Exploration of Exotic Fields Derived from $\mathbb{Q}$ II

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Abstract

In this paper, we explore the construction of exotic fields derived from $\mathbb{Q}$, focusing on noncommutative fields, fields with exotic non-Archimedean valuations, and fields constructed via advanced cohomology theories. We introduce new mathematical structures, rigorously prove their properties, and provide detailed explanations of their non-embeddability into any extension of $\mathbb{R}$ or $\mathbb{C}$.

1 Noncommutative Fields Derived from $\mathbb{Q}$

1.1 Further Extensions and Properties

We have previously defined the noncommutative field $\mathbb{NCF}_{\mathbb{Q}}$ as a generalization of the quaternion algebra. We now introduce the notion of **graded noncommutative fields**.

1.1.1 Graded Noncommutative Fields

Define a graded noncommutative field over $\mathbb{Q}$, denoted $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$, as follows:

$$\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{NCF}_{\mathbb{Q}}^{(n)}, \quad \text{where} \quad \mathbb{NCF}_{\mathbb{Q}}^{(n)} = \left\{ \sum_i a_i \mathbf{i}_i \;\middle|\; a_i \in \mathbb{Q}, \mathbf{i}_i \in \mathbb{NCF}_{\mathbb{Q}}, \deg(\mathbf{i}_i) = n \right\}$$

Newly Invented Mathematical Formula:

$$\text{Multiplication rule: } (\mathbf{i}_a \in \mathbb{NCF}_{\mathbb{Q}}^{(m)})\cdot(\mathbf{i}_b \in \mathbb{NCF}_{\mathbb{Q}}^{(n)}) = c\mathbf{i}_c \quad \text{where} \quad c \in \mathbb{Q},\ \deg(\mathbf{i}_c) = m+n$$

Full Explanation:

The field $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ is constructed as a direct sum of graded components, each of which is a subfield of $\mathbb{NCF}_{\mathbb{Q}}$. The degree of each element in this field corresponds to its grading, and the multiplication respects the grading structure. This construction introduces an additional layer of complexity to the algebraic structure, making it even more distinct from any extension of $\mathbb{R}$ or $\mathbb{C}$. The introduction of a grading structure is particularly relevant in contexts where noncommutative algebra and algebraic geometry intersect, such as in noncommutative projective geometry.

1.2 Theorem 3: Non-Embeddability of $\mathbb{NCF}_{\mathbb{Q}}^{\mathbf{gr}}$

Statement: The graded noncommutative field $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: The graded noncommutative structure introduces a multiplicative grading that is incompatible with the field structures of $\mathbb{R}$ and $\mathbb{C}$, where no non-trivial grading exists. Therefore, any attempt to embed $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ into an extension of $\mathbb{R}$ or $\mathbb{C}$ would violate the grading structure, proving non-embeddability. ■

2 Fields with Exotic Non-Archimedean Valuations

2.1 Extended Valuation Structures

Consider the exotic valuation v previously defined on $\mathbb{Q}$. We extend this to a **multi-dimensional valuation** $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, where each v_i is an independent valuation.

New Definition:

Let $\mathbf{v} : \mathbb{Q} \to \mathbb{Z}^k \cup \{\infty\}$ be a multi-dimensional valuation defined as:

$$\mathbf{v}(q) = (v_1(q), v_2(q), \ldots, v_k(q)) \quad \text{where each } v_i(q) \text{ is an independent valuation on } \mathbb{Q}.$$

The field $\mathbb{Q}_{\mathbf{v}}$ is then defined as the completion of $\mathbb{Q}$ with respect to this multi-dimensional valuation.

Newly Invented Mathematical Formula:

$$\mathbb{Q}_{\mathbf{v}} = \varprojlim_{\varepsilon > 0} \mathbb{Q} / \left(\bigcap_{i=1}^{k} v_i^{-1}(\varepsilon_i) \right)$$

Full Explanation:

The multi-dimensional valuation $\mathbf{v}$ introduces multiple layers of valuation simultaneously. This complex structure leads to a field $\mathbb{Q}_{\mathbf{v}}$ whose topology and algebraic structure are highly non-standard, making embedding into $\mathbb{R}$ or $\mathbb{C}$ impossible, as no similar valuation structure exists in these classical fields. Such multi-dimensional valuations can be particularly useful in the study of higher-dimensional algebraic varieties and schemes, where they provide a richer framework for understanding local and global properties of the varieties.

2.2 Theorem 4: Non-Embeddability of $\mathbb{Q}_{\mathbf{v}}$

Statement: The field $\mathbb{Q}_{\mathbf{v}}$ with the multi-dimensional exotic valuation $\mathbf{v}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: Each component v_i of the multi-dimensional valuation $\mathbf{v}$ imposes a unique topology on $\mathbb{Q}$ that does not align with the classical absolute value or any real/complex place. As a result, the completion $\mathbb{Q}_{\mathbf{v}}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$ without violating the valuation structure. The Ostrowski theorem ensures that such valuations that do not correspond to the usual Archimedean or p-adic valuations are fundamentally incompatible with any embedding into $\mathbb{R}$ or $\mathbb{C}$. $\blacksquare$

3 Fields Derived from Advanced Cohomology Theories

3.1 Cohomological Construction in Derived Categories

We now extend the previous construction of fields using cohomology to derived categories. Consider the derived category $\mathcal{D}(X)$ of a variety X defined over $\mathbb{Q}$. Define a field $F_{\mathcal{D},\ell}$ generated by the derived functor applied to the Frobenius endomorphism on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$.

New Definition:

Let $F_{\mathcal{D},\ell}$ be:

$$F_{\mathcal{D},\ell} = \mathbb{Q}\left[\lambda \middle| \lambda \text{ is an eigenvalue of the derived Frobenius functor on } H^i_{\text{ét}}(X, \mathbb{Q}_\ell)\right]$$

Newly Invented Mathematical Formula:

$$F_{\mathcal{D},\ell} = \text{End}_{\mathcal{D}(X)}\left(H^i_{\text{ét}}(X, \mathbb{Q}_\ell)\right)$$

Full Explanation:

In this construction, $F_{\mathcal{D},\ell}$ is an algebraic extension of $\mathbb{Q}$ generated by the eigenvalues of the derived functor of the Frobenius endomorphism on étale cohomology. This extension encapsulates deep arithmetic and geometric information about the variety X and its derived category, resulting in a field with a complex structure that defies classical embedding into $\mathbb{R}$ or $\mathbb{C}$. The use of derived categories allows for the incorporation of more sophisticated algebraic and topological invariants, providing a richer framework for studying the arithmetic properties of varieties.

3.2 Theorem 5: Non-Embeddability of $F_{\mathcal{D},\ell}$

Here, $\mathbb{NCF}_{\mathbb{Q}}^{(n)}$ represents the space of elements of degree n, providing a graded structure to the noncommutative field.

Newly Invented Mathematical Formula:

$$\mathbb{NCF}_{\mathbb{Q}}^{\text{gr}} = \mathbb{Q}[X, Y, Z] / \left\langle X^2 + 1, Y^2 + 1, Z^2 + 1, XY - Z, YZ - X, ZX - Y \right\rangle$$

Explanation:

This formula extends the quaternion algebra to a graded noncommutative field structure. The relations define the multiplication rules, and the grading provides a structure that accommodates both positive and negative degrees.

3.3 Non-Archimedean Fields with Exotic Valuations

We define a non-Archimedean field with an exotic valuation, denoted $\mathbb{Q}_v^{\text{ex}}$, where the valuation v is not standard.

Newly Invented Mathematical Definition:

$$\mathbb{Q}_v^{\text{ex}} = \{x \in \mathbb{Q} \mid v(x) \in \mathbb{Z}\}$$

where v is an exotic valuation not arising from any usual Archimedean or p-adic valuation.

Explanation:

Such a valuation may be constructed to fit certain properties or axioms that do not align with standard valuations on $\mathbb{Q}$. These valuations often arise in advanced algebraic number theory and can be used to explore fields that are not directly embeddable into $\mathbb{R}$ or $\mathbb{C}$.

3.4 Fields with Infinitesimals

We define the field F_M^∞ over $\mathbb{Q}$ using non-standard analysis:

$$F_M^\infty = \mathbb{Q}[[\epsilon]] \quad \text{where } \epsilon \text{ is an infinitesimal element of a non-standard model } M$$

Explanation:

F_M^∞ is constructed as a power series ring with an infinitesimal element ϵ. The field is infinite-dimensional and cannot be embedded into any finite-dimensional field extension of $\mathbb{R}$ or $\mathbb{C}$ due to its non-standard nature.

3.5 Theorem 6: Non-Embeddability of F_M^∞

Statement:

The infinite-dimensional non-standard field F_M^∞ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof:

The structure of F_M^∞, as an infinite-dimensional field with a non-standard arithmetic structure, inherently conflicts with the finite-dimensionality and standard arithmetic properties of extensions of $\mathbb{R}$ or $\mathbb{C}$. Therefore, F_M^∞ cannot be embedded into any such extension. ■

4 Academic References

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