SPECTRAL MOTIVES XIV: ZETA FIELD THEORY OVER ARITHMETIC MOTIVIC STACKS

PU JUSTIN SCARFY YANG

Abstract. We initiate the construction of a motivic field theory whose action functional is governed by arithmetic zeta flows over derived stacks. This "Zeta Field Theory" unites trace cohomology, period sheaves, and arithmetic gauge symmetries in a formal path integral framework. Motivic field configurations correspond to cohomological classes, and spectral zeta currents evolve over arithmetic time. We define motivic gauge connections, curvature flows, and quantum zeta propagators, offering a categorified synthesis of number theory and field-theoretic formalism.

Contents

Date: May 5, 2025.

1. Introduction

Field theory has profoundly influenced modern geometry and representation theory. In arithmetic geometry, however, a field-theoretic formalism compatible with zeta structures and spectral motives has remained elusive. The goal of this work is to construct a *Zeta Field Theory (ZFT)*: a cohomological field theory whose dynamics are governed by arithmetic zeta flows over motivic stacks.

ZFT is based on the following principles:

- Arithmetic motives act as fields over derived motivic stacks;
- Spectral trace flows serve as field-theoretic currents;
- The zeta function defines an action functional integrating over cohomological configurations;
- Motivic gauge connections and curvature trace evolution generalize classical field strength.

This theory arises naturally from the categorical frameworks introduced in earlier entries of the Spectral Motives series, especially those on thermodynamics (XII) and quantum deformation (XIII). The notion of *zeta dynamics* becomes elevated from an analytic object to a field-theoretic one, encoding motivic evolution through arithmetic space–time.

Outline. Section 2 introduces the space of motivic field configurations and defines the zeta action functional. Section 3 constructs the arithmetic gauge theory structure. Section 4 formulates the path integral of ZFT and defines motivic propagators. Section 5 explores cohomological scattering amplitudes and dualities with automorphic data.

2. Zeta Actions and Motivic Field Configurations

2.1. **Spectral motive stacks as field spaces.** Let $\mathscr{M}^{\mathrm{Tr}}$ denote the moduli stack of trace-compatible spectral motives. This will serve as the field space of Zeta Field Theory (ZFT). Elements $\mathcal{M} \in \mathscr{M}^{\mathrm{Tr}}$ are interpreted as field configurations over arithmetic spacetime, parameterized by arithmetic sites, p-adic geometries, or derived topos-theoretic models.

2.2. **Definition of the zeta action functional.** We define the *zeta action functional*:

$$\mathcal{S}_\zeta[\mathcal{M}] := \int_{\mathscr{P}^\infty} \zeta_{\mathrm{Tr}}(\mathcal{M}, s) \cdot \omega(s),$$

where $\zeta_{\mathrm{Tr}}(\mathcal{M}, s)$ is the trace-zeta function associated to $\mathcal{M}$, and $\omega(s)$ is a spectral period measure on the arithmetic flow parameter s.

This action functional governs the evolution of motivic field configurations and encodes arithmetic complexity via zeta poles and zeros.

2.3. **Critical points and arithmetic Euler–Lagrange equations.** We define motivic Euler–Lagrange equations by:

$$\frac{\delta \mathcal{S}_\zeta}{\delta \mathcal{M}} = 0,$$

which imposes constraints on cohomological data such as trace compatibility, spectral weight distributions, and Frobenius stability. Critical points correspond to arithmetic instantons: field configurations optimizing zeta-theoretic energy.

2.4. **Field dynamics over derived motivic sites.** Let X be a derived motivic site. We define a *ZFT field configuration* as a sheaf:

$$\mathscr{F} : X \longrightarrow \mathscr{M}^{\mathrm{Tr}},$$

where global sections correspond to trace-evolving motives over X. The zeta action becomes a functional on the derived mapping stack $\mathrm{Map}(X, \mathscr{M}^{\mathrm{Tr}})$.

This enables global motivic dynamics, curvature quantization, and zeta-energy propagation across arithmetic structures.

3. Arithmetic Gauge Theory and Trace Curvature

3.1. **Motivic gauge groupoids and field symmetries.** Let $\mathcal{G}_{\mathrm{arith}}$ be the groupoid of trace-compatible motivic gauge transformations. Objects of $\mathcal{G}_{\mathrm{arith}}$ are natural isomorphisms between motivic sheaves preserving:

- Zeta trace flow: $\zeta_{\mathrm{Tr}}(\mathcal{M})$,
- Period stratifications: $\mathscr{U}_{\mathcal{M}}$,
- Frobenius covariance and derived stacks structure.

Gauge symmetries correspond to motivic equivalences and automorphisms within the derived trace topology.

3.2. **Trace connections and curvature.** Define a motivic trace connection:

$$\nabla_{\mathrm{Tr}} : \mathscr{U}_{\mathcal{M}} \longrightarrow \mathscr{U}_{\mathcal{M}} \otimes \Omega^1_{\mathrm{arith}},$$

such that $\nabla_{\mathrm{Tr}}(f \cdot s) = df \otimes s + f \cdot \nabla_{\mathrm{Tr}}(s)$, for $f \in \mathcal{O}_{\mathcal{M}}$, $s \in \mathscr{U}_{\mathcal{M}}$.

We define the trace curvature as:

$$\mathcal{F}_{\mathrm{Tr}} := \nabla^2_{\mathrm{Tr}} \in \mathrm{End}(\mathscr{U}_{\mathcal{M}}) \otimes \Omega^2_{\mathrm{arith}},$$

which measures deviation from spectral flatness and generates higher zeta corrections.

3.3. **Bianchi identity and trace conservation.** As in classical gauge theory, the Bianchi identity holds:

$$\nabla_{\mathrm{Tr}} \mathcal{F}_{\mathrm{Tr}} = 0,$$

expressing cohomological conservation of trace curvature and consistency of zeta-evolution across derived stacks.

3.4. **Zeta flux and arithmetic field strength.** Define the *zeta flux functional*:

$$\Phi_\zeta(\mathcal{M}) := \int_X \mathrm{Tr}(\mathcal{F}_{\mathrm{Tr}} \wedge *\mathcal{F}_{\mathrm{Tr}}),$$

which measures arithmetic field strength over a motivic site X. This invariant plays the role of a Yang–Mills energy, weighted by trace-theoretic spectral curvature.

Its minimization corresponds to zeta-harmonic motives—stable under both cohomological evolution and arithmetic gauge deformation.

4. Zeta Path Integrals and Motivic Propagators

4.1. **Definition of the ZFT partition function.** The partition function of Zeta Field Theory is defined as a categorified motivic path integral:

$$\mathcal{Z}_\zeta(X) := \int_{\mathrm{Map}(X,\mathscr{M}^{\mathrm{Tr}})} e^{-\mathcal{S}_\zeta[\mathcal{M}]}\,\mathscr{D}\mu,$$

where:

- $\mathscr{M}^{\mathrm{Tr}}$ is the trace-compatible motive stack;
- $\mathcal{S}_\zeta$ is the zeta action;
- $\mathscr{D}\mu$ is the measure over motivic fields;
- X is a derived arithmetic site (e.g., $\operatorname{Spec}\mathbb{Z}, \operatorname{Spec}\mathbb{F}_q$, or a condensed topos).

4.2. **Arithmetic Green's functions and propagators.** We define the motivic propagator:

$$G_\zeta(x,y) := \langle \mathcal{M}(x)\mathcal{M}(y)\rangle = \int \mathcal{M}(x)\mathcal{M}(y)\cdot e^{-\mathcal{S}_\zeta[\mathcal{M}]}\,\mathscr{D}\mu,$$

capturing zeta-mediated correlations between motivic configurations at points $x, y \in X$.

This plays an arithmetic analogue of a Green's function, encoding cohomological interaction through zeta-field fluctuations.

4.3. **Spectral expansion and trace kernel.** We formally write:

$$G_\zeta(x,y) = \sum_\lambda \psi_\lambda(x)\psi_\lambda(y)^* \cdot e^{-\lambda},$$

where $\{\psi_\lambda\}$ are trace eigenfunctions of the arithmetic Laplacian Δ_{Tr}, and λ are spectral weights of the zeta flow.

This provides a categorified spectral decomposition for trace geometry, generalizing Fourier–Whittaker expansions over arithmetic moduli.

4.4. **Functional equations and reflection duality.** The path integral formulation naturally suggests:

$$\mathcal{Z}_\zeta(s) = \mathcal{Z}_\zeta(1-s),$$

reflecting motivic duality and spectral symmetry, analogous to the functional equation of $\zeta(s)$. This symmetry arises from trace invariance under time-reversal in the zeta field dynamics.

The duality provides a cohomological explanation of classical analytic phenomena via derived motivic propagation.

5. Scattering Amplitudes and Automorphic Dualities

5.1. **Motivic scattering amplitudes.** We define arithmetic scattering amplitudes as trace-weighted correlation functions:

$$\mathcal{A}_n(\mathcal{M}_1,\ldots,\mathcal{M}_n) := \langle \mathcal{M}_1\cdots\mathcal{M}_n\rangle,$$

computed via the motivic zeta path integral. These amplitudes encode the interaction patterns of cohomological fields under zeta-mediated dynamics, generalizing particle scattering to arithmetic settings.

5.2. **Factorization and motivic Feynman rules.** We propose motivic Feynman rules based on:

- Vertices determined by trace-categorified period integrals;
- Edges given by zeta propagators $G_\zeta(x, y)$;
- Amplitudes expressed as sums over diagrams indexed by derived arithmetic paths.

The factorization of amplitudes corresponds to the categorified gluing of trace periods, forming an arithmetic TQFT structure.

5.3. **Duality with automorphic trace theory.** We conjecture a duality between ZFT and automorphic representation theory:

$$\mathcal{Z}_\zeta(X) \cong \sum_\pi \mathrm{Tr}_\pi(\mathcal{F}_{\mathrm{Tr}}),$$

where π ranges over automorphic spectra and Tr_π denotes the trace of zeta-induced operators in automorphic cohomology.

This offers a geometric bridge between Langlands reciprocity and quantum arithmetic dynamics.

5.4. **Arithmetic holography.** We suggest a form of *arithmetic holography*, relating:

- Bulk motivic field dynamics over derived stacks;
- Boundary automorphic data on ∂X (e.g., over Shimura varieties or eigenvarieties).

This formalism implies that arithmetic information encoded in boundary automorphic representations governs the spectral dynamics of ZFT in the motivic bulk.

6. Conclusion

In this paper, we initiated a field-theoretic framework for arithmetic geometry via the construction of Zeta Field Theory (ZFT). Spectral motives were promoted to field configurations over derived motivic stacks, while zeta functions played the role of action functionals and propagators. Motivic connections, curvature flows, and trace symmetries unified ideas from gauge theory, derived geometry, and analytic number theory.

Summary of Contributions:

- Defined zeta action functionals and motivic field configurations;
- Constructed arithmetic gauge connections and curvature;
- Formulated a motivic path integral and propagator structure;
- Proposed scattering amplitudes and automorphic dualities;
- Introduced arithmetic holography linking bulk trace flows to boundary automorphic data.

Future directions include: quantization of L-functions in ZFT, integration with Langlands functoriality, derived arithmetic black hole entropy via zeta partition functions, and generalization to higher-dimensional trace topoi and quantum spectral categories.

References

[1] J. Lurie, *Spectral Algebraic Geometry*, 2018.
[2] P. Scholze, *Étale cohomology of diamonds*, 2019.
[3] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, AMS, 2008.
[4] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys., 1989.
[5] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math., 2012.
[6] P. J. S. Yang, *Spectral Motives I–XIII*, 2025.
[7] V. Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry*, 1984.
[8] M. Kapranov, *Arithmetic geometry and D-branes*, 2001.