NEW FIELDS CONSTRUCTIBLE ONLY VIA SYMBOLIC INVERSE LIMITS

PU JUSTIN SCARFY YANG

Abstract. Traditional field constructions via algebraic extensions, completions, and closures leave out a vast space of symbolic, computable, and inverse-limit-driven number systems. Symbolic Profinite Fields (SPFs) provide a robust, AI-compatible framework to generate entirely new fields not accessible via classical methods. This article explores the types, structures, and implications of such novel fields.

Contents

1. Limitations of Classical Field Constructions

Traditional field theory offers:

- Algebraic closures: $\overline{\mathbb{Q}}, \overline{\mathbb{F}_q}$;
- Completions: $\mathbb{Q}_p, \mathbb{R}$;
- Function fields and formal series: $\mathbb{F}_q(t), \mathbb{F}_q((t))$;
- Model-theoretic extensions (e.g. ultraproducts).

Yet these lack mechanisms for:

- Encoding symbolic truncation or precision;
- Cross-metric or hybrid local-global limits;
- Hierarchies with dyadic, epistemic, or AI-learnable structure.

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2. Symbolic Profinite Inverse Limits

Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of symbolic fields with truncation maps $\pi_n^{n+1} : F_{n+1} \to F_n$. Then:

$$\widehat{F}^{\mathrm{sym}} := \varprojlim F_n$$

is a symbolic profinite field, capable of hosting new algebraic, geometric, or analytic properties not captured in any individual F_n.

3. Constructing Previously Unknown Fields

3.1. Heterogeneous Hybrid Fields.

$$F_n = \begin{cases} \mathbb{Q}_2, & n \text{ even} \\ \mathbb{Q}_3, & n \text{ odd} \end{cases} \Rightarrow \widehat{F}^{\mathrm{sym}} \not\subseteq \mathbb{Q}_p$$

This field interpolates nontrivially between multiple p-adic structures.

3.2. Dyadic-Transcendental Fields.

$$F_n = \mathbb{Q}(\sqrt[n]{2}, \pi_n), \quad \text{with } \pi_n \text{ symbolic transcendental approximations}$$

Produces a field containing both algebraic towers and symbolic transcendental layers.

3.3. Epistemic Approximation Fields. Define F_n to be rational numbers expressible with 2^{-n} epistemic precision. Then:

$$\widehat{F}^{\mathrm{sym}} = \text{epistemically bounded field}$$

This field models computable human cognition of quantities.

4. Comparison Table

Field Type	**Classical**	**SPF Constructible**	**Description**
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$	Yes	Yes	Standard reinterpreted
$\mathbb{Q}_p, \mathbb{F}_q((t))$	Yes	Yes	Completions & formal te
$\mathbb{C}_p, \overline{\mathbb{Q}}$	Limited	Yes (via skeleton)	Symbolic approximation
Heterogeneous Hybrids	No	Yes	Cross-p interpolated fi
Dyadic-Epistemic	No	Yes	Symbolic human-perceivable pr
AI-generative Fields	No	Yes	Symbolically learnable str

5. Implications and Future Work

- Symbolic fields provide new coordinates for arithmetic geometry and motives;
- Open door to AI-discoverable number theory;
- Introduce novel symbolic class field theory and Galois stacks;
- Reframe complexity theory and logical decidability in field-theoretic terms.

References

[1] N. Bourbaki, *Algebra II: Chapters 4–7*, Springer, 1990.
[2] S. Lang, *Algebra*, 3rd Edition, Springer, 2002.
[3] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, 1979.
[4] I. B. Fesenko and S. V. Vostokov, *Local Fields and Their Extensions*, American Mathematical Society, 2002.
[5] J.-M. Fontaine, *Représentations p-adiques des corps locaux I*, in *The Grothendieck Festschrift*, Vol. II, Birkhäuser, 1990.
[6] P. Scholze, *Perfectoid Spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.
[7] P. J. S. Yang, *Symbolic Profinite Fields and Constructible Number Systems via AI-Compatible Inverse Limits*, 2025. *[This document]*.
[8] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, 1992.