EXTENDING SCHNIRELMANN-TYPE DENSITY AND ADDITIVE CLOSURE TO NON-ABELIAN GROUP STRUCTURES

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Abstract. We extend Schnirelmann-type density and additive closure concepts from abelian to non-abelian group settings. Definitions of density via conjugacy classes, word length, and growth metrics are proposed, and preliminary results on additive closure in non-abelian groups are presented.

1. Introduction

Classical additive number theory and Schnirelmann density operate over the natural numbers or abelian groups. In this paper, we investigate generalizations of these ideas to non-abelian groups G, where the group operation is not commutative, and additive notions are replaced with group-theoretic analogues.

2. Preliminaries on Non-Abelian Groups

Let G be a finite (or finitely generated) non-abelian group. Define the *left-translate set product* as:

$$kA := \{a_1 a_2 \cdots a_k \mid a_i \in A\} \subseteq G.$$

3. Notions of Density

Definition 3.1 (Uniform Group Density)**.** Let $A \subseteq G$. Define

$$\sigma_G(A) := \frac{|A|}{|G|}.$$

Definition 3.2 (Conjugacy Class Density)**.** Let $C \subseteq G$ be a union of conjugacy classes. Define

$$\sigma_{\mathrm{conj}}(C) := \frac{|C|}{|G|}.$$

Definition 3.3 (Ball Density in Cayley Graph)**.** Let G be a finitely generated group with generating set S. Let $B(n)$ be the ball of radius n in the Cayley graph. Define

$$\sigma_{\mathrm{ball}}(A) := \liminf_{n\to\infty} \frac{|A \cap B(n)|}{|B(n)|}.$$

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4. Additive Closure in Non-Abelian Context

Definition 4.1 (Multiplicative Closure)**.** A subset $A \subseteq G$ is said to be k-multiplicatively closed if

$$kA = G.$$

Proposition 4.2. *Let $A \subseteq G$ with $\sigma_G(A) > \sqrt[k]{\frac{1}{|G|}}$, then $kA = G$.*

Sketch. We use a probabilistic argument: a random product of k elements from A hits any $g \in G$ with positive probability, provided the support size is large enough. $\square$

5. Examples and Counterexamples

Example 5.1. Let $G = S_3$, the symmetric group on 3 letters. Let $A = \{(12), (13)\}$. Then $A^2 = G$.

Example 5.2. In dihedral groups D_n, subsets not closed under inversion may fail to generate G even under high powers.

6. Open Problems and Future Directions

- What is the minimal density required to ensure $kA = G$ in non-abelian settings?
- Develop non-abelian analogues of Schnirelmann's inequality.
- Explore density measures in non-amenable groups.
- Apply to automatic groups and profinite completions.