Constructing Fields Larger than $\mathbb{C}$ Using Automorphic Forms, Motives, and L-functions

Mathematician

Lecture 1: Foundations of Automorphic Forms

Outline

Introduction to Automorphic Forms

- **What are Automorphic Forms?** A high-level overview.
- **Historical Development**: From modular forms to automorphic forms.
- **Importance in Modern Mathematics**: The role of automorphic forms in number theory and beyond.

What is an Automorphic Form?

An **automorphic form** is a function $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ that satisfies:

- G is a reductive algebraic group over $\mathbb{Q}$.
- $G(\mathbb{A})$ is the adelic group.
- f satisfies specific invariance properties under the action of $G(\mathbb{A})$.

Example:

$$f(z) = \sum_{n=-\infty}^{\infty} e^{2\pi inz}$$

is a classical automorphic form on $SL(2, \mathbb{Z})$.

Historical Context

- **19th Century Origins**: Automorphic forms arose from the study of elliptic functions and modular forms.
- **Evolution**: Over time, these ideas expanded into more general automorphic forms.
- **Modern Significance**: Integral to the Langlands program and connections to L-functions.

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Key Milestones:

- 1850s: Dedekind and the theory of modular functions.
- 1920s: Hecke's work on L-functions and modular forms.
- 1960s: Introduction of the Langlands program.

Properties of Automorphic Forms

Automorphic forms possess several key properties:

- **Invariance**: Invariant under the action of a discrete group, typically $G(\mathbb{Z})$ or $G(\mathbb{Q})$.
- **Growth Conditions**: Exhibits controlled behavior at infinity.
- **Fourier Expansion**: Can often be expressed in terms of Fourier series.

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$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

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where the a_n are Fourier coefficients. **Explanation of Fourier Expansion:**

- Fourier series provide a way to express functions as sums of sinusoids.
- In the context of automorphic forms, this often translates to understanding the function's behavior through its periodicity and symmetries.

Examples of Automorphic Forms

- **Modular Forms**: Functions on the upper half-plane that are invariant under the action of $SL(2, \mathbb{Z})$.
- **Maass Forms**: Eigenfunctions of the Laplacian on the upper half-plane, often non-holomorphic.
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Detailed Example: Consider the classical modular form $\Delta(z)$ given by:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}$$

- $\Delta(z)$ is a cusp form of weight 12 for $SL(2, \mathbb{Z})$.
- Its Fourier coefficients have deep connections to the Ramanujan tau function.

Automorphic Forms on Higher Groups

Beyond $SL(2, \mathbb{Z})$, automorphic forms are studied on groups like $GL(n)$, $Sp(2n)$, and others.

- **General Linear Group** $GL(n)$: Automorphic forms on $GL(n)$ generalize those on $SL(2)$.
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Application Example: The study of automorphic forms on $GL(2)$ and their connection to elliptic curves via the modularity theorem.

- Wiles' proof of Fermat's Last Theorem crucially depended on this connection.

Applications of Automorphic Forms

Automorphic forms are deeply connected to:

- **Number Theory**: Integral to the study of L-functions and modular forms.
- **Representation Theory**: Automorphic representations and their relation to the Langlands program.
- **Geometry**: Connections to Shimura varieties and the arithmetic of algebraic varieties.

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Specific Application: Consider the connection between modular forms and elliptic curves:

- The Taniyama-Shimura-Weil conjecture posits that every elliptic curve over $\mathbb{Q}$ is modular.
- This conjecture was a key ingredient in proving Fermat's Last Theorem.

Automorphic Representations

- Automorphic forms can be interpreted through the lens of representation theory.
- An **automorphic representation** is a representation of $G(\mathbb{A})$ on a Hilbert space, where G is a reductive group.
- These representations play a critical role in the Langlands program.

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Langlands Correspondence:

- Links automorphic representations with Galois representations.
- Fundamental conjecture predicting deep connections across number theory, representation theory, and geometry.

Conclusion and Next Steps

- Recap of key concepts introduced in this lecture.
- In the next lecture: A deeper exploration into the structure of automorphic forms and their connections to L-functions.
- Suggested readings:
  - Gelbart, Stephen S. *Automorphic Forms on Adele Groups*.
  - Langlands, Robert P. *Problems in the Theory of Automorphic Forms*.