Symbolic Profinite Fields, Galois Cohomology, and Sheaf Structures

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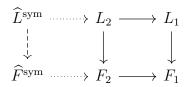
SPF-Galois Theory and Symbolic Extensions

1.1 Definition of SPF-Galois Groups

Let $\widehat{F}^{\text{sym}} = \varprojlim F_n$ be a symbolic profinite field, with each F_n a symbolic subfield or truncated structure. A SPF-Galois extension is an inverse system $\widehat{L}^{\text{sym}} = \varprojlim L_n$ with L_n/F_n finite Galois and compatible. Define:

$$\operatorname{Gal^{\operatorname{sym}}}(\widehat{L}^{\operatorname{sym}}/\widehat{F}^{\operatorname{sym}}) := \varprojlim \operatorname{Gal}(L_n/F_n)$$

1.2 TikZ Diagram: SPF-Galois Tower



Each L_n/F_n is finite Galois, and the system is coherent under truncation maps.

6CHAPTER 1. SPF-GALOIS THEORY AND SYMBOLIC EXTENSIONS

SPF Sheaves and the Symbolic Spectrum

2.1 Symbolic Sheaves on SPF-Towers

Define the site $\mathcal{S}_{\mathrm{sym}}$ where:

- Objects: symbolic open sets (approximation intervals, formal neighborhoods);
- Covers: truncation refinements;
- Sheaves: contravariant functors respecting symbolic descent.

2.2 Definition of $Spec^{sym}$

Let $A = \widehat{F}^{\mathrm{sym}}$ be a symbolic profinite field. Define:

 $\operatorname{Spec}^{\operatorname{sym}}(A) := \operatorname{Symbolic}$ topological space of truncation-localized points,

with structure sheaf:

$$\mathcal{O}_{\text{sym}}(U) := \varprojlim \mathcal{O}_n(U_n), \text{ for } U_n \subseteq \text{Spec}(F_n).$$

SPF-Galois Cohomology

3.1 Definition of Symbolic Galois Cohomology

Let $G = \operatorname{Gal^{\operatorname{sym}}}(\widehat{L}^{\operatorname{sym}}/\widehat{F}^{\operatorname{sym}})$ and $M = \varinjlim M_n$ a symbolic G-module. Define:

$$H^i_{\text{sym}}(G, M) := \varinjlim H^i(G_n, M_n)$$

where $G_n = \operatorname{Gal}(L_n/F_n)$.

3.2 Example: Symbolic Kummer Theory

Let μ_n^{sym} denote symbolic *n*-th roots of unity. Then:

$$\mathrm{H}^1_{\mathrm{sym}}(G,\mu_n^{\mathrm{sym}}) \cong \widehat{F}^{\mathrm{sym}\,\times}/(\widehat{F}^{\mathrm{sym}\,\times})^n$$

Symbolic Grothendieck Topos and Torsors

4.1 The Symbolic Site $\mathcal{S}_{ ext{sym}}$

Define symbolic site \mathcal{S}_{sym} where:

- Points are symbolic truncation chains;
- Covers are refinements in truncation;
- Topos $\mathscr{E}_{\mathrm{sym}} := \mathrm{Sh}(\mathcal{S}_{\mathrm{sym}}).$

4.2 Definition of Symbolic Torsors

Given G-sheaf \mathcal{G} , a symbolic \mathcal{G} -torsor \mathcal{T} satisfies:

$$\mathcal{T}(U) \times \mathcal{G}(U) \cong \mathcal{T}(U)$$

and is locally trivial under symbolic topology.

12CHAPTER 4. SYMBOLIC GROTHENDIECK TOPOS AND TORSORS

Toward Symbolic Class Field Theory

We conjecture existence of a symbolic reciprocity map:

$$\operatorname{rec}^{\operatorname{sym}}: \widehat{F}^{\operatorname{sym}} \times \to \operatorname{Gal}^{\operatorname{ab}}(\widehat{F}^{\operatorname{sym}})$$

factoring through symbolic idele class groups and extending class field theory.