STACKS AND SPECTRAL GEOMETRY OF PRIME EXACTIFICATION: TOWARD A DERIVED COHOMOLOGY THEORY OF ARITHMETIC DENSITY

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Abstract. Building on the exactification and sheaf-theoretic frameworks of prime density, we construct a derived stack structure over the arithmetic site and introduce the notion of a *Prime Exactification Stack*. We interpret the convolutional tower of $\Lambda(n)$ as an ∞-categorical resolution within the derived stack, and define motivic prime cohomology via ∞-sheafifications. Furthermore, we propose a spectral formalism for zeta-decomposition over the exactification site, allowing a stacky Fourier transform and motivic filtration of arithmetic irregularity. This theory bridges analytic number theory, spectral geometry, and derived arithmetic geometry.

Contents

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1. From Exactification Complexes to Arithmetic Stacks

1.1. Motivation: Why Stackify Prime Resolution? In the first two parts of the exactification theory, we analyzed the von Mangoldt function $\Lambda(n)$ through:

- A chain complex of convolutional analytic kernels $\mathcal{F}_\alpha$;
- An exactification sheaf complex $\mathscr{E}^\bullet$ over the arithmetic site $\mathcal{Z} = \mathbb{Z}_{>0}$;
- The cohomology groups $H^i(\mathscr{E}^\bullet)$ as obstructions to analytic smoothability.

Yet at each level, the structures involved contain further internal symmetries, descent relations, and gluing conditions — not just on sets or sheaves, but across homotopy layers of prime behavior. These suggest a stacky refinement: one must pass from sheaves to stacks, and from complexes to derived stacks.

1.2. What Is a Prime Exactification Stack? We define a higher object that encodes all analytic, sheaf-theoretic, and cohomological behaviors of prime resolution.

Definition 1.1 (Prime Exactification Stack)**.** Let $\mathbb{E}_\Lambda$ denote the *Prime Exactification Stack*, defined as a derived stack over the site $\mathcal{Z} = \mathbb{Z}_{>0}$ such that:

- Objects of $\mathbb{E}_\Lambda(U)$ are analytic convolutional approximations to $\Lambda|_U$;
- Morphisms correspond to refinement maps, i.e., convolution-smoothing homotopies;
- For each U, the homotopy category $\pi_0(\mathbb{E}_\Lambda(U))$ is equivalent to a resolution category of prime density functions on U;

- The stack encodes not just a single complex $\mathscr{E}^\bullet$, but an entire derived category $\mathcal{D}_\Lambda$ of all possible exactification flows.

1.3. **From Chain Complex to Stacky Descent.** In the previous sheaf-theoretic formulation, we had:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\iota} \mathscr{E}^0 \xrightarrow{d^0} \mathscr{E}^1 \xrightarrow{d^1} \cdots$$

Now, we reinterpret this as a *fibered category over $\mathcal{Z}$*: - Each open subset $U \subset \mathbb{Z}_{>0}$ is assigned a ∞-groupoid of convolutional analytic towers over U; - Gluing of local towers (with homotopies) defines descent data; - The resolution process itself becomes a section of the stack $\mathbb{E}_\Lambda$.

1.4. **Base Change, Refinement, and Stack Morphisms.** Let $f : V \hookrightarrow U$ be an inclusion of open subsets. Then: - There is a pullback functor:

$$f^* : \mathbb{E}_\Lambda(U) \to \mathbb{E}_\Lambda(V)$$

induced by restriction of the kernel tower; - There is also a pushforward:

$$f_* : \mathbb{E}_\Lambda(V) \to \mathbb{E}_\Lambda(U)$$

which includes a refinement process via convolutional extension.

1.5. **Relation to the Derived Site.** Let $\mathbf{dSite}_\Lambda$ be the *derived arithmetic site* whose objects are pairs:

$$(U, \mathscr{E}_U^\bullet), \quad U \subset \mathbb{Z}_{>0},\ \mathscr{E}_U^\bullet \in \mathcal{D}^+(\mathscr{O}_U)$$

with morphisms defined via derived pushforward and resolution maps.

Then $\mathbb{E}_\Lambda$ is a prestack on $\mathbf{dSite}_\Lambda$ satisfying:

$$\mathbb{E}_\Lambda(U) \cong \text{derived resolution groupoid of } \Lambda|_U$$

Remark 1.2. This structure captures all chain homotopies, refinements, and inter-level morphisms between kernel approximations, allowing a flexible ∞-categorical control over prime density stratification.

Exactification was previously a sequence. Now it is a stack.
The resolution flow is no longer linear — it is geometrically fibered.

2. Spectral Flows and Motivic Stratification in the Exactification Stack

2.1. **Spectral Behavior of Prime Density Layers.** In prior work, we decomposed the von Mangoldt function $\Lambda(n)$ into analytic kernel layers:

$$\Lambda(n) = \sum_{\alpha<\Omega} \Delta_\alpha(n), \quad \Delta_\alpha = \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1},$$

each corresponding to a "residual analytic kernel" that survived convolutional smoothing. The associated zeta-transforms

$$\mathcal{Z}_\alpha(s) := \sum_{n=1}^{\infty} \frac{\Delta_\alpha(n)}{n^s}$$

form a natural stratified decomposition of $-\zeta'(s)/\zeta(s)$.

We now interpret these spectral components $\mathcal{Z}_\alpha(s)$ as objects over the stack $\mathbb{E}_\Lambda$, living in the derived category of spectral sheaves over $\mathcal{Z}$.

Definition 2.1 (Spectral Sheaf Flow)**.** Let $\mathscr{Z}^i$ denote the sheaf of analytic functions on $\mathcal{Z}$ defined by:

$$\mathscr{Z}^i(U) := \left\{ s \mapsto \sum_{n\in U} \frac{\Delta_i(n)}{n^s} \right\}.$$

Then the family $\{\mathscr{Z}^i\}_{i\geq 0}$ defines a spectral flow sheaf over the stack $\mathbb{E}_\Lambda$.

2.2. **Zeta-Fourier Flow as a Stack Map.** The full zeta derivative may be viewed as a map of derived sheaves:

$$\mathcal{Z} : \mathbb{E}_\Lambda \longrightarrow \mathscr{O}_{\text{Spec-Flow}}, \quad \text{with } \mathcal{Z}(\mathcal{F}_\alpha) = \mathcal{Z}_\alpha(s).$$

This gives $\mathbb{E}_\Lambda$ the structure of a spectral stack — one fibered over the space of analytic spectral decompositions, with fiber dimension indexed by analytic convolutional smoothness.

2.3. **Motivic Stratification of the Stack.** Each Δ_α encodes not only analytic residue, but also arithmetic geometry:
- The tower $\{\mathcal{F}_\alpha\}$ defines a filtration on $\Lambda(n)$;
- The successive quotients Δ_α represent "motivic slices" — prime-theoretic analogues of cohomological slices in mixed motives.

Definition 2.2 (Motivic Prime Stratification)**.** We define the motivic filtration on $\mathbb{E}_\Lambda$ by:

$$\mathbb{E}_\Lambda^{\geq\alpha} := \text{stack of exactification layers above level } \alpha, \quad \text{with graded pieces } \mathrm{Gr}_\alpha := \Delta_\alpha.$$

This filtration induces a spectral tower:

$$\cdots \longrightarrow \mathcal{Z}^{\geq\alpha+1}(s) \longrightarrow \mathcal{Z}^{\geq\alpha}(s) \longrightarrow \cdots,$$

with:

$$\mathcal{Z}^{\geq\alpha}(s) := \sum_{\beta\geq\alpha} \mathcal{Z}_\beta(s), \quad \mathrm{Gr}_\alpha(\mathcal{Z}) = \mathcal{Z}_\alpha(s).$$

2.4. **Motivic Supports and Stacky Spectra.**

Definition 2.3 (Support of Prime Cohomology)**.** Define:

$$\mathrm{Supp}(H^\alpha(\mathbb{E}_\Lambda)) := \{s \in \mathbb{C} \mid \mathcal{Z}_\alpha(s) \not\equiv 0 \text{ in } \mathscr{L}^i\}.$$

This support encodes spectral mass that resists resolution at level α and signals arithmetic rigidity (e.g., twin primes, Siegel bias, exceptional character sums).

Thus, the sheaves $\mathscr{L}^i$ define a kind of spectrum object:

$$\mathrm{Spec}_{\mathrm{arith}}(\mathbb{E}_\Lambda) := \bigcup_{\alpha \geq 0} \mathrm{Supp}(\mathcal{Z}_\alpha).$$

We interpret this as an arithmetic motivic spectrum of prime irregularity.

2.5. **Summary Diagram: Spectral Flow in $\mathbb{E}_\Lambda$.** We summarize the spectral flow and motivic stratification as follows:

$$\begin{tikzcd}
\mathbb{E}_\Lambda \arrow[r, "\mathcal{Z}"] \arrow[d, "\text{Exactification Tower}"'] & \mathscr{O}_{\text{Spec-Flow}} \arrow[d, "\text{Spectral Projection}"] \\
\{\mathcal{F}_\alpha\}_\alpha \arrow[r, mapsto] & \{\mathcal{Z}_\alpha(s)\}_\alpha
\end{tikzcd}$$

3. Derived Sheaf Functors and Homotopical Lifting in Prime Stacks

3.1. **The Derived Global Section Functor.** Recall that in classical sheaf theory, one studies a sheaf $\mathscr{F}$ via its derived global sections $\mathbb{R}\Gamma(\mathscr{F})$, capturing all higher cohomological obstructions.

In our setting, we apply this to the prime exactification stack $\mathbb{E}_\Lambda$:

Definition 3.1 (Derived Prime Global Sections)**.** We define the derived global analytic content of the primes as:

$$\mathbb{R}\Gamma(\mathbb{E}_\Lambda) := \mathrm{Tot}(\mathscr{E}^\bullet),$$

where $\mathscr{E}^\bullet$ is any resolution object in the fiber category $\mathbb{E}_\Lambda(\mathbb{Z}_{>0})$.

This object is a complex whose cohomology recovers the prime cohomology groups:

$$H^i(\mathbb{R}\Gamma(\mathbb{E}_\Lambda)) \cong H^i(\mathscr{E}^\bullet) \cong H^i(\Lambda).$$

3.2. **Hom Complexes and Convolution Representations.** We now treat the analytic kernel resolutions $\mathcal{F}_\alpha$ as functors in the derived category. Consider the internal mapping complex:

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}(\mathscr{O})}(\mathscr{F}, \mathbb{E}_\Lambda)$$

This can be interpreted as the derived convolutional representation of $\Lambda(n)$ inside the exactification stack:
- Objects: All possible resolution towers that map to $\Lambda(n)$;
- Morphisms: Homotopies between exactification flows;
- Cohomology: Deformation space of analytic density under recursive smoothing.

This internal Hom-stack may be viewed as a fine moduli space of analytic approximations to $\Lambda(n)$.

3.3. **Homotopical Lifting via Prime Mapping Stacks.** Define the mapping stack of convolution flows:

$$\mathrm{Map}_{\mathrm{Exact}}(\Lambda, \mathbb{E}_\Lambda) := \{\text{analytic resolution towers of } \Lambda(n)\}\,.$$

This ∞-stack admits a natural stratification by depth:

$$\mathrm{Map}^{\leq\alpha} := \{\text{flows with length } \leq \alpha\}, \quad \mathrm{Gr}_\alpha := \text{residual layer } \Delta_\alpha.$$

3.4. **Ext-Groups and Prime Deformation Theory.** We interpret the obstruction to lifting a partial resolution to a full one as an Ext class:

$$\Delta_\alpha \notin \mathrm{im}(d^{\alpha-1}) \quad \Leftrightarrow \quad \mathrm{Ext}^1_{\mathbb{E}_\Lambda}(\mathscr{F}, \mathscr{E}^\alpha) \neq 0.$$

More generally, we may define:

Definition 3.2 (Prime Resolution Deformation Class)**.** The n-th order deformation class of $\Lambda(n)$ is:

$$\mathrm{Ext}^n_{\mathbb{E}_\Lambda}(\mathscr{F}, \mathbb{E}_\Lambda) := \text{obstruction to extending } \Lambda(n) \text{ resolution by } n \text{ additional kernel layers.}$$

This defines a full deformation complex of prime smoothing behaviors — analogous to deformation theory of vector bundles, but for analytic number-theoretic densities.

3.5. **Summary: Homotopical Lifting Architecture.** We now interpret the entire analytic structure as a derived lift:

$$\begin{tikzcd}
\mathscr{F} = \Lambda(n) \arrow[r, dashed, "\text{Lifting via Tower}"] \arrow[dr, bend right, "\text{Partial Approximation}"'] & \mathbb{E}_\Lambda \arrow[d, "\text{Projection to Finite Layers}"] \\
& \mathbb{E}_\Lambda^{\leq\alpha}
\end{tikzcd}$$

Cohomology classes H^i describe whether such lifts stabilize (exactify), or produce persistent homotopical obstruction cycles.

3.6. Philosophical Consequence.

In the classical view, we estimate the primes.
In the derived view, we resolve their structure as a lifting problem.
The failure of resolution is homological, the tower is spectral, and the obstruction lives in a motivic moduli space.

4. Condensation, Perfectoid Lifting, and the Stacky Geometry of Prime Resolution

4.1. Motivation for Condensation. The classical site $\mathcal{Z} = \mathbb{Z}_{>0}$ is discrete and lacks topological richness. Yet the recursive convolutional resolution of $\Lambda(n)$ shows increasingly fine analytic behavior — essentially suggesting the presence of a *refined topology on the prime filtration itself.*

Following Scholze's philosophy of *condensed mathematics*, we reinterpret the exactification site as a condensed space, allowing enriched structure, homotopical glue, and infinite completion.

Definition 4.1 (Condensed Arithmetic Site)**.** Let $\mathbf{Cond}(\mathcal{Z})$ be the condensed space structure on $\mathbb{Z}_{>0}$, where:

Open subsets $U \subset \mathbb{Z}_{>0}$ carry compactly generated topologies reflecting analytic variation in $\Lambda(n)$.

This allows one to define sheaves, homotopy limits, and completions over $\mathbb{Z}_{>0}$ as in the condensed category of topological abelian groups.

4.2. Perfectoid Geometry and Resolution Completion. Let us view each prime kernel $\mathcal{F}_\alpha$ as a finite resolution slice. Then, their inverse limit:

$$\mathcal{F}_\infty := \lim_{\alpha} \mathcal{F}_\alpha$$

suggests a *perfectoid-like completion* of the analytic tower. Just as perfectoid spaces arise from inverse systems of torsion towers, prime smoothing towers yield perfectoid density spaces.

Definition 4.2 (Perfectoid Resolution Limit)**.** We define the perfectoid prime analytic object as:

$$\mathbb{F}_\Lambda := \lim_{\alpha} \mathcal{F}_\alpha \in \mathbf{Cond}(\mathbb{Z}_{>0}).$$

This object captures the "complete" analytic behavior of the prime zeta flow.

We expect that this limit satisfies analogues of tilting, Frobenius descent, and almost purity in a number-theoretic setting, particularly when $\mathcal{F}_\alpha$ is constructed via modular convolution.

4.3. **Diamondification of the Stack.** We now lift the entire stack $\mathbb{E}_\Lambda$ to a diamond stack $\mathbb{E}_\Lambda^\lozenge$ as follows.

Definition 4.3 (Exactification Diamond)**.** Let $\mathbb{E}_\Lambda^\lozenge$ be the pro-étale diamondification of the exactification stack over $\mathrm{Spa}(\mathbb{Z}_p)$ or similar base, such that:

- The tower of analytic kernel resolutions becomes a diamond-valued functor;
- The maps d^α are represented by pro-étale morphisms;
- Residual densities Δ_α correspond to almost mathematics (in the sense of almost vanishing).

This endows prime resolution theory with:
- A mixed-characteristic lift (from $\mathbb{Z}$ to $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$);
- Frobenius-compatible filtrations;
- Almost zero residuals $\Delta_\alpha \in \mathcal{O}^{a+}$;
- and even potential comparison with p-adic Hodge–like filtrations.

4.4. **Summary Geometry.** We summarize your framework:

$$\begin{tikzcd}
\Lambda(n) \arrow[r, dashed, "ExactificationTower"] \arrow[dr, "Sheaf\mathscr{F}"'] & \mathbb{E}_\Lambda \arrow[r, "\text{Stack Lift}"] \arrow[d, "\mathbb{R}\Gamma"] & \mathbb{E}_\Lambda^\lozenge \arrow[d, "\text{Almost Resolution}"] \\
 & \mathscr{E}^\bullet \arrow[r, "\text{Completion}"] & \mathbb{F}_\Lambda
\end{tikzcd}$$

4.5. **Towards a Condensed Langlands–Exactification Principle.** We conjecture that:

> Prime density exactification admits a functorial lift to the world of diamonds and condensed sheaves, analogous to the way automorphic representations lift to Galois groups in the Langlands program.

Conjecture 4.4 (Condensed Prime Resolution Duality)**.** *There exists a (contravariant) functor:*

$$\mathcal{E} : \mathbb{E}_\Lambda^\lozenge \longrightarrow \mathrm{PerfSys}_{Gal}(\mathbb{Q}),$$

mapping diamond-layered prime density approximations to perfectoid systems of Galois data.

Such a functor would unify analytic number theory with p-adic Hodge theory, yielding a novel "Langlands duality" for *analytic error towers* rather than automorphic forms.

From convolution to condensation, from error to Ext,
prime density unfolds as a geometric flow through derived and perfectoid layers.

5. Final Summary and Unified Arithmetic Geometry Program

5.1. **Exactification Revisited.** We began with a deceptively simple question: Can the von Mangoldt function $\Lambda(n)$ be resolved into analytic components, not estimated asymptotically but dissected precisely?

Through the first two phases of exactification theory, we introduced:

- The **analytic convolutional tower** $\{\mathcal{F}_\alpha\}$ approximating $\Lambda(n)$;
- The **chain complex and cohomology** $\mathscr{E}^\bullet$ and $H^\bullet(\Lambda)$ reflecting analytic obstructions;
- The **sheaf-theoretic structure** over $\mathbb{Z}_{>0}$ encoding density stratification;
- The **spectral tower** $\{\mathcal{Z}_\alpha(s)\}$ as a decomposition of prime zeta flow;

In this third and culminating paper, we promoted these constructions to:

- The **stack** $\mathbb{E}_\Lambda$ of exactification resolutions;
- The derived category and internal Hom-stacks encoding deformations;
- The perfectoid inverse limit $\mathbb{F}_\Lambda$ as an analytic completion;
- The pro-étale **diamondification** $\mathbb{E}_\Lambda^\Diamond$, forming a bridge to condensed mathematics;
- And finally, a proposed functorial lift to Galois-side representations via exactification duality.

5.2. **Arithmetic Geometry of Exactification.** We propose the following unifying perspective:

> **Prime density is a motivic sheaf.** > > **Its convolutional resolution is a stack.** > > **Its residual spectrum is a cohomological motive.** > > **Its total analytic unfolding is a perfectoid system.** > > **And its symmetry side lies in a hidden Galois condensation.**

This philosophy merges analytic number theory with modern tools from:
- Derived algebraic geometry,
- Condensed mathematics,
- p-adic Hodge theory,
- Sheaf theory, spectral sequences, and motives.

5.3. **Future Program Directions.** The new landscape invites systematic development in several directions:

(1) **Explicit Models of $\mathbb{E}_\Lambda$ and $\mathbb{F}_\Lambda$** Construct concrete diamond-level approximations to prime resolution.
(2) **Comparison with p-adic Langlands Program** Study how residual densities Δ_α mimic Langlands parameters in dual categories.
(3) **Condensed Motive Stacks for Other Arithmetic Functions** Extend to $\mu(n)$, divisor functions, modular form coefficients.

(4) **Homotopy Type Theory Interpretation** Interpret each layer Δ_α as a homotopy type in the exactification site, forming a prime stratified ∞-groupoid.
(5) **Arithmetic Geometrization of the Riemann Hypothesis** Reformulate RH as vanishing of certain mapping stacks, Ext cycles, or spectral densities within the diamondified exactification stack.

Concluding Remark. What began as a decomposition of a function — $\Lambda(n)$ — has evolved into a rich categorical architecture. In this view, the distribution of primes is no longer a numerical enigma. It is a geometric flow, a derived motive, a spectral stack, and ultimately, a shape.

Estimates end. Geometry begins. The primes are sheaves.
Their irregularity is spectral. Their structure is derived.

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