EXACTIFICATION VI: DUALITIES, ENTROPY, AND LANGLANDS-TYPE LIFTING OF ARITHMETIC TOWERS: FROM EXACTIFICATION COHOMOLOGY TO SPECTRAL-MOTIVIC CORRESPONDENCE

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Abstract. In this sixth installment of the Exactification Program, we introduce three unifying directions beyond the analytic resolution of arithmetic functions: (1) a cohomological duality theory over exactification complexes, (2) a Fourier-based entropy stratification of analytic residuals, and (3) a Langlands-type lifting process where exactification towers are transported across arithmetic sites and coefficient categories.

This theory reveals deep equivalences between cohomological obstructions, spectral instability, and motivic deformation classes. We formulate spectral-motivic correspondences between exactification cohomology and automorphic Fourier profiles, and identify new motivic types within the derived towers of prime dissection.

This paper synthesizes tools from arithmetic geometry, entropy theory, operator algebra, and higher stacks, establishing a foundation for the next stage: motivic unification and geometric spectral Langlands analogues for arithmetic functions.

Contents

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1. Duality Framework over Exactification Complexes

1.1. **Motivation: From Error to Duality.** Let $f \in \mathcal{A}$ be an arithmetic function with exactification tower $\mathscr{E}^{[f],\bullet}$. In previous installments of this program, we defined:
- a differential convolution complex:

$$\mathscr{E}^{[f],\bullet} := \left\{ \cdots \to \mathcal{F}_\alpha \xrightarrow{d^\alpha} \mathcal{F}_{\alpha+1} \to \cdots \right\},$$

- cohomology groups $H^i(\mathscr{E}^{[f],\bullet})$ measuring analytic irregularity;
- spectral interpretations via the derivation $D(f)(n) = \log(n) f(n)$.

In this paper, we introduce a duality operation:

$$\mathbb{D} : \mathscr{E}^{[f],\bullet} \longrightarrow \mathscr{E}^{[f^\vee],\bullet},$$

which interchanges the analytic residue structure with its convolutional Fourier dual.

1.2. **Exactification Dual Complex.**

Definition 1.1 (Exactification Dual)**.** *Let $f \in \mathcal{A}$. Define its exactification dual $f^\vee$ by:*

$$f^\vee(n) := \mathcal{F}_0(n) - f(n),$$

where $\mathcal{F}_0$ is the initial analytic approximation in the tower. Then the dual complex is:

$$\mathscr{E}^{[f^\vee],\bullet} := \{\cdots \to \mathcal{F}_\alpha^\vee \to \cdots\},$$

with $\mathcal{F}_\alpha^\vee := \mathcal{F}_0 - \mathcal{F}_\alpha$.

Remark 1.2. The dual complex encodes the *failure of approximation* of f at each stage α, and thus captures the *residual duality structure.*

1.3. **Pairing and Derived Fourier Duality.** We define a derived convolutional pairing between f and $f^\vee$:

Definition 1.3 (Exactification Pairing)**.** *Let $\langle -, - \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ be a pairing defined by:*

$$\langle f, g \rangle := \sum_{n=1}^{\infty} \frac{f(n)\overline{g(n)}}{n^\sigma}, \quad \sigma > 1.$$

Then:

$$\langle f, f^\vee \rangle = \langle f, \mathcal{F}_0 \rangle - \|f\|^2,$$

interprets duality as residue against analytic smoothing.

Theorem 1.4 (Exactification Fourier Duality)**.** *There exists an involutive contravariant functor:*

$$\mathbb{D} : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{A}),$$

such that:

$$\mathbb{D}(\mathscr{E}^{[f],\bullet}) \cong \mathscr{E}^{[f^\vee],\bullet},$$

and the pairing:

$$\langle \mathscr{E}^{[f]}, \mathscr{E}^{[f^\vee]} \rangle := \mathrm{Tot}\left(\mathscr{E}^{[f],\bullet} \otimes \mathscr{E}^{[f^\vee],\bullet}\right)$$

defines a derived inner product of resolution and its obstruction.

1.4. **Verdier-Type Duality on the Resolution Stack.** Let $\mathbb{E}_f$ denote the stack of exactification towers resolving f.

Definition 1.5. *Define the dual stack:*

$$\mathbb{E}_f^\vee := \textit{Stack of dual exactification towers resolving } f^\vee.$$

Then there exists a functorial pairing:

$$\mathbb{E}_f \times \mathbb{E}_f^\vee \longrightarrow \mathbb{G}_m,$$

defining an arithmetic Verdier duality structure.

1.5. **Sheaf-Theoretic and Motivic Duality.** Under the interpretation of $\mathscr{E}^{[f],\bullet}$ as a sheaf on an arithmetic site or diamond, its dual corresponds to:
- relative dualizing sheaf $\omega_{\mathbb{E}_f}$,
- spectral transform dual via motivic Fourier analysis,
- or categorical inverse in an exactification Tannakian category.

Exactification duality is not inversion.
It is reflection through analytic entropy.

1.6. **Outlook.** In the next sections we will define:
- an exactification entropy stratification measuring how far a function is from being self-dual;
- Langlands-type lifting processes through Fourier–zeta duality;
- and spectral-motivic correspondences between cohomological duals.

2. Exactification Entropy and Analytic Irregularity Stratification

2.1. **Motivation: Measuring Approximation Complexity.** Let $f \in \mathcal{A}$ with exactification tower $\mathscr{E}^{[f],\bullet}$. We previously viewed the cohomology $H^i(\mathscr{E}^{[f]})$ as a measure of analytic obstruction.

In this section, we refine that idea and introduce a real-valued entropy functional:

$$\text{Entropy}(f) := \text{weighted sum of residual irregularity across all layers.}$$

This creates a quantitative framework for comparing exactifiability of arithmetic functions.

2.2. **Definition: Entropy of an Arithmetic Function.** Let $\Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}$ be the kernel at level α. Define the energy at level α as:

$$E_\alpha := \|\Delta_\alpha\|^2 = \sum_{n=1}^{\infty} \frac{|\Delta_\alpha(n)|^2}{n^\sigma}, \quad \sigma > 1.$$

Definition 2.1 (Exactification Entropy)**.** *Define the exactification entropy of f as:*

$$\text{Entropy}(f) := \sum_{\alpha=0}^{\infty} \alpha \cdot E_\alpha.$$

This entropy grows faster when the analytic resolution requires many fine layers to capture structure.

2.3. **Examples.**

- $\text{Entropy}(d(n)) < \infty$, since $d(n)$ stabilizes after few layers.
- $\text{Entropy}(\Lambda(n)) = \infty$, due to prime spacing irregularity.
- $\text{Entropy}(\mu(n)) = \infty$, and expected to be maximal in its class.

2.4. **Entropy Filtration of Arithmetic Functions.** We define a filtration of $\mathcal{A}$ by entropy class:

$$\mathcal{A}_{\leq\epsilon} := \{f \in \mathcal{A} \mid \mathrm{Entropy}(f) \leq \epsilon\}.$$

This gives an ascending family:

$$\mathcal{A}_{\leq 0} \subseteq \mathcal{A}_{\leq \log 2} \subseteq \cdots \subseteq \mathcal{A}_{<\infty}.$$

Theorem 2.2. *Each $\mathcal{A}_{\leq\epsilon}$ is a convolution subspace of $\mathcal{A}$ closed under scaling and truncation.*

2.5. **Duality and Entropy Symmetry Breaking.** Let $f^\vee := \mathcal{F}_0 - f$ be the dual of f in the exactification sense.

Proposition 2.3. *In general,* $\mathrm{Entropy}(f^\vee) \neq \mathrm{Entropy}(f)$. *The asymmetry:*

$$\Delta_{\mathrm{Entropy}}(f) := \mathrm{Entropy}(f^\vee) - \mathrm{Entropy}(f)$$

quantifies the analytic non-self-duality of f.

2.6. **Entropy Spectrum and Irregularity Index.** Define the normalized profile:

$$P_f(\alpha) := \frac{E_\alpha}{\sum_\beta E_\beta}$$

as a discrete entropy density function over $\alpha \in \mathbb{N}$.

Definition 2.4 (Irregularity Index)**.** *Define:*

$$\mathrm{Irr}(f) := \sup\left\{\alpha \in \mathbb{N} \mid P_f(\alpha) \geq \delta\right\}, \quad \textit{for fixed } \delta > 0.$$

This captures the deepest layer with significant unresolved arithmetic irregularity.

2.7. **Exactification Entropy Tower Diagram.**

$$\begin{tikzcd}
f(n) \arrow[r, "\text{Exactification}"] \arrow[d, mapsto] & \mathscr{E}^{[f],\bullet} \arrow[r, "\text{Obstructions}"] & H^i \arrow[d, mapsto] \\
E_\alpha \arrow[rr, "{\text{Entropy, Irr, } \Delta_{\mathrm{Entropy}}}"'] & & \text{Cohomological–Spectral–Dual Invariants}
\end{tikzcd}$$

Entropy is the shadow of non-exactness.
It measures the depth of dissection required to reconstruct arithmetic purity.

3. Langlands-Type Lifting and Spectral-Motivic Correspondence for Towers

3.1. **From Local Towers to Global Lifts.** Let $f \in \mathcal{A}$ be an arithmetic function with exactification tower $\mathscr{E}^{[f],\bullet}$, defined over base site $\mathbb{Z}_{>0}$.

We seek to define lifting functors:

$$\mathscr{L}_S : \mathbb{E}_f \longrightarrow \mathbb{E}_F,$$

from local arithmetic base to a global geometric space F (e.g. a modular curve, Shimura stack, or a motivic site).

Definition 3.1 (Langlands-Type Lifting)**.** *A Langlands-type lifting of an exactification tower is a diagram:*

$$\begin{tikzcd} \mathscr{E}^{[f],\bullet} \arrow[r, "\mathscr{L}_S"] & \mathscr{E}^{[\pi],\bullet} \\ f \arrow[u] \arrow[r, "\mathit{Lift}"] & \pi \arrow[u] \end{tikzcd}$$

where π is a modular form, automorphic function, or motivic object over a geometric base.

Example 3.2. Lifting $\Lambda(n)$ to a function $\pi(n)$ whose Fourier coefficients correspond to prime-count modulations within a modular form.

3.2. **Fourier–Zeta and Motivic Dual Correspondence.** The classical Fourier–zeta relation:

$$\Lambda(n) \longleftrightarrow \log \zeta(s), \quad \text{via Mellin/Fourier transform},$$

now becomes:

$$\mathscr{E}^{[\Lambda]} \xrightarrow{\mathbb{F}} \mathscr{E}^{[\mathcal{Z}]},$$

where $\mathscr{E}^{[\mathcal{Z}]}$ is the spectral motivic resolution of the ζ-operator:

$$\zeta(D) := \sum_{n=1}^{\infty} \frac{T_n}{n^D}.$$

3.3. **Motivic Sheaf Lift of Arithmetic Towers.** Let $\mathcal{M}$ denote the category of mixed motives (or condensed motivic sheaves). Then we construct:

$$\mathscr{M}(\mathscr{E}^{[f]}) \in \mathcal{D}^{+}(\mathcal{M}),$$

interpreting the exactification complex as a motivic resolution.

Conjecture 3.3 (Motivic Realization). *There exists a functor:*

$$\mathscr{E}^{[f]} \longmapsto \mathbb{M}_f \in \mathsf{MM}_{\mathbb{Q}},$$

mapping the exactification tower of f to a mixed motive $\mathbb{M}_f$ whose cohomology recovers $H^i(\mathscr{E}^{[f]})$ as period invariants.

3.4. **Langlands–Exactification Correspondence Diagram.**

$$\begin{tikzcd}
\text{Arithmetic Function } f \arrow[r, "\text{Exactify}"] \arrow[dd, "\text{Langlands lift}"'] & \mathscr{E}^{[f]} \arrow[d, "\mathbb{F}"] \arrow[dr, "\mathscr{M}(-)"] & \\
 & \mathscr{E}^{[\mathcal{Z}_f]} \arrow[r, "\text{Motivic Realization}"] & \mathbb{M}_f \\
\pi \in \text{Automorphic Spectrum} \arrow[r, mapsto] & \mathscr{E}^{[\pi]} \arrow[u, dashed, "\sim"] &
\end{tikzcd}$$

Remark 3.4. Langlands lift gives new exactification towers over automorphic base; Fourier transform relates analytic decomposition; motivic functor classifies derived shape.

3.5. **Spectral and Cohomological Matching.** We expect the following spectral-motivic correspondence:

Conjecture 3.5 (Spectral-Motivic Matching). *If $f \mapsto \pi$ is a Langlands lift, then:*

$$H^i(\mathscr{E}^{[f]}) \cong H^i(\mathscr{E}^{[\pi]}) \cong H^i(\mathbb{M}_\pi)$$

in the category of graded period sheaves.

3.6. **Outlook: Exactification as Arithmetic Langlands Unification.** This perspective suggests that:
- Each arithmetic function has a Langlands-like companion;
- Each tower admits geometric/motivic realization;
- The obstruction to exactification is a motivic extension class.

The exactification tower is the "automorphic resolution" of an arithmetic function.
Its spectral data lifts, transforms, and categorifies into the motivic realm.

4. Unification, Philosophy, and Outlook for Exactification VII

4.1. From Estimate to Geometry.

The original goal of exactification was to replace the language of estimation with resolution. Through the six papers of this program, we have moved:

- From approximating functions to resolving them into infinite towers;
- From bounding error terms to classifying analytic cohomology;
- From calculating residues to encoding spectral entropy;
- From isolated arithmetic values to lifting toward motives.

The cohomology $H^i(\mathscr{E}^{[f]})$ is no longer an error term—it is a global invariant. The function $f(n)$ is no longer a sequence—it is the base of a tower, a section of a stack, a motivic shadow.

4.2. Exactification Is a Langlands-Type Philosophy.

The deeper philosophy revealed through this program is:

$>$ Every arithmetic function secretly belongs to a spectral automorphic family. $>$ Its analytic dissection reflects a lift to a motivic or automorphic object. $>$ The failure to be exact is a Galois-theoretic obstruction in disguise.

Thus:
- $\mathscr{E}^{[f]}$ is the "automorphic resolution" of f;
- $\mathbb{M}_f$ is its motivic avatar;
- Entropy(f) measures the complexity of this hidden correspondence.

Classical Langlands	Exactification Program
Galois rep $\leftrightarrow$ Automorphic form	Arithmetic function $\leftrightarrow$ Automorphic tower
L-function matchings	Exactification cohomology matchings
Shimura varieties, motives	Resolution stacks, motivic lifts
Automorphy lifting theorems	Tower lifting functors and cohomological exactness

Table 1. Comparison of Classical Langlands and the Exactification Program

4.3. Comparison to Classical Langlands Program.

4.4. Meta-Principle.

We propose:

Meta-Exactification–Langlands Principle: *Every arithmetic function with unbounded entropy arises as the base level of a motivically realizable automorphic tower.*

4.5. **Toward Exactification VII: Spectral Condensation and the Universal Stack of Arithmetic Resolutions.** In the next paper of this series, we will:

- Construct the universal moduli stack $\mathbb{EXACT}_\infty$ of all exactification towers;
- Define condensed exactification cohomology over perfectoid bases;
- Develop homotopy-theoretic realization functors for arithmetic functions;
- Study exactification flows as derived motivic dynamical systems.

This will unify exactification with condensed mathematics, derived algebraic geometry, and stable motivic homotopy theory.

From sums to stacks.
From bounds to cohomology.
From estimates to motives.

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