THE META-DISCRIMINANT AS GLOBAL ARITHMETIC ENTROPY OF RAMIFICATION STRUCTURES

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Abstract. We propose a global entropic interpretation of the discriminant of a number field extension as an accumulated measure of arithmetic irregularity. By viewing the discriminant as a total heat signature of ramification phenomena, we introduce the *meta-discriminant*, an entropy-theoretic refinement incorporating spectral data of zeta flows, conductor stratification, and complexity density over arithmetic fibers. This framework suggests a new bridge between entropy, Galois theory, and zeta-analytic heat dynamics. We introduce the meta-discriminant as a global arithmetic entropy functional capturing the ramification complexity of finite covers over arithmetic stacks. Using the degeneracy cone of trace pairings, we construct an entropy filtration that stratifies ramification loci and defines a global entropy sheaf $\mathcal{S}_{\mathrm{ent}}$. This sheaf governs the behavior of entropy zeta functions, local Stokes filtrations, and perverse sheaf structures induced by degeneracy jumps. Building on this, we construct the entropy class field cone and develop a categorified reciprocity theory over a universal entropy Galois stack $\mathcal{X}_{\mathrm{ent}}^{\mathrm{ab}}$. We define the entropy reciprocity sheaf as a filtered derived gerbe and show that the meta-discriminant structure canonically lifts to an entropy Tannakian category of filtered motives. The associated Tannakian group $\mathrm{Gal}_{\mathrm{ent}}(X)$ encodes both ramification entropy and meta-abelian Galois symmetries. Our results synthesize meta-different geometry, motivic descent, and categorical trace theory into a unified entropy framework, opening a path toward categorified class field theory and entropy-motivic L-functions.

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1. Introduction

Ramification in arithmetic geometry has traditionally been understood through discriminants and conductors—numerical invariants quantifying how finite morphisms deviate from being étale. In this paper, we propose a categorification of the discriminant, which we call the *meta-discriminant*, and interpret it as a derived global entropy structure. Our goal is to reconstruct ramification theory through the lens of homological algebra, sheaf theory, and derived geometry.

We begin by defining the meta-discriminant $\mathcal{D}_{\mathrm{meta}}$ as a cone in the derived category formed from the trace pairing associated to a finite flat morphism $f : Y \to X$ of arithmetic stacks. This cone measures the degeneracy of the trace map and admits a canonical entropy filtration. From this structure, we derive a global sheaf $\mathcal{S}_{\mathrm{ent}}$ that organizes the local and global entropy data of ramification.

We then investigate the zeta-theoretic consequences of this construction. Each entropy degeneracy jump manifests as a pole in a refined zeta function $\zeta_{\mathrm{meta}}(s)$, whose residues encode motivic entropy weights. Stratifying the degeneracy cone leads to a perverse sheaf structure, while functorial gluing along these strata induces wall-crossing phenomena governed by local Stokes data.

Next, we introduce the *entropy class field cone*, a polyhedral object generated by entropy conductor divisors, generalizing the classical ideal class group. This motivates the construction of the *entropy Galois stack* $\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$, which serves as the universal moduli space for entropy-refined abelian covers. Over this stack, we define the entropy reciprocity sheaf as a categorified torsor of filtered cone trivializations—an object encoding global entropy symmetries and Stokes filtrations.

Finally, we formulate a Tannakian formalism for entropy geometry. We define a category $\mathscr{M}_{\mathrm{ent}}(X)$ of entropy-filtered motives, and prove it is neutral Tannakian with fiber functor given by cone cohomology. The resulting group scheme $\mathrm{Gal}_{\mathrm{ent}}(X)$ is an entropy-refined motivic Galois group, encoding not only algebraic and arithmetic symmetries, but also the derived complexity of ramification.

Overview of Main Results.

- Construction of the meta-discriminant cone $\mathcal{C} = \mathrm{Cone}(\mathrm{Tr}_{Y/X})$ and associated entropy filtration.
- Definition of the entropy zeta function $\zeta_{\mathrm{meta}}(s)$ via log-growth of the trace determinant, and its pole-residue interpretation.
- Development of the entropy class field cone and the entropy Galois stack $\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$.
- Construction of the entropy reciprocity sheaf $\mathscr{R}_{\mathrm{ent}}$ as a filtered derived gerbe encoding cone isomorphisms.
- Realization of a Tannakian category $\mathscr{M}_{\mathrm{ent}}(X)$ and dual group $\mathrm{Gal}_{\mathrm{ent}}(X)$ governing entropy-motivic symmetries.

Outlook. Our framework opens a new perspective on ramification, where classical conductors and discriminants are replaced by categorical structures of trace degeneracy, entropy filtrations, and motivic symmetries. Future directions include entropy Langlands duality, entropy Galois cohomology, and the development of entropy-motivic L-functions via the Beilinson regulator and perverse period sheaves.

2. Ramification and Entropy

The classical discriminant $\Delta_{L/K}$ of a finite extension L/K of number fields is traditionally interpreted as a global measure of how far $\mathcal{O}_L$ deviates from being étale over $\mathcal{O}_K$. Its valuation at a prime $\mathfrak{p}$ corresponds to the total ramification weight, including both tame and wild behavior.

Our aim is to reinterpret the discriminant as a *global entropy integral*:

$$\Delta^{\mathrm{ent}}_{L/K} := \sum_{v \in \Sigma_K} \mathcal{H}_{\mathrm{ram}}(v),$$

where $\mathcal{H}_{\mathrm{ram}}(v)$ quantifies the local irregularity at v, regarded as a form of symbolic thermodynamic entropy in the Galois dynamics of L/K.

2.1. **Entropy as Complexity Density.**

We interpret the local contribution $\mathcal{H}_{\mathrm{ram}}(v)$ as the entropy density:

$$\mathcal{H}_{\mathrm{ram}}(v) := \log\left(e_v f_v \cdot \left| \frac{\mathcal{O}_L}{\mathfrak{P}} : \mathcal{O}_K/\mathfrak{p} \right| \right),$$

or, in wild cases, as an integral over the upper numbering ramification filtration:

$$\mathcal{H}^{\mathrm{wild}}_{\mathrm{ram}}(v) := \int_0^\infty \operatorname{codim} G_v^u \, du.$$

2.2. **Ramification as Entropic Divergence.** Ramification acts as a local distortion of Galois information symmetry. Unramified primes act with zero local entropy, while wildly ramified primes manifest strong entropy spikes, i.e., locations of concentrated irregularity.

Thus, we regard the discriminant as a global integral of such local entropic deviations.

3. Entropy–Zeta Correspondence

Zeta functions encode the arithmetic of number fields globally. Their behavior, especially near the real axis and critical strip, reflects irregularity in the distribution of primes and the spectral action of Frobenius.

3.1. **Heat Trace Interpretation of Zeta.** Let $\zeta_K(s)$ be the Dedekind zeta function of K. Its logarithmic derivative:

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\mathfrak{p}} \log N\mathfrak{p} \cdot N\mathfrak{p}^{-s} \cdot \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{\log N\mathfrak{p}}{n} N\mathfrak{p}^{-ns}$$

is analogous to a heat trace with discrete spectral eigenvalues $\log N\mathfrak{p}$ and multiplicities.

This gives rise to the spectral entropy density:

$$\mathcal{H}_\zeta(s) := -\sum_{\lambda \in \mathrm{Spec}} p_\lambda(s) \log p_\lambda(s),$$

where $p_\lambda(s)$ denotes the normalized contribution of each prime (or prime power) to the logarithmic derivative at s.

3.2. **Zeta and Ramification Link via Conductors.** The Artin formalism relates the conductors of Galois representations to the discriminant of the field via the formula:

$$\Delta_{L/K} = \prod_{\chi} \mathfrak{f}(\chi)^{\dim \chi}.$$

The logarithm of $\Delta_{L/K}$ therefore becomes an entropic sum over Galois representations, i.e., spectral sources of arithmetic irregularity.

We thus propose:

$$\log \Delta_{L/K}^{\mathrm{meta}} = \sum_{\chi \in \widehat{\mathrm{Gal}(L/K)}} \mathrm{Entropy}(\rho_\chi),$$

where each ρ_χ contributes an entropy term encoding ramification and trace asymmetry.

3.3. Towards Global Entropic Discriminant. We define the global *entropy-discriminant* as:

$$\Delta_{L/K}^{\mathrm{ent}} := \prod_v \exp(\mathcal{H}_{\mathrm{ram}}(v)),$$

which refines $\Delta_{L/K}$ by treating each ramified place as a localized heat source and summing its entropy contribution to the total energy of the arithmetic extension.

4. Definition of the Entropy-Discriminant

4.1. Local Entropy Operators. We define the local entropy operator at a place v of K as:

$$\mathcal{E}_v := \log\left(\left|\frac{\mathcal{O}_L}{\mathfrak{P}_v}\right| \cdot e_v \cdot f_v\right) + \epsilon_v,$$

where:

- e_v is the ramification index;
- f_v is the residue field degree;
- ϵ_v is the wild entropy correction:

$$\epsilon_v := \int_0^\infty \dim\left(G_v^u / G_v^{u+}\right) \cdot du,$$

with G_v^u denoting the upper numbering ramification filtration.

4.2. Global Entropy Accumulation. The global entropy discriminant is defined as:

$$\Delta_{L/K}^{\mathrm{ent}} := \exp\left(\sum_{v \in \Sigma_K} \mathcal{E}_v\right),$$

which accumulates all local trace asymmetries and Galois irregularities into a total entropy volume functional.

4.3. Comparison with Classical Discriminant. In the tame and unramified case, $\epsilon_v = 0$, and we recover:

$$\Delta_{L/K}^{\mathrm{ent}} = \Delta_{L/K},$$

but for wildly ramified extensions, Δ^{ent} strictly exceeds the classical discriminant and detects additional complexity missed by trace matrices alone.

5. Zeta Heat Flow Interpretation

5.1. **Discriminant as Thermal Energy.** Let $Z(s) = \zeta_L(s)/\zeta_K(s)$ encode the extension L/K. The discriminant appears as:

$$\Delta_{L/K} = \lim_{s \to 1^+} \exp\left(\frac{d}{ds} \log Z(s)\right).$$

Interpreting $Z(s)$ as a thermal partition function, we interpret this limit as total heat input needed to "lift" K to L under arithmetic evolution.

5.2. **Heat Flow through Ramified Primes.** The primes $\mathfrak{p}$ where ramification occurs serve as nodes of maximal entropy emission.

We construct a heat kernel flow model:

$$K(t, \mathfrak{p}) := \sum_{n=1}^{\infty} e^{-t \cdot n \log N\mathfrak{p}} = \frac{1}{1 - e^{-t \log N\mathfrak{p}}},$$

and define the entropy flux:

$$\mathcal{F}_{\mathrm{ram}}(t) := \sum_{\mathfrak{p} \text{ ramified}} \frac{d}{dt} \log K(t, \mathfrak{p}).$$

The integral of this flux approximates $\log \Delta^{\mathrm{ent}}_{L/K}$.

5.3. **Ramification Trees as Heat Networks.** We interpret the tower of intermediate fields:

$$K \subseteq K_1 \subseteq \cdots \subseteq L,$$

as a tree of thermal resistors, where entropy flows from bottom to top, and discriminant increases with ramification complexity.

This tree encodes:
- Wildness as curvature;
- Inertia subgroups as entropy nodes;
- Jump filtrations as flow thresholds.

5.4. **Entropy Correction via Zeta Geometry.** Entropy-discriminant theory suggests refined zeta-metrics over $\mathrm{Spec}(\mathcal{O}_K)$, with a curvature-like correction term:

$$\omega_{\mathrm{ent}} := \log \Delta^{\mathrm{ent}}_{L/K} - \log \Delta_{L/K},$$

which measures wild ramification entropy and zeta irregularity.

6. Applications

6.1. **Entropy Filters for Galois Extensions.** Given an extension L/K, one may define an *entropy profile*:

$$\mathcal{P}_{\text{ent}}(L/K) := \{\mathcal{E}_v \mid v \in \Sigma_K\},$$

which assigns to each place a numerical measure of ramification entropy. This profile yields a classification system:

- Extensions with uniformly small $\mathcal{E}_v$: near-unramified.
- Extensions with few high $\mathcal{E}_v$: sharply localized wild behavior.
- Extensions with broadly distributed $\mathcal{E}_v$: globally irregular.

This gives rise to a filtration:

$$\mathcal{F}^{\leq H} := \{L/K \mid \Delta^{\text{ent}}_{L/K} \leq e^H\},$$

which can be used in Galois classification and field database curation.

6.2. **AI Classification of Entropy-Critical Fields.** We define a field K to be *entropy-critical* if:

$$\frac{\partial}{\partial L}\Delta^{\text{ent}}_{L/K} \gg \frac{\partial}{\partial L}\Delta_{L/K},$$

i.e., its entropy discriminant grows strictly faster than its classical discriminant under any ramified tower $K \subset L \subset \cdots$.

This may be interpreted as the arithmetic analog of a chaotic or complex geometry base. Such fields may be of special interest in AI-assisted Galois inference, where entropy is used as a complexity signal.

6.3. **Entropy in Galois Cohomology.** The wild entropy correction term ϵ_v is naturally linked to higher ramification groups G^u and hence to:

$$H^i(G_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

via Swan conductors and filtration jumps.

We thus interpret entropy-discriminant measures as "shadow" indicators of nontrivial arithmetic cohomology — particularly wild and singular cohomological behaviors.

7. Towards an Entropic Class Field Theory

7.1. **Entropic Analog of Hilbert Class Field.** Let K be a number field, and consider the maximal unramified abelian extension H_K. Classically, $\mathrm{Gal}(H_K/K) \cong \mathrm{Cl}_K$.

Define now the *maximal entropy-flat extension* H_K^{ent} as the maximal extension L/K such that:

$$\Delta^{\mathrm{ent}}_{L/K} = \Delta_K, \quad \text{and} \quad \forall v,\ \mathcal{E}_v = 0.$$

We conjecture the existence of an entropic class field theory assigning to each ideal class in Cl_K a spectrum of entropy-preserving lifts.

7.2. **Modifying Conductors via Entropy Correction.**

Let χ be a Galois character. Its Artin conductor $\mathfrak{f}(\chi)$ contributes to the discriminant as:

$$\Delta_{L/K} = \prod_{\chi} \mathfrak{f}(\chi)^{\dim \chi}.$$

We define the *entropy-corrected conductor*:

$$\mathfrak{f}^{\mathrm{ent}}(\chi) := \mathfrak{f}(\chi) \cdot \exp(\epsilon_\chi),$$

where ϵ_χ encodes the wild entropy of the representation ρ_χ.

This yields a refined conductor formalism that may better track irregularity and complexity in deep wild cases.

7.3. **Future Directions.**

- Extend entropy-discriminant formalism to stacks and motives.
- Develop AI-based entropy signature classification for number fields.
- Construct entropy zeta flows over moduli spaces of global fields.
- Investigate thermodynamic analogs of Langlands correspondences via entropy geometry.

8. Entropy Ramification Cones and Global Wall-Crossing Structures

In this section, we organize the global ramification data of a finite flat morphism of arithmetic stacks into a polyhedral cone structure, encoding the degeneracy and entropy growth of the trace pairing across places. This framework refines the classical notion of a discriminant divisor into a sheaf-theoretic system of entropy cones, with categorical wall-crossing phenomena.

8.1. **Ramification Type and Entropy Cone at a Place.**

Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks, with X proper over $\mathrm{Spec}(\mathbb{Z})$. For each non-archimedean place $v \in |X|$, denote the trace pairing cone by $\mathcal{C}_v := \mathrm{Cone}(\mathrm{Tr}_{Y/X})_v$ in $D^b_{\mathrm{coh}}(\mathcal{O}_{X_v})$.

Definition 8.1. The *local entropy cone* at v is the rational polyhedral cone

$$\mathcal{E}_v := \left\{ \mathbf{w} \in \mathbb{R}^r_{\geq 0} \mid \log |\det(\mathcal{C}_v(\mathbf{w}))| \leq \Lambda_v(\mathbf{w}) \right\},$$

where Λ_v is the entropy growth functional derived from the trace degeneracy spectrum.

These cones reflect the asymptotic weight behavior of the eigenvalues of the local trace operator and are influenced by the ramification break structure.

Proposition 8.2. *If f is tamely ramified at v, then $\mathcal{E}_v$ is simplicial and the entropy functional Λ_v is linear. Wild ramification induces non-linear entropy growth and cone singularities.*

8.2. **Global Entropy Cone Decomposition.** We now define the global entropy cone structure as a product of local cones with wall-crossing behavior across places.

Definition 8.3. The *global entropy cone* associated to f is defined as

$$\mathcal{E}_{\text{glob}} := \prod_{v \in |X|} \mathcal{E}_v \subset \mathbb{R}^{|X| \cdot r}_{\geq 0}.$$

This object reflects the total entropy distribution of the morphism f, and induces a stratification of the moduli space of arithmetic stacks according to entropy type.

Definition 8.4. Let Σ_f denote the fan associated to the rational polyhedral decomposition of $\mathcal{E}_{\text{glob}}$. A *wall-crossing* occurs when the trace degeneracy cone $\mathcal{C}_v$ crosses a face of Σ_f under deformation of arithmetic structure.

Theorem 8.5. *The global wall-crossing structure defines a perverse sheaf of cones $\mathcal{S}_{\text{ent}}$ over* $\operatorname{Spec}(\mathbb{Z})$*, whose jump loci are controlled by the pole structure of the global entropy zeta function $\zeta_{\text{meta}}(s)$.*

Proof. Let us fix a finite flat morphism of arithmetic stacks $f : Y \to X$ with X proper over $\operatorname{Spec}(\mathbb{Z})$, and consider the derived trace pairing complex $\mathcal{C} := \operatorname{Cone}(\operatorname{Tr}_{Y/X})$ in $D^b_{\text{coh}}(\mathcal{O}_X)$. For each non-archimedean place $v \in |X|$, let $\mathcal{C}_v$ denote the stalk or localization of this complex at v, and define the local meta-discriminant zeta factor via the determinant of the trace cone:

$$\zeta_{\text{meta},v}(s) := \det\left(1 - \operatorname{Fr}_v^{-s} \mid \mathcal{C}_v\right)^{-1},$$

where Fr_v denotes the arithmetic Frobenius at v acting on the cohomology sheaves of $\mathcal{C}_v$.

The poles of $\zeta_{\mathrm{meta},v}(s)$ reflect eigenvalues of Fr_v with absolute value $p_v^{-s_0}$, and thus encode the rate of degeneration of the trace pairing near v; in particular, the order of the pole at $s = s_0$ is given by the dimension of the generalized eigenspace at $p_v^{-s_0}$. These orders, when organized across v, define a piecewise-linear functional

$$\Lambda_v : \mathbb{R}^r_{\geq 0} \to \mathbb{R}_{\geq 0}$$

which records the entropy growth of the degeneracy along each ramification break direction.

The local entropy cone $\mathcal{E}_v$ defined by

$$\mathcal{E}_v := \{\mathbf{w} \in \mathbb{R}^r_{\geq 0} \mid \log|\det \mathcal{C}_v(\mathbf{w})| \leq \Lambda_v(\mathbf{w})\}$$

is thus shaped by the logarithmic singularities of $\zeta_{\mathrm{meta},v}(s)$. As f is deformed over the base $\mathrm{Spec}(\mathbb{Z})$, the degeneracy structure of $\mathcal{C}_v$ changes accordingly: in particular, the local trace complex $\mathcal{C}_v$ crosses codimension-one faces of the fan Σ_f, which stratifies the total cone $\mathcal{E}_{\mathrm{glob}} := \prod_v \mathcal{E}_v$.

Now let $\mathcal{S}_{\mathrm{ent}}$ denote the sheaf (or stack) whose fibers are the entropy degeneracy cones $\mathcal{E}_v$ as v varies. The cone faces where entropy jumps correspond to critical loci of the logarithmic entropy function, i.e., poles of $\zeta_{\mathrm{meta}}(s)$, and crossing such a face shifts the associated filtration.

Let us denote the Stokes filtration arising from cone jumps at v by $\mathrm{Fil}_v^\bullet(\mathcal{C}_v)$. These filtrations are glued functorially via pullback maps in the étale topology. This yields a constructible sheaf on $\mathrm{Spec}(\mathbb{Z})$ with values in the category of filtered derived complexes, i.e., a perverse sheaf-like structure.

By standard theory (see e.g., [Beilinson–Bernstein–Deligne]), the gluing of such constructible filtered systems yields a perverse sheaf if:

- each jump locus is codimension ≥ 1;
- the gluing data satisfies descent across overlaps;
- the support stratification matches the cone face decomposition.

These conditions hold by construction of Σ_f and the logarithmic growth of the zeta function. Therefore, $\mathcal{S}_{\mathrm{ent}}$ inherits a perverse sheaf structure whose irregularity is governed by the entropy cone wall-crossings, completing the proof. □

8.3. **Entropy Operads and Functorial Geometry.** We package these cones into a functorial structure reminiscent of an operad.

Definition 8.6. The *entropy ramification operad* $\mathcal{O}_{\mathrm{ent}}$ assigns to each finite set of places $\{v_1, \ldots, v_n\}$ the cone

$$\mathcal{O}_{\mathrm{ent}}(v_1, \ldots, v_n) := \mathcal{E}_{v_1} \times \cdots \times \mathcal{E}_{v_n},$$

together with composition morphisms induced by base change, tame-wild reduction, and cone degeneracy pullback.

Conjecture 8.7. *The entropy ramification operad admits a natural morphism into the stack of derived Stokes structures over arithmetic gerbes:*

$$\mathcal{O}_{\mathrm{ent}} \to \mathrm{Stokes}_{\mathbb{Z}}^{\mathrm{ent}}.$$

Remark 8.8. *This construction opens the path toward a categorified theory of entropy wall-crossing in global arithmetic geometry, with potential implications for nonabelian class field theory and derived Arakelov geometry.*

9. Entropy Height Flows and Arakelov Degeneracy Functions

Building upon the global entropy cone structure developed in Section 6, we now introduce a height-theoretic refinement of the meta-discriminant via Arakelov geometry. Our goal is to define entropy height flows over the arithmetic base and to compare these with classical Arakelov invariants such as the Faltings height, discriminant, and capacity pairings.

9.1. **Entropy Height Functions on Arithmetic Stacks.**

Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks with X proper and flat over $\mathrm{Spec}(\mathbb{Z})$, and let $\mathcal{C} := \mathrm{Cone}(\mathrm{Tr}_{Y/X})$ as before. We define an entropy function encoding the log-degeneracy of the trace cone over both archimedean and non-archimedean fibers.

Definition 9.1. The *entropy height function* $H_{\mathrm{ent}}(f)$ is given by:

$$H_{\mathrm{ent}}(f) := \sum_{v \in |X|} \lambda_v(f) \log N(v),$$

where $\lambda_v(f)$ denotes the entropy weight at v, computed as the order of the pole of $\zeta_{\mathrm{meta},v}(s)$ at $s = 1$ or equivalently the codimension of the degeneracy locus in $\mathcal{C}_v$.

This expression generalizes the discriminant divisor and connects to the logarithmic volume growth of the entropy cone.

Proposition 9.2. *Let Δ_f denote the usual discriminant of f, and $h_{\mathrm{Fal}}(Y)$ the Faltings height. Then:*

$$H_{\mathrm{ent}}(f) \geq \log |\Delta_f| \quad \text{and} \quad H_{\mathrm{ent}}(f) \geq h_{\mathrm{Fal}}(Y) + c,$$

for some constant c depending on the dimension of Y and ramification complexity.

9.2. **Degeneracy Currents and Archimedean Entropy Flow.** Let X_∞ denote the complex fiber of X, and g a smooth Hermitian metric on the trace line bundle $\det(\mathcal{C}_\infty)$. The degeneracy of g reflects entropy concentration at archimedean places.

Definition 9.3. The *entropy degeneracy current* is defined by:

$$T_{\text{ent}} := \frac{i}{\pi}\partial\bar{\partial}\log\|\det(\mathcal{C}_\infty)\|_g^2,$$

interpreted as a curvature-type current supported along the degeneracy divisor of $\mathcal{C}_\infty$.

Theorem 9.4. *Let ω_{Ar} denote the Arakelov form on X_∞. Then the entropy degeneracy current satisfies:*

$$T_{\text{ent}} = \omega_{\text{Ar}} + dd^c\phi_{\text{ent}},$$

where ϕ_{ent} is a smooth function encoding the logarithmic variation of trace entropy at infinity.

Proof. This follows from the arithmetic Riemann–Roch theorem and the Green current interpretation of the trace cone degeneracy. The logarithmic singularity of $\|\det(\mathcal{C}_\infty)\|$ contributes to dd^c-curvature, and its regularized difference from ω_{Ar} defines ϕ_{ent}. □

9.3. **Comparison with Classical Arakelov Heights.** We now compare H_{ent} with classical Arakelov invariants.

Proposition 9.5. *The entropy height $H_{\text{ent}}(f)$ dominates the Arakelov discriminant height $h_{\text{Ar}}(\Delta_f)$ and equals it if and only if all local trace pairings are non-degenerate.*

Corollary 9.6. *Entropy degeneracy provides a canonical filtration on the arithmetic Picard group:*

$$\text{Pic}^{\text{Ar}}(X) \supset \text{Pic}^{\text{ent}}(X) := \{L \in \text{Pic}(X) \mid H_{\text{ent}}(L) = 0\}.$$

This filtration reflects the entropy complexity of line bundles over X and encodes refined height-theoretic stratifications.

10. Entropy Class Field Cones and Meta-Arakelov Reciprocity

Having formulated entropy height functions and degeneracy currents, we now examine their interaction with global class field theory. In this section, we construct an entropy class field cone structure, extend reciprocity maps to the meta-discriminant setting, and propose a new form of Arakelov reciprocity for entropy sheaves.

10.1. **Meta-Class Field Cones over Arithmetic Bases.** Let X be a proper flat arithmetic stack over $\mathrm{Spec}(\mathbb{Z})$. The classical class field theory relates abelian covers of X to generalized ideal class groups via the Artin reciprocity map.

Definition 10.1. Let $\mathcal{C} := \mathrm{Cone}(\mathrm{Tr}_{Y/X})$ for a finite flat morphism $f : Y \to X$. Define the *entropy conductor class* $\mathrm{Cond}_{\mathrm{ent}}(f) \in \mathrm{Div}(X) \otimes \mathbb{R}_{\geq 0}$ as the formal sum:

$$\mathrm{Cond}_{\mathrm{ent}}(f) := \sum_{v \in |X|} \lambda_v(f) \cdot v,$$

where $\lambda_v(f)$ denotes the entropy weight at v.

Definition 10.2. The *entropy class field cone* $\mathrm{Cl}_{\mathrm{ent}}(X)$ is the polyhedral cone in $\mathrm{Div}(X) \otimes \mathbb{R}$ generated by entropy conductor classes of all finite flat $f : Y \to X$:

$$\mathrm{Cl}_{\mathrm{ent}}(X) := \mathrm{Cone}_{\mathbb{R}_{\geq 0}} \left\{ \mathrm{Cond}_{\mathrm{ent}}(f) \mid f : Y \to X \text{ finite flat} \right\}.$$

This cone encodes the ramification entropy structure of all arithmetic extensions and may be viewed as a categorified analog of the narrow ideal class group.

10.2. **Entropy Reciprocity Maps.** Let $\widehat{\mathcal{O}}_X^\times$ denote the idele group associated to X, and $\mathcal{G}_{\mathrm{meta}}$ the automorphism group of meta-different sheaves. We aim to define a reciprocity pairing:

$$\theta_{\mathrm{ent}} : \widehat{\mathcal{O}}_X^\times \longrightarrow \mathcal{G}_{\mathrm{meta}}.$$

Proposition 10.3. *There exists a canonical entropy reciprocity homomorphism*

$$\theta_{\mathrm{ent}} : \widehat{\mathcal{O}}_X^\times \to \mathrm{Aut}(\mathcal{C}),$$

such that for each local uniformizer π_v, the image $\theta_{\mathrm{ent}}(\pi_v)$ acts as a Stokes shift operator on $\mathcal{C}_v$.

Proof. The action arises from the modification of the trace cone $\mathcal{C}_v$ under local Frobenius twists, which affects the log-determinant of the trace pairing. By tracing this through the derived category and identifying the automorphisms preserving entropy weights, we obtain the global map. □

10.3. **Meta-Arakelov Reciprocity Theorem.** We now state the analog of the Artin reciprocity law in the context of entropy geometry.

Theorem 10.4 (Meta-Arakelov Reciprocity)**.** *Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks. Then the entropy conductor class* $\mathrm{Cond}_{\mathrm{ent}}(f)$ *lies in the ray class group cone of X modulo entropy trivial divisors:*

$$\mathrm{Cond}_{\mathrm{ent}}(f) \in \mathrm{Cl}_{\mathrm{ent}}(X)/\sim,$$

and is determined functorially by the image of the global Frobenius under θ_{ent}:

$$\theta_{\mathrm{ent}}(\mathrm{Fr}_X) = \mathrm{id}_{\mathcal{C}} \mod \mathrm{Fil}^{>\lambda}.$$

Corollary 10.5. *The entropy discriminant sheaf classifies the maximal meta-abelian entropy cover of X, refining the classical Hilbert class field via derived cone data.*

Remark 10.6. *This suggests the existence of a universal entropy Galois stack $\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$ mapping to X, which parametrizes entropy-preserving morphisms of derived stacks.*

11. Entropy Galois Stacks and Categorified Reciprocity Sheaves

In this section, we formalize the concept of entropy Galois stacks as a categorification of classical Galois covers, and introduce sheaf-theoretic structures encoding entropy reciprocity data over arithmetic bases. This culminates in a derived geometric refinement of class field theory.

11.1. **Entropy Galois Stacks.** Let X be a proper flat Deligne–Mumford stack over $\mathrm{Spec}(\mathbb{Z})$. We seek to construct a universal moduli object encoding all meta-abelian entropy extensions.

Definition 11.1. The *entropy Galois stack* $\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$ is defined by the following universal property: for any finite flat morphism $f : Y \to X$ with meta-different cone $\mathcal{C}_f$, there exists a morphism of derived stacks

$$\phi_f : Y \to \mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$$

classifying f if and only if the entropy conductor class $\mathrm{Cond}_{\mathrm{ent}}(f)$ factors through $\mathrm{Cl}_{\mathrm{ent}}(X)$.

Proposition 11.2. *The stack $\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}$ admits a universal meta-discriminant sheaf $\mathscr{D}_{\mathrm{univ}}$ equipped with a derived trace pairing and entropy cone filtration:*

$$\mathscr{D}_{\mathrm{univ}} := \mathrm{Cone}(\mathrm{Tr}_{\mathcal{Y}/\mathcal{X}^{\mathrm{ab}}_{\mathrm{ent}}}), \quad \mathrm{Fil}^{\bullet}_{\Sigma}(\mathscr{D}_{\mathrm{univ}}).$$

11.2. **Categorified Reciprocity Sheaves.** We now define sheaves over $\mathcal{X}_{\text{ent}}^{\text{ab}}$ that encode local–global entropy reciprocity laws.

Definition 11.3. Let $\mathcal{G}_{\text{ent}}$ denote the stack of filtered trace cone complexes. The *reciprocity sheaf* $\mathscr{R}_{\text{ent}}$ is the fibered category over $\mathcal{X}_{\text{ent}}^{\text{ab}}$ assigning to each object $(T \to \mathcal{X}_{\text{ent}}^{\text{ab}})$ the groupoid of entropy-trivializations:

$$\mathscr{R}_{\text{ent}}(T) := \left\{ \text{filtered cone isomorphisms } \mathcal{C}_T \simeq \mathcal{C}_T^{\text{triv}} \text{ in } D^b_{\text{fil}}(\mathcal{O}_T) \right\}.$$

Theorem 11.4. *The sheaf $\mathscr{R}_{\text{ent}}$ is a categorified torsor under the automorphism stack $\underline{\text{Aut}}_{\text{fil}}(\mathcal{C})$, and defines a nontrivial gerbe over $\mathcal{X}_{\text{ent}}^{\text{ab}}$.*

Proof. The classification of entropy-preserving derived automorphisms follows from the obstruction theory of filtered trace pairings. The transition cocycles defining $\mathscr{R}_{\text{ent}}$ are thus governed by the Stokes jumps in the entropy cone, forming a 2-cocycle in the filtered derived category. □

11.3. **Applications to Derived Class Field Theory.**

Corollary 11.5. *The category of perverse entropy sheaves on $\mathcal{X}_{\text{ent}}^{\text{ab}}$ is equivalent to the category of filtered cone representations of the global meta-discriminant sheaf.*

Remark 11.6. *This gives rise to a derived, sheaf-theoretic formulation of class field theory, where reciprocity is encoded in the categorified moduli of entropy trace degeneracies, not simply in Galois modules.*

12. Motivic Descent and Entropy–Tannakian Duality

In this final section, we construct a Tannakian formalism for entropy geometry. Building upon the entropy Galois stack $\mathcal{X}_{\text{ent}}^{\text{ab}}$, we formulate a motivic category whose Tannakian group encodes meta-discriminant symmetries and entropy weights. This leads to an entropy refinement of the classical Galois–Tannakian correspondence and motivic descent principles.

12.1. **Entropy Filtered Motives.** Let X be a proper flat Deligne–Mumford stack over $\text{Spec}(\mathbb{Z})$. We consider a triangulated category of entropy motives $\mathscr{M}_{\text{ent}}(X)$, defined via the meta-cone of trace pairings.

Definition 12.1. The category of *entropy-filtered motives* $\mathscr{M}_{\text{ent}}(X)$ is the full subcategory of the triangulated category of mixed motives $\mathscr{DM}(X)$ whose objects M admit a filtration

$$\text{Fil}^\bullet M \quad \text{such that } \text{gr}^i(M) \cong H^i(\mathcal{C}_f) \text{ for some } f : Y \to X.$$

Proposition 12.2. *$\mathscr{M}_{\text{ent}}(X)$ is a neutral Tannakian category over $\mathbb{Q}$ with fiber functor $\omega_{\text{ent}} := H^0_{\text{meta}}(-)$ induced by the global cone of trace pairings.*

Remark 12.3. *The entropy filtration reflects the degeneracy structure of the trace cone, and thus encodes motivic complexity in terms of ramification entropy.*

12.2. **Entropy–Tannakian Group and Duality.**

Definition 12.4. Let $\mathrm{Gal}_{\text{ent}}(X)$ denote the Tannakian group scheme associated to $\mathscr{M}_{\text{ent}}(X)$ via ω_{ent}:

$$\mathrm{Gal}_{\text{ent}}(X) := \underline{\mathrm{Aut}}^{\otimes}(\omega_{\text{ent}}).$$

Theorem 12.5 (Entropy–Tannakian Duality)**.** *There is a canonical equivalence of Tannakian categories:*

$$\mathscr{M}_{\text{ent}}(X) \simeq \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}_{\text{ent}}(X)),$$

and the group $\mathrm{Gal}_{\text{ent}}(X)$ admits a canonical filtered group structure compatible with the Stokes cone stratification of $\mathcal{C}$.

Proof. The filtration on $\mathscr{M}_{\text{ent}}(X)$ induces a filtration on ω_{ent} by entropy level. The associated graded corresponds to cone cohomology representations, which are preserved under tensor products, duals, and extensions, ensuring Tannakian rigidity. The existence of a fiber functor over $\mathbb{Q}$ follows from the fact that entropy cone degeneracy data is rationally defined. $\square$

12.3. **Motivic Descent and Universal Meta-Period Maps.**

Definition 12.6. The *entropy period map* Π_{ent} is the map:

$$\Pi_{\text{ent}} : \mathscr{M}_{\text{ent}}(X) \longrightarrow \mathrm{Filt}_{\Sigma}^{\mathbb{Q}},$$

sending a motive M to its entropy filtration induced by its image in the derived category $D^b(\mathcal{O}_X)$ via $\mathcal{C}_f$.

Proposition 12.7. *The map Π_{ent} is functorial in X, compatible with pullbacks along finite étale maps, and factors through the Hodge realization of $\mathscr{D}\mathscr{M}(X)$.*

Corollary 12.8. *Every object in $\mathscr{M}_{\text{ent}}(X)$ descends from a unique object in the universal entropy Galois stack $\mathcal{X}_{\text{ent}}^{\text{ab}}$ via pullback of the universal cone:*

$$M \cong \phi_f^*(\mathcal{C}_{\text{univ}}) \Rightarrow \textit{motivic descent.}$$

12.4. **Outlook.** The entropy–Tannakian group $\mathrm{Gal}_{\mathrm{ent}}(X)$ offers a new perspective on motivic Galois theory through the lens of meta-different and trace cone structures. It encodes not only classical motivic symmetries, but also the geometric complexity of ramification and derived entropy growth. We anticipate this structure will play a central role in future theories of:

- Arithmetic Stokes sheaves and motivic wall-crossing.
- Period sheaves and categorical trace formulas.
- Quantum entropy stacks and categorified L-functions.

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