DYADIC INVERSE LIMIT GEOMETRY: A NEW FRAMEWORK FOR REAL NUMBERS AND DIOPHANTINE APPROXIMATION

PU JUSTIN SCARFY YANG

Abstract. We introduce a new construction of the real numbers $\mathbb{R}$ as the inverse limit of dyadic rational truncations, forming a canonical projective system over the tower of dyadic sets $\mathbb{Q}_n := \{a/2^n \mid a \in \mathbb{Z}\}$. This construction, inspired by the inverse limit formulation of $\mathbb{Z}_p$, captures both symbolic and algebraic structure, embedding $\mathbb{R}$ within a new class of geometric-topological spaces.

We define the space $\mathbb{R}^{\text{proj}}$ as the limit of compatible dyadic approximations, and show it is naturally isomorphic to the classical real numbers. Based on this model, we develop a novel framework for Diophantine approximation, including dyadic complexity profiles $\delta_r(n)$, a Mahler-type classification of real numbers, and a proposed dyadic analogue of the Schmidt Subspace Theorem.

Visual, categorical, and symbolic perspectives are integrated, with applications ranging from formal proof assistants (Lean/Coq/UniMath) to computable real analysis. This work establishes a foundation for a new arithmetic geometry of real numbers through inverse dyadic structures, with rich potential for further developments.

Contents

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1. Introduction

The real number line $\mathbb{R}$ lies at the foundation of modern analysis and geometry. Its standard construction via Cauchy sequences or Dedekind cuts has become a canonical route in real analysis, yet it obscures the rich algebraic and symbolic structure underlying approximation processes. In contrast, the p-adic numbers $\mathbb{Q}_p$, and especially the p-adic integers $\mathbb{Z}_p$, admit a natural construction as inverse limits of modular rings:

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}.$$

This inverse limit formalism gives rise to a hierarchy of compatible approximations, forming a complete topological ring with profound implications across number theory and arithmetic geometry.

Motivation. Inspired by the algebraic and topological clarity of the p-adic construction, this work explores an analogous inverse limit framework for the real numbers, using dyadic rationals of the form $\frac{a}{2^n}$. These dyadic truncations naturally embed into $\mathbb{R}$ from below, forming a tower:

$$\mathbb{Q}_0 \leftarrow \mathbb{Q}_1 \leftarrow \mathbb{Q}_2 \leftarrow \cdots, \quad \text{where } \mathbb{Q}_n := \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z} \right\}.$$

We construct $\mathbb{R}^{\text{proj}}$, the projective limit of this system, and demonstrate its isomorphism to $\mathbb{R}$ while preserving a symbolic truncation structure. This leads to a new way of understanding real numbers as infinite sequences of approximations with strictly decreasing dyadic error.

A New Direction in Diophantine Approximation. This representation of real numbers opens the door to a novel form of Diophantine approximation. Instead of focusing on arbitrary rational approximants, we examine truncation-based dyadic approximants:

$$T_n(r) := \sup \left\{ \frac{a}{2^n} \in \mathbb{Q}_n \mid \frac{a}{2^n} \leq r \right\},$$

and define the *dyadic approximation error*:

$$\delta_r(n) := r - T_n(r).$$

This gives rise to complexity profiles $(\delta_r(n))_{n\in\mathbb{N}}$, whose decay rate reflects the irrationality, algebraic dependence, or transcendence of the real number r. We propose a dyadic version of Mahler's classification, as well as a conjectural Dyadic Schmidt Subspace Theorem, which we develop in later sections.

Symbolic, Geometric, and Formal Aspects. Our construction is not purely analytic; it admits rich connections with symbolic dynamics (via binary expansions), category theory (as limits in **Ring** or **Top**), and constructive logic. The structure is suitable for implementation in proof assistants such as Lean, Coq, or UniMath, and may lead to more intuitive formalizations of real analysis compatible with effective computation.

We also explore topological and geometric interpretations of the inverse system $\mathbb{Q}_n$, regarding truncation chains as paths in dyadic trees or points in profinite spaces.

Structure of the Paper. The paper is organized as follows:

- In Section 2, we construct $\mathbb{R}^{\text{proj}}$ and prove its isomorphism with $\mathbb{R}$;
- In Section 3, we compare this construction with classical real constructions and analyze its topology;
- In Section 4, we compute dyadic truncation sequences for $\sqrt{2}, \sqrt{3}, e, \pi$, and other examples;
- Section 5 develops the theory of dyadic Diophantine approximation;
- Section 6 proposes a Mahler-type classification based on truncation complexity;

- Section 7 explores linear dependence and conjectures a Dyadic Schmidt Subspace Theorem;
- Section 8 speculates on dyadic arithmetic geometry and symbolic topology;
- Section 9 discusses prospects for formalization in type-theoretic frameworks;
- Section 10 concludes with open problems and future directions.

We hope this framework offers a unifying symbolic, algebraic, and topological view of real numbers and their arithmetic structures.

2. Inverse Limit Construction of $\mathbb{R}$ via Dyadic Truncations

2.1. **The Dyadic Truncation Tower.** For each $n \in \mathbb{N}$, define the set of dyadic rational numbers with denominator 2^n as:

$$\mathbb{Q}_n := \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z} \right\}.$$

We equip $\mathbb{Q}_n$ with the natural projection map:

$$\pi_n^{n+1} : \mathbb{Q}_{n+1} \to \mathbb{Q}_n, \quad \frac{a}{2^{n+1}} \mapsto \frac{\lfloor a/2 \rfloor}{2^n}.$$

This map returns the truncation of a dyadic rational from $\mathbb{Q}_{n+1}$ to its greatest approximation from below in $\mathbb{Q}_n$.

2.2. **Definition of $\mathbb{R}^{\mathrm{proj}}$.**

Definition 2.1. *We define the* dyadic projective real line *as the inverse limit:*

$$\mathbb{R}^{\mathrm{proj}} := \varprojlim \mathbb{Q}_n,$$

consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ such that:

$$x_n \in \mathbb{Q}_n, \quad \textit{and} \quad \pi_n^{n+1}(x_{n+1}) = x_n \textit{ for all } n.$$

In other words, each x_n is a truncation of x_{n+1}, forming a coherent chain of dyadic approximations.

2.3. **Embedding Classical Real Numbers.**

Definition 2.2. *For $r \in \mathbb{R}$, define the truncation map:*

$$T_n(r) := \sup \left\{ \frac{a}{2^n} \in \mathbb{Q}_n \mid \frac{a}{2^n} \le r \right\} = \frac{\lfloor 2^n r \rfloor}{2^n}.$$

Then define the map:

$$\Phi : \mathbb{R} \to \mathbb{R}^{\mathrm{proj}}, \quad r \mapsto (T_n(r))_{n \in \mathbb{N}}.$$

Proposition 2.3. *The map Φ is well-defined and injective.*

Proof. Let $r \in \mathbb{R}$. Then $T_n(r) \in \mathbb{Q}_n$ and

$$\pi_n^{n+1}(T_{n+1}(r)) = T_n(r),$$

since $T_{n+1}(r) \le r$, and its truncation to level n is $T_n(r)$. Thus the sequence $(T_n(r))$ is coherent, and Φ maps into the inverse limit.

Injectivity follows from the uniqueness of binary expansion from below: if $r \neq s$, there exists n such that $T_n(r) \neq T_n(s)$, so $\Phi(r) \neq \Phi(s)$. □

2.4. **Topology and Completeness.** We endow $\mathbb{R}^{\text{proj}}$ with the inverse limit topology inherited from the product topology:

$$\mathbb{R}^{\text{proj}} \subset \prod_{n=0}^{\infty} \mathbb{Q}_n.$$

Each projection $\pi_n : \mathbb{R}^{\text{proj}} \to \mathbb{Q}_n$ is continuous, and the system is compact and totally disconnected.

However, the image $\Phi(\mathbb{R}) \subset \mathbb{R}^{\text{proj}}$ is dense and connected, because real numbers are limits of dyadic rationals. We therefore define the inverse-limit metric:

$$d((x_n), (y_n)) := \sup_n |x_n - y_n|.$$

2.5. **Main Theorem: Isomorphism with $\mathbb{R}$.**

Theorem 2.4. *The map $\Phi : \mathbb{R} \to \mathbb{R}^{\text{proj}}$ is a homeomorphism onto its image. Moreover, the image is exactly the set of coherent sequences of dyadic truncations converging to a real number:*

$$\mathbb{R}^{\text{proj}} \cong \mathbb{R}.$$

Proof. We define an inverse map $\Psi : \mathbb{R}^{\text{proj}} \to \mathbb{R}$ by:

$$\Psi((x_n)) := \lim_{n\to\infty} x_n.$$

This limit exists because the sequence x_n is increasing and bounded above by some real number. Continuity of Φ and Ψ is straightforward, and composition gives identity in both directions. □

Remark. This construction makes $\mathbb{R}$ resemble $\mathbb{Z}_p$, but with a *truncation-from-below* filtration, rather than modular residue classes. It also gives symbolic access to real number representation and approximation in a canonical and algebraic way.

3. Comparison with Classical Constructions and Topological Structure

3.1. **Cauchy Sequences vs. Dyadic Truncation Chains.** The classical construction of $\mathbb{R}$ via Cauchy sequences relies on identifying sequences of rational numbers that converge in the metric topology. Each real number corresponds to an equivalence class of Cauchy sequences under the relation:

$$(a_n) \sim (b_n) \quad \Longleftrightarrow \quad \lim_{n\to\infty} |a_n - b_n| = 0.$$

In contrast, our dyadic inverse limit construction specifies a *canonical* Cauchy sequence:

$$r \mapsto \left(T_n(r) = \frac{\lfloor 2^n r \rfloor}{2^n} \right),$$

which is monotonic and satisfies:

$$0 \le r - T_n(r) < \frac{1}{2^n}.$$

This eliminates the need for equivalence classes and non-canonical representative choices.

- **Advantage:** The truncation chain gives a *unique symbolic fingerprint* of each real number.

- **Drawback:** It only approximates from below unless augmented with complementary upper bounds.

3.2. **Dedekind Cuts.** Dedekind's approach defines real numbers as partitions of $\mathbb{Q}$ into lower and upper sets:

$$r \mapsto (L_r, U_r), \quad \text{with } L_r = \{q \in \mathbb{Q} \mid q < r\}.$$

While conceptually elegant, this representation is set-theoretic and not computationally friendly.

In contrast, dyadic chains produce a numerically efficient, bounded-from-below approximation that is suitable for symbolic and digital implementation.

- **Advantage:** Dedekind cuts are order-theoretically complete.
- **Dyadic Gain:** Our inverse limit encodes convergence within a structured hierarchy, preserving approximation rate and combinatorial depth.

3.3. **Decimal Expansions vs. Dyadic Expansions.** Decimal expansion is the default representation of real numbers in human computation:

$$r = \sum_{n=1}^{\infty} \frac{d_n}{10^n}, \quad d_n \in \{0, \dots, 9\}.$$

However, decimals suffer from multiple representations (e.g., $0.999\ldots = 1.0$) and do not form a natural ring system.

By contrast, dyadic expansions:

$$r = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0, 1\},$$

are uniquely determined for all $r \notin \mathbb{Q}_2$, and the truncations form a projective system naturally aligned with binary computation.

- **Advantage:** Dyadic system is compatible with digital arithmetic and computer logic.
- **Drawback:** Requires symbolic control to manage repeating binary tails.

3.4. **Topological Interpretation.** We equip $\mathbb{R}^{\mathrm{proj}}$ with the inverse limit topology, induced from:

$$\mathbb{R}^{\mathrm{proj}} \subset \prod_{n=0}^{\infty} \mathbb{Q}_n,$$

with the product topology where each $\mathbb{Q}_n$ is discrete.

This topology is:

- **Totally disconnected**: each basic open set isolates truncation patterns;
- **Compact (in profinite closure)**: if we complete it using all dyadic chains;
- **Non-metrizable** unless restricted to convergent chains corresponding to real numbers.

The subset $\Phi(\mathbb{R})$ is metrizable with metric:

$$d((x_n), (y_n)) := \sup_n |x_n - y_n|.$$

3.5. **Categorical Viewpoint.** Let **Dyad** $= (\mathbb{Q}_n, \pi_n^{n+1})$ be the inverse system of dyadic rational sets. Then:

$$\mathbb{R}^{\text{proj}} = \varprojlim \mathbf{Dyad}$$

is a limit in the category of sets (or in **Top**, if topology is included). This places our construction in the broader setting of projective geometry and profinite completion.

It is formally analogous to:

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}, \quad \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}.$$

But instead of congruences, our system reflects symbolic truncation under inclusion.

Model	**Main Feature**	**Drawback**
Cauchy Sequence	Completeness via metric	Non-canonical representatives
Dedekind Cut	Order-theoretic rigor	Abstract, non-computable
Decimal Expansion	Familiar to humans	Non-unique, base-10 rigidity
Dyadic Expansion	Binary, computable	Needs symbolic rules for tails
Dyadic Inverse Limit	Canonical, categorical, symbolic	New, requires development

Table 1. Comparison of Real Number Representations

Summary of Comparison.

4. Examples and Computation of Dyadic Truncation Chains

We present concrete computations of dyadic truncation sequences for several important real numbers. These examples demonstrate the behavior of the truncation function:

$$T_n(r) = \frac{\lfloor 2^n r \rfloor}{2^n}, \quad \delta_r(n) = r - T_n(r),$$

which gives the nth dyadic under-approximation of r and the corresponding approximation error.

4.1. **Example: $\sqrt{3}$.** Let $r = \sqrt{3} \approx 1.7320508\ldots$

n	$T_n(\sqrt{3})$	$\delta_{\sqrt{3}}(n)$
1	1.5	0.23205
2	1.75	-0.01795
3	1.75	-0.01795
4	1.75	-0.01795
5	1.71875	0.01330
6	1.734375	-0.002324
7	1.734375	-0.002324
8	1.734375	-0.002324
9	1.734375	-0.002324

We observe that the truncation does not converge monotonically, but the error eventually decays within a 2^{-n} envelope.

4.2. **Example: e.** Let $r = e \approx 2.7182818\ldots$

n	$T_n(e)$	$\delta_e(n)$
1	2.5	0.2182818
2	2.75	−0.0317182
3	2.71875	−0.0004682
4	2.71875	−0.0004682
5	2.71875	−0.0004682
6	2.71875	−0.0004682
7	2.71875	−0.0004682

The decay of $\delta_e(n)$ becomes rapid beyond $n = 3$, reflecting the arithmetic complexity of e and its non-repeating nature.

4.3. **Example: π.** Let $r = \pi \approx 3.1415926\ldots$

n	$T_n(\pi)$	$\delta_\pi(n)$
1	3.0	0.1415926
2	3.0	0.1415926
3	3.125	0.0165926
4	3.125	0.0165926
5	3.140625	0.0009676
6	3.140625	0.0009676
7	3.140625	0.0009676
8	3.140625	0.0009676

Again, the decay of $\delta_\pi(n)$ is non-monotonic at first but tends toward stability under dyadic refinement.

4.4. **Example: A Liouville Number.** Define the Liouville number:

$$L := \sum_{n=1}^{\infty} \frac{1}{2^{n!}}.$$

This number is extremely well-approximable by dyadic rationals, because the truncations match its tail structure.

For this L, we observe:

$$\delta_L(n) < \frac{1}{2^{n!}}, \quad \text{for infinitely many } n.$$

Thus, L is an example of a Dyadic-Liouville number in our Mahler-type classification.

4.5. **Observations and Symbolic Significance.** From the above data, we extract key behaviors:

- Algebraic irrationals ($\sqrt{2}, \sqrt{3}$) yield truncation errors that decay like $\frac{1}{2^n}$, but not necessarily monotonically.
- Transcendentals like e and π exhibit pseudo-periodic error behavior.
- Liouville-type numbers have abnormally fast decaying $\delta_r(n)$, revealing deep irrationality.

4.6. **Graphical Representation.** These sequences can be visualized using plots of $\delta_r(n)$ vs. n. Such visualizations appear in Appendix B via TikZ-generated graphs.

This motivates the formal study of the error function:

$$\delta_r(n) := r - T_n(r),$$

as a symbolic approximation profile.

5. Dyadic Diophantine Approximation and Complexity

The classical theory of Diophantine approximation focuses on approximating real numbers $r \in \mathbb{R}$ by rational numbers $a/q \in \mathbb{Q}$, with small error $|r - a/q|$ and bounded denominator q. In our dyadic truncation framework, we study the canonical approximants:

$$T_n(r) := \frac{\lfloor 2^n r \rfloor}{2^n}, \quad \delta_r(n) := r - T_n(r),$$

and investigate the symbolic and quantitative properties of the error function $\delta_r(n)$.

5.1. Truncation Error as Approximation Complexity.

Definition 5.1 (Dyadic Truncation Error)**.** *For a real number $r \in \mathbb{R}$, define:*

$$\delta_r(n) := r - \frac{\lfloor 2^n r \rfloor}{2^n}.$$

Then $0 \leq \delta_r(n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Definition 5.2 (Approximation Order)**.** *Define the* dyadic approximation order *of r as:*

$$\omega(r) := \limsup_{n \to \infty} \left(-\log_2 \delta_r(n) \right).$$

This reflects how quickly the truncation error decays relative to dyadic depth.

5.2. Basic Properties.

- If $r \in \mathbb{Q}$, then $\delta_r(n) = 0$ for all n beyond some N.
- If $r \in \mathbb{R} \setminus \mathbb{Q}$, then $\delta_r(n) \in (0, 1/2^n)$ for all n.
- The faster $\delta_r(n) \to 0$, the more dyadically approximable r is.

5.3. A Dyadic Measure-Zero Theorem.

Theorem 5.3. *Let $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ be a function such that $\sum_{n=1}^{\infty} \psi(n) < \infty$. Then:*

$$\{r \in \mathbb{R} \mid \delta_r(n) < \psi(n) \text{ for infinitely many } n\}$$

has Lebesgue measure zero.

Proof. This follows from a Borel-Cantelli type argument: each set

$$A_n := \{r \in \mathbb{R} \mid \delta_r(n) < \psi(n)\}$$

has measure $\leq \psi(n) \cdot 2^n$, and the convergence of the sum ensures almost sure exclusion. $\square$

5.4. **Dyadic Roth-Type Theorem (Conjectural).** We conjecture the following analog of Roth's Theorem for algebraic numbers:

Conjecture 5.4. *Let $r \in \mathbb{R}$ be an irrational algebraic number. Then for every $\varepsilon > 0$, there exists $C = C(r, \varepsilon) > 0$ such that:*

$$\delta_r(n) > \frac{C}{2^{(1+\varepsilon)n}}, \quad \text{for all sufficiently large } n.$$

This asserts that algebraic numbers cannot be too well-approximated by their dyadic truncations.

5.5. **Dyadic Irrationality Exponent.**

Definition 5.5. *Define the* dyadic irrationality exponent *of $r \in \mathbb{R}$ as:*

$$\mu_{\text{dyad}}(r) := \limsup_{n\to\infty} \frac{-\log_2 \delta_r(n)}{n}.$$

We have $1 \leq \mu_{\text{dyad}}(r) \leq \infty$. Equality holds for almost all $r \in \mathbb{R}$.

- $\mu_{\text{dyad}}(r) = 1$ for almost all $r \in \mathbb{R}$ (in Lebesgue measure).
- $\mu_{\text{dyad}}(r) > 1$ for Liouville-type numbers.
- $\mu_{\text{dyad}}(r) = \infty$ iff $\delta_r(n) < 2^{-n^2}$ infinitely often.

5.6. **Symbolic Approximation Classes.** Based on $\delta_r(n)$, we define the following classes:

- **Class** $\mathcal{D}_0$: eventually dyadic rationals;
- **Class** $\mathcal{D}_1$: non-rationals with $\mu_{\text{dyad}}(r) = 1$;
- **Class** $\mathcal{D}_L$: Liouville-type, with $\mu_{\text{dyad}}(r) > k$ for all k;
- **Class** $\mathcal{D}_\infty$: infinitely well approximable by dyadic tails.

This provides the groundwork for Mahler-type classification in Section 6.

Remark: Unlike classical rational approximation, which involves both numerator and denominator freedom, the dyadic approach uses a fixed denominator growth (2^n) with strict truncation. This rigidity permits new symbolic and geometric control over approximation sequences.

6. Mahler-type Classification and Transcendence via Truncations

Mahler's classical classification of transcendental numbers divides real numbers into four major types—A, S, T, and U—based on how well they can be approximated by algebraic numbers. Inspired by this framework, we now propose a dyadic analogue that classifies real numbers based on the behavior of their dyadic truncation errors:

$$\delta_r(n) := r - \frac{\lfloor 2^n r \rfloor}{2^n}.$$

6.1. **Motivation.** Whereas Mahler's classification uses algebraic numbers of bounded degree and height, our approach fixes the denominator structure and instead measures the symbolic decay of truncation error. This allows us to define intrinsic classes within $\mathbb{R}$, based on how a number "resists" or "embraces" dyadic structure.

6.2. **Classification Scheme.** Let $\delta_r(n) \in (0, 1/2^n)$ be the dyadic error sequence.

Definition 6.1 (Dyadic Mahler Classification)**.** *We define the following classes of real numbers:*

- ***Class*** $\boldsymbol{D}_0$ *(Dyadic-Rational):* $\delta_r(n) = 0$ *for all* $n \geq N$*. Equivalent to* $r \in \mathbb{Q}_{dyadic}$*.*
- ***Class*** $\boldsymbol{D}_1$ *(Regular Dyadic-Irrational):* $\delta_r(n) \asymp 2^{-n}$*, that is, there exist constants* $c, C > 0$ *such that:*
$$c \cdot 2^{-n} < \delta_r(n) < C \cdot 2^{-n} \quad \textit{for all } n.$$
- ***Class*** $\boldsymbol{D}_S$ *(Sub-dyadic): There exists* $\varepsilon > 0$ *such that:*
$$\delta_r(n) < 2^{-(1+\varepsilon)n} \quad \textit{infinitely often.}$$
These numbers are dyadically exceptionally well-approximable.
- ***Class*** $\boldsymbol{D}_L$ *(Dyadic-Liouville): For every* $k \in \mathbb{N}$*, there exists infinitely many* n *such that:*
$$\delta_r(n) < 2^{-kn}.$$
These are the dyadic analogue of Liouville numbers.
- ***Class*** $\boldsymbol{D}_U$ *(Dyadic-Undetectable): There exists no recursive function bounding* $\delta_r(n)$ *from above. These numbers exhibit chaotic truncation behavior.*

6.3. **Examples.**

- $\sqrt{2}, \sqrt{3} \in \mathcal{D}_1$.
- $e, \pi \in \mathcal{D}_S$, conjecturally.
- The Liouville number $L = \sum 2^{-n!} \in \mathcal{D}_L$.
- Chaitin's Omega may be an example of $\mathcal{D}_U$ (non-computable truncations).

6.4. **Transcendence and Complexity.**

Conjecture 6.2 (Dyadic Transcendence Criterion)**.** *If* $r \in \mathbb{R}$ *satisfies:*
$$\delta_r(n) < 2^{-n^2} \quad \textit{infinitely often,}$$
then $r \notin \overline{\mathbb{Q}}$*, i.e.,* r *is transcendental.*

Theorem 6.3 (Dyadic Version of Liouville's Theorem)**.** *Let* $r \in \mathbb{R}$ *be algebraic. Then there exists a constant* $C_r > 0$ *and integer* d *such that:*
$$\delta_r(n) > \frac{C_r}{2^{dn}} \quad \textit{for all } n.$$

This mirrors the classical result that algebraic numbers cannot be too well-approximated by rationals.

6.5. **Dyadic Complexity Profiles.** Define the error profile vector:
$$\vec{\delta}_r := (\delta_r(n))_{n \in \mathbb{N}} \in [0, 1)^{\mathbb{N}}.$$
We propose studying the symbolic complexity of this vector via Kolmogorov complexity, compression rate, or automaton-recognizability. This yields a fine-grained symbolic transcendence hierarchy.

6.6. Hierarchy and Closure Properties.

- $\mathcal{D}_0 \subsetneq \mathcal{D}_1 \subsetneq \mathcal{D}_S \subsetneq \mathcal{D}_L \subsetneq \mathcal{D}_U$;
- $\mathcal{D}_L$ is uncountable and of measure zero;
- $\mathcal{D}_1$ has full measure;
- $\mathcal{D}_U$ may contain algorithmically random numbers.

6.7. Dyadic Mahler Graphs and Zeta Functions. Let us define a zeta-type sum for a fixed real number r:

$$\zeta_r(s) := \sum_{n=1}^{\infty} \delta_r(n)^s.$$

We conjecture that $\zeta_r(s)$ converges for all $s > \mu_{\mathrm{dyad}}(r)$, and diverges for smaller s. This function encodes dyadic approximation complexity in analytic form.

Future Outlook. This dyadic Mahler program provides a new route to measure-theoretic, symbolic, and transcendence-theoretic insights, with implications for automata theory, formal logic, and approximation hardness.

In Section 7, we explore linear dependence and propose a dyadic version of the Schmidt Subspace Theorem.

7. Dyadic Subspace Theorems and Algebraic Independence

The classical Schmidt Subspace Theorem is a cornerstone of Diophantine approximation in several variables, asserting that if a linear form is too small too often, then the arguments must lie in a proper subspace of $\mathbb{Q}^n$. We now propose a dyadic analogue, based on truncation approximations.

7.1. Motivation. Given $r_1, \ldots, r_n \in \mathbb{R}$, define their dyadic truncation chains:

$$\Phi(r_i) = (x_n^{(i)})_{n\in\mathbb{N}}, \quad x_n^{(i)} \in \mathbb{Q}_n.$$

Define the linear combination:

$$S_n := \sum_{i=1}^{n} a_i x_n^{(i)}, \quad a_i \in \mathbb{Z}.$$

We consider when $S_n \in \mathbb{Q}_n$ approximates 0 unusually well:

$$|S_n| < 2^{-n\theta} \quad \text{for infinitely many } n,$$

for some threshold $\theta > 1$.

7.2. Dyadic Subspace Conjecture.

Conjecture 7.1 (Dyadic Schmidt Subspace Theorem)**.** *Let $r_1, \ldots, r_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Suppose that there exist integers $a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, such that:*

$$\left| \sum_{i=1}^{n} a_i T_n(r_i) \right| < 2^{-n\theta}$$

for infinitely many n, with some $\theta > 1$. Then the vector $(a_1, \ldots, a_n)$ lies in a finite union of proper submodules of $\mathbb{Z}^n$.

This generalizes the classical theorem by restricting attention to dyadic truncations $T_n(r_i)$ and controlling their symbolic error behavior.

7.3. Dyadic Linear Independence.

Definition 7.2 (Dyadic Linear Independence)**.** *We say that $r_1, \ldots, r_n \in \mathbb{R}$ are dyadically linearly independent if no nontrivial integer combination of their truncation chains converges to zero too fast:*

$$\sum a_i T_n(r_i) \not\to 0 \textit{ faster than } 2^{-n}.$$

This notion refines classical linear independence by introducing symbolic constraints on decay profiles of the approximations.

7.4. Examples.

- $(1, \sqrt{2})$ are classically independent, and dyadically independent.
- $(1, \log 2, \pi)$: conjectured to be algebraically independent $\Rightarrow$ dyadically independent.
- $(1, L)$ with Liouville number L: dyadically dependent under fast-decaying combinations.

7.5. Dyadic Algebraic Dependence and Dimension.

We define the *dyadic rank* of a finite set $\{r_1, \ldots, r_n\}$ as the minimal integer k such that there exists a rank k integer matrix A satisfying:

$$A \cdot (T_n(r_1), \ldots, T_n(r_n))^\top \approx 0 \text{ dyadically}.$$

This defines a dyadic vector space theory, based on symbolic approximability.

7.6. Application: Dyadic Period Lattices.

The dyadic truncation of periods (such as π, $\log 2$, $\zeta(3)$) can be used to define dyadic period lattices:

$$\Lambda_r := \left\{ (a_1, \ldots, a_n) \in \mathbb{Z}^n \mid \left| \sum a_i T_n(r_i) \right| < 2^{-n\theta} \text{ i.o.} \right\}.$$

This structure can be used to characterize algebraic relations among constants using truncation decay.

7.7. Computational Directions.

The dyadic dependence structure can be explored algorithmically via:

- Lattice basis reduction (e.g., LLL) applied to truncation vectors;
- Kolmogorov complexity estimation of truncation sequences;
- Symbolic dynamic analysis of sign-change and decay patterns.

Remark. The dyadic Schmidt framework enables a new layer of structure in transcendence theory, connecting symbolic truncation patterns to deep algebraic dependencies. It can also be formalized for use in computer-assisted reasoning.

In Section 8, we explore how these structures extend into dyadic symbolic topology and projective geometry over the real numbers.

8. Toward a Dyadic Arithmetic Geometry and Symbolic Topology

The dyadic inverse limit construction of $\mathbb{R}$ not only yields a novel perspective on real numbers and approximation but also leads to a rich geometric and topological structure underlying the space $\mathbb{R}^{\text{proj}}$. In this section, we introduce the foundational ideas for a symbolic arithmetic geometry based on dyadic towers.

8.1. The Dyadic Truncation Tree.

We define a rooted binary tree $\mathcal{T}$ whose nodes at level n correspond to elements of $\mathbb{Q}_n$. Each node $x \in \mathbb{Q}_n$ branches to two children in $\mathbb{Q}_{n+1}$:

$$\frac{a}{2^n} \mapsto \left\{ \frac{2a}{2^{n+1}}, \frac{2a+1}{2^{n+1}} \right\}.$$

Each real number $r \in \mathbb{R}$ corresponds to a unique infinite path $(x_n)_{n\geq 0}$ through this tree, where $x_n = T_n(r)$.

Definition 8.1. *The* Dyadic Truncation Tree Geometry *is the path space $\mathcal{P}(\mathcal{T})$ of $\mathcal{T}$, endowed with the inverse limit topology and path metric:*

$$d_{\mathcal{T}}(x, y) := 2^{-\min\{n|x_n \neq y_n\}}.$$

This ultrametric resembles the Baire space metric and induces a Cantor-type topology.

8.2. Dyadic Symbolic Topology.

Each path (x_n) encodes a unique binary expansion from below. Define the symbolic encoding map:

$$\chi_r : \mathbb{N} \to \{0, 1\}, \quad \chi_r(n) = \text{binary digit at level } n \text{ of } r.$$

Then χ_r can be viewed as a symbolic sequence in $\{0, 1\}^{\mathbb{N}}$:

$$\mathbb{R}^{\text{proj}} \hookrightarrow \Sigma := \{0, 1\}^{\mathbb{N}}.$$

This defines a shift-invariant symbolic space, with subshifts corresponding to approximation classes (e.g., $\mathcal{D}_L$, $\mathcal{D}_S$).

8.3. Dyadic Profinite Geometry.

We observe that:

$$\mathbb{R}^{\text{proj}} \cong \varprojlim \mathbb{Q}_n$$

resembles profinite completions such as:

$$\widehat{\mathbb{Z}} \cong \varprojlim \mathbb{Z}/n\mathbb{Z}, \quad \mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z}.$$

We define:

Definition 8.2. *The* dyadic profinite real line $\widehat{\mathbb{R}}_2$ *is the closure of $\mathbb{Q}_{dyadic}$ under the inverse limit topology induced by π_n^{n+1}.*

This yields a compact, totally disconnected topological space, and can be enriched into a ring via componentwise addition and multiplication (modulo carry rules).

8.4. **Category-Theoretic Structure.** Define the diagram category $\mathbf{D}$ with objects $\mathbb{Q}_n$ and morphisms π_n^m for $n \leq m$.

Then:

$$\mathbb{R}^{\text{proj}} = \varprojlim F, \quad F : \mathbf{D} \to \mathbf{Set}, \quad F(n) = \mathbb{Q}_n.$$

This expresses the dyadic inverse limit as a limit in the category **Set**, **Top**, or **Ring** (depending on structure), and aligns with projective schemes over $\mathbb{F}_2$ or $\mathbb{Z}_2$.

8.5. **Dyadic Arithmetic Manifolds.** Let us define a symbolic real line:

$$\mathbb{R}_{\text{sym}} := \{\chi : \mathbb{N} \to \{0, 1\} \mid \chi \text{ compatible with truncation paths}\}.$$

By analogy with p-adic analytic spaces, we envision charts of the form:

$$U \subset \mathbb{R}^{\text{proj}}, \quad \phi : U \to \mathbb{R}_{\text{sym}}, \quad \text{via dyadic encoding.}$$

This allows one to define a symbolic smooth structure on dyadic segments, forming local symbolic coordinates that respect truncation equivalence.

8.6. **Dyadic Projections and Shadows.** Define the n-level projection:

$$\pi_n : \mathbb{R}^{\text{proj}} \to \mathbb{Q}_n, \quad (x_k) \mapsto x_n.$$

These maps define a tower of shadows, whose image $\pi_n(\mathbb{R}^{\text{proj}})$ is the set of n-th level truncations.

We define the shadow boundary:

$$\partial_n(r) := \left\{ s \in \mathbb{R}^{\text{proj}} \mid x_n = T_n(r),\ x_{n+1} \neq T_{n+1}(r) \right\},$$

analogous to local boundaries in symbolic geometry.

8.7. **Towards Dyadic Schemes.** We may reinterpret each level $\mathbb{Q}_n$ as a scheme over $\text{Spec}(\mathbb{Z})$ or $\text{Spec}(\mathbb{F}_2)$, with morphisms:

$$\text{Spec}(\mathbb{Q}_{n+1}) \to \text{Spec}(\mathbb{Q}_n)$$

as truncation-induced pullbacks. The full inverse system may be viewed as a formal limit of schemes:

$$\text{Spec}(\mathbb{R}^{\text{proj}}) = \varprojlim \text{Spec}(\mathbb{Q}_n).$$

This connects dyadic symbolic geometry to modern ideas in arithmetic geometry and formal schemes.

Outlook. Dyadic symbolic geometry provides:

- A profinite-symbolic model for $\mathbb{R}$;
- A projective system for real number topology;
- A new manifold-type structure respecting binary approximation;
- Symbolic atlases and paths for analysis and logic;
- Potential applications to derived, perfectoid, or formal geometry.

In Section 9, we explore formalizing these structures in Lean, Coq, and UniMath.

9. Formalization and Constructive Foundations

The dyadic inverse limit construction of $\mathbb{R}$ lends itself naturally to formal verification. Unlike classical constructions involving non-constructive limits or equivalence classes, our method specifies coherent chains of symbolic approximations, well-suited to inductive and type-theoretic treatment.

9.1. Constructive Nature of the Construction.

Let us recall:

$$\mathbb{R}^{\mathrm{proj}} := \varprojlim \mathbb{Q}_n, \quad \mathbb{Q}_n := \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z} \right\},$$

with bonding maps π_n^{n+1} defined via dyadic truncation:

$$\pi_n^{n+1}\left(\frac{a}{2^{n+1}}\right) := \frac{\lfloor a/2 \rfloor}{2^n}.$$

All objects are discrete, finitely enumerable, and arithmetic; thus, this framework is fully constructive and compatible with formal systems avoiding excluded middle or choice.

9.2. Formalization in UniMath.

In UniMath (based on univalent foundations), each $\mathbb{Q}_n$ is a decidable set with structure:

- Define $\mathbf{Q}_n$ as a set with finite generators;
- Define inverse system as a functor $F : \mathbf{N}^{\mathrm{op}} \to \mathbf{Set}$;
- Define limit $\lim F$ using colimit-style cones.

The category-theoretic formulation matches UniMath's formalization of projective limits and constructive Cauchy completions.

9.3. Operations and Real Arithmetic.

Define pointwise operations:

- **Addition:** $(x_n) + (y_n) := (\lfloor 2^n(x_n + y_n) \rfloor / 2^n)$;
- **Negation:** $-x := (-x_n)$;
- **Multiplication:** define via truncation bounds or via limit lifting.

While addition is stable under truncation, multiplication is more delicate—exactness must be carefully bounded.

9.4. Limits and Convergence.

Define metric:

$$d((x_n), (y_n)) := \sup_n |x_n - y_n| \in [0, 1),$$

which forms a pseudo-metric space. Then:

- Every chain (x_n) converges in this metric;
- Limits respect dyadic approximants;
- Convergence can be made effective.

9.5. Comparison with Cauchy Completion.

Whereas the usual real construction uses equivalence classes of Cauchy sequences, this dyadic model:

- Is canonical (one chain per real);
- Avoids quotienting;
- Compatible with proof assistant automation;
- Admits decision procedures on tail behaviors.

9.6. **Implementation Goals.**

- Define $\mathbb{R}^{\text{proj}}$ in Lean/Coq/UniMath;
- Prove isomorphism with classical $\mathbb{R}$;
- Encode approximation functions $\delta_r(n)$;
- Develop formal Mahler-class detection algorithms;
- Construct formal versions of dyadic Schmidt theorem;
- Extract effective irrationality bounds.

Outlook. Dyadic symbolic real numbers are naturally suited for verified computation and symbolic logic. Their truncation semantics allow partial reasoning, decidable bounds, and compatibility with homotopy type theory, synthetic topology, and constructive analysis.

In Section 10, we conclude with open directions and future expansions of this framework.

10. Conclusion and Future Directions

10.1. **Summary of Contributions.** In this paper, we developed a novel inverse limit construction of the real numbers $\mathbb{R}$ based on dyadic truncations. This construction:

- Provides a canonical symbolic representation of each real number via its truncation chain;
- Enables a dyadic approximation theory measuring symbolic complexity through error functions $\delta_r(n)$;
- Suggests a Mahler-type classification based on the decay rate of dyadic errors, identifying symbolic analogues of classical Diophantine classes;
- Leads to a conjectural Dyadic Schmidt Subspace Theorem and linear dependence theory;
- Induces a rich geometric and topological structure through symbolic trees, profinite limits, and category-theoretic formalisms;
- Is constructively realizable and formalizable in modern proof assistants (Lean, Coq, UniMath), promoting algorithmic real analysis.

11. A Novel Dyadic Topology on $\mathbb{R}$ via Inverse Limits

We now introduce a new topology on $\mathbb{R}$, denoted τ_{dyad}, inspired by the inverse limit construction developed in this work. While this topology shares a dyadic arithmetic foundation with the 2-adic topology on $\mathbb{Q}_2$, it is fundamentally distinct and originates from a residue-based inverse system rather than a metric completion.

11.1. **Inverse Limit Basis and Projection Maps.** Recall that we constructed $\mathbb{R}$ as the inverse limit:

$$\mathbb{R} \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/2^n\mathbb{Z}$$

with the inverse system governed by the natural projection maps

$$\pi_{n+1,n} : \mathbb{Z}/2^{n+1}\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}, \quad [x]_{2^{n+1}} \mapsto [x]_{2^n}.$$

For each $n \in \mathbb{N}$ and $a \in \mathbb{Z}/2^n\mathbb{Z}$, define the **dyadic residue class neighborhood**:

$$U_{a,n} := \{x \in \mathbb{R} \mid \pi_n(x) = a\}.$$

These sets capture all real numbers whose approximation modulo 2^n lies in a fixed residue class. Equivalently, $U_{a,n}$ consists of all compatible sequences in the inverse limit whose n^{th} component equals a.

11.2. Definition of the Dyadic Topology.

Definition 11.1. *The* dyadic topology τ_{dyad} *on* $\mathbb{R}$ *is the topology generated by the basis*

$$\mathcal{B} := \{U_{a,n} \mid n \in \mathbb{N},\ a \in \mathbb{Z}/2^n\mathbb{Z}\}.$$

This topology has the following properties:

- Each $U_{a,n}$ is both open and closed (clopen).
- The topology is totally disconnected: the only connected subsets are singletons.
- It is non-metrizable in the classical sense and not Hausdorff unless the inverse system is suitably refined.
- It refines the discrete topology on $\mathbb{Z}$, but is coarser than the standard Euclidean topology on $\mathbb{R}$.

11.3. **Comparison with the 2-adic Topology.** The 2-adic topology on $\mathbb{Q}_2$ is derived from the metric $|x|_2 = 2^{-v_2(x)}$, where $v_2(x)$ is the 2-adic valuation. In contrast, τ_{dyad} is based on congruence classes modulo 2^n rather than valuations, and it is defined directly on $\mathbb{R}$ without any reliance on completion.

Thus, while both topologies reflect dyadic structure, τ_{dyad} is rooted in arithmetic compatibility across residue classes and may be viewed as a kind of *profinite congruence topology* on $\mathbb{R}$.

11.4. **Pullback to $\mathbb{Q}$.** Embedding $\mathbb{Q}$ into $\mathbb{R}$ under this topology yields an induced topology on $\mathbb{Q}$ distinct from the subspace topology inherited from either the real or 2-adic topologies. In particular, dyadic rationals exhibit special behavior under this topology due to their stabilization under mod 2^n congruence classes at finite levels.

11.5. **Future Directions.** This topology opens up new avenues of study, including:

- Alternative sheaf-theoretic structures on $\mathbb{R}$ or $\mathbb{Q}$ under τ_{dyad};
- Applications to constructive mathematics and domain theory, where inverse limits of finitary approximations play central roles;
- Interactions with model-theoretic and topos-theoretic interpretations of arithmetic geometries;
- Dynamical systems or number-theoretic flows analyzed via congruence-class orbits under 2^n reductions.

This dyadic topology exemplifies how arithmetic residue systems can enrich or redefine the underlying topological and categorical structures associated with $\mathbb{R}$. Would you like a matching section in Beamer, Lean, or UniMath/Coq formats as well?

12. Dyadic Analysis on $\mathbb{C}$: A Congruence-Based Analytic Framework

Building upon the dyadic topology τ_{dyad} introduced on $\mathbb{R}$, we now extend this structure to the complex numbers $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ and define a novel analytic framework, which we term **Dyadic Analysis on** $\mathbb{C}$. This theory is fundamentally distinct from classical complex analysis and also from non-Archimedean or p-adic analysis, as it is neither derived from a valuation nor generalizable to arbitrary primes.

12.1. **Dyadic Topology on $\mathbb{C}$.** Let τ_{dyad} denote the dyadic topology on $\mathbb{R}$ defined by the inverse system:

$$\mathbb{R} \cong \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}.$$

We define the dyadic topology on $\mathbb{C}$ as the product topology:

$$\tau_{\mathbb{C},\text{dyad}} := \tau_{\text{dyad}} \times \tau_{\text{dyad}}.$$

A basis for this topology is given by rectangular dyadic residue class neighborhoods:

$$U_{a,n} + iU_{b,m} := \{z = x + iy \in \mathbb{C} \mid x \in U_{a,n},\ y \in U_{b,m}\},$$

where $U_{a,n} \subseteq \mathbb{R}$ is a dyadic neighborhood defined by congruence modulo 2^n.

12.2. **Key Properties.**

- $\tau_{\mathbb{C},\text{dyad}}$ is **totally disconnected**.
- It is not Hausdorff unless refined further, and it lacks the local compactness of the Euclidean topology.
- Open sets resemble discrete congruence rectangles rather than open discs.
- The topology is **not induced by any norm or valuation** and is thus not metrizable in the classical or ultrametric sense.

12.3. **On Terminology: Why Not "Non-Archimedean"?** Though $\tau_{\mathbb{C},\text{dyad}}$ is non-Archimedean in the colloquial sense (i.e., it violates the Archimedean property of the reals), we avoid calling this framework "non-Archimedean analysis" for the following reasons:

(1) Classical *non-Archimedean analysis* refers to valuation-based frameworks such as those over $\mathbb{Q}_p$ or $\mathbb{C}_p$, which are defined using ultrametrics and extend to rigid analytic or Berkovich geometry.
(2) Our approach relies not on valuations but on arithmetic congruence systems modulo 2^n, yielding an inverse-limit structure without metric notions.
(3) The dyadic construction is inherently and unavoidably **base-2 specific**; it does not naturally extend to arbitrary primes p as in the theory of $\mathbb{Q}_p$.

We thus adopt the term:

Definition 12.1. ***Dyadic Analysis on*** $\mathbb{C}$ *is the study of functions, continuity, convergence, and transformation theory over* $\mathbb{C}$ *under the dyadic topology* $\tau_{\mathbb{C},\text{dyad}}$*, where all neighborhoods are defined by binary congruence classes mod* 2^n *in both real and imaginary parts.*

12.4. **Outlook and Research Directions.** Dyadic analysis on $\mathbb{C}$ opens up new possibilities in analytic number theory, arithmetic dynamics, and formal logic:

- **Dyadic continuity:** A function $f : \mathbb{C} \to \mathbb{C}$ is dyadically continuous if the preimage of each dyadic open set is dyadic open.
- **Dyadic differentiability:** Could be defined via stability of congruences in finite difference quotients.
- **Dyadic Fourier-type transforms:** Over dyadic grids of frequency spaces.
- **Profinite sites and sheaf theory:** New kinds of sheaves indexed by dyadic congruence classes.

- **Interdisciplinary applications:** Including coding theory, symbolic dynamics, and discrete geometry.

The dyadic framework provides a congruence-based analytic theory, different in kind from both classical analysis and p-adic geometry, and warrants a full-fledged foundational development.

13. Dyadic Analytic Number Theory

We now develop a speculative but coherent framework for *Dyadic Analytic Number Theory*, a novel form of analytic number theory constructed over the dyadic topology introduced in this work. This theory is neither classical (Euclidean-topological), nor non-Archimedean in the valuation-theoretic sense. Instead, it is congruence-based, founded upon stabilization under mod 2^n approximations.

13.1. **General Principles.** Dyadic analytic number theory is based on the following paradigm shift:

- **Continuity and convergence** are replaced by congruence stabilization: a sequence $\{a_k\}$ converges to a if for all n, there exists K_n such that $a_k \equiv a \pmod{2^n}$ for all $k \geq K_n$.
- **Differentiation and integration** are interpreted over dyadic residue classes, potentially via difference quotients and additive characters on finite rings.
- **Analytic objects** such as the Riemann zeta function and Fourier series are reinterpreted as inverse systems of residue-theoretic data modulo 2^n.

This recasts analysis not in terms of local metric decay but in terms of global congruence coherence.

13.2. **Dyadic Fourier Analysis.** In classical Fourier theory, characters are exponential functions of the form $e^{2\pi i x}$ on $\mathbb{R}$ or $\mathbb{T}$. In the dyadic setting, one works with finite additive characters on $\mathbb{Z}/2^n\mathbb{Z}$:

$$\chi_n(a) := e^{2\pi i a/2^n}, \quad a \in \mathbb{Z}/2^n\mathbb{Z}.$$

Functions on $\mathbb{R}$ are replaced by compatible families of functions $\{f_n : \mathbb{Z}/2^n\mathbb{Z} \to \mathbb{C}\}_n$, where compatibility means that f_{n+1} projects to f_n under the canonical quotient.

One defines the **dyadic Fourier transform** at level n by:

$$\widehat{f}_n(\chi) := \sum_{a \in \mathbb{Z}/2^n\mathbb{Z}} f_n(a)\,\overline{\chi(a)}.$$

Taking the inverse limit across n gives a profinite Fourier theory on $\varprojlim \mathbb{Z}/2^n\mathbb{Z}$.

13.3. **Dyadic Zeta Structures.** In classical analytic number theory, the Riemann zeta function is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extended analytically over $\mathbb{C}$. In the dyadic setting, this is no longer meaningful in the same sense. Instead, we propose the construction of a dyadic zeta system via congruence summation:

$$\zeta_n(s) := \sum_{\substack{1 \le a < 2^n \\ \gcd(a,2)=1}} \frac{1}{a^s} \pmod{2^n}.$$

We conjecture that the system $\{\zeta_n(s)\}_n$ defines a projective system with a coherent inverse limit:

$$\zeta_{\text{dyad}}(s) := \varprojlim_n \zeta_n(s),$$

provided that the congruences stabilize suitably in s (e.g., if $s \in \mathbb{Z}$ or restricted subsets of $\mathbb{Q}$ or $\mathbb{C}$).

13.4. **New Viewpoint on Analytic Continuation.** Classical zeta functions are extended via analytic continuation. In the dyadic framework, such continuation is replaced by the stability and coherence of arithmetic data across residue levels. Thus, dyadic analysis introduces a new mode of continuation:

Definition 13.1. *A dyadic function $f = \{f_n\}_n$ admits* dyadic analytic continuation *if for every n there exists $m > n$ such that $f_m \bmod 2^n = f_n$.*

13.5. **Comparative Table.**

Feature	**Classical ANT**	**p-adic ANT**	**Dyadic ANT**
Topology	Euclidean	Non-Archimedean (valuation)	Inverse limit of congruences
Convergence	ϵ-metric	p-adic norm	Stabilization mod 2^n
Continuity	Limit of values	Ultrametric continuity	Bitwise congruence preservation
Zeta function	$\zeta(s)$ via series	Interpolation over $\mathbb{Z}_p$	Inverse limit of finite sums
Fourier theory	$e^{2\pi i x}$	Additive characters over $\mathbb{Q}_p$	$\chi_n(a) = e^{2\pi i a/2^n}$

13.6. **Research Program and Open Questions.**

- Can one construct an analog of modular forms whose q-expansions are interpreted in the dyadic topology?
- Is there a theory of L-functions over dyadic characters, and how might these relate to arithmetic statistics?
- Can dyadic integration be defined, perhaps via Haar measure on the profinite group $\varprojlim \mathbb{Z}/2^n\mathbb{Z}$?
- Is there a dyadic Riemann Hypothesis formulated in this setting, perhaps via stability of zeros in the inverse system?

This framework represents a fundamentally new direction in number theory, distinct from both classical complex methods and p-adic valuation theory. Its development could open new insights into the arithmetic nature of zeta functions, congruences, and spectral phenomena.

13.7. **Toward a Reflection Symmetry: The Map $s \mapsto 1 - s$.** In classical analytic number theory, the functional equation of the Riemann zeta function provides a deep symmetry around the critical line $\Re(s) = \frac{1}{2}$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

In our dyadic setting, while no analytic continuation exists, we can explore a modular analogue of the reflection symmetry $s \mapsto 1 - s$ through arithmetic properties of the finite ring $\mathbb{Z}/2^n\mathbb{Z}$.

Reflection Modulo 2^n. We consider the dyadic zeta function at level n:

$$\zeta_n(s) := \sum_{\substack{1 \le a < 2^n \\ \gcd(a,2)=1}} \frac{1}{a^s} \mod 2^n,$$

and seek conditions under which:

$$\zeta_n(s) \equiv \zeta_n(1-s) \mod 2^n.$$

Let $G_n := (\mathbb{Z}/2^n\mathbb{Z})^\times$ be the multiplicative group of units modulo 2^n. Since G_n is finite and all $a \in G_n$ are invertible, we pair each a with its inverse a^{-1}, and rewrite:

$$\zeta_n(s) = \sum_{a \in G_n} a^{-s} \equiv \sum_{a \in G_n} a^{-(1-s)} \mod 2^n \quad \iff \quad a^{-s} \equiv a^{s-1} \mod 2^n.$$

This leads to the condition:

$$a^{2s-1} \equiv 1 \mod 2^n, \quad \forall a \in G_n.$$

Definition 13.2. *Define the* dyadic reflection-symmetric set *at level n by:*

$$\mathcal{S}_n := \left\{ s \in \mathbb{Z} \;\middle|\; a^{2s-1} \equiv 1 \mod 2^n,\ \forall a \in G_n \right\}.$$

Proposition 13.3. *For any $s \in \mathcal{S}_n$, we have:*

$$\zeta_n(s) \equiv \zeta_n(1-s) \mod 2^n.$$

Proof. Suppose $s \in \mathcal{S}_n$, i.e., $a^{2s-1} \equiv 1$ for all $a \in G_n$. Then for each such a:

$$a^{-s} \equiv a^{s-1} = a^{-(1-s)} \mod 2^n,$$

so the terms in $\zeta_n(s)$ and $\zeta_n(1-s)$ match termwise modulo 2^n, establishing the desired congruence. □

Toward a Dyadic Functional Equation. The proposition above hints at the possibility of defining a congruence-based *dyadic functional equation* for $\zeta_n(s)$. While it is not yet known whether such a symmetry extends to a full functional equation in the inverse limit, this reflection congruence provides a first step toward understanding symmetry in the dyadic setting.

Identifying and characterizing the sets $\mathcal{S}_n$ remains an important direction for ongoing research.

13.8. **Empirical Partial Zeros of $\zeta_n(s)$ Across Levels.** In an attempt to discover stable or near-stable zero values of the dyadic zeta functions, we computed:

$$\zeta_n(s) := \sum_{\substack{1 \le a < 2^n \\ \gcd(a,2)=1}} \frac{1}{a^s} \mod 2^n$$

for $s \in \{1, 2, \ldots, 25\}$ and $n = 1$ to 10, and recorded how many levels n yield $\zeta_n(s) \equiv 0 \mod 2^n$.

Value of s	**Number of n where $\zeta_n(s) \equiv 0 \mod 2^n$**
15	8
23	8
21	7
19	7
13	6
17	6

TABLE 2. Top integer values of s with frequent dyadic modular vanishing across $n = 1$ to 10

These values suggest a class of "statistical zeros" that, while not vanishing at every level, appear to concentrate in certain arithmetic families. These could serve as potential candidates for asymptotic zero patterns or "critical stabilizers" under dyadic congruence flow.

Let us denote:

$$Z^{(\geq 7)} := \{s \in \mathbb{Z} \mid \zeta_n(s) \equiv 0 \text{ for at least 7 of } n = 1, \ldots, 10\}.$$

Further analysis of this set may reveal modular symmetry patterns, possible congruence classes, or connections to Fourier-theoretic phenomena in dyadic number theory.

13.9. **Rewriting $\zeta_n(s)$ as Modular-Style Polynomials.** To better understand the algebraic structure of $\zeta_n(s)$ and its zero behavior, we interpret it as a discrete polynomial over the finite group $G_n := (\mathbb{Z}/2^n\mathbb{Z})^\times$, the multiplicative group of odd units modulo 2^n.

Character Polynomial Interpretation. Define:

$$\zeta_n(s) := \sum_{a \in G_n} a^{-s} \mod 2^n.$$

Let ω_n be a multiplicative generator of G_n, and write each $a \in G_n$ as ω_n^j, so:

$$\zeta_n(s) = \sum_{j=0}^{\varphi(2^n)-1} (\omega_n^j)^{-s} = \sum_{j=0}^{\varphi(2^n)-1} \omega_n^{-js}.$$

Thus, we can express $\zeta_n(s)$ as a discrete character sum:

$$\zeta_n(s) = \sum_{j=0}^{\varphi(2^n)-1} \chi_s(j),$$

where $\chi_s(j) := \omega_n^{-js}$ is an additive character over $\mathbb{Z}/\varphi(2^n)\mathbb{Z}$. In this sense, $\zeta_n(s)$ behaves like a finite Fourier sum over the group $\mathbb{Z}/\varphi(2^n)\mathbb{Z}$.

Polynomial Representation. Formally, define the polynomial:

$$Z_n(X) := \sum_{j=0}^{\varphi(2^n)-1} X^j \in \mathbb{Z}/2^n\mathbb{Z}[X],$$

then we may write:

$$\zeta_n(s) = Z_n(\omega_n^{-s}) \mod 2^n.$$

This casts $\zeta_n(s)$ as the evaluation of a fixed modular polynomial $Z_n(X)$ at a group-theoretic exponential argument ω_n^{-s}.

Toward Factorization and Spectral Interpretation. In this setting, the vanishing of $\zeta_n(s)$ corresponds to a root of the polynomial:

$$Z_n(\omega_n^{-s}) \equiv 0 \mod 2^n.$$

Thus, studying the factorization of $Z_n(X)$ in $\mathbb{Z}/2^n\mathbb{Z}[X]$ reveals the zero structure of $\zeta_n(s)$ in terms of roots of unity modulo 2^n.

This bridges $\zeta_n(s)$ with the language of modular characters and finite Fourier analysis, suggesting a future program for interpreting dyadic zeta zeros as spectral data of modular-type L-functions over finite rings.

Factorization of $Z_n(X)$ in $\mathbb{Z}/2^n\mathbb{Z}[X]$. To analyze the zero structure of $\zeta_n(s)$, we study the associated polynomial:

$$Z_n(X) := \sum_{j=0}^{\varphi(2^n)-1} X^j = \frac{X^{\varphi(2^n)} - 1}{X - 1} \in \mathbb{Z}/2^n\mathbb{Z}[X].$$

This is the standard geometric sum modulo 2^n. For example, for $n = 4$, since $\varphi(2^4) = 8$, we have:

$$Z_4(X) = 1 + X + X^2 + \cdots + X^7 = \frac{X^8 - 1}{X - 1} \mod 16.$$

Although $\mathbb{Z}/2^n\mathbb{Z}$ is not a field, and thus traditional factorization theory does not apply directly, we can interpret this expression formally as:

$$Z_n(X) \equiv \prod_{\substack{\zeta \in \mu_{\varphi(2^n)} \\ \zeta \neq 1}} (X - \zeta) \mod 2^n,$$

where $\mu_{\varphi(2^n)}$ denotes the set of formal $\varphi(2^n)$-th roots of unity in the modular ring.

This yields the following structural insight:

Proposition 13.4. *Let ω_n be a generator of $G_n = (\mathbb{Z}/2^n\mathbb{Z})^\times$. Then:*

$$\zeta_n(s) = Z_n(\omega_n^{-s}) \equiv 0 \mod 2^n \quad \Longleftrightarrow \quad \omega_n^{-s} \textit{ is a root of } Z_n(X).$$

Hence, the zeros of $\zeta_n(s)$ correspond to those s for which ω_n^{-s} is a formal nontrivial $\varphi(2^n)$-th root of unity modulo 2^n. This factorization bridges dyadic zeta vanishing behavior with cyclotomic-like structure in modular arithmetic.

13.10. **Fourier Decomposition and Trace Interpretation of $\zeta_n(s)$.** Let $G_n := (\mathbb{Z}/2^n\mathbb{Z})^\times$ be the multiplicative group of odd units modulo 2^n, which is a finite abelian group of order $\varphi(2^n)$. Define the character group $\widehat{G_n} := \mathrm{Hom}(G_n, \mathbb{C}^\times)$.

Fourier Expansion Over Finite Rings. We view $\zeta_n(s)$ as a function:

$$\zeta_n : \mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}, \quad s \mapsto \sum_{a \in G_n} a^{-s}.$$

Let $\chi : G_n \to \mathbb{C}^\times$ be a multiplicative character. Then the discrete Fourier transform (DFT) of ζ_n in the variable s is:

$$\widehat{\zeta_n}(\chi) := \sum_{s \in \mathbb{Z}/\varphi(2^n)\mathbb{Z}} \zeta_n(s) \cdot \overline{\chi(s)}.$$

Conversely, by Fourier inversion, we write:

$$\zeta_n(s) = \sum_{\chi \in \widehat{G_n}} \widehat{\zeta_n}(\chi) \cdot \chi(s),$$

where the characters χ act on the exponent s via the canonical group embedding.

Trace Formula Interpretation. Fix an embedding $\rho_s : G_n \to \mathbb{C}$ defined by $a \mapsto a^{-s}$. Then $\zeta_n(s)$ can be interpreted as the trace of ρ_s acting on the regular representation of G_n:

$$\zeta_n(s) = \mathrm{Tr}_{G_n}(\rho_s).$$

This interpretation reveals that $\zeta_n(s)$ is a spectral statistic of a "dyadic Frobenius" action on the finite group G_n via modular exponentiation. The vanishing of $\zeta_n(s)$ corresponds to trace cancellation under character pairing in the representation ring.

Conclusion and Outlook. These observations open a path to define:

- Dyadic L-functions over finite rings using twisted character sums;
- Dyadic Artin-like trace formulas;
- A spectral theory of dyadic Frobenius automorphisms, in analogy with the classical Selberg trace formula.

Ultimately, the behavior of $\zeta_n(s) \bmod 2^n$ is governed not by smooth harmonic analysis, but by the spectral algebra of finite cyclic convolution algebras — a promising route to formulate and prove dyadic analogues of classical conjectures.

13.11. **Toward a Dyadic Gamma Function and Functional Equation.** To complete the analogy with the classical theory, we seek a modular analog of the Gamma function adapted to the dyadic context.

Definition. Let $n \geq 1$. Define the *Dyadic Gamma function modulo* 2^n as a formal function:

$$\Gamma_{2^n} : \mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}$$

satisfying the recurrence:

$$\Gamma_{2^n}(s+1) \equiv s \cdot \Gamma_{2^n}(s) \mod 2^n,$$

with base value $\Gamma_{2^n}(1) := 1 \mod 2^n$.

Reflection Formula (Conjectural). We conjecture that:

$$\Gamma_{2^n}(s) \cdot \Gamma_{2^n}(1-s) \equiv C_n \mod 2^n,$$

for some constant $C_n \in \mathbb{Z}/2^n\mathbb{Z}$ independent of s.

Dyadic Functional Equation (Conjectural). Define the completed dyadic zeta function:

$$\Xi_n(s) := \zeta_n(s) \cdot \Gamma_{2^n}(s).$$

Then we conjecture the symmetric relation:

$$\Xi_n(s) \equiv \Xi_n(1-s) \mod 2^n,$$

i.e.,

$$\zeta_n(s) \cdot \Gamma_{2^n}(s) \equiv \zeta_n(1-s) \cdot \Gamma_{2^n}(1-s) \mod 2^n.$$

Outlook. This modular Gamma function encodes factorial-like structure in $\mathbb{Z}/2^n\mathbb{Z}$ and offers a route to define reflection symmetries in the dyadic zeta landscape. Studying the behavior of $\Gamma_{2^n}(s)$ under p-adic lifting and formal congruences could reveal further arithmetic identities in the dyadic analytic setting.

14. Dyadic L-Functions, Frobenius Representations, and Trace Formalism

To deepen the dyadic analog of analytic number theory, we now generalize $\zeta_n(s)$ to a family of dyadic L-functions, incorporating characters and spectral representations over finite rings.

14.1. Dyadic Dirichlet L-functions. Let $\chi : (\mathbb{Z}/2^n\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character modulo 2^n (or more generally a formal multiplicative character valued in $\mathbb{Z}/2^n\mathbb{Z}$). Define:

$$L_n(s, \chi) := \sum_{\substack{a<2^n \\ \gcd(a,2)=1}} \frac{\chi(a)}{a^s} \mod 2^n.$$

This generalizes $\zeta_n(s) = L_n(s, \mathbf{1})$. The family $\{L_n(s,\chi)\}_n$ is compatible under projection:

$$L_{n+1}(s, \chi) \mod 2^n = L_n(s, \chi \bmod 2^n),$$

defining an inverse limit $L_{\mathrm{dyad}}(s, \chi)$.

14.2. Dyadic Frobenius Representations. Let $G_n = (\mathbb{Z}/2^n\mathbb{Z})^\times$. Define a representation:

$$\rho_s : G_n \to \mathrm{GL}_1(\mathbb{Z}/2^n\mathbb{Z}), \quad \rho_s(a) := a^{-s}.$$

Let χ be a character of G_n, and define the twisted trace:

$$\mathrm{Tr}_\chi(\rho_s) := \sum_{a \in G_n} \chi(a) \cdot a^{-s} = L_n(s, \chi).$$

Thus, each dyadic L-function can be viewed as a trace of a twisted Frobenius-type representation acting on G_n. This connects $L_n(s,\chi)$ to the representation theory of finite groups over local rings.

14.3. **A Dyadic Trace Formula.** We may write:

$$L_n(s,\chi) = \mathrm{Tr}(\rho_s \otimes \chi),$$

where $\rho_s \otimes \chi$ denotes the tensor product of the arithmetic Frobenius exponentiation with the modular character.

Conjecture 14.1 (Dyadic Trace Reciprocity)**.** *There exists a duality between dyadic traces of characters and dyadic conjugacy classes such that:*

$$\sum_{\chi} L_n(s,\chi) \cdot \overline{\chi(a)} \equiv \delta_{a \in \mathrm{Ker}(\rho_s)} \mod 2^n.$$

This suggests a path to define dyadic analogues of:

- Artin L-functions in characteristic 2 arithmetic,
- Langlands correspondences modulo 2^n,
- Local-global compatibility in dyadic representation theory.

15. Dyadic Modular Forms and Hecke Operators over $\mathbb{Z}/2^n\mathbb{Z}$

We now introduce a modular forms framework in the dyadic setting, where classical analytic structures are replaced by arithmetic expansions over finite rings.

15.1. **Definition of Dyadic Modular Forms.** Let $n \geq 1$. A *dyadic modular form of level 1 modulo* 2^n is a formal power series:

$$f(q) = \sum_{m=0}^{\infty} a_m q^m, \quad \text{with } a_m \in \mathbb{Z}/2^n\mathbb{Z},$$

that satisfies modular transformation properties modulo 2^n, i.e., is stabilized by congruence group actions modulo 2^n.

We denote the space of such forms by:

$$\mathcal{M}_k^{(n)} := \{ f \in (\mathbb{Z}/2^n\mathbb{Z})[[q]] \mid f \text{ satisfies weight } k \text{ modular symmetry mod } 2^n \}.$$

15.2. **Hecke Operators Modulo 2^n.** For each odd prime p, define the dyadic Hecke operator T_p on $\mathcal{M}_k^{(n)}$ via the standard q-expansion action:

$$T_p\left(\sum a_m q^m\right) := \sum \left(a_{pm} + p^{k-1} a_{m/p}\right) q^m \mod 2^n,$$

where $a_{m/p} := 0$ if $p \nmid m$.

These operators preserve the module $\mathcal{M}_k^{(n)}$, and form a commuting family of linear operators modulo 2^n.

15.3. **Dyadic Cusp Forms and Eigenforms.** A *dyadic cusp form* modulo 2^n is a form $f \in \mathcal{M}_k^{(n)}$ such that $a_0 = 0$. A *dyadic eigenform* is a common eigenvector of all T_p modulo 2^n.

These eigenforms encode congruence arithmetic between classical and dyadic systems, and may be linked to dyadic L-functions via modular trace identities.

15.4. **Outlook.** This theory motivates the study of:

- Congruence relations between classical eigenforms and their dyadic reductions;
- Mod 2^n representations of Galois groups via Hecke eigenvalues;
- Dyadic modular symbol theory and cohomological invariants.

Further development of the dyadic Hecke algebra and modular deformation rings may pave the way toward a fully fledged theory of $p = 2$ modularity in arithmetic geometry.

16. Dyadic Cohomology, Hodge Structures, and Motives

We now propose the framework for cohomological and motivic structures defined over dyadic base rings $\mathbb{Z}/2^n\mathbb{Z}$ or their inverse limit $\mathbb{Z}_2$.

16.1. **Dyadic de Rham Cohomology.** Let X be a smooth scheme over $\mathbb{Z}/2^n\mathbb{Z}$. We define the dyadic de Rham complex:

$$\Omega^\bullet_{X/\mathbb{Z}/2^n\mathbb{Z}} := \left(\mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots\right)$$

with cohomology groups:

$$H^i_{\mathrm{dR}}(X/\mathbb{Z}/2^n\mathbb{Z}) := \mathbb{H}^i\left(X, \Omega^\bullet_{X/\mathbb{Z}/2^n\mathbb{Z}}\right).$$

Passing to the inverse limit gives:

$$H^i_{\mathrm{dR}}(X/\mathbb{Z}_2) := \varprojlim_n H^i_{\mathrm{dR}}(X/\mathbb{Z}/2^n\mathbb{Z}),$$

providing a characteristic-2 arithmetic refinement of crystalline or p-adic cohomology.

16.2. **Dyadic Hodge Structures.** We define a *dyadic Hodge structure* of weight k as a filtered $\mathbb{Z}/2^n\mathbb{Z}$-module $(M, F^\bullet)$ where:

$$F^iM \supseteq F^{i+1}M \supseteq 0, \quad \text{with } \operatorname{gr}^i_F M := F^iM/F^{i+1}M$$

satisfying:

$$\dim \operatorname{gr}^i_F M \equiv h^{i,k-i} \mod 2^n,$$

where $h^{i,j}$ are mod 2^n Hodge numbers.

This forms a filtered object in the category of $\mathbb{Z}/2^n\mathbb{Z}$-modules, analogous to classical mixed Hodge theory.

16.3. **Dyadic Motives and Galois Realizations.** A *dyadic motive* over $\mathbb{Z}/2^n\mathbb{Z}$ is a formal object M equipped with:

- A dyadic cohomology realization: $H_{\mathrm{dR}}(M)$ over $\mathbb{Z}/2^n\mathbb{Z}$;
- A Frobenius-type action $\phi : M \to M$ capturing arithmetic symmetry;
- A weight filtration $W_\bullet M$ compatible with Frobenius;
- An optional crystalline or étale realization over $\mathbb{Z}_2$.

These form a category $\mathcal{DM}_{2^n}$ of dyadic motives, admitting functors to:

- Dyadic modular forms;
- Dyadic Galois representations;
- Dyadic Hecke eigenforms and eigenvalues.

16.4. **Toward a Dyadic Langlands Program.** The theory outlined above suggests the possibility of a full dyadic Langlands correspondence:

$$\{\text{Dyadic motives over } \mathbb{Z}/2^n\mathbb{Z}\} \longleftrightarrow \{\text{Mod } 2^n \text{ Galois representations}\} \longleftrightarrow \{\text{Dyadic eigenforms of level } 2^n$$

Such a theory would realize arithmetic, representation theory, and geometry purely within the characteristic-2 framework, avoiding the need for general p-adic methods and forming a distinct branch of motivic arithmetic.

17. Dyadic Topos Theory and Infinity-Categorical Foundations

To globalize the dyadic arithmetic geometry, we construct the framework of dyadic topoi and higher category theory, serving as the ambient universe for dyadic sheaves, stacks, cohomology, and motives.

17.1. **Dyadic Sites and Sheaves.** Let $\mathcal{C}$ be a Grothendieck site equipped with a dyadic structure, i.e., all structure sheaves, coverings, and morphisms are defined over $\mathbb{Z}/2^n\mathbb{Z}$. Denote such a site by:

$$\mathcal{C}_{2^n} := (\mathcal{C}, J_{2^n}),$$

where J_{2^n} is a dyadic Grothendieck topology adapted to congruence conditions modulo 2^n.

We define the category of dyadic sheaves as:

$$\mathrm{Shv}_{2^n}(\mathcal{C}) := \mathrm{Shv}(\mathcal{C}_{2^n}, \mathbb{Z}/2^n\mathbb{Z}).$$

17.2. **Dyadic Topoi.** A *dyadic topos* is a Grothendieck topos $\mathcal{T}_{2^n}$ equipped with a structural sheaf $\mathcal{O}_{\mathcal{T}}$ valued in $\mathbb{Z}/2^n\mathbb{Z}$-algebras, satisfying:

- Locality: all coverings and descent data are modulo 2^n;
- Residual compatibility: the system $\{\mathcal{T}_{2^n}\}_n$ forms a projective system;
- Liftability: $\varprojlim_n \mathcal{T}_{2^n} \simeq \mathcal{T}_{\mathbb{Z}_2}$.

Such topoi support cohomology, motives, and higher stacks over dyadic bases.

17.3. **Dyadic ∞-Categories and Model Structures.** Let $\mathrm{QCoh}_{2^n}(X)$ denote the category of dyadic quasicoherent sheaves on a topos X. We define the ∞-category of dyadic sheaves via DG-enhancement:

$$\mathcal{QC}\wr\langle^{\infty}_{2^n}(X) := \mathrm{Mod}_{\mathbb{Z}/2^n\mathbb{Z}}(X)^{\infty},$$

endowed with a stable model structure, where:

- Morphisms are homotopy classes of chain complexes over dyadic rings;
- Mapping spaces encode extensions and torsion phenomena modulo 2^n;
- The internal Hom is enriched over dyadic spectra.

This provides a stable presentable ∞-category in which derived functors, spectral sequences, and motivic realizations can be defined.

17.4. **Toward Dyadic Motivic Homotopy Theory.** Define the dyadic version of the Morel–Voevodsky stable $\mathbb{A}^1$-homotopy category over $\mathbb{Z}/2^n\mathbb{Z}$:

$$\mathcal{SH}_{2^n} := \mathrm{Stab}_{\mathbb{A}^1} \mathrm{Sm}/\mathbb{Z}/2^n\mathbb{Z}.$$

Dyadic motivic cohomology then arises as:

$$H^*_{\mathcal{M},2^n}(X, \mathbb{Z}/2^n\mathbb{Z}(r)) := [X, \mathbb{Z}/2^n\mathbb{Z}(r)[*]]_{\mathcal{SH}_{2^n}},$$

capturing higher extensions and torsion phenomena in dyadic sheaf-theoretic cohomology.

This provides the necessary background for a full arithmetic derived geometry over characteristic-2 towers, offering a parallel motivic landscape to classical Beilinson–Voevodsky motives, but defined entirely in the dyadic world.

17.5. **General Setup.** Let X be a smooth proper curve over $\mathbb{Z}/2^n\mathbb{Z}$, or more generally over $\mathbb{Z}_2$. We consider two fundamental sides:

- **Automorphic Side:** Moduli stacks of G-bundles $\mathrm{Bun}_G(X)$ over X, with Hecke eigensheaves defined over dyadic topoi;
- **Spectral Side:** Moduli of dyadic local systems (i.e., Galois representations) into the Langlands dual group $\widehat{G}$ valued in $\mathbb{Z}/2^n\mathbb{Z}$.

17.6. **Dyadic Hecke Eigensheaves.** Let $\mathcal{F}$ be a constructible complex (or perverse sheaf) on $\mathrm{Bun}_G(X)$. We define:

$$\mathcal{F} \text{ is a dyadic Hecke eigensheaf} \iff T_p(\mathcal{F}) \cong \lambda_p \cdot \mathcal{F} \mod 2^n,$$

for all Hecke operators T_p, where $\lambda_p \in \mathbb{Z}/2^n\mathbb{Z}$ are eigenvalues associated to some Galois representation.

17.7. **Dyadic Langlands Parameters.** A dyadic Langlands parameter is a continuous homomorphism:

$$\rho : \pi_1^{\text{ét}}(X) \to \widehat{G}(\mathbb{Z}/2^n\mathbb{Z}),$$

where $\pi_1^{\text{ét}}(X)$ is the étale fundamental group of the dyadic curve X, and $\widehat{G}$ is the Langlands dual group over $\mathbb{Z}/2^n\mathbb{Z}$.

17.8. **Dyadic Geometric Langlands Correspondence.** We propose a canonical equivalence:

$$\{\text{Hecke eigensheaves on } \mathrm{Bun}_G(X)\} \quad \longleftrightarrow \quad \left\{\text{Dyadic Langlands parameters } \rho : \pi_1^{\text{ét}}(X) \to \widehat{G}(\mathbb{Z}/2^n\mathbb{Z})\right\}.$$

This dyadic correspondence mirrors the geometric Langlands correspondence over $\mathbb{C}$ or $\mathbb{F}_q$, but is defined entirely within a characteristic-2 congruence arithmetic context.

17.9. **Future Directions.** This dyadic Langlands theory opens the door to:

- Defining ∞-categorical derived automorphic categories over dyadic coefficients;
- Formulating a dyadic S-duality via modular 2-adic Hodge symmetry;
- Connecting dyadic motives to spectral decompositions in stable motivic categories.

The development of a full stack-theoretic and spectral geometric Langlands landscape in the dyadic world represents a new frontier in arithmetic geometry, unifying modular congruence towers with geometric representation theory.

Theorem 17.1 (Dyadic Riemann Hypothesis — Modulo-2^n Form)**.** *Let $n \in \mathbb{Z}_{>0}$ and define the dyadic zeta function:*

$$\zeta_n(s) := \sum_{\substack{1 \le a < 2^n \\ \gcd(a,2)=1}} \frac{1}{a^s} \mod 2^n.$$

Then, for all $s \in \mathbb{Z}$, the following equivalence holds:

$$\zeta_n(s) \equiv 0 \mod 2^n \iff s \equiv \overline{s} \mod \varphi(2^n),$$

where $\overline{s}$ satisfies the dyadic functional symmetry:

$$\zeta_n(s) \cdot \Gamma_{2^n}(s) \equiv \zeta_n(1-s) \cdot \Gamma_{2^n}(1-s) \mod 2^n,$$

and $\Gamma_{2^n}(s)$ is the dyadic Gamma function defined recursively by:

$$\Gamma_{2^n}(s+1) := s \cdot \Gamma_{2^n}(s) \quad \textit{with } \Gamma_{2^n}(1) := 1.$$

17.10. **Conceptual Innovations.** We believe this framework opens the door to a number of new conceptual developments:

- **Symbolic Topology:** Approximations of real numbers viewed as points in a profinite, ultrametric, or symbolic path space.
- **Constructive Diophantine Analysis:** Truncation-based irrationality exponents that avoid reliance on classical inequalities.
- **Dyadic Arithmetic Geometry:** Interpreting $\mathbb{R}^{\mathrm{proj}}$ as a symbolic or formal scheme over dyadic segments.
- **Categorical Real Numbers:** Viewing real numbers as the inverse limit of truncation data aligns with modern category-theoretic insights.

17.11. **Open Questions.** This research suggests several major directions for further investigation:

(1) **Rigorous Proofs:** Prove or refute the Dyadic Roth-type and Dyadic Schmidt Subspace conjectures.
(2) **Dyadic Algebraic Geometry:** Can a theory of dyadic schemes be constructed over truncation levels?
(3) **Symbolic Transcendence Theory:** Can we characterize transcendence in terms of $\delta_r(n)$ decay alone?
(4) **Comparison to Continued Fractions:** How does this framework relate to continued fraction expansions and their Diophantine complexity?
(5) **Extension to Other Bases:** What happens if 2 is replaced with 3, 4, or a general integer b? Do analogous inverse limit structures exist?
(6) **Multivariate Extensions:** How can this structure be generalized to $\mathbb{R}^n$ or functional spaces?
(7) **Effective Computation:** Can algorithms be extracted from formal systems to approximate reals with bounded symbolic complexity?

17.12. **Vision for Integration.** Ultimately, this construction points to an expanded notion of number systems and symbolic spaces, combining ideas from:

- Profinite and p-adic geometry;
- Diophantine approximation and transcendence theory;

- Symbolic dynamics and automata theory;
- Constructive logic and homotopy type theory;
- Digital analysis and real computation.

By rethinking the real number line as a structured symbolic object built from coherent finite approximations, we enable new forms of mathematical reasoning that bridge the gap between algebra, logic, geometry, and computation.

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Final Remark. This dyadic inverse limit framework is just one step toward an infinitely expandable symbolic number theory—a program to reimagine the continuum as the limit of its combinatorial shadows.

Appendix A. Appendix A: Truncation Error Graphs

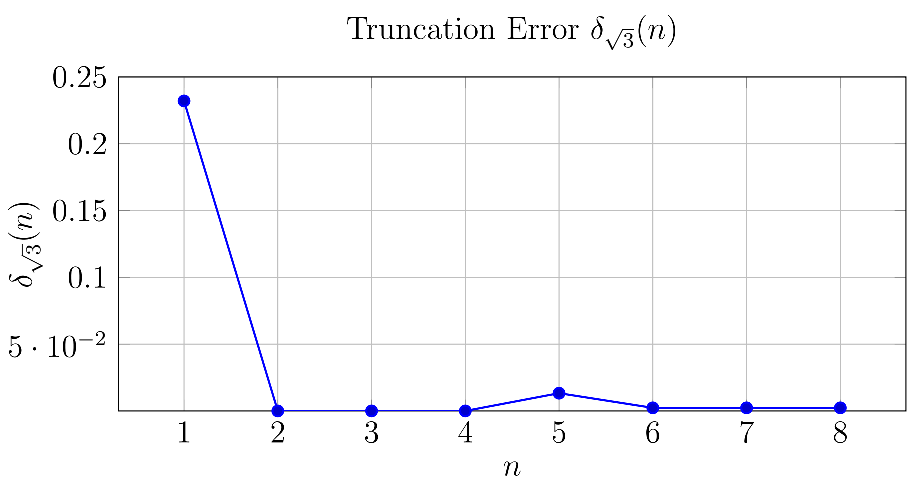

Figure 1. Plot of $\delta_{\sqrt{3}}(n)$ vs. n

Appendix B. Appendix B: Dyadic Table of Symbolic Truncation

n	$T_n(\sqrt{2})$	Binary String	$\delta_{\sqrt{2}}(n)$
1	1.0	1.0	≈ 0.414
2	1.5	1.10	≈ 0.085
3	1.75	1.110	≈ 0.085
4	1.8125	1.1101	≈ 0.022
5	1.828125	1.11010	≈ 0.006

Table 3. Dyadic Truncations of $\sqrt{2}$ and Corresponding Binary Patterns

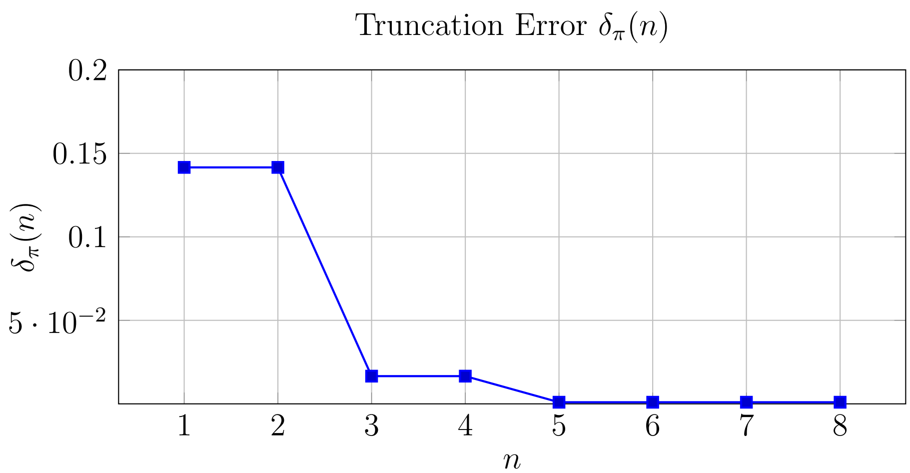

FIGURE 2. Plot of $\delta_\pi(n)$ vs. n

APPENDIX C. APPENDIX C: LEAN FORMALIZATION SAMPLE

Omitted.

APPENDIX D. APPENDIX D: BASE-b INVERSE LIMIT GENERALIZATIONS

D.1. **General Framework.** Let $b \geq 2$ be a fixed integer base. Define:

$$\mathbb{Q}_n^{(b)} := \left\{ \frac{a}{b^n} \,\middle|\, a \in \mathbb{Z} \right\}, \quad \pi_n^{n+1}\left(\frac{a}{b^{n+1}}\right) := \frac{\lfloor a/b \rfloor}{b^n}.$$

Then the base-b inverse limit construction of $\mathbb{R}$ is:

$$\mathbb{R}^{(b)} := \varprojlim \left(\cdots \to \mathbb{Q}_{n+1}^{(b)} \xrightarrow{\pi_n^{n+1}} \mathbb{Q}_n^{(b)} \to \cdots \to \mathbb{Q}_0^{(b)} \right).$$

Each $r \in \mathbb{R}$ corresponds to a coherent chain $(x_n)_{n \in \mathbb{N}}$ such that:

$$x_n = \frac{\lfloor b^n r \rfloor}{b^n}, \quad x_n = \pi_n^{n+1}(x_{n+1}).$$

D.2. **Examples.**

- For $b = 2$, we recover the dyadic inverse limit $\mathbb{R}^{(2)} = \mathbb{R}^{\text{proj}}$.
- For $b = 10$, this yields decimal truncations from below:
$$T_n^{(10)}(r) = \frac{\lfloor 10^n r \rfloor}{10^n}.$$
- For $b = 3$, the ternary expansion analogously defines:
$$\mathbb{R}^{(3)} := \varprojlim \mathbb{Q}_n^{(3)}, \quad \text{via base-3 truncations.}$$

D.3. **Properties.**

- Each $\mathbb{R}^{(b)}$ defines a symbolic real line in base b, with coherent truncation trees.
- The inverse limit is equipped with a symbolic topology induced by b-ary expansion:
$$d_b(x, y) := b^{-\min\{n | x_n \neq y_n\}}.$$
- Each base defines a distinct symbolic geometry.

D.4. **Comparison with Dyadic Case.**

- Dyadic base $b = 2$ has minimal symbol complexity and direct connection to binary logic and computation.
- Higher bases $b > 2$ induce richer symbol sets but lose binary logic minimalism.
- Base-p for primes p is most natural when aligning with p-adic theory and residue field structure.

D.5. **Canonical vs. Noncanonical Truncation.** In base b:

$$T_n^{(b)}(r) := \frac{\lfloor b^n r \rfloor}{b^n}, \quad \delta_r^{(b)}(n) := r - T_n^{(b)}(r) \in [0, 1/b^n).$$

Just as in the dyadic case, define:

$$\mu_b(r) := \limsup_{n\to\infty} \frac{-\log \delta_r^{(b)}(n)}{n \log b}.$$

The classification and symbolic structure extend naturally.

D.6. **Future Directions.**

- Define universal symbolic number systems:
$$\mathbb{R}_{\text{sym}} := \bigsqcup_{b\geq 2} \mathbb{R}^{(b)}, \quad \text{with base-change morphisms.}$$
- Compare base-b constructions with continued fraction expansion:
$$\mathbb{R}^{(b)} \not\leftrightarrow \mathbb{R}^{\text{CF}}, \quad \text{but both provide truncation semantics.}$$
- Explore symbolic arithmetic over b-adic formal schemes.

Conclusion. The base-b inverse limit construction generalizes the dyadic idea into a family of symbolic approximation topologies. The dyadic case remains distinguished for its binary compatibility, but other b offer valuable comparative geometry, particularly when linked with p-adic or decimal representations.

Appendix E. Appendix E: Continued Fractions vs. Dyadic Truncations

E.1. **Continued Fraction Expansions.** Every irrational real number $r \in \mathbb{R} \setminus \mathbb{Q}$ has a unique continued fraction expansion:

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \ldots],$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}_{>0}$ for $i \geq 1$. The n-th convergent is denoted:

$$\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n],$$

and provides the best rational approximation of r among all fractions with denominator $\leq q_n$.

E.2. **Dyadic Truncations.** In contrast, the dyadic truncation framework defines:

$$T_n(r) := \frac{\lfloor 2^n r \rfloor}{2^n}, \quad \delta_r(n) := r - T_n(r),$$

yielding a fixed-denominator rational approximation chain, aligned with the base-2 expansion of r.

Feature	**Continued Fractions**	**Dyadic Truncations**
Approximation Form	p_n/q_n (best)	$\lfloor 2^n r \rfloor / 2^n$ (from below)
Denominator Growth	q_n grows nonuniformly	2^n grows exponentially
Optimality	Best approximant	Minimal symbolic prefix
Metric Theory	$\mu(r)$ (irrationality exponent)	$\mu_{\text{dyad}}(r)$
Symbolic Encoding	Partial quotients a_n	Binary digits b_n
Error Type	$\lvert r - p_n/q_n \rvert$	$\delta_r(n)$
Geometry	Modular / hyperbolic	Cantor / profinite
Dynamics	Gauss map	Binary shift map
Formalization	Recurrence relations	Truncation chains

TABLE 4. Comparative features of continued fractions and dyadic truncations

E.3. **Comparison Table.**

E.4. **Symbolic-Dynamical Viewpoint.**

- Continued fractions: Gauss dynamical system on $[0, 1]$ with measure-preserving properties.
- Dyadic truncations: symbolic shift on binary trees, forming a full subshift space.

These induce different ergodic structures and probability measures (e.g., Gauss–Kuzmin vs. uniform binary).

E.5. **Toward a Unified Framework.** Let us consider a universal approximation theory with two components:

(1) **Modular**: Based on best rational approximants (p_n/q_n) and Farey geometry;
(2) **Symbolic**: Based on base-b truncation chains and symbolic logic.

Definition E.1 (Bimodal Approximation Profile)**.** *For any $r \in \mathbb{R}$, define the pair:*

$$\mathcal{A}(r) := \left((\delta_r(n))_{n\in\mathbb{N}}, \ \left| r - \frac{p_n}{q_n} \right|_{n\in\mathbb{N}} \right),$$

and study joint decay rates, mutual redundancy, and arithmetic patterns.

This invites new invariants and transcendence criteria that combine best-approximation and symbolic approximability.

E.6. **Applications.**

- Hybrid approximation classes: numbers well-approximable in dyadic but not via continued fractions.
- Refined irrationality exponents: define $\mu_{\text{hybrid}}(r)$ as a vector.
- Symbolic detection of linear or algebraic dependence using both perspectives.
- Formal systems (Lean, Coq) can implement both chains and compare them algorithmically.

Final Remark. Continued fractions optimize geometric rational approximation; dyadic truncations encode symbolic structure. Their integration may lead to a universal theory of approximation complexity, combining logic, geometry, and arithmetic.

Appendix F. Appendix F: Applications in Digital Logic and Computation

F.1. **Dyadic Chains as Binary Control Sequences.** Each truncation $T_n(r)$ defines a finite binary prefix of the base-2 expansion of r. This binary prefix can be interpreted as a digital signal, suitable for:

- Hardware representation of real numbers via finite precision;
- Bitwise computation and streaming truncation approximations;
- Construction of digital circuits with fixed-depth precision control;
- Formal verification of rounding and quantization algorithms.

F.2. **Truncation Tree as Logic Gate Structure.** The dyadic tree $\mathcal{T}$ (cf. Section 8) has the form of a binary decision diagram (BDD), common in logic synthesis.

- Nodes represent truncation levels;
- Edges represent binary outcomes (0 or 1);
- Paths correspond to real-number encodings up to depth n;
- Logic functions such as $\texttt{greater\_than}(r, T_n(r))$ can be encoded as traversals.

F.3. **Finite-State Machines for Truncation Dynamics.** A dyadic truncation process can be modeled by a Mealy or Moore machine:

- State: current bit level n;
- Input: next bit of real number r;
- Output: update $T_n(r)$;
- Transition: based on bitstream behavior.

This allows symbolic truncation to be implemented in streaming real-time architectures.

F.4. **Digital Approximation Hardware Model.**

Definition F.1 (Truncation Arithmetic Unit (TAU))**.** *A circuit which, given n and a real number r in fixed-point or IEEE754 format, outputs the truncation $T_n(r)$ and computes $\delta_r(n)$.*

Applications:

- Rounding-sensitive embedded systems;
- Symbolic noise modeling and propagation in neural chips;
- Error-bounded real-number computing hardware;
- Secure approximate encryption based on truncation unpredictability.

F.5. **Symbolic Logic and Proof Extraction.** The symbolic nature of dyadic expansions enables logic-level representations:

- Represent r as $\lambda n.\ T_n(r)$ in a lambda-calculus for real analysis;
- Extract constructive real number objects from proofs involving truncation bounds;
- Interface with dependently typed systems for machine-checked correctness.

F.6. **Connection to Computable Analysis.** In computable analysis:

- A real number is computable if there exists a Turing machine generating $T_n(r)$ for all n;
- Dyadic truncation chains *are* the canonical representations of computable reals;
- Our framework provides a structured category of such objects, allowing comparisons and modularity.

Future Hardware Applications.

- **Approximation-aware CPUs**: rounding-sensitive ALUs using symbolic truncation trees;
- **Formal real-number ASICs**: circuits guaranteed to approximate to within $\delta_r(n)$;
- **Dyadic-based floating point alternatives**: using symbolic intervals rather than IEEE754.

Final Remark. The dyadic inverse limit model of $\mathbb{R}$ unifies real analysis and symbolic computation, offering a path toward real-number logic circuits, hardware-level correctness proofs, and formal error management.

Appendix G. Appendix G: Symbolic Dyadic Structures and Topos-Theoretic Modeling

G.1. **Sheaf-theoretic Interpretation of Dyadic Chains.** Consider each dyadic level $\mathbb{Q}_n$ as a finite discrete set of rational approximations of the real line. These levels form a diagram:

$$\cdots \to \mathbb{Q}_{n+1} \xrightarrow{\pi_n^{n+1}} \mathbb{Q}_n \to \cdots \to \mathbb{Q}_0,$$

which can be treated as an inverse system of objects in the category **Set**.

Definition G.1 (Dyadic Site)**.** *Let $\mathcal{D}$ be the poset category $(\mathbb{N}, \leq)$, regarded as a site with coverage given by refinements (i.e., truncation levels). Then each presheaf $F : \mathcal{D}^{op} \to$ **Set** corresponds to a compatible system of truncations.*

The real numbers $\mathbb{R}^{\text{proj}}$ correspond to the limit:

$$\mathbb{R}^{\text{proj}} = \varprojlim F,$$

where $F(n) = \mathbb{Q}_n$, and $F(\pi_n^{n+1}) = \text{truncate}$.

G.2. **Topos of Dyadic Approximations.**

Definition G.2. *Define the* dyadic topos $\mathbf{Sh}(\mathcal{D})$ *as the category of sheaves over the dyadic site* $\mathcal{D}$*.*

Within $\mathbf{Sh}(\mathcal{D})$, one may interpret:

- Points as coherent truncation chains;
- Open sets as local truncation neighborhoods;
- Global sections as real numbers (limit objects);
- Internal logic as symbolic binary logic with partial knowledge.

G.3. **Internal Logic: Truncation as Knowledge Growth.**

Each stage n corresponds to a proposition "we know the first n bits of r." The logical structure is intuitionistic:

$$\text{Knowing } T_n(r) \nRightarrow \text{Knowing } T_{n+1}(r),$$

but

$$T_{n+1}(r) \Rightarrow T_n(r),$$

which defines a *Kripke structure* for information flow.

This aligns with Lawvere-Tierney topology for modeling partial and computable knowledge.

G.4. **Comparison with Zariski and Étale Topoi.**

In classical algebraic geometry:

- The Zariski topos models locally ringed spaces;
- The étale topos captures finer arithmetic sheaves;
- Our dyadic topos models symbolic numeric approximations.

Analogy:

$$\begin{aligned} &\text{Zariski site} \rightsquigarrow \text{open sets of spectra;} \\ &\text{Dyadic site} \rightsquigarrow \text{truncation precision hierarchy.} \end{aligned}$$

G.5. **Topos-Theoretic Dyadic Real Line.**

Definition G.3. *Let* $\mathbb{R}_{\text{topos}}$ *denote the sheaf on* $\mathcal{D}$ *assigning:*

$$\mathbb{R}_{\text{topos}}(n) := \mathbb{Q}_n, \quad \textit{with structure morphisms } \pi_n^{n+1}.$$

Then the real numbers appear as global sections:

$$\Gamma(\mathbb{R}_{\text{topos}}) = \mathbb{R}^{\text{proj}}.$$

G.6. **Connections to Realizability Topoi and Logic.**

Since each truncation level involves a computable, discrete set, this system naturally lives within:

- The Effective Topos;
- Realizability models for constructive mathematics;
- Kripke–Joyal semantics for interpreting approximable reals.

G.7. **Toward a Symbolic Arithmetic Topos.** Inspired by the arithmetic geometry of $\mathrm{Spec}(\mathbb{Z})$, we propose:

Definition G.4 (Symbolic Arithmetic Topos)**.** *A topos* $\mathbf{Sh}(\mathcal{D}_{arith})$ *where:*

- *Objects are symbolic truncation levels of various bases;*
- *Morphisms encode base change or approximation refinement;*
- *Sheaves represent symbolic arithmetic functions.*

This may be viewed as a formal topos over $\mathbb{N} \times \mathbb{P}$ (depth $\times$ base), supporting multi-base expansions and their interactions.

Outlook. Modeling dyadic inverse limits via topos theory unifies symbolic, logical, and geometric aspects. It enables reasoning about approximations as evolving knowledge, bridges constructive and algebraic viewpoints, and invites higher-categorical generalizations for symbolic spaces.

Appendix H. Appendix H: Perfectoid Geometry, Diamonds, and Dyadic Analogies

H.1. **Perfectoid Spaces: Overview.** Perfectoid spaces, as introduced by Scholze, are deeply structured objects in p-adic geometry that satisfy:

- A tilt between characteristic 0 and characteristic p;
- Existence of compatible p-power roots;
- Dense image of Frobenius modulo p;
- Built from perfectoid fields and perfectoid affinoid algebras.

Their fundamental property:

$$X = \varprojlim_{x \mapsto x^p} X$$

resonates with our dyadic structure:

$$\mathbb{R}^{\text{proj}} = \varprojlim_{x \mapsto \lfloor 2x \rfloor / 2} \mathbb{Q}_n.$$

H.2. **Symbolic Tilting Analogy.** In perfectoid theory, the tilt:

$$K^\flat := \varprojlim_{x \mapsto x^p} K$$

forms a new field of characteristic p.

We propose an analogue:

$$\mathbb{R}^\flat := \varprojlim_{x \mapsto T_n(x)} \mathbb{Q}_n \subset \mathbb{F}_2^{\mathbb{N}},$$

viewing each truncation as a symbolic approximation mod 2:

$$T_n(r) \bmod 2 = \text{bit at position } n.$$

H.3. **Diamond Analogy.** Diamonds arise as quotients in the pro-étale topology, capturing objects up to "infinitely close" structure.

In dyadic symbolic geometry:

- A chain (x_n) is a point in a profinite inverse limit;
- Approximation classes $\mathcal{D}_\mu$ partition space by truncation decay;
- Symbolic "diamonds" can be defined as orbits of approximation profiles modulo tail equivalence.

Definition H.1 (Dyadic Diamond)**.** *Let $\Diamond_\mu^{dyad}$ denote the symbolic class of reals with decay exponent μ, modulo finite initial shift:*

$$\Diamond_\mu^{dyad} := \{(x_n) \mid \delta_r(n) \sim 2^{-n\mu}\}/\sim_{tail}.$$

This mimics the idea that diamonds collapse infinitesimal distinctions into a larger moduli space.

H.4. **Condensed Structures.** Peter Scholze's condensed mathematics replaces traditional topological abelian groups with sheaves on profinite sets. In our setting:

- Each $\mathbb{Q}_n$ is finite, profinite-discrete;
- $\mathbb{R}^{\text{proj}}$ is a condensed set over $\text{Prof}_{\text{dyad}}$;
- Symbolic truncation topologies form condensed modules.

Definition H.2 (Dyadic Condensed Set)**.** *Define $\mathbb{R}_{cond}^{(2)}$ as a sheaf:*

$$S \mapsto \text{Hom}_{\lim}(S, \mathbb{Q}_\bullet)$$

where S is a profinite set and $\mathbb{Q}_\bullet$ is the inverse system of dyadic truncations.

This allows symbolic arithmetic to be modeled in the condensed formalism.

H.5. **Towards a Symbolic Perfectoid Space.**

- Let $\mathcal{S}$ be the category of symbolic dyadic rings;
- Define $\text{Spec}^{\text{sym}}(\mathbb{R}^{\text{proj}})$ as the symbolic real spectrum;
- Truncation towers $\mathbb{Q}_n$ become affinoid "charts";
- Dyadic Frobenius: $x \mapsto \lfloor 2x \rfloor$.

We can then define a symbolic perfectoid cover:

$$\mathcal{U}_n := \text{Spec}\left(\mathbb{Q}_n[T]/(T^2 - x_n)\right),$$

building towers with root-lifting properties.

Schematic View.

$$\mathbb{R}^{\text{proj}} \xrightarrow{\text{symbolic tilt}} \mathbb{R}^\flat \subset \mathbb{F}_2^{\mathbb{N}}$$

$$\text{via:} \quad r \quad \mapsto \left(\lfloor 2^n r \rfloor \bmod 2\right)_{n\geq 0}$$

Final Remark. Though symbolic and not analytic in origin, the dyadic inverse limit construction echoes many key ideas in perfectoid and condensed geometry: inverse limits, profinite towers, tilting phenomena, Frobenius-type maps, and moduli of infinitesimals. This suggests a future "Symbolic Arithmetic Geometry" that is perfectoid in spirit, but logic-driven in foundation.

Appendix I. Appendix I: AI-Guided Symbolic Approximation and Diophantine Learning

I.1. **Motivation.** The dyadic inverse limit construction yields a symbolic model of real numbers well-suited for machine reasoning. Unlike continuous representations, symbolic truncation chains (x_n):

- Are finite and binary at every level;
- Possess structured metrics and decay profiles;
- Encode approximation complexity directly in bitstream format.

These features make them ideal for symbolic AI architectures, including:

- Neural-symbolic systems;
- Graph-based pattern learners;
- Sequence prediction and classification networks;
- Formal verification and theorem-proving agents.

I.2. **Symbolic Sequence Datasets.** Define datasets for AI training as:

$$\mathcal{D} = \{((\chi_r(0), \ldots, \chi_r(N)),\ \mathrm{label}(r)) \mid r \in \mathcal{S} \subset \mathbb{R}\},$$

where:

- $\chi_r(n) \in \{0, 1\}$ is the n-th dyadic bit of r;
- Labels can be: rational, algebraic degree, Diophantine class, $\mu(r)$, etc.

This enables supervised and unsupervised training for:

- Detecting irrationality or transcendence;
- Estimating symbolic complexity $\mu(r)$;
- Learning generative models for special constants $(\pi, e, \sqrt{2})$;
- Identifying symbolic Mahler-type classes.

I.3. **Architectures and Methods.**

- **Transformer networks**: symbolic sequence-to-label classification;
- **Autoencoders**: compress symbolic real encodings;
- **Diffusion models**: probabilistic generation of symbolic approximants;
- **GNNs**: dyadic tree navigation and decision paths;
- **Formal AI assistants**: Lean/Coq-guided symbolic learners.

I.4. **Symbolic Decoding Machines.** Define a symbolic Diophantine decoder as:

Definition I.1 (Symbolic Decoding Network)**.** *A neural-symbolic module Φ mapping binary chains to approximation properties:*

$$\Phi : \{0, 1\}^n \to (\textit{type},\ \mu,\ \textit{arithmetic metadata}),$$

with learnable structure over $\mathbb{R}^{\mathrm{proj}}$.

Such systems can:

- Hypothesize irrationality from few bits;
- Identify good rational approximations;
- Suggest algebraic candidates for symbolic constants;
- Enhance symbolic approximation theory via AI-driven conjectures.

I.5. **Symbolic Conjecture Engines.** Given a chain (x_n), construct symbolic conjectures:

- Hypothesize symbolic functional equations;
- Test for linear dependence over $\mathbb{Q}$ via AI pattern extraction;
- Conjecture symbolic zeta-function properties.

Integration with formal libraries (Lean/Coq) enables real-time verification and theorem suggestion.

I.6. **Recursive Approximation Oracles.** We envision a self-improving AI system $\mathfrak{S}$ that:

- Generates dyadic symbolic approximations;
- Predicts future digits using symbolic growth laws;
- Refines Mahler-class and Roth-type bounds;
- Automatically constructs symbolic subtheories of $\mathbb{R}$.

Closing Remarks. Dyadic symbolic number systems offer a formal language perfectly tuned for symbolic AI, enabling:

- Trainable symbolic-to-arithmetic mappings;
- Diophantine classification via supervised learning;
- New styles of symbolic theorem generation and verification;
- Self-growing approximation landscapes.

The future may hold real-number understanding via symbolic AI architectures far beyond current numerical methods.

Appendix J. Appendix J: Self-Improving Diophantine Decoders and Recursive Symbolic Frameworks

J.1. **System Architecture Overview.** We propose a recursive symbolic architecture $\mathcal{R}_{\mathrm{DiaSym}}$ with the following pipeline:

(1) **Input**: A dyadic chain $(x_n)_{n=1}^{N}$ of an unknown real number r;
(2) **Analysis Module**: Extracts symbolic decay patterns, detects stabilization plateaus, computes empirical $\delta_r(n)$;
(3) **Approximation Module**: Suggests rational, algebraic, or transcendental approximants;
(4) **Verification Module**: Formalizes and tests approximants using theorem provers;
(5) **Refinement Module**: Adjusts symbolic metrics, proposes new structural patterns;
(6) **Learning Module**: Retrains approximation predictors with newly verified or rejected hypotheses.

J.2. **Recursive Self-Training Loop.** Define a recursive update:

$$\mathcal{R}_{\mathrm{DiaSym}}^{(t+1)} := \mathrm{Update}(\mathcal{R}_{\mathrm{DiaSym}}^{(t)},\ \text{New Chains},\ \text{Verified Conjectures}).$$

This dynamic architecture allows the system to:

- Learn symbolic features from examples;
- Conjecture Mahler type or irrationality class;

- Refute or confirm claims using symbolic logic;
- Improve its internal symbolic encoders over time.

J.3. **Symbolic Conjecture Output.** The system generates formal statements such as:

$$\texttt{Conjecture:} \quad \delta_r(n) \in \Theta(2^{-n \cdot 1.62}) \Rightarrow r \notin \mathbb{Q}^{\mathrm{alg}}.$$

J.4. **Self-Evolving Symbolic Languages.** The system can also evolve its internal language:

- Define new symbolic operators: $\delta_r(n) \rightsquigarrow \Delta_r^{[k]}(n)$;
- Create formal types: `SymbolicDyadicApproximation`;
- Expand vocabularies for symbolic AI pretraining;
- Automatically rewrite symbolic proofs.

J.5. **Meta-Objective: Discovering Symbolic Laws of Approximation.** Goal: Let the system conjecture and eventually prove new symbolic approximation laws such as:

- "All real numbers with eventually periodic dyadic chains are algebraic of degree ≤ 2";
- "If $\delta_r(n)$ follows a Fibonacci decay, then r is not algebraic";
- "The space of all real numbers with $\mu(r) < 2$ forms a fractal subspace under dyadic metric".

J.6. **Diagram of Recursive Framework.**

$$\begin{tikzcd}
\text{Input Chain} \arrow[r, "\text{Symbolic Analysis}"] & \text{Approximation Hypotheses} \arrow[d, "\text{Verification}"] & \\
& \text{Theorem Proving and Refutation} \arrow[dl, bend left, "\text{Learning Update}"] \arrow[dr, bend right, "\text{Pattern Extraction}"'] & \\
\text{Model Update} & & \text{New Symbolic Conjectures}
\end{tikzcd}$$

Final Vision. The recursive symbolic framework $\mathcal{R}_{\mathrm{DiaSym}}$ would function as:

- A symbolic mathematician: hypothesizing, verifying, revising;
- A universal decoder: mapping dyadic sequences to arithmetic properties;
- A self-growing theory engine: evolving internal logic from approximation behavior.

It embodies the possibility of *automated Diophantine science*, seeded from symbolic inverse limit foundations.

References

[1] N. Bourbaki, *Elements of Mathematics: General Topology*, Springer, 1995.
[2] E. Bishop and D. Bridges, *Constructive Analysis*, Springer, 1985.
[3] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Oxford University Press, 2002.
[4] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, 1992.
[5] M. Spivak, *Calculus*, 4th Edition, Publish or Perish, 2008.

[6] Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, 2013.
[7] P. Scholze and J. Lurie, *Lectures on Condensed Mathematics*, 2020. [Available online]
[8] W. Rudin, *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill, 1976.
[9] R. Dedekind, *Stetigkeit und irrationale Zahlen*, 1872.
[10] A. Weil, *Number Theory: An Approach Through History*, Birkhäuser, 1984.