ARITHMETIC FUNCTION CALCULUS UNDER DIRICHLET CONVOLUTION

PU JUSTIN SCARFY YANG

Abstract. We formalize the structure and operations of the ring of arithmetic functions under Dirichlet convolution. We explicitly define and analyze the four arithmetic operations–pointwise addition, Dirichlet convolution, Dirichlet inverse (division), and convolution exponentiation–as well as the logarithmic and exponential maps that bridge additive and multiplicative behavior. This framework sets the stage for algebraic exploration of this ring using modern ring theory.

Contents

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1. Introduction and Motivation

Let $\mathcal{A}$ denote the set of all arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. We endow $\mathcal{A}$ with two operations:

- Pointwise addition: $(f + g)(n) := f(n) + g(n)$
- Dirichlet convolution: $(f * g)(n) := \sum_{d|n} f(d)g(n/d)$

With these operations, $(\mathcal{A}, +, *)$ becomes a commutative ring with unity ε, where

$$\varepsilon(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

Despite being widely known, the internal arithmetic structure of this ring remains underdeveloped in the literature. We initiate a systematic calculus of arithmetic functions under these operations, with analogues of addition, subtraction, multiplication, and division, as well as functional logarithms and exponentials under convolution.

2. The Basic Ring Structure

Definition 2.1. *The set $\mathcal{A}$ of all arithmetic functions is a commutative ring with:*

- *Additive identity:* $0(n) := 0$ *for all* n*;*
- *Multiplicative identity:* $\varepsilon(n) := \delta_{n,1}$*;*
- *Additive inverse:* $(-f)(n) := -f(n)$*;*
- *Multiplicative inverse:* f^{-1} *exists if and only if* $f(1) \neq 0$*.*

Definition 2.2 (Convolution Exponentiation)**.** *Let* $f \in \mathcal{A}$ *and* $k \in \mathbb{N}$*, define* f^{*k} *recursively by:*

$$f^{*0} := \varepsilon, \quad f^{*k} := f * f^{*(k-1)}.$$

Definition 2.3 (Dirichlet Logarithm)**.** *If* $f(1) = 1$*, define*

$$\log^* f := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (f - \varepsilon)^{*k}.$$

Definition 2.4 (Dirichlet Exponential)**.** *For* $g \in \mathcal{A}$ *with* $g(1) = 0$*, define*

$$\exp^*(g) := \sum_{k=0}^{\infty} \frac{1}{k!} g^{*k}.$$

Proposition 2.5. *If* $f = \exp^*(g)$*, then* $g = \log^*(f)$*, and vice versa.*

Example 2.6. *Let* $f(n) = 1$ *for all* n*. Then* $\log^* f = \Lambda(n)$*, the von Mangoldt function.*

3. Algebraic Operations and Convolution Calculus

3.1. **Formal Arithmetic Operations.**

We define a full calculus system of arithmetic functions under Dirichlet convolution by explicitly formalizing the following operations:

- **Addition:** Pointwise, $(f + g)(n) = f(n) + g(n)$.
- **Subtraction:** Pointwise, $(f - g)(n) = f(n) - g(n)$.
- **Multiplication:** Dirichlet convolution, $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$.
- **Division:** If $g(1) \neq 0$, define $f/g := f * g^{-1}$, where g^{-1} is the Dirichlet inverse of g.

3.2. **Functional Logarithms and Exponentials.**

Arithmetic functions may also be analyzed via their logarithmic and exponential images:

- **Logarithmic Rule:** $\log^*(f * g) = \log^* f + \log^* g$ if $f(1) = g(1) = 1$.
- **Exponential Rule:** $\exp^*(A + B) = \exp^* A * \exp^* B$.
- **Inverse Identity:** If $f = \exp^*(g)$, then $g = \log^*(f)$.

3.3. **Examples and Applications.**

Example 3.1. *Let* $f(n) = n^s$ *for a fixed complex number* s*. Then* f *is completely multiplicative, and*

$$\log^* f(n) = s \log n.$$

Example 3.2. *Let* $f(n) = \mu(n)$*, the Möbius function. Then* f *is multiplicative but not completely multiplicative. Its logarithmic image is highly non-linear and reflects cancellation over divisors.*

Example 3.3. *Let $f(n) = \Lambda(n)$, the von Mangoldt function. Then*

$$\exp^*(\Lambda)(n) = 1(n),$$

so $\log^*(1) = \Lambda$.

4. Logarithmic Structure and Multiplicativity Classification

4.1. **Motivation.** The Dirichlet logarithm $\log^*$ transforms the multiplicative structure of arithmetic functions into an additive one. We ask: what algebraic properties of $\log^* f$ can detect whether f is multiplicative or completely multiplicative?

4.2. **Log-linear Spectrum.**

Definition 4.1 (Log-linear Spectrum)**.** *Let f be a Dirichlet-logarithmic arithmetic function with $f(1) = 1$. The* log-linear spectrum *of f is defined as the set:*

$$\mathrm{Spec}_{\log}(f) := \left\{ \frac{\log^* f(p^k)}{k} : p \in \mathbb{P},\ k \in \mathbb{N} \right\}.$$

Definition 4.2 (Log-linearly Additive Function)**.** *We say that f is* log-linearly additive *if there exists a function $A : \mathbb{P} \to \mathbb{C}$ such that for all prime powers p^k,*

$$\log^* f(p^k) = k \cdot A(p).$$

4.3. **Main Theorem.**

Theorem 4.3. *Let f be an arithmetic function with $f(1) = 1$. Then f is completely multiplicative if and only if* $\log^* f$ *is log-linearly additive.*

Proof.

($\Rightarrow$) Suppose f is completely multiplicative. Then $f(p^k) = f(p)^k$. Taking Dirichlet logarithm:

$$\log^* f(p^k) = \log(f(p^k)) = k \cdot \log(f(p)).$$

Hence $\log^* f(p^k) = k \cdot A(p)$ with $A(p) := \log f(p)$.

($\Leftarrow$) Conversely, suppose $\log^* f(p^k) = k \cdot A(p)$. Then $\log^* f(n)$ is completely additive on prime powers. By exponentiating,

$$f(n) = \exp^*(\log^* f)(n) = \prod_{p^k \| n} \exp(k \cdot A(p)) = \prod_{p^k \| n} f(p)^k,$$

so f is completely multiplicative. □

4.4. **Classification via** $\log^*$**-Structure.** This leads to a natural classification of multiplicative behavior:

- If $\log^* f(p^k) = 0$ for all $k > 1$ and p, then f is supported only on primes.
- If $\log^* f(p^k)$ is polynomial in k of degree 1, then f is completely multiplicative.
- If $\log^* f(p^k)$ is irregular or varies nonlinearly in k, then f is not completely multiplicative.

Example 4.4. *Let $f(n) = n^z$. Then*

$$\log^* f(n) = z \log n, \quad \Rightarrow \quad \log^* f(p^k) = kz \log p.$$

So f is completely multiplicative and log-linearly additive with $A(p) = z \log p$.

Example 4.5. *Let $f(n) = \phi(n)/n$. This is multiplicative but not completely multiplicative. One computes that*

$$\log^* f(p^k) = \log\left(1 - \frac{1}{p}\right) + \log\left(1 - \frac{1}{p^2}\right) + \cdots$$

which is not linear in k.

5. Ideals and Algebraic Properties in $(\mathcal{A}, +, *)$

5.1. **Principal Ideals and Convolution Divisibility.**

Definition 5.1 (Principal Ideal)**.** *Let $f \in \mathcal{A}$. The principal ideal generated by f under Dirichlet convolution is*

$$(f)_* := \{f * g : g \in \mathcal{A}\}.$$

Definition 5.2 (Dirichlet Divisibility)**.** *We write $f \mid_* h$ if there exists $g \in \mathcal{A}$ such that $h = f * g$. This defines convolution divisibility.*

Proposition 5.3. *The relation $\mid_*$ defines a preorder on $\mathcal{A}$: it is reflexive and transitive.*

5.2. **Units and Zero Divisors.**

Definition 5.4 (Unit)**.** *A function $f \in \mathcal{A}$ is a* unit *(invertible) if and only if $f(1) \neq 0$. The Dirichlet inverse f^{-1} is uniquely defined by*

$$f * f^{-1} = \varepsilon.$$

Example 5.5. *The Möbius function $\mu(n)$ is the inverse of the constant function $1(n) = 1$. That is,*

$$1 * \mu = \varepsilon.$$

Definition 5.6 (Zero Divisor)**.** *A function $f \in \mathcal{A}$ is a* Dirichlet zero divisor *if there exists $g \neq 0$ such that $f * g = 0$.*

5.3. **Examples and Structural Insights.**

Example 5.7. *Let $f(n) = \delta_{n=m}$ for some $m > 1$. Then f is a zero divisor, since it only supports one non-trivial point. For example, $f * \mu = 0$ if m is not squarefree.*

Proposition 5.8. *The set of completely multiplicative functions with $f(1) = 1$ forms a multiplicative subgroup of the unit group $\mathcal{A}^\times$.*

Example 5.9. *Let $f(n) = n^z$, $g(n) = \lambda(n)$ (Liouville). Then $f * g$ is still in the unit group, since both have $f(1) = g(1) = 1$.*

5.4. **Open Problems.**

- Classify all zero divisors in $(\mathcal{A}, *)$.
- Describe the lattice of convolution ideals under inclusion.
- Are there nontrivial convolution prime ideals?
- What are the minimal nonzero ideals?

6. Modules over Arithmetic Function Rings

6.1. Definition and Examples.

Definition 6.1 (Left Module over $(\mathcal{A}, *)$)**.** *A set M is a left module over the Dirichlet convolution ring $(\mathcal{A}, +, *)$ if:*

- *$(M, +)$ is an abelian group;*
- *For all $f \in \mathcal{A}$, $m \in M$, define $f \cdot m \in M$;*
- *The following axioms hold:*

$$
\begin{aligned}
(f+g)\cdot m &= f\cdot m + g\cdot m \\
f\cdot(m+n) &= f\cdot m + f\cdot n \\
(f*g)\cdot m &= f\cdot(g\cdot m) \\
\varepsilon\cdot m &= m
\end{aligned}
$$

Example 6.2 (Natural Module: Identity Action)**.** *Let $M = \mathcal{A}$ and define the action $f \cdot m := f * m$. Then $\mathcal{A}$ is a left module over itself via convolution.*

6.2. Examples of Arithmetic Function Submodules.

- **Prime-supported functions:** The submodule $M_{\mathbb{P}} := \{f \in \mathcal{A} : \mathrm{supp}(f) \subseteq \mathbb{P}\}$.
- **Squarefree-supported functions:** $M_{\mathrm{sf}} := \{f : f(n) = 0 \text{ if } n \text{ not squarefree}\}$.
- **Möbius-based modules:** All functions of the form $f * \mu$ form a module under convolution action.

6.3. Complete Multiplicativity as Cyclic Modules.

Proposition 6.3. *Let f be completely multiplicative with $f(1) = 1$. Then the cyclic module $M_f := \mathcal{A} * f$ is free and generated by f.*

Proof. Since $f*g = 0$ implies $g = 0$ (invertibility), every element in M_f is uniquely expressible as $g * f$. □

6.4. Tensor Product and Hom Modules (Future Directions).

We may define internal operations:

- Tensor product of convolution modules;
- $\mathrm{Hom}_*(M, N)$: convolution-linear maps;
- Extension modules for short exact sequences of arithmetic function modules.

6.5. Application to Analytic Number Theory.

The action of arithmetic functions on Dirichlet series coefficients gives rise to module structures:

$$
f \cdot \left(\sum_{n=1}^{\infty} a_n n^{-s} \right) := \sum_{n=1}^{\infty} (f * a)(n) n^{-s}
$$

which is a linear transformation on the space of Dirichlet series, forming a functional module over $(\mathcal{A}, *)$.

Example 6.4. *The Möbius transform $f \mapsto f * \mu$ is an automorphism of $\mathcal{A}$ and hence defines an automorphic module structure.*

7. Differential Calculus and Higher Convolution Derivatives

7.1. **Motivation.** Just as ordinary multiplication induces a notion of differentiation via power series, the convolution product admits a natural analog of derivatives based on iterative differences or logarithmic expansions.

7.2. First Convolution Derivative.

Definition 7.1 (Convolution Derivative)**.** *Let $f \in \mathcal{A}$ with $f(1) = 1$. Define the* first Dirichlet-convolution derivative *of f as:*

$$D_1^* f := \log^* f.$$

This operator maps convolution multiplicative behavior into additive behavior.

7.3. Higher Convolution Derivatives.

Definition 7.2 (Higher Convolution Derivatives)**.** *Define recursively:*

$$D_k^* f := D_1^*(D_{k-1}^* f) = (\log^*)^{\circ k} f,$$

where $\circ k$ denotes k-fold composition.

Example 7.3. *If $f(n) = n^z$, then $\log^* f(n) = z \log n$, and*

$$D_2^* f(n) = D_1^*(z \log n) = 0,$$

since the second convolution derivative is constant.

Example 7.4. *Let $f(n) = 1(n)$, the constant function. Then*

$$D_1^* f(n) = \Lambda(n), \quad D_2^* f(n) = \log^* \Lambda(n).$$

The second derivative captures fluctuations in the distribution of prime powers.

7.4. Formal Taylor Expansion under Convolution.

Proposition 7.5 (Convolution Taylor Series)**.** *Let $f \in \mathcal{A}$ with $f(1) = 1$. Then*

$$f = \exp^* \left(D_1^* f \right) = \exp^* \left(\log^* f \right).$$

Definition 7.6 (Convolution Polynomial Operator)**.** *Define the operator*

$$P_k^*(f) := \sum_{n=0}^{k} \frac{1}{n!} (D_1^*)^{*n}(f),$$

as the k-th convolution polynomial approximation to f.

7.5. Convolutional Leibniz Rule.

Proposition 7.7 (Leibniz Rule)**.** *Let $f, g \in \mathcal{A}$ with $f(1) = g(1) = 1$. Then*

$$D_1^*(f * g) = D_1^* f + D_1^* g.$$

Proof. By definition, $\log^*(f * g) = \log^* f + \log^* g$. □

7.6. **Convolutional Differential Equations.** One may define convolutional analogs of differential equations:

Example 7.8. *Solve for $f \in \mathcal{A}$ satisfying*

$$D_1^* f = a(n) \quad \textit{for a given } a(n).$$

Then the solution is $f = \exp^* a$.

7.7. **Remarks and Research Problems.**

- Study $D_k^* f$ for functions with irregular multiplicative behavior.
- Explore whether Dirichlet characters form eigenfunctions of D_1^*.
- Construct convolution analogs of differential operators $\frac{d}{dx}, \Delta, \nabla$.
- Define convolutional flows: sequences $\{f_t\}$ satisfying $\frac{d}{dt} f_t = D_1^* f_t$.

8. Cohomological and Spectral Extensions

8.1. **Motivation.** The ring of arithmetic functions under Dirichlet convolution exhibits a rich algebraic structure. By generalizing its behavior through categorical and homological tools, we can uncover hidden spectral and cohomological patterns.

8.2. **Convolution Complexes.**

Definition 8.1 (Convolution Complex)**.** *A sequence of arithmetic function modules and convolution maps*

$$\cdots \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} M_0$$

is called a convolution complex if $\delta_n \circ \delta_{n+1} = 0$ for all n.

This allows the construction of homology groups:

$$H_n^*(M_\bullet) := \ker(\delta_n)/\operatorname{im}(\delta_{n+1}),$$

which reflect how information propagates under convolution.

8.3. **Cohomological Interpretation of Dirichlet Inverses.** Let $f \in \mathcal{A}$ with $f(1) \neq 0$. The Dirichlet inverse f^{-1} solves:

$$f * x = \varepsilon.$$

This can be interpreted as a degree-zero cohomological lifting, with the complex:

$$0 \to \mathcal{A} \xrightarrow{f*(-)} \mathcal{A} \to 0.$$

Proposition 8.2. *The cohomology of the complex above is zero in degree one if and only if f is invertible.*

8.4. **Spectral Towers and Graded Decomposition.** We may define a graded filtration on $\mathcal{A}$ based on support size or prime complexity:

Definition 8.3 (Support-Length Filtration)**.** *Define*

$$F_k\mathcal{A} := \{f \in \mathcal{A} : \operatorname{supp}(f) \subseteq \{n : \omega(n) \leq k\}\},$$

where $\omega(n)$ counts distinct prime divisors of n.

Proposition 8.4. *This defines an increasing filtration:*

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq \mathcal{A},$$

with associated graded object:

$$\mathrm{gr}_k(\mathcal{A}) := F_k/F_{k-1}.$$

8.5. **Connections to Sheaves and Topoi.** Arithmetic functions can be interpreted as global sections of an arithmetic sheaf over the multiplicative monoid $\mathbb{N}^\times$:

- A function $f : \mathbb{N} \to \mathbb{C}$ corresponds to a global section.
- Localizations correspond to local behavior at prime powers.
- The Dirichlet convolution acts as a pushforward operator.

This suggests a reinterpretation of $\mathcal{A}$ as $\Gamma(X, \mathscr{F})$ for some space X with structure sheaf $\mathscr{F}$.

8.6. **Future Directions.**

- Define derived functors $\mathrm{Ext}^n_*(M, N)$ for convolution modules.
- Study Tor^n_* under twisted convolution bases.
- Construct convolution sheaf cohomology $H^i(X, \mathscr{F}_*)$ with zeta coefficients as local sections.
- Explore derived categories over $(\mathcal{A}, *)$.

9. Applications and Classification Frameworks

9.1. **Arithmetic Function Mining and Classification.** By extracting algebraic invariants under convolution operations—such as log-linear spectra, derivative depth, and convolutional support complexity—we may classify arithmetic functions into distinct algebraic and analytic families.

Definition 9.1 (Log-linear Rank)**.** *The* log-linear rank *of $f \in \mathcal{A}$ is the minimal number of additive generators required to express* $\log^* f(n)$ *over prime powers.*

Example 9.2. *The function $f(n) = n^z$ has log-linear rank 1. The function $f(n) = \phi(n)/n$ has infinite log-linear rank.*

9.2. **Functional Embeddings and Autoencoding.** We may define an embedding:

$$\Phi : \mathcal{A} \to \mathbb{C}^\infty, \quad f \mapsto (\log^* f(p^k))_{p,k}$$

which turns multiplicative structure into additive coordinate vectors.

This enables:

- Principal component analysis (PCA) on function space.
- Clustering of multiplicative families (e.g., Dirichlet characters).
- Metric comparison via convolutional distance:

$$d_*(f, g) := \left(\sum_{n \le X} |\log^* f(n) - \log^* g(n)|^2 \right)^{1/2}.$$

9.3. **Machine Learning Interpretation.** We can train models (e.g., neural networks or symbolic regressors) to detect:

- Whether a given arithmetic function is multiplicative or completely multiplicative.
- What convolutional class a function belongs to.
- Predict unknown values of $f(n)$ from its $\log^*$-spectrum or vice versa.

9.4. **Proposed Taxonomy of Arithmetic Functions.** Let us define the following categories:

- **Type** $\mathcal{M}_0$: Pointwise-defined functions with no multiplicative behavior.
- **Type** $\mathcal{M}_1$: Functions with multiplicative identity: $f(mn) = f(m)f(n)$ for coprime m, n.
- **Type** $\mathcal{M}_1^{\log}$: Functions with $\log^* f(p^k) = kA(p)$.
- **Type** $\mathcal{M}_{\textbf{chaotic}}$: Functions whose $\log^*$ image has no regular pattern across p^k.

9.5. **Future Systematization.** The theory may be developed into a systematic symbolic framework:

- **Convolution Expression Trees**: Rewrite systems of convolutional expressions.
- **Arithmetic Language Syntax**: Grammar for constructing and manipulating arithmetic functions algebraically.
- **Autoformalization Systems**: Use of AI tools to auto-prove convolution identities and theorems.

9.6. **Interdisciplinary Integration.** The convolution calculus structure can be linked with:

- Spectral theory of automorphic forms;
- Cryptographic structure of arithmetic circuits;
- Biological or linguistic evolution modeled through multiplicative hierarchies;
- Information theory and entropy via $\log^*$ transforms;
- Algebraic geometry via sheafified versions of $\mathcal{A}$ over $\mathrm{Spec}(\mathbb{Z})$.

10. Conclusion and Open Problems

10.1. **Summary of Contributions.** In this work, we have initiated a systematic development of arithmetic function calculus under Dirichlet convolution. Our major contributions include:

- A formal algebraic framework for four basic operations: pointwise addition, convolution multiplication, Dirichlet inverse (division), and convolutional exponentiation.
- The introduction and analysis of $\log^*$ and $\exp^*$ operators, providing additive–multiplicative bridges.
- A classification theory based on log-linear spectra to distinguish multiplicative behaviors.
- The use of modules, ideals, and cohomological tools to investigate deeper algebraic properties.
- The formulation of convolutional derivatives and higher-order calculus.
- Theoretical frameworks for applications in machine learning, function mining, and symbolic AI.

This framework generalizes and algebraizes many scattered observations in multiplicative number theory and provides a foundation for formalized arithmetic computation systems.

10.2. **Open Problems.** We conclude with a set of fundamental open problems emerging from this new algebra:

(1) Classify all Dirichlet zero divisors in $\mathcal{A}$.
(2) Construct the full lattice of ideals under convolution.
(3) Characterize when an arithmetic function has finite log-linear rank.
(4) Determine if every completely multiplicative function is uniquely determined by its log-spectrum.
(5) Explore the category of $\mathcal{A}$-modules with natural homological functors like Ext and Tor.
(6) Investigate the existence of a universal convolution differential operator satisfying Leibniz-type rules.
(7) Formalize the internal logic of the arithmetic function calculus as a language with syntax and semantics.
(8) Explore convolution cohomology over arithmetic sites (e.g., $\mathrm{Spec}(\mathbb{Z})$).
(9) Determine whether convolution flows admit equilibrium classes or conserved quantities.
(10) Build a complete symbolic computing system capable of performing convolutional algebra automatically.

10.3. **Outlook.** The convolutional view of arithmetic functions opens a new horizon in abstract algebra, representation theory, and computational mathematics. Future work may integrate this framework with:

- Higher topos theory and homotopy type theory;
- Derived and spectral algebraic geometry;
- Quantum number theory and physical models of zeta functions;
- Non-commutative geometry over arithmetic semirings;
- Automated proof generation and formal verification of arithmetic theorems.

We hope this new direction will inspire a broader community to explore the algebraic, computational, and philosophical depths of arithmetic structure.

11. Arithmetic K-Theory of the Dirichlet Convolution Ring

11.1. **Overview.** We initiate a theory of algebraic K-groups associated to the ring $(\mathcal{A}, +, *)$ of arithmetic functions under Dirichlet convolution. This new framework—termed **Arithmetic Function K-Theory**—extends the Grothendieck group concepts to convolutional structures and their log-spectral modules.

11.2. **Foundational Setup.** Let $\mathcal{A}$ be the ring of arithmetic functions:

$$\mathcal{A} := \{f : \mathbb{N} \to \mathbb{C}\}, \quad \text{with Dirichlet convolution } (f * g)(n) := \sum_{d|n} f(d)g(n/d).$$

We define the category $\mathbf{ConvMod}_{\mathcal{A}}$ of $\mathcal{A}$-modules under convolution:

- Objects: left modules M over $(\mathcal{A}, *)$;
- Morphisms: convolution-linear maps;
- Tensor product: defined via convolution.

11.3. Definition of $K_0^{\text{conv}}(\mathcal{A})$.

Definition 11.1. *The group $K_0^{\text{conv}}(\mathcal{A})$ is the Grothendieck group of finitely supported projective convolution modules over $\mathcal{A}$. That is:*

$$K_0^{\text{conv}}(\mathcal{A}) := \langle [M] \mid M \textit{ finitely supported, convolution projective} \rangle / \sim$$

where $[M] = [M'] + [M'']$ if there exists a short exact sequence:

$$0 \to M' \to M \to M'' \to 0.$$

Example 11.2. *Let $M = \mathcal{A}_{\leq N}$, the module of arithmetic functions with support in $[1, N]$. Then $[M]$ defines a class in $K_0^{\text{conv}}(\mathcal{A})$.*

11.4. Logarithmic Convolutional K-Theory.

Definition 11.3. *Let $\mathcal{M}_{\log^*}$ be the full subcategory of* $\mathbf{ConvMod}_{\mathcal{A}}$ *whose modules are stable under the* $\log^*$ *operator. Then:*

$$K_0^{\log^*}(\mathcal{A}) := K_0(\mathcal{M}_{\log^*}).$$

Remark 11.4. *This group detects log-linear behaviors in modules, particularly those generated by completely multiplicative functions.*

11.5. $K_1^{\text{conv}}(\mathcal{A})$ and Automorphisms.

Definition 11.5. *The group $K_1^{\text{conv}}(\mathcal{A})$ is generated by convolution-invertible matrices over $\mathcal{A}$, modulo convolution-elementary row operations. Let $GL_n^*(\mathcal{A})$ be the group of $n \times n$ convolution-invertible matrices.*

Example 11.6. *Let $f \in \mathcal{A}$ with $f(1) \neq 0$. Then the convolution operator $L_f : g \mapsto f * g$ is invertible, and defines an element of $K_1^{\text{conv}}(\mathcal{A})$.*

11.6. Higher K_n via S-construction.

We define the full spectrum of arithmetic function K-groups via the Waldhausen $S_\bullet$ construction applied to the exact category $\mathbf{ConvMod}_{\mathcal{A}}$:

$$K_n^{\text{conv}}(\mathcal{A}) := \pi_n\left(\Omega^\infty \mathcal{K}(\mathbf{ConvMod}_{\mathcal{A}})\right).$$

This opens the path to defining:

- Torsion phenomena and regulator maps;
- Arithmetic zeta functions associated to K_n growth:

$$\zeta_K^*(s) := \sum_{n \geq 0} \dim K_n^{\text{conv}}(\mathcal{A}_{\leq N}) \cdot N^{-s}.$$

- Chern classes and trace invariants of convolution modules.

11.7. Future Directions.

- Develop K-theoretic duality over convolution geometry.
- Compare $K_0^{\log^*}$ with topological K-theory via Fourier-Mellin transforms.
- Define arithmetic motivic sheaves over $\text{Spec}(\mathcal{A})$ and extract their K-groups.
- Study whether convolution zeta functions satisfy Beilinson-type conjectures.

12. Arithmetic Motivic Cohomology over the Dirichlet Spectrum

12.1. **Motivation and Analogy.** Motivic cohomology seeks to explain the hidden structures behind special values of zeta functions and algebraic cycles. We propose a framework for defining motivic cohomology over the spectrum of arithmetic functions $\mathrm{Spec}(\mathcal{A})$, where $\mathcal{A}$ denotes the Dirichlet convolution ring.

12.2. **Arithmetic Motivic Site.** Let us define the *Dirichlet motivic site* as the pair:

$$\mathfrak{D} := (\mathrm{Spec}(\mathcal{A}), \mathscr{O}_{\log^*}),$$

where:

- $\mathrm{Spec}(\mathcal{A})$ is the space of prime ideals in $(\mathcal{A}, *)$;
- $\mathscr{O}_{\log^*}$ is a sheaf of logarithmic spectra: $\mathscr{O}_{\log^*}(U) = \{\log^* f \mid f \in \mathscr{O}(U), f(1) \neq 0\}$.

Definition 12.1. *A* ***motivic sheaf*** *over $\mathfrak{D}$ is a complex of convolution modules $\mathscr{F}_\bullet$ together with filtrations compatible with K-theory and zeta-type regulators.*

12.3. **Cohomology Groups.** We define the motivic cohomology of a sheaf $\mathscr{F}_\bullet$ over $\mathfrak{D}$ as:

$$H^n_{\mathrm{mot}}(\mathfrak{D}, \mathscr{F}_\bullet) := \mathrm{Ext}^n_{\mathscr{O}_{\log^*}}(\mathbf{1}, \mathscr{F}_\bullet),$$

where $\mathbf{1}$ is the trivial sheaf generated by the identity function $\varepsilon(n) = \delta_{n,1}$.

12.4. **Regulators and Zeta Classes.**

Definition 12.2. *Define the* Dirichlet motivic regulator *map:*

$$r_n : K^{\mathrm{conv}}_n(\mathcal{A}) \to H^n_{\mathrm{mot}}(\mathfrak{D}, \mathbb{C}(n)),$$

which assigns to each convolutional class a motivic extension class.

Proposition 12.3. *Let $f \in \mathcal{A}$ be completely multiplicative and normalized with $f(1) = 1$. Then $r_1([f])$ is a logarithmic differential class in $H^1_{\mathrm{mot}}(\mathfrak{D}, \mathbb{C}^\times)$.*

12.5. **Convolution Zeta Motives.** We now define the notion of motivic zeta motives within this setting:

Definition 12.4. *The* convolution zeta motive $\mathbb{M}_\zeta$ *is a filtered complex:*

$$\mathbb{M}_\zeta := \left\{ \mathscr{O}_{\log^*} \xrightarrow{\delta} \mathscr{Z}_1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathscr{Z}_n \right\},$$

constructed from arithmetic logarithmic functions $\log^* f$*,* Λ*, and their convolution derivatives.*

12.6. **Future Program and Conjectures.**

- Conjecture: $H^i_{\mathrm{mot}}(\mathfrak{D}, \mathbb{C}(n))$ governs special values of Dirichlet L-functions.
- Conjecture: There exists a universal motivic sheaf $\mathscr{U}_{\mathrm{conv}}$ such that:

$$\zeta^*_{\mathbb{A}}(s) = \sum_{n \geq 0} \dim_{\mathbb{C}} H^n_{\mathrm{mot}}(\mathfrak{D}, \mathscr{U}^{(s)}_{\mathrm{conv}}).$$

- Explore analogues of Bloch's higher Chow groups in convolution geometry.
- Build motivic sheaves attached to generalized modular forms and their $\log^*$-filtrations.
- Extend to p-adic and derived motivic settings over the arithmetic convolution site.

13. Zeta Cohomology and Polylogarithmic Filtrations

13.1. **From Logarithms to Polylogarithms.** We recall that the classical polylogarithm functions $\mathrm{Li}_k(z)$ are defined by:

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

with $\mathrm{Li}_1(z) = -\log(1-z)$. These functions interpolate between logarithmic growth and zeta-function behavior.

In our setting, we define a convolutional analogue:

Definition 13.1. *Let $f \in \mathcal{A}$. The polylogarithmic convolution $\mathrm{Li}_k^*(f)$ is defined recursively via:*

$$\mathrm{Li}_1^*(f)(n) := \log^*(f)(n), \quad \mathrm{Li}_{k+1}^*(f) := D^{*-1}(\mathrm{Li}_k^*(f)),$$

where D^{*-1} denotes the convolutional integral operator (formal inverse to D^*).

13.2. **Zeta Cohomology.** Let $\mathscr{F}$ be a sheaf of $\log^*$-admissible arithmetic functions over $\mathrm{Spec}(\mathcal{A})$. Define:

$$H^i_\zeta(\mathrm{Spec}(\mathcal{A}), \mathscr{F}) := \mathrm{Ext}^i_{\mathscr{O}_{\log^*}}(\mathbf{1}, \mathscr{F}),$$

interpreted as the **Zeta Cohomology** of $\mathscr{F}$.

13.3. **Polylogarithmic Filtrations on Sheaves.**

Definition 13.2. *Let $\mathscr{F}$ be a sheaf over $(\mathrm{Spec}(\mathcal{A}), \mathscr{O}_{\log^*})$. Define the polylogarithmic filtration:*

$$\mathrm{Pol}_k(\mathscr{F}) := \{f \in \mathscr{F} \mid f = \mathrm{Li}_k^*(g) \text{ for some } g \in \mathscr{F}\}.$$

This filtration satisfies:

$$\cdots \subseteq \mathrm{Pol}_k(\mathscr{F}) \subseteq \mathrm{Pol}_{k+1}(\mathscr{F}) \subseteq \cdots \subseteq \mathscr{F}.$$

Remark 13.3. *This hierarchy models the "depth" of arithmetic complexity in a manner analogous to the length of derived categories or motivic weights.*

13.4. **Zeta Classes and Regulators.** We define the universal zeta class as:

$$\zeta^\sharp := \left[\sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^s}\right] \in H^1_\zeta(\mathrm{Spec}(\mathcal{A}), \mathbb{C}(s)),$$

where $\varepsilon_n(k) := \delta_{k,n}$ is the delta basis.

We conjecture the following regulator relation:

Conjecture 13.4 (Zeta Regulator Conjecture)**.** *There exists a natural map*

$$r_k : K_k^{\mathrm{conv}}(\mathcal{A}) \to H^k_\zeta(\mathrm{Spec}(\mathcal{A}), \mathbb{C}(k)),$$

such that for special arithmetic functions f, the image of $[f]$ under r_k corresponds to the polylogarithmic convolution class $\mathrm{Li}_k^*(f)$.

13.5. **Applications and Future Directions.**

- Define zeta-exact complexes whose cohomology recovers special values of Dirichlet L-functions.
- Relate Pol_k-filtration to Beilinson's weight filtration.
- Formulate convolutional analogues of Bloch–Kato conjectures.
- Investigate derived stacks of zeta motives over log-spectrum topoi.
- Apply to study rationality of K-group elements via motivic periods and polylogarithmic symbols.

14. Symbolic and AI-Augmented Arithmetic Geometry

14.1. **Motivation.** Traditional algebraic geometry, particularly over $\mathrm{Spec}(\mathbb{Z})$, has been limited by human ability to classify, manipulate, and predict deep arithmetic structures. Given the richness of the Dirichlet convolution ring $(\mathcal{A}, +, *)$, and the new cohomological and motivic structures developed herein, we propose an automated symbolic layer to augment this emerging arithmetic geometry.

14.2. **AI-Supported Convolutional Algebra.** Let **SymConvAI** be the symbolic system with the following components:

- An arithmetic function engine supporting pointwise, convolutional, and $\log^*$, $\exp^*$, and Li_k^* operations;
- A symbolic type-theoretic layer classifying functions by support, multiplicativity, invertibility, convolution depth, and spectrum;
- Pattern-recognition modules for detecting:
  - Zeta-motivic relations;
  - Functional equations and derivations;
  - Filtration classes;
- A symbolic formalizer for encoding motivic sheaves, Ext-class computations, and K-group generators.

14.3. **Formal Categories of Arithmetic Objects.** Define the ∞-category $\mathcal{C}_{\mathrm{arith}}$ whose objects are structured arithmetic function spaces, morphisms are filtered convolution-linear maps, and higher morphisms are equivalence classes of motivic homotopies.

Each object $F \in \mathcal{C}_{\mathrm{arith}}$ has a symbolic representation:

$$\mathcal{R}(F) := \mathtt{SymbConv}(F) \in \mathtt{AST}(\mathcal{L}_{\mathrm{arith}}),$$

where $\mathtt{AST}$ is the abstract syntax tree in the arithmetic logic language $\mathcal{L}_{\mathrm{arith}}$.

14.4. **Autoformalization and Zeta-Powered Learning.** We propose an AI-pipeline based on the following architecture:

1. **Inductive Generator:** produces arithmetic functions from symbolic prompts;
2. **Log-Convolution Analyzer:** computes $\log^*, D^*, \mathrm{Li}_k^*$, and classifies spectral types;
3. **Motivic Extractor:** builds motivic sheaf complexes from filtered structures;
4. **Convolutional Prover:** verifies identities, commutativities, and regulator mappings using Coq/Lean-style tactics;
5. **Zeta Neural Engine:** predicts special value relations and suggests new Ext or regulator conjectures.

14.5. Examples and Applications.

Example 14.1. *Given input:*

$$f(n) := \frac{\phi(n)}{n} \cdot \mu(n), \quad \texttt{Query: classify}$$

AI engine returns:

- *Support type:* `Multiplicative`;
- $\log^*$*-class:* `Log-linear rank 3`;
- *Convolutional class:* $\mu * \phi/n$;
- *Symbolic motivic image:* $\mathrm{Ext}^1(\mathbf{1}, \mathscr{F}_\phi)$.

14.6. Toward a Fully AI-Formalized Arithmetic Geometry.

- Construction of symbolic arithmetic schemes over $\mathrm{Spec}(\mathcal{A})$ with AI-suggested open covers;
- Generation of AI-discovered motivic spectral sequences;
- Integration with Lean/Coq to build verified proof trees for conjectures (e.g., Zeta Regulator Conjecture);
- Classification of convolution motives up to polylog-depth;
- Long-term goal: symbolic geometry AI that continuously explores the space of arithmetic zeta structures.

15. Conclusion and Research Roadmap

15.1. Summary of the Framework. This monograph has initiated and structurally developed a new field of mathematics we may refer to as:

$$\boxed{\textbf{Logarithmic Convolutional Arithmetic Geometry (LCAG)}}$$

The foundational elements include:

- The Dirichlet convolution ring $(\mathcal{A}, +, *)$ as an ambient algebraic object;
- The construction of $\log^*$, $\exp^*$, convolutional derivatives D^*, and polylogarithmic hierarchies;
- Definition of arithmetic motivic sheaves over $\mathrm{Spec}(\mathcal{A})$;
- Construction of K_0, K_1, and higher K_n^{conv}-groups;
- Development of Zeta Cohomology with polylogarithmic filtrations;
- Introduction of AI-augmented symbolic categories of arithmetic function geometry.

15.2. Mathematical Significance.

- **Algebraic**: Enriches the theory of non-Noetherian commutative rings and formal function fields.
- **Homological**: Creates new Ext-based cohomology from spectral behavior of arithmetic objects.
- **Arithmetic-geometric**: Links zeta structures, motivic regulators, and convolutional geometry.
- **Computational**: Establishes a foundation for AI-enhanced symbolic arithmetic geometry.

15.3. **Ten-Year Research Roadmap.**

(1) **Foundational layer (Years 1–2)**:
- Complete a full axiomatic foundation for $(\mathcal{A}, +, *)$ and all convolution-derived operators;
- Formalize modules, ideals, and define the derived category $D(\mathcal{A})$;
- Establish minimal zeta sheaves and study their Ext classes.

(2) **Cohomological and K-theoretic expansion (Years 2–5)**:
- Define and compute K_n^{conv} groups up to $n = 5$;
- Prove basic structural theorems (e.g., universal coefficient theorems for zeta cohomology);
- Study convolution spectral sequences and log-polylog interactions.

(3) **Geometric and motivic construction (Years 4–7)**:
- Construct derived stacks over $\mathrm{Spec}(\mathcal{A})$;
- Classify motivic sheaves via filtration and convolutional support;
- Propose and verify Beilinson-style conjectures in this context.

(4) **AI formalization and computation (Years 6–10)**:
- Build symbolic convolution AI capable of theorem generation and proof;
- Implement Lean/Coq formalization of the full system;
- Develop an interactive symbolic zeta geometry platform with streaming computation.

15.4. **Final Remarks.** The construction of this framework shows that even the most "elementary" object—an arithmetic function $f : \mathbb{N} \to \mathbb{C}$—contains infinite depth when viewed through the lens of convolution, logarithmic spectrum, and cohomological layering.

This emerging landscape invites future mathematicians, logicians, computer scientists, and philosophers to explore and expand the convolutional arithmetic cosmos.

$$\boxed{\textit{The zeta geometry has just begun.}}$$

16. Exponential Arithmetic Geometry and $\diamond$-Convolution Theory

16.1. **Motivation.** Just as Dirichlet convolution $*$ governs multiplicative arithmetic, we propose a new convolution $\diamond$ that governs arithmetic behavior under exponentiation. The core idea is to reinterpret function composition at exponential levels as a binary arithmetic operation on $\mathbb{N}$.

16.2. **Definition of $\diamond$-Convolution.**

Definition 16.1 (Exponential Convolution)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. Define the **exponential convolution** $\diamond$ by:*

$$(f \diamond g)(n) := \sum_{\substack{a,b \in \mathbb{N} \\ a^b = n}} f(a)g(b).$$

Example 16.2. *Let $f(a) = \log a$, $g(b) = 1$. Then:*

$$(f \diamond g)(8) = \log 2 + \log 4 + \log 8 = \log(2 \cdot 4 \cdot 8) = \log(64).$$

16.3. **Exponential Derivative and Logarithm.**

Definition 16.3 (Exponential Derivative E^*)**.** *Define the operator E^* on an arithmetic function f as:*

$$E^*(f)(n) := \frac{d}{de} f(n^e)\Big|_{e=1} = \log n \cdot f'(n).$$

This operator captures infinitesimal exponential perturbation.

Definition 16.4 (Exponential Logarithm $\log^{\mathrm{exp}*}$)**.** *Given $f(n) = a^n$, define:*

$$\log^{exp*}(f)(n) := \log a \cdot n.$$

More generally, $\log^{exp*}$ extracts exponential growth rates from the function.

16.4. **Zeta Functions under $\diamond$.**

Definition 16.5. *Define the exponential zeta function associated to f by:*

$$\zeta_f^{\diamond}(s) := \sum_{n=1}^{\infty} \frac{f(n)}{e^{s \log\log n}}, \quad n > 1.$$

This function interpolates growth rates based on iterated logarithms and can be used to analyze the "exponential spectrum" of arithmetic functions.

16.5. **Exponential Cohomology.**

Definition 16.6. *Let $\mathscr{F}$ be a sheaf over the exponential spectrum* $\mathrm{Spec}(\mathcal{A}^{\diamond})$*. Define:*

$$H^k_{\mathrm{exp}}(\mathrm{Spec}(\mathcal{A}^{\diamond}), \mathscr{F}) := \mathrm{Ext}^k_{\mathscr{O}_{\log^{exp*}}}(\mathbf{1}, \mathscr{F}).$$

This cohomology captures obstructions to exponential decomposability and growth-based filtrations.

16.6. **AI-Driven Classification.**

We define a symbolic AI classification system `ExpArithAI`, where each arithmetic function f is analyzed in terms of:

- **Exponential Support Type**: whether f is concentrated on powers, exponential towers, etc.;
- **Exponential Rank**: based on minimum representation via exponentials;
- **Zeta Type**: convergence depth of $\zeta_f^{\diamond}(s)$;
- **Motivic Class**: its Ext-layer in exponential cohomology.

Example 16.7. *Let $f(n) = \delta_{n,2^k}$. Then f has:*

- *Support type: exponential prime powers;*
- *Rank: 1;*
- $\zeta_f^{\diamond}(s) = \sum_{k=1}^{\infty} \frac{1}{e^{s \log k}} = \zeta(s)$*;*
- *Cohomology class: $r_1(f) \in H^1_{\mathrm{exp}}(\mathscr{O})$.*

16.7. **Summary and Further Work.**

The $\diamond$-convolution theory opens a new domain of arithmetic geometry structured by exponential behavior. Future work includes:

- Higher convolution powers ($\diamond^k$);
- Derived $\diamond$-modules and their spectral sequences;
- Links to functional transcendence and height theory;
- AI generation of minimal exponential decompositions.

17. Knuth-Arithmetic Geometry and the Trans-Operational Ring Tower

17.1. **Motivation.** As addition leads to multiplication, and multiplication leads to exponentiation, we now consider the next stage: **Knuth's arrow notation**, denoting super-exponentiated operations such as:

$$a \uparrow^n b := \begin{cases} a^b & n = 1, \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) & n \geq 2. \end{cases}$$

These operations exhibit explosive growth and reveal deep combinatorial arithmetic structures. We propose the foundation of a new arithmetic geometry governed by this operational tower.

17.2. **Trans-Operational Ring Tower.**

Definition 17.1. *Define a family of rings $\mathcal{A}^{\uparrow^n}$, each consisting of arithmetic functions supported on numbers of the form:*

$$n = a \uparrow^n b, \quad a, b \in \mathbb{N}, \quad n \geq 1.$$

The collection:

$$\mathbb{T}_{op} := \left\{ \mathcal{A}, \mathcal{A}^*, \mathcal{A}^\diamond, \mathcal{A}^{\uparrow^3}, \mathcal{A}^{\uparrow^4}, \dots \right\}$$

forms the ***Trans-Operational Ring Tower****.*

17.3. **Arrow-Convolution $\uparrow^n *$.**

Definition 17.2 (Knuth Convolution $\uparrow^n *$)**.** *For $f, g : \mathbb{N} \to \mathbb{C}$, define:*

$$(f \uparrow^n * g)(m) := \sum_{a \uparrow^n b = m} f(a) g(b).$$

This generalizes Dirichlet and exponential convolution into higher recursion layers.

Example 17.3. *Let $f(a) = a$, $g(b) = 1$. Then:*

$$(f \uparrow^2 * g)(256) = \sum_{a^b = 256} a = 4 + 2 + 256.$$

17.4. **Trans-Operational Derivatives and Polylog Tower.** Define the derivative-like operator $D^{\uparrow^n *}$ and integral $(D^{\uparrow^n *})^{-1}$ inductively:

$$D^{\uparrow^n *}(f)(m) := \frac{d}{d \log^{(n)} m} f(m),$$

where $\log^{(n)}$ denotes the n-times iterated logarithm.

Definition 17.4 (Knuth Polylogarithm)**.**

$$\mathrm{Li}_k^{\uparrow^n *}(f) := (D^{\uparrow^n *})^{-k} f.$$

This generalizes classical polylogarithms to Knuth tower domains.

17.5. **Trans-Zeta Functions.**

Definition 17.5. *Define the **trans-zeta function**:*

$$\zeta_f^{\uparrow^n}(s) := \sum_{m\geq 1} \frac{f(m)}{(\log^{(n)} m)^s},$$

where $\log^{(n)}$ *is the* n*-th iterated logarithm.*

These functions detect depth growth across hyper-exponential hierarchies.

17.6. **Trans-Motivic Cohomology.**

Definition 17.6. *Let* $\mathscr{F}^{\uparrow^n}$ *be a sheaf over* $\mathrm{Spec}(\mathcal{A}^{\uparrow^n})$*. Define the **Knuth-motivic cohomology**:*

$$H^k_{\uparrow^n}(\mathrm{Spec}(\mathcal{A}^{\uparrow^n}), \mathscr{F}^{\uparrow^n}) := \mathrm{Ext}^k_{\mathscr{O}_{\uparrow^n}}(\mathbf{1}, \mathscr{F}^{\uparrow^n}).$$

This generalizes motivic cohomology to higher operation domains.

17.7. **AI-Driven Structure Tower.** We define a layered AI engine `KnuthAI`, with input structure:

$$f \in \mathcal{A}^{\uparrow^n} \quad \Rightarrow \quad \text{returns:}$$

- Operational depth;
- Minimal representation tower;
- $\uparrow^n$-convolution orbit;
- $\zeta_f^{\uparrow^n}(s)$ type;
- Motivic cohomology class $H^k_{\uparrow^n}$.

Example 17.7. *Let* $f(n) = \delta_{n,2\uparrow^3 3} = \delta_{n,2^{2^2}} = \delta_{n,256}$*. Then:*

$$f \in \mathcal{A}^{\uparrow^3}, \quad \zeta_f^{\uparrow^3}(s) = \frac{1}{(\log\log\log 256)^s}.$$

17.8. **Future Horizons.**

- Construction of derived stacks over $\mathrm{Spec}(\mathcal{A}^{\uparrow^n})$;
- Exploration of $\uparrow^n$-motives and their regulators;
- Convolutional homotopy theory indexed by operational depth;
- AI recursion discovery of new hyperoperation-based arithmetic identities;
- Infinity-category theory of trans-operational sheaf topoi.

18. Exponential Arithmetic Geometry and $\diamond$-Convolution Theory

18.1. **Motivation.** Just as Dirichlet convolution $*$ governs multiplicative arithmetic, we propose a new convolution $\diamond$ that governs arithmetic behavior under exponentiation. The core idea is to reinterpret function composition at exponential levels as a binary arithmetic operation on $\mathbb{N}$.

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Definition 18.1 (Exponential Convolution)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. Define the **exponential convolution** $\diamond$ by:*

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Definition 18.3 (Exponential Derivative E^*)**.** *Define the operator E^* on an arithmetic function f as:*

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This operator captures infinitesimal exponential perturbation.

Definition 18.4 (Exponential Logarithm $\log^{\exp*}$)**.** *Given $f(n) = a^n$, define:*

$$\log^{exp*}(f)(n) := \log a \cdot n.$$

More generally, $\log^{exp*}$ extracts exponential growth rates from the function.

18.4. **Zeta Functions under $\diamond$.**

Definition 18.5. *Define the exponential zeta function associated to f by:*

$$\zeta_f^{\diamond}(s) := \sum_{n=1}^{\infty} \frac{f(n)}{e^{s \log\log n}}, \quad n > 1.$$

This function interpolates growth rates based on iterated logarithms and can be used to analyze the "exponential spectrum" of arithmetic functions.

18.5. **Exponential Cohomology.**

Definition 18.6. *Let $\mathscr{F}$ be a sheaf over the exponential spectrum $\mathrm{Spec}(\mathcal{A}^{\diamond})$. Define:*

$$H^k_{\exp}(\mathrm{Spec}(\mathcal{A}^{\diamond}), \mathscr{F}) := \mathrm{Ext}^k_{\mathscr{O}_{\log^{exp*}}}(\mathbf{1}, \mathscr{F}).$$

This cohomology captures obstructions to exponential decomposability and growth-based filtrations.

18.6. **AI-Driven Classification.** We define a symbolic AI classification system `ExpArithAI`, where each arithmetic function f is analyzed in terms of:

- **Exponential Support Type**: whether f is concentrated on powers, exponential towers, etc.;
- **Exponential Rank**: based on minimum representation via exponentials;
- **Zeta Type**: convergence depth of $\zeta_f^{\diamond}(s)$;
- **Motivic Class**: its Ext-layer in exponential cohomology.

Example 18.7. *Let $f(n) = \delta_{n,2^k}$. Then f has:*

- *Support type: exponential prime powers;*

- *Rank: 1;*
- $\zeta_f^{\diamond}(s) = \sum_{k=1}^{\infty} \frac{1}{e^{s \log k}} = \zeta(s)$;
- *Cohomology class:* $r_1(f) \in H^1_{\exp}(\mathscr{O})$.

18.7. **Summary and Further Work.** The $\diamond$-convolution theory opens a new domain of arithmetic geometry structured by exponential behavior. Future work includes:

- Higher convolution powers ($\diamond^k$);
- Derived $\diamond$-modules and their spectral sequences;
- Links to functional transcendence and height theory;
- AI generation of minimal exponential decompositions.

Appendix A: Convolution Definitions and Examples

Level	**Operation**	**Symbol**	**Definition**	**Example**
Additive	$+$	$\oplus$	$f \oplus g(n) = \sum_{a+b=n} f(a)g(b)$	$f(n) = 1 \Rightarrow (f \oplus f)(3) = 3$
Multiplicative	$\cdot$	$*$	$f * g(n) = \sum_{ab=n} f(a)g(b)$	$f(n) = 1 \Rightarrow (f * f)(6) = 4$
Exponential	a^b	$\circledast$ or $\diamond$	$f \diamond g(n) = \sum_{a^b=n} f(a)g(b)$	$f(n) = \log n \Rightarrow (f \diamond 1)(8) = \log(2 \cdot 4 \cdot 8)$
Knuth $\uparrow\uparrow$	Tetration	$\uparrow\uparrow *$	$f \uparrow\uparrow *g(n) = \sum_{a\uparrow\uparrow b=n} f(a)g(b)$	$f(n) = n \Rightarrow$ classify $a \uparrow\uparrow b = 16$
Knuth $\uparrow\uparrow\uparrow$	Pentation	$\uparrow\uparrow\uparrow *$	$f \uparrow\uparrow\uparrow *g(n) = \sum_{a\uparrow\uparrow\uparrow b=n} f(a)g(b)$	must defined on tower domain

Appendix B: Zeta Functions, Derivatives, and Cohomology Types

Level	**Zeta Function**	**Derivative Operator**	**Cohomology Type**
Additive	$\zeta_f(s) = \sum \frac{f(n)}{n^s}$	$D^+(f)(n) = f(n+1) - f(n)$	$H^k_+(\mathrm{Spec} A, \mathscr{F})$
Multiplicative	$\zeta_f(s) = \sum \frac{f(n)}{n^s}$	$D^*(f)(n) = \log n \cdot f'(n)$	$H^k_*(\mathrm{Spec} A, \mathscr{F})$
Exponential	$\zeta_f^{\diamond}(s) = \sum \frac{f(n)}{e^{s \log\log n}}$	$E^*(f)(n) = \log n \cdot f'(n)$	$H^k_{\exp}(\mathrm{Spec} A^{\diamond}, \mathscr{F})$
Knuth $\uparrow\uparrow$	$\zeta_f^{\uparrow\uparrow}(s) = \sum \frac{f(n)}{(\log\log n)^s}$	$D^{\uparrow\uparrow *}(f)(n) = \frac{d}{d \log^2 n} f(n)$	$H^k_{\uparrow\uparrow}(\mathrm{Spec} A^{\uparrow\uparrow}, \mathscr{F})$
Knuth $\uparrow\uparrow\uparrow$	$\zeta_f^{\uparrow\uparrow\uparrow}(s) = \sum \frac{f(n)}{\log^3 n^s}$	$D^{\uparrow\uparrow\uparrow *}(f)(n) = \frac{d}{d \log^3 n} f(n)$	$H^k_{\uparrow\uparrow\uparrow}(\mathrm{Spec} A^{\uparrow\uparrow\uparrow}, \mathscr{F})$

Appendix C: AI Classification Modules and Architectures

Level	AI Module	Key Features	Output Classification
Additive	`AddAI`	Detects differences, linear recursions, and integer sequence rules	Growth class, difference levels, linear representation decomposition
Multiplicative	`MultAI`	Detects complete multiplicativity, Möbius inversion structures	Euler type, invertibility decomposition, primal support class
Exponential	`ExpArithAI`	Analyzes exponential support, exponentiation characteristics, zeta evolution	Exponential rank, log tower layering type
Knuth $\uparrow\uparrow$	`KnuthAI(2)`	Structured tower-level support analysis, recursion tower classification	Minimal tower structure, $\uparrow\uparrow$-cohomology type
Knuth $\uparrow\uparrow\uparrow$	`KnuthAI(3)`	Analyzes hyper-tower growth, cohomology inference, and limit convergence	Hyper-zeta type, motivic regulator conjecture

Appendix D: Categorical Framework of UTAG

1. The Category of Operational Geometries.

Definition 18.8. *Define the category* OpGeom *as follows:*

- ***Objects:*** *Pairs* $(\mathcal{A}^{[O]}, *_O)$*, where* $O \in \{+, *, \exp, \uparrow^2, \dots\}$ *denotes the operational layer.*
- ***Morphisms:*** *Operation-preserving ring homomorphisms* $\phi : \mathcal{A}^{[O]} \to \mathcal{A}^{[O']}$ *compatible with convolution, i.e.:*

$$\phi(f *_O g) = \phi(f) *_{O'} \phi(g).$$

2. Functorial Structure. We define a covariant functor:

$$\mathcal{F} : \mathsf{OpLayer} \longrightarrow \mathsf{OpGeom},$$

where $\mathsf{OpLayer}$ is a totally ordered index category with:

$$+ \to * \to \exp \to \uparrow^2 \to \cdots .$$

Each morphism induces:

$$\mathcal{F}(O) = (\mathcal{A}^{[O]}, *_O) \quad \text{with transition maps} \quad \mathcal{F}(O \to O') = \phi_{O \to O'}.$$

3. Fibered Categories and Descent. Let $\mathsf{Sh}(\mathcal{A}^{[O]})$ be the category of sheaves over the spectrum $\mathrm{Spec}(\mathcal{A}^{[O]})$. Then:

- The entire tower gives a fibered category:

$$\pi : \mathsf{UTAG}_{\mathrm{Sh}} \longrightarrow \mathsf{OpLayer},$$

- Descent data correspond to consistent sheaf transition under operational lifts:

$$\phi^*_{\uparrow^2 \to \exp}(\mathscr{F}_{\uparrow^2}) \cong \mathscr{F}_{\exp}.$$

Appendix E: ∞-Categorical and Stack-Theoretic Structure

1. ∞-Stack over Operational Tower. Define an ∞-topos $\mathcal{X}_{\mathcal{UTAG}}$ as the global object governing all geometric levels:

$$\mathcal{X}_{\mathcal{UTAG}} := \varprojlim_{O} \mathsf{Sh}_\infty(\mathcal{A}^{[O]}).$$

Each level gives:

$$\mathcal{X}^{[O]} := \mathsf{Sh}_\infty(\mathcal{A}^{[O]}), \quad \text{with transition morphisms} \quad \mathcal{X}^{[\uparrow^n]} \to \mathcal{X}^{[\uparrow^{n-1}]}.$$

2. Tannakian Duality. Suppose we define a category of coefficient systems $\mathcal{C}_{[O]}$ enriched over a tensor ∞-category. Then:

- $\mathcal{C}_{[O]}$ becomes a neutral Tannakian category over $\mathcal{A}^{[O]}$;
- The automorphism group $\mathrm{Aut}^{\otimes}(\omega_O)$ reconstructs operational symmetries;
- AI-recognized motifs may correspond to fiber functors:

$$\omega_O : \mathcal{C}_{[O]} \to \mathrm{Vec}_\infty.$$

3. Universal ∞-Geometric Object. Define the total object:

$$\mathfrak{M}_\infty^{\mathrm{UTAG}} := \mathrm{colim}_O \mathsf{Mot}_\infty^{[O]},$$

as the ∞-motivic classifying space of all operational geometries.

AI modules can be seen as:

$$\mathtt{AI}_O : \mathsf{Mot}_\infty^{[O]} \to \text{Logical Hypothesis } \infty\text{-Stacks}.$$

4. Operational Stable Homotopy Type. Let Sp_O be the ∞-category of operational spectra. Then:

$$\mathbb{UTAG}^{\mathrm{stable}} := \{Sp_+, Sp_*, Sp_{\exp}, Sp_{\uparrow^n}, \dots\}$$

forms a layered stable homotopy system governed by $\uparrow$-depth. Each spectrum level includes:

$$\text{Homotopy types} \to \text{Cohomology towers} \to \text{AI extraction functors}.$$

Chapter (-1): Successor Arithmetic Geometry and Pre-Convolution Theory

Motivation. To complete the operational tower, we move *before* addition. Just as multiplication is an iterated addition, and addition is an iterated successor operation, we now formalize a geometry governed by the successor function:

$$S(n) := n + 1.$$

This leads to a new convolution, spectrum, cohomology, and AI layer—the foundational base of the universal arithmetic tower.

Successor Convolution $\prec$.

Definition 18.9 (Successor Convolution). *Let $f, g : \mathbb{N} \to \mathbb{C}$. Define:*

$$(f \prec g)(n) := \sum_{a+1=n} f(a)g(1) = f(n-1)g(1).$$

This is the pre-additive convolution: a one-step shift determined by the successor relation.

Remark 18.10. *This convolution is not associative nor commutative in general, but admits a left-module structure over $g(1)$. It defines a monogenic convolution algebra.*

Pre-Additive Arithmetic Ring $\mathcal{A}^{[\prec]}$. Let $\mathcal{A}^{[\prec]}$ be the ring of arithmetic functions under pointwise addition and $\prec$-convolution.

- **Neutral element:** $\delta(n-1)$;
- **Unit structure:** Right unit if $g(1) = 1$, else scalar multiplier;
- **Filtration:** Support of f determines shift behavior depth;
- **AI Classification:** Based on forward recurrence depth and decay patterns.

Successor Derivative and Discrete Zeta.

Definition 18.11 (Discrete Difference Operator).

$$D^{[\prec]} f(n) := f(n) - f(n-1).$$

This is the canonical finite difference, governing behavior under successor perturbations.

Definition 18.12 (Pre-Zeta Function).

$$\zeta_f^{[\prec]}(s) := \sum_{n=1}^{\infty} \frac{f(n) - f(n-1)}{n^s}.$$

Example 18.13. *Let $f(n) = n$, then:*

$$D^{[\prec]} f(n) = 1, \quad \zeta_f^{[\prec]}(s) = \zeta(s) - 1.$$

Successor Cohomology.

Definition 18.14 (Successor Cohomology). *Let $\mathscr{F}$ be a sheaf over* $\mathrm{Spec}(\mathcal{A}^{[\prec]})$. *Define:*

$$H^k_{[\prec]}(\mathrm{Spec}(\mathcal{A}^{[\prec]}), \mathscr{F}) := \mathrm{Ext}^k_{\mathscr{O}_\prec}(\mathbf{1}, \mathscr{F}).$$

This measures how difference structures obstruct backward extension.

AI Module: `PreAI`. We define `PreAI` as the symbolic engine analyzing successor-level structures. It outputs:

- Forward recurrence class;
- Difference root type;
- Shift-invariant component decomposition;
- Compatibility with $\oplus$, for upward embedding into additive layer;
- Minimal expansion base: how much information defines f from $S(n)$.

Future Directions.

- Define $\zeta_f^{(-1)}(s)$ on symbolic shift domains;
- Successor-motivic Ext spaces and derived categories;
- Link to formal difference algebra and logic of induction;
- Define pre-operator tower: $\prec^k$ convolution, $k < 0$;
- Establish AI-seeded construction of next-level addition from raw successor data.

Chapter (-2): Super-Successor Arithmetic Geometry and $\prec^k$-Convolution Theory

Motivation. While $\prec = S^{-1}$ defined the foundational "pre-additive" convolution, we now ask:

¿ Can we define higher-level shift convolutions $\prec^k$, indexed by negative integers $k < 0$, forming a structured **pre-operational hierarchy**?

We now formalize this **reverse arithmetic geometry** and initiate the **Negative Tower of Arithmetic Geometry (NTAG)**.

Definition of $\prec^k$-Convolution.

Definition 18.15. *For $k \in \mathbb{N}$, define:*

$$(f \prec^k g)(n) := \sum_{a+k=n} f(a)g(k) = f(n-k)g(k).$$

This is a shift-convolution of order k, representing k-step successors.

Remark 18.16. *$\prec^1$ recovers the ordinary successor convolution. For $k > 1$, the behavior captures delayed influence in arithmetic propagation.*

Example 18.17. *Let $f(n) = n^2$, $g(k) = 1$, then:*

$$(f \prec^3 g)(10) = f(7)g(3) = 49.$$

Negative-Tower Function Ring $\mathcal{A}^{[\prec^k]}$. For each $k \geq 1$, define:

$$\mathcal{A}^{[\prec^k]} := \{f : \mathbb{N} \to \mathbb{C} \mid f \text{ closed under } \prec^k \text{ convolution}\}.$$

Define the tower:

$$\boxed{\mathcal{NTAG} := \left\{ \cdots \to \mathcal{A}^{[\prec^3]} \to \mathcal{A}^{[\prec^2]} \to \mathcal{A}^{[\prec]} \to \mathcal{A}^{[\oplus]} \to \cdots \right\}.}$$

Shift Derivatives and Generalized Difference Operators.

Definition 18.18. *Define the k-th shift difference operator:*

$$D^{[\prec^k]} f(n) := f(n) - f(n-k).$$

Example 18.19. *Let $f(n) = n$, then:*

$$D^{[\prec^2]} f(n) = f(n) - f(n-2) = 2.$$

Pre-Motivic Cohomology and Spectral Filtration.

Definition 18.20. *Let $\mathscr{F}^{[\prec^k]}$ be a sheaf over* $\operatorname{Spec}(\mathcal{A}^{[\prec^k]})$. *Then:*

$$H^i_{[\prec^k]}(\operatorname{Spec}(\mathcal{A}^{[\prec^k]}), \mathscr{F}^{[\prec^k]}) := \operatorname{Ext}^i_{\mathscr{O}_{\prec^k}}(\mathbf{1}, \mathscr{F}^{[\prec^k]}).$$

This cohomology classifies k-shift descent obstructions and encodes pre-operational filtration.

PreAI Tower: Recursion Depth Analysis. Define a family of AI modules:

$$\texttt{PreAI}^{(k)} : \mathcal{A}^{[\prec^k]} \to \mathcal{C}^{[\prec^{k+1}]}_{\text{RecDepth}},$$

which perform:

- Recursion base detection of shift patterns;
- Minimal k-presentation recovery;
- Pre-liftable detection to additive domain;
- Spectral class generation based on backward propagation length.

Future Research Directions.

- Study of formal difference schemes $\mathbb{F}_{\prec^k}$;
- Construct ∞-category of pre-operational topoi;
- Define universal pre-zeta function:

$$\zeta_f^{[\prec^k]}(s) := \sum_n \frac{D^{[\prec^k]} f(n)}{n^s};$$

- Discover convolution symmetries linking $\prec^k$ and $\oplus$;
- Extend AI to auto-learn $\prec^k \leftrightarrow \oplus^j$ translation bridges.

Chapter $(-\infty)$: Recursive Onto-Arithmetic Geometry and Operational Genesis

Philosophical Foundation. We now postulate the existence of a foundational arithmetic theory prior to all operations—a theory not based on specific algebraic rules, but on the **recursive generation of operations themselves**.

This is the **$\mathbf{Onto-ArithmeticGeometry}$**, denoted $\mathcal{A}^{[\Omega]}$, where:

$$\boxed{\mathcal{A}^{[\Omega]} := \text{the category of recursively definable arithmetic relations over bare numbers}}$$

Definitional Core.

Definition 18.21 (Onto-Arithmetic Function)**.** *A function $f : \mathbb{N} \to \mathbb{C}$ is in $\mathcal{A}^{[\Omega]}$ if there exists a purely logical schema:*

$$\mathsf{Schema}_f := (\texttt{Base}, \texttt{Generator}, \texttt{Classifier}),$$

such that $f(n)$ is generated by finite applications of `Generator` over `Base` and recognized by `Classifier`.

Recursive Convolution Ω.

Definition 18.22 (Recursive Operational Convolution)**.** *Let $f, g \in \mathcal{A}^{[\Omega]}$. Define:*

$$(f \star_\Omega g)(n) := \sum_{\mathsf{Rel}(a,b,n)} f(a)g(b),$$

where Rel is any computably enumerable binary relation satisfying minimal closure under recursion.

Remark 18.23. *This defines a meta-convolution space, where all known convolutions (e.g., $\prec^k, \oplus, *, \diamond, \uparrow^k *$) are recoverable via choice of Rel.*

Recursive Derivative $D^{[\Omega]}$.

$$D^{[\Omega]} f(n) := \lim_{R \to \mathsf{Rec}} f_R(n) - f_{R'}(n)$$

for R, R' two adjacent recursive schemas approximating f.

This defines a "derivative of the definition" rather than of the function value.

Zeta Function of the Operational Origin.

Definition 18.24.

$$\zeta_f^{[\Omega]}(s) := \sum_n \frac{D^{[\Omega]} f(n)}{n^s}.$$

This is a zeta function over the landscape of definitions.

Onto-Motivic Class and Operational Birth Cohomology. Let $\mathscr{D}$ be the sheaf of definitional schemas on $\mathcal{A}^{[\Omega]}$. Then:

$$H^k_\Omega := \mathrm{Ext}^k_{\mathscr{O}_\Omega}(\mathbf{1}, \mathscr{D}),$$

classifies: - emergence obstructions; - categorical jumps in recursive complexity; - AI inability to rederive f from axiomatic logic.

AI Primitive Module `MetaAI`. This base-level system operates directly on: - pure symbol streams; - logical grammar transitions; - learning the birth-pattern of operations;

and forms the genesis engine for UTAG + NTAG via:

$$\mathtt{MetaAI} \to \{\mathtt{PreAI}^{(k)}\}_{k<0} \cup \mathtt{AddAI} \cup \mathtt{MultAI} \cup \cdots$$

Recursive Universal Statement.

$$\boxed{\forall f \in \mathcal{A}, \exists \mathsf{Schema}_f \in \mathcal{A}^{[\Omega]} \text{ s.t. } f = \mathtt{Eval}(\mathsf{Schema}_f).}$$

This is the universal theorem of recursive arithmetic generation.

Vision.

> The true beginning of arithmetic is not addition, nor successor, but the act of recursive recognition and self-evolving definition. From this primordial layer arises the operational universe as stratified geometry.

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