PROJECTIVE SYSTEMS OF MULTIRADICAL EXTENSIONS OF $\mathbb{Q}$ AND THEIR GALOIS GROUPS

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Abstract. We construct and study the projective system of Galois groups arising from arbitrary (including infinite) radical extensions of the rational number field $\mathbb{Q}$. Specifically, we consider the infinite radical closure generated by adjoining all radicals $\sqrt[n]{q}$ for integers $n \geq 2$ and rational numbers $q \in \mathbb{Q}^\times$, and define the associated profinite Galois group as an inverse limit. We examine structural properties, continuity, abelian quotients, and propose generalizations toward a multiradical class field theory.

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1. Introduction

One of the central themes in modern algebraic number theory is the study of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the understanding of how specific infinite towers of field extensions reflect the algebraic, arithmetic, and cohomological structure of $\mathbb{Q}$. Classical work in Kummer theory provides detailed knowledge of extensions of the form $\mathbb{Q}(\sqrt[n]{q})$ for fixed $n \in \mathbb{Z}_{\geq 2}$ and $q \in \mathbb{Q}^{\times}$, particularly under the assumption that suitable roots of unity are adjoined.

In this paper, we move beyond individual radical extensions and even finite tuples thereof. We initiate the systematic development of the Galois-theoretic and projective framework associated to the infinite multiradical closure of $\mathbb{Q}$, defined by

$$\mathbb{Q}_\infty^{\mathrm{rad}} := \mathbb{Q}\left(\{\sqrt[n_i]{q_i}\}_{i\in I}\right),$$

for arbitrary index sets $I \subset \mathbb{Z}_{\geq 2} \times \mathbb{Q}^\times$, possibly infinite.

Our goal is to define a projective system of Galois groups over finite subextensions and to study the structure of the resulting profinite group:

$$\mathcal{G}_{\mathrm{rad}} := \varprojlim_{J \subset_{\mathrm{fin}} I} \mathrm{Gal}\left(\mathbb{Q}\left(\left\{\sqrt[n_j]{q_j}\right\}_{j\in J}\right)/\mathbb{Q}\right),$$

which encodes the arithmetic symmetries of all radical extensions simultaneously. We show that this group is a topological profinite group, explore its torsion and solvable properties, and study the nature of its abelianizations and limits.

This construction provides a natural radical analogue to the cyclotomic $\mathbb{Z}_p$-extension and opens a path toward a broader radical class field theory.

2. Preliminaries and Notation

Throughout, let $\mathbb{Q}$ denote the rational number field and $\overline{\mathbb{Q}}$ its fixed algebraic closure. We define:

Definition 2.1. Let $I \subset \mathbb{Z}_{\geq 2} \times \mathbb{Q}^\times$ be any (finite or infinite) index set. Define the *infinite multiradical extension* of $\mathbb{Q}$ associated to I as

$$K_I := \mathbb{Q}\left(\{\sqrt[n_i]{q_i}\}_{(n_i,q_i)\in I}\right) \subset \overline{\mathbb{Q}}.$$

Definition 2.2. Let $\mathcal{F}(I)$ denote the set of all finite subsets $J \subset I$, partially ordered by inclusion. Then $\{K_J\}_{J\in\mathcal{F}(I)}$ forms a directed system of fields, with $K_J \subset K_{J'}$ for $J \subset J'$.

Definition 2.3. We define the *radical closure field* of $\mathbb{Q}$ (with respect to I) as the union

$$K_I^{\mathrm{rad}} := \bigcup_{J\in\mathcal{F}(I)} K_J,$$

and its associated *projective Galois group* as

$$\mathcal{G}_I := \varprojlim_{J\in\mathcal{F}(I)} \mathrm{Gal}(K_J/\mathbb{Q}).$$

We will often write $\mathbb{Q}_\infty^{\mathrm{rad}} := K_{\mathbb{Z}_{\geq 2}\times\mathbb{Q}^\times}$ and denote its projective Galois group by $\mathcal{G}_{\mathrm{rad}}$.

Remark 2.4. Note that each $\mathrm{Gal}(K_J/\mathbb{Q})$ is finite and typically abelian or solvable, especially when all relevant roots of unity are adjoined. The inverse limit $\mathcal{G}_{\mathrm{rad}}$ is therefore a compact, totally disconnected, profinite group.

In subsequent sections, we will study the algebraic structure, functoriality, cohomological properties, and representation theory of $\mathcal{G}_{\mathrm{rad}}$.

3. Structure of the Infinite Radical Galois Group $\mathcal{G}_{\mathrm{rad}}$

In this section, we study the algebraic and topological structure of the profinite group

$$\mathcal{G}_{\mathrm{rad}} := \varprojlim_{J \subset_{\mathrm{fin}} \mathbb{Z}_{\geq 2} \times \mathbb{Q}^\times} \mathrm{Gal}\left(\mathbb{Q}\left(\left\{\sqrt[n_j]{q_j}\right\}_{(n_j,q_j)\in J}\right)/\mathbb{Q}\right).$$

This group encodes all possible radical symmetries of $\mathbb{Q}$ under arbitrary root extractions and plays a central role in the radical Galois theory we propose.

3.1. **Topological and Algebraic Properties.**

Proposition 3.1. *$\mathcal{G}_{\mathrm{rad}}$ is a compact, totally disconnected, profinite group.*

Proof. Each $K_J/\mathbb{Q}$ is a finite Galois extension. Thus, each $\mathrm{Gal}(K_J/\mathbb{Q})$ is a finite group with the discrete topology. The inverse limit of finite discrete groups is profinite, and inherits compactness and total disconnectedness from the limit topology. Hence, $\mathcal{G}_{\mathrm{rad}}$ is profinite. □

Proposition 3.2. *If all roots of unity ζ_n are contained in K_J for each J, then $\mathrm{Gal}(K_J/\mathbb{Q})$ is abelian, and $\mathcal{G}_{\mathrm{rad}}$ is a pro-abelian group.*

Proof. This follows from classical Kummer theory: if $\mathbb{Q}(\zeta_n) \subseteq K_J$, then $\mathrm{Gal}(K_J/\mathbb{Q})$ is a subgroup of $\mathrm{Hom}(\mathbb{Q}^\times/(\mathbb{Q}^\times)^n, \mu_n)$, which is abelian. Since inverse limits preserve abelianity, $\mathcal{G}_{\mathrm{rad}}$ is pro-abelian. □

Remark 3.3. In general, when roots of unity are not included, the extensions $K_J/\mathbb{Q}$ may not be normal, and their Galois closures may introduce non-abelian groups. Thus, the full group $\mathcal{G}_{\mathrm{rad}}$ is pro-solvable but not necessarily pro-abelian.

3.2. **Decomposition via Tensor Systems.** Let us write

$$\mathcal{G}_{\mathrm{rad}} \cong \varprojlim_{J} \bigotimes_{(n,q)\in J} \mathrm{Gal}\left(\mathbb{Q}\left(\sqrt[n]{q}\right)/\mathbb{Q}\right),$$

when K_J is a compositum of linearly disjoint radical extensions. This suggests a form of profinite tensor product structure. We now define the key building blocks.

Definition 3.4. Let $\mathcal{K}_{n,q} := \mathbb{Q}(\sqrt[n]{q})$. We define its Galois group $G_{n,q} := \mathrm{Gal}(\mathcal{K}_{n,q}/\mathbb{Q})$, and its Galois closure $\widetilde{\mathcal{K}}_{n,q}$ with corresponding group $\widetilde{G}_{n,q}$.

Lemma 3.5. *For fixed $n \in \mathbb{Z}_{\geq 2}$ and $q \in \mathbb{Q}^\times$, $\mathcal{K}_{n,q}/\mathbb{Q}$ is Galois iff either:*

(1) *$n = 2$, or*
(2) *$q \in \mathbb{Q}$ is such that $\zeta_n \in \mathbb{Q}$.*

Proof. By elementary Galois theory, $\mathbb{Q}(\sqrt[n]{q})/\mathbb{Q}$ is Galois iff all n-th roots of unity appear in the normal closure. For example, $\mathbb{Q}(\sqrt[3]{2})$ is not Galois since $\zeta_3 \notin \mathbb{Q}$. $\square$

3.3. Filtered Subgroups and Universal Property.

Definition 3.6. For each finite $J \subset I$, define

$$G_J := \mathrm{Gal}\left(K_J/\mathbb{Q}\right), \quad \text{and} \quad N_J := \ker\left(\mathcal{G}_{\mathrm{rad}} \to G_J\right).$$

Then the family $\{N_J\}$ forms a fundamental system of open neighborhoods of the identity.

Proposition 3.7 (Universal Property)**.** *Let G be a profinite group and $\phi_J : G \to \mathrm{Gal}(K_J/\mathbb{Q})$ be a compatible family of continuous surjective maps for each finite J. Then there exists a unique continuous homomorphism*

$$\phi : G \to \mathcal{G}_{\mathrm{rad}}$$

such that for all J, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \mathcal{G}_{\mathrm{rad}} \\ & \searrow \phi_J & \downarrow \\ & & \mathrm{Gal}(K_J/\mathbb{Q}) \end{array}$$

commutes. That is, $\mathcal{G}_{\mathrm{rad}}$ is the projective limit in the category of profinite groups.

3.4. Canonical Quotients and Abelianization.

Definition 3.8. Let $\mathcal{G}_{\mathrm{rad}}^{\mathrm{ab}} := \mathcal{G}_{\mathrm{rad}}/[\mathcal{G}_{\mathrm{rad}}, \mathcal{G}_{\mathrm{rad}}]$ denote the abelianization. We define the abelian radical Galois character group as

$$\mathrm{RadChar}(\mathbb{Q}) := \mathrm{Hom}_{\mathrm{cont}}\left(\mathcal{G}_{\mathrm{rad}}^{\mathrm{ab}}, \widehat{\mathbb{Z}}\right).$$

Remark 3.9. The group $\mathrm{RadChar}(\mathbb{Q})$ can be interpreted as the space of continuous characters arising from infinite multiradical descent, generalizing cyclotomic characters.

4. Examples and Computations

To illustrate the structure and richness of the profinite radical Galois group $\mathcal{G}_{\mathrm{rad}}$, we compute explicit cases for selected index sets $I \subset \mathbb{Z}_{\geq 2} \times \mathbb{Q}^{\times}$, focusing on cases with arithmetic and Galois-theoretic interest. These computations provide insight into the torsion, abelian, and non-abelian characteristics of finite and infinite radical substructures.

4.1. **Example 1: All Square Roots of Rational Primes.**

Let $I = \{(2, p) : p \text{ prime}\}$. Then

$$K_I = \mathbb{Q}\left(\{\sqrt{p}\}_{p \text{ prime}}\right),$$

and

$$\mathcal{G}_I = \varprojlim_{J \subset_{\mathrm{fin}} I} \mathrm{Gal}\left(\mathbb{Q}\left(\{\sqrt{p}\}_{p \in J}\right)/\mathbb{Q}\right).$$

Proposition 4.1. *Each subextension $\mathbb{Q}\left(\left\{\sqrt{p}\right\}_{p\in J}\right)/\mathbb{Q}$ is a Galois extension with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where $r = |J|$.*

Proof. The field $\mathbb{Q}\left(\sqrt{p_1}, \ldots, \sqrt{p_r}\right)$ is the compositum of r quadratic extensions with discriminants that are pairwise coprime. Since each $\sqrt{p_i}$ defines a quadratic extension, the total Galois group is a product $(\mathbb{Z}/2\mathbb{Z})^r$. □

Corollary 4.2. *The profinite group $\mathcal{G}_I$ is isomorphic to the product*

$$\prod_{p \text{ prime}} \mathbb{Z}/2\mathbb{Z}.$$

This example demonstrates that $\mathcal{G}_I$ is an infinite product of elementary abelian 2-groups, and hence a compact abelian profinite group of exponent 2.

4.2. **Example 2: All Cube Roots of Squarefree Integers.**

Let $I = \{(3, a) : a \in \mathbb{Q}^{\times} \text{ squarefree and } a > 0\}$.

The field $K_I = \mathbb{Q}\left(\{\sqrt[3]{a}\}_{a \text{ squarefree}}\right)$ is not a normal extension of $\mathbb{Q}$, but we may take the Galois closures K_J^{gal} for finite J to ensure that each $\mathrm{Gal}(K_J^{\mathrm{gal}}/\mathbb{Q})$ is well-defined.

Proposition 4.3. *Let $J = \{(3, a_1), \ldots, (3, a_r)\}$. Then*

$$\mathrm{Gal}(K_J^{\mathrm{gal}}/\mathbb{Q}) \hookrightarrow S_3^r,$$

where each factor corresponds to the Galois group of a non-cyclotomic cubic extension.

Proof. Each $\mathbb{Q}(\sqrt[3]{a_i})$ is a non-normal extension of degree 3. Its normal closure contains ζ_3 and has Galois group isomorphic to S_3. Hence, the Galois group of the compositum of such fields embeds in S_3^r. □

Corollary 4.4. *The group $\mathcal{G}_I$ is a closed subgroup of the profinite group $\prod_a S_3$ indexed over squarefree $a \in \mathbb{Q}^\times$.*

This example shows that $\mathcal{G}_I$ is profinite, non-abelian, and has rich ramification and solvable substructure.

4.3. Example 3: All $\sqrt[n]{q}$ for Fixed q.

Let $q \in \mathbb{Q}^\times$ be fixed. Let $I = \{(n, q) : n \in \mathbb{Z}_{\geq 2}\}$.

Then

$$K_I = \mathbb{Q}\left(\{\sqrt[n]{q}\}_{n \in \mathbb{Z}_{\geq 2}}\right).$$

This is the tower of all radical extensions of a fixed element q.

Proposition 4.5. *If $\zeta_n \in \mathbb{Q}(\sqrt[n]{q})$ for all n, then the field K_I is a subfield of the maximal abelian extension of $\mathbb{Q}$.*

Remark 4.6. This setting relates to Iwasawa-theoretic $\mathbb{Z}_p$-extensions, especially for $q = p$ and $n = p^r$, where the Galois group becomes isomorphic to $\mathbb{Z}_p$ or its finite approximations.

4.4. Example 4: Total Radical Closure.

Let $I = \mathbb{Z}_{\geq 2} \times \mathbb{Q}^\times$. Then

$$K_I = \mathbb{Q}\left(\{\sqrt[n]{q}\}_{n \geq 2, q \in \mathbb{Q}^\times}\right) =: \mathbb{Q}_\infty^{\mathrm{rad}},$$

and we are computing the full object of our study:

$$\mathcal{G}_{\mathrm{rad}} := \varprojlim_{J \in \mathcal{F}(I)} \mathrm{Gal}(K_J/\mathbb{Q}).$$

This group includes all the previous examples as closed subgroups and serves as a universal radical Galois group for $\mathbb{Q}$.

Remark 4.7. While $\mathbb{Q}_\infty^{\mathrm{rad}}$ is not Galois over $\mathbb{Q}$, its normal closure is, and $\mathcal{G}_{\mathrm{rad}}$ can be viewed as the projective limit of finite solvable groups arising from multiradical composita.

5. Connections to Classical and Modern Theory

Our construction of the profinite group $\mathcal{G}_{\mathrm{rad}}$ as the projective limit of all multiradical Galois groups over $\mathbb{Q}$ unifies and extends several classical frameworks in algebraic number theory. This section explores its relationship with Kummer theory, cyclotomic fields, Iwasawa theory, and the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

5.1. **Relation to Kummer Theory.** Kummer theory provides an explicit description of extensions of the form

$$K := \mathbb{Q}(\zeta_n, \sqrt[n]{q})/\mathbb{Q},$$

where ζ_n is a primitive n-th root of unity and $q \in \mathbb{Q}^\times$.

Theorem 5.1 (Kummer)**.** *Let K be a field containing the n-th roots of unity, and let $q \in K^\times$. Then the extension $K(\sqrt[n]{q})/K$ is Galois, with Galois group isomorphic to a subgroup of μ_n.*

In our case, $\mathbb{Q}$ does not contain all roots of unity, but the system of fields $\mathbb{Q}(\sqrt[n]{q})$ with varying n, q can be embedded into their normal closures containing roots of unity. Thus, $\mathcal{G}_{\mathrm{rad}}$ may be interpreted as a global object interpolating the Kummer theory over varying roots.

Remark 5.2. Our projective system incorporates not only abelian extensions (where roots of unity are included), but also non-abelian radical extensions, such as $\mathbb{Q}(\sqrt[3]{2})$, whose normal closure is not abelian. Hence, $\mathcal{G}_{\mathrm{rad}}$ generalizes the Kummer theory to non-abelian multiradical settings.

5.2. **Relation to the Cyclotomic Tower and Iwasawa Theory.** The classical cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is the union

$$\mathbb{Q}_\infty := \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n}),$$

with Galois group $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p^\times$.

Our construction includes the fields $\mathbb{Q}(\sqrt[p^n]{p})$, which are deeply tied to the cyclotomic tower via Kummer theory:

$$\mathbb{Q}(\sqrt[p^n]{p}) \subseteq \mathbb{Q}(\zeta_{p^n}, \sqrt[p^n]{p}).$$

Proposition 5.3. *Let $q \in \mathbb{Q}^\times$ and fix p prime. The tower $\mathbb{Q}(\sqrt[p^n]{q})$ is closely related to the cyclotomic $\mathbb{Z}_p$-extension when $\zeta_{p^n} \in \mathbb{Q}(\sqrt[p^n]{q})$.*

This shows that parts of $\mathcal{G}_{\mathrm{rad}}$ are intertwined with the cyclotomic Galois group and can be viewed as a radical analogue of the Iwasawa tower.

5.3. **Embedding into the Absolute Galois Group.** Let $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$. Then for each $J \in \mathcal{F}(I)$, there is a natural surjection

$$G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(K_J/\mathbb{Q}).$$

Proposition 5.4. *The canonical projection maps induce a continuous group homomorphism:*

$$\varphi : G_{\mathbb{Q}} \longrightarrow \mathcal{G}_{\mathrm{rad}}.$$

Definition 5.5. We define the *radical kernel* as

$$\ker(\varphi) := \bigcap_J \mathrm{Gal}(\overline{\mathbb{Q}}/K_J).$$

Remark 5.6. The radical kernel consists of automorphisms in $G_{\mathbb{Q}}$ that fix all radicals of all rational numbers. Understanding this kernel may shed light on the non-radical part of $G_{\mathbb{Q}}$, and potentially lead to new filtrations of $G_{\mathbb{Q}}$ based on radical vs. non-radical descent.

5.4. Galois Cohomology and Future Arithmetic Interpretations.

Given the profinite nature of $\mathcal{G}_{\mathrm{rad}}$, we may consider its continuous cohomology groups:

$$H^i_{\mathrm{cont}}(\mathcal{G}_{\mathrm{rad}}, M)$$

for various discrete or profinite $\mathcal{G}_{\mathrm{rad}}$-modules M.

Potential applications include:

- Radical analogues of Hilbert 90 and cohomological duality theorems.
- Interpretations of multiradical descent as torsors or gerbes.
- Classifying continuous characters as maps $\mathcal{G}_{\mathrm{rad}} \to \widehat{\mathbb{Z}}$ or $\mathbb{C}^\times$.

Such cohomological approaches may lead to a "radical class field theory" in analogy with the classical abelian case.

6. Future Directions

The construction of $\mathcal{G}_{\mathrm{rad}}$ opens a number of new avenues in algebraic number theory, arithmetic geometry, and Galois representation theory. This section outlines some potential research trajectories.

6.1. Toward a Radical Class Field Theory.

Class field theory classifies abelian extensions of number fields using the idele class group and reciprocity maps. It is natural to ask:

Question 6.1. *Can the projective system of radical extensions be described via a suitable* radical class field theory*, possibly through a novel reciprocity map involving a radicalized class group?*

Some directions include:

- Define a *radical idele group* or *radicalized Picard group* whose abelianized quotient relates to $\mathcal{G}_{\mathrm{rad}}^{\mathrm{ab}}$.
- Construct an explicit Artin-type reciprocity map for multiradical extensions.
- Develop a theory of radical ramification and unramified radicals analogous to unramified abelian extensions.

6.2. **Geometric and Motivic Analogues.** The multiradical field extension $\mathbb{Q}_\infty^{\mathrm{rad}}$ can be viewed as analogous to the universal covering of $\mathbb{G}_m$, the multiplicative group scheme. We propose:

Conjecture 6.2. *There exists a geometric object or stack $\mathcal{R}$ over* $\mathrm{Spec}\mathbb{Q}$ *such that:*

$$\pi_1^{\acute{e}t}(\mathcal{R}) \cong \mathcal{G}_{\mathrm{rad}}.$$

This motivates potential connections to:

- Étale fundamental groups of stacks and log schemes.
- Tannakian categories generated by radical motives or rank-one Kummer sheaves.
- Periods of multiradical structures and their comparison isomorphisms.

6.3. **Radical Anabelian Geometry and Beyond.** Given Grothendieck's anabelian philosophy — that the structure of certain schemes is determined by their étale fundamental group — we ask:

Question 6.3. *Does $\mathcal{G}_{\mathrm{rad}}$ retain enough information to reconstruct radical subschemes or classifying spaces over $\mathbb{Q}$?*

If so, this would yield a radical analogue of the anabelian program and support the vision of encoding arithmetic data in the tower of all radical field extensions.

6.4. **Non-Abelian and Higher Structures.** The majority of known radical extensions are solvable (though not abelian). We may ask:

- Can $\mathcal{G}_{\mathrm{rad}}$ be realized as a non-abelian geometric fundamental group?
- Does a higher stack structure or derived groupoid formulation apply?
- Can we categorify $\mathcal{G}_{\mathrm{rad}}$ to obtain a higher Galois theory of radicals?

These directions point toward a potential new theory: a radical-based motivic or categorical Galois theory.

6.5. **Arithmetic Statistics and Probabilistic Properties.** The study of random field extensions — inspired by Bhargava, Ellenberg–Venkatesh, et al. — can be adapted to this context:

Question 6.4. *What is the statistical distribution of Galois groups* $\mathrm{Gal}(\mathbb{Q}(\{\sqrt[n_i]{q_i}\}_{i=1}^r)/\mathbb{Q})$ *as $r \to \infty$?*

This raises probabilistic and asymptotic questions for radical towers and suggests applications in computational number theory and cryptography.

7. Level I: Radical Numbers and Abelian Galois Groups

This section explores the first and most classical layer in the hierarchy of algebraic numbers: those expressible by radicals over $\mathbb{Q}$. These are precisely the algebraic numbers whose minimal polynomials define finite abelian Galois extensions of $\mathbb{Q}$, and whose roots can be explicitly constructed using root extractions and field operations.

7.1. Definition and Construction.

Definition 7.1. A number $\alpha \in \overline{\mathbb{Q}}$ is called a *radical number* if there exists a finite sequence of field extensions

$$\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n$$

such that $K_i = K_{i-1}(\sqrt[r_i]{a_i})$ for some $a_i \in K_{i-1}$ and $r_i \in \mathbb{Z}_{\geq 2}$, and $\alpha \in K_n$.

These extensions are called *radical extensions*. The field $\mathbb{Q}^{\mathrm{rad}}$ is the union of all such finite radical extensions:

$$\mathbb{Q}^{\mathrm{rad}} := \bigcup_{\text{finite radical } K/\mathbb{Q}} K.$$

Example 7.2. The numbers $\sqrt{2}$, $\sqrt[3]{5}$, and $\zeta_5 = e^{2\pi i/5}$ are all radical numbers. The field $\mathbb{Q}(\zeta_5)$ is a cyclotomic field of degree 4 over $\mathbb{Q}$, with abelian Galois group isomorphic to $(\mathbb{Z}/5\mathbb{Z})^\times$.

7.2. Kummer Theory and Abelian Extensions.

Theorem 7.3 (Kummer). *Let K be a field of characteristic not dividing n such that K contains the n-th roots of unity. Then for any $a \in K^\times$, the extension $K(\sqrt[n]{a})/K$ is Galois with Galois group a subgroup of the cyclic group μ_n, and*

$$\mathrm{Gal}(K(\sqrt[n]{a})/K) \cong \mathrm{Hom}(K^\times/(K^\times)^n, \mu_n).$$

Applying Kummer theory over $\mathbb{Q}$ requires adjoining the appropriate roots of unity first. For instance, the field

$$\mathbb{Q}(\zeta_n, \sqrt[n]{a})/\mathbb{Q}$$

is Galois and typically abelian when ζ_n is included.

7.3. Cyclotomic Fields and Kronecker–Weber Theorem.

Theorem 7.4 (Kronecker–Weber). *Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$ for some $n \geq 1$.*

Corollary 7.5. *Every radical extension of $\mathbb{Q}$ with abelian Galois group is contained in some $\mathbb{Q}(\zeta_n, \sqrt[n]{a_1}, \ldots, \sqrt[n]{a_k})$.*

7.4. **Structure of Galois Groups.** Let us consider the radical field

$$K = \mathbb{Q}(\sqrt[n_1]{q_1}, \dots, \sqrt[n_r]{q_r}),$$

where each $q_i \in \mathbb{Q}^\times$ and $n_i \in \mathbb{Z}_{\geq 2}$, and assume $\zeta_{n_i} \in K$ for all i.

Proposition 7.6. *If the q_i are multiplicatively independent modulo n_i-th powers in $\mathbb{Q}^\times$, and if all required roots of unity are adjoined, then*

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \prod_{i=1}^{r} \mathbb{Z}/n_i\mathbb{Z}.$$

Proof. Each $\mathbb{Q}(\sqrt[n_i]{q_i})$ is a Kummer extension with Galois group $\mathbb{Z}/n_i\mathbb{Z}$ under the assumption that $\zeta_{n_i} \in \mathbb{Q}$. The compositum of linearly disjoint Kummer extensions has Galois group the product. □

7.5. **Projective Systems and the Radical Galois Group.** Let $\mathcal{I}$ be the set of all finite subsets of $\mathbb{Z}_{\geq 2} \times \mathbb{Q}^\times$ and define for each $J \in \mathcal{I}$ the radical field

$$K_J := \mathbb{Q}\left(\{\sqrt[n]{q} \mid (n, q) \in J\}\right).$$

Definition 7.7. The *radical Galois group* is defined as the projective limit

$$\mathcal{G}_{\mathrm{rad}} := \varprojlim_{J \in \mathcal{I}} \mathrm{Gal}(K_J/\mathbb{Q}).$$

Proposition 7.8. $\mathcal{G}_{\mathrm{rad}}$ *is a profinite, abelian, compact topological group.*

Proof. Each $\mathrm{Gal}(K_J/\mathbb{Q})$ is finite and abelian (assuming roots of unity are present). The inverse limit of finite abelian groups is profinite and abelian. □

7.6. **Conclusion of Level I.** We have now completed the analysis of Level I algebraic numbers. These are the most explicitly understood: constructible via radicals, governed by abelian Galois theory, and embedded in the framework of cyclotomic and Kummer extensions.

8. Level II: Nested Radical Numbers and Solvable Galois Groups

This section develops the theory of algebraic numbers constructed via *nested radicals*, i.e., radical expressions built iteratively upon one another. While the resulting field extensions may no longer be abelian, they remain solvable in the Galois-theoretic sense. This corresponds to the second level in our classification hierarchy.

8.1. Definition and Examples.

Definition 8.1. A number $\alpha \in \overline{\mathbb{Q}}$ is called a *nested radical number* if it lies in a finite tower of radical extensions of $\mathbb{Q}$ of the form

$$\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n, \quad \text{with } K_i = K_{i-1}(\sqrt[r_i]{a_i}),$$

where each $a_i \in K_{i-1}$ and $r_i \in \mathbb{Z}_{\geq 2}$.

Example 8.2. The number $\sqrt{1+\sqrt{2}}$ lies in the tower:

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{1+\sqrt{2}}).$$

Another example: $\sqrt[3]{2+\sqrt{5}} \in \mathbb{Q}(\sqrt{5}, \sqrt[3]{2+\sqrt{5}})$.

8.2. General Theory of Nested Radical Fields.

Nested radicals lead to composita of multiple radical extensions. While each individual radical extension may be abelian, their composition often is not.

Proposition 8.3. *Let $K = \mathbb{Q}(\sqrt{a})$, and let $L = K(\sqrt{b})$ for $a \in \mathbb{Q}$, $b \in K$. Then $L/\mathbb{Q}$ is solvable of degree at most 4, but not necessarily abelian.*

Proof. Each step is a radical extension of degree at most 2. The full extension may involve non-commuting automorphisms, making the Galois group non-abelian but still solvable. □

Remark 8.4. The Galois group of a nested radical extension may be a non-abelian solvable group, such as the dihedral group D_4 or the quaternion group Q_8.

8.3. Solvable Galois Groups in Nested Radical Towers.

Definition 8.5. A group G is *solvable* if it has a finite derived series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

such that each quotient G_i/G_{i+1} is abelian.

Theorem 8.6. *If a number $\alpha \in \overline{\mathbb{Q}}$ lies in a nested radical tower over $\mathbb{Q}$, then its minimal polynomial has a solvable Galois group.*

Proof. By construction, each step is a radical extension, corresponding to an abelian quotient. The composition yields a solvable extension, hence a solvable Galois group. □

8.4. **Examples of Solvable but Non-Abelian Galois Groups.**

Example 8.7. Let $K = \mathbb{Q}(\sqrt{2})$, and consider the field

$$L = K(\sqrt[3]{\sqrt{2}+1}).$$

The Galois group of the normal closure of $L/\mathbb{Q}$ may be a non-abelian solvable group such as a semidirect product of cyclic groups.

Another classic example:

$$f(x) = x^4 - 2x^2 + 9$$

has Galois group isomorphic to the dihedral group D_4, which is solvable but non-abelian.

8.5. **Projective Structure of Nested Radical Galois Groups.** Let $\mathcal{I}_{\text{nest}}$ denote the category of nested radical field extensions of $\mathbb{Q}$ of bounded depth. Each such extension corresponds to a solvable Galois group.

Definition 8.8. We define the *nested radical Galois group* as

$$\mathcal{G}_{\text{nest}} := \varprojlim_{K \in \mathcal{I}_{\text{nest}}} \operatorname{Gal}(K/\mathbb{Q}),$$

where each K is a finite nested radical field.

Proposition 8.9. *$\mathcal{G}_{nest}$ is a profinite, solvable topological group containing $\mathcal{G}_{\text{rad}}$ as a dense subgroup.*

Remark 8.10. While $\mathcal{G}_{\text{rad}}$ is abelian, $\mathcal{G}_{\text{nest}}$ is generally not. The complexity of nested radicals grows rapidly with depth.

8.6. **Solvability versus Explicitness.** Even though nested radicals give solvable extensions, they are still not always *explicitly* computable:
- Radicals of radicals may require **irrational or nonconstructible subradicals**.
- Complexity in determining normal closures, discriminants, and automorphism group actions increases rapidly.

Example 8.11. $\sqrt[4]{2+\sqrt{3}}$ lies in a degree-8 extension of $\mathbb{Q}$, but its Galois group is not obviously abelian.

8.7. **Conclusion of Level II.** Nested radical numbers generalize radicals by allowing composition of radical steps. Though still solvable, the Galois groups encountered in this level include non-abelian solvable groups, and the associated field extensions become highly structured and layered. This level marks the transition point beyond fully explicit formulas, as we move into territory where radical construction no longer guarantees simplicity.

9. Level III: Solvable but Non-Radical Algebraic Numbers

This section addresses algebraic numbers that lie in **solvable field extensions** of $\mathbb{Q}$ but **cannot be expressed using radicals alone**. These numbers extend beyond nested radicals and require **transcendental special functions**, such as elliptic or modular functions, to describe their minimal polynomial roots. This is the third layer in our classification hierarchy.

9.1. Definition and Examples.

Definition 9.1. An algebraic number $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ lies in *Level III* if its minimal polynomial $f(x) \in \mathbb{Q}[x]$ has a *solvable* Galois group, but α is not expressible using radicals (finite nested root extractions over $\mathbb{Q}$).

Example 9.2. The general quintic equation

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

is not solvable by radicals unless its Galois group is solvable. However, certain special quintics have solvable Galois groups (e.g., F_{20}, the Frobenius group), yet their roots cannot be written using radicals.

A concrete example:

$$f(x) = x^5 - 10x + 5$$

has a solvable but non-abelian Galois group. Its roots can be expressed using **Bring radicals** or elliptic modular functions.

9.2. Abel–Ruffini Theorem and Solvable Galois Groups.

Theorem 9.3 (Abel–Ruffini)**.** *There is no general solution in radicals to polynomial equations of degree five or higher. That is, not all quintic equations are solvable by radicals.*

Remark 9.4. This does not contradict the existence of solvable Galois groups in degree 5. Rather, it shows that solvability by radicals is not guaranteed, even when a polynomial has a solvable Galois group.

9.3. Non-Radical Solvable Numbers and Transcendental Solutions.

Although radicals fail, some quintic equations can still be solved by transforming them into special forms and applying transcendental functions.

Definition 9.5. A *Bring radical* is a root of a polynomial of the form:

$$x^5 + ax + b = 0,$$

solved by applying a reduction to the Bring–Jerrard form and solving using modular functions or generalized hypergeometric series.

Theorem 9.6 (Hermite–Kronecker)**.** *There exists a general solution to the Bring–Jerrard quintic using theta functions and modular invariants, though not expressible by radicals.*

Example 9.7. The equation

$$x^5 + 5x + 12 = 0$$

has Galois group isomorphic to the Frobenius group F_{20}, which is solvable but non-abelian. Its roots are not radical-expressible, but can be constructed using special transcendental functions.

9.4. **Constructive and Computational Barriers.**

- Solvable polynomials may require **elliptic modular functions** to describe their roots.
- No universal "closed form" exists for these solutions.
- Symbolic manipulation software (e.g., Mathematica, Magma) represents such roots symbolically (e.g., $\mathrm{Root}[f, i]$), reflecting their non-explicit nature.

Remark 9.8. These cases show that solvability (in the Galois-theoretic sense) does not equate to expressibility using radicals. The solvable hierarchy extends deeper than radical closure.

9.5. **Field Extensions Beyond the Radical Closure.** Let $\mathbb{Q}^{\mathrm{rad}}$ denote the radical closure of $\mathbb{Q}$, and let $\mathbb{Q}^{\mathrm{solv}}$ denote the compositum of all number fields whose normal closures over $\mathbb{Q}$ have solvable Galois groups.

Definition 9.9. Define:

$$\mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}},$$

where $\mathbb{Q}^{\mathrm{nest}}$ is the field generated by all nested radical numbers, and $\mathbb{Q}^{\mathrm{solv}}$ is generated by all algebraic numbers with solvable Galois groups.

Proposition 9.10. $\mathbb{Q}^{\mathrm{solv}}$ *is a strictly larger field than* $\mathbb{Q}^{\mathrm{nest}}$ *and contains numbers not constructible via radicals or nested radicals.*

Remark 9.11. The field $\mathbb{Q}^{\mathrm{solv}}$ contains all Level I–III numbers. Level IV begins precisely where solvability fails.

9.6. **Conclusion of Level III.** This level marks a profound transition in the expressibility of algebraic numbers: while the Galois groups remain solvable, the numbers themselves are no longer expressible by radicals or nested radicals. Their construction requires transcendental methods, and their Galois-theoretic description is subtler than what radical towers can capture.

10. Level IV: Non-Solvable Algebraic Numbers

This section develops the fourth layer in our classification of algebraic numbers: the domain of irreducible polynomials over $\mathbb{Q}$ whose Galois groups are *non-solvable*. The roots of such polynomials cannot be expressed in radicals, nested radicals, or even by transcendental functions such as modular forms. Their only general representation is symbolic, typically as specific roots of explicitly known polynomials. This level represents the structural boundary between solvability and the intrinsic symmetry complexity of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

10.1. Definition and Examples.

Definition 10.1. An algebraic number $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ lies in *Level IV* if it is a root of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is *non-solvable*.

Example 10.2. Let $f(x) = x^5 - x - 1$. This polynomial is irreducible over $\mathbb{Q}$, and its Galois group is isomorphic to S_5, the symmetric group on five letters. Since S_5 is non-solvable, no root of f is expressible using radicals, nested radicals, or known transcendental functions.

10.2. Irreducibility and the Full Symmetric Group.

Proposition 10.3. *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible degree-n polynomial. If its discriminant is not a square and its modulo-p reductions exhibit cycle types corresponding to a transposition and an n-cycle, then* $\mathrm{Gal}(f) = S_n$.

Corollary 10.4. *The Galois group of $x^5 - x - 1$ is S_5, which is not solvable.*

Proof. Standard discriminant and reduction modulo p arguments show the group is transitive, contains a 5-cycle and a transposition, and hence must be the full symmetric group S_5. □

10.3. Limits of Expression: Abel–Ruffini and Beyond.

Theorem 10.5 (Abel–Ruffini, refined)**.** *Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree $n \geq 5$. If* $\mathrm{Gal}(f)$ *is not solvable, then its roots are not expressible by radicals, nor by nested radicals or compositions thereof.*

Remark 10.6. This result shows that there exists no general formula, even involving transcendental functions, for the roots of such polynomials. The roots can only be represented symbolically or numerically.

10.4. **Classification of Non-Solvable Galois Groups.** The set of non-solvable finite groups includes:

- The symmetric groups S_n for $n \geq 5$.
- The alternating groups A_n for $n \geq 5$.
- Simple non-abelian groups such as $PSL_2(p)$.
- Non-solvable semidirect products and other complex extensions.

Example 10.7. Let $f(x) = x^6 - x + 1$. Then f is irreducible and its Galois group is isomorphic to S_6, which is non-solvable and not simple. The roots of f cannot be expressed by any known explicit closed-form expression.

10.5. **Symbolic Representation of Non-Solvable Roots.** For a non-solvable irreducible polynomial $f(x) \in \mathbb{Q}[x]$ and a root α, one typically writes:

$$\alpha := \mathrm{Root}_k(f),$$

where $1 \leq k \leq \deg(f)$ specifies the root within the complex embedding. Such symbolic notation is standard in computational algebra systems and is used in theoretical number theory when closed-form expressions are unavailable.

Remark 10.8. Although α cannot be explicitly expressed, arithmetic in $\mathbb{Q}(\alpha)$ is fully defined via operations modulo the minimal polynomial $f(x)$. Thus, $\mathbb{Q}(\alpha)$ remains an effective number field.

10.6. **Field of Non-Solvable Extensions.**

Definition 10.9. Let $\mathbb{Q}^{\mathrm{sym}}$ denote the union of all normal extensions of $\mathbb{Q}$ with non-solvable Galois groups. Then we have the field tower:

$$\mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}} \subset \mathbb{Q}^{\mathrm{sym}} \subset \overline{\mathbb{Q}}.$$

Proposition 10.10. *There exist algebraic numbers in $\mathbb{Q}^{\mathrm{sym}} \setminus \mathbb{Q}^{\mathrm{solv}}$, whose minimal polynomials have non-solvable Galois groups.*

Remark 10.11. This level introduces truly generic field extensions, reflecting the full symmetry and complexity of root permutations beyond solvable systems.

10.7. **Conclusion of Level IV.** This level contains the first genuinely uncomputable (in a closed-form sense) algebraic numbers. Although we can symbolically and numerically describe them, there is no general radical or transcendental expression for their roots. The non-solvability of the Galois group introduces a permanent barrier to elementary construction.

In the final section, we formalize the symbolic representation of all algebraic numbers and show how arbitrary algebraic elements are organized under a naming system that includes all prior levels as definable cases.

11. Level V: Symbolically Named Algebraic Numbers

This final section establishes the fifth and most general level in our classification: the set of all algebraic numbers not lying in $\mathbb{Q}$, represented by their minimal polynomials and root indices. While some of these numbers fall into Levels I–IV and have explicit or constructive descriptions, most elements of this level require purely symbolic representation. This level exhausts $\overline{\mathbb{Q}} \setminus \mathbb{Q}$.

11.1. **Symbolic Root Notation.**

Definition 11.1. Let $f(x) \in \mathbb{Q}[x]$ be a monic irreducible polynomial of degree $n \geq 2$, and let $1 \leq k \leq n$. The *k-th root of f* is the k-th complex root of $f(x)$ under some fixed total ordering of $\overline{\mathbb{Q}}$ (e.g., lexicographic ordering on real and imaginary parts). We denote this root symbolically as

$$\alpha := \mathrm{Root}_k(f).$$

Remark 11.2. This naming convention is standard in symbolic computation and abstract algebra. It allows manipulation and referencing of specific algebraic numbers even in the absence of explicit expressions.

Example 11.3. Let $f(x) = x^5 - x - 1$. Define $\alpha := \mathrm{Root}_1(f)$ to mean "the unique real root of f". Then $\mathbb{Q}(\alpha)$ is a degree-5 number field, and all arithmetic is governed by operations modulo $f(x)$.

11.2. **Universal Coverage of Symbolic Roots.**

Proposition 11.4. *Every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ can be written uniquely in the form $Root_k(f)$, where $f(x) \in \mathbb{Q}[x]$ is monic and irreducible of degree at least 2, and $1 \leq k \leq \deg f$.*

Proof. By the fundamental theorem of algebra, every algebraic number is a root of some nonzero polynomial in $\mathbb{Q}[x]$, and can be isolated as a single root of its minimal polynomial. Fixing an ordering on roots provides uniqueness. □

11.3. **Symbolic Closure and Field Inclusion.**

Definition 11.5. Let $\mathbb{Q}^{\mathrm{sym}}$ denote the field generated by all symbolically named algebraic numbers. Then

$$\mathbb{Q}^{\mathrm{sym}} := \bigcup_{f \in \mathbb{Q}[x],\ \mathrm{irr}} \mathbb{Q}(\mathrm{Root}_k(f)),$$

where f runs over all monic irreducible polynomials of degree ≥ 2 and k over root indices.

Remark 11.6. This field coincides with $\overline{\mathbb{Q}}$, and thus we have:

$$\mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}} \subset \mathbb{Q}^{\mathrm{sym}} = \overline{\mathbb{Q}}.$$

11.4. **Galois Groups and Generality.** Since symbolic algebraic numbers encompass all roots of irreducible polynomials over $\mathbb{Q}$, their Galois groups can be *arbitrary finite groups*.

Remark 11.7. This level contains:

- Numbers with known radical expressions (Level I),
- Nested radical constructions (Level II),
- Solvable-by-transcendental-function roots (Level III),
- Non-solvable root fields (Level IV),
- And all remaining roots of irreducible polynomials over $\mathbb{Q}$.

11.5. **Computational and Theoretical Role.** While symbolic naming does not provide closed-form expressions, it has deep applications:

- Enables field and ring operations in computer algebra systems.
- Supports manipulation and reasoning in number field computations.
- Provides a canonical placeholder for unsolvable root values.
- Forms the foundation of constructive algebraic number theory.

Example 11.8. Let $f(x) = x^7 - 3x^3 + 7x + 2$. Then $\mathrm{Root}_4(f)$ is a symbol for the 4th complex root of $f(x)$. Its field $\mathbb{Q}(\mathrm{Root}_4(f))$ is a subfield of $\overline{\mathbb{Q}}$ with full structure determined by f.

11.6. **Conclusion of Level V.** This level concludes the classification of algebraic numbers in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$. Every such number may be symbolically represented as a specific root of an irreducible rational polynomial. Levels I–IV identify cases with increasing expressive power and decreasing explicitness, but all ultimately reside within the symbolic closure formalized here. This establishes a complete, layered theory of algebraic numbers based on their Galois-theoretic and representational complexity.

Meta-Theorem: The Five-Level Classification of Algebraic Numbers in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$

Theorem 11.9 (Hierarchical Structure of Algebraic Numbers)**.** *Every element $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ lies uniquely in one of the following five strictly* *nested classes, each corresponding to a class of Galois groups and level of expressibility:*

$$\mathbb{Q} \subset \mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}} \subset \mathbb{Q}^{\mathrm{sym}} = \overline{\mathbb{Q}}.$$

The corresponding algebraic numbers fall into five levels:

Level I *Radical numbers — Galois group abelian.*

Level II *Nested radical numbers — Galois group solvable and possibly non-abelian.*

Level III *Solvable but non-radical numbers — solvable Galois group, not radical-expressible.*

Level IV *Non-solvable numbers — Galois group not solvable (e.g., S_n).*

Level V *General symbolic algebraic numbers — defined as roots of irreducible polynomials.*

Remark 11.10. This hierarchy provides a complete classification of the algebraic numbers in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$, organized by the expressibility of their defining field extensions and the solvability of their associated Galois groups.

Visual Diagram (Conceptual)

$$\begin{array}{c}
\boxed{\mathbb{Q}} \\
\Big\downarrow \quad \text{(adjoin radicals)} \\
\boxed{\mathbb{Q}^{\mathrm{rad}}} \\
\Big\downarrow \quad \text{(compose nested radicals)} \\
\boxed{\mathbb{Q}^{\mathrm{nest}}} \\
\Big\downarrow \quad \text{(allow all solvable Galois groups)} \\
\boxed{\mathbb{Q}^{\mathrm{solv}}} \\
\Big\downarrow \quad \text{(include non-solvable roots)} \\
\boxed{\mathbb{Q}^{\mathrm{sym}} = \overline{\mathbb{Q}}}
\end{array}$$

12. Stratification of the Absolute Galois Group of $\mathbb{Q}$

This section introduces a new viewpoint on the absolute Galois group of the rational numbers,

$$G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

based on the five-level classification of algebraic numbers developed in this manuscript. We present a filtration of $G_{\mathbb{Q}}$ by nested closed normal subgroups associated with radical, nested, and solvable extensions of $\mathbb{Q}$, culminating in the full algebraic closure $\overline{\mathbb{Q}}$.

12.1. **The Tower of Fixed Fields.** Recall the strictly increasing tower of intermediate fields:

$$\mathbb{Q} \subset \mathbb{Q}^{\text{rad}} \subset \mathbb{Q}^{\text{nest}} \subset \mathbb{Q}^{\text{solv}} \subset \mathbb{Q}^{\text{sym}} := \overline{\mathbb{Q}},$$

where:

- $\mathbb{Q}^{\text{rad}}$ is the radical closure of $\mathbb{Q}$,
- $\mathbb{Q}^{\text{nest}}$ is the closure under nested radicals,
- $\mathbb{Q}^{\text{solv}}$ is the union of all solvable normal extensions of $\mathbb{Q}$,
- $\mathbb{Q}^{\text{sym}} = \overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.

Each inclusion corresponds by Galois duality to a descending chain of closed subgroups of $G_{\mathbb{Q}}$:

$$G_{\mathbb{Q}} \trianglerighteq G^{\text{rad}} \trianglerighteq G^{\text{nest}} \trianglerighteq G^{\text{solv}} \trianglerighteq \{1\},$$

where:

$$G^{\text{rad}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\text{rad}}), \quad G^{\text{nest}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\text{nest}}), \quad G^{\text{solv}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\text{solv}}).$$

12.2. **Interpretation of the Quotients.** Each successive quotient encodes a distinct layer of symmetry complexity:

- $G_{\mathbb{Q}}/G^{\text{rad}}$: governs the abelian radical part — cyclotomic and Kummer theory.
- $G^{\text{rad}}/G^{\text{nest}}$: measures the added structure from nested radical towers — non-abelian but solvable.
- $G^{\text{nest}}/G^{\text{solv}}$: corresponds to non-radical solvable extensions — includes Bring-type quintics.
- G^{solv}: the kernel of all solvable representations — detects non-solvable Galois actions (e.g., S_n).

Definition 12.1 (Solvability Filtration of $G_{\mathbb{Q}}$)**.** Define the solvability filtration of the absolute Galois group $G_{\mathbb{Q}}$ as:

$$G_{\mathbb{Q}} \supseteq G^{\text{rad}} \supseteq G^{\text{nest}} \supseteq G^{\text{solv}} \supseteq \{1\},$$

where each subgroup corresponds to the fixing field of the respective level in the radical hierarchy.

Theorem 12.2. *The filtration $\{G_{\mathbb{Q}} \supseteq G^{\text{rad}} \supseteq G^{\text{nest}} \supseteq G^{\text{solv}}\}$ is strictly descending and consists entirely of closed normal subgroups. The corresponding quotient groups reflect abelian, solvable non-abelian, and non-solvable symmetry, respectively.*

12.3. **A Profinite Layered Perspective on $G_{\mathbb{Q}}$.** This filtration offers a structured lens through which to study $G_{\mathbb{Q}}$:

- It provides a natural decomposition of $G_{\mathbb{Q}}$ by expressibility depth of its fixed fields.
- It aligns with classical results (Kronecker–Weber, Kummer theory) and modern transcendence phenomena (Abel–Ruffini, inverse Galois problem).
- It defines potential new objects such as the *radical quotient group* $G_{\mathbb{Q}}/G^{\mathrm{rad}}$ and the *solvable residual group* G^{solv}.

Remark 12.3. While the full structure of $G_{\mathbb{Q}}$ remains mysterious, this filtration highlights conceptually tractable slices of it, each with its own arithmetic and cohomological character.

12.4. **Future Directions.** This perspective invites further developments:

- Defining Galois cohomology theories relative to each subgroup in the filtration.
- Investigating the automorphism towers and deformation theory of each quotient.
- Formulating a "radical class field theory" parallel to abelian CFT.
- Studying how this filtration interacts with anabelian phenomena, motives, or Galois representations.

Conjecture 12.4. *The filtration $\mathbb{Q} \subset \mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}} \subset \overline{\mathbb{Q}}$ is canonical and functorial among all global fields admitting radical towers, and induces a natural solvability stratification on their absolute Galois groups.*

12.5. **Galois Cohomology Theories Relative to the Solvability Filtration.** Let $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, and consider the descending filtration:

$$G_{\mathbb{Q}} \trianglerighteq G^{\mathrm{rad}} \trianglerighteq G^{\mathrm{nest}} \trianglerighteq G^{\mathrm{solv}} \trianglerighteq \{1\}.$$

Each subgroup in this filtration corresponds to a fixed field in $\overline{\mathbb{Q}}$ and defines a tower of field extensions

$$\mathbb{Q} \subset \mathbb{Q}^{\mathrm{rad}} \subset \mathbb{Q}^{\mathrm{nest}} \subset \mathbb{Q}^{\mathrm{solv}} \subset \overline{\mathbb{Q}}.$$

Definition 12.5. Let M be a discrete or profinite $G_{\mathbb{Q}}$-module. For each subgroup $G^* \trianglelefteq G_{\mathbb{Q}}$ in the filtration, define the relative Galois cohomology group:

$$H^i(G_{\mathbb{Q}}/G^*, M^{G^*}),$$

where M^{G^*} denotes the invariants of M under G^*.

Remark 12.6. This defines new cohomological invariants that measure the obstruction to lifting or descending arithmetic data across successive layers of the solvability filtration.

Example 12.7.

- $H^1(G_{\mathbb{Q}}/G^{\mathrm{rad}}, \mathbb{Q}/\mathbb{Z})$ classifies continuous abelian characters factoring through the radical quotient.
- $H^2(G_{\mathbb{Q}}/G^{\mathrm{nest}}, \mu_n)$ detects nontrivial central extensions and relations among nested radicals.
- $H^1(G^{\mathrm{solv}}, \mathbb{Q}_\ell)$ encodes Galois representations with image in a non-solvable group.

Proposition 12.8. *These cohomology groups interpolate classical results such as Kummer theory, Hilbert 90, and Tate cohomology, and provide a refined view of the radical versus non-radical arithmetic in* $\overline{\mathbb{Q}}$.

Future Work 12.9. *A future theory may define a full derived category of solvability-filtered Galois representations, leading to new spectral sequences and long exact sequences interpolating cohomological data at each level.*

12.6. **Automorphism Towers and Deformation Theory of Solvability Quotients.** Each quotient group in the solvability filtration of $G_{\mathbb{Q}}$ reflects a layer of symmetry complexity in the absolute Galois group:

$$G_{\mathbb{Q}} \trianglerighteq G^{\mathrm{rad}} \trianglerighteq G^{\mathrm{nest}} \trianglerighteq G^{\mathrm{solv}} \trianglerighteq \{1\}.$$

These quotients are:

$$\begin{aligned} Q_1 &:= G_{\mathbb{Q}}/G^{\mathrm{rad}}, \\ Q_2 &:= G^{\mathrm{rad}}/G^{\mathrm{nest}}, \\ Q_3 &:= G^{\mathrm{nest}}/G^{\mathrm{solv}}, \\ Q_4 &:= G^{\mathrm{solv}}. \end{aligned}$$

We propose a systematic study of the *automorphism towers* and *deformations* of these quotient groups to reveal their intrinsic group-theoretic rigidity, flexibility, and interrelations.

Automorphism Towers.

Definition 12.10. Let G be a finite or profinite group. The *automorphism tower* of G is the sequence:

$$G_0 := G, \quad G_1 := \mathrm{Aut}(G_0), \quad G_2 := \mathrm{Aut}(G_1), \quad \ldots$$

continued as long as $G_{n+1} \neq G_n$ (up to isomorphism). If this stabilizes, we say the tower *terminates*.

Conjecture 12.11. *Each quotient Q_i arising from the solvability filtration has a terminating automorphism tower. In particular:*

$$\mathrm{Aut}^*(Q_1),\ \mathrm{Aut}^*(Q_2),\ \mathrm{Aut}^*(Q_3),\ \mathrm{Aut}^*(Q_4)$$

are definable, algebraically structured profinite groups whose invariants encode symmetries of the radical/nested/solvable Galois action.

Remark 12.12. These automorphism towers reflect the internal redundancy or stability of the layers of $G_{\mathbb{Q}}$ under its outer symmetries and may detect hidden descent or rigidity phenomena.

Deformation Theory of Quotient Representations. Let $\rho : G_{\mathbb{Q}} \twoheadrightarrow Q_i$ be the canonical projection. We may consider deformations of ρ in the sense of Mazur or Schlessinger.

Definition 12.13. Let $\mathcal{C}$ be the category of complete Noetherian local rings with residue field $\mathbb{F}_p$. A *deformation functor* of ρ is:

$$D_\rho : \mathcal{C} \to \mathrm{Sets},$$

assigning to each $R \in \mathcal{C}$ the set of lifts:

$$\tilde{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_n(R), \quad \text{with } \tilde{\rho} \mod \mathfrak{m}_R = \rho.$$

Future Work 12.14. *Study the representability, tangent space, and obstruction theory of the deformation functors of each solvability quotient. In particular:*

- $D_{\rho^{\mathrm{rad}}}$ *captures abelian Galois characters;*
- $D_{\rho^{\mathrm{nest}}}$ *reflects layered radical extensions;*
- $D_{\rho^{\mathrm{solv}}}$ *probes the boundary between solvable and non-solvable images.*

Remark 12.15. This deformation-theoretic perspective allows a geometric and modular interpretation of each level of the filtration, possibly linking it to arithmetic moduli spaces or stacks parametrizing radical field extensions.

12.7. **Toward a Radical Class Field Theory (RCFT).** Classical class field theory (CFT) describes the abelian extensions of a number field K via the arithmetic of its idele class group. In the case $K = \mathbb{Q}$, the Kronecker–Weber theorem states that:

$$\text{Every finite abelian extension of } \mathbb{Q} \text{ lies in some } \mathbb{Q}(\zeta_n).$$

This provides a global reciprocity law linking Galois groups and arithmetic class structures.

We now propose a generalization: a theory of radical extensions — potentially non-abelian but still constructed through successive radical operations — governed by a new class-like group structure and reciprocity framework.

Radical Field Tower and Its Galois Group. Let us consider the nested tower:

$$\mathbb{Q} \subset \mathbb{Q}^{\text{rad}} \subset \mathbb{Q}^{\text{nest}} \subset \mathbb{Q}^{\text{solv}} \subset \overline{\mathbb{Q}},$$

and the associated radical Galois group:

$$\mathcal{G}_{\text{rad}} := \varprojlim_{J \in \mathcal{I}} \operatorname{Gal}\left(\mathbb{Q}\left(\{\sqrt[n]{q}\}_{(n,q)\in J}\right)/\mathbb{Q}\right),$$

as introduced earlier.

Conjecture 12.16 (Radical Class Field Theory)**.** *There exists a functorial assignment:*

$$\mathsf{RadicalClassGroup}(K) \longleftrightarrow \operatorname{Gal}(K^{\text{rad}}/K),$$

generalizing the abelian class group, where K^{rad} is the radical closure of K and the group encodes information about radical composita over K.

Remark 12.17. Unlike classical class field theory, the radical class group need not be abelian, but its structure should reflect multiplicative relations among radicals and their Galois symmetries.

Class Field Theory (CFT)	**Radical Class Field Theory (RCFT)**
Abelian extensions	Radical extensions
Idele class group C_K	Radical class group $\mathcal{R}_K$ (to be defined)
Artin reciprocity map	Radical reciprocity map
Local-global principles	Radical-local patching of radicals
Ray class fields	Constrained radical fields with ramification control

TABLE 1. Heuristic analogy between CFT and the envisioned RCFT

Structural Analogies with Abelian CFT.

Goals and Directions.

- Define a radical ideal class group $\mathcal{R}_K$ measuring the failure of radical generation in a field K.
- Construct radical analogues of the Hilbert class field and ray class fields.
- Describe radical unramified extensions using suitable local conditions on radicals.

- Investigate whether K^{rad}/K is a maximal radical extension with prescribed ramification and Galois group structure.

Future Work 12.18. *A global theory of radical class fields would yield a non-abelian generalization of class field theory, capturing fields generated by radicals and nested radicals, potentially leading to new arithmetic reciprocity laws, radical modular forms, and local-global principles beyond the abelian setting.*

12.8. **Interaction with Anabelian Geometry, Motives, and Galois Representations.** The solvability filtration:

$$G_{\mathbb{Q}} \trianglerighteq G^{\text{rad}} \trianglerighteq G^{\text{nest}} \trianglerighteq G^{\text{solv}} \trianglerighteq \{1\}$$

reflects increasingly complex symmetries in the absolute Galois group of $\mathbb{Q}$. This stratification may admit meaningful interaction with major arithmetic theories such as *anabelian geometry*, *motivic Galois theory*, and *automorphic Galois representations*.

Anabelian Geometry and Solvability Layers. Grothendieck's anabelian program suggests that certain hyperbolic schemes over number fields are completely determined by their étale fundamental groups and Galois actions.

Question 12.19. *Can the layers of the solvability filtration detect or reflect anabelian behavior in towers of fields or schemes constructed via radical or solvable covers?*

Proposal 12.20. Define a *radical anabelian category* of arithmetic schemes whose function fields lie within $\mathbb{Q}^{\text{rad}}$ or $\mathbb{Q}^{\text{nest}}$, and study whether their arithmetic can be recovered from their associated Galois quotients $G_{\mathbb{Q}}/G^{*}$.

Motivic Galois Groups and Radical/Non-Radical Structure. In the theory of motives, one associates to a category of cohomological realizations a *motivic Galois group* — a pro-algebraic group capturing hidden symmetries among periods and cohomologies.

Conjecture 12.21. *There exists a radical motivic category* $\mathsf{Mot}^{\text{rad}}$ *whose Tannakian Galois group is naturally identified with a solvable quotient of* $G_{\mathbb{Q}}$*, such as* $G_{\mathbb{Q}}/G^{\text{nest}}$*.*

Remark 12.22. Such a category may include rank-one Kummer motives, iterated extensions by radicals, and periods generated by multiradical integrals. Its study would yield a motive-theoretic analog of radical class field theory.

Galois Representations and Solvable Lifts. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a continuous Galois representation. The image $\rho(G_{\mathbb{Q}})$ reflects deep arithmetic and geometric information.

Question 12.23. *When can ρ be factored through $G_{\mathbb{Q}}/G^{\mathrm{rad}}$ or G^{nest}? What geometric or automorphic objects correspond to such representations?*

Definition 12.24. Define a *radical Galois representation* to be a continuous homomorphism:

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \quad \text{that factors through } G_{\mathbb{Q}}/G^{\mathrm{rad}}.$$

Future Work 12.25. *Investigate whether radical or nested-solvable representations arise from specific subcategories of:*

- *Elliptic curves with constrained torsion fields,*
- *Tame ramification behavior in radical towers,*
- *Moduli of curves admitting radical function field equations.*

Explore whether these representations are modular, and if so, how their corresponding automorphic forms encode radical structural data.

Outlook. The solvability filtration of $G_{\mathbb{Q}}$ provides a new approach to dissecting and reconstructing the deep structure of number-theoretic symmetry. Its compatibility with anabelian principles, motivic categories, and Galois representations opens avenues for connecting classical field theory with modern arithmetic geometry, and may serve as a new organizing principle across several foundational areas.

Future Work 12.26. *A future theory may define a full derived category of solvability-filtered Galois representations, leading to new spectral sequences and long exact sequences interpolating cohomological data at each level.*

12.9. **Derived Categories and Spectral Sequences of the Solvability Filtration.** The solvability filtration

$$G_{\mathbb{Q}} \trianglerighteq G^{\mathrm{rad}} \trianglerighteq G^{\mathrm{nest}} \trianglerighteq G^{\mathrm{solv}} \trianglerighteq \{1\}$$

naturally stratifies the category of continuous representations of $G_{\mathbb{Q}}$ into subcategories according to their level of descent or factorization.

We propose a future framework wherein this stratification is extended into a full derived categorical and homological structure.

Solvability-Filtered Galois Representations.

Definition 12.27. A *solvability-filtered Galois representation* is a continuous representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(R),$$

for R a topological ring, equipped with a filtration of closed normal subgroups:

$$G_{\mathbb{Q}} \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_r = \{1\},$$

such that the restriction of ρ to each G_i induces a well-defined quotient representation

$$\rho_i : G_i/G_{i+1} \to \mathrm{GL}_n(R).$$

In the case of the solvability filtration, we take:

$$G_i := G^{(i)} \in \{G_{\mathbb{Q}}, G^{\mathrm{rad}}, G^{\mathrm{nest}}, G^{\mathrm{solv}}, \{1\}\}.$$

Remark 12.28. This framework enables us to isolate and study the contribution of each level of Galois-theoretic complexity to the behavior of a representation.

Derived Category of Filtered Representations. Let $\mathsf{Rep}_{G_{\mathbb{Q}}}$ denote the abelian category of continuous representations of $G_{\mathbb{Q}}$ over a fixed ring R (e.g., $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$). We define a triangulated subcategory:

Definition 12.29. Let $\mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}(G_{\mathbb{Q}}, R)$ be the derived category of solvability-filtered representations, i.e., complexes of representations with a filtration induced by the solvability tower and differentials compatible with the filtration layers.

Conjecture 12.30. *There exists a canonical t-structure on $\mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}(G_{\mathbb{Q}}, R)$ with cohomological truncation functors reflecting radical, nested, and solvable cohomology layers.*

Spectral Sequences Associated to the Filtration. For any object $V^\bullet \in \mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}(G_{\mathbb{Q}}, R)$, we expect a spectral sequence

$$E_1^{i,j} = H^j(G^{(i)}/G^{(i+1)}, V^i) \Rightarrow H^{i+j}(G_{\mathbb{Q}}, V),$$

where:

- V^i is the associated graded component for the i-th layer of the filtration,
- $G^{(i)}$ ranges over the successive subgroups: $G_{\mathbb{Q}} \supset G^{\mathrm{rad}} \supset G^{\mathrm{nest}} \supset G^{\mathrm{solv}}$.

Remark 12.31. This spectral sequence measures how cohomological data is "assembled" from radical, nested, and solvable contributions — analogously to the Hochschild–Serre or coniveau spectral sequences.

Long Exact Sequences of Layered Cohomology. Each short exact sequence of Galois groups:

$$1 \to G_{i+1} \to G_i \to Q_i \to 1$$

gives rise to a long exact sequence in cohomology:

$$\cdots \to H^n(Q_i, M^{G_{i+1}}) \to H^n(G_i, M) \to H^n(G_{i+1}, M) \to H^{n+1}(Q_i, M^{G_{i+1}}) \to \cdots$$

Definition 12.32. We call the collection of such long exact sequences across the filtration the *solvability cohomological ladder*.

Future Work 12.33. *Develop this ladder into a full cohomological machine — possibly a mixed filtration compatible with weights and levels of complexity in the arithmetic fundamental group. Explore compatibility with:*

- *Étale cohomology of arithmetic schemes,*
- *Motives filtered by radical solvability,*
- *Galois deformation rings with radical height stratifications.*

This concludes the first major future direction you asked to develop. It forms the seed of a radical homological algebra for filtered Galois theory.

Future Work 12.34. *Study the representability, tangent space, and obstruction theory of the deformation functors of each solvability quotient. In particular:*

- $D_{\rho^{\mathrm{rad}}}$ *captures abelian Galois characters;*
- $D_{\rho^{\mathrm{nest}}}$ *reflects layered radical extensions;*
- $D_{\rho^{\mathrm{solv}}}$ *probes the boundary between solvable and non-solvable images.*

12.10. **Deformation Theory of Solvability-Quotient Galois Representations.** Let $\rho : G_{\mathbb{Q}} \twoheadrightarrow Q$ be a continuous, finite-dimensional Galois representation. Classical deformation theory, due to Mazur and Schlessinger, studies lifts of ρ to representations valued in Artinian local rings, organizing these lifts into a deformation functor whose geometry encodes the arithmetic of the representation.

In the context of the solvability filtration:

$$G_{\mathbb{Q}} \trianglerighteq G^{\mathrm{rad}} \trianglerighteq G^{\mathrm{nest}} \trianglerighteq G^{\mathrm{solv}} \trianglerighteq \{1\},$$

we examine the deformation theory of the induced quotient representations

$$\rho^{\mathrm{rad}},\ \rho^{\mathrm{nest}},\ \rho^{\mathrm{solv}},$$

obtained by composing ρ with the canonical projections to the radical, nested, and solvable Galois quotients.

Deformation Functors for Quotient Representations. Let $\mathcal{C}$ denote the category of complete Noetherian local $\mathbb{Z}_\ell$-algebras with residue field $\mathbb{F}_\ell$. For a fixed representation $\rho_i : G_{\mathbb{Q}} \twoheadrightarrow Q_i \hookrightarrow \mathrm{GL}_n(\mathbb{F}_\ell)$, define the functor:

$$D_{\rho_i} : \mathcal{C} \to \mathrm{Sets},$$

where $D_{\rho_i}(A)$ consists of equivalence classes of deformations

$$\tilde{\rho}_i : G_{\mathbb{Q}} \to \mathrm{GL}_n(A), \quad \tilde{\rho}_i \mod \mathfrak{m}_A = \rho_i.$$

Definition 12.35. We define the deformation functors:

$$D_{\rho^{\mathrm{rad}}}, \quad D_{\rho^{\mathrm{nest}}}, \quad D_{\rho^{\mathrm{solv}}}$$

to classify deformations of $G_{\mathbb{Q}}$-representations factoring through $G_{\mathbb{Q}}/G^{\mathrm{rad}}$, $G_{\mathbb{Q}}/G^{\mathrm{nest}}$, and $G_{\mathbb{Q}}/G^{\mathrm{solv}}$, respectively.

Tangent Spaces and Obstruction Theory.

Proposition 12.36. *For each deformation functor D_{ρ_i}, the tangent space is canonically isomorphic to*

$$t_{D_{\rho_i}} := D_{\rho_i}(\mathbb{F}_\ell[\epsilon]/(\epsilon^2)) \cong H^1(G_{\mathbb{Q}}, \mathrm{ad}(\rho_i)),$$

where $\mathrm{ad}(\rho_i)$ is the adjoint representation on the Lie algebra of GL_n twisted by ρ_i.

Proposition 12.37. *Obstructions to lifting a deformation over a small extension $A \to A/I$ with square-zero ideal I lie in*

$$\mathrm{Ob}_{\rho_i} \in H^2(G_{\mathbb{Q}}, \mathrm{ad}(\rho_i) \otimes I).$$

Remark 12.38. The size and structure of these H^1 and H^2 cohomology groups reflect the richness and rigidity of radical/nested/solvable Galois representations. The functor $D_{\rho^{\mathrm{rad}}}$ typically has smooth, pro-representable hulls (moduli of characters), while $D_{\rho^{\mathrm{solv}}}$ may be obstructed or singular.

Characterization of Solvability Levels via Deformation Theory.

- $D_{\rho^{\mathrm{rad}}}$: Governs deformations of abelian characters; representable by formal Lie groups; corresponds to unramified characters or cyclotomic twists.
- $D_{\rho^{\mathrm{nest}}}$: Encodes higher radical complexity; tangent spaces reflect commutator layers and layered Kummer extensions.
- $D_{\rho^{\mathrm{solv}}}$: Contains obstructions arising from non-nilpotent solvable groups; its image is where radical solvability ends and general solvable deformations emerge.

Functoriality and Stratification.

Definition 12.39. Let $\mathcal{D}_{\mathrm{solv}} := \{D_{\rho_i}\}_{i\in\{\mathrm{rad},\mathrm{nest},\mathrm{solv}\}}$ be the tower of deformation functors associated to the solvability filtration.

Define morphisms of functors:

$$D_{\rho^{\mathrm{rad}}} \to D_{\rho^{\mathrm{nest}}} \to D_{\rho^{\mathrm{solv}}} \to D_{\rho},$$

compatible with inclusion of Galois image quotients and satisfying naturality under base change.

Future Work 12.40. *Study the representability and obstruction towers of $\mathcal{D}_{\mathrm{solv}}$. In particular:*

- *Determine whether $D_{\rho^{\mathrm{rad}}}$ is formally smooth;*
- *Characterize the singular loci of $D_{\rho^{\mathrm{nest}}}$;*
- *Study the jumping behavior of tangent dimensions between $D_{\rho^{\mathrm{nest}}}$ and $D_{\rho^{\mathrm{solv}}}$;*
- *Link the geometry of these functors to modular and motivic deformation spaces.*

This completes the second development, establishing a deformation-theoretic backbone for radical and solvable Galois representations.

Goals and Directions.

- Define a radical ideal class group $\mathcal{R}_K$ measuring the failure of radical generation in a field K.
- Construct radical analogues of the Hilbert class field and ray class fields.
- Describe radical unramified extensions using suitable local conditions on radicals.
- Investigate whether K^{rad}/K is a maximal radical extension with prescribed ramification and Galois group structure.

Future Work 12.41. *A global theory of radical class fields would yield a non-abelian generalization of class field theory, capturing fields generated by radicals and nested radicals, potentially leading to new arithmetic reciprocity laws, radical modular forms, and local-global principles beyond the abelian setting.*

Goals and Directions of Radical Class Field Theory (RCFT). To formulate a radical class field theory parallel to the classical abelian case, we propose the following programmatic goals.

1. Radical Ideal Class Group $\mathcal{R}_K$.

Definition 12.42. Let K be a number field. Define the *radical ideal class group* $\mathcal{R}_K$ as the group of fractional ideals modulo those that become principal after adjoining a finite tower of radicals over K:

$$\mathcal{R}_K := \frac{\{\text{fractional ideals of } K\}}{\{\text{ideals generated by elements expressible via radicals over } K\}}.$$

Remark 12.43. This generalizes the usual ideal class group Cl_K by incorporating not just multiplicative relations among ideals, but also their realizability via radical extensions. One may also define $\mathcal{R}_K^{\text{nest}}$ for nested radical closures.

2. Radical Hilbert and Ray Class Fields.

Definition 12.44. Let K be a number field. Define the *radical Hilbert class field* H_K^{rad} to be the maximal unramified radical extension of K, i.e.,

$$H_K^{\text{rad}} := \text{maximal unramified subextension of } K^{\text{rad}}.$$

Definition 12.45. Given a modulus $\mathfrak{m}$, define the *radical ray class field* $K^{\text{rad}}(\mathfrak{m})$ to be the maximal radical extension of K unramified outside $\mathfrak{m}$ and satisfying prescribed radical ramification at the primes dividing $\mathfrak{m}$.

Remark 12.46. Unlike the abelian case, these fields may have non-abelian, though solvable or metabelian, Galois groups. Their structure would depend on local radical ramification, nesting depth, and group-theoretic composition.

3. Local Conditions and Radical Unramified Extensions. We propose the definition of *radical inertia* and *radical decomposition* groups in analogy with local class field theory.

Definition 12.47. Let v be a place of K, and let L/K be a radical extension. Define the *radical inertia group* I_v^{rad} to consist of those automorphisms in $\mathrm{Gal}(L/K)$ acting trivially on the radicals whose exponents are unramified at v.

Remark 12.48. Local radical unramifiedness corresponds to the preservation of radical-generating exponents under completions at v. This should be encoded by local ramification data on Kummer or Artin–Schreier-type generators.

4. Maximal Radical Extension and Galois Description.

Question 12.49. *Is K^{rad}/K the maximal radical extension of K subject to the constraint that its Galois group remains solvable and generated by radical operations?*

Conjecture 12.50. *There exists a universal radical Galois group $\mathcal{G}_K^{\text{rad}} := \text{Gal}(K^{\text{rad}}/K)$ satisfying a functorial property across radical base changes and preserving local ramification filtrations.*

Definition 12.51. A field extension L/K is called *radically maximal* if it is not contained in any further radical extension of K unless that extension ceases to be solvable.

Remark 12.52. Understanding the structure of $\mathcal{G}_K^{\text{rad}}$ would generalize the role of $\text{Gal}(K^{\text{ab}}/K)$ in class field theory, and provide a solvable closure object with concrete Galois-theoretic stratification.

Future Work 12.53. *A global theory of radical class fields would yield a non-abelian generalization of class field theory, capturing fields generated by radicals and nested radicals, potentially leading to:*

- *Radical reciprocity laws,*
- *Modular forms with radical coefficients,*
- *New types of conductors and discriminants adapted to radical ramification,*
- *Stratified ray class theory indexed by radical nesting depth,*
- *A radical Langlands correspondence interpolating solvable symmetries.*

This completes the third full development. Next, shall we continue with the final integration step: compiling all four expanded future directions into a unified

Future Work 12.54. *Investigate whether radical or nested-solvable representations arise from specific subcategories of:*

- *Elliptic curves with constrained torsion fields,*
- *Tame ramification behavior in radical towers,*
- *Moduli of curves admitting radical function field equations.*

Explore whether these representations are modular, and if so, how their corresponding automorphic forms encode radical structural data.

Geometric Origins and Modularity of Radical Galois Representations. Having constructed the solvability filtration on the absolute Galois group

$$G_{\mathbb{Q}} \trianglerighteq G^{\text{rad}} \trianglerighteq G^{\text{nest}} \trianglerighteq G^{\text{solv}} \trianglerighteq \{1\},$$

we now ask: do geometric or automorphic objects naturally give rise to Galois representations factoring through these layers?

We focus on three primary geometric sources of such representations.

1. Elliptic Curves with Constrained Torsion Fields. Let $E/\mathbb{Q}$ be an elliptic curve. For each integer $n \geq 1$, the action of $G_{\mathbb{Q}}$ on $E[n]$ defines a Galois representation:

$$\rho_{E,n} : G_{\mathbb{Q}} \to \mathrm{Aut}(E[n]) \cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

Definition 12.55. We say E has *radically constrained torsion* if the fields $\mathbb{Q}(E[n])$ lie in $\mathbb{Q}^{\mathrm{rad}}$ or $\mathbb{Q}^{\mathrm{nest}}$ for all n in some infinite set.

Conjecture 12.56. *There exist infinite families of elliptic curves $E/\mathbb{Q}$ whose torsion fields are contained entirely in solvable extensions of radical type. These give rise to Galois representations:*

$$\rho_E : G_{\mathbb{Q}} \to \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

whose image lies in a subgroup factoring through G^{nest} or G^{solv}.

Remark 12.57. The image of ρ_E reflects the arithmetic of division polynomials and the algebraic structure of torsion points. Radical constraints may force the image to lie in solvable subgroups of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

2. Tame Ramification in Radical Towers. Radical extensions are often built from roots of unity and radicals $\sqrt[n]{q}$ with controlled ramification. The associated Galois groups are frequently tamely ramified at a finite set of primes.

Definition 12.58. We define the *radical ramification set* $\mathrm{Ram}_{\mathrm{rad}}(K)$ of a number field K as the set of primes where any generator $\sqrt[n]{q} \in K$ has wild ramification.

Future Work 12.59. *Investigate the class of Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ whose restriction to inertia groups I_p is unipotent and tame, and whose image lies in a solvable radical subgroup of GL_n. Such representations may descend from radical field towers.*

3. Moduli of Curves with Radical Function Fields. Let $C/\mathbb{Q}$ be a smooth projective curve such that its function field $\mathbb{Q}(C)$ is generated by radical or nested radical extensions over $\mathbb{Q}(x)$. For example:

$$y^2 = x(x-1)(x-\sqrt{2}) \quad \text{or} \quad y^3 = x^2 + \sqrt[4]{3}.$$

Definition 12.60. A curve C is said to have a *radical function field* if $\mathbb{Q}(C)$ is generated by variables satisfying radical equations over $\mathbb{Q}$.

Conjecture 12.61. *There exists a stratification of the moduli space $\mathcal{M}_g$ of genus-g curves by radical complexity, where each stratum corresponds* *to the Galois image of the fundamental group $\pi_1^{ét}(C/\mathbb{Q})$ lying within G^{rad}, G^{nest}, or G^{solv}.*

Modularity of Radical Representations. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a Galois representation arising from one of the above constructions.

Question 12.62. *If ρ factors through G^{rad} or G^{nest}, is ρ modular? If so, do the associated modular forms reflect radical structure in their Fourier coefficients, level, or weight?*

Future Work 12.63. *Develop a theory of* radical modularity*, associating automorphic forms to radical Galois representations. In particular:*

- *Identify congruence relations among modular forms reflecting radical descent.*
- *Study Hecke eigenforms whose associated Galois representations lie in solvable radical subgroups.*
- *Explore whether newtypes of L-functions — defined via radical field towers — satisfy functional equations and meromorphic continuation.*

Outlook. These investigations link the solvability filtration to concrete arithmetic geometry, enabling the extraction of new structures from classical objects like elliptic curves, fundamental groups of curves, and modular forms. This suggests a potential radical–modular correspondence, refining the classical Langlands paradigm through the lens of radical constructibility.

13. Future Directions

The solvability filtration of the absolute Galois group

$$G_{\mathbb{Q}} \trianglerighteq G^{\mathrm{rad}} \trianglerighteq G^{\mathrm{nest}} \trianglerighteq G^{\mathrm{solv}} \trianglerighteq \{1\}$$

opens multiple deep directions for further exploration, spanning cohomological algebra, deformation theory, class field theory generalizations, and connections to arithmetic geometry. We outline here four major research programs for future development.

13.1. **Derived Categories and Spectral Sequences of the Solvability Filtration.**

We propose defining a full derived category of Galois representations filtered by radical/nested/solvable descent levels, together with associated homological structures.

Definition 13.1. Let $\mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}(G_{\mathbb{Q}}, R)$ be the derived category of solvability-filtered Galois representations over a topological ring R. Objects consist of complexes of $G_{\mathbb{Q}}$-representations with compatible solvability filtrations.

Conjecture 13.2. *There exists a canonical t-structure on* $\mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}(G_{\mathbb{Q}}, R)$ *with cohomological truncation functors interpolating radical, nested, and solvable cohomological data.*

For $V^\bullet \in \mathsf{D}^{\mathrm{fil}}_{\mathrm{solv}}$, a spectral sequence of the form

$$E_1^{i,j} = H^j(G^{(i)}/G^{(i+1)}, V^i) \Rightarrow H^{i+j}(G_{\mathbb{Q}}, V)$$

may be constructed, assembling cohomology across the filtration layers.

Definition 13.3. We define the resulting long exact sequences as the *solvability cohomological ladder* of $G_{\mathbb{Q}}$.

This structure may give rise to a solvability-weighted version of Galois cohomology, offering a new stratification of arithmetic obstructions, deformation classes, and cohomological dualities.

13.2. Deformation Theory of Solvability-Quotient Galois Representations.

We define deformation functors

$$D_{\rho^{\mathrm{rad}}}, \quad D_{\rho^{\mathrm{nest}}}, \quad D_{\rho^{\mathrm{solv}}}$$

classifying deformations of representations factoring through the successive quotients $G_{\mathbb{Q}}/G^*$.

Each tangent space satisfies:

$$t_{D_{\rho_i}} \cong H^1(G_{\mathbb{Q}}, \mathrm{ad}(\rho_i)),$$

with obstructions in $H^2(G_{\mathbb{Q}}, \mathrm{ad}(\rho_i))$. These measure the extent to which Galois representations with radical or nested solvability constraints can deform, and what types of singularities or rigidity they exhibit.

Future Work 13.4. *Study the geometry of the tower of functors* $\mathcal{D}_{\mathrm{solv}}$*:*

$$D_{\rho^{\mathrm{rad}}} \to D_{\rho^{\mathrm{nest}}} \to D_{\rho^{\mathrm{solv}}} \to D_{\rho},$$

examining their representability, lifting properties, tangent behavior, and compatibility with modular and motivic deformation spaces.

13.3. Radical Class Field Theory (RCFT).

We propose a generalization of class field theory from the abelian to the solvable-radical domain. Let $\mathcal{R}_K$ denote the radical ideal class group, defined via ideals that become principal after adjoining radicals.

Definition 13.5. The radical Hilbert class field H_K^{rad} is the maximal unramified subfield of K^{rad}, and the radical ray class field $K^{\mathrm{rad}}(\mathfrak{m})$ is defined by radical ramification constraints.

Conjecture 13.6. *There exists a global reciprocity theory assigning to each radical modulus a class field with Galois group in* $\mathcal{G}_K^{\mathrm{rad}} := \mathrm{Gal}(K^{\mathrm{rad}}/K)$*, functorial under base change and compatible with local radical ramification filtrations.*

This theory would encompass radical analogues of Hilbert and ray class fields, radical conductors, and moduli, potentially extending to a radical Langlands program with solvable symmetry.

13.4. **Modularity and Geometry of Radical Galois Representations.** We examine whether radical and nested-solvable Galois representations arise from geometric or automorphic sources:

- **Elliptic curves** with torsion fields inside $\mathbb{Q}^{\mathrm{rad}}$ or $\mathbb{Q}^{\mathrm{nest}}$;
- **Tame ramification** in radical towers;
- **Moduli of curves** with radical function fields.

Conjecture 13.7. *There exists a radical stratification of the moduli space of curves, and radical Galois representations correspond to Hecke eigenforms with solvable images in* GL_n*.*

Future Work 13.8. *Define a* radical modularity theory *wherein automorphic forms correspond to radical Galois representations, encode radical descent in their coefficients, and admit radical Langlands duals.*

Summary. These future directions point toward a vast expansion of algebraic number theory beyond the abelian case, connecting Galois stratification, representation theory, deformation geometry, reciprocity laws, and modular forms into a radical solvability framework. The program opens a new vision for classifying and understanding the full complexity of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through a rich solvability lens.

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