Zero-Crossings in the Riemann Zeta Function and Hardy's $Z(t)$ Function: An Extension with Zerotrix

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Introduction

The study of zero-crossings in complex functions, such as the Riemann zeta function $\zeta(s)$ and Hardy's $Z(t)$ function, is fundamental in understanding the distribution of prime numbers and the validity of the Riemann Hypothesis. In this document, we extend this theory by introducing the concept of *Zerotrix*, which represents a structured approach to analyze and categorize zero-crossings.

Riemann Zeta Function

The Riemann zeta function $\zeta(s)$ is a complex function defined for $s = \sigma + it$, where σ and t are real numbers. It is initially defined for $\Re(s) > 1$ by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

and by analytic continuation to other values of s, except for $s = 1$ where it has a simple pole.

The Riemann Hypothesis conjectures that all non-trivial zeros of $\zeta(s)$ lie on the critical line:

$$\Re(s) = \frac{1}{2}. \tag{2}$$

These zeros are the points where $\zeta(s) = 0$.

Hardy's $Z(t)$ Function

Hardy's $Z(t)$ function is a real-valued function defined on the critical line $s = \frac{1}{2} + it$. It is given by:

$$Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right), \tag{3}$$

where $\theta(t)$ is the Riemann-Siegel theta function:

$$\theta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2}\log\pi. \tag{4}$$

Here, Γ is the Gamma function.

The function $Z(t)$ simplifies the study of the zeros of $\zeta(s)$ on the critical line because the zeros of $\zeta\left(\frac{1}{2} + it\right)$ correspond to the zeros of $Z(t)$. Specifically, $Z(t) = 0$ if and only if $\zeta\left(\frac{1}{2} + it\right) = 0$.

Zero-Crossings and Zerotrix

The zero-crossings of Hardy's $Z(t)$ function are the points where $Z(t)$ changes sign, i.e., where $Z(t) = 0$. These zero-crossings correspond to the non-trivial zeros of the Riemann zeta function on the critical line.

We introduce the concept of *Zerotrix* to provide a structured approach for analyzing and categorizing these zero-crossings.

Definition of Zerotrix

A *Zerotrix* is defined as a matrix-like structure that organizes the zero-crossings of a function within a specified domain. For Hardy's $Z(t)$ function, the Zerotrix can be represented as follows:

$$\text{Zerotrix}(Z(t)) = \{t_i \mid Z(t_i) = 0,\ t_i \in \mathbb{R}\}. \tag{5}$$

Here, t_i are the values of t at which $Z(t)$ crosses zero.

Properties of Zerotrix

1. **Ordered Sequence**: The entries in the Zerotrix for $Z(t)$ are ordered according to the values of t. 2. **Density**: The distribution of zero-crossings can be analyzed by examining the density of entries in the Zerotrix. 3. **Symmetry**: The Zerotrix may exhibit symmetry properties depending on the underlying function and its domain.

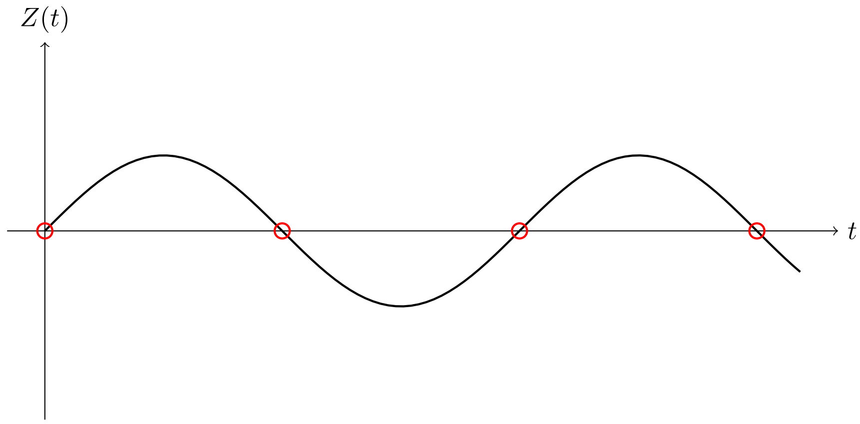
Example of Zerotrix for $Z(t)$

Consider the first few zero-crossings of Hardy's $Z(t)$ function. The Zerotrix can be represented as:

$$\text{Zerotrix}(Z(t)) = \{t_1, t_2, t_3, \ldots\}, \tag{6}$$

where $t_1, t_2, t_3, \ldots$ are the points where $Z(t) = 0$.

Graphical Representation of Zero-Crossings



In this plot, the points where the curve intersects the t-axis are the zero-crossings of $Z(t)$, indicating the non-trivial zeros of $\zeta\left(\frac{1}{2}+it\right)$.

Conclusion

The concept of zero-crossings is crucial in understanding the zeros of the Riemann zeta function and verifying the Riemann Hypothesis. The introduction of Zerotrix provides a structured approach to analyzing these zero-crossings, offering new insights into their distribution and properties. This framework can be extended to other complex functions to further our understanding of their zeros and the underlying mathematical structures.