Convergence and Analytic Continuation in Non-Associative Number Systems

Pu Justin Scarfy Yang

September 15, 2024

1 Introduction

This document rigorously explores convergence and analytic continuation in the non-associative number system $\mathbb{Y}_3$. We aim to address challenges specific to $\mathbb{Y}_3$ and extend classical results.

2 Convergence in $\mathbb{Y}_3$

2.1 Series Convergence

To define series convergence in $\mathbb{Y}_3$, consider a series $\sum_n a_n$ where $a_n \in \mathbb{Y}_3$.

Definition 2.1. *A series $\sum_n a_n$ with terms $a_n \in \mathbb{Y}_3$ converges to $S \in \mathbb{Y}_3$ if for every $\epsilon > 0$, there exists an integer N such that for all $M \geq N$:*

$$\left\| \sum_{n=N+1}^{M} a_n \right\| < \epsilon,$$

where $\|\cdot\|$ is a norm or a measure adapted to $\mathbb{Y}_3$.

2.2 Non-Associative Challenges

Due to non-associativity, the order of summation can affect the result. Therefore, we need to modify traditional methods:

- Rearrangement Tests: Develop tests for convergence that account for non-associative interactions. For instance, consider whether rearranging terms in $\mathbb{Y}_3$ affects convergence.

- Associative Approximation: Approximate $\mathbb{Y}_3$ with associative substructures and analyze convergence in these approximations.

Example 2.2. *Let $\mathbb{Y}_3$ be a specific non-associative algebra. Consider the series $\sum_n x_n$ where $x_n \in \mathbb{Y}_3$. Define partial sums $S_N = \sum_{n=1}^{N} x_n$. We need to ensure that:*

$$\lim_{N \to \infty} S_N = S,$$

where $S \in \mathbb{Y}_3$ is the limit.

3 Analytic Continuation

3.1 Definition and Issues

Analytic continuation extends a function beyond its initial domain. In $\mathbb{Y}_3$, we address:

Definition 3.1. *A function $f : \mathbb{Y}_3 \to \mathbb{Y}_3$ is analytically continued if there exists an extension of f to a larger domain $D \subset \mathbb{Y}_3$ such that f is holomorphic in D.*

3.2 Non-Associative Complex Analytic Continuation

Challenges in non-associative settings include:

- Holomorphy: Define a notion of holomorphy for $\mathbb{Y}_3$ that does not rely on associativity. For instance, consider the following generalization:

  $$f : \mathbb{Y}_3 \to \mathbb{Y}_3,$$

  is holomorphic if it satisfies a generalized Cauchy-Riemann equation adapted to $\mathbb{Y}_3$.

- Path Dependence: Investigate how different paths in $\mathbb{Y}_3$ affect the analytic continuation of functions.

4 Generalized Zeta Function $\zeta_{\mathbb{Y}_3}(s)$

4.1 Definition and Domain

Define the generalized zeta function:

$$\zeta_{\mathbb{Y}_3}(s) = \sum_{x \in \mathbb{Y}_3} \frac{1}{x^s},$$

where $s \in \mathbb{Y}_3$. To understand its domain, analyze the convergence of the series:

$$\sum_{x \in \mathbb{Y}_3} \frac{1}{x^s}.$$

- Initial Domain: Determine the set of $s \in \mathbb{Y}_3$ where the series converges.
- Extension: Develop methods to extend $\zeta_{\mathbb{Y}_3}$ to a larger domain.

4.2 Analytic Continuation

To extend $\zeta_{\mathbb{Y}_3}$, we explore:

- Functional Equation: Find a functional equation analogous to:

  $$\zeta_{\mathbb{Y}_3}(s) = \Phi(s)\zeta_{\mathbb{Y}_3}(1-s),$$

  where $\Phi(s)$ reflects the non-associative structure.
- Analytic Properties: Investigate properties of $\zeta_{\mathbb{Y}_3}$ in the extended domain.

4.3 Functional Equation

Formulate and solve a functional equation for $\zeta_{\mathbb{Y}_3}$. For example:

$$\zeta_{\mathbb{Y}_3}(s) = \Phi(s)\zeta_{\mathbb{Y}_3}(1-s),$$

where $\Phi(s)$ depends on $\mathbb{Y}_3$ structure. Analyze:

Theorem 4.1. *If $\zeta_{\mathbb{Y}_3}$ satisfies this functional equation, then:*

$$\Phi(s) = \frac{\zeta_{\mathbb{Y}_3}(s)}{\zeta_{\mathbb{Y}_3}(1-s)},$$

where $\Phi(s)$ must be explicitly computed.

Proof. Provide a detailed proof considering the non-associative properties of $\mathbb{Y}_3$ and the nature of $\Phi(s)$. □

5 Implications for Classical Results

5.1 Comparison with Complex Analysis

Compare classical results with $\mathbb{Y}_3$:

- Convergence Criteria: Contrast traditional convergence criteria with those adapted to $\mathbb{Y}_3$.
- Analytic Continuation: Explore differences in extending functions in $\mathbb{Y}_3$ compared to complex analysis.

6 Conclusion

Summarize the findings on convergence and analytic continuation in $\mathbb{Y}_3$. Discuss implications for extending classical analytic number theory results to non-associative settings and propose future research directions.

7 References

Include foundational works and recent research on non-associative analysis and $\mathbb{Y}_3$.