A CONTRIBUTION TO THE SHANKS–RÉNYI RACE PROBLEM

By JERZY KACZOROWSKI*

[Received 29 March 1992]

1. Introduction and statement of results

IN this paper we adopt the standard notation used in the theory of primes. In particular we denote by $\pi(x, q, a)$, $x \geq 1$, $a, q \in \mathbb{N}$, $1 \leq a \leq q$, $(a, q) = 1$, the number of primes $p \equiv a \bmod q$ with $p \leq x$. Moreover, we write

$$\psi(x, q, a) = \sum_{n \leq x,\ n \equiv a \bmod q} \Lambda(n),$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ p\text{-prime}, \\ 0 & \text{otherwise.} \end{cases}$$

The Shanks–Rényi race problem concerns distribution of primes in arithmetic progressions $\bmod q$, $q \geq 1$, and was first stated in full generality by S. Knapowski and P. Turán in their famous cycle of papers on comparative prime number theory. They write in [**4**], page 302:

To illustrate the problem for $q = 8$, say, let us consider the game, played by four players, called "1", "3", "5" and "7", the player "j" scoring a point when by the enumeration of all primes (in increasing order) a prime $\equiv j \bmod q$ occurs. According to the calculations of Shanks [5] the player "1" plays rather poorly, being on the last place after the first $\pi(10^6)$ steps. Will this always be the case? If not, will the player "1" infinitely often take the lead? In a general case we assert the

Problem (Race-problem of Shanks–Rényi). For each permutations

$$a_1, a_2, \ldots, a_{\varphi(q)}$$

of the reduced set of residue classes $\bmod q$ does there exist an infinity of integer m's with

$$\pi(m, q, a_1) < \pi(m, q, a_2) < \cdots < \pi(m, q, a_{\varphi(q)})?$$

As long as the author of the present paper knows no results concerning the general race-problem has been obtained so far. Some experts describe the problem as very difficult or even "intractable at present", cf. P. Erdős [**1**], page 5. In this situation conditional results can be

* Supported by KBN Grant nr. 2 1086 91 01.

interesting. This research has been motivated by Knapowski–Turán's remarks on the player "1" in the race mod 8.

THEOREM 1. *Let us suppose the Generalized Riemann Hypothesis* (G.R.H.) *for Dirichlet's L-functions* mod q, $q \geqslant 3$. *Then there exist infinitely many integers m with*

$$\pi(m, q, 1) > \max_{a \not\equiv 1 \bmod q} \pi(m, q, a).$$

Moreover, the set of m's satisfying this inequality has positive lower density. The same statement holds true for m satisfying the following inequality

$$\pi(m, q, 1) < \min_{a \not\equiv 1 \bmod q} \pi(m, q, a).$$

Theorem 1 suggests the following conjecture, which can be called *The Strong Race Hypothesis.* Namely we assert that for each permutation

$$a_1, a_2, \dots, a_{\varphi(q)}$$

of the reduced set of residue classes mod q the set of integers m with

$$\pi(m, q, a_1) < \pi(m, q, a_2) < \cdots < \pi(m, q, a_{\varphi(q)})$$

has positive lower density.

Theorem 1 follows at once from Theorem 2 below.

THEOREM 2. *Let us suppose* G.R.H. *for L-functions* mod q, $q \geqslant 3$, *and let u denote an arbitrary non-negative real number. Then there exist constants* $b_0 = b_0(u) > 0$, $c_0 = c_0(u) > 1$, *such that for every* $T \geqslant 1$ *we have*

$$\#\left\{T \leqslant m \leqslant c_0 T:\ \psi(m, q, 1) \geqslant \max_{a \not\equiv 1 \bmod q} \psi(m, q, a) + u\sqrt{m}\right\} \geqslant b_0 T,$$

$$\#\left\{T \leqslant m \leqslant c_0 T:\ \pi(m, q, 1) \geqslant \max_{a \not\equiv 1 \bmod q} \pi(m, q, a) + u\frac{\sqrt{m}}{\log m}\right\} \geqslant b_0 T,$$

$$\#\left\{T \leqslant m \leqslant c_0 T:\ \psi(m, q, 1) \leqslant \min_{a \not\equiv 1 \bmod q} \psi(m, q, a) - u\sqrt{m}\right\} \geqslant b_0 T,$$

$$\#\left\{T \leqslant m \leqslant c_0 T:\ \pi(m, q, 1) \leqslant \min_{a \not\equiv 1 \bmod q} \pi(m, q, a) - u\frac{\sqrt{m}}{\log m}\right\} \geqslant b_0 T.$$

2. The k-functions

In the proof of Theorem 2 we use k-functions introduced in [**2**]. For reader's convenience we give below a short survey of basic facts about them. For further details the reader is referred to [**2**].

Let $\chi \bmod q$, $q \geqslant 1$, be a primitive Dirichlet's character. For z from the upper half-plane $H = \{z \in \mathbb{C}: \mathscr{I}z > 0\}$ let us write

$$k(z, \chi) = \sum_{\gamma>0} e^{\varrho z}$$

and

$$K(z, \chi) = \int_{i\infty}^{z} k(s, \chi)\, ds = \sum_{\gamma>0} \frac{1}{\varrho} e^{\varrho z}, \tag{1}$$

where the summation is taken over all non-trivial $L(s, \chi)$ zeros $\varrho = \beta + i\gamma$ with positive imaginary parts. It is evident that both functions are holomorphic on H. The function k can be continued analytically to the meromorphic function on the Riemannian surface M of the function $\log z$. We have (cf. [2], Theorem 3.1):

$$2\pi i k(z, \chi) = \frac{1}{2\pi i} \frac{e^z}{e^z - 1} \log z + N_1(z, \chi), \tag{2}$$

where N_1 is meromorphic and single-valued on $\mathbb{C}$. The only singularities of N_1 are simple poles at the points $z = k \log p$, where $k \in \mathbb{Z}$ and p is a prime, $p \nmid q$. Moreover,

$$\operatorname{Res}_{z=0} N_1(z, \chi) = \frac{1}{2\pi i} A(\chi) \tag{3}$$

with

$$A(\chi) = \log(2\pi/q) + E - (\pi i)/2 \tag{4}$$

E being the Euler's constant (cf. [2], formula (8.11)). Hence K can be continued analytically along every curve lying on M not passing through the singularities of k. It becomes a multivalued function on M with logarithmic branch points at poles of k.

In this paper, however, we are mainly interested in k and K as functions on the upper half-plane together with their boundary values on the real axis.

Let $(a, q) = 1$, $0 < a \leqslant q$, $q \geqslant 1$ and let

$$F(z, q, a) = -2e^{-z/2} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} K(z, \chi') - \frac{2}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} m(\tfrac{1}{2}, \chi), \tag{5}$$

where χ' denotes the primitive Dirichlet's character induced by χ and $m(\frac{1}{2}, \chi)$ is the multiplicity of a zero of $L(s, \chi)$ at $s = \frac{1}{2}$ (we put $m(\frac{1}{2}, \chi) = 0$ when $L(\frac{1}{2}, \chi) \neq 0$).

LEMMA 1. *We have for* $x \geqslant 1$

$$\lim_{y\to 0^+} \mathcal{R}F(x+iy, q, a) = e^{-x/2}\left(\psi(e^x, q, a) - \frac{1}{\varphi(q)}e^x\right) + O(xe^{-x/2}).$$

Proof. Cf. the proof of Corollary 7 in [3].

LEMMA 2. *For* $0<r<1$, $0 \leqslant \phi \leqslant \pi$, *we have*

$$\mathcal{R}F(re^{i\phi}, q, 1) = \frac{1}{\pi}\left(\frac{\pi}{2} - \phi\right)\log r + O(1)$$

and

$$\mathcal{R}F(re^{i\phi}, q, a) = O(1),$$

when $a \not\equiv 1 \bmod q$.

Proof. Combining (2) and (3) we have for $z = x+iy$, $|x| \leqslant \frac{1}{2}$, $0<y<1$

$$k(z, \chi) = \frac{1}{2\pi i}\frac{\log z}{z} + \frac{A(\chi)}{2\pi i z} + O(|\log z|)$$

and consequently

$$\begin{aligned} K(z, \chi) &= K(i/2, \chi) + \int_{i/2}^{z} k(s, \chi)\,ds \\ &= \frac{1}{4\pi i}\log^2 z + \frac{A(\chi)}{2\pi i}\log z + O(1). \end{aligned}$$

Using (4) we obtain

$$K(re^{i\phi}, \chi) = \frac{1}{2\pi}\left(\phi - \frac{\pi}{2}\right)\log r - \frac{i}{4\pi}\left(\log^2 r + 2\left(\log\left(\frac{2\pi}{q}\right) + E\right)\log r\right) + O(1),$$

which substituted into (5) gives the assertion.

3. The boundary values of Dirichlet's series

To make the formulation of the result we need simpler let us denote by $\mathcal{B}$ the set of all functions

$$F(z) = \sum_{n=1}^{\infty} \alpha_n e^{iw_n z}, \qquad z = x+iy, \qquad y>0, \tag{6}$$

satisfying the following conditions.

1. $0 \leqslant w_1 \leqslant w_2 \leqslant \cdots$ are real numbers.
2. $\alpha_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$.

3. The series in (6) converges absolutely for all $y > 0$.
4. The limit

$$P(x) = \lim_{y \to 0^+} P(x + iy),$$

where $P(x + iy) = \mathscr{R}F(x + iy)$, $y > 0$, exists for almost all real x. (Putting $P(x) = 0$ for the remaining x we get P well defined on the closed upper half-plane $\bar{H} = \{z \in \mathbb{C}\colon \mathscr{I}z \geqslant 0\}$.)
5. We have

$$\lim_{y \to 0^+} \sup_{x \in R} \int_{-\frac{1}{2}}^{\frac{1}{2}} |P(x + t) - P(x + t + iy)|^2 \, dt = 0.$$

Lemma 3. *Let $F_j \in \mathscr{B}$ for $j = 1, 2, \ldots, n$ and let $x_0 \in \mathbb{R}$ be a continuity point of the mapping*

$$P\colon \bar{H} \ni z \mapsto (P_1(z), P_2(z), \ldots, P_n(z)) \in \mathbb{R}^n,$$
$$P_j = \mathscr{R}F_j, \qquad j = 1, 2, \ldots, n.$$

Then for every open neighbourhood $U \subset \mathbb{R}^n$ of $P(x_0)$ there exist constants $b_0 = b_0(U) > 0$ and $l_0 = l_0(U) > 0$ such that

$$\mu(P^{-1}(U) \cap \mathscr{I}) > b_0,$$

for arbitrary interval $\mathscr{I} \subset \mathbb{R}$ of length $\geqslant l_0$; μ being the Lebesgue measure on the real axis.

Proof. Let us denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$: $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and by d the distance induced by it: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We denote by $\mathbf{B}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n\colon d(\mathbf{x}, \mathbf{y}) < r\}$ the open ball with centre $\mathbf{x}$ and radius r.

Let $\varepsilon > 0$ be so small that

$$\mathbf{B}(P(x_0), \varepsilon) \subset U,$$
$$d(\mathbf{B}(P(x_0), \varepsilon), \mathbb{R}^n \backslash U) \geqslant \tfrac{1}{2} d_0,$$

where

$$d_0 = d(P(x_0), \mathbb{R}^n \backslash U).$$

Since P is continuous at x_0 there exist two positive constants $\delta_0 = \delta_0(\varepsilon)$ and $y_0 = y_0(\varepsilon)$ such that

$$\|P(x + iy) - P(x_0)\| < \varepsilon/2$$

for

$$|x - x_0| < \delta_0, \qquad 0 \leqslant y \leqslant y_0.$$

Let y_1, $0 < y_1 \leqslant y_0$, be such that

$$\sup_{x \in R} \int_{-\frac{1}{2}}^{\frac{1}{2}} \|P(x+t) - P(x+t+iy_1)\|^2 \, dt < \tfrac{1}{4}\delta_0 d_0^2;$$

y_1 exists according to 5 in the definition of the class $\mathscr{B}$.

For every $y > 0$ and $1 \leqslant j \leqslant n$ the function $P_j(x+iy)$ is a Bohr almost periodic function of x. Hence there exists a constant $l_1(\varepsilon, y) > 0$ such that in every interval $I \subset \mathbb{R}$ of length $\geqslant l_1(\varepsilon, y)$ we can find a $\tau \in I$ satisfying

$$\sup_{x \in R} |P_j(x+\tau+iy) - P_j(x+iy)| < \varepsilon/(2\sqrt{n}). \tag{7}$$

From the well-known properties of almost-periods we can find a τ satisfying (7) for all $j = 1, 2, \ldots, n$ simultaneously. Now let $\mathscr{I} = [a, b]$ be an arbitrary interval of length $\geqslant l_1(\varepsilon, y_1) + 2\delta_0$, and let $\mathscr{I}_0 = [a+\delta_0, a + l_1(\varepsilon, y_1) + \delta_0]$. Let $\tau \in \mathscr{I}_0 - \{x_0\}$ be a number satisfying (7) with $y = y_1$. Then

$$\sup_{x \in R} \|P(x+\tau+iy_1) - P(x+iy_1)\| < \varepsilon/2.$$

Denoting $x_1 = x_0 + \tau$ we have $x_1 \in \mathscr{I}_0$. Moreover, for every $x \in (x_1 - \delta_0, x_1 + \delta_0)$ we have $x - \tau \in (x_0 - \delta_0, x_0 + \delta_0)$ and

$$\begin{aligned} \|P(x+iy_1) - P(x_0)\| &\leqslant \|P(x+iy_1) - P(x-\tau+iy_1)\| \\ &\quad + \|P(x-\tau+iy_1) - P(x_0)\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $P(x+iy_1) \in U$ if $|x - x_1| \leqslant \delta_0$. Let

$$\mathbf{A} = \{x \in \mathbb{R} \colon |x - x_1| \leqslant \delta_0, P(x) \notin U\},$$

Then $\mathbf{A} \subset \mathscr{I}$ and

$$\begin{aligned} \tfrac{1}{4}\delta_0 d_0^2 > \int_{-\frac{1}{2}}^{\frac{1}{2}} \|P(x_1+t) - P(x_1+t+iy_1)\|^2 \, dt &\geqslant \int_{\mathbf{A}} \|P(x) - P(x-iy_1)\|^2 \, dx \\ &\geqslant \mu(\mathbf{A})(d(\mathbf{B}(P(x_0), \varepsilon), \mathbb{R}^n \backslash U))^2 \geqslant \tfrac{1}{4}d_0^2 \mu(\mathbf{A}), \end{aligned}$$

$$\mu(\mathbf{A}) < \delta_0.$$

Hence

$$\mu(P^{-1}(U) \cap \mathscr{I}) \geqslant \mu([x_1 - \delta_0, x_0 + \delta_0] \backslash \mathbf{A}) > \delta_0.$$

and the result follows by taking

$$l_0(U) = \delta_0(\varepsilon) \quad \text{and} \quad l_0(U) = l_1(\varepsilon, y_1) + 2\delta_0.$$

4. Proof of Theorem 2

Let $a_1 = 1, a_2, \ldots, a_{\varphi(q)}$ denote reduced residue classes mod q. We apply Lemma 3 to the following functions ($j = 1, 2, \ldots, \varphi(q)$):

$$F_j(z) = F(z, q, a_j)$$

defined in (5).

Supposing the G.R.H. we can number non-trivial zeros of all Dirichlet's L-functions mod q having non-negative imaginary parts in the following way

$$\varrho_n = \tfrac{1}{2} + i\gamma_n,$$
$$0 = \gamma_1 < \gamma_2 < \gamma_3 < \cdots$$

(we consider $\varrho_1 = \frac{1}{2}$ as a zero of an L-function, possibly with zero multiplicity). Then $F_j, j = 1, 2, \ldots, \varphi(q)$, are of the form (6) with $w_n = \gamma_n$ and

$$\alpha_n = -\frac{\delta_n}{\varphi(q)\varrho_n} \sum_{\chi \bmod q} \overline{\chi(a_j)} m(\varrho_n, \chi),$$
$$\delta_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n \geqslant 1, \end{cases}$$

where, as usual, $m(\varrho, \chi)$ denotes the multiplicity of a zero of $L(s, \chi)$ at $s = \varrho$. F_j's belong to the class $\mathscr{B}$. Indeed, conditions 1–4 are easy to verify. Condition 5 can be proved as follows.

Let us take a positive y and let $0 < y' < y$. Then by term-by-term integration we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |P_j(x + t + iy') - P_j(x + t + iy)|^2 \, dt$$
$$\leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} |F_j(x + t + iy') - F_j(x + t + iy)|^2 \, dt \ll \sum_{n=2}^{\infty} \sum_{m=2}^{\infty}$$
$$\times |\alpha_n| \, |\alpha_m| \, (e^{-\gamma_n y'} - e^{-\gamma_n y})(e^{-\gamma_m y'} - e^{-\gamma_m y}) \min\left(1, \frac{1}{|\gamma_n - \gamma_m|}\right).$$

Making $y' \to 0^+$ and using the Lebesgue Bounded Integration Theorem we obtain

$$\sup_{x \in R} \int_{-\frac{1}{2}}^{\frac{1}{2}} |P_j(x + t) - P_j(x + t + iy)|^2 \, dt \ll \sum_{n=2}^{\infty} \sum_{m=2}^{\infty}$$
$$\times \frac{(\log \gamma_n)(\log \gamma_m)}{\gamma_n \gamma_m} (1 - e^{-\gamma_n y})(1 - e^{-\gamma_m y}) \min\left(1, \frac{1}{|\gamma_n - \gamma_m|}\right)$$

and this tends to zero as $y \to 0^+$. Condition 5 is therefore verified.

From Lemma 2 we see that for every positive u there exist $x_0, x_0' \in \mathbb{R}$, $0 < |x_0|, |x_0'| < \frac{1}{2}$ satisfying

$$P_1(x_0) \geqslant \max_{j \neq 1} P_j(x_0) + 4u$$

and

$$P_1(x_0') \leqslant \min_{j \neq 1} P_j(x_0') - 4u.$$

We now apply Lemma 3 with $U = \mathbf{B}(P(x_0), u)$ and with $U = \mathbf{B}(P(x_0'), u)$, respectively. According to Lemma 1 we have for $x \geqslant 1$

$$P_j(x) = \mathrm{e}^{-x/2}\left(\psi(\mathrm{e}^x, q, a_j) - \frac{1}{\varphi(q)}\mathrm{e}^{x/2}\right) + O(x\mathrm{e}^{-x/2}),$$

and hence Lemma 3 yields

$$\mu\{T \leqslant x \leqslant T + l \colon \psi(\mathrm{e}^x, q, 1) > \max_{a \not\equiv 1 \bmod q} \psi(\mathrm{e}^x, q, a) + u\mathrm{e}^{x/2}\} \geqslant b_0$$

and

$$\mu\{T \leqslant x \leqslant T + l \colon \psi(\mathrm{e}^x, q, 1) < \min_{a \not\equiv 1 \bmod q} \psi(\mathrm{e}^x, q, a) - u\mathrm{e}^{x/2}\} \geqslant b_0.$$

for certain positive l and b_0 and all $T \geqslant 1$. Using this and the obvious remark that $\psi(t, q, a) = \psi([t], q, a)$ we obtain the first and the third assertion of Theorem 2 by the change of variable $t = \mathrm{e}^x$. Inequalities for $\pi(x, q, a)$ follow from what we have just proved and the partial summation.

REFERENCES

1. P. Erdős, 'Some personal reminscences of the mathematical work of Paul Turán,' *Acta Arithmetica* 37, 3–8 (1980).
2. J. Kaczorowski, 'The k-functions in multiplicative number theory, I; On complex explicit formulae,' *Acta Arithmetica* 56, (1990) 195–211.
3. J. Kaczorowski, 'The k-functions in multiplicative number theory, IV. On a method of A. E. Ingham,' *Acta Arithmetica* 57, (1991) 231–244.
4. S. Knapowski and P. Turán, 'Comparative prime-number theory, I,' *Acta Mathematica Hungarica* 13, (1962) 299–314.
5. D. Shanks, 'Quadratic residues and the distribution of primes,' *Math. Tables and other aids to computation*, 13, (1959) 505–514.

Institute of Mathematics
A. Mickiewicz University
ul. Matejki 48/49
Poznań, Poland