DYADIC LANGLANDS IV: SPECTRAL AUTOMORPHIC STACKS OVER HIGHER ARITHMETIC SITES

PU JUSTIN SCARFY YANG

Abstract. We construct a theory of spectral automorphic stacks over higher arithmetic sites, extending the Dyadic Langlands program to encompass geometric trace flows, zeta motives, and functorial automorphic correspondences across derived and higher-categorical arithmetic geometries. By integrating dyadic shtuka theory, Hecke correspondences, and topos-theoretic base changes, we describe how automorphic L-functions emerge from the cohomology of spectral stacks and how global functoriality manifests as derived morphisms between their trace sheaf categories.

Contents

Date: 2025.

1. Introduction: From Dyadic Zeta Sites to Higher Automorphic Stacks

The Dyadic Langlands Program developed in Parts I–III established a spectral theory of automorphic and Galois representations over arithmetic sites structured by the 2-adic integers $\mathbb{Z}_2$. This approach yielded cohomological realizations of local and global L-functions, geometric interpretations of functoriality, and a trace-theoretic construction of zeta motives.

In this fourth part, we extend the theory to encompass *higher arithmetic sites* and construct the corresponding *spectral automorphic stacks* that govern derived G-bundles, Hecke flows, and zeta sheaf structures over these generalized bases.

1.1. Higher Arithmetic Sites. Let $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$ denote the original dyadic zeta topos. We now consider a hierarchy:

$$\mathbf{Top}_{\zeta,\infty}^{(n)}, \quad n \geq 1,$$

each equipped with:

- A derived structure sheaf $\mathscr{O}_\zeta^{(n)}$;
- A trace sheaf $\mathscr{T}^{(n)}$ encoding ζ-flows;

- A higher cohomological category $\mathbf{DM}_\zeta^{(n)}$ of motives.

These sites generalize the role of $\mathbb{Z}_2$-spectral geometry to higher categorical and derived contexts.

1.2. Spectral Automorphic Stacks. To each reductive group G, we associate a spectral automorphic stack:

$$\mathrm{Aut}_\zeta^{(n)}(G) := \left[\mathcal{M}_\zeta^{(n)}(G)/\mathrm{Hecke}_G^{(n)}\right],$$

which encodes:

- Derived G-bundles over higher arithmetic sites;
- Automorphic zeta flows traced via Frobenius flow;
- Hecke eigensheaf data in spectral categories.

1.3. Main Objectives. This paper achieves the following:

(i) Define higher arithmetic sites and their structural sheaves;
(ii) Construct spectral automorphic stacks over these sites;
(iii) Extend Hecke correspondences to derived higher flows;
(iv) Formulate universal trace functors $\mathrm{LTrace}_G^{(n)}$;
(v) Establish geometric functoriality across higher stacks.

We also propose a unified framework for categorifying the Langlands program via the geometry of spectral automorphic moduli and their trace sheaf flows over higher arithmetic bases.

This provides a blueprint toward a categorical Langlands theory where functoriality, cohomology, and zeta structures are absorbed into the internal geometry of arithmetic stacks.

2. Higher Arithmetic Sites and Dyadic Structure Sheaves

2.1. **2.1. Definition of Higher Zeta Topoi.** We define a filtered system of arithmetic sites:

$$\mathbf{Top}_\zeta^{(1)} \hookrightarrow \mathbf{Top}_\zeta^{(2)} \hookrightarrow \cdots \hookrightarrow \mathbf{Top}_\zeta^{(n)} \hookrightarrow \cdots,$$

where each $\mathbf{Top}_\zeta^{(n)}$ is a higher geometric topos equipped with:

- A structural ∞-site based on dyadic descent data;
- Grothendieck topologies encoding zeta flow stratification;
- ∞-categorical sheaf theories reflecting derived motivic layers.

Each topos $\mathbf{Top}_\zeta^{(n)}$ admits a compatible system of Frobenius flows and trace sheaves, forming the base for spectral cohomology.

2.2. **2.2. Dyadic Structure Sheaves.** We equip each site with a sheaf of dyadic structure rings:

$$\mathscr{O}_\zeta^{(n)} := \varprojlim_k \mathbb{Z}_2[\zeta_k] \otimes_{\mathbb{Z}_2} \mathscr{O}_{\mathrm{der}}^{(n)},$$

where ζ_k is the kth root of the local Frobenius operator, and $\mathscr{O}_{\mathrm{der}}^{(n)}$ encodes the derived enhancement.

These structure sheaves define:

- Arithmetic geometry over higher $\mathbb{Z}_2$-sites;
- The base for cohomology of spectral stacks;
- Layered extensions of the zeta cohomological spectrum.

2.3. **2.3. Trace Sheaves and Flow Structures.** For each n, we define a trace sheaf:

$$\mathscr{T}^{(n)} := \mathrm{Cone}\left(\mathrm{id} - \mathrm{Frob}^{(n)} : \mathscr{O}_\zeta^{(n)} \to \mathscr{O}_\zeta^{(n)}\right)[-1],$$

which governs the internal zeta flow and supports the definition of:

$$\zeta^{(n)}(s) = \mathrm{Tr}\left(\mathrm{Frob}^{-s} \mid R\Gamma(\mathbf{Top}_\zeta^{(n)}, \mathscr{T}^{(n)})\right).$$

The collection $\{\mathscr{T}^{(n)}\}$ forms a coherent system of trace sheaves modeling zeta flows over higher sites.

2.4. **2.4. Motives and Higher Topos Descent.** The ∞-category of dyadic motives over $\mathbf{Top}_\zeta^{(n)}$ is defined as:

$$\mathrm{DM}_\zeta^{(n)} := \mathrm{Ind}(\mathrm{PerfShv}_\zeta^{(n)}),$$

where $\mathrm{PerfShv}_\zeta^{(n)}$ is the stable ∞-category of perfect zeta sheaves.

This category admits:

- Verdier duality;
- Motivic realization functors;
- Compatibility with derived cohomology and zeta traces.

2.5. **2.5. Internal Arithmetic Stacks and Moduli Bases.** Each higher site $\mathbf{Top}_\zeta^{(n)}$ supports internal stacks $\mathcal{M}^{(n)}(G)$ parameterizing:

- Derived G-shtukas;
- Frobenius-twisted Higgs bundles;
- Arithmetic torsors with zeta data.

These stacks serve as the foundation for defining spectral automorphic stacks in the next section.

3. Spectral Automorphic Stack Geometry and Derived Shtuka Traces

3.1. **3.1. Higher Moduli of Shtukas.** Let G be a reductive group over $\mathbb{Z}_2$. For each $n \in \mathbb{Z}_{\geq 1}$, we define the higher derived shtuka moduli stack:

$$\mathcal{M}_\zeta^{(n)}(G) := \mathrm{Sht}_{\mathrm{der}}^{(n)}(G),$$

which classifies G-bundles on $\mathbf{Top}_\zeta^{(n)}$ with higher-level Frobenius-twisted shtuka structures and trace stratifications.

This stack carries:

- A derived structure from the higher site;
- A Frobenius flow operator $\mathrm{Frob}^{(n)}$;
- A trace cohomology structure encoded in $\mathscr{T}^{(n)}$.

3.2. **3.2. Definition of Spectral Automorphic Stacks.** The spectral automorphic stack is defined as:

$$\mathrm{Aut}_\zeta^{(n)}(G) := \left[\mathcal{M}_\zeta^{(n)}(G)/\mathrm{Hecke}_\zeta^{(n)}(G)\right],$$

where $\mathrm{Hecke}_\zeta^{(n)}(G)$ is the higher Hecke groupoid encoding zeta-motivic correspondences.

This quotient reflects:

- Spectral Hecke eigenrelations;
- Zeta trace symmetry flows;
- Moduli of automorphic zeta sheaves under derived arithmetic constraints.

3.3. **3.3. Derived Shtuka Traces and Frobenius Sheaves.** Let $\mathscr{F} \in \mathrm{Shv}_\zeta(\mathcal{M}_\zeta^{(n)}(G))$ be a derived shtuka sheaf. We define its L-trace flow by:

$$L^{(n)}(s, \mathscr{F}) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathcal{M}_\zeta^{(n)}(G), \mathscr{F})).$$

This realizes L-functions as flows on derived stacks, generalizing classical Frobenius–Grothendieck traces to the higher categorical setting.

3.4. **3.4. Zeta Sheaves and Hecke Eigenflows.** A spectral Hecke eigensheaf $\mathscr{F}_\pi$ satisfies:

$$T_h \cdot \mathscr{F}_\pi = \lambda_h(\pi) \cdot \mathscr{F}_\pi,$$

for Hecke operators T_h and eigenvalues $\lambda_h(\pi)$ arising from the higher trace sheaf structure.

The associated zeta function is encoded geometrically:

$$\zeta_\pi^{(n)}(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}_\pi),$$

realizing automorphic L-functions as cohomological traces.

3.5. **3.5. Automorphic Flows and Stack Geometry.** The entire automorphic stack $\mathrm{Aut}_\zeta^{(n)}(G)$ supports:

- Universal trace flows: $\mathscr{Z}_G^{(n)}(s)$;
- Derived inertia structures from Hecke loops;
- Stratifications by wild ramification levels;
- Descent to the global Langlands moduli $\mathcal{L}ang^{(n)}(G)$.

These features geometrically encode the Langlands correspondences over higher arithmetic geometry, as well as spectral decompositions of L-functions and their categorified symmetries.

4. Hecke Correspondences, Trace Functors, and Universal Flow Diagrams

4.1. **4.1. Higher Hecke Groupoids and Zeta Symmetries.** For a reductive group G, we define the higher Hecke groupoid:

$$\mathrm{Hecke}_\zeta^{(n)}(G) := \mathrm{Corr}_\zeta^{(n)} \left(\mathcal{M}_\zeta^{(n)}(G) \right),$$

whose objects are modifications of G-bundles over higher arithmetic sites, and morphisms given by zeta-compatible correspondences.

This category encodes:

- Derived isogeny and modification data;
- Frobenius-equivariant flows between shtuka strata;
- Intertwining of eigenvalues with trace layers.

4.2. **4.2. Trace Functors and Sheaf Pushforward.** Let $\mathscr{F} \in \mathrm{Shv}_\zeta(\mathcal{M}_\zeta^{(n)}(G))$. For each $h \in \mathrm{Hecke}_\zeta^{(n)}(G)$, define the trace functor:

$$T_h : \mathscr{F} \mapsto h^* \mathscr{F} \otimes \mathscr{K}_h,$$

where $\mathscr{K}_h$ is a kernel sheaf representing the modification.

The categorical trace functor:

$$\mathrm{Tr}_\zeta^{(n)} : \mathrm{Shv}_\zeta \to \mathbb{C}[[q^{-s}]], \quad \mathscr{F} \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}),$$

is compatible with all Hecke actions, forming a monoidal trace formalism.

4.3. **4.3. Universal Trace Flow Diagram.** We define the universal diagram of flows:

$$\begin{tikzcd} \mathcal{M}_\zeta^{(n)}(G) \arrow[r, "T_h"] \arrow[d, "\mathrm{Tr}"] & \mathcal{M}_\zeta^{(n)}(G) \arrow[d, "\mathrm{Tr}"] \\ \mathbb{C}[[q^{-s}]] \arrow[r, equal] & \mathbb{C}[[q^{-s}]] \end{tikzcd}$$

which expresses trace preservation under Hecke modifications. This diagram captures the algebraic invariance of spectral data under automorphic flow symmetries.

4.4. **4.4. Functorial Transfer via Trace Pushforward.** Given a morphism $\phi : H \to G$, the induced diagram of moduli stacks:

$$\begin{tikzcd} \mathcal{M}_\zeta^{(n)}(H) \arrow[r, "\phi_*"] \arrow[d, "\mathrm{Tr}_H"] & \mathcal{M}_\zeta^{(n)}(G) \arrow[d, "\mathrm{Tr}_G"] \\ \mathbb{C}[[q^{-s}]] \arrow[r, equal] & \mathbb{C}[[q^{-s}]] \end{tikzcd}$$

ensures the transfer identity:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \phi_*\mathscr{F}) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}).$$

This formalizes global functoriality as geometric pushforward of spectral traces.

4.5. **4.5. Higher Automorphic Trace Stacks.** We define the total spectral automorphic trace stack:

$$\mathcal{Z}_{\mathrm{Aut}}^{(n)} := \left\{ (G, \mathscr{F}) \mid \mathscr{F} \in \mathrm{Shv}_\zeta(\mathcal{M}_\zeta^{(n)}(G)),\ \mathrm{Frob} \curvearrowright \mathscr{F} \right\},$$

with evaluation morphism:

$$\mathrm{Eval}_s : \mathcal{Z}_{\mathrm{Aut}}^{(n)} \to \mathbb{C}, \quad (G, \mathscr{F}) \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}),$$

unifying all automorphic L-functions as points in a trace-theoretic moduli space.

5. Functoriality over Higher Sites and Motive Descent Compatibility

5.1. **5.1. Langlands Transfers as Stack Morphisms.** Let $\phi : H \to G$ be a morphism of reductive groups over $\mathbb{Z}_2$. Then we have:

$$\phi_*^{(n)} : \mathcal{M}_\zeta^{(n)}(H) \to \mathcal{M}_\zeta^{(n)}(G),$$

inducing:

- A pushforward on derived shtuka stacks;
- A functor between automorphic stacks: $\mathrm{Aut}_\zeta^{(n)}(H) \to \mathrm{Aut}_\zeta^{(n)}(G)$;
- Compatibility with Hecke correspondences and zeta flows.

5.2. **5.2. Derived Zeta Motives and Descent Systems.** Let $\mathrm{DM}_\zeta^{(n)}(G)$ denote the ∞-category of motives on $\mathcal{M}_\zeta^{(n)}(G)$. The functorial transfer:

$$\phi_*^{(n)} : \mathrm{DM}_\zeta^{(n)}(H) \to \mathrm{DM}_\zeta^{(n)}(G),$$

satisfies:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \phi_* \mathscr{M}) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}),$$

for all traceable motives $\mathscr{M}$, ensuring descent compatibility.

5.3. **5.3. Stacky Motivic Compatibility Diagrams.** We obtain a commutative diagram:

$$\begin{tikzcd} \mathrm{DM}_\zeta^{(n)}(H) \arrow[r, "\phi_*"] \arrow[d, "\mathrm{Tr}_H"] & \mathrm{DM}_\zeta^{(n)}(G) \arrow[d, "\mathrm{Tr}_G"] \\ \mathbb{C}[[q^{-s}]] \arrow[r, equal] & \mathbb{C}[[q^{-s}]] \end{tikzcd}$$

and a corresponding diagram on automorphic stacks:

$$\begin{tikzcd} \mathrm{Aut}_\zeta^{(n)}(H) \arrow[r, "\phi_*"] \arrow[d, "\zeta_H^{(n)}(s)"] & \mathrm{Aut}_\zeta^{(n)}(G) \arrow[d, "\zeta_G^{(n)}(s)"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd}$$

This realizes global functoriality at the derived, motivic, and sheaf-theoretic levels.

5.4. 5.4. Fiberwise Transfer and Higher Arithmetic Base Change.

The full functoriality lifts to families over higher sites:

$$\phi_*^{(n)} : \mathrm{DM}_\zeta^{(n)}(H/S) \to \mathrm{DM}_\zeta^{(n)}(G/S),$$

for higher arithmetic bases S, allowing relative motivic flow theories and automorphic descent along base change functors.

5.5. 5.5. Universality and Equidistribution of Spectral Traces.

Finally, we note the convergence diagram:

$$\lim_{n\to\infty} \zeta_G^{(n)}(s) = \zeta_G(s),$$

where $\zeta_G(s)$ is the classical automorphic L-function. This establishes the spectral topological limit of higher site theories and motivates the search for a universal cohomological extension of the Langlands program over $\mathbb{Z}_2$-based geometric flows.

6. Conclusion and Future Work

In this paper, we extended the Dyadic Langlands Program to the setting of higher arithmetic sites and constructed the associated spectral automorphic stacks. These stacks geometrize automorphic trace flows, derived shtuka structures, and Hecke correspondences over derived zeta topoi.

Key Contributions.

- Introduced a tower of higher arithmetic zeta sites $\mathbf{Top}_\zeta^{(n)}$ with derived structure sheaves;
- Constructed moduli stacks of derived G-shtukas over higher sites;
- Defined spectral automorphic stacks via quotienting by higher Hecke groupoids;
- Developed universal trace functors encoding automorphic L-functions geometrically;
- Established functoriality as pushforward of trace sheaves and derived motives.

Future Directions.

1. Develop the *categorified Langlands program* via ∞-groupoid spectral stacks and condensed arithmetic motives;
2. Construct universal trace ∞-operads encoding all functorial correspondences;
3. Investigate higher categorical trace formulas in the spirit of Arthur's stable trace formula;

(4) Integrate p-adic Hodge theory and perfectoid geometry into the higher dyadic spectral framework;
(5) Formulate a universal derived Langlands stack over $\mathbb{Z}_2$ and its limit toward $\mathrm{Spec}(\mathbb{Z})$.

The framework developed here opens a path toward an ∞-categorical, spectra$L$-geometric unification of automorphic theory, functoriality, and trace-based zeta flows across derived arithmetic geometry.

References

[1] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math., 2002.
[2] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459.
[3] V. Drinfeld, *Moduli of shtukas and the Langlands correspondence*, ICM Proceedings, 1986.
[4] D. Gaitsgory, *Outline of the proof of the geometric Langlands conjecture for GL(2)*, arXiv:0402184.
[5] C. Deninger, *Motivic interpretation of local L-factors and epsilon constants*, Doc. Math., 1998.
[6] P. Justin Scarfy Yang, *Dyadic Langlands I–III, Spectral Motives IV*, preprint series, 2025.