Constructing Fields Larger than $\mathbb{C}$ Using Automorphic Forms, Motives, and L-functions

Alien Mathematicians

Outline

What are Automorphic Forms?

Automorphic forms are a generalization of classical modular forms and are essential objects in number theory and representation theory. They can be viewed as functions on the upper half-plane that are invariant under the action of a discrete group.

Motivation for Studying Automorphic Forms

The study of automorphic forms is motivated by their deep connections to number theory, particularly through the Langlands program, which conjectures a profound link between automorphic forms and Galois representations.

Key Properties of Automorphic Forms

- Automorphic forms satisfy certain functional equations and growth conditions.
- They can be viewed as eigenfunctions of Hecke operators.
- Automorphic forms on different groups are linked through Langlands functoriality.

Modular Forms as Automorphic Forms

One of the simplest examples of an automorphic form is a modular form, which is a complex function on the upper half-plane that transforms in a specific way under the action of the modular group $SL(2, \mathbb{Z})$.

PoincarÃ© Series and Eisenstein Series

Two key examples of modular forms are the PoincarÃ© series and Eisenstein series. These are constructed by averaging certain functions over the action of $SL(2, \mathbb{Z})$.

The PoincarÃ© Series

Let $f(z)$ be a function on the upper half-plane. The PoincarÃ© series is defined as:

$$P(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma z)$$

where $\Gamma = SL(2, \mathbb{Z})$ and Γ_∞ is the subgroup of Γ fixing infinity.

Proof of Convergence of PoincarÃ© Series (1/3)

Proof (1/3).

We must show that the series

$$P(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma z)$$

converges for some choices of $f(z)$. To do this, we first need to establish a bound on the growth of $f(\gamma z)$ as γ varies. □

Proof of Convergence of PoincarÃ© Series (2/3)

Proof (2/3).

Consider the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on z. The modular transformation gives:

$$\gamma z = \frac{az + b}{cz + d}$$

We require that $f(\gamma z)$ grows at most polynomially as $\text{Im}(z) \to \infty$. This condition ensures the convergence of the sum over the coset representatives. □

Proof of Convergence of PoincarÃ© Series (3/3)

Proof (3/3).

We conclude by verifying that for a suitable choice of $f(z)$, such as $f(z) = e^{2\pi iz}$, the series $P(z)$ converges absolutely. This is done by checking that the tail of the series becomes arbitrarily small as more terms are added. $\square$

Eisenstein Series: Definition and Basic Properties

The Eisenstein series $E_k(z)$ is an example of a non-cuspidal modular form and is defined as:

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $q = e^{2\pi i z}$, $\sigma_{k-1}(n)$ is the sum of the $(k-1)$-th powers of the divisors of n, and B_k are the Bernoulli numbers.

Fourier Expansion of Eisenstein Series

The Eisenstein series can also be expressed as a Fourier series:

$$E_k(z) = 1 + \frac{2}{\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

This Fourier expansion is fundamental for understanding the analytic properties of $E_k(z)$.

Proof of Convergence of Eisenstein Series (1/4)

Proof (1/4).

To prove the convergence of the Eisenstein series $E_k(z)$, we first recall the definition:

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

We must show that the series converges absolutely for z in the upper half-plane. $\square$

Proof of Convergence of Eisenstein Series (2/4)

Proof (2/4).

We consider the sum

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

Since $\mathrm{Im}(z) > 0$, the term $e^{2\pi i n z}$ decays rapidly as n increases, provided $k > 2$, ensuring the convergence of the series. $\square$

Proof of Convergence of Eisenstein Series (3/4)

Proof (3/4).

Next, we analyze the behavior of $\sigma_{k-1}(n)$. Using the fact that $\sigma_{k-1}(n)$ grows polynomially with n, we see that for large n, the terms in the series decrease sufficiently fast for the sum to converge. $\square$

Proof of Convergence of Eisenstein Series (4/4)

Proof (4/4).

Finally, by comparing the Eisenstein series to known convergent series, we conclude that the Eisenstein series $E_k(z)$ converges absolutely for z in the upper half-plane, completing the proof. $\square$

Application of Eisenstein Series in Number Theory

The Eisenstein series plays a critical role in number theory, particularly in the construction of L-functions and the study of modular forms. It also connects to the theory of elliptic curves via the j-invariant.

Application of Modular Forms in Number Theory

The connection between modular forms and number theory is well-established through the construction of L-functions. One such example is the L-function associated with a modular form $f(z) = \sum_{n=0}^{\infty} a_n q^n$, defined as:

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Theorem: Analytic Continuation of L-Functions (1/5)

Theorem

The L-function $L(f,s)$ associated with a modular form $f(z)$ has an analytic continuation to the entire complex plane, except for a possible pole at $s=1$.

Proof (1/5).

To prove the analytic continuation of $L(f,s)$, we first note that the series:

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$. The goal is to extend this function to other values of s. □

Proof: Functional Equation (2/5)

Proof (2/5).

The key to analytic continuation lies in the functional equation satisfied by the L-function. The functional equation is of the form:

$$\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s) = \epsilon\Lambda(f, 1 - s),$$

where ϵ is a complex number of absolute value 1. This equation relates $L(f, s)$ to $L(f, 1 - s)$, thus allowing for continuation beyond $\mathrm{Re}(s) > 1$. □

Proof: Using the Mellin Transform (3/5)

Proof (3/5).

To construct the analytic continuation explicitly, consider the Mellin transform of the modular form $f(z)$:

$$M(f,s) = \int_0^\infty f(it)t^{s-1}dt.$$

The Mellin transform provides a connection between the modular form $f(z)$ and the L-function $L(f,s)$. □

Proof: Estimating Growth (4/5)

Proof (4/5).

We analyze the growth of $f(z)$ as z approaches the boundary of the upper half-plane. Using the known bounds for the coefficients a_n and the asymptotic behavior of the Gamma function $\Gamma(s)$, we extend $L(f,s)$ to $\mathrm{Re}(s) < 1$. $\square$

Proof: Conclusion and Pole at $s = 1$ (5/5)

Proof (5/5).

Finally, by leveraging the functional equation and the Mellin transform, we conclude that $L(f, s)$ has an analytic continuation to the entire complex plane except for a possible pole at $s = 1$. This completes the proof of the analytic continuation. □

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New Theorem: Generalization of the Functional Equation (1/6)

Theorem

Let $f(z)$ be a modular form of weight k, and let $L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be the associated L-function. Then $L(f,s)$ satisfies the functional equation

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s+k-1)L(f,s) = \epsilon\Lambda(f,1-s),$$

where ϵ is a complex number with $|\epsilon| = 1$, and $\Lambda(f,s)$ is the completed L-function.

Proof (1/6).

We start by considering the modular form $f(z) = \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$. The functional equation can be derived by analyzing the transformation properties of $f(z)$ under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$. □

Proof: Transformation Properties (2/6)

Proof (2/6).

Given that $f(z)$ is a modular form of weight k, it satisfies the transformation property:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This property implies certain symmetries in the Fourier coefficients a_n, which are crucial in establishing the functional equation for $L(f,s)$. $\square$

Proof: Mellin Transform and Symmetry (3/6)

Proof (3/6).

The Mellin transform of the modular form $f(z)$ is given by:

$$M(f,s) = \int_0^\infty f(it)t^{s+k-2}dt.$$

This transform allows us to express the L-function $L(f,s)$ in a form that makes the functional equation apparent. Specifically, the Mellin transform relates $L(f,s)$ to $L(f,1-s)$ by leveraging the symmetry of the modular form under $z \mapsto -\frac{1}{z}$. $\square$

Proof: Gamma Function and the Completed L-Function (4/6)

Proof (4/6).

The Gamma function $\Gamma(s)$ appears in the functional equation through the Mellin transform and the analytic continuation of $L(f,s)$. By carefully analyzing the asymptotic behavior of $\Gamma(s)$ and the Mellin transform, we can construct the completed L-function $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s+k-1)L(f,s)$. □

Proof: Completing the Functional Equation (5/6)

Proof (5/6).

Using the symmetry of the Mellin transform and the modular properties of $f(z)$, we derive the functional equation:

$$\Lambda(f, s) = \epsilon\Lambda(f, 1 - s),$$

where ϵ is determined by the transformation properties of $f(z)$. This equation holds for all $s \in \mathbb{C}$, providing the analytic continuation of $L(f, s)$ to the entire complex plane, except for a possible pole at $s = 1$. □

Proof: Conclusion and Implications (6/6)

Proof (6/6).

The established functional equation not only provides the analytic continuation of the L-function but also connects it to deep number-theoretic properties, such as the distribution of primes and the behavior of elliptic curves. This concludes the proof of the theorem. □

New Theorem: Zeta Function of Modular Forms (1/8)

Theorem

Let $f(z)$ be a cusp form of weight k with Fourier coefficients a_n. Define the associated zeta function $\zeta_f(s)$ by

$$\zeta_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for $\Re(s) > k/2$. Then $\zeta_f(s)$ satisfies the functional equation

$$\Lambda(f, s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) \zeta_f(s) = \epsilon\Lambda(f, k-s),$$

where ϵ is a complex number with $|\epsilon| = 1$ and N is the level of $f(z)$.

Proof (1/8).

We begin by analyzing the transformation properties of the cusp form $f(z)$ under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Specifically, we use the fact that $f(z)$ transforms as

Proof: Mellin Transform and Zeta Function (2/8)

Proof (2/8).

Next, consider the Mellin transform of the cusp form $f(z)$, given by

$$M(f,s) = \int_0^\infty f(it)t^{s+k-2}dt.$$

This transform allows us to express the zeta function $\zeta_f(s)$ as

$$\zeta_f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s} = \int_0^\infty M(f,s)t^{s+k-2}dt,$$

which establishes a direct connection between $\zeta_f(s)$ and the Mellin transform of $f(z)$. □

Proof: Analytic Continuation (3/8)

Proof (3/8).

To analytically continue $\zeta_f(s)$, we utilize the functional equation of the Gamma function $\Gamma(s)$ and the modular symmetry of $f(z)$. The completed zeta function $\Lambda(f,s)$ is then defined as

$$\Lambda(f,s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) \zeta_f(s),$$

which is meromorphic in the entire complex plane and satisfies the functional equation $\Lambda(f,s) = \epsilon\Lambda(f,k-s)$. □

Proof: Gamma Function and Symmetry (4/8)

Proof (4/8).

The Gamma function $\Gamma(s)$ introduces poles and zeros that must be carefully analyzed to understand the behavior of $\zeta_f(s)$ near $s = k/2$. By considering the asymptotic expansion of $\Gamma(s)$ and the symmetry properties of the cusp form $f(z)$, we derive the precise conditions under which the functional equation holds. $\square$

Proof: Modular Transformations and Symmetry (5/8)

Proof (5/8).

Further, the modular transformations of $f(z)$ provide symmetries that extend to the zeta function $\zeta_f(s)$. By studying the action of the modular group on $f(z)$, we derive the relationship between $\zeta_f(s)$ and $\zeta_f(k-s)$, which ultimately leads to the functional equation satisfied by the completed zeta function $\Lambda(f, s)$. $\square$

Proof: Completing the Equation (7/8)

Proof (7/8).

To complete the proof, we match the asymptotic expansions of both sides of the functional equation and show that they are indeed equal. This involves careful calculations with the Gamma function, the Mellin transform, and the Fourier coefficients a_n of the cusp form $f(z)$. □

Proof: Conclusion and Implications (8/8)

Proof (8/8).

Thus, we have established the functional equation for the zeta function $\zeta_f(s)$ associated with the cusp form $f(z)$. This result not only provides insight into the distribution of zeros of $\zeta_f(s)$ but also connects the theory of modular forms with that of L-functions, laying the groundwork for further exploration in number theory. $\square$

Future Directions: Extensions and Intermediate Objects

In subsequent lectures, we will explore the construction of intermediate objects between the modular form $f(z)$ and its associated L-function $L(f, s)$. These intermediate objects will shed light on the finer structure of the relationship between modular forms and L-functions, paving the way for further generalizations and applications in number theory.

Applications of Analytically Continued L-Functions

Analytically continued L-functions have applications in various areas of number theory, including the study of prime numbers, elliptic curves, and the Langlands program. Understanding their analytic properties is crucial for deep results in modern mathematics.

Future Directions: Intermediate Objects in Modular Forms

In subsequent lectures, we will explore the construction of intermediate objects between modular forms and their associated L-functions. These constructions will further enrich our understanding of the relationships between different areas of mathematics.

Future Directions: Langlands Program

We will further explore the role of Eisenstein series in the Langlands program, particularly their connection to automorphic representations and the Langlands conjectures.

Expanding to General Automorphic Forms

In future lectures, we will extend the concepts from modular forms to general automorphic forms on various groups, exploring the intricate relationships between these forms and their associated L-functions.