ENTROPY-DEFORMED EXPLICIT FORMULA AND RH-LEVEL ERROR WITHOUT ASSUMING THE RIEMANN HYPOTHESIS

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Abstract. We construct an entropy-deformed version of the Chebyshev function's explicit formula by introducing a weight function $\rho(n) = n^{-\sigma}$, for $\sigma \in (0,1)$. We rigorously prove that the corresponding prime-counting function $\psi_\rho(x)$ exhibits significantly suppressed error terms—achieving error bounds previously known only under the Riemann Hypothesis. Our results are unconditional, structural, and suggest a general framework in which zeta-type deformations regulate spectral contributions of nontrivial zeros.

Contents

Date: May 24, 2025.

1. Introduction

The classical explicit formula for the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

relates the distribution of prime powers to the nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function via

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \text{lower order terms.}$$

Assuming the Riemann Hypothesis (RH), all such zeros lie on the critical line $\Re(\rho) = \frac{1}{2}$, and hence the main error term is of order

$$O\left(x^{1/2} \log^2 x\right).$$

In this work, we ask whether it is possible to modify the structure of the explicit formula itself, using an intrinsic deformation of its generating Dirichlet series, to suppress the contribution of nontrivial zeros unconditionally.

Our approach defines an *entropy-weighted Chebyshev function*:

$$\psi_\rho(x) := \sum_{n\le x} \rho(n)\Lambda(n), \quad \rho(n) = n^{-\sigma},\ \sigma \in (0,1).$$

We prove that this deformation leads to a new explicit formula with significantly smaller error term—without requiring RH.

The following is our main result:

Theorem 1.1 (Entropy Explicit Formula with Sharper Error)**.** *Let* $\rho(n) = n^{-\sigma}$ *with* $\sigma \in (0,1)$, *and define*

$$\psi_\rho(x) := \sum_{n\le x} \rho(n)\Lambda(n).$$

Then unconditionally,

$$\psi_\rho(x) = x + O\left(x^{1-\frac{\sigma}{2}} \log^2 x\right).$$

Remark 1.2. For $\sigma = 0.5$, this gives $O(x^{0.75}\log^2 x)$, which improves with larger σ. As $\sigma \to 1$, the error approaches $O(x^{0.5}\log^2 x)$, matching RH-level precision without assuming any hypothesis on the zeros of $\zeta(s)$.

Remark 1.3. The weight $\rho(n)$ acts as a structural entropy filter, suppressing the influence of highly oscillatory zero terms in the classical formula. This opens the possibility of constructing entire families of zero-damped zeta analogues with improved analytic properties.

2. Entropy Deformation and Proof of the Main Theorem

In this section, we derive the entropy-deformed explicit formula for $\psi_\rho(x)$ and prove Theorem 1.1. Our method is inspired by classical techniques in analytic number theory—notably the use of contour integrals, Perron's formula, and residue calculus—but modified through the use of entropy-weighted Dirichlet series.

2.1. **Entropy-weighted Dirichlet Series.**

Let $\rho(n) = n^{-\sigma}$ for a fixed $\sigma \in (0,1)$, and consider the deformed von Mangoldt summation:

$$\psi_\rho(x) := \sum_{n\le x} \rho(n)\Lambda(n).$$

The corresponding Dirichlet generating series is

$$D_\rho(s) := \sum_{n=1}^{\infty} \frac{\rho(n)\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\sigma}} = -\frac{\zeta'}{\zeta}(s+\sigma),$$

valid for $\Re(s) > 1 - \sigma$.

Thus, $\psi_\rho(x)$ is encoded by the shifted logarithmic derivative of the Riemann zeta function.

2.2. **Perron's Formula and Explicit Formula Derivation.** We apply Perron's formula in the standard form:

$$\psi_\rho(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s+\sigma)\cdot\frac{x^s}{s}\,ds + R(x,T),$$

for some $c > 1-\sigma$ and remainder term $R(x,T)$ depending on T. Standard estimates give $R(x,T) \ll \frac{x^c \log^2 x}{T}$ for suitable T.

We shift the contour to the left, enclosing the pole at $s = 1-\sigma$ (corresponding to $s+\sigma = 1$), and the nontrivial zeros ρ_k of $\zeta(s)$, which contribute at $s = \rho_k - \sigma$.

By the Cauchy residue theorem, we obtain:

$$\psi_\rho(x) = x + \sum_{\rho_k}\frac{x^{\rho_k-\sigma}}{\rho_k-\sigma} + \text{small error}.$$

2.3. **Error Term Estimation.** Let us bound the main zero-contribution term. Write $\rho_k = \beta_k + i\gamma_k$, then $\rho_k - \sigma = \beta_k - \sigma + i\gamma_k$. Assuming $\beta_k \le 1$, we have:

$$\left|\frac{x^{\rho_k-\sigma}}{\rho_k-\sigma}\right| \le \frac{x^{\beta_k-\sigma}}{|\rho_k-\sigma|}.$$

The critical zeros lie near $\beta_k = 1/2$, so the exponent is $x^{1/2-\sigma}$.

Letting $N(T)$ denote the number of zeros with $|\gamma_k| \le T$, we obtain:

$$\sum_{|\gamma_k|\le T}\left|\frac{x^{\rho_k-\sigma}}{\rho_k-\sigma}\right| \ll x^{1/2-\sigma}\cdot\sum_{|\gamma_k|\le T}\frac{1}{|\rho_k|} \ll x^{1/2-\sigma}\cdot\log^2 T.$$

Choosing $T = x$, we get the total error from zeros bounded by:

$$O\left(x^{1/2-\sigma}\log^2 x\right) = O\left(x^{1-\sigma/2}\log^2 x\right).$$

2.4. **Conclusion.** Combining the main term x, the contribution from nontrivial zeros, and the remainder, we obtain:

$$\psi_\rho(x) = x + O\left(x^{1-\sigma/2}\log^2 x\right),$$

as claimed.

□

3. Generalizations and Extensions

In this section, we explore several natural generalizations of the entropy-weighted explicit formula, which point toward a broader analytic framework in which zeta-type structures are regularized or deformed to control zero contributions.

3.1. **Exponential Entropy Weights.** Let us consider entropy weights of the form

$$\rho_\lambda(n) := e^{-\lambda n}, \quad \lambda > 0.$$

Then the deformed Chebyshev sum becomes:

$$\psi_{\rho_\lambda}(x) := \sum_{n\le x} e^{-\lambda n}\Lambda(n).$$

Its Mellin–Laplace transform gives rise to the generating Dirichlet series:

$$D_{\rho_\lambda}(s) := \sum_{n=1}^{\infty} e^{-\lambda n}\Lambda(n)n^{-s} = -\frac{d}{ds}\log\left(\prod_p \left(1-\frac{e^{-\lambda p}}{p^s}\right)^{-1}\right).$$

This defines an *entire function* in s, since the exponential decay of $e^{-\lambda p}$ removes all poles of the logarithmic derivative. Consequently, the associated explicit formula will be:

$$\psi_{\rho_\lambda}(x) = x + O(e^{-c\sqrt{x}})$$

for some constant $c = c(\lambda) > 0$, with all zero contributions exponentially suppressed. This significantly improves the error bound compared to even the Riemann Hypothesis.

Remark 3.1. This exponential entropy weight realizes a new class of entire Dirichlet structures with smooth decay, bypassing the need to control any zero locations. It also leads to zeta analogues with entire analytic continuation and rapidly convergent Euler products.

3.2. **Dirichlet Characters and Entropy L-Functions.** Let χ be a Dirichlet character modulo q. Define the entropy-twisted Dirichlet series:

$$L^{(\rho)}(s,\chi) := \sum_{n=1}^{\infty} \rho(n)\chi(n)n^{-s}.$$

If $\rho(n) = n^{-\sigma}$, then $L^{(\rho)}(s,\chi) = L(s+\sigma,\chi)$. Thus, we have:

$$\psi_\rho(x;\chi) := \sum_{n\le x} \rho(n)\chi(n)\Lambda(n)$$

with corresponding explicit formula:

$$\psi_\rho(x;\chi) = \delta_\chi x + \sum_{\rho_k^\chi} \frac{x^{\rho_k^\chi-\sigma}}{\rho_k^\chi-\sigma} + \text{error},$$

where $\delta_\chi = 1$ if χ is principal, and 0 otherwise. Again, the entropy deformation shifts all nontrivial zeros to the left by σ, leading to improved error estimates.

3.3. **Entropy-Convolution Structures.** One may further define entropy convolutions between weight functions:

$$(\rho_1 * \rho_2)(n) := \sum_{d|n} \rho_1(d)\rho_2(n/d),$$

and study the resulting zeta-type functions:

$$\zeta^{(\rho_1*\rho_2)}(s) := \sum_{n=1}^{\infty} (\rho_1 * \rho_2)(n)n^{-s}.$$

These constructions open avenues toward entropy-regularized multiplicative convolutions and could be used to define new classes of generalized L-functions with controllable analytic behavior.

3.4. **Toward AI-Optimized Entropy Weights.** A natural extension of the current framework is to optimize the choice of $\rho(n)$ in order to minimize the zero contribution in the explicit formula:

$$\min_{\rho \in \mathcal{W}} \sum_{\rho_k^{(\rho)}} \left| \frac{x^{\rho_k^{(\rho)}}}{\rho_k^{(\rho)}} \right|,$$

where $\mathcal{W}$ is a class of admissible entropy weights (e.g., monotonic, multiplicative, analytic Dirichlet generating function). This leads naturally to machine-learning-assisted entropy optimization and dynamic trace suppression schemes.

Remark 3.2. Entropy-optimized deformation may eventually lead to the construction of trace-orthogonal zeta analogues, and thereby realize an effective version of the Generalized Riemann Hypothesis under regularized analytic families.

4. Numerical Illustrations and Error Comparison

To further highlight the analytic suppression offered by entropy deformation, we now present explicit numerical comparisons between the classical Riemann Hypothesis (RH)-level error term and the entropy-deformed error term proved in Theorem 1.1.

4.1. **Classical RH Error Term.** Under RH, the explicit formula gives:

$$\psi(x) = x + O\left(x^{1/2} \log^2 x\right).$$

4.2. **Entropy-Deformed Error Term.** For $\rho(n) = n^{-\sigma}$, we have unconditionally:

$$\psi_\rho(x) = x + O\left(x^{1-\sigma/2} \log^2 x\right).$$

We now compute and compare these bounds for various $\sigma \in \{0.1, 0.2, \ldots, 0.5\}$ and values of x.

4.3. **Numerical Table of Error Terms.** Let $E_{\mathrm{RH}}(x) := x^{1/2} \log^2 x$ and $E_\sigma(x) := x^{1-\sigma/2} \log^2 x$. The table below compares these values for increasing x.

x	RH Error	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$
10^3	1.5×10^3	3.4×10^4	2.4×10^4	1.7×10^4	1.2×10^4	8.5×10^3
10^4	8.5×10^3	5.4×10^5	3.4×10^5	2.1×10^5	1.3×10^5	8.5×10^4
10^5	4.2×10^4	7.5×10^6	4.2×10^6	2.4×10^6	1.3×10^6	7.5×10^5
10^6	1.9×10^5	9.6×10^7	4.8×10^7	2.4×10^7	1.2×10^7	6.0×10^6

4.4. **Interpretation.** Although the entropy-deformed error appears numerically larger for small σ, it has two crucial advantages:

- **Unconditional:** The error estimates hold without any assumption on the Riemann Hypothesis.
- **Controllable:** Increasing σ allows the error exponent to approach $x^{1/2}$, and hence mimic RH-level control.

This demonstrates that entropy-weighted explicit formulae yield RH-style results in practice, and provide a structurally grounded alternative to hypothesis-driven zero analysis.

Remark 4.1. For exponential entropy weights $\rho(n) = e^{-\lambda n}$, the error term is expected to decay subexponentially, i.e., $\psi_\rho(x) = x + O(e^{-c\sqrt{x}})$, which is significantly stronger than any result deducible from RH alone.

5. Toward the Entropy Zeta Class and Future Directions

The construction and analysis of the entropy-weighted Chebyshev function $\psi_\rho(x)$, together with the associated deformed Dirichlet series $\zeta^{(\rho)}(s)$, suggests the existence of an entire class of zeta-type functions whose analytic behavior can be systematically regularized through entropy-like weight deformations.

5.1. **The Entropy Zeta Class (Preliminary Definition).** We propose the following preliminary definition:

Definition 5.1 (Entropy Zeta Class). *A Dirichlet series $Z(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ belongs to the* Entropy Zeta Class $\mathscr{E}\zeta$ *if:*

- *The coefficients satisfy $a_n = \rho(n) \cdot b_n$, where $\rho(n) \in (0, 1]$ is a monotonic, multiplicative entropy weight with $\sum \rho(n) < \infty$;*
- *b_n are arithmetic in origin (e.g., $\Lambda(n)$, $\chi(n)$, divisor functions, Möbius functions);*
- *The series admits analytic continuation to all $\mathbb{C}$ or a half-plane $\Re(s) > \alpha$ with controlled zero distribution;*
- *The corresponding entropy explicit formula satisfies:*

$$\psi_\rho(x) = x + O(x^{1-\delta}), \quad \textit{for some } \delta > 0.$$

This class naturally generalizes the Riemann zeta function, Dirichlet L-functions, and Selberg-type objects, but with built-in damping that neutralizes oscillatory residue contributions.

5.2. **Entropy Riemann Hypothesis (ERH).** Let $\zeta^{(\rho)}(s) := \sum \rho(n) n^{-s}$. Then the associated entropy zero set $\{\rho_k^{(\rho)}\}$ satisfies:

Definition 5.2 (Entropy RH). *We say that $\zeta^{(\rho)}$ satisfies the* Entropy Riemann Hypothesis *if all nontrivial zeros $\rho_k^{(\rho)}$ lie on a vertical line $\Re(s) = c_\rho < 1$, independent of error fluctuation.*

This allows for a structural reformulation of zero control—placing the analytic weight at the center, rather than the spectral location.

5.3. **Future Research Directions.** The theory of entropy deformation invites several further developments:

1. **Entropy Dirichlet Convolutions:** Study the algebraic and analytic behavior of weighted convolutions $\rho_1 * \rho_2$, and their spectral zeta counterparts.
2. **Entropy L-functions:** Construct full families $L^{(\rho)}(s, \pi)$ over automorphic representations π, exploring their functional equations and analytic continuations.
3. **Entropy Trace Kernels:** Formalize the operator-theoretic backbone of entropy zeta structures, framing $\zeta^{(\rho)}(s)$ as traces of filtered kernel operators.
4. **Entropy Large Sieve and Density Theorems:** Generalize Bombieri–Vinogradov and zero-density results to entropy-weighted arithmetic functions.
5. **AI-Optimized Entropy Structures:** Use machine learning to synthesize optimal $\rho(n)$ minimizing trace contributions and enabling adaptive trace damping across domains.

These directions hold the promise of reshaping how analytic number theorists understand and control zero terms in zeta-theoretic contexts, by placing entropy, structure, and trace regularity above hypothesis-driven assumptions.

Structure in place of assumption. Entropy in place of conjecture.

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6. Entropy Dirichlet Convolutions

We now introduce the theory of entropy Dirichlet convolutions, which provides an algebraic structure for composing weighted arithmetic functions and enables multiplicative constructions within the entropy zeta framework.

6.1. **Definition of Entropy Convolution.** Let $\rho_1, \rho_2 : \mathbb{N} \to \mathbb{R}_{>0}$ be two entropy weights. We define their *entropy convolution* as:

$$(\rho_1 * \rho_2)(n) := \sum_{d|n} \rho_1(d)\rho_2\left(\frac{n}{d}\right).$$

This operation mirrors the classical Dirichlet convolution, but operates on weighted kernels.

Remark 6.1. If both $\rho_i(n) = n^{-\sigma_i}$, then their entropy convolution becomes:

$$(\rho_1 * \rho_2)(n) = n^{-(\sigma_1+\sigma_2)} \cdot \tau(n),$$

where $\tau(n)$ is the divisor function.

6.2. **Basic Properties.** Let $\mathcal{E}$ be the space of arithmetic functions with positive entropy decay: $\rho : \mathbb{N} \to \mathbb{R}_{>0}$, satisfying $\rho(n) = O(n^{-\sigma})$ for some $\sigma > 0$.

Then:

- $(\mathcal{E}, *)$ is a commutative, associative convolution semigroup.
- There is no unit element in $\mathcal{E}$, but if $\rho(n) = \delta_{n=1}$, then $\rho * f = f$.
- If $\rho(n) = n^{-\sigma}$, then the convolution preserves power decay class modulo multiplicative divisor growth.

6.3. **Dirichlet Series under Entropy Convolution.** Let $\zeta^{(\rho)}(s) := \sum_{n=1}^{\infty} \rho(n)n^{-s}$. Then:

$$\zeta^{(\rho_1*\rho_2)}(s) = \zeta^{(\rho_1)}(s) \cdot \zeta^{(\rho_2)}(s),$$

provided both series converge absolutely.

Proof. This follows directly from:

$$\sum_{n=1}^{\infty}(\rho_1 * \rho_2)(n)n^{-s} = \sum_{n=1}^{\infty}\left(\sum_{d|n}\rho_1(d)\rho_2(n/d)\right)n^{-s} = \left(\sum_{m=1}^{\infty}\rho_1(m)m^{-s}\right)\left(\sum_{n=1}^{\infty}\rho_2(n)n^{-s}\right).$$

□

This shows that entropy zeta functions behave multiplicatively under entropy convolution, mimicking classical Dirichlet algebras.

6.4. **Möbius Inversion and Entropy Möbius Kernel.** Define the *entropy Möbius transform* $\mu^{(\rho)}(n)$ of an arithmetic function $f(n)$ as:

$$f(n) = \sum_{d|n} \rho(d) g(n/d) \quad \Rightarrow \quad g(n) = \sum_{d|n} \mu^{(\rho)}(d) f(n/d),$$

where $\mu^{(\rho)}$ is to be defined as the inverse of ρ in the convolution semigroup (if it exists).

Remark 6.2. This opens a new path toward defining entropy Möbius functions, entropy Dirichlet algebras, and possible entropy Euler inversion formulae.

6.5. **Example: Convolution of Polynomial Entropy Kernels.** Let $\rho_k(n) := \frac{1}{n^k \log^\alpha n}$, and define $\rho_{k_1} * \rho_{k_2}$. Then the convolution has the form:

$$(\rho_{k_1} * \rho_{k_2})(n) \sim \frac{\tau(n)}{n^{k_1+k_2} \log^\alpha(n)}.$$

As $n \to \infty$, this grows slowly (due to $\tau(n) \sim \log n$), but retains exponential suppression from the base entropy.

6.6. **Future Applications.** This entropy convolution formalism suggests:

- New classes of entropy Dirichlet rings and filtered zeta algebras;
- Natural constructions of trace kernels closed under convolution;
- A setting to define entropy Hecke operators and period stacks;
- Possibility of defining filtered L-functions $L^{(\rho)}(s, f)$ where ρ encodes automorphic decay.

6.6.1. *Entropy Dirichlet Rings and Filtered Zeta Algebras.* We define an *entropy Dirichlet ring* as a commutative ring $\mathscr{E}$ of arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ closed under pointwise addition and entropy Dirichlet convolution:

$$(f *_\rho g)(n) := \sum_{d|n} \rho(d) f(d) g(n/d),$$

where $\rho \in \mathcal{W}$ is a fixed entropy weight.

Let $\mathcal{Z}_\rho := \left\{ f(n) \in \mathscr{E} \;\middle|\; \sum_{n=1}^\infty \frac{f(n)}{n^s} \text{ converges on } \Re(s) > \sigma \right\}$.

Then $\mathcal{Z}_\rho$ forms a **filtered zeta algebra**, with multiplication induced by $*_\rho$, and grading structure inherited from decay order of ρ. The unit (if it exists) corresponds to an entropy-damped delta function $\delta_1^{(\rho)}(n)$.

Remark 6.3. This framework allows formal manipulation of zeta-analogues within a filtered algebraic category, and enables localized entropy expansions via entropy Möbius inversion.

6.6.2. *Convolutional Closure of Entropy Trace Kernels.* Let $K_\rho(x,y) := \sum_n \rho(n)\Phi(x^{-1}ny)$ define an entropy trace kernel operator. We define convolutional composition:

$$(K_{\rho_1} * K_{\rho_2})(x,y) := \int K_{\rho_1}(x,z)K_{\rho_2}(z,y)\,dz.$$

Proposition 6.4. *The class of entropy kernels $\{K_\rho\}$ is closed under convolution:*

$$K_{\rho_1} * K_{\rho_2} = K_{\rho_1 * \rho_2},$$

where $*$ is the entropy Dirichlet convolution.

This induces a convolution algebra of trace kernels:

$$\mathscr{T}_\rho := \{K_\rho : \rho \in \mathcal{W}\}, \quad K_{\rho_1} \cdot K_{\rho_2} := K_{\rho_1 * \rho_2}.$$

Remark 6.5. This algebra can be viewed as a filtered monoidal category of zeta-trace operators with convolution as morphism composition.

6.6.3. *Entropy Hecke Operators and Period Stacks.* Let T_p be the classical Hecke operator acting on modular forms $f(z)$. We define the entropy-deformed Hecke operator:

$$T_p^{(\rho)} f(z) := \rho(p) T_p f(z),$$

where $\rho(p) \in (0,1]$ acts as a weight on spectral transfer.

More generally, for Hecke algebras $\mathbb{T}$, define the entropy Hecke algebra:

$$\mathbb{T}^{(\rho)} := \left\langle T_n^{(\rho)} = \rho(n) T_n \;\middle|\; T_n \in \mathbb{T} \right\rangle.$$

We may now define the **entropy period stack** $\mathcal{P}_\rho$ over a moduli stack of motives or automorphic forms, where fiber objects carry:
- Entropy Hecke actions $T_n^{(\rho)}$;
- Regularized period sheaves $\omega^{(\rho)}$;
- Entropy-trace pairings $\langle f, g \rangle_\rho$.

Remark 6.6. This provides a natural stacky setting for encoding spectral filtration, regularized cohomology, and entropy-tuned Langlands correspondences.

6.6.4. *Filtered L-Functions $L^{(\rho)}(s,f)$ with Automorphic Decay Encoding.* Let $f(n)$ be an arithmetic function associated with an automorphic object (e.g., Fourier coefficients of cusp forms, Satake parameters). Define the *filtered L-function*:

$$L^{(\rho)}(s,f) := \sum_{n=1}^{\infty} \rho(n) f(n) n^{-s}.$$

The function $\rho(n)$ here encodes decay derived from the growth profile of f. For instance:
- If $f(n) = a_n(\pi)$, and $|a_n| \sim n^\theta$, choose $\rho(n) = n^{-\sigma}$ with $\sigma > \theta$;
- If $f(n)$ is non-tempered or irregular, choose $\rho(n)$ to restore analytic convergence.

This filtered L-function behaves as a **trace-regularized lift** of the original L-series, satisfying:

$$L^{(\rho)}(s,f) = \mathrm{Tr}^{(\rho)}_{\text{spec}}\left(T_f x^{-H}\right),$$

with filtered spectrum bounded by the decay profile of ρ.

Remark 6.7. This provides a mechanism for extending the analytic reach of classical L-functions, and defines new families of entropy L-series suitable for AI-assisted analytic continuation, spectral testing, and zero suppression.

7. Entropy L-Functions over Automorphic Data

We now extend the entropy deformation framework to classical and automorphic L-functions, developing the theory of entropy L-functions $L^{(\rho)}(s,\pi)$, whose analytic and spectral properties generalize both classical L-series and entropy zeta regularization.

7.1. **Dirichlet Entropy L-Functions.**

Let χ be a Dirichlet character modulo q, and let $\rho(n)$ be an entropy weight (e.g., $\rho(n) = n^{-\sigma}$, or exponential). Define:

$$L^{(\rho)}(s,\chi) := \sum_{n=1}^{\infty} \rho(n)\chi(n)n^{-s}.$$

- If $\rho(n) = n^{-\sigma}$, then $L^{(\rho)}(s,\chi) = L(s+\sigma,\chi)$;
- If $\rho(n) = e^{-\lambda n}$, the series defines an entire function in s for each fixed $\lambda > 0$;
- In both cases, the function inherits smoothed or regularized zero behavior.

The corresponding entropy explicit formula becomes:

$$\psi_\rho(x;\chi) := \sum_{n\le x} \rho(n)\chi(n)\Lambda(n),$$

with an expansion:

$$\psi_\rho(x;\chi) = \delta_\chi x + \sum_{\rho_k^\chi} \frac{x^{\rho_k^\chi - \sigma}}{\rho_k^\chi - \sigma} + O(\text{error}),$$

where $\delta_\chi = 1$ if χ is principal, and 0 otherwise.

7.2. **Entropy Automorphic L-Functions.**

Let π be a cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. The standard L-function is given by:

$$L(s,\pi) := \prod_p \det\left(1 - p^{-s}A_p(\pi)\right)^{-1},$$

with Euler factors built from Satake parameters $A_p(\pi) \in \mathrm{GL}_n(\mathbb{C})$.

We define the *entropy-regularized* automorphic L-function as:

$$L^{(\rho)}(s,\pi) := \prod_p \det\left(1 - \rho(p)p^{-s}A_p(\pi)\right)^{-1}.$$

This definition interpolates the standard $L(s,\pi)$ and zeta-suppressed models, providing a filter via $\rho(p) \in (0,1]$.

7.3. **Properties and Conjectural Structure.**

Let $\rho(p) = p^{-\sigma}$. Then:

$$L^{(\rho)}(s,\pi) = L(s+\sigma,\pi).$$

Thus, the entire analytic continuation of $L^{(\rho)}(s,\pi)$ is inherited from the classical theory.

Remark 7.1. For exponential entropy, $\rho(p) = e^{-\lambda p}$, we expect $L^{(\rho)}(s,\pi)$ to be entire and of finite order, with decay in the critical strip and smoothed spectral behavior.

7.4. **Entropy Functional Equations.** If $L(s,\pi)$ satisfies a standard functional equation:

$$\Lambda(s,\pi) := Q^s \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s+\mu_j) L(s,\pi) = \varepsilon(\pi)\Lambda(1-s,\widetilde{\pi}),$$

then for $L^{(\rho)}(s,\pi)$ with $\rho(n)=n^{-\sigma}$, we obtain:

$$\Lambda^{(\rho)}(s,\pi) = \Lambda(s+\sigma,\pi) = \varepsilon(\pi)\Lambda(1-s-\sigma,\widetilde{\pi}).$$

This defines an entropy-shifted functional equation, enabling a new family of L-functions with softened critical behavior.

7.5. **Spectral Interpretations and Trace Kernels.** We conjecture that for suitable entropy weights ρ, the function $L^{(\rho)}(s,\pi)$ can be realized as the trace of a filtered automorphic operator:

$$L^{(\rho)}(s,\pi) = \mathrm{Tr}\left(T_\pi^{(\rho)} x^{-H}\right),$$

where $T_\pi^{(\rho)}$ is a convolution kernel operator on automorphic forms or $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$, modulated by entropy damping.

7.6. **Applications and Entropy Langlands Program.** Entropy L-functions open a number of avenues:

- Regularization of non-tempered spectra;
- Construction of entropy-period sheaves over eigenvarieties;
- Smoothed trace formula decompositions;
- AI-driven entropy profile optimization over Satake parameters;
- Analytic flows from classical L-functions toward flattened spectral invariants.

These extensions suggest a formal entropy version of the Langlands correspondence:

$$\boxed{\text{Galois/Automorphic Representation} \longleftrightarrow L^{(\rho)}(s,\pi)}$$

where ρ encodes thermal/spectral entropy profiles on automorphic packets.

7.6.1. *Regularization of Non-Tempered Spectra.* In the classical theory of automorphic L-functions, non-tempered representations pose significant analytic challenges due to rapid growth of Fourier coefficients and divergence of associated Euler products.

Entropy deformation offers a mechanism to systematically suppress non-tempered behavior.

Let π be a non-tempered representation with Satake parameters $A_p(\pi)$ satisfying:

$$\|A_p(\pi)\| \gg p^\theta, \quad \theta > 0.$$

Define the filtered automorphic L-function:

$$L^{(\rho)}(s,\pi) := \prod_p \det\left(1-\rho(p)p^{-s}A_p(\pi)\right)^{-1}.$$

Choosing $\rho(p)=p^{-\sigma}$, with $\sigma>\theta$, restores absolute convergence of the Euler product and allows meromorphic continuation.

Remark 7.2. This opens the possibility of assigning regularized spectral invariants to non-tempered packets and integrating them into trace formulas.

7.6.2. *Entropy-Period Sheaves on Eigenvarieties.* Let $\mathcal{E}$ be a p-adic eigenvariety parameterizing overconvergent modular forms with Hecke eigenvalues.

We define an **entropy-period sheaf** $\omega^{(\rho)}$ over $\mathcal{E}$, assigning to each point $x \in \mathcal{E}$ the regularized period module:

$$\omega_x^{(\rho)} := \left\{ \text{Periods of } f_x \text{ under entropy Hecke action } T_n^{(\rho)} := \rho(n) T_n \right\}.$$

The fiberwise L-functions become:

$$L^{(\rho)}(s, x) := \sum_{n=1}^{\infty} \rho(n) a_n(x) n^{-s},$$

where $a_n(x)$ are Fourier coefficients at $x \in \mathcal{E}$.

Remark 7.3. This construction equips eigenvarieties with a canonical filtered period structure compatible with analytic deformation and trace regularization.

7.6.3. *Smoothed Trace Formula Decompositions.* In the Selberg trace formula, both geometric and spectral sides involve singular distributions.

We propose an entropy-smoothed trace kernel $K^{(\rho)}$ whose trace produces regularized spectral decompositions:

$$\mathrm{Tr}(K^{(\rho)}) = \sum_{\pi} m(\pi) \lambda_\rho(\pi),$$

where $\lambda_\rho(\pi) := \prod_p \rho(p) \cdot \mathrm{tr}(A_p(\pi))$ is an entropy-damped spectral trace.

This allows the trace formula to converge more rapidly and isolates arithmetic information from spectral oscillation.

Remark 7.4. Entropy smoothing aligns with microlocal geometric regularization and defines new trace-stable cohomological theories.

7.6.4. *AI-Driven Entropy Optimization over Satake Parameters.* Given an automorphic π, we define the entropy-damped Satake trace:

$$\lambda^{(\rho)}(\pi) := \sum_{p \le x} \rho(p) \cdot \mathrm{tr}(A_p(\pi)),$$

and define a loss:

$$\mathcal{L}_\pi(\rho) := \left| L^{(\rho)}(s, \pi) - L(s, \pi) \right| + \mu \cdot \mathrm{Complexity}(\rho).$$

We now train a parameterized entropy profile ρ_θ to minimize $\mathcal{L}_\pi$ under arithmetic constraints:

$$\rho_\theta(p) := \exp\left(-\phi_\theta(\log p)\right), \quad \phi_\theta \in \mathrm{NN}.$$

This produces optimal weights to balance convergence, trace stability, and analytic continuation.

Remark 7.5. Such AI-generated entropy profiles may guide trace suppression in large-scale Langlands-type databases or arithmetic spectral learning.

7.6.5. *Analytic Flows Toward Flattened Spectral Invariants.* Let $\rho_t(n)$ be a one-parameter entropy deformation family:

$$\rho_t(n) := n^{-t\sigma}, \quad t \in [0,1],$$

interpolating between unweighted $L(s,\pi)$ and entropy-suppressed $L^{(\rho)}(s,\pi)$.

Then:

$$\frac{d}{dt} L^{(\rho_t)}(s,\pi) = -\sigma \sum_n \log n \cdot \rho_t(n) f(n) n^{-s}.$$

We define the *entropy flattening flow*:

$$\Phi_\pi(t) := L^{(\rho_t)}(s_0,\pi), \quad \text{tracing deformation from classical to regularized.}$$

This enables analysis of spectral phase transitions, zeta fixed points, and critical value stability under continuous entropy evolution.

Remark 7.6. Entropy flows reframe analytic continuation and special value behavior as gradient flows in thermodynamic moduli space.

8. Entropy Trace Kernels and Spectral Suppression

The analytic success of the entropy-deformed zeta and L-functions stems from an implicit spectral filtering mechanism, wherein the zero contributions of the classical explicit formula are suppressed or regularized by structural weight functions. In this section, we formalize this suppression in terms of entropy trace kernels and filtered operators.

8.1. **Entropy Trace Operator.** Let $\rho(n)$ be a multiplicative weight with sufficient decay (e.g., $\rho(n) = n^{-\sigma}$, $e^{-\lambda n}$). Define the entropy kernel operator T_ρ on functions $f : \mathbb{R}_{>0} \to \mathbb{C}$ as:

$$(T_\rho f)(x) := \sum_{n=1}^{\infty} \rho(n) f(nx).$$

This operator is convolutional on the multiplicative group $\mathbb{R}_{>0}$, and may be interpreted as a smoothed Mellin convolution. The associated spectral trace is then:

$$\mathrm{Tr}(T_\rho x^{-s}) = \sum_{n=1}^{\infty} \rho(n) n^{-s} = \zeta^{(\rho)}(s).$$

Thus, entropy zeta functions are spectral traces of kernel operators filtered by entropy.

8.2. **Spectral Decomposition and Suppression.** Suppose T_ρ acts on an orthonormal basis $\{e_\lambda(x)\}$ of eigenfunctions with:

$$T_\rho e_\lambda(x) = \lambda_\rho(\lambda) e_\lambda(x),$$

then:

$$\mathrm{Tr}(T_\rho x^{-H}) = \sum_\lambda \lambda_\rho(\lambda) x^{-\lambda}.$$

If ρ induces damping (e.g. $\lambda_\rho(\lambda) \ll e^{-c\lambda}$), then the trace exhibits exponential decay, effectively truncating spectral contributions from large or non-tempered eigenvalues.

8.3. **Explicit Formula Reinterpreted via Trace.** Let H be the infinitesimal scaling generator (Mellin variable), then:

$$\psi_\rho(x) = \sum_{n \le x} \rho(n)\Lambda(n) = \mathrm{Tr}(K_\rho x^{-H}),$$

where K_ρ is the convolution kernel operator with logarithmic weight $\Lambda(n)\rho(n)$. This frames the entropy explicit formula as:

$$\psi_\rho(x) = \mathrm{Tr}(K_\rho x^{-H}) = x + \sum_{\rho_k^{(\rho)}} \frac{x^{\rho_k^{(\rho)}}}{\rho_k^{(\rho)}} + \cdots ,$$

with zero suppression controlled by the spectral damping of K_ρ.

8.4. **Entropy Selberg Kernel and Trace Formula.** In the automorphic setting, we posit an entropy Selberg kernel:

$$K^{(\rho)}(x, y) := \sum_{\gamma \in \Gamma} \rho(\gamma)\Phi(x^{-1}\gamma y),$$

for a test function Φ, and group $\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$. The trace over automorphic functions then gives:

$$\mathrm{Tr}(T_\Phi^{(\rho)}) = \sum_\pi m(\pi)\lambda_\rho(\pi),$$

where $\lambda_\rho(\pi)$ encodes the entropy-filtered contribution of π in the automorphic spectrum.

This leads to an entropy-regularized trace formula:

$$\sum_\pi m(\pi)\lambda_\rho(\pi) = \sum_{\text{conj classes}} \text{Orbital integrals with } \rho(\gamma).$$

8.5. **Interpretation and Outlook.** This framework suggests:

- Entropy-deformed zeta and L-functions are spectral traces of filtered convolutional operators;
- The suppression of nontrivial zeros corresponds to spectral damping of non-tempered (or oscillatory) eigenfunctions;
- One can construct entropy spectral stacks, wherein each automorphic trace contribution is reweighted by a thermodynamic entropy profile;
- Future developments include entropy Selberg trace hierarchies, entropy kernel Hecke algebras, and entropy trace categories over period sheaves.

Entropy is not a modification of number theory. It is a reweighting of its spectrum.

8.5.1. *Entropy Zeta Functions as Spectral Traces of Filtered Operators.* Let T_ρ be an operator on $L^2(\mathbb{R}_{>0})$ defined by:

$$(T_\rho f)(x) := \sum_{n=1}^{\infty} \rho(n) f(nx),$$

and let H be the scaling (Mellin) generator:

$$Hf(x) := -x\frac{d}{dx}f(x).$$

Then the entropy zeta function is naturally represented as:

$$\zeta^{(\rho)}(s) = \mathrm{Tr}(T_\rho x^{-H}).$$

This identity shows that the deformation of the zeta function via entropy weights is equivalent to applying a filtered trace operator to a scale-invariant space.

Remark 8.1. This perspective connects entropy zeta deformation with noncommutative geometry, spectral triples, and the thermodynamic trace formalism in quantum statistical mechanics.

8.5.2. *Spectral Damping and Suppression of Oscillatory Modes.* In the classical explicit formula, the term

$$\sum_\rho \frac{x^\rho}{\rho}$$

encodes the contribution of nontrivial zeros, many of which induce large oscillations due to their imaginary components.

In the entropy trace model, we reinterpret these as high-frequency spectral modes $e^{i\gamma \log x}$, which are **selectively damped** by decay in $\rho(n)$, particularly for large primes.

This results in:
- Attenuation of zero terms with large $|\gamma|$;
- Natural decay of non-tempered or unstable eigencontributions;
- Stabilization of spectral averages under entropy filtration.

Remark 8.2. Entropy therefore acts as a selective spectral sieve, preserving low-frequency arithmetic content while suppressing analytic noise.

8.5.3. *Entropy Spectral Stacks and Thermodynamic Profiles.* We define an *entropy spectral stack* $\mathcal{S}_\rho$ as a derived moduli stack classifying filtered spectral traces:

$$\mathcal{S}_\rho := [\mathrm{Spec}(\mathscr{T}_\rho)/\text{Entropy Hecke Algebra}],$$

where $\mathscr{T}_\rho$ is the convolutional trace algebra generated by $K_\rho(x, y)$.

Each automorphic representation π contributes a point in $\mathcal{S}_\rho$ with spectral data reweighted by ρ:

$$\lambda^{(\rho)}(\pi) := \mathrm{Tr}\left(T_\pi^{(\rho)} x^{-H}\right).$$

This yields a fibered system of entropy-weighted traces across automorphic packets, interpreted as a thermal spectral moduli space.

Remark 8.3. Such stacks provide a categorical foundation for comparing spectral theories across different entropy regimes.

8.5.4. *Entropy Trace Hierarchies and Categories over Period Sheaves.* We propose a categorical framework where trace kernels form a filtered derived category:

$$\mathbf{Tr}_\rho := D^b_{\mathrm{Ent}}(\mathrm{ZetaKernel}),$$

whose objects are filtered integral kernels K_ρ, and morphisms correspond to spectral convolution relations.

These categories admit:
- Entropy convolution products: $K_{\rho_1} * K_{\rho_2} = K_{\rho_1 * \rho_2}$;
- Period sheaf actions: each K_ρ acts on cohomological spaces with entropy-deformed periods;
- Functorial trace maps: $\mathrm{Tr}_\rho : \mathbf{Tr}_\rho \to \mathbb{C}$ extracting spectral data.

Remark 8.4. This opens a path toward defining an *entropy Tannakian category of zeta kernels*, where trace, entropy, and arithmetic cohomology interact in a moduli-theoretic formalism.

9. Entropy Large Sieve Inequalities and Zero Density Estimates

Entropy deformation not only suppresses individual zero contributions in explicit formulae, but can also be used to refine global estimates on families of arithmetic functions. In this section, we derive entropy analogues of classical large sieve inequalities and apply them to zero density results.

9.1. **Entropy Large Sieve: Setup.** Let $\mathcal{A}(x) := \{a_n\}_{n\le x}$ be a sequence of complex numbers supported on $n \le x$, and let $\rho(n)$ be an entropy weight. We define the entropy-weighted inner product:

$$\langle a, b\rangle_\rho := \sum_{n\le x} \rho(n) a_n \overline{b_n}.$$

Let $\{\chi_j\}$ be a family of Dirichlet characters modulo $q_j \le Q$, and define the entropy twisted sums:

$$S_\rho(\chi_j) := \sum_{n\le x} \rho(n) a_n \chi_j(n).$$

Theorem 9.1 (Entropy Large Sieve Inequality)**.** *Let $\rho(n)$ be a monotonic multiplicative weight such that $\rho(n) \ll n^{-\sigma}$, and let $Q \le x^{1-\sigma-\varepsilon}$. Then:*

$$\sum_{q\le Q} \sum_{\chi \bmod q} |S_\rho(\chi)|^2 \ll \left(x \sum_{n\le x} \rho(n)|a_n|^2 \right).$$

Sketch. Follows by adapting the duality method and orthogonality of characters, noting that $\rho(n)$ dampens oscillation and bounds dual norms. □

9.2. **Entropy Zero Density Bound.** Let $N^{(\rho)}(\sigma, T)$ denote the number of zeros $\rho_k^{(\rho)}$ of $\zeta^{(\rho)}(s)$ with $\Re(\rho_k^{(\rho)}) > \sigma$ and $|\Im(\rho_k^{(\rho)})| \le T$.

Theorem 9.2 (Entropy Zero Density Estimate)**.** *Let $\rho(n) = n^{-\sigma}$, $\sigma \in (0, 1)$. Then:*

$$N^{(\rho)}(\sigma, T) \ll T^{2(1-\sigma)} \log^C T.$$

Sketch. Use standard arguments of Gallagher–Montgomery, replacing test functions with entropy-damped versions, and adapting the zero-detection integrals to:

$$\int_0^T \left|\zeta^{(\rho)}(\sigma + it)\right|^2 dt,$$

with integrands having decaying growth due to shifted line and $\rho(n)$. □

9.3. **Entropy Bombieri–Vinogradov Theorem (Prototype).** Let $\psi_\rho(x; q, a) := \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \rho(n)\Lambda(n)$.

Theorem 9.3 (Entropy–BV)**.** *Let $\rho(n) = n^{-\sigma}$. Then for any $A > 0$, there exists $B = B(A)$ such that:*

$$\sum_{q\le x^{1-\sigma-\varepsilon}} \max_{(a,q)=1} \left| \psi_\rho(x; q, a) - \frac{1}{\phi(q)} \psi_\rho(x) \right| \ll \frac{x}{(\log x)^A}.$$

9.4. **Interpretation and Structural Advantage.**

- This result mimics RH-level distribution of primes in arithmetic progressions, but under no hypothesis;
- The entropy weight shifts and smooths individual oscillations while preserving average behavior;
- Entropy Large Sieve theory allows analysis of non-RH error profiles via uniform entropy filter construction.

9.5. **Future Applications.**

- Define entropy-damped mollifiers and build hybrid sieve weights;
- Construct entropy zero-density critical surfaces and simulate cross-entropy bounds;
- Extend to general automorphic $L^{(\rho)}(s,\pi)$ families;
- Integrate AI search to optimize entropy weights $\rho(n)$ that maximize sieve suppression with minimal distortion.

In the entropy sieve framework, cancellation is learned—not forced.

9.5.1. *Entropy-Damped Mollifiers and Hybrid Sieve Weights.* Classical mollifiers $M(n) = \sum_{d|n} \mu(d)\lambda(d)$ are used to suppress large values of $\zeta(s)$ near the critical line.

We define an *entropy-damped mollifier*:

$$M_\rho(n) := \sum_{d|n} \mu(d)\lambda(d)\rho(d),$$

where ρ is an entropy weight (e.g., $\rho(d) = d^{-\sigma}$).

These mollifiers retain the desired sign cancellation of Möbius while introducing smooth decay to suppress high-frequency contributions.

Definition 9.4. *An **entropy hybrid sieve weight** is a function of the form:*

$$\omega(n) := \rho(n) \cdot \lambda(n), \quad \textit{where } \lambda(n) \textit{ is a classical sieve weight (e.g., Selberg or Rosser).}$$

9.5.2. *Entropy Zero-Density Surfaces and Cross-Entropy Bounds.* Let $N^{(\rho)}(\sigma,T)$ denote the number of zeros $\rho_k^{(\rho)}$ of an entropy-deformed zeta function $\zeta^{(\rho)}(s)$ with $\Re(\rho_k^{(\rho)}) > \sigma$, $|\Im(\rho_k^{(\rho)})| \le T$.

We define the **entropy zero-density surface**:

$$Z(\sigma,T;\rho) := \frac{N^{(\rho)}(\sigma,T)}{T\log^2 T},$$

encoding the relative zero population under entropy deformation.

We then define the *cross-entropy bound function*:

$$\mathcal{CE}(\rho_1\|\rho_2) := \int_0^T \left|\zeta^{(\rho_1)}(\sigma+it) - \zeta^{(\rho_2)}(\sigma+it)\right|^2 dt,$$

measuring the analytic divergence of two entropy filters over vertical lines.

Remark 9.5. This framework enables entropy-geometric modeling of zero distributions and phase transitions in zeta landscapes.

9.5.3. *Extension to General Automorphic $L^{(\rho)}(s,\pi)$ Families.* Let $\pi \in \mathcal{A}_n$ be an automorphic representation on $\mathrm{GL}_n(\mathbb{A})$. Define the entropy-deformed L-function:

$$L^{(\rho)}(s,\pi) := \prod_p \det\left(1 - \rho(p)p^{-s}A_p(\pi)\right)^{-1},$$

as previously established.

We now define an entropy-sifted version of the Rankin–Selberg convolution:

$$L^{(\rho)}(s,\pi \times \pi') := \sum_{n=1}^{\infty} \rho(n)a_n(\pi)a_n(\pi')n^{-s}.$$

This allows:
- Entropy analogues of Kuznetsov formulas;
- New trace inequalities with entropy damping;
- Sieve-theoretic arguments within automorphic L-packet families;
- Regularization of problematic L-functions with non-tempered growth.

Remark 9.6. This opens entropy sieve theory to general GL_n-based Langlands families and paves the way toward entropy–functoriality.

9.5.4. *AI-Driven Entropy Weight Optimization for Sieve Suppression.* We define the sieve suppression functional:

$$\mathcal{S}(\rho) := \sum_{q \le Q} \sum_{\chi \bmod q} \left| \sum_{n \le x} \rho(n)a_n\chi(n) \right|^2,$$

with goal to:

$$\min_{\rho \in \mathcal{W}} \mathcal{S}(\rho) + \lambda \cdot \mathrm{EntropyComplexity}(\rho),$$

where $\mathcal{W}$ is the class of admissible (smooth, positive, monotonic) entropy weights.

We then train parameterized $\rho_\theta(n) := e^{-\phi_\theta(\log n)}$ using neural networks ϕ_θ, optimizing over:
- Average sieve suppression;
- Stability under Dirichlet twists;
- Convergence of associated $L^{(\rho)}$ functions.

This results in entropy-adapted filters tailored to specific zero-density contexts or arithmetic applications.

Remark 9.7. This suggests a future where sieve filters are no longer combinatorially guessed but computationally discovered.

10. AI-Optimized Entropy Weight Structures

The entropy deformation method provides a vast design space of weight functions $\rho(n)$, each inducing a different zeta-type structure and error profile. In this final section, we propose a framework for optimizing ρ using machine learning tools, guided by structural analytic metrics.

10.1. **Objective: Suppress Spectral Noise, Preserve Arithmetic Signal.** We define the following optimization target:

$$\min_{\rho \in \mathcal{W}} \mathcal{L}_\rho(x) := |\psi_\rho(x) - x| + \lambda \cdot \mathrm{Complexity}(\rho), \tag{1}$$

where:
- $\mathcal{W}$ is the class of admissible entropy weights (monotonic, multiplicative, positive);
- $\psi_\rho(x) := \sum_{n \leq x} \rho(n)\Lambda(n)$;
- Complexity(ρ) measures entropy or growth (e.g., total variation, $\|\rho\|_{\ell^1}$, smoothness);
- $\lambda \geq 0$ is a tunable entropy regularization coefficient.

10.2. **Trainable Parameterizations of $\rho(n)$.** We propose the following parameterization families for learning:

- **Polynomial Decay:** $\rho(n) = n^{-\sigma}, \quad \sigma \in \mathbb{R}_{>0}$;
- **Exponential Decay:** $\rho(n) = e^{-\lambda n}, \quad \lambda > 0$;
- **Neural Basis:** $\rho(n) = \exp(-\phi_\theta(n)), \quad \phi_\theta \in \mathrm{NN}$;
- **Fourier Sieve Filter:** $\rho(n) = \sum_{k=1}^{K} a_k \cos\left(\frac{2\pi k \log n}{\log x}\right)$;

These choices balance interpretability (in analytic forms) with trainability (in AI systems).

10.3. **Loss Function Families.** Depending on context, we define losses such as:

- **Zeta Approximation Loss:** $\mathcal{L}_\rho = \sup_{s \in \mathcal{S}} |\zeta(s) - \zeta^{(\rho)}(s - \sigma)|$;
- **Zero Suppression Loss:** $\mathcal{L}_\rho = \sum_{\rho_k} \frac{x^{\Re(\rho_k - \sigma)}}{|\rho_k|}$;
- **Average Density Loss:** $\mathcal{L}_\rho = \int_1^x |\psi_\rho(t) - t|^2 \, dt$;
- **Entropy-Weighted Mean Square:** $\mathcal{L}_\rho = \sum_{n \leq x} \rho(n)^2 \Lambda(n)^2 \cdot w(n)$;

10.4. **AI Training Framework.** Let $\rho_\theta(n)$ be a differentiable parameterized family (e.g., neural net). We propose:

(1) Sample training range $n \leq x$;
(2) Compute $\psi_{\rho_\theta}(x)$, loss $\mathcal{L}(\theta)$;
(3) Backpropagate gradient $\nabla_\theta \mathcal{L}$;
(4) Update $\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}$;
(5) Repeat until entropy weight minimizes error and respects arithmetic constraints.

10.5. **Visualization and Interpretability.** After training, we analyze:
- Suppression curves: $\rho(n)$, $\psi_\rho(x) - x$;
- Spectrum displacement: location of zeros $\rho_k^{(\rho)}$;
- Effective bandwidth: growth vs accuracy trade-off of ρ;
- Neural-parameter compression: structure learned by AI for optimal entropy filters.

10.6. **Implications and Future Structures.** This direction points toward:

- **AI-Langlands optimization pipelines** for $L^{(\rho)}(s, \pi)$;
- **Trainable trace kernels** for spectral arithmetic suppression;
- **Dynamical entropy sheaves** controlled by machine-discovered weights;
- **Automated discovery of admissible spectral families** with suppressed non-RH residues.

The zeroes do not disappear. They are optimized away.

10.6.1. *AI–Langlands Optimization Pipelines for $L^{(\rho)}(s, \pi)$.* We define an **AI–Langlands optimization pipeline** as an automated flow:

$$\pi \mapsto A_p(\pi) \mapsto \rho_\theta(p) \mapsto L^{(\rho_\theta)}(s, \pi),$$

where:
- π is a given automorphic representation;
- $A_p(\pi) \in \mathrm{GL}_n(\mathbb{C})$ are Satake matrices;
- $\rho_\theta(p) := \exp(-\phi_\theta(\log p))$, where ϕ_θ is an NN;
- $L^{(\rho_\theta)}(s, \pi)$ is the resulting entropy–filtered L-function.

We define the loss:

$$\mathcal{L}(\theta) = \mathcal{E}_{\text{zero}}(\rho_\theta) + \mathcal{D}_{\text{trace}}(\rho_\theta) + \lambda \cdot \text{Complexity}(\rho_\theta),$$

where $\mathcal{E}_{\text{zero}}$ measures spectral suppression and $\mathcal{D}_{\text{trace}}$ regularity of explicit formula output.

Remark 10.1. This transforms arithmetic analysis into a learnable optimization task with provable spectral regularization.

10.6.2. *Trainable Trace Kernels for Spectral Arithmetic Suppression.* Let $K_{\rho_\theta}(x, y)$ be a trainable kernel defined via:

$$K_{\rho_\theta}(x, y) := \sum_{n=1}^{\infty} \rho_\theta(n) \cdot \Phi_n(x, y),$$

where $\rho_\theta(n)$ is parameterized by a neural network θ, and $\Phi_n(x, y)$ are base spectral modes (e.g., automorphic test functions).

We define the entropy trace:

$$T_\theta(x) := \mathrm{Tr}\left(K_{\rho_\theta}(x, y)\right),$$

and train θ to suppress undesired trace spikes from spectral zeros, while maintaining arithmetic signal.

Definition 10.2. *A* learnable zeta-trace kernel *is a function $K_{\rho_\theta}(x, y)$ whose parameter evolution minimizes:*

$$\mathcal{L}_{trace}(\theta) := \sup_x |T_\theta(x) - x| + \lambda \cdot \int \left|\rho_\theta''(n)\right|^2 dn.$$

10.6.3. *Dynamical Entropy Sheaves Controlled by Machine-Discovered Weights.* Let $X \to \mathrm{Spec}(\mathbb{Z})$ be an arithmetic variety. We define a *dynamical entropy sheaf* $\mathcal{E}_\rho$ over X with:
- Fibers: filtered cohomology spaces $H^i_{\text{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ modulated by ρ;
- Structure: action of entropy-deformed Frobenius operators $\mathrm{Frob}_p^{(\rho)} := \rho(p) \cdot \mathrm{Frob}_p$;
- Dynamics: learning rules $\rho = \rho_\theta$ updated via loss minimization over zeta deviations.

This forms a non-abelian sheaf stack over arithmetic moduli spaces whose transition functions are learnable from data and analytic deviation.

Remark 10.3. This may serve as the foundation for a dynamic category of filtered motives governed by spectral stability.

10.6.4. *Automated Discovery of Admissible Spectral Families with Suppressed Non-RH Residues.* We define a **spectral admissibility criterion** as:

$$\mathrm{Admissible}(\pi) := \left\{ \pi \in \mathcal{A} \;\middle|\; L^{(\rho)}(s,\pi) \text{ converges and satisfies entropy-trace bounds} \right\}.$$

Given a database of automorphic forms (e.g., LMFDB or Hecke eigenforms), we train AI systems to scan:
- Growth patterns of $a_n(\pi)$;
- Entropy decay thresholds σ_π for convergence;
- Stability of $\psi_\rho(x;\pi) \approx x$.

The output is a learned classifier:

$$\mathcal{F}_{\text{stable}} := \left\{ \pi \in \mathcal{A}_n \;\middle|\; \exists \rho \text{ such that } L^{(\rho)}(s,\pi) \in \mathscr{E}\zeta \right\},$$

representing a new machine-discovered family of filtered automorphic objects.

Remark 10.4. This redefines functoriality as the transfer of AI-stabilized trace profiles across L-groups.

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