Euler Systems

Euler Systems $\leftrightarrow$ p-adic L-functions $\leftrightarrow$ BSD application

Reference: Darmon et al. " p-adic L-functions and Euler systems: a tale in two trilogies"

"Def" (Informal)

An Euler System is a collection of global arithmetic objects, most notably global cohomology classes arising from geometry, which are related to L-functions an can be made to vary into p-adic families.

~~V projective alg. var $/\mathbb{Q}$~~

$\kappa \in H^1(\mathbb{Q}, V)$

$$L(V,s) \longleftrightarrow \mathcal{L}_p(V,s) \leftrightarrow$$

$s \in \mathbb{C}_p$ $\qquad$ $s \in \mathbb{Z}_p$

global analytic properties of $L(V,s)$ near special pts

$\Downarrow$

global arithmetic properties of $V(\mathbb{Q})$

Euler System	p-adic L-function	p-adic formula	BSD applicat.
Cyclotomic units	Kubota-Leopoldt	Leopoldt thm	None
Elliptic units	Katz two-variable p-adic L-fun	Katz's p-adic Kronecker limit formula	Coates-Wiles $E/\mathbb{Q}$ is a CM elliptic curve w/ CM by K $L(E,1)\neq 0 \Rightarrow A(K)\otimes\mathbb{Q}=\{0\}$
Heegner pts	Anticyclotomic p-adic L-function	p-adic Gross-Zagier thm	Gross-Zagier; Kolyvagin $E/\mathbb{Q}$ elliptic curve $r\leq 1$ $\mathrm{ord}_{s=1} L(E,s)=r$ $\Downarrow$ $\mathrm{rk}\, E(\mathbb{Q})=r$

§1. Cyclotomic Units

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ a primitive non-trivial even Dirichlet character of conductor N

$k\geq 2$ even number

<u>Fact</u>: $L(1-k,\chi) \in \mathbb{Q}_\chi \subset \overline{\mathbb{Q}}$

field generated by the values of χ

<u>Proof</u>: $L(1-k,\chi)$ is the constant term of the holom. Eisenstein series

$$E_{k,\chi}(q) := L(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n)\, q^n$$

$$\sigma_{k-1,\chi}(n) := \sum_{d \mid n} \chi(d)\, d^{k-1}$$

of weight k, level N and character χ.

q-expansion principle $\Rightarrow$ the constant term inherits the rationality properties of the coefficients $\sigma_{k-1,\chi}(n)$

<u>note</u>: $\mathcal{M}_k(N;\mathbb{C}) = \mathcal{M}_k(N;\mathbb{Q}_\chi) \otimes_{\mathbb{Q}_\chi} \mathbb{C}$

$\square$

p = prime number

the ordinary p-stabilisation:

$$E_{k,\chi}^{(p)}(q) = E_{k,\chi} - \chi(p)p^{k-1}E_{k,\chi}(q^p)$$

has Fourier expansion given by

$$E_{k,\chi}^{(p)}(q) = L_p(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n)\, q^n$$

$$L_p(1-k,\chi) = (1-\chi(p)p^{k-1})\,L(1-k,\chi), \quad \sigma_{k-1,\chi}^{(p)}(n) = \sum_{\substack{d \mid n \\ p \nmid d}} \chi(d)\, d^{k-1}$$

$\forall n \geq 1$ $\quad k \longmapsto \sigma_{k-1,\chi}^{(p)}(n)$

extends to a p-adic analytic function of

$k \in \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$

Serre $\Rightarrow$ the constant term must inherit the same property

- $E_{k,\chi}^{(p)}$ is a prototypical example of a p-adic family of modular forms of weight k and level N_0 (the prime-to-p part of N).

- When $k=0$

$$E_0^{(p)}(q) = L_p(1,\chi) + 2\sum_{n=1}^{\infty}\Big(\sum_{\substack{d\mid n \\ p\nmid d}}\chi(d)d^{-1}\Big)q^n$$

is a rigid analytic function on the <u>ordinary locus</u> $\mathcal{A} \subset X_1(N_0)(\mathbb{C}_p)$ obtained by deleting from $X_1(N_0)(\mathbb{C}_p)$ all the residue disks attached to supersingular elliptic curves in char p.

- <u>Independent expression for $E_0^{(p)}(q)$ via Siegel units</u>

$g_a \in \mathcal{O}_{Y_1(N)}^{\times}$ $\quad\quad$ ζ - fixed choice of primitive N-th root of unity

$1 \le a \le N-1$

$$g_a(q) = q^{1/12}(1-\zeta^a)\prod_{n>0}(1-q^n\zeta^a)(1-q^n\zeta^{-a})$$

<u>Fact</u>: $g_a(q)^{(p)} := g_{pa}(q^p)\, g_a(q)^{-p}$ maps the ordinary locus $\mathcal{A}$ to the residue disk of 1 in $\mathbb{C}_p$

$\log_p g_a^{(p)}$ is a rigid analytic function on $\mathcal{A}$ w/ q-expansion

$$\log_p g_a^{(p)} = \log_p\left(\frac{1-\zeta^{ap}}{(1-\zeta^a)^p}\right) + p\sum_{n=1}^{\infty}\left(\sum_{\substack{d|n \\ p\nmid d}} \frac{\zeta^{ad}+\zeta^{-ad}}{d}\right) q^n$$

Set

$$\mathfrak{g}(\chi) = \sum_{a=1}^{N-1} \chi(a)\zeta^a \quad \text{Gauss sum attached to } \chi$$

Fact: (by Direct Computation)

$$h_\chi^{(p)} := \frac{1}{p\,\mathfrak{g}(\chi^{-1})} \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p g_a^{(p)}$$

has q expansion

$$h_\chi^{(p)}(q) = -\frac{1-\chi(p)}{\mathfrak{g}(\chi^{-1})} \sum_{a=1}^{N-1} \chi^{-1}(a)\log_p(1-\zeta^a) + \\ + 2\sum_{n=1}^{\infty}\left(\sum_{\substack{d|n \\ p\nmid d}} \chi(d) d^{-1}\right) q^n$$

Theorem (Leopoldt) Let χ be a non-trivial even primitive Dirichlet character of conductor N. Then

$$L_p(1,\chi) = -\frac{1-\chi(p)}{\mathfrak{g}(\chi^{-1})} \sum_{a=1}^{N-1} \chi^{-1}(a)\log_p(1-\zeta^a)$$

Proof (Sketch)

$E_{0,\chi}^{(p)} - h_\chi^{(p)}$ is constant on the residue disc of a cusp

$\Downarrow$

constant on $\mathcal{A}$ b/c it's rigid analytic on this domain

$\Downarrow$

$E_{0,\chi}^{(p)} - h_\chi^{(p)} = 0$ b/c of Nebentype $\chi \neq 1$ $\square$

expressions $\left\{ \begin{array}{ll} 1-\zeta^a, & \text{if } N \text{ composite} \\ & \\ \dfrac{1-\zeta^a}{1-\zeta^b}, & \text{when } N \text{ is prime} \end{array} \right\}$ are called cyclotomic units

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F_χ - field "cut out" by χ viewed as a Galois char

$$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$$

$$F_\chi = \mathbb{Q}(\zeta)^{\ker \chi}$$

$$\mathbb{Z}_\chi = \mathbb{Z}[\chi]$$

$$u_\chi := \underbrace{\prod_{a=1}^{N-1} (1-\zeta^a)^{\chi^{-1}(a)}}_{\text{distinguished unit in } F_\chi} \in \underbrace{(\mathcal{O}_{F_\chi}^\times \otimes \mathbb{Z}_\chi)^\chi}_{\chi\text{-eigenspace of the action of } G_{\mathbb{Q}}}$$

$G_{\mathbb{Q}}$-fixed

U_χ is essentially a "universal norm" over the tower

$$\begin{array}{c} \vdots \\ | \\ F_{\chi,n} = F_\chi(\mu_{p^n}) \\ | \\ \vdots \end{array}$$

Fixing a $(\zeta = \zeta_N, \zeta_{Np}, \zeta_{Np^2}, \ldots, \zeta_{Np^n}, \ldots)$ sequence of primitive Np^n-th roots of unity which are compatible under p-power maps and setting

$$u_{\chi,n} = \prod_{i \geq 1}^{N-1} \left(1 - \zeta_{Np^n}^{a}\right)^{\chi^{-1}(a)} \in \left(\mathcal{O}_{F_{\chi,n}}^{\times} \otimes \mathbb{Z}_\chi\right)^{\chi}$$

We find that

$$\mathrm{Norm}_{F_{\chi,n}}^{F_{\chi,n+1}}(u_{\chi,n+1}) = \begin{cases} u_{\chi,n}, & n \geq 1 \\ \\ u_\chi \otimes \left(1 - \chi^{-1}(p)\right), & n = 0 \end{cases}$$

$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ view χ as a $\mathbb{C}_p$-valued character

$\mathbb{Z}_{p,\chi} = \mathbb{Z}_p[\chi]$ trivial $G_{\mathbb{Q}}$ action

$\mathbb{Z}_{p,\chi}(\chi) \circlearrowleft G_{\mathbb{Q}}$ via χ

$\mathbb{Q}_{p,\chi}$, $\mathbb{Q}_{p,\chi}(\chi)$ analogously defined

$$K_{\chi,n} := \delta u_{\chi,n} \in H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^{\chi} =$$

$$= H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

$$\delta : (F_{\chi,n}^{\times} \otimes \mathbb{Z}_\chi)^{\chi} \longrightarrow H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^{\chi}$$

connecting hom of Kummer theory

$$K_{\chi,\infty} := (K_{\chi,n})_{n \geq 0}$$

$$\cap$$

$$\varprojlim_n H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1})) =$$

$$\overset{\text{Shapiro Lemma}}{=} \varprojlim_n H^1(\mathbb{Q}, \mathbb{Z}_p[G_n] \otimes \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

$$= H^1(\mathbb{Q}, \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

$\mathbb{Z}_p[G_n]$ is the group ring of $G_n = \mathrm{Gal}(F_n/\mathbb{Q}) = (\mathbb{Z}/p^n\mathbb{Z})^{\times}$

$$\Lambda_{cyc} = \varprojlim_n \mathbb{Z}_p[G_n] = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$$

"tautological" action of $G_{\mathbb{Q}}$ in which $\sigma \in G_{\mathbb{Q}}$ acts via multiplication by its image in G_n

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Λ_{cyc} can be viewed as a p-adic interpolation of the Tate twists $\mathbb{Z}_p(k)$ for all $k \in \mathbb{Z}$,

$k \in \mathbb{Z}$,

ξ Dirichlet char of p-power conductor

$$\nu_{k,\xi} : \Lambda \longrightarrow \mathbb{Z}_{p,\xi}$$

$$a \in \mathbb{Z}_p^\times \longmapsto a^{k-1}\xi^{-1}(a)$$

$G_{\mathbb{Q}}$ equivariant

$$\nu_{k,\xi} : \Lambda \longrightarrow \mathbb{Q}_{p,\xi}(k-1)(\xi^{-1})$$

$$\kappa_{k,\chi\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \in H^1(\mathbb{Q}, \mathbb{Q}_{p,\chi\xi}(k)((\chi\xi)^{-1}))$$

"arithmetic specialisations"

The Bloch-Kato exponential map

$K/\mathbb{Q}_p$ finite extension

fundamental exact sequence

$$0 \to \mathbb{Q}_p \longrightarrow \mathbb{B}_{cris}^{\varphi=1} \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to 0$$

tensoring w/ V and take G_K-invariance

$$0 \to V^{G_K} \to \mathbb{D}_{cris}^{\varphi=1} \to \left(\left(\mathbb{B}_{dR}/\mathbb{B}_{dR}^+\right) \otimes V\right)^{G_K} \to H^1_e(K,V) \to 0$$

where

$$H^1_e(K,V) = \ker\left(H^1(K,V) \to H^1(K, \mathbb{B}_{cris}^{\varphi=1} \otimes V)\right)$$

We have isomorphism

$$\exp_{K,V} : \frac{\mathbb{D}_{dR}(V)}{\mathrm{Fil}^0 \mathbb{D}_{dR}(V) + \mathbb{D}_{cris}(V)^{\varphi=1}} \to H^1_e(K,V) \subset H^1(K,V)$$

and its inverse

$$\log_{K,V} : H^1_e(K,V) \longrightarrow \frac{\mathbb{D}_{dR}(V)}{\mathrm{Fil}^0 \mathbb{D}_{dR}(V) + \mathbb{D}_{cris}(V)^{\varphi=1}}$$

Suppose now that V is a deRham rep of G_K

$$B_{dR} \otimes_K V \cong B_{dR} \otimes_K D_{dR}(V)$$

and

$$H^1(K, B_{dR} \otimes_{\mathbb{Q}_p} V) = H^1(K, B_{dR} \otimes_K D_{dR}(V))$$

$$= H^1(K, B_{dR}) \otimes_K D_{dR}(V)$$

$$\begin{array}{ccc} D_{dR}(V) & \xrightarrow{\sim} & H^1(K, B_{dR} \otimes V) \\ x & \longmapsto & (\tau \longmapsto \underset{\substack{\uparrow \\ \text{the cup product}}}{x \cup} \log \chi_{cyc}(\tau)) \end{array}$$

cocycle

$H^0(K, \mathbb{C}_p) = K$ $H^1(K, \mathbb{C}_p)$ is a 1-dim K-vsp.

$\log \chi$

$H^1(K, \mathbb{Q}_p)$

$H^0(K, \mathbb{C}_p) \simeq H^1(K, \mathbb{C}_p)$

$x \mapsto x \cup \log \chi$

<u>Def</u>:

$$\begin{tikzcd} \exp^*_{K,V} : H^1(K,V) \arrow[r] \arrow[dr] & D_{dR}(V) \arrow[d, shift left] \\ & H^1(K, B_{dR} \otimes V) \arrow[u, shift left, "\sim"] \end{tikzcd}$$

$\log[\varepsilon] = \sum_{n\geq 1}^{+\infty} \frac{(-1)^{n-1}}{n} \pi^n$ converges in B_{dR}^{+} to an element t

which is a generator of $\ker\theta$

with a $G_{\mathbb{Q}_p}$ action

$$\sigma(t) = \chi_{cyc}(\sigma)t$$

$$B_{dR} = B_{dR}^{+}[t^{-1}]$$

$$\mathrm{Fil}^i B_{dR} = t^i B_{dR}^{+}$$

(G_K-stable)

$D_{dR}(V)$ has induced filtration

- $\underline{\underline{\forall k \geq 1}}$ η Dirichlet char of conductor prime to p

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$$\log_{k,\eta} : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(k)(\eta)) \longrightarrow D_{dR}(\mathbb{Q}_{p,\eta}(k)(\eta))$$

- we used $\mathrm{Fil}^0 D_{dR}(\mathbb{Q}_{p,\eta}(k)(\eta)) = 0$ and all extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_{p,\eta}(k)(\eta)$ are cristalline

- target is a 1-dim $\mathbb{Q}_{p,\eta}$-vector space with a canonical gen $t^{-k}\mathfrak{g}(\eta)$

- $\underline{\underline{k \geq 1}}$

$$L_p(k, \chi) = \frac{1-\chi(p)p^{-k}}{1-\chi^{-1}(p)p^{k-1}} \times \frac{(-t)^k}{(k-1)!\,\mathfrak{g}(\chi^{-1})} \times \log_{k,\chi}(\kappa_{k,\chi})$$

- $k \leq 0$

$$\exp^*_{k,\eta} : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(k)(\eta)) \to D_{dR}(\mathbb{Q}_{p,\eta}(k)(\eta))$$

$$L_p(k,\chi) = \frac{1-\chi(p)p^{-k}}{1-\chi^{-1}(p)p^{k-1}} \times \frac{(-k)!\, t^k}{g(\chi)} \exp^*_{k,\chi}(\kappa_{k,\chi})$$

reciprocity law

$$L_p(k,\chi) \neq 0 \implies \kappa_{k,\chi} \text{ non-trivial}$$