\ell)$. (All of these correspond to the direct integral decomposition $\tau \simeq \int_{\mathbb{R}} \tau^\alpha d\alpha$.)

Hereafter we assume $\mathfrak{z} \subset \mathfrak{h}$. Provided $\mathfrak{z} \cap \ker f \neq \{0\}$, everything passes to the quotient space and the desired result is immediately obtained. Hence it remains for us to examine the case where $\dim \mathfrak{z} = 1$ and $f|_{\mathfrak{z}} \neq 0$. Take as usual a Heisenberg triplet $(\tilde{X}, Y, Z)$ such that $\mathfrak{z} = \mathbb{R}Z, [\tilde{X}, Y] = Z, Y \in \mathfrak{g}_0$ and that $\mathfrak{g} = \mathbb{R}\tilde{X} + \mathfrak{k}$, where $\mathfrak{k}$ denotes the centralizer of Y in $\mathfrak{g}$.

Assume $\mathfrak{h} \subset \mathfrak{k}$. If $Y \in \mathfrak{h}$, $\tau' = \mathrm{ind}_H^K \chi$ ($K = \exp \mathfrak{k}$) has infinite multiplicities. Since $\tau_0' = \mathrm{ind}_H^{G_0 \cap K} \chi$ for the given $\mathfrak{g}_0$ has finite multiplicities, it is enough to apply the induction hypothesis to this τ'. If $Y \notin \mathfrak{h}$, $T = Y \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$ fulfils the request of the theorem when $\mathfrak{g}_0 = \mathfrak{k}$. So, let us assume $\mathfrak{g}_0 \neq \mathfrak{k}$. If $W \notin \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, we can always take $T = Y$. Hence assume $W \in \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Since $\tau_0 = \mathrm{ind}_{G_0 \cap K}^{G_0} \left(\mathrm{ind}_H^{G_0 \cap K} \chi\right)$ has finite multiplicities, τ' must have infinite multiplicities. Otherwise $W \in \mathcal{U}(\mathfrak{k} \cap \mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, which contradicts the assumption. Now it is enough to apply the induction hypothesis to $\mathfrak{k}$.

Finally, assume $\mathfrak{h} \not\subset \mathfrak{k}$. We consider $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{k}, H_0 = \exp(\mathfrak{h}_0)$ and $f_0 = f|_{\mathfrak{k}} \in \mathfrak{k}^*$. Then, we know that $\tau_1 = \mathrm{ind}_{H_0}^K \chi_{f_0}$ has infinite multiplicities. We set $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$, $M = \exp \mathfrak{m}$ and notice that $\tau_2 = \mathrm{ind}_{H_0}^M \chi_{f_0}$ has finite multiplicities. Since W is represented by an element of $\mathcal{U}(\mathfrak{k})$, the induction hypothesis applied to $\mathfrak{k}$ assures that there exists $T \in \mathcal{U}(\mathfrak{m}, \tau_2)$ such that $[W, T] \notin \mathcal{U}(\mathfrak{k})\mathfrak{a}_{\tau_1}$. Here, we may think that T is an element of a system of rational generators of $\mathcal{U}(\mathfrak{m}, \tau_2)$ studied in Corwin and Greenleaf [17], while, utilizing a weak Malcev basis adapted to $\mathfrak{h}$ in $\mathfrak{g}_0$, it is easily checked that $\{Y, \gamma_2, \ldots, \gamma_q\}$ form a system of rational generators for $\mathcal{U}(\mathfrak{m}, \tau_2)$. Here $\{\gamma_j\}_{j=2}^q$ constitute rational generators for $\mathcal{U}(\mathfrak{g}_0, \tau_0)$. As $[W, Y] = 0$, we recognize the existence of such an element T in $\mathcal{U}(\mathfrak{g}_0, \tau_0)$. ∎

Let us use here again the notations introduced at the beginning of Chap. 9. However, $\{X_r\}_{1 \le r \le q}$ denotes a coexponential basis to $\mathfrak{h}$ in $\mathfrak{g}$ adapted to the sequence of Lie subalgebras (9.2.3), i.e. $X_r \in \mathfrak{k}_r \setminus \mathfrak{k}_{r-1}$ $(1 \le r \le q)$. In what follows, if $\mathfrak{g} \neq \mathfrak{h}$, $\mathfrak{g}'$ always denotes an ideal of codimension 1 in $\mathfrak{g}$ which contains $\mathfrak{h}$. Moreover, we choose the composition series of ideals (9.1.1) in such a way that $\mathfrak{g}_{n-1} = \mathfrak{g}'$. Likewise, when $\dim \mathfrak{h} \ge 1$, $\mathfrak{h}'$ always denotes a Lie subalgebra of codimension 1 in $\mathfrak{h}$ and the composition series (9.2.4) satisfies $\mathfrak{h}_{d-1} = \mathfrak{h}'$. Then, when $\mathfrak{g}'$, $\mathfrak{h}'$ both exist, $\dim \mathfrak{g} \ge 2$ and $\mathfrak{g} \supsetneq \mathfrak{g}' \supset \mathfrak{h} \supsetneq \mathfrak{h}'$.

Further, we make the following convention. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. We consider the elements $\{X_r\}_{1\le r\le q}, \{Y_s\}_{1\le s\le d}$ introduced above and just before Lemma 9.2.14. For each q-tuple $J = (j_1, j_2, \dots, j_q) \in \mathbb{N}^q$, d-tuple $K = (k_1, k_2, \dots, k_d) \in \mathbb{N}^d$ and $(d-1)$-tuple $L = (\ell_1, \ell_2, \dots, \ell_{d-1}) \in \mathbb{N}^{d-1}$, we denote by X^J, Y^K and Y'^L respectively the elements $X^J = X_q^{j_q} \cdots X_2^{j_2} X_1^{j_1}$, $Y^K = Y_d^{k_d} \cdots Y_2^{k_2} Y_1^{k_1}$ and $Y'^L = Y_{d-1}^{\ell_{d-1}} \cdots Y_2^{\ell_2} Y_1^{\ell_1}$ of $\mathcal{U}(\mathfrak{g})$. As in the common usage, $|J|$ (resp. $|K|$, $|L|$) signifies the sum $j_1 + j_2 + \cdots + j_q$ (resp. $k_1 + k_2 + \cdots + k_d$, $\ell_1 + \ell_2 + \cdots + \ell_{d-1}$). Similarly, we consider the elements $\hat{Y}_s = Y_s + if(Y_s)$, $1 \le s \le d$ of $\mathcal{U}(\mathfrak{g})$. Hence $\hat{Y}^K = \hat{Y}_d^{k_d} \cdots \hat{Y}_2^{k_2} \hat{Y}_1^{k_1}$, and $\hat{Y'}^L = \hat{Y}_{d-1}^{\ell_{d-1}} \cdots \hat{Y}_2^{\ell_2} \hat{Y}_1^{\ell_1}$.

By the Poincaré–Birkhoff–Witt theorem, the families

$$\{X^J \hat{Y}^K; (J, K) \in \mathbb{N}^q \times \mathbb{N}^d\},\ \{X^J \hat{Y}^K; (J, K) \in \mathbb{N}^q \times \mathbb{N}^d, |K| > 0\}$$

form respectively bases of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. The elements $\{X^J; J \in \mathbb{N}^q\}$ make a basis of a linear complement S of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ in $\mathcal{U}(\mathfrak{g})$. Assuming $\mathfrak{h} \neq \{0\}$, we set $H' = \exp(\mathfrak{h}')$ and $\tau' = \mathrm{ind}_{H'}^G \chi_f$. Put $\hat{Y}^K = \hat{Y}_d^k \hat{Y'}^L$. Thus, the families

$$\{X^J \hat{Y}_d^k \hat{Y'}^L; (J, k, L) \in \mathbb{N}^q \times \mathbb{N} \times \mathbb{N}^{d-1}, |L| > 0\},\ \{X^J \hat{Y}_d^k; (J, k) \in \mathbb{N}^q \times \mathbb{N}\}$$

constitute bases respectively of $U(\mathfrak{g})\mathfrak{a}_{\tau'}$ and a linear complement of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ in $\mathcal{U}(\mathfrak{g})$.

We mention some properties of $\mathcal{U}(\mathfrak{g}, \tau)$ obtained in [5].

Lemma 12.1.2. (i) $\mathfrak{g}_{i_d-1} S \subset S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.
(ii) $[\mathfrak{h}, S] \subset S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.

Proof. (i) For $0 \le r \le q$ and $k \in \mathbb{N}$, we denote by $S_{r,k}$ the subspace of S generated by the elements $X^J = X_r^{j_r} \cdots X_1^{j_1}$, $J \in \mathbb{N}^r$, satisfying $|J| \le k$. In particular, $S_{0,k} = S_{r,0} = \mathbb{C}$. In order to confirm that many monomials are well ordered, the index r is useful. Now, it suffices to show the following by induction on k.

$(*)$ $Y' X^J \in S_{r,k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}, \forall\, Y' \in \mathfrak{h}',\ \forall\, X^J \in S_{r,k}$;
$(**)$ $X_s X^J \in S_{\max(s,r),k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ for any s such that $j_s \le i_d - 1$ and $\forall\, X^J \in S_{r,k}$.

This is clear when $k = 0$. Assume $k > 0$ and that the result is correct until the step $k - 1$. If we take X_s such that $s \ge r$, then $X_s X^J \in S_{s,k+1}$ and the result is clear. If we take $T = Y' \in \mathfrak{h}'$ or $T \in \mathfrak{g}_{i_d-1}$ such that $T = X_s$ with $s < r$, then we can write X^J as $X^J = X_r X^{J'}$ using $X^{J'} = X_r^{j_r-1} \cdots X_1^{j_1}$ and make use of the equality

$$TX^J = [T, X_r]X^{J'} + X_r TX^{J'}.$$

$[T, X_r] \in \mathfrak{g}_{j_r-1} \cap \mathfrak{g}_{i_d-1}$ and $X^{J'} \in S_{r,k-1}$. Hence,

$$[T, X_r]X^{J'} \in S_{r,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$$

by induction. Similarly, $TX^{J'} \in S_{r,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. Thus $X_r TX^{J'} \in S_{r,k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.

(ii) We show by induction on k that

$$[Y, X^J] \in S_{r,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$$

for $\forall\, Y \in \mathfrak{h}$, $\forall\, X^J \in S_{r,k}$. This is clear when $k = 0$. Assume that the claim is correct at the step $k-1$, $k \neq 0$. First of all

$$[Y, X^J] = [Y, X_r]X^{J'} + X_r[Y, X^{J'}].$$

Since $[Y, X_r] \in \mathfrak{g}_{j_r-1} \cap \mathfrak{g}_{i_d-1}$ and $X^{J'} \in S_{r,k-1}$, we know from the claim (i) $[Y, X_r]X^{J'} \in S_{r,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. Finally, $[Y, X^{J'}] \in S_{r,k-1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ by induction and $X_r[Y, X^{J'}] \in S_{r,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. ■

Lemma 12.1.3. *Let W be an element of $\mathcal{U}(\mathfrak{g})$ and assume that $W \equiv \sum_{k \le k'} A_k \hat{Y}_d^k$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$, where A_k are elements of S. Assume $Y \in \mathfrak{h}$. By Lemma 12.1.2, we obtain elements $(B_k)_{k \le k'}$ of S so that $[Y, A_k] \equiv B_k$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. Then, modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$,*

$$[Y, W] \equiv [Y, \sum_{k \le k'} A_k \hat{Y}_d^k] \equiv \sum_{k \le k'} [Y, A_k]\hat{Y}_d^k \equiv \sum_{k \le k'} B_k \hat{Y}_d^k.$$

Proof. The first and the third equalities are obtained from the inclusion $\mathfrak{h} \subset \mathcal{U}(\mathfrak{g}, \tau')$ and the second from the relations $[Y, A_k \hat{Y}_d^k] = [Y, A_k]\hat{Y}_d^k + A_k[Y, \hat{Y}_d^k]$ and $[Y, \hat{Y}_d] \in \mathfrak{h}' \cap \ker f \subset \mathfrak{a}_{\tau'}$. ■

Proposition 12.1.4. *(i) $[\mathfrak{h}, \mathcal{U}(\mathfrak{g}, \tau) \cap S] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. Especially,*

$$\mathcal{U}(\mathfrak{g}, \tau) \cap S \subset \mathcal{U}(\mathfrak{g}, \tau').$$

(ii) We get the decomposition $\mathcal{U}(\mathfrak{g}, \tau) = (\mathcal{U}(\mathfrak{g}, \tau) \cap S) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and the restriction to $\mathcal{U}(\mathfrak{g}, \tau) \cap S$ of the projection from $\mathcal{U}(\mathfrak{g}, \tau)$ onto $D_\tau(G/H)$ is a bijection. Thus, each element of

$$D_\tau(G/H) \cong \mathcal{U}(\mathfrak{g}, \tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$$

has a unique representative in S. This representative belongs to $\mathcal{U}(\mathfrak{g}, \tau')$.

Proof. (i) By (ii) of Lemma 12.1.2,

$$[\mathfrak{h}, \mathcal{U}(\mathfrak{g}, \tau) \cap S] \subset (S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}) \cap \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

(ii) The result follows from the decomposition $\mathcal{U}(\mathfrak{g}) = S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and the inclusion relation $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \subset \mathcal{U}(\mathfrak{g}, \tau)$. ∎

Proposition 12.1.5. *Let $(A_k)_{0\le k\le k'}$ be a family of elements of S, then*

$$\sum_{k\le k'} A_k \hat{Y}_d^k \in \mathcal{U}(\mathfrak{g}, \tau') \Longleftrightarrow A_k \in \mathcal{U}(\mathfrak{g}, \tau'),\ \forall\, k \le k'.$$

Proof. Let $Y' \in \mathfrak{h}'$. We define for all k the element B_k of S so that $[Y', A_k] = B_k$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. By Lemma 12.1.3,

$$\Big[Y', \sum_{k\le k'} A_k \hat{Y}_d^k\Big] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} \Longleftrightarrow B_k = 0, \forall\, k \le k'.$$

∎

Proposition 12.1.6. *Let $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{h}$ and $\mathfrak{h}'$ be as before.*

(i) The following equivalence holds:

$$\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} \Longleftrightarrow \mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

(ii) Besides, if we assume $\mathcal{U}(\mathfrak{g}, \tau') \subset \mathcal{U}(\mathfrak{g}, \tau)$,

$$\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} \Longleftrightarrow \mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

Proof. (i) $\Leftarrow$ is clear. We show $\Rightarrow$. Let W' be an element of $\mathcal{U}(\mathfrak{g}, \tau') \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'})$. W' is written modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ as $W' \equiv \sum_{k\le k'} A_k \hat{Y}_d^k$. Here A_k are elements of S. Because W' does not belong to $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$, one of the A_k, say A_{k_0}, does not belong to $\mathcal{U}(\mathfrak{g}')$. In other words, X_q appears in A_{k_0}. Then, by Proposition 12.1.5, $A_{k_0} \in \mathcal{U}(\mathfrak{g}, \tau')$. Since $(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau) \cap S = \mathcal{U}(\mathfrak{g}') \cap S$ and $A_{k_0} \in S \setminus \mathcal{U}(\mathfrak{g}')$, A_{k_0} does not belong to $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Thus we get $\Rightarrow$.

(ii) $\Rightarrow$ is directly obtained from (i) and our assumption. We show $\Leftarrow$. From the decomposition

$$\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau = \big(\mathcal{U}(\mathfrak{g}') \cap S\big) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau,\ \mathcal{U}(\mathfrak{g}, \tau) = (\mathcal{U}(\mathfrak{g}, \tau) \cap S) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$$

and our assumption, $\mathcal{U}(\mathfrak{g}, \tau) \cap S \not\subset \mathcal{U}(\mathfrak{g}') \cap S = (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau) \cap S$. The result follows from the inclusion relation $\mathcal{U}(\mathfrak{g}, \tau) \cap S \subset \mathcal{U}(\mathfrak{g}, \tau')$ of Proposition 12.1.4. ∎

Proposition 12.1.7. *Assume* $\mathcal{U}(\mathfrak{g}, \tau') \cap S \subset \mathcal{U}(\mathfrak{g}, \tau) \cap S$. *Then,*

$$[\mathfrak{h}, \mathcal{U}(\mathfrak{g}, \tau')] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

Proof. We write an element W of $\mathcal{U}(\mathfrak{g}, \tau')$ as

$$W \equiv \sum_{k \le k'} A_k \hat{Y}_d^k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'},$$

where A_k are elements of S. By Proposition 12.1.5, $A_k \in \mathcal{U}(\mathfrak{g}, \tau') \cap S$ and $A_k \in \mathcal{U}(\mathfrak{g}, \tau) \cap S$ by assumption. Finally, from the first assertion of Proposition 12.1.4,

$$[\mathfrak{h}, W] \subset \sum_{k \le k'} \left([\mathfrak{h}, A_k]\hat{Y}_d^k + A_k[\mathfrak{h}, \hat{Y}_d^k] \right) \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

■

Proposition 12.1.8. *Let* $\mathfrak{g}, \mathfrak{g}'$ *and* $\mathfrak{h}$ (*also* $\mathfrak{h}'$ *in the third assertion*) *be as before and suppose that* X_q *exists.*

(i) *Let* $W = \sum_{k=0}^{m} X_q^k A_k \in \mathcal{U}(\mathfrak{g}, \tau)$, *where* $m \ge 1$ *and* $A_k \in \mathcal{U}(\mathfrak{g}')$. *Then,* $A_m \in \mathcal{U}(\mathfrak{g}, \tau)$ *and* $m X_q A_m + A_{m-1} \in \mathcal{U}(\mathfrak{g}, \tau)$.

(ii) *Let* $W = X_q U + V$ *with* $U, V \in \mathcal{U}(\mathfrak{g}')$. *Then,*

$$W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \iff U \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau.$$

(iii) *Assume* $\mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ (*resp.* $\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$). *Then, there exists an element*

$$W = X_q U + V \in \mathcal{U}(\mathfrak{g}, \tau) \setminus \left(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right)$$
$$(\textit{resp.} \in \mathcal{U}(\mathfrak{g}, \tau') \setminus \left(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right))$$

with $U \in (\mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ (*resp.* $U \in (\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$) *and* $V \in \mathcal{U}(\mathfrak{g}')$.

Proof. (i) Immediately from the Poincaré–Birkhoff–Witt theorem,

$$\mathcal{U}(\mathfrak{g}) = \oplus_j X_q^j \mathcal{U}(\mathfrak{g}'), \ \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau = \oplus_j X_q^j \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau. \tag{12.1.1}$$

Set $\mathfrak{i}_{m-2} = \oplus_{j=0}^{m-2} X_q^j \mathcal{U}(\mathfrak{g}')$. Then for arbitrary $Y \in \mathfrak{h}$,

$$\begin{aligned}[W, Y] &\equiv X_q^m [A_m, Y] + \sum_{j=1}^{m} X_q^{j-1}[X_q, Y] X_q^{m-j} A_m \\ &\quad + X_q^{m-1}[A_{m-1}, Y] \bmod \mathfrak{i}_{m-2} \\ &\equiv X_q^m [A_m, Y] + X_q^{m-1}(m[X_q, Y]A_m + [A_{m-1}, Y]) \bmod \mathfrak{i}_{m-2}\end{aligned}$$

belongs to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Therefore, $A_m \in \mathcal{U}(\mathfrak{g}, \tau)$ and $m X_q A_m + A_{m-1} \in \mathcal{U}(\mathfrak{g}, \tau)$.

(ii) Making use of expression (12.1.1), $W \in \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ when and only when $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$. Because,

$$W \in \big(X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')\big) \cap \Big(\oplus_j X_q^j\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau \oplus \mathcal{U}(\mathfrak{g}')\Big) = X_q\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau \oplus \mathcal{U}(\mathfrak{g}').$$

(iii) With A_k belonging to $\mathcal{U}(\mathfrak{g}')$, we put

$$W' = \sum_{k=0}^{m} X_q^k A_k \in \mathcal{U}(\mathfrak{g}, \tau) \setminus \big(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\big)$$

$$\Big(\text{resp. } W' = \sum_{k=0}^{m} X_q^k A_k \in \mathcal{U}(\mathfrak{g}, \tau') \setminus \big(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\big)\Big).$$

From assertion (i), $A_m \in \mathcal{U}(\mathfrak{g}, \tau)$ and $W = mX_qA_m + A_{m-1} \in \mathcal{U}(\mathfrak{g}, \tau)$ (resp. $A_m \in \mathcal{U}(\mathfrak{g}, \tau')$ and $W = mX_qA_m + A_{m-1} \in \mathcal{U}(\mathfrak{g}, \tau')$). Without loss of generality, we may assume $A_m \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ and, taking (ii) into account, W is provided with the required properties. ∎

Theorem 12.1.9. *With the notation and assumption of Proposition 12.1.8, assume*

$$\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} \text{ and } \mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}') \not\subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}').$$

Then $\mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

Proof. We first show that there exists an element $W = X_qU + V \in \mathcal{U}(\mathfrak{g}, \tau') \cap S$, where $U, V \in \mathcal{U}(\mathfrak{g}') \cap S$, so that:

(a) $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, or equivalently for $U \neq 0$,
(b) $(\mathrm{ad} Y_d)W \in \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

This fact will be utilized later in various situations in order to construct an element of $\mathcal{U}(\mathfrak{g}, \tau)$ not belonging to $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. From assertion (i) of Proposition 12.1.6, we know that $\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Thus, from the third assertion of Proposition 12.1.8, there exists $W'' = X_qU'' + V'' \in \mathcal{U}(\mathfrak{g}, \tau') \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau)$ with $U'' \in (\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ and $V'' \in \mathcal{U}(\mathfrak{g}')$. Next let $m' \in \mathbb{N}$ be the maximal integer such that $W' = (\mathrm{ad}\, Y_d)^{m'}(W'') \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Using the fact that $\mathcal{U}(\mathfrak{g}) = S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, let W be the unique element of S such that $W \equiv W'$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Then W satisfies the conditions (a) and (b).

Now let us introduce the notation used only in this proof. For $A, B \in \mathcal{U}(\mathfrak{g})$, set $\{A, B\} = AB + BA$. If we write $A_s = (\mathrm{ad}\, Y_d)^s A$ with $s \in \mathbb{N}$, then $A_0 = A$. Next, for $r \in \mathbb{N} \setminus \{0\}$, we define

$$\mathcal{T}_r(A) = \{A_0, A_{2r}\} - \{A_1, A_{2r-1}\} + \cdots + (-1)^{r-1}\{A_{r-1}, A_{r+1}\} + (-1)^r A_r^2.$$

Since Y_d belongs to $\mathcal{U}(\mathfrak{g}, \tau')$, if A belongs to $\mathcal{U}(\mathfrak{g}, \tau')$, so does A_s and $\mathcal{T}_r(A)$. Furthermore, if r is large enough to get $A_{2r+1} \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$, then $\mathcal{T}_r(A) \in \mathcal{U}(\mathfrak{g}, \tau)$. In order to see this, it suffices to show that this new condition implies $(\text{ad } Y_d)\mathcal{T}_r(A) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. In fact,

$$(\text{ad } Y_d)\mathcal{T}_r(A) = A_{2r+1}A + AA_{2r+1} \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\mathcal{U}(\mathfrak{g}, \tau') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} = \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

Let m be the minimal integer such that $W_m = (\text{ad } Y_d)^m(W) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. By means of the induction on s and assertion (ii) of Lemma 12.1.2, we see that W_s belongs to $S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ and hence $W_m \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. We separate the cases by the value of m.

- When $m = 1$, from $W \in \mathcal{U}(\mathfrak{g}, \tau) \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau)$ the result is obvious.
- Let $r \geq 1$ and $m = 2r + 1$. The above remark says that

$$\mathcal{T}_r(W) = \{W_0, W_{2q}\} - \{W_1, W_{2q-1}\} + \cdots + (-1)^{r-1}\{W_{r-1}, W_{r+1}\} + (-1)^r W_r^2$$

belongs to $\mathcal{U}(\mathfrak{g}, \tau)$. We want to show $\mathcal{T}_r(W) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Modulo $\mathcal{U}(\mathfrak{g}')$, $\mathcal{T}_r(W) \equiv 2WW_{2r} \equiv 2X_qUW_{2r}$. Neither U nor W_{2r} belongs to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ and the quotient ring $(\mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}'))\,/\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ has no non-trivial zero divisor. So, the result follows.
- Let $r > 1$ and $m = 2r$. For arbitrary $c \in \mathbb{C}$, we know

$$(\text{ad}Y_d)^{2r+1}\left(W(W_{2r-2} + cW_{2r-1})\right) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

If we put $\tilde{W}(c) = W(W_{2r-2} + cW_{2r-1})$, the remark mentioned before gives $\mathcal{T}_r\left(\tilde{W}(c)\right) \in \mathcal{U}(\mathfrak{g}, \tau)$. We want to show $\mathcal{T}_r\left(\tilde{W}(c)\right) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ for some $c \in \mathbb{C}$. For a while we regard $\tilde{W}(c)$ and $\mathcal{T}\left(\tilde{W}(c)\right)$ as elements of the polynomial ring $\mathcal{U}(\mathfrak{g})[c]$ of c with coefficients in $\mathcal{U}(\mathfrak{g})$. For any $r \geq 1$,

$$\begin{aligned}
\tilde{W}(c)_0 &= W(W_{2r-2} + cW_{2r-1}) \\
\tilde{W}(c)_1 &\equiv W_1(W_{2r-2} + cW_{2r-1}) + WW_{2r-1} \\
\tilde{W}(c)_2 &\equiv W_2(W_{2r-2} + cW_{2r-1}) + 2W_1W_{2r-1} \in \mathcal{U}(\mathfrak{g}')[c] \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\tilde{W}(c)_{2r-1} &\equiv W_{2r-1}(W_{2r-2} + cW_{2r-1}) + (2r-1)W_{2r-2}W_{2r-1} \in \mathcal{U}(\mathfrak{g}')[c] \\
\tilde{W}(c)_{2r} &\equiv 2rW_{2r-1}^2 \in \mathcal{U}(\mathfrak{g}')
\end{aligned}$$

modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]$. Now, let us show

$$\mathcal{T}_r\left(\tilde{W}(c)\right) \in cX_q\mathcal{U}(\mathfrak{g}') \oplus X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] \bmod \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]$$

and draw from it the component relative to $cX_q\mathcal{U}(\mathfrak{g}')$. The following relations hold for $r > 1$ but not for $r = 1$:

$$\begin{aligned}\mathcal{T}_r\left(\tilde{W}(c)\right) \equiv &\{\tilde{W}(c)_0, \tilde{W}(c)_{2r}\} - \{\tilde{W}(c)_1, \tilde{W}(c)_{2r-1}\} \\ &\bmod \left(\mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]\right) \\ \equiv &2\tilde{W}(c)_0\tilde{W}(c)_{2r} - 2\tilde{W}(c)_1\tilde{W}(c)_{2r-1} \bmod \left(\mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]\right) \\ \equiv &2c(2r-1)WW_{2r-1}^3 \bmod \left(X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]\right) \\ \equiv &2c(2r-1)X_qUW_{2r-1}^3 \bmod \left(X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]\right).\end{aligned}$$

Now, if we regard $\mathcal{T}_r\left(\tilde{W}(c)\right)$ as an element of $\mathcal{U}(\mathfrak{g})$, there exists from what we mentioned above $\tilde{U} \in \mathcal{U}(\mathfrak{g}')$ so that

$$\mathcal{T}_r\left(\tilde{W}(c)\right) \equiv X_q\left(2c(2r-1)UW_{2r-1}^3 + \tilde{U}\right) \bmod \left(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right).$$

Here, UW_{2r-1}^3 does not belong to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$. Because, modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c]$ neither U nor W_{2r-1} belongs there and the ring $\left(\mathcal{U}(\mathfrak{g},\tau) \cap \mathcal{U}(\mathfrak{g}')\right)/\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ has no non-trivial zero divisor. Paying attention to $X_q\mathcal{U}(\mathfrak{g}') \cap (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau) = X_q\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$, for all c satisfying

$$2c(2r-1)UW_{2r-1}^3 + \tilde{U} \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau,$$

$\mathcal{T}_r\left(\tilde{W}(c)\right) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

Remark that this method produces no result for $m = 2$. Because, in this case

$$\mathcal{T}_1\left(\tilde{W}(c)\right) \in X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau[c].$$

– Let $m = 2$. Let us use for the first time the assumption $\mathcal{U}(\mathfrak{g},\tau') \cap \mathcal{U}(\mathfrak{g}') \not\subset \mathcal{U}(\mathfrak{g},\tau) \cap \mathcal{U}(\mathfrak{g}')$. Immediately from this, there exists $T \in (\mathcal{U}(\mathfrak{g},\tau') \cap \mathcal{U}(\mathfrak{g}') \cap S) \setminus (\mathcal{U}(\mathfrak{g},\tau) \cap \mathcal{U}(\mathfrak{g}'))$ so that $(\mathrm{ad}\, Y_d)T \in (\mathcal{U}(\mathfrak{g},\tau) \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Then, $WT \in \mathcal{U}(\mathfrak{g},\tau')$, $(\mathrm{ad}\, Y_d)^3(WT) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ and

$$\mathcal{T}_1(WT) = \{(WT)_0, (WT)_2\} - (WT)_1^2 \in \mathcal{U}(\mathfrak{g},\tau).$$

Therefore we want to show $\mathcal{T}_1(WT) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. To begin with,

$$\begin{aligned}\mathcal{T}_1(WT) &\equiv \{WT, W_1T_1 + T_1W_1\} - (W_1T + WT_1)^2 \bmod \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \\ &\equiv -W^2T_1^2 \equiv -X_q^2(UT_1)^2 \bmod \left(X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right).\end{aligned}$$

Remarking that $X_q^2\mathcal{U}(\mathfrak{g}') \cap \left(X_q\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right) = X_q^2\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$, the desired result comes from the fact that neither U nor T_1 belongs to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$. ■

12.2 Corwin–Greenleaf Functions

Theorem 12.1.1 and Theorem 12.1.9 offer nice tools in many situations to attack the commutativity conjecture, nevertheless there is still a case out of their range. In terms of Theorem 12.1.9, it is the case where

$$\mathcal{U}(\mathfrak{g},\tau') \cap \mathcal{U}(\mathfrak{g}') \subset \mathcal{U}(\mathfrak{g},\tau) \cap \mathcal{U}(\mathfrak{g}').$$

In order to treat this case, we need still to dig into the Corwin–Greenleaf e-central elements introduced in Chap. 9 (cf. [36]).

An element $A \in \mathcal{U}(\mathfrak{g})$ is said to be Γ_τ-central if $\pi_\ell(A)$ is a scalar operator at any ℓ in a non-empty Zariski open set $\mathcal{O}$ of Γ_τ. Namely, there is a function φ_A on $G \cdot \mathcal{O}$ so that

$$\pi_\ell(A) = \varphi_A(\ell)Id, \quad \forall\, \ell \in G \cdot \mathcal{O}. \tag{12.2.1}$$

Theorem 9.2.1 of Chap. 9 is extended to this situation: $\pi_\ell(A)$ are scalars at all $\ell \in \Gamma_\tau$ and the function φ_A is an H-invariant polynomial function. We denote by $\mathcal{U}(\mathfrak{g},\Gamma_\tau)$ the algebra of all Γ_τ-central elements in $\mathcal{U}(\mathfrak{g})$ and by

$$Z(\mathfrak{g},\Gamma_\tau) = \{\varphi_A;\, A \in \mathcal{U}(\mathfrak{g},\Gamma_\tau)\}$$

the algebra of all functions satisfying (12.2.1). We designate by α the mapping

$$\mathcal{U}(\mathfrak{g},\Gamma_\tau) \ni A \mapsto \varphi_A \in Z(\mathfrak{g},\Gamma_\tau)$$

and by $CD_\tau(G/H)$ the centre of the algebra $D_\tau(G/H)$; we set

$$\mathcal{U}_C(\mathfrak{g},\tau) = R^{-1}\left(CD_\tau(G/H)\right) = \{A \in \mathcal{U}(\mathfrak{g},\tau);\, [A,\mathcal{U}(\mathfrak{g},\tau)] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\}.$$

If we write as ϖ the restriction of the mapping R to $\mathcal{U}_C(\mathfrak{g},\tau)$,

$$\varpi : \mathcal{U}_C(\mathfrak{g},\tau) \ni A \mapsto R(A) \in CD_\tau(G/H).$$

We indicate by L (resp. R as before) the left (resp. right) action of $\mathcal{U}(\mathfrak{g})$: for $X \in \mathfrak{g}$ and $\psi \in C^\infty(G)$,

$$\left(L(X)\psi\right)(g) = \frac{d}{dt}\psi(\exp(-tX)g)|_{t=0};$$

$$\left(R(X)\psi\right)(g) = \frac{d}{dt}\psi(g\exp(tX))|_{t=0}.$$

The principal anti-automorphism $\Diamond$ sends $\mathcal{U}(\mathfrak{g}, \Gamma_\tau)$ to $\mathcal{U}_C(\mathfrak{g}, \tau)$ and the mappings $\alpha,\ \varpi$ are surjective. Besides, it turns out that there is an injection $\delta : Z(\mathfrak{g}, \Gamma_\tau) \to CD_\tau(G/H)$ so that $\delta(\varphi_A) = L(A) = R(\Diamond(A))$.

Let us recall the sequence of ideals

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_k) = k\ (0 \le k \le n). \tag{9.1.1}$$

Put $\mathfrak{g}_{n-1} = \mathfrak{g}'$ and $\ell' = \ell|_{\mathfrak{g}'}$ for $\ell \in \mathfrak{g}^*$. For a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, we denote respectively by $\mathfrak{m}(\ell)$ and $\mathfrak{m}(\ell')$ Lie subalgebras $\mathfrak{m} \cap \mathfrak{g}(\ell)$ and $\mathfrak{m} \cap (\mathfrak{g}')^\ell$. In Chap. 9, we took the uniquely determined multi-indices $e \in \mathcal{E}$ and introduced the index sets $T(e), T(e_H)$ and $U(e) = T(e) \backslash T(e_H)$. Besides, as

$$T(e) = \{m_1 < m_2 < \cdots < m_t\},$$

making use of a Malcev basis $\{X_j\}_{1 \le j \le n}$ associated with the sequence of ideals (9.1.1), we introduced the Corwin–Greenleaf e-central element

$$\sigma_k = A_j = P_j X_j + Q_j,\ P_j,\ Q_j \in \mathcal{U}(\mathfrak{g}_{j-1})$$

for $j = m_k \in T(e)$. There, P_j itself was e-central and we wrote simply as $\varphi_j(\ell)$ the polynomial function $\varphi_{A_j}(\ell)$ given by the above formula (12.2.1). We agree to call these $\varphi_j(\ell)$ **Corwin–Greenleaf functions**.

Furthermore, for $0 \le i \le n$, we set $T_i(e) = T(e) \cap \{0, \ldots, i\},\ T_i(e_H) = T(e_H) \cap \{0, \ldots, i\}$ and $U_i(e) = U(e) \cap \{1, \ldots, i\}$. Finally, let us define the algebra

$$Z_i(\mathfrak{g}, \tau) = \{\varphi_A; A \in \mathcal{U}(\mathfrak{g}_i) \cap \mathcal{U}(\mathfrak{g}, \Gamma_\tau)\},\ 0 \le i \le n.$$

Thus,

$$\{0\} = Z_0(\mathfrak{g}, \tau) \subseteq Z_1(\mathfrak{g}, \tau) \subseteq \cdots \subseteq Z_n(\mathfrak{g}, \tau) = Z(\mathfrak{g}, \Gamma_\tau).$$

The main results of [36] are as follows:

$(*)$ For all integers $0 \le k \le n$, the family $\{\varphi_j; j \in T_k(e)\}$ makes a system of **rational generators** of $Z_k(\mathfrak{g}, \tau)$.

$(**)$ For all integers $0 \le k \le n$, the family $\{\varphi_j; j \in U_k(e)\}$ makes a **transcendental basis** of the algebra $Z_k(\mathfrak{g}, \tau)$.

Toward these results, let us follow [36] for a while. We indicate the cardinal of a set A by $\sharp(A)$. When a property holds for all elements of some non-empty Zariski open set of a vector space V, we shall say that this property holds generally on V or for general elements of V. Let us first show $(*)$.

System of Rational Generators of $Z_k(\mathfrak{g}, \tau)$

Theorem 12.2.1. *Let* $0 \leq k \leq n$.

(1) *For all integers* $0 \leq k \leq n$*, the family* $\{\varphi_j;\, j \in T_k(e)\}$ *makes a system of rational generators of* $Z_k(\mathfrak{g}, \tau)$.
(2) *If* $k \in S(e)$*, then* $Z_k(\mathfrak{g}, \tau) = Z_{k-1}(\mathfrak{g}, \tau)$.

We define the elements $\{X_j^\dagger\}_{1 \leq j \leq n}$ in $\mathcal{U}(\mathfrak{g})$ by $X_j^\dagger = X_j$ if $j \in S(e)$ and by $X_j^\dagger = A_j$ if $j \in T(e)$. Here $A_j = P_j X_j + Q_j$ is the Corwin–Greenleaf e-central element.

Lemma 12.2.2. *Let* $W \in \mathcal{U}(\mathfrak{g}_k)$, $1 \leq k \leq n$,*. Then there exist an element* $(\alpha_j)_{j \in T_k(e)}$ *of* $\mathbb{N}^{\sharp(T_k(e))}$ *and a mapping* $(\beta_1, \ldots, \beta_k) \mapsto \lambda_{\beta_1 \ldots \beta_k}$ *from* $\mathbb{N}^k$ *to* $\mathbb{C}$*, whose values are* 0 *except at finite points, so that we have*

$$\pi_\ell\left(\prod_{j \in T_k(e)} P_j^{\alpha_j} \cdot W\right) = \pi_\ell\left(\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{N}^k} \lambda_{\beta_1\ldots\beta_k}\left(X_k^\dagger\right)^{\beta_k} \cdots \left(X_1^\dagger\right)^{\beta_1}\right)$$

for arbitrary $\ell \in \Gamma_\tau$.

Proof. Our proof is a little technical but its idea is simple. Let us show by induction on k' that the following holds for $1 \leq k' \leq k$: there exist an element

$$(\alpha_j)_{j \in \left(T_k(e) \setminus T_{k-k'}(e)\right)}$$

of $\mathbb{N}^{\sharp(T_k(e)) - \sharp(T_{k-k'}(e))}$ and a mapping

$$(\beta_{k-k'+1}, \ldots, \beta_k) \mapsto V_{\beta_{k-k'+1}\cdots\beta_k}$$

from $\mathbb{N}^{k'}$ to $\mathcal{U}(\mathfrak{g}_{k-k'})$, whose values are 0 except at finite points, so that we have

$$\pi_\ell\Bigg(\prod_{j \in \left(T_k(e)\setminus T_{k-k'}(e)\right)} P_j^{\alpha_j} \cdot W - \sum_{(\beta_{k-k'+1},\ldots,\beta_k)\in\mathbb{N}^{k'}} \left(X_k^\dagger\right)^{\beta_k} \cdots \left(X_{k-k'+1}^\dagger\right)^{\beta_{k-k'+1}} V_{\beta_{k-k'+1}\cdots\beta_k}\Bigg) = 0 \tag{12.2.2}$$

on Γ_τ.

Notice that, when $k' = k$, this property is nothing but the formula of the lemma since $\mathcal{U}(\mathfrak{g}_{k-k'}) = \mathcal{U}(\mathfrak{g}_0) = \mathbb{C}$. Each element of $\mathcal{U}(\mathfrak{g}_{k-k'+1})$ is written as a linear combination of elements having the form $X_{k-k'+1}^\beta B$, $B \in \mathcal{U}(\mathfrak{g}_{k-k'})$. Hence, when $k' = 1$, we get a relation of the form

$$\pi_\ell\left(W - \sum_{\gamma\in\mathbb{N}} X_k^\gamma B_\gamma\right) = 0,\ \ B_\gamma \in \mathcal{U}(\mathfrak{g}_{k-1}).$$

Even when $k' > 1$, the following analogous relation is obtained:

$$\pi_\ell\Bigg(\prod_{j\in\left(T_k(e)\setminus T_{k-k'}(e)\right)} P_j^{\alpha_j}\cdot W \\ - \sum_{\gamma=(\gamma_{k-k'+1},\ldots,\gamma_k)\in\mathbb{N}^{k'}} \left(X_k^\dagger\right)^{\gamma_k}\cdots\left(X_{k-k'+2}^\dagger\right)^{\gamma_{k-k'+2}} X_{k-k'+1}^{\gamma_{k-k'+1}} B_\gamma\Bigg) = 0, \tag{12.2.3}$$

where B_γ ($\gamma \in \mathbb{N}^{k'}$) is an element of $\mathcal{U}(\mathfrak{g}_{k-k'})$.

If $k - k' + 1 \in S(e)$, then $X_{k-k'+1} = X_{k-k'+1}^\dagger$ and $T_{k-k'}(e) = T_{k-k'+1}(e)$. Thus, if we put

$$\gamma_k = \beta_k,\ \ldots,\ \gamma_{k-k'+1} = \beta_{k-k'+1},\ B_\gamma = V_{\beta_{k-k'+1}\cdots\beta_k}$$

in relation (12.2.3), formula (12.2.2) is directly obtained.

If $k - k' + 1 \in T(e)$, we put $\alpha = \max\{\gamma_{k-k'+1}; \gamma \in \mathbb{N}^{k'}, B_\gamma \neq 0\}$. Operating $\pi_\ell(P_{k-k'+1})^\alpha$ on both sides of Eq. (12.2.3), it is enough to see that the product $\pi_\ell(P_{k-k'+1})^\alpha X_{k-k'+1}^\beta$ ($\beta \le \alpha$) is written as a linear combination of elements having the form $\left(X_{k-k'+1}^\dagger\right)^a R$ ($a \in \mathbb{N},\ R \in \mathcal{U}(\mathfrak{g}_{k-k'})$). To simplify the notation, we put $P_{k-k'+1} = P,\ X_{k-k'+1} = X,\ Q_{k-k'+1} = Q$ and $X^\dagger = PX + Q$. By $\alpha \ge \beta$,

$$\pi_\ell(P^\alpha X^\beta) = \pi_\ell(P^\beta X^\beta P^{\alpha-\beta}) = \pi_\ell\left((X^\dagger - Q)^\beta P^{\alpha-\beta}\right)$$

and, since $P,\ Q$ belong to $\mathcal{U}(\mathfrak{g}_{k-k'})$, we get the desired result. ■

When we take the product of the family of elements $(a_j)_{j\in T}$, not necessarily commutative, indexed by an ordered set T, $\prod_{j\in T}\uparrow a_j$ (resp. $\prod_{j\in T}\downarrow a_j$) denotes the product where j increases (resp. decreases). For $S = (S_j)_{j\in S_k(e)} \in \mathbb{N}^{\sharp(S_k(e))}$, we put $X^S = \prod_{j\in S_k(e)}\downarrow X_j^{S_j}$. Then, the previous lemma is rewritten in the following form.

Lemma 12.2.3. *Let $W \in \mathcal{U}(\mathfrak{g}_k)$, $1 \le k \le n$. Then, there exist an element $\alpha = (\alpha_j)_{j\in T_k(e)}$ of $\mathbb{N}^{\sharp(T_k(e))}$ and a mapping $S \mapsto a_S = a_S\left((A_j)_{j\in T_k(e)}\right)$, whose values are 0 except at finite points, from $\mathbb{N}^{\sharp(S_k(e))}$ to the subalgebra of $\mathcal{U}(\mathfrak{g}_k)$ generated by $\{A_j\}_{j\in T_k(e)}$ so that*

$$\pi_\ell(P^\alpha W) = \pi_\ell\left(\sum_{S\in\mathbb{N}^{\sharp(S_k(e))}} a_S X^S\right), \quad P^\alpha = \prod_{j\in T_k(e)} P_j^{\alpha_j},$$

holds for $\ell \in \Gamma_\tau$.

In this lemma, if we replace W by an Γ_τ-central element σ of $\mathcal{U}(\mathfrak{g}_k)$:

Proposition 12.2.4. *Let* $1 \le k \le n$ *and* σ *an* Γ_τ*-central element of* $\mathcal{U}(\mathfrak{g}_k)$*. Then, there exist an element* $\alpha = (\alpha_j)_{j\in T_k(e)}$ *of* $\mathbb{N}^{\sharp(T_k(e))}$ *and a polynomial expression* $a = a\left((A_j)_{j\in T_k(e)}\right)$ *so that* $\pi_\ell(P^\alpha\sigma) = \pi_\ell(a)$ *holds for all* $\ell \in \Gamma_\tau$*. Here,* $P^\alpha = \prod_{j\in T_k(e)} P_j^{\alpha_j}$.

Proof. Using the notations of the last lemma, we have

$$\pi_\ell\left(a_0 - P^\alpha\sigma + \sum_{S\in\mathbb{N}^{\sharp(S_k(e))}\setminus\{0\}} a_S X^S\right) = 0.$$

Here $a_0,\ a_S$ are Γ_τ-central elements of $\mathcal{U}(\mathfrak{g}_k)$ and, if we supply the functions $\beta_0(\ell),\ \beta_S(\ell)$ on Γ_τ by

$$\pi_\ell(a_0 - P^\alpha\sigma) = \beta_0(\ell)Id, \quad \pi_\ell(a_S) = \beta_S(\ell)Id,$$

then

$$\sum_{S\in\mathbb{N}^{\sharp(S_k(e))}} \beta_S(\ell)\pi_\ell(X^S) = 0.$$

According to Pedersen [58], the restriction of the representation π_ℓ at a general point ℓ of Γ_τ to the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by X^S's is faithful, namely injective. Since X^S's are mutually linearly independent by the Poincaré–Birkhoff–Witt theorem mentioned in the first chapter, $\beta_S(\ell) = 0$ for arbitrary $S \in \mathbb{N}^{\sharp(S_k(e))}$. In particular, taking $S = 0,\ a = a_0$, we get the proposition. ■

Let us begin a proof of Theorem 12.2.1. We first show assertion (2). So, assume $k \in S(e)$ and take an Γ_τ-central element σ in $\mathcal{U}(\mathfrak{g}_k)$. Then, let us see that there exists an Γ_τ-central element σ_0 in $\mathcal{U}(\mathfrak{g}_{k-1})$ so that $\pi_\ell(\sigma) = \pi_\ell(\sigma_0)$ on Γ_τ. Again by the Poincaré–Birkhoff–Witt theorem, we write $\sigma = \sigma_0 + X_k\sigma_1$ using $\sigma_0 \in \mathcal{U}(\mathfrak{g}_{k-1})$ and $\sigma_1 \in \mathcal{U}(\mathfrak{g}_k)$. If we replace W by σ in Lemma 12.2.3, we have

$$\pi_\ell(P^\alpha\sigma) = \pi_\ell\left(\sum_{S\in\mathbb{N}^{\sharp(S_k(e))}} a_S X^S\right)$$

at a general point ℓ of Γ_τ. On the other hand,

$$\pi_\ell(P^\alpha\sigma) = \pi_\ell(P^\alpha\sigma_0) + \pi_\ell(P^\alpha X_k\sigma_1).$$

Hence, using again the above-mentioned Pedersen's result,

$$\pi_\ell(P^\alpha X_k \sigma_1) = \pi_\ell \left(\sum_{S \in \mathbb{N}^{\sharp(S_k(e))}, S_k > 0} a_S X^S \right) = 0$$

and $\pi_\ell(X_k \sigma_1) = 0$ on Γ_τ. Therefore $\pi_\ell(\sigma) = \pi_\ell(\sigma_0)$.

Let us next show (1) by induction on k. For this aim, an element θ of $Z_k(\mathfrak{g}, \tau)$ being given, we show that θ is written as a fractional expression of $\{\varphi_j\}_{j \in T_k(e)}$. When $k = 0$, since $\mathfrak{g}_0 = \{0\}$, $\mathcal{U}(\mathfrak{g}_0) = Z_0(\mathfrak{g}, \tau) = \mathbb{C}$, θ is a constant and the assertion is clear.

When $k > 0$, we take an Γ_τ-central element $\sigma \in \mathcal{U}(\mathfrak{g}_k)$ so that $\pi_\ell(\sigma) = \theta(\ell) Id$ on Γ_τ. Choosing $a = a\left((A_j)_{j \in T_k(e)}\right)$ and $\alpha \in \mathbb{N}^{\sharp(T_k(e))}$ as in the previous proposition, we have $\pi_\ell(P^\alpha \sigma) = \pi_\ell(a)$. Hence, if we put

$$\Phi_0 = a\left((\varphi_j)_{j \in T_k(e)}\right) \in \mathbb{C}\left[(\varphi_j)_{j \in T_k(e)}\right],$$

then $\pi_\ell(a) = \Phi_0(\ell) Id$ on Γ_τ. Let $\Psi_0 \in Z_{k-1}(\mathfrak{g}, \tau)$ be such that $\pi_\ell(P^\alpha) = \Psi_0(\ell) Id$ on Γ_τ, then $\Psi_0 \theta = \Phi_0$. Since P_j, $j \in T_k(e)$, is a polynomial of A_j, $j \in T_{k-1}(e)$ as we saw at the beginning of Chap. 9, the desired result follows. ■

Transcendental Basis of $Z_k(\mathfrak{g}, \tau)$ Now we pass to $(**)$.

Theorem 12.2.5. *Let* $0 \leq k \leq n$.

(1) *The family* $\{\varphi_j; j \in U_k(e)\}$ *makes a transcendental basis of the algebra* $Z_k(\mathfrak{g}, \tau)$.
(2) *At a general point* ℓ *of* Γ_τ, *the transcendental degree of the algebra* $Z_k(\mathfrak{g}, \tau)$ *is equal to dim* $(\mathfrak{g}_k(\ell)/\mathfrak{h}_k(\ell))$.

We need some preparations to prove this theorem. To simplify the notation, we simply write Γ instead of Γ_τ. Related to the sequence of ideals (9.1.1), let $p_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ and $p'_j : (\mathfrak{g}')^* = \mathfrak{g}_{n-1}^* \to \mathfrak{g}_j^*$ be the restriction maps, and put $\Gamma_j = p_j(\Gamma)$. Let $0 \leq k \leq j \leq n-1$. We write $Z(\mathfrak{g}_k, \Gamma_j)$ to indicate the algebra of the functions θ on Γ_j given by

$$\pi_\ell(\sigma) = \theta(\ell) Id, \ \ell \in \Gamma_j$$

with Γ_j-central element σ of $\mathcal{U}(\mathfrak{g}_k)$.

Assume in general that a nilpotent Lie group $K = \exp \mathfrak{k}$ acts on a finite-dimensional real vector space V. Let $\mathcal{L} = (\mathfrak{k}_j)_{j \in J}$ be a sequence of ideals of $\mathfrak{k}$ beginning with $\mathfrak{k}_0 = \{0\}$, $K_j = \exp(\mathfrak{k}_j)$ and P an affine subspace of V. In this situation,

$$\mathcal{O}(P, V, K, \mathcal{L}) = \left\{ \ell \in P; \dim\left(K_j \cdot \ell\right) = \max_{\lambda \in P} \dim\left(K_j \cdot \lambda\right), \forall j \in J \right\}$$

is a Zariski open set of P. We put

$$S(e(P,V,K,\mathcal{L})) = \{j \in J \setminus \{0\}; \mathfrak{k}_j(\ell) \subset \mathfrak{k}_{j-1},\ \forall\, \ell \in \mathcal{O}(P,V,K,\mathcal{L})\},$$

$$T(e(P,V,K,\mathcal{L})) = \{j \in J \setminus \{0\}; \mathfrak{k}_j = \mathfrak{k}_{j-1} + \mathfrak{k}_j(\ell),\ \forall\, \ell \in \mathcal{O}(P,V,K,\mathcal{L})\}.$$

By $Z(\mathfrak{g}_k, \Gamma_k)^G$ we denote the subring of $Z(\mathfrak{g}_k, \Gamma_k)$ composed of the elements extensible to G-invariant functions on $G{\cdot}\Gamma_k$.

Proposition 12.2.6. *Let $0 \le k \le n, \ell \in G{\cdot}\Gamma$. When we put $\ell_k = \ell|_{\mathfrak{g}_k}$, there exists an injection ι_k from $Z_k(\mathfrak{g}, \tau)$ to $Z(\mathfrak{g}_k, \Gamma_k)^G$, which is characterized by $\iota_k(\theta)(\ell_k) = \theta(\ell)$ for $\theta \in Z_k(\mathfrak{g}, \tau)$.*

Proof. Let us first show the next lemma.

Lemma 12.2.7. *Let $\mathfrak{b}$ be the Vergne polarization at ℓ of $\mathfrak{g}$ constructed from the composition series (9.1.1). Hence, $\mathfrak{b}_k = \mathfrak{b} \cap \mathfrak{g}_k$ is a polarization at ℓ_k of $\mathfrak{g}_k$. Put $B = \exp \mathfrak{b},\ B_k = \exp(\mathfrak{b}_k)$ and $\pi_\ell = \mathrm{ind}_B^G \chi_\ell,\ \pi_{\ell_k} = \mathrm{ind}_{B_k}^{G_k} \chi_{\ell_k}$. Then, the restriction map $\Phi \to \Phi|_{G_k}$ is a surjection from $\mathcal{H}_{\pi_\ell}^\infty$ to $\mathcal{H}_{\pi_{\ell_k}}^\infty$.*

Proof. With respect to the composition series (9.1.1), let $S(\ell)$ and $T(\ell)$ be respectively the set of the jump and the non-jump indices of the orbit $G{\cdot}\ell$. We set $S_k(\ell) = S(\ell) \cap \{1, \ldots, k\},\ T_k(\ell) = T(\ell) \cap \{1, \ldots, k\}$. If $\{X_j\}_{1 \le j \le n}$ is a Malcev basis of $\mathfrak{g}$ relative to the composition series (9.1.1), then $\{X_j\}_{j \in T(\ell)}$ is a Malcev basis of $\mathfrak{g}(\ell)$. Using the notations introduced before, we define by

$$\beta\left(\{s_j\}_{j \in S(\ell)\setminus S_k(\ell)}, g_k, \{t_j\}_{j \in T(\ell)\setminus T_k(\ell)}\right) = \prod_{j \in S(\ell)\setminus S_k(\ell)} \downarrow \exp(s_j X_j){\cdot}g_k{\cdot} \prod_{j \in T(\ell)\setminus T_k(\ell)} \uparrow \exp(t_j X_j)$$

a polynomial map

$$\beta : \mathbb{R}^{\sharp(S(\ell)\setminus S_k(\ell))} \times G_k \times \mathbb{R}^{\sharp(T(\ell)\setminus T_k(\ell))} \to G,$$

which is a bijection. We fix $\varphi \in \mathcal{S}\left(\mathbb{R}^{\sharp(S(\ell)\setminus S_k(\ell))}\right)$ such that $\varphi(0) = 1$. For $\Phi_k \in \mathcal{H}_{\pi_{\ell_k}}^\infty$, if we give the function Φ on G by

$$\Phi\left(\beta\left(\{s_j\}_{j \in S(\ell)\setminus S_k(\ell)}, g_k, \{t_j\}_{j \in T(\ell)\setminus T_k(\ell)}\right)\right) = \varphi\left(\{s_j\}_{j \in (S(\ell)\setminus S_k(\ell))}\right){\cdot}e^{-i \sum_{j \in (T(\ell)\setminus T_k(\ell))} \ell(t_j X_j)}{\cdot}\Phi_k(g_k),$$

then $\Phi \in \mathcal{H}_{\pi_\ell}^\infty$ and $\Phi|_{G_k} = \Phi_k$. ∎

Let $\theta \in Z_k(\mathfrak{g}, \tau)$ and take a Γ-central element σ in $\mathcal{U}(\mathfrak{g}_k)$ such that $\pi_\ell(\sigma) = \theta(\ell)Id$ for $\ell \in G{\cdot}\Gamma$. Then, with the notations of the lemma,

$$\pi_{\ell_k}(\sigma)\left(\Phi|_{G_k}\right)(g_k) = \pi_\ell(\sigma)(\Phi)(g_k) = \theta(\ell)\left(\Phi|_{G_k}\right)(g_k)$$

for arbitrary $\Phi \in \mathcal{H}_{\pi_\ell}, g_k \in G_k$. Namely, $\pi_{\ell_k}(\sigma) = \theta(\ell)Id$ for $\ell \in G{\cdot}\Gamma$. Because the G-invariance of θ in $G{\cdot}\Gamma$ induces the G-invariance of $\iota_k(\theta)$ in $G{\cdot}\Gamma_k$, the proposition holds. ∎

We indicate by $\mathcal{L}$ the composition series (9.1.1) of $\mathfrak{g}$ and, for a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, put $\mathfrak{k}_j = \mathfrak{k} \cap \mathfrak{g}_j$ $(0 \le j \le n)$. We set $\mathcal{I}_K = \{0\} \cup \{1 \le j \le n; \mathfrak{k}_{j-1} \ne \mathfrak{k}_j\}$ and designate by $\mathcal{L}_K$ the composition series $\{\mathfrak{k}_j\}_{j \in \mathcal{I}_K}$ of $\mathfrak{k}$. As before, let $\mathfrak{g}' = \mathfrak{g}_{n-1}$, $G' = \exp(\mathfrak{g}')$, $\mathfrak{k}' = \mathfrak{k}_{n-1}$, $K' = \exp(\mathfrak{k}')$ and $\mathcal{L}' = \mathcal{L}_{G'}$, $\Gamma' = \Gamma_{n-1}$. Moreover, set $p = p_{n-1}$, $\ell' = p(\ell)$, $\mathcal{I}_H = \mathcal{I}$, $\mathcal{I}' = \mathcal{I}_{H'}$.

If we set

$$\mathcal{O} = \mathcal{O}(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K) \cap p^{-1}\left(\mathcal{O}(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'})\right),$$

$\mathcal{O}$ and its projection $\mathcal{O}' = p(\mathcal{O})$ are Zariski open sets respectively in Γ and Γ'. To say more precisely, $\mathcal{O}$ is composed of all $\ell \in \Gamma$ satisfying

$$\dim(\mathfrak{k}_j(\ell)) = \min_{\ell \in \Gamma} \dim(\mathfrak{k}_j(\ell)), \quad \forall\, j \in \mathcal{I}_K,$$

$$\dim(\mathfrak{k}_j(\ell')) = \min_{\ell \in \Gamma} \dim(\mathfrak{k}_j(\ell')) = \min_{\ell' \in \Gamma'} \dim(\mathfrak{k}_j(\ell')), \quad \forall\, j \in \mathcal{I}_{K'}.$$

Taking into account whether the K_j-orbits are saturated or not relative to $\mathfrak{g}'$, we get the following situation.

Proposition 12.2.8. *There occurs the following alternative of* (1) *or* (2).

(1) $T\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'}\right)\right) = T_{n-1}\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right)$.
(2) *There exists an integer* $r_K \in \mathcal{I}_K \cap \{1, \ldots, n-1\} = \mathcal{I}_{K'} \setminus \{0\}$ *so that*

$$T\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}'_{K'}\right)\right) = T_{n-1}\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right) \cup \{r_K\}.$$

In order that eventuality (1) *occurs it is necessary and sufficient that* $\mathfrak{g} = \mathfrak{g}' + \mathfrak{k}^\ell$ *at any* $\ell \in \mathcal{O}$.

In order that eventuality (2) *occurs it is necessary and sufficient that* $\mathfrak{k}^\ell \subset \mathfrak{g}'$ *at any* $\ell \in \mathcal{O}$.

Proof. Let $j \in \mathcal{I}_{K'} \setminus \{0\}$. Taking into account whether the K_j-orbits are saturated or not relative to $\mathfrak{g}'$, we have the alternative of the following situations.

(a) $\mathfrak{k}_j(\ell) = \mathfrak{k}_j(\ell')$ at any $\ell \in \mathcal{O}$, in other words $\mathfrak{g} = \mathfrak{g}' + \mathfrak{k}_j^\ell$. In this case, if $j' \in \mathcal{I}_{K'}$, $j' \le j$, then $\mathfrak{k}_{j'}(\ell) = \mathfrak{k}_{j'}(\ell')$ at any $\ell \in \mathcal{O}$.

(b) $\mathfrak{k}_j(\ell) \subsetneq \mathfrak{k}_j(\ell')$ at any $\ell \in \mathcal{O}$, in other words $\dim(\mathfrak{k}_j(\ell)) = \dim(\mathfrak{k}_j(\ell')) - 1$, or equivalently $\mathfrak{k}_j^{\ell} \subset \mathfrak{g}'$. In this case, if $j' \in \mathcal{I}_{K'}$, $j' \geq j$, then $\dim(\mathfrak{k}_{j'}(\ell)) = \dim(\mathfrak{k}_{j'}(\ell')) - 1$ at any $\ell \in \mathcal{O}$.

Therefore, taking $j = n-1$, we find the following two possibilities.

(a) At any $\ell \in \mathcal{O}$,

$$\mathfrak{k}'(\ell) = \mathfrak{k}_{n-1}(\ell) = \mathfrak{k}_{n-1}(\ell') = \mathfrak{k}'(\ell').$$

Thus, $\mathfrak{k}(\ell) = \mathfrak{k}(\ell')$. In fact, this is clear if $\mathfrak{k} = \mathfrak{k}'$. Otherwise, $\mathfrak{g} = \mathfrak{g}' + \mathfrak{k}$ and

$$\mathfrak{k}(\ell) = \mathfrak{k} \cap \mathfrak{g}^{\ell} = \mathfrak{k} \cap (\mathfrak{g}')^{\ell} \cap \mathfrak{k}^{\ell} = \mathfrak{k}(\ell').$$

Hence necessarily we are in situation (1).

(b) The case where $\dim(\mathfrak{k}'(\ell)) = \dim(\mathfrak{k}'(\ell')) - 1$ at any $\ell \in \mathcal{O}$. In this case, there exists the minimal element $r_K \in \mathcal{I}_{K'} \setminus \{0\}$ such that

$$\dim(\mathfrak{k}_{r_K}(\ell)) = \dim(\mathfrak{k}_{r_k}(\ell')) - 1,\ \forall\, \ell \in \mathcal{O}.$$

Then, if $j < r_K$, $\dim(\mathfrak{k}_j(\ell)) = \dim(\mathfrak{k}_j(\ell'))$ at any $\ell \in \mathcal{O}$ and therefore

$$T_{r_K-1}\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'}\right)\right) = T_{r_K-1}\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right).$$

Similarly, if $j > r_K$, $\dim(\mathfrak{k}_j(\ell)) = \dim(\mathfrak{k}_j(\ell')) - 1$ at any $\ell \in \mathcal{O}$ and therefore

$$\begin{aligned}&T\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'}\right)\right) \setminus T_{r_K}\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'}\right)\right)\\ &\qquad = T\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right) \setminus T_{r_K}\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right).\end{aligned}$$

Besides, at any $\ell \in \mathcal{O}$,

$$\dim(\mathfrak{k}_{r_K-1}(\ell)) = \dim(\mathfrak{k}_{r_K-1}(\ell')) \text{ and } \dim(\mathfrak{k}_{r_K}(\ell)) = \dim(\mathfrak{k}_{r_K}(\ell')) - 1.$$

Hence,

$$\dim(\mathfrak{k}_{r_K}(\ell)) - \dim(\mathfrak{k}_{r_K-1}(\ell)) = \dim(\mathfrak{k}_{r_K}(\ell')) - \dim(\mathfrak{k}_{r_K-1}(\ell')) - 1,$$

while, since the difference of the two terms concerning the dimension in both sides must be equal to either 0 or 1, it is necessarily the case that

$$\dim(\mathfrak{k}_{r_K}(\ell')) = \dim(\mathfrak{k}_{r_K}(\ell')) + 1,\ \dim(\mathfrak{k}_{r_K}(\ell)) = \dim(\mathfrak{k}_{r_K-1}(\ell)).$$

Thus,

$$r_K \in T\left(e\left(\Gamma', (\mathfrak{g}')^*, K', \mathcal{L}_{K'}\right)\right),\ r_K \notin T\left(e\left(\Gamma, \mathfrak{g}^*, K, \mathcal{L}_K\right)\right).$$

Consequently, we are inevitably in situation (2) and the proposition is established. ∎

Following our usage of the notations, we set

$$e_{H'} = e\left(\Gamma', (\mathfrak{g}')^*, H', \mathcal{L}_{H'}\right),\ \ e' = e\left(\Gamma', (\mathfrak{g}')^*, G', \mathcal{L}'\right).$$

Thus,

$$T(e_{H'}) = \{j \in \mathcal{I}'; \mathfrak{h}_j = \mathfrak{h}_{j-1} + \mathfrak{h}_j(\ell')\},$$

$$T(e') = \{1 \leq j \leq n-1; \mathfrak{g}_j = \mathfrak{g}_{j-1} + \mathfrak{g}_j(\ell')\}$$

for general elements ℓ' of Γ'. When we apply the induction on $n \geq 1$, it is often useful to compare $T(e)$ and $T(e')$, likewise $T(e_H)$ and $T(e_{H'})$.

From now on, we put $r = r_H,\ q = r_G$. Begin by rewriting the previous proposition in the cases where $\mathfrak{k} = \mathfrak{h}$ and $\mathfrak{k} = \mathfrak{g}$.

Proposition 12.2.9. *In Γ we put*

$$\mathcal{O} = \mathcal{O}\left(\Gamma, \mathfrak{g}^*, H, \mathcal{L}_H\right) \cap p^{-1}\left(\mathcal{O}\left(\Gamma', (\mathfrak{g}')^*, H', \mathcal{L}_{H'}\right)\right).$$

Then we have the following alternative situations:

(1) *$T(e_{H'}) = T_{n-1}(e_H)$.*
(2) *There exists $r \in \mathcal{I}' \setminus \{0\}$ such that $T(e_{H'}) = T_{n-1}(e_H) \cup \{r\}$.*

In order that eventuality (1) *occurs it is necessary and sufficient that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}^\ell$ at any $\ell \in \mathcal{O}$, or equivalently $\mathfrak{h}(\ell) = \mathfrak{h}(\ell'),\ \ell' = p(\ell)$.*

In order that eventuality (2) *occurs it is necessary and sufficient that $\mathfrak{h}^\ell \subset \mathfrak{g}'$ at any $\ell \in \mathcal{O}$, or equivalently $\mathfrak{h}(\ell) \subsetneq \mathfrak{h}(\ell'),\ \ell' = p(\ell)$. In this case, $dim(\mathfrak{h}(\ell)) = dim(\mathfrak{h}(\ell')) - 1$. Again in this situation, r is the element of $\mathcal{I}'$ uniquely determined by the following condition. At any $\ell \in \mathcal{O}$, if $j < r$, $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}_j^\ell$, or equivalently $\mathfrak{h}_j(\ell) = \mathfrak{h}_j(\ell')$ holds, and if $r \leq j \leq n-1$, $\mathfrak{h}_j^\ell \subset \mathfrak{g}'$, or equivalently*

$$\mathfrak{h}_j(\ell) \subsetneq \mathfrak{h}_j(\ell'),\ dim(\mathfrak{h}_j(\ell)) = dim(\mathfrak{h}_j(\ell')) - 1.$$

Proposition 12.2.10. *In Γ we put*

$$\mathcal{O} = \mathcal{O}\left(\Gamma, \mathfrak{g}^*, G, \mathcal{L}\right) \cap p^{-1}\left(\mathcal{O}\left(\Gamma', (\mathfrak{g}')^*, G', \mathcal{L}'\right)\right).$$

Then we have the following alternative situations:

(1) *$T(e') = T_{n-1}(e)$.*
(2) *There exists $1 \leq q \leq n-1$ such that $T(e') = T_{n-1}(e) \cup \{q\}$.*

In order that eventuality (1) *occurs it is necessary and sufficient that* $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}^{\ell}$ *at any* $\ell \in \mathcal{O}$, *or equivalently* $\mathfrak{g}(\ell) = \mathfrak{g}(\ell')$, $\ell' = p(\ell)$.

In order that eventuality (2) *occurs it is necessary and sufficient that* $\mathfrak{g}^{\ell} \subset \mathfrak{g}'$ *at any* $\ell \in \mathcal{O}$, *or equivalently* $\mathfrak{g}(\ell) \subsetneq \mathfrak{g}(\ell')$, $\ell' = p(\ell)$. *In this case,* $dim(\mathfrak{g}(\ell)) = dim(\mathfrak{g}(\ell')) - 1$. *Again in this situation,* q *is the element of* $\mathcal{I}'$ *uniquely determined by the following condition. At any* $\ell \in \mathcal{O}$, *if* $j < q$, $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}_j^{\ell}$, *or equivalently* $\mathfrak{g}_j(\ell) = \mathfrak{g}_j(\ell')$ *holds, and if* $q \leq j \leq n-1$, $\mathfrak{g}_j^{\ell} \subset \mathfrak{g}'$, *or equivalently*

$$\mathfrak{g}_j(\ell) \subsetneq \mathfrak{g}_j(\ell'),\ dim(\mathfrak{g}_j(\ell)) = dim(\mathfrak{g}_j(\ell')) - 1.$$

As a result of these two propositions:

Proposition 12.2.11. *When* $T(e_{H'}) = T_{n-1}(e_H) \cup \{r\}$, *there exists an integer* q *such that* $T(e') = T(e) \cup \{q\}$, $1 \leq q \leq r$.

Proof. Indeed, since $\mathfrak{g}_r^{\ell} \subset \mathfrak{h}_r^{\ell} \subset \mathfrak{g}'$, we obtain $q \leq r$ from the above result. ■

When $n > 1$, we put

$$U_k(e') = U(e') \cap \{1, \dots, k\} = T_k(e') \backslash T_k(e_{H'})$$

for $0 < k \leq n-1$. Let us compare $U_k(e)$ and $U_k(e')$.

Proposition 12.2.12. *Let* $0 \leq k \leq n-1$. *Only one of the following situations occurs:*

(1) $U_k(e') = U_k(e)$.
(2) $U_k(e') = U_k(e) \cup \{q\}$.
(3) $U_k(e') \cup \{r\} = U_k(e) \cup \{q\}$.

In order that eventuality (1) *occurs it is necessary and sufficient that one of*

$$(1a)\ q \text{ does not exist};\quad (1b)\ k < q;\quad (1c)\ q = r \leq k$$

holds.

In order that eventuality (2) *occurs it is necessary and sufficient that either*

$$(2a)\ q \leq k \text{ and } r \text{ does not exist or } (2b)\ q \leq k < r.$$

And as for (3) *it is necessary and sufficient that* $(3a)$ $q < r \leq k$ *holds.*

Now we prepare a kind of differentiable curve in Γ_k.

Lemma 12.2.13. *Let* V *be a differentiable manifold,* Ω *and* P *submanifolds embedded in* V *and* $\ell \in \Omega \cap P$. *Suppose that there exist an open set* $\mathcal{U}$ *of* V *and an open set* $\mathcal{V}$ *of* P *such that* $\ell \in \mathcal{V} \subset \mathcal{U} \cap P$, *and further a differentiable map* F *from* $\mathcal{U}$ *to a differentiable manifold* W *satisfying the following conditions: putting* $\varphi = F|_{\mathcal{V}}$,

(1) $\varphi^{-1}(\varphi(\ell)) = \Omega \cap \mathcal{V}$;
(2) $F^{-1}(\varphi(\ell)) \supset \Omega \cap \mathcal{U}$;
(3) *the rank of* $(d\varphi)_{\ell'}$ *is constant independent of* $\ell' \in \mathcal{V}$.

Then, $T_\ell(\Omega) \cap T_\ell(P) = T_\ell(\Omega \cap P)$. *In other words, the intersection of* Ω *and* P *is transversal at the point* ℓ.

Proof. No matter what the conditions, it is clear that $T_\ell(\Omega \cap P) \subset T_\ell(\Omega) \cap T_\ell(P)$. In order to show the inverse inclusion, let $Y \in T_\ell(\Omega) \cap T_\ell(P)$. Since $Y \in T_\ell(\Omega)$, $(dF)_\ell(Y) = 0$ by (2). Besides, as $Y \in T_\ell(P)$, $(d\varphi)_\ell(Y)$ has a meaning and $(d\varphi)_\ell(Y) = (dF)_\ell(Y) = 0$. Therefore, $Y \in \ker(d\varphi)_\ell = T_\ell(\Omega \cap P)$ follows from (1) and (3). ■

Returning to the former situation, assume that a nilpotent Lie group $K = \exp \mathfrak{k}$ of dimension d acts linearly on a finite-dimensional real vector space V, and let P be an affine subspace of V. In these circumstances, the following holds.

Theorem 12.2.14 (Transversality of Orbits). *The interior* $\mathcal{O}_T(P, V, K)$ *relative to the Zariski topology of the set of all* $\ell \in P$ *satisfying* $T_\ell(K\cdot\ell) \cap T_\ell(P) = T_\ell((K\cdot\ell) \cap P)$ *is not a empty set.*

Proof. Let us make use of the following lemma.

Lemma 12.2.15 (Cf. [16, pp. 88–93]). *There exist a subset* U *of* V*, a mapping* F_0 *from* U *to a vector space* W *and a family* $\{\mathcal{U}_j, p_j, q_j\}_{1\le j\le p}$ *of a Zariski open set* $\mathcal{U}_j$ *of* V *and polynomial maps* p_j, q_j *on* $\mathcal{U}_j$ *in such a manner that:*

(1) U *is* K*-invariant;*
(2) $U \cap P$ *is a Zariski open set of* P*;*
(3) $F_0(\ell_1) = F_0(\ell_2) \iff \ell_1$ *and* ℓ_2 *belong to the same* K*-orbit;*
(4) $U = \bigcup_{1\le j\le p}(U \cap \mathcal{U}_j)$*;*
(5) q_j *does not vanish on* $\mathcal{U}_j$ *and* $F_0|_{U\cap\mathcal{U}_j} = \frac{p_j}{q_j}$.

We look for appropriate $\mathcal{U}$, F, $\mathcal{V}$ and, putting $\Omega = K\cdot\ell$ for any $\ell \in \mathcal{V}$, let us apply Lemma 12.2.13. In Lemma 12.2.15, one of the Zariski open sets $\mathcal{U}_j$ of V, say $\mathcal{U}_1$, satisfies $P \cap \mathcal{U}_1 \neq \emptyset$. Take $\mathcal{U} = \mathcal{U}_1$ and set $F = \frac{p_1}{q_1}$ on $\mathcal{U}$. In fact, on the non-empty Zariski open set $\mathcal{U} \cap U \cap P$ of P, the restriction $F|_{\mathcal{U}\cap U\cap P}$ is well defined and becomes a rational mapping. Besides, take $\mathcal{V}$ as the Zariski open set of P composed of the points where the differential of this restriction has the maximal rank. Then, if we put $\Omega = K\cdot\ell$ for any $\ell \in \mathcal{V}$, Lemma 12.2.13 holds and $\mathcal{V} \subset \mathcal{O}_T(P, V, K)$. So, the theorem is settled. ■

Let us abuse a little bit the notations used until now. Let $\mathfrak{a} \subset \mathfrak{g}$ and $\mathfrak{p}$ a vector subspace of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{p}$. For $\ell \in \mathfrak{p}^*$, we set $\mathfrak{a}^\ell = \{X \in \mathfrak{g}; \ell([X, \mathfrak{a}]) = \{0\}\}$.

Corollary 12.2.16. *Let* $\Gamma_k = p_k(\Gamma) \subset \mathfrak{g}_k^*$, $0 \le k < n$, ℓ *be an element of the Zakiski open set* $\mathcal{O}_T(\Gamma_k, \mathfrak{g}_k^*, G)$ *of* Γ_k *and finally* $X \in \mathfrak{h}_k^\ell \setminus \mathfrak{g}(\ell)$. *Then there exists a differentiable curve* $t \mapsto g(t)$ *from* $(-\varepsilon, \varepsilon)$ *to* G *satisfying the following conditions:*

(1) $g'(0) \equiv X \ mod \ \mathfrak{g}(\ell)$;
(2) $g(t) \cdot \ell \in \Gamma_k, \ -\varepsilon < t < \varepsilon$.

Proof. Obviously $X \cdot \ell \neq 0$ and $X \cdot \ell|_{\mathfrak{h}_k} = 0$. Since $\ell \in \mathcal{O}_T(\Gamma_k, \mathfrak{g}_k^*, G)$, the last theorem gives

$$X \cdot \ell \in (T_\ell(G \cdot \ell) \cap T_\ell(\Gamma_k)) \setminus \{0\} = T_\ell((G \cdot \ell) \cap \Gamma_k) \setminus \{0\}.$$

Therefore, there exists a differentiable curve $\mu : t \mapsto \mu(t)$ from $(-\varepsilon, \varepsilon)$ to $(G \cdot \ell) \cap \Gamma_k$ so that $\mu(0) = \ell, \ \mu'(0) = X \cdot \ell$. Then, let s be a differentiable section from $G/G(\ell) \simeq G \cdot \ell$ to G. The mapping $t \mapsto (s \circ \mu)(t)$ is provided with the desired properties. ∎

With these preparations we start to prove Theorem 12.2.5. The proof of the theorem will be done for each case listed in the next proposition which is immediately seen.

Proposition 12.2.17. *Among the cases listed below, only one occurs:*

(1) $0 = k = n$.
(2) $0 \leq k < n$:

 (a) $U_k(e) = U_k(e')$;
 (b) $U_k(e) \cup \{q\} = U_k(e')$;
 (c) $U_k(e) \cup \{q\} = U_k(e') \cup \{r\}$.

(3) $0 < k = n$:

 (a) $n \in S(e)$;
 (b) $n \notin \mathcal{I}, \ n \in T(e)$;
 (c) $n \in \mathcal{I}, \ n \in U(e)$;
 (d) $n \in \mathcal{I}, \ n \in T(e_H)$.

Here, the integers $r, \ q$ are respectively introduced by Propositions 12.2.9 and 12.2.10. The different cases of situation (2) were mentioned in Proposition 12.2.11. In the proof of Theorem 12.2.5 which we begin now, the proof for (2) will be done by induction on n. $n \geq 2$ being given, the proof for cases (3)(a), (b) presumes the theorem in situation (2) for the same n. Further, the proof for (3)(c), (d) presumes the theorem for (3)(a), (b). Therefore, n being given, we need to establish the theorem in this order for each case listed up in this proposition.

Proof of Theorem 12.2.5. For the brevity of the notation, we simply write Z_k instead of $Z_k(\mathfrak{g}, \tau)$. Likewise, Z'_k for $Z(\mathfrak{g}_k, \Gamma')$ and Z' for Z'_{n-1}. Further, we write ι for $\iota_{n-1} : Z_{n-1} \to (Z')^G$ in Proposition 12.2.6. Finally, we introduce the injection ι'_k from Z'_k to $(Z^k)^{G'}$ given by the same proposition at the level of G' through $\iota'_k(\theta)(\ell'_k) = \theta(\ell')$ for $\theta \in Z'_k, \ \ell' \in G' \cdot \Gamma', \ell'_k = \ell'|_{\mathfrak{g}_k}$, where Z^k stands for $Z(\mathfrak{g}_k, \Gamma_k)$.

(1) The case $0 = k = n$. We have $\mathcal{U}(\mathfrak{g}_0) = \mathbb{C}$, while, since $U_0(e) = \emptyset$, the transcendental degree of Z_0 and $\sharp(U_0(e))$ are both 0. The assertion holds clearly.

(2) The case $0 \le k < n$. We employ induction on n. In these cases, we make $\theta \in Z_k$ correspond to $\theta' = \iota(\theta) \in Z'_k$. Especially, we make the family $\{\varphi_j\}_{j \in U_k(e)}$ of Corwin–Greenleaf functions in Z_k correspond to the family $\varphi'_j = \iota(\varphi_j)$ of those in Z'_k.

(a) Let $U_k(e) = U_k(e')$. By the induction hypothesis, the functions $\{\varphi'_j\}_{j \in U_k(e')}$ make a transcendental basis of Z'_k and belong to the subalgebra $\iota(Z_k)$ of Z'_k. Hence these make a transcendental basis of $\iota(Z_k)$ too. As Z_k is isomorphic to $\iota(Z_k)$, $\{\varphi_j\}_{j \in U_k(e)}$ is a transcendental basis of Z_k.

(b) Let $U_k(e) \cup \{q\} = U_k(e')$. If we take in Z'_k the Corwin–Greenleaf function φ_q corresponding to the index q, the induction hypothesis assures that

$$\{\varphi'_j\}_{j \in U_k(e)} \cup \{\varphi_q\}$$

makes a transcendental basis of Z'_k. Hence $\{\varphi'_j\}_{j \in U_k(e)}$ are algebraically independent in $\iota(Z_k) \subset Z'_k$ and $\{\varphi_j\}_{j \in U_k(e)}$ are algebraically independent in Z_k. Let us see that those are algebraic generators. Let $\theta \in Z_k$. By the induction hypothesis, θ' satisfies a non-trivial algebraic relation with coefficients in $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}, \varphi_q\right]$. Arranging this relation according to the power of φ_q,

$$\sum_{0 \le \beta \le \alpha} P_\beta\left(\{\varphi_j(\ell)\}_{j \in U_k(e)}, \theta(\ell)\right) \varphi_q(\ell')^\beta = 0$$

at any $\ell \in \Gamma$. Here $P_\beta \in \mathbb{C}\left[\{\varphi_j\}_{j \in U_k(e)}, \theta\right]$ and $\ell' = p(\ell)$. By the assumption, the power of θ in at least one of the P_β, say $P = P_\gamma$, is larger than or equal to 1. Therefore, if we can show that all P_β $(0 \le \beta \le \alpha)$ are 0, we will have in particular $P\left(\{\varphi_j\}_{j \in U_k(e)}, \theta\right) = 0$ and a desired algebraic dependence.

Recall that $\{\varphi_j\}_{j \in U_k(e)}$ depend only on $\ell_k = p_k(\ell)$, More precisely,

$$\varphi_j(\ell) = \iota_k(\varphi_j)(\ell_k),\ \forall\, j \in U_k(e),\ \forall\, \ell \in G \cdot \Gamma,$$

$$\varphi_q(\ell') = \iota'_k(\varphi_q)(\ell_k),\ \forall\, \ell \in G' \cdot \Gamma,\ \ell' = p(\ell).$$

Let us consider the following functions. For $0 \le \beta \le \alpha$, the functions

$$Q_\beta = \iota_k\left(P_\beta\left(\{\varphi_j\}_{j \in U_k(e)}, \theta\right)\right)$$

on $G \cdot \Gamma_k$. These are elements of $(Z^k)^G$. Besides, $\vartheta = \iota'_k(\varphi_q)$ and $R = \sum_{0 \le \beta \le \alpha} Q_\beta \vartheta^\beta$ on $G' \cdot \Gamma_k$ are elements of $(Z^k)^{G'}$. The restrictions of these functions to $\Gamma_k = p_k(\Gamma)$ are polynomial functions.

Using the notations used before in Theorem 12.2.14 and others, we define a non-empty Zariski open set of Γ_k by

$$\mathcal{O} = p_k\left(\mathcal{O}(\Gamma, \mathfrak{g}^*, G, \mathcal{L})\right) \cap p_k'\left(\mathcal{O}(\Gamma', (\mathfrak{g}')^*, G, \mathcal{L})\right)$$
$$\cap\, p_k\left(\mathcal{O}(\gamma, \mathfrak{g}^*, H, \mathcal{L}_H)\right) \cap p_k'\left(\mathcal{O}(\Gamma', (\mathfrak{g}')^*, H, \mathcal{L}_H)\right) \cap \mathcal{O}_T(\Gamma_k, \mathfrak{g}_k^*, G).$$

Let us show that the polynomial

$$x \mapsto \sum_{0\le\beta\le\alpha} Q_\beta(\ell)x^\beta$$

has infinite zero points for $\ell \in \mathcal{O}$. If so, it follows that Q_β and hence P_β is 0. The theorem will be settled in this situation. We make use of the form of the Corwin–Greenleaf function φ_j and the following facts:

(i) Q_β is G-invariant;
(ii) ϑ is G'-invariant;
(iii) R is identically 0 on $G' \cdot \Gamma_k$.

For arbitrary $\ell \in \mathfrak{g}_k^*$, we put $\ell_{q-1} = \ell|_{\mathfrak{g}_{q-1}}$. From the properties of the Corwin–Greenleaf function described in Chap. 9, ϑ is written as

$$\vartheta(\ell) = \tilde{p}_q(\ell_{q-1})\ell(X_q) + \tilde{q}_q(\ell_{q-1}),\ \forall\, \ell \in \Gamma_k,$$

where $X_q \in \mathfrak{g}_q \backslash \mathfrak{g}_{q-1}$ and $\tilde{p}_q$ is an G-invariant polynomial function not identically 0.

Let us utilize Corollary 12.2.16 and the notations employed there. Let $\ell \in \mathcal{O}$. Since $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}_k^\ell$, take an element X_ℓ of $\mathfrak{h}_k^\ell \setminus \mathfrak{g}'$. As $q \le k$, $\mathfrak{g}(\ell) \subset \mathfrak{g}'$ and therefore $X_\ell \notin \mathfrak{g}(\ell)$. Let us consider a differentiable curve $t \mapsto g_\ell(t)$ from $(-\varepsilon, \varepsilon)$ to G such that $g_\ell'(0) \equiv X_\ell$ modulo $\mathfrak{g}'$ and that $g_\ell(t) \cdot \ell \in \Gamma_k$ for all t.

Next, since $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}_{q-1}^\ell$, there exists an element Z_ℓ of $\mathfrak{g}'$ so that $Y_\ell = g_\ell'(0) + Z_\ell \in \mathfrak{g}_{q-1}^\ell \backslash \mathfrak{g}'$. Now if we give a differentiable curve $t \mapsto h_\ell(t)$ from $(-\varepsilon, \varepsilon)$ to G by $h_\ell(t) = \exp(tZ_\ell) \cdot g_\ell(t)$, then $Y_\ell = h_\ell'(0)$. If we put here $\ell_t = h_\ell(t) \cdot \ell$, $\ell_t \in G' \cdot \Gamma_k$ for any t.

Hence $R(\ell_t) = 0$ by (iii). Secondly by (i),

$$\sum_{0\le\beta\le\alpha} Q_\beta(\ell_t)\vartheta(\ell_t)^\beta = \sum_{0\le\beta\le\alpha} Q_\beta(\ell)\vartheta(\ell_t)^\beta = 0.$$

To show $Q_\beta = 0$ from this, if we see that the value at the point 0 of the derivative of the differentiable function $t \mapsto \vartheta(\ell_t)$ is not equal to 0, this function will have infinite values on an open interval $(-\varepsilon, \varepsilon)$ and we should have $Q_\beta = 0$.

Since $Y_\ell \in \mathfrak{g}(\ell_{q-1}) = \mathfrak{g}_{q-1}^\ell$, $Y_\ell \cdot \ell_{q-1} = 0$, while $\ell([X_q, Y_\ell]) \ne 0$. In fact, if $\ell([X_q, Y_\ell]) = 0$, then $\ell([\mathfrak{g}_q, Y_\ell]) = \{0\}$, which contradicts $\mathfrak{g}_q^\ell \subset \mathfrak{g}'$, $Y_\ell \notin \mathfrak{g}'$. Therefore, on a non-empty Zariski open set of Γ_k,

$$\frac{d}{dt}\vartheta(\ell_t)|_{t=0} = \frac{d}{dt}\tilde{p}_q\left(h_\ell(t)\cdot\ell_{q-1}\right)|_{t=0}\ell(X_q)$$
$$+\ \tilde{p}_q(\ell_{q-1})\frac{d}{dt}(h_\ell(t)\cdot\ell)|_{t=0}(X_q) + \frac{d}{dt}\tilde{q}_q(h_\ell(t)\cdot\ell_{q-1})|_{t=0}$$
$$=(d\,\tilde{p}_q)_{\ell_{q-1}}(Y_\ell\cdot\ell_{q-1})\ell(X_q) + \tilde{p}_q(\ell_{q-1})\ell([X_q, Y_\ell]) + (d\tilde{q}_q)_{\ell_{q-1}}(Y_\ell\cdot\ell_{q-1})$$
$$=\tilde{p}_q(\ell_{q-1})\ell([X_q, Y_\ell]) \neq 0.$$

In consequence this case has been settled.

(c) The case where $U_k(e) \cup \{q\} = U_k(e') \cup \{r\},\ q < r$. In this case, $q, r \leq k,\ r \in T_k(e) \cap T_k(e_{H'})$ and $r \notin T_k(e_H)$. If we denote again by φ_q the Corwin–Greenleaf function in Z'_k corresponding to the index q, the treatment of this case is due to the following result.

Lemma 12.2.18. *In these circumstances, φ_q is algebraic over $\mathbb{C}\left[\{\varphi'_j\}_{j\in U_r(e)}\right]$.*

Proof. $\varphi'_r \in Z'_r$ and $U_r(e') = U_{r-1}(e) \cup \{q\}$. By the induction hypothesis,

$$\{\varphi'_j\}_{j\in U_{r-1}(e)} \cup \{\varphi_q\}$$

makes a transcendental basis of Z'_r. Especially, φ'_r is algebraic on

$$\mathbb{C}\left[\{\varphi'_j\}_{j\in U_{r-1}(e)}, \varphi_q\right].$$

Hence, using $Q \in \mathbb{C}\left[\{\varphi'_j\}_{j\in U_{r-1}(e)}, \varphi_q, \varphi'_r\right]$, there exists an algebraic relation $Q = 0$ and the coefficient of the term in Q having the maximal degree for φ'_r is an element of $\mathbb{C}\left[\{\varphi'_j\}_{j\in U_{r-1}(e)}, \varphi_q\right]$ not equal to 0. Arranging Q according to the power of φ_q, two cases are possible:

(1) Among the coefficients of φ_q^β in Q, there is one not equal to 0. Then φ_q becomes algebraic on

$$\mathbb{C}\left[\{\varphi'_j\}_{j\in U_{r-1}(e)}, \varphi'_r\right] = \mathbb{C}\left[\{\varphi'_j\}_{j\in U_r(e)}\right]$$

and the lemma follows.

(2) The coefficients of φ_q^β in Q are all 0. Then φ'_r is algebraic on

$$\mathbb{C}\left[\{\varphi'_j\}_{j\in U_{r-1}(e)}\right].$$

In other words, φ_r is algebraic on $\mathbb{C}\left[\{\varphi_j\}_{j\in U_{r-1}(e)}\right]$.

In order to show the lemma, it is enough to see that this second possibility supplies a contradiction. The relation

$$\sum_{0\le\beta\le\alpha} P_\beta \varphi_r^\beta = 0,\ P_\beta \in \mathbb{C}\left[\{\varphi_j\}_{j\in U_{r-1}(e)}\right]$$

being given, we show $P_\alpha = 0$. Then, we are led to a contradiction.

The function φ_r depends only on ℓ_r and the family of functions $\{\varphi_j\}_{j\in U_{r-1}(e)}$ depend only on ℓ_{r-1}. More precisely, putting $\ell_r = p_r(\ell),\ \ell_{r-1} = p_{r-1}(\ell)$ for arbitrary $\ell \in G\cdot\Gamma$,

$$\varphi_r(\ell) = \iota_r(\varphi_r)(\ell_r),\ \ \varphi_j(\ell) = \iota_{r-1}(\varphi_j)(\ell_{r-1}),\ \forall\, j \in U_{r-1}(e).$$

We take $X_r \in \mathfrak{g}_r\backslash\mathfrak{g}_{r-1}$ in $\mathfrak{h}$ and put $\gamma = f(X_r)$. Let $\kappa : \ell \mapsto \ell|_{\mathfrak{g}_{r-1}}$ be the projection from $\mathfrak{g}_r^*$ to $\mathfrak{g}_{r-1}^*$. Then, we can write

$$\iota_r(\varphi_r)(\ell) = \tilde{p}_r(\kappa(\ell))\ell(X_r) + \tilde{q}_r(\kappa(\ell)),\ \forall\,\ell \in G\cdot\Gamma_r,$$

as a Corwin–Greenleaf function.

From now on, we identify $\kappa^{-1}(\Gamma_{r-1})$ and $\Gamma_{r-1}\times\mathbb{R}$ by the bijection $\ell \mapsto \left(\ell|_{\mathfrak{g}_{r-1}}, \ell(X_r)\right)$. Then, $\Gamma_r \simeq \Gamma_{r-1}\times\{\gamma\}$. The restriction of the mapping κ gives a bijection from Γ_r onto Γ_{r-1} and $\kappa(\ell,\gamma) = \ell$ for $\ell \in \Gamma_{r-1}$. Utilizing the function ϑ given by

$$\vartheta(\ell,x) = \tilde{p}_r(\ell)x + \tilde{q}_r(\ell)$$

on $\kappa^{-1}(\Gamma_{r-1}) = \Gamma_{r-1}\times\mathbb{R}$, $\varphi_r|_{\Gamma_r} = \vartheta|_{\Gamma_{r-1}\times\{\gamma\}}$ and

$$\varphi_r(\ell,\gamma) = -\tilde{p}_r(\ell)x + \vartheta(\ell,x) + \tilde{p}_r(\ell)\gamma$$

for $(\ell,x) \in \Gamma_{r-1}\times\mathbb{R}$. If we substitute this right member into the expression

$$\ell \mapsto \sum_{0\le\beta\le\alpha} P_\beta\left(\{\iota_{r-1}(\varphi_j)(\ell)\}_{j\in U_{r-1}(e)}\right)\varphi_r(\ell,\gamma)^\beta,$$

which is by assumption identically 0 on Γ_{r-1}, we obtain the identity

$$\sum_{0\le\beta\le\alpha} Q_\beta^{(1)}\left(\{\iota_{r-1}(\varphi_j)(\ell)\}_{j\in U_{r-1}(e)}, \tilde{p}_r(\ell), \vartheta(\ell,x)\right)x^\beta = 0$$

for all $\ell \in \Gamma_{r-1},\ x \in \mathbb{R}$. If we put

$$Q_\beta(\ell,x) = Q_\beta^{(1)}\left(\{\iota_{r-1}(\varphi_j)(\ell)\}_{j\in U_{r-1}(e)}, \tilde{p}_r(\ell), \vartheta(\ell,x)\right),$$

$$Q_\alpha(\ell,x) = P_\alpha\left(\{\iota_{r-1}(\varphi_j)(\ell)\}_{j\in U_{r-1}(e)}\right)(-\tilde{p}_r(\ell))^\alpha$$

does not depend on x.

Using the notations given before in Theorem 12.2.14 and others, we define a non-empty Zariski open set of Γ_{r-1} by

$$\mathcal{O} = p_{r-1}\left(\mathcal{O}(\Gamma, \mathfrak{g}^*, G, \mathcal{L})\right) \cap p'_{r-1}\left(\mathcal{O}(\Gamma', (\mathfrak{g}')^*, G, \mathcal{L})\right)$$
$$\cap\, p_{r-1}\left(\mathcal{O}(\Gamma, \mathfrak{g}^*, H, \mathcal{L}_H)\right) \cap p'_{r-1}\left(\mathcal{O}(\Gamma', (\mathfrak{g}')^*, H, \mathcal{L}_H)\right) \cap \mathcal{O}_T(\Gamma_{r-1}, \mathfrak{g}^*_{r-1}, G).$$

Let $\ell \in \mathcal{O}$ and utilize Corollary 12.2.16 and the notations employed there. In our case, since $\mathfrak{g} = \mathfrak{g}' + (\mathfrak{g}_{r-1} \cap \mathfrak{h})^\ell$, we take $X_\ell \in (\mathfrak{g}_{r-1} \cap \mathfrak{h})^\ell$ in the outside of $\mathfrak{g}'$. As $q \leq r-1$, $\mathfrak{g}(\ell) \subset \mathfrak{g}'$ and $X_\ell \notin \mathfrak{g}(\ell)$. Take a differentiable curve $t \mapsto g_\ell(t)$ from $(-\varepsilon, \varepsilon)$ to G such that $g'_\ell(0) \equiv X_\ell$ modulo $\mathfrak{g}(\ell)$ and $\ell_t = g_\ell(t) \cdot \ell \in \Gamma_{r-1}$ for any t. Taking into account that $g_\ell(t)$ keeps $\kappa^{-1}(\Gamma_{r-1})$ stable, put $(\ell_t, x_t) = g_\ell(t) \cdot (\ell, \gamma)$.

Since the functions $\varphi_j,\ j \in U_r(e)$, and $\tilde{p}_r$ are invariant by the action of G, the function $t \mapsto Q_\beta(\ell_t, x_t)$ is constant. Therefore, we know that

$$t \mapsto \sum_{0 \leq \beta \leq \alpha} Q_\beta(\ell, \gamma) x_t^\beta = 0,\ \forall\, t \in (-\varepsilon, \varepsilon).$$

Now, let us see that the mapping $t \mapsto x_t$ admits infinite number of values. If so, it follows that $Q_\alpha(\ell, \gamma) = 0$ for $\ell \in \mathcal{O}$. Since $\tilde{p}_r$ is not identically 0 on Γ_{r-1}, we conclude that $P_\alpha = 0$ and the lemma holds. For this aim, it suffices to show $x'_t(0) \neq 0$. So, let us remark that $\ell([X_r, X_\ell]) \neq 0$. Indeed, if $\ell([X_r, X_\ell]) = 0$, $\ell([\mathfrak{g}_r, X_\ell]) = \{0\}$, which contradicts $\mathfrak{g}_r^\ell \subset \mathfrak{g}'$, $X_\ell \notin \mathfrak{g}'$. Hence $x'_t(0) = (g'_t(0) \cdot \ell)(X_r) = \ell([X_r, X_\ell]) \neq 0$ and we get the lemma. ∎

In order to finish the treatment of this situation in the proof of the theorem, we start from the presumed fact by induction that $\{\varphi_j\}_{j \in U_k(e)\setminus\{r\}} \cup \{\varphi_q\}$ is a transcendental basis of Z'_k. From this:

(1) The transcendental degree of Z'_k is equal to $\sharp(U_k(e))$.
(2) Z'_k is algebraic on $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}, \varphi_q\right]$.

By the last lemma, φ_q is algebraic on $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}\right]$ and $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}, \varphi_q\right]$ is algebraic on $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}\right]$. Hence, Z'_k becomes algebraic on $\mathbb{C}\left[\{\varphi'_j\}_{j \in U_k(e)}\right]$ and $\{\varphi'_j\}_{j \in U_k(e)}$ makes a system of algebraic generators of Z'_k. Since the transcendental degree of Z'_k is equal to $\sharp(U_k(e))$, $\{\varphi'_j\}_{j \in U_k(e)}$ is a transcendental basis of Z'_k contained in $\iota(Z_k)$. In consequence, $\{\varphi_j\}_{j \in U_k(e)}$ becomes a transcendental basis of Z_k and this case is settled.

(3) The case where $0 < k = n$. We use the just-established fact that $\{\varphi_j\}_{j \in U_{n-1}(e)}$ makes a transcendental basis of Z_{n-1}. Besides, we will utilize twice the following lemma.

Lemma 12.2.19. *Let* $0 < k = n$ *and assume* $n \in U(e)$. *Then,* $\{\varphi_j\}_{j \in U(e)}$ *is a system of algebraic generators of* Z.

Proof. $T(e) = T_{n-1}(e) \cup \{n\}$ and $U(e) = U_{n-1}(e) \cup \{n\}$. By Theorem 12.2.1, Z is algebraic on

$$\mathbb{C}\left[\{\varphi_j\}_{j \in T(e)}\right] = \mathbb{C}\left[\{\varphi_j\}_{j \in T_{n-1}(e)}\right][\varphi_n],$$

while, since $\mathbb{C}\left[\{\varphi_j\}_{j \in T_{n-1}(e)}\right]$ is algebraic on $\mathbb{C}\left[\{\varphi_j\}_{j \in U_{n-1}(e)}\right]$, $\mathbb{C}\left[\{\varphi_j\}_{j \in T(e)}\right]$ becomes algebraic on $\mathbb{C}\left[\{\varphi_j\}_{j \in U(e)}\right]$ and the system $\{\varphi_j\}_{j \in U(e)}$ is a system of algebraic generators. ■

(a) The case where $0 < k = n,\ n \in S(e)$. In this case, $U(e) = U_{n-1}(e)$, while, by Theorem 12.2.1, $Z = Z_{n-1}$ and the theorem follows.

(b) The case where $0 < k = n,\ n \notin \mathcal{I},\ n \in T(e)$. By Lemma 12.2.19, the system $\{\varphi_j\}_{j \in U(e)}$ is a system of algebraic generators of Z. It remains to see that this system is algebraically free. So, let us see that the relation of the form

$$\sum_{0 \le \beta \le \alpha} P_\beta \varphi_n^\beta = 0,\ P_\beta \in \mathbb{C}\left[\{\varphi_j\}_{j \in U_{n-1}(e)}\right]$$

means $P_\beta = 0,\ 0 \le \beta \le \alpha$,. Then such a relation reduces to the trivial one and the system $\{\varphi_j\}_{j \in U(e)}$ turns out to be algebraically free in Z. For this aim, it suffices to see that, for a general element ℓ of Γ, the polynomial

$$x \mapsto \sum_{0 \le \beta \le \alpha} P_\beta(\ell) x^\beta$$

has infinite number of zero points.

Let $\ell \in \Gamma,\ X_n \in \mathfrak{g} \setminus \mathfrak{g}'$ and $\ell' = p(\ell)$. We consider the differentiable curve $t \mapsto \ell_t$ from $\mathbb{R}$ to Γ given by $p(\ell_t) = \ell',\ \ell_t(X_n) = t$. Because $P_\beta(\ell_t) = \iota(P_\beta)(p(\ell_t)) = P_\beta(\ell)$ for all β,

$$\sum_{0 \le \beta \le \alpha} P_\beta(\ell) \varphi_n(\ell_t)^\beta = 0.$$

Finally, it suffices to see that, for a general element ℓ of Γ, the mapping $t \mapsto \varphi_n(\ell_t)$ admits an infinite number of values. This is derived from the expression of the Corwin–Greenleaf function φ_n. Indeed,

$$\varphi_n(\ell_t) = \tilde{p}_n(\ell') \cdot t + \tilde{q}_n(\ell')$$

and the function $\ell \mapsto \tilde{p}_n(\ell')$ does not vanish on a Zariski open set of Γ. Thus this situation has been treated.

In order to treat the remaining cases, let us begin with a simple lemma which we utilized several times.

Lemma 12.2.20. *Provided $\mathfrak{h} \neq \mathfrak{g}$, there is a composition series of ideals $\tilde{\mathcal{L}} = \{\tilde{\mathfrak{g}}_j\}_{0 \leq j \leq n}$ in $\mathfrak{g}$ such that $\mathfrak{h} \subset \tilde{\mathfrak{g}}_{n-1}$.*

Proof. Take an ideal $\tilde{\mathfrak{g}}_{n-1}$ of codimension 1 in $\mathfrak{g}$, which contains $\mathfrak{h}$. Then it suffices to take a composition series of the $\mathfrak{g}$-module $\tilde{\mathfrak{g}}_{n-1}$. ∎

Lemma 12.2.21. *The transcendental degree of Z is equal to $\sharp(U(e))$.*

Proof. This result has just been proved in all circumstances where $\mathfrak{h} \subset \mathfrak{g}'$. So, let us assume $n \in \mathcal{I}$ and use the fact that

$$\sharp(U(e)) = \min_{\ell \in \Gamma} \dim(\mathfrak{g}(\ell)) - \min_{\ell \in \Gamma} \dim(\mathfrak{h}(\ell)). \tag{12.2.4}$$

The lemma is evident if $\mathfrak{h} = \mathfrak{g}$. Indeed, in this case, $\Gamma = \{f\}$, $\mathfrak{g}(f) = \mathfrak{h}(f) = \mathfrak{g}$. Hence the transcendental degree of Z and $\sharp(U(e))$ are both 0. Let $\mathfrak{h} \neq \mathfrak{g}$. Because the right member of Eq. (12.2.4) does not depend on the choice of the composition series $\mathcal{L}$, it suffices to replace $\mathcal{L}$ by $\tilde{\mathcal{L}}$ so that $\mathfrak{h} \subset \tilde{\mathfrak{g}}_{n-1}$. ∎

(c) The case where $0 < k = n$, $n \in \mathcal{I}$, $n \in T(e)$ and $n \notin T(e_H)$. In this case, $n \in U(e)$. Taking into account the algebraic dimension, it suffices that $\{\varphi_j\}_{j \in U(e)}$ is a system of algebraic generators of Z. However, this is already shown in Lemma 12.2.19.
(d) The case where $0 < k = n$, $n \in \mathcal{I}$, $n \in T(e)$ and $n \in T(e_H)$. In this case, $n \notin U(e)$. By the same reasoning as in the preceding situation, it suffices that $\{\varphi_j\}_{j \in U(e)}$ is an algebraically free system in Z. Since $U(e) = U_{n-1}(e)$, this system is a transcendental basis of Z_{n-1} and clearly satisfies the desired property.

Theorem 12.2.5 concerning the transcendental basis of Z bas been proved.

∎

Key Lemmas

Now we are ready to explain in detail Lemma 9.2.14 whose concrete examples were computed in Chap. 9.

Lemma 12.2.22. *Assume $\dim \mathfrak{h} \geq 1$ and let $i_s \in T(e_H)$. That is to say, at a general $\ell \in \Gamma_\tau$, $\mathfrak{h}_s = \mathfrak{h}_{s-1} + \mathfrak{h}_s(\ell)$ and there exists k $(1 \leq k \leq t)$ such that $m_k = i_s$. Then, the following assertions hold.*

(i) *There exists a polynomial P satisfying*

$$P\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_k)\right) \equiv 0 \ \ mod\, \mathcal{U}(\mathfrak{g}_{m_k})\mathfrak{a}_s, \tag{12.2.5}$$

where the coefficient of the maximal power of $\diamondsuit(\sigma_k)$ does not belong to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

(ii) *There exists a polynomial Q satisfying*

$$Q\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_k), Y_s\right) \equiv 0 \ \ mod\, \mathcal{U}(\mathfrak{g}_{m_k-1})\mathfrak{a}_{s-1},$$

where the coefficient of the maximal power of Y_s does not belong to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

Proof. As $m_k \in T(e_H)$, $m_k \notin U(e)$. Hence by Theorem 12.2.5, the family $\{\varphi_{\sigma_j};\, j \in U_{m_k-1}(e)\}$ makes a transcendental basis of $Z_{m_k}(\mathfrak{g}, \tau)$. Especially, the element φ_{σ_k} of $Z_{m_k}(\mathfrak{g}, \tau)$ is algebraic on the ring generated by this family and hence naturally on the ring generated by the family $\{\varphi_{\sigma_j};\, j \in T_{m_k-1}(e)\}$. In other words, there exists a polynomial P of k-variables so that

$$P(\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_k}) = \sum_{j=0}^{m} P_j(\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_{k-1}})\varphi_{\sigma_k}^j = 0$$

with $P_m(\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_{k-1}}) \neq 0$. From this,

$$\varpi\left(P\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_k)\right)\right) = \delta(P(\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_k})) = 0$$

and $\varpi\left(P_m\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_{k-1})\right)\right) \neq 0$. From this,

$$P\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_k)\right) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \cap \mathcal{U}(\mathfrak{g}_{m_k}) = \mathcal{U}(\mathfrak{g}_{m_k})\mathfrak{a}_s$$

and $P_m\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_{k-1})\right) \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

(ii) First, remark that $\hat{Y}_s \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \subset \mathcal{U}_C(\mathfrak{g}, \tau)$. We write $\sigma_k = \xi_k Y_s + R_k$, where $\xi_k,\ R_k \in \mathcal{U}(\mathfrak{g}_{m_k-1})$. Thus,

$$\diamondsuit(\sigma_k) = -Y_s\diamondsuit(\xi_k) + \diamondsuit(R_k).$$

In this manner Eq. (12.2.5) is rewritten as

$$P\left(\diamondsuit(\sigma_1), \ldots, \diamondsuit(\sigma_k) + \left(Y_s + \sqrt{-1}f(Y_s)\right)\diamondsuit(\xi_k)\right) \equiv 0 \ \ \mathrm{mod}\, \mathcal{U}(\mathfrak{g}_{m_k-1})\mathfrak{a}_{s-1} \tag{12.2.6}$$

and Y_s disappears. By the way, let us notice that in relation (12.2.6) the elements $\diamondsuit(\sigma_r)$ $(1 \leq r \leq k)$, ξ_k and Y_s belong to $\mathcal{U}_C(\mathfrak{g}, \tau)$. Developing P with respect to Y_s,

$$\varpi\big(P_m(\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1}))(\diamondsuit(\xi_k)Y_s)^m + \sum_{j=0}^{m-1}\tilde{Q}_j(\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_k),\diamondsuit(\xi_k))Y_s^j\big) = 0, \tag{12.2.7}$$

where $\tilde{Q}_j$ are polynomials whose degree relative to $\diamondsuit(\xi_k)$ is less than or equal to m. Now, by Theorem 12.2.1, there exist two polynomials S, T of $k-1$ variables so that

$$\varpi\ (S\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1}))\ \diamondsuit(\xi_k)) = \varpi\ (T\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1})))$$

and the both members of this equation are not 0. We multiply (12.2.7) by $\varpi\ (S\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1}))^m)$ to find that

$$\varpi\left(\big(T\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1}))\big)^m P_m\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_{k-1}))\ Y_s^m + \sum_{j=0}^{m-1} Q_j\ (\diamondsuit(\sigma_1),\ldots,\diamondsuit(\sigma_k))\ Y_s^j\right) = 0$$

holds with some polynomials Q_j and (ii) follows. ■

Lemma 12.2.23. *Suppose that dim* $\mathfrak{h} \geq 1$ *and* $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell)$ *at a general* $\ell \in \Gamma_\tau$. *Then,* $R(Y_d)$ *is algebraic on* $CD_{\tau'}(G/H)$. *In other words, provided* $i_d \in T(e_H)$, *there exists a polynomial* Q *of* Y_d *with coefficients in* $\mathcal{U}_C(\mathfrak{g},\tau')$ *satisfying* $Q(Y_d) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$, *and such that the coefficient of the maximal power of* Y_d *does not belong to* $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.

Proof. Let $e' \in \mathcal{E}$ be the multi-index such that $\Gamma_{\tau'} \cap U_{e'}$ becomes a non-empty Zariski open set of $\Gamma_{\tau'}$. Let us begin by showing $T_{i_d}(e') = T_{i_d}(e)$. Introducing the restriction map $p : \mathfrak{g}^* \to \mathfrak{g}^*_{i_d-1}$, p is a continuous open map with respect to the Zariski topology. Let us remark that $p(\Gamma_\tau) \subset p(\Gamma_{\tau'}) \cap \mathfrak{g}^*_{i_d-1}$ and $p^{-1}\ (p(\Gamma_\tau)) \subset \Gamma_{\tau'}$. Take $1 \leq r \leq i_d$. Since $[\mathfrak{g},\mathfrak{g}_{i_d}] \subset \mathfrak{g}_{i_d-1}$, the restriction of arbitrary $\ell \in \mathfrak{g}^*$ to $[\mathfrak{g},\mathfrak{g}_{i_d}]$ is not changed when we replace ℓ by an arbitrary element of $p^{-1}(p(\ell))$. The same applies for $\mathfrak{g}_r(\ell)$.

Assume $r \in T_{i_d}(e)$ (resp. $r \notin T_{i_d}(e)$). Namely, $\mathfrak{g}_r = \mathfrak{g}_{r-1} + \mathfrak{g}_r(\ell)$ (resp. $\mathfrak{g}_r(\ell) \subset \mathfrak{g}_{r-1}$) on a non-empty Zariski open set $\mathcal{O}$ of Γ_τ. The same relation holds on a non-empty Zariski open set $p^{-1}\ (p(\mathcal{O}))$ of $\Gamma_{\tau'}$. Thus, $r \in T(e')$ (resp. $r \notin T(e')$) and our assertion follows.

In this way, there exists a sub-sequence $(\sigma_r)_{1\leq r\leq k}$ of $\Gamma_{\tau'}$-central elements. For arbitrary $\Gamma_{\tau'}$-central element σ, $\pi_\ell(\sigma)$ is a scalar at all $\ell \in \Gamma_{\tau'}$ hence at all $\ell \in \Gamma_\tau \subset \Gamma_{\tau'}$. If so, $(\sigma_r)_{1\leq r\leq k}$ is a sequence of Γ_τ-central elements too. We take this sequence, choose $s = d$ and can apply Lemma 12.2.22 (ii). The desired result comes from the fact that $\diamondsuit(\sigma_r)$ belongs to $\mathcal{U}_C(\mathfrak{g},\tau) \cap \mathcal{U}_C(\mathfrak{g},\tau')$. ■

Example 12.2.24. We take up once again

$$\mathfrak{g} = \langle X_1, \dots, X_5 \rangle_{\mathbb{R}} : \ [X_5, X_k] = X_{k-1} (2 \le k \le 4),$$

which was treated in Example 9.2.15 of Chap. 9. Let $f = a_3 X_3^* + a_4 X_4^*$ with $a_3, a_4 \in \mathbb{R}$ and $\mathfrak{h} = \mathbb{R}X_3 \oplus \mathbb{R}X_4$. Then, $\mathfrak{h} \in S(f, \mathfrak{g})$. We write our affine space as

$$\Gamma_\tau = \{\ell = f + \ell_1 X_1^* + \ell_2 X_2^* + \ell_5 X_5^*; (\ell_1, \ell_2, \ell_5) \in \mathbb{R}^3\} \cong \mathbb{R}^3.$$

Setting $\mathfrak{g}_j = \langle X_1, \dots, X_j \rangle_{\mathbb{R}}$ as before, we consider a composition series

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \mathfrak{g}_4 \subset \mathfrak{g}_5 = \mathfrak{g}$$

of $\mathfrak{g}$. If we introduce a Zariski open set

$$\mathcal{O} = \{\ell \in \Gamma_\tau; \ell_1 \ne 0, \ \ell_2 \ne 0\}$$

of Γ_τ, at arbitrary $\ell \in \mathcal{O}$, $e = e(\ell) = (0, 1, 1, 1, 2)$ and $T(e) = \{1, 3, 4\}$, $S(e) = \{2, 5\}$, $T(e_H) = \{4\}$, $U(e) = \{1, 3\}$. According to this, we obtained as Corwin–Greenleaf e-central elements

$$A_1 = \sigma_1 = X_1, \ A_3 = \sigma_2 = 2X_1X_3 - X_2^2, \ A_4 = \sigma_3 = X_1^2 X_4 - X_1 X_2 X_3 + \frac{X_2^3}{3}.$$

They belong to the centre of $\mathcal{U}(\mathfrak{g})$. In order to make more precise the corresponding Corwin–Greenleaf functions $\{\varphi_j\}_{j=1,3,4}$ and the operators $\{\pi_\ell(X_j)\}_{1 \le j \le 4}$, let us realize the representation π_ℓ of $\mathcal{U}(\mathfrak{g})$ in $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^*(\mathbb{R})$ by means of the common polarization $\mathfrak{b} = \sum_{j=1}^4 \mathbb{R}X_j$ at $\ell \in \mathcal{O}$ and the coexponential basis X_5 to $\mathfrak{b}$ in $\mathfrak{g}$. By a simple computation, for arbitrary $\psi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
-i\,(\pi_\ell(X_1)\psi)(x) &= \ell_1 \psi(x) \\
-i\,(\pi_\ell(X_2)\psi)(x) &= (\ell_2 - \ell_1 x)\psi(x) \\
-i\,(\pi_\ell(X_3)\psi)(x) &= \left(a_3 - \ell_2 x + \ell_1 \frac{x^2}{2}\right)\psi(x) \\
-i\,(\pi_\ell(X_4)\psi)(x) &= \left(a_4 - a_3 x + \ell_2 \frac{x^2}{2} - \ell_1 \frac{x^3}{3!}\right)\psi(x).
\end{aligned}$$

So, if we set

$$\begin{aligned}
F_1(\ell_1, \ell_2; x) &= \ell_1 \\
F_2(\ell_1, \ell_2; x) &= \ell_2 - \ell_1 x \\
F_3(\ell_1, \ell_2; x) &= a_3 - \ell_2 x + \ell_1 \frac{x^2}{2} \\
F_4(\ell_1, \ell_2; x) &= a_4 - a_3 x + \ell_2 \frac{x^2}{2} - \ell_1 \frac{x^3}{3!},
\end{aligned}$$

then each Z_k, $1 \le k \le 4$, is a subring of the polynomial ring generated by $\{F_j\}_{1\le j\le k}$ and composed of functions independent of x. Especially,

$$-i\varphi_1(\ell_1, \ell_2, \ell_5) = \ell_1$$
$$-\varphi_3(\ell_1, \ell_2, \ell_5) = 2a_3\ell_1 - \ell_2^2$$
$$i\varphi_4(\ell_1, \ell_2, \ell_5) = a_4\ell_1^2 - a_3\ell_1\ell_2 + \frac{\ell_2^3}{3}.$$

As we have seen until now, $Z_1 = Z_2 = \mathbb{C}[\varphi_1]$ and $\{\varphi_1,\ \varphi_3\}$ makes a system of rational generators of Z_3 and at the same time a transcendental basis of Z_4, while $\{\varphi_1,\ \varphi_3,\ \varphi_4\}$ makes a system of rational generators of Z_4 and $Z = Z_5 = Z_4$.

Proposition 12.2.25. *In particular, let $f = 0$, i.e. $a_3 = a_4 = 0$. Then,*

(1) $Z_3 = \mathbb{C}[\ell_1, \ell_2^2]$*;*
(2) $Z = Z_4 = Z_5$ *is isomorphic to the subalgebra of* $\mathbb{C}[\ell_1, \ell_2]$ *spanned linearly by the monomials* $\ell_1^p\ell_2^q$ *verifying* $q \ne 1$.

Proof. (1) The elements of Z_3 are polynomials of $\ell_1,\ \ell_2$ and rational fractional functions of $\ell_1,\ \ell_2^2$. Hence, by the parity of the degree, they are elements of $\mathbb{C}[\ell_1, \ell_2^2]$.

(2) Let Z' denote the subalgebra in question. Since $-i\varphi_1 = \ell_1$, $\varphi_3 = \ell_2^2$, $3i\varphi_4 = \ell_2^3$ in our situation, evidently $Z' \subset Z$. To show the inverse inclusion, we give $\varphi \in Z$ arbitrarily. Then, there exists a polynomial P such that

$$\varphi = P(F_1, F_2, F_3, F_4)$$

is independent of x. When we develop P by means of concrete expressions of F_j, $1 \le j \le 4$, the monomials of F_j, $1 \le j \le 4$, which give monomials of $\ell_1,\ \ell_2, x$ having the forms $\ell_1^p\ell_2$ and $\ell_1^{p+1}x$, are only $F_1^pF_2$. Hence, in the development of φ by $\ell_1,\ \ell_2$, the coefficient of the monomial $\ell_1^p\ell_2$ is 0, as is that of $-\ell_1^{p+1}x$. Thus $\varphi \in Z'$. ∎

Remark 12.2.26. Let $k \in T(e_H)$, then $\{\varphi_j\}_{j\in U_{k-1}(e)} = \{\varphi_j\}_{j\in U_k(e)}$ and $\{\varphi_j\}_{j\in T_{k-1}(e)}$ containing $\{\varphi_j\}_{j\in U_{k-1}(e)}$ is a system of rational generators of Z_{k-1} and also a system of algebraic generators of Z_k. But in general it is not a system of rational generators of Z_k. Further, Z_{k-1} and Z_k admit the same transcendental basis but in general they are not equal. Indeed, let $k = 4$ for instance in the previous proposition. Then, we saw that

$$-i\varphi_1 = \ell_1,\ \varphi_3 = \ell_2^2,\ i\varphi_4 = \frac{\ell_2^3}{3}.$$

Thus $9\varphi_4^2 + \varphi_3^3 = 0$ and φ_4 is algebraically dependent of φ_3. However, if there exist polynomials $S(\varphi_1, \varphi_3),\ T(\varphi_1, \varphi_3)$ satisfying

$$S(\varphi_1, \varphi_3)\varphi_4 = T(\varphi_1, \varphi_3),$$

the parities of the degree of ℓ_2 will be different in the both members of this equation. That is to say, there are no such polynomials $S,\ T$. Besides, since $\varphi_4 \in Z_4 \backslash Z_3$, it is clear that $Z_4 \neq Z_3$.

Remark 12.2.27. In general, Z is not a polynomial ring. In fact, let us consider for example the situation of the last proposition. Let Z^* be the ideal of Z composed of all functions which vanish at $\ell_1 = \ell_2 = 0$. Besides, we denote by I the ideal of Z spanned linearly by the elements of the form $\ell_1^p \ell_2^q$ where $p \geq 2,\ q = 0$ or $p \geq 1,\ q \geq 2$ or $q \geq 4$. Then, since

$$Z^* = \mathbb{C}\ell_1 \oplus \mathbb{C}\ell_2^2 \oplus \mathbb{C}\ell_2^3 \oplus I \text{ and } Z^* \cdot Z^* = I,$$

$\{\ell_1,\ \ell_2^2,\ \ell_2^3\}$ makes a basis of the vector space Z^*/I and $\dim(Z^*/I) = 3$.

Since the transcendental degree of Z is 2, assume that

$$\{\alpha_1 = \alpha_1(\ell_1, \ell_2),\ \alpha_2 = \alpha_2(\ell_1, \ell_2)\}$$

form Z as polynomial ring. Without loss of generality, we may suppose that $\alpha_1,\ \alpha_2 \in Z^*$. Then, all elements of Z^* are written as polynomials of $\alpha_1,\ \alpha_2$ and, since $Z^* \cdot Z^* = I$, it follows that $\{\alpha_1,\ \alpha_2\}$ makes a basis of the vector space Z^*/I. But this contradicts $\dim(Z^*/I) = 3$.

12.3 Proof of the Commutativity Conjecture

We are now ready to establish the commutativity conjecture. Let us begin with a simple lemma which will be repeatedly utilized in the proof of the next theorem.

Lemma 12.3.1. *Let* $\mathfrak{k},\ \mathfrak{k}'$ *be two Lie subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{h} \subset \mathfrak{k}' \subset \mathfrak{k}$. *Let* $\mathfrak{h}''$ *be a Lie subalgebra of* $\mathfrak{h}$. *Then, the following properties are mutually equivalent:*

(i) *For a general* $\ell \in \Gamma_\tau$, $\mathfrak{h}'' \cap \mathfrak{k}^\ell = \mathfrak{h}'' \cap \mathfrak{k}'^\ell$ (*resp.* $dim(\mathfrak{h}'' \cap \mathfrak{k}^\ell) = dim(\mathfrak{h}'' \cap \mathfrak{k}'^\ell) - 1$).
(ii) *At a general* $\ell \in \Gamma_{\mathfrak{k},\mathfrak{h}''} = \{\ell \in \mathfrak{k}^*; \ell|_{\mathfrak{h}''} = f|_{\mathfrak{h}''}\}$, $\mathfrak{h}'' \cap \mathfrak{k}^\ell = \mathfrak{h}'' \cap \mathfrak{k}'^\ell$ (*resp.* $dim(\mathfrak{h}'' \cap \mathfrak{k}^\ell) = dim(\mathfrak{h}'' \cap \mathfrak{k}'^\ell) - 1$).

Proof. Let $\mathfrak{k}''$ be the ideal of $\mathfrak{k}$ generated by $\mathfrak{h}''$. Let $p : \mathfrak{g}^* \to \mathfrak{k}''^*$ and $q : \mathfrak{k}^* \to \mathfrak{k}''^*$ be the restriction maps. Let us make use of the fact that these are both continuous open maps with respect to the Zariski topology. Using the notations similar to $\Gamma_{\mathfrak{k},\mathfrak{h}''}$, $p(\Gamma_\tau) \subset \Gamma_{\mathfrak{k}'',\mathfrak{h}''}$, $q^{-1}(p(\Gamma_\tau)) \subset \Gamma_{\mathfrak{k},\mathfrak{h}''}$ and $p^{-1}\left(q(\Gamma_{\mathfrak{k},\mathfrak{h}''})\right) \subset \Gamma_{\mathfrak{g},\mathfrak{h}''}$.

Let $\ell \in \mathfrak{g}^*$ and $\lambda \in q^{-1}(p(\ell))$. Since $[\mathfrak{k}, \mathfrak{h}''] \subset \mathfrak{k}''$, $\ell|_{[\mathfrak{k},\mathfrak{h}'']} = \lambda|_{[\mathfrak{k},\mathfrak{h}'']}$. From this, $\mathfrak{h}'' \cap \mathfrak{k}^\ell = \mathfrak{h}'' \cap \mathfrak{k}^\lambda$ and $\mathfrak{h}'' \cap \mathfrak{k}'^\ell = \mathfrak{h}'' \cap \mathfrak{k}'^\lambda$. Hence, if one of the two claims of (i) holds on a non-empty Zariski open set $\mathcal{O}$ of Γ_τ, then it holds on the non-empty Zariski open set $q^{-1}(p(\mathcal{O}))$ of $\Gamma_{\mathfrak{k},\mathfrak{h}''}$ too. In this way, (i) means (ii).

Conversely, let $\ell \in \mathfrak{k}^*$ and $\lambda \in p^{-1}(q(\ell)) \subset \mathfrak{g}^*$. By the same arguments as above, if one of the two claims of (ii) holds on a non-empty Zariski open set $\mathcal{O}$ of $\Gamma_{\mathfrak{k},\mathfrak{h}''}$, it holds on the non-empty Zariski open set $p^{-1}(q(\mathcal{O})) \cap \Gamma_\tau$ of Γ_τ. Thus, (ii) means (i). ■

The commutativity conjecture is a by-product of the following theorem.

Theorem 12.3.2. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and assume $\dim \mathfrak{g} \geq 1$. Let H be a closed connected proper subgroup of G with Lie algebra $\mathfrak{h}$, f an element of the linear dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ such that $f([\mathfrak{h},\mathfrak{h}]) = \{0\}$ and $\mathfrak{g}'$ an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{h}$. Then, the following properties are mutually equivalent:*

(i) $\mathcal{U}(\mathfrak{g},\tau) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.
(ii) *At a general $\ell \in \Gamma_\tau$, the H-orbit $H \cdot \ell$ is saturated in the direction of $\mathfrak{g}'^{\perp}$.*

Proof. It is convenient to show the following equivalence:

$$\mathcal{U}(\mathfrak{g},\tau) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \iff \dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1 \text{ for a general } \ell \in \Gamma_\tau.$$

Let us use the inductions on the dimension n of $\mathfrak{g}$ and the dimension d of $\mathfrak{h}$. When $d = 0$, clearly at all $\ell \in \mathfrak{g}^*$, $\mathfrak{h}(\ell) = \mathfrak{h}(\ell') = \{0\}$. Besides, $\mathcal{U}(\mathfrak{g},\tau)$ (resp. $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$) is equal to $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{g}')$). Hence, the theorem is trivial in this case.

Let $n > d \geq 1$. Assume that by the induction hypothesis the theorem is correct for $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}', \tilde{\mathfrak{h}}, \tilde{f})$ with required properties, where $\dim \tilde{\mathfrak{g}} < n$ or $\dim \tilde{\mathfrak{g}} = n$ and $\dim \tilde{\mathfrak{h}} < d$. Choosing $\mathfrak{h}'$ as before, $\mathfrak{h}' \subsetneq \mathfrak{h} \subseteq \mathfrak{g}' \subsetneq \mathfrak{g}$. There are various cases.

At a general $\ell \in \Gamma_\tau$, $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$ or $\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1$.

At a general $\ell \in \Gamma_{\tau'}$, $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ or $\dim(\mathfrak{h}'(\ell)) = \dim(\mathfrak{h}'(\ell')) - 1$.

If we replace in Lemma 12.3.1 $\mathfrak{k}$ by $\mathfrak{g}$, $\mathfrak{k}'$ by $\mathfrak{g}'$ and $\mathfrak{h}''$ by $\mathfrak{h}'$, we know that

$$\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell') \text{ (resp. } \dim(\mathfrak{h}'(\ell)) = \dim(\mathfrak{h}'(\ell')) - 1)$$

at a general $\ell \in \Gamma_{\tau'}$ when and only when the same property holds at a general $\ell \in \Gamma_\tau$. Moreover, if $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$ at a general $\ell \in \Gamma_\tau$, then evidently $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ at a general $\ell \in \Gamma_\tau$.

These remarks lead us to the following three cases.

Case 1. At a general $\ell \in \Gamma_\tau$, $\dim(\mathfrak{h}'(\ell)) = \dim(\mathfrak{h}'(\ell')) - 1$.

As we already remarked, in this case $\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1$ generally. First, by the induction hypothesis applied to $(\mathfrak{g}, \mathfrak{h}')$, we have

$$\mathcal{U}(\mathfrak{g},\tau') \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

Here,

$$\mathcal{U}(\mathfrak{g},\tau) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

In fact, suppose that there exists an element $W \in \mathcal{U}(\mathfrak{g}, \tau)$ such that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, then from the second assertion of Proposition 12.1.4, there exists $W' \in \mathcal{U}(\mathfrak{g}, \tau') \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'})$ satisfying $W' \equiv W$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. This contradicts the induction hypothesis.

Case 2. At a general $\ell \in \Gamma_\tau$, $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$.

In this case, at a general $\ell \in \Gamma_{\tau'}$, $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$. By the induction hypothesis applied to $(\mathfrak{g}, \mathfrak{h}')$, we have

$$\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

We divide this case further into two subcases.

(2a) $\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}') \not\subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}')$.

Taking what we have seen until now into account, we know immediately from Theorem 12.1.9 that

$$\mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

(2b) $\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}') \subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}')$.

Our first objective is to show that $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell)$, i.e. $i_d \in T(e_H)$, at a general $\ell \in \Gamma_\tau$ and to make use of Lemma 12.2.23. First of all, we show by induction that, for $0 \le r \le q-1$, $\mathfrak{h} = \mathfrak{h}' + (\mathfrak{h} \cap \mathfrak{k}_r^\ell)$ at a general $\ell \in \Gamma_\tau$. This is evident for $r = 0$ because of $\mathfrak{h} \cap \mathfrak{k}_0^\ell = \mathfrak{h}$. Let $r > 0$ and assume that the property is true until the step of $r-1$. Recall our assumption $\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{k}_r) \subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{k}_r)$. Then, replacing $\mathfrak{g}$ by $\mathfrak{k}_r$ in assertion (ii) of Proposition 12.1.6, we have

$$\mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{k}_r) \not\subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_\tau,$$

or

$$\mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{k}_r) \subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_\tau \text{ and } \mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{k}_r) \subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_{\tau'}.$$

Hence, by the induction hypothesis on the dimension of G, $\mathfrak{h} \cap \mathfrak{k}_r^\ell = \mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell$ at a general $\ell \in \Gamma_\tau \cap \mathfrak{k}_r^*$, or $\dim\left(\mathfrak{h} \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell\right) - 1$ at a general $\ell \in \Gamma_\tau \cap \mathfrak{k}_r^*$ and $\dim\left(\mathfrak{h}' \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h}' \cap \mathfrak{k}_{r-1}^\ell\right) - 1$ at a general $\ell \in \Gamma_{\tau'} \cap \mathfrak{k}_r^*$. Replacing $\mathfrak{k}$ by $\mathfrak{k}_r$, $\mathfrak{k}'$ by $\mathfrak{k}_{r-1}$ and $\mathfrak{h}''$ by $\mathfrak{h}$ or $\mathfrak{h}'$ in Lemma 12.3.1, this is the same as saying that $\mathfrak{h} \cap \mathfrak{k}_r^\ell = \mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell$ at a general $\ell \in \Gamma_\tau$, or $\dim\left(\mathfrak{h} \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell\right) - 1$ and $\dim\left(\mathfrak{h}' \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h}' \cap \mathfrak{k}_{r-1}^\ell\right) - 1$ at a general $\ell \in \Gamma_\tau$.

In the first case, by induction on $r = \dim(\mathfrak{k}_r/\mathfrak{h})$, we clearly have $\mathfrak{h} = \mathfrak{h}' + (\mathfrak{h} \cap \mathfrak{k}_r^\ell)$ at a general $\ell \in \Gamma_\tau$, while, in the second case, the assumption $\mathfrak{h} \cap \mathfrak{k}_r^\ell \subset \mathfrak{h}'$ leads us to a contradiction. In fact, verifying by induction that $\dim\left(\mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell\right) = \dim\left(\mathfrak{h}' \cap \mathfrak{k}_{r-1}^\ell\right) + 1$, we deduce

$$\dim\left(\mathfrak{h} \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h}' \cap \mathfrak{k}_r^\ell\right) = \dim\left(\mathfrak{h}' \cap \mathfrak{k}_{r-1}^\ell\right) - 1 = \dim\left(\mathfrak{h} \cap \mathfrak{k}_{r-1}^\ell\right) - 2$$

from $\mathfrak{h} \cap \mathfrak{k}_r^\ell = \mathfrak{h}' \cap \mathfrak{k}_r^\ell$. But this is impossible since $\mathfrak{k}_{r-1}$ has the codimension 1 in $\mathfrak{k}_r$.

Thus, in all cases, $\mathfrak{h} = \mathfrak{h}' + (\mathfrak{h} \cap \mathfrak{k}_r^\ell)$ at a general $\ell \in \Gamma_\tau$. Applying this result as $r = q - 1$, $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell')$ at a general element $\ell \in \Gamma_\tau$, we get $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell)$ by $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$.

Now, let us show $\mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. From the first assertion of Proposition 12.1.6, $\mathcal{U}(\mathfrak{g}, \tau') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and there exists

$$W = X_q U + V \in \mathcal{U}(\mathfrak{g}, \tau') \setminus \big(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\big)$$

from the third assertion of Proposition 12.1.8. Here $U \in (\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ and $V \in \mathcal{U}(\mathfrak{g}')$. It remains to see $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Indeed, let us show $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.

First, $Y_d \in \mathcal{U}(\mathfrak{g}, \tau')$ and replacing $\mathfrak{g}$ by $\mathfrak{g}'$ in Proposition 12.1.7, we know that

$$[Y_d, \mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}')] \subset \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\tau'}.$$

Therefore,

$$\begin{aligned}[W, Y_d] = [X_q U + V, Y_d] &\in \big(\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} + \mathcal{U}(\mathfrak{g}')\big) \cap \mathcal{U}(\mathfrak{g}, \tau') \\ &= \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} + (\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}')),\end{aligned}$$

and from this $[[W, Y_d], Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$.

Now, by Lemma 12.2.23, there exist $m > 0$ and $Q_j \in \mathcal{U}_C(\mathfrak{g}, \tau')$, $0 \le j \le m$, such that $Q_m \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ and

$$\sum_{j=0}^{m} Q_j Y_d^j \equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$$

holds. Among these equalities, we choose one where m becomes minimal. The adjoint action of W is written as

$$\left(\sum_{j=1}^{m} j Q_j Y_d^{j-1}\right)[W, Y_d] \equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}.$$

Here, $\left(\sum_{j=1}^{m} j Q_j Y_d^{j-1}\right) \not\equiv 0$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$. This comes from the minimality of m when $m > 1$ and the fact that $Q_1 \not\equiv 0$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ when $m = 1$. Because the ring $\mathcal{U}_C(\mathfrak{g}, \tau')/\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ is an integral domain, we get $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'}$ and finish the proof of this case.

Case 3. At a general $\ell \in \Gamma_\tau$, $\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1$ and $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$.

Let us remark that the centre $\mathfrak{z}$ of $\mathfrak{g}$ is contained in $\mathfrak{g}'$ if $\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1$ at a certain point ℓ. The assumption $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ will be utilized only in situation (3c)(i).

Let us show the inclusion $\mathcal{U}(\mathfrak{g}, \tau) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. For this aim, it is enough by assertion (iii) of Proposition 12.1.8 to show that, if $W = X_q U + V \in \mathcal{U}(\mathfrak{g}, \tau)$ using $U \in \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}')$ and $V \in \mathcal{U}(\mathfrak{g}')$, then inevitably $U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

According to $\mathfrak{z}$ and $\tilde{\mathfrak{z}} = \mathfrak{z} \cap \mathfrak{h} \cap \ker f$, let us divide the case into three subcases.

(3a) $\tilde{\mathfrak{z}} \neq \{0\}$.

This case is easily settled by applying the induction hypothesis to $(\mathfrak{g}/\tilde{\mathfrak{z}}, \mathfrak{h}/\tilde{\mathfrak{z}})$.

(3b) $\tilde{\mathfrak{z}} = \{0\}$ and $\dim \mathfrak{z} \geq 2$.

In this case, $\dim(\mathfrak{h} \cap \mathfrak{z}) = 1$ or $\mathfrak{h} \cap \mathfrak{z} = \{0\}$. Because these two possibilities are treated in the same way, we treat here only the second possibility. So, let us assume $\mathfrak{h} \cap \mathfrak{z} = \{0\}$. Let us recall the elements $\{X_r\}_{1 \leq r \leq q}$ and $\{Y_s\}_{1 \leq s \leq d}$ introduced, relating to the composition series (9.1.1), just before Lemma 9.2.14 in Chap. 9. We suppose that X_1, X_2 belong to $\mathfrak{z}$, and put $\hat{\mathfrak{h}} = \mathfrak{k}_2$. For each $(q-3)$-tuple $J = (j_3, j_4, \ldots, j_{q-1}) \in \mathbb{N}^{q-3}$, set $X^J = X_{q-1}^{j_{q-1}} \cdots X_4^{j_4} X_3^{j_3}$. Let us consider the vector subspace S_1 of $\mathcal{U}(\mathfrak{g}')$ spanned by the family $(X^J \hat{Y}^K)_{J \in \mathbb{N}^{q-3},\ K \in \mathbb{N}^d}$ and the vector subspace S_1^* of $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ spanned by $(X^J \hat{Y}^K)_{J \in \mathbb{N}^{q-3},\ K \in \mathbb{N}^d,\ |K|>0}$. In this proof, we denote by

$$\{X_1^*, X_2^*, Y_1^*, \ldots, Y_d^*, X_3^*, \ldots, X_{q-1}^*, X_q^*\}$$

the basis of $\mathfrak{g}^*$ dual to the basis

$$\{X_1, X_2, Y_1, \ldots, Y_d, X_3, \ldots, X_{q-1}, X_q\}$$

of $\mathfrak{g}$.

Because $\hat{f}([\hat{\mathfrak{h}}, \hat{\mathfrak{h}}]) = \{0\}$ at arbitrary $\hat{f} \in \Gamma_\tau$, we set $\hat{\tau} = \mathrm{ind}_{\hat{H}}^G \chi_{\hat{f}}$ with $\hat{H} = \exp \hat{\mathfrak{h}}$. Then, the family

$$\Big\{ X^J \hat{Y}^K \left(X_1 + \sqrt{-1}\hat{f}(X_1)\right)^j \left(X_2 + \sqrt{-1}\hat{f}(X_2)\right)^k ; \quad J \in \mathbb{N}^{q-3},\ K \in \mathbb{N}^d,\ j, k \in \mathbb{N},\ |K| + j + k > 0 \Big\}$$

makes a basis of $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\hat{\tau}}$. In particular, any element U_* of $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\hat{\tau}}$ is uniquely written as

$$U_* = \sum_{j,k} U_*^{(j,k)} \left(X_1 + \sqrt{-1}\hat{f}(X_1)\right)^j \left(X_2 + \sqrt{-1}\hat{f}(X_2)\right)^k, \tag{12.3.1}$$

where $U_*^{(j,k)} \in S_1$ if $j+k \neq 0$ and $U_*^{(0,0)} \in S_1^*$.

Now, $W \in \mathcal{U}(\mathfrak{g}, \hat{\tau})$ and $U \in \mathcal{U}(\mathfrak{g}, \hat{\tau}) \cap \mathcal{U}(\mathfrak{g}')$. There exists a non-empty Zariski open set $\mathcal{O}_0$ of Γ_τ, whose elements ℓ verify that $\dim(\mathfrak{h}(\ell))$ and $\dim(\mathfrak{h}'(\ell'))$ are minimal. Take $\hat{f}$ in $\mathcal{O}_0$ and replace $\mathfrak{h}$ by $\hat{\mathfrak{h}}$, f by $\hat{f}$ and Γ_τ by $\Gamma_{\hat{\tau}}$. Then, we recognize that the condition of case (3a) is satisfied, and we deduce $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\hat{\tau}}$.

Now fix $\hat{f}$ in $\mathcal{O}_0$ and put $\hat{X}_j = X_j + \sqrt{-1}\hat{f}(X_j)$ $(j = 1, 2)$. Replacing U_* by U in the expression (12.3.1),

$$U = \sum_{j,k} U^{(j,k)} \hat{X}_1^j \hat{X}_2^k, \tag{12.3.2}$$

where $U^{(j,k)} \in S_1$ if $j+k \neq 0$ and $U^{(0,0)} \in S_1^*$.

As is easily seen, there exists a non-empty Zariski open set $\mathcal{O}$ of $\mathbb{R}^2$, whose elements (u, v) satisfy $\hat{f}_{u,v} = \hat{f} + uX_1^* + vX_2^* \in \mathcal{O}_0$. Replacing U_* by U and $\hat{f}$ by $\hat{f}_{u,v}$ in expression (12.3.1), we get

$$U = \sum_{j,k} U_{u,v}^{(j,k)} \left(\hat{X}_1 + \sqrt{-1}u\right)^j \left(\hat{X}_2 + \sqrt{-1}v\right)^k, \tag{12.3.3}$$

where $U_{u,v}^{(j,k)} \in S_1$ if $j+k \neq 0$ and $U_{u,v}^{(0,0)} \in S_1^*$.

From the equation

$$U = \sum_{j,k} U^{(j,k)} \left(\hat{X}_1 + \sqrt{-1}u - \sqrt{-1}u\right)^j \left(\hat{X}_2 + \sqrt{-1}v - \sqrt{-1}v\right)^k$$

and formula (12.3.3),

$$U_{u,v}^{(0,0)} = \sum_{j,k} (-\sqrt{-1}u)^j (-\sqrt{-1}v)^k U^{(j,k)} \in S_1^* \subset \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau, \quad \forall\, (u, v) \in \mathcal{O}.$$

From this relation, for all $(j, k) \in \mathbb{N}^2$ and $J \in \mathbb{N}^{q-3}$, the coefficient of X^J in $U^{(j,k)}$ vanishes and $U^{(j,k)} \in S_1^* \subset \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$. In particular, $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$ and the theorem holds in this case.

Now, if we cut the composition series (9.1.1) at the stage $\mathfrak{g}' = \mathfrak{g}_{n-1}$ which contains $\mathfrak{h}$,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} = \mathfrak{g}'.$$

Let $p_j^* : (\mathfrak{g}')^* \to \mathfrak{g}_j^*$ $(1 \leq j \leq n-1)$ be the restriction map. As before, we define $e_j^*(\ell) = \dim\left(G' \cdot p_j^*(\ell)\right), e^*(\ell) = (e_1^*(\ell), \ldots, e_{n-1}^*(\ell))$ at $\ell \in (\mathfrak{g}')^*$ and set $\mathcal{E}^* = \{e^*(\ell); \ell \in (\mathfrak{g}')^*\}$. Let $e^* \in \mathcal{E}^*$ and agree that $e_0 = 0$. We define the G'-invariant layer $U_{e^*}^* = \{\ell \in (\mathfrak{g}')^*; e^*(\ell) = e^*\}$, the set of the jump indices

$S'(e^*) = \{1 \leq j \leq n-1; e_j^* = e_{j-1}^* + 1\}$ and the set of the non-jump indices $T'(e^*) = \{1 \leq j \leq n-1; e_j^* = e_{j-1}^*\}$. We take the multi-index $e^* \in \mathcal{E}^*$ such that the layer $U_{e^*}^*$ encounters $\Gamma'_\tau = \{\ell' \in (\mathfrak{g}')^*; \ell'|_{\mathfrak{h}} = f|_{\mathfrak{h}}\}$ in a non-empty Zariski open set of Γ'_τ. Finally, we denote by $T'(e_H^*)$ the set of the indices $i_s \in \mathcal{I}$ such that $\mathfrak{h}_s = \mathfrak{h}_{s-1} + \mathfrak{h}(\ell')$ at general elements $\ell' \in \Gamma'_\tau$.

Before we continue the study of case 3, let us notice that from our assumptions $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell')$ and $\mathfrak{h}(\ell) = \mathfrak{h}'(\ell)$ at general elements $\ell \in \Gamma_\tau$. In other words, $i_d \in T'(e_H^*)$ and $i_d \notin T(e_H)$. In fact, from our assumptions

$$\dim(\mathfrak{h}(\ell)) - \dim(\mathfrak{h}'(\ell)) = \dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}'(\ell')) - 1$$

at general elements $\ell \in \Gamma_\tau$, while $\dim(\mathfrak{h}(\ell)) - \dim(\mathfrak{h}'(\ell))$ and $\dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}'(\ell'))$ are either 0 or 1. Hence, $\dim(\mathfrak{h}(\ell)) - \dim(\mathfrak{h}'(\ell)) = 0$ and $\dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}'(\ell')) = 1$ at general elements $\ell \in \Gamma_\tau$. Besides, by Proposition 12.2.9, the difference between $T'(e_H^*)$ and $T_{n-1}(e_H)$ is at most one element. Thus, in the present situation,

$$T'(e_H^*) = T_{n-1}(e_H) \cup \{i_d\}. \tag{12.3.4}$$

(3c) $\dim \mathfrak{z} = 1$ and $\tilde{\mathfrak{z}} = \{0\}$.

Let $\mathfrak{z}'$ be the centre of $\mathfrak{g}'$. Take $Y \in \mathfrak{g}_2 \backslash \mathfrak{g}_1$ and $Z \in \mathfrak{z}\backslash\{0\}$ such that $\mathfrak{z} = \mathfrak{g}_1 = \mathbb{R}Z$ and denote by $\tilde{\mathfrak{g}}$ the centralizer of $\mathfrak{g}_2$ in $\mathfrak{g}$. In what follows we shall examine four cases.

(i) $\mathfrak{g}' = \tilde{\mathfrak{g}}$.

This is equivalent to saying $\mathfrak{g}_2 \subset \mathfrak{z}'$. In this case, $f([X_q, Y]) \neq \{0\}$ and $\mathfrak{h} \subset \tilde{\mathfrak{g}}$. From the assumption, $2 \in T'(e^*)$ and $2 \notin T(e)$. By Proposition 12.2.10, the difference between $T'(e^*)$ and $T_{n-1}(e)$ is at most one element. Hence, in this case,

$$T'(e^*) = T_{n-1}(e) \cup \{2\}. \tag{12.3.5}$$

Our first objective is to show that $R(Y)$ is algebraic on the centre $CD_\tau(G/H)$ of $D_\tau(G/H)$. In other words, let us see that there exists a polynomial P of Y so that

$$P(Y) = \sum_{j=0}^{m} P_j Y^j \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau,$$

where the coefficients P_j belong to $\mathcal{U}_C(\mathfrak{g}, \tau)$ and $P_m \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. This result is easily seen when $2 \in \mathcal{I}$. In fact, in this case there is a real number a such that $Y + aZ \in \mathfrak{h}$. Obviously $Y + aZ + if(Y + aZ) \in \mathfrak{a}_\tau$ and $Z \in \mathcal{U}_C(\mathfrak{g}, \tau)$. This supplies a desired polynomial relation.

Next assume $2 \notin \mathcal{I}$. In this case, $i_d \neq 2$ hence $i_d > 2$. By the facts (12.3.4), (12.3.5), $i_d \in T'(e_H^*) \subset T'(e^*)$ and $i_d \in T(e)\backslash T(e_H) = U(e)$. This is also directly proved as follows. $\dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}'(\ell')) = 1$ holds generally on Γ_τ. From this, $Y(\ell) \in \mathfrak{h}(\ell') \setminus \mathfrak{h}'$, namely there is an $Y(\ell) \in \mathfrak{g}_{i_d}(\ell') \setminus \mathfrak{g}_{i_d-1}$. Then, from $i_d \notin T(e_H)$, $\ell([X_q, Y(\ell)]) \neq 0$. Therefore,

$$\ell\left(\left[X_q, \ell([X_q, Y])Y(\ell) - \ell([X_q, Y(\ell)])Y\right]\right) = 0.$$

We know from this that

$$\ell([X_q, Y])Y(\ell) - \ell([X_q, Y(\ell)])Y \in \mathfrak{g}_{i_d}(\ell) \setminus \mathfrak{g}_{i_d-1}$$

and that $i_d \in T(e)$.

Let $m_k = i_d$. From the preceding results, we get a subsequence $(\sigma_r)_{1 \le r \le k}$ of Corwin–Greenleaf Γ_τ-central elements. As we already saw, the family $(\varpi(\lozenge(\sigma_r)))_{1 \le r \le k}$ generates algebraically the subring $\delta\left(Z_{m_k}(\mathfrak{g}, \tau)\right)$ of $CD_\tau(G/H)$. Then, we know also that σ_r is at the same time a Γ'_τ-central element. Furthermore, Y is a Γ'_τ-central element. In this way, the family $\{Y\} \cup (\sigma_r)_{1 \le r \le k}$ makes a subsequence of Corwin–Greenleaf Γ'_τ-central elements and, with $\tau^* = \mathrm{ind}_H^{G'} \chi_f$, $\varpi(Y) \cup (\varpi(\lozenge(\sigma_r)))_{1 \le r \le k}$ generate algebraically the subring $\delta\left(Z_{m_k}(\mathfrak{g}', \tau^*)\right)$ of $CD_{\tau^*}(G'/H)$.

As $m_k \in T'(e_H^*)$, applying the first assertion of Lemma 12.2.22 to the stage of $\mathfrak{g}'$ $\lozenge(\sigma_k)$ is algebraic on the family $\{Y\} \cup (\lozenge(\sigma_k))_{1 \le r \le k-1}$ modulo $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau$. Consequently, we obtain a polynomial P so that

$$P\left(\lozenge(\sigma_1), \ldots, \lozenge(\sigma_{k-1}), Y, \lozenge(\sigma_k)\right) \equiv 0 \mod \mathcal{U}(\mathfrak{g}_{i_d})\mathfrak{a}_\tau, \tag{12.3.6}$$

where the coefficient of the maximal power of $\lozenge(\sigma_k)$ does not belong to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Equation (12.3.6) is rewritten with certain polynomials

$$P_j = P_j\left(\lozenge(\sigma_1), \ldots, \lozenge(\sigma_k)\right), \; 0 \le j \le m,$$

as

$$\sum_{j=0}^{m} P_j\left(\lozenge(\sigma_1), \ldots, \lozenge(\sigma_k)\right) Y^j \equiv 0 \tag{12.3.7}$$

modulo $\mathcal{U}(\mathfrak{g}_{i_d})\mathfrak{a}_\tau$. These P_j are elements of $\mathcal{U}_C(\mathfrak{g}, \tau)$.

Since $m_k \in T(e) \setminus T(e_H)$, from Theorem 12.2.5 $\lozenge(\sigma_k)$ is algebraically independent of the family $(\lozenge(\sigma_r))_{1 \le r \le k-1}$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. If so, we may assume without loss of generality that, in relation (12.3.7), P_m does not belong to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and $m \ge 1$. Hence (12.3.7) is a non-trivial relation. Henceforth, we choose the polynomial P in such a fashion that the degree $m \ge 1$ in (12.3.7) will be minimal.

As $[W, Y] = ZU = UZ$, applying the adjoint action of W to formula (12.3.7), we obtain

$$\left(\sum_{j=1}^{m} jP_j Y^{j-1}\right) UZ \equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

Then, it turns out that $\left(\sum_{j=1}^{m} jP_j Y^{j-1}\right) \not\equiv 0$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. This follows from the minimality of m if $m > 1$ and, if $m = 1$ from the fact that $P_1 \not\equiv 0$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Besides, clearly $Z \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Since the ring $\mathcal{U}(\mathfrak{g}, \tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ has no non-trivial zero divisor, $U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and we finish the proof of this case.

(ii) At general elements $\ell \in \Gamma_\tau$, $\mathfrak{g}' \neq \tilde{\mathfrak{g}}$, $\mathfrak{h} \subset \tilde{\mathfrak{g}}$ and $\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell)$.

First, we can choose $X_q \in \tilde{\mathfrak{g}}$ and $X \in \mathfrak{g}'$ so that

$$\mathfrak{g}' = (\mathfrak{g}' \cap \tilde{\mathfrak{g}}) \oplus \mathbb{R}X \ \text{ and } \ \tilde{\mathfrak{g}} = (\mathfrak{g}' \cap \tilde{\mathfrak{g}}) \oplus \mathbb{R}X_q.$$

Since $\mathfrak{g}' \cap \tilde{\mathfrak{g}}$ has the codimension 2 in $\mathfrak{g}$, the value of

$$\dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim\left(\mathfrak{h} \cap \mathfrak{g}^\ell\right) = \dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim(\mathfrak{h}(\ell))$$

might be 0, 1 or 2. From the assumptions,

$$\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1 \ \text{ and } \ \dim(\mathfrak{h}(\ell)) = \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right)$$

at general elements $\ell \in \Gamma_\tau$. By the way, let us pay attention also to

$$\begin{aligned} &\dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim(\mathfrak{h}(\ell)) \\ &= \dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim(\mathfrak{h}(\ell')) + \dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}(\ell)) \\ &= \dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) + \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) - \dim(\mathfrak{h}(\ell)). \end{aligned}$$

We know from this that

$$\dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim(\mathfrak{h}(\ell)) = 1.$$

Take $W = X_q U + V$ as before. Let us see that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ means a contradiction. In fact, putting $\tilde{G} = \exp \tilde{\mathfrak{g}}$ and $\tilde{\tau} = \mathrm{ind}_H^{\tilde{G}} \chi_f$, this condition implies the existence of an element $\tilde{W} = X_q \tilde{U} + \tilde{V} \in \mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau})$ with $\tilde{U} \in (\mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau}) \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{g}')\mathfrak{a}_\tau$ and $\tilde{V} \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{g}')$. However, since

$$\dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) - \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) = 1,$$

this contradicts the induction hypothesis concerning $n = \dim G$.

For $b \in \mathbb{N}$, we define the subspace S_b of $\mathcal{U}(\mathfrak{g}')$ by $S_b = \sum_{i=0}^{b-1} X^i \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$ if $b \geq 1$ and by $S_0 = \{0\}$. It is easy to rewrite W as

$$W = \sum_{i=0}^{a} X^i X_q U_i + \sum_{i=0}^{b} X^i V_i = \sum_{i=0}^{a} X^i X_q U_i + X^b V_b + W_b \tag{12.3.8}$$

with some integers a, b. Here $U_i, V_i \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$ and $W_b \in S_b$. Without loss of generality, we may choose W in such a manner that $U_i, V_i \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) \setminus \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})\mathfrak{a}_\tau$ and $b \leq a$. In fact, assume $b > a$. In the first assertion of Proposition 12.1.8, if we replace $\mathfrak{g}$ by $\mathfrak{g}'$, $\mathfrak{g}'$ by $\mathfrak{g}' \cap \tilde{\mathfrak{g}}$ and X_q by X, it results that $V_b \in (\mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau}) \cap \mathcal{U}(\mathfrak{g}'))$. Next, taking $\dim\left(\mathfrak{h} \cap (\mathfrak{g}' \cap \tilde{\mathfrak{g}})^\ell\right) = \dim(\mathfrak{h}(\ell'))$ into account, the induction hypothesis gives an element $XA + B \in \mathcal{U}(\mathfrak{g}', \tau^*)$ using $A \in (\mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau}) \cap \mathcal{U}(\mathfrak{g}')) \setminus \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{g}')\mathfrak{a}_\tau$ and $B \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{g}')$. Then,

$$W' = WA^b - (XA + B)^b V_b = X^a X_q U_a A^b + X^{b'} V_{b'} + W_{b'}$$

is an element of $\mathcal{U}(\mathfrak{g}, \tau) \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau)$ and it turns out that $b' < b$, $V_{b'} \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$ and $W_{b'} \in S_{b'}$. In consequence, by repeating this process we may assume in (12.3.8) that $b \leq a$. Provided $b = a$ (resp. $b < a$), applying again Proposition 12.1.8, we see that $\tilde{W} = X_q U_a + V_a \in \mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau})$ (resp. $\tilde{W} = X_q U_a \in \mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau})$) using $U_a \notin \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})\mathfrak{a}_\tau$. This contradicts the fact that

$$\dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) = \dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - 1$$

and the induction hypothesis.

(iii) At general elements $\ell \in \Gamma_\tau$, $\mathfrak{g}' \neq \tilde{\mathfrak{g}}$, $\mathfrak{h} \subset \tilde{\mathfrak{g}}$ and $\dim(\mathfrak{h}(\ell)) = \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) - 1$.

From the assumption $\mathfrak{h}^\ell \subset \mathfrak{g}'$, $\mathfrak{h}^\ell \subset \tilde{\mathfrak{g}}$ and hence $\mathfrak{h}^\ell \subset \mathfrak{g}' \cap \tilde{\mathfrak{g}}$. By the same reason, this means

$$\dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim(\mathfrak{h}(\ell')) = 1$$

and

$$\dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) = 1.$$

Take $W = X_q U + V$ as before. Let us see that $U \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ gives a contradiction. We know $U \in \mathcal{U}(\mathfrak{g}', \tau^*)$. Using $U_i \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$, we write U in the form $\sum_{i=0}^{m} X^i U_i$. Replacing $\mathfrak{g}$ by $\mathfrak{g}'$ and $\mathfrak{g}'$ by $\mathfrak{g}' \cap \tilde{\mathfrak{g}}$, it results that

$$\dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim(\mathfrak{h}(\ell')) = 1$$

at general elements $\ell \in \Gamma_{\tau^*}$. Taking this into consideration and applying the induction hypothesis concerning $\dim \mathfrak{g}$, we see that

$$\mathcal{U}(\mathfrak{g}', \tau^*) \subset \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) + \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau.$$

Finally, $U_i \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ for $i \neq 0$. Thus, we may assume $U = U_0 \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$. Likewise, we may assume without loss of generality that, using $V_i \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) \setminus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, V is written in the form $V = \sum_{i=0}^m X^i V_i$. Here if $m \geq 1$, then we are led to a contradiction. In fact, under this assumption, the first assertion of Proposition 12.1.8 gives

$$mXV_m + V_{m-1} \in \mathcal{U}(\mathfrak{g}', \tau^*) \setminus \left(\mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) + \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\tau\right),$$

but this is impossible as we saw above.

In consequence, we may choose $W = X_q U_0 + V_0$ using $U_0,\ V_0 \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$ with $U_0 \notin \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})\mathfrak{a}_\tau$. Finally, making use of the fact that

$$\dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right) = 1$$

at general elements $\ell \in \Gamma_\tau$, it results by induction that

$$\mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau}) \subset \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) + \mathcal{U}(\tilde{\mathfrak{g}})\mathfrak{a}_\tau.$$

This is absurd and the theorem has been settled in this case.

(iv) $\mathfrak{h} \not\subset \tilde{\mathfrak{g}}$.

In this case, we set $\tilde{\mathfrak{h}} = \mathfrak{h} \cap \tilde{\mathfrak{g}}$, $\tilde{H} = \exp \tilde{\mathfrak{h}}$ and choose $X \in \mathfrak{h}$ so that $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathbb{R}X$. Obviously, $\mathfrak{g} = \tilde{\mathfrak{g}} + \mathbb{R}X$ and $\mathfrak{h} \subset (\mathbb{R}X)^\ell$ at any $\ell \in \Gamma_\tau$. From this,

$$\dim(\mathfrak{h}(\ell)) = \dim(\mathfrak{h}(\ell')) - 1, \quad \dim(\mathfrak{h}(\ell)) = \dim\left(\mathfrak{h} \cap \tilde{\mathfrak{g}}^\ell\right)$$

at general elements $\ell \in \Gamma_\tau$. If so, by arguments analogous to those developed in case (3)(ii) above it results that

$$\dim\left(\mathfrak{h} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim(\mathfrak{h}(\ell)) = 1$$

at general elements $\ell \in \Gamma_\tau$. $\mathfrak{g}_*$ being any ideal of $\mathfrak{g}$ containing Y, we see that $\mathfrak{h} \cap \mathfrak{g}_*^\ell = \tilde{\mathfrak{h}} \cap \mathfrak{g}_*^\ell$ at general elements $\ell \in \Gamma_\tau$. So,

$$\dim\left(\tilde{\mathfrak{h}} \cap \left(\mathfrak{g}' \cap \tilde{\mathfrak{g}}\right)^\ell\right) - \dim\left(\tilde{\mathfrak{h}} \cap \tilde{\mathfrak{g}}^\ell\right) = 1 \tag{12.3.9}$$

generally in Γ_τ.

Let us see that assuming $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ we are led to a contradiction. Indeed, using $U_i,\ V_i \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$, we can write $W = X_q(\sum_i U_i X^i) + \sum_i V_i X^i$ in this case and

$$W \equiv X_q \sum_i \left(-\sqrt{-1} f(X)\right)^i U_i + \sum_i \left(-\sqrt{-1} f(X)\right)^i V_i \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

Therefore, we may assume without loss of generality that $W = X_q U + V$ with $U,\ V \in \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}})$. Then, $W \in \mathcal{U}(\tilde{\mathfrak{g}}, \tilde{\tau}),\ \tilde{\tau} = \mathrm{ind}_{\tilde{H}}^{\tilde{G}} \chi_f$, and $W \notin \mathcal{U}(\mathfrak{g}' \cap \tilde{\mathfrak{g}}) + \mathcal{U}(\tilde{\mathfrak{g}})\mathfrak{a}_{\tilde{\tau}}$. Replacing $\mathfrak{g}$ by $\tilde{\mathfrak{g}}$, $\mathfrak{g}'$ by $\mathfrak{g}' \cap \tilde{\mathfrak{g}}$ and $\mathfrak{h}$ by $\tilde{\mathfrak{h}}$, and using the induction hypothesis concerning dim $\mathfrak{g}$, we see that this is incompatible with (12.3.9).

Thus the proof of the theorem is complete. ■

Now we arrive at the commutativity conjecture.

Corollary 12.3.3 ([37]). *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h} \in S(f, \mathfrak{g})$. We keep the previous notations. The algebra $D_\tau(G/H)$ is commutative if and only if $\tau = ind_H^G \chi_f$ has finite multiplicities.*

Proof. Corwin and Greenleaf [17] already showed one direction: if τ has finite multiplicities, the algebra $D_\tau(G/H)$ is commutative. So, it remains to show the inverse direction. Assuming that τ has infinite multiplicities, let us show by induction on dim $\mathfrak{g}$ that $D_\tau(G/H)$ is non-commutative. Let us first recall the following [14]: generally on Γ_τ,

$$\begin{aligned} \tau \text{ has finite multiplicities} &\iff \dim(H \cdot \ell) = \frac{1}{2}\dim(G \cdot \ell) \\ &\iff 2(\dim \mathfrak{h} - \dim(\mathfrak{h}(\ell))) = \dim \mathfrak{g} - \dim(\mathfrak{g}(\ell)). \end{aligned}$$

Therefore, it suffices to show that $D_\tau(G/H)$ is non-commutative if

$$2(\dim \mathfrak{h} - \dim(\mathfrak{h}(\ell))) < \dim \mathfrak{g} - \dim(\mathfrak{g}(\ell))$$

at general elements $\ell \in \Gamma_\tau$. In this situation, clearly $\mathfrak{h} \neq \mathfrak{g}$. Let $\mathfrak{g}'$ be an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{h}$.

If $D_{\tau*}(G'/H) \subset D_\tau(G/H)$ is already non-commutative, there is nothing to do. Hence, suppose that $D_{\tau*}(G'/H)$ is commutative. This is the same as saying that $2(\dim \mathfrak{h} - \dim(\mathfrak{h}(\ell'))) = \dim(\mathfrak{g}') - \dim(\mathfrak{g}'(\ell'))$ at general elements $\ell \in \Gamma_\tau$. Hence, it results that

$$2(\dim(\mathfrak{h}(\ell')) - \dim(\mathfrak{h}(\ell))) < 1 + \dim(\mathfrak{g}'(\ell')) - \dim(\mathfrak{g}(\ell)) \leq 2$$

and hence $\mathfrak{h}(\ell') = \mathfrak{h}(\ell)$ at general elements $\ell \in \Gamma_\tau$. If so, by Theorem 12.3.2, there exists an element $W \in \mathcal{U}(\mathfrak{g}, \tau)$ such that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. In consequence, by Theorem 12.1.1, there exists an element $T \in \mathcal{U}(\mathfrak{g}', \tau^*)$ so that $[W, T] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Thus we can conclude that the algebra $D_\tau(G/H)$ is non-commutative. ■

Example 12.3.4. Let $\mathfrak{g}$ be the nilpotent Lie algebra of dimension 7 with a basis $\{X_i; 1 \le i \le 7\}$ which satisfies the following nonzero commutation relations:

$$[X_6, X_2] = X_1,\ [X_6, X_4] = X_2,\ [X_6, X_5] = X_4,\ [X_6, X_7] = X_3,$$
$$[X_4, X_5] = X_3,\ [X_5, X_3] = X_1,\ [X_4, X_7] = -X_1.$$

The centre of $\mathfrak{g}$ is clearly $\mathfrak{z} = \mathbb{R}X_1$. We choose the composition series of ideals (9.1.1) as follows:

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_6 \subset \mathfrak{g}_7 = \mathfrak{g},$$

where

$$\mathfrak{g}_j = \langle X_1, \dots, X_j \rangle_{\mathbb{R}} = \sum_{k=1}^{j} \mathbb{R}X_k \ (1 \le j \le 7).$$

We consider in $\mathfrak{g}^*$ the basis $\{X_j^*; 1 \le j \le 7\}$ dual to the basis $\{X_j; 1 \le j \le 7\}$ of $\mathfrak{g}$ and put $f = \lambda X_4^*$. Taking $\mathfrak{h} = \mathbb{R}X_4$,

$$\Gamma_\tau = \left\{ \sum_{j=1}^{7} \xi_j X_j^*; \xi_4 = \lambda \right\}.$$

Let us describe general H-orbits in Γ_τ, namely H-orbits of maximal dimension. They are contained in the Zariski open set

$$\mathcal{O} = \left\{ \sum_{j=1}^{7} \xi_j X_j^* \in \Gamma_\tau; \xi_3 \neq 0 \right\}.$$

By a simple direct computation, if

$$\ell = \sum_{j=1}^{7} \xi_j X_j^* \in \Gamma_\tau,$$

$$\mathrm{Ad}^*\,(\exp(-tX_4))\,(\ell) = \sum_{j=1}^{7} \xi_j(t) X_j^* \ (\forall\, t \in \mathbb{R}),$$

where

$$\xi_1(t) = \xi_1,\ \xi_2(t) = \xi_2,\ \xi_3(t) = \xi_3,\ \xi_4(t) = \lambda,$$

$$\xi_5(t) = \xi_5 + t\xi_3,\ \xi_6(t) = \xi_6 - t\xi_2,\ \xi_7(t) = \xi_7 - t\xi_1. \tag{12.3.10}$$

The index sets $\mathcal{I},\ \mathcal{J}$ defined just after Theorem 9.2.1 become respectively $\mathcal{I} = \{4\}$ and $\mathcal{J} = \{1, 2, 3, 5, 6, 7\}$. Besides, the sequence of Lie subalgebras (12.2.3) becomes

$$\mathfrak{k}_0 = \mathfrak{h} = \mathbb{R}X_4,\ \ \mathfrak{k}_1 = \mathbb{R}X_1 \oplus \mathbb{R}X_4,$$
$$\mathfrak{k}_2 = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_4,\ \ \mathfrak{k}_j = \mathfrak{g}_{j+1}\ (3 \le j \le 6).$$

Setting $\tau_j = \mathrm{ind}_H^{K_j} \chi_f,\ K_j = \exp(\mathfrak{k}_j)$, we consider the associated sequence of subalgebras:

$$D_{\tau_1}(K_1/H) \subseteq D_{\tau_2}(K_2/H) \subseteq \cdots \subseteq D_{\tau_5}(K_5/H) \subseteq D_{\tau_6}(K_6/H) = D_\tau(G/H). \tag{12.3.11}$$

Theorem 12.3.2 tells us which of these inclusions are proper and which are equalities.

Using the computations (12.3.10) of H-orbits and putting $\ell_j = \ell|_{\mathfrak{k}_j}$ $(1 \le j \le 6)$ for any $\ell \in \mathfrak{g}^*$,

$$\dim(H \cdot \ell_j) = 0,\ \ \forall\, \ell \in \mathcal{O},\ \ 0 \le j \le 3,$$

$$\dim(H \cdot \ell_j) = 1,\ \ \forall\, \ell \in \mathcal{O},\ \ 4 \le j \le 6. \tag{12.3.12}$$

According to the sequence $\mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_6$, the jump of the dimensions of general H-orbits in Γ_τ happens at only one place, the passage from $\mathfrak{k}_3$ to $\mathfrak{k}_4$. Hence, by Theorem 12.3.2, $D_{\tau_j}(K_j/H)$ is properly contained in $D_{\tau_{j+1}}(K_{j+1}/H)$ except $j = 3$ and the equality $D_{\tau_3}(K_3/H) = D_{\tau_4}(K_4/H)$ holds when $j = 3$. Thus, if we describe (12.3.11) in detail,

$$\begin{aligned} D_{\tau_1}(K_1/H) &\subsetneq D_{\tau_2}(K_2/H) \subsetneq D_{\tau_3}(K_3/H) \\ &= D_{\tau_4}(K_4/H) \subsetneq D_{\tau_5}(K_5/H) \subsetneq D_\tau(G/H). \end{aligned} \tag{12.3.13}$$

In other words, except $j = 3$ there exist nonzero elements of $D_{\tau_{j+1}}(K_{j+1}/H)$ not belonging to $D_{\tau_j}(K_j/H)$. To confirm it, let us construct concretely these new comers. In the computations (12.3.10), if we put $u = \xi_5 + t\xi_3$, we can parameterize general H-orbits in Γ_τ as follows: provided $\ell = \sum_{j=1}^{7} \xi_j X_j^* \in \mathcal{O}$,

$$\mathrm{Ad}^*\,(\exp(-tX_4))\,(\ell) = \sum_{j=1}^{7} r_j(u) X_j^*\ (\forall\, t \in \mathbb{R}),$$

where

$$r_1(u) = \xi_1,\ \ r_2(u) = \xi_2,\ \ r_3(u) = \xi_3,\ \ r_4(u) = \lambda,$$
$$r_5(u) = u,\ \ r_6(u) = \frac{\xi_6\xi_3 + \xi_2\xi_5 - u\xi_2}{\xi_3},$$
$$r_7(u) = \frac{\xi_7\xi_3 + \xi_1\xi_5 - u\xi_1}{\xi_3}.$$

This gives H-invariant polynomials

$$\xi_1,\ \xi_2,\ \xi_3,\ \xi_6\xi_3 + \xi_2\xi_5,\ \xi_7\xi_3 + \xi_1\xi_5$$

on $\mathfrak{g}^*$. Applying to these polynomials the symmetrization map $S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, we procure elements

$$X_6X_3 + X_2X_5,\ X_1,\ X_2,\ X_3,\ X_7X_3 + X_1X_5$$

of $\mathcal{U}(\mathfrak{g}, \tau)$. Precisely, we get X_1 at the passage from $\mathbb{C} = \mathcal{U}(\mathfrak{k}_0, \chi_f)$ to $\mathcal{U}(\mathfrak{k}_1, \tau_1)$, X_2 from $\mathcal{U}(\mathfrak{k}_1, \tau_1)$ to $\mathcal{U}(\mathfrak{k}_2, \tau_2)$, X_3 from $\mathcal{U}(\mathfrak{k}_2, \tau_2)$ to $\mathcal{U}(\mathfrak{k}_3, \tau_3)$, $X_6X_3 + X_2X_5$ from $\mathcal{U}(\mathfrak{k}_4, \tau_4)$ to $\mathcal{U}(\mathfrak{k}_5, \tau_5)$ and $X_7X_3 + X_1X_5$ from $\mathcal{U}(\mathfrak{k}_5, \tau_5)$ to $\mathcal{U}(\mathfrak{g}, \tau)$.

Let us confirm that no newcomer appears at the passage from $D_{\tau_3}(K_3/H)$ to $D_{\tau_4}(K_4/H)$. Assume that there exists $A \in \mathcal{U}(\mathfrak{k}_4, \tau_4)$ not belonging to $\mathcal{U}(\mathfrak{k}_3) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. From assertion (iii) of Proposition 12.1.8, we may assume that $A = X_5U + V$ with $U,\ V \in \mathcal{U}(\mathfrak{k}_3)$. Since $\mathfrak{k}_3$ is commutative, $[A, X_4] = -X_3U$. By $X_3 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, it results that $U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and $V \in \mathcal{U}(\mathfrak{k}_3, \tau_3)$. Thus, $A \in \mathcal{U}(\mathfrak{k}_3, \tau_3) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ and hence $D_{\tau_3}(K_3/H) = D_{\tau_4}(K_4/H)$.

Now let us turn toward the question of the commutativity of $D_\tau(G/H)$. By a direct calculation, general G-orbits in Γ_τ are of dimension 6 and general H-orbits are of dimension 1 by (12.3.12). Hence $\tau = \mathrm{ind}_H^G \chi_f$ has infinite multiplicities. Therefore, the algebra $D_\tau(G/H)$ should be non-commutative. Indeed, the groups $K_j = \exp(\mathfrak{k}_j)$ are commutative for $1 \le j \le 3$. Hence, the representation $\tau_j = \mathrm{ind}_H^{K_j} \chi_{f_j}$ has finite multiplicities and the algebra $D_{\tau_j}(K_j/H)$ is commutative.

Let $j = 5$ and consider a general element $\ell \in \Gamma_\tau$. The orbit $K_5{\cdot}\ell_5$ is of dimension 4 and the orbit $H \cdot \ell_5$ is of dimension 1 by (12.3.12). Therefore, $\tau_5 = \mathrm{ind}_H^{K_5} \chi_{f_5}$ has infinite multiplicities. At the same time, X_2 and $X_6X_3 + X_2X_5$ are two elements of $\mathcal{U}(\mathfrak{k}_5, \tau_5)$ and, since

$$[X_2, X_6X_3 + X_2X_5] = -X_1X_3,\ \ X_1X_3 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau,$$

$$[X_2, X_6X_3 + X_2X_5] \not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

In this way, we confirm that $D_{\tau_5}(K_5/H)$ is non-commutative.

Chapter 13
Commutativity Conjecture: Restriction Case

13.1 Kernel of Representation

As Frobenius reciprocity suggests, there is a kind of duality between the induction and the restriction of representations. Under this guiding principle, let us formulate and prove for the restriction of representations the counterpart of the commutativity conjecture. These results were obtained by [6]. As usual, let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $K = \exp \mathfrak{k}$ an analytic subgroup of G with Lie algebra $\mathfrak{k}$. For $\pi \in \hat{G}$, $\Omega(\pi) = \Omega_G(\pi)$ denotes the corresponding coadjoint orbit of G. When $\pi \in \hat{G}$ is given, we examine the restriction $\pi|_K$ of π to K. First of all, its canonical irreducible decomposition is described by Theorem 8.2.4 in Chap. 8. We denote by

$$\ker\pi = \{X \in \mathcal{U}(\mathfrak{g}); \pi(X) = 0\}$$

the primitive ideal of $\mathcal{U}(\mathfrak{g})$ associated with π. Besides, for two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{U}(\mathfrak{g})$, we denote the centralizer of $\mathcal{M}$ modulo $\mathcal{N}$ by

$$\mathfrak{c}(\mathcal{M}, \mathcal{N}) = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathcal{M}] \subset \mathcal{N}\}.$$

Let us set $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \mathfrak{c}(\mathfrak{k}, \ker\pi)$. Clearly, $\ker\pi$ is a two-sided ideal of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. When we define the algebra $D_\pi(G)^K$ as the image of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ by the homomorphism π,

$$D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}/\ker\pi = (\mathcal{U}(\mathfrak{g})/\ker\pi)^K,$$

where the last algebra is the algebra composed of all K-invariant elements of $\mathcal{U}(\mathfrak{g})/\ker\pi$. The algebra $D_\pi(G)^K$ keeps $\mathcal{H}_\pi^\infty$ stable and is regarded as the algebra composed of all differential operators on it which are commutative with the action of K. We would like to understand the structure of the algebra $D_\pi(G)^K$ in connection with the irreducible decomposition of $\pi|_K$. To say more, when the

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representation $\pi|_K$ is irreducible, $D_\pi(G)^K$ is the trivial algebra composed of scalar operators. Exactly as in the case of monomial representations, there occurs the following alternative: in Theorem 8.2.4 of Chap. 8, the multiplicities $n_\pi(\sigma)$ are either uniformly bounded almost everywhere with respect to ν or $n_\pi(\sigma) = \infty$ almost everywhere with respect to ν. According to these two eventualities, we say that $\pi|_K$ has finite or infinite multiplicities.

Let us take up once again the composition series of ideals (9.1.1):

$$\mathcal{S} \,:\, \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_k) = k\ (0 \le k \le n), \tag{9.1.1}$$

and recall the notations introduced at the beginning of Chap. 9 :$X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1},\ p_j : \mathfrak{g}^* \to \mathfrak{g}_j^*,\ \mathcal{E},\ e \in \mathcal{E},\ U_e,\ S(e),\ T(e)$ etc. If we set

$$\mathfrak{g}_S^* = \sum_{j \in S(e)} \mathbb{R}X_j^* \text{ and } \mathfrak{g}_T^* = \sum_{j \in T(e)} \mathbb{R}X_j^*,$$

then $\mathfrak{g}^* = \mathfrak{g}_S^* \oplus \mathfrak{g}_T^*$. In $\mathcal{E}$ there exists a total ordering $\mathcal{E} = \left\{e^{(1)} > \cdots > e^{(k)}\right\}$ such that $U_{e^{(k)}}$ and $\cup_{j \le i} U_{e^{(j)}}$ for any i become Zariski open sets of $\mathfrak{g}^*$. Hence, each layer U_e is an algebraic set, namely a difference of two Zariski open sets of $\mathfrak{g}^*$.

We mention here some results of Pedersen [58]. Let U_e be an arbitrary layer and $S(e) = \{j_1 < \cdots < j_m\}$, where m is the dimension of G-orbits in U_e. Applying at the same time Theorem 6.1.15 of Chap. 6, there exist functions $R_j^e : U_e \times \mathbb{R}^m \to \mathbb{R},\ 1 \le j \le n$:

- when we fix $\ell \in U_e$, $x = (x_1, \ldots, x_m) \mapsto R_j^e(\ell, x)$ is a polynomial map of $x \in \mathbb{R}^m$ and their coefficients are G-invariant functions on U_e;
- $R_j^e(\ell, x) = x_k$ for $j = j_k \in S(e)$, $\ell \in U_e$;
- when $j_k \le j \le j_{k+1}$, $R_j^e(\ell, x)$ depends only on $x_1, \ldots, x_k$;
- the coadjoint orbit $G \cdot \ell$ is given by

$$G \cdot \ell = \left\{ \sum_{j=1}^n R_j^e(\ell, x) X_j^*; x \in \mathbb{R}^m \right\}$$

  for any $\ell \in U_e$.

We replace the variables x_k by $-iX_{j_k}$ in $R_j^e(\ell, x)$, form

$$R_j^e(\ell, -iX_{j_1}, \ldots, -iX_{j_m})$$

in the symmetric algebra $S(\mathfrak{g})$ and write its image in $\mathcal{U}(\mathfrak{g})$ by the symmetrization map as $r_j^e(\ell)$. In particular, $r_{j_k}^e(\ell) = -iX_{j_k}$. Let V_e be the subspace of $S(\mathfrak{g})$ spanned by the elements of the form

$$X_{j_1}^{k_1} \cdots X_{j_m}^{k_m},\ k_1, \ldots, k_m \in \mathbb{N},$$

F_e the image of V_e in $\mathcal{U}(\mathfrak{g})$ by the symmetrization map and E_e the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by the elements of the form

$$X_{j_1}^{k_1}\cdots X_{j_m}^{k_m},\ k_1,\ldots,k_m \in \mathbb{N}.$$

When $e = \emptyset$, put $V_e = F_e = E_e = \mathbb{C}{\cdot}1$. Pedersen [58] showed that the primitive ideal $\ker\pi_\ell$, $\ell \in U_e$, is generated by

$$u_j^e(\ell) = X_j - i r_j^e(\ell),\quad j \in T(e)$$

and that

$$\mathcal{U}(\mathfrak{g}) = \ker\pi_\ell \oplus E_e = \ker\pi_\ell \oplus F_e.$$

Here, π is faithful on E_e and on F_e. In this way, E_e and F_e are identified with $\mathcal{U}(\mathfrak{g})/\ker\pi_\ell$. Abusing a little the notations,

$$D_{\pi_\ell}(G)^K \cong E_e^K \cong F_e^K \cong \mathbb{C}[X_{j_1},\ldots,X_{j_m}]^K.$$

Of course, these isomorphisms are simply as vector spaces.

Removing the dependence on ℓ from these results of Pedersen on $\ker\pi_\ell$, Corwin and Greenleaf [17] constructed their e-central elements. These e-central elements are useful to construct elements of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ and play a principal role to prove the commutativity.

We fix the coadjoint orbit $\Omega = \Omega(\pi) \subset \mathfrak{g}^*$ corresponding to $\pi \in \hat{G}$. We further fix an element $\ell \in \Omega$ and introduce the coordinates into Ω by means of a Malcev basis $\{\tilde{X}_k; 1 \le k \le q\}$, $q = \dim(\mathfrak{g}/\mathfrak{g}(\ell)) = \dim \Omega$ relative to $\mathfrak{g}(\ell)$. That is to say,

$$\mathfrak{g}(\ell) + \sum_{k=1}^{r} \mathbb{R}\tilde{X}_k,\ 1 \le r \le q,$$

are Lie subalgebras of $\mathfrak{g}$. The mapping

$$\mathbb{R}^q \ni T = (t_1,\ldots,t_q) \mapsto \Phi(T) = \exp(t_q\tilde{X}_q)\cdots\exp(t_1\tilde{X}_1)\cdot\ell \in \Omega$$

is a diffeomorphism and through this mapping the measure $\tilde{\nu}_\pi$ turns out to be equivalent to the Lebesgue measure on $\mathbb{R}^q$. Let

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k},\ \dim(\mathfrak{k}_s) = s,\ 0 \le s \le d,$$

be a Jordan–Hölder sequence of $\mathfrak{k}$ and $\{Y_1,\ldots,Y_d\}$ a basis of $\mathfrak{k}$ adapted to this sequence. We define a matrix M_r, $1 \le r \le d$, of type (r,d) by

$$M_r = \big(\Phi(T)\left([Y_i, Y_j]\right)\big)_{1\le i\le r,\ 1\le j\le d}$$

and indicate the rank of M_r by $e_r\left(\Phi(T)|_{\mathfrak{k}}\right)$. The set

$$e\left(\Phi(T)|_{\mathfrak{k}}\right) = \left(e_1\left(\Phi(T)|_{\mathfrak{k}}\right), \ldots, e_d\left(\Phi(T)|_{\mathfrak{k}}\right)\right)$$

is nothing but the set of dimension indices of $\Phi(T)|_{\mathfrak{k}} \in \mathfrak{k}^*$ given from the above Jordan–Hölder sequence. For $1 \leq i \leq d$, put $e_i^0 = \max_{T\in\mathbb{R}^q} e_i\left(\Phi(T)|_{\mathfrak{k}}\right)$ and $e^0 = \left(e_1^0, \ldots, e_d^0\right)$. Evidently, $\mathcal{D} = \{T \in \mathbb{R}^q; e\left(\Phi(T)|_{\mathfrak{k}}\right) = e^0\}$ is a Zariski open set of $\mathbb{R}^q$. Hence, if we introduce a layer $U_{\mathfrak{k}}(\pi) = U_{e^0} = \{\zeta \in \mathfrak{k}^*; e(\zeta) = e^0\}$ in $\mathfrak{k}^*$, it results that

$$\nu\left(\hat{K} \setminus \hat{\rho}_K\left(U_{\mathfrak{k}}(\pi)\right)\right) = 0.$$

For $A \in S(\mathfrak{k})$, we define the principal closed set $\mathcal{F}(A)$ of $\mathfrak{k}^*$ by $\mathcal{F}(A) = \{\zeta \in \mathfrak{k}^*; A(\zeta) = 0\}$. Then, $\mathcal{F}_A = \{T \in \mathbb{R}^q; \Phi(T)|_{\mathfrak{k}} \in \mathcal{F}(A)\}$ becomes a Zariski closed set of $\mathbb{R}^q$. Owing to the construction of e-central elements (cf. [17]), there is a Zariski open set $\mathcal{Z}$ of $\mathfrak{k}^*$ such that $\mathcal{Z} \cap U_{e^0}$ is a non-empty K-invariant set where this construction is practised. Let $p_{\mathfrak{k}}$ be the restriction map $\mathfrak{g}^* \to \mathfrak{k}^*$. If $\mathcal{Z} \cap p_{\mathfrak{k}}(\Omega) = \emptyset$, the construction is practised replacing U_{e^0} by the sub-layer $U_{e^0} \setminus (\mathcal{Z} \cap U_{e^0})$, and it continues likewise. In this fashion this process supplies Corwin–Greenleaf e^0-central elements effectively ν-almost everywhere in $\hat{K}$.

Let

$$\mathcal{I} = \{i_1 < i_2 < \cdots < i_d\}$$

be the set of indices $1 \leq i \leq n$ such that $\mathfrak{g}_i \cap \mathfrak{k} \neq \mathfrak{g}_{i-1} \cap \mathfrak{k}$ in the composition series of ideals (9.1.1) and set

$$\mathcal{J} = \{j_1 < j_2 < \cdots < j_p\} = \{1, 2, \ldots, n\} \setminus \mathcal{I} \ \ (p = \dim(\mathfrak{g}/\mathfrak{k})).$$

Setting $\mathfrak{l}_0 = \mathfrak{k}$ and $\mathfrak{l}_r = \mathfrak{k} + \mathfrak{g}_{j_r}$ for $1 \leq r \leq p$, we obtain the sequence

$$\mathfrak{k} = \mathfrak{l}_0 \subset \mathfrak{l}_1 \subset \cdots \subset \mathfrak{l}_{p-1} \subset \mathfrak{l}_p = \mathfrak{g}, \ \ \dim(\mathfrak{l}_r/\mathfrak{l}_{r-1}) = 1,$$

of Lie subalgebras of $\mathfrak{g}$. On the other hand, considering $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_{i_s}$ $(1 \leq s \leq d)$, we get a composition series

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k}, \ \ \dim(\mathfrak{k}_s) = s, \ 0 \leq s \leq d,$$

of ideals in $\mathfrak{k}$. Extracting vectors $Y_s \in \mathfrak{k}_s \setminus \mathfrak{k}_{s-1}$ for $1 \leq s \leq d$ we make a Jordan–Hölder basis $\{Y_1, \ldots, Y_d\}$ of $\mathfrak{k}$. Likewise, extracting vectors $X_r \in \mathfrak{l}_r \setminus \mathfrak{l}_{r-1}$ for $1 \leq r \leq p$, we make a Malcev basis $\{X_1, \ldots, X_p\}$ of $\mathfrak{g}$ relative to $\mathfrak{k}$. Let us begin with extending the result of Pedersen mentioned above.

Theorem 13.1.1. *We suppose that* $\mathfrak{g}(\ell) \cap \mathfrak{k}_k \neq \mathfrak{g}(\ell) \cap \mathfrak{k}_{k-1}$ *at almost all* $\ell \in \Omega$ *relative to* $\tilde{\nu}_\pi$. *Then, using* $P_j \in \mathcal{U}(\mathfrak{k}_{k-1})$ $(0 \leq j \leq m)$ *with* $\pi(P_m) \neq 0$, *there exists an element* $W \in \mathcal{U}(\mathfrak{k}_k) \cap \ker \pi$ *of the form* $W = \sum_{j=0}^{m} P_j Y_k^j$.

Proof. We put

$$d_j = \max_{\ell \in \Omega} \dim(K_j \cdot \ell)$$

with $K_j = \exp(\mathfrak{k}_j)$, $1 \leq j \leq d$, and

$$\mathcal{O} = \{\ell \in \Omega; \dim(K_k \cdot \ell) = d_k,\ \dim(K_{k-1} \cdot \ell) = d_{k-1}\}.$$

Clearly, $\mathcal{O}$ is a non-empty Zariski open set of Ω, while, as is easily seen, $\mathfrak{g}(\ell) \cap \mathfrak{k}_k \neq \mathfrak{g}(\ell) \cap \mathfrak{k}_{k-1}$ when and only when $\dim(K_k \cdot \ell) = \dim(K_{k-1} \cdot \ell)$. Now, if we put

$$\mathcal{Z} = \{\ell \in \Omega; \mathfrak{g}(\ell) \cap \mathfrak{k}_k \neq \mathfrak{g}(\ell) \cap \mathfrak{k}_{k-1}\},$$

$\mathcal{Z}$ contains a non-empty Zariski open set of Ω provided $\mathcal{Z} \cap \mathcal{O} \neq \emptyset$, otherwise $\tilde{\nu}_\pi(\mathcal{Z}) = 0$. Similarly, letting $K'_{k-1} = \exp(\mathfrak{k}_{k-1} + \mathbb{R}X_1)$ and considering the maximal values of $\dim(K_{k-1} \cdot \ell)$ and $\dim(K'_{k-1} \cdot \ell)$ when ℓ moves in Ω, we see that the set $\mathcal{V} = \{\ell \in \Omega; X_1 \in \mathfrak{k}_{k-1} + \mathfrak{g}(\ell)\}$ contains a non-empty Zariski open set of Ω, unless $\tilde{\nu}_\pi(\mathcal{V}) = 0$.

Let us proceed by induction on $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{k})$. When $\dim(\mathfrak{g}/\mathfrak{k}) = 0$, the assertion is nothing but the well-known result of N. Pedersen. As above, we take a Jordan–Hölder sequence of $\mathfrak{g}$ passing the centre $\mathfrak{z}$ and put $\mathfrak{l} = \mathfrak{l}_1$, $L = \exp \mathfrak{l}$. Since $\dim(\mathfrak{g}/\mathfrak{l}) = \dim(\mathfrak{g}/\mathfrak{k}) - 1$, our assertion is established for $\mathfrak{l}$. Besides, we may assume that $\mathfrak{k}$ contains $\mathfrak{z}$ and further, if $\mathfrak{a} = \mathfrak{z} \cap \ker \ell \neq \{0\}$ at $\ell \in \Omega$, we may descend to the quotient Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ and apply the induction hypothesis. Thus, as usual we are brought to the case where $\dim \mathfrak{z} = 1$, $\ell|_{\mathfrak{z}} \neq 0$.

We assume $\mathfrak{g}(\ell) \cap \mathfrak{k}_k \neq \mathfrak{g}(\ell) \cap \mathfrak{k}_{k-1}$ at almost all $\ell \in \Omega$ relative to the measure $\tilde{\nu}_\pi$. From the induction hypothesis applied to $\mathfrak{l}$, there exists $W' \in \mathcal{U}(\mathfrak{l} \cap \mathfrak{g}_{i_k}) \cap \ker \pi$ which is written as

$$W' = \sum_{j=0}^{m} a'_j Y_k^j, \tag{13.1.1}$$

where $a'_j \in \mathcal{U}(\mathfrak{l} \cap \mathfrak{k}_{i_k - 1})$ $(0 \leq j \leq m)$ with $\pi(a'_m) \neq 0$. Hence, if $i_k < j_1$, the result follows immediately. Assume $j_1 < i_k$ from now on. By means of $a_j \in \mathcal{U}(\mathfrak{k}_k)$ $(0 \leq j \leq m')$ we rewrite (13.1.1) as

$$W' = \sum_{j=0}^{m'} a_j X_1^j \tag{13.1.2}$$

and choose this element W' in such a manner that m' might be minimal. When j_1 is a non-jump index in the dimension indices $e(\ell) = (e_1(\ell), \dots, e_n(\ell))$ of $\ell \in \Omega$ relative to the sequence of ideals $\mathcal{S}$, there exists

$$\hat{W} = X_1 + \hat{Q} \in \ker \pi$$

due to Pedersen [58]. Here, $\hat{Q} \in \mathcal{U}(\mathfrak{g}_{j_1-1})$. Substituting this expression of X_1 into formula (13.1.2) of W' and eliminating X_1, we obtain the desired result.

In what follows, we assume that j_1 is a jump-index in $e(\ell)$, $\ell \in \Omega$. Let $p_{\mathfrak{l}} : \mathfrak{g}^* \to \mathfrak{l}^*$ be the canonical projection and $\hat{\rho}_L : \mathfrak{l}^* \to \hat{L}$ the Kirillov map of L. From the irreducible decomposition of the restriction $\pi|_K$, the property $\pi(W') = 0$ means $\sigma(W') = 0$ for almost all $\sigma \in \hat{L}$ with respect to the measure $\nu_L^\pi = (\hat{\rho}_L \circ p_{\mathfrak{l}})_* (\tilde{\nu}_\pi)$.

Let us examine separately two cases mentioned above. First consider the case where $\mathcal{V}$ contains a non-empty Zariski open set of Ω. Since $X_1 \in \mathfrak{k}_{k-1} + \mathfrak{g}(\ell)$ at a general $\ell \in \Omega$, we take in $\ker \pi$ another element

$$W'' = \sum_{j=0}^{n''} b'_j Y_s^j, \tag{13.1.3}$$

where $b'_j \in \mathcal{U}(\mathfrak{l} \cap \mathfrak{g}_{i_s-1})$ $(0 \le j \le n'')$, $\pi(b'_{n''}) \neq 0$. Besides, $j_1 < i_s \le i_{k-1}$ and $X_1 \in \mathfrak{k}_s + \mathfrak{g}(\ell)$ almost everywhere relative to $\tilde{\nu}_\pi$. We rewrite the expression (13.1.3) as

$$W'' = \sum_{j=0}^{n'} X_1^j b_j, \tag{13.1.4}$$

where $b_j \in \mathcal{U}(\mathfrak{k}_s)$ $(0 \le j \le n')$. Let us remark that X_1 appears effectively in this expression (13.1.4), in other words, $\pi(b_{n'}) \neq 0$. In fact, if all the $b'_j \in \mathcal{U}(\mathfrak{k}_s)$ $(0 \le j \le n'')$ belong to $\mathcal{U}(\mathfrak{k}_{s-1})$, then $(\pi|_K)(W'') = (\pi|_{K_s})(W'') = 0$, while, for almost all irreducible representations ρ of K_s relative to $\nu_{K_s}^\pi$, their coadjoint orbits are saturated in the direction of $(\mathfrak{k}_{s-1})^\perp$, or there appear in the irreducible decomposition of $\pi|_{K_s}$ a one-parameter family of irreducible representations which are mutually inequivalent but whose restrictions to K_{s-1} are all equivalent to $\rho|_{K_{s-1}}$. This means as in the second case mentioned below that $\pi\left(b'_{n''}\right) = 0$, but this is absurd.

Here again we choose W'' given by formula (13.1.3) so that n' in expression (13.1.4) will be minimal. In order to get the desired result, let us eliminate X_1 from formulas (13.1.2) and (13.1.4). Combining (13.1.2) with (13.1.4), we can write

$$a_{m'} W'' = W' X_1^{n'-m'} b_{n'} + \sum_{j=0}^{n'-1} X_1^j c'_j = W' b_{n'} X_1^{n'-m'} + \sum_{j=0}^{n'-1} X_1^j c_j$$

when $m' \le n'$. Here $c'_j,\ c_j \in \mathcal{U}(\mathfrak{k}_{k-1})$ $(0 \le j \le n'-1)$. Next, putting $W''' = \sum_{j=0}^{n'-1} X_1^j c_j$ and repeating this manipulation,

$$\begin{aligned} a_{m'}^2 W'' &= a_{m'} W' b_{n'} X_1^{n'-m'} + a_{m'} W''' \\ &= a_{m'} W' b_{n'} X_1^{n'-m'} + W' c_{n'-1} X_1^{n'-m'-1} + \sum_{j=0}^{n'-2} X_1^j d_j, \end{aligned}$$

where $d_j \in \mathcal{U}(\mathfrak{k}_{k-1})$ $(0 \le j \le n'-2)$. If we continue likewise, we get

$$a_{m'}^p W'' = a_{m'}^{p-1} W' b_{n'} X_1^{n'-m'} + a_{m'}^{p-2} W' c_{n'-1} X_1^{n'-m'-1} + \cdots + \tilde{W} \tag{13.1.5}$$

for some $p \in \mathbb{N}$. Here, $\tilde{W} = \sum_{j=0}^{r} X_1^j \tilde{b}_j \in \ker \pi$ and $r < m'$, $\tilde{b}_j \in \mathcal{U}(\mathfrak{k}_k)$ $(0 \le j \le r)$. If $\tilde{W}$ contains effectively Y_k, this contradicts the minimality of m'. Hence $\tilde{W}$ does not contain Y_k effectively. Moreover, it does not happen that all the $\tilde{b}_j$ $(0 \le j \le r)$ are contained in $\ker \pi$. Otherwise, if we write

$$W'' = \sum_{j=0}^{n'} b''_j X_1^j \quad \left(b''_j \in \mathcal{U}(\mathfrak{k}_s),\ 0 \le j \le n'\right),$$

we see that the constant term $a_{m'}^p b''_0$ of Eq. (13.1.5) belongs to $\ker \pi$. Thus, $b''_0 \in \ker \pi$ and $W'' - b''_0$ becomes divisible by X_1, what contradicts the choice of W''.

From these observations, we may assume $m' > n'$ in the following arguments. Assume $m' > 0$. From the expressions (13.1.2) and (13.1.4), we obtain

$$W' b_{n'} = a_{m'} X_1^{m'-n'} W'' + \sum_{j=0}^{m'-1} a''_j X_1^j,$$

where $a''_j \in \mathcal{U}(\mathfrak{k}_k)$ $(0 \le j \le m'-1)$. In the expression of $W' b_{n'}$, there is a power of Y_k with degree larger than or equal to 1 and whose coefficient does not belong to $\ker \pi$. Since $W'' \in \ker \pi$, it follows that $\tilde{W} = \sum_{j=0}^{m'-1} a''_j X_1^j$ contains effectively Y_k but this contradicts the choice of W'.

Next, let us examine the case where $\tilde{\nu}_\pi(\mathcal{V}) = 0$. Let us assume $m' > 0$ and show $\pi(a_{m'}) = 0$. Indeed, either the corresponding coadjoint orbits $\omega = (\hat{\rho}_L)^{-1}(\sigma)$ of L are saturated in the direction of $\mathfrak{k}^\perp$ for almost all $\sigma \in \hat{L}$ relative to $\nu_L^\pi = (\hat{\rho}_L \circ p_\mathfrak{l})_* (\tilde{\nu}_\pi)$ or there appear in the irreducible decomposition of $\pi|_L$ a one-parameter family of irreducible representations which are mutually inequivalent but whose restrictions to K_k are all equivalent to $\sigma|_{K_k}$. When $\omega \subset \mathfrak{l}^*$ is saturated in the direction of $\mathfrak{k}^\perp$, σ is realized by means of a polarization contained in $\mathfrak{k}$ and $\sigma(X_1)$ is nothing but the partial differential operator with respect to the first coordinate introduced along X_1. From this, we know that $\sigma(a_{m'}) = 0$ in this situation.

Henceforth, we suppose that $\omega \subset \mathfrak{l}^*$ is non-saturated in the direction of $\mathfrak{k}^{\perp}$. Then, in the irreducible decomposition of $\pi|_L$, the contribution of the subset $\hat{\rho}_L\left(\omega + (\mathfrak{k}_k)^{\perp}\right) \subset \hat{L}$ is not negligible. We find therein a one-parameter family of irreducible representations σ_ℓ constructed at $\ell \in \omega + (\mathfrak{k}_k)^{\perp}$. They admit the same e-central element $\xi = \bar{P}X_1 + \bar{Q} \in \mathcal{U}(\mathfrak{l})$ as those having the dimension indices e of L-orbits relative to the sequence of ideals

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_k \subset \mathfrak{k}_k + \mathbb{R}X_1 \subset \mathfrak{k}_{k+1} + \mathbb{R}X_1 \subset \cdots \subset \mathfrak{l}$$

of $\mathfrak{l}$, and whose sub-layer Ω_e encounters effectively the support of ν_L^{π}. In more detail, $\bar{P},\ \bar{Q} \in \mathcal{U}(\mathfrak{k}_k)$ and $\sigma_\ell(\xi)$ is a scalar operator whose value is given by $\bar{p}(\ell')\ell(X_1) + \bar{q}(\ell')$. Further, with $\ell_i = \ell(Y_i)$ $(1 \le i \le k)$, $\bar{p},\ \bar{q}$ are rational functions of $\ell' = (\ell_1, \ldots, \ell_k)$ and $\bar{P}$ is also an e-central element such that $\sigma_\ell(\bar{P}) = \bar{p}(\ell')Id \neq 0$.

Bringing these facts into expression (13.1.2) of $W' \in \ker \pi$ and dividing W' by ξ as above, we get

$$\bar{P}^{m'}W' = \bar{P}^{m'}\sum_{j=0}^{m'} X_1^j \hat{a}_j = \xi^{m'}\hat{a}_{m'} + \sum_{j=0}^{m'-1} X_1^j b_j,$$

where $\hat{a}_j,\ b_j \in \mathcal{U}(\mathfrak{k}_k)$ and $\hat{a}_{m'} = a_{m'}$. Multiplying this equation by $\bar{P}^{m'-1}$ and repeating the process, we obtain

$$\begin{aligned}
\bar{P}^{2m'-1}W' &= \bar{P}^{m'-1}\left(\xi^{m'}a_{m'} + \sum_{j=0}^{m'-1} X_1^j b_j\right)\\
&= \bar{P}^{m'-1}\xi^{m'}a_{m'} + \bar{P}^{m'-1}\left(\sum_{j=0}^{m'-1} X_1^j b_j\right)\\
&= \bar{P}^{m'-1}\xi^{m'}a_{m'} + \xi^{m'-1}b_{m'-1} + \sum_{j=0}^{m'-2} X_1^j c_j,
\end{aligned}$$

where $c_j \in \mathcal{U}(\mathfrak{k}_k)$ $(0 \le j \le m'-2)$. Continuing this manipulation, there exists a certain $r \in \mathbb{N}$ so that

$$\bar{P}^r W' = \sum_{j=0}^{m'} \bar{P}^{r_j}\xi^j h_j,$$

where $h_j \in \mathcal{U}(\mathfrak{k})$ $(0 \le j \le m')$, $h_{m'} = a_{m'}$ and $0 = r_0 < r_1 < \cdots < r_{m'}$ is a increasing sequence of integers. Taking $\pi(W') = 0$ into account, we procure

$$\sum_{j=0}^{m'} \bar{p}^{r_j}(\ell')\left(\bar{p}(\ell')\ell(X_1)+\bar{q}(\ell')\right)^j \sigma_\ell(h_j)=0.$$

Since this last expression is a polynomial of the variable $\ell(X_1)$, the coefficient of its maximal power is inevitably 0 and hence $\sigma_\ell(h_{m'}) = \sigma_\ell(a_{m'}) = 0$ at almost all $\ell \in \mathfrak{l}^*$ relative to ν_L^π. In this way, $\pi(a_{m'}) = 0$ is deduced and we meet a contradiction. In consequence, we have shown that $m' = 0$. ■

13.2 Commutativity of the Algebra $D_\pi(G)^K$

Let us prove the first result concerning the commutativity of the algebra $D_\pi(G)^K$.

Theorem 13.2.1. *Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group, $K = \exp \mathfrak{k}$ an analytic subgroup of G and π an irreducible unitary representation of G. If $\pi|_K$ has finite multiplicities, the algebra $D_\pi(G)^K$ is commutative.*

Proof. Let us employ the induction on dim $\mathfrak{g}$. As is immediately seen, we may assume that $\mathfrak{k}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$, and if $\mathfrak{a} = \mathfrak{z} \cap \ker \ell$, $\ell \in \Omega$, is not trivial, we may descend to the quotient Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ and apply the induction hypothesis. Thus as usual we are brought to the case where dim $\mathfrak{z} = 1$ and the elements of Ω do not vanish on $\mathfrak{z}$. Let $\{X, Y, Z\}$ be a Heisenberg triplet such that $\mathfrak{z} = \mathbb{R}Z$, $\ell(Z) = 1$ for $\ell \in \Omega$, $[X, Y] = Z$ and $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}'$ with the centralizer $\mathfrak{g}'$ of Y in $\mathfrak{g}$. We put $G' = \exp(\mathfrak{g}')$ and consider the following two possible cases.

The first case occurs when $\mathfrak{k} \subset \mathfrak{g}'$. From the description of the irreducible decomposition of $\pi|_K$, if $\pi|_K$ has finite multiplicities, almost all K-orbits in Ω are saturated in the direction of $(\mathfrak{g}')^\perp$. Hence, $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \subset \mathcal{U}(\mathfrak{g}') + \ker \pi$. In fact, this is clear if $Y \in \mathfrak{k}$. If $Y \notin \mathfrak{k}$, put $\mathfrak{l} = \mathfrak{k} \oplus \mathbb{R}Y$. From the saturation of orbits, $\mathfrak{l} \cap \mathfrak{g}(\ell) \neq \mathfrak{k} \cap \mathfrak{g}(\ell)$ at almost all $\ell \in \Omega$, and by Theorem 13.1.1, there exists

$$W = \sum_{j=0}^{m'} a_j Y_s^j = \sum_{r=0}^{m} W_r Y^r \in \mathcal{U}(\mathfrak{l}) \cap \ker \pi$$

for a certain s $(1 \leq s \leq d)$. Here $a_j \in \mathcal{U}(\mathfrak{l} \cap \mathfrak{g}_{i_s-1})$ $(0 \leq j \leq m')$, $\pi(a_{m'}) \neq 0$ and $W_r \in \mathcal{U}(\mathfrak{k}_s)$ $(0 \leq r \leq m)$. The K_s-coadjoint orbits in $(\mathfrak{k}_s)^*$ belong, almost everywhere relative to $(p_{\mathfrak{k}_s})_*(\tilde{\nu}_\pi)$, to a one-parameter family non-saturated in the direction of $(\mathfrak{k}_{s-1})^\perp$ and the corresponding irreducible unitary representations appear in the irreducible decomposition of $\pi|_{K_s}$ continuously in this parameter. From this, exactly as in the proof of Theorem 13.1.1, we see that W contains effectively Y. Namely, $m > 0$ and $W_m \notin \ker \pi$.

Let us assume for a while that $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \not\subset \mathcal{U}(\mathfrak{g}') + \ker \pi$. Then, there exists $V = Xa + b \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ with $a, b \in \mathcal{U}(\mathfrak{g}')$ and $a \notin \ker \pi$. From this,

$$(\mathrm{ad}V)^m(W) = m!a^m W_m Z^m \in \ker \pi,$$

which contradicts the fact that $\pi\,(\mathcal{U}(\mathfrak{g}))$ has no non-trivial zero divisor. This last fact comes from Theorem 6.2.13 of Chap. 6 which says that the action of $\pi(A) = d\pi(A),\ A \in \mathcal{U}(\mathfrak{g})$, on $\mathcal{H}_\pi^\infty$, realized as $\mathcal{S}(\mathbb{R}^k)$ for some k, is a linear differential operator with polynomial coefficients.

Let us realize as $\pi = \mathrm{ind}_B^G \chi_\ell$ using a polarization $\mathfrak{b} \subset \mathfrak{g}'$ at $\ell \in \Omega$. Here $B = \exp \mathfrak{b}$ and χ_ℓ is as usually the unitary character of B defined by $\chi_\ell(\exp X) = e^{i\ell(X)},\ X \in \mathfrak{b}$. We write the restriction of π to K as

$$\pi|_K = (\pi|_{G'})\,|_K \simeq \int_{\mathbb{R}}^{\oplus} \pi_t|_K dt,$$

where π_t denotes the irreducible unitary representation of G' corresponding to

$$(\exp(tX)\cdot\ell)\,|_{\mathfrak{g}'}.$$

Since $\pi|K$ has finite multiplicities, the representation $\pi_t|_K$ too has finite multiplicities at almost all $t \in \mathbb{R}$, and hence $D_{\pi_t}(G')^K$ is commutative by the induction hypothesis. On the other hand, it is easily seen that

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \bigcap_{t\in\mathbb{R}} \mathcal{U}_{\pi_t}(\mathfrak{g}')^{\mathfrak{k}},$$

and $D_\pi(G)^K$ is commutative.

The second case occurs when $\mathfrak{k} \not\subset \mathfrak{g}'$. In this case, we choose X in $\mathfrak{k}$. If almost all K-orbits in Ω are saturated in the direction of $(\mathfrak{g}')^\perp$. The commutativity in question is shown exactly as in the first case. Therefore, it is enough to examine the case where almost all K-orbits in Ω are non-saturated in the direction of $(\mathfrak{g}')^\perp$. In this case, $\pi|_K$ having finite multiplicities, almost all K-orbits in $\mathfrak{k}^*$ relative to the measure $(p_{\mathfrak{k}})_*(\tilde{\nu}_\pi)$ are non-saturated in the direction of $(\mathfrak{k} \cap \mathfrak{g}')^\perp$. From the preceding study on the Corwin–Greenleaf elements, there exists $\xi = aX + b \in \mathcal{U}(\mathfrak{k}),\ a,\ b \in \mathcal{U}(\mathfrak{k} \cap \mathfrak{g}')$, such that $\rho(\xi)$ is a scalar operator for almost all $\rho \in \hat{K}$ relative to $(\hat{\rho}_K \circ p_{\mathfrak{k}})_*\,(\tilde{\nu}_\pi)$, and $a \notin \ker \pi$ too has the same property. Hence, $a,\ \xi \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

Let $A,\ B \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. When $A,\ B$ belong to $\mathcal{U}(\mathfrak{g}')$, considering $\mathfrak{k}' = (\mathfrak{k} \cap \mathfrak{g}') \oplus \mathbb{R}Y$ instead of $\mathfrak{k}$ and reasoning exactly as in the first case, we see $[\pi(A), \pi(B)] = 0$. We write $A = \sum_{j=0}^m X^j A_j$ and $B = \sum_{j=0}^{m'} X^j B_j$ with $A_j,\ B_j \in \mathcal{U}(\mathfrak{g}')$. Then, $A_m,\ B_{m'} \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. In order to show the commutativity in question, let us assume for instance $m > 0$ and employ a new induction on $m + m'$. The degree of the element

$$\tilde{A} = a^m A - \xi^m A_m \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$$

with respect to X is smaller than m. Besides, $[\xi, B]$ and $[A_m, B]$ belong to $\ker \pi$. So,

$$\left[\tilde{A}, B\right] \equiv a^m [A, B] \in \ker \pi.$$

Thus, $[A, B] \in \ker \pi$ and the proof of the theorem is complete. ■

Definition 13.2.2. Let $\ell \in \mathfrak{g}^*$ and B_ℓ the bilinear form defined as before. We say that $\mathfrak{k}$ is **co-isotropic** at ℓ if the subspace $\mathfrak{k}^\ell = \{X \in \mathfrak{g}; B_\ell(X, \mathfrak{k}) = \{0\}\}$ is isotropic regarding B_ℓ.

By means of this concept of being co-isotropic, we are able to characterize the case where $\pi|_K$ has finite multiplicities.

Proposition 13.2.3. *When and only when $\mathfrak{k}$ is co-isotropic almost everywhere on Ω, $\pi|_K$ has finite multiplicities.*

Proof. We proceed by induction on the dimension of G. Since we may assume $\mathfrak{g} \neq \mathfrak{k}$, we take an ideal $\mathfrak{g}_0$ of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{k}$ and put $G_0 = \exp(\mathfrak{g}_0)$. When Ω is non-saturated with respect to $\mathfrak{g}_0$, $\mathfrak{g}(\ell)$ at $\ell \in \Omega$ is not contained in $\mathfrak{g}_0$ and the restriction $\pi|_{G_0}$ is irreducible. Let $\tilde{\Omega} \subset \mathfrak{g}_0^*$ be the coadjoint orbit of G_0 corresponding to $\pi|_{G_0} \in \widehat{G_0}$.

Since $\pi|_K = (\pi|_{G_0})|_K$, the induction hypothesis assures that $\pi|_K$ has finite multiplicities when and only when $\mathfrak{k}$ is co-isotropic almost everywhere on $\tilde{\Omega}$, while, if we set $\ell_0 = \ell|_{\mathfrak{g}_0}$ for $\ell \in \Omega$, there exists a certain element $X(\ell) \in \mathfrak{g}(\ell)$ not belonging to $\mathfrak{g}_0$ so that $\mathfrak{k}^\ell = \mathfrak{k}^{\ell_0} \oplus X(\ell)$. Hence, $\mathfrak{k}$ is co-isotropic at ℓ if and only if it is co-isotropic at ℓ_0.

Supposing that Ω is saturated relative to $\mathfrak{g}_0$, let us examine two possibilities. First, general K-orbits in Ω are supposed saturated relative to $\mathfrak{g}_0$. The restriction $\pi|_{G_0}$ is decomposed into a one-parameter family,

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} \pi_t \, dt.$$

Moreover, with $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$, $\pi_t = \exp(tX)\cdot\pi_0$ and $\pi \simeq \mathrm{ind}_{G_0}^G \pi_t$ for any $t \in \mathbb{R}$. Since general K-orbits in Ω are saturated in the direction of $(\mathfrak{g}_0)^\perp \subset \mathfrak{g}^*$, the set

$$\{t \in \mathbb{R}; (\exp(tX)\cdot\ell_0)|_{\mathfrak{k}} = (k\cdot\ell_0)|_{\mathfrak{k}}, \exists k \in K\}$$

is a finite set for a general $\ell \in \Omega$. Hence, $\pi|_K$ has finite multiplicities if and only if $(\pi_t)|_K$ has finite multiplicities at almost all $t \in \mathbb{R}$. We denote by $\Omega_t \subset \mathfrak{g}_0^*$ the coadjoint orbit of G_0 corresponding to $\pi_t \in \widehat{G_0}$ $(t \in \mathbb{R})$. From what we have just seen and the induction hypothesis, $\pi|_K$ has finite multiplicities if and only if $\mathfrak{k} \subset \mathfrak{g}_0$ is co-isotropic almost everywhere on Ω_t for almost all $t \in \mathbb{R}$. But the latter condition is equivalent to that $\mathfrak{k}$ is co-isotropic almost everywhere on Ω since $\mathfrak{k}^\ell \subset \mathfrak{g}_0$ at a general $\ell \in \Omega$.

Lastly, suppose that almost all K-orbits in Ω are non-saturated relative to $\mathfrak{g}_0$. Take such a general element $\ell \in \Omega$. Then, there exist $X(\ell) \in \mathfrak{k}^{\ell}$ not belonging to $\mathfrak{g}_0$ and $Y(\ell) \in \mathfrak{g}_0(\ell_0)$ not belonging to $\mathfrak{g}(\ell)$. Because $Y(\ell)$ belongs to $\mathfrak{k}^{\ell}$, $\mathfrak{k}$ is not co-isotropic. Here, if we look at the K-orbit

$$\omega = K{\cdot}p_{\mathfrak{k}}(\ell) \subset p_{\mathfrak{k}}(\Omega) \subset \mathfrak{k}^*,$$

it is clear that $p_{\mathfrak{k}}^{-1}(\omega) \cap \Omega$ contains infinite K-orbits, and $\pi|_K$ has infinite multiplicities. ∎

Remark 13.2.4. As in the case of induced representations, Proposition 13.2.3 no longer holds for exponential solvable Lie groups. For example, let $\mathfrak{g} = \langle A, X, Y\rangle_{\mathbb{R}} : [A, X] = X - Y,\ [A, Y] = X + Y,\ \mathfrak{k} = \mathbb{R}X, \pi$ an infinite-dimensional irreducible unitary representation of G and Ω the corresponding coadjoint orbit of G. Then, $\mathfrak{k}^{\ell} = \mathbb{R}X \oplus \mathbb{R}Y$ at generic $\ell \in \Omega$ and $\mathfrak{k}$ is co-isotropic at ℓ. However, it is immediately seen that $\pi|_K$ has infinite multiplicities.

Let us give another interpretation of this fact by a general principle due to Michel Duflo. Let $\ell \in p_{\mathfrak{k}}(\Omega) \subset \mathfrak{k}^*$, ω the coadjoint orbit of K passing ℓ and σ the irreducible unitary representation of K corresponding to ω. Then, Proposition 13.2.3 is interpreted as follows:

Proposition 13.2.5. *When and only when $K(\ell)$-orbits in $\Omega \cap p_{\mathfrak{k}}^{-1}(\ell)$ are open sets almost everywhere, the representation $\pi|_K$ has finite multiplicities. In this situation, the multiplicity $n_\pi(\sigma)$ in Theorem 8.2.4 is equal to the number of $K(\ell)$-orbits contained in $\Omega \cap p_{\mathfrak{k}}^{-1}(\ell)$.*

Writing $\mathfrak{k}_{d+r}$ instead of $\mathfrak{l}_r$ $(1 \le r \le p)$, we get a sequence of Lie subalgebras

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k} = \mathfrak{k}_d \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g}$$

of $\mathfrak{g}$. Here $\dim(\mathfrak{k}_r) = r,\ 1 \le r \le n$ and $\mathfrak{k}_r$ is an ideal of $\mathfrak{k}$ for $1 \le r \le d$.

Let us extract a vector $X_s \in \mathfrak{k}_s \setminus \mathfrak{k}_{s-1}$ for $1 \le s \le n$. In reality, the dual basis $\{X_1^*, \ldots, X_n^*\}$ is a Jordan–Hölder basis of $\mathfrak{g}^*$ relative to the unipotent coadjoint action $\mathrm{Ad}^*(K)$. The projections $p_j : \mathfrak{g}^* \to \mathfrak{k}_j^*$ $(1 \le j \le n)$ commute with the action of K and exactly as before, the linear form $\ell \in \mathfrak{g}^*$ gives an n-tuple of non-negative integers

$$e(\ell) = (e_1(\ell), \ldots, e_n(\ell))$$

defined by $e_k(\ell) = \dim(K{\cdot}p_k(\ell))$ $(1 \le k \le n)$. Conversely, each n-tuple $e = (e_1, \ldots, e_n)$ of non-negative integers gives a layer U_e of K-orbits in $\mathfrak{g}^*$ defined by

$$U_e = \{\ell \in \mathfrak{g}^*; e_k(\ell) = e_k,\ 1 \le k \le n\}.$$

Then, there exists a unique layer U_e such that $U_e \cap \Omega$ becomes a non-empty Zariski open set of Ω. We consider the set $S(e)$ of the jump indices:

$$S(e) = \{1 \le k \le n; e_k = e_{k-1} + 1\}$$

and the set $T(e)$ of the non-jump indices:

$$T(e) = \{1 \le k \le n; e_k = e_{k-1}\},$$

where we agree that $e_0 = 0$. Put $K_j = \exp(\mathfrak{k}_j)$, $1 \le j \le n-1$. In particular, $K_d = K$.

Theorem 13.2.6. *When we pass from $\mathfrak{k}_{j-1}$ to $\mathfrak{k}_j$, the algebra $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ enlarges if $j \in T(e)$, and does not enlarge if $j \in S(e)$. More precisely, if we set $\mathcal{U}_\pi(\mathfrak{k}_j)^{\mathfrak{k}} = \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \cap \mathcal{U}(\mathfrak{k}_j)$ for $1 \le j \le n$:*

(1) *If $j \in T(e)$, $\mathcal{U}_\pi(\mathfrak{k}_j)^{\mathfrak{k}} \ne \mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}} + \mathcal{U}(\mathfrak{k}_j)\big(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker\pi\big)$ and there exists $W \in \mathcal{U}_\pi(\mathfrak{k}_j)^{\mathfrak{k}}$ of the form $W = aX_j + b$ $(a,\ b \in \mathcal{U}(\mathfrak{k}_{j-1}))$, $\pi(a) \ne 0$.*
(2) *If $j \in S(e)$, $\mathcal{U}_\pi(\mathfrak{k}_j)^{\mathfrak{k}} = \mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}} + \mathcal{U}(\mathfrak{k}_j)\big(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker\pi\big)$.*

Proof. First we treat the case where $j \in T(e)$. The claim for the first index $1 \in T(e)$ is automatically obtained. Assume that the statements are true for the indices less than or equal to $j-1$. When $1 \le j \le d$, with $e' = (e_1, \dots, e_d)$, the Corwin–Greenleaf e'-central element satisfies the condition of the theorem. Let $d+1 \le j$. When $\mathfrak{k} = \{0\}$, the claim is evident. So, let us employ induction on dim $\mathfrak{k}$.

From the induction hypothesis for $\mathfrak{k}_{d-1}$, there exists

$$W = aX_j + b \ \ (a,\ b \in \mathcal{U}(\mathfrak{k}_{j-1})), \tag{13.2.1}$$

which satisfies the conditions for $\mathfrak{k}_{d-1}$. Namely, $\pi([Y, W]) = 0$ for arbitrary $Y \in \mathfrak{k}_{d-1}$ and $\pi(a) \ne 0$. Let us first remark that we may assume $[\mathfrak{k}_{d-1}, a] \subset \ker\pi$. In fact, if there exists $Y \in \mathfrak{k}_{d-1}$ such that $[Y, a] \notin \ker\pi$, the element

$$[Y, W] = [Y, a]X_j + b',\ b' \in \mathcal{U}(\mathfrak{k}_{j-1}),$$

of ker π is already our desired element.

We operate $\mathrm{ad}(Y_d)$ on W several times so that $(\mathrm{ad}(Y_d))^m(W)$ belongs for the first time to ker π. Then, as is immediately seen,

$$W_r = (\mathrm{ad}(Y_d))^{r-1}(W),\ r \ge 1$$

too belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}_{d-1}}$ and has the same form as in expression (13.2.1). Besides, using the maximal power m' such that

$$(\mathrm{ad}(Y_d))^{m'}(W) \notin \mathcal{U}(\mathfrak{k}_{j-1})$$

modulo ker π, we replace W by $(\mathrm{ad}(Y_d))^{m'}(W)$. Then, we may suppose that

$$(\mathrm{ad}(Y_d))(W) \in \mathcal{U}(\mathfrak{k}_{j-1})$$

modulo ker π.

If $m = 1$, then $\pi([\mathfrak{k}, W]) = \{0\}$ and there is nothing left to do. Hence, we assume $m \geq 2$. We first treat the case where $m = 2q + 1 \geq 3$. In this case, if we set

$$\begin{aligned}\tilde{W} = &(W_1 W_m + W_m W_1) - (W_2 W_{m-1} + W_{m-1} W_2) + \cdots \\ &+ (-1)^{q-2}(W_{q-1} W_{q+3} + W_{q+3} W_{q-1}) \\ &+ (-1)^{q-1}(W_q W_{q+2} + W_{q+2} W_q) + (-1)^q W_{q+1}^2,\end{aligned}$$

$[Y_d, \tilde{W}] \in \ker \pi$ and $[\mathfrak{k}, \tilde{W}] \subset \ker \pi$. Let us notice that we can write

$$\tilde{W} = (aW_m + W_m a)X_j + b'$$

with a certain $b' \in \mathcal{U}(\mathfrak{k}_{j-1})$. As we have used this fact several times until now, when we realize π as a monomial representation in $L^2(\mathbb{R}^k)$ starting from a polarization, the space $\mathcal{H}_\pi^{+\infty}$ of the C^∞-vectors coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^k)$, where $\pi(a),\ \pi(W_m)$ act as differential operators with polynomial coefficients. Hence, it results that

$$\pi(aW_m + W_m a) \neq 0$$

and $\tilde{W}$ possesses the required properties.

Secondly, suppose that m is an even number. If we replace W by

$$W' = W(W_{m-1} + cW_m)\ (0 \neq c \in \mathbb{C})$$

and set $W'_r = (\mathrm{ad}(Y_d))^{r-1}(W'),\ 1 \leq r \leq m+1$, then

$$\begin{aligned}&W'_2 \equiv W_2(W_{m-1} + cW_m) + W_1 W_m,\ \cdots, \\ &\qquad W'_m \equiv W_m(W_{m-1} + cW_m) + (m-1)W_{m-1}W_m,\ W'_{m+1} \equiv mW_m^2\end{aligned}$$

modulo $\ker \pi$. Therefore, our new $\tilde{W}$ is obtained by

$$\begin{aligned}\tilde{W} = &m\left\{W_1(W_{m-1} + cW_m)W_m^2 + W_m^2 W_1(W_{m-1} + cW_m)\right\} \\ &- W_1 W_m \{W_m(W_{m-1} + cW_m) + (m-1)W_{m-1}W_m\} \\ &- \{W_m(W_{m-1} + cW_m) + (m-1)W_{m-1}W_m\} W_1 W_m + \tilde{V},\end{aligned}$$

where $\tilde{V}$ is a certain element of $\mathcal{U}(\mathfrak{k}_{j-1})$. We have

$$\begin{aligned}\tilde{W} \equiv &m(W_1 W_{m-1} W_m^2 + W_m^2 W_1 W_{m-1}) \\ &- (m-1)(W_1 W_m W_{m-1} + W_{m-1} W_m W_1) W_m \\ &- (W_1 W_m^2 W_{m-1} + W_m W_{m-1} W_1 W_m) + c(m-1)(W_1 W_m^2 + W_m^2 W_1) W_m,\end{aligned}$$

modulo $\mathcal{U}(\mathfrak{k}_{j-1})$. Besides, since $\pi\left((W_1W_m^2 + W_m^2 W_1)W_m\right) \neq 0$ as we saw above, it is clear that we can choose $0 \neq c \in \mathbb{C}$ so that $\tilde{W} \notin \mathcal{U}(\mathfrak{k}_{j-1}) + \ker\pi$.

This procedure does not work well when $m = 2$. Recall the definition

$$\mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}_{d-1}} = \left\{\hat{W} \in \mathcal{U}(\mathfrak{k}_{j-1}); \pi\left(\left[\hat{W}, \mathfrak{k}_{d-1}\right]\right) = \{0\}\right\}.$$

Provided $\left[Y_d, \mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}_{d-1}}\right] \not\subset \ker\pi$, we can increase m by multiplying W by

$$\hat{W} \in \mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}_{d-1}}$$

such that $\pi\left(\left[Y_d, \hat{W}\right]\right) \neq 0$. Indeed, take such a $\hat{W}$ satisfying

$$\hat{W}_1 = \left[Y_d, \hat{W}\right] \notin \ker\pi,\ \hat{W}_2 = \left[Y_d, \hat{W}_1\right] \in \ker\pi.$$

Then, by the change from $W = W_1$ to $\tilde{W}_1 = W_1\hat{W}$, we get

$$\tilde{W}_2 = \left[Y_d, \tilde{W}_1\right] = W_2\hat{W} + W_1\hat{W}_1,\ \tilde{W}_3 = \left[Y_d, \tilde{W}_2\right] \equiv 2W_2\hat{W}_1 \ (\mathrm{mod}\ \ker\pi).$$

Here, if we produce $\tilde{W} = \tilde{W}_1\tilde{W}_3 + \tilde{W}_3\tilde{W}_1 - \tilde{W}_2^2$, regarding X_j the first two terms are of degree 1 but the last term is of degree 2. In fact, we can write

$$\tilde{W} = -\left(a\hat{W}_1\right)^2 X_j^2 + a_1X_j + a_2$$

with some $a_1,\ a_2 \in \mathcal{U}(\mathfrak{k}_{j-1})$. Let us remark that $\pi\left(a\hat{W}_1\right) \neq 0$.

Next, in order to decrease the degree let us make use of the element $X_{d+1} \in \mathfrak{k}_{d+1}$ belonging to the normalizer of $\mathfrak{k}$. Then, $\left[X_{d+1}, \tilde{W}\right]$ also satisfies the relation

$$\pi\left(\left[\mathfrak{k}, \left[X_{d+1}, \tilde{W}\right]\right]\right) = \{0\}.$$

Choosing X_{d+1} freely, we may suppose $[X_{d+1}, X_j] \notin \ker\pi$. Otherwise, it results that $[\mathfrak{k}, X_j] \subset \ker\pi$ and X_j fulfils the request. Hence by continuity, letting ϵ be a sufficiently small positive real number,

$$\left[X_{d+1} + \sum_{i=1}^{d} \alpha_i X_i, X_j\right] \notin \ker\pi$$

for $\alpha_i \in \mathbb{C}$ such that $|\alpha_i| < \epsilon\ (1 \leq i \leq d)$. If $\left[X_{d+1}, a\hat{W}_1\right] \notin \ker\pi$, by replacing $\tilde{W}$ by $(\mathrm{ad}(X_{d+1}))^q\,(\tilde{W})$ with an appropriate $q \in \mathbb{N}$, we may assume

$$\tilde{W} = a'X_j^2 + a''X_j + a''',$$

where $a',\ a'',\ a''' \in \mathcal{U}(\mathfrak{k}_{j-1}),\ \pi(a') \neq 0$ and $[X_{d+1} + \mathfrak{k}, a'] \subset \ker \pi$. Now, if there exist complex numbers $\hat{\alpha}_i\ (1 \leq i \leq d)$ with $|\hat{\alpha}_i| < \epsilon$ satisfying

$$\left[X_{d+1} + \sum_{i=1}^{d} \hat{\alpha}_i X_i, 2a' X_j + a''\right] \notin \ker \pi,$$

then, $W^* = \left[X_{d+1} + \sum_{i=1}^{d} \hat{\alpha}_i X_i, \tilde{W}\right]$ fulfils our request. If there are no such $\hat{\alpha}_i\ (1 \leq i \leq d)$, it turns out that $[\mathfrak{k}, 2a' X_j + a''] \subset \ker \pi$ and $2a' X_j + a''$ is our desired element.

Provided $\left[Y_d, \mathcal{U}_\pi(\mathfrak{k}_{j-1})^{\mathfrak{k}_{d-1}}\right] \subset \ker \pi$, from the induction hypothesis for the index $j-1$,

$$\mathfrak{k} \cap \left(\mathfrak{k}_{j-1}(\ell|_{\mathfrak{k}_{j-1}})\right) \neq \mathfrak{k}_{d-1} \cap \left(\mathfrak{k}_{j-1}(\ell|_{\mathfrak{k}_{j-1}})\right)$$

at a general $\ell \in \Omega$. From this, taking $j \in T(e)$ into account,

$$\mathfrak{k} \cap \left(\mathfrak{k}_j(\ell|_{\mathfrak{k}_j})\right) \neq \mathfrak{k}_{d-1} \cap \left(\mathfrak{k}_j(\ell|_{\mathfrak{k}_j})\right).$$

Using the multiplicity $m(\rho)$ and the measure ν_j explained in Theorem 8.2.4 of Chap. 8, we write the canonical irreducible decomposition formula as

$$\pi|_{K_j} \simeq \int_{\widehat{K_j}}^{\oplus} m(\rho)\rho d\nu_j(\rho).$$

By Theorem 13.1.1, for almost all $\rho \in \widehat{K_j}$ relative to ν_j, there exists an element $V_\rho \in \mathcal{U}(\mathfrak{k}) \cap \ker \rho$ of the form

$$V_\rho = \sum_{j=0}^{r} P_j Y_d^j,$$

where $P_j \in \mathcal{U}(\mathfrak{k}_{d-1})\ (0 \leq j \leq r)$ and $\rho(P_r) \neq 0$. We choose V_ρ in such a manner that $r \geq 1$ might be as small as possible. Then, if $(\mathrm{ad}(Y_d))^2\,(W)$ belongs to $\ker \pi$,

$$[W, V_\rho] \equiv \left(\sum_{j=1}^{r} j P_j Y_d^{j-1}\right)[W, Y_d] \bmod \ker \rho.$$

Since $\sum_{j=1}^{r} j P_j Y_d^{j-1}$ does not belong to $\ker \rho$ and the actions in $\mathcal{H}_\rho^{+\infty}$ have no non-trivial zero divisor, we get $\rho\left([W, Y_d]\right) = 0$ from $\rho\left([W, V_\rho]\right) = 0$. Consequently, $\pi\left([Y_d, W]\right) = 0$ and $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

Next, let us examine the case where $j \in S(e)$. We employ a new induction on dim $\mathfrak{g}$. First, let us remark that the study is reduced to the case where $j = n$.

Indeed, if $j \leq n-1$, let us consider $\pi' = \pi|_{K_{n-1}}$. When π' is irreducible, we apply the induction hypothesis. If not, π' is decomposed into a one-parameter family π'_t ($t \in \mathbb{R}$) of irreducible unitary representations of K_{n-1}. Now, if there exists $W \in \mathcal{U}(\mathfrak{k}_j) \setminus \mathcal{U}(\mathfrak{k}_{j-1})$ such that $\pi([\mathfrak{k}, W]) = \{0\}$, then $\pi'_t([\mathfrak{k}, W]) = \{0\}$. Hence $W \in \mathcal{U}(\mathfrak{k}_j) \cap \ker \pi'_t$ by the induction hypothesis. In consequence, $W \in \mathcal{U}(\mathfrak{k}_j) \cap \ker \pi$.

Henceforth, we suppose $j = n$ and that the K_{d-1}-orbit passing a general $\ell \in \Omega$ is non-saturated relative to $\mathfrak{k}_{n-1}$. We may assume without loss of generality that $\mathfrak{k}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. When $\dim \mathfrak{z} \geq 2$, we may pass to the quotient Lie algebra of $\mathfrak{g}$ by $\mathfrak{z} \cap \ker \pi$ and apply the induction hypothesis. In this way, we are as usual brought to the following case: $\dim \mathfrak{z} = 1$, $\pi|_{\mathfrak{z}} \neq 0$, namely $\mathfrak{z} = \mathfrak{g}_1 = \mathbb{R}Z$, $\pi(Z) \neq 0$. Let $\mathfrak{g}_2 = \mathbb{R}Y \oplus \mathfrak{z}$. We denote by $\tilde{\mathfrak{g}}$ the centralizer of Y in $\mathfrak{g}$. Finally, taking $X \in \mathfrak{g} \setminus \tilde{\mathfrak{g}}$ such that $[X, Y] = Z$ we write $\mathfrak{g} = \mathbb{R}X + \tilde{\mathfrak{g}}$.

First, assume $\tilde{\mathfrak{g}} = \mathfrak{k}_{n-1}$. Provided $Y \in \mathfrak{k}$, the condition $\pi([Y, W]) = 0$ for an element W of $\mathcal{U}(\mathfrak{g})$ leads to $W \in \mathcal{U}(\tilde{\mathfrak{g}}) + \ker \pi$. In fact, taking a polarization $\mathfrak{b}$ at $f \in \Omega$ of $\mathfrak{g}$ contained in $\tilde{\mathfrak{g}}$, we realize

$$\pi \simeq \mathrm{ind}_B^G \chi_f \simeq \mathrm{ind}_{\tilde{G}}^G \left(\mathrm{ind}_B^{\tilde{G}} \chi_f \right).$$

Here, of course, $B = \exp \mathfrak{b}$, $\tilde{G} = \exp \tilde{\mathfrak{g}}$. Now, we write

$$W = \sum_{\gamma=0}^{m} a_\gamma X^\gamma,\ a_\gamma \in \mathcal{U}(\tilde{\mathfrak{g}}),\ 0 \leq \gamma \leq m, \tag{13.2.2}$$

with $\pi(a_m) \neq 0$. If $m \geq 1$, we have

$$\pi([Y, W]) = \sum_{\gamma=1}^{m} \gamma \pi(a_\gamma)\pi(Z)\pi(X)^{\gamma-1} = 0.$$

However, since $\pi(X)$ is nothing but the partial differential operator relative to the first coordinate introduced along X, this is a contradiction.

If $Y \notin \mathfrak{k}$, setting $\mathfrak{k}' = \mathfrak{k} \oplus \mathbb{R}Y = \mathfrak{k} + \mathfrak{g}_2$,

$$\mathfrak{k}' \cap \mathfrak{g}(\ell) \neq (\mathfrak{k}_{d-1} + \mathfrak{g}_2) \cap \mathfrak{g}(\ell)$$

at almost all $\ell \in \Omega$. Therefore, taking Theorem 13.1.1 into account, there exists $V = \sum_{j=0}^{r} P_j Y_d^j \in \ker \pi$, where $P_j \in \mathcal{U}(\mathfrak{k}_{d-1} + \mathfrak{g}_2)$ $(0 \leq j \leq r)$ with $\pi(P_r) \neq 0$. If we rewrite V as

$$V \equiv \sum_{k=0}^{q} b_k Y^k,\ b_k \in \mathcal{U}(\mathfrak{k})\ (0 \leq k \leq q) \tag{13.2.3}$$

modulo $\ker \pi$ with $\pi(b_q) \neq 0$, then $q \geq 1$. In fact, in the irreducible decomposition

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m(\sigma)\sigma d\nu(\sigma),$$

σ is written in the form $\chi_t \otimes \tilde{\sigma},\ t \in \mathbb{R}$, where χ_t is a unitary character of K trivial on K_{d-1} and where $\tilde{\sigma}$ is an irreducible unitary representation of K with irreducible restriction to K_{d-1}. Besides, there exists a measure ν' on $\widehat{K_{d-1}}$ so that the measure $d\nu(\sigma)$ is equivalent to the product measure $dt \times d\nu'(\tilde{\sigma})$. From this, if $q = 0$ or $V = \sum_{j=0}^{r} P_j Y_d^j \in \ker \pi,\ P_j \in \mathcal{U}(\mathfrak{k}_{d-1})$ $(0 \leq j \leq r)$, it follows that $\pi(P_r) = 0$. This is contradictory. We choose V in such a manner that $q \geq 1$ will be minimal in expression (13.2.3).

Lastly, if there exists $W \in \mathcal{U}(\mathfrak{g})$ of the form in expression (13.2.2) with $m \geq 1$ such that $\pi([\mathfrak{k}, W]) = \{0\}$, we compute

$$0 = \pi([W, V]) = \sum_{k=0}^{q} \pi(b_k)\pi\left(\left[W, Y^k\right]\right) = \sum_{k=0}^{q} \pi(b_k)\left(\sum_{\gamma=0}^{m} \pi(a_\gamma)\pi\left(\left[X^\gamma, Y^k\right]\right)\right)$$

$$= \left(m\pi(Z)\left(\sum_{k=1}^{q} k\pi(b_k)\pi(Y)^{k-1}\right)\pi(X)^{m-1}\right) + \sum_{j=0}^{m-2} \pi(c_j)\pi(X)^j$$

with $c_j \in \mathcal{U}(\tilde{\mathfrak{g}})$ $(0 \leq j \leq m-2)$. We derive from this that $V' = \sum_{k=1}^{q} kb_k Y^{k-1} \in \ker \pi$. But this contradicts the choice of V.

Secondly, let us examine the case where $\tilde{\mathfrak{g}} \neq \mathfrak{k}_{n-1}$. Suppose that $W \in \mathcal{U}(\mathfrak{g})$ satisfies $\pi([\mathfrak{k}, W]) = \{0\}$. Since $Y \in \mathfrak{k}_{n-1}$, the $(K \cap \tilde{G})$-orbits are already saturated with respect to $\mathfrak{k}_{n-1}$ almost everywhere in Ω. When $\mathfrak{k} \not\subset \tilde{\mathfrak{g}}$, W belongs to $\mathcal{U}(\mathfrak{k}_{n-1}) + \ker \pi$ by the induction hypothesis. Next assume $\mathfrak{k} \subset \tilde{\mathfrak{g}}$. If general K-orbits in Ω are saturated relative to $\tilde{\mathfrak{g}}$, it follows from what we have seen until now that $W \in \mathcal{U}(\tilde{\mathfrak{g}}) + \ker \pi$, and then from the induction hypothesis we conclude that $W \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1}) + \ker \pi$.

As the last possibility, let us study the case where general K-orbits in Ω are non-saturated with respect to $\tilde{\mathfrak{g}}$. This means that, at a general $\ell \in \Omega$, the orbit $K \cdot (\ell|_{\mathfrak{k}_{n-1}}) \subset \mathfrak{k}_{n-1}^*$ is non-saturated with respect to $\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1}$. Therefore, there exists $V = aX + b \in \mathcal{U}(\mathfrak{k}_{n-1})$ such that $\pi([\mathfrak{k}, V]) = \{0\}$. Here, $a,\ b \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1})$ and $\pi(a) \neq 0$. Now, we choose in $\mathfrak{k}_{n-1}$ the element $X \in \mathfrak{g} \setminus \tilde{\mathfrak{g}}$ introduced above. We write $W = \sum_{k=0}^{m} X^k c_k,\ c_k \in \mathcal{U}(\tilde{\mathfrak{g}})$ $(0 \leq k \leq m)$ and assume $m \geq 1$. Otherwise, it follows from the induction hypothesis applied to $\tilde{\mathfrak{g}}$ that $W \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1}) + \ker \pi$. Because

$$W' = [Y, W] = \sum_{k=1}^{m} kX^{k-1}Zc_k$$

also satisfies the condition $\pi([\mathfrak{k}, W']) = \{0\}$, the induction hypothesis on the degree of X in W implies $c_k \in \mathcal{U}(\mathfrak{k}_{n-1})$ $(1 \leq k \leq m)$ modulo $\ker \pi$. Finally, we write

$$V^m = a^m X^m + \sum_{j=0}^{m-1} X^j b_j$$

with $b_j \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1})$ $(0 \le j \le m-1)$. Taking $\pi([\mathfrak{k}, a]) = \{0\}$ into consideration,

$$W'' = a^m W - V^m c_m = \sum_{j=0}^{m-1} \left(a^m X^j c_j - X^j b_j c_m\right) = \sum_{j=0}^{m-1} X^j \tilde{c}_j$$

also satisfies $\pi([\mathfrak{k}, W'']) = \{0\}$. Here, $\tilde{c}_j$ $(0 \le j \le m-1)$ are all elements of $\mathcal{U}(\tilde{\mathfrak{g}})$. Since the degree of X in W'' is less than or equal to $m-1$, the induction hypothesis asserts that $\tilde{c}_j$ all belong to $\mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1})$ modulo $\ker \pi$. Because $\tilde{c}_0 = a^m c_0 + c'$ with a certain $c' \in \mathcal{U}(\tilde{\mathfrak{g}} \cap \mathfrak{k}_{n-1}) + \ker \pi$ and $\pi(a) \neq 0$, we conclude that $c_0 \in \mathcal{U}(\mathfrak{k}_{n-1}) + \ker \pi$. Finally, $W \in \mathcal{U}(\mathfrak{k}_{n-1}) + \ker \pi$ as expected. ■

We are now ready to prove the main theorem of this chapter, the commutativity theorem for restrictions of representations.

Theorem 13.2.7. *Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, $\pi \in \hat{G}$ and $K = \exp \mathfrak{k}$ an analytic subgroup of G. The algebra $D_\pi(G)^K$ is commutative if and only if the restriction $\pi|_K$ has finite multiplicities.*

Proof. We already showed in Theorem 13.2.1 that $D_\pi(G)^K$ is commutative if $\pi|_K$ has finite multiplicities. Here, we suppose that $\pi|_K$ has infinite multiplicities and show that $D_\pi(G)^K$ is non-commutative. Owing to Proposition 13.2.3, there exist two non-jump indices $j_1, j_2 \in T(e)$ $(j_1 < j_2)$ which are mutually dual for the alternating bilinear form B_ℓ at a general $\ell \in \Omega$. In order to show the non-commutativity in question, letting $d + 1 \le j$, we may replace K by K_j if $\pi|_{K_j}$ already has infinite multiplicities. So, the study is reduced to the situation where $\pi|_K$ has infinite multiplicities but $\pi|_{K_{d+1}}$ has finite multiplicities. From now on we assume this situation. Then, it follows that there exists only one pair of the non-jump indices $j_1 < j_2$ mentioned above and $j_1 = d+1$. In particular, $\mathfrak{k}$ contains the centre $\mathfrak{z}$ of $\mathfrak{g}$. By Theorem 13.2.6, there exist two elements $W_k = a_k X_{j_k} + b_k$ $(k = 1, 2)$ so that $\pi([\mathfrak{k}, W_k]) = \{0\}$, where $a_k, b_k \in \mathcal{U}(\mathfrak{k}_{j_k-1})$ and $\pi(a_k) \neq 0$. Especially $a_1, b_1 \in \mathcal{U}(\mathfrak{k})$. What we should show is $\pi([W_1, W_2]) \neq 0$.

By means of the Vergne polarization $\mathfrak{b}$ of $\mathfrak{g}$ constructed at $f \in \Omega$ from the Jordan–Hölder sequence given at the beginning, we realize $\pi \simeq \mathrm{ind}_B^G \chi_f$ in $L^2(\mathbb{R}^q)$ $(q = \dim(\mathfrak{g}/\mathfrak{b}))$ as a monomial representation induced from the corresponding analytic subgroup $B = \exp \mathfrak{b}$. The position of the index j_1 is found in B and that of the index j_2 in $\mathbb{R}^q$. Hence, $\pi([\mathfrak{k}, a_2]) = \{0\}$. At a general $\ell \in \Omega$, we denote by $\tilde{\ell}$ the restriction of ℓ to $\mathfrak{k}_{j_1}$ and by ℓ' that to $\mathfrak{k}_{j_1-1}$. Then $\dim(G \cdot \tilde{\ell}) = \dim(G \cdot \ell') + 1$. Besides, since the orbit $K_{j_1} \cdot \tilde{\ell}$ is non-saturated with respect to $\mathfrak{k}_{j_1-1}$, we see $\pi([\mathfrak{k}, a_1]) = \{0\}$ by changing the value $\tilde{\ell}(X_{j_1})$ independently of ℓ'.

Put as usual $\mathfrak{a} = \mathfrak{z} \cap \ker f$. If $\mathfrak{a} \neq \{0\}$, we may descend to the quotient Lie algebra $\mathfrak{g}/\mathfrak{a}$ and apply there the induction hypothesis. Assume $\mathfrak{a} = \{0\}$. Then, $\dim \mathfrak{z} = 1$, namely $\mathfrak{z} = \mathfrak{g}_1 = \mathfrak{k}_1$ and $f|_{\mathfrak{z}} \neq 0$. Setting $\mathfrak{g}_2 = \mathbb{R}Y + \mathfrak{z}$ and $\tilde{\mathfrak{g}}$ as the centralizer of $\mathfrak{g}_2$ in $\mathfrak{g}$, it follows that $\tilde{\mathfrak{g}}$ is an ideal of codimension 1 in $\mathfrak{g}$ containing the polarization $\mathfrak{b}$. Let $\mathfrak{g} = \mathbb{R}X \oplus \tilde{\mathfrak{g}}$.

Let us separate two cases. First, the case where $\mathfrak{g}_2 \subset \mathfrak{k}$. By Theorem 13.2.6, $W_1,\ W_2$ are included in $\mathcal{U}(\tilde{\mathfrak{g}})$ modulo $\ker\pi$, while the restriction of π to $\tilde{G} = \exp(\tilde{\mathfrak{g}})$ is decomposed into a one-parameter family of irreducible representations $\{\pi_t; t \in \mathbb{R}\}$ as

$$\pi|_{\tilde{G}} \simeq \int_{\mathbb{R}}^{\oplus} \pi_t dt.$$

Indeed, $\pi_t = \exp(tX)\cdot\pi_0$ with $\pi_0 = \mathrm{ind}_B^{\tilde{G}} \chi_f$. This means that, if $t_1 \neq t_2$, $\pi_{t_1}|_K$ and $\pi_{t_2}|_K$ are disjointly decomposed. Thus, $\pi_t|_K$ has infinite multiplicities for almost all $t \in \mathbb{R}$. So, the desired result is derived from the induction hypothesis applied to these π_t.

Secondly, the case where $\mathfrak{g}_2 \not\subset \mathfrak{k}$. In this case, $\mathfrak{k}_{j_1} = \mathfrak{k}_{d+1} = \mathfrak{k} + \mathfrak{g}_2$ and $W_1 = a_1 Y + b_1'$ with a new $b_1' \in \mathcal{U}(\mathfrak{k})$. Moreover, $\mathfrak{k}_{j_2-1} \subset \tilde{\mathfrak{g}}$ and $X_{j_2} \notin \tilde{\mathfrak{g}}$. Taking these into account, since $0 \neq [Y, X_{j_2}] \in \mathfrak{z}$ and $\pi|_{\mathfrak{z}} \neq 0$, we have

$$\begin{aligned}\pi\left([W_1, W_2]\right) &= \pi\left(\left[a_1 Y + b_1', a_2 X_{j_2} + b_2\right]\right) = \pi\left(\left[a_1 Y, a_2 X_{j_2} + b_2\right]\right)\\ &= \pi\left(a_1\left[Y, a_2 X_{j_2} + b_2\right]\right) = \pi(a_1)\pi(a_2)\pi\left(\left[Y.X_{j_2}\right]\right) \neq 0.\end{aligned}$$

■

13.3 Commutativity of the Centralizer of Lie Subalgebras

Keeping the notations, let $G = \exp\mathfrak{g}$ be a connected and simply connected nilpotent Lie group, $K = \exp\mathfrak{k}$ a connected closed subgroup of G and $p = p_{\mathfrak{k}} : \mathfrak{g}^* \to \mathfrak{k}^*$ the canonical projection. Let us study the commutativity of the centralizer $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ in $\mathcal{U}(\mathfrak{g})$. To begin with, we denote by $\mathcal{O}$ the set of the linear forms on $\mathfrak{g}$ whose coadjoint G-orbits have maximal dimension. $\mathcal{O}$ is a non-empty Zariski open set of $\mathfrak{g}^*$. For $\phi \in \mathcal{O}$, we set

$$d_K(\phi) = \max_{\ell\in G\cdot\phi} \dim(K\cdot\ell),\ \ d_K'(\phi) = \max_{\ell\in G\cdot\phi} \dim\left(K\cdot p_{\mathfrak{k}}(\ell)\right).$$

By definition, $d_K,\ d_K'$ are G-invariant functions on $\mathcal{O}$ taking non-negative integers as their values. Now, we put

$$d_K = \max_{\phi\in\mathcal{O}} d_K(\phi)\ \ d_K' = \max_{\phi\in\mathcal{O}} d_K'(\phi)$$

and

$$\mathcal{Z} = \left\{\phi \in \mathcal{O}; d_K(\phi) = d_K,\ d_K'(\phi) = d_K'\right\}.$$

$\mathcal{Z}$ is G-invariant and contains a non-empty Zariski open set $\tilde{\mathcal{Z}}$ with following properties. The dimensions of the G-orbits and the K-orbits of the elements of $\tilde{\mathcal{Z}}$ are maximal and the dimensions of K-coadjoint orbits of their projections onto $\mathfrak{k}^*$ are also maximal. We say that the irreducible unitary representations of G associated with the elements of $\tilde{\mathcal{Z}}$ are K-generic.

Let $\pi \in \hat{G}$ and $\ell \in p(\Omega(\pi)) \subset \mathfrak{k}^*$. We denote by ω the coadjoint orbit of K passing ℓ, by σ the irreducible unitary representation of K corresponding to ω and by $K(\ell)$ the stabilizer of ℓ in K. By Proposition 13.2.5, when and only when the $K(\ell)$-orbits in $\Omega(\pi) \cap p^{-1}(\ell)$ are generally open sets, the representation $\pi|_K$ has finite multiplicities. This is also equivalent to that the K-orbits in $\Omega(\pi) \cap p^{-1}(\omega)$ are generally open sets. In other words, when and only when

$$\dim(\Omega(\pi)) = 2\dim(K \cdot \phi) - \dim(K \cdot p(\phi)),\ \phi \in \Omega(\pi)$$

holds generally on $\Omega(\pi)$, the representation $\pi|_K$ has finite multiplicities. When the representation π is K-generic, $\pi|_K$ has finite multiplicities if and only if

$$\dim(\Omega(\pi)) = 2d_K - d'_K.$$

Remarking that $\dim(\Omega(\pi)) - 2d_K + d'_K$ is a non-negative invariant quantity on a Zariski open set $\tilde{\mathcal{Z}}$, we admit the following alternative: $\pi_\ell|_K$ has finite multiplicities for all $\ell \in \tilde{\mathcal{Z}}$ or $\pi_\ell|_K$ has generally infinite multiplicities on $\mathcal{Z}$.

Concerning the restriction to K of the coadjoint action of G, the maximal layer and $\mathcal{Z}$ intersect necessarily in a non-empty Zariski open set. Let $S(e),\ T(e)$ be the sets of corresponding jump indices and non-jump indices. For a finite-dimensional vector space V, $\mathbb{C}[V^*]$ denotes the algebra of all polynomial functions on the dual vector space V^*. If we represent by the superscript K the totality of K-invariant elements, it is well known that the symmetrization map gives an isomorphism between $\mathbb{C}[\mathfrak{g}^*]^K$ and $\mathcal{U}(\mathfrak{g})^K$ as vector spaces. Let us make use of this fact taking some $\mathfrak{k}_j$, $j \geq d+1$ as $\mathfrak{g}$.

When $j \in T(e)$, the general K-orbits in ${\mathfrak{k}_j}^*$ are non-saturated with respect to $\mathfrak{k}_{j-1}$ and there exists a K-invariant polynomial function on ${\mathfrak{k}_j}^*$, having the form $P = P_1 X_j + P_2$, where $P_1,\ P_2$ are elements of $\mathbb{C}[{\mathfrak{k}_{j-1}}^*]$ and P_1 is not identically 0. Thus, transferring by the symmetrization map, we could find an element of $\mathfrak{c}(\mathfrak{k}, \{0\}) = \mathfrak{c}(\mathfrak{k})$ having the form $W = aX_j + b$ with $a \neq 0,\ b$ in $\mathcal{U}(\mathfrak{k}_{j-1})$.

Next, let μ be the Plancherel measure for the regular representation of G and assume that $\pi|_K$ has finite multiplicities for almost all $\pi \in \hat{G}$ relative to μ. Then,

$$\mathfrak{c}(\mathfrak{k}) = \bigcap_{\pi \in \hat{G}} \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$$

is commutative.

Conversely, let $\mathcal{A}$ be the set of all $\pi \in \hat{G}$ such that $\pi|_K$ has infinite multiplicities. If $\mu(\mathcal{A}) \neq 0$, $\mathcal{A} \cap \hat{\rho}_G(\mathcal{Z}) \neq \emptyset$ and $\pi_\ell|_K$ has infinite multiplicities for almost all $\ell \in \mathcal{Z}$. When we consider a sequence of Lie subalgebras

$$\mathfrak{k} = \mathfrak{k}_d \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g},$$

there exists some index $j_0 \geq d$ such that, for almost all $\pi \in \hat{G}$ relative to μ, $\pi|_{K_{j_0}}$ has infinite multiplicities but $\pi|_{K_{j_0+1}}$ has finite multiplicities. Then, it suffices to show the non-commutativity of $\mathfrak{c}(\mathfrak{k}_{j_0})$. That is to say, we may replace $\mathfrak{k}$ by $\mathfrak{k}_{j_0}$. This means that, using the reasoning of Theorem 13.2.7 on the intersection of $\mathcal{Z}$ and another non-empty Zariski open set if necessary, there exists only one pair of non-jump indices $(j_1,\ j_2)$ which are mutually dual for the alternating bilinear forms at almost all $\ell \in \mathcal{Z}$. Then, there exist elements $W_k = a_k X_{j_k} + b_k$ $(k = 1,\ 2)$ of $\mathcal{U}(\mathfrak{k}_{j_k})^K$ such that $a_k,\ b_k$ are elements of $\mathcal{U}(\mathfrak{k}_{j_k-1})$ satisfying $\pi(a_k) \neq 0$ for at least some K-generic $\pi \in \hat{G}$. Further, W_k is an element of $\mathfrak{c}(\mathfrak{k})$ and also of $\mathcal{U}_\pi(\mathfrak{k}_k)^{\mathfrak{k}}$. Fixing such a K-generic irreducible representation π and following the reasoning in the proof of Theorem 13.2.7, we verify $\pi([W_1, W_2]) \neq 0$. Hence $\mathfrak{c}(\mathfrak{k})$ is not commutative. In this way, we obtain the following result.

Theorem 13.3.1. *Let G be a connected and simply connected nilpotent Lie group and K a connected closed subgroup of G. Then, the following assertions are equivalent to each other:*

(1) *$\pi|_K$ has finite multiplicities for almost all $\pi \in \hat{G}$.*
(2) *There exists a K-generic representation $\pi \in \hat{G}$ such that $\pi|_K$ has finite multiplicities.*
(3) *The centralizer of $\mathfrak{k}$ in $\mathcal{U}(\mathfrak{g})$ is commutative.*

Remark 13.3.2. The polynomial conjecture is also interpreted for restrictions of representations. However, the obtained conjecture is proved only in particular cases, for instance when K is a normal subgroup (cf. [7]).

Example 13.3.3 (Non-nilpotent Case). Let $\mathfrak{g}_\alpha = \langle A,\ X,\ Y\rangle_{\mathbb{R}} : [A, X] = X - \alpha Y,\ [A, Y] = \alpha X + Y$ $(0 \neq \alpha \in \mathbb{R})$. Then, $G_\alpha = \exp(\mathfrak{g}_\alpha)$ is an exponential solvable but not completely solvable Lie group. Let $\{A^*,\ X^*,\ Y^*\}$ be the dual basis of $\{A,\ X,\ Y\}$ in $\mathfrak{g}_\alpha^*$. The coadjoint orbits of G_α, which do not degenerate to one point, are those Ω_θ passing

$$f_\theta = (\cos\theta)X^* + (\sin\theta)Y^* \ (0 < \theta \leq 2\pi).$$

We denote by π_θ the irreducible unitary representation of G_α associated with f_θ. Then,

$$\Omega_\theta = \left\{sA^* + e^{-t}\cos(\theta - \alpha t)X^* + e^{-t}\sin(\theta - \alpha t)Y^*; s,\ t \in \mathbb{R}\right\}.$$

The polarization $\mathfrak{b} = \mathbb{R}X \oplus \mathbb{R}Y$ at f_θ satisfies the Pukanszky condition. Through this realization, the representation π_θ acts on $L^2(\mathbb{R})$ and we get a self-adjoint operator

$$i\pi(A) = -i\frac{d}{dt},\ i\pi(X) = -e^{-t}\cos(\theta - \alpha t),\ i\pi(Y) = -e^{-t}\sin(\theta - \alpha t).$$

(1) Take $\mathfrak{k} = \mathbb{R}X \oplus \mathbb{R}Y$. Then,

$$\pi_\theta|_K \simeq \int_{\mathbb{R}}^{\oplus} \chi_{\exp(tA)\cdot f_\theta}\, dt \simeq \int_{\mathbb{R}}^{\oplus} \chi_{e^{-t}\cos(\theta-\alpha t)X^* + e^{-t}\sin(\theta-\alpha t)Y^*}\, dt,$$

while $\mathcal{U}_{\pi_\theta}(\mathfrak{g})^{\mathfrak{k}}$ is nothing but the centralizer of $\mathfrak{k}$ in $\mathcal{U}(\mathfrak{g})$ and $D_\pi(G)^K \cong \mathcal{U}(\mathfrak{g})^K$ is commutative. In fact, if

$$W = \sum_{j=0}^{m} A^j C_j,\quad C_j \in \mathcal{U}(\mathfrak{k})$$

is an element of $\mathcal{U}(\mathfrak{g})^K$,

$$0 = [W, X + iY] = m(1 + i\alpha)A^{m-1}C_m + P,$$

where the degree with respect to A of the element P of $\mathcal{U}(\mathfrak{g})$ is less than or equal to $m - 2$. Hence it follows that $m = 0$ and $W \in \mathcal{U}(\mathfrak{k})$. In consequence,

$$D_\pi(G)^K \cong \mathbb{C}[X, Y] \cong \mathbb{C}\left[e^{-t}\cos(\theta - \alpha t), e^{-t}\sin(\theta - \alpha t)\right].$$

(2) Next take $\mathfrak{k} = \mathbb{R}X$. In this case, $\pi_\theta|_K$ has infinite multiplicities but $D_\pi(G)^K \cong \mathbb{C}[X, Y]$ is commutative. As this example shows, Theorems 13.2.7, 13.3.1 no longer hold for exponential solvable Lie groups.

Remark 13.3.4. As we easily become aware, at present many of the detailed analyses are possible only for nilpotent Lie groups, Even if we limit our study to exponential solvable Lie groups, there remains much to be clarified. In order to open a view on prospective developments, it may be instructive to deal with completely solvable Lie groups whose Lie algebras are normal j-algebras in the Pjatetskii–Shapiro sense, when we think about their rich algebraic structures.

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List of Notations

Chapter 1.

Section 1.1. G, H (Lie group), $\mathfrak{g}$, $\mathfrak{h}$ (Lie algebra), $[X, Y]$ (Lie bracket), exp (exponential map), Ad (adjoint representation of Lie group), ad (adjoint representation of Lie algebra), $D^k\mathfrak{g}$ (derived sequence), $C^k\mathfrak{g}$ (descending central series), $\mathfrak{g}_{\mathbb{C}}$ (complexification), $\sigma_T(\pi)$, $\sigma_T(\mathrm{ad})$, $CB(X, Y)$
Section 1.2. $\mathcal{U}(\mathfrak{g})$ (universal enveloping algebra), $\mathcal{U}_q(\mathfrak{g})$, $\diamondsuit$ (principal anti-isomorphism), $S(\mathfrak{g})$ (symmetric algebra)
Section 1.3. $\|A\|_{\mathrm{HS}}$ (Hilbert–Schmidt norm), $\mathrm{Tr}(A)$ (trace), $\mathcal{H}_\pi$, $\pi \simeq \rho$ (equivalence), $\hat{G}$ (unitary dual), $\sum_i \pi_i$ (direct sum), $\int_X \pi_x d\mu(x)$ (direct integral)

Chapter 2.

Section 2.1. $C_c(G)$, $\|f\|_\infty$ (uniform norm), $\mathcal{V}$, $\mathrm{supp}(f)$ (support), $\lambda(s)$ (left translation), $f * g$ (convolution), $\Delta_G(g)$ (modular function)
Section 2.2. $L^p(G)$, $\rho(s)$ (right translation)
Section 2.3. $B(X)$, $\pi(f)$, $BL_G(\mathcal{H}, \mathcal{H}')$, $C^*(G)$ (C^*-algebra), $\|\cdot\|_{\mathrm{op}}$ (operator norm)

Chapter 3.

Section 3.1. $\Delta_{G,H}(h)$, $\mathcal{E}(G/H)$, $R_{G/H}$, $Q_{G/H}$, $\oint_{G/H} k(x) d\mu_{G,H}(x)$
Section 3.2. $\mathcal{E}(G/H, \rho)$, $L^2(G/H, \rho)$, $\tau_{\rho,H} = \mathrm{ind}_H^G \rho$ (induced representation), $CB(G/H)$, $M_\rho(\phi)$
Section 3.3. π^a
Section 3.4. $\pi|_C$ (restriction)

Chapter 4.

Section 4.1. $\mathbb{R}^n$ (abelian Lie group), $\hat{\varphi}$ (Fourier transform), $\mathcal{S}_n$
Section 4.2. H_n (Heisenberg group)

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Section 4.3. $\tau_{\pm}$
Section 4.4. G_θ (Grélaud group), $\mathfrak{g}_\theta$

Chapter 5.

Section 5.1. Ad^* (coadjoint representation), $G(f)$ (stabilizer), $\mathfrak{g}(f)$, B_f, $\mathfrak{a}^{\perp,\mathfrak{g}^*}$, $\mathfrak{a}^\perp$ (annihilator), $\mathfrak{a}^f$, $S(f,\mathfrak{g})$, $M(f,\mathfrak{g})$, $P(f,G)$, $\mathfrak{d}$, $\mathfrak{e}$, J (complex structure), S_f, $P^+(f,G)$, χ_f (unitary character), $\rho(f,\eta_f,\mathfrak{p},G)$ (holomorphically induced representation), $\mathcal{H}(f,\eta_f,\mathfrak{p},G)$
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Section 5.3. $\mathrm{pker}(\pi)$, $\hat{\rho}(f,\mathfrak{h},G)$, $\hat{\mathcal{H}}(f,\mathfrak{h},G)$, $\mathfrak{z}$ (centre), $\mathfrak{a}$ (minimal non-central ideal), $I(f,\mathfrak{g})$, $m(f,\mathfrak{g})$, $\mathcal{F}$ (Fourier transformation), $\hat{\rho} = \hat{\rho}_G$ (Kirillov–Bernat mapping), $\gamma(\mathfrak{g})$ (pedestal)
Section 5.4. $T_{\mathfrak{h}_2\mathfrak{h}_1}$ (formal intertwining operator), τ_{ijk} (Maslov index)

Chapter 6.

Section 6.1. $\mathcal{Z}$ (Jordan–Hölder basis), I^ℓ (index set), $d\mu_O$ (invariant measure), $\tilde{E}_\mathcal{X}$, $E_\mathcal{Y}$, $E_{\mathcal{Y},H}$, $E_{G/H}$, $\pi_{\ell,\mathfrak{h}}$, $\rho_{\ell,\mathfrak{h}}$
Section 6.2. $\mathcal{PD}(V)$, X^α, D^α, $\mathcal{S}(V)$, $w(x)$, $\omega(x)$, $\mathcal{H}^\infty$, $\mathcal{S}(G/H,\chi)$, $\mathcal{SK}(G/P,\chi_\ell)$
Section 6.4. $d\nu_O$ (invariant measure)
Section 6.5. Q_I, $I_{\min}$, $\mathfrak{g}^*_{\min}$, $\mathfrak{v}^*_{\mathrm{gen}}$, ρ_j $(j \notin I_{\min})$

Chapter 7.

Section 7.2. $(\mathfrak{g},j,\beta)$ (exponential Kähler algebra), $(\mathfrak{g},j,\omega)$ (exponential j-algebra), $\eta^{\frac{1}{2}(\alpha_m-\alpha_k)}$, $\eta^{\frac{1}{2}\alpha_m}$, $\eta^{\frac{1}{2}(\alpha_m+\alpha_k)}$
Section 7.3. $(abc)_x$
Section 7.5. $D(\Omega,Q)$ (Siegel domain), $H(D,\Psi,\Phi)$
Section 7.7. $U(f,\mathfrak{h})$, $c(\sigma,f,\mathfrak{h})$, $V(f,\mathfrak{h})$

Chapter 8.

Section 8.1. τ (monomial representation), Γ_τ (affine space), $m(\pi)$, $n_\pi(\sigma)$ (multiplicity)
Section 8.2. $\pi|_K$ (restriction), $p(\mathfrak{g},\mathfrak{k})$ (projection), $\pi_1 \otimes \pi_2$ (tensor product)

Chapter 9.

Section 9.1. $e_j(\ell)$, $e(\ell)$ (dimension index), $\mathcal{E}$, U_e (layer), $S(e)$ (jump index set), $T(e)$ (non-jump index set), $D_\tau(G/H)$ (algebra of invariant differential operators), $\mathfrak{a}_\tau$ (kernel), $\mathcal{U}(\mathfrak{g},\tau)$, $\mathbb{C}[\Gamma_\tau]^H$
Section 9.2. $M(\mathfrak{g}_\mathbb{C},\mathfrak{b},\lambda)$, $\rho(\lambda,\mathfrak{b},\mathfrak{g}_\mathbb{C})$, $\mathrm{Gr}(\mathfrak{g}_\mathbb{C},r)$ (Grassmannian manifold), $\mathrm{Gr}(\mathfrak{g}_\mathbb{C})$, $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$, $I(\lambda)$, $\ker\rho$, $\mathbb{C}[X_1,\dots,X_p]$ (polynomial ring), $T(e_H)$, $U(e)$, $Z(\mathfrak{g},\Gamma_\tau)$

Chapter 10.

Section 10.1. $\mathcal{H}_\pi^{-\infty}$, $\left(\mathcal{H}_\pi^{-\infty}\right)^{K,\chi}$
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Chapter 11.

Section 11.1. δ_τ (Dirac distribution), ϕ_H^f
Section 11.2. $U(f, \mathfrak{h})$, $c(\Omega, f, \mathfrak{h}, \mathfrak{g})$
Section 11.4. (μ, U), ν_Ω

Chapter 12.

Section 12.1. $\{A, B\}$, $\mathcal{T}_r(A)$
Section 12.2. $\varphi_A(\ell)$, $CD_\tau(G/H)$, $\mathcal{U}_C(\mathfrak{g}, \tau)$, $\mathcal{O}(P, V, K, \mathcal{L})$, $S(e(P, V, K, \mathcal{L}))$, $T(e(P, V, K, \mathcal{L}))$

Chapter 13.

Section 13.1. $\mathfrak{c}(\mathcal{M}, \mathcal{N})$, $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, $D_\pi(G)^K$
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