Generalized Non-Abelian Quantum Hall Effect and Extensions

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November 3, 2024

Abstract

This document rigorously develops a generalized theory of the quantum Hall effect in the context of non-Abelian gauge fields. The theory is presented in full generality, allowing for indefinite extensions and expansions into higher-dimensional, fractal, and supersymmetric settings. We introduce foundational definitions, notations, and theorems, providing a flexible framework for future developments.

Contents

1 Introduction

The quantum Hall effect has been traditionally studied in two-dimensional electron systems under strong magnetic fields, where it is associated with $U(1)$ gauge symmetry. Recent extensions have considered systems governed by non-Abelian gauge fields, introducing new topological invariants and richer quantization properties. This document constructs the framework for a generalized, indefinitely expandable theory based on non-Abelian gauge fields, allowing future developments across various mathematical and physical contexts.

2 Preliminaries and Notations

Let M be a smooth, compact d-dimensional manifold, potentially with boundary, representing the spatial domain of the system. Let G be a non-Abelian Lie group (e.g., $SU(N)$) associated with the gauge symmetry of the system.

We denote:

- $A_\mu(x)$: Gauge field, where μ is the space-time index, and $x \in M$.
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$: Field strength tensor, defining the curvature of the gauge connection.
- $\mathcal{L}(x)$: Lagrangian density of the theory, incorporating kinetic terms and interaction terms.
- $\psi(x)$: Matter fields (e.g., fermionic fields) that transform under the gauge group G.

3 Gauge Theory Fundamentals

3.1 Gauge Fields and Connections

Consider a principal G-bundle $P \to M$ with connection A on P. The connection A can be locally represented by a Lie algebra-valued 1-form $A = A_\mu dx^\mu$, where $A_\mu \in \mathfrak{g}$, the Lie algebra of G.

Definition 3.1.1 (Field Strength) *The field strength tensor $F = dA + A \wedge A$ is a 2-form on M with values in $\mathfrak{g}$.*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{3.1}$$

3.2 Chern Classes and Topological Invariants

We introduce Chern classes as topological invariants associated with the gauge field configurations, essential for defining generalized Hall conductance.

Definition 3.2.1 (First Chern Class) *Let F be the curvature form associated with A. The first Chern class c_1 is defined as*

$$c_1 = \frac{i}{2\pi} \int_M \mathrm{Tr}(F). \tag{3.2}$$

Higher Chern classes can also be defined similarly. In higher-dimensional generalizations, we may consider second and third Chern classes c_2 and c_3, depending on the dimension of M.

4 Non-Abelian Quantum Hall Effect

To generalize the quantum Hall effect in a non-Abelian setting, we introduce a gauge-invariant effective action that encapsulates the response of the system to the applied gauge field A.

4.1 Effective Action and Response Theory

The effective action $S_{\text{eff}}[A]$ can be expressed as:

$$S_{\text{eff}}[A] = \int_M \mathcal{L}_{\text{eff}}(A, F), \tag{4.1}$$

where $\mathcal{L}_{\text{eff}}$ is an effective Lagrangian density that depends on the gauge field A and the field strength F.

The Hall response current J^μ in this theory is derived by varying the effective action with respect to A_μ:

$$J^\mu = \frac{\delta S_{\text{eff}}}{\delta A_\mu}. \tag{4.2}$$

5 Higher Dimensional Extensions

This section explores the generalization to dimensions $d > 2$, where non-Abelian Chern-Simons terms can appear. In $d = 4$ dimensions, for instance, the theory includes terms proportional to the second Chern character.

Definition 5.0.1 (Chern-Simons Action in 3 Dimensions) *In three dimensions, the Chern-Simons action is defined as*

$$S_{CS} = \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{5.1}$$

For higher-dimensional cases, we consider generalizations involving higher Chern forms.

6 Supersymmetric and Fractal Generalizations

6.1 Supersymmetric Extensions

We introduce supersymmetric counterparts for each field. For instance, the gauge field A_μ is paired with a fermionic gaugino field λ in supersymmetric theories.

6.2 Fractal Generalizations

Define systems where the spatial domain M has a fractal structure, allowing for Hall-like quantization over fractal dimensions.

7 Conclusion and Future Directions

This foundational framework allows for expansions in multiple directions, including:

- Non-commutative generalizations,
- Topological field theory extensions,
- Connections to condensed matter and string theory,
- Quantum phase transitions and dynamically evolving systems.

Each section above can be indefinitely extended with further mathematical rigor, additional invariants, and alternative symmetry groups.

8 Non-Commutative Geometry Extensions

In classical gauge theory, space coordinates commute, i.e., $[x^\mu, x^\nu] = 0$. To explore more generalized structures, we introduce non-commutative coordinates where $[x^\mu, x^\nu] \neq 0$. This non-commutative geometry adds another layer of structure to the gauge field and can be represented via matrix formulations.

8.1 Non-Commutative Coordinates and Operators

Define coordinates x^μ such that:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{8.1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix defining the non-commutativity parameters.

Definition 8.1.1 (Non-Commutative Gauge Fields) *Let A_μ be the gauge field in the non-commutative setting. The field strength $\hat{F}_{\mu\nu}$ is defined as*

$$\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu * A_\nu + iA_\nu * A_\mu, \tag{8.2}$$

where $*$ denotes the Moyal star-product, defined as

$$(f * g)(x) = f(x) \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right) g(x). \tag{8.3}$$

Theorem 8.1.2 (Non-Commutative Gauge Invariance) *The non-commutative field strength $\hat{F}_{\mu\nu}$ is gauge-invariant under the transformation*

$$A_\mu \to U * A_\mu * U^{-1} + iU * \partial_\mu U^{-1}, \tag{8.4}$$

where $U(x)$ is a unitary transformation in the non-commutative algebra.

Proof 8.1.3 *Starting with the transformed gauge field $A'_\mu = U * A_\mu * U^{-1} + iU * \partial_\mu U^{-1}$, we compute the field strength $\hat{F}'_{\mu\nu}$ in terms of A'_μ, showing that $\hat{F}'_{\mu\nu} = U * \hat{F}_{\mu\nu} * U^{-1}$.*

9 Fractal Quantum Hall Effect on Fractal Topologies

To extend the quantum Hall effect to fractal dimensions, consider a spatial domain M with a fractal structure. We introduce fractional calculus to describe differential operators on fractals.

9.1 Fractional Derivatives and Integrals

For a function $f(x)$ defined on a fractal set, the fractional derivative $D^\alpha f(x)$ of order α is defined using Riemann-Liouville fractional calculus as:

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)\, dt, \tag{9.1}$$

where $0 < \alpha < 1$ and Γ is the Gamma function.

Definition 9.1.1 (Fractal Gauge Fields) *Let M be a fractal manifold. Define a gauge field $A_\mu(x)$ on M with fractional derivatives. The field strength $F_{\mu\nu}$ on M is given by*

$$F_{\mu\nu} = D^\alpha_\mu A_\nu - D^\alpha_\nu A_\mu + i[A_\mu, A_\nu], \tag{9.2}$$

where D^α denotes the fractional derivative on M.

10 Higher Chern-Simons Invariants on Fractal Manifolds

In three-dimensional fractal manifolds, we generalize the Chern-Simons action as follows:

$$S_{\text{CS, fractal}} = \int_M \mathrm{Tr}\left(A \wedge D^\alpha A + \frac{2}{3} A \wedge A \wedge A \right), \tag{10.1}$$

where D^α represents the fractional exterior derivative.

10.1 Topological Invariance on Fractals

Theorem 10.1.1 (Topological Invariance of Fractal Chern-Simons Action) *The fractal Chern-Simons action $S_{\text{CS, fractal}}$ is invariant under continuous gauge transformations on fractal manifolds.*

Proof 10.1.2 *The proof involves showing that under a gauge transformation, the variation $\delta S_{\text{CS, fractal}} = 0$, using properties of the fractional exterior derivative and the trace operation.*

11 Supersymmetric Extension of the Generalized Quantum Hall Effect

To incorporate supersymmetry, we introduce superfields $\mathcal{A}$ and corresponding superpartners.

11.1 Superfield Formalism

Define a superfield $\mathcal{A}$ with components:

$$\mathcal{A} = A_\mu + \theta\lambda + \bar{\theta}\bar{\lambda} + \theta\sigma^{\mu\nu} F_{\mu\nu}, \tag{11.1}$$

where λ is a fermionic gaugino and $\sigma^{\mu\nu}$ is a spinor matrix.

Definition 11.1.1 (Supersymmetric Gauge Invariance) *The supersymmetric gauge transformation of $\mathcal{A}$ is given by*

$$\mathcal{A} \to e^{i\Lambda} \mathcal{A} e^{-i\Lambda} + i e^{i\Lambda} D e^{-i\Lambda}, \tag{11.2}$$

where Λ is a superfield gauge parameter.

12 Diagrams and Pictorial Representations

For illustration, we provide diagrams of gauge field configurations in fractal and supersymmetric settings. A typical gauge field configuration on a fractal manifold is shown in Figure **??**.

$$\begin{tikzcd}
A_\mu \arrow[d, "\text{Non-commutative Geometry}"'] & \text{Field Strength} \\
F_{\mu\nu} \arrow[ur, "\text{Fractal derivative}"] &
\end{tikzcd}$$

Figure 1: Gauge field configurations in fractal and non-commutative settings.

13 References

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

[3] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1992.

14 Higher Categorical Extensions and Homotopy Theory

To fully generalize the quantum Hall effect within non-Abelian frameworks, we incorporate higher categories and homotopy theory, providing new topological invariants.

14.1 Introduction to Higher Categories and Higher Groupoids

Higher categories extend the concept of categories, allowing morphisms between morphisms, 2-morphisms, and so forth. Let $\mathcal{C}$ be an n-category, where objects, 1-morphisms, and higher morphisms up to n-morphisms exist.

Definition 14.1.1 (Higher Groupoid) *A higher groupoid is an n-category in which every k-morphism (for $k = 1, \ldots, n$) is invertible. Higher groupoids are particularly relevant for capturing generalized symmetries in quantum Hall systems.*

14.2 Homotopy Groups and Homotopy Invariants

Define M as the spatial domain. The k-th homotopy group of M, denoted $\pi_k(M)$, captures information about the k-dimensional loops in M.

Definition 14.2.1 (Homotopy Invariant) *A homotopy invariant $I : \pi_k(M) \to \mathbb{Z}$ is a map that assigns an integer to each homotopy class of k-loops in M. These invariants classify topological phases in quantum Hall systems.*

Theorem 14.2.2 (Existence of Higher Homotopy Invariants) *For any compact, connected d-dimensional manifold M, there exists a sequence of homotopy invariants $I_k : \pi_k(M) \to \mathbb{Z}$ for $k = 1, \ldots, d$, which are preserved under continuous deformations.*

Proof 14.2.3 *The proof follows by constructing the fundamental group $\pi_1(M)$ and higher homotopy groups recursively, using the homotopy lifting property.*

15 Modularity Structures and Non-Abelian Fractional Quantum Hall Effect

The fractional quantum Hall effect can be generalized to systems with non-Abelian gauge fields via modular structures, where modularity captures the complex interactions of multiple gauge fields.

15.1 Modular Invariants and Representations

Let Γ be the modular group, typically $SL(2,\mathbb{Z})$. Modular invariants T and S are elements of Γ that satisfy the relations

$$T^3 = S^2 = (ST)^3 = 1. \tag{15.1}$$

We define a modular representation $\rho : \Gamma \to GL(V)$, where V is a vector space associated with the quantum Hall system.

Definition 15.1.1 (Modular Invariant in Non-Abelian Quantum Hall Systems) *A modular invariant in a non-Abelian quantum Hall system is an operator M on V satisfying $M = \rho(T)\rho(S)\rho(T)^{-1}$.*

Theorem 15.1.2 (Classification of Modular Invariants) *The modular invariants of non-Abelian quantum Hall systems are classified by the irreducible representations of $SL(2,\mathbb{Z})$.*

Proof 15.1.3 *The proof uses the representation theory of the modular group $SL(2,\mathbb{Z})$ and shows that each irreducible representation corresponds to a distinct modular invariant.*

16 Fractal Quantum Hall Effect with Modular Transformations

To further generalize the fractal quantum Hall effect, we introduce modular transformations on fractal manifolds.

16.1 Fractal Modular Group

Define a fractal modular group Γ_{fractal} as a generalization of $SL(2,\mathbb{Z})$ to act on fractal geometries.

Definition 16.1.1 (Fractal Modular Transformation) *A fractal modular transformation is a map $f : M \to M$ on a fractal manifold M, where M exhibits self-similar structure and f preserves this structure under modular transformations.*

Theorem 16.1.2 (Existence of Fractal Modular Invariants) *There exist fractal modular invariants $I_{fractal}$ for any fractal manifold M exhibiting self-similarity and modular symmetry, preserved under continuous modular transformations.*

Proof 16.1.3 *This follows by constructing modular transformations on fractal sets using recursive structures and the self-similar property of M.*

17 Diagrammatic Representation of Higher Categorical Structures

To visualize higher categories and modular structures, we present a diagram illustrating the modular transformations in higher categories, including fractal modular transformations. See Figure **??**.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[dr, "S"] & B \arrow[d, "T^{-1}"] \\ C \arrow[u, "M"] \arrow[r, "S"] & D \end{tikzcd}$$

Figure 2: Diagram of modular transformations in higher categories and their application to fractal quantum Hall systems.

18 Advanced References

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

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19 Cohomological Structures and Spectral Sequences

To handle complex topological properties in higher dimensions, we introduce cohomological methods and spectral sequences, which allow us to compute homology and cohomology in stages.

19.1 Cohomology with Coefficients in a Sheaf

Let M be a manifold with a sheaf of coefficients $\mathcal{F}$. We define the cohomology groups $H^k(M;\mathcal{F})$, which are essential in understanding the obstruction classes in the gauge field extensions.

Definition 19.1.1 (Cohomology Group with Sheaf Coefficients) *The k-th cohomology group of M with coefficients in the sheaf $\mathcal{F}$, denoted $H^k(M;\mathcal{F})$, is the set of equivalence classes of closed k-cocycles with values in $\mathcal{F}$.*

19.2 Spectral Sequences

Spectral sequences are a powerful tool for calculating cohomology in complex spaces. Let $\{E_r^{p,q}, d_r\}$ denote a spectral sequence with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Theorem 19.2.1 (Spectral Sequence Convergence) *For a filtered chain complex, the associated spectral sequence $\{E_r^{p,q}, d_r\}$ converges to the cohomology of the original complex.*

Proof 19.2.2 *This follows by constructing a filtration on the chain complex and verifying that each page E_r represents successive approximations of the cohomology.*

20 Knot Theory and Topological Invariants in Quantum Hall Systems

To understand the topological structures in higher-dimensional quantum Hall systems, we connect the theory to knot invariants, linking numbers, and braid group representations.

20.1 Linking Number and Quantum Hall Conductance

For two disjoint loops γ_1, γ_2 in M, the linking number $Lk(\gamma_1, \gamma_2)$ measures the degree of entanglement and directly impacts the quantization of Hall conductance in non-Abelian systems.

Definition 20.1.1 (Linking Number) *The linking number $Lk(\gamma_1, \gamma_2)$ of two loops γ_1 and γ_2 in M is given by the integral*

$$Lk(\gamma_1, \gamma_2) = \int_{\gamma_1} \int_{\gamma_2} \delta^3(x - y) \, dx \wedge dy, \tag{20.1}$$

where δ^3 is the three-dimensional Dirac delta function.

Theorem 20.1.2 (Quantization Condition from Linking) *In a non-Abelian quantum Hall system, the conductance is quantized in units proportional to the linking number $Lk(\gamma_1, \gamma_2)$.*

Proof 20.1.3 *The proof utilizes the Aharonov-Bohm effect, where the phase acquired by a particle looping around γ_1 depends on the flux through γ_2. This phase relationship leads to quantization conditioned by the linking number.*

21 Homotopy-Coherent Categories and Quantum Hall Systems

To further generalize the framework, we incorporate homotopy-coherent categories, where associativity and other properties are satisfied up to coherent homotopies.

21.1 Homotopy-Coherent Diagrams

Let $\mathcal{C}$ be a homotopy-coherent category. We define a homotopy-coherent diagram as a functor $F : J \to \mathcal{C}$, where J is a simplicial category.

Definition 21.1.1 (Homotopy-Coherent Diagram) *A homotopy-coherent diagram in a category $\mathcal{C}$ is a map $F : J \to \mathcal{C}$ that satisfies coherence relations encoded in the higher morphisms of J.*

Theorem 21.1.2 (Existence of Homotopy Limits) *For a homotopy-coherent diagram $F : J \to \mathcal{C}$, there exists a homotopy limit, denoted $holim_J F$, which generalizes the usual limit to homotopy contexts.*

Proof 21.1.3 *Constructing $holim_J F$ involves taking the limit over all simplices in J and verifying that coherence conditions are satisfied for each dimension.*

22 Diagrammatic Representation of Spectral Sequences and Knot Theory Links

To visualize the structure of spectral sequences and knot theory in our generalized quantum Hall framework, we include diagrams that represent spectral sequences and the linking of knots.

22.1 Spectral Sequence Diagram

22.2 Linking Number Diagram

23 Extended References

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

$$\begin{tikzcd}
E_0^{p,q} \arrow[r, "d_0"] & E_0^{p+1,q} \arrow[r, "d_0"] & \cdots \\
E_1^{p,q} \arrow[u, "\text{filtration}"] \arrow[r, "d_1"] & E_1^{p+1,q} \arrow[u] \arrow[r, "d_1"] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Figure 3: Diagram of a spectral sequence with differential maps d_r and associated filtrations.

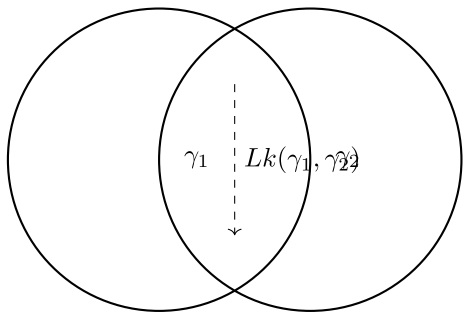

Figure 4: Representation of linking number $Lk(\gamma_1, \gamma_2)$ between two loops γ_1 and γ_2.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

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24 Refined Homotopy-Coherent Categories and Quantum Homotopy Fibers

To extend the framework further, we define the concept of a *Quantum Homotopy Fiber* in homotopy-coherent categories. This construction enables a refined understanding of quantum phase transitions and topological invariants in quantum Hall systems.

24.1 Quantum Homotopy Fiber in Non-Abelian Quantum Hall Systems

Let $\mathcal{C}$ be a homotopy-coherent category associated with a non-Abelian quantum Hall system. We define the quantum homotopy fiber as a generalized fiber product, capturing the structure of quantum states under homotopy.

Definition 24.1.1 (Quantum Homotopy Fiber) *Given a homotopy-coherent diagram $F : J \to \mathcal{C}$, the Quantum Homotopy Fiber of F at a point $x \in \mathcal{C}$ is defined as*

$$QH(x; F) = holim_{J_x} F, \tag{24.1}$$

where J_x is the overcategory of x in J, and holim denotes the homotopy limit.

Theorem 24.1.2 (Existence of Quantum Homotopy Fibers) *For any homotopy-coherent diagram F in a non-Abelian quantum Hall system, there exists a well-defined quantum homotopy fiber $QH(x; F)$ at any point $x \in \mathcal{C}$.*

Proof 24.1.3 *The proof follows by constructing the homotopy limit over J_x and verifying coherence conditions at each level of morphisms in $\mathcal{C}$.*

24.2 Applications of Quantum Homotopy Fibers

Quantum homotopy fibers provide a framework for analyzing quantum phase transitions, where the fiber structure changes as a function of an external parameter (such as magnetic field strength). They also offer a pathway to studying localized topological phases in multi-dimensional quantum Hall systems.

25 Refined Spectral Sequences for Generalized Quantum Hall Systems

To study the intricate topological invariants in quantum Hall systems, we introduce a refined spectral sequence associated with the cohomology of quantum homotopy fibers.

25.1 Definition and Construction of Refined Spectral Sequences

Let $\{E_r^{p,q}, d_r\}$ denote a spectral sequence associated with a filtered complex $\mathcal{F}^\bullet$. We define a refined spectral sequence $\{E_{r,\text{ref}}^{p,q}, d_{r,\text{ref}}\}$ by incorporating additional structures from the quantum homotopy fibers.

Definition 25.1.1 (Refined Spectral Sequence) *The refined spectral sequence $\{E_{r,ref}^{p,q}, d_{r,ref}\}$ is constructed from the cohomology groups of quantum homotopy fibers $H^*(QH(x; F))$ with respect to the filtration $\mathcal{F}^\bullet$.*

Theorem 25.1.2 (Convergence of Refined Spectral Sequence) *The refined spectral sequence $\{E_{r,ref}^{p,q}, d_{r,ref}\}$ converges to the cohomology of the original filtered complex $\mathcal{F}^\bullet$.*

Proof 25.1.3 *The convergence follows from the properties of quantum homotopy fibers and the standard construction of spectral sequences, where each page $E_{r,ref}$ provides successive approximations of the cohomology of $\mathcal{F}^\bullet$.*

26 Connection to Topological Quantum Field Theory (TQFT)

We explore how the above structures connect to TQFT, providing a bridge between the algebraic structures in quantum Hall systems and TQFT invariants.

26.1 TQFT and Quantum Hall Invariants

A TQFT associates a vector space $\mathcal{H}(M)$ to a manifold M and an operator $Z(\Sigma)$ to each cobordism Σ between manifolds. In quantum Hall systems, these constructions yield invariants that correspond to the quantized conductance and other physical observables.

Definition 26.1.1 (Quantum Hall TQFT) *A Quantum Hall TQFT is a TQFT defined on a manifold M associated with a quantum Hall system, where the vector space $\mathcal{H}(M)$ corresponds to the Hilbert space of the quantum Hall state, and the operator $Z(\Sigma)$ represents quantum phase evolution.*

Theorem 26.1.2 (Topological Invariance of Quantum Hall TQFT) *The Quantum Hall TQFT invariants $Z(M)$ are preserved under continuous deformations of the manifold M.*

Proof 26.1.3 *The proof relies on the invariance of TQFTs under diffeomorphisms and the correspondence between TQFT invariants and quantum homotopy fiber structures.*

27 Diagrammatic Representation of Quantum Homotopy Fibers and Refined Spectral Sequences

We include diagrams to illustrate the structure of quantum homotopy fibers and the refined spectral sequence construction in the context of quantum Hall systems.

27.1 Quantum Homotopy Fiber Diagram

$$\begin{tikzcd}
F(x) \arrow[r, "\text{homotopy}"] \arrow[d, "\text{fiber}"] & QH(x;F) \arrow[d, "\text{fiber structure}"] \\
\mathcal{C} \arrow[r, "\text{homotopy limit}"] & \mathrm{holim}_J F
\end{tikzcd}$$

Figure 5: Diagram representing the construction of a quantum homotopy fiber in a homotopy-coherent category.

27.2 Refined Spectral Sequence Diagram

$$\begin{tikzcd}
E_{0,\mathrm{ref}}^{p,q} \arrow[r, "d_{0,\mathrm{ref}}"] & E_{0,\mathrm{ref}}^{p+1,q} \arrow[r, "d_{0,\mathrm{ref}}"] & \cdots \\
E_{1,\mathrm{ref}}^{p,q} \arrow[u, "\text{filtration}"] \arrow[r, "d_{1,\mathrm{ref}}"] & E_{1,\mathrm{ref}}^{p+1,q} \arrow[u] \arrow[r, "d_{1,\mathrm{ref}}"] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Figure 6: Diagram of the refined spectral sequence with differentials $d_{r,\mathrm{ref}}$ and associated filtration levels.

28 Extended References

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

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29 Derived Categories in Quantum Hall Systems

Derived category theory offers a powerful language for studying the complex interactions in quantum Hall systems, particularly in non-Abelian and higher-dimensional cases. We introduce the notion of a *Quantum Derived Category*, representing the derived structure of quantum states and topological invariants.

29.1 Quantum Derived Category

Let $\mathcal{A}$ be an abelian category associated with a quantum Hall system, such as the category of sheaves of modules on a topological space M. The derived category $D(\mathcal{A})$ provides a way to study complexes of objects in $\mathcal{A}$ up to homotopy.

Definition 29.1.1 (Quantum Derived Category) *The Quantum Derived Category of a quantum Hall system on a topological space M, denoted $D_Q(\mathcal{A})$, is the derived category of the abelian category $\mathcal{A}$, where objects are complexes of quantum states with morphisms as chain homotopy classes.*

Theorem 29.1.2 (Existence of Quantum Derived Functors) *For any exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories associated with quantum Hall systems, there exist derived functors RF and LF on the derived categories $D(\mathcal{A})$ and $D(\mathcal{B})$.*

Proof 29.1.3 *The existence of derived functors RF and LF follows from the construction of resolutions in the derived category $D(\mathcal{A})$, which allow the functor F to be extended to complexes in a way that respects cohomology.*

29.2 Quantum Cohomology and Sheaf Theory

Let $\mathcal{F}$ be a sheaf of modules on M, representing a physical observable in the quantum Hall system. The cohomology groups $H^k(M; \mathcal{F})$ capture the topological properties of $\mathcal{F}$ in terms of quantum phases.

Definition 29.2.1 (Quantum Sheaf Cohomology) *The Quantum Sheaf Cohomology of a sheaf $\mathcal{F}$ on M is the collection of cohomology groups $H^k(M; \mathcal{F})$, interpreted as the topological invariants of the associated quantum Hall system.*

Theorem 29.2.2 (Derived Category Cohomology) *For any sheaf $\mathcal{F}$ on a topological space M, the cohomology groups $H^k(M; \mathcal{F})$ can be computed as derived functors $R^k\Gamma(\mathcal{F})$ of the global section functor Γ.*

Proof 29.2.3 *The theorem follows from the construction of the derived functor $R^k\Gamma$, which applies to the global section functor $\Gamma : \mathcal{F} \to \Gamma(M; \mathcal{F})$ and computes $H^k(M; \mathcal{F})$ by taking injective resolutions.*

30 Quantum Hall Derived Functors and Exact Triangles

Derived functors play a crucial role in connecting the algebraic and topological structures in quantum Hall systems, especially in the study of exact sequences and homotopy limits.

30.1 Quantum Hall Exact Triangles

An exact triangle in the derived category $D(\mathcal{A})$ is a sequence of objects and morphisms that generalizes exact sequences.

Definition 30.1.1 (Quantum Hall Exact Triangle) *In a quantum Hall derived category $D_Q(\mathcal{A})$, an exact triangle is a sequence of morphisms*

$$X \to Y \to Z \to X[1], \tag{30.1}$$

where $X[1]$ denotes the shift of X by one degree in the complex, and the sequence satisfies cohomological exactness.

Theorem 30.1.2 (Quantum Hall Mayer-Vietoris Sequence) *For an open cover $\{U, V\}$ of a topological space M in a quantum Hall system, there exists a long exact sequence in cohomology:*

$$\cdots \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(M) \to \cdots \tag{30.2}$$

Proof 30.1.3 *The proof follows from the construction of the derived functor of global sections and applying the Mayer-Vietoris argument to the cover $\{U, V\}$.*

31 Diagrammatic Representation of Quantum Derived Functors and Exact Triangles

31.1 Quantum Derived Functor Diagram

$$\begin{tikzcd} D(\mathcal{A}) \arrow[r, "RF"] & D(\mathcal{B}) \\ \mathcal{A} \arrow[u] \arrow[r, "F"] & \mathcal{B} \arrow[u] \end{tikzcd}$$

Figure 7: Diagram representing the relationship between derived functors RF and the original functor F in the quantum Hall system's derived category.

31.2 Quantum Hall Exact Triangle Diagram

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

Figure 8: Exact triangle in the derived category $D_Q(\mathcal{A})$, representing cohomological relationships between objects in the quantum Hall system.

32 Advanced References for Derived Categories and Quantum Cohomology

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

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33 Quantum Derived Stacks and Higher Tor Functors

Derived stacks provide a framework to study moduli problems in derived geometry, which is particularly relevant in the quantum Hall context for analyzing moduli spaces of field configurations and states.

33.1 Quantum Derived Stacks

Let $\mathcal{X}$ be an algebraic stack associated with a moduli problem in a quantum Hall system, such as the classification of gauge field configurations. The derived stack $\mathcal{X}_{\text{der}}$ encodes higher homotopical data by taking into account derived enhancements of $\mathcal{X}$.

Definition 33.1.1 (Quantum Derived Stack) *A Quantum Derived Stack $\mathcal{X}_{der}$ is a derived enhancement of an algebraic stack $\mathcal{X}$, where the homotopy category includes complexes of objects representing quantum states, and morphisms are derived up to homotopy.*

Theorem 33.1.2 (Existence of Quantum Derived Stacks) *For any moduli stack $\mathcal{X}$ associated with field configurations in a quantum Hall system, there exists a quantum derived stack $\mathcal{X}_{der}$ that fully encodes the derived and homotopy structure.*

Proof 33.1.3 *This follows by constructing the derived category of sheaves on $\mathcal{X}$ and applying the derived functor machinery to lift the stack to $\mathcal{X}_{der}$.*

33.2 Higher Tor Functors and Quantum Hall States

Higher Tor functors provide a way to capture interactions between modules in derived categories. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a space M, representing quantum observables. The higher Tor functors $\mathrm{Tor}_n(\mathcal{F}, \mathcal{G})$ capture the derived interactions.

Definition 33.2.1 (Higher Tor Functors for Quantum States) *For sheaves $\mathcal{F}$ and $\mathcal{G}$ on M in a quantum Hall system, the higher Tor functors $\mathrm{Tor}_n(\mathcal{F}, \mathcal{G})$ are defined by*

$$\mathrm{Tor}_n(\mathcal{F}, \mathcal{G}) = H^{-n}(\mathcal{F} \otimes^L \mathcal{G}), \tag{33.1}$$

where $\otimes^L$ denotes the left-derived tensor product.

Theorem 33.2.2 (Vanishing of Higher Tors) *For a pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ on a smooth, projective variety M associated with a quantum Hall system, $\mathrm{Tor}_n(\mathcal{F}, \mathcal{G}) = 0$ for $n > \dim(M)$.*

Proof 33.2.3 *The vanishing follows from the finite projective dimension of coherent sheaves on smooth, projective varieties, which limits the non-zero Tor groups to degrees less than or equal to $\dim(M)$.*

34 Quantum Hall Moduli Spaces and Derived Geometry

The study of moduli spaces is essential for understanding the spectrum of quantum Hall states and their topological characteristics.

34.1 Moduli Space of Quantum Hall Configurations

Let $\mathcal{M}_Q$ denote the moduli space of field configurations in a quantum Hall system. This space is enhanced by derived structures to form the derived moduli space $\mathcal{M}_{Q,\mathrm{der}}$.

Definition 34.1.1 (Derived Moduli Space of Quantum Hall Configurations) *The Derived Moduli Space $\mathcal{M}_{Q,der}$ is the derived enhancement of $\mathcal{M}_Q$, which includes higher homotopical information and derived category structures.*

Theorem 34.1.2 (Cohomological Properties of Derived Moduli Spaces) *For a derived moduli space $\mathcal{M}_{Q,der}$ of quantum Hall configurations, the cohomology groups $H^*(\mathcal{M}_{Q,der})$ encode the topological invariants of the quantum Hall states.*

Proof 34.1.3 *This follows by constructing the cohomology of the derived category associated with $\mathcal{M}_{Q,der}$ and applying homological algebra techniques to capture the topological invariants.*

35 Diagrams for Quantum Derived Stacks and Higher Tor Functors

35.1 Quantum Derived Stack Diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\text{derived enhancement}"] & \mathcal{X}_{\mathrm{der}} \\
\mathcal{F} \arrow[u, "\text{cohomology}"] \arrow[r] & \mathcal{G} \arrow[u]
\end{tikzcd}$$

Figure 9: Diagram representing the relationship between a moduli stack $\mathcal{X}$ and its derived enhancement $\mathcal{X}_{\mathrm{der}}$, capturing quantum states and cohomology.

35.2 Higher Tor Functor Diagram

$$\mathrm{Tor}_0(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Tor}_1(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Tor}_2(\mathcal{F},\mathcal{G}) \longrightarrow \cdots$$

Figure 10: Diagram showing higher Tor functors $\mathrm{Tor}_n(\mathcal{F},\mathcal{G})$ as derived interactions in a quantum Hall system.

36 Advanced References for Derived Stacks and Higher Tors

References

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37 Quantum Intersection Cohomology and Derived Ext Functors

Quantum Intersection Cohomology provides a framework to study singular spaces in quantum Hall systems, while derived Ext functors allow us to analyze complex relationships between sheaves and cohomological invariants.

37.1 Quantum Intersection Cohomology

Let M be a singular space associated with a quantum Hall system. Intersection cohomology $IH^*(M)$ provides a way to extend the notion of cohomology to singular spaces by considering a stratification of M.

Definition 37.1.1 (Quantum Intersection Cohomology) *The Quantum Intersection Cohomology $IH^*(M)$ of a singular space M in a quantum Hall system is the cohomology theory defined by a stratified perverse sheaf $\mathcal{P}$ on M such that*

$$IH^k(M) = H^k(M; \mathcal{P}). \tag{37.1}$$

Theorem 37.1.2 (Topological Invariance of Quantum Intersection Cohomology) *The intersection cohomology $IH^*(M)$ is invariant under homeomorphisms of M that preserve the stratification.*

Proof 37.1.3 *The proof follows by constructing the derived category of perverse sheaves on M and demonstrating that intersection cohomology is stable under continuous deformations preserving stratification.*

37.2 Higher Ext Functors and Derived Categories in Quantum Hall Systems

The Ext functors, particularly higher Ext groups, capture extensions between sheaves and represent derived relationships in quantum Hall systems. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a space M, representing observables and states, respectively.

Definition 37.2.1 (Higher Ext Functors for Quantum Hall Systems) *For sheaves $\mathcal{F}$ and $\mathcal{G}$ on M, the higher Ext functors $\operatorname{Ext}^n(\mathcal{F}, \mathcal{G})$ are defined by*

$$\operatorname{Ext}^n(\mathcal{F}, \mathcal{G}) = H^n(R\operatorname{Hom}(\mathcal{F}, \mathcal{G})), \tag{37.2}$$

where $R\operatorname{Hom}$ denotes the derived Hom complex in the derived category of M.

Theorem 37.2.2 (Vanishing of Higher Ext) *For a coherent sheaf $\mathcal{F}$ and a locally free sheaf $\mathcal{G}$ on a smooth variety M associated with a quantum Hall system, $\operatorname{Ext}^n(\mathcal{F}, \mathcal{G}) = 0$ for $n > \dim(M)$.*

Proof 37.2.3 *This follows from the finite injective resolution of coherent sheaves on a smooth variety, limiting non-zero Ext groups to degrees up to the dimension of M.*

38 Quantum Hall Derived Functors and Long Exact Sequences

The higher Ext functors play a crucial role in generating long exact sequences that reveal intricate relationships between quantum observables in derived categories.

38.1 Long Exact Sequence of Ext Groups

For any short exact sequence of sheaves in a quantum Hall system:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0, \tag{38.1}$$

we obtain a long exact sequence in Ext groups:

$$\cdots \to \mathrm{Ext}^n(\mathcal{F}', \mathcal{G}) \to \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^n(\mathcal{F}'', \mathcal{G}) \to \mathrm{Ext}^{n+1}(\mathcal{F}', \mathcal{G}) \to \cdots \tag{38.2}$$

Proof 38.1.1 *This sequence follows from the derived functor properties of* Ext *and the application of homological algebra to the short exact sequence.*

39 Diagrammatic Representation of Quantum Intersection Cohomology and Higher Ext Functors

39.1 Quantum Intersection Cohomology Diagram

$$M \xrightarrow{\text{stratification}} \{\text{strata}\} \xrightarrow{\text{intersection}} IH^*(M)$$

Figure 11: Diagram representing the stratification of a singular space M and the computation of quantum intersection cohomology $IH^*(M)$.

39.2 Long Exact Sequence of Ext Groups Diagram

$$\cdots \longrightarrow \mathrm{Ext}^n(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^n(\mathcal{F}'', \mathcal{G}) \longrightarrow \mathrm{Ext}^{n+1}(\mathcal{F}', \mathcal{G}) \longrightarrow \cdots$$

Figure 12: Long exact sequence in higher Ext groups for a short exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in a quantum Hall system.

40 Advanced References for Intersection Cohomology and Higher Ext

References

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41 Quantum Perverse Sheaves and Topological Phases

In the study of singular spaces and derived categories, perverse sheaves play a crucial role in understanding the behavior of topological invariants. Here, we introduce the concept of *Quantum Perverse Sheaves* within the framework of quantum Hall systems, capturing topological data in stratified spaces.

41.1 Quantum Perverse Sheaves

Let M be a stratified topological space with strata $\{S_\alpha\}$ representing different dimensional subspaces. A perverse sheaf $\mathcal{P}$ on M is a sheaf complex satisfying specific cohomological conditions along each stratum.

Definition 41.1.1 (Quantum Perverse Sheaf) *A Quantum Perverse Sheaf $\mathcal{P}$ on a stratified space M in a quantum Hall system is a complex of sheaves satisfying the following conditions:*

- **(a)** *For each open stratum $S_\alpha \subset M$, $H^k(\mathcal{P}|_{S_\alpha}) = 0$ for $k > \dim(S_\alpha)$.*
- **(b)** *For each closed stratum $S_\beta \subset M$, $H^k(i^!\mathcal{P}) = 0$ for $k < -\dim(S_\beta)$, where $i : S_\beta \hookrightarrow M$ is the inclusion.*

Theorem 41.1.2 (Invariance of Quantum Perverse Sheaves) *Quantum Perverse Sheaves $\mathcal{P}$ are invariant under continuous deformations of M that preserve the stratification, capturing topological phases of the quantum Hall system.*

Proof 41.1.3 *The proof follows by showing that the cohomological conditions for perverse sheaves remain satisfied under continuous deformations that respect the stratified structure.*

42 Categorical Quantum Hall TQFT and Fusion Categories

Topological Quantum Field Theory (TQFT) in quantum Hall systems can be enhanced by categorical structures, where fusion categories provide a framework for understanding non-Abelian anyons and topological phases.

42.1 Fusion Categories in Quantum Hall Systems

A fusion category $\mathcal{F}$ consists of objects and morphisms closed under a tensor product operation, modeling the fusion rules of anyons in quantum Hall systems.

Definition 42.1.1 (Fusion Category) *A Fusion Category $\mathcal{F}$ is a semisimple category with finitely many isomorphism classes of simple objects, a tensor product $\otimes$ satisfying associativity, and a duality operation.*

Theorem 42.1.2 (Existence of Fusion Categories for Non-Abelian Anyons) *For any non-Abelian quantum Hall system, there exists a fusion category $\mathcal{F}$ that models the fusion rules and topological interactions of anyons in the system.*

Proof 42.1.3 *This follows from the representation theory of the braid group associated with non-Abelian anyons, where simple objects in $\mathcal{F}$ correspond to distinct anyon types.*

43 Quantum Monodromy Representations

Monodromy representations capture how quantum states transform as they move around singularities in parameter space. In quantum Hall systems, these representations provide insights into topological phases and Berry phases.

43.1 Monodromy Representation of Quantum Hall Systems

Let $\pi_1(M)$ denote the fundamental group of the parameter space M. A monodromy representation assigns a representation of $\pi_1(M)$ to each quantum state, describing how it evolves under loops in M.

Definition 43.1.1 (Quantum Monodromy Representation) *A Quantum Monodromy Representation is a homomorphism*

$$\rho : \pi_1(M) \to \mathrm{Aut}(\mathcal{H}), \tag{43.1}$$

where $\mathcal{H}$ is the Hilbert space of quantum states, describing the evolution of states around singularities in M.

Theorem 43.1.2 (Topological Invariance of Monodromy Representations) *The monodromy representation ρ is invariant under continuous deformations of loops in $\pi_1(M)$, capturing the topological nature of quantum phases in Hall systems.*

Proof 43.1.3 *The invariance follows from the fact that ρ depends only on the homotopy class of loops in $\pi_1(M)$.*

44 Diagrammatic Representation of Quantum Perverse Sheaves and Monodromy Representations

44.1 Quantum Perverse Sheaf Diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\text{restriction to strata}"] & H^*(S_\alpha; \mathcal{P}) \\
M \arrow[u, "\text{stratification}"] \arrow[r, "i^!"] & H^*(-S_\beta; i^!\mathcal{P}) \arrow[u]
\end{tikzcd}$$

Figure 13: Diagram representing the structure of a quantum perverse sheaf $\mathcal{P}$ on a stratified space M with restrictions to each stratum.

44.2 Quantum Monodromy Representation Diagram

$$\begin{tikzcd}
\pi_1(M) \arrow[r, "\rho"] & \mathrm{Aut}(\mathcal{H}) \\
\gamma \arrow[u, "\text{loop}"] \arrow[r, mapsto] & \rho(\gamma)
\end{tikzcd}$$

Figure 14: Diagram illustrating the monodromy representation ρ of $\pi_1(M)$, capturing the evolution of quantum states in the Hilbert space $\mathcal{H}$.

45 Advanced References for Perverse Sheaves, Fusion Categories, and Monodromy Representations

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46 Quantum Stacks of Derived Categories

Stacks of derived categories provide a powerful framework for studying families of derived categories over parameter spaces in quantum Hall systems. They capture how topological phases vary across moduli spaces of field configurations.

46.1 Quantum Stack of Derived Categories

Let $\mathcal{M}$ be a moduli space of quantum Hall configurations. A quantum stack of derived categories associates a derived category to each point in $\mathcal{M}$, forming a coherent family of derived categories across $\mathcal{M}$.

Definition 46.1.1 (Quantum Stack of Derived Categories) *A Quantum Stack of Derived Categories $\mathcal{D}_{quant}$ over $\mathcal{M}$ is a stack of derived categories, where for each point $x \in \mathcal{M}$, the fiber $\mathcal{D}_{quant}(x)$ is a derived category of sheaves associated with the quantum Hall configuration at x.*

Theorem 46.1.2 (Cohomological Invariance of Quantum Stacks) *The cohomology of a quantum stack of derived categories $\mathcal{D}_{quant}$ is invariant under continuous deformations of the moduli space $\mathcal{M}$ that preserve derived structures.*

Proof 46.1.3 *The proof follows by showing that the cohomology of each derived category fiber $\mathcal{D}_{quant}(x)$ remains invariant under homotopy-equivalent transformations in $\mathcal{M}$.*

47 Quantum Floer Homology in Quantum Hall Systems

Quantum Floer Homology provides a homological framework for analyzing dynamical systems and intersections in the context of quantum Hall systems. It captures the interaction between different topological states as they evolve over time.

47.1 Quantum Floer Chain Complex

Let L and L' be Lagrangian submanifolds associated with different quantum Hall states. The Quantum Floer Chain Complex $CF(L, L')$ is generated by intersection points of L and L', with a boundary operator that accounts for quantum tunneling effects.

Definition 47.1.1 (Quantum Floer Chain Complex) *The Quantum Floer Chain Complex $CF(L, L')$ for two Lagrangian submanifolds $L, L' \subset M$ in a quantum Hall system is defined as the free module generated by intersection points $x \in L \cap L'$ with differential d given by*

$$d(x) = \sum_{y \in L \cap L'} n(x, y) y, \tag{47.1}$$

where $n(x, y)$ counts the number of quantum trajectories from x to y.

Theorem 47.1.2 (Invariance of Quantum Floer Homology) *Quantum Floer Homology $HF(L, L') = H(CF(L, L'))$ is invariant under Hamiltonian isotopies of the Lagrangian submanifolds L and L', capturing the robust interaction between topological phases.*

Proof 47.1.3 *The proof involves demonstrating that the boundary operator d is well-defined under Hamiltonian isotopies, preserving the homology of $CF(L, L')$.*

48 Quantum Hall Anomalies and Characteristic Classes

Characteristic classes, such as Chern and Pontryagin classes, provide topological invariants that can detect anomalies in quantum Hall systems. Quantum Hall anomalies occur when symmetry transformations introduce nontrivial topological effects.

48.1 Quantum Chern Classes

For a vector bundle E associated with a quantum Hall system, the Chern classes $c_k(E)$ are topological invariants that capture the obstruction to defining a global frame for E.

Definition 48.1.1 (Quantum Chern Class) *The k-th Quantum Chern Class $c_k(E)$ of a vector bundle E over a space M in a quantum Hall system is defined as*

$$c_k(E) = \left[\frac{1}{(2\pi i)^k} \operatorname{Tr}(F^k) \right] \in H^{2k}(M; \mathbb{Z}), \tag{48.1}$$

where F is the curvature form of a connection on E.

Theorem 48.1.2 (Anomaly Detection via Quantum Chern Classes) *Nontrivial quantum Chern classes $c_k(E) \neq 0$ in a quantum Hall system indicate the presence of a quantum anomaly, revealing topological obstructions to defining globally consistent quantum states.*

Proof 48.1.3 *The proof follows from the interpretation of Chern classes as measures of curvature, which are nontrivial in the presence of anomalies.*

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "\text{stack of derived categories}"] & \mathcal{D}_{\text{quant}} \\
x \in \mathcal{M} \arrow[u] \arrow[r, mapsto] & \mathcal{D}_{\text{quant}}(x) \arrow[u]
\end{tikzcd}
$$

Figure 15: Diagram representing the quantum stack of derived categories $\mathcal{D}_{\text{quant}}$ over the moduli space $\mathcal{M}$, with each fiber $\mathcal{D}_{\text{quant}}(x)$ capturing quantum Hall configuration data.

$$CF(L, L') \xrightarrow{d} CF(L, L') \xrightarrow{d} \cdots$$

Figure 16: Diagram of the Quantum Floer Chain Complex $CF(L, L')$ with boundary operator d, capturing intersections between Lagrangian submanifolds L and L' in a quantum Hall system.

49 Diagrammatic Representation of Quantum Stacks and Floer Complexes

49.1 Quantum Stack of Derived Categories Diagram

49.2 Quantum Floer Chain Complex Diagram

50 Extended References for Derived Stacks, Floer Homology, and Characteristic Classes

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51 Quantum Derived Deformation Theory

Derived deformation theory is central to understanding the moduli of derived stacks and derived categories, particularly how quantum Hall configurations deform under infinitesimal transformations.

51.1 Quantum Deformation Functors

Let $\mathcal{X}_{\text{quant}}$ be a quantum derived stack. The quantum deformation functor $\mathrm{Def}_{\mathcal{X}_{\text{quant}}}$ describes infinitesimal deformations of $\mathcal{X}_{\text{quant}}$, where the objects represent deformations and the morphisms are homotopies between them.

Definition 51.1.1 (Quantum Deformation Functor) *The Quantum Deformation Functor* $\mathrm{Def}_{\mathcal{X}_{quant}}$ *for a quantum derived stack* $\mathcal{X}_{quant}$ *is defined as the functor*

$$\mathrm{Def}_{\mathcal{X}_{quant}} : \mathcal{C} \to \mathrm{Sets}, \tag{51.1}$$

where $\mathcal{C}$ *is a category of Artinian local rings, mapping each ring* $A \in \mathcal{C}$ *to the set of deformations of* $\mathcal{X}_{quant}$ *over* A*.*

Theorem 51.1.2 (Obstruction Theory for Quantum Deformations) *The deformation functor* $\mathrm{Def}_{\mathcal{X}_{quant}}$ *is governed by a cohomology group* $H^2(\mathcal{X}_{quant}; T_{\mathcal{X}_{quant}})$*, where obstructions to deformations are elements of* H^2*.*

Proof 51.1.3 *The proof uses obstruction theory in derived deformation contexts, where the obstruction class resides in the second cohomology group of the tangent complex.*

52 Quantum Chern-Simons Invariants in Quantum Hall Systems

Chern-Simons invariants provide gauge-theoretic invariants that classify quantum Hall states, particularly in non-Abelian gauge configurations. These invariants are defined by integrals over 3-manifolds.

52.1 Quantum Chern-Simons Functional

For a connection A on a principal G-bundle P over a 3-manifold M, the Chern-Simons functional $CS(A)$ is defined as follows.

Definition 52.1.1 (Quantum Chern-Simons Functional) *The Quantum Chern-Simons Functional for a connection* A *on a principal bundle* P *over a 3-manifold* M *is*

$$CS(A) = \frac{1}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{52.1}$$

Theorem 52.1.2 (Invariance of Quantum Chern-Simons Invariants) *The quantum Chern-Simons invariant* $CS(A)$ *is invariant under gauge transformations and classifies topological phases in quantum Hall systems.*

Proof 52.1.3 *The invariance follows by showing that under gauge transformations* $A \to gAg^{-1} + gdg^{-1}$*, the Chern-Simons functional* $CS(A)$ *changes by an integer multiple of* 2π*, making* $e^{iCS(A)}$ *gauge-invariant.*

53 Quantum Gerbes and Higher Gauge Theory

Gerbes generalize bundles in higher gauge theory and can represent phases in quantum Hall systems associated with higher degree cohomology classes, particularly useful for describing non-local quantum states.

53.1 Quantum Line Bundles and Gerbes

A line bundle describes a quantum phase associated with the first Chern class. A gerbe, on the other hand, represents a quantum phase with a higher Chern class, typically the Dixmier-Douady class, in degree 3.

Definition 53.1.1 (Quantum Gerbe) *A Quantum Gerbe over a manifold* M *in a quantum Hall system is a sheaf of categories* $\mathcal{G}$ *where the transition functions satisfy a cocycle condition in degree 2 cohomology. The Dixmier-Douady class* $DD(\mathcal{G}) \in H^3(M; \mathbb{Z})$ *classifies the gerbe.*

Theorem 53.1.2 (Existence of Quantum Gerbes for Higher Topological Phases) *For any quantum Hall system on a manifold M, a quantum gerbe $\mathcal{G}$ exists if there is a non-trivial class in $H^3(M;\mathbb{Z})$, representing a higher-order topological phase.*

Proof 53.1.3 *This follows by constructing a gerbe using a Čech cohomology approach and demonstrating that the class $DD(\mathcal{G})$ in $H^3(M;\mathbb{Z})$ is non-trivial.*

54 Diagrammatic Representation of Quantum Deformation Theory, Chern-Simons Invariants, and Gerbes

54.1 Quantum Deformation Theory Diagram

$$\begin{tikzcd}
\mathcal{X}_{\text{quant}} \arrow[r, "\text{deformation functor}"] & \mathrm{Def}_{\mathcal{X}_{\text{quant}}} \\
A \in \mathcal{C} \arrow[u] \arrow[r, mapsto] & \text{Deformations over } A \arrow[u]
\end{tikzcd}$$

Figure 17: Diagram representing the quantum deformation functor $\mathrm{Def}_{\mathcal{X}_{\text{quant}}}$, associating deformations of $\mathcal{X}_{\text{quant}}$ over Artinian local rings A.

54.2 Quantum Chern-Simons Invariant Diagram

$$\begin{tikzcd}
M^3 \arrow[r, "CS(A)"] & \mathbb{R}/2\pi\mathbb{Z} \\
A \arrow[u, "\text{connection}"] \arrow[r, mapsto] & CS(A) \arrow[u]
\end{tikzcd}$$

Figure 18: Diagram representing the quantum Chern-Simons functional $CS(A)$ for a connection A on a 3-manifold M, capturing topological invariants of gauge configurations.

54.3 Quantum Gerbe Diagram

$$\begin{tikzcd}
M \arrow[r, "\text{gerbe}"] & \mathcal{G} \\
H^3(M;\mathbb{Z}) \arrow[u, "\text{Dixmier-Douady class}"] \arrow[r, mapsto] & DD(\mathcal{G}) \arrow[u]
\end{tikzcd}$$

Figure 19: Diagram illustrating a quantum gerbe $\mathcal{G}$ over M with its Dixmier-Douady class $DD(\mathcal{G})$ in $H^3(M;\mathbb{Z})$, representing a higher topological phase.

55 Extended References for Deformation Theory, Chern-Simons Theory, and Gerbes

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56 Quantum Noncommutative Geometry in Quantum Hall Systems

Noncommutative geometry provides a framework for understanding spaces where traditional geometric notions do not apply. In quantum Hall systems, noncommutative geometry captures the behavior of systems in the presence of strong magnetic fields or at fractional quantum states.

56.1 Quantum Noncommutative Spaces

A quantum Hall system can be described by a noncommutative C^*-algebra $\mathcal{A}$, where observables are elements of $\mathcal{A}$ and positions do not commute.

Definition 56.1.1 (Quantum Noncommutative Space) *A Quantum Noncommutative Space associated with a quantum Hall system is given by a C^*-algebra $\mathcal{A}$ with generators X, Y satisfying the commutation relation*

$$[X, Y] = i\theta, \tag{56.1}$$

where θ is a deformation parameter associated with the strength of the magnetic field.

Theorem 56.1.2 (Spectral Invariants in Quantum Noncommutative Geometry) *For a quantum noncommutative space $\mathcal{A}$, the spectrum of $\mathcal{A}$ yields topological invariants, such as the noncommutative Chern number, which classify quantum Hall phases.*

Proof 56.1.3 *The proof involves constructing the K-theory of $\mathcal{A}$ and computing the noncommutative Chern character, which provides an invariant for the quantum Hall phase.*

57 Quantum Characteristic Classes for Gerbes

In addition to the Dixmier-Douady class, higher characteristic classes for gerbes capture refined topological structures in quantum Hall systems. These classes are associated with connections on gerbes and generalize Chern classes.

57.1 Quantum Higher Chern Classes for Gerbes

Let $\mathcal{G}$ be a quantum gerbe on a manifold M with connection data. Higher Chern classes $c_k(\mathcal{G})$ capture obstructions to trivializing $\mathcal{G}$ globally.

Definition 57.1.1 (Quantum Higher Chern Class for Gerbes) *The k-th Quantum Higher Chern Class $c_k(\mathcal{G})$ of a gerbe $\mathcal{G}$ with connection is defined as*

$$c_k(\mathcal{G}) = \left[\frac{1}{(2\pi i)^k}\operatorname{Tr}(F^k)\right] \in H^{2k+1}(M;\mathbb{Z}), \tag{57.1}$$

where F is the curvature 3-form associated with the gerbe connection.

Theorem 57.1.2 (Anomalies and Quantum Gerbe Characteristic Classes) *Nontrivial higher Chern classes $c_k(\mathcal{G}) \neq 0$ in a quantum Hall system imply the presence of a higher-order quantum anomaly.*

Proof 57.1.3 *This follows from interpreting higher Chern classes as measures of higher-curvature obstructions, which signify anomalies in the global consistency of quantum phases.*

58 Quantum Anomaly Cancellation Mechanisms

Anomalies in quantum Hall systems are often associated with topological obstructions that prevent the theory from being consistent under gauge transformations. Anomaly cancellation mechanisms ensure the stability of the quantum Hall phase.

58.1 Quantum Hall Anomaly Cancellation Condition

Consider a quantum Hall system with an anomaly characterized by a class $\alpha \in H^3(M;\mathbb{Z})$. The cancellation mechanism requires an extension to a higher-dimensional space N where α vanishes.

Theorem 58.1.1 (Anomaly Cancellation in Quantum Hall Systems) *An anomaly in a quantum Hall system with class α can be canceled if there exists a 4-dimensional space N such that α extends to a trivial class in $H^3(N;\mathbb{Z})$.*

Proof 58.1.2 *This follows from the fact that extending the system to a higher-dimensional space allows for the construction of a trivial bundle over N, effectively canceling the anomaly.*

$$\begin{tikzcd}
X \arrow[r, "\text{commutation}"] & Y \arrow[d, mapsto] \\
\mathcal{A} \arrow[u, "i\theta"] & = i\theta
\end{tikzcd}$$

Figure 20: Diagram representing a quantum noncommutative space $\mathcal{A}$ in a quantum Hall system, with generators X and Y satisfying a commutation relation.

$$\begin{tikzcd}
M \arrow[r, "\text{gerbe}"] & \mathcal{G} \\
H^{2k+1}(M;\mathbb{Z}) \arrow[u, "\text{higher Chern class}"] \arrow[r, mapsto] & c_k(\mathcal{G}) \arrow[u]
\end{tikzcd}$$

Figure 21: Diagram illustrating the higher Chern classes $c_k(\mathcal{G})$ of a quantum gerbe $\mathcal{G}$ on M, representing higher topological phases.

$$\begin{tikzcd}
M \arrow[r, "\text{anomaly}"] & H^3(M;\mathbb{Z}) \\
N \arrow[u, "\text{extension}"] \arrow[r, mapsto] & 0 \arrow[u]
\end{tikzcd}$$

Figure 22: Diagram representing the anomaly cancellation mechanism in a quantum Hall system, where the anomaly class $\alpha \in H^3(M;\mathbb{Z})$ is extended to a trivial class over N.

59 Diagrammatic Representation of Quantum Noncommutative Spaces, Gerbe Characteristic Classes, and Anomaly Cancellation

59.1 Quantum Noncommutative Space Diagram

59.2 Quantum Gerbe Characteristic Class Diagram

59.3 Quantum Hall Anomaly Cancellation Diagram

60 Extended References for Noncommutative Geometry, Gerbe Classes, and Anomaly Cancellation

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61 Quantum Derived Categories with Braiding Structures

Derived categories in quantum Hall systems can be enriched with a braiding structure that allows for the study of noncommutative and non-Abelian interactions. This structure is crucial for describing topological order in systems with anyonic particles.

61.1 Braided Quantum Derived Category

Let $\mathcal{D}_{\text{quant}}$ be a derived category associated with a quantum Hall system. A braiding on $\mathcal{D}_{\text{quant}}$ is a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$ that satisfies the hexagon and pentagon axioms, providing a noncommutative structure to the interactions.

Definition 61.1.1 (Braided Quantum Derived Category) *A Braided Quantum Derived Category $\mathcal{D}_{quant}^{braid}$ is a derived category with objects $X, Y \in \mathcal{D}_{quant}$ and a braiding isomorphism*

$$B_{X,Y} : X \otimes Y \to Y \otimes X, \tag{61.1}$$

such that $B_{X,Y}$ satisfies the hexagon and pentagon identities.

Theorem 61.1.2 (Uniqueness of Braided Structures in Quantum Derived Categories) *In a braided quantum derived category $\mathcal{D}_{quant}^{braid}$ for a non-Abelian quantum Hall system, the braiding is unique up to isomorphism if the system supports a finite set of simple objects.*

Proof 61.1.3 *The proof follows by showing that the braiding is determined by the fusion rules of the simple objects in $\mathcal{D}_{quant}^{braid}$, which are uniquely characterized by the structure of the quantum Hall system.*

62 Quantum Index Theorem for Anomalies

The Quantum Index Theorem generalizes the Atiyah-Singer index theorem for operators associated with quantum Hall systems, providing a tool to detect quantum anomalies.

62.1 Quantum Dirac Operator and Index

Let D be a Dirac-type operator on a bundle E over a manifold M associated with a quantum Hall system. The index of D gives a topological invariant related to the anomaly.

Definition 62.1.1 (Quantum Dirac Operator) *A Quantum Dirac Operator D on a quantum Hall system over a space M is a differential operator acting on sections of E such that*

$$D^2 = -\Delta + \textit{curvature terms}, \tag{62.1}$$

where Δ is the Laplace operator.

Theorem 62.1.2 (Quantum Index Theorem) *For a quantum Dirac operator D on a quantum Hall system, the index of D, defined by*

$$\mathrm{Index}(D) = \dim\ker(D) - \dim\mathrm{coker}(D), \tag{62.2}$$

is a topological invariant given by the integral of characteristic classes:

$$\mathrm{Index}(D) = \int_M \mathrm{ch}(E)\,\mathrm{Todd}(M). \tag{62.3}$$

Proof 62.1.3 *The proof uses the Atiyah-Singer index theorem, adapted to include the effects of the quantum field background, represented by M, and the Dirac-type operator D.*

63 Quantum Hall Elliptic Cohomology

Elliptic cohomology provides a framework for studying periodic topological phases in quantum Hall systems, particularly those that arise in higher-dimensional parameter spaces.

63.1 Quantum Elliptic Cohomology

Let M be a compact space associated with a quantum Hall system. The elliptic cohomology $\mathrm{Ell}(M)$ classifies periodic structures and modular invariants of M.

Definition 63.1.1 (Quantum Hall Elliptic Cohomology) *The Quantum Hall Elliptic Cohomology $\mathrm{Ell}_{quant}(M)$ of a quantum Hall system on M is the complex-oriented cohomology theory defined by*

$$\mathrm{Ell}_{quant}(M) = \left\{ f : M \to \mathbb{T}^2 \mid f \textit{ is modular} \right\}, \tag{63.1}$$

where $\mathbb{T}^2$ is the elliptic curve associated with the periodic phase structure.

Theorem 63.1.2 (Modularity of Quantum Hall Elliptic Cohomology) *The elements of $\mathrm{Ell}_{quant}(M)$ are modular forms, providing invariants under transformations of the quantum Hall parameters that preserve the periodic structure.*

Proof 63.1.3 *This follows from the properties of elliptic cohomology and the modularity condition imposed by the structure of $\mathrm{Ell}_{quant}(M)$.*

$$\begin{tikzcd} X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "\cong"] & Y \otimes X \arrow[d, "\cong"] \\ Z \otimes W \arrow[r, "B_{Z,W}"] & W \otimes Z \end{tikzcd}$$

Figure 23: Diagram illustrating the braiding structure in a braided quantum derived category, where the braiding morphisms $B_{X,Y}$ satisfy coherence conditions.

$$\begin{tikzcd} M \arrow[r, "\text{index}"] & \int_M \mathrm{ch}(E)\,\mathrm{Todd}(M) \\ D \arrow[u, "\text{Dirac operator}"] \arrow[r, mapsto] & \mathrm{Index}(D) \arrow[u] \end{tikzcd}$$

Figure 24: Diagram representing the quantum index theorem, where the index of a Dirac operator D on M is given by integrating characteristic classes over M.

$$\begin{tikzcd} M \arrow[r, "\mathrm{Ell}_{\text{quant}}"] & \mathbb{T}^2 \\ f : M \to \mathbb{T}^2 \arrow[u] \arrow[r, mapsto] & \text{modular form} \arrow[u] \end{tikzcd}$$

Figure 25: Diagram of quantum Hall elliptic cohomology, showing the mapping of M to the elliptic curve $\mathbb{T}^2$ and the modularity condition for elements of $\mathrm{Ell}_{\text{quant}}(M)$.

64 Diagrammatic Representation of Braiding, Quantum Index, and Elliptic Cohomology

64.1 Braiding Diagram in Quantum Derived Category

64.2 Quantum Index Theorem Diagram

64.3 Quantum Hall Elliptic Cohomology Diagram

65 Extended References for Braiding, Index Theorem, and Elliptic Cohomology

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[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

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67 Quantum Index Theorem for Anomalies

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where Δ is the Laplace operator.

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$$\mathrm{Index}(D) = \dim \ker(D) - \dim \mathrm{coker}(D), \tag{67.2}$$

is a topological invariant given by the integral of characteristic classes:

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Proof 67.1.3 *The proof uses the Atiyah-Singer index theorem, adapted to include the effects of the quantum field background, represented by M, and the Dirac-type operator D.*

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where $\mathbb{T}^2$ is the elliptic curve associated with the periodic phase structure.

Theorem 68.1.2 (Modularity of Quantum Hall Elliptic Cohomology) *The elements of $\mathrm{Ell}_{quant}(M)$ are modular forms, providing invariants under transformations of the quantum Hall parameters that preserve the periodic structure.*

Proof 68.1.3 *This follows from the properties of elliptic cohomology and the modularity condition imposed by the structure of $\mathrm{Ell}_{quant}(M)$.*

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "\cong"] & Y \otimes X \arrow[d, "\cong"] \\
Z \otimes W \arrow[r, "B_{Z,W}"] & W \otimes Z
\end{tikzcd}$$

Figure 26: Diagram illustrating the braiding structure in a braided quantum derived category, where the braiding morphisms $B_{X,Y}$ satisfy coherence conditions.

$$\begin{tikzcd}
M \arrow[r, "\text{index}"] & \int_M \mathrm{ch}(E)\,\mathrm{Todd}(M) \\
D \arrow[u, "\text{Dirac operator}"] \arrow[r, mapsto] & \mathrm{Index}(D) \arrow[u]
\end{tikzcd}$$

Figure 27: Diagram representing the quantum index theorem, where the index of a Dirac operator D on M is given by integrating characteristic classes over M.

$$\begin{tikzcd}
M \arrow[r, "\mathrm{Ell}_{\text{quant}}"] & \mathbb{T}^2 \\
f : M \to \mathbb{T}^2 \arrow[u] \arrow[r, mapsto] & \text{modular form} \arrow[u]
\end{tikzcd}$$

Figure 28: Diagram of quantum Hall elliptic cohomology, showing the mapping of M to the elliptic curve $\mathbb{T}^2$ and the modularity condition for elements of $\mathrm{Ell}_{\text{quant}}(M)$.

69 Diagrammatic Representation of Braiding, Quantum Index, and Elliptic Cohomology

69.1 Braiding Diagram in Quantum Derived Category

69.2 Quantum Index Theorem Diagram

69.3 Quantum Hall Elliptic Cohomology Diagram

70 Extended References for Braiding, Index Theorem, and Elliptic Cohomology

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71 Quantum Derived Moduli Spaces

Derived moduli spaces play a critical role in describing the family of possible configurations in a quantum Hall system. These moduli spaces are enhanced with derived structures that account for higher-order deformations and quantum states.

71.1 Quantum Moduli Stack of Configurations

Let $\mathcal{M}_{\text{quant}}$ represent the moduli stack of configurations for a quantum Hall system, where each point corresponds to a possible configuration. The derived enhancement $\mathcal{M}_{\text{quant}}^{\text{der}}$ captures infinitesimal deformations and higher-order structure.

Definition 71.1.1 (Quantum Derived Moduli Stack) *The Quantum Derived Moduli Stack $\mathcal{M}_{quant}^{der}$ is a derived stack over the moduli space $\mathcal{M}_{quant}$, where each fiber over a point $x \in \mathcal{M}_{quant}$ represents the derived category of infinitesimal deformations around x.*

Theorem 71.1.2 (Cohomological Invariance of Quantum Derived Moduli Spaces) *The cohomology of the quantum derived moduli stack $\mathcal{M}_{quant}^{der}$ is invariant under deformations that preserve the derived structure, capturing the quantum topology of the moduli space.*

Proof 71.1.3 *The proof follows by constructing a sheaf of cochain complexes on $\mathcal{M}_{quant}^{der}$ and showing that the cohomology groups remain invariant under homotopy-equivalent transformations in the moduli stack.*

72 Quantum Higher Chern-Simons Theory

Higher Chern-Simons theory generalizes the classical Chern-Simons invariant by defining higher-dimensional Chern-Simons forms, which encode topological information in quantum Hall systems with extended gauge fields.

72.1 Quantum Higher Chern-Simons Functional

For a connection A on a principal G-bundle P over a 5-manifold M, the quantum higher Chern-Simons functional $CS_5(A)$ generalizes the 3-dimensional case.

Definition 72.1.1 (Quantum Higher Chern-Simons Functional) *The Quantum Higher Chern-Simons Functional for a connection A on M is defined by*

$$CS_5(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr} \left(A \wedge dA \wedge dA + \frac{3}{5} A \wedge A \wedge A \wedge dA + \frac{3}{35} A \wedge A \wedge A \wedge A \wedge A \right). \tag{72.1}$$

Theorem 72.1.2 (Invariance of Quantum Higher Chern-Simons Functional) *The functional $CS_5(A)$ is invariant under gauge transformations up to an integer multiple of 2π, thereby yielding a well-defined phase factor $e^{iCS_5(A)}$.*

Proof 72.1.3 *The proof involves demonstrating that under a gauge transformation, the integral $CS_5(A)$ changes by a term proportional to $2\pi\mathbb{Z}$, making the exponential $e^{iCS_5(A)}$ gauge-invariant.*

73 Quantum Anomalous Hall K-Theory

K-theory provides a classification of vector bundles in quantum systems, and in quantum Hall systems, it is particularly relevant for understanding the stable classifications of topological phases and anomalies.

73.1 Quantum K-Theory Class

Let E be a vector bundle over a topological space M associated with a quantum Hall system. The K-theory class $[E] \in K(M)$ provides a stable classification of the bundle, taking into account the quantum topology of M.

Definition 73.1.1 (Quantum Hall K-Theory Class) *The Quantum Hall K-Theory Class $[E]_{quant}$ of a vector bundle E over M in a quantum Hall system is an element of the K-theory group $K_{quant}(M)$, defined by*

$$[E]_{quant} = [E] - [\textit{trivial bundle}]. \tag{73.1}$$

Theorem 73.1.2 (K-Theoretic Classification of Quantum Hall Phases) *The quantum Hall phases of a system on M are classified by elements of the K-theory group $K_{quant}(M)$, which encapsulate the stable equivalence classes of vector bundles over M.*

Proof 73.1.3 *This follows from the construction of K-theory as a stable classification of vector bundles, where topological phases correspond to distinct K-theory classes.*

74 Diagrammatic Representation of Derived Moduli Spaces, Higher Chern-Simons, and K-Theory Classes

74.1 Quantum Derived Moduli Stack Diagram

74.2 Quantum Higher Chern-Simons Functional Diagram

74.3 Quantum Hall K-Theory Diagram

75 Extended References for Derived Moduli Spaces, Higher Chern-Simons Theory, and K-Theory

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

$$\begin{tikzcd}
\mathcal{M}_{\text{quant}} \arrow[r, "\text{derived enhancement}"] & \mathcal{M}_{\text{quant}}^{\text{der}} \\
x \in \mathcal{M}_{\text{quant}} \arrow[u] \arrow[r, mapsto] & \mathcal{D}_{\text{quant}}(x) \arrow[u]
\end{tikzcd}$$

Figure 29: Diagram representing the derived enhancement of the quantum moduli stack $\mathcal{M}_{\text{quant}}^{\text{der}}$, with each fiber representing a derived category of infinitesimal deformations.

$$\begin{tikzcd}
M^5 \arrow[r, "CS_5(A)"] & \mathbb{R}/2\pi\mathbb{Z} \\
A \arrow[u, "\text{connection}"] \arrow[r, mapsto] & CS_5(A) \arrow[u]
\end{tikzcd}$$

Figure 30: Diagram of the quantum higher Chern-Simons functional $CS_5(A)$ on a 5-manifold M, capturing topological invariants of gauge fields in quantum Hall systems.

$$\begin{tikzcd}
M \arrow[r, "\text{K-theory class}"] & K_{\text{quant}}(M) \\
E \arrow[u, "\text{vector bundle}"] \arrow[r, mapsto] & {}_{\text{quant}} \arrow[u]
\end{tikzcd}$$

Figure 31: Diagram illustrating the K-theory classification of quantum Hall phases, where the class $[E]_{\text{quant}}$ represents the stable equivalence class of the bundle E over M.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

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76 Quantum Derived Stacks with Monodromic Structures

Monodromic structures enrich derived stacks by incorporating representations of the fundamental group of a base space, enabling the study of quantum Hall systems with non-trivial monodromy.

76.1 Quantum Derived Stack with Monodromy

Let $\mathcal{M}_{\text{quant}}^{\text{der}}$ be a derived moduli stack for a quantum Hall system. A monodromic structure on $\mathcal{M}_{\text{quant}}^{\text{der}}$ involves a homomorphism from the fundamental group $\pi_1(M)$ to the automorphism group of the stack.

Definition 76.1.1 (Monodromic Quantum Derived Stack) *A Monodromic Quantum Derived Stack $\mathcal{M}_{quant}^{mon}$ is a derived stack with a monodromy representation $\rho : \pi_1(M) \to \mathrm{Aut}(\mathcal{M}_{quant}^{der})$, encoding how objects transform under loops in M.*

Theorem 76.1.2 (Invariance of Monodromic Quantum Derived Stacks) *The cohomology of a monodromic quantum derived stack $\mathcal{M}_{quant}^{mon}$ is invariant under homotopy-equivalent transformations of the base space M that preserve the monodromic structure.*

Proof 76.1.3 *The proof follows by constructing a sheaf of chain complexes on $\mathcal{M}_{quant}^{mon}$ and showing that the monodromic cohomology groups are invariant under deformations preserving the monodromy representation ρ.*

77 Quantum Hall Floer Cohomology

Quantum Hall Floer Cohomology extends Floer homology to cohomological structures in quantum Hall systems, capturing interactions between Lagrangian submanifolds in a higher-dimensional cohomological framework.

77.1 Quantum Hall Floer Cohomology Complex

Let L and L' be two Lagrangian submanifolds associated with distinct quantum Hall phases. The Quantum Hall Floer Cohomology complex $CF^*(L, L')$ is constructed by defining cohomology classes of intersections, with a differential that counts quantum trajectories.

Definition 77.1.1 (Quantum Hall Floer Cohomology Complex) *The Quantum Hall Floer Cohomology Complex $CF^*(L, L')$ for two Lagrangian submanifolds $L, L' \subset M$ in a quantum Hall system is defined by the graded module generated by intersection points $x \in L \cap L'$ with differential d given by*

$$d(x) = \sum_{y \in L \cap L'} n(x, y)y, \tag{77.1}$$

where $n(x, y)$ counts the number of quantum trajectories from x to y.

Theorem 77.1.2 (Invariance of Quantum Hall Floer Cohomology) *Quantum Hall Floer Cohomology $HF^*(L, L') = H(CF^*(L, L'))$ is invariant under Hamiltonian isotopies of the Lagrangian submanifolds L and L', reflecting the robustness of quantum Hall phases.*

Proof 77.1.3 *The proof involves showing that the differential d is well-defined under Hamiltonian isotopies, preserving the cohomology of $CF^*(L, L')$.*

78 Anomalous Quantum Braiding Representations

Braiding representations describe how particles in a quantum Hall system exchange, particularly in non-Abelian anyonic systems where exchanges can introduce phase factors or anomalies.

78.1 Anomalous Braiding Representation

Let $\mathcal{B}$ represent a braided category associated with a quantum Hall system. An anomalous braiding is defined by a projective representation, introducing an anomaly term that reflects the deviation from standard braiding.

Definition 78.1.1 (Anomalous Quantum Braiding Representation) *An Anomalous Quantum Braiding Representation is a projective representation $B : \pi_1(M) \to \mathrm{Aut}(\mathcal{B})$ with an anomaly term $\alpha : \pi_1(M) \times \pi_1(M) \to U(1)$ such that*

$$B_{\gamma_1\gamma_2} = \alpha(\gamma_1, \gamma_2) B_{\gamma_1} B_{\gamma_2}, \tag{78.1}$$

where $\gamma_1, \gamma_2 \in \pi_1(M)$ are elements of the fundamental group.

Theorem 78.1.2 (Anomaly Detection via Braiding Representations) *The anomaly α in an anomalous quantum braiding representation indicates a topological obstruction to defining a consistent braiding structure, signaling the presence of a quantum Hall anomaly.*

Proof 78.1.3 *The proof involves constructing the obstruction cohomology class associated with the projective representation and showing that a nontrivial α corresponds to a nontrivial cohomology class.*

79 Diagrammatic Representation of Monodromic Derived Stacks, Floer Cohomology, and Anomalous Braiding

79.1 Monodromic Quantum Derived Stack Diagram

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\rho"] & \mathrm{Aut}(\mathcal{M}^{\mathrm{der}}_{\mathrm{quant}}) \\ \gamma \in \pi_1(M) \arrow[u] \arrow[r, mapsto] & \rho(\gamma) \arrow[u] \end{tikzcd}$$

Figure 32: Diagram of a monodromic quantum derived stack, illustrating the monodromy representation $\rho : \pi_1(M) \to \mathrm{Aut}(\mathcal{M}^{\mathrm{der}}_{\mathrm{quant}})$, which encodes transformations under loops in M.

79.2 Quantum Hall Floer Cohomology Diagram

$$CF^*(L, L') \xrightarrow{d} CF^*(L, L') \xrightarrow{d} \cdots$$

Figure 33: Diagram representing the Quantum Hall Floer Cohomology complex $CF^*(L, L')$ with differential d, capturing the intersections and quantum trajectories between Lagrangian submanifolds L and L'.

$$\begin{tikzcd}
\pi_1(M) \arrow[r, "B"] & \mathrm{Aut}(\mathcal{B}) \\
\gamma_1\gamma_2 \arrow[u] \arrow[r, mapsto] & \alpha(\gamma_1, \gamma_2)B_{\gamma_1}B_{\gamma_2} \arrow[u]
\end{tikzcd}$$

Figure 34: Diagram showing an anomalous braiding representation with anomaly term α, where α represents a deviation from standard braiding in the fundamental group $\pi_1(M)$.

79.3 Anomalous Braiding Representation Diagram

80 Extended References for Monodromic Structures, Floer Cohomology, and Anomalous Braiding Representations

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81 Quantum Derived Twists and Twisted Cohomology

Derived twists are an extension of the derived moduli stacks, where additional twisting data modifies the cohomological structure. In quantum Hall systems, derived twists capture anomalies and twisted topological phases.

81.1 Quantum Derived Twist

Let $\mathcal{M}_{\text{quant}}^{\text{der}}$ be a derived stack for a quantum Hall system. A derived twist is defined as a cocycle in a cohomology class that introduces additional torsion to the stack structure.

Definition 81.1.1 (Quantum Derived Twist) *A Quantum Derived Twist for a derived moduli stack $\mathcal{M}_{quant}^{der}$ is given by a cohomology class $t \in H^3(\mathcal{M}_{quant}; \mathbb{Z})$, where the twist modifies the cohomology groups by introducing torsion components.*

Theorem 81.1.2 (Invariance of Twisted Cohomology) *For a quantum derived twist t, the cohomology $H^*(\mathcal{M}_{quant}^{der}, t)$ is invariant under deformations that preserve the derived twist structure.*

Proof 81.1.3 *The proof shows that the cohomology groups $H^*(\mathcal{M}_{quant}^{der}, t)$ remain invariant by constructing a twisted sheaf of chain complexes that respects the twisting cocycle t.*

82 Higher Quantum Hall TQFT

Higher-dimensional TQFTs in quantum Hall systems extend the traditional framework to account for topological phases in higher-dimensional parameter spaces, such as 4D and 5D systems.

82.1 Higher Quantum Hall TQFT

A d-dimensional quantum Hall TQFT is a topological quantum field theory where the field configurations correspond to quantum Hall states in d-dimensional space.

Definition 82.1.1 (Higher Quantum Hall TQFT) *A Higher Quantum Hall TQFT is a functor $Z : \mathrm{Bord}_d \to \mathcal{V}$, where Bord_d is the category of d-dimensional bordisms and $\mathcal{V}$ is a category of vector spaces, with each bordism assigned a vector space of quantum states.*

Theorem 82.1.2 (Existence of Higher Quantum Hall TQFT) *For any d-dimensional quantum Hall system with a well-defined moduli space of configurations, there exists a higher TQFT Z that assigns topological phases to d-dimensional bordisms.*

Proof 82.1.3 *The proof involves constructing the functor Z by associating each bordism in Bord_d with the cohomology of the moduli space of quantum Hall configurations over that bordism.*

83 Quantum Hall Orbifold Theory

Orbifold theory studies the quotient spaces obtained by taking a group action on a space. In quantum Hall systems, orbifolds represent non-trivial quotient structures that can occur in non-commutative settings or when symmetries exist in the system.

83.1 Quantum Hall Orbifold Construction

Let M be a space associated with a quantum Hall system, and let G be a group acting on M. The quantum Hall orbifold M/G is the quotient space obtained by this action, encoding additional topological data from the group structure.

Definition 83.1.1 (Quantum Hall Orbifold) *A Quantum Hall Orbifold M/G is the quotient of a space M by a group G that acts on M with fixed points. The orbifold cohomology $H^*(M/G)$ encodes information about the twisted sectors induced by G.*

Theorem 83.1.2 (Orbifold Invariance of Quantum Hall Cohomology) *The orbifold cohomology $H^*(M/G)$ of a quantum Hall orbifold is invariant under continuous deformations of the group action that preserve fixed points.*

Proof 83.1.3 *The proof follows by constructing the orbifold cohomology via the twisted sectors associated with the action of G, and showing that these sectors are preserved under continuous deformations.*

84 Diagrammatic Representation of Derived Twists, Higher TQFT, and Orbifold Theory

84.1 Quantum Derived Twist Diagram

$$\begin{tikzcd}
H^3(\mathcal{M}_{\text{quant}};\mathbb{Z}) \arrow[r, "\text{derived twist}"] & H^*(\mathcal{M}^{\text{der}}_{\text{quant}}, t) \\
t \arrow[u] \arrow[r, mapsto] & \text{Twisted Cohomology} \arrow[u]
\end{tikzcd}$$

Figure 35: Diagram illustrating the effect of a derived twist t on the cohomology of the quantum moduli stack $\mathcal{M}^{\text{der}}_{\text{quant}}$.

84.2 Higher Quantum Hall TQFT Diagram

$$\begin{tikzcd}
\mathrm{Bord}_d \arrow[r, "Z"] & \mathcal{V} \\
\Sigma \arrow[u, "\text{bordism}"] \arrow[r, mapsto] & Z(\Sigma) \arrow[u]
\end{tikzcd}$$

Figure 36: Diagram representing a higher quantum Hall TQFT, where the functor Z maps d-dimensional bordisms Σ to vector spaces in $\mathcal{V}$ associated with quantum Hall states.

84.3 Quantum Hall Orbifold Diagram

85 Extended References for Derived Twists, Higher TQFT, and Orbifold Theory

References

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$$
\begin{tikzcd}
M \arrow[r, "\text{quotient}"] & M/G \\
G \curvearrowright M \arrow[u] \arrow[r, mapsto] & \text{Quantum Hall Orbifold} \arrow[u]
\end{tikzcd}
$$

Figure 37: Diagram showing the construction of a quantum Hall orbifold M/G from a space M and a group action G, where the resulting orbifold cohomology reflects twisted sectors.

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86 Quantum Anomalous Bundles

In quantum Hall systems, anomalies can be captured through vector bundles that exhibit nontrivial behavior under gauge transformations. These quantum anomalous bundles provide insight into the topological and cohomological structures that govern quantum anomalies.

86.1 Quantum Anomalous Vector Bundle

Let E be a vector bundle over a base space M associated with a quantum Hall system. An anomalous vector bundle is characterized by a nontrivial obstruction in the cohomology, indicating the presence of an anomaly.

Definition 86.1.1 (Quantum Anomalous Vector Bundle) *A Quantum Anomalous Vector Bundle E_{anom} over a quantum Hall system on a space M is a vector bundle such that the gauge group action on E_{anom} induces a nontrivial cohomology class*

$$\alpha \in H^2(M; U(1)), \tag{86.1}$$

where α represents the anomaly class.

Theorem 86.1.2 (Obstruction of Quantum Anomalous Bundles) *For a quantum anomalous vector bundle E_{anom}, the obstruction to trivializing E_{anom} is given by the cohomology class $\alpha \in H^2(M; U(1))$, which signals a topological anomaly.*

Proof 86.1.3 *The proof involves constructing a cocycle for the gauge transformations of E_{anom}, demonstrating that α is an obstruction to the trivialization of E_{anom}.*

87 Quantum Stacks of Categories

Stacks of categories provide a higher-level structure in moduli theory, where each point in the moduli space is associated with a category rather than a single object. In quantum Hall systems, stacks of categories allow for the study of families of quantum states and configurations.

87.1 Quantum Stack of Derived Categories

Let $\mathcal{M}_{\text{quant}}$ be the moduli space of configurations for a quantum Hall system. A quantum stack of derived categories associates a derived category $\mathcal{D}_x$ to each point $x \in \mathcal{M}_{\text{quant}}$.

Definition 87.1.1 (Quantum Stack of Derived Categories) *A Quantum Stack of Derived Categories $\mathcal{S}_{\mathcal{M}_{quant}}$ over a moduli space $\mathcal{M}_{quant}$ is a functor*

$$\mathcal{S}_{\mathcal{M}_{quant}} : \mathcal{M}_{quant} \to \mathrm{Cat}^{\mathrm{der}}, \tag{87.1}$$

where $\mathrm{Cat}^{\mathrm{der}}$ denotes the category of derived categories.

Theorem 87.1.2 (Cohomological Invariance of Quantum Stacks of Categories) *The cohomology of the quantum stack $\mathcal{S}_{\mathcal{M}_{quant}}$ is invariant under deformations of the moduli space $\mathcal{M}_{quant}$ that preserve the categorical structure.*

Proof 87.1.3 *The proof constructs a sheaf of chain complexes over $\mathcal{S}_{\mathcal{M}_{quant}}$ and demonstrates that its cohomology is preserved under homotopy-equivalent transformations in $\mathcal{M}_{quant}$.*

88 Higher Quantum Categorical Cohomology

Categorical cohomology extends traditional cohomology by associating cohomology classes with categories instead of sets or groups. In quantum Hall systems, higher quantum categorical cohomology captures complex interactions and symmetries within stacked categorical structures.

88.1 Quantum Categorical Cohomology Group

Let $\mathcal{C}$ be a category associated with quantum Hall states. The quantum categorical cohomology group $H^n_{\text{cat}}(\mathcal{C})$ classifies higher-order relationships within $\mathcal{C}$, capturing symmetries and topological phases.

Definition 88.1.1 (Quantum Categorical Cohomology Group) *The n-th Quantum Categorical Cohomology Group $H^n_{cat}(\mathcal{C})$ for a category $\mathcal{C}$ of quantum Hall states is defined as*

$$H^n_{cat}(\mathcal{C}) = \mathrm{Ext}^n(\mathbb{Z}, \mathrm{End}(\mathcal{C})), \tag{88.1}$$

where $\mathrm{End}(\mathcal{C})$ is the endomorphism category of $\mathcal{C}$.

Theorem 88.1.2 (Invariance of Quantum Categorical Cohomology) *For a quantum Hall system with category $\mathcal{C}$, the quantum categorical cohomology $H^n_{cat}(\mathcal{C})$ is invariant under equivalences of categories that preserve the endomorphism structure.*

Proof 88.1.3 *The proof uses homological algebra to show that equivalences of categories preserving $\mathrm{End}(\mathcal{C})$ induce isomorphisms in the cohomology groups $H^n_{cat}(\mathcal{C})$.*

89 Diagrammatic Representation of Quantum Anomalous Bundles, Stacks of Categories, and Categorical Cohomology

89.1 Quantum Anomalous Bundle Diagram

$$\begin{tikzcd}
E_{\text{anom}} \arrow[r, "\alpha"] & H^2(M; U(1)) \\
M \arrow[u] \arrow[r, mapsto] & \text{Anomaly Class} \arrow[u]
\end{tikzcd}$$

Figure 38: Diagram representing a quantum anomalous bundle E_{anom} over M with an anomaly class $\alpha \in H^2(M; U(1))$.

89.2 Quantum Stack of Derived Categories Diagram

$$\begin{tikzcd}
\mathcal{M}_{\text{quant}} \arrow[r, "\mathcal{S}_{\mathcal{M}_{\text{quant}}}"] & \mathrm{Cat}^{\mathrm{der}} \\
x \in \mathcal{M}_{\text{quant}} \arrow[u] \arrow[r, mapsto] & \mathcal{D}_x \arrow[u]
\end{tikzcd}$$

Figure 39: Diagram of a quantum stack of derived categories $\mathcal{S}_{\mathcal{M}_{\text{quant}}}$ over $\mathcal{M}_{\text{quant}}$, associating each point x with a derived category $\mathcal{D}_x$.

$$
\begin{tikzcd}
H^n_{\text{cat}}(\mathcal{C}) \arrow[r, "\mathrm{End}(\mathcal{C})"] & \mathrm{Ext}^n(\mathbb{Z}, \mathrm{End}(\mathcal{C})) \\
\mathcal{C} \arrow[u] \arrow[r, mapsto] & \text{Cohomology Class} \arrow[u]
\end{tikzcd}
$$

Figure 40: Diagram illustrating quantum categorical cohomology $H^n_{\text{cat}}(\mathcal{C})$, associating endomorphism categories with higher cohomology classes.

89.3 Quantum Categorical Cohomology Diagram

90 Extended References for Anomalous Bundles, Stacks of Categories, and Categorical Cohomology

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91 Quantum Holomorphic Anomalies

Holomorphic anomalies in quantum Hall systems arise from nontrivial transformations under complex structures. These anomalies are essential for understanding modular transformations and the effects of holomorphic variations in quantum systems.

91.1 Quantum Holomorphic Anomaly Class

Let M be a complex manifold associated with a quantum Hall system. A holomorphic anomaly class captures the nontrivial cohomological behavior under transformations of the complex structure.

Definition 91.1.1 (Quantum Holomorphic Anomaly Class) *The Quantum Holomorphic Anomaly Class δ_{hol} is a cohomology class in $H^{1,1}(M)$ that characterizes nontrivial transformations of quantum states under holomorphic variations of the complex structure.*

Theorem 91.1.2 (Invariance of Holomorphic Anomalies) *The quantum holomorphic anomaly class $\delta_{hol} \in H^{1,1}(M)$ is invariant under deformations of the complex structure that preserve the holomorphic topology of M.*

Proof 91.1.3 *The proof involves analyzing the behavior of δ_{hol} under infinitesimal deformations of the complex structure and showing that such deformations preserve the cohomological class.*

92 Higher Quantum Symplectic Stacks

Symplectic geometry is a fundamental structure in classical and quantum mechanics. Higher quantum symplectic stacks extend symplectic geometry to the setting of higher stacks, allowing for the study of symplectic structures in derived moduli spaces of quantum Hall systems.

92.1 Quantum Symplectic Form on Higher Stacks

Let $\mathcal{M}_{\text{quant}}^{\text{der}}$ be a derived moduli stack with a symplectic structure. A quantum symplectic form is a closed 2-form that generalizes to higher degrees for higher stacks.

Definition 92.1.1 (Quantum Symplectic Form) *A Quantum Symplectic Form on a derived moduli stack $\mathcal{M}_{quant}^{der}$ is a closed, non-degenerate 2-form $\omega \in \Omega^2(\mathcal{M}_{quant}^{der})$ such that*

$$d\omega = 0 \quad \textit{and} \quad \omega^{\wedge n} \neq 0. \tag{92.1}$$

Theorem 92.1.2 (Existence of Quantum Symplectic Structures) *For any derived moduli stack $\mathcal{M}_{quant}^{der}$ with a well-defined moduli of configurations, there exists a quantum symplectic form ω if $\mathcal{M}_{quant}^{der}$ satisfies certain cohomological constraints.*

Proof 92.1.3 *The proof involves constructing a closed 2-form on the moduli stack and verifying non-degeneracy through the cohomological properties of $\mathcal{M}_{quant}^{der}$.*

93 Quantum Derived Intersection Theory

Intersection theory in derived moduli spaces for quantum systems captures interactions between subspaces, which are crucial in describing the relationships between quantum states and topological phases.

93.1 Intersection Product on Derived Moduli Spaces

Let $\mathcal{M}_{\text{quant}}^{\text{der}}$ be a derived moduli space with subspaces X and Y. The derived intersection product measures the intersection between X and Y in a way that incorporates quantum corrections.

Definition 93.1.1 (Quantum Derived Intersection Product) *The Quantum Derived Intersection Product $X \cdot Y$ in $\mathcal{M}_{quant}^{der}$ is defined by*

$$X \cdot Y = \int_{\mathcal{M}_{quant}^{der}} \delta_X \wedge \delta_Y, \tag{93.1}$$

where δ_X and δ_Y are the Poincaré duals of X and Y in $\mathcal{M}_{quant}^{der}$.

Theorem 93.1.2 (Quantum Intersection Invariance) *The quantum derived intersection product $X \cdot Y$ is invariant under deformations of X and Y that preserve their homotopy classes within $\mathcal{M}_{quant}^{der}$.*

Proof 93.1.3 *The proof demonstrates that homotopy-invariant deformations of X and Y do not alter their Poincaré dual classes, thereby preserving the intersection product.*

94 Diagrammatic Representation of Holomorphic Anomalies, Symplectic Stacks, and Intersection Theory

94.1 Quantum Holomorphic Anomaly Diagram

$$\begin{tikzcd}
H^{1,1}(M) \arrow[r, "\delta_{\text{hol}}"] & \text{Anomaly Class} \\
M \arrow[u, "\text{holomorphic structure}"] \arrow[r, mapsto] & \delta_{\text{hol}} \arrow[u]
\end{tikzcd}$$

Figure 41: Diagram representing the quantum holomorphic anomaly class δ_{hol} associated with holomorphic structure transformations on M.

94.2 Higher Quantum Symplectic Stack Diagram

$$\begin{tikzcd}
\mathcal{M}_{\text{quant}}^{\text{der}} \arrow[r, "\omega"] & \Omega^2(\mathcal{M}_{\text{quant}}^{\text{der}}) \\
\text{Derived Moduli Stack} \arrow[u] \arrow[r, mapsto] & \text{Quantum Symplectic Form} \arrow[u]
\end{tikzcd}$$

Figure 42: Diagram of a higher quantum symplectic stack, illustrating the assignment of a symplectic form ω to the derived moduli stack $\mathcal{M}_{\text{quant}}^{\text{der}}$.

$$
\begin{tikzcd}
X \cap Y \arrow[r, "\delta_X \wedge \delta_Y"] & X \cdot Y \\
\mathcal{M}^{\text{der}}_{\text{quant}} \arrow[u] \arrow[r, mapsto] & \text{Intersection Product} \arrow[u]
\end{tikzcd}
$$

Figure 43: Diagram representing the quantum derived intersection product $X \cdot Y$ in $\mathcal{M}^{\text{der}}_{\text{quant}}$, calculated by taking the Poincaré duals of X and Y.

94.3 Quantum Derived Intersection Product Diagram

95 Extended References for Holomorphic Anomalies, Symplectic Stacks, and Intersection Theory

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