TOWARDS A FUNCTIONAL EQUATION FOR DEFORMED EULER ZETA FAMILIES

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Abstract. We study a family of deformed Dirichlet series defined by

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0,1],$$

where $t = 0$ corresponds to the constant function 1, and $t = 1$ recovers the classical Riemann zeta function $\zeta(s)$. This family interpolates between trivial and full Euler structure, and exhibits a progressive concentration of modulus minima toward the critical line $\Re(s) = \frac{1}{2}$ as $t \to 1^-$. Our goal is to formalize a functional equation-like structure for $L_t(s)$ to explain the emergence of critical symmetry.

Contents

1. Motivation

The Riemann zeta function satisfies a well-known functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

This deep symmetry implies that nontrivial zeros are reflected across the line $\Re(s) = \frac{1}{2}$. However, this identity is specific to $t = 1$. We ask:

> *Can a deformation parameter $t \in [0,1]$ continuously evolve into this symmetry, and is $\Re(s) = \frac{1}{2}$ the inevitable fixed point of analytic concentration?*

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2. Constructing a Deformation Analogue

We define the deformation family:

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0,1].$$

Let us define an auxiliary function $\Xi_t(s)$ modeled on the completed zeta function:

$$\Xi_t(s) := \phi_t(s) \cdot L_t(s),$$

where $\phi_t(s)$ is a deformation of the classical Γ-type pre-factor. At $t = 1$, we expect:

$$\phi_1(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad \Xi_1(s) = \Xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and

$$\Xi(s) = \Xi(1-s).$$

We conjecture:

(Deformation Symmetry Hypothesis) There exists a smooth family $\phi_t(s)$ such that:

$$\lim_{t\to 1^-} \Xi_t(s) = \Xi(s), \quad \text{and} \quad \Xi_t(s) \neq \Xi_t(1-s) \text{ for } t < 1.$$

3. Goal

We aim to:

1. Define $\phi_t(s)$ explicitly or through variational conditions.
2. Understand how functional symmetry emerges at $t = 1$ as a nontrivial limit.
3. Study whether the zero structure of $\Xi_t(s)$ converges to that of $\Xi(s)$.

4. Next Steps

In subsequent sections, we will explore:

- The modulus field $\mathcal{F}_t(s) := \log |L_t(s)|^2$ and its variational properties.
- Zero focusing flows under the gradient of $\mathcal{F}_t(s)$.
- Whether $\Re(s) = \frac{1}{2}$ is the unique zero attractor in the $t \to 1^-$ limit.