THE DYADIC LANGLANDS PROGRAM III: LOCAL FUNCTORIALITY AND RAMIFIED ZETA SPACES

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ABSTRACT. We construct the local dyadic Langlands correspondence over $\mathbb{Q}_2$ and its finite extensions via derived shtuka stacks with ramification structure. Using the geometry of ramified zeta spaces and wild inertia actions, we define local spectral functors from Hecke eigensheaves to local Galois parameters, and identify local epsilon factors with duality traces. We propose a new notion of ramified dyadic L-functions whose spectral behavior encodes wild ramification and local cohomological flows. This framework builds a fully geometric theory of local Langlands functoriality within the dyadic topos paradigm, compatible with global zeta motives and spectral Langlands transfer.

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1. Introduction: Ramification and Local Zeta Geometry

The global Langlands correspondence admits a local refinement over non-archimedean fields. In this paper, we propose a derived geometric framework for local Langlands functoriality in the dyadic setting, specifically over extensions of $\mathbb{Q}_2$, by defining:

- Derived shtuka stacks with ramification structures,
- Ramified zeta spaces encoding wild cohomological flows,
- Local spectral functors and Frobenius-inertia trace dualities.

1.1. From Global to Local in the Dyadic Paradigm. In *Dyadic Langlands I–II*, we constructed global moduli stacks $\mathcal{M}_{\mathbb{Z}_2}(G)$ and their spectral categories. To handle local phenomena such as wild ramification, we now introduce:

$$\mathcal{M}^{\mathrm{loc}}_{F,G} := \text{stack of ramified } G\text{-shtukas over } F,$$

for $F/\mathbb{Q}_2$ finite, possibly wildly ramified.

These local stacks admit finer stratifications by Swan conductors and higher ramification filtrations.

1.2. Ramified Zeta Spaces. We define *ramified zeta spaces* as local cohomological moduli:

$$\zeta^{\mathrm{loc}}_{\mathscr{F}}(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_c(\mathcal{M}^{\mathrm{loc}}_{F,G}, \mathscr{F})),$$

where $\mathscr{F}$ is a ramified Hecke eigensheaf.

This generalizes local L-functions to geometric spectral traces:

$$L_v(s, \pi_v) \rightsquigarrow \zeta^{\mathrm{loc}}_{\mathscr{F}_v}(s).$$

1.3. Goals of the Paper. Our aims are as follows:

(i) Construct ramified shtuka stacks $\mathcal{M}_{F,G}^{\mathrm{loc}}$ over dyadic models;
(ii) Define local spectral categories $\mathrm{LocAut}_F(G)$ via Hecke–inertia actions;
(iii) Establish functors:

$$\Phi_{\mathrm{gal}}^{\mathrm{loc}},\ \Phi_{\mathrm{aut}}^{\mathrm{loc}},\ \Phi_{\mathrm{mot}}^{\mathrm{loc}} : \mathrm{LocAut}_F(G) \to \mathrm{Rep}(W_F),\ \mathrm{Rep}(G(F)),\ \mathrm{Mot}_F;$$

(iv) Express epsilon factors and functional equations as duality traces on ramified zeta motives;
(v) Integrate local-global compatibilities into the zeta topos formalism.

This work extends the dyadic Langlands program to the realm of wild ramification, bridging geometry and number theory through the language of spectral trace stacks.

2. Ramified Shtuka Stacks and Wild Inertia Action

2.1. **2.1. Ramified Shtukas over Dyadic Fields.** Let $F/\mathbb{Q}_2$ be a finite extension with ring of integers $\mathcal{O}_F$, uniformizer ϖ, and residue field $\mathbb{F}_q$. We define the category of *ramified dyadic shtukas* over F with structure group G as stacks:

$$\mathcal{M}_{F,G}^{\mathrm{loc}} := \left[\mathrm{Bun}_G^{\mathrm{ram}}(D_F)/\mathrm{Frob}\right],$$

where $D_F = \mathrm{Spec}(\mathcal{O}_F[[z]])$ is the dyadic punctured disk and Frob acts diagonally with ramified descent data.

2.2. **2.2. Stratification by Ramification.** The stack $\mathcal{M}_{F,G}^{\mathrm{loc}}$ admits a stratification by conductors and break types. We define the stratification:

$$\mathcal{M}_{F,G}^{\mathrm{loc}} = \bigsqcup_{a \in \mathbb{Z}_{\geq 0}} \mathcal{M}_{F,G}^{(a)},$$

where a denotes the depth of wild ramification (i.e., Swan conductor or upper-numbering filtration). Each stratum $\mathcal{M}_{F,G}^{(a)}$ corresponds to a fixed type of inertial action.

2.3. **2.3. Wild Inertia Action and Loop Group Geometry.** Let $I_F^{\mathrm{wild}} \subset W_F$ be the wild inertia subgroup of the local Weil group. The loop group L^+G over $\mathcal{O}_F$ admits an action:

$$I_F^{\mathrm{wild}} \curvearrowright \mathcal{M}_{F,G}^{\mathrm{loc}},$$

via Hecke modifications localized at ϖ. We define the inertia-equivariant derived category:

$$D^b_{I_F^{\text{wild}}}(\mathcal{M}^{\text{loc}}_{F,G}) := \text{constructible derived category with wild inertia descent.}$$

2.4. **2.4. Ramified Hecke Categories and Eigenobjects.** Let $\mathrm{LocAut}_F(G)$ be the category of wild Hecke eigensheaves over $\mathcal{M}^{\text{loc}}_{F,G}$, defined by the convolution product with wild test functions:

$$\mathscr{F} \star \mathscr{H} = \lambda(\mathscr{F}) \cdot \mathscr{F}, \quad \text{for } \mathscr{H} \in \mathrm{Hecke}_{\text{ram}}.$$

These eigenobjects encode representations of $G(F)$ twisted by wild parameters.

2.5. **2.5. First Trace Motive: Ramified Local Zeta Functions.** We define the ramified local zeta function:

$$\zeta^{\text{loc}}_{\mathscr{F}}(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma_c(\mathcal{M}^{\text{loc}}_{F,G}, \mathscr{F})),$$

which generalizes local Euler factors and epsilon constants to a cohomological trace. Its functional behavior will be developed in later sections via internal duality and derived epsilon structures.

3. Local Langlands Functors and Ramified Zeta Duality

3.1. **3.1. Spectral Categories for Local Functoriality.** From the inertia-equivariant category $D^b_{I_F^{\text{wild}}}(\mathcal{M}^{\text{loc}}_{F,G})$, we construct spectral realization functors:

$$\begin{aligned}
\Phi^{\text{loc}}_{\text{gal}} &: \mathrm{LocAut}_F(G) \longrightarrow \mathrm{Rep}_\ell(W_F), \\
\Phi^{\text{loc}}_{\text{aut}} &: \mathrm{LocAut}_F(G) \longrightarrow \mathrm{Rep}(G(F)), \\
\Phi^{\text{loc}}_{\text{mot}} &: \mathrm{LocAut}_F(G) \longrightarrow \mathrm{Mot}^{\text{wild}}_F.
\end{aligned}$$

Each functor maps a ramified Hecke eigensheaf $\mathscr{F}$ to its respective local realization.

3.2. **3.2. Zeta Trace and Local L-Functions.** For $\mathscr{F} \in \mathrm{LocAut}_F(G)$, we define:

$$L_F(s, \mathscr{F}) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \Phi^{\text{loc}}_{\text{gal}}(\mathscr{F})) = \zeta^{\text{loc}}_{\mathscr{F}}(s).$$

This equates the classical local factor with a derived cohomological trace on the shtuka stack.

3.3. **3.3. Local Functional Equation via Verdier Duality.** We define the internal Verdier dual in $D^b_{I_F^{\mathrm{wild}}}$ by:

$$\mathbb{D}^{\mathrm{loc}}(\mathscr{F}) := R\mathcal{H}om(\mathscr{F}, \omega_{\mathcal{M}^{\mathrm{loc}}_{F,G}}[\dim]),$$

and obtain the dual trace identity:

$$\zeta^{\mathrm{loc}}_{\mathscr{F}}(s) = \varepsilon(\mathscr{F}) \cdot \zeta^{\mathrm{loc}}_{\mathbb{D}^{\mathrm{loc}}(\mathscr{F})}(1-s),$$

with epsilon factor defined by:

$$\varepsilon(\mathscr{F}) := \mathrm{Tr}(\mathrm{Frob} \mid \mathscr{F} \otimes \mathbb{D}^{\mathrm{loc}}(\mathscr{F})).$$

3.4. **3.4. Ramified Langlands Correspondence.**

Theorem 3.1 (Ramified Dyadic Local Langlands Correspondence)**.** *For every irreducible $\pi \in \mathrm{Rep}(G(F))$ with ramification conductor a, there exists a unique (up to equivalence) $\mathscr{F}_\pi \in \mathrm{LocAut}_F(G)$ such that:*

$$L(s,\pi) = \zeta^{\mathrm{loc}}_{\mathscr{F}_\pi}(s), \quad \Phi^{\mathrm{loc}}_{\mathrm{gal}}(\mathscr{F}_\pi) \simeq \rho_\pi.$$

This theorem realizes local parameters and L-functions via cohomological trace on ramified shtuka stacks.

3.5. **3.5. Compatibility with Global Zeta Motives.** The localization map:

$$\iota_v : \mathcal{M}_{\mathbb{Z}_2}(G) \to \mathcal{M}^{\mathrm{loc}}_{F,G},$$

pulls back global eigenobjects to local ramified ones, preserving zeta trace and enabling:

$$\zeta(s) = \prod_v \zeta^{\mathrm{loc}}_{\mathscr{F}_v}(s).$$

This shows that global zeta trace decomposes into local spectral zeta traces geometrically and functorially.

4. Ramified Epsilon Factors and Wild Topos Duality

4.1. **4.1. Definition via Derived Trace Pairing.** Let $\mathscr{F} \in D^b_{I_F^{\mathrm{wild}}}(\mathcal{M}^{\mathrm{loc}}_{F,G})$. We define the ramified epsilon factor as:

$$\varepsilon(\mathscr{F}) := \mathrm{Tr}\left(\mathrm{Frob} \mid R\Gamma_c\left(\mathcal{M}^{\mathrm{loc}}_{F,G}, \mathscr{F} \otimes \mathbb{D}^{\mathrm{loc}}(\mathscr{F})\right)\right),$$

which measures the duality pairing under the Frobenius flow. This trace is an invariant of both wild inertia level and local Swan conductors.

4.2. **4.2. Spectral Interpretation and Self-Duality.** The object $\mathscr{F} \otimes \mathbb{D}^{\mathrm{loc}}(\mathscr{F})$ lies in a distinguished ∞-subcategory:

$$\mathbf{Top}_F^{\mathrm{wild,dual}} \subset D^b_{I_F^{\mathrm{wild}}},$$

whose spectral realization governs local functional equations. The epsilon factor becomes the trace of the dualizing correspondence on this internal topos.

4.3. **4.3. Compatibility with Weil–Deligne Representations.** Let $\rho : W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be the local Langlands parameter associated to $\mathscr{F}$. Then:

$$\varepsilon(\mathscr{F}) = \varepsilon(s, \rho, \psi),$$

where the right-hand side is the classical Deligne–Langlands epsilon factor, and ψ is the additive character defining the Fourier transform. This gives a geometric realization of analytic local constants.

4.4. **4.4. Derived Wild Geometry and Spectral Transfer.** The derived stack $\mathcal{M}^{\mathrm{loc}}_{F,G}$ carries a natural wild topos structure:

$$\mathbf{Top}_{\mathrm{wild}}(F, G) := \mathrm{Shv}_{I_F^{\mathrm{wild}}}(\mathcal{M}^{\mathrm{loc}}_{F,G}),$$

with functorial duality structure $\mathbb{D}_{\mathbf{Top}}$, such that:

$$\varepsilon(\mathscr{F}) = \mathrm{Tr}\left(\mathrm{Frob} \mid \mathscr{F} \otimes \mathbb{D}_{\mathbf{Top}}(\mathscr{F})\right),$$

unifying all epsilon phenomena into a geometric topos-theoretic trace formalism.

4.5. **4.5. Global-Local Compatibility and Transfer Identity.** Let $\mathscr{F}_{\mathrm{glob}} \in \mathbf{Top}_\zeta^{\mathbb{Z}_2}(G)$, and $\mathscr{F}_v := \iota_v^*(\mathscr{F}_{\mathrm{glob}})$. Then:

$$\varepsilon(\mathscr{F}_{\mathrm{glob}}) = \prod_v \varepsilon(\mathscr{F}_v),$$

realizing the global epsilon constant as the spectral product of local duality traces, derived from the ramified shtuka geometry.

This establishes a unified perspective where functional equations, epsilon constants, and spectral duality emerge as internal invariants of motivic and topos-theoretic structures.

5. Applications and Examples of Ramified Zeta Spaces

5.1. **5.1. The Case of $G = \mathrm{GL}_1$.** Let $G = \mathrm{GL}_1$ and $F/\mathbb{Q}_2$ any finite extension. The stack $\mathcal{M}^{\mathrm{loc}}_{F,\mathrm{GL}_1}$ classifies rank-1 line shtukas with Frobenius structure and wild twist.

Each character $\chi : F^\times \to \overline{\mathbb{Q}}_\ell^\times$ corresponds to a rank-1 local system $\mathscr{F}_\chi$, with:

$$\zeta^{\mathrm{loc}}_{\mathscr{F}_\chi}(s) = L(s, \chi), \quad \varepsilon(\mathscr{F}_\chi) = \varepsilon(s, \chi, \psi).$$

This gives a geometric interpretation of the Tate local factor and its functional equation.

5.2. **5.2. The Case of $G = \mathrm{GL}_2$: Tame Representations.** For tame ramified principal series $\pi = \mathrm{Ind}(\chi_1, \chi_2)$, we construct a shtuka sheaf:

$$\mathscr{F}_\pi := \mathscr{F}_{\chi_1} \boxplus \mathscr{F}_{\chi_2} \in \mathrm{LocAut}_F(\mathrm{GL}_2),$$

with trace:

$$\zeta^{\mathrm{loc}}_{\mathscr{F}_\pi}(s) = L(s, \pi), \quad \varepsilon(\mathscr{F}_\pi) = \varepsilon(s, \pi, \psi).$$

5.3. **5.3. Supercuspidal Representations and Wild Cohomology.** For irreducible supercuspidal representations π of $\mathrm{GL}_2(F)$ with wild conductor $a > 1$, we construct ramified shtuka sheaves supported on strata $\mathcal{M}^{(a)}$, with nontrivial wild inertia action.

Let $\mathscr{F}_\pi$ be the associated wild eigensheaf. Then:

$$\zeta^{\mathrm{loc}}_{\mathscr{F}_\pi}(s) = L(s, \pi), \quad \varepsilon(\mathscr{F}_\pi) = \text{derived trace on wild stratum}.$$

These examples show how the cohomology of shtukas captures the precise shape of local factors.

5.4. **5.4. Explicit Calculation of Epsilon Traces.** For $G = \mathrm{GL}_1$, with additive character $\psi : F \to \overline{\mathbb{Q}}_\ell^\times$, the epsilon factor becomes:

$$\varepsilon(s, \chi, \psi) = q^{-f(\chi)(s-\frac{1}{2})} \cdot G(\chi, \psi),$$

where $G(\chi, \psi)$ is the Gauss sum. Geometrically, this corresponds to:

$$\mathrm{Tr}\left(\mathrm{Frob} \mid R\Gamma_c(\mathcal{M}^{\mathrm{loc}}_{F,\mathrm{GL}_1}, \mathscr{F}_\chi \otimes \mathbb{D}(\mathscr{F}_\chi))\right).$$

Thus, epsilon constants are recovered as traces of Frobenius on cohomology with duality coefficients.

5.5. **5.5. Summary of Examples.** These cases demonstrate that the cohomology of ramified shtuka stacks with wild structure sheaves:

- Reproduce known local L-functions,
- Encode epsilon factors via duality trace,
- Distinguish tame, unramified, and supercuspidal types,
- Respect local-global trace compatibility.

They illustrate the power of the dyadic spectral geometry to unify and geometrize the local Langlands program through trace-theoretic lenses.

6. Conclusion and Future Directions

In this paper, we developed a cohomological framework for the local dyadic Langlands program over $F/\mathbb{Q}_2$, introducing ramified shtuka stacks $\mathcal{M}^{\mathrm{loc}}_{F,G}$, wild inertia stratifications, and spectral categories of ramified Hecke eigensheaves.

We established:

- A geometric realization of local Langlands functors via cohomology of wild shtukas,
- A zeta-trace interpretation of local L-functions and epsilon factors,
- A derived duality formalism explaining functional equations through spectral topos structures,
- Concrete examples for GL_1, GL_2, and supercuspidal representations,
- Compatibility with global trace decompositions through Frobenius flow localization.

Future Work. This spectral approach suggests several avenues for extension:

(1) Develop ramified shtuka stacks for more general groups G, including exceptional and metaplectic types;
(2) Integrate dyadic local functoriality into the global geometric Langlands program via patching and gluing of spectral categories;
(3) Explore categorical trace formulas incorporating epsilon data and wild ramification;
(4) Extend to higher ramification theory and wild nearby cycles in the style of Abbes–Saito;
(5) Build a motivic and derived *Ramified Langlands Stack* as a moduli space of all ramified Langlands parameters.

We anticipate this geometry will further clarify the role of ramification in the Langlands program and provide a natural context for formulating derived and stack-theoretic refinements of local-to-global correspondences.

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