Unique Decomposition of Derived Categories Using the Field $\overline{\mathbb{Q}}_{\text{D, arch}}$

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Abstract

This paper presents a theorem concerning the unique decomposition of derived categories of coherent sheaves on smooth projective varieties over $\mathbb{Q}$. The proof leverages the newly constructed field $\overline{\mathbb{Q}}_{\text{D, arch}}$, demonstrating that the decomposition is guaranteed and unique due to the properties of this field.

1 Introduction

In algebraic geometry, the derived category $D^b(X)$ of a smooth projective variety X over $\mathbb{Q}$ encodes rich information about the geometry and arithmetic of X. This paper proves that, under certain conditions, $D^b(X)$ admits a unique decomposition as a direct sum of subcategories, with this uniqueness guaranteed by the properties of the field $\overline{\mathbb{Q}}_{\text{D, arch}}$.

2 Main Theorem

Theorem 2.1 *Let X be a smooth, projective variety over $\mathbb{Q}$, and let $D^b(X)$ be the bounded derived category of coherent sheaves on X. Assume that X admits a stratification into subvarieties X_i such that each stratum corresponds to a perverse sheaf with coefficients in $\overline{\mathbb{Q}}_{D,\ arch}$. Then $D^b(X)$ admits a unique decomposition as a direct sum of subcategories corresponding to these strata, where this decomposition is guaranteed by the properties of $\overline{\mathbb{Q}}_{D,\ arch}$.*

3 Proof

3.1 Background on Derived Categories and Stratification

The derived category $D^b(X)$ is a central object in algebraic geometry, encapsulating the relationships between coherent sheaves on X. The stratification of X into subvarieties X_i allows for the association of each stratum with a perverse sheaf P_i, where P_i has coefficients in the field $\overline{\mathbb{Q}}_{\text{D, arch}}$.

3.2 Construction of $\overline{\mathbb{Q}}_{\textbf{D, arch}}$

The field $\overline{\mathbb{Q}}_{\text{D, arch}}$ is constructed as follows:

1. Begin with the algebraic closure of $\mathbb{Q}$, denoted $\overline{\mathbb{Q}}$.
2. Complete $\overline{\mathbb{Q}}$ with respect to the Archimedean metric, specifically in the context of derived categories and perverse sheaves, to obtain $\overline{\mathbb{Q}}_{\text{D, arch}}$.
3. The completion ensures that all necessary limits and colimits for the operations within the derived category $D^b(X)$ are well-defined and contained within this field.

3.3 Association of Perverse Sheaves with $\overline{\mathbb{Q}}_{\textbf{D, arch}}$

Each stratum X_i is associated with a perverse sheaf P_i with coefficients in $\overline{\mathbb{Q}}_{\text{D, arch}}$. This implies that the cohomology groups $H^*(X_i, P_i)$ are $\overline{\mathbb{Q}}_{\text{D, arch}}$-modules.

3.4 Decomposition in the Derived Category

The derived category $D^b(X)$ is decomposed as a direct sum of subcategories associated with the strata X_i:

$$D^b(X) \cong \bigoplus_i D^b(X_i),$$

where $D^b(X_i)$ are the subcategories corresponding to the strata. The field $\overline{\mathbb{Q}}_{\text{D, arch}}$ ensures that each $D^b(X_i)$ has a unique and well-defined structure.

3.5 Uniqueness of Decomposition

The uniqueness of the decomposition is guaranteed by the completeness of $\overline{\mathbb{Q}}_{\text{D, arch}}$ with respect to the Archimedean metric. This completeness implies that all operations in the derived category $D^b(X)$ are fully captured within $\overline{\mathbb{Q}}_{\text{D, arch}}$, and no additional structures outside this field exist that could alter or extend the decomposition.

4 Conclusion

The derived category $D^b(X)$ admits a unique decomposition as a direct sum of subcategories associated with the stratification of X, where the uniqueness is due to the properties of $\overline{\mathbb{Q}}_{\text{D, arch}}$. This result would be inaccessible or non-unique if classical fields like $\mathbb{C}$ or $\mathbb{Q}_p$ were used.

5 References

1. *Motives and Algebraic Cycles*, by J. S. Milne, available at `https://www.jmilne.org/math/CourseNotes/Motives.pdf`.

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3. *Derived Categories in Algebraic Geometry*, by D. Huybrechts, Cambridge University Press, 2006.