# SPECTRAL MOTIVES AND ZETA TRANSFER IV: GLOBAL FUNCTORIALITY AND UNIVERSAL L-TRACES

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Abstract. We develop a theory of global functoriality for spectral zeta motives over dyadic arithmetic topoi. Using the geometry of spectral stacks and derived Hecke eigenflows, we construct universal L-trace functors and prove they encode global Langlands transfers through derived pushforward of zeta cohomology. This framework extends the classical theory of L-functions and functoriality into a geometric formalism that unifies motive, automorphic, and categorical zeta invariants across arithmetic sites.

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## 1. Introduction: Global Zeta Transfers and Langlands Flows

The Langlands program predicts a deep functorial relation between automorphic forms on different groups via transfer of $L$-functions. In this paper, we extend the dyadic spectral zeta framework to formulate a global geometric version of Langlands functoriality as a derived trace transformation on spectral motives.

**1.1. Spectral Motives and Zeta Sites.** In previous work, we constructed the global arithmetic zeta topos $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$, together with spectral stacks $\mathcal{M}_{\mathbb{Z}_2}(G)$ classifying derived $G$-shtukas with zeta trace structures.

Let $\mathscr{M}_\zeta$ denote the universal zeta motive over $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$. This object controls global zeta flows and their trace realizations:

$$\zeta(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}_\zeta).$$

**1.2. Universal L-Trace Functor.** We define a universal functorial L-trace:

$$\mathrm{LTrace}_G^{\mathbb{Z}_2} : \mathrm{LocSys}_{\mathbb{Z}_2}(G) \to \mathrm{Shv}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(G)),$$

that assigns to each $G$-local system a spectral sheaf with derived Frobenius trace encoding $L$-function data.

This trace realizes $L(s,\pi)$ as:

$$L(s,\pi) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathrm{LTrace}_G^{\mathbb{Z}_2}(\rho_\pi)).$$

**1.3. Global Functoriality as Pushforward of Zeta Flows.** Given a morphism of reductive groups $\phi : H \to G$, we define a spectral transfer map:

$$\phi_*^\zeta : \mathcal{M}_{\mathbb{Z}_2}(H) \to \mathcal{M}_{\mathbb{Z}_2}(G),$$

and show that pushforward of zeta motives yields Langlands functoriality:

$$\phi_*^\zeta(\mathrm{LTrace}_H(\rho)) = \mathrm{LTrace}_G(\phi \circ \rho).$$

**1.4. Summary of Goals.** This paper develops:

(i) The universal L-trace functor $\mathrm{LTrace}_G^{\mathbb{Z}_2}$;
(ii) Zeta flow pushforwards along group morphisms and arithmetic correspondences;
(iii) Spectral automorphic stacks with trace sheaves encoding global Langlands lifts;
(iv) Cohomological realization of L-functions via derived geometric traces;
(v) Compatibility with previous constructions in *Dyadic Langlands I–III*.

These structures provide a unified motivic interpretation of $L$-functions and functoriality over derived arithmetic sites, extending the reach of spectral zeta motives into the categorical and automorphic landscape.

## 2. Universal L-Trace Functors and Derived Spectral Sheaves

2.1. **2.1. Moduli of Spectral Local Systems.** Let $G$ be a reductive group over $\mathbb{Z}_2$. We define the stack of $G$-local systems over the dyadic arithmetic topos:

$$\mathrm{LocSys}_{\mathbb{Z}_2}(G) := \mathrm{Hom}(\pi_1^{\mathrm{et}}(\mathbf{Top}_\zeta^{\mathbb{Z}_2}), G).$$

This space classifies spectral Galois parameters, possibly varying in derived families.

2.2. **2.2. Zeta Sheaves and Spectral Realizations.** We define $\zeta$-sheaves as spectral sheaves on $\mathcal{M}_{\mathbb{Z}_2}(G)$ equipped with Frobenius trace data:

$$\mathrm{Shv}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(G)) := \left\{ \mathscr{F} \in D^b(\mathcal{M}_{\mathbb{Z}_2}(G)) \mid \mathrm{Frob} \curvearrowright \mathscr{F},\ \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}) \in \mathbb{C}[[q^{-s}]] \right\}.$$

These sheaves generalize the classical notion of Hecke eigensheaves with trace parameters replacing eigenvalues.

2.3. **2.3. Construction of the L-Trace Functor.** We define the functor:

$$\mathrm{LTrace}_G^{\mathbb{Z}_2} : \mathrm{LocSys}_{\mathbb{Z}_2}(G) \to \mathrm{Shv}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(G)),$$

as follows:

(i) Given $\rho : \pi_1^{\mathrm{et}}(\mathbf{Top}_\zeta^{\mathbb{Z}_2}) \to G$, pull it back to define a local system $\mathscr{L}_\rho$;
(ii) Extend $\mathscr{L}_\rho$ to a derived shtuka sheaf $\widetilde{\mathscr{L}_\rho}$ over $\mathcal{M}_{\mathbb{Z}_2}(G)$;
(iii) Define:

$$\mathrm{LTrace}_G^{\mathbb{Z}_2}(\rho) := \widetilde{\mathscr{L}_\rho},$$

equipped with derived Frobenius trace structure and cohomological flow.

2.4. **2.4. Functorial Properties and Derived Traces.** The functor $\mathrm{LTrace}_G^{\mathbb{Z}_2}$ satisfies:

- **Exactness:** It preserves quasi-isomorphisms;
- **Frobenius Linearity:** $\mathrm{LTrace}(\rho \otimes \chi) = \mathrm{LTrace}(\rho) \otimes \mathscr{L}_\chi$;
- **Trace Realization:** For each $\rho$, we have:

$$L(s, \rho) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma_c(\mathcal{M}_{\mathbb{Z}_2}(G), \mathrm{LTrace}_G^{\mathbb{Z}_2}(\rho))).$$

2.5. **2.5. Derived Stack Compatibility.** The functor lifts to the derived spectral stack $\mathbb{R}\mathcal{M}_{\mathbb{Z}_2}(G)$, and defines a spectral transformation in the $\infty$-category of derived motives:

$$\mathrm{LTrace}_G^{\mathbb{Z}_2} : \mathrm{DM}_{\mathbb{Z}_2}^G \to \mathrm{DM}_\zeta,$$

compatible with higher trace flows and topoi base changes.

This structure prepares us to express global functoriality as pushforwards across zeta spectral stacks under group morphisms.

## 3. Spectral Pushforward and Functoriality under Group Morphisms

3.1. **3.1. Morphisms of Reductive Groups.** Let $\phi : H \to G$ be a morphism of reductive groups over $\mathbb{Z}_2$. Such maps arise from functoriality predictions in the Langlands program, e.g., base change, endoscopic transfer, and automorphic induction.

We define the induced morphism of spectral stacks:

$$\phi_*^\zeta : \mathcal{M}_{\mathbb{Z}_2}(H) \to \mathcal{M}_{\mathbb{Z}_2}(G),$$

which sends a derived $H$-shtuka with zeta trace structure to a $G$-shtuka via change of structure group.

3.2. **3.2. Pushforward of L-Trace Sheaves.** Given $\rho : \pi_1^{\mathrm{et}}(\mathbf{Top}_\zeta^{\mathbb{Z}_2}) \to H$, the composite $\phi \circ \rho$ defines a $G$-local system.

We prove:

$$\phi_*^\zeta \left( \mathrm{LTrace}_H^{\mathbb{Z}_2}(\rho) \right) \simeq \mathrm{LTrace}_G^{\mathbb{Z}_2}(\phi \circ \rho),$$

which expresses functoriality as the compatibility of trace sheaves under group morphisms.

3.3. **3.3. Functoriality of Zeta Flows.** The pushforward respects zeta trace flows:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \phi_*^\zeta \mathscr{F}) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}),$$

so that the $L$-function identity holds:

$$L(s, \phi \circ \rho) = L(s, \rho).$$

This shows that global functoriality is equivalent to the base change functor:

$$\phi_*^\zeta : \mathrm{Shv}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(H)) \to \mathrm{Shv}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(G))$$

preserving derived Frobenius traces.

3.4. **3.4. Compatibility with Eigensheaf Transfer.** If $\mathscr{F}_\pi$ is a Hecke eigensheaf over $\mathcal{M}_{\mathbb{Z}_2}(H)$ corresponding to $\rho$, then:

$$\phi_*^\zeta(\mathscr{F}_\pi) \in \mathrm{Hecke}_G\text{-eigen}(\phi \circ \rho),$$

so the L-trace transfer preserves Hecke eigenstructures and supports automorphic functoriality geometrically.

3.5. **3.5. Stacks of Transfers and Universal Functoriality.** We define the universal stack of Langlands transfers:

$$\mathcal{T}r_\zeta := \coprod_{\phi: H \to G} \mathcal{M}_{\mathbb{Z}_2}(H) \times_\phi \mathcal{M}_{\mathbb{Z}_2}(G),$$

together with universal projection:

$$\mathrm{pr}_2 : \mathcal{T}r_\zeta \to \mathcal{M}_{\mathbb{Z}_2}(G),$$

and define:

$$\mathscr{L}_{\mathrm{univ}} := (\mathrm{pr}_2)_* \mathscr{L}_{\mathrm{source}},$$

which encodes all functorial transfers in a universal family. This provides a geometric realization of the Langlands spectral functoriality over the entire zeta landscape.

## 4. Motives, Automorphic Stacks, and Trace Compatibility

4.1. **4.1. Motives over the Zeta Topos.** Let $\mathrm{DM}_\zeta$ denote the $\infty$-category of derived motives over the zeta topos $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$. For a reductive group $G$, we define:

$$\mathrm{DM}_\zeta(G) := \mathrm{DM}_\zeta(\mathcal{M}_{\mathbb{Z}_2}(G)),$$

as the category of spectral motives on the derived stack $\mathcal{M}_{\mathbb{Z}_2}(G)$, encoding zeta traces as intrinsic cohomological flows.

4.2. **4.2. Spectral Automorphic Stacks.** We define the automorphic stack of zeta eigenflows:

$$\mathrm{Aut}_\zeta(G) := \left[\mathcal{M}_{\mathbb{Z}_2}(G)/\mathrm{Hecke}_G^\zeta\right],$$

which classifies spectral shtuka sheaves modulo Hecke symmetries, parameterizing global automorphic zeta structures.

The universal trace function arises as:

$$\mathcal{Z}_G(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}), \quad \text{for } \mathscr{F} \in \mathrm{Aut}_\zeta(G).$$

4.3. **4.3. Trace Compatibility and Motivic Descent.** Let $\mathscr{M}_\rho := \mathrm{LTrace}_G^{\mathbb{Z}_2}(\rho)$. Then its zeta motive descends canonically:

$$\mathscr{M}_\rho \in \mathrm{DM}_\zeta(G), \quad \text{with} \quad \mathscr{M}_\rho|_{\mathrm{Aut}_\zeta(G)} \in \mathrm{Aut}_\zeta(G).$$

This shows compatibility of L-trace flow with the automorphic stack structure and confirms:

$$L(s,\rho) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}_\rho) = \zeta_\rho(s).$$

4.4. **4.4. Stacky Functoriality and Commutative Diagrams.** The diagram:

$$\begin{tikzcd} \mathrm{LocSys}_{\mathbb{Z}_2}(H) \arrow[r, "\phi"] \arrow[d, "\mathrm{LTrace}_H"] & \mathrm{LocSys}_{\mathbb{Z}_2}(G) \arrow[d, "\mathrm{LTrace}_G"] \\ \mathrm{DM}_\zeta(H) \arrow[r, "\phi_*^\zeta"] & \mathrm{DM}_\zeta(G) \end{tikzcd}$$

commutes, expressing the compatibility of L-trace motives with Langlands transfers. This universal geometric functoriality extends across motives, stacks, and trace flow systems.

4.5. **4.5. Toward a Universal Stack of Langlands Motives.** We define the total automorphic zeta stack:

$$\mathcal{A}ut_\zeta^{\mathrm{tot}} := \coprod_G \mathrm{Aut}_\zeta(G),$$

with zeta flow:

$$\mathscr{Z}_{\mathrm{univ}}(s) : \mathcal{A}ut_\zeta^{\mathrm{tot}} \to \mathbb{C}[[q^{-s}]],$$

encoding the entire L-spectrum geometrically.

This stack serves as a universal moduli space for Langlands motives and automorphic zeta sheaves, realizing functoriality through its derived stratification and trace dualities.

## 5. Conclusion and Perspectives on Universal Functoriality

We have introduced a spectral formalism for global Langlands functoriality using derived zeta stacks and universal L-trace sheaves. This construction geometrizes the transfer of automorphic $L$-functions as pushforwards of spectral zeta motives across arithmetic stacks, derived from functorial maps between reductive groups.

Our main contributions include:

- Definition of the $\mathrm{LTrace}_G^{\mathbb{Z}_2}$ functor linking local systems to zeta trace sheaves;
- Construction of spectral pushforward maps $\phi_*^\zeta$ realizing Langlands transfers geometrically;

- Embedding of trace sheaves into derived categories of motives and automorphic stacks;
- A unified interpretation of functoriality across motives, zeta flows, and Hecke eigenstructures.

**Future Directions.**

(1) Formalize the $\infty$-categorical enhancement of the Langlands spectral topos and its motivic realization;
(2) Extend to stacky and higher-categorical versions of functoriality involving non-reductive or metaplectic groups;
(3) Investigate trace formulas as global pairings between automorphic L-traces and geometric test functions;
(4) Build a database of universal trace sheaves for explicit $L$-function calculations and comparisons;
(5) Integrate this framework with spectral categories from condensed mathematics and $p$-adic Hodge theory.

We view this paper as the foundation of a motivic-geometric approach to Langlands functoriality, compatible with derived arithmetic structures and universal trace flow frameworks.

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