CATEGORICAL AND HIGHER-ORDER LOGICAL INTERPRETATIONS OF SCHNIRELMANN-TYPE DENSITY

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ABSTRACT. We develop categorical and higher-order logical foundations for Schnirelmann-type density by interpreting additive closure as colimits, supports as sieves, and densities as functorial invariants. The goal is to reframe additive number theory within topos-theoretic and homotopy-theoretic contexts.

1. Additive Closure as a Categorical Colimit

Definition 1.1 (Additive Diagram)**.** Let $\mathcal{A}$ be a category with objects indexed by $k \in \mathbb{N}$ and morphisms corresponding to additive sums $A^{\oplus k} \to G$. Define a diagram $\mathcal{D}_A : \mathbb{N} \to \mathbf{Set}$ by $k \mapsto kA$.

Definition 1.2 (Additive Colimit Closure)**.** A is said to be *colimit-additively closed* if $\operatorname{colim}(\mathcal{D}_A) = G$.

2. Topos-Theoretic Density

Definition 2.1 (Sieve Support)**.** Let $A \in \mathcal{E}$, a Grothendieck topos. Define the sieve $\mathcal{S}_A$ on $1 \to G$ such that $\mathcal{S}_A(U) = \{u \in U : u \in A\}$.

Definition 2.2 (Topos Density)**.** The density of A in $\mathcal{E}$ is the degree to which $\mathcal{S}_A$ covers G. Formally, $\sigma_{\mathcal{E}}(A)$ is a morphism in Ω.

3. Higher-Order Logical Extensions

Definition 3.1 (Second-Order Schnirelmann Density Predicate)**.** In second-order arithmetic, define a predicate:

$$\Sigma(A) := \forall X \subseteq \mathbb{N}\ \exists k \text{ such that } kA \supseteq X \text{ or } X \cap \mathbb{N} \setminus kA \text{ is finite.}$$

Proposition 3.2. *$\Sigma(A)$ holds iff A is cofinal-dense in the arithmetic hierarchy.*

4. Homotopy and ∞-Categorical Viewpoint

Definition 4.1 (Additive Simplicial Object)**.** Let $A_\bullet$ be a simplicial object where $A_n := kA$. Define the augmentation map $A_\bullet \to G$ and say A is dense if this map is an effective epimorphism.

Remark 4.2. This aligns additive closure with Kan complexes and classifying spaces.

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5. Future Work

- Model Schnirelmann closure in cohesive toposes
- Use modal logic to quantify closure under internal homs
- Extend to sheaf-theoretic and stack-theoretic density
- Relate additive colimits to spectral sequences and descent