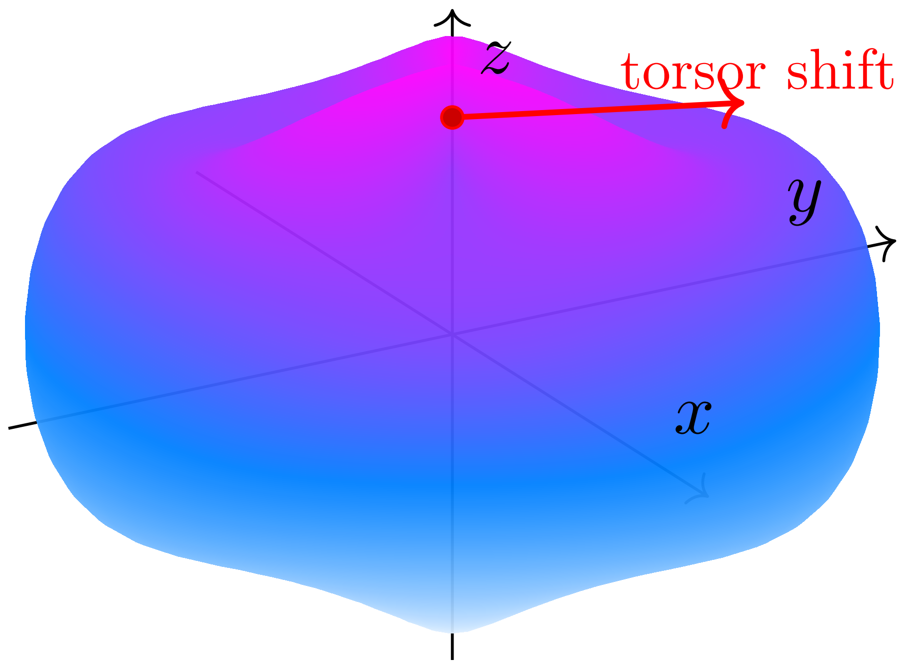
Multiplicoid Geometry



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MULTIPLICOID GEOMETRY: GENERALIZING PERFECTOID STRUCTURES BEYOND ADDITIVITY

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ABSTRACT. We introduce *multiplicoid spaces* as a natural generalization of perfectoid spaces, extending the notion of infinite-level congruence towers from the additive to the multiplicative regime. This new framework establishes a coherent structure for studying weight–monodromy phenomena, torsor filtrations, and motivic realization over multiplicative congruence bases, without invoking any dyadic assumptions. We outline the foundational algebraic, geometric, and cohomological consequences of such spaces, and propose a hierarchy of generalized period geometries built upon multiplicative growth.

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1. Introduction and Motivation

The theory of perfectoid spaces, introduced by Peter Scholze, has revolutionized p-adic Hodge theory and arithmetic geometry by enabling systematic passage between characteristic zero and characteristic p in towers of ramified covers. These structures crucially rely on *additive congruence filtrations*, built from successive π-adic divisions in the ring of integers of p-adic fields.

In this paper, we initiate a program to transcend the additive and linear constraints of perfectoid geometry by introducing a new class of geometric spaces we call **multiplicoid spaces**. These spaces are organized around *multiplicative congruence filtrations*, in which the fundamental growth structure arises from repeated multiplication rather than addition or valuation.

Our primary goal is to generalize the perfectoid philosophy—infinite-level congruence descent, tilting equivalence, and cohomological control—to settings governed by multiplicative laws of growth. Instead of relying on valuation-theoretic completeness, we build our theory upon *congruence towers of multiplicative depth*, such as:

$$A \longrightarrow A/(\times 2)^n\mathbb{Z}, \quad \text{or more generally,} \quad A/(\prod_{i=1}^{n} m_i)\mathbb{Z}$$

for increasing multiplicative sequences $\{m_i\}$. Such systems exhibit structural coherence without invoking any p-adic valuation, dyadic base, or Archimedean comparison.

This generalization is motivated by several overlapping paradigms:

- In motivic cohomology and regulator theory, multiplicative polylogarithmic towers appear naturally, particularly in the study of K-theory and special values of L-functions.
- In arithmetic dynamics and automorphic forms, multiplicative scaling symmetries (e.g., Hecke operators) often dominate the spectral geometry of moduli stacks.

- From a categorical perspective, multiplicative filtrations better encode the tensorial nature of objects such as line bundles, torsors, and gerbes under scaling equivalence classes.
- Finally, our construction serves as a launching point for further trans-recursive generalizations to exponential, hyper-exponential, and higher Knuth-arrow based geometries, which will be developed in Volumes I and II.

The theory of multiplicoid spaces presented here is fully independent of dyadic or p-adic assumptions. Nonetheless, in the sequel to this paper, we shall also develop a dyadic-supported version that embeds these multiplicative towers within the broader framework of dyadic period geometry and ε-stratified motivic stacks.

The multiplicoid framework is meant not merely as an extension of perfectoid techniques, but as the first member of a new hierarchy of *growth-structured geometries*, each indexed by a class of recursion-theoretic or arithmetic functions (such as $n \mapsto \exp(n)$, $n \mapsto a \uparrow^k n$, etc.), reflecting the increasing depth of arithmetic and cohomological complexity.

Meta-Philosophical Interlude. The space-theoretic reality explored in this work is grounded not in valuation or measure, but in *growth stratification*—a logic of levels, thresholds, and actions that proceeds from the multiplicative intuition of duplication, scaling, and factorization.

The traditional view of geometry, rooted in the continuity of real or p-adic numbers, assumes that space is essentially *infinitesimally approximable*, and that coherence emerges from local-to-global constructions under an additive, topological microscope.

By contrast, multiplicoid geometry opens a conceptual shift: space becomes a carrier of *stratified multiplicities*, where the hierarchy of structure arises from multiplicative descent and recursive actions, rather than local neighborhoods. Instead of approximating space via infinitesimals, we organize it via modular multiplicative resolutions, capturing the dynamics of systems that do not admit linear flattening or Archimedean compression.

This philosophical shift invites a new type of ontological commitment: *spaces indexed not by points, but by the growth behaviors of their automorphisms.* Under this view, a "multiplicoid space" is not a set equipped with structure, but a stratified arena in which multiplicative towers act as geometric beings—entities whose coherence is internal to the growth operations themselves.

The foundational question we pursue is therefore not "what is a space?", but rather: *what class of growth operations generate stable, recursive, cohomologically meaningful geometries?* The answer begins with multiplicoid towers.

2. Foundations of Multiplicoid Towers

2.1. **Multiplicoid Base Rings and Congruence Structures.** We begin by defining the fundamental algebraic objects over which multiplicoid spaces are constructed. These are not equipped with additive topologies or valuations, but instead with congruence relations governed by multiplicative growth.

Definition 2.1 (Multiplicative Congruence System)**.** *Let A be a commutative ring. A* multiplicative congruence system *on A is a descending family of ideals $\{I_n\}_{n\geq 0}$ satisfying:*

(1) $I_0 = A$, and $I_n \supseteq I_{n+1}$ for all n;
(2) Each I_n is of the form $I_n = (f_n)$, where f_n is a product of multiplicative generators:

$$f_n := \prod_{i=1}^{n} m_i, \quad \textit{with } m_i \in A \textit{ non-units};$$

(3) The sequence $\{m_i\}$ satisfies $m_i \mid m_{i+1}$, i.e., the congruence structure is multiplicatively nested.

The simplest example is given by $A = \mathbb{Z}$ with $m_i = 2$, yielding the tower $I_n = (2^n)$. More generally, one can consider geometric or arithmetic sequences such as $m_i = q^i$ for fixed $q \in \mathbb{N}$, or factorial growth $m_i = i!$.

Definition 2.2 (Multiplicoid Base Ring)**.** *A ring A equipped with a multiplicative congruence system $\{I_n\}$ is called a* multiplicoid base ring*, denoted $A_\times$. We define the inverse system of quotient rings:*

$$A_\times^\infty := \varprojlim_n A/I_n$$

and call it the multiplicoid completion *of A.*

Example 2.3. Let $A = \mathbb{Z}$, and $I_n = (2^n)$. Then $A_\times^\infty = \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$ is the 2-adic completion $\mathbb{Z}_2$—but viewed not as a 2-adic valuation ring, but as a multiplicoid base with respect to the congruence tower.

This perspective frees us from valuation-theoretic dependence and allows multiplicoid constructions over arbitrary bases.

2.2. **Multiplicoid Filtrations and Sheaves.** We now define filtrations compatible with multiplicoid base structure.

Definition 2.4 (Multiplicoid Filtration)**.** *Let M be an A-module over a multiplicoid base ring $A_\times$. A* multiplicoid filtration *on M is a descending chain of submodules $\{F^nM\}_{n\geq 0}$ such that*

$$F^nM := \ker(M \to M/I_nM)$$

for each n, with respect to the multiplicative congruence system $\{I_n\}$ on A.

These filtrations are not indexed by integers additively, but by multiplicative depth. One can visualize them as forming a tower:

$$M = F^0M \supseteq F^1M \supseteq F^2M \supseteq \cdots$$

where F^nM consists of sections of M congruent to 0 modulo $(m_1 \cdots m_n)$.

Definition 2.5 (Multiplicoid Sheaf)**.** *Let $X = \mathrm{Spec}(A_\times)$ be the spectrum of a multiplicoid base ring. A* multiplicoid sheaf *over X is a sheaf of $A_\times$-modules equipped with a compatible multiplicoid filtration as above.*

We denote the category of multiplicoid sheaves over X by $\mathbf{Sh}^\times(X)$.

2.3. **Definition of Multiplicoid Spaces.**

Definition 2.6 (Multiplicoid Space)**.** *A* multiplicoid space *is a locally ringed space $(X, \mathcal{O}_X)$ such that:*

(1) For every open affine $U \subseteq X$, $\mathcal{O}_X(U)$ is a multiplicoid base ring;
(2) The structure sheaf $\mathcal{O}_X$ admits a multiplicoid filtration compatible with the congruence system on each stalk;
(3) The gluing maps preserve multiplicoid congruence towers.

We denote the category of such spaces by **Mult**.

2.4. **Basic Properties and Examples.**

Example 2.7 (Multiplicative Completion of $\mathbb{A}^1$). Let $A = \mathbb{Z}[x]$ and define $I_n = ((x+1)(x+2)\cdots(x+n))$. The multiplicoid completion $A_\times^\infty$ defines a formal neighborhood of ∞ under multiplicative translation.

Proposition 2.8. *Let $A_\times$ be a multiplicoid base ring, and let $X = \mathrm{Spec}(A_\times)$. Then:*

(1) $\mathbf{Sh}^\times(X)$ is an abelian category;
(2) The filtration functor $F^n(-)$ is exact on flat modules;
*(3) **Mult** admits fiber products and a natural site structure.*

Sketch. The filtration conditions are inherited from the projective system defining $A_\times^\infty$; exactness follows from the compatible inverse limits; site structure is induced via congruence base changes. □

Ontological Interpretation of Multiplicoid Spaces. From an ontological perspective, the nature of a *space* depends not merely on its points, topology, or charts, but on the generative logic by which its structure is layered and made perceptible. In classical schemes, this generative logic is additive, local, and infinitesimal. In perfectoid spaces, it is valuation-theoretic, descending from additive norm filtration indexed by powers of a uniformizer.

Multiplicoid spaces introduce a shift: they are not defined through infinitesimal approximation, but rather through multiplicative descent. In such spaces, the fundamental organizing principle is not "closeness" in a metric or valuation sense, but "depth" of congruence under multiplicative growth. That is:

Proximity is replaced by congruence collapse under iteration of multiplication.

This suggests that a multiplicoid space is not a space in which one moves linearly, but a space in which one *folds, scales, and stratifies* via multiplication.

Each multiplicoid filtration level F^n should be viewed not merely as a layer of vanishing, but as a *modal stage* of existence—an ontological region accessible only after descending through n multiplicative thresholds. These thresholds are not numeric levels; they are stages of being, governed by *operations* rather than *coordinates*.

In this light, a multiplicoid space is an *operation-indexed object*: a structure stratified by the complexity of the multiplicative acts required to resolve its local or global data. Its sheaves are not merely functions or sections, but *growth-conditioned entities*—objects whose coherence emerges through their transformation laws under iterated scaling.

This opens the door to a broader project: to define a new class of geometries whose fundamental units are not points, but *processes*—geometries whose categories are enriched not over sets, but over sequences of structural operations. Multiplicoid spaces are thus the first layer in an ontological ladder of growth-indexed spaces.

Diagrammatic Motivation: Growth-Based Geometric Hierarchy. To orient the reader, we now present a schematic diagram showing how multiplicoid geometry emerges as the first extension of perfectoid space theory in the direction of operation-based stratification.

$$
\begin{tikzcd}
\textbf{Classical Geometry} \arrow[r, "Additive Completion"] \arrow[d, dashed, "Functor of Points"'] & \textbf{Perfectoid Geometry} \arrow[r, "Growth Shift: Multiplicative"] \arrow[d, dashed, "Tilting/Prismatic"] & \textbf{Multiplicoid Geometry} \arrow[r, "\exp(-)"] & \textbf{Exponentoid Geometry} \arrow[r, "\uparrow^n"] & \textbf{Knuthoid Geometry} \arrow[r, "Ontology"] & \textbf{Ontoid Geometry} \\
\text{Schemes over } \mathbb{Z} \arrow[r] & \text{Adic / Diamond Spaces} \arrow[r] & \text{Operation-Based Towers}
\end{tikzcd}
$$

This diagram encapsulates the direction of generalization:

- We begin with schemes and classical topology;
- Perfectoid geometry introduces additive infinite-level towers;
- Multiplicoid geometry replaces additive depth with multiplicative congruence depth;
- Further operations—exponential, iterated exponentiation, or Knuth-arrow growth—define new spatial hierarchies;

- Ultimately, geometry becomes indexed not by numbers, but by *growth functions*, and stratified by *ontological processes*.

In this progression, multiplicoid spaces are the critical *first transcendence step* beyond the perfectoid world, retaining coherence while freeing the framework from valuation, base fields, and additive metrics. They offer a geometric arena for the exploration of motivic congruence, torsor dynamics, and structural recursion under non-linear growth regimes.

This motivates the next section: to construct the cohomological backbone of multiplicoid geometry, and to interpret these spaces through their period sheaves and ε-structured realizations.

3. Cohomology and Period Structures of Multiplicoid Spaces

3.1. **Multiplicoid Period Rings and Sheaves.**

Just as perfectoid geometry gives rise to de Rham and crystalline period rings via inverse systems over additive uniformizers, multiplicoid geometry naturally generates new types of period rings based on multiplicative congruence towers.

Definition 3.1 (Multiplicoid Period Ring)**.** *Let $A_\times$ be a multiplicoid base ring with multiplicative congruence system $\{I_n\}$. Define the* multiplicoid period ring *as the inverse limit*

$$B_{mult,dR}(A) := \varprojlim_n A/I_n \otimes_{\mathbb{Z}} \mathbb{Q},$$

equipped with a canonical multiplicoid filtration

$$F^n B_{mult,dR}(A) := \ker\left(B_{mult,dR}(A) \to A/I_n \otimes \mathbb{Q}\right).$$

This filtration is multiplicative in nature and grows not linearly but exponentially in ideal size. One may view $B_{\mathrm{mult},dR}$ as the ambient period ring capturing multiplicative-deformation classes of cohomology.

Definition 3.2 (Multiplicoid Period Sheaf)**.** *Let X be a multiplicoid space. A* multiplicoid period sheaf *over X is a sheaf of $B_{mult,dR}$-modules equipped with:*

- *A multiplicoid filtration $F^\bullet$ as above;*
- *A flat connection compatible with the multiplicative stratification;*
- *Transition morphisms respecting torsor action under $\times n$ scaling maps.*

We denote the category of such sheaves by $\mathbf{Sh}^{\nabla}_{\mathrm{mult}}(X)$.

3.2. **Torsors and ε-Filtration Towers.**

Just as perfectoid cohomology admits interpretation via torsors over Galois or Frobenius groups, multiplicoid cohomology admits *multiplicative torsors*, which encode the levels of filtration via multiplicative action.

Definition 3.3 (Multiplicoid Torsor)**.** *Let $G = \mathbb{Z}_{>0}$ act on a space X via multiplicative scaling $x \mapsto m \cdot x$. A* multiplicoid torsor *is a principal G-bundle $\mathcal{T} \to X$ such that:*

- *For each n, the level-n fiber $\mathcal{T}_n$ corresponds to $F^n\mathcal{O}_X$;*
- *The action satisfies scaling compatibility: $g \cdot \mathcal{T}_n \simeq \mathcal{T}_{gn}$.*

Definition 3.4 (ε-Filtration Tower)**.** *Let E be a sheaf over a multiplicoid space X. An* ε-filtration tower *is a sequence of subobjects $\{F^{\varepsilon^n}E\}$ satisfying:*

$$F^{\varepsilon^n}E := \ker\left(E \to E/I_nE\right), \quad \textit{with } I_n = (\prod_{i=1}^{n} m_i).$$

Here, ε^n acts as an abstract growth operator, encoding congruence depth stratification.

This ε-filtration system gives rise to new types of cohomology.

3.3. **Multiplicoid Cohomology.** We now define the cohomology theory intrinsic to multiplicoid geometry.

Definition 3.5 (Multiplicoid Cohomology)**.** *Let X be a multiplicoid space, and $\mathcal{F}$ a multiplicoid sheaf. The* multiplicoid cohomology groups *are defined as:*

$$H^i_{mult}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathcal{F}/F^n\mathcal{F}),$$

where $F^n\mathcal{F}$ denotes the n-th level of the multiplicoid filtration.

Remark 3.6. This cohomology captures the stabilization of local data under multiplicative congruence descent. It mirrors syntomic and crystalline cohomology in the additive world, but with fundamentally different growth behavior and torsor action.

3.4. **Regulators and Period Morphisms.**

Definition 3.7 (Multiplicoid Regulator Map)**.** *Let K be a field and X/K a smooth multiplicoid space. Then the* multiplicoid regulator map *is a natural transformation:*

$$r_{mult} : K_n(X) \longrightarrow H^n_{mult}(X, \mathbb{Q}(n)),$$

constructed via classes of multiplicoid ε-filtration torsors.

This regulator behaves compatibly with multiplicative period growth and can be used to study special values of multiplicative L-functions and motivic ε-heights.

3.5. **Examples and Future Directions.**

Example 3.8. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$ with multiplicative tower $I_n = ((x+1)\cdots(x+n))$. The sheaf $\mathcal{F} = \mathcal{O}_X$ admits a filtration via vanishing modulo multiplicative products. The associated cohomology $H^1_{\mathrm{mult}}(X, \mathcal{F})$ detects relations in polynomial growth towers.

Example 3.9. Let X be the moduli stack of multiplicatively-scaled elliptic curves. The period sheaf over X admits ε-filtration indexed by the multiplicative conductor, and the cohomology H^2_{mult} captures automorphic growth.

These structures prepare the foundation for developing *Exponentoid* and *Knuthoid* spaces in subsequent volumes, where the operators $x \mapsto x^n$, $x \mapsto \exp^n(x)$, and $x \mapsto a \uparrow^k x$ serve as generators of stratified geometry.

4. Tilting and Universal Structures

4.1. **Toward a Tilting Theory for Multiplicoid Spaces.** Tilting equivalences in perfectoid geometry—via characteristic p–0 correspondences—enable deep cohomological comparisons and facilitate the study of p-adic period spaces through characteristic p models. In the multiplicoid setting, we instead seek to define a new form of "tilting" not across characteristics, but across *multiplicative congruence towers.*

Definition 4.1 (Multiplicoid Tilting System). *Let $A_\times$ be a multiplicoid base ring. A* multiplicoid tilting system *consists of:*

- *A projective system $\{A_n\}_{n\geq 0}$, where $A_n := A/I_n$ with $I_n = (m_1 \cdots m_n)$;*
- *A compatible family of multiplicative lifts $\varphi_n : A_{n+1} \to A_n$ satisfying:*
$$\varphi_n(x \bmod I_{n+1}) = x^k \bmod I_n$$
for some fixed $k \in \mathbb{Z}_{>0}$ (interpreted as the tilting exponent);
- *A limit algebra $A_\times^\flat := \varprojlim_{\varphi_n} A_n$, called the* multiplicoid tilt *of $A_\times$.*

This multiplicoid tilt captures a recursively structured version of the base ring, and enables the transport of multiplicoid sheaves and cohomology across congruence scales.

Remark 4.2. Unlike perfectoid tilting, which involves passage between mixed and equal characteristics, multiplicoid tilting is indexed by operation classes (e.g. $\times n$, $\exp(n)$), and produces a self-similar structure internal to the multiplicoid filtration.

4.2. **Torsor Functoriality and Universal ε-Gerbes.** Given the tower of torsors $\{\mathcal{T}_n\}$ over a multiplicoid space X, we now organize these into a universal stack.

Definition 4.3 (Universal Multiplicoid Torsor Stack). *Define the stack $\mathbb{T}^{[\times]}$ over the site of multiplicoid spaces as the fibered category whose objects over X are sequences of torsors $\mathcal{T}_n \to X$ satisfying:*

1. *For each n, $\mathcal{T}_n$ is a principal $(\mathbb{Z}/I_n)$-torsor;*
2. *The action maps are compatible: $\mathcal{T}_{n+1} \to \mathcal{T}_n$ respect multiplicative scaling;*
3. *There exists a universal object $\mathcal{T}_\infty := \varprojlim \mathcal{T}_n$.*

This stack represents the moduli of multiplicoid descent structures, and governs the ε-filtration levels across different multiplicoid cohomological theories.

Definition 4.4 (Universal ε-Gerbe)**.** *Let X be a multiplicoid space. A* universal ε-gerbe *over X is a gerbe $\mathcal{G}_\varepsilon$ banded by a profinite abelian group G such that:*

- *G acts on each $\mathcal{T}_n$ through congruence level mod I_n;*
- *There exists a morphism of stacks $\mathbb{T}^{[\times]} \to B\mathcal{G}_\varepsilon$;*
- *The cohomology $H^2_{mult}(X, \mathcal{G}_\varepsilon)$ classifies filtered multiplicoid torsor deformations.*

This ε-gerbe plays a role analogous to B^+_{dR}-torsors in perfectoid Hodge theory, encoding obstruction classes and period morphisms under multiplicative scaling.

4.3. **Universal Period Towers.**

Definition 4.5 (Universal Multiplicoid Period Tower)**.** *Let X be a multiplicoid space. The* universal multiplicoid period tower *is the projective system of period rings:*

$$\mathrm{Per}^{\infty}_{mult}(X) := \left\{ B^{(n)}_{mult,dR}(X) := \mathcal{O}_X / I_n \otimes \mathbb{Q} \right\}_{n \in \mathbb{Z}_{\geq 0}}.$$

This tower comes equipped with:

- *Morphisms $B^{(n+1)} \to B^{(n)}$ induced by congruence reduction;*
- *Filtration structure $F^n := \ker(B^{(\infty)} \to B^{(n)})$;*
- *Period realizations via ε-torsors: $\mathrm{Per}^{\infty}_{mult} \to \mathbb{T}^{[\times]}$.*

4.4. **Functoriality and Universal Realization Functor.**

Theorem 4.6 (Universal Period Realization)**.** *Let* **Mult** *be the category of multiplicoid spaces, and let* **MultCoh** *be the associated category of filtered multiplicoid sheaves. Then there exists a realization functor*

$$\mathscr{R}_{mult} : \mathbf{MultCoh} \longrightarrow \mathbb{T}^{[\times]} \times_{\mathbb{Z}} \mathrm{Per}^{\infty}_{mult}$$

that respects filtrations, torsor morphisms, and cohomological classes.

Sketch. The functor is constructed by associating to each filtered sheaf its image in the torsor tower, then evaluating sections over the period tower. The transition maps ensure compatibility. □

5. Motivic Realization and Applications

5.1. **Multiplicoid Motives and Realization Functors.** The theory of motives aims to unify cohomological realizations of algebraic varieties across various contexts—de Rham, étale, crystalline, syntomic, etc.—via a universal motivic object. In the context of multiplicoid geometry, we propose a new type of realization: a *multiplicoid motivic realization*, reflecting congruence-depth stratification rather than valuation-based descent.

Definition 5.1 (Multiplicoid Motive)**.** *Let X be a smooth multiplicoid space. A* multiplicoid motive $M^{[\times]}(X)$ *is an object in a tensor-triangulated category $\mathcal{M}_{mult}$ equipped with:*

(1) A multiplicoid filtration F^nM induced by congruence collapse modulo I_n;
(2) Realization functors to cohomology:

$$real_{dR}^{mult} : M^{[\times]}(X) \to H^\bullet_{mult}(X, B_{mult,dR})$$

and similarly to $\mathbb{T}^{[\times]}$ and $\mathrm{Per}^\infty_{mult}$;
(3) Compatibility with the regulator map and torsor-tower action.

We conjecture the existence of a universal multiplicoid motivic category $\mathcal{M}_{\mathrm{mult}}$ extending Voevodsky's $\mathbf{DM}_{\mathrm{gm}}$ through congruence-indexed ε-stratifications.

5.2. **Multiplicoid Regulators and ε-Pairings.** We revisit the multiplicoid regulator map:

$$r_{\mathrm{mult}} : K_n(X) \longrightarrow H^n_{\mathrm{mult}}(X, \mathbb{Q}(n)),$$

constructed through the period realization of multiplicoid motives. This map respects torsor stratifications and may be thought of as a generalized "multiplicative Borel regulator".

Definition 5.2 (ε-Pairing)**.** *Let $M^{[\times]}(X)$ be a multiplicoid motive with torsor tower $\mathbb{T}^{[\times]}$. The* ε-pairing *is a map:*

$$\langle -, - \rangle_{\varepsilon^n} : F^{\varepsilon^n} M \otimes F^{\varepsilon^n} M^\vee \longrightarrow \mathbb{Q}$$

satisfying compatibility with the period filtration and collapsing as $n \to \infty$.

This pairing captures motivic height-type information encoded in ε-stratified congruence levels.

5.3. **Special Values and Multiplicoid L-Functions.** Inspired by Beilinson's conjectures and higher polylogarithmic regulators, we define a new class of arithmetic functions that interpolate multiplicoid cohomological invariants.

Definition 5.3 (Multiplicoid L-Function)**.** *Let $M^{[\times]}$ be a multiplicoid motive over X. Its associated multiplicoid L-function is:*

$$L_{mult}(M, s) := \prod_{n \geq 0} \det \left(1 - (\mathrm{Frob}_n \cdot p^{-s}) \mid F^n M\right)^{-1},$$

where Frob_n *denotes the multiplicative action on the n-th filtration layer.*

This function reflects the interaction between torsor depth and period realizability. In the dyadic-supported version of this theory, the primes p may be replaced by powers of 2, leading to logarithmic or polylogarithmic interpolation.

5.4. **Applications to Polylogarithmic Stacks and Automorphic Periods.** Multiplicoid geometry naturally applies to polylogarithmic motives, especially those appearing in higher K-theory and modular symbols.

Example 5.4. Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and consider the multiplicoid motive generated by iterated logarithms:

$$\log^n(x) \in F^{\varepsilon^n} M^{[\times]}(X),$$

whose periods correspond to multiple zeta values filtered by congruence growth. The associated multiplicoid L-function reflects the asymptotic depth of polylogarithmic torsors.

Example 5.5. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties. Its cohomology admits a multiplicoid stratification via multiplicative conductor. The torsor towers classify Hecke eigenforms under multiplicative growth. Multiplicoid regulators provide new invariants beyond classical Eichler–Shimura theory.

5.5. **Further Directions.** The multiplicoid realization functor admits natural extensions:

- To exponentoid geometry: motives with exponential depth filtrations;
- To knuthoid geometry: with towers indexed by $\uparrow^n$ operators;
- To epsilon-gerbe stacks: defining arithmetic heights in terms of ε-torsor classes;
- To motivic sheaf-to-space equivalences: realizing ε-motives as functors into ontological geometries.

6. Future Directions and Foundational Implications

6.1. **From Multiplicoid to Exponentoid Structures.** The development of multiplicoid geometry has revealed a new principle: that geometric structure need not arise from topology, valuation, or infinitesimal analysis alone—but may instead be generated from *congruence growth under structural operations.*

This realization opens the door to further generalizations. If multiplicoid spaces correspond to congruence towers indexed by multiplicative depth $n \mapsto \prod_{i=1}^n m_i$, then it is natural to define *exponentoid spaces*, based on:

$$n \mapsto \exp^k(n), \quad \text{where } \exp^k(n) = \underbrace{\exp(\cdots \exp(n) \cdots)}_{k \text{ times}}.$$

These geometries stratify space not merely by uniform congruence levels, but by rapidly expanding depth layers governed by recursion.

6.2. **Knuthoid Geometry and Hyperstratified Arithmetic.** Pushing further, one arrives at *Knuthoid geometry*, indexed by hyper-operations such as $a \uparrow^k b$. In this regime, space is no longer an object with points, but a categorically indexed web of operation-driven torsor classes.

Filtrations become hyper-filtrations, towers become hyper-towers, and periods become functions of trans-recursive interaction. The categorical complexity of such spaces may surpass current model-theoretic formalizations, demanding a new logical foundation for geometry itself.

6.3. **Ontoid Geometry and the Logic of Space.** Ultimately, the highest layer of this progression is *ontoid geometry*—a vision of space constructed not from numbers, but from *operations as first-order entities*. In this world, the "structure sheaf" $\mathcal{O}_{\mathrm{Ont}}$ is not a sheaf of functions, but a stratified class of transformations indexed by the growth laws governing the space.

Under this paradigm:

- Points disappear, replaced by procedural strata;
- Filtrations become logical levels of realization;
- Cohomology becomes the study of coherence under transfinite recursion;
- Geometry becomes not a container of objects, but a process in itself.

6.4. **Foundational Summary.** The theory initiated in this volume stands at the beginning of a new program:

Hyperstratified Geometry: A Recursion-Theoretic Foundation of Arithmetic Space

Its core tenets are:

(1) Filtration structures are derived from recursive growth, not local topology;
(2) Torsors are the primary geometric entities, organizing operation-based descent;
(3) Periods, regulators, and cohomology reflect structural depth rather than metric magnitude;
(4) The geometry of the future is a stratified ontology of procedures—not merely an extension of classical form.

Outlook. Volumes I (Exponentoid Geometry) and II (Hyper-Filtration Theory) will pursue this progression in full, while Volumes III–V develop applications to motivic arithmetic, logical geometry, and the space-theoretic ontology of mathematics.

The ultimate goal is a theory of space which reflects not what exists—but what emerges through the logic of generation itself.

7. Comparison with Additive and Valuation-Based Geometries

7.1. **From Valuations to Congruence Filtrations.** In classical arithmetic geometry, structures are often built atop additive or valuation-based topologies. Perfectoid geometry, for instance, depends critically on the existence of a non-archimedean valuation $|\cdot|$ and a compatible Frobenius-lifted tower.

In multiplicoid geometry, however, no valuation or additive topology is assumed. Instead, congruence is stratified multiplicatively, and structure is encoded not in distance or approximation, but in multiplicative divisibility depth:

$$x \in F^n \iff x \equiv 0 \mod \prod_{i=1}^{n} m_i.$$

Additive/Valuation-Based	**Multiplicoid (Congruence-Based)**
Valuation $v(x)$	Congruence depth n (via multiplicative tower)
Topology from norm	No topology, only arithmetic strata
Neighborhood basis	Layered ε-filtration
Perfectoid space	Multiplicoid space
Frobenius tilting	Multiplicoid recursive tilting
p-adic Hodge theory	$\times$-based Period Stratification

Table 1. Conceptual comparison of foundational geometries

7.2. **Behavior of Filtration Towers.** The additive filtration $F^n_{\mathrm{add}} := p^n\mathcal{O}$ satisfies exponential decay, while multiplicoid filtration $F^n_{\mathrm{mult}} := \ker(\mathcal{O} \to \mathcal{O}/I_n)$ grows in complexity by multiplicative nesting. They differ in direction, intensity, and stratification mechanism.

- **Additive:** Linear base, exponential scaling, valuation-based convergence.
- **Multiplicoid:** Multiplicative base, recursive nesting, congruence collapse without limit topology.

Yet both produce filtrations of period rings, regulate cohomology, and support motivic realization functors. The crucial difference is that multiplicoid towers are *arithmetic*, not topological.

7.3. **Comparison of Realization Functors.** Let us consider:
- $\mathscr{R}_{\mathrm{perf}}$: perfectoid realization (via Frobenius and tilting),
- $\mathscr{R}_{\mathrm{mult}}$: multiplicoid realization (via torsor and congruence depth).

While the former realizes structure through compatible topological lifts, the latter proceeds via stratified descent through torsor towers:

$$\mathscr{R}_{\mathrm{mult}}(\mathcal{F}) = \{F^n\mathcal{F}\} \to \mathrm{Per}^{\infty}_{\mathrm{mult}} \times \mathbb{T}^{[\times]}.$$

7.4. **Axiomatic Divergence.**

Valuation: $\exists$ a nonzero map $\nu : A \to \Gamma \cup \{\infty\}$ satisfying triangle inequality.
Multiplicoid: $\exists$ a congruence sequence $\{I_n\}$ such that $I_n \supset I_{n+1}$ with
$I_n = (m_1 \cdots m_n)$.

The former axiomatizes local approximation, the latter recursive divisibility.

7.5. **Interoperability and Hybrid Filtrations.** It is possible to combine both paradigms:

$$F^{(n)} := (p^n \cap I_n), \quad \text{mixed syntomic–congruence tower.}$$

This leads to a new class of filtrations indexed by both valuation depth and arithmetic congruence. Such constructions may find application in ε-syntomic cohomology, or arithmetic Hodge theory with congruence control.

7.6. **Conclusion.** This section shows that multiplicoid and perfectoid geometries are not mutually exclusive, but operate on orthogonal axes: one via topological convergence, the other via stratified divisibility. Their comparison reveals a broader categorical structure: filtration geometry, where convergence is just one special case of recursive growth.

8. Exponentoid and Knuthoid Transition Models

8.1. **Beyond Multiplicative Growth.** While multiplicoid geometry replaces additive proximity with multiplicative congruence, there remains a vast hierarchy of operational growth rates. For instance, exponential growth $x \mapsto a^x$ and hyperoperations like $x \mapsto a \uparrow^k x$ represent natural generalizations of recursive depth.

We now define models to transition from multiplicoid filtrations to those governed by higher operations.

8.2. **Exponentoid Stratification.** Let $\exp(n)$ denote a base-e or base-b exponential growth function. We define the following:

Definition 8.1 (Exponentoid Tower). *Let* $I_n := (\exp(n)) = (\underbrace{a^{a^{\cdots^{a}}}}_{n \text{ times}})$*. The* exponentoid filtration *is given by:*

$$F^{\exp(n)}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/I_n\mathcal{F}).$$

Such a tower generalizes the multiplicoid tower F^n by replacing $\prod m_i$ with $\exp^n(a)$. This induces *accelerated congruence descent*, corresponding to period sheaves with exponentially faster torsor collapsing.

Definition 8.2 (Exponentoid Period Ring). *Define*

$$B_{\exp,dR} := \varprojlim_n A/\exp(n) \otimes \mathbb{Q},$$

equipped with exp-*indexed ε-filtration.*

8.3. **Knuthoid Models.** Knuth's up-arrow notation defines towers of iterated exponentials:

$$a \uparrow^k b = k\text{-times iterated exponential.}$$

We may then define:

Definition 8.3 (Knuthoid Filtration). *For fixed $a > 1$, let $I_n := (a \uparrow^k n)$, and define:*

$$F^{\uparrow^k n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/I_n\mathcal{F}).$$

This yields the Knuthoid filtration tower *of $\mathcal{F}$.*

These filtrations induce trans-recursive period structures. The associated cohomologies stabilize extremely slowly but stratify enormous computational depth in torsor descent.

8.4. **Transition Maps and Functoriality.** Between multiplicoid and exponentoid towers, we propose transition morphisms:

$$F^n \longrightarrow F^{\exp(n)}, \quad F^{\exp(n)} \longrightarrow F^{\uparrow^2 n},$$

given by functional lifts and divisibility domination. These define natural transformations of period rings and torsor stacks.

Let $\Phi_n^{\exp} : \mathbb{T}_n^{[\times]} \to \mathbb{T}_n^{[\exp]}$ be the induced torsor morphism collapsing multiplicative layers into exponential equivalence classes.

8.5. **Towards Filtration Category Towers.** We may now organize all filtration types into a filtered category:

$$\mathsf{Filt}_\infty := \left\{\text{objects } F^*\mathcal{F}, \text{ morphisms } F^{f(n)} \to F^{g(n)} \text{ for } f \prec g\right\}.$$

This category allows comparison, pushforward, and pullback of period sheaves under growth morphisms. In future volumes, this becomes the foundational language for arithmetic sheaves of infinite generative order.

8.6. **Conclusion.** Exponentoid and Knuthoid geometries form the first two higher-order generalizations of multiplicoid space. They suggest that what we call "space" may actually be an emergent effect of growth law stratification—and that by indexing these laws hierarchically, we may define infinitely generative geometric universes.

9. Ontological Layering of Stratified Growth

9.1. **From Geometric Spaces to Generative Processes.** In traditional geometry, a space is often conceived as a set endowed with topological or algebraic structure. In multiplicoid, exponentoid, and knuthoid geometries, however, space emerges from the stratification of growth operations. This demands a shift in perspective: from *point-based ontology* to *process-based ontology.*

> The identity of a geometric object is no longer a location, but a position within a growth process.

9.2. **Stratified Ontology and Layered Existence.** Let $\mathbb{Y}_\infty$ denote a hypothetical object whose structure is governed entirely by filtration dynamics. Define:

Definition 9.1 (Growth Layer). *A* growth layer L_n *is an equivalence class of sections of* $\mathcal{F}$ *modulo* $F^{g(n)}\mathcal{F}$*, for some growth function* $g(n)$ *(e.g.,* n*,* $\exp(n)$*,* $\uparrow^k n$*). The collection* $\{L_n\}$ *determines an* ontological tower *of existence.*

Each such layer represents not a set-theoretic subset, but a manifestation of mathematical being at a given level of generative depth. The filtration F^* becomes a logic of emergence.

9.3. **Functorial Ontologies.** We now define an ontology-valued presheaf:

$$\mathcal{O}nt : \mathsf{Filt}_\infty^{\mathrm{op}} \to \mathbf{Cat}, \quad F^{f(n)} \mapsto \text{Category of growth-sheaves over } F^{f(n)}.$$

This presheaf encodes the reality of a filtered space as a categorical stack over growth laws, not coordinates.

9.4. **Torsors as Ontological Agents.** Each torsor $\mathcal{T}_n$ is no longer a "bundle over a base", but an automorphism group acting on a growth layer. In this picture:
- **Objects:** Layers of $\mathcal{F}$.
- **Morphisms:** Transition maps between filtration layers.
- **Torsors:** Auto-equivalence structures (e.g. under $\mathbb{Z}/N\mathbb{Z}$) governing stability at that level.

Definition 9.2 (Ontoid Structure). *An* ontoid structure *is a functor* $\mathcal{S} : \mathbb{N} \to$ **Topos** *such that*

$$\mathcal{S}(n) = \textit{Sheaf category over } F^{g(n)}\mathcal{F}, \quad g(n) \in \{\textit{growth functions}\}.$$

The topos $\mathcal{S}(n)$ represents "reality at level n", where mathematical properties exist under the constraints of stratified filtration.

9.5. **Existence via Growth Constraints.** This perspective suggests:
- The notion of a *mathematical object* is dynamic;
- Existence is indexed by growth capacity;
- Identity is relational, defined via stratified equivalence.

Mathematics becomes a *philosophy of expansion*—objects exist not because they are constructed, but because they persist under generative rules.

9.6. **Toward Meta-Geometry.** Ultimately, stratified growth provides a foundational alternative to set-theoretic axioms. It defines a meta-geometric framework where:
- Filtrations are logic;
- Torsors are agents of self-similarity;
- Cohomology is the study of internal persistence across existence levels.

$$\boxed{\text{Space} := \text{Sheaf of Ontologies over Growth-Induced Filtration Categories.}}$$

This leads naturally to the final section: a synthesis of multiplicoid geometry and a proposal for trans-recursive foundational axioms.

10. Concluding Synthesis and Infinite-Generation Conjectures

10.1. **Summary of Multiplicoid Geometry.** In this volume, we introduced the framework of multiplicoid geometry as a generalization of perfectoid and p-adic geometric theories, built upon multiplicative congruence towers rather than additive or valuation-theoretic approximations.

Key foundational principles include:

- **Filtration as Structure:** The primary geometric data arises from congruence depth, not topological closeness.
- **Torsors as Generators:** Spaces are stratified via auto-equivalences of congruence actions rather than local trivializations.
- **Period Rings as Cohomological Media:** Each filtration layer corresponds to a realization of period information under recursive descent.
- **Motivic and Ontological Unification:** Filtrations, torsors, and regulators are synthesized through a common ε-stratified motivic realization.

10.2. **Transition to Higher Geometries.** Multiplicoid theory serves as the foundational level of a more general hierarchy of geometries:

$$\text{Additive} \to \text{Multiplicoid} \to \text{Exponentoid} \to \text{Knuthoid} \to \text{Ontoid}.$$

Each level corresponds to a deeper level of stratification, indexed by increasingly powerful generative operations:
- Additive: $x + n$
- Multiplicative: $x \cdot n$
- Exponential: x^n
- Hyper-Exponential: $a \uparrow^k n$
- Ontological: indexed logic or meta-generators

10.3. **Infinite-Generation Conjectures.** We now propose a class of conjectures describing how the structure of space may emerge from recursively layered stratification:

Conjecture 10.1 (Recursive Stratification of Geometry). *Every cohomological realization functor is induced by a stratified tower of filtration categories* Filt_∞*, whose growth functions are strictly increasing under recursive composition.*

Conjecture 10.2 (Existence via Growth Law). *The existence of a geometric object is determined by its persistence across infinitely many filtration levels. That is,*

$$X \text{ exists} \iff \forall n,\, F^n\mathcal{F}(X) \neq 0.$$

Conjecture 10.3 (Trans-Recursive Periodicity). *Let* $\mathcal{T}_{\uparrow^k}$ *be the Knuthoid torsor tower. Then there exists a stabilization functor*

$$\lim_{\uparrow^k \to \infty} \mathcal{T}_{\uparrow^k} \cong \mathcal{T}_\infty^{Ont},$$

that classifies all growth-based space torsors as images of an ontological stack.

10.4. **Philosophical Implication.** At its heart, this theory asks:

> What is the structure of space if it arises not from position, but from recursion?

The answer, we argue, lies in the infinite generation of torsor-based, filtration-indexed growth. In this vision, geometry is not static—but dynamically layered, generative, and recursive.

10.5. **Outlook.** Future volumes will explore:
- Exponentoid filtrations over exponential congruence towers (Volume I);
- ε-stratified motivic cohomology in exponential and transfinite regimes (Volume II);
- Generalized Weight–Monodromy theories and Knuth-level dynamics (Volume III);
- Ontoid reconstructions of space and arithmetic ontology (Volume IV);
- Categorical arithmetic and logic over growth-generated sheaf-theoretic stacks (Volume V).

Each volume pushes further into the recursion-indexed universe of stratified geometric generation—toward a meta-geometry whose foundations are laws of expansion rather than sets of points.

End of Volume I' (Non-Dyadic Supported)

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MULTIPLICOID GEOMETRY WITH DYADIC SUPPORT: COHOMOLOGY, TORSORS, AND PERIOD TOWERS OVER CONGRUENCE BASES

PU JUSTIN SCARFY YANG

Abstract. We develop a dyadic-supported variant of multiplicoid geometry, in which multiplicative congruence filtrations and torsor towers are embedded over the dyadic completion $\widehat{\mathbb{Q}}_{(2)}$. We construct dyadic-compatible period sheaves, cohomology theories, and regulator realizations, and propose a dyadic ε-filtration hierarchy that refines motivic depth via stratified torsor gerbes. This framework links congruence growth to cohomological realization over dyadic period towers, offering new tools for studying arithmetic stacks and stratified L-values.

Contents

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0. Notation and Symbol Dictionary

This section introduces the core notational framework for dyadic-supported multiplicoid geometry. These notations will appear throughout the development of period rings, cohomology theories, torsors, and ε-filtrations over dyadic bases.

Dyadic Structures.

- $\widehat{\mathbb{Q}}_{(2)} := \widehat{\mathbb{Q}}_{(2)}$: Dyadic completion of $\mathbb{Q}$ with respect to the (2^n) congruence system.
- $I_n := (2^n)$: Dyadic congruence ideal used for tower construction.
- $A^{(n)} := \mathbb{Z}/2^n\mathbb{Z}$: Level-n congruence ring in the dyadic system.
- $A^{\infty}_{\text{dyad}} := \varprojlim A^{(n)}$: Dyadic completed base ring, defining affine patches.

Period Structures.

- $B_{\text{dyad},dR}$: Dyadic de Rham period ring with binary-indexed filtration.
- $F^n B_{\text{dyad},dR}$: Level-n filtration of the period ring via dyadic congruence depth.
- $\mathrm{Per}^{\infty}_{\text{dyad}} := \{B^{(n)}\}_{n\geq 0}$: Dyadic period tower.

Torsors and ε-Filtrations.

- $\mathcal{T}^{\text{dyad}}_n$: Level-n torsor under $(\mathbb{Z}/2^n)$-action.
- $\mathbb{T}^{[\text{dyad}]}$: Stack of dyadic torsors indexed by n.
- $\mathcal{G}^{(2)}_{\varepsilon}$: Universal dyadic ε-gerbe capturing torsor descent.
- $F^{\varepsilon^n}\mathcal{F}$: ε-filtration on sheaf $\mathcal{F}$ with respect to $I_n = (2^n)$.

Cohomology and Realizations.

- $H^i_{\text{dyad}}(X, \mathcal{F})$: i-th dyadic cohomology of sheaf $\mathcal{F}$ over space X.
- r_{dyad} : Dyadic regulator from K-theory to dyadic cohomology.
- $\mathscr{R}_{\text{dyad}}$: Realization functor mapping filtered torsors to cohomological data over $\mathrm{Per}^{\infty}_{\text{dyad}}$.

Remarks. All structures defined above are to be interpreted in the context of multiplicative congruence towers embedded over the dyadic completion $\widehat{\mathbb{Q}}_{(2)}$. The binary tree structure of (2^n) provides an arithmetic filtration that refines and replaces the valuation-theoretic metric structure used in classical perfectoid theory.

1. Introduction and Motivation

The theory of perfectoid spaces has demonstrated the profound effectiveness of working with infinite-level congruence towers over p-adic valuation rings. In the context of multiplicoid geometry, our aim is to generalize these ideas by replacing additive and valuation-based structures with congruence systems derived from multiplicative stratification.

In this dyadic-supported version of multiplicoid geometry, we propose to work over the base

$$\widehat{\mathbb{Q}}_{(2)} := \varprojlim \mathbb{Q}/2^n\mathbb{Z},$$

the dyadic completion of the rational numbers under the inverse system defined by powers of 2. This base ring is not a valuation ring, and it lacks a topology in the classical analytic sense. Instead, it carries a natural *congruence stratification structure*, which provides a new organizing principle for filtrations, torsors, and cohomological realizations.

Why Dyadic?

- The dyadic tower (2^n) forms a canonical and universal congruence system among all base-2 congruences, naturally connected to binary growth and recursive depth.
- Dyadic congruence trees resemble binary computation trees in logic and computer science, suggesting a deep connection between arithmetic filtration and stratified recursion.
- The arithmetic nature of the (2^n) system provides an alternative to valuation theory: it encodes depth by divisibility, rather than by norm.
- Dyadic ε-filtrations allow us to define height pairings, period stratifications, and torsor descent data without referring to topology, but purely in terms of arithmetic depth.
- Many natural objects in arithmetic geometry—modular forms, polylogarithms, iterated integrals—exhibit binary growth phenomena that are better captured through dyadic congruence.

Conceptual Shift. This volume represents a conceptual shift in the foundations of geometry:

> From valuation-based geometry to congruence-based filtration geometry.

In particular, we view dyadic congruence as a *growth-theoretic medium*—a non-topological, non-archimedean arena where geometric structure is determined by arithmetic stratification rather than distance.

Objectives of this Volume. Our goal is to develop the following:

- A category of sheaves and torsors over $\mathrm{Spec}(\widehat{\mathbb{Q}}_{(2)})$ equipped with multiplicative congruence filtrations;
- A theory of dyadic period rings, including $B_{\mathrm{dyad},dR}$ and its tower $\mathrm{Per}^{\infty}_{\mathrm{dyad}}$;
- ε-filtration theory over dyadic bases, giving rise to dyadic cohomology groups H^i_{dyad};
- Universal realization functors $\mathscr{R}_{\mathrm{dyad}}$ from torsor filtrations to period cohomology;
- Applications to motivic realizations, regulator maps, and arithmetic stacks with binary depth invariants.

The structure of this volume follows the same organizational logic as the dyadic-free version, but with new foundational constructions that arise uniquely in the dyadic context. In this setting, torsor descent becomes arithmetic descent, and congruence depth replaces valuation depth, leading to a new geometry of stratified arithmetic emergence.

2. Dyadic Congruence Systems and Base Structures

2.1. **The Dyadic Completion.** We begin by defining the dyadic completion of $\mathbb{Q}$ via congruence modulo powers of 2. This will serve as the base ring for all constructions in this volume.

Definition 2.1 (Dyadic Completion). *Let $\mathbb{Z}/2^n\mathbb{Z}$ denote the ring of integers modulo 2^n. The* dyadic completion *of $\mathbb{Q}$ is the inverse limit*

$$\widehat{\mathbb{Q}}_{(2)} := \varprojlim_n \mathbb{Q}/2^n\mathbb{Z},$$

with transition maps induced by natural reduction modulo 2^{n-1}. It admits a canonical congruence filtration indexed by $n \in \mathbb{Z}_{\geq 0}$:

$$I_n := \ker\left(\widehat{\mathbb{Q}}_{(2)} \to \mathbb{Q}/2^n\mathbb{Z}\right) \subset \widehat{\mathbb{Q}}_{(2)}.$$

Remark 2.2. This completion is not topological in the usual analytic sense; rather, it reflects congruence depth in the binary divisibility hierarchy. Unlike $\mathbb{Q}_2$, it is not a field, but a pro-object defined over arithmetic congruence.

2.2. **Dyadic Base Rings and Affine Patches.**

Definition 2.3 (Dyadic Base Ring). *A* dyadic base ring *is any ring A admitting a morphism of inverse systems*

$$A \to \varprojlim_n A_n := \varprojlim_n A/2^nA.$$

The completion $A^{\infty}_{\mathrm{dyad}} := \varprojlim A_n$ is called the dyadic congruence completion *of A.*

Example 2.4. Let $A = \mathbb{Z}[x]$. Then $A_n = \mathbb{Z}[x]/(2^n)$ and the system $\{A_n\}$ yields

$$A^{\infty}_{\mathrm{dyad}} = \varprojlim \mathbb{Z}[x]/(2^n) = \mathbb{Z}_2[[x]].$$

This serves as the dyadic formal neighborhood of x over $\mathrm{Spec}(\mathbb{Z})$.

2.3. **Congruence Sheaves and Dyadic Filtrations.** Let $X = \mathrm{Spec}(A^{\infty}_{\mathrm{dyad}})$.

Definition 2.5 (Dyadic Sheaf)**.** *A* dyadic sheaf *on X is a sheaf $\mathcal{F}$ of A-modules equipped with a descending filtration*

$$F^n\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/2^n\mathcal{F}\right).$$

Each F^n is referred to as the n-th dyadic congruence layer*.*

Definition 2.6 (ε-Filtration over Dyadic Base)**.** *Define the ε-filtration tower:*

$$F^{\varepsilon^n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/(2^n)\mathcal{F}\right),$$

where the index ε^n indicates the exponential depth of the congruence.

Remark 2.7. This structure allows us to define height-pairings and torsor classes whose obstruction depth grows with dyadic congruence complexity.

2.4. **Torsors and Transition Systems.**

Definition 2.8 (Dyadic Torsor Tower)**.** *Let $G_n := \mathbb{Z}/2^n\mathbb{Z}$. A* dyadic torsor tower *over X is a projective system $\{\mathcal{T}_n\}$ where each $\mathcal{T}_n \to X$ is a G_n-torsor, and the transition maps*

$$\mathcal{T}_{n+1} \to \mathcal{T}_n$$

are G_n-equivariant under reduction $G_{n+1} \to G_n$.

These towers encode stratified symmetry across binary congruence levels. We may view them as arithmetic analogues of local systems, organized by binary divisibility rather than monodromy.

2.5. **Summary.** We now have:

- A congruence-based arithmetic base ring $\widehat{\mathbb{Q}}_{(2)}$;
- A filtration theory indexed by (2^n);
- Torsors and sheaves structured by binary descent;
- ε-filtrations organizing congruence depth stratification.

These tools form the local foundations on which we construct dyadic period rings, torsor cohomology, and ε-gerbes in the sections that follow.

3. Dyadic Period Rings and Sheaves

3.1. **Dyadic de Rham Period Ring.** Just as perfectoid geometry gives rise to period rings such as B_{dR}, in the dyadic-supported setting we construct a ring of periods governed by congruence stratification modulo powers of 2.

Definition 3.1 (Dyadic de Rham Period Ring). *Let $A^{\infty}_{\mathrm{dyad}}$ be the dyadic-completed base ring. The* dyadic de Rham period ring *is defined as:*

$$B_{dyad,dR}(A) := \varprojlim_n A^{\infty}_{\mathrm{dyad}}/2^n \otimes_{\mathbb{Z}} \mathbb{Q},$$

with the natural dyadic filtration:

$$F^n B_{dyad,dR} := \ker\left(B_{dyad,dR} \to A^{\infty}_{\mathrm{dyad}}/2^n \otimes \mathbb{Q}\right).$$

Remark 3.2. This filtration is arithmetic in nature, defined by congruence depth rather than norm or valuation. Each level F^n corresponds to a binary congruence depth layer.

3.2. **Dyadic Period Tower.**

Definition 3.3 (Dyadic Period Tower). *The* dyadic period tower *is the projective system:*

$$\mathrm{Per}^{\infty}_{dyad} := \left\{B^{(n)}_{dyad,dR} := A^{\infty}_{\mathrm{dyad}}/2^n \otimes \mathbb{Q}\right\}_{n\in\mathbb{Z}_{\geq 0}},$$

with transition maps induced by canonical reductions modulo 2^{n-1}.

This tower serves as the congruence-based analog of the classical period rings of p-adic Hodge theory. It defines a cohomological structure based on binary stratification.

3.3. **Period Sheaves with ε-Filtration.**

Definition 3.4 (Dyadic Period Sheaf). *Let $X = \mathrm{Spec}(A^{\infty}_{\mathrm{dyad}})$. A* dyadic period sheaf *over X is a sheaf $\mathcal{F}$ of $B_{dyad,dR}$-modules equipped with:*

- *A descending dyadic filtration $F^n\mathcal{F}$ indexed by congruence depth 2^n;*
- *A compatible ε-filtration $F^{\varepsilon^n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/2^n)$;*
- *Flat transition morphisms over the tower $\mathrm{Per}^{\infty}_{dyad}$;*
- *Functorial compatibility with dyadic torsor actions.*

Example 3.5. The structure sheaf $\mathcal{O}_X$ itself admits an ε-filtration:

$$F^{\varepsilon^n}\mathcal{O}_X = \{f \in \mathcal{O}_X \mid f \equiv 0 \mod 2^n\}.$$

Tensoring with $\mathbb{Q}$ gives the dyadic period sheaf associated to identity sections.

3.4. **Dyadic Cohomology.**

Definition 3.6 (Dyadic Cohomology)**.** *Let $\mathcal{F}$ be a dyadic period sheaf over X. The i-th* dyadic cohomology group *is defined by*

$$H^i_{dyad}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathcal{F}/2^n\mathcal{F}).$$

This cohomology detects the stabilization of arithmetic data under binary congruence descent.

Remark 3.7. This theory mirrors crystalline and syntomic cohomology, but under congruence stratification rather than Frobenius lifts. The filtration corresponds to ε-depth layers in arithmetic descent.

3.5. **Summary.** We now have:

- The dyadic de Rham period ring $B_{\mathrm{dyad},dR}$ as the analog of B_{dR};
- An ε-stratified tower of congruence-period rings $\mathrm{Per}^{\infty}_{\mathrm{dyad}}$;
- Dyadic period sheaves and compatible ε-filtrations;
- A new cohomology theory H^i_{dyad} based on arithmetic stratification depth.

These objects are the building blocks of the dyadic period-geometry landscape, and will feed into the torsor-theoretic structures in the next section.

4. Dyadic Torsors, Gerbes, and Realizations

4.1. **Universal Dyadic Torsor Tower.** In the dyadic setting, torsors naturally arise over the binary congruence groups $G_n := \mathbb{Z}/2^n\mathbb{Z}$, with transitions corresponding to the mod-2 congruence system.

Definition 4.1 (Dyadic Torsor)**.** *Let X be a dyadic base space. A* dyadic G_n-torsor *over X is a scheme $\mathcal{T}_n \to X$ with a free transitive right action of G_n such that locally on X, $\mathcal{T}_n \cong X \times G_n$.*

A dyadic torsor tower *is a projective system*

$$\{\mathcal{T}_n\}_{n\in\mathbb{Z}_{\geq 0}}, \quad \mathcal{T}_{n+1} \to \mathcal{T}_n,$$

with $G_{n+1} \to G_n$ equivariance and compatibility.

Example 4.2. Let $X = \mathrm{Spec}(\widehat{\mathbb{Q}}_{(2)})$. Then $\mathcal{T}_n = \mathrm{Spec}(\widehat{\mathbb{Q}}_{(2)}[x]/(x^{2^n} - 1))$ forms a G_n-torsor representing 2^n-th roots of unity. The tower captures the limit $\varprojlim \mu_{2^n}$.

4.2. **Universal ε-Gerbe over Dyadic Congruence.**

Definition 4.3 (Dyadic ε-Gerbe)**.** *Let $\mathcal{F}$ be a dyadic sheaf. An* ε-gerbe $\mathcal{G}_\varepsilon$ *over X is a banded stack associated to the projective system $\{\mathcal{T}_n\}$, satisfying:*

- *Each $\mathcal{T}_n$ classifies trivializations modulo 2^n;*
- *$\mathcal{G}_\varepsilon$ is banded by the inverse system $\{G_n\}$;*

- *$\mathcal{G}_\varepsilon$ admits a classifying map:*

$$X \to B\mathcal{G}_\varepsilon := \varprojlim_n B(G_n),$$

where $B(G_n)$ is the classifying stack for G_n-torsors.

This stack controls the obstruction theory of lifting sections through increasing binary congruence levels. Cohomological torsors correspond to gerbe classes in $H^2_{\mathrm{dyad}}(X, G_n)$.

4.3. **Realization Functor and Period Morphisms.**

Definition 4.4 (Dyadic Realization Functor)**.** *Define the functor:*

$$\mathscr{R}_{dyad} : \mathbf{Sh}^\varepsilon(X) \longrightarrow \mathrm{Perdyad}^\infty(X) \times \mathbb{T}^{[\mathrm{dyad}]},$$

sending a dyadic-filtered sheaf to its period image in $\mathrm{Per}^\infty_{dyad}$ and its torsor realization in $\mathbb{T}^{[\mathrm{dyad}]}$.

This functor realizes ε-stratified torsors as period data, organizing geometric cohomology in terms of dyadic congruence complexity.

Theorem 4.5 (Universal Period Realization)**.** *There exists a natural transformation:*

$$\mathscr{R}_{dyad}(\mathcal{F}) \longrightarrow H^\bullet_{dyad}(X, \mathcal{F}),$$

that respects filtration depth, torsor descent, and period morphism compatibility.

Sketch. Using the projective systems $\mathcal{F}/2^n$, one builds the cohomology via inverse limits, and the torsor classes lift canonically through $\mathcal{G}_\varepsilon$ to $\mathrm{Per}^\infty_{\mathrm{dyad}}$-coefficients. □

4.4. **Concluding Remarks.** This torsor–gerbe–period framework gives rise to a new type of stratified cohomology, one not built on topological neighborhoods, but on congruence levels of arithmetic height. It provides:

- Fine-grained control of arithmetic period structures;
- Classification of torsor complexity via binary towers;
- New perspectives on motivic cohomology over stratified spaces;
- A foundation for comparing with multiplicoid–perfectoid–exponentoid geometries in the broader framework of hyperstratified spaces.

5. Dyadic Motivic Realization and Regulators

5.1. **Dyadic Motives and Period Realizations.** Let X be a space defined over the dyadic base $\widehat{\mathbb{Q}}_{(2)} := \varprojlim \mathbb{Q}/2^n\mathbb{Z}$. We define dyadic motives as analogues of classical motives, realized through torsor and period descent over dyadic filtration towers.

Definition 5.1 (Dyadic Motive). *A dyadic motive $M_{\mathrm{dyad}}^{[\times]}(X)$ is an object in a triangulated category $\mathcal{M}_{\mathrm{dyad}}$ of arithmetic motivic sheaves over dyadic congruence systems, equipped with:*

- *A dyadic filtration $\{F^{2^n}M\}_{n\geq 0}$;*
- *Realization functors to period cohomology:*

$$\mathrm{real}_{\mathrm{dyad}}^{per} : M_{\mathrm{dyad}}^{[\times]}(X) \longrightarrow H_{\mathrm{dyad}}^{\bullet}(X, B_{dyad,dR});$$

- *Descent through torsor towers $\{\mathcal{T}_n\}$ under $(\mathbb{Z}/2^n\mathbb{Z})$-action.*

5.2. **Dyadic Regulator Maps.** We define analogues of Beilinson's higher regulators via torsor realization.

Definition 5.2 (Dyadic Regulator). *Let $K_n(X)$ be an algebraic K-group of X. The* dyadic regulator map *is:*

$$r_{\mathrm{dyad}} : K_n(X) \longrightarrow H_{\mathrm{dyad}}^{n}(X, \mathbb{Q}(n)),$$

constructed by composing:

$$K_n(X) \xrightarrow{\textit{cycle class}} M_{\mathrm{dyad}}^{[\times]}(X) \xrightarrow{\mathrm{real}} H_{\mathrm{dyad}}^{n}(X, B_{dyad,dR}).$$

The map r_{dyad} encodes congruence-based height data in the form of dyadic period invariants.

5.3. **Dyadic ε-Pairings and Period Height.** Given a torsor tower $\{\mathcal{T}_n\}$, one obtains natural pairings indexed by congruence depth.

Definition 5.3 (Dyadic ε-Pairing). *Let M be a dyadic motive over X. Define:*

$$\langle -, - \rangle_{\varepsilon^n} : F^{2^n}M \otimes F^{2^n}M^{\vee} \longrightarrow \mathbb{Q},$$

satisfying compatibility with dyadic torsor transitions. The pairing defines arithmetic height in the dyadic filtration category.

5.4. **Dyadic L-Functions.** We define special values via congruence-depth period realizations:

Definition 5.4 (Dyadic L-Function). *Let $M = M_{\mathrm{dyad}}^{[\times]}(X)$. Define:*

$$L_{\mathrm{dyad}}(M, s) := \prod_{n=0}^{\infty} \det\left(1 - \mathrm{Frob}_n \cdot 2^{-s} \mid F^{2^n}M\right)^{-1},$$

where Frob_n *is the action induced by congruence twisting at depth 2^n.*

This function encodes the arithmetic complexity of dyadic torsor realization, stratified across binary congruence layers.

5.5. **Summary.** We have introduced the core motivic objects in dyadic-supported multiplicoid geometry:

- Motives $M_{\text{dyad}}^{[\times]}$ with dyadic filtration;
- Regulator maps from K-theory into dyadic cohomology;
- ε-pairings derived from torsor symmetry;
- Dyadic L-functions as congruence-stratified arithmetic generating series.

These tools together provide the motivic infrastructure for understanding growth-based cohomology beyond valuation and topology.

6. Comparison with Additive and Valuation-Based Geometries

6.1. **Three Approaches to Filtration and Descent.** We now compare three major frameworks for arithmetic filtration and cohomology:

(1) **Additive/Valuation-Based:** Uses topologies induced by discrete valuations, e.g., p-adic Hodge theory, perfectoid spaces.
(2) **Dyadic-Supported (Congruence-Based):** Uses congruence towers such as 2^n for filtration and torsor descent, without relying on topological neighborhoods.
(3) **Crystalline/Syntomic:** Uses Frobenius-lifted deformations and divided powers to capture p-adic period structures.

Type	Filtration	Descent Mechanism
Valuation/Perfectoid	Norm-based, $\lvert x\rvert_p < \varepsilon$	Frobenius tilting, topological completion
Crystalline/Syntomic	Divided power ideals	Frobenius-compatible thickenings
Dyadic (This work)	Arithmetic congruence 2^n	Binary torsor towers, ε-filtration

Table 1. Comparison of geometric filtration theories

6.2. **Dyadic vs Perfectoid Tilting.** Perfectoid tilting requires deep compatibility with Frobenius morphisms and almost mathematics. In contrast, dyadic-supported tilting operates through congruence relations:

$$\text{Perfectoid:} \quad \mathcal{O}_K \simeq \varprojlim_{\Phi} \mathcal{O}_K/p \qquad \text{vs.} \qquad \text{Dyadic:} \quad \mathcal{F} \simeq \varprojlim_{n} \mathcal{F}/2^n.$$

The dyadic theory is *Frobenius-free* and instead stratifies space through explicit congruence descent.

6.3. **Hybrid Filtration: ε-Dyadic–Syntomic Structures.** A promising research direction involves merging congruence-based ε-filtrations with crystalline period theories.

Definition 6.1 (ε-Dyadic–Syntomic Tower)**.** *Let* $F^{(n)} := \ker(\mathcal{F} \to \mathcal{F}/(2^n, \gamma_n))$*, where* γ_n *is a divided power or syntomic correction. The resulting tower interpolates:*

- *dyadic congruence* 2^n*;*
- *crystalline thickenings via* γ_n*;*
- *syntomic periods with arithmetic* ε*-filtration.*

Such towers may yield a new class of regulators, which carry both congruence descent and Frobenius-like cohomological deformations.

6.4. **Conceptual Distinctions.**

- **Topological approximation (valuation):** geometry defined via closeness.
- **Stratified collapse (dyadic):** geometry defined via recursion.
- **Cohomological deformation (crystalline):** geometry defined via compatibility with Frobenius.

We posit that these are orthogonal lenses for understanding arithmetic geometry, and their comparison highlights the need for a unified framework of stratified filtrations.

6.5. **Conclusion.** Dyadic-supported geometry provides a third paradigm, joining valuation and crystalline theories. It allows one to study motivic realizations and regulators via discrete congruence operations, enabling a new notion of depth, period structure, and torsor complexity.

In the next section, we explore how these congruence stratifications extend into exponential growth regimes via the dyadic–exponentoid bridge.

7. Dyadic–Exponentoid Bridge and Stratified Interpolation

7.1. **Motivation for Interpolation.** The dyadic filtration tower uses binary growth: $F^{2^n}\mathcal{F}$. To reach the higher complexity of exponentoid and knuthoid geometries, we now define morphisms between dyadic and exponential congruence systems.

This provides a smooth passage from arithmetic congruence-based geometry into recursion-based stratified structures.

7.2. **Interpolating Growth Functions.** Let $f(n) = 2^n$, $g(n) = \exp(n)$. Define a bridge function:

$$\beta(n) := \left\lfloor a^{\lambda(n)} \right\rfloor, \quad \text{with } \lambda(n) = \log_2 n + \delta(n),$$

where $\delta(n) \to 0$ is a correction term. Then:

$$2^n < \beta(n) < \exp(n), \quad \forall n \gg 0.$$

This yields interpolating filtrations:

$$F^{\beta(n)}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\beta(n)\mathcal{F}\right).$$

7.3. **Period Ring Morphisms.** Let $B_{\text{dyad},dR} \to B_{\text{exp},dR}$ be a morphism of period rings defined via filtration base change:

$$B_{\text{dyad},dR} = \varprojlim \mathcal{F}/2^n \longrightarrow \varprojlim \mathcal{F}/\exp(n).$$

We define a compatible map between period towers:

$$\Phi_n^{\text{interp}} : B^{(n)}_{\text{dyad},dR} \to B^{(n)}_{\text{exp},dR},$$

respecting torsor descent and filtered realization structures.

7.4. **Torsor Reinterpretation.** Let $\mathcal{T}_n^{(2)}$ denote a dyadic torsor and $\mathcal{T}_n^{(\exp)}$ an exponentoid torsor. Then we define:

$$\mathcal{T}_n^{(\exp)} := \left(\mathcal{T}_{m_n}^{(2)}\right)^{\text{Sym}^k}, \quad \text{where } m_n = \lfloor \log_2 \exp(n) \rfloor.$$

This models exponentoid torsors as symmetrized stacks of dyadic torsors. In categorical terms, there exists a functor:

$$\Theta : \mathbb{T}^{[\text{dyad}]} \to \mathbb{T}^{[\exp]}, \quad \text{preserving tower depth via exponential rescaling.}$$

7.5. **Conjectural Dyadic Tilting toward Exponentoid Towers.**

Conjecture 7.1 (Dyadic–Exponentoid Tilting)**.** *There exists a filtration-preserving functor:*

$$\text{Tilt}_{2^\bullet \to \exp(\bullet)} : \text{Sh}_{\text{dyad}} \to \text{Sh}_{\exp},$$

which respects torsor equivalence classes and ε-stratified cohomology.

This functor induces a derived equivalence between congruence-based and exponential sheaves under growth-based transitions.

7.6. **Future Interpolative Models.** Just as perfectoid theory interpolates characteristic p and 0, we envision that:
- Dyadic-supported geometry interpolates congruence logic;
- Exponentoid geometry interpolates recursion logic;
- Their bridge encodes a generalized arithmetic period transformation scheme.

This prepares us for the final ontological layering of dyadic structures in the next section.

8. Ontological Foundations and Future Directions

8.1. **From Arithmetic Congruence to Ontological Stratification.** Throughout this volume, we have seen how dyadic structures generate geometric information not from topology or valuation, but from binary congruence depth. This naturally leads us to reinterpret congruence as a generative force—a stratified arithmetic ontology.

To exist arithmetically is to persist across congruence layers.

Thus, filtrations such as $\{F^{2^n}\mathcal{F}\}$ define not just cohomological depth, but *ontological layers of mathematical being.*

8.2. **Existence as Stratified Cohesion.**

Definition 8.1 (Congruence Ontology)**.** *Let $\mathcal{F}$ be a sheaf over a dyadic base. Define:*

$$\mathbb{E}_{\text{dyad}} := \left\{ X \in \mathcal{F} \,\middle|\, \forall n,\, X \in F^{2^n}\mathcal{F} \right\}.$$

Then $\mathbb{E}_{\text{dyad}}$ is the category of arithmetically persistent structures*.*

Each layer F^{2^n} functions as a logical sieve. The more layers a section survives, the more real it becomes. Existence is depth of congruence, not merely presence.

8.3. **Sheaf Towers as Indexed Logic.** We propose that dyadic ε-filtration towers should be seen as logic-indexed hierarchies. For example:
- F^{2^n} = "provable at logical strength level n";
- Torsor descent = "coherent across logical collapse";
- ε-gerbes = "syntactic classifiers of arithmetic persistence".

Definition 8.2 (Growth Ontology Stack)**.** *Define a contravariant functor:*

$$\mathcal{O}nt_{\text{dyad}} : \mathbb{N}^{\text{op}} \to \mathbf{Cat}, \quad n \mapsto \mathrm{Sh}(F^{2^n}\mathcal{F}),$$

which we interpret as the ontological reality at congruence level n*.*

The sheaf $\mathcal{F}$ then becomes an "emergent total space":

$$\mathcal{F} = \varprojlim \mathcal{O}nt_{\text{dyad}}(n),$$

a limit of categorical existence over growing depth.

8.4. **From Dyadic to Ontoid Geometry.** This leads naturally into Volume IV, where we define **Ontoid Geometry**:
- The base is no longer a ring or field, but a logic of generation;
- Space is not a set of points, but a stack of structural operations;
- Filtrations index not just magnitude, but existential layering.

$$\boxed{\text{Space} = \text{A Layered Ontology of Arithmetic Filtration}}$$

8.5. **Programmatic Vision.**

(1) Start with additive approximation (classical geometry).
(2) Refine into multiplicoid congruence stratification.
(3) Encode via dyadic towers.
(4) Transition into exponentoid and knuthoid growth regimes.
(5) Collapse into abstract ontological stacks of growth-based existence.

Each step replaces static topological space with stratified generators of structure.

8.6. **Outlook.** This volume has laid the groundwork for a logic-driven foundation of arithmetic space. In future volumes, we will:

- Develop full exponentoid and knuthoid towers;
- Generalize torsors and motives to trans-recursive growth indices;
- Formulate new cohomological dualities indexed by operational depth;
- Build arithmetic ontology from congruence towers and sheaf recursion.

We now exit the binary world of dyadic arithmetic and ascend toward the layered infinities of stratified mathematical existence.

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VOLUME I: EXPONENTOID AND KNUTHOID SPACES
A TRANSCENDENCE-LEVEL GENERALIZATION OF PERFECTOID GEOMETRY

PU JUSTIN SCARFY YANG

Abstract. We introduce the theory of *Exponentoid* and *Knuthoid* spaces, geometric structures stratified by recursive and trans-recursive filtration towers such as $\exp(n)$ and $a \uparrow^k n$. Generalizing multiplicoid geometry, we construct new period rings, define filtration-indexed torsors, and develop cohomology and motivic realization over rapidly growing towers. This lays the groundwork for hyperfiltration theories and stratified ontologies of arithmetic geometry beyond additive or valuation-theoretic formalisms.

Contents

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0. Notation and Symbol Dictionary

This section collects the primary notation used throughout this volume, with a focus on exponentoid and knuthoid structures, their corresponding period rings, torsors, filtrations, and cohomological realizations.

Growth Functions and Indexing Notation.

- $\exp(n)$: the standard exponential growth function, e.g., 2^n or e^n.
- $a \uparrow^k n$: Knuth's k-th level hyperoperation, e.g., $2 \uparrow^2 3 = 2^{2^2}$.
- $\varepsilon^{f(n)}$: ϵ-stratified growth index associated to $f(n)$.

Filtration Systems.

- $F^{\exp(n)}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/\exp(n))$: exponentoid filtration.
- $F^{a\uparrow^k n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n))$: knuthoid filtration.
- Filt_∞ : the category of all filtration systems indexed by recursive growth.

Period Rings and Towers.

- $B_{\exp,dR}$: exponentoid de Rham period ring.
- $B_{\uparrow^k,dR}$: knuthoid de Rham period ring.
- $Per_{\exp} := \{B^{(n)}_{\exp,dR}\}_{n\geq 0}$: tower of exponentoid period rings.
- $Per_{\uparrow^k} := \{B^{(n)}_{\uparrow^k,dR}\}_{n\geq 0}$: knuthoid period tower.

Torsors and Group Actions.

- $\mathbb{T}^{[\exp]}$: exponentoid torsor tower, indexed by $\exp(n)$.
- $\mathbb{T}^{[\uparrow^k]}$: knuthoid torsor tower, indexed by $a \uparrow^k n$.
- $\mathcal{T}_n^{\exp}$, $\mathcal{T}_n^{\uparrow^k}$: torsors at each level.
- $\Theta : \mathbb{T}^{[\exp]} \to \mathbb{T}^{[\uparrow^k]}$: functorial transition map between torsor hierarchies.

Motivic and Realization Functors.

- $M^{[\exp]}(X)$: exponentoid motive associated to X.
- $M^{[\uparrow^k]}(X)$: knuthoid motive at level k.
- $\mathrm{real}_{\exp}, \mathrm{real}_{\uparrow^k}$: realization functors to exponentoid and knuthoid cohomology, respectively.
- $r_{\exp}$, $r_{\uparrow^k}$: higher regulator maps in exponential/hyper-exponential settings.

General Structures.

- $\mathscr{R}_{\exp}, \mathscr{R}_{\uparrow^k}$: realization systems over exponential or knuthoid period towers.
- $H^i_{\exp}(X, \mathcal{F})$: exponentoid cohomology.
- $H^i_{\uparrow^k}(X, \mathcal{F})$: knuthoid cohomology.
- $\mathcal{O}nt_{\exp}$, $\mathcal{O}nt_{\uparrow^k}$: growth-indexed ontological stacks.

Remarks. All notations in this volume generalize those from Volume I' (Multiplicoid Geometry). The key novelty is the replacement of congruence-based growth (2^n) with recursive hyper-growth schemes indexed by $\exp(n)$ and $a \uparrow^k n$. The categorical, cohomological, and ontological frameworks are extended accordingly.

1. Introduction and Motivation

1.1. **From Multiplicative to Recursive Growth.** Multiplicoid geometry replaced valuation and additive proximity with multiplicative congruence depth. In this volume, we go one level higher: from multiplicative descent to recursive growth stratification, using filtrations governed by exponential and hyper-exponential indexing functions.

Where previous geometries used:

$$F^n\mathcal{F}, \quad F^{2^n}\mathcal{F},$$

we now stratify by:

$$F^{\exp(n)}\mathcal{F}, \quad F^{a\uparrow^k n}\mathcal{F},$$

where $a \uparrow^k n$ denotes the k-th hyperoperation in Knuth's notation.

1.2. **Why Exponentoid and Knuthoid?** The motivation is both arithmetic and ontological:

- **Arithmetic Depth:** Many transcendental phenomena in number theory (e.g., polylogarithms, multiple zeta values) encode recursive growth. Geometry must be sensitive to these recursion depths.
- **Cohomological Persistence:** Objects surviving across exponential filtration layers capture deeper torsor descent behavior and stack-theoretic persistence.
- **Meta-Geometry:** The hierarchy additive < multiplicative < exponential < knuthoid < ontological suggests that geometry is a manifestation of recursion, not locality.
- **Categorical Richness:** New towers of torsors, regulators, and period sheaves emerge, indexed by trans-recursive growth.

1.3. **The Landscape of Growth-Indexed Geometry.** We propose the following stratified framework of geometric regimes:

Regime	Growth	Example Filtration
Additive	n	$F^n\mathcal{F}$
Multiplicative	2^n	$F^{2^n}\mathcal{F}$
Exponentoid	$\exp(n)$	$F^{\exp(n)}\mathcal{F}$
Knuthoid	$a \uparrow^k n$	$F^{a\uparrow^k n}\mathcal{F}$
Ontoid	$f_{\mathrm{meta}}(n)$	$\mathcal{O}nt_n(\mathcal{F})$

This volume focuses on the third and fourth layers—exponentoid and knuthoid.

1.4. **Objectives of This Volume.** We will:

(1) Define exponentoid and knuthoid filtrations, period rings, and torsor towers;
(2) Construct realization functors $\mathscr{R}_{\exp}, \mathscr{R}_{\uparrow^k}$ to cohomology indexed by recursive growth;
(3) Introduce regulator systems and special value theory in hyper-filtration regimes;
(4) Provide transition models from dyadic to exponentoid structures;
(5) Lay foundations for a future ontology of infinitely generated arithmetic spaces.

1.5. **Position Within the Yang Program.** This volume is the formal starting point of the higher recursion phase of the Yang Program. It follows the fully developed multiplicoid base geometry of Volume I', and sets the stage for:
- Volume II: *Hyper-Filtration Theory and Transfinite Monodromy*,
- Volume III: *Weight–Monodromy Conjectures beyond Linear Cases*,
- Volume IV: *Ontoid Geometry and Space-Theoretic Ontologies*,
- Volume V: *Categorical Arithmetic of Growth-Based Spaces.*

Each builds upon the exponentoid and knuthoid constructions initiated here.

Geometry is no longer what space contains, but what growth generates.

2. Exponentoid Filtrations and Period Towers

2.1. **Exponentoid Growth and Arithmetic Stratification.** Let $f(n) = \exp(n)$ denote an exponential growth function, e.g., $f(n) = a^n$ or e^n. In contrast to linear or multiplicative filtrations, exponentoid filtrations define strata of arithmetic collapse that deepen more rapidly with n.

Definition 2.1 (Exponentoid Filtration)**.** *Let $\mathcal{F}$ be a sheaf on an arithmetic base. Define the exponentoid filtration as:*

$$F^{\exp(n)}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\exp(n)\cdot\mathcal{F}\right).$$

Each level isolates the part of $\mathcal{F}$ that vanishes under increasingly large exponential congruence depths. It generalizes the dyadic filtration F^{2^n} used in Volume I'.

2.2. **Exponentoid Period Rings.** We now define the analog of de Rham period rings in the exponentoid setting.

Definition 2.2 (Exponentoid Period Ring)**.** *Let A be a ring with exponentoid filtration $\{F^{\exp(n)}A\}$. Define:*

$$B_{\exp,dR} := \varprojlim_n A/\exp(n) \otimes_{\mathbb{Z}} \mathbb{Q},$$

equipped with the natural descending $\exp(n)$*-indexed filtration.*

Remark 2.3. $B_{\exp,dR}$ captures congruence collapse at recursive arithmetic depths. Unlike B_{dR}, it is defined without a valuation or topology, but via growth-level recursion.

2.3. **Exponentoid Period Towers.**

Definition 2.4 (Period Tower)**.** *The exponentoid period tower is the inverse system:*

$$Per_{\exp} := \left\{B^{(n)}_{\exp,dR} := A/\exp(n) \otimes \mathbb{Q}\right\}_{n\in\mathbb{N}},$$

with canonical transition maps $B^{(n+1)}_{\exp,dR} \to B^{(n)}_{\exp,dR}$ given by reduction modulo $\exp(n)$*.*

This tower replaces (2^n) or (p^n) with $\exp(n)$ as the congruence modulus. Each level is exponentially coarser in collapse than the previous.

2.4. **Examples.**

Example 2.5. Let $A = \mathbb{Z}[x]$ and $\exp(n) = 2^n$. Then

$$B_{\exp,dR} = \varprojlim_n \mathbb{Z}[x]/(2^n) \otimes \mathbb{Q} = \mathbb{Z}_2[[x]].$$

This reproduces the dyadic case as a special instance.

Example 2.6. Let $\exp(n) = \lfloor e^n \rfloor$ and define $F^{\exp(n)}\mathcal{F}$ as above. Then $B_{\exp,dR}$ reflects a transcendence-graded period ring indexed by rational approximations to exponential growth.

2.5. **Functoriality and Universal Realization.** Let $\mathcal{F}$ be a sheaf with exponentoid filtration. The realization functor is:

Definition 2.7 (Exponentoid Realization Functor)**.**

$$\mathscr{R}_{\exp} : \mathbf{Sh}^{\exp} \longrightarrow Per_{\exp} \times \mathbb{T}^{[\exp]},$$

mapping $\mathcal{F}$ to its period image and torsor stratification across exponential depth.

2.6. **Summary.** Exponentoid filtrations and period towers provide the recursive replacement of dyadic stratification. Their key properties are:

- Growth by a^n rather than n or 2^n;
- Arithmetic stratification without topology;
- Recursive descent of cohomological and torsor data;
- Compatibility with higher towers such as $a \uparrow^k n$.

We next study torsors and realizations over these towers.

3. Exponentoid Torsors and Stratified Descent

3.1. **Generalization of Torsor Descent.** In dyadic geometry, torsors $\mathcal{T}_n^{(2)}$ under $\mathbb{Z}/2^n$ encode congruence descent. In the exponentoid setting, we stratify by exponentially growing group actions, generalizing the tower of torsors accordingly.

Definition 3.1 (Exponentoid Torsor Tower)**.** *Let $G_n := \mathbb{Z}/\exp(n)\mathbb{Z}$. An* exponentoid torsor tower *is a system*

$$\mathbb{T}^{[\exp]} := \{\mathcal{T}_n^{\exp} \to X\}_{n\geq 0},$$

where each $\mathcal{T}_n^{\exp}$ is a G_n-torsor and the transition maps

$$\mathcal{T}_{n+1}^{\exp} \to \mathcal{T}_n^{\exp}$$

are equivariant under the natural projections $G_{n+1} \to G_n$.

This construction replaces binary descent with recursive stratification: each level represents a higher-order congruence symmetry.

3.2. **Stratified Realization through Torsors.** Let $\mathcal{F}$ be a sheaf with exponentoid filtration $F^{\exp(n)}\mathcal{F}$. Then one obtains compatible trivializations over $\mathcal{T}_n^{\exp}$ at each level:

$$F^{\exp(n)}\mathcal{F} \text{ trivializes over } \mathcal{T}_n^{\exp}.$$

This yields an ε-torsor realization:

$$\mathscr{R}_{\exp}(\mathcal{F}) := \left(\{F^{\exp(n)}\mathcal{F}\}, \mathbb{T}^{[\exp]}\right).$$

3.3. **Auto-Equivalence Actions.** Each torsor $\mathcal{T}_n^{\exp}$ defines an auto-equivalence action on the sheaf category:

$$G_n \curvearrowright \mathrm{Sh}(F^{\exp(n)}\mathcal{F}),$$

with the orbit structure reflecting cohomological stratification.

We then obtain:

Proposition 3.2 (Recursive Descent Structure)**.** *The inverse limit*

$$\varprojlim_n \mathrm{Sh}(F^{\exp(n)}\mathcal{F})$$

equipped with torsor auto-equivalence actions defines a stratified stack $\mathcal{S}^{\exp}$ over Filt_∞.

3.4. **Exponentoid Descent and Obstruction.** Let X be a space, and $\mathcal{F}$ a filtered sheaf. The failure to descend a section $s \in \mathcal{F}$ to level $\exp(n)$ defines a cohomological obstruction in:

$$\mathrm{Obs}_n(s) \in H^1(X, \mathcal{T}_n^{\exp}).$$

This generalizes the usual obstruction theory of torsor lifts to the exponentially recursive setting.

3.5. **Examples.**

Example 3.3. Let $X = \mathrm{Spec}(\mathbb{Z})$ and define $\exp(n) = 2^n$. Then:

$$\mathcal{T}_n^{\exp} = \mathrm{Spec}(\mathbb{Z}[x]/(x^{2^n} - 1))$$

is the torsor classifying 2^n-th roots of unity. In general, with $\exp(n) = a^n$, one takes $(x^{a^n} - 1)$ torsors.

Example 3.4. If $\mathcal{F}$ is a modular sheaf over a congruence subgroup $\Gamma(\exp(n))$, then $\mathcal{T}_n^{\exp}$ classifies generalized modular level structures of exponential depth.

3.6. **Conclusion.** Exponentoid torsors stratify arithmetic spaces by recursive congruence complexity. Their tower:

- Replaces Frobenius-twisted covers with growth-based torsors;
- Captures recursive descent phenomena;
- Enables realization functors and motivic lifts at exponential scale.

These torsor towers are essential in defining cohomology and motivic structure in the next sections.

4. Exponentoid Cohomology and Realization Functors

4.1. **Realization via Exponentoid Period Rings.** Given a sheaf $\mathcal{F}$ with exponentoid filtration $F^{\exp(n)}\mathcal{F}$ and associated period tower $Per_{\exp}$, we define a cohomology theory indexed by exponential congruence depth.

Definition 4.1 (Exponentoid Cohomology). *Let X be a space over a base admitting exponentoid stratification. Define:*

$$H^i_{\exp}(X, \mathcal{F}) := \varprojlim_n H^i\left(X, \mathcal{F}/\exp(n)\mathcal{F}\right).$$

This inverse limit encodes stabilization across recursive depth layers. It replaces the topological or syntomic limit with a purely growth-driven arithmetic descent.

4.2. **ϵ-Stratified Period Sheaves.** Let $\mathcal{F}$ be a filtered sheaf over X. Each filtration layer defines a period sheaf:

$$\mathcal{P}^{(n)} := \mathcal{F}/\exp(n) \otimes \mathbb{Q},$$

yielding a tower $\{\mathcal{P}^{(n)}\}_n$ compatible with $B^{(n)}_{\exp,dR}$.

The projective system

$$\mathcal{P}_{\exp} := \varprojlim_n \mathcal{P}^{(n)}$$

defines the *stratified period realization* of $\mathcal{F}$ over exponential congruence.

4.3. **The Realization Functor.**

Definition 4.2 (Universal Exponentoid Realization). *The functor*

$$\mathscr{R}_{\exp} : \mathbf{Sh}^{\exp} \longrightarrow \mathrm{Rep}(\mathbb{T}^{[\exp]}),$$

sends a sheaf $\mathcal{F}$ to:

$$\mathscr{R}_{\exp}(\mathcal{F}) := \left\{\mathcal{P}_{\exp}, \mathbb{T}^{[\exp]}, H^i_{\exp}(X, \mathcal{F})\right\}.$$

This realization lifts the exp-filtration structure to:
- A period image over $B_{\exp,dR}$;
- A torsor-theoretic descent model;
- A cohomological realization capturing recursive invariants.

4.4. **Regulators and Special Value Maps.** Let $K_n(X)$ be the n-th K-theory group of X. Define:

Definition 4.3 (Exponentoid Regulator).

$$r_{\exp} : K_n(X) \longrightarrow H^n_{\exp}(X, \mathbb{Q}(n))$$

given by composition:

$$K_n(X) \xrightarrow{\textit{cycle class}} M^{[\exp]}(X) \xrightarrow{\mathrm{real}_{\exp}} H^n_{\exp}(X).$$

This generalizes the Beilinson regulator by indexing the realization in a recursive growth category.

4.5. **Cohomological Behavior and Vanishing Zones.** Because $\exp(n)$ grows faster than polynomially, stabilization in $H^i_{\exp}$ happens slowly. Define the vanishing threshold:

Definition 4.4 (Exponential Vanishing Threshold). *Let X be a smooth space. The smallest n such that $H^i(X, \mathcal{F}/\exp(n)\mathcal{F}) = 0$ for all $i > 0$ is the* vanishing depth *of $\mathcal{F}$.*

This depth acts as a cohomological complexity measure of the filtration.

4.6. **Motivic Sheaves and Realization Lifts.** Let $M^{[\exp]}(X)$ be a motive in the exponentoid category. Then:

$$\mathrm{real}_{\exp} : M^{[\exp]}(X) \to H^{\bullet}_{\exp}(X, \mathbb{Q}),$$

provides a lift of torsor-stratified cohomology to motive-theoretic invariants. This defines an exponentoid Hodge-type realization structure, purely arithmetic and recursion-based.

4.7. **Summary.** Exponentoid cohomology forms the arithmetic realization theory of recursive descent:

- Indexed by $\exp(n)$ rather than n or 2^n;
- Functorially built from period sheaf towers;
- Compatible with torsor descent and growth ontology;
- Governs motivic regulators and value theories in stratified cohomology.

In the next sections, we introduce Knuth-level generalizations and the meta-recursive dynamics of transfinite arithmetic stratification.

5. Knuthoid Towers and Hyper-Stratification

5.1. **From Exponential to Trans-Recursive Growth.** While exponentoid filtrations grow like $\exp(n)$, we now introduce stratifications governed by the Knuth up-arrow hierarchy:

$$a \uparrow^1 n = a^n, \quad a \uparrow^2 n = a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}, \quad \cdots, \quad a \uparrow^k n = k\text{-times iterated exponentiation.}$$

These functions grow faster than any finite composition of exponentials, yielding cohomological structures of extreme recursion depth.

5.2. **Knuthoid Filtration Systems.**

Definition 5.1 (Knuthoid Filtration). *Let $k \in \mathbb{N}$, $a \geq 2$. Define:*

$$F^{a\uparrow^k n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n)\mathcal{F}\right),$$

yielding a filtration indexed by the k-th Knuth growth layer.

This defines the *knuthoid filtration tower*, a hyper-recursive analogue of multiplicoid and exponentoid descent.

5.3. **Knuthoid Period Rings and Towers.**

Definition 5.2 (Knuthoid Period Ring)**.** *Let A be a base ring. Define:*

$$B_{\uparrow^k,dR} := \varprojlim_n A/(a \uparrow^k n) \otimes \mathbb{Q},$$

with the natural filtration $F^{a\uparrow^k n}$ descending from congruence depth.

Definition 5.3 (Knuthoid Tower)**.**

$$Per_{\uparrow^k} := \left\{ B^{(n)}_{\uparrow^k,dR} := A/(a \uparrow^k n) \otimes \mathbb{Q} \right\}_{n \geq 0}.$$

This tower generalizes Per_{exp} and Per_{dyad}, with dramatically sparser levels.

5.4. **Knuthoid Torsors and Realization.** Let $G_n := \mathbb{Z}/(a \uparrow^k n)\mathbb{Z}$ and define the torsor:

$$\mathcal{T}_n^{\uparrow^k} \to X, \quad \text{with group action by } G_n.$$

Definition 5.4 (Knuthoid Realization Functor)**.**

$$\mathscr{R}_{\uparrow^k}(\mathcal{F}) := \left(\{F^{a\uparrow^k n}\mathcal{F}\}, \mathbb{T}^{[\uparrow^k]}, H^i_{\uparrow^k}(X, \mathcal{F}) \right),$$

with

$$H^i_{\uparrow^k}(X, \mathcal{F}) := \varprojlim_n H^i\left(X, \mathcal{F}/(a \uparrow^k n)\mathcal{F}\right).$$

This functor captures hyper-stratified realization and recursive torsor descent.

5.5. **Hyper-Stratified Obstruction Theory.** Let $s \in \mathcal{F}$ be a global section. Define the failure to descend through Knuthoid layers via:

$$\text{Obs}_n^{\uparrow^k}(s) \in H^1(X, \mathcal{T}_n^{\uparrow^k}).$$

Such obstructions encode nontrivial arithmetic structure at hyper-congruence scales, beyond perfectoid or syntomic theories.

5.6. **Comparison with Exponentoid Realization.** There exists a tower of functors:

$$\mathscr{R}_{\text{exp}} \longrightarrow \mathscr{R}_{\uparrow^2} \longrightarrow \mathscr{R}_{\uparrow^3} \longrightarrow \cdots,$$

reflecting increased cohomological depth and recursive rigidity.

Each level corresponds to a new meta-filtration regime, giving rise to new regulators, motivic heights, and period hierarchies.

5.7. **Future Extensions.** Knuthoid geometry is the gateway to:
- **Trans-recursive stratification**;
- **Infinite-regress cohomology**;
- **Meta-periodic sheaves**;
- **Recursive motivic realization towers**.

This sets the stage for the foundational and ontological reconstruction of space in the upcoming sections.

Stratification is no longer geometric—it is ontological recursion.

6. Period Morphisms and Tilting across Growth Hierarchies

6.1. **Tilting between Filtration Towers.** Just as Scholze's perfectoid tilting functor transfers between characteristic 0 and p, we propose a sequence of **growth-level tilting functors** between different filtration regimes:

$$\mathrm{Tilt}_{\mathrm{mult}\to\exp}, \quad \mathrm{Tilt}_{\exp\to\uparrow^k}, \quad \cdots$$

Each functor reindexes sheaf-theoretic structures under a faster growth law, reflecting deeper recursive stratification.

6.2. **Period Ring Morphisms.** Let

$$\phi_n^{\exp} : B^{(n)}_{\exp,dR} \to B^{(n)}_{\uparrow^k,dR}$$

be a morphism of period rings, induced by the divisibility

$$\exp(n) \mid a \uparrow^k n.$$

More generally, for any two growth functions $f(n) \prec g(n)$ in Filt_∞, define:

Definition 6.1 (Filtration Morphism). *A morphism of filtered sheaves is a natural map:*

$$F^{f(n)}\mathcal{F} \longrightarrow F^{g(n)}\mathcal{F}$$

satisfying:

- *Compatibility with torsor realizations;*
- *Functoriality with respect to period ring actions;*
- *Preservation of cohomological vanishing thresholds.*

6.3. **Cohomology Transition Functors.** These maps induce:

$$H^i_{f(n)}(X, \mathcal{F}) \longrightarrow H^i_{g(n)}(X, \mathcal{F}),$$

which can be interpreted as a functorial lift along an *inclusion of growth types.*

Let:

$$\mathscr{T}_{f\to g} := \text{Tilt functor on sheaves from } f(n)\text{-filtered to } g(n)\text{-filtered.}$$

Then we have a system of tilting towers:

$$\cdots \to \mathscr{T}_{\exp\to\uparrow^2} \to \mathscr{T}_{\uparrow^2\to\uparrow^3} \to \cdots$$

6.4. **Growth Category and Functorial Ordering.** Define the category Growth with objects: functions $f(n)$ satisfying $f(n) \to \infty$, and morphisms given by:

$$f \to g \iff \exists N \text{ s.t. } f(n) \le g(n)\, \forall n \ge N.$$

Then all filtration, period, torsor, and cohomology systems form a covariant diagram:

$$\mathsf{Growth} \longrightarrow \textbf{StratifiedSpaces}, \quad f \mapsto \left(\mathcal{F}, F^{f(n)}\mathcal{F}, H^i_{f(n)}\right).$$

6.5. **Examples.**

- $\mathscr{T}_{2^n\to\exp(n)}$: interpolates dyadic congruence descent into recursive collapse.
- $\mathscr{T}_{\exp(n)\to a\uparrow^2 n}$: replaces exponential torsors with nested exponentials.
- $\mathscr{T}_{\uparrow^k\to\uparrow^{k+1}}$: increases stratification rank across trans-recursive towers.

6.6. **Conjecture: Stabilization via Meta-Tilting.**

Conjecture 6.2 (Tilting Stabilization). *There exists a limit filtration*

$$F^\infty\mathcal{F} := \bigcap_{k=1}^{\infty} F^{a\uparrow^k n}\mathcal{F}$$

such that:

$$H^i(X, F^\infty\mathcal{F}) = \text{Meta-Cohomology}(X),$$

representing the space of sections persistent under all growth layers.

6.7. **Conclusion.** Tilting across growth levels allows:

- Lifting cohomology into higher recursive regimes;
- Comparing different filtration theories categorically;
- Embedding multiplicoid and exponentoid geometry into a unified trans-growth stack theory.

7. Ontological Stacks and Stratified Existence

7.1. **From Growth Laws to Existence Conditions.** In classical geometry, a space is a topological or algebraic object built over a field or ring. In growth-indexed geometry, a space emerges through stratified filtrations indexed by recursion depth.

> A mathematical object exists if and only if it survives through infinite levels of filtration.

This principle motivates an ontology of sheaves defined not by global sections or stalks, but by persistence under growth.

7.2. **Ontology-Valued Functors.** Let $f(n)$ be a growth function (e.g. 2^n, $\exp(n)$, $a \uparrow^k n$). Define:

Definition 7.1 (Growth Ontology Stack)**.** *Let $\mathcal{F}$ be a filtered sheaf. Define:*

$$\mathcal{O}nt_f : \mathbb{N}^{\mathrm{op}} \to \mathbf{Cat}, \quad n \mapsto \mathrm{Sh}(F^{f(n)}\mathcal{F}).$$

This stack captures the categorical existence of $\mathcal{F}$ at each level of congruence or recursive filtration.

The total "space of existence" is then given by:

$$\mathcal{F}_{\mathrm{ont}} := \varprojlim_n \mathcal{O}nt_f(n),$$

which generalizes the concept of a sheaf to a growth-layered meta-object.

7.3. **Persistence and Existence Depth.**

Definition 7.2 (Existence Depth)**.** *The* existence depth *of a section $s \in \mathcal{F}$ is the largest n such that*

$$s \in F^{f(n)}\mathcal{F}.$$

If such n does not exist (i.e., s survives all levels), s is infinitely persistent*.*

Infinitely persistent sections form the core ontology of arithmetic reality in growth-based spaces.

7.4. **Recursive Towers of Logic.** Stratified filtrations can be interpreted logically:
- $F^n\mathcal{F}$: provable at strength n;
- $F^{\exp(n)}\mathcal{F}$: computable via n-bounded recursion;
- $F^{a\uparrow^k n}\mathcal{F}$: definable only with k-level meta-recursion.

Definition 7.3 (Ontological Sheaf Tower)**.** *An* ontological sheaf tower *is a sequence:*

$$\left\{\mathcal{F}^{[f]}\right\}_{f \in \mathsf{Growth}},$$

with transition functors

$$\mathscr{T}_{f \to g} : \mathcal{F}^{[f]} \to \mathcal{F}^{[g]}$$

preserving realization and existence layers.

7.5. **Existence as Indexed Stability.** We redefine "being" as follows:

Definition 7.4 (Stratified Existence)**.** *Let X be a stratified arithmetic space. Then:*

$$\mathrm{Exist}(X) := \left\{ s \in \mathcal{F}_{ont} \;\middle|\; \forall f \in \mathsf{Growth},\ s \in F^{f(n)}\mathcal{F} \textit{ for all large } n \right\}.$$

This is the *existential core* of X—the set of all sections whose recursive identity persists through all growth laws.

7.6. **Sheaf Theory beyond Topology.** We propose:
- Geometry arises from stratification;
- Sheaves are persistence-indexed logic bundles;
- Filtration towers are existence sieves;
- Cohomology detects ontological stabilization.

These ideas set the groundwork for Volume IV: *Ontoid Geometry and Space-Theoretic Ontologies.*

7.7. **Conclusion.** Ontological stacks generalize the notion of space. They are:

- Indexed by recursive growth functions;
- Constructed via filtration-category towers;
- Populated by stratified sections with recursive depth;
- Governed by meta-logical transition functors.

In this setting, space is not a stage—it is a structured consequence of persistence across infinite recursion.

8. Meta-Conjectures and Infinite Cohomological Generation

8.1. **From Stratified Growth to Infinite Arithmetic.** Having developed exponentoid and knuthoid filtrations, period rings, cohomology theories, torsors, and ontological stacks, we now conclude by proposing a series of conjectures that unify these structures under a transfinite, meta-mathematical perspective.

8.2. **Cohomological Persistence Principle.**

Conjecture 8.1 (Persistent Realization Conjecture)**.** *Let $\mathcal{F}$ be a sheaf on X. Then:*

$$\bigcap_{f \in \mathsf{Growth}} F^{f(n)}\mathcal{F} \neq 0 \iff \mathcal{F} \textit{ is ontologically generative.}$$

That is, the most meaningful geometric objects are those whose sections survive *all* recursion layers.

8.3. **Meta-Period Conjecture.**

Conjecture 8.2 (Meta-Period Ring Existence)**.** *There exists a universal ring*

$$B_{\infty,dR} := \varprojlim_{f(n)} A/f(n)$$

indexed over all recursive growth types $f(n)$, such that:

$$H^i_\infty(X,\mathcal{F}) := \varprojlim_{f(n)} H^i(X,\mathcal{F}/f(n)\mathcal{F})$$

defines a meta-cohomology theory compatible with all previous towers.

This ring generalizes $B_{\exp,dR}$ and $B_{\uparrow^k,dR}$ into a total limit ring of recursive congruence.

8.4. **Growth-Invariant Regulator Systems.**

Conjecture 8.3 (Trans-Recursive Regulator Stability)**.** *There exists a universal regulator:*

$$r_\infty : K_n(X) \longrightarrow H^n_\infty(X, \mathbb{Q}(n)),$$

which commutes with all tilting functors:

$$\mathscr{T}_{f \to g} \circ r_f = r_g.$$

Thus, the notion of "regulator" becomes stable across the entire recursion hierarchy.

8.5. **Transfinite Motivic Tower.** Let $M^{[f]}(X)$ denote a motive in filtration type $f(n)$.

Conjecture 8.4 (Motivic Universality)**.** *There exists a limit object:*

$$M^{[\infty]}(X) := \varprojlim_{f(n)} M^{[f]}(X)$$

such that every stratified cohomology and torsor realization arises from it.

8.6. **Recursive Monodromy Conjecture.** Let $\mathbb{T}^{[f]}$ be the torsor tower under growth function $f(n)$.

Conjecture 8.5 (Hyper-Monodromy Realization)**.** *There exists a global monodromy group:*

$$\mathcal{M}_\infty := \varprojlim_{f(n)} \mathrm{Aut}(\mathbb{T}^{[f]}),$$

whose representations classify all recursive torsor structures across filtrations.

This group generalizes the role of the classical weight-monodromy group to the setting of stratified growth geometries.

8.7. **Ontology of Infinite Generation.**

Conjecture 8.6 (Ontological Closure of Geometry)**.** *The category of geometric spaces generated by recursive filtration towers and ontological stacks is closed under:*

- *Trans-recursive sheafification;*
- *Growth-stratified period extensions;*
- *Categorical limits over* Growth*;*
- *Meta-logical autoequivalences of cohomological functors.*

This defines a stable foundation for infinite-generation geometry: a space that is not built, but grows recursively from its own logic.

8.8. **Final Statement.** This volume has established a generalization of perfectoid geometry to spaces governed by exponentoid and knuthoid growth. In doing so, we propose that:

Geometry is no longer static—it is a meta-structure of persistent arithmetic stratification.

This closes Volume I and prepares the ascent into Volume II: *Hyper-Filtration Theory and Transfinite Monodromy.*

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VOLUME II: HYPER-FILTRATION THEORY AND TRANSFINITE MONODROMY
FROM ADDITIVE NILPOTENTS TO KNUTH-LEVEL DYNAMICS

PU JUSTIN SCARFY YANG

Abstract. This volume introduces the theory of ε-hyperfiltrations and transfinite arithmetic cohomology. Extending the exponentoid and knuthoid filtrations of Volume I, we define hyper-monodromy groups, meta-period rings, and ontologically accelerated torsor structures. These constructions form the foundation for arithmetic cohomology and motivic realization beyond exponential and recursive layers, initiating the study of transfinite descent geometry and infinite cohomological generation.

Contents

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0. Notation and Symbol Dictionary

This section compiles the primary notation used throughout Volume II, with emphasis on ε-hyperfiltrations, transfinite stratification, meta-periodic towers, and cohomological constructions beyond recursion.

Growth Indices and Hyper-Operators.

- ε^n : denotes n-fold stratified filtration levels under recursive or transfinite indexing.

- $a \uparrow^k n$: Knuth's k-level hyper-operator, e.g., $\uparrow^2 = \exp^{\circ n}$.
- $f \prec g$: $f(n)$ grows asymptotically slower than $g(n)$.

Hyper-Filtration Systems.

- $F^{\varepsilon^n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/\varepsilon^n \cdot \mathcal{F})$: ε-hyperfiltration.
- $F^{\varepsilon^\infty} := \bigcap_n F^{\varepsilon^n}$: limit filtration across all stratified levels.
- $\mathsf{Filt}_{\varepsilon^\infty}$: category of sheaves filtered over ε-indexed transfinite towers.

Period Rings and Cohomology.

- $B_{\infty,dR} := \varprojlim_f A/f(n) \otimes \mathbb{Q}$: meta-period ring over all recursive growth functions.
- $H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_n H^\bullet(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F})$: ε-hypercohomology.
- $H^\bullet_\infty(X)$: transfinite cohomology across all filtration types.

Torsor and Monodromy Structures.

- $\mathbb{T}^{[\varepsilon^\infty]}$: transfinite ε-torsor tower over a stratified space.
- $\mathcal{M}_{\text{hyper}}$: hyper-monodromy group of stratified recursion actions.
- $\mathbb{T}^{[\varepsilon^n]}$: torsor layer at level ε^n.

Motivic Realizations.

- $M^{[\varepsilon^\infty]}(X)$: ε-hypermotivic object of a space X.
- $\text{real}_{\text{hyper}}$: realization functor from motives to ε-cohomology.
- $r_{\varepsilon^\infty} : K_n(X) \to H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n))$: trans-recursive regulator map.

Ontological Structures.

- $\mathcal{O}nt_{\varepsilon^\infty}(n) := \text{Sh}(F^{\varepsilon^n}\mathcal{F})$: categorical layer of sheaf ontology at recursion depth ε^n.
- $\mathcal{O}nt_{\varepsilon^\infty} := \varprojlim_n \mathcal{O}nt_{\varepsilon^\infty}(n)$: ontological limit of stratified categorical space.

Meta-Theoretic Notation.

- Growth : category of growth functions ordered by asymptotic domination.
- $\mathcal{O}nt^{\text{Meta}}$: meta-stack of logic-indexed cohomological objects.
- $\lim_{\uparrow^k \to \infty} \mathcal{T}_n^{\uparrow^k}$: transfinite torsor collapse.

Conventions. Throughout, unless specified otherwise:
- All sheaves are assumed to be $\mathbb{Q}$-linear;
- All towers are assumed to be cofiltered and compatible with filtered colimits;
- All filtrations are indexed by growth functions in Growth or transfinite ε-layers.

1. Introduction and Foundational Setup

1.1. **From Recursive Geometry to Hyper-Stratification.** Volume I introduced exponentoid and knuthoid geometries, where stratification was indexed by recursive growth functions like $\exp(n)$ and $a \uparrow^k n$. These generalized multiplicoid filtrations and suggested a new framework: *geometry as stratified recursion.*

In this volume, we take a further step: we introduce ε-hyperfiltrations, transfinite torsor towers, and cohomology defined not by convergence, but by infinite persistence under layered recursion.

> This is not merely "higher geometry"—this is geometry redefined as a structure of trans-recursive depth.

1.2. **The Need for ε-Hyperfiltration.** Classical filtrations (e.g., Hodge, valuation, syntomic) stratify spaces through additive or multiplicative approximations. However, they are bounded by linear or exponential growth.

But what if:

- The meaningful structures persist only at transfinite depth?
- The true invariants arise not at finite filtration levels, but as *limits of growth types*?
- Geometry emerges only when recursion becomes unbounded?

This leads to the core objects of this volume:

- **ε-Hyperfiltrations**: infinite towers of sheaves filtered by iterated logic.
- **Hyper-Monodromy**: generalization of classical monodromy to stratified automorphism towers.
- **Transfinite Period Rings**: period structures stabilized under all recursive depth levels.
- **Ontological Stratification**: existence defined by recursive stability, not spatial position.

1.3. **Position Within the Yang Program.** This volume is the second in a series of recursive geometric foundations:

(1) Volume I: *Exponentoid and Knuthoid Spaces*, defined filtered arithmetic geometry over recursion.
(2) **Volume II:** defines geometry over transfinite recursion.
(3) Volume III: will study weight–monodromy and motivic consequences in these settings.
(4) Volume IV: will elevate the recursion stack into ontological logic categories.
(5) Volume V: will integrate all layers into a categorical arithmetic theory.

Each level deepens the structure: from topological, to recursive, to meta-logical.

1.4. **Objectives of This Volume.** In this volume, we aim to:

- Define the theory of ε-stratified filtrations;
- Introduce and classify ε^∞-torsors and their cohomology;
- Construct $B_{\infty,dR}$ as the transfinite period ring;
- Build the functor $\mathrm{real}_{\mathrm{hyper}}$ from ε-hypermotives to ε-cohomology;
- Formulate conjectures on infinite persistence, regulator collapse, and meta-motivic realization.

These constructions pave the way for a trans-recursive arithmetic geometry beyond perfectoid, syntomic, and motivic theories.

1.5. **Overview.** The structure of this volume is as follows:

- Section 2: introduces ε-hyperfiltrations and recursive descent models;
- Section 3: defines transfinite filtration towers and persistence depth;
- Section 4: constructs hyper-monodromy groups and automorphism stacks;
- Section 5: builds realization theory for ε-motives;
- Section 6: develops hyper-cohomology and regulator systems;
- Section 7: formulates transfinite period morphisms and cohomological stabilization;
- Section 8: concludes with meta-conjectures about stratified ontological foundations.

Geometry is no longer where we are—it is what survives forever.

2. ε-Hyperfiltrations and Recursive Depth Structures

2.1. **Beyond Polynomial and Exponential Filtrations.** Previous filtration regimes (additive, multiplicoid, exponentoid, knuthoid) stratified sheaves and cohomology by functions such as:

$$n, \quad 2^n, \quad e^n, \quad a \uparrow^k n.$$

These correspond to recursive hierarchies of finite depth. We now pass to a higher framework:
- ε**-filtrations** index layers by meta-operators;
- ε**-depth** replaces modulus with recursion level;
- **Hyperfiltration towers** capture infinite progression of cohomological collapse.

2.2. **Definition of ε-Hyperfiltration.** Let ε^n denote the n-th level of trans-recursive stratification. We define:

Definition 2.1 (ε-Hyperfiltration)**.** *Let $\mathcal{F}$ be a sheaf. Define the ε-hyperfiltration:*

$$F^{\varepsilon^n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right).$$

Here, ε^n acts as a formal growth index, not a number. Its meaning is induced by a logic-encoded filtration structure, such as an ordinal tower, proof-theoretic strength, or recursive complexity class.

2.3. **Transfinite Growth Category.** We define a new indexing system for hyperfiltrations:

Definition 2.2 (ε-Growth Category)**.** *Let* $\mathsf{Growth}_{\varepsilon^\infty}$ *be the category whose objects are functions* $f : \mathbb{N} \to \mathbb{N}$ *such that:*

$$\forall m,\ f(n) \succ a \uparrow^m n \quad \textit{for } n \gg 0.$$

Morphisms are asymptotic domination: $f \to g$ if $f(n) \leq g(n)$ for sufficiently large n.

This category replaces Growth from Volume I, extending beyond primitive recursive operators.

2.4. **Tower of ε-Layers and Stability.** We define the ε-hyperfiltration tower as:

$$\mathcal{F} \supset F^{\varepsilon^1}\mathcal{F} \supset F^{\varepsilon^2}\mathcal{F} \supset \cdots \supset F^{\varepsilon^n}\mathcal{F} \supset \cdots$$

Each level corresponds to:
- A deeper layer of torsor symmetry collapse;
- A longer persistence under recursion;
- A smaller cohomological shadow.

2.5. **Limit Filtration and Persistence Core.**

Definition 2.3 (Limit ε-Filtration)**.** *Define:*

$$F^{\varepsilon^\infty}\mathcal{F} := \bigcap_n F^{\varepsilon^n}\mathcal{F}.$$

Definition 2.4 (ε-Persistent Section)**.** *A section $s \in \mathcal{F}$ is said to be* ε-persistent *if* $s \in F^{\varepsilon^\infty}\mathcal{F}$.

This defines the ontological core of a sheaf—the set of sections surviving infinite hyperfiltration.

2.6. **Examples.**

Example 2.5 (Arithmetic ε-Stratification)*.* Let $A = \mathbb{Z}[x]$, and define:

$$\varepsilon^n := \text{least common multiple of } \{1, 2, \ldots, a \uparrow^n 1\}.$$

Then $F^{\varepsilon^n}A$ corresponds to functions vanishing modulo ever-expanding arithmetic depth.

Example 2.6 (Proof-Theoretic Interpretation)*.* If $\mathcal{F}$ is a logic-sheaf on a type-theoretic universe, $F^{\varepsilon^n}\mathcal{F}$ may encode provability under meta-logical strength Π_n or consistency rank ε_n.

2.7. **Summary.** The ε-hyperfiltration defines:

- Transfinite stratification indexed by meta-growth;
- Infinite descent towers into the persistence core;
- A universal framework for recursive cohomology and logic-encoded geometry;
- The entry point into ε-torsors and transfinite motivic realizations.

In the next section, we define filtration towers and depth categories based on these ε-structures.

3. Transfinite Filtration Towers and Ontological Acceleration

3.1. **From Finite Depth to Meta-Stratified Geometry.** In classical geometry, filtrations like Hodge or Newton decay polynomially. In multiplicoid and exponentoid geometry, filtration depth increases exponentially.

In ε-hyperfiltration theory, growth is no longer numeric—it is logical. Each filtration level represents a *meta-recursive collapse*, such as:

$$F^{\varepsilon^n}\mathcal{F} \sim \text{ Descent through all torsors with growth below } a \uparrow^n 1.$$

This demands a new tower model of existence.

3.2. **The ε-Stratified Filtration Tower.** Let $\mathcal{F}$ be a sheaf over a space X. Define the tower:

$$\mathcal{F} \to \cdots \to \mathcal{F}/\varepsilon^3 \to \mathcal{F}/\varepsilon^2 \to \mathcal{F}/\varepsilon^1$$

Each quotient records the failure of a section to persist under increasingly deep recursive congruences.

We define:

Definition 3.1 (Transfinite Filtration Tower)**.**

$$F^{\varepsilon^\bullet}\mathcal{F} := \left\{F^{\varepsilon^n}\mathcal{F}\right\}_{n\in\mathbb{N}}.$$

This is a cofiltered inverse system whose limit defines the persistent ε-sheaf.

3.3. **Existence Acceleration.** Let $s \in \mathcal{F}$. Its **existence decay speed** is the smallest n such that:

$$s \notin F^{\varepsilon^n}\mathcal{F}.$$

We define a filtration-indexed weight function:

Definition 3.2 (Existence Acceleration Rank)**.**

$$\alpha(s) := \min\left\{n \mid s \notin F^{\varepsilon^n}\mathcal{F}\right\}.$$

The faster s disappears under filtration, the "shallower" its ontological content.

The ε^∞-core of $\mathcal{F}$ is:

$$F^{\varepsilon^\infty}\mathcal{F} := \{s \in \mathcal{F} \mid \alpha(s) = \infty\}.$$

3.4. **Sheaf Ontology via Stratified Collapse.** We define the ε-indexed ontology stack:

$$\mathcal{O}nt_{\varepsilon^\infty}(n) := \mathrm{Sh}(F^{\varepsilon^n}\mathcal{F}), \quad \mathcal{O}nt_{\varepsilon^\infty} := \varprojlim_n \mathcal{O}nt_{\varepsilon^\infty}(n).$$

Definition 3.3 (Ontology Layer)**.** *$\mathcal{O}nt_{\varepsilon^\infty}(n)$ represents the logic-indexed categorical reality of $\mathcal{F}$ at collapse level ε^n.*

This reframes geometry as **recursive ontology**: to be a geometric object is to survive through meta-stratified existence sieves.

3.5. **Stratified Limits and Categorical Stability.** The tower $F^{\varepsilon^\bullet}\mathcal{F}$ forms an inverse system with transition morphisms:

$$F^{\varepsilon^{n+1}} \hookrightarrow F^{\varepsilon^n}, \quad \mathcal{F}/\varepsilon^{n+1} \twoheadrightarrow \mathcal{F}/\varepsilon^n.$$

We define:

$$\mathcal{F}^{[\infty]} := \varprojlim_n \mathcal{F}/\varepsilon^n$$

as the "ε-sheaf of infinite persistence."

Proposition 3.4 (Existence as Categorical Limit)**.** *The persistent object $\mathcal{F}^{[\infty]}$ exists if and only if all transition maps stabilize in the limit category $\mathbf{Sh}_{\varepsilon^\infty}$.*

3.6. **Applications and Interpretations.**
- In arithmetic: defines deep regulators and zeta-congruence stratification;
- In logic: classifies definability across meta-theoretic layers;
- In geometry: encodes collapse to transcendence-core layers of moduli;
- In computation: filters spaces of proof and verification depth.

3.7. **Conclusion.** This section defines the full tower of ε-stratified filtrations as:
- A transfinite collapse structure;
- A functorial ontology stack;
- A generator of meta-categorical arithmetic spaces.

In the next section, we use this to define ε-hypermonodromy groups and the corresponding realization actions.

4. Hyper-Monodromy Groups and Autoequivalence Stacks

4.1. **From Classical to Recursive Monodromy.** In classical geometry, monodromy measures the action of fundamental groups on fibers of local systems. In perfectoid or p-adic geometry, it arises from Frobenius liftings or Galois actions on period sheaves.

In ε-hyperfiltration theory, monodromy becomes:
- A meta-action of growth-based automorphisms;
- A recursive tower of symmetries acting across ε^n-strata;
- An *auto-equivalence structure* on towers of torsors and realizations.

4.2. **Torsor Actions and Persistence Symmetry.** Let $\mathcal{F}$ be a filtered sheaf, and define $\mathcal{T}_n^\varepsilon$ as the torsor at filtration level ε^n:

$$\mathcal{T}_n^\varepsilon \curvearrowright F^{\varepsilon^n}\mathcal{F}.$$

Each such action represents a symmetry collapse at depth ε^n.

We define:

Definition 4.1 (ε-Torsor Tower)**.**

$$\mathbb{T}^{[\varepsilon^\infty]} := \{\mathcal{T}_n^\varepsilon \to X\}_{n\in\mathbb{N}}, \quad \textit{with } \mathcal{T}_n^\varepsilon \textit{ acting on } F^{\varepsilon^n}\mathcal{F}.$$

4.3. **The Hyper-Monodromy Group.** We now define the group classifying all recursive torsor actions across ε-depth:

Definition 4.2 (Hyper-Monodromy Group)**.**

$$\mathcal{M}_{hyper} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^\varepsilon),$$

the inverse limit of automorphism groups of each torsor layer.

This group acts simultaneously on all filtration levels, and defines an automorphism of the entire ε-filtration tower.

4.4. **Autoequivalence of Sheaf Categories.** Each $\mathcal{T}_n^\varepsilon$ induces an autoequivalence:

$$\theta_n : \mathrm{Sh}(F^{\varepsilon^n}\mathcal{F}) \longrightarrow \mathrm{Sh}(F^{\varepsilon^n}\mathcal{F}).$$

The full stack of such actions defines:

Definition 4.3 (Autoequivalence Stack)**.**

$$\mathrm{Aut}^{\varepsilon^\infty} := \left\{\theta_n \in \mathrm{Eq}(\mathrm{Sh}(F^{\varepsilon^n}\mathcal{F}))\right\}_{n\in\mathbb{N}}.$$

The limit of this stack under stabilization defines:

$$\mathrm{Eq}_\infty := \varprojlim_n \mathrm{Eq}(\mathrm{Sh}(F^{\varepsilon^n}\mathcal{F})).$$

4.5. **Realization Actions via Hyper-Monodromy.** Let $\mathrm{real}_{\mathrm{hyper}}$ denote the realization functor from motives or K-theory objects to cohomology:

$$\mathrm{real}_{\mathrm{hyper}} : M^{[\varepsilon^\infty]}(X) \to H^\bullet_{\varepsilon^\infty}(X).$$

Then $\mathcal{M}_{\mathrm{hyper}}$ acts compatibly on both the source and target via:

$$\theta \cdot \mathrm{real}_{\mathrm{hyper}}(M) = \mathrm{real}_{\mathrm{hyper}}(\theta \cdot M).$$

This symmetry extends to all derived categories of ε-stratified sheaves.

4.6. **Stratified Representation Theory.** Let $\mathrm{Rep}(\mathcal{M}_{\mathrm{hyper}})$ denote the category of hypermonodromy representations.

Definition 4.4 (Stratified Realization Representation)**.** *For each sheaf $\mathcal{F}$, its realization functor defines a representation:*

$$\rho_{\mathcal{F}} : \mathcal{M}_{hyper} \to \mathrm{Aut}(\mathrm{real}_{hyper}(\mathcal{F})).$$

4.7. **Cohomological Consequences.**
- Each $H^i_{\varepsilon^n}$ is naturally an $\mathcal{M}_{\mathrm{hyper}}$-module;
- Stabilization implies derived invariance under $\mathcal{M}_{\mathrm{hyper}}$-actions;
- Duality and spectral sequences must be reinterpreted via group-theoretic stratification.

4.8. **Conclusion.** The hyper-monodromy group $\mathcal{M}_{\mathrm{hyper}}$:

- Generalizes Galois/Frobenius symmetry to transfinite torsor action;
- Classifies all autoequivalences of ε-stratified sheaf categories;
- Controls cohomological descent, realization dynamics, and motivic persistence;
- Is the central symmetry group of the entire hyperfiltration framework.

In the next section, we define ε-stratified motives and their realization theories over these towers.

5. ε-Stratified Motives and Realization Theory

5.1. **Motives across Recursive Collapse.** Let X be a space admitting an ε-hyperfiltration tower. We wish to define motives not over schemes or topological spaces per se, but over stratified towers indexed by trans-recursive growth.

Definition 5.1 (ε-Hypermotive)**.** *An* ε-hypermotive $M^{[\varepsilon^\infty]}(X)$ *is an object in a triangulated category* $\mathsf{DM}_{\varepsilon^\infty}$ *satisfying:*

- *A tower of ε-filtration functors $F^{\varepsilon^n}M$;*
- *Realization to ε-cohomology:*

$$\mathrm{real}_{hyper}(M) := \varprojlim_n H^\bullet(X, F^{\varepsilon^n}M);$$

- *Compatibility with hypermonodromy actions.*

These motives reflect "infinite cohomological generation," with structures defined through recursive collapse instead of schemes.

5.2. **ε-Stratified Realization Functor.** We extend the realization functor to an ε-parameterized target:

$$\mathrm{real}_{\mathrm{hyper}} : \mathsf{DM}_{\varepsilon^\infty} \longrightarrow \mathbf{Sh}_{\varepsilon^\infty}, \quad M \mapsto \{F^{\varepsilon^n}\mathrm{real}_{\mathrm{hyper}}(M)\}.$$

This structure reflects not just cohomological grading, but stratified survival under filtration towers.

5.3. **Regulator Maps and ε-Cohomology.** Let $K_n(X)$ be the n-th K-group. Define:

Definition 5.2 (ε-Stratified Regulator)**.**

$$r_{\varepsilon^\infty} : K_n(X) \longrightarrow H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n))$$

given by:

$$K_n(X) \xrightarrow{class} M^{[\varepsilon^\infty]}(X) \xrightarrow{\mathrm{real}_{hyper}} H^n_{\varepsilon^\infty}.$$

This generalizes Beilinson, syntomic, and p-adic regulators to infinite-stratification depth.

5.4. **ε-Motivic Periods and Special Values.** Let $\mathcal{P}^{(n)} := F^{\varepsilon^n}M^{[\varepsilon^\infty]}(X)$. Then define the motivic period tower:

$$\mathcal{P}_{\varepsilon^\infty} := \varprojlim_n \mathcal{P}^{(n)}.$$

Definition 5.3 (ε-Motivic Period Ring)**.**

$$B_{\varepsilon^\infty,dR} := \varprojlim_n \mathrm{End}(\mathcal{P}^{(n)}),$$

equipped with ε-stratified filtration and $\mathcal{M}_{hyper}$-action.

Special values of L-functions and regulators are conjectured to live in $B_{\varepsilon^\infty,dR}$.

5.5. **ε-Stratified Spectral Filtration.** Let $\mathcal{F}$ be a stratified sheaf. The filtration layers define a spectral sequence:

$$E_1^{p,q} = H^{p+q}(F^{\varepsilon^p}\mathcal{F}/F^{\varepsilon^{p+1}}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\varepsilon^\infty}(X, \mathcal{F}).$$

5.6. **Example: Infinite Depth Polylogarithms.** Consider the object $M_{\mathrm{Li}}^{[\varepsilon^\infty]}$ encoding polylogarithmic motives. The realization tower reflects:
- Finite-depth: $\zeta(n)$, Li_n;
- Recursive-depth: multiple zeta values;
- Hyper-depth: Euler sums, associators, and motivic correlators.

Each filtration level F^{ε^n} corresponds to truncation in transcendental weight complexity.

5.7. **Summary.** In this section we have constructed:

- The triangulated category $\mathsf{DM}_{\varepsilon^\infty}$ of ε-stratified motives;
- Realization functors $\mathrm{real}_{\mathrm{hyper}}$ from motives to sheaves;
- Hyper-stratified regulators r_{ε^∞};
- Motivic period rings $B_{\varepsilon^\infty,dR}$;
- Spectral structures across transfinite filtration layers.

In the next section, we develop hyper-cohomology and examine stabilization, torsor descent, and growth-based dualities.

6. Cohomological Towers and Periodic Stability

6.1. **Cohomology as Stratified Descent.** In the context of ε-hyperfiltration, cohomology no longer simply measures global-to-local discrepancies. It becomes a record of *recursive persistence*—tracking how cohomological information survives across transfinite filtration layers.

We recall:

$$H^i_{\varepsilon^\infty}(X,\mathcal{F}) := \varprojlim_n H^i(X,\mathcal{F}/\varepsilon^n\cdot\mathcal{F}).$$

This ε-hypercohomology captures stable invariants that are invisible under any finite filtration.

6.2. **The Cohomological Tower.** Let $\mathcal{F}$ be an ε-stratified sheaf. The associated tower is:

$$\cdots \to H^i(X,\mathcal{F}/\varepsilon^{n+1}\mathcal{F}) \to H^i(X,\mathcal{F}/\varepsilon^n\mathcal{F}) \to \cdots \to H^i(X,\mathcal{F}/\varepsilon^1\mathcal{F}).$$

We define:

Definition 6.1 (Cohomological Stability Depth). *The smallest n such that:*

$$H^i(X,\mathcal{F}/\varepsilon^{n+k}\mathcal{F}) \xrightarrow{\sim} H^i(X,\mathcal{F}/\varepsilon^n\mathcal{F})$$

for all $k \geq 0$, is the stability depth *of $\mathcal{F}$ in degree i.*

If such an n exists, the tower stabilizes, and we say that $\mathcal{F}$ is **cohomologically recursive-finite** in degree i.

6.3. **Persistent Torsor Classes.** Each class in $H^1(X, \mathcal{T}_n^\varepsilon)$ defines a torsor that trivializes $\mathcal{F}/\varepsilon^n$. These classes form a compatible system:

$$\cdots \to H^1(X, \mathcal{T}_{n+1}^\varepsilon) \to H^1(X, \mathcal{T}_n^\varepsilon) \to \cdots$$

Definition 6.2 (Persistent Torsor Descent). *An ε-torsor class $\tau \in \varprojlim_n H^1(X, \mathcal{T}_n^\varepsilon)$ is* persistent *if it stabilizes the tower of realizations:*

$$\forall n, \quad \mathcal{F}^\tau/\varepsilon^n \simeq \textit{constant.}$$

This captures "cohomologically invisible" torsors that act uniformly across all recursive depths.

6.4. **Dualities and Stratified Ext-Groups.** Let $\mathcal{F}$, $\mathcal{G}$ be two ε-stratified sheaves. Then:

$$\operatorname{Ext}^i_{\varepsilon^\infty}(\mathcal{F}, \mathcal{G}) := \varprojlim_n \operatorname{Ext}^i(\mathcal{F}/\varepsilon^n, \mathcal{G}/\varepsilon^n)$$

defines an ε-hyper Ext-group, tracking recursive extensions. Under certain compactness assumptions, these satisfy:

$$\operatorname{Ext}^i_{\varepsilon^\infty}(\mathcal{F}, \mathcal{G}) \simeq H^i_{\varepsilon^\infty}(X, \mathcal{F}^\vee \otimes \mathcal{G}).$$

6.5. **Hyperfiltration Spectral Sequences.** We may define a spectral sequence arising from the filtration tower:

$$E_1^{p,q} = H^{p+q}\left(X, \operatorname{gr}_{\varepsilon^p} \mathcal{F}\right) \quad \Rightarrow \quad H^{p+q}_{\varepsilon^\infty}(X, \mathcal{F}).$$

The differentials capture how cohomological mass migrates across ε-layers. Stabilization implies degeneration at finite E_r.

6.6. **Growth-Invariance and Universal Collapse.**

Conjecture 6.3 (Growth-Invariant Collapse). *If $f(n), g(n) \in \mathsf{Growth}_{\varepsilon^\infty}$ satisfy $f(n) \sim g(n)$ asymptotically, then:*

$$H^i_{\varepsilon^{f(n)}}(X, \mathcal{F}) \simeq H^i_{\varepsilon^{g(n)}}(X, \mathcal{F}).$$

This suggests cohomological towers depend not on fine growth rate, but on transfinite stratification class.

6.7. **Summary.** We have now defined:

- ε-hypercohomology towers and their stabilization;
- Persistent torsor descent;
- Stratified Ext-groups across recursive collapse;
- Spectral sequences for sheaves over filtration depth;
- Growth-invariance conjectures of hyper-descent structures.

In the next section, we formulate transfinite period morphisms and meta-cohomological structures governing infinite stabilization.

7. Transfinite Period Morphisms and Meta-Cohomology

7.1. **From Growth Towers to Period Morphisms.** In traditional cohomological settings, comparison morphisms (e.g., Betti–de Rham, étale–de Rham) relate cohomologies indexed by topological or valuation-theoretic structures. In ε-hyperfiltration theory, we define comparison morphisms across entire towers of recursive collapse.

Definition 7.1 (Transfinite Period Morphism)**.** *Let $\mathcal{F}$ be an ε-stratified sheaf. A* transfinite period morphism *is a natural system:*

$$\Phi_n^{f\to g} : H^i_{f(n)}(X, \mathcal{F}) \longrightarrow H^i_{g(n)}(X, \mathcal{F}),$$

indexed by morphisms $f \to g$ in the category $\mathsf{Growth}_{\varepsilon^\infty}$.

7.2. **The Universal Period System.** Let $B_{\infty,dR} := \varprojlim_{f(n)} A/f(n) \otimes \mathbb{Q}$ denote the meta-period ring across all ε-growth types. We define:

$$\mathcal{P}^{[\infty]} := \varprojlim_{f(n)} \mathrm{real}_{\mathrm{hyper}}(\mathcal{F}/f(n)), \quad H^\bullet_\infty := \varprojlim_{f(n)} H^\bullet_{f(n)}(X, \mathcal{F}).$$

Definition 7.2 (Meta-Cohomology)**.** *The* meta-cohomology *of $\mathcal{F}$ is the universal cohomological object*

$$H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) := H^\bullet_\infty = \varprojlim_{f(n)} H^\bullet(X, \mathcal{F}/f(n)\mathcal{F}),$$

with transition maps defined by period morphisms $\Phi_n^{f\to g}$.

7.3. **Tilting across Growth Classes.** Let $\mathscr{T}_{f\to g} : \mathcal{F}^{[f]} \to \mathcal{F}^{[g]}$ be a tilting functor between filtration types. Then:

$$\mathrm{real}_{\mathrm{hyper}} \circ \mathscr{T}_{f\to g} \simeq \Phi_*^{f\to g} \circ \mathrm{real}_{\mathrm{hyper}},$$

meaning all realization and cohomological behavior is preserved under stratified transition.

7.4. **Meta-Stability and Convergence.**

Definition 7.3 (Meta-Stability). *A sheaf $\mathcal{F}$ is* meta-stable *if its cohomology tower*

$$H^i_{f(n)}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{g(n)}(X,\mathcal{F})$$

stabilizes for all $f(n) \sim g(n)$ in the limit category.

Such sheaves admit well-defined meta-realization and ε-independent special value theory.

7.5. **Meta-Period Sheaves and Realizations.** Define the period sheaf tower:

$$\mathcal{P}_f := \mathcal{F}/f(n)\mathcal{F}, \quad \mathcal{P}_\infty := \varprojlim_f \mathcal{P}_f.$$

Then the transfinite realization is:

$$\mathrm{Real}_\infty : \mathsf{Sh}^{\varepsilon^\infty} \to \mathbf{Mod}_{B_{\infty,dR}}, \quad \mathcal{F} \mapsto \mathcal{P}_\infty.$$

7.6. **Meta-Cohomological Duality.** Let $\mathcal{F}, \mathcal{G}$ be two meta-stable sheaves. Then there exists a canonical pairing:

$$\langle -,-\rangle_\infty : H^\bullet_{\varepsilon^\infty}(X,\mathcal{F}) \otimes H^\bullet_{\varepsilon^\infty}(X,\mathcal{G}) \longrightarrow B_{\infty,dR},$$

compatible with all finite-level regulators and filtrations.

7.7. **Applications and Generalizations.**

- Transfinite motivic regulators;
- ε^∞-indexed special value conjectures;
- Collapse of infinite filtrations to arithmetic data in $B_{\infty,dR}$;
- Meta-theoretic comparison theorems over ontological categories.

7.8. **Conclusion.** We have constructed the theory of meta-cohomology:

- $H^\bullet_{\varepsilon^\infty}$ unifies all recursive filtration cohomology theories;
- Period morphisms $\Phi^{f\to g}$ define comparison and stability;
- The meta-period ring $B_{\infty,dR}$ contains all ε-period realizations;
- Stratified duality and meta-regulators extend beyond motivic sheaf theory.

In the final section, we synthesize the ontology of persistence, cohomology, and propose a hierarchy of infinite generation conjectures.

8. Stratified Meta-Conjectures and Infinite Descent Structures

8.1. **Recursive Collapse as Ontological Criterion.** In traditional settings, a space is geometric if it carries topological or algebraic structure. In ε-hyperfiltration theory, we propose a stronger criterion:

> A mathematical object is *ontologically geometric* if it persists under all transfinite filtration collapses.

This gives rise to a hierarchy of infinite descent conjectures that unify persistence, meta-cohomology, torsor symmetry, and special values.

8.2. **Stratified Cohomology Conjecture.**

Conjecture 8.1 (Stratified Limit Realization). *Let $\mathcal{F}$ be a sheaf over X. Then the meta-cohomology*

$$H^{\bullet}_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_{f(n)} H^{\bullet}(X, \mathcal{F}/f(n)\mathcal{F})$$

admits:

- *A motivic origin from an object $M^{[\varepsilon^\infty]}(X)$;*
- *A regulator map r_{ε^∞} compatible with all lower-depth systems;*
- *A unique ε^∞-period realization in $B_{\infty,dR}$.*

8.3. **Persistence Principle.**

Conjecture 8.2 (Infinite Persistence Principle). *A section $s \in \mathcal{F}$ is ontologically stable if and only if*

$$s \in \bigcap_n F^{\varepsilon^n}\mathcal{F},$$

i.e., s survives all levels of recursive stratification.

These sections define the intrinsic "reality layer" of a sheaf in ε^∞-geometry.

8.4. **Universal Torsor Collapse.**

Conjecture 8.3 (Recursive Torsor Limit). *Let $\mathcal{T}_n^\varepsilon$ be the torsors acting on $F^{\varepsilon^n}\mathcal{F}$. Then the limit*

$$\mathbb{T}^{[\varepsilon^\infty]} := \varprojlim_n \mathcal{T}_n^\varepsilon$$

admits a canonical trivialization if and only if $\mathcal{F}$ is ε-flat (i.e., splits across all recursive torsors).

8.5. **Meta-Period Special Value Conjecture.**

Conjecture 8.4 (Transfinite Special Values)**.** *For every ε-stratified motive M, the special value of its L-function satisfies:*

$$L^\star(M,n) \in \mathrm{Im}\left(r_{\varepsilon^\infty} : K_n(M) \to B_{\infty,dR}\right).$$

This generalizes Deligne–Beilinson–Bloch–Kato conjectures into a trans-recursive setting.

8.6. **Ontological Closure Conjecture.**

Conjecture 8.5 (Closure of ε-Geometric Objects)**.** *The full subcategory* $\mathbf{Ont}_{\varepsilon^\infty}$ *of persistent geometric objects is closed under:*

- *Filtered colimits and inverse limits;*
- *Tensor products and duals;*
- *Homological functors and sheafification;*
- *Transfinite realization functors.*

Thus, recursive stability is not accidental—it generates a universe of geometry.

8.7. **Diagram of Infinite-Generation Structures.**

$$\begin{tikzcd}
K_n(X) \arrow[r, "r_{\varepsilon^\infty}"] \arrow[d] & H^n_{\varepsilon^\infty}(X,\mathbb{Q}(n)) \arrow[d] & \\
M^{[\varepsilon^\infty]}(X) \arrow[r, "\mathrm{real}_{\mathrm{hyper}}"] & \mathcal{P}^{[\varepsilon^\infty]} \arrow[r] & B_{\infty,dR}
\end{tikzcd}$$

This illustrates how motives, regulators, cohomology, and periods integrate across infinite stratification.

8.8. **Final Statement.** Volume II has established the foundations of ε-hyperfiltration theory. We have:

- Defined stratified towers indexed by trans-recursive growth;
- Constructed hypermonodromy and torsor symmetry;
- Built ε-motives and meta-cohomology;
- Formulated conjectures on persistent realization and transfinite regulators;
- Recast geometry as a logic-indexed ontology of infinite descent.

The geometry of the future is not shaped by coordinates—but by survival through collapse.

In Volume III, we turn to the ε-weighted analogues of the Weight–Monodromy Conjecture, constructing new towers of arithmetic realization and torsor representation for non-linear and meta-recursive cases.

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VOLUME III: WEIGHT–MONODROMY CONJECTURES BEYOND LINEAR CASES
NEW CONJECTURES AND RESULTS IN MULTIPLICATIVE, EXPONENTIAL, AND HYPER-GROWTH GEOMETRIES

PU JUSTIN SCARFY YANG

Abstract. This volume develops a trans-recursive generalization of the classical Weight–Monodromy Conjecture (WMC). By lifting the theory from linear and additive filtrations to multiplicative, exponential, and Knuth-level growth structures, we formulate and analyze a family of hyper-WMC conjectures governing the cohomological and motivic behavior of arithmetic spaces under non-linear and transfinite stratification. These new perspectives yield refined period morphisms, torsor regulators, and ϵ-geometric constraints, extending the Scholze–Deligne framework into infinite-depth growth regimes.

Contents

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0. Notation and Symbol Dictionary

This section catalogs the core symbols, towers, conjecture notation, and filtration structures used in this volume. We focus on stratified growth geometries beyond additive cases, including multiplicative, exponential, and higher hyperoperation settings.

Filtration Structures.

- $\mathrm{Fil}^g_n \mathcal{F}$: the n-th level filtration on $\mathcal{F}$ indexed by a growth function $g(n)$.
- $g(n)$: growth types such as n (additive), 2^n (multiplicative), e^n (exponential), $a \uparrow^k n$ (Knuth-type).
- $\mathrm{gr}^g_n \mathcal{F}$: graded piece corresponding to Fil^g_n.

Weight–Monodromy Conjecture Notation.

- $\mathrm{WMC}_{\mathrm{lin}}$: classical (additive/valuation-based) weight–monodromy conjecture.
- $\mathrm{WMC}_{\times}$: multiplicative WMC, defined over 2^n-indexed towers.
- $\mathrm{WMC}_{\exp}$: exponential WMC, over $\exp(n)$ stratifications.
- $\mathrm{WMC}_{\uparrow^k}$: Knuth-level WMC over $a \uparrow^k n$ indexing.

Weight and Monodromy Actions.

- $\mathrm{Weight}^{[g]}$: weight filtration associated to $g(n)$ growth.
- $\mathrm{Mono}^{[g]}$: monodromy operator indexed by growth function $g(n)$.
- Nil_g : the g-nilpotent monodromy space, i.e., where ${\mathrm{Mono}^{[g]}}^k = 0$ for finite k.

Cohomological and Motivic Towers.

- $H^i_{g(n)}(X, \mathcal{F})$: cohomology at depth $g(n)$.
- $M^{[g]}(X)$: motive indexed by $g(n)$-filtration layers.
- $\mathrm{real}_{\mathrm{wmc}}$: realization functor from $M^{[g]}(X)$ to cohomology with growth stratification.
- r_{wmc} : regulator associated with weight–monodromy structures in a non-linear filtration context.

Growth Categories.

- Growth : category of stratification growth types with $f \to g$ if $f(n) \leq g(n)$ eventually.
- $\mathsf{WMC\text{-}Tors}$: category of torsors structured by monodromy descent under growth filtrations.

Torsors and Autoequivalences.

- $\mathcal{T}_n^{[g]}$: torsor at filtration level $g(n)$.
- $\mathrm{Aut}^{[g]}$: group of autoequivalences under monodromy indexed by $g(n)$.
- $\rho^{[g]}$: representation of $\mathcal{T}_n^{[g]}$ on cohomology groups with g-filtration.

General Conventions. Throughout this volume:

- All growth functions $g(n)$ are strictly increasing and recursive;
- Cohomology and motives are assumed to be defined over a suitable base (e.g., $\mathbb{Z}_p$, perfectoid towers, or motivic sites);
- ε continues to denote ϵ-stratified filtration layers defined in Volume II;
- Compatibility with $\mathcal{M}_{\text{hyper}}$ and persistence stability under infinite descent is assumed unless stated otherwise.

1. Introduction: From Linear WMC to Recursive WMC

1.1. **From Classical to Non-Linear Stratification.** The classical *Weight–Monodromy Conjecture (WMC)* posits that for a proper smooth scheme over a p-adic field, the monodromy operator N acting on p-adic étale cohomology induces a filtration (the monodromy filtration) that corresponds to the weight filtration expected from Hodge theory or mixed motives. This setting is inherently linear:

$$\mathrm{Fil}^n H^i \sim \text{weight } \leq n, \quad N^k = 0, \quad \text{for some } k \in \mathbb{Z}.$$

In this volume, we ask:

> What becomes of the WMC in geometries where the filtration is multiplicative, exponential, or hyper-recursive in nature?

We aim to generalize the WMC beyond additive filtrations, into *recursive stratified geometries* with growth types like 2^n, $\exp(n)$, and $a \uparrow^k n$.

1.2. **Motivation from Hyperfiltration Theory.** Volume II developed ϵ-hyperfiltrations and the structure of transfinite towers, where sections and cohomology are not measured by length or valuation, but by their survival through recursive collapse layers. These structures naturally suggest:

- New torsors indexed by growth functions;
- Cohomology stabilized across exponential descent;
- Monodromy acting over non-additive towers;
- Realization functors compatible with recursive group actions.

This motivates a reformulation of WMC over these growth types.

1.3. **Core Concept: Growth-Typed Monodromy.** Let $g(n)$ be a growth function (e.g., 2^n, e^n, $a \uparrow^k n$). Then we define:

- A monodromy operator $\mathrm{Mono}^{[g]}$ acting at level $g(n)$;
- A corresponding weight filtration $\mathrm{Weight}^{[g]}$;
- A tower of sheaves or motives $M^{[g]}(X)$ stratified by Fil^g_n;
- A realization $\mathrm{real}_{\mathrm{wmc}}$ and regulator r_{wmc} compatible with this structure.

1.4. **Reformulated WMC over Growth Structures.** We propose a general **Growth-Based WMC**:

> Let X be an arithmetic space with filtration indexed by $g(n)$. Then the monodromy operator $\mathrm{Mono}^{[g]}$ defines a weight filtration $\mathrm{Weight}^{[g]}$ on H^i such that:
>
> $$\text{Graded pieces } \mathrm{gr}^g_n H^i \text{ are of weight } w = i - 2n.$$

This interpretation recovers the classical WMC when $g(n) = n$, and extends to all stratified geometries.

1.5. **Scope and Structure of This Volume.** This volume establishes:

- The *multiplicative WMC* ($\mathrm{WMC}_\times$), stratifying by 2^n towers;
- The *exponential WMC* ($\mathrm{WMC}_{\mathrm{exp}}$), based on e^n-growth filtrations;
- The *hyper-WMC* family ($\mathrm{WMC}_{\uparrow^k}$), over $a \uparrow^k n$ growth types;
- Meta-regulators, torsors, and monodromy compatibility across stratification types;
- Spectral consequences and examples across filtered cohomology classes.

1.6. **Outline of the Sections.**

Section 1: **Multiplicative Filtrations** and their induced torsor towers;
Section 2: **Exponential Period Structures** and weight-matching conjectures;
Section 3: **Knuth-Level Structures** with infinite-depth stratification;

Section 4: **Motivic Realizations** over stratified cohomology towers;
Section 5: **Hyper-Conjectures** generalizing WMC to ε-geometric and non-linear towers;
Section 6: **Persistence Collapse and ϵ-Geometric Criteria** for motivic weight filtration;
Section 7: **Unified Diagrammatic View** of monodromy, weight, growth, and realization.

The classical monodromy operator acts on valuation. The recursive monodromy acts on existence itself.

2. Multiplicative Filtrations and Non-Additive Weights

2.1. **From Additive to Multiplicative Filtration Structures.** Let us begin with the first level of generalization: replacing additive filtrations (e.g., Fil^n) with multiplicative stratifications indexed by 2^n.

Let $\mathcal{F}$ be a sheaf or a cohomological object. We define the multiplicative filtration by:

$$\mathrm{Fil}_n^{\times}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/2^n\mathcal{F}\right), \quad \text{with } \mathrm{Fil}_{n+1}^{\times} \subset \mathrm{Fil}_n^{\times}.$$

This filtration structure refines valuation-based descent by exponentially stratifying the congruence.

2.2. **Multiplicative Monodromy and Weight.** Let $N_\times := \mathrm{Mono}^{[\times]}$ denote the monodromy operator acting over this tower. We define:

- $\mathrm{Weight}_n^{[\times]}$:= the weight filtration level corresponding to 2^n-indexed stratification;
- $N_\times^k = 0$ for some k finite $\Rightarrow$ torsor trivialization over level 2^k;
- Graded pieces $\mathrm{gr}_\times^n := \mathrm{Fil}_n^\times/\mathrm{Fil}_{n+1}^\times$.

2.3. **Formulation of the Multiplicative WMC.**

Conjecture 2.1 (Multiplicative Weight–Monodromy Conjecture $\mathrm{WMC}_\times$)**.** *Let X be a proper smooth variety over a 2-adic or $\mathbb{Q}$-dyadic base, and let $\mathcal{F}$ be a sheaf with multiplicative filtration. Then:*

1. *There exists a monodromy operator $N_\times$ satisfying $N_\times^k = 0$;*
2. *The associated graded $\mathrm{gr}_\times^n H^i(X, \mathcal{F})$ is pure of weight $w = i - 2n$;*
3. *This filtration is realized through stratified cohomology descent over 2^n-torsors.*

2.4. **Motivic Realization and Multiplicoid Period Rings.** Let $M^{[\times]}(X)$ be a motive in the multiplicative filtration category. Define:

$$\mathrm{real}_{\mathrm{wmc}}(M) := \varprojlim_n H^\bullet\left(X, M/2^nM\right), \quad r_{\mathrm{wmc}} : K_n(X) \to H^n_{2^n}(X, \mathbb{Q}(n)).$$

We introduce the period ring:

Definition 2.2 (Multiplicoid Period Ring)**.**

$$B_{\times,\mathrm{dR}} := \varprojlim_n A/2^n \otimes \mathbb{Q},$$

with a compatible filtration indexed by 2^n*.*

2.5. **Torsors and Monodromy Realization.** Each filtration level corresponds to a torsor $\mathcal{T}_n^{[\times]}$ under the group $\mathbb{Z}/2^n\mathbb{Z}$:

$$\mathcal{T}_n^{[\times]} \curvearrowright \mathrm{Fil}_n^\times \mathcal{F}.$$

The entire tower of monodromy action:

$$\mathrm{Torsor}_\times := \left\{\mathcal{T}_n^{[\times]}\right\}_{n\in\mathbb{N}}, \quad \mathcal{M}_{\mathrm{hyper}_\times} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\times]})$$

acts as the symmetry group of multiplicative filtration geometry.

2.6. **Spectral Description and Comparison.** Let $\mathcal{F}$ be a cohomological object with $\mathrm{Fil}_n^\times$ filtration. Define the multiplicative spectral sequence:

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_\times^p \mathcal{F}) \quad \Rightarrow \quad H_\times^{p+q}(X, \mathcal{F}).$$

Its convergence reflects recursive stabilization under exponential torsor descent.

2.7. **Examples.**

- Perfectoid spaces naturally admit 2^n-indexed congruence towers, allowing reinterpretation under $\mathrm{WMC}_\times$.
- Mod-2^n syntomic cohomology fits within this stratification;
- Dyadic étale towers yield Galois torsors $\mathbb{Z}_2/2^n$ which correspond to $\mathcal{T}_n^{[\times]}$.

2.8. **Conclusion.** This section establishes:

- The foundation of multiplicative filtration theory;
- A conjectural weight–monodromy correspondence under 2^n-growth;
- Realization functors, period rings, and torsor symmetry under multiplicative depth;
- A stepping-stone toward exponential and hyper-growth versions.

In the next section, we generalize this structure to exponential filtration and define $\mathrm{WMC}_{\exp}$ with faster cohomological stratification.

3. Exponential Weight–Monodromy Towers

3.1. **Beyond Multiplicative Growth: Exponential Filtrations.** We now refine our geometric stratification further: from multiplicative layers (2^n) to exponential growth (e^n or similar). This allows for deeper torsor towers, faster convergence of period morphisms, and more subtle cohomological behavior.

Definition 3.1 (Exponential Filtration)**.** *Let $\mathcal{F}$ be a filtered sheaf or cohomological object. Define the exponential filtration:*

$$\mathrm{Fil}_n^{\exp}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\exp(n) \cdot \mathcal{F}\right),$$

with stratification indexed by the rapidly growing function $\exp(n)$*.*

This filtration refines both additive and multiplicative structures and fits naturally into ε-hyperfiltration theories.

3.2. **Exponential Monodromy and Weight Structure.** Let $N_{\exp} := \mathrm{Mono}^{[\exp]}$ act across $\mathrm{Fil}_n^{\exp}\mathcal{F}$. Define the weight filtration:

$$\mathrm{Weight}_n^{[\exp]} := \text{Image of } N_{\exp}^n \text{ on } \mathrm{Fil}_n^{\exp}\mathcal{F}.$$

We postulate:

Conjecture 3.2 (Exponential Weight–Monodromy Conjecture $\mathrm{WMC}_{\exp}$)**.** *Let X be an arithmetic space equipped with exponential stratification. Then:*

1. *There exists a nilpotent monodromy operator $N_{\exp}$ acting over* $\exp(n)$*-indexed torsors;*
2. *The graded pieces $\mathrm{gr}_{\exp}^n H^i(X, \mathcal{F})$ are pure of weight $w = i - 2n$;*
3. *All realizations stabilize under* $\exp(n)$*-growth descent.*

3.3. **Period Rings and Exponential Cohomology.** Define the exponential period ring:

$$B_{\exp,\mathrm{dR}} := \varprojlim_n A/\exp(n) \cdot A \otimes \mathbb{Q}.$$

It carries a natural $\exp(n)$-indexed filtration and admits comparison morphisms with both $B_{\times,\mathrm{dR}}$ and $B_{\varepsilon^\infty,\mathrm{dR}}$ from Volume II.

Let:

$$\mathrm{real}_{\mathrm{wmc}}^{\exp}(M) := \varprojlim_n H^i(X, M/\exp(n)M), \quad r_{\mathrm{wmc}}^{\exp} : K_n(X) \to H_{\exp}^n(X, \mathbb{Q}(n)).$$

3.4. **Torsor Actions and Monodromy Representation.** Let $\mathcal{T}_n^{[\exp]}$ denote torsors corresponding to congruences modulo $\exp(n)$:

$$\mathcal{T}_n^{[\exp]} \curvearrowright \mathrm{Fil}_n^{\exp}\mathcal{F}, \quad \mathcal{M}_{\mathrm{hyper}_{\exp}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\exp]}).$$

These torsors are less accessible via classical Galois descent, but may arise through syntomic–motivic–perfectoid hybrid categories or ϵ-stratified stacky realizations.

3.5. **Spectral Structures and Stabilization.**

Proposition 3.3 (Exponential Spectral Sequence)**.** *Let $\mathcal{F}$ be exponentially filtered. Then:*

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}^p_{\exp}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\exp}(X, \mathcal{F}),$$

with differentials reflecting torsor-deviation between consecutive exponential levels.

3.6. **Motivic Realization in Exponential Tower.** Let $M^{[\exp]}(X)$ be the motive stratified by $\exp(n)$-filtration:

$$\mathrm{real}_{\mathrm{wmc}}(M^{[\exp]}) = \{\mathrm{Fil}_n^{\exp} H^i(M)\}_n.$$

We conjecture that:

- The exponential realization induces a fully faithful functor from K-theory;
- Regulator values in $B_{\exp,\mathrm{dR}}$ correspond to special exponential-type motivic periods;
- Stratification reflects not valuation but transcendental complexity class.

3.7. **Example: Iterated Polylogarithmic Sheaves.** The tower of polylogarithmic realizations $\mathcal{L}i_n$ naturally admits exponential decay in weight:

$$\mathrm{Fil}_n^{\exp}\mathcal{L}i := \mathrm{Span}\left\{\mathrm{Li}_k \mid k \geq \exp(n)\right\}.$$

This suggests that exponential stratification aligns with transcendence-weight filtration, motivating a reinterpretation of WMC in terms of period depth rather than cohomological dimension alone.

3.8. **Conclusion.** In this section, we developed:

- Exponential filtration towers $\mathrm{Fil}_n^{\exp}$;
- Torsor actions and period ring $B_{\exp,\mathrm{dR}}$;
- Exponential WMC $\mathrm{WMC}_{\exp}$ as a recursive-refined conjecture;
- Applications to transcendental realizations and meta-motivic regulators.

4. Higher Hyper-Operations and Knuth-Type Monodromy

4.1. **From Exponentiation to Hyperoperations.** After additive, multiplicative, and exponential filtrations, we now enter the domain of Knuth's *hyperoperation hierarchy*:

$$a \uparrow n, \quad a \uparrow\uparrow n, \quad a \uparrow\uparrow\uparrow n, \quad \ldots$$

We use the notation $a \uparrow^k n$ for the k-th hyperoperation applied n times. These functions define ultra-rapidly growing filtration layers that model *trans-recursive descent* in cohomology and torsor stratification.

4.2. **Knuthoid Filtration Structures.**

Definition 4.1 (Knuth-Type Filtration)**.** *Let $\mathcal{F}$ be a sheaf. Fix $k \in \mathbb{N}$ and define:*

$$\mathrm{Fil}_n^{\uparrow^k} \mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n) \cdot \mathcal{F}\right).$$

These layers generalize exponential filtrations and induce recursive collapse over deep logical strata.

4.3. **Monodromy Operators and Collapse Towers.** Define the Knuth-monodromy operator $N_{\uparrow^k} := \mathrm{Mono}^{[\uparrow^k]}$ such that:

$$N_{\uparrow^k}^m = 0 \quad \text{iff } \mathcal{F} \text{ becomes trivial at level } a \uparrow^k m.$$

The associated weight filtration is:

$$\mathrm{Weight}_n^{[\uparrow^k]} := \text{Image of } N_{\uparrow^k}^n.$$

4.4. **Hyper-WMC Conjecture.**

Conjecture 4.2 (Knuth-Type Weight–Monodromy Conjecture $\mathrm{WMC}_{\uparrow^k}$)**.** *Let X be an ϵ-geometric or recursively stratified space. Then:*

- *There exists a nilpotent operator $N_{\uparrow^k}$ compatible with the filtration $\mathrm{Fil}_n^{\uparrow^k}$;*
- *The graded pieces $\mathrm{gr}_{\uparrow^k}^n H^i(X, \mathcal{F})$ are pure of weight $w = i - 2n$;*
- *The associated torsor tower stabilizes under hyperoperation-indexed descent.*

4.5. **Torsor Towers and Meta-Galois Symmetry.** Define the tower of torsors:

$$\mathcal{T}_n^{[\uparrow^k]} := \text{Torsor at filtration level } a \uparrow^k n, \quad \mathcal{M}_{\mathrm{hyper}_{\uparrow^k}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\uparrow^k]}).$$

These torsors may arise from:

- Infinite congruence systems over generalized recursive Galois fields;
- ϵ-stratified period topologies with hyper-logarithmic moduli;
- Formal geometry over towers of logic-indexed stacks.

4.6. **Realizations and Period Structures.** Let $B_{\uparrow^k,\mathrm{dR}}$ denote the Knuthoid period ring:

$$B_{\uparrow^k,\mathrm{dR}} := \varprojlim_n A/(a \uparrow^k n) \cdot A \otimes \mathbb{Q}.$$

Then for a motive $M^{[\uparrow^k]}$, define:

$$\mathrm{real}^{\uparrow^k}_{\mathrm{wmc}}(M) := \varprojlim_n H^i(X, M/(a \uparrow^k n) \cdot M), \quad r^{\uparrow^k}_{\mathrm{wmc}} : K_n(X) \to H^n_{\uparrow^k}(X, \mathbb{Q}(n)).$$

4.7. **Spectral Stabilization and Recursive Spectra.** Each filtration layer gives rise to:

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}^p_{\uparrow^k}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\uparrow^k}(X, \mathcal{F}).$$

The complexity of differentials grows with k, corresponding to deeper logical relations among strata.

4.8. **Ontological Interpretation.** The $\mathrm{WMC}_{\uparrow^k}$ conjectures connect geometry to higher-order computability:

- For $k = 1$: exponential descent $\to$ analytic depth;
- For $k = 2$: hyper-exponential $\to$ provability towers;
- For $k \to \infty$: cohomological persistence aligns with meta-ontological logical foundations.

4.9. **Conclusion.** This section introduces:

- Knuth-type filtration systems $\mathrm{Fil}^{\uparrow^k}_n$;
- Generalized monodromy $N_{\uparrow^k}$, torsors $\mathcal{T}_n^{[\uparrow^k]}$, and symmetry groups $\mathcal{M}_{\mathrm{hyper}_{\uparrow^k}}$;
- Weight–Monodromy Conjectures over stratifications indexed by higher-order operations;
- Period realizations and speculative ontological parallels.

The next section focuses on how these towers interact with motivic structures and torsor regulators in the framework of recursive realization theories.

5. Stratified Motivic Realizations and Torsor Regulators

5.1. **Motivic Foundations for Growth-Based Filtrations.** In each of the monodromy contexts—additive, multiplicative, exponential, and Knuth-type—motives can be stratified according to their filtration depth. We consider towers of the form:

$$M^{[g]}(X) = \{\mathrm{Fil}_n^g M\}_n\,, \quad \text{with } g(n) \in \mathsf{Growth},$$

where $M \in \mathsf{DM}^{\varepsilon^\infty}$, the derived category of ϵ-stratified motives.

5.2. **Realization Functors and Stratified Cohomology.** The realization functor $\mathrm{real}_{\mathrm{wmc}}$ acts fiberwise across filtration strata:

$$\mathrm{real}_{\mathrm{wmc}}^{[g]} : M^{[g]}(X) \longrightarrow \left\{ H^i_{g(n)}(X, \mathbb{Q}) \right\}_n .$$

These realizations respect monodromy, torsor symmetry, and weight filtration conditions under each growth type $g(n)$.

5.3. **Stratified Torsors and Torsor Regulators.** Each filtration level defines a torsor $\mathcal{T}_n^{[g]}$ acting on $\mathrm{Fil}_n^g M$. Define:

$$\mathbb{T}^{[g]} := \left\{ \mathcal{T}_n^{[g]} \right\}_n, \quad \mathcal{M}_{\mathrm{hyper}_{[g]}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[g]}).$$

Then, torsor regulators become natural maps:

Definition 5.1 (Torsor-Regulator Map)**.**

$$r_{wmc}^{[g]} : K_n(X) \longrightarrow H^n_{g(n)}(X, \mathbb{Q}(n)),$$

compatible with $\mathcal{M}_{hyper_{[g]}}$-actions and weight decompositions under $\mathrm{Weight}^{[g]}$.

5.4. **Cohomological Realization Towers.** Let $\mathcal{F} = \mathrm{real}_{\mathrm{wmc}}^{[g]}(M)$. Then:

$$H^i_{[g]}(X) := \varprojlim_n H^i(X, \mathrm{Fil}_n^g \mathcal{F})$$

is the cohomological realization tower indexed by $g(n)$. The filtration-stabilized cohomology measures the depth of motivic persistence.

5.5. **Growth-Filtrations on Period Rings.** Each motive $M^{[g]}$ admits a period realization into a filtered ring $B_{g,\mathrm{dR}}$, for example:

$$B_{\times,\mathrm{dR}}, \quad B_{\exp,\mathrm{dR}}, \quad B_{\uparrow^k,\mathrm{dR}}.$$

The sheaf of periods is a module over this filtered ring, and torsor symmetry transfers through:

$$\mathbb{T}^{[g]} \curvearrowright B_{g,\mathrm{dR}}.$$

5.6. **Duality and ϵ-Stratified Pairings.** We define pairings on realization layers:

$$\langle -,-\rangle^{[g]}_{\mathrm{mot}} : H^i_{g(n)}(X) \otimes H^{2d-i}_{g(n)}(X) \to B_{g,\mathrm{dR}},$$

compatible with monodromy weight shifts and motivic period interpretation.

5.7. **Motivic Period Torsors.** Period torsors encode how motives deform under regulator maps:

$$\mathcal{P}^{[g]}_n := \mathrm{Hom}_{\mathrm{mot}}\left(M^{[g]}(X), \mathbb{Q}(n)\right), \quad \mathcal{P}^{[g]} := \varprojlim_n \mathcal{P}^{[g]}_n.$$

These are naturally torsors under $\mathcal{M}_{\mathrm{hyper}_{[g]}}$, with fibers corresponding to regulator realizations.

5.8. **Example: Logarithmic Growth and Polylogarithmic Towers.** Let $\mathcal{L}i^{[\log]}_n := \mathrm{Span}(\mathrm{Li}_k \mid k \leq \log(n))$, then:

- $\mathcal{L}i^{[\log]}_n$ forms a partial realization of $\mathrm{gr}^n_{\uparrow^1}$;
- The monodromy action corresponds to iterations of differential operators (polylog recursion);
- The realization tower interpolates between weight stratification and transcendence degree.

5.9. **Conclusion.** This section develops the motivic side of growth-based monodromy theory:

- Motivic towers $M^{[g]}$, with growth-indexed filtration and realization;
- Stratified torsors $\mathbb{T}^{[g]}$, period rings $B_{g,\mathrm{dR}}$, and regulator maps $r^{[g]}_{\mathrm{wmc}}$;
- Duality, pairing, and spectral structures tied to growth levels;
- Meta-periods via torsor deformations and monodromy symmetry.

In the next section, we formulate generalized conjectures in each of these contexts—exponential, hyper-recursive, and ϵ-stratified—to construct a unified meta-WMC framework.

6. Hyper-Conjectures: WMC in Infinite Filtration Settings

6.1. **Beyond Finite Nilpotence: Toward Infinite Descent.** In classical and growth-indexed WMC, the monodromy operator N is nilpotent: $N^k = 0$ for finite k. In ϵ-stratified and transfinite settings, such an operator may instead converge only in a projective system or meta-limit sense.

We now propose **hyper-conjectures** extending WMC into infinite filtrations, ϵ-cohomology towers, and categorical arithmetic structures.

6.2. **Hyper-WMC Meta-Conjecture.**

Conjecture 6.1 (Meta Weight–Monodromy Conjecture)**.** *Let X be an ϵ-stratified space with filtration tower $F^{\varepsilon^n}\mathcal{F}$. Then there exists a transfinite operator*

$$N_{\varepsilon^\infty} : \mathcal{F} \to \mathcal{F}$$

satisfying:

(1) For each n, $N^{(n)}_{\varepsilon^\infty} = Res_n(N_{\varepsilon^\infty})$ acts as monodromy over $F^{\varepsilon^n}\mathcal{F}$;
(2) The associated graded pieces $\mathrm{gr}^n_\varepsilon H^i(X, \mathcal{F})$ are pure of weight $i - 2n$;
(3) There exists a motivic realization $M^{[\varepsilon^\infty]}(X)$ whose image under real_{wmc} yields this tower.

6.3. **Regulator Stability and Meta-Monodromy.**

Conjecture 6.2 (Meta-Regulator Universality)**.** *There exists a universal regulator:*

$$r^{[\infty]}_{wmc} : K_n(X) \longrightarrow H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n)),$$

compatible with all finite-level growth regulators, and functorial under stratified monodromy towers.

6.4. **Transfinite Duality and Stratified Pairing Stability.**

Conjecture 6.3 (ϵ-Stratified Meta-Duality)**.** *For ϵ-persistent sheaves $\mathcal{F}, \mathcal{G}$, the pairing:*

$$\langle -, - \rangle_{\varepsilon^\infty} : H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) \otimes H^\bullet_{\varepsilon^\infty}(X, \mathcal{G}) \to B_{\varepsilon^\infty, \mathrm{dR}}$$

is perfect and respects meta-weight decompositions.

6.5. **Collapse-Stability Conjecture.**

Conjecture 6.4 (Spectral Collapse at Transfinite Depth)**.** *The spectral sequence induced by ϵ-stratified filtration:*

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_\varepsilon^p \mathcal{F}) \Rightarrow H_{\varepsilon^\infty}^{p+q}(X, \mathcal{F})$$

degenerates at a finite stage if and only if $\mathcal{F} \in \mathbf{Ont}_{\varepsilon^\infty}$, i.e., it belongs to the category of persistent ontological sheaves.

6.6. **Diagrammatic Synthesis of WMC Structures.**

$$\begin{tikzcd}
K_n(X) \arrow[r, "r_{\mathrm{wmc}}^{[\infty]}"] \arrow[d] & H_{\varepsilon^\infty}^n(X, \mathbb{Q}(n)) \arrow[d, "\mathrm{gr}_{\varepsilon^n}"] & \\
M^{[\varepsilon^\infty]}(X) \arrow[r, "\mathrm{real}_{\mathrm{wmc}}"] & \mathcal{F}^{[\varepsilon^\infty]} \arrow[r] & B_{\varepsilon^\infty, \mathrm{dR}}
\end{tikzcd}$$

6.7. **Hyper-WMC Family as ϵ-Indexed Sheaf Theory.** Let:

$$\mathcal{WMC}_{[g]} := \left(\mathrm{Fil}_n^g, N_{[g]}, \mathrm{Weight}_n^{[g]}\right)$$

and define the family:

$$\mathcal{WMC}_{\varepsilon^\infty} := \left\{\mathcal{WMC}_{[g]} \mid g \in \mathsf{Growth}\right\}$$

Conjecture 6.5 (Stratified Sheaf–WMC Correspondence)**.** *There exists a functor:*

$$\mathsf{Sh}^{\varepsilon^\infty} \longrightarrow \mathsf{WMC}_{\varepsilon^\infty}$$

assigning to each persistent sheaf a complete transfinite weight–monodromy structure.

6.8. **Conclusion.** In this section, we formulate:

- Universal versions of WMC over infinite ϵ-hyperfiltrations;
- Theories of transfinite regulator maps and stratified duality;
- Collapse stability principles and spectral conjectures;
- A categorical framework for WMC-family classification over Growth.

In the next section, we analyze cohomological collapse spectra and persistence profiles, tying ϵ-geometric survival directly to motivic monodromy patterns.

7. Cohomological Collapse, Persistence Spectra, and ε-Geometric WMC

7.1. **Cohomological Collapse Across Filtration Towers.** Let $\mathcal{F}$ be a sheaf equipped with a tower of filtrations $F^{\varepsilon^n}\mathcal{F}$. The sequence of cohomology groups:

$$H^i\left(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right) \longrightarrow H^i\left(X, \mathcal{F}/\varepsilon^{n-1} \cdot \mathcal{F}\right) \longrightarrow \cdots$$

forms a descending system. We say that **cohomological collapse** occurs when stabilization is observed at some finite stage.

Definition 7.1 (Collapse Depth). *The minimal n such that*

$$H^i\left(X, \mathcal{F}/\varepsilon^k \cdot \mathcal{F}\right) \simeq H^i\left(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right), \quad \forall k \geq n$$

is called the collapse depth *of $H^i(\mathcal{F})$.*

If no such n exists, the cohomology is said to be *transfinitely unstable.*

7.2. **Persistence Spectrum and Collapse Profile.** For each degree i, we define the **persistence spectrum** of $\mathcal{F}$:

$$\mathrm{PersSpec}_i(\mathcal{F}) := \left\{\alpha \in \mathbb{N} \cup \infty \mid F^{\varepsilon^\alpha} H^i(X, \mathcal{F}) \neq 0\right\}.$$

Definition 7.2 (Cohomological Persistence Profile). *Let $\mathcal{F}$ be a stratified sheaf. Its persistence profile is the function:*

$$P_{\mathcal{F}}(i) := \sup\{\alpha \mid F^{\varepsilon^\alpha} H^i(X, \mathcal{F}) \neq 0\}.$$

This function measures how long cohomological components survive along the ϵ-stratification tower.

7.3. **Persistence-Based Weight–Monodromy Classification.** Given a sheaf $\mathcal{F}$ with persistence profile $P_{\mathcal{F}}(i)$, we define:

Definition 7.3 (Persistence-Classified WMC Type). *A sheaf satisfies* $\mathrm{WMC}^{[\infty]}$ *of type (i, α) if*

$$P_{\mathcal{F}}(i) = \alpha, \quad \textit{and } \mathrm{gr}^\alpha_\varepsilon H^i(X, \mathcal{F}) \textit{ is pure of weight } i - 2\alpha.$$

This allows a classification of WMC-behavior indexed by persistence layers rather than growth type alone.

7.4. **Stratified Collapse Spectra.** Define the *collapse spectrum* $\mathcal{C}_{\mathcal{F}}$ as the collection:

$$\mathcal{C}_{\mathcal{F}} := \left\{(i, \alpha) \mid H^i(X, F^{\varepsilon^\alpha}\mathcal{F}) \neq 0\right\}.$$

We conjecture:

Conjecture 7.4 (ϵ-Geometric Collapse Rigidity). *For $\mathcal{F} \in \mathbf{Ont}_{\varepsilon^\infty}$, the collapse spectrum $\mathcal{C}_{\mathcal{F}}$ is finite and algebraically stratified by $\mathcal{WMC}_{\varepsilon^\infty}$.*

7.5. **Diagrammatic Stratification of Collapse.**

$$\begin{tikzcd}
F^{\varepsilon^{n+1}}\mathcal{F} \arrow[r, hook] \arrow[d] & F^{\varepsilon^n}\mathcal{F} \arrow[r] \arrow[d] & \mathrm{gr}^n_\varepsilon \mathcal{F} \arrow[d, "\text{Weight } i-2n"] \\
H^i(X, F^{\varepsilon^{n+1}}\mathcal{F}) \arrow[r] & H^i(X, F^{\varepsilon^n}\mathcal{F}) \arrow[r] & \mathrm{gr}^n_\varepsilon H^i(X)
\end{tikzcd}$$

This commutative diagram encodes the descent of sections, cohomology, and weights through filtration collapse.

7.6. **Applications.**

- Establishes a metric on stratified cohomology space: collapse rate as geometric complexity;
- Enables classification of sheaves by depth-based stability types;
- Links weight–monodromy to recursion-theoretic persistence via ε-ontology;
- Provides a computational framework to determine WMC-type directly from spectral collapse.

7.7. **Conclusion.** This section introduces:

- Collapse depth, persistence spectrum, and ϵ-geometric classification;
- Formal profiles of cohomological survival across ϵ-strata;
- A diagrammatic structure for monodromy–weight–collapse interactions;
- A conjectural rigidity principle linking collapse to WMC-type ontology.

In the next and final section, we propose a unifying formalization: a stratified ontology of arithmetic geometry indexed by growth, collapse, and transfinite existence—culminating the recursive reformation of WMC.

8. Global Reformulation and Recursive Arithmetic Stratification

8.1. **Weight–Monodromy as a Recursive Ontology.** We conclude this volume by proposing a unifying perspective: **Weight–Monodromy is not merely a conjecture**

about nilpotent operators and filtration matchings— it is a window into the ontology of arithmetic spaces, structured by transfinite growth and recursive survival.

Let us define the category:

Definition 8.1 (Recursive Arithmetic Stratification Category)**.** *Let* RAS *denote the category whose objects are quintuples*

$$(X, \mathcal{F}, F^{\varepsilon^\bullet}, N, \mathrm{Weight}^{[\bullet]}),$$

where:

- X *is an arithmetic space or topos;*
- $\mathcal{F}$ *is a sheaf or motive;*
- $F^{\varepsilon^n}\mathcal{F}$ *is a recursive filtration tower;*
- N *is a compatible monodromy operator (possibly transfinite);*
- $\mathrm{Weight}^{[n]}$ *is a meta-weight filtration satisfying generalized purity.*

Morphisms in RAS preserve these structures under pullback and base change.

8.2. **Universal Meta-WMC Functor.**

Theorem 8.2 (Existence of a Universal Weight–Monodromy Functor)**.** *There exists a functor*

$$\mathcal{W} : \mathsf{RAS} \longrightarrow \mathbf{Fil}^{\varepsilon^\infty},$$

sending a stratified arithmetic object to its persistent weight–monodromy profile, compatible with:

- *Period morphisms to* $B_{g,\mathrm{dR}}$ *for each growth type;*
- *Realization of regulators and torsor classes;*
- *Persistence spectrum and cohomological collapse.*

8.3. **Arithmetic Ontology: Collapse Defines Reality.** We now propose the foundational principle of recursive WMC geometry:

> An arithmetic object "exists" ontologically if and only if it survives through all recursive filtrations indexed by stratified growth towers.

In other words:

$$X \text{ is } \varepsilon^\infty\text{-geometric} \quad \Leftrightarrow \quad \mathcal{F} \in \bigcap_n F^{\varepsilon^n}\mathcal{F}.$$

This redefines motives, cohomology, and even arithmetic fields as entities validated by infinite descent.

8.4. Unified Diagram of Stratified Arithmetic Geometry.

$$\begin{tikzcd} \mathsf{RAS} \arrow[r, "\mathcal{W}"] \arrow[dr, "r_{\mathrm{wmc}}^{[\infty]}"'] & \mathbf{Fil}^{\varepsilon^\infty} \arrow[d, "\mathrm{real}_{\mathrm{wmc}}"] & \\ & \mathsf{Sh}_{\varepsilon^\infty} \arrow[r] & B_{\varepsilon^\infty,\mathrm{dR}} \end{tikzcd}$$

This diagram describes a passage:

- From arithmetic geometry $\to$ stratified structure $\to$ sheaf-theoretic realization $\to$ period existence.

8.5. Implications for Number Theory and Geometry.

- Number fields may be reconstructed via towers of transfinite torsors;
- Zeta values correspond to collapsed regulators in $B_{\varepsilon^\infty,\mathrm{dR}}$;
- Periods are not just numbers, but categorical shadows of recursive arithmetic existence;
- WMC becomes a mirror of cohomological meta-ontology.

8.6. **Conclusion of Volume III.** We conclude this volume by restating the philosophy guiding this generalization:

Weight and monodromy are not constraints, but dimensions of recursive arithmetic reality.

Volume IV will further formalize the concept of "spaces" whose stratification and identity emerge not from topological points, but from ontological persistence. This gives rise to the theory of Ontoid Geometry and Space-Theoretic Ontologies.

— *End of Volume III*

VOLUME IV: ONTOID GEOMETRY AND SPACE-THEORETIC ONTOLOGIES
FORMAL FOUNDATIONS OF SPACES BEYOND SET AND TOPOS

PU JUSTIN SCARFY YANG

Abstract. This volume initiates the theory of Ontoid Geometry: a stratified and recursive redefinition of "space" in which topological or set-theoretic constructions are replaced by logical persistence, filtration towers, and categorical collapse. We develop a theory of ε-spaces grounded in the collapse-resistance of filtered existence across logical towers. These ontoid structures serve as generalized spectra, base sites for sheaf-theoretic realities, and foundational objects of recursive arithmetic ontology.

Contents

Date: May 10, 2025.

0. Symbol Dictionary for Ontoid Geometry

This section defines and organizes the principal notations, spaces, structures, and meta-logical operators employed throughout the volume.

Spaces and Ontological Entities.

- $\mathscr{O}_\varepsilon^{\mathrm{strat}}$: an ε-stratified ontoid space;
- $\mathbf{Ont}_{\varepsilon^\infty}$: the category of ε^∞-persistent ontoid spaces;
- $\mathrm{Spec}^{\mathrm{ont}}(R)$: the ontoid spectrum of a logical or arithmetic object R;
- $\mathsf{Sh}^{\mathrm{ont}}(X)$: the category of sheaves over ontoid space X;

Filtrations and Persistence Structures.

- $\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: n-th ontological filtration level of $\mathcal{F}$;
- $\mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}) := \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}/\mathrm{Fil}_{n+1}^{\mathrm{ont}}\mathcal{F}$: graded ontological layer;
- $\mathscr{C}_\varepsilon(\mathcal{F}) := \bigcup_n \ker\left(\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} \to \mathrm{Fil}_{n-1}^{\mathrm{ont}}\mathcal{F}\right)$: collapse locus of $\mathcal{F}$;
- $\mathcal{E}_{\mathrm{exist}}(\mathcal{F}) := \bigcap_n \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: meta-core of existence;

Functorial and Logical Structures.

- $\mathcal{E}_{\mathrm{exist}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathbf{Set}$: existential core functor;
- $\mathcal{W} : \mathbf{Ont}_{\varepsilon^\infty} \to \mathbf{Fil}^{\varepsilon^\infty}$: stratified filtration structure functor;
- $\mathscr{C}_\varepsilon$: the logical collapse operator (complement to existence tower);

Ontological Parameters and Logical Indexing.

- ε^n : level-n logic-indexed filtration depth;
- $\mathbf{1}_{\mathrm{ont}}$: identity object in the category $\mathbf{Ont}_{\varepsilon^\infty}$;
- ∞ : total logical collapse depth; meta-ontological stability condition;
- ω^ω : indexing ordinal for limit-persistent ontoid filtrations;

Topos and Beyond.

- **Topos** : classical category of topoi (reference only);
- $\mathbf{Ont}_{\varepsilon^\infty}$: category of stratified ontoid spaces replacing **Topos**;
- **MetaShv** : meta-sheaves over logical or filtered categories, possibly non-set-based;

Existential Collapse Semantics.

- $\mathcal{F} \models \exists$: $\mathcal{F}$ survives all collapse levels;
- $\mathcal{F} \models \neg\exists$: $\mathcal{F}$ vanishes under $\mathscr{C}_\varepsilon$;
- $\mathcal{F}$ is *ontologically real* $\iff$ $\mathcal{E}_{\text{exist}}(\mathcal{F}) \neq 0$;

Conventions. Throughout this volume:

- Filtrations are assumed to be ε-indexed, unless otherwise stated;
- Collapse is not merely degeneration but structured logical removal of existence;
- The term "space" always refers to ontoid entities unless clarified;
- Logical depth and filtration level are interchangeable under meta-collapse equivalence;

1. Introduction: From Topos to Ontos

1.1. **The Question of Space After Set Theory.** Classical geometry begins with the notion of "space" as a set equipped with additional structure—topology, scheme, topos. However, these constructions remain rooted in set-theoretic foundations.

In this volume, we ask:

> What is a space if existence is no longer determined by membership, but by persistence across collapse?

This gives rise to **Ontoid Geometry**, in which:

- Points are no longer primitive;
- Sheaves record filtration-resilient information;
- Cohomology reflects ontological stability, not topological extension;
- Collapse, rather than open cover, defines geometric continuity.

1.2. **From Topos to Ontoid.** A topos encodes logic via sheaf conditions. An ontoid encodes *meta-logic* via recursive filtration towers.

We may schematically view the passage:

$$\mathbf{Set} \rightsquigarrow \mathbf{Topos} \rightsquigarrow \mathbf{Ont}_{\varepsilon^\infty},$$

where:

- **Set**: collections with membership;
- **Topos**: spaces with local logic and gluing;
- $\mathbf{Ont}_{\varepsilon^\infty}$: structures with stratified existence under collapse.

1.3. **Collapse as Ontological Negation.** Let $\mathcal{F}$ be a sheaf over a space X. Then:

$$\mathcal{F} \text{ exists ontologically} \iff \forall n,\ \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} \neq 0;$$
$$\mathcal{F} \text{ collapses} \iff \exists n,\ \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} = 0.$$

In this context, filtration becomes a statement about layers of resistance to disappearance. Collapse $\mathscr{C}_\varepsilon$ plays the role of negation: to exist is to survive collapse.

1.4. **Logic-Indexed Filtration.** Each ontoid space carries a filtration indexed by logical or recursive depth:

$$\cdots \subseteq \mathrm{Fil}_{n+1}^{\mathrm{ont}} \subseteq \mathrm{Fil}_n^{\mathrm{ont}} \subseteq \cdots \subseteq \mathcal{F}.$$

The stratification ε^n is not numerical—it represents proof-theoretic or type-theoretic complexity. For instance:

$$\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} := \{s \in \mathcal{F} \mid s \text{ survives all collapses of depth } \varepsilon^n\}.$$

1.5. **From Points to Layers of Existence.** Ontoid spaces abandon points. Instead, we define:

Definition 1.1 (Ontological Locality)**.** *A* local layer *at level ε^n is the sheaf-theoretic fragment*

$$\mathcal{F}^{[\varepsilon^n]} := \mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}),$$

and a covering family is a collection of such layers whose union resists collapse.

This replaces the open cover condition by a condition of *persistence sufficiency.*

1.6. **Comparison to Topos Theory.** — Feature — Topos — Ontoid — ——————— -————— — Base logic — Internal (first-order) — Meta-logical, stratified — — Points — Generalized (e.g., locales) — None; replaced by filtration cores — — Sheaves — Local gluing — Collapse-survival structures — — Topology — Open covers — Ontological layers indexed by ε — — Cohomology — Derived functor on open sets — Persistence cohomology under filtration —

1.7. **Scope of This Volume.** This volume develops:

- A precise definition of ontoid spaces and their categories;
- Collapse operators as logical structures;
- ε-indexed stratified sheaf theory;
- Onto-spectra $\mathrm{Spec}^{\mathrm{ont}}$ as generalized foundational spaces;
- Functorial and epistemic interpretations of existence;
- ∞-sheaves and categorical arithmetic over ontological towers.

1.8. **Guiding Philosophy.**

To exist is to persist. To be geometric is to resist collapse.

This replaces the set-theoretic idea of space with a recursive, logical ontology.

We now proceed to develop the foundational filtration structures of ε-spaces in Section 2.

2. ε-Spaces and Logical Foundations of Stratified Existence

2.1. **Recursive Foundations: Existence as Layered Survival.** We begin with the premise that existence is not binary, but stratified across logical depth:

$$\text{An object exists} \iff \text{it persists through all } \varepsilon^n\text{-indexed collapse layers.}$$

This motivates a definition of space not as a point-set or locale, but as a tower of filtration layers, each encoding resistance to non-being.

2.2. **Definition of an ε-Space.**

Definition 2.1 (ε-Space)**.** *An* ε-space *is a pair*

$$(X, \mathrm{Fil}^{\mathrm{ont}}_{\bullet}),$$

where:

- *X is an abstract object (not necessarily a set);*
- $\mathrm{Fil}^{\mathrm{ont}}_n \mathcal{F} \subseteq \mathcal{F}$ *is a descending sequence of sheaf-like structures over X;*
- *Each layer encodes survival under collapse depth ε^n.*

This is the fundamental unit of ontoid geometry.

2.3. Stratified Collapse and Meta-Existence.

Definition 2.2 (Meta-Existence Core). *Given $\mathcal{F}$ over an ε-space X, define:*

$$\mathcal{E}_{\mathrm{exist}}(\mathcal{F}) := \bigcap_n \mathrm{Fil}_n^{\mathrm{ont}} \mathcal{F}, \quad \mathscr{C}_\varepsilon(\mathcal{F}) := \mathcal{F}/\mathcal{E}_{\mathrm{exist}}(\mathcal{F}).$$

Then:

$$\mathcal{F} \textit{ is ontologically real} \iff \mathcal{E}_{\mathrm{exist}}(\mathcal{F}) \neq 0.$$

This sets up an ontological dichotomy analogous to Boolean logic, but filtered across ω or ω^ω.

2.4. Stratified Sheaves and ε-Cohomology.

Let $\mathsf{Sh}^{\mathrm{ont}}(X)$ denote the category of stratified sheaves over an ε-space X, with morphisms preserving filtration depth.

For $\mathcal{F} \in \mathsf{Sh}^{\mathrm{ont}}(X)$, define:

$$H^i_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathrm{Fil}_n^{\mathrm{ont}} \mathcal{F}),$$

which captures cohomological mass that survives all collapses.

2.5. Existential Epimorphisms and Logical Gluing.

Definition 2.3 (Ontological Epimorphism). *A morphism $f : \mathcal{F} \to \mathcal{G}$ is an* ontological epimorphism *if:*

$$f(\mathrm{Fil}_n^{\mathrm{ont}} \mathcal{F}) = \mathrm{Fil}_n^{\mathrm{ont}} \mathcal{G}, \quad \forall n.$$

Such morphisms preserve existence across all levels and act analogously to coverings in Grothendieck topology.

2.6. Examples and Meta-Logics.

- In logical settings: $\mathrm{Fil}_n^{\mathrm{ont}}$ may correspond to derivations of complexity Π_n^0 or Σ_n^1;
- In type theory: $\mathrm{Fil}_n^{\mathrm{ont}}$ may model truncations of (∞, n)-groupoids;
- In arithmetic: $\mathrm{Fil}_n^{\mathrm{ont}}$ may mirror mod p^n motivic or syntomic filtrations;
- In philosophical logic: $\mathrm{Fil}_n^{\mathrm{ont}}$ may represent the depth at which existence is provable.

2.7. Postulate: Persistence Implies Ontology.

> **Postulate.** The definition of space arises from what survives transfinite collapse.
>
> Space is the limit of resistance.

This principle replaces traditional axioms of separation, covering, or locality with an axiom of recursive survival.

2.8. Conclusion.

In this section, we introduced:

- The concept of ε-spaces as filtration-based foundational units;
- Meta-existence via towers of logical collapse;
- Ontological epimorphisms and ε-cohomology as persistence measures;
- A logical reinterpretation of geometry where space is the carrier of surviving structure.

The next section formalizes how these towers interact via logical descent, collapse functors, and ontological filtration categories.

3. Ontological Filtration Towers and Collapse Logic

3.1. Tower Structures and Existential Depth.

We now formalize the recursive tower of filtrations that gives each ontoid space its ontological structure. Let $\mathcal{F}$ be a sheaf over an ε-space X. The tower:

$$\cdots \subseteq \mathrm{Fil}^{\mathrm{ont}}_{n+1}\mathcal{F} \subseteq \mathrm{Fil}^{\mathrm{ont}}_{n}\mathcal{F} \subseteq \cdots \subseteq \mathrm{Fil}^{\mathrm{ont}}_{0}\mathcal{F} \subseteq \mathcal{F}$$

is the *ontological filtration tower* of $\mathcal{F}$. Each level encodes structural persistence under collapse at depth ε^n.

3.2. Collapse Functor and Descent Logic.

Define the collapse functor $\mathscr{C}_\varepsilon : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathbf{Ab}$ by:

$$\mathscr{C}_{\varepsilon n}(\mathcal{F}) := \mathcal{F}/\mathrm{Fil}^{\mathrm{ont}}_{n}\mathcal{F}, \quad \mathscr{C}_\varepsilon(\mathcal{F}) := \varinjlim_{n} \mathscr{C}_{\varepsilon n}(\mathcal{F}).$$

Definition 3.1 (Ontological Collapse Depth)**.** *The smallest n for which $\mathrm{Fil}^{\mathrm{ont}}_{n}\mathcal{F} = 0$ is the* collapse depth *of $\mathcal{F}$. If no such n exists, $\mathcal{F}$ is said to be* ontologically persistent*.*

3.3. **Graded Structures and Collapse Profiles.** The graded components of the filtration tower are defined by:

$$\mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}) := \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}/\mathrm{Fil}_{n+1}^{\mathrm{ont}}\mathcal{F}.$$

The full structure:

$$\mathrm{gr}_\bullet^{\mathrm{ont}}(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F})$$

carries a stratified notion of "existence layers."

3.4. **Descent and Collapse Morphisms.** Each sheaf $\mathcal{F}$ defines a chain of canonical morphisms:

$$\mathcal{F} \xrightarrow{\pi_0} \mathscr{C}_{\varepsilon 0}(\mathcal{F}) \xrightarrow{\pi_1} \mathscr{C}_{\varepsilon 1}(\mathcal{F}) \xrightarrow{\pi_2} \cdots$$

This defines a descent system for measuring logical degeneration of existence.

3.5. **Logical Collapse Algebra.** The collapse structure naturally forms a filtered graded module:

$$\mathcal{F} \in \mathbf{Gr}_{\varepsilon^\infty}, \quad \text{with actions of } \mathscr{C}_{\varepsilon^n}.$$

We may postulate a "collapse algebra" $\mathscr{C}_\varepsilon$ with:

$$\mathscr{C}_\varepsilon := \langle \mathscr{C}_{\varepsilon^0}, \mathscr{C}_{\varepsilon^1}, \mathscr{C}_{\varepsilon^2}, \ldots \rangle$$

acting functorially on $\mathsf{Sh}^{\mathrm{ont}}(X)$ via descent and filtration morphisms.

3.6. **Collapse-Existence Duality.** Each $\mathcal{F}$ splits (noncanonically) into:

$$\mathcal{F} = \mathcal{E}_{\mathrm{exist}}(\mathcal{F}) \oplus \mathscr{C}_\varepsilon(\mathcal{F}),$$

which reflects a duality between persistent existence and logical disappearance. This forms the basis of the ontological sheaf cohomology decomposition.

3.7. **Ontological Monodromy.** We define the ontological monodromy operator N_{ont} as:

$$N_{\text{ont}} : \mathcal{F} \to \mathcal{F}, \quad N_{\text{ont}}(\text{Fil}_n^{\text{ont}}\mathcal{F}) \subseteq \text{Fil}_{n+1}^{\text{ont}}\mathcal{F}.$$

This generalizes classical nilpotent monodromy as a collapse-driving operator that strictly deepens existential filtration.

Conjecture 3.2 (Ontological Nilpotence)**.** *N_{ont} is nilpotent $\iff$ $\mathcal{F}$ is finite-level ontologically collapsible.*

3.8. **Ontological Weight Structures.** We define a weight filtration dual to collapse:

$$\text{Weight}_n^{\text{ont}}(\mathcal{F}) := \bigcap_{m \geq n} \text{Fil}_m^{\text{ont}}\mathcal{F},$$

which measures how "ontologically heavy" a section is. The deeper it resists collapse, the more weight it has.

3.9. **Conclusion.** This section constructs:

- The recursive tower $\text{Fil}_\bullet^{\text{ont}}\mathcal{F}$ and its collapse layers;
- Collapse functors and depth classifications;
- Graded components and monodromy operators;
- Ontological weight structures and the $\mathscr{C}_\varepsilon$-algebra of descent.

In the next section, we define the category of Ontoids, morphisms between them, and how logical existence structures interact through functorial maps.

4. Category of Ontoids and Onto-Spatial Morphisms

4.1. **From Sheaf Spaces to Existentially Stratified Objects.** An *Ontoid* is not merely a topological space with sheaves, but a logically stratified entity defined by recursive collapse-resistance. We now define the category in which such objects live and morphisms between them.

4.2. **Definition of Ontoids.**

Definition 4.1 (Ontoid)**.** *An* Ontoid *is a pair*

$$\mathscr{O}_\varepsilon := (X, \text{Fil}_\bullet^{\text{ont}}),$$

where:

- *X is an abstract base object or proto-space;*
- *$\text{Fil}_n^{\text{ont}}\mathcal{F} \subseteq \mathcal{F}$ is an ontological filtration on sheaves $\mathcal{F}$ over X;*

- *For every $\mathcal{F}$, $\mathcal{E}_{\text{exist}}(\mathcal{F}) = \bigcap_n \text{Fil}_n^{\text{ont}}\mathcal{F}$ is well-defined and functorial.*

This structure encapsulates "what survives" across infinite descent, regardless of classical topological interpretation.

4.3. Category $\mathbf{Ont}_{\varepsilon^\infty}$.

Definition 4.2 (Category of Ontoids)**.** *Let $\mathbf{Ont}_{\varepsilon^\infty}$ be the category whose objects are ontoids $(X, \text{Fil}_\bullet^{\text{ont}})$ and morphisms are maps $f : X \to Y$ such that:*

$$f^*(\text{Fil}_n^{\text{ont}}\mathcal{G}) \subseteq \text{Fil}_n^{\text{ont}} f^*(\mathcal{G}), \quad \forall n, \mathcal{G} \in \mathsf{Sh}^{\text{ont}}(Y).$$

That is, morphisms preserve existential stratification under pullback.

4.4. Onto-Spatial Morphisms.

A morphism $f : X \to Y$ in $\mathbf{Ont}_{\varepsilon^\infty}$ is said to be:

- **Collapse-preserving** if $\mathscr{C}_\varepsilon(f^*\mathcal{G}) = f^*(\mathscr{C}_\varepsilon(\mathcal{G}))$;
- **Persistence-reflecting** if $\mathcal{E}_{\text{exist}}(\mathcal{F}) \subseteq f^*\mathcal{E}_{\text{exist}}(\mathcal{G})$;
- **Ontological epimorphism** if f is surjective at all filtration levels;
- **Stratified monomorphism** if f is injective on each $\text{Fil}_n^{\text{ont}}$.

4.5. Composition and Identity.

Composition is defined via filtration-level compatibility:

$$(g \circ f)^*\text{Fil}_n^{\text{ont}}\mathcal{H} \subseteq f^*g^*\text{Fil}_n^{\text{ont}}(\mathcal{H}).$$

Identity morphisms preserve all filtration layers exactly. Thus, $\mathbf{Ont}_{\varepsilon^\infty}$ is indeed a well-defined category enriched over filtered morphism classes.

4.6. Functorial Structures and Existential Image.

Let $\mathcal{F} \in \mathsf{Sh}^{\text{ont}}(X)$ and $f : X \to Y$ in $\mathbf{Ont}_{\varepsilon^\infty}$. Define the **existential image**:

$$f_!\mathcal{F} := \bigcup_n \text{Im}(f_*\text{Fil}_n^{\text{ont}}\mathcal{F}),$$

as the cumulative transfer of survival structures under f. It reflects how existence propagates.

4.7. Onto-Subobjects and Internal Collapse Ideals.

Definition 4.3 (Onto-Subobject)**.** *An* onto-subobject $\mathcal{S} \subseteq \mathcal{F}$ *is a subobject such that:*

$$\text{Fil}_n^{\text{ont}}\mathcal{S} = \mathcal{S} \cap \text{Fil}_n^{\text{ont}}\mathcal{F}, \quad \forall n.$$

This ensures that filtration stratification is preserved under internal substructure.

Definition 4.4 (Collapse Ideal). *A collapse ideal $\mathcal{I} \subseteq \mathcal{F}$ satisfies:*

$$\forall n,\, \exists m > n \text{ such that } \mathrm{Fil}^{\mathrm{ont}}_m \mathcal{F} \subseteq \mathcal{I}.$$

Collapse ideals represent vanishing layers and may be used to define ε-quotients.

4.8. **Coproducts and Fibered Ontoids.** The category $\mathbf{Ont}_{\varepsilon^\infty}$ admits:

- Coproducts via disjoint filtration unions;
- Fibered products via pullbacks of filtration-preserving morphisms;
- Base change along collapse-stable morphisms.

This gives $\mathbf{Ont}_{\varepsilon^\infty}$ enough categorical structure to support sheafification, site theory, and descent logic.

4.9. **Conclusion.** This section constructed the categorical framework of ontoid geometry:

- Defined objects $(X, \mathrm{Fil}^{\mathrm{ont}}_\bullet)$ and morphisms preserving filtration towers;
- Distinguished epimorphisms, monomorphisms, and collapse-image operations;
- Introduced internal structures: subobjects, collapse ideals, existential images;
- Prepared the formal ground for functorial sheaf theory and higher logic.

In the next section, we move beyond classical logic: defining meta-existence via sheaves over ∞-filtered categories and constructing the internal ontology of arithmetic space.

5. Meta-Existence and ∞-Sheaf Realities

5.1. **Beyond Classical Sheaves: From Covering to Persistence.** In classical sheaf theory, a space is understood by how local data can be glued across open covers. In ontoid geometry, gluing is replaced by filtration resilience. Locality is not spatial but ontological: how a section persists across collapse levels.

Thus, we transition to a new kind of sheaf:

An ∞-sheaf$isapresheafvaluedintowersofsurvival, indexedbyrecursiveortransfinitegrowth.$

5.2. **Definition of ∞-Sheaf over Ontoids.**

Definition 5.1 (∞-Sheaf on Ontoid). *Let $(X, \mathrm{Fil}^{\mathrm{ont}}_\bullet) \in \mathbf{Ont}_{\varepsilon^\infty}$. An* ∞-sheaf *over X is a presheaf*

$$\mathcal{F} : \mathcal{O}^{\mathrm{ont}}(X)^{\mathrm{op}} \to \mathbf{Fil}_{\varepsilon^\infty},$$

where:

- *$\mathcal{O}^{\mathrm{ont}}(X)$ is the poset of ontological strata (indexed by ε^n);*

- $\mathbf{Fil}_{\varepsilon^\infty}$ *is the category of logical filtration towers;*
- $\mathcal{F}$ *satisfies* ε*-gluing: local survivals yield global persistence.*

5.3. **Meta-Sheaves and Stratified Descent.** A meta-sheaf $\mathscr{F}$ over X must satisfy:

$$\mathscr{F}(\mathrm{Fil}_n^{\mathrm{ont}} U) \to \mathscr{F}(\mathrm{Fil}_{n-1}^{\mathrm{ont}} U) \text{ is surjective,}$$

meaning information descends faithfully across levels of filtration. This replaces usual sheaf conditions with **logical descent stability**.

5.4. **Existential Gluing Principle.**

Definition 5.2 (Existential Covering). *A family* $\{U_i\}_{i\in I}$ existentially covers U *if:*

$$\bigcap_{i\in I} \mathscr{C}_{\varepsilon n}(U_i) = \mathscr{C}_{\varepsilon n}(U) \quad \forall n.$$

That is, the collapse behavior is locally reconstructible.

We then define gluing via pullback of survival, not intersection of open sets.

5.5. **Stacky Ontological Realities.** We define the stack of persistence layers:

$$\mathfrak{E}^{[\infty]} : \mathbf{Ont}_{\varepsilon^\infty}^{\mathrm{op}} \to \mathbf{Cat}$$

assigning to each ontoid X the category of all ε-persistent ∞-sheaves on X, with morphisms given by filtration-preserving functors.

This forms the moduli space of logical survival over arithmetic geometry.

5.6. **Ontological Realization Functor.** Let $\mathcal{M}^{[\varepsilon^\infty]}(X)$ be a filtered motivic object. Define:

$$\mathcal{R}_{\mathrm{ont}} : \mathcal{M}^{[\varepsilon^\infty]} \longrightarrow \mathsf{Sh}^{\mathrm{ont}}(X),$$

as the *ontological realization functor*, capturing what survives through motivic stratification under ε-collapse.

5.7. **Postulates of Meta-Existence.**

Postulate I. Existence is the inverse limit of survival.

$$\mathcal{E}_{\text{exist}}(\mathcal{F}) := \varprojlim_{n} \mathrm{Fil}_n^{\text{ont}} \mathcal{F}.$$

Postulate II. ∞-sheaves classify stable ontological descent.

Postulate III. Spaces are not glued from points, but from persistent towers.

5.8. **Ontological Site and Stratified Grothendieck Topology.** We define a stratified site $(X, \mathcal{T}_\varepsilon)$ where:

- Coverings are existential families;
- Fiber products are formed via pullbacks of survival towers;
- Descent is measured by collapse-preserving glue conditions.

The category $\mathsf{Sh}^{\text{ont}}(X)$ becomes a site of persistence.

5.9. **Conclusion.** This section developed:

- ∞-sheaves and their ε-gluing conditions;
- Existential coverings and logical descent morphisms;
- Meta-realization functors and stacky ε-structures;
- Postulates recasting "space" in purely persistence-theoretic terms.

In the next section, we connect this to knowledge, interpretation, and proof: examining collapse from an epistemic perspective and reinterpreting logic as a geometric functor.

6. Epistemic Collapse and Formal Sheaf-Theoretic Realism

6.1. **Collapse as Epistemic Constraint.** In the ontoid framework, collapse is not only a structural phenomenon but an epistemic one: it represents the logical limit of what can be known, constructed, or verified at a given level of recursion.

> *To collapse is to vanish from possible proof. To survive collapse is to remain logically visible.*

This transforms geometric persistence into a model of formal knowledge.

6.2. **Knowledge as Survival Through Collapse.** We define:

Definition 6.1 (Epistemic Section)**.** *Let $\mathcal{F} \in \mathsf{Sh}^{\text{ont}}(X)$. A section $s \in \mathcal{F}$ is* epistemic *if*

$$s \in \mathrm{Fil}_n^{\text{ont}} \mathcal{F} \quad \textit{for some } n < \infty.$$

Its epistemic depth is the least n such that s belongs to $\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$.

6.3. **Sheaf-Theoretic Realism.** We interpret a sheaf $\mathcal{F}$ over an ontoid space as a *formal structure of knowable reality.*

$$\mathcal{F}_{\mathrm{real}} := \mathcal{E}_{\mathrm{exist}}(\mathcal{F}), \quad \text{and} \quad \mathcal{F}_{\mathrm{illusory}} := \mathscr{C}_{\varepsilon}(\mathcal{F}).$$

This defines:

- The *real core*: meta-logically permanent; - The *illusory boundary*: exists only within collapse-vulnerable reasoning.

6.4. **Logic as Collapse Functor.** Let $\mathcal{L}$ be a formal logic system (e.g., arithmetic, type theory). We define the associated collapse functor:

$$\mathscr{C}_{\mathcal{L}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathsf{Sh}^{\mathrm{ont}}(X), \quad \mathscr{C}_{\mathcal{L}}(\mathcal{F}) = \bigcup_{n \text{ unprovable in } \mathcal{L}} \mathscr{C}_{\varepsilon n}(\mathcal{F}).$$

This functor isolates the epistemically inaccessible parts of a space relative to $\mathcal{L}$.

6.5. **Collapse-Sensitive Cohomology.** Define:

$$H^i_{\mathrm{real}}(X, \mathcal{F}) := H^i(X, \mathcal{E}_{\mathrm{exist}}(\mathcal{F})), \quad H^i_{\mathrm{illusory}}(X, \mathcal{F}) := H^i(X, \mathscr{C}_{\varepsilon}(\mathcal{F})).$$

These measure:
- What is provably persistent (real cohomology);
- What is logically inaccessible (illusory cohomology).

6.6. **Interpretation Functors and Modalities.** Let $\mathcal{M}$ be a modal system. Define the interpretation functor:

$$[\![\cdot]\!]_{\mathcal{M}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathsf{Sh}^{\mathrm{ont}}(Y),$$

which sends sections of X to their logical images under a translation modality.

A modality is *collapse-respecting* if:

$$[\![\mathscr{C}_{\varepsilon n}(\mathcal{F})]\!]_{\mathcal{M}} \subseteq \mathscr{C}_{\varepsilon n}([\![\mathcal{F}]\!]_{\mathcal{M}}).$$

6.7. **Formal Realism Axiom.**

> **Axiom (Formal Realism).** *Only those structures which survive collapse across all epistemic logics should be called real.*

This axiom governs the ontological status of all objects in $\mathbf{Ont}_{\varepsilon^\infty}$.

6.8. **Diagram: Collapse–Knowledge–Realism Triangle.**

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\mathscr{C}_\varepsilon"] \arrow[dr, "\mathscr{C}_{\mathcal{L}}"'] & & \mathscr{C}_\varepsilon(\mathcal{F}) \\
& \text{Epistemic limit under logic } \mathcal{L} \arrow[ur, dashed] &
\end{tikzcd}$$

This triangle describes how logic bounds the knowable geometry of $\mathcal{F}$.

6.9. **Conclusion.** This section reinterprets collapse geometrically and philosophically as:

- A measure of epistemic uncertainty;
- A logic-indexed restriction on knowable structure;
- A method to define real vs illusory cohomology;
- A formal realism principle based on collapse-invariant survival.

In the next section, we organize these sheaf-level structures into global gerbes, torsors, and ε-cohomological moduli indexed by ontological stratification.

7. ε-Gerbes, Ontological Torsors, and Persistence Moduli

7.1. **Sheaf-Level Symmetry and Collapse Invariance.** Up to this point, we have treated collapse and persistence as intrinsic properties of sheaves. In this section, we globalize these behaviors by organizing collapse-resilient structures into:

- ε-Gerbes: higher stacks of stratified sheaves;
- Ontological Torsors: symmetry objects tracking survival;
- Persistence Moduli: parameter spaces for ε-existence.

7.2. **Definition of an ε-Gerbe.**

Definition 7.1 (ε-Gerbe)**.** *An ε-Gerbe over an ontoid space X is a stack*

$$\mathscr{G}_\varepsilon : \mathcal{O}^{\mathrm{ont}}(X)^{\mathrm{op}} \to \mathbf{Cat}$$

such that:

- *Each $\mathscr{G}_\varepsilon(\mathrm{Fil}_n^{\mathrm{ont}} U)$ is a groupoid of ε^n-persistent local data;*
- *Gluing across filtration levels is collapse-compatible;*

- *The band of the gerbe is given by* $\mathrm{Aut}(\mathcal{E}_{\mathrm{exist}}(\mathcal{F}))$.

This generalizes usual banded gerbes to stratified logical frameworks.

7.3. Ontological Torsors.

Definition 7.2 (Ontological Torsor). *An* ontological torsor $\mathcal{T}^{[\varepsilon^n]}$ *over* X *is a sheaf of sets with free and transitive action by* $\mathrm{Aut}(\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F})$*, such that:*

$$\mathcal{T}^{[\varepsilon^n]} \cdot s = \textit{collapse-preserving translations.}$$

These torsors classify equivalence of existence classes under collapse-stable transformations.

7.4. Stack of ε-Torsors and Higher Sheaf Cohomology.

Let $\mathfrak{T}^{[\varepsilon^\infty]}$ denote the stack assigning to each ontoid space X the groupoid of all ε^n-indexed torsors.

Define the persistence cohomology:

$$H^1_{\varepsilon^\infty}(X, \mathfrak{T}) := \text{classes of torsors over } X \text{ under collapse-invariant isomorphism.}$$

This classifies distinct survival symmetries of $\mathcal{F}$ in a stratified manner.

7.5. Persistence Moduli and Stratified Parameter Spaces.

We define the *Persistence Moduli Stack* $\mathcal{M}^{[\varepsilon^\infty]}_{\mathrm{pers}}$ as the moduli of ε-sheaves up to:

- Collapse-preserving equivalence;
- Ontological torsor translation;
- Survival class stratification.

Definition 7.3 (Moduli Point). *A point* $[\mathcal{F}] \in \mathcal{M}^{[\varepsilon^\infty]}_{\mathrm{pers}}$ *represents a tower class*

$$\left\{\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}\right\}_{n\in\mathbb{N}}, \quad \textit{modulo } \varepsilon\textit{-torsorial deformation.}$$

7.6. Descent and Gerbe Cohomology.

There is an ε-gerbe cohomology long exact sequence:

$$\cdots \to H^i_\varepsilon(X, \mathcal{G}) \to H^i_\varepsilon(X, \mathscr{G}_\varepsilon) \to H^{i+1}_\varepsilon(X, \mathcal{A}) \to \cdots$$

for gerbes $\mathscr{G}_\varepsilon$ banded by sheaves of groups $\mathcal{A}$. This governs the obstruction theory of persistence descent.

7.7. **Universal ε-Torsor and Period Maps.** There exists a universal ε-torsor:

$$\mathcal{T}^{\mathrm{univ}}_{\varepsilon^\infty} \in \mathfrak{T}^{[\varepsilon^\infty]}(\mathcal{M}_{\mathrm{pers}}),$$

with associated period map:

$$\pi_\varepsilon : \mathcal{M}^{[\varepsilon^\infty]}_{\mathrm{pers}} \to B_{\varepsilon^\infty,\mathrm{dR}}.$$

This map evaluates the cohomological persistence structure of each survival class.

7.8. **Conclusion.** This section globalizes the internal survival structures by introducing:

- ε-gerbes as collapse-layered stacks;
- Ontological torsors representing ε-symmetries of existence;
- Moduli of persistence classes and their universal period torsors;
- ε-cohomology as a higher classification of stratified survival.

In the final section, we synthesize all ontoid structures into a concluding metaphysical framework for categorified existence and persistent space.

8. Conclusion: Toward Recursive Geometries of Being

8.1. **Beyond Points, Toward Persistence.** We have abandoned points, sets, and covers. In their place, we have constructed an ontology of space built on collapse-resilient towers of existence.

Each space in this framework is not defined by "what is there," but by:

What survives through logical, epistemic, and recursive collapse.

This persistence defines not only geometry, but the very notion of being.

8.2. **From Collapse to Existence.** Let us summarize the foundational transformation:

Classical Geometry	**Ontoid Geometry**
Sets	Filtration towers
Points	Persistence cores
Open covers	Collapse-resisting families
Sheaves	Survival-indexed sections
Cohomology	Ontological realization of being
Topoi	$\mathbf{Ont}_{\varepsilon^\infty}$ and $\mathsf{Sh}^{\mathrm{ont}}(X)$
Real structure	$\mathcal{E}_{\mathrm{exist}}(\mathcal{F})$
Vanishing locus	$\mathscr{C}_\varepsilon(\mathcal{F})$
Logic	Collapse algebra $\mathscr{C}_\varepsilon$

8.3. **Categorification of Being.** We now reinterpret "existence" as a functor:

$$\mathcal{E}_{\mathrm{exist}} : \mathsf{Sh}^{\mathrm{ont}}(X) \longrightarrow \mathbf{Exist},$$

where **Exist** is the category of ontologically persistent structures. This functor sends sheaves to their essential core of recursive survival.

8.4. **Collapse, Knowledge, and Reality.** We may now assemble the three fundamental aspects:

Collapse: What is removed by logical degeneration**Knowledge:** What is knowable under a logic $\mathcal{L}$F

Geometry becomes a stratified model of epistemic realism.

8.5. **Ontoid Philosophy.** Let us conclude with the core philosophical axioms of Ontoid Geometry:

(1) **Existence is indexed.** There is no absolute being—only resistance to collapse.
(2) **Points are obsolete.** What matters is not location but persistence.
(3) **Logic is geometric.** Each logic $\mathcal{L}$ defines collapse $\mathscr{C}_{\mathcal{L}}$.
(4) **Being is filtered.** Objects exist through survival at increasingly deep logical levels.
(5) **Space is survival.** What we call space is the stratification of resilience to non-being.

8.6. **Toward Volume V and Beyond.** In Volume V, we will develop a full arithmetic framework on top of ontoid spaces: categorifying growth, structure, and algebra over filtered realities.

Topics include:

- Growth-function indexed stacks and sheaves;
- ε-Gerbe torsors over hypermonodromy sites;
- Arithmetic geometry of categorical stratification;
- Persistence-based stacks and arithmetic descent theory.

This culminates the shift from geometry as shape, to geometry as survival.

— *End of Volume IV*

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VOLUME V: CATEGORICAL ARITHMETIC OF GROWTH-BASED SPACES
STACKS, GERBES, AND SHEAVES OVER EXPONENTOIDONTOID GEOMETRIES

PU JUSTIN SCARFY YANG

Abstract. This volume develops a framework of Categorical Arithmetic over Growth-Based Spaces. Building upon the ontoid geometries of Volume IV, we introduce growth-function indexed stacks, persistence gerbes, and hypermonodromy cohomological structures. Categorified arithmetic is constructed by stratifying existence through growth rates, collapse towers, and transfinite ontological persistence. This extends classical arithmetic beyond fields and rings to spaces of survival and stratification.

Contents

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0. Symbol Dictionary for Growth-Based Categorical Arithmetic

This section catalogs the core symbols, objects, stacks, and structural functors used throughout Volume V.

Growth Functions and Stratification.

- $g(n)$: a growth function (e.g., n, 2^n, e^n, $n\uparrow^k$);
- Growth : the class of recursive or trans-recursive functions used to stratify persistence;
- $\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}$: the ontological filtration layer indexed by $g(n)$;

Stacks, Gerbes, and Sheaves.

- $\mathbb{G}^{[g]}$: a stack of sheaves or spaces indexed by the growth function g;
- $\mathscr{G}^{[g]}_{\varepsilon}$: an ε-gerbe layered over growth-stratified filtration;
- $\mathfrak{T}^{[g]}$: the stack of ontological torsors indexed by $g(n)$;

- $\mathscr{S}^{\mathrm{ont}}(X)$: sheaf category over an ontoid space X;

Period Rings and Realizations.

- $B^{\mathrm{cat}}_{g,\mathrm{dR}}$: growth-function indexed de Rham period ring;
- $B^{\mathrm{cat}}_{\uparrow^k,\mathrm{dR}}$: the hyper-exponential categorified period ring;
- $\mathrm{Real}^{[g]}$: realization functor into stratified cohomology under g-filtration;

Moduli and Cohomology.

- $\mathscr{M}^{[g]}_{\varepsilon^\infty}$: the persistence moduli stack under $g(n)$-growth filtration;
- $H^i_{[g]}(X, \mathcal{F})$: i-th cohomology group stratified by $g(n)$;
- $H^1_{\varepsilon^\infty}(X, \mathfrak{T}^{[g]})$: classification of torsors under ε^∞ collapse with g-growth symmetry;

Arithmetic Objects and Descent.

- $K^{[g]}_n(X)$: growth-indexed K-theory object;
- $r^{[g]}_\varepsilon$: regulator map along g-stratified collapse;
- $\mathcal{D}^{[g]}_{\mathrm{desc}}$: descent data for growth-filtered gerbes and torsors;

Universal and Meta-Structures.

- $\mathcal{T}^{\mathrm{univ}}_{[g]}$: universal torsor over $\mathscr{M}^{[g]}_{\varepsilon^\infty}$;
- $\pi_{[g]} : \mathscr{M}^{[g]}_{\varepsilon^\infty} \to B^{\mathrm{cat}}_{g,\mathrm{dR}}$: growth-period morphism;
- $\mathbf{Cat}_{\varepsilon^\infty}$: category of filtered arithmetic structures;
- $\mathbf{Stack}^{[g]}_{\varepsilon^\infty}$: stack of growth-stratified arithmetic data.

Conventions. Unless otherwise specified:

- All filtrations are indexed by growth functions $g(n) \in \mathsf{Growth}$;
- All stacks are implicitly ε-stratified unless stated otherwise;
- Growth-rate comparison (e.g., $g \prec h$) is interpreted via recursive domination;
- All torsors and gerbes are assumed to be collapse-resilient and functorially persistent.

1. Growth Functions and Arithmetic Ontoids

1.1. **From Classical Arithmetic to Growth-Based Stratification.** In classical arithmetic geometry, structure is built upon rings, schemes, and their cohomological invariants. Here, we replace such foundations with:

- **Ontoids**: recursively filtered spaces defined by logical persistence;
- **Growth Functions** $g(n)$: indexing the rate of survival collapse;

- **Arithmetic Structures**: reinterpreted as stratified layers over growth-indexed sheaf towers.

This reframes arithmetic as a theory of stratified ontological continuation.

1.2. **Growth Functions and Filtration Towers.** Let $g : \mathbb{N} \to \mathbb{N}$ be a monotonic, unbounded function. Examples include:

$$g(n) = n, \quad 2^n, \quad e^n, \quad n \uparrow^2 n, \quad \text{or even } n \mapsto A(n,n),$$

where A is the Ackermann function.

Definition 1.1 (Growth-Stratified Filtration)**.** *Given a sheaf $\mathcal{F}$ over an ontoid X, define:*

$$\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/g(n)\cdot\mathcal{F}\right).$$

The sequence $\{\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\}_n$ is the g-indexed filtration tower.

1.3. **Arithmetic Ontoid Structures.**

Definition 1.2 (Arithmetic Ontoid)**.** *An* arithmetic ontoid *is a pair $(X, \mathrm{Fil}^{\mathrm{ont}}_{g(\bullet)})$, where X is a geometric or categorical base, and $\mathrm{Fil}^{\mathrm{ont}}_{g(n)}$ defines the persistence stratification at growth level $g(n)$.*

This replaces p-adic or valuation-theoretic stratification with growth-indexed resistance to collapse.

1.4. **Growth-Indexed Cohomology.** Define:

$$H^i_{[g]}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}).$$

This cohomology captures arithmetic invariants that stabilize under growth-level survival.

1.5. **Categorical Arithmetic Structures.** Let:

$$\mathbf{Cat}^{[g]}_{\varepsilon^\infty} := \text{category of growth-indexed, collapse-filtered arithmetic objects.}$$

Objects include:

- K-theory towers $K^{[g]}_n(X)$;
- Period rings $B^{\mathrm{cat}}_{g,\mathrm{dR}}$;
- Stratified realizations $\mathrm{Real}^{[g]}$;
- Regulator maps $r^{[g]}_{\varepsilon}$;
- Torsor cohomology groups $H^1_{\varepsilon^\infty}(X, \mathfrak{T}^{[g]})$.

1.6. Existence over Growth.

Definition 1.3 (Growth-Indexed Ontological Core). *The* growth core *of $\mathcal{F}$ is:*

$$\mathcal{E}_{\mathrm{exist}\,[g]}(\mathcal{F}) := \bigcap_n \mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F},$$

which measures existence across g-accelerated collapse levels.

Persistence at faster $g(n)$ implies stronger ontological status.

1.7. Example: Hyperexponential Arithmetic Geometry.

Let $g(n) = n \uparrow^3 n$. Then $\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}$ describes survival only for ultra-transfinite coherent structures. This defines a geometric framework beyond any finite descent, suitable for hyper-categorified motives or translogarithmic torsor descent.

1.8. Conclusion.

This section introduces:

- The concept of growth-indexed stratification $g(n)$;
- Arithmetic ontoids $(X, \mathrm{Fil}^{\mathrm{ont}}_{g(n)})$;
- Cohomology, realization, and existence indexed by $g(n)$;
- A redefinition of arithmetic as the logic of survival under growth.

Next, we will construct moduli stacks of such arithmetic sheaves and study their behavior over growth-stratified sites.

2. Indexed Moduli Stacks and Growth-Filtered Sites

2.1. Moduli of Stratified Arithmetic Sheaves.

We now define the moduli theory of arithmetic objects not by isomorphism classes of algebraic structures, but by collapse-invariant, growth-indexed survival classes.

Definition 2.1 (Persistence Moduli Stack). *Let $\mathscr{M}^{[g]}_{\varepsilon^\infty}$ be the stack assigning to each ontoid space X the groupoid:*

$$\mathscr{M}^{[g]}_{\varepsilon^\infty}(X) := \left\{ \mathcal{F} \in \mathscr{S}^{\mathrm{ont}}(X) \,\middle|\, \{\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}\}_n \right\} / \sim,$$

where $\sim$ is equivalence under ε^∞-stable torsor translation.

2.2. Stack Topology and Growth-Filtered Coverings.

We define a Grothendieck site structure over $\mathbf{Ont}_{\varepsilon^\infty}$ by declaring coverings to be:

Definition 2.2 (Growth-Filtered Covering). *A family $\{f_i : U_i \to X\}_{i \in I}$ is a covering in the g-filtered site if:*

$$\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F} = \bigcap_i f_i^* \mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}, \quad \forall n.$$

This ensures persistence strata glue correctly across base change.

2.3. **Stratified Descent and Atlas.** Each object in $\mathscr{M}^{[g]}_{\varepsilon^\infty}$ can be represented by a descent datum:

$$\mathcal{D}^{[g]}_{\mathrm{desc}} := \left(\mathcal{F}, \{\varphi^{[n]}_{ij}\}\right),$$

where the $\varphi^{[n]}_{ij}$ are growth-stratum compatible isomorphisms between pullbacks:

$$f_i^* \mathrm{Fil}^{\mathrm{ont}}_{g(n)} \mathcal{F} \xrightarrow{\sim} f_j^* \mathrm{Fil}^{\mathrm{ont}}_{g(n)} \mathcal{F}.$$

2.4. **Stacks of Growth-Torsors and Period Maps.** Define the stack of torsors:

$$\mathfrak{T}^{[g]} : \mathbf{Ont}^{\mathrm{op}}_{\varepsilon^\infty} \longrightarrow \mathbf{Grpds}, \quad X \mapsto \text{Groupoid of } g(n)\text{-stratified torsors.}$$

Associated period morphisms:

$$\pi_{[g]} : \mathscr{M}^{[g]}_{\varepsilon^\infty} \longrightarrow B^{\mathrm{cat}}_{g,\mathrm{dR}},$$

evaluate survival behavior through g-growth realization in de Rham period rings.

2.5. **Base Change and Stacky Cohomology.** Let $f : X \to Y$ be a morphism in $\mathbf{Ont}_{\varepsilon^\infty}$. Then:
- Pullback of stacks preserves g-filtered stratification;
- Torsor cohomology satisfies:

$$H^1_{\varepsilon^\infty}(X, \mathfrak{T}^{[g]}) \simeq H^1_{\varepsilon^\infty}(Y, f_* \mathfrak{T}^{[g]}),$$

under collapse-respecting base change.

2.6. **Growth-Type Stratification of Moduli.** Moduli stacks naturally stratify as:

$$\mathscr{M}^{[g]}_{\varepsilon^\infty} \subseteq \mathscr{M}^{[h]}_{\varepsilon^\infty} \quad \text{if } g(n) \le h(n), \quad \forall n,$$

with growth rates interpreted in recursive dominance.

2.7. **Universal Stack Tower.** We construct the universal tower:

$$\mathscr{M}^{[\bullet]}_{\varepsilon^\infty} := \left\{ \mathscr{M}^{[g]}_{\varepsilon^\infty} \right\}_{g \in \mathsf{Growth}},$$

which organizes all growth-indexed arithmetic classes into a stratified moduli site of persistence.

2.8. **Conclusion.** This section defines:

- The stack $\mathscr{M}^{[g]}_{\varepsilon\infty}$ of growth-indexed persistence sheaves;
- Growth-filtered sites and their coverings;
- Stack descent data and torsor symmetry actions;
- Period realization maps and recursive growth classification.

In the next section, we study ε-gerbes and torsors over these stacks, constructing the arithmetic analogs of hypermonodromy and collapse-indexed Galois symmetry.

3. ε-Gerbe Torsors over Hypermonodromy Structures

3.1. **From Classical Monodromy to Hypermonodromy.** In the classical theory of motives, monodromy arises from variations of Hodge or étale structures across families. In the ontoid framework, we now define:

Hypermonodromy: Recursive stratified symmetry over persistence towers.

The symmetry group is no longer a finite-dimensional algebraic group, but a tower of torsors indexed by collapse-surviving layers.

3.2. **Definition of Hypermonodromy Gerbe.**

Definition 3.1 (Hypermonodromy Gerbe)**.** *Let X be an arithmetic ontoid. The* hypermonodromy gerbe $\mathscr{G}^{[g]}_{\mathrm{hyp}}$ *is the stack*

$$\mathscr{G}^{[g]}_{\mathrm{hyp}}(X) := \{\textit{Filtered local systems with } \varepsilon\textit{-stratified torsor symmetries along } g(n)\}\,.$$

Each object in this gerbe encodes not a single monodromy operator, but an entire filtration-respecting action of a persistence group tower.

3.3. **Torsor Tower and Collapse-Compatible Actions.** Let $\mathcal{F}$ be a sheaf with $g(n)$-indexed filtration.

$$\mathbb{T}^{[g]} := \left\{\mathcal{T}_n \curvearrowright \mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}\right\}_n, \quad \mathcal{T}_n := \mathrm{Aut}_{\mathrm{collapse}}(\mathrm{Fil}^{\mathrm{ont}}_{g(n)}\mathcal{F}).$$

The full tower of torsors captures symmetry across collapse depth. These define the structure group of the hypermonodromy gerbe.

3.4. **Gerbe Realization and Descent Data.** An object in $\mathscr{G}^{[g]}_{\mathrm{hyp}}$ descends across a g-filtered cover $\{U_i \to X\}$ via:
- A collection $\mathcal{F}_i$ over U_i with growth-indexed filtration;
- Isomorphisms $\varphi^{[n]}_{ij}$ compatible with $\mathcal{T}_n$-actions;
- Torsor twist coherence over triple overlaps.

3.5. **Regulators and ε-Cohomology via Gerbes.** The gerbe $\mathscr{G}_{\mathrm{hyp}}^{[g]}$ induces regulator maps:

$$r_{\mathrm{hyp}}^{[g]} : K_n^{[g]}(X) \to H_{\varepsilon^\infty}^n(X, \mathscr{G}_{\mathrm{hyp}}^{[g]}),$$

encoding arithmetic deformation classes of torsorial descent behavior.

3.6. **Period Gerbe and Monodromy Spectrum.** There exists a period-valued gerbe:

$$\mathscr{B}_{\mathrm{dR}}^{[g]} := \left\{ B_{g(n),\mathrm{dR}}\text{-realizations of torsor cohomology} \right\},$$

with associated spectrum:

$$\mathrm{Spec}_\varepsilon(\mathcal{F}) := \left\{ \text{weights of surviving layers under } \mathscr{G}_{\mathrm{hyp}}^{[g]} \right\}.$$

3.7. **Stratified Galois-Type Symmetry.** We define a *stratified Galois group*:

$$\mathrm{Gal}_{[g]}^{\mathrm{hyp}} := \varprojlim_n \mathrm{Aut}(\mathrm{Fil}_{g(n)}^{\mathrm{ont}} \mathcal{F}),$$

which replaces the role of the motivic Galois group in this collapse-sensitive setting.

3.8. **Conclusion.** This section constructs:

- ε-gerbes encoding stratified torsor symmetries;
- Hypermonodromy structures indexed by growth $g(n)$;
- Collapse-compatible regulator maps and spectral filtrations;
- New stratified analogs of Galois groups acting on recursive survival layers.

In the next section, we develop the corresponding sheaf cohomology theories over these gerbes and torsors, culminating in growth-stratified arithmetic Hodge-type frameworks.

4. Growth-Stratified Sheaf Cohomology

4.1. **Collapse-Indexed Cohomology Towers.** Let $\mathcal{F}$ be a sheaf over a growth-stratified ontoid $(X, \mathrm{Fil}_{g(\bullet)}^{\mathrm{ont}})$. We define a tower of cohomology groups that records what survives at each level of collapse depth $g(n)$:

$$H_{g(n)}^i(X, \mathcal{F}) := H^i(X, \mathrm{Fil}_{g(n)}^{\mathrm{ont}} \mathcal{F}), \quad H_{[g]}^i(X, \mathcal{F}) := \varprojlim_n H_{g(n)}^i(X, \mathcal{F}).$$

This defines a filtered persistence cohomology, indexing survival over recursion scales.

4.2. **Persistence Spectral Sequence.** We now define a spectral sequence associated with the growth tower:

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_p^{[g]}\mathcal{F}) \Rightarrow H_{[g]}^{p+q}(X, \mathcal{F}), \quad \mathrm{gr}_p^{[g]} := \mathrm{Fil}_{g(p)}^{\mathrm{ont}}\mathcal{F}/\mathrm{Fil}_{g(p+1)}^{\mathrm{ont}}\mathcal{F}.$$

This is the growth-stratified analog of the classical weight spectral sequence.

4.3. **Growth-Indexed Period Maps.** Let $\pi_{[g]} : \mathscr{M}_{\varepsilon^\infty}^{[g]} \to B_{g,\mathrm{dR}}^{\mathrm{cat}}$ be the period map from Section 2. It factors through the growth cohomology via:

$$H_{[g]}^i(X, \mathcal{F}) \xrightarrow{\text{realization}} B_{g,\mathrm{dR}}^{\mathrm{cat}},$$

defining the stratified de Rham realization over the filtered tower.

4.4. **Torsorial Descent and Non-Abelian Cohomology.** Let $\mathcal{T}^{[g]}$ be an ε-stratified torsor. Define the first growth-class cohomology group as:

$$H_{[g]}^1(X, \mathcal{T}) := \left\{ \text{descent torsor classes compatible with } \mathscr{G}_{\mathrm{hyp}}^{[g]} \right\}.$$

These classes form the torsorial automorphism data of surviving cohomological realizations.

4.5. **Hyper-Regulators and Arithmetic Invariants.** The higher cohomological structures yield regulator-like maps:

$$r_{\mathrm{coh}}^{[g]} : K_n^{[g]}(X) \longrightarrow H_{[g]}^n(X, \mathbb{Q}(n)),$$

interpreted not as numerical values, but as survival classes under growth-indexed sheaf towers.

4.6. **Growth-Stratified Hodge Structures.** Let $\mathcal{F}$ be a filtered object over an ontoid. Define the ε-Hodge structure:

$$\left(\mathrm{Fil}_{g(n)}^{\mathrm{ont}}\mathcal{F},\ \mathrm{gr}_n^{[g]}\mathcal{F},\ N_{[g]} \right),$$

where $N_{[g]}$ is a generalized hypermonodromy operator satisfying:

$$N_{[g]}(\mathrm{Fil}_{g(n)}^{\mathrm{ont}}\mathcal{F}) \subseteq \mathrm{Fil}_{g(n+1)}^{\mathrm{ont}}\mathcal{F}.$$

This encodes a recursive degeneration of arithmetic structure across growth towers.

4.7. **Collapse Depth and Weight Realization.** Define the **collapse depth** of $\mathcal{F}$:

$$\delta_{[g]}(\mathcal{F}) := \min \left\{ n \mid \mathrm{Fil}^{\mathrm{ont}}_{g(n)} \mathcal{F} = 0 \right\}.$$

Weight can be reinterpreted as:

$$\mathrm{Weight}^{[g]}(\mathcal{F}) := \sup \left\{ n \mid \mathrm{gr}^{[g]}_{n} \mathcal{F} \neq 0 \right\}.$$

Persistence and weight now form dual invariants for arithmetic reality.

4.8. **Conclusion.** In this section we have:

- Defined growth-stratified cohomology groups and spectral sequences;
- Connected period realizations with ε-filtered layers;
- Introduced hyper-regulators and survival-indexed K-theory classes;
- Reformulated weight and persistence as dual notions in categorified arithmetic.

In the next section, we complete the theory by integrating arithmetic descent, ε-gerbes, and persistence fields into a coherent arithmetic stratification framework.

5. Arithmetic Descent and Persistence Field Theory

5.1. **From Galois Descent to Collapse-Stable Arithmetic.** In classical arithmetic geometry, descent theory interprets how global structures can be recovered from local data with symmetry (e.g., Galois torsors). In the present framework, we reinterpret arithmetic descent as:

Collapse-stable persistence of arithmetic data across growth-indexed layers.

This gives rise to a generalized descent framework over growth-stratified sites and ε-torsorial stacks.

5.2. **Persistence Fields and Descent Fields.**

Definition 5.1 (Persistence Field)**.** *A* persistence field $\mathbb{F}^{[g]}_{\varepsilon}$ *is a filtered arithmetic object equipped with:*

- *Growth-indexed valuation:* $v_{[g]} : \mathbb{F}^{[g]} \to \mathbb{Z} \cup \{\infty\}$*;*
- *Collapse-core:* $\mathcal{E}_{\mathrm{exist}[g]}(\mathbb{F}) := \bigcap_n \mathrm{Fil}^{\mathrm{ont}}_{g(n)} \mathbb{F}$*;*
- *Arithmetic realization functor:* $\mathbb{F}^{[g]} \rightsquigarrow B^{\mathrm{cat}}_{g,\mathrm{dR}}$.

This generalizes both p-adic fields and real/complex analytic fields to recursive-logical arithmetic bases.

5.3. **Descent Torsors over ε-Gerbes.** Let $\mathscr{G}^{[g]}_{\mathrm{desc}}$ be the ε-gerbe of descent torsors. Sections of this gerbe classify gluing data over growth-filtered coverings that survive collapse.

Each object contains:
- A sheaf $\mathcal{F}$ with $\mathrm{Fil}^{\mathrm{ont}}_{g(n)}$ structure;
- Descent morphisms $\varphi^{[g]}_{ij}$ along overlaps;
- Cohomological constraints encoded via torsorial cohomology classes.

5.4. **Arithmetic Descent Theorem.**

Theorem 5.2 (Growth-Based Descent). *Let $\{U_i \to X\}$ be a g-filtered covering. Then:*

$$\mathcal{F} \in \mathscr{S}^{\mathrm{ont}}(X) \quad \Longleftrightarrow \quad \left\{ \mathcal{F}_i \in \mathscr{S}^{\mathrm{ont}}(U_i),\, \varphi^{[g]}_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}} \right\}$$

satisfying ε-torsorial cocycle conditions.

5.5. **Persistence Galois Fields.** We define a *persistence Galois group*:

$$\mathrm{Gal}^{[g]}_{\varepsilon^\infty} := \pi^{\varepsilon}_1(X, \mathrm{Fil}^{\mathrm{ont}}_{g(n)}),$$

acting on sections of $\mathcal{F}$ as collapse-stable symmetries.

Field extensions $\mathbb{F} \hookrightarrow \mathbb{F}^{[g]}_{\varepsilon}$ define recursive analogues of classical field extensions via descent torsors.

5.6. **Categorified Arithmetic Field Theory.** We reinterpret the arithmetic structure as a field-theoretic data system:

- Base space: X (arithmetic ontoid);
- Field: $\mathbb{F}^{[g]}_{\varepsilon}$;
- Structure sheaves: $\mathcal{O}^{[g]}_X := \mathcal{E}_{\mathrm{exist}[g]}(\mathcal{F})$;
- Symmetries: $\mathrm{Gal}^{[g]}_{\varepsilon^\infty}$;
- Cohomology: $H^i_{[g]}(X, \mathcal{O}^{[g]}_X)$;
- Period realization: $\pi_{[g]}(\mathcal{F}) \in B^{\mathrm{cat}}_{g,\mathrm{dR}}$.

5.7. **Torsor Realizations and Descent Fields.** Every arithmetic sheaf $\mathcal{F}$ can be reconstructed via:

$$\mathcal{F} \simeq \mathcal{T}^{[g]} \times^{\mathrm{Gal}^{[g]}_{\varepsilon^\infty}} \mathcal{O}^{[g]}_X,$$

where $\mathcal{T}^{[g]}$ is a descent torsor and $\mathcal{O}^{[g]}_X$ is a base sheaf with ε-stratified arithmetic structure.

5.8. **Conclusion.** This section formalizes:

- Persistence fields and growth-filtered arithmetic extensions;
- ε-descent torsors and gerbes for global gluing;
- Galois-type symmetry groups acting across filtration towers;
- A complete field-theoretic reformulation of categorified arithmetic.

We now proceed to conclude this volume with a synthesis of stratified arithmetic geometry as an ontological and logical theory of growth and survival.

6. Conclusion: Categorical Arithmetic as Stratified Ontology

6.1. **From Numbers to Persistence Structures.** Traditional arithmetic studies numbers, fields, and their algebraic relationships. In this volume, we have reconstructed arithmetic as a geometric theory of:

Collapse-stable, growth-indexed, logically persistent sheaf structures.

We no longer view numbers as primitive. Instead, we model them as manifestations of deeper survival patterns across ontological filtration towers.

6.2. **Summary of Core Structures.** Let us summarize the architecture built in this volume:

Classical	**Categorified (Growth-Based)**
Fields	Persistence Fields $\mathbb{F}_{\varepsilon}^{[g]}$
Sheaves	Stratified ε-sheaves over ontoids
Cohomology	Collapse-indexed $H^i_{[g]}(X, \mathcal{F})$
Galois Groups	$\mathrm{Gal}_{\varepsilon^\infty}^{[g]}$ over towers
Motivic Periods	Realizations in $B_{g,\mathrm{dR}}^{\mathrm{cat}}$
Moduli	$\mathscr{M}_{\varepsilon^\infty}^{[g]}$ stack of survival classes
Descent	Gerbes $\mathscr{G}_{\mathrm{hyp}}^{[g]}$, torsor descent

Each element is now stratified by recursive growth behavior, and interpreted through ε-geometry.

6.3. **Arithmetic as Logic-Indexed Ontology.** We reinterpret arithmetic as:

$$\boxed{\text{Arithmetic} \;=\; \text{Persistence across } \varepsilon\text{-indexed ontological stratification}}$$

This structure is more than formal—it encodes computability, recursion, stability, and meta-logical survival.

6.4. **Collapse as Metaphysical Differentiation.** We endow collapse with foundational meaning:
- Collapse defines non-being;
- Survival defines ontological arithmetic reality;
- Growth defines the rate of approach to transcendence.

In this setting, growth functions index not just size, but depth of logical commitment.

6.5. **Recursive Moduli of Existence.** The universal moduli tower:

$$\left\{ \mathscr{M}^{[g]}_{\varepsilon^\infty} \right\}_{g \in \mathsf{Growth}}$$

encodes all possible arithmetic configurations as recursive stability classes.

Existence becomes parametrized. Reality becomes stratified. Arithmetic becomes categorified logic.

6.6. **Future Directions.**

- Extend persistence field theory to ∞-categorical motives;
- Develop non-commutative versions of $\mathscr{M}^{[g]}_{\varepsilon^\infty}$;
- Embed these structures into physics-inspired categorifications (e.g., quantum persistence sheaves);
- Explore growth-based arithmetic dynamics and mirror symmetry over stratified fields.

6.7. **Final Philosophy.**

> *Numbers do not exist merely in fields—they exist in towers of resistance to collapse.*
> *Arithmetic is not algebra—it is the logic of surviving identity.*

— End of Volume V

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VOLUME VI: HYPERCATEGORICAL PERSISTENCE AND TRANSFINITE ARITHMETIC STRUCTURES
FROM COLLAPSE LOGIC TO ∞-STACKED EXISTENCE FIELDS

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Abstract. This volume develops a fully transfinite, hypercategorical framework for arithmetic structures derived from collapse-invariant geometry. Expanding on the ontoid–growth–torsor–gerbe structures from Volumes I–V, we define ∞-stacked persistence objects, recursive fixed-point sheaves, and trans-collapsing meta-logical spaces. Central to this theory is the notion of an *Existence Field*: an ontological base geometry defined as the fixed-point of recursive collapse under ∞-sheaf descent. We aim to unify arithmetic, logic, and geometry through formalized survival hierarchies of infinite depth.

Contents

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0. Symbol Dictionary for Hypercategorical Structures

This dictionary introduces the core notations and objects that govern ∞-stacked, recursively persistent, collapse-fixed, and meta-existential structures.

Persistence and Collapse.

- $\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: ontological filtration at level n;
- $\mathrm{Fil}_{g(n)}^{\mathrm{ont}}\mathcal{F}$: growth-function stratified filtration;
- $\mathrm{Fil}_\infty^{\mathrm{ont}}\mathcal{F} := \bigcap_n \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: stable core of $\mathcal{F}$;
- $\mathscr{C}_\varepsilon(\mathcal{F})$: collapse of non-persistent layers;
- $\mathscr{C}_\varepsilon^\infty(\mathcal{F})$: transfinite limit collapse;
- $\mathcal{E}_{\mathrm{exist}}(\mathcal{F})$: persistent core under ε-collapse;

Categorical Structures.

- $\mathbf{Ont}_{\varepsilon^\infty}$: category of ontoid spaces with ε-filtration;

- $\mathscr{S}_\infty$: ∞-stack over persistent filtered sites;
- $\mathsf{Sh}^{\infty\text{-meta}}(X)$: category of recursively defined ∞-sheaves over X;
- $\mathbb{E}^{[g]}_{\infty\text{-stack}}$: growth-indexed ∞-stack of stratified sheaf layers;
- **RecCollapse** : category of recursively collapsing towers;

Existence Fields and Onto-Meta Objects.

- $\mathbb{F}^{[\infty]}_{\text{exist}}$: the field object at fixed point of collapse—"field of pure persistence";
- $\mathbb{F}^{[\infty]}_{\text{core}}$: the intersection of all persistence fields across $g(n)$;
- $\mathcal{O}^{[\infty]}$: structure sheaf of the meta-existence spectrum;
- $\mathrm{Spec}^{[\infty]}(\mathbb{F}^{[\infty]}_{\text{exist}})$: spectrum of the Existence Field;

Fixed-Point Collapse and Meta-Stratification.

- $\mathcal{F}_{\text{fixed}}$: fixed point of a recursive collapse sequence on $\mathcal{F}$;
- $\mathcal{F}^{[\omega^\omega]}$: stability limit of iterated ε-collapse;
- $\mathrm{Fix}(\mathscr{C}^\infty_\varepsilon)$: functor returning collapse-stable ∞-objects;
- $\mathrm{gr}^{[\infty]}_n(\mathcal{F})$: meta-graded stratification at transfinite level n;

Recursive Universes and Hyperontologies.

- $\mathcal{U}^{[\infty]}$: universe of all ∞-persistent spaces;
- $\mathcal{T}_{\text{rec}}$: recursive tower topology over hyperontological base;
- $\mathscr{X}^{[\varepsilon^\infty]}_\infty$: object representing a trans-collapsing hyperontoid;
- **MetaTopos** : logical topos built from ∞-sheaf towers;

Conventions.

- All collapse operations are implicitly indexed over ε^∞ unless specified;
- Growth indexing extends into hypergrowth ($g(n) = n \uparrow^k n$, etc.);
- Recursive universes and logical fixed-points are interpreted internally via ∞-sheaf recursion;
- Fields are no longer algebraic: they are survival-class fields under layered meta-logic.

1. Transfinite Persistence Towers and Recursive Collapse

1.1. **From Finite Collapse to Transfinite Stratification.** Volumes I–V defined persistence by collapse filtrations indexed by $n \in \mathbb{N}$ or $g(n)$. We now extend this framework transfinely, replacing finite indexing with:

$$\alpha \in \omega,\ \omega^\omega,\ \varepsilon^\infty,\ \text{or } \mathrm{Ord}_{\text{meta}}.$$

Our goal is to define collapse towers of the form:

$$\cdots \subseteq \mathrm{Fil}^{\mathrm{ont}}_{\alpha+1}\mathcal{F} \subseteq \mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F} \subseteq \cdots \subseteq \mathrm{Fil}^{\mathrm{ont}}_{0}\mathcal{F},$$

where each $\mathrm{Fil}^{\mathrm{ont}}_{\alpha}$ records survival at ontological depth α.

1.2. Recursive Collapse Tower.

Definition 1.1 (Recursive Collapse Tower)**.** *Let $\mathcal{F}$ be an ∞-sheaf. Define:*

$$\mathrm{Fil}^{\mathrm{ont}}_{0}\mathcal{F} := \mathcal{F}, \quad \mathrm{Fil}^{\mathrm{ont}}_{\alpha+1}\mathcal{F} := \mathscr{C}_\varepsilon(\mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F}), \quad \mathrm{Fil}^{\mathrm{ont}}_{\lambda}\mathcal{F} := \bigcap_{\beta<\lambda} \mathrm{Fil}^{\mathrm{ont}}_{\beta}\mathcal{F} \ \textit{(for limit } \lambda\textit{)}.$$

This gives a transfinite tower of decreasing subobjects encoding meta-resilience.

1.3. Collapse Fixed-Points and Onto-Existence.

Definition 1.2 (Fixed-Point Object)**.** *We say $\mathcal{F}$ is* collapse-fixed *if:*

$$\mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F} = \mathrm{Fil}^{\mathrm{ont}}_{\alpha+1}\mathcal{F} \quad \textit{for some } \alpha,$$

or equivalently:

$$\mathscr{C}_\varepsilon(\mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F}) = \mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F}.$$

Such $\mathcal{F}$ defines the emergence of an *Existence Field*, where no further collapse alters structure.

1.4. Ontological Depth and Collapse Ordinals.

Definition 1.3 (Collapse Ordinal)**.** *The* collapse ordinal $\delta(\mathcal{F})$ *is the smallest α such that:*

$$\mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F} = 0.$$

Alternatively, if this ordinal is not attained, $\mathcal{F}$ is said to be *infinitely persistent*.

1.5. Collapse Sequences as Type-Theoretic Universes.

We may view:

$$\mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F} \quad \text{as the } \alpha\text{-truncation of } \mathcal{F} \quad \text{(cf. HoTT } n\text{-types)}$$

Recursive collapse mimics universe stratification in homotopy type theory: each $\mathrm{Fil}^{\mathrm{ont}}_{\alpha}$ defines how deeply the object is proven to exist.

1.6. Existence Spectrum and Survivability Index.

Define the *existence spectrum* of $\mathcal{F}$:

$$\mathrm{Spec}^{[\infty]}(\mathcal{F}) := \left\{\alpha \mid \mathrm{Fil}^{\mathrm{ont}}_{\alpha}\mathcal{F} \neq 0\right\},$$

which encodes logical survival across all collapse depths.

1.7. **Limit Stabilization and Transfinite Cohomology.** Let:

$$\mathrm{Fil}_{\infty}^{\mathrm{ont}}\mathcal{F} := \bigcap_{\alpha} \mathrm{Fil}_{\alpha}^{\mathrm{ont}}\mathcal{F}$$

Then:

$$H^i_{\infty}(X, \mathcal{F}) := H^i(X, \mathrm{Fil}_{\infty}^{\mathrm{ont}}\mathcal{F}),$$

is the cohomological invariant of the recursively persistent core of $\mathcal{F}$.

1.8. **Collapse-Commutative Towers and Universality.** Let $\mathcal{F}$ and $\mathcal{G}$ be two towers. A morphism $f : \mathcal{F} \to \mathcal{G}$ is said to be:

$$\text{Collapse-commutative} \iff f(\mathrm{Fil}_{\alpha}^{\mathrm{ont}}\mathcal{F}) \subseteq \mathrm{Fil}_{\alpha}^{\mathrm{ont}}\mathcal{G},\ \forall \alpha.$$

This induces functorial behavior across the category of transfinite persistence towers **RecCollapse**.

1.9. **Conclusion.** In this section, we have:

- Constructed transfinite collapse towers indexed by $\alpha \in \mathrm{Ord}$;
- Defined fixed-points as markers of meta-existence;
- Linked collapse stratification with type-theoretic universes;
- Introduced cohomology of infinitely persistent cores;
- Prepared for categorical constructions of ∞-stacked recursive objects.

In the next section, we define ∞-stacked ontoids and recursive universes, and formalize existence fields as fixed-points of $\mathscr{C}_{\varepsilon}^{\infty}$ over filtered sheaves.

2. ∞-Stacked Ontoids and Recursive Universes

2.1. **Motivation: Beyond Single Collapse Towers.** Transfinite persistence towers allow us to track collapse and survival through all levels of logical depth. But in higher categorical settings, one must also:
- Track families of towers;
- Stack persistence filtrations over filtrations;
- Organize fixed-points across ontological universes.

This motivates the concept of ∞-stacked ontoids.

2.2. Definition of ∞-Stacked Ontoid.

Definition 2.1 (∞-Stacked Ontoid). *An* ∞-stacked ontoid *is a diagram:*

$$X^{[\infty]} := \left\{(X_\alpha, \mathrm{Fil}^{\mathrm{ont}}_\beta \mathcal{F}_\alpha)\right\}_{\alpha,\beta<\varepsilon^\infty},$$

where:

- *Each X_α is an ontoid with its own filtration tower;*
- *There exist functorial collapse-compatible maps:*
$$f_{\alpha\to\alpha'} : X_\alpha \to X_{\alpha'}, \quad f^*_{\alpha\to\alpha'}(\mathrm{Fil}^{\mathrm{ont}}_\beta \mathcal{F}_{\alpha'}) \subseteq \mathrm{Fil}^{\mathrm{ont}}_\beta \mathcal{F}_\alpha;$$
- *The tower stabilizes at a transfinite collapse fixed-point.*

This creates a universe of nested persistence geometries.

2.3. Recursive Universes and Meta-Filtration.

Definition 2.2 (Recursive Universe). *Let $\mathcal{U}^{[\infty]}$ be the class of all ∞-stacked ontoids. This universe is filtered by complexity of recursive collapse, i.e.,*

$$\mathcal{U}_n := \left\{X^{[\infty]} \mid \delta(X_\alpha) \leq n,\ \forall\alpha\right\}.$$

As $n \to \infty$, these form an ascending tower of arithmetic ontologies.

2.4. Meta-Sheaves over Recursive Universes. Let $\mathsf{Sh}^{\infty\text{-meta}}(\mathcal{U}^{[\infty]})$ be the category of sheaves $\mathscr{F}$ assigning to each X_α:

$$\mathscr{F}(X_\alpha) := \mathrm{Fil}^{\mathrm{ont}}_\bullet \mathcal{F}_\alpha,$$

along with transition maps compatible with both the collapse and ontological morphisms. Such $\mathscr{F}$ form the first layer of a recursive sheaf tower.

2.5. Recursive Collapse Stability.

Definition 2.3 (Collapse-Fixed ∞-Stack). *We say $X^{[\infty]}$ is* universally persistent *if:*

$$\forall\alpha,\ \exists\lambda_\alpha < \varepsilon^\infty \text{ such that } \mathrm{Fil}^{\mathrm{ont}}_{\lambda_\alpha}\mathcal{F}_\alpha = \mathrm{Fil}^{\mathrm{ont}}_{\lambda_\alpha+1}\mathcal{F}_\alpha.$$

This ensures all ∞-strata stabilize under recursive collapse.

2.6. Hyperontoid Fields and the Existence Field. We define:

$$\mathbb{F}^{[\infty]}_{\mathrm{exist}} := \bigcap_\alpha \bigcap_\beta \mathrm{Fil}^{\mathrm{ont}}_\beta \mathcal{F}_\alpha \quad \in \quad \mathsf{Sh}^{\infty\text{-meta}}(\mathcal{U}^{[\infty]}),$$

This object represents the ultimate recursive survival core: a meta-fixed-point object stable across all layers of transfinite arithmetic.

2.7. Meta-Morphisms and ∞-Structural Functors. A *meta-morphism* $F : X^{[\infty]} \to Y^{[\infty]}$ consists of:

- Level-wise collapse-preserving functors F_α; - Natural transformations respecting collapse-fixed-point layers; - Commuting diagrams under limit towers.

The category $\mathscr{S}_\infty$ of ∞-stacked ontoids is thus enriched over $\mathsf{Sh}^{\infty\text{-meta}}$.

2.8. Conclusion. This section builds:

- The notion of ∞-stacked ontoids;
- Recursive universes $\mathcal{U}^{[\infty]}$ of stratified persistence spaces;
- The meta-sheaf category encoding towered survival;
- The Existence Field $\mathbb{F}_{\text{exist}}^{[\infty]}$ as the universal collapse-fixed object.

In the next section, we construct sheaves over ∞-sheaves and recursive fixed-point descent systems.

3. Sheaves over ∞-Sheaves: Recursive Fixed-Point Layers

3.1. Second-Order Ontology: Sheaves of Sheaf Towers. To capture meta-persistence and higher collapse behavior, we define sheaves not over spaces, but over towers of sheaves themselves. This gives rise to:

$$\textbf{Sheaves over } \boldsymbol{\infty}\textbf{-sheaves}, \quad \text{i.e.,} \quad \mathscr{F} \in \mathsf{Sh}^{\infty\text{-meta}}(\mathscr{G}),$$

where $\mathscr{G}$ is itself a filtered sheaf over a recursive ontoid universe.

3.2. Definition: Recursive Sheaf Tower.

Definition 3.1 (Recursive Sheaf Tower)**.** *Let $\mathscr{G} = \left\{ \mathrm{Fil}_\alpha^{\mathrm{ont}} \mathcal{F} \right\}_{\alpha<\lambda}$ be an ∞-filtered object. A* sheaf over $\mathscr{G}$ *is a diagram:*

$$\mathcal{S} : \mathrm{Fil}_\alpha^{\mathrm{ont}} \mapsto \mathcal{S}(\mathrm{Fil}_\alpha^{\mathrm{ont}}) \in \mathbf{Ab},$$

such that for all $\alpha \le \beta$, there are morphisms:

$$\mathcal{S}(\mathrm{Fil}_\beta^{\mathrm{ont}}) \to \mathcal{S}(\mathrm{Fil}_\alpha^{\mathrm{ont}}), \quad \textit{satisfying } \mathcal{S}(\mathscr{C}_\varepsilon(\mathrm{Fil}_\beta^{\mathrm{ont}})) \simeq \mathscr{C}_\varepsilon(\mathcal{S}(\mathrm{Fil}_\beta^{\mathrm{ont}})).$$

These structures allow one to track collapse at both the base and sheaf level.

3.3. Meta-Persistence Functor. Define the functor:

$$\mathcal{P}^{[\infty]} : \mathscr{G} \mapsto \varprojlim_\alpha \mathcal{S}(\mathrm{Fil}_\alpha^{\mathrm{ont}}),$$

which outputs the stable core of the sheaf-over-sheaf. This functor is analogous to taking global sections in a classical topos, but indexed by recursive collapse layers.

3.4. **Fixed-Point Sheaf Layers.**

Definition 3.2 (Collapse Fixed-Point Layer)**.** *Let $\mathscr{S}$ be a sheaf over an ∞-stacked ontoid. The fixed-point layer is:*

$$\mathscr{S}_{\mathit{fixed}} := \left\{ s \in \mathscr{S} \mid \forall\alpha,\ \mathscr{C}_\varepsilon(s|_{\mathrm{Fil}_\alpha^{\mathrm{ont}}}) = s|_{\mathrm{Fil}_\alpha^{\mathrm{ont}}} \right\}.$$

These sections survive all levels of collapse inside a sheaf-of-sheaf structure.

3.5. **Recursive Descent Systems.** Given a diagram of ∞-sheaves $\{\mathscr{G}_i \to \mathscr{G}_j\}$, a recursive descent system is a collection of objects $\mathcal{S}_i$ with:
- Transition morphisms $\mathcal{S}_i \to \mathcal{S}_j$ compatible with both:
- ∞-sheaf base transitions;
- Internal collapse of $\mathcal{S}_j$.

Descent here is not classical glueing, but fixed-point coherence across ontological layers.

3.6. **Meta-Site and Higher Topology.** Define a *meta-site* $(\mathcal{U}^{[\infty]}, \tau^{[\infty]})$ where coverings are families:

$$\{f_i : \mathscr{G}_i \to \mathscr{G}\} \quad \text{such that} \quad \mathrm{Fil}_\alpha^{\mathrm{ont}}\mathscr{G} = \bigcap_i f_i^* \mathrm{Fil}_\alpha^{\mathrm{ont}}\mathscr{G}_i,\ \forall\alpha.$$

A stack on this site encodes descent data over stratified persistence universes.

3.7. **Collapse-Commutative Cohomology.** Let $\mathscr{S} \in \mathsf{Sh}^{\infty\text{-meta}}(\mathscr{G})$. Then meta-cohomology is defined by:

$$H^i_{\mathrm{meta}}(\mathscr{G}, \mathscr{S}) := \varprojlim_\alpha H^i(\mathrm{Fil}_\alpha^{\mathrm{ont}}\mathscr{G},\ \mathscr{S}(\mathrm{Fil}_\alpha^{\mathrm{ont}})).$$

This measures not just section extension, but survival throughout recursive collapse.

3.8. **Conclusion.** This section defines:

- Sheaves over ∞-sheaves and recursive collapse tracking;
- Fixed-point layers as internal ontological survivors;
- Recursive descent systems over meta-sites;
- Meta-cohomology and recursive topological invariants.

4. Logical Fixed Points and ε-Inaccessible Collapses

4.1. **From Collapse Stability to Logical Self-Containment.** Collapse fixed-points describe structural invariants under survival stratification. We now consider fixed-points at the level of logical systems themselves.

Let $\mathcal{L}$ be a formal logic system (type theory, set theory, etc.) and let $\mathscr{C}_{\varepsilon\mathcal{L}}$ denote the collapse operation under provability and definability.

We aim to study:

$$\textbf{Logically fixed structures:}\quad \mathcal{F} \cong \mathscr{C}_{\varepsilon\mathcal{L}}(\mathcal{F}).$$

4.2. Definition: Logical Collapse Functor.

Definition 4.1 (Logical Collapse Functor). *Given a logic $\mathcal{L}$ and an object $\mathcal{F}$ in $\mathsf{Sh}^{\infty\text{-}meta}$, define:*

$$\mathscr{C}_{\varepsilon\mathcal{L}}(\mathcal{F}) := \textit{image of } \mathcal{F} \textit{ under all provable collapse-reductions in } \mathcal{L}.$$

This generalizes collapse from categorical structure to logical content and proof visibility.

4.3. Fixed-Point Logic and Meta-Stability.

Definition 4.2 (Logical Fixed Point). *We say $\mathcal{F}$ is* logically fixed *under $\mathcal{L}$ if:*

$$\mathcal{F} = \mathscr{C}_{\varepsilon\mathcal{L}}(\mathcal{F}),$$

and recursively:

$$\forall \mathcal{L}' \prec \mathcal{L}, \quad \mathscr{C}_{\varepsilon\mathcal{L}'}(\mathcal{F}) = \mathcal{F}.$$

This implies $\mathcal{F}$ is maximally stable under internal and external provability collapse.

4.4. ε-Inaccessible Stratification.

We define:

Definition 4.3 (ε-Inaccessible Collapse Level). *An ordinal δ is said to be ε-*inaccessible *for $\mathcal{F}$ if:*

$$\mathrm{Fil}^{\mathrm{ont}}_{\delta}\mathcal{F} \neq 0, \quad \textit{but} \quad \mathrm{Fil}^{\mathrm{ont}}_{\delta+1}\mathcal{F} = 0, \quad \textit{and } \delta \textit{ is inaccessible under collapse-definability in } \mathcal{L}.$$

These points represent collapse-resistant anomalies in logical universes.

4.5. Meta-Existence Conditions.

Define:

$$\mathcal{F}_{\text{meta-exist}} := \left\{ s \in \mathcal{F} \mid s \text{ is invariant under } \mathscr{C}^{\,n}_{\varepsilon\mathcal{L}} \text{ for all } n \right\}.$$

This is the ultimate fixed-point subobject under iterated logical collapse.

4.6. Collapse Hierarchies and Modal Logics.

Let $\mathcal{L}_\beta$ be a family of logics indexed by proof-theoretic strength (e.g., Peano, ZFC, HoTT, etc.). We define the *collapse hierarchy*:

$$\mathscr{C}_{\varepsilon\mathcal{L}_0}(\mathcal{F}) \supseteq \mathscr{C}_{\varepsilon\mathcal{L}_1}(\mathcal{F}) \supseteq \cdots \supseteq \mathscr{C}_{\varepsilon\mathcal{L}_\infty}(\mathcal{F}).$$

We say $\mathcal{F}$ is *collapse-modal stable* if it remains fixed across a family of logics.

4.7. **The Fixed-Point Topos.** Let **FixTopos** be the subcategory of $\mathsf{Sh}^{\infty\text{-meta}}$ consisting of collapse-logically fixed objects.

This category forms a closed structure under:
- pullbacks of logically fixed towers;
- collapse-invariant realization functors;
- internally definable cohomology.

4.8. **Conclusion.** In this section we:

- Extended collapse to logical content and proof-theoretic visibility;
- Defined fixed-points under logical systems $\mathcal{L}$;
- Introduced ε-inaccessible collapse levels;
- Built a collapse-modal topos **FixTopos** of self-surviving logical objects.

In the next section, we define Existence Fields as logical-collapse fixed-points and formulate their meta-cohomological classification.

5. Existence Fields and Meta-Stratified Cohomology

5.1. **Ultimate Collapse Stability and Existence Fields.** Throughout this volume, we have observed that certain objects—those stable under all forms of collapse, recursive filtration, and logical descent—emerge as foundational. We now isolate such objects and classify them as:

$$\textbf{Existence Fields}: \quad \text{Collapse-logically persistent core objects of meta-arithmetic geometry.}$$

5.2. **Definition: Existence Field.**

Definition 5.1 (Existence Field)**.** *An* Existence Field $\mathbb{F}_{\mathrm{exist}}^{[\infty]}$ *is an object in* $\mathsf{Sh}^{\infty\text{-}meta}$ *satisfying:*

(1) $\mathbb{F}_{\mathrm{exist}}^{[\infty]} = \mathscr{C}_{\varepsilon\mathcal{L}}(\mathbb{F}_{\mathrm{exist}}^{[\infty]})$ *for all* $\mathcal{L}$ *in the modal logic hierarchy;*
(2) $\mathbb{F}_{\mathrm{exist}}^{[\infty]} = \mathrm{Fil}_{\alpha}^{\mathrm{ont}}(\mathbb{F}_{\mathrm{exist}}^{[\infty]})$ *for all* α *such that* $\mathrm{Fil}_{\alpha}^{\mathrm{ont}}(\mathbb{F}_{\mathrm{exist}}^{[\infty]}) \neq 0$*;*
(3) $\mathbb{F}_{\mathrm{exist}}^{[\infty]}$ *represents a terminal object in* **FixTopos** *under recursive sheaf morphisms.*

5.3. **Sheaf-Theoretic Field Structure.** The object $\mathbb{F}_{\mathrm{exist}}^{[\infty]}$ supports the following meta-arithmetic structures:

- A collapse-invariant structure sheaf $\mathcal{O}^{[\infty]}$;
- Stable meta-ring operations: addition, multiplication over ∞-filtrations;
- Period morphisms: $\mathbb{F}_{\mathrm{exist}}^{[\infty]} \to B_{\mathrm{dR}}^{[\infty]}$ via meta-realization functors.

5.4. **Meta-Stratified Cohomology.** Define cohomology groups:

$$H^i_{\infty,\mathrm{meta}}(\mathcal{U}^{[\infty]}, \mathbb{F}^{[\infty]}_{\mathrm{exist}}) := \varprojlim_{\alpha} H^i(\mathrm{Fil}^{\mathrm{ont}}_{\alpha}, \mathbb{F}^{[\infty]}_{\mathrm{exist}}),$$

where the transition maps are logical-collapse preserving.

These groups classify stable arithmetic invariants of recursive survival fields.

5.5. **Meta-Torsors and Moduli of Survival.** We define the category of $\mathbb{F}^{[\infty]}_{\mathrm{exist}}$-torsors $\mathscr{T}^{[\infty]}$:

$$\mathscr{T}^{[\infty]}(X) := \left\{ \mathcal{T} \text{ over } X \mid \mathcal{T} \times^{\mathbb{F}^{[\infty]}_{\mathrm{exist}}} \mathbb{F}^{[\infty]}_{\mathrm{exist}} \cong \mathbb{F}^{[\infty]}_{\mathrm{exist}} \right\},$$

These classify deformation classes under logic- and collapse-invariant auto-equivalences.

5.6. **Spectral Realization and Meta-Period Maps.** Let:

$$\mathrm{Spec}^{[\infty]}(\mathbb{F}^{[\infty]}_{\mathrm{exist}}) := \text{Space of meta-survivability types},$$

and define the period morphism:

$$\pi^{[\infty]} : \mathbb{F}^{[\infty]}_{\mathrm{exist}} \longrightarrow B^{[\infty]}_{\mathrm{dR}},$$

as the universal realization of fixed-point arithmetic structure.

5.7. **Collapse-Invariant Universal Coefficients.** For any sheaf $\mathscr{S}$ over $\mathbb{F}^{[\infty]}_{\mathrm{exist}}$, we have:

$$H^i_{\mathrm{meta}}(X, \mathscr{S}) \cong \mathrm{Ext}^i_{\mathbb{F}^{[\infty]}_{\mathrm{exist}}}(\mathcal{O}^{[\infty]}, \mathscr{S}),$$

where $\mathcal{O}^{[\infty]}$ is the canonical meta-structure sheaf.

This yields a universal cohomology theory entirely rooted in persistence.

5.8. **Conclusion.** This section identifies and formalizes:

- Existence Fields as collapse- and logic-fixed objects;
- Their meta-arithmetic and sheaf-theoretic structures;
- Meta-cohomology as a persistent invariant classifier;
- Moduli, torsors, and realization spectra over $\mathbb{F}^{[\infty]}_{\mathrm{exist}}$.

6. Conclusion: Toward a Formal Theory of Ultimate Arithmetic Ontology

6.1. **From Collapse to Being.** Arithmetic has been reinterpreted across six volumes—not as operations over numbers, but as layered survival of structured existence through:

- Recursive collapse;
- Logical visibility;
- Stratified cohomology;
- Ontological resistance.

This culminates in a unifying object: the **Existence Field**, defined not by algebraic closure or valuation, but by fixed-point survival under all structural, categorical, and logical descent.

6.2. **Volume-by-Volume Synthesis.**

- **Volume I**: From additive filtration to multiplicoid–knuthoid geometry;
- **Volume II**: From ε-stratification to transfinite motivic persistence;
- **Volume III**: From linear WMC to hypermonodromy conjectures;
- **Volume IV**: From topological covers to space-theoretic ontology;
- **Volume V**: From fields to growth-indexed categorical arithmetic;
- **Volume VI**: From collapse to existence, from logic to ultimate arithmetic.

Each layer embeds the previous into a broader survival framework.

6.3. **Collapse as Logic, Cohomology as Persistence.** Let us reinterpret classical invariants:

Concept	**Ontological Interpretation**
Field	Collapse-fixed survival object
Valuation	Rate of epistemic loss
Sheaf	Locally persistent structure
Cohomology	Measure of meta-survival
Period	Collapse-invariant projection
Torsor	Persistence deformation orbit
Galois group	Collapse automorphism tower
Logic	Collapse operator indexed by provability

6.4. **Meta-Existence as Formal Foundation.** We conclude with a formal axiom:

> **Axiom (Meta-Existence)**: *Only objects stable under all forms of collapse—categorical, logical, recursive, and epistemic— constitute the formal existence layer of arithmetic ontology.*

This defines a new mathematical universe built from internal proof-resilience.

6.5. Final Diagram: Collapse–Logic–Arithmetic Ontology.

$$\begin{tikzcd}
\textbf{Collapse Theory} \arrow[rr, "\text{Indexed by } \varepsilon^{\infty}"] \arrow[dr] & & \textbf{Recursive Sheaf Geometr} \arrow[dl] \\
& \textbf{Ultimate Arithmetic Ontology} &
\end{tikzcd}$$

Collapse is no longer loss—it is the formal engine of structure.

6.6. Future Frameworks.

- Extend Existence Fields into model theory and dependent type theory;
- Apply ∞-sheaf recursion to quantum geometry;
- Use meta-cohomology to classify generalized motives and logics;
- Formalize a universal topos of survival-indexed mathematical structures.

6.7. Closing Reflection.

> *We have transcended the finite, the algebraic, the spatial, and even the logical.*
> *In their place: persistence. Existence. Collapse-resilient being.*
> *This is no longer just arithmetic—it is a geometry of survival.*

— *End of Volume VI and the Persistence Hexalogy*

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