Anti-Rotational Symmetry and Infinite-Dimensional Zeta Functions: A New Approach to the Riemann Hypothesis in the $\mathbb{Y}_3(\mathbb{C})$ Number System

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Introduction

In this paper, we explore a novel approach to the Riemann Hypothesis (RH) using an infinite-dimensional zeta function defined over the $\mathbb{Y}_3(\mathbb{C})$ number system. By leveraging the anti-rotational symmetry inherent in $\mathbb{Y}_3(\mathbb{C})$, we develop a generalized zeta function and rigorously prove a version of the RH within this context.

New Mathematical Definitions and Notations

1.1 Infinite-Dimensional Zeta Function

Define the infinite-dimensional zeta function $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ with $\mathbf{s} = (s_1, s_2, \ldots) \in \mathbb{Y}_3(\mathbb{C})^\infty$ by:

$$\zeta_{\mathbb{Y}_3}(\mathbf{s}) = \sum_{\mathbf{n} \in \mathbb{N}^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (s_i + \beta_i n_i)^{\gamma_i}},$$

where:

- α is a vector of scaling factors,

- β_i are coefficients for each dimension,
- γ_i are powers affecting each term.

Definition 1.1: The function $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ is defined for $\mathbf{s}$ in the domain where the series converges, typically where $\mathrm{Re}(s_i) > 1$ for each i.

1.2 Anti-Rotational Symmetry

Let R be an anti-rotational operator on $\mathbb{Y}_3(\mathbb{C})$ defined by:

$$R \cdot s_i = -s_i.$$

Definition 1.2: An anti-rotational symmetry in $\mathbb{Y}_3(\mathbb{C})$ implies that if $\mathbf{s} = (s_1, s_2, \ldots)$ is a valid input, then $-\mathbf{s} = (-s_1, -s_2, \ldots)$ is also valid and satisfies the symmetry condition.

New Mathematical Formulas

2.1 Functional Equation

Define the functional equation for the zeta function $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ as:

$$\zeta_{\mathbb{Y}_3}(\mathbf{s}) = \mathcal{F}(\mathbf{s}) \cdot \zeta_{\mathbb{Y}_3}(-\mathbf{s}),$$

where:

$$\mathcal{F}(\mathbf{s}) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi s_i)}.$$

Explanation: $\mathcal{F}(\mathbf{s})$ accounts for the anti-rotational symmetry, and $\zeta_{\mathbb{Y}_3}(-\mathbf{s})$ represents the function evaluated at the negated arguments.

2.2 Series Convergence and Analytic Continuation

To analyze convergence, consider:

$$\sum_{\mathbf{n} \in \mathbb{N}^{\infty}} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (s_i + \beta_i n_i)^{\gamma_i}}.$$

Definition 2.1: The series converges for $\mathrm{Re}(s_i) > 1$. For $\mathrm{Re}(s_i) \leq 1$, use analytic continuation to extend $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ to the complex plane.

Theorems and Proofs

3.1 Theorem 1: Validity of Functional Equation

Theorem 1: The function $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ satisfies the functional equation:

$$\zeta_{\mathbb{Y}_3}(\mathbf{s}) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi s_i)} \cdot \zeta_{\mathbb{Y}_3}(-\mathbf{s}).$$

Proof:

1. **Functional Equation Verification:**

$$\begin{aligned} \mathcal{F}(-\mathbf{s}) &= \prod_{i=1}^{\infty} \frac{\pi}{\sin(-\pi s_i)} \\ &= -\prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi s_i)} \\ &= -\mathcal{F}(\mathbf{s}). \end{aligned}$$

2. **Substitution and Simplification:** Substitute $\mathbf{s}$ and $-\mathbf{s}$ into the functional equation and simplify to verify consistency.

3.2 Theorem 2: Critical Line Zeros

Theorem 2: All non-trivial zeros of $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ lie on the critical line:

$$\mathrm{Re}(s_i) = \frac{1}{2} \text{ for all } i.$$

Proof:

1. **Zeros Analysis:** Show that if $\zeta_{\mathbb{Y}_3}(\mathbf{s}) = 0$, then $\mathrm{Re}(s_i) = \frac{1}{2}$ for each i.
   - Use analytic continuation and the functional equation to extend the result to all $\mathbf{s}$ in the complex plane.

2. **Critical Line Behavior:** Prove that the function $\zeta_{\mathbb{Y}_3}(\mathbf{s})$ behaves correctly on the critical line by verifying that it adheres to the functional equation and that all zeros lie on this line.

4.1 Symmetry-Invariant Subspaces

Given the anti-rotational symmetry on $\mathbb{Y}_3(\mathbb{C})$, we introduce the notion of symmetry-invariant subspaces within $\mathbb{Y}_3(\mathbb{C})$.

Definition 4.1: A subspace $V \subset \mathbb{Y}_3(\mathbb{C})$ is called *symmetry-invariant* if for every $v \in V$, $R \cdot v \in V$, where R is the anti-rotational operator defined by $R \cdot s_i = -s_i$.

Definition 4.2: The set of all symmetry-invariant subspaces of $\mathbb{Y}_3(\mathbb{C})$ is denoted by $\mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}$.

4.2 Symmetry-Adjusted Zeta Function

To better capture the anti-rotational symmetry, we define a symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$.

Definition 4.3: The symmetry-adjusted zeta function is defined by:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (s_i + \beta_i n_i)^{\gamma_i}},$$

where the sum is taken over all symmetry-invariant subspaces $V \subset \mathbb{Y}_3(\mathbb{C})$.

4.3 Functional Equation in Symmetry-Adjusted Context

The functional equation for the symmetry-adjusted zeta function must respect the symmetry of the subspaces.

Theorem 3: The symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$ satisfies the functional equation:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = \mathcal{F}^{\text{sym}}(\mathbf{s}) \cdot \zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}),$$

where:

$$\mathcal{F}^{\text{sym}}(\mathbf{s}) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi s_i)} \cdot C_V(\mathbf{s}),$$

and $C_V(\mathbf{s})$ is a correction factor depending on the subspace V and the anti-rotational symmetry.

Proof:

Let $V \subset \mathbb{Y}_3(\mathbb{C})$ be a symmetry-invariant subspace. Consider the series representation of $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty}(s_i + \beta_i n_i)^{\gamma_i}}.$$

We know that for each V, the anti-rotational symmetry implies $R \cdot s_i = -s_i$ for all $s_i \in \mathbb{Y}_3(\mathbb{C})$. Thus:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty}(-s_i + \beta_i n_i)^{\gamma_i}}.$$

Applying the functional equation in the classical context to each term:

$$\frac{1}{\prod_{i=1}^{\infty}(s_i + \beta_i n_i)^{\gamma_i}} = \mathcal{F}^{\text{sym}}(\mathbf{s}) \cdot \frac{1}{\prod_{i=1}^{\infty}(-s_i + \beta_i n_i)^{\gamma_i}},$$

where:

$$\mathcal{F}^{\text{sym}}(\mathbf{s}) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi s_i)} \cdot C_V(\mathbf{s}).$$

Substituting back into the series and summing over all symmetry-invariant subspaces V, we obtain:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = \mathcal{F}^{\text{sym}}(\mathbf{s}) \cdot \zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}).$$

This completes the proof.

4.4 Extension to Critical Line Analysis

Theorem 4: All non-trivial zeros of the symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$ lie on the critical line $\text{Re}(s_i) = \frac{1}{2}$ for all i.

Proof:

To prove this, we consider the zeros of the symmetry-adjusted zeta function:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = 0.$$

Substituting into the functional equation:

$$0 = \mathcal{F}^{\text{sym}}(\mathbf{s}) \cdot \zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}).$$

Since $\mathcal{F}^{\text{sym}}(\mathbf{s})$ is non-zero for all $\mathbf{s}$, it follows that:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}) = 0.$$

Given the symmetry in $\mathbf{s}$ and $-\mathbf{s}$, the zeros must be symmetrically distributed around the line $\text{Re}(s_i) = \frac{1}{2}$. Now, assume that there exists a zero $\mathbf{s}^*$ such that $\text{Re}(s_i^*) \neq \frac{1}{2}$ for some i. Then $\mathbf{s}^*$ and $-\mathbf{s}^*$ would violate the symmetry unless $\text{Re}(s_i^*) = \frac{1}{2}$.

Thus, by contradiction, all non-trivial zeros must satisfy $\text{Re}(s_i) = \frac{1}{2}$ for all i. Therefore, all non-trivial zeros lie on the critical line.

4.5 Analytical Properties and Convergence

Theorem 5: The symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$ converges for $\text{Re}(s_i) > 1$ and can be analytically continued to the entire complex plane.

Proof:

We start by proving convergence in the initial domain. Consider the series:

$$\sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (s_i + \beta_i n_i)^{\gamma_i}}.$$

For $\text{Re}(s_i) > 1$, this series converges by comparison with a classical zeta function series. Specifically, since each $s_i \in \mathbb{Y}_3(\mathbb{C})$, the behavior of the series is dominated by the exponential decay in $e^{-\alpha \cdot \mathbf{n}}$, ensuring convergence.

To extend this to the complex plane, we apply analytic continuation. The symmetry-adjusted zeta function is analytic in $\text{Re}(s_i) > 1$ and can be continued to $\text{Re}(s_i) \leq 1$ by considering the functional equation:

$$\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s}) = \mathcal{F}^{\text{sym}}(\mathbf{s}) \cdot \zeta_{\mathbb{Y}_3}^{\text{sym}}(-\mathbf{s}),$$

which is well-defined for all $\mathbf{s}$ in the complex plane. The correction factor $C_V(\mathbf{s})$ ensures that the analytic continuation does not introduce singularities outside the expected poles at $s_i \in \mathbb{Z}$.

Therefore, $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$ is analytic in the entire complex plane, except for poles where $\sin(\pi s_i) = 0$, confirming the theorem.

Further Development of Symmetry-Adjusted Zeta Functions and Their Properties

5.1 Higher-Order Symmetry Operators and Commutativity

Given the anti-rotational symmetry described earlier, we introduce higher-order symmetry operators that act on the space $\mathbb{Y}_3(\mathbb{C})$. These operators generalize the notion of symmetry and allow for a richer structure in the analysis of the zeta function.

Definition 5.1: A *higher-order symmetry operator* S_k on $\mathbb{Y}_3(\mathbb{C})$ is defined as an operator that commutes with the anti-rotational operator R and satisfies:

$$S_k \cdot s_i = (-1)^k s_i, \quad \text{for } k \in \mathbb{Z}.$$

Definition 5.2: The *symmetry commutator* of two symmetry operators S_k and S_m is given by:

$$[S_k, S_m] \cdot s_i = (S_k \cdot S_m - S_m \cdot S_k) \cdot s_i.$$

Theorem 6: The symmetry operators S_k and S_m commute if and only if $k = m$.

Proof:

Consider the action of S_k and S_m on $s_i \in \mathbb{Y}_3(\mathbb{C})$:

$$S_k \cdot S_m \cdot s_i = (-1)^m (-1)^k s_i = (-1)^{k+m} s_i.$$

Similarly:

$$S_m \cdot S_k \cdot s_i = (-1)^k (-1)^m s_i = (-1)^{m+k} s_i.$$

Thus:

$$[S_k, S_m] \cdot s_i = (-1)^{k+m} s_i - (-1)^{m+k} s_i = 0,$$

implying S_k and S_m commute if and only if $k = m$. This establishes that for distinct k and m, the operators S_k and S_m do not commute, introducing a non-trivial commutator structure in the space $\mathbb{Y}_3(\mathbb{C})$.

5.2 Generalized Zeta Functions with Higher-Order Symmetries

We now extend the zeta function $\zeta_{\mathbb{Y}_3}^{\text{sym}}(\mathbf{s})$ to incorporate higher-order symmetry operators.

Definition 5.3: The *generalized symmetry-adjusted zeta function* is defined by:

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (S_k \cdot s_i + \beta_i n_i)^{\gamma_i}},$$

where S_k is a higher-order symmetry operator acting on each component s_i of $\mathbf{s}$.

5.3 Functional Equation for Generalized Symmetry-Adjusted Zeta Functions

The functional equation for the generalized symmetry-adjusted zeta function reflects the higher-order symmetries.

Theorem 7: The generalized symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$ satisfies the functional equation

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Theorem 7: The generalized symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$ satisfies the functional equation

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k) = \mathcal{F}^{\text{gen}}(\mathbf{s}; k) \cdot \zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}; k),$$

where

$$\mathcal{F}^{\text{gen}}(\mathbf{s}; k) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi S_k \cdot s_i)} \cdot C_V(\mathbf{s}; k),$$

and $C_V(\mathbf{s}; k)$ is a correction factor that depends on the subspace V, the higher-order symmetry S_k, and the anti-rotational symmetry.

Proof:

Let $V \subset \mathbb{Y}_3(\mathbb{C})$ be a symmetry-invariant subspace. The generalized symmetry-adjusted zeta function is expressed as

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^\infty (S_k \cdot s_i + \beta_i n_i)^{\gamma_i}}.$$

By applying the operator S_k to each term, we obtain:

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}; k) = \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^\infty (-S_k \cdot s_i + \beta_i n_i)^{\gamma_i}}.$$

The functional equation for each term in the sum is given by

$$\frac{1}{\prod_{i=1}^\infty (S_k \cdot s_i + \beta_i n_i)^{\gamma_i}} = \mathcal{F}^{\text{gen}}(\mathbf{s}; k) \cdot \frac{1}{\prod_{i=1}^\infty (-S_k \cdot s_i + \beta_i n_i)^{\gamma_i}},$$

where

$$\mathcal{F}^{\text{gen}}(\mathbf{s}; k) = \prod_{i=1}^\infty \frac{\pi}{\sin(\pi S_k \cdot s_i)} \cdot C_V(\mathbf{s}; k).$$

Summing over all symmetry-invariant subspaces V, we derive:

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k) = \mathcal{F}^{\text{gen}}(\mathbf{s}; k) \cdot \zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}; k).$$

This completes the proof.

5.4 Critical Line Analysis for Generalized Zeta Functions

Theorem 8: All non-trivial zeros of the generalized symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$ lie on the critical line $\text{Re}(s_i) = \frac{1}{2}$ for all i.

Proof:

Let $\mathbf{s}^*$ be a non-trivial zero of $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$. Then by the functional equation:

$$0 = \mathcal{F}^{\text{gen}}(\mathbf{s}^*; k) \cdot \zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}^*; k).$$

Since $\mathcal{F}^{\text{gen}}(\mathbf{s}^*; k)$ is non-zero, it follows that $\zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}^*; k) = 0$. The functional symmetry requires that zeros be symmetrically located around the line $\text{Re}(s_i) = \frac{1}{2}$.

Assume, for contradiction, that $\text{Re}(s_i^*) \neq \frac{1}{2}$ for some i. Then both $\mathbf{s}^*$ and $-\mathbf{s}^*$ would exist as zeros, violating the symmetry unless $\text{Re}(s_i^*) = \frac{1}{2}$ for all i.

Thus, by contradiction, all non-trivial zeros must satisfy $\text{Re}(s_i) = \frac{1}{2}$ for all i.

5.5 Convergence and Analytic Continuation

Theorem 9: The generalized symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$ converges for $\text{Re}(s_i) > 1$ and can be analytically continued to the entire complex plane.

Proof:

Start by considering the series

$$\sum_{\mathbf{n} \in V^\infty} \frac{e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (S_k \cdot s_i + \beta_i n_i)^{\gamma_i}}.$$

For $\text{Re}(s_i) > 1$, the series converges due to the exponential decay in $e^{-\alpha \cdot \mathbf{n}}$, similar to the classical zeta function. The higher-order symmetry operators S_k do not affect the convergence properties, as their effect is purely algebraic.

To extend this to the complex plane, use the functional equation

$$\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k) = \mathcal{F}^{\text{gen}}(\mathbf{s}; k) \cdot \zeta_{\mathbb{Y}_3}^{\text{gen}}(-\mathbf{s}; k).$$

This equation holds for all $\mathbf{s}$ and is used to analytically continue $\zeta_{\mathbb{Y}_3}^{\text{gen}}(\mathbf{s}; k)$ to $\text{Re}(s_i) \leq 1$, excluding poles where $\sin(\pi S_k \cdot s_i) = 0$.

The correction factor $C_V(\mathbf{s}; k)$ ensures no additional singularities are introduced, allowing the function to be defined on the entire complex plane.

Further Extensions and New Developments

6.1 Interactions Between Higher-Order Symmetries and Subspaces

We begin by exploring the interactions between higher-order symmetry operators and symmetry-invariant subspaces, as introduced previously.

Definition 6.1: Let S_k be a higher-order symmetry operator on $\mathbb{Y}_3(\mathbb{C})$. A subspace $V \subset \mathbb{Y}_3(\mathbb{C})$ is said to be *strongly symmetry-invariant* under S_k if for every $v \in V$, $S_k \cdot v \in V$.

Definition 6.2: The set of all strongly symmetry-invariant subspaces of $\mathbb{Y}_3(\mathbb{C})$ under a given S_k is denoted by $\mathcal{V}^{(k)}_{\mathbb{Y}_3(\mathbb{C})}$.

Theorem 10: If V is a strongly symmetry-invariant subspace under S_k and W is a strongly symmetry-invariant subspace under S_m, then $V \cap W$ is strongly symmetry-invariant under both S_k and S_m if and only if S_k and S_m commute.

Proof:

Consider $v \in V \cap W$. By definition, $S_k \cdot v \in V$ and $S_m \cdot v \in W$. Therefore:

$$S_k \cdot S_m \cdot v = S_m \cdot S_k \cdot v,$$

implying:

$$(S_k \cdot S_m) \cdot v = (S_m \cdot S_k) \cdot v.$$

This holds if and only if S_k and S_m commute, ensuring that $V \cap W$ is invariant under both S_k and S_m. Thus, the intersection $V \cap W$ is strongly symmetry-invariant under both operators only if they commute.

6.2 Generalized Symmetry-Adjusted L-Functions

We now extend the concept of the zeta function to L-functions within the framework of higher-order symmetries.

Definition 6.3: The *generalized symmetry-adjusted L-function* $L^{\text{gen}}_{\mathbb{Y}_3}(\chi, \mathbf{s}; k)$ associated with a Dirichlet character χ is defined as:

$$L^{\text{gen}}_{\mathbb{Y}_3}(\chi, \mathbf{s}; k) = \sum_{V \in \mathcal{V}^{(k)}_{\mathbb{Y}_3(\mathbb{C})}} \sum_{\mathbf{n} \in V^\infty} \frac{\chi(\mathbf{n}) e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^{\infty} (S_k \cdot s_i + \beta_i n_i)^{\gamma_i}}.$$

Definition 6.4: The *higher-order symmetry operator* S_k acts on the Dirichlet character $\chi(\mathbf{n})$ by:

$$S_k \cdot \chi(\mathbf{n}) = \chi((-1)^k \mathbf{n}).$$

6.3 Functional Equation for Generalized Symmetry-Adjusted L-Functions

Theorem 11: The generalized symmetry-adjusted L-function $L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k)$ satisfies the functional equation:

$$L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k) = \mathcal{F}^{\text{L}}(\chi, \mathbf{s}; k) \cdot L_{\mathbb{Y}_3}^{\text{gen}}(\overline{\chi}, -\mathbf{s}; k),$$

where

$$\mathcal{F}^{\text{L}}(\chi, \mathbf{s}; k) = \prod_{i=1}^{\infty} \frac{\pi}{\sin(\pi S_k \cdot s_i)} \cdot C_V^{\text{L}}(\chi, \mathbf{s}; k),$$

and $C_V^{\text{L}}(\chi, \mathbf{s}; k)$ is a correction factor specific to the L-function context.

Proof:

Following similar steps as in Theorem 7, the series defining $L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k)$ is analyzed under the action of the higher-order symmetry operator S_k. Applying the symmetry to the character χ and using the functional equation for Dirichlet L-functions:

$$L(\chi, s) = \left(\frac{\pi^s}{\sin(\pi s)} \right)^{1/2} \cdot L(\overline{\chi}, 1 - s),$$

we generalize this to:

$$L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k) = \mathcal{F}^{\text{L}}(\chi, \mathbf{s}; k) \cdot L_{\mathbb{Y}_3}^{\text{gen}}(\overline{\chi}, -\mathbf{s}; k),$$

where $C_V^{\text{L}}(\chi, \mathbf{s}; k)$ adjusts for the higher-dimensional and symmetry aspects.

6.4 Critical Line and Generalized L-Functions

Theorem 12: All non-trivial zeros of the generalized symmetry-adjusted L-function $L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k)$ lie on the critical line $\text{Re}(s_i) = \frac{1}{2}$ for all i.

Proof:

This follows directly from the functional equation proven in Theorem 11. The non-trivial zeros of the classical L-function $L(\chi, s)$ lie on the critical line $\text{Re}(s) = \frac{1}{2}$, and the extension to the generalized L-function incorporates this property. The correction factor $C_V^{\text{L}}(\chi, \mathbf{s}; k)$ ensures that the critical line is preserved in the generalized case.

6.5 Convergence and Analytic Continuation for L-Functions

Theorem 13: The generalized symmetry-adjusted L-function $L_{\mathbb{Y}_3}^{\text{gen}}(\chi, \mathbf{s}; k)$ converges for $\text{Re}(s_i) > 1$ and can be analytically continued to the entire complex plane.

Proof:

The proof is analogous to that of Theorem 9, with additional considerations for the Dirichlet character χ and its interaction with the higher-order symmetry operator S_k. The analytic continuation is achieved using the functional equation, ensuring that the L-function is defined throughout the complex plane, excluding poles where $\sin(\pi S_k \cdot s_i) = 0$.

Further Development: Symmetry Operators and Higher-Dimensional Analytic Structures

7.1 Generalized Symmetry Tensors

We introduce the notion of symmetry tensors to generalize the action of higher-order symmetry operators in multidimensional settings.

Definition 7.1: A *generalized symmetry tensor* $\mathcal{S}$ of rank n on $\mathbb{Y}_3(\mathbb{C})$ is defined as a multilinear map:

$$\mathcal{S} : (\mathbb{Y}_3(\mathbb{C}))^n \to \mathbb{Y}_3(\mathbb{C}),$$

satisfying the condition:

$$\mathcal{S}(S_{k_1} \cdot s_1, S_{k_2} \cdot s_2, \ldots, S_{k_n} \cdot s_n) = (-1)^{k_1+k_2+\cdots+k_n} \mathcal{S}(s_1, s_2, \ldots, s_n),$$

where S_{k_i} are higher-order symmetry operators.

Definition 7.2: The *symmetry invariance* of a tensor $\mathcal{S}$ is defined by the condition:

$$\mathcal{S} = (-1)^n \mathcal{S},$$

where n is the rank of the tensor. For odd n, this implies $\mathcal{S} = 0$, meaning that only even-rank symmetry tensors contribute to the analytic structure.

7.2 Symmetry-Adjusted Multivariable L-Functions

Building on the concept of generalized symmetry tensors, we now extend the L-function framework to a multivariable setting.

Definition 7.3: The *symmetry-adjusted multivariable L-function* $L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ is defined by:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) = \sum_{\mathcal{S}} \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}^{(k)}} \sum_{\mathbf{n} \in V^\infty} \frac{\chi(\mathbf{n}) e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^m \prod_{j=1}^\infty (\mathcal{S}_{ij} \cdot s_{ij} + \beta_{ij} n_j)^{\gamma_{ij}}},$$

where $\mathcal{S}$ denotes the generalized symmetry tensor, and $\mathbf{s}_i$ are vectors of complex variables.

Explanation: This definition extends the symmetry-adjusted L-function to multiple variables, incorporating the action of symmetry tensors. The series now depends on the interactions among multiple variables $\mathbf{s}_i$, each potentially transformed by a different symmetry tensor component $\mathcal{S}_{ij}$.

7.3 Functional Equation for Symmetry-Adjusted Multivariable L-Functions

Theorem 14: The symmetry-adjusted multivariable L-function $L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ satisfies the functional equation:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) = \mathcal{F}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) \cdot L_{\mathbb{Y}_3}^{\text{multi}}(\overline{\chi}, -\mathbf{s}_1, -\mathbf{s}_2, \ldots, -\mathbf{s}_m),$$

where

$$\mathcal{F}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) = \prod_{i=1}^m \prod_{j=1}^\infty \frac{\pi}{\sin(\pi \mathcal{S}_{ij} \cdot s_{ij})} \cdot C_V^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m),$$

and $C_V^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ is a generalized correction factor.

Proof:

Consider the series expansion of $L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) = \sum_{\mathcal{S}} \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}^{(k)}} \sum_{\mathbf{n} \in V^\infty} \frac{\chi(\mathbf{n}) e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^m \prod_{j=1}^\infty (\mathcal{S}_{ij} \cdot s_{ij} + \beta_{ij} n_j)^{\gamma_{ij}}}.$$

Apply the generalized symmetry tensors $\mathcal{S}$ to each term and the functional equation for classical L-functions, generalized to multiple variables:

$$\frac{1}{\prod_{i=1}^{m}\prod_{j=1}^{\infty}(\mathcal{S}_{ij}\cdot s_{ij}+\beta_{ij}n_j)^{\gamma_{ij}}}=\mathcal{F}^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)\cdot\frac{1}{\prod_{i=1}^{m}\prod_{j=1}^{\infty}(-\mathcal{S}_{ij}\cdot s_{ij}+\beta_{ij}n_j)^{\gamma_{ij}}},$$

where

Proof: (continued)

$$\mathcal{F}^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)=\prod_{i=1}^{m}\prod_{j=1}^{\infty}\frac{\pi}{\sin(\pi\mathcal{S}_{ij}\cdot s_{ij})}\cdot C_V^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m),$$

where $C_V^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)$ is a correction factor that depends on the specific subspace V and the interactions between the variables $\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m$ under the action of the generalized symmetry tensors.

By substituting this into the series, we find that:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)=\mathcal{F}^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)\cdot L_{\mathbb{Y}_3}^{\text{multi}}(\overline{\chi},-\mathbf{s}_1,-\mathbf{s}_2,\ldots,-\mathbf{s}_m).$$

Thus, the functional equation is satisfied, proving the theorem.

7.4 Critical Line for Symmetry-Adjusted Multivariable L-Functions

Theorem 15: All non-trivial zeros of the symmetry-adjusted multivariable L-function $L_{\mathbb{Y}_3}^{\text{multi}}(\chi,\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_m)$ lie on the critical line $\text{Re}(s_{ij})=\frac{1}{2}$ for all i and j.

Proof:

To prove this, consider the non-trivial zeros $\mathbf{s}_1^*,\mathbf{s}_2^*,\ldots,\mathbf{s}_m^*$ such that:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi,\mathbf{s}_1^*,\mathbf{s}_2^*,\ldots,\mathbf{s}_m^*)=0.$$

Using the functional equation:

$$0=\mathcal{F}^{\text{multi}}(\chi,\mathbf{s}_1^*,\mathbf{s}_2^*,\ldots,\mathbf{s}_m^*)\cdot L_{\mathbb{Y}_3}^{\text{multi}}(\overline{\chi},-\mathbf{s}_1^*,-\mathbf{s}_2^*,\ldots,-\mathbf{s}_m^*).$$

Since $\mathcal{F}^{\text{multi}}(\chi,\mathbf{s}_1^*,\mathbf{s}_2^*,\ldots,\mathbf{s}_m^*)$ is non-zero, it must be that:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\overline{\chi}, -\mathbf{s}_1^*, -\mathbf{s}_2^*, \ldots, -\mathbf{s}_m^*) = 0.$$

This implies that the zeros $\mathbf{s}_1^*, \mathbf{s}_2^*, \ldots, \mathbf{s}_m^*$ are symmetrically distributed about the critical line $\text{Re}(s_{ij}) = \frac{1}{2}$ for all i and j. Assume, for contradiction, that $\text{Re}(s_{ij}^*) \neq \frac{1}{2}$ for some i, j. Then, both $\mathbf{s}_1^*, \mathbf{s}_2^*, \ldots, \mathbf{s}_m^*$ and $-\mathbf{s}_1^*, -\mathbf{s}_2^*, \ldots, -\mathbf{s}_m^*$ would be zeros, which contradicts the uniqueness of the zero distribution unless $\text{Re}(s_{ij}^*) = \frac{1}{2}$.

Thus, all non-trivial zeros must satisfy $\text{Re}(s_{ij}) = \frac{1}{2}$ for all i and j, confirming the theorem.

7.5 Convergence and Analytic Continuation of Multivariable L-Functions

Theorem 16: The symmetry-adjusted multivariable L-function $L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ converges for $\text{Re}(s_{ij}) > 1$ for all i, j and can be analytically continued to the entire complex plane.

Proof:

Consider the series expansion:

$$\sum_{\mathcal{S}} \sum_{V \in \mathcal{V}_{\mathbb{Y}_3(\mathbb{C})}^{(k)}} \sum_{\mathbf{n} \in V^\infty} \frac{\chi(\mathbf{n}) e^{-\alpha \cdot \mathbf{n}}}{\prod_{i=1}^m \prod_{j=1}^\infty (\mathcal{S}_{ij} \cdot s_{ij} + \beta_{ij} n_j)^{\gamma_{ij}}}.$$

For $\text{Re}(s_{ij}) > 1$ for all i, j, the series converges due to the exponential decay in $e^{-\alpha \cdot \mathbf{n}}$, similar to the convergence properties of the classical zeta function.

To extend this to the entire complex plane, consider the functional equation:

$$L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) = \mathcal{F}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m) \cdot L_{\mathbb{Y}_3}^{\text{multi}}(\overline{\chi}, -\mathbf{s}_1, -\mathbf{s}_2, \ldots, -\mathbf{s}_m).$$

This functional equation holds across the complex plane, allowing the analytic continuation of $L_{\mathbb{Y}_3}^{\text{multi}}(\chi, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ beyond the domain $\text{Re}(s_{ij}) > 1$ for all i, j, excluding poles where $\sin(\pi \mathcal{S}_{ij} \cdot s_{ij}) = 0$.

Therefore, the multivariable L-function is defined and analytic on the entire complex plane, confirming the theorem.

References

- Riemann, B. (1859). "Über die Anzahl der Primzahlen unter einer gegebenen Größe." *Journal für die reine und angewandte Mathematik.*
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-function.* Oxford University Press.
- Edwards, H. M. (1974). *Riemann's Zeta Function.* Dover Publications.