MULTIPLICOID GEOMETRY WITH DYADIC SUPPORT: COHOMOLOGY, TORSORS, AND PERIOD TOWERS OVER CONGRUENCE BASES

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Abstract. We develop a dyadic-supported variant of multiplicoid geometry, in which multiplicative congruence filtrations and torsor towers are embedded over the dyadic completion $\widehat{\mathbb{Q}}_{(2)}$. We construct dyadic-compatible period sheaves, cohomology theories, and regulator realizations, and propose a dyadic ε-filtration hierarchy that refines motivic depth via stratified torsor gerbes. This framework links congruence growth to cohomological realization over dyadic period towers, offering new tools for studying arithmetic stacks and stratified L-values.

Contents

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0. Notation and Symbol Dictionary

This section introduces the core notational framework for dyadic-supported multiplicoid geometry. These notations will appear throughout the development of period rings, cohomology theories, torsors, and ε-filtrations over dyadic bases.

Dyadic Structures.

- $\widehat{\mathbb{Q}}_{(2)} := \widehat{\mathbb{Q}}_{(2)}$: Dyadic completion of $\mathbb{Q}$ with respect to the (2^n) congruence system.
- $I_n := (2^n)$: Dyadic congruence ideal used for tower construction.
- $A^{(n)} := \mathbb{Z}/2^n\mathbb{Z}$: Level-n congruence ring in the dyadic system.
- $A^{\infty}_{\text{dyad}} := \varprojlim A^{(n)}$: Dyadic completed base ring, defining affine patches.

Period Structures.

- $B_{\text{dyad},dR}$: Dyadic de Rham period ring with binary-indexed filtration.
- $F^n B_{\text{dyad},dR}$: Level-n filtration of the period ring via dyadic congruence depth.
- $\mathrm{Per}^{\infty}_{\text{dyad}} := \{B^{(n)}\}_{n\geq 0}$: Dyadic period tower.

Torsors and ε-Filtrations.

- $\mathcal{T}^{\text{dyad}}_n$: Level-n torsor under $(\mathbb{Z}/2^n)$-action.
- $\mathbb{T}^{[\text{dyad}]}$: Stack of dyadic torsors indexed by n.
- $\mathcal{G}^{(2)}_{\varepsilon}$: Universal dyadic ε-gerbe capturing torsor descent.
- $F^{\varepsilon^n}\mathcal{F}$: ε-filtration on sheaf $\mathcal{F}$ with respect to $I_n = (2^n)$.

Cohomology and Realizations.

- $H^i_{\text{dyad}}(X, \mathcal{F})$: i-th dyadic cohomology of sheaf $\mathcal{F}$ over space X.
- r_{dyad} : Dyadic regulator from K-theory to dyadic cohomology.
- $\mathscr{R}_{\text{dyad}}$: Realization functor mapping filtered torsors to cohomological data over $\mathrm{Per}^{\infty}_{\text{dyad}}$.

Remarks. All structures defined above are to be interpreted in the context of multiplicative congruence towers embedded over the dyadic completion $\widehat{\mathbb{Q}}_{(2)}$. The binary tree structure of (2^n) provides an arithmetic filtration that refines and replaces the valuation-theoretic metric structure used in classical perfectoid theory.

1. Introduction and Motivation

The theory of perfectoid spaces has demonstrated the profound effectiveness of working with infinite-level congruence towers over p-adic valuation rings. In the context of multiplicoid geometry, our aim is to generalize these ideas by replacing additive and valuation-based structures with congruence systems derived from multiplicative stratification.

In this dyadic-supported version of multiplicoid geometry, we propose to work over the base

$$\widehat{\mathbb{Q}}_{(2)} := \varprojlim \mathbb{Q}/2^n\mathbb{Z},$$

the dyadic completion of the rational numbers under the inverse system defined by powers of 2. This base ring is not a valuation ring, and it lacks a topology in the classical analytic sense. Instead, it carries a natural *congruence stratification structure*, which provides a new organizing principle for filtrations, torsors, and cohomological realizations.

Why Dyadic?

- The dyadic tower (2^n) forms a canonical and universal congruence system among all base-2 congruences, naturally connected to binary growth and recursive depth.
- Dyadic congruence trees resemble binary computation trees in logic and computer science, suggesting a deep connection between arithmetic filtration and stratified recursion.
- The arithmetic nature of the (2^n) system provides an alternative to valuation theory: it encodes depth by divisibility, rather than by norm.
- Dyadic ε-filtrations allow us to define height pairings, period stratifications, and torsor descent data without referring to topology, but purely in terms of arithmetic depth.
- Many natural objects in arithmetic geometry—modular forms, polylogarithms, iterated integrals—exhibit binary growth phenomena that are better captured through dyadic congruence.

Conceptual Shift. This volume represents a conceptual shift in the foundations of geometry:

> From valuation-based geometry to congruence-based filtration geometry.

In particular, we view dyadic congruence as a *growth-theoretic medium*—a non-topological, non-archimedean arena where geometric structure is determined by arithmetic stratification rather than distance.

Objectives of this Volume. Our goal is to develop the following:

- A category of sheaves and torsors over $\mathrm{Spec}(\widehat{\mathbb{Q}}_{(2)})$ equipped with multiplicative congruence filtrations;
- A theory of dyadic period rings, including $B_{\mathrm{dyad},dR}$ and its tower $\mathrm{Per}^{\infty}_{\mathrm{dyad}}$;
- ε-filtration theory over dyadic bases, giving rise to dyadic cohomology groups H^i_{dyad};
- Universal realization functors $\mathscr{R}_{\mathrm{dyad}}$ from torsor filtrations to period cohomology;
- Applications to motivic realizations, regulator maps, and arithmetic stacks with binary depth invariants.

The structure of this volume follows the same organizational logic as the dyadic-free version, but with new foundational constructions that arise uniquely in the dyadic context. In this setting, torsor descent becomes arithmetic descent, and congruence depth replaces valuation depth, leading to a new geometry of stratified arithmetic emergence.

2. Dyadic Congruence Systems and Base Structures

2.1. The Dyadic Completion.

We begin by defining the dyadic completion of $\mathbb{Q}$ via congruence modulo powers of 2. This will serve as the base ring for all constructions in this volume.

Definition 2.1 (Dyadic Completion)**.** *Let $\mathbb{Z}/2^n\mathbb{Z}$ denote the ring of integers modulo 2^n. The* dyadic completion *of $\mathbb{Q}$ is the inverse limit*

$$\widehat{\mathbb{Q}}_{(2)} := \varprojlim_n \mathbb{Q}/2^n\mathbb{Z},$$

with transition maps induced by natural reduction modulo 2^{n-1}. It admits a canonical congruence filtration indexed by $n \in \mathbb{Z}_{\geq 0}$:

$$I_n := \ker\left(\widehat{\mathbb{Q}}_{(2)} \to \mathbb{Q}/2^n\mathbb{Z}\right) \subset \widehat{\mathbb{Q}}_{(2)}.$$

Remark 2.2. This completion is not topological in the usual analytic sense; rather, it reflects congruence depth in the binary divisibility hierarchy. Unlike $\mathbb{Q}_2$, it is not a field, but a pro-object defined over arithmetic congruence.

2.2. Dyadic Base Rings and Affine Patches.

Definition 2.3 (Dyadic Base Ring)**.** *A* dyadic base ring *is any ring A admitting a morphism of inverse systems*

$$A \to \varprojlim_n A_n := \varprojlim_n A/2^n A.$$

The completion $A^{\infty}_{\mathrm{dyad}} := \varprojlim A_n$ is called the dyadic congruence completion *of A.*

Example 2.4. Let $A = \mathbb{Z}[x]$. Then $A_n = \mathbb{Z}[x]/(2^n)$ and the system $\{A_n\}$ yields

$$A_{\text{dyad}}^{\infty} = \varprojlim \mathbb{Z}[x]/(2^n) = \mathbb{Z}_2[[x]].$$

This serves as the dyadic formal neighborhood of x over $\operatorname{Spec}(\mathbb{Z})$.

2.3. **Congruence Sheaves and Dyadic Filtrations.**

Let $X = \operatorname{Spec}(A_{\text{dyad}}^{\infty})$.

Definition 2.5 (Dyadic Sheaf). *A* dyadic sheaf *on X is a sheaf $\mathcal{F}$ of A-modules equipped with a descending filtration*

$$F^n\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/2^n\mathcal{F}\right).$$

Each F^n is referred to as the n-th dyadic congruence layer*.*

Definition 2.6 (ε-Filtration over Dyadic Base). *Define the ε-filtration tower:*

$$F^{\varepsilon^n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/(2^n)\mathcal{F}\right),$$

where the index ε^n indicates the exponential depth of the congruence.

Remark 2.7. This structure allows us to define height-pairings and torsor classes whose obstruction depth grows with dyadic congruence complexity.

2.4. **Torsors and Transition Systems.**

Definition 2.8 (Dyadic Torsor Tower). *Let $G_n := \mathbb{Z}/2^n\mathbb{Z}$. A* dyadic torsor tower *over X is a projective system $\{\mathcal{T}_n\}$ where each $\mathcal{T}_n \to X$ is a G_n-torsor, and the transition maps*

$$\mathcal{T}_{n+1} \to \mathcal{T}_n$$

are G_n-equivariant under reduction $G_{n+1} \to G_n$.

These towers encode stratified symmetry across binary congruence levels. We may view them as arithmetic analogues of local systems, organized by binary divisibility rather than monodromy.

2.5. **Summary.**

We now have:

- A congruence-based arithmetic base ring $\widehat{\mathbb{Q}}_{(2)}$;
- A filtration theory indexed by (2^n);
- Torsors and sheaves structured by binary descent;
- ε-filtrations organizing congruence depth stratification.

These tools form the local foundations on which we construct dyadic period rings, torsor cohomology, and ε-gerbes in the sections that follow.

3. Dyadic Period Rings and Sheaves

3.1. **Dyadic de Rham Period Ring.** Just as perfectoid geometry gives rise to period rings such as B_{dR}, in the dyadic-supported setting we construct a ring of periods governed by congruence stratification modulo powers of 2.

Definition 3.1 (Dyadic de Rham Period Ring). *Let $A^{\infty}_{\mathrm{dyad}}$ be the dyadic-completed base ring. The* dyadic de Rham period ring *is defined as:*

$$B_{dyad,dR}(A) := \varprojlim_{n} A^{\infty}_{\mathrm{dyad}}/2^n \otimes_{\mathbb{Z}} \mathbb{Q},$$

with the natural dyadic filtration:

$$F^n B_{dyad,dR} := \ker\left(B_{dyad,dR} \to A^{\infty}_{\mathrm{dyad}}/2^n \otimes \mathbb{Q}\right).$$

Remark 3.2. This filtration is arithmetic in nature, defined by congruence depth rather than norm or valuation. Each level F^n corresponds to a binary congruence depth layer.

3.2. **Dyadic Period Tower.**

Definition 3.3 (Dyadic Period Tower). *The* dyadic period tower *is the projective system:*

$$\mathrm{Per}^{\infty}_{dyad} := \left\{B^{(n)}_{dyad,dR} := A^{\infty}_{\mathrm{dyad}}/2^n \otimes \mathbb{Q}\right\}_{n\in\mathbb{Z}_{\geq 0}},$$

with transition maps induced by canonical reductions modulo 2^{n-1}.

This tower serves as the congruence-based analog of the classical period rings of p-adic Hodge theory. It defines a cohomological structure based on binary stratification.

3.3. **Period Sheaves with ε-Filtration.**

Definition 3.4 (Dyadic Period Sheaf). *Let $X = \mathrm{Spec}(A^{\infty}_{\mathrm{dyad}})$. A* dyadic period sheaf *over X is a sheaf $\mathcal{F}$ of $B_{dyad,dR}$-modules equipped with:*

- *A descending dyadic filtration $F^n\mathcal{F}$ indexed by congruence depth 2^n;*
- *A compatible ε-filtration $F^{\varepsilon^n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/2^n)$;*
- *Flat transition morphisms over the tower $\mathrm{Per}^{\infty}_{dyad}$;*
- *Functorial compatibility with dyadic torsor actions.*

Example 3.5. The structure sheaf $\mathcal{O}_X$ itself admits an ε-filtration:

$$F^{\varepsilon^n}\mathcal{O}_X = \{f \in \mathcal{O}_X \mid f \equiv 0 \mod 2^n\}.$$

Tensoring with $\mathbb{Q}$ gives the dyadic period sheaf associated to identity sections.

3.4. **Dyadic Cohomology.**

Definition 3.6 (Dyadic Cohomology). *Let $\mathcal{F}$ be a dyadic period sheaf over X. The i-th* dyadic cohomology group *is defined by*

$$H^i_{dyad}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathcal{F}/2^n\mathcal{F}).$$

This cohomology detects the stabilization of arithmetic data under binary congruence descent.

Remark 3.7. This theory mirrors crystalline and syntomic cohomology, but under congruence stratification rather than Frobenius lifts. The filtration corresponds to ε-depth layers in arithmetic descent.

3.5. **Summary.** We now have:

- The dyadic de Rham period ring $B_{\text{dyad},dR}$ as the analog of B_{dR};
- An ε-stratified tower of congruence-period rings $\text{Per}^{\infty}_{\text{dyad}}$;
- Dyadic period sheaves and compatible ε-filtrations;
- A new cohomology theory H^i_{dyad} based on arithmetic stratification depth.

These objects are the building blocks of the dyadic period-geometry landscape, and will feed into the torsor-theoretic structures in the next section.

4. Dyadic Torsors, Gerbes, and Realizations

4.1. **Universal Dyadic Torsor Tower.** In the dyadic setting, torsors naturally arise over the binary congruence groups $G_n := \mathbb{Z}/2^n\mathbb{Z}$, with transitions corresponding to the mod-2 congruence system.

Definition 4.1 (Dyadic Torsor). *Let X be a dyadic base space. A* dyadic G_n-torsor *over X is a scheme $\mathcal{T}_n \to X$ with a free transitive right action of G_n such that locally on X, $\mathcal{T}_n \cong X \times G_n$.*

A dyadic torsor tower *is a projective system*

$$\{\mathcal{T}_n\}_{n\in\mathbb{Z}_{\geq 0}}, \quad \mathcal{T}_{n+1} \to \mathcal{T}_n,$$

with $G_{n+1} \to G_n$ equivariance and compatibility.

Example 4.2. Let $X = \text{Spec}(\widehat{\mathbb{Q}}_{(2)})$. Then $\mathcal{T}_n = \text{Spec}(\widehat{\mathbb{Q}}_{(2)}[x]/(x^{2^n} - 1))$ forms a G_n-torsor representing 2^n-th roots of unity. The tower captures the limit $\varprojlim \mu_{2^n}$.

4.2. **Universal ε-Gerbe over Dyadic Congruence.**

Definition 4.3 (Dyadic ε-Gerbe). *Let $\mathcal{F}$ be a dyadic sheaf. An* ε-gerbe $\mathcal{G}_\varepsilon$ *over X is a banded stack associated to the projective system $\{\mathcal{T}_n\}$, satisfying:*

- *Each $\mathcal{T}_n$ classifies trivializations modulo 2^n;*
- *$\mathcal{G}_\varepsilon$ is banded by the inverse system $\{G_n\}$;*

- *$\mathcal{G}_\varepsilon$ admits a classifying map:*

$$X \to B\mathcal{G}_\varepsilon := \varprojlim_n B(G_n),$$

where $B(G_n)$ is the classifying stack for G_n-torsors.

This stack controls the obstruction theory of lifting sections through increasing binary congruence levels. Cohomological torsors correspond to gerbe classes in $H^2_{\text{dyad}}(X, G_n)$.

4.3. Realization Functor and Period Morphisms.

Definition 4.4 (Dyadic Realization Functor)**.** *Define the functor:*

$$\mathscr{R}_{dyad} : \mathbf{Sh}^{\varepsilon}(X) \longrightarrow \mathrm{Perdyad}^{\infty}(X) \times \mathbb{T}^{[\text{dyad}]},$$

sending a dyadic-filtered sheaf to its period image in $\mathrm{Per}^{\infty}_{dyad}$ *and its torsor realization in* $\mathbb{T}^{[\text{dyad}]}$.

This functor realizes ε-stratified torsors as period data, organizing geometric cohomology in terms of dyadic congruence complexity.

Theorem 4.5 (Universal Period Realization)**.** *There exists a natural transformation:*

$$\mathscr{R}_{dyad}(\mathcal{F}) \longrightarrow H^{\bullet}_{dyad}(X, \mathcal{F}),$$

that respects filtration depth, torsor descent, and period morphism compatibility.

Sketch. Using the projective systems $\mathcal{F}/2^n$, one builds the cohomology via inverse limits, and the torsor classes lift canonically through $\mathcal{G}_\varepsilon$ to $\mathrm{Per}^{\infty}_{\text{dyad}}$-coefficients. □

4.4. Concluding Remarks.

This torsor–gerbe–period framework gives rise to a new type of stratified cohomology, one not built on topological neighborhoods, but on congruence levels of arithmetic height. It provides:

- Fine-grained control of arithmetic period structures;
- Classification of torsor complexity via binary towers;
- New perspectives on motivic cohomology over stratified spaces;
- A foundation for comparing with multiplicoid–perfectoid–exponentoid geometries in the broader framework of hyperstratified spaces.

5. Dyadic Motivic Realization and Regulators

5.1. Dyadic Motives and Period Realizations.

Let X be a space defined over the dyadic base $\widehat{\mathbb{Q}}_{(2)} := \varprojlim \mathbb{Q}/2^n\mathbb{Z}$. We define dyadic motives as analogues of classical motives, realized through torsor and period descent over dyadic filtration towers.

Definition 5.1 (Dyadic Motive). *A* dyadic motive $M_{\mathrm{dyad}}^{[\times]}(X)$ *is an object in a triangulated category* $\mathcal{M}_{\mathrm{dyad}}$ *of arithmetic motivic sheaves over dyadic congruence systems, equipped with:*

- *A dyadic filtration* $\{F^{2^n}M\}_{n\geq 0}$*;*
- *Realization functors to period cohomology:*

$$\mathrm{real}_{\mathrm{dyad}}^{per} : M_{\mathrm{dyad}}^{[\times]}(X) \longrightarrow H_{\mathrm{dyad}}^{\bullet}(X, B_{dyad,dR});$$

- *Descent through torsor towers* $\{\mathcal{T}_n\}$ *under* $(\mathbb{Z}/2^n\mathbb{Z})$*-action.*

5.2. **Dyadic Regulator Maps.** We define analogues of Beilinson's higher regulators via torsor realization.

Definition 5.2 (Dyadic Regulator). *Let* $K_n(X)$ *be an algebraic* K*-group of* X*. The* dyadic regulator map *is:*

$$r_{\mathrm{dyad}} : K_n(X) \longrightarrow H_{\mathrm{dyad}}^n(X, \mathbb{Q}(n)),$$

constructed by composing:

$$K_n(X) \xrightarrow{\textit{cycle class}} M_{\mathrm{dyad}}^{[\times]}(X) \xrightarrow{\mathrm{real}} H_{\mathrm{dyad}}^n(X, B_{dyad,dR}).$$

The map r_{dyad} encodes congruence-based height data in the form of dyadic period invariants.

5.3. **Dyadic ε-Pairings and Period Height.** Given a torsor tower $\{\mathcal{T}_n\}$, one obtains natural pairings indexed by congruence depth.

Definition 5.3 (Dyadic ε-Pairing). *Let* M *be a dyadic motive over* X*. Define:*

$$\langle -, - \rangle_{\varepsilon^n} : F^{2^n}M \otimes F^{2^n}M^\vee \longrightarrow \mathbb{Q},$$

satisfying compatibility with dyadic torsor transitions. The pairing defines arithmetic height in the dyadic filtration category.

5.4. **Dyadic L-Functions.** We define special values via congruence-depth period realizations:

Definition 5.4 (Dyadic L-Function). *Let* $M = M_{\mathrm{dyad}}^{[\times]}(X)$*. Define:*

$$L_{\mathrm{dyad}}(M, s) := \prod_{n=0}^{\infty} \det\left(1 - \mathrm{Frob}_n \cdot 2^{-s} \mid F^{2^n}M\right)^{-1},$$

where Frob_n *is the action induced by congruence twisting at depth* 2^n*.*

This function encodes the arithmetic complexity of dyadic torsor realization, stratified across binary congruence layers.

5.5. **Summary.** We have introduced the core motivic objects in dyadic-supported multiplicoid geometry:

- Motives $M_{\text{dyad}}^{[\times]}$ with dyadic filtration;
- Regulator maps from K-theory into dyadic cohomology;
- ε-pairings derived from torsor symmetry;
- Dyadic L-functions as congruence-stratified arithmetic generating series.

These tools together provide the motivic infrastructure for understanding growth-based cohomology beyond valuation and topology.

6. Comparison with Additive and Valuation-Based Geometries

6.1. **Three Approaches to Filtration and Descent.** We now compare three major frameworks for arithmetic filtration and cohomology:

(1) **Additive/Valuation-Based:** Uses topologies induced by discrete valuations, e.g., p-adic Hodge theory, perfectoid spaces.
(2) **Dyadic-Supported (Congruence-Based):** Uses congruence towers such as 2^n for filtration and torsor descent, without relying on topological neighborhoods.
(3) **Crystalline/Syntomic:** Uses Frobenius-lifted deformations and divided powers to capture p-adic period structures.

Type	**Filtration**	**Descent Mechanism**
Valuation/Perfectoid	Norm-based, $\lvert x\rvert_p < \varepsilon$	Frobenius tilting, topological completion
Crystalline/Syntomic	Divided power ideals	Frobenius-compatible thickenings
Dyadic (This work)	Arithmetic congruence 2^n	Binary torsor towers, ε-filtration

Table 1. Comparison of geometric filtration theories

6.2. **Dyadic vs Perfectoid Tilting.** Perfectoid tilting requires deep compatibility with Frobenius morphisms and almost mathematics. In contrast, dyadic-supported tilting operates through congruence relations:

$$\text{Perfectoid:}\quad \mathcal{O}_K \simeq \varprojlim_{\Phi} \mathcal{O}_K/p \qquad \text{vs.} \qquad \text{Dyadic:}\quad \mathcal{F} \simeq \varprojlim_{n} \mathcal{F}/2^n.$$

The dyadic theory is *Frobenius-free* and instead stratifies space through explicit congruence descent.

6.3. **Hybrid Filtration: ε-Dyadic–Syntomic Structures.** A promising research direction involves merging congruence-based ε-filtrations with crystalline period theories.

Definition 6.1 (ε-Dyadic–Syntomic Tower)**.** *Let* $F^{(n)} := \ker(\mathcal{F} \to \mathcal{F}/(2^n, \gamma_n))$*, where* γ_n *is a divided power or syntomic correction. The resulting tower interpolates:*

- *dyadic congruence* 2^n*;*
- *crystalline thickenings via* γ_n*;*
- *syntomic periods with arithmetic* ε*-filtration.*

Such towers may yield a new class of regulators, which carry both congruence descent and Frobenius-like cohomological deformations.

6.4. **Conceptual Distinctions.**

- **Topological approximation (valuation):** geometry defined via closeness.
- **Stratified collapse (dyadic):** geometry defined via recursion.
- **Cohomological deformation (crystalline):** geometry defined via compatibility with Frobenius.

We posit that these are orthogonal lenses for understanding arithmetic geometry, and their comparison highlights the need for a unified framework of stratified filtrations.

6.5. **Conclusion.** Dyadic-supported geometry provides a third paradigm, joining valuation and crystalline theories. It allows one to study motivic realizations and regulators via discrete congruence operations, enabling a new notion of depth, period structure, and torsor complexity.

In the next section, we explore how these congruence stratifications extend into exponential growth regimes via the dyadic–exponentoid bridge.

7. Dyadic–Exponentoid Bridge and Stratified Interpolation

7.1. **Motivation for Interpolation.** The dyadic filtration tower uses binary growth: $F^{2^n}\mathcal{F}$. To reach the higher complexity of exponentoid and knuthoid geometries, we now define morphisms between dyadic and exponential congruence systems.

This provides a smooth passage from arithmetic congruence-based geometry into recursion-based stratified structures.

7.2. **Interpolating Growth Functions.** Let $f(n) = 2^n$, $g(n) = \exp(n)$. Define a bridge function:

$$\beta(n) := \left\lfloor a^{\lambda(n)} \right\rfloor, \quad \text{with } \lambda(n) = \log_2 n + \delta(n),$$

where $\delta(n) \to 0$ is a correction term. Then:

$$2^n < \beta(n) < \exp(n), \quad \forall n \gg 0.$$

This yields interpolating filtrations:

$$F^{\beta(n)}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\beta(n)\mathcal{F}\right).$$

7.3. **Period Ring Morphisms.** Let $B_{\mathrm{dyad},dR} \to B_{\exp,dR}$ be a morphism of period rings defined via filtration base change:

$$B_{\mathrm{dyad},dR} = \varprojlim \mathcal{F}/2^n \longrightarrow \varprojlim \mathcal{F}/\exp(n).$$

We define a compatible map between period towers:

$$\Phi_n^{\mathrm{interp}} : B^{(n)}_{\mathrm{dyad},dR} \to B^{(n)}_{\exp,dR},$$

respecting torsor descent and filtered realization structures.

7.4. **Torsor Reinterpretation.** Let $\mathcal{T}_n^{(2)}$ denote a dyadic torsor and $\mathcal{T}_n^{(\exp)}$ an exponentoid torsor. Then we define:

$$\mathcal{T}_n^{(\exp)} := \left(\mathcal{T}_{m_n}^{(2)}\right)^{\mathrm{Sym}^k}, \quad \text{where } m_n = \lfloor \log_2 \exp(n) \rfloor.$$

This models exponentoid torsors as symmetrized stacks of dyadic torsors. In categorical terms, there exists a functor:

$$\Theta : \mathbb{T}^{[\mathrm{dyad}]} \to \mathbb{T}^{[\exp]}, \quad \text{preserving tower depth via exponential rescaling.}$$

7.5. **Conjectural Dyadic Tilting toward Exponentoid Towers.**

Conjecture 7.1 (Dyadic–Exponentoid Tilting)**.** *There exists a filtration-preserving functor:*

$$\mathrm{Tilt}_{2^\bullet \to \exp(\bullet)} : \mathrm{Sh}_{\mathrm{dyad}} \to \mathrm{Sh}_{\exp},$$

which respects torsor equivalence classes and ε-stratified cohomology.

This functor induces a derived equivalence between congruence-based and exponential sheaves under growth-based transitions.

7.6. **Future Interpolative Models.** Just as perfectoid theory interpolates characteristic p and 0, we envision that:
- Dyadic-supported geometry interpolates congruence logic;
- Exponentoid geometry interpolates recursion logic;
- Their bridge encodes a generalized arithmetic period transformation scheme.

This prepares us for the final ontological layering of dyadic structures in the next section.

8. Ontological Foundations and Future Directions

8.1. **From Arithmetic Congruence to Ontological Stratification.** Throughout this volume, we have seen how dyadic structures generate geometric information not from topology or valuation, but from binary congruence depth. This naturally leads us to reinterpret congruence as a generative force—a stratified arithmetic ontology.

To exist arithmetically is to persist across congruence layers.

Thus, filtrations such as $\{F^{2^n}\mathcal{F}\}$ define not just cohomological depth, but *ontological layers of mathematical being.*

8.2. **Existence as Stratified Cohesion.**

Definition 8.1 (Congruence Ontology)**.** *Let $\mathcal{F}$ be a sheaf over a dyadic base. Define:*

$$\mathbb{E}_{\text{dyad}} := \left\{ X \in \mathcal{F} \;\middle|\; \forall n,\, X \in F^{2^n}\mathcal{F} \right\}.$$

Then $\mathbb{E}_{\text{dyad}}$ is the category of arithmetically persistent structures*.*

Each layer F^{2^n} functions as a logical sieve. The more layers a section survives, the more real it becomes. Existence is depth of congruence, not merely presence.

8.3. **Sheaf Towers as Indexed Logic.** We propose that dyadic ε-filtration towers should be seen as logic-indexed hierarchies. For example:
- F^{2^n} = "provable at logical strength level n";
- Torsor descent = "coherent across logical collapse";
- ε-gerbes = "syntactic classifiers of arithmetic persistence".

Definition 8.2 (Growth Ontology Stack)**.** *Define a contravariant functor:*

$$\mathcal{O}nt_{\text{dyad}} : \mathbb{N}^{\text{op}} \to \mathbf{Cat}, \quad n \mapsto \mathrm{Sh}(F^{2^n}\mathcal{F}),$$

which we interpret as the ontological reality at congruence level n*.*

The sheaf $\mathcal{F}$ then becomes an "emergent total space":

$$\mathcal{F} = \varprojlim \mathcal{O}nt_{\text{dyad}}(n),$$

a limit of categorical existence over growing depth.

8.4. **From Dyadic to Ontoid Geometry.** This leads naturally into Volume IV, where we define **Ontoid Geometry**:
- The base is no longer a ring or field, but a logic of generation;
- Space is not a set of points, but a stack of structural operations;
- Filtrations index not just magnitude, but existential layering.

$$\boxed{\text{Space} = \text{A Layered Ontology of Arithmetic Filtration}}$$

8.5. Programmatic Vision.

(1) Start with additive approximation (classical geometry).
(2) Refine into multiplicoid congruence stratification.
(3) Encode via dyadic towers.
(4) Transition into exponentoid and knuthoid growth regimes.
(5) Collapse into abstract ontological stacks of growth-based existence.

Each step replaces static topological space with stratified generators of structure.

8.6. **Outlook.** This volume has laid the groundwork for a logic-driven foundation of arithmetic space. In future volumes, we will:

- Develop full exponentoid and knuthoid towers;
- Generalize torsors and motives to trans-recursive growth indices;
- Formulate new cohomological dualities indexed by operational depth;
- Build arithmetic ontology from congruence towers and sheaf recursion.

We now exit the binary world of dyadic arithmetic and ascend toward the layered infinities of stratified mathematical existence.

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