FOUNDATIONS OF DYADIC DIFFERENTIAL TOPOLOGY IN THE SENSE OF BOTT AND TU

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Abstract. We initiate the formal development of dyadic differential topology from first principles, in the sense of Bott and Tu, by introducing dyadic structures on smooth manifolds, the dyadic analogue of differential forms, and integration theory over dyadic chains. This foundational framework builds on *dyadic topological analysis* and provides the basis for a generalized de Rham theory in dyadic settings.

Contents

1. Introduction

Dyadic topology, as outlined in *Foundations of Dyadic Topological Analysis*, provides a multiscalar and locally binary decomposition of topological data. However, a rigorous differential-topological framework analogous to Bott and Tu's differential forms is absent. This work defines

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and studies dyadic analogues of smooth manifolds, tangent bundles, differential forms, exterior derivatives, and cohomology theories in a way that remains compatible with classical smooth theory.

2. Dyadic Smooth Structures

Definition 2.1 (Dyadic Manifold)**.** A *dyadic manifold* is a topological space M equipped with an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that each chart $\phi_\alpha : U_\alpha \to \mathbb{D}^n$ maps homeomorphically to an open subset of the *dyadic space* $\mathbb{D}^n$, where $\mathbb{D} = \bigcup_{k\in\mathbb{N}} 2^{-k}\mathbb{Z}$ is the dyadic rationals.

Remark 2.2. $\mathbb{D}$ is a dense subring of $\mathbb{Q}$, and its topology is induced from $\mathbb{R}$. Thus, charts map into a totally disconnected, yet dense, setting—encoding fine multiscale information.

Definition 2.3 (Dyadic Smooth Function)**.** A function $f : M \to \mathbb{R}$ is *dyadically smooth* if, in every dyadic chart ϕ_α, the pullback $f \circ \phi_\alpha^{-1}$ can be approximated arbitrarily closely (in uniform norm) by piecewise-linear functions with dyadic coefficients and slopes.

3. Dyadic Tangent Spaces and Forms

Definition 3.1 (Dyadic Tangent Vector)**.** At a point $p \in M$, the *dyadic tangent space* $T_p^{\mathbb{D}} M$ is the set of derivations acting on dyadically smooth functions at p, i.e., $\mathbb{D}$-linear maps $v : C^\infty_{\mathbb{D}}(M) \to \mathbb{D}$ satisfying the Leibniz rule.

Definition 3.2 (Dyadic Differential Forms)**.** A dyadic k-form ω on M assigns to each $p \in M$ an alternating multilinear map

$$\omega_p : \underbrace{T_p^{\mathbb{D}} M \times \cdots \times T_p^{\mathbb{D}} M}_{k \text{ times}} \to \mathbb{D}$$

that varies dyadically smoothly with p.

4. Dyadic Exterior Derivative and Cohomology

Definition 4.1 (Dyadic Exterior Derivative)**.** The operator $d : \Omega^k_{\mathbb{D}}(M) \to \Omega^{k+1}_{\mathbb{D}}(M)$ is defined using the dyadic analogue of Cartan's formula and satisfies $d \circ d = 0$.

Theorem 4.2 (Dyadic de Rham Complex)**.** *The sequence*

$$0 \longrightarrow \Omega^0_{\mathbb{D}}(M) \xrightarrow{d} \Omega^1_{\mathbb{D}}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathbb{D}}(M) \longrightarrow 0$$

is a complex, and its cohomology groups $H^k_{\mathrm{dR},\mathbb{D}}(M)$ generalize the classical de Rham cohomology in a dyadic setting.

5. Integration on Dyadic Chains

Definition 5.1 (Dyadic Singular Chains)**.** A *dyadic singular k-chain* is a formal linear combination of maps $\sigma : \Delta^k_{\mathbb{D}} \to M$, where $\Delta^k_{\mathbb{D}}$ denotes the standard dyadic k-simplex.

Definition 5.2 (Dyadic Integration)**.** The integral of a dyadic k-form ω over a dyadic k-chain $c = \sum_i a_i \sigma_i$ is defined as

$$\int_c \omega := \sum_i a_i \int_{\sigma_i} \omega,$$

using dyadic Riemann integration over simplices with dyadic vertices.

Theorem 5.3 (Dyadic Stokes' Theorem)**.** *Let* $\omega \in \Omega_{\mathbb{D}}^{k-1}(M)$ *and* c *a dyadic* k*-chain. Then:*

$$\int_c d\omega = \int_{\partial c} \omega.$$

6. Future Directions

This initial foundation opens the door to:

- Dyadic characteristic classes;
- Dyadic Morse theory;
- Dyadic Hodge theory;
- Comparison with p-adic and tropical geometries.

7. Dyadic Vector Bundles

Definition 7.1 (Dyadic Vector Bundle)**.** A *dyadic vector bundle* of rank k over a dyadic manifold M is a topological space E together with a continuous surjection $\pi : E \to M$ such that for each $p \in M$, there exists a neighborhood U of p and a homeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{D}^k$$

making the diagram commute and such that the induced maps on fibers are dyadic-linear.

Example 7.2. The dyadic tangent bundle $T^{\mathbb{D}}M$ is a canonical example of a dyadic vector bundle.

8. Dyadic Connections

Definition 8.1 (Dyadic Connection)**.** Let $E \to M$ be a dyadic vector bundle. A *dyadic connection* ∇ is a $\mathbb{D}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes_{\mathbb{D}} E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all dyadically smooth functions f and dyadic sections s.

Remark 8.2. The dyadic exterior derivative d acts on coefficients in $\mathbb{D}$, and hence ∇ encodes dyadic variation across fibers.

9. Dyadic Curvature and Characteristic Classes

Definition 9.1 (Dyadic Curvature)**.** The curvature R^∇ of a dyadic connection ∇ is defined by the composition

$$R^\nabla = \nabla^2 : \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E).$$

Theorem 9.2. *The dyadic curvature* R^∇ *measures the failure of second covariant derivatives to commute in the dyadic differential framework.*

Definition 9.3 (Dyadic Chern Class)**.** Let E be a rank k dyadic vector bundle with connection ∇. The dyadic total Chern class is defined as

$$c^{\mathbb{D}}(E) = \det\left(I + \frac{i}{2\pi} R^\nabla\right)$$

where the determinant is taken over the dyadic algebra of endomorphisms.

10. Dyadic de Rham–Chern–Weil Theory

Theorem 10.1 (Dyadic Chern–Weil). *The dyadic characteristic classes $c_i^{\mathbb{D}}(E)$ derived from curvature forms R^∇ are closed under dyadic exterior derivative and define cohomology classes in $H^{2i}_{\mathrm{dR},\mathbb{D}}(M)$ independent of the choice of connection.*

Proof. Follows from the trace invariance of the curvature polynomial under dyadic gauge transformations and the closedness of invariant polynomials. □

Remark 10.2. This dyadic version of Chern–Weil theory opens the door to dyadic topological invariants for quantum field models and dyadic index theorems.

11. Dyadic Morse Theory

Definition 11.1 (Dyadic Morse Function). A dyadically smooth function $f : M \to \mathbb{R}$ is a *dyadic Morse function* if at each critical point $p \in M$, the dyadic Hessian $H_f^{\mathbb{D}}(p)$ is nondegenerate, i.e., its determinant in the dyadic sense is non-zero.

Definition 11.2 (Dyadic Critical Point). A point $p \in M$ is a *dyadic critical point* of f if $df(p) = 0$, where df is the dyadic differential.

Definition 11.3 (Dyadic Index). The *dyadic index* of a critical point p is the number of negative eigenvalues (over $\mathbb{D}$) of the dyadic Hessian matrix $H_f^{\mathbb{D}}(p)$.

Theorem 11.4 (Dyadic Morse Lemma). *Let f be a dyadic Morse function. Then near a critical point p, there exist dyadic coordinates $(x_1, \dots, x_n)$ such that*

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

with λ the dyadic index of p.

12. Dyadic Hodge Theory

Let M be a compact oriented dyadic Riemannian manifold of dimension n.

Definition 12.1 (Dyadic Metric). A *dyadic Riemannian metric g* on M is a smooth section of the symmetric positive-definite dyadic bilinear form

$$g_p : T_p^{\mathbb{D}} M \times T_p^{\mathbb{D}} M \to \mathbb{D}$$

varying smoothly over M.

Definition 12.2 (Dyadic Hodge Star). The dyadic Hodge star operator $\star_{\mathbb{D}} : \Omega^k_{\mathbb{D}}(M) \to \Omega^{n-k}_{\mathbb{D}}(M)$ is defined using the dyadic volume form and dyadic inner product.

Definition 12.3 (Dyadic Laplacian). Define the dyadic Laplacian $\Delta_{\mathbb{D}} = dd^* + d^*d$, where $d^* = \star_{\mathbb{D}} d \star_{\mathbb{D}}^{-1}$ is the dyadic codifferential.

Definition 12.4 (Dyadic Harmonic Form). A form $\omega \in \Omega^k_{\mathbb{D}}(M)$ is *dyadic harmonic* if $\Delta_{\mathbb{D}}\omega = 0$.

Theorem 12.5 (Dyadic Hodge Theorem). *Every dyadic de Rham cohomology class in $H^k_{\mathrm{dR},\mathbb{D}}(M)$ contains a unique dyadic harmonic representative.*

13. Dyadic Poincaré Duality

Theorem 13.1. *If M is a compact oriented dyadic manifold of dimension n, then the bilinear pairing*

$$H^k_{\mathrm{dR},\mathbb{D}}(M) \times H^{n-k}_{\mathrm{dR},\mathbb{D}}(M) \to \mathbb{D}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

is non-degenerate.

14. Dyadic Loop Spaces

Definition 14.1 (Dyadic Loop Space)**.** Let M be a dyadic manifold. The *dyadic loop space $\mathcal{L}_{\mathbb{D}}M$* is the space of all dyadically smooth maps

$$\gamma : S^1_{\mathbb{D}} \to M,$$

where $S^1_{\mathbb{D}}$ denotes the dyadic unit circle, constructed as the limit of dyadic polygonal approximations.

Definition 14.2 (Dyadic Energy Functional)**.** Given a dyadic Riemannian metric g, define the energy of a dyadic loop $\gamma \in \mathcal{L}_{\mathbb{D}}M$ by

$$E_{\mathbb{D}}(\gamma) := \frac{1}{2} \int_{S^1_{\mathbb{D}}} \left\| \frac{d\gamma}{dt} \right\|_g^2 dt,$$

where the derivative is interpreted in the dyadic sense.

Remark 14.3. Critical points of $E_{\mathbb{D}}$ correspond to dyadic geodesics, suggesting a dyadic version of the classical path integral formulation.

15. Dyadic Spectral Theory

Definition 15.1 (Dyadic Spectrum of the Laplacian)**.** Let $\Delta_{\mathbb{D}}$ be the dyadic Laplacian on $\Omega^k_{\mathbb{D}}(M)$. The set of $\lambda \in \mathbb{D}$ such that

$$\Delta_{\mathbb{D}}\omega = \lambda\omega$$

for some non-zero $\omega \in \Omega^k_{\mathbb{D}}(M)$ is called the *dyadic spectrum.*

Proposition 15.2. *The dyadic spectrum of $\Delta_{\mathbb{D}}$ is a discrete subset of $\mathbb{D} \subset \mathbb{Q}$ and accumulates only at infinity.*

Definition 15.3 (Dyadic Heat Kernel)**.** Let $K_{\mathbb{D}}(t, x, y)$ be the fundamental solution to the dyadic heat equation:

$$\left(\frac{\partial}{\partial t} + \Delta_{\mathbb{D}} \right) K_{\mathbb{D}}(t, x, y) = 0.$$

Theorem 15.4 (Dyadic Heat Trace Expansion)**.** *The trace of the dyadic heat kernel admits an asymptotic expansion as $t \to 0^+$:*

$$\mathrm{Tr}(e^{-t\Delta_{\mathbb{D}}}) \sim \sum_{j=0}^{\infty} a_j^{\mathbb{D}} t^{j-n/2},$$

where the coefficients $a_j^{\mathbb{D}}$ are dyadic spectral invariants.

16. Towards a Dyadic Index Theorem

Definition 16.1 (Dyadic Elliptic Operator)**.** A differential operator $D_{\mathbb{D}} : \Gamma(E) \to \Gamma(F)$ between dyadic vector bundles is *dyadic elliptic* if its dyadic symbol $\sigma_{D_{\mathbb{D}}}$ is invertible away from the zero section.

Definition 16.2 (Dyadic Analytical Index)**.** The *dyadic analytic index* of an elliptic operator $D_{\mathbb{D}}$ is defined as

$$\mathrm{ind}_{\mathrm{an},\mathbb{D}}(D_{\mathbb{D}}) := \dim_{\mathbb{D}} \ker D_{\mathbb{D}} - \dim_{\mathbb{D}} \mathrm{coker} D_{\mathbb{D}}.$$

Conjecture 16.3 (Dyadic Atiyah–Singer Index Theorem (Preliminary))**.** Let $D_{\mathbb{D}}$ be a dyadic elliptic differential operator on a compact dyadic manifold M. Then

$$\mathrm{ind}_{\mathrm{an},\mathbb{D}}(D_{\mathbb{D}}) = \int_M \mathrm{ch}_{\mathbb{D}}(E) \cdot \hat{A}_{\mathbb{D}}(TM),$$

where $\mathrm{ch}_{\mathbb{D}}$ and $\hat{A}_{\mathbb{D}}$ are dyadic characteristic classes.

17. Dyadic Cobordism Theory

Definition 17.1 (Dyadic Cobordism)**.** Let M_0 and M_1 be compact dyadic n-manifolds. A dyadic cobordism from M_0 to M_1 is a compact dyadic $(n+1)$-manifold W such that

$$\partial W = M_1 \sqcup (-M_0),$$

where the boundary is taken in the dyadic sense and $-M_0$ denotes the manifold with reversed dyadic orientation.

Proposition 17.2. *Dyadic cobordism defines an equivalence relation on the set of compact dyadic n-manifolds.*

Definition 17.3 (Dyadic Cobordism Ring)**.** The set of dyadic cobordism classes of n-manifolds forms an abelian group $\Omega_n^{\mathbb{D}}$ under disjoint union, and the graded ring

$$\Omega_*^{\mathbb{D}} = \bigoplus_{n=0}^{\infty} \Omega_n^{\mathbb{D}}$$

is the *dyadic cobordism ring*.

18. Dyadic K-Theory

Definition 18.1 (Dyadic Topological K-Theory)**.** Let M be a compact dyadic manifold. The *dyadic topological K-theory group* $K_{\mathbb{D}}^0(M)$ is the Grothendieck group of isomorphism classes of dyadic vector bundles over M.

Definition 18.2 (Dyadic Reduced K-Theory)**.** Define $\widetilde{K}_{\mathbb{D}}^0(M)$ as the kernel of the rank homomorphism:

$$\mathrm{rank} : K_{\mathbb{D}}^0(M) \to \mathbb{D}.$$

Definition 18.3 (Dyadic K-Theoretic Chern Character)**.** The dyadic Chern character is a natural transformation:

$$\mathrm{ch}_{\mathbb{D}} : K_{\mathbb{D}}^0(M) \to H_{\mathrm{dR},\mathbb{D}}^{2*}(M)$$

that respects tensor products and exterior powers.

19. Dyadic Classifying Spaces and Characteristic Classes

Definition 19.1 (Dyadic Classifying Space)**.** Let $\mathrm{GL}_n(\mathbb{D})$ be the dyadic general linear group. Define the dyadic classifying space $B\mathrm{GL}_n(\mathbb{D})$ as the geometric realization of the nerve of the category of rank-n dyadic vector bundles.

Theorem 19.2. *Every dyadic vector bundle $E \to M$ corresponds to a homotopy class of maps:*

$$[M, B\mathrm{GL}_n(\mathbb{D})].$$

Definition 19.3 (Dyadic Universal Chern Class)**.** The dyadic universal Chern classes $c_i^{\mathbb{D}} \in H^{2i}(B\mathrm{GL}_n(\mathbb{D}); \mathbb{D})$ pull back under the classifying map to the dyadic Chern classes of any vector bundle.

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