MULTI-LAYERED YANG-GALOIS GROUPS AND YANG-LANGLANDS PROGRAM

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Contents

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1. Introduction

The Yang-Langlands Program extends the classical Langlands program by introducing the $\mathbb{Y}_n(F)$ number systems, which allow for the definition of new arithmetic structures, including a hierarchy of Galois and Yang-Galois groups.

2. Hierarchy of Yang-Galois Groups

We define the following Galois-type structures:

2.1. **Absolute Yang-Galois Group.**

$$\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))$$

This is the fundamental Galois group that governs the full arithmetic structure of $\mathbb{Y}_n(F)$. It serves as an analogue to the classical absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

2.2. **Intermediate Yang-Galois Group.**

$$\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(\overline{F}))$$

This group captures the extension between $\mathbb{Y}_n(\overline{F})$, the closure of $\mathbb{Y}_n(F)$ over $\overline{F}$, and the full closure $\overline{\mathbb{Y}_n(F)}$.

2.3. **Arithmetic Yang-Galois Group.**

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F))$$

This group governs the behavior of $\mathbb{Y}_n(F)$ within the algebraic closure $\overline{F}$ of F, forming an arithmetic substructure of the absolute Yang-Galois group.

2.4. **Universal Yang-Galois Group.**

$$\mathrm{Gal}(Q_{\mathbb{Y}_n,v\alpha}/\mathbb{Y}_n(F))$$

where $Q_{\mathbb{Y}_n,v\alpha}$ is the universal valuation field associated with $\mathbb{Y}_n(F)$.

3. Yang-Langlands Correspondence

With the Yang-Galois structures defined, we establish an extended correspondence:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \longleftrightarrow A_{\mathbb{Y}_n(F)}$$

where $A_{\mathbb{Y}_n(F)}$ denotes the automorphic representation space associated with $\mathbb{Y}_n(F)$. This extends the classical Langlands duality:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longleftrightarrow \mathrm{GL}_n(\mathbb{A})$$

4. Yang-Langlands L-Functions

Extending classical L-functions, we define:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_{v\alpha} \zeta_{Q_{\mathbb{Y}_n,v\alpha}}(s)$$

which generalizes the classical Dirichlet series into the Yang-Langlands framework.

5. Geometric Interpretation

We introduce a moduli interpretation for the Yang-Langlands program:

$$D^b(Y_{\mathbb{Y}_n}) \longleftrightarrow \mathrm{Rep}(G_{\mathbb{Y}_n})$$

where $D^b(Y_{\mathbb{Y}_n})$ is the bounded derived category of coherent sheaves over a Nocturnis moduli space.

6. Future Directions

- Classification of Yang-Galois representations.
- Higher categorical structures for $\mathbb{Y}_n(F)$ arithmetic.
- p-adic and geometric extensions of Yang-Langlands.

7. Hierarchy of Yang-Galois Groups

We define a multi-layered hierarchy of Yang-Galois groups associated with $\mathbb{Y}_n(F)$:

7.1. Absolute Yang-Galois Group.

$$\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))$$

This group serves as an analogue to the classical absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, governing the full arithmetic structure of $\mathbb{Y}_n(F)$.

7.2. Intermediate Yang-Galois Group.

$$\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(\overline{F}))$$

This structure encodes the extension between $\mathbb{Y}_n(\overline{F})$, the closure of $\mathbb{Y}_n(F)$ over $\overline{F}$, and the full closure $\overline{\mathbb{Y}_n(F)}$.

7.3. Arithmetic Yang-Galois Group.

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F))$$

This group governs the behavior of $\mathbb{Y}_n(F)$ within the algebraic closure $\overline{F}$ of F, forming an arithmetic substructure of the absolute Yang-Galois group.

7.4. Universal Yang-Galois Group.

$$\mathrm{Gal}(Q_{\mathbb{Y}_n,v\alpha}/\mathbb{Y}_n(F))$$

where $Q_{\mathbb{Y}_n,v\alpha}$ is the universal valuation field associated with $\mathbb{Y}_n(F)$. This group represents a global arithmetic structure extending beyond classical number fields.

8. Extended Yang-Langlands Correspondence

Given these new Yang-Galois groups, we define the following extended Langlands-type dualities:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \longleftrightarrow A_{\mathbb{Y}_n(F)}$$

where $A_{\mathbb{Y}_n(F)}$ denotes the automorphic representation space associated with $\mathbb{Y}_n(F)$. This extends the classical Langlands correspondence:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longleftrightarrow \mathrm{GL}_n(\mathbb{A})$$

by incorporating the $\mathbb{Y}_n(F)$ number systems and their associated valuation structures.

9. Generalized Yang-Langlands L-Functions

With these new Yang-Galois structures, we propose a refined L-function framework:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_{v\alpha} \zeta_{Q_{\mathbb{Y}_n,v\alpha}}(s)$$

which extends the classical Dirichlet series into the higher-dimensional Yang-Langlands setting, capturing deeper arithmetic properties.

10. Refined Structure of Yang-Galois Groups

Building upon the previous hierarchy, we introduce further refinements that capture deeper arithmetic and geometric aspects of the Yang-Langlands program.

10.1. **Derived Yang-Galois Group.** We define the derived version of the Yang-Galois group, capturing higher homotopical structures:

$$D_{\mathrm{Gal}}(\mathbb{Y}_n(F)) = \pi_1^{\mathrm{hom}}(\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)))$$

where π_1^{hom} represents the fundamental group in the homotopical category. This provides a connection between the representation theory of Yang-Galois groups and derived algebraic geometry.

10.2. **Cohomological Yang-Galois Group.** We extend the cohomological framework by defining:

$$H^i_{\mathrm{Gal}}(\mathbb{Y}_n(F), M) = H^i(\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)), M)$$

for an arbitrary module M, allowing us to explore a cohomological perspective of the Yang-Langlands program.

10.3. **Geometric Yang-Galois Group.** For the geometric counterpart, we propose:

$$\pi_1^{\mathrm{arith}}(\mathbb{Y}_n(F)) \longrightarrow \pi_1^{\mathrm{geom}}(\mathbb{Y}_n(\overline{F}))$$

which describes the distinction between the arithmetic fundamental group and the geometric fundamental group in the Yang-Langlands setting.

10.4. **Hecke Operators and Yang-Hecke Algebra.** The presence of multiple Yang-Galois groups suggests an underlying algebraic structure governed by a Yang-Hecke algebra:

$$\mathcal{H}_{\mathbb{Y}_n(F)} = \bigoplus_{g \in G_{\mathbb{Y}_n(F)}} \mathbb{C}[g]$$

which generalizes the classical Hecke algebra to capture automorphic structures of $\mathbb{Y}_n(F)$.

11. Expanded Yang-Langlands Correspondence

With these refined structures, we now formulate an extended version of the Yang-Langlands correspondence:

$$\mathrm{Hom}_{\mathrm{Rep}}(G_{\mathbb{Y}_n(F)}, A_{\mathbb{Y}_n(F)}) \longleftrightarrow \mathrm{Coh}(\mathbb{Y}_n(F))$$

where $\mathrm{Coh}(\mathbb{Y}_n(F))$ is the category of coherent sheaves associated with the moduli space of Yang-Langlands representations.

12. Higher-Dimensional Yang-Langlands L-Functions

We extend the previous L-function construction by introducing a multi-variable Yang-zeta function:

$$\zeta_{\mathbb{Y}_n}(s_1, s_2, \dots, s_k) = \prod_{v\alpha} L_{\mathbb{Y}_n, v\alpha}(s_1, s_2, \dots, s_k)$$

where each variable s_i corresponds to a different layer of the Yang-Galois hierarchy.

13. Geometric Moduli of Yang-Langlands Objects

We propose a derived category formulation:

$$D^b(Y_{\mathbb{Y}_n}) \longleftrightarrow \mathrm{Rep}_{\mathrm{coh}}(G_{\mathbb{Y}_n})$$

establishing a geometric Langlands-type duality between the derived category of coherent sheaves and the category of coherent representations of the Yang-Galois group.

14. Future Perspectives

- Defining explicit automorphic sheaves for the Yang-Langlands framework.
- Extending the program to an ∞-categorical setting.
- Developing explicit moduli stacks governing the representation theory of $G_{\mathbb{Y}_n(F)}$.
- Establishing connections with arithmetic dynamics and motivic homotopy theory.

15. Spectral and Adelic Yang-Langlands Theory

We introduce the spectral perspective on the Yang-Langlands program, incorporating the adelic and harmonic analysis structures.

15.1. Adelic Yang-Galois Representations.

For a global field F and its adeles $\mathbb{A}_F$, we define the adelic Yang-Galois group:

$$G_{\mathbb{Y}_n}(\mathbb{A}_F) = \varprojlim \mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \times \mathbb{A}_F^\times$$

which extends the classical Langlands framework into the adelic setting of the Yang-Langlands correspondence.

15.2. Spectral Decomposition and Automorphic Forms.

Given a Hecke algebra $\mathcal{H}_{\mathbb{Y}_n(F)}$, we define the spectral decomposition of automorphic forms:

$$L^2(G_{\mathbb{Y}_n}(\mathbb{A}_F)) = \bigoplus_{\pi} m(\pi)\pi,$$

where π ranges over automorphic representations and $m(\pi)$ denotes the multiplicity.

16. Functoriality in the Yang-Langlands Framework

We propose an extended functoriality principle:

$$\mathbb{Y}_n(F) \to \mathbb{Y}_m(F) \Rightarrow \mathrm{Gal}(\mathbb{Y}_m(\overline{F})/\mathbb{Y}_m(F)) \to \mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F))$$

which induces a natural transformation on the automorphic side:

$$A_{\mathbb{Y}_m(F)} \to A_{\mathbb{Y}_n(F)}$$

governing the behavior of representations under field extensions.

17. Motivic and Derived Yang-Langlands Program

17.1. Motivic Yang-Galois Extensions.

We define the motivic Galois group associated with the $\mathbb{Y}_n(F)$ number system:

$$G_{\mathbb{Y}_n}^{\mathrm{mot}} = \mathrm{Aut}(\mathcal{T}_{\mathbb{Y}_n}),$$

where $\mathcal{T}_{\mathbb{Y}_n}$ is the Tannakian category of pure motives over $\mathbb{Y}_n(F)$.

17.2. **Derived Automorphic Categories.** We extend automorphic sheaf theory to a derived categorical framework:

$$D^b_{\mathrm{coh}}(Y_{\mathbb{Y}_n}) \longleftrightarrow D^b(\mathrm{Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n})).$$

This establishes an ∞-categorical interpretation of the Yang-Langlands program.

18. Higher Reciprocity Laws in the Yang Framework

We generalize reciprocity laws to the Yang-Langlands setting:

$$\prod_v \mathrm{Art}_{\mathbb{Y}_n(F_v)} = \mathrm{Id},$$

where $\mathrm{Art}_{\mathbb{Y}_n(F_v)}$ denotes the Yang-Artin reciprocity map at each place v.

19. Higher p-adic Deformations in the Yang-Langlands Program

For a fixed prime p, we study the deformation theory of Yang-Galois representations:

$$\mathcal{D}_{\mathbb{Y}_n,p} = \mathrm{Hom}(W(\mathbb{F}_p), G_{\mathbb{Y}_n}).$$

This controls the infinitesimal structure of Yang-Galois representations in the p-adic setting.

20. Future Research Directions

- Investigate the role of perfectoid spaces in the Yang-Langlands theory.
- Develop explicit instances of functoriality in the Yang framework.
- Explore the relationship between Yang-Galois groups and non-commutative Iwasawa theory.
- Construct motivic L-functions within the Yang-Langlands framework.

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21. Yang-Langlands Dual Group and Tannakian Formalism

21.1. **Definition of the Yang-Langlands Dual Group.** We define the dual group $G^\vee_{\mathbb{Y}_n}$ as the Langlands dual of the Yang-Galois group:

$$G^\vee_{\mathbb{Y}_n} = \mathrm{Hom}_{\mathrm{Grp}}(G_{\mathbb{Y}_n}, \mathbb{C}^\times).$$

This governs the representation-theoretic duality between the automorphic and arithmetic sides of the Yang-Langlands program.

21.2. **Tannakian Interpretation.** The category of Yang-Galois representations is naturally a Tannakian category:

$$\mathrm{Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}) \cong \text{Ind-Tann}(\mathbb{Y}_n(F)).$$

This provides a bridge between the classical Langlands formalism and categorical structures in non-abelian Hodge theory.

22. Yang-Langlands for Reductive Algebraic Groups

For a connected reductive algebraic group G over $\mathbb{Y}_n(F)$, we formulate the correspondence:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \longleftrightarrow {}^L G.$$

Here ${}^L G$ is the L-group of G, encoding the arithmetic properties of Yang-Langlands representations.

23. Geometric Realization via Fargues-Fontaine Yang Space

23.1. Definition of the Yang-Fargues-Fontaine Curve. We define a non-Archimedean analytic space associated with $\mathbb{Y}_n(F)$:

$$\mathcal{X}_{\mathbb{Y}_n} = \varprojlim \mathrm{Spec}(\mathbb{Y}_n(\overline{F}) \otimes \mathbb{Z}_p).$$

This space generalizes the Fargues-Fontaine curve into the Yang-theoretic framework.

23.2. Geometric Langlands over $\mathcal{X}_{\mathbb{Y}_n}$. We propose a geometric Langlands-type correspondence:

$$D^b_{\mathrm{coh}}(\mathcal{X}_{\mathbb{Y}_n}) \longleftrightarrow D^b(\text{Ind-Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n})).$$

This extends the categorification of the classical Langlands program into an analytic-geometric setting.

24. p-adic and Perfectoid Extensions

24.1. Yang-Perfectoid Theory. For a perfectoid field extension $K/\mathbb{Y}_n(F)$, we introduce:

$$\mathbb{Y}_n^{\mathrm{perf}}(K) = \varprojlim \mathbb{Y}_n(K^{1/p^n}).$$

This structure is expected to play a crucial role in p-adic Hodge theory and p-adic automorphic forms.

24.2. p-adic Local Langlands and Higher Rigid Spaces. The p-adic version of the Yang-Langlands program can be formulated using rigid geometry:

$$\mathrm{Gal}(\mathbb{Y}_n(K)/\mathbb{Y}_n(F)) \longleftrightarrow \mathrm{GL}_n(D_K),$$

where D_K is a division algebra over the p-adic period ring.

25. Yang-Langlands L-Functions and Spectral Expansion

25.1. Higher-Dimensional Zeta and L-Functions. We propose a new class of L-functions associated with Yang-Galois groups:

$$L_{\mathbb{Y}_n}(s_1, s_2, \ldots, s_k) = \prod_{v\alpha} L_{Q_{\mathbb{Y}_n}, v\alpha}(s_1, s_2, \ldots, s_k).$$

These encode deeper spectral properties beyond classical automorphic L-functions.

25.2. Yang-Langlands Spectral Decomposition. The space of automorphic functions admits a spectral decomposition:

$$L^2(G_{\mathbb{Y}_n}(\mathbb{A})) = \bigoplus_{\pi} m(\pi)\pi,$$

where π runs over irreducible automorphic representations.

26. Applications and Future Research

- Explore the interactions between Yang-Langlands and the geometric Satake equivalence.
- Investigate potential links between Yang-Langlands L-functions and quantum field theory.
- Develop p-adic and perfectoid extensions to capture new aspects of arithmetic geometry.
- Construct explicit automorphic sheaves within the geometric Yang-Langlands framework.

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27. Categorification of the Yang-Langlands Program

27.1. **Infinity-Categorical Yang-Langlands Correspondence.** We extend the Yang-Langlands program to an ∞-categorical setting, defining an ∞-groupoid of Yang-Galois representations:

$$\mathrm{Sh}(\mathbb{Y}_n(F)) \simeq \mathrm{Fun}(BG_{\mathbb{Y}_n}, \mathcal{S}).$$

Here, $\mathrm{Sh}(\mathbb{Y}_n(F))$ is the derived stack of sheaves over $\mathbb{Y}_n(F)$, and $\mathcal{S}$ denotes the ∞-category of spaces.

27.2. **Higher Tannakian Duality.** We propose a higher-categorical extension of Tannakian duality:

$$\mathrm{Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}) \simeq \text{Ind-Pro-Tann}(\mathbb{Y}_n(F)),$$

where the right-hand side represents the category of ind-pro-motives associated with $\mathbb{Y}_n(F)$.

28. Adelic Yang-Langlands via Factorization Spaces

We introduce the factorization space formalism:

$$\mathcal{F}_{\mathbb{Y}_n} = \varprojlim \mathcal{M}_{G_{\mathbb{Y}_n}}.$$

This defines a geometric object encoding the adelic Langlands duality for $\mathbb{Y}_n(F)$.

28.1. **Hecke Eigenobjects and Factorization Categories.** The category of Hecke eigensheaves in the Yang setting satisfies:

$$D^b(\mathrm{Coh}(Bun_{G_{\mathbb{Y}_n}})) \simeq D^b(\mathrm{LocSys}_{G^\vee_{\mathbb{Y}_n}}).$$

This provides a categorified version of the classical Langlands correspondence.

29. Non-Archimedean and Perfectoid Geometry in Yang-Langlands

29.1. **Perfectoid Moduli of Automorphic Representations.** For a perfectoid field K over $\mathbb{Y}_n(F)$, we define:

$$\mathcal{M}^{\mathrm{perf}}_{G_{\mathbb{Y}_n}}(K) = \varprojlim \mathcal{M}_{G_{\mathbb{Y}_n}}(K^{1/p^n}).$$

This moduli space encodes the rigid-analytic structure of Yang automorphic forms.

29.2. **p-adic Geometric Satake Equivalence.** The p-adic Satake category in the Yang framework satisfies:

$$\mathrm{Sat}_{\mathbb{Y}_n} \simeq \mathrm{Rep}_{\mathbb{Y}_n}(G^{\vee}_{\mathbb{Y}_n}).$$

This establishes a deeper connection between Yang automorphic representations and the dual group.

30. Quantum Aspects of the Yang-Langlands Program

30.1. **Yang-Langlands and Quantum Groups.** We formulate a quantum extension of the Yang-Langlands program:

$$U_q(\mathfrak{g}_{\mathbb{Y}_n}) \longleftrightarrow \mathcal{D}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}).$$

This relates quantum groups to the derived geometric Langlands formalism.

30.2. **Topological Field Theory and Yang-Langlands.** We propose a TQFT realization:

$$\mathcal{Z}_{\mathbb{Y}_n}(\Sigma) = \int_{\mathbb{Y}_n(F)} \mathcal{A}_{\mathrm{Yang}}.$$

Here, $\mathcal{Z}_{\mathbb{Y}_n}(\Sigma)$ represents a quantum Yang-Langlands partition function associated with a surface Σ.

31. Concluding Remarks and Future Directions

- Explore the relationship between Yang-Langlands and the geometric Langlands program for algebraic loop groups.
- Develop connections between Yang-Langlands automorphic sheaves and logarithmic motives.
- Investigate the role of derived symplectic geometry in the representation theory of $G_{\mathbb{Y}_n}$.
- Formulate an explicit arithmetic application of the Yang-Langlands functoriality principle.

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32. Higher-Dimensional Noncommutative Yang-Langlands Program

32.1. **Noncommutative Geometry and Yang-Galois Theory.** We introduce a noncommutative version of the Yang-Galois group using derived noncommutative algebraic geometry:

$$G^{\mathrm{nc}}_{\mathbb{Y}_n} = \mathrm{Aut}_{\mathrm{NC}}(\mathbb{Y}_n(F)),$$

where $\mathrm{Aut}_{\mathrm{NC}}$ denotes the category of noncommutative automorphisms over $\mathbb{Y}_n(F)$.

32.2. **Yang-Langlands for Derived Affine Stacks.** Extending to the category of derived stacks, we define:

$$D^b_{\mathrm{NC}}(\mathbb{Y}_n) \longleftrightarrow D^b_{\mathrm{NC}}(\mathrm{Rep}(G_{\mathbb{Y}_n})).$$

This provides a new duality between derived noncommutative moduli spaces and Yang-Galois representations.

32.3. **Derived and Deformed Yang-Galois Representations.** For a formal deformation parameter $\hbar$, we introduce:

$$\mathrm{Def}_{\hbar} G_{\mathbb{Y}_n} = \mathrm{Spec}\, H^{\bullet}(\mathbb{Y}_n, \mathcal{O}_{G_{\mathbb{Y}_n}})_{\hbar}.$$

This controls infinitesimal deformations of Yang-Galois representations in the presence of a quantized parameter.

33. Yang-Langlands and Higher-Category Theory

33.1. **Infinity-Operads in the Yang Framework.** We define a homotopical generalization of the Yang-Langlands correspondence:

$$\mathcal{C}_{\infty}(\mathbb{Y}_n) = \varprojlim \mathrm{Fun}(BG_{\mathbb{Y}_n}, \mathcal{S}_{\infty}).$$

Here, $\mathcal{S}_{\infty}$ is the ∞-category of derived spectra.

33.2. **Extended Functoriality in Higher-Category Theory.** The Yang-Langlands functoriality principle extends to the ∞-categorical setting as:

$$\mathrm{Map}_{\infty}(X, Y_{\mathbb{Y}_n}) \longleftrightarrow \mathrm{Hom}_{\infty}(\mathrm{Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}), \mathrm{QCoh}(Y)).$$

34. Arithmetic Topology and Yang-Langlands

34.1. **Yang-Fundamental Groups in Arithmetic Topology.** We define an arithmetic homotopy group structure:

$$\pi_1^{\mathrm{arith}}(\mathbb{Y}_n(F)) = \varprojlim \mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)).$$

34.2. **Cohomological Yang Reciprocity Laws.** We propose a reciprocity law for higher arithmetic Yang-Galois groups:

$$\prod_v \mathrm{Art}_{\mathbb{Y}_n(F_v)} = \mathrm{Id}.$$

This extends class field theory to the Yang framework.

35. Yang-Langlands in Topological Quantum Field Theory

35.1. **2D and 3D TQFT Interpretations.** The partition function of a Yang-Langlands TQFT is given by:

$$Z_{\mathbb{Y}_n}(\Sigma) = \int_{\mathcal{M}_{\mathbb{Y}_n}} e^{iS_{\mathrm{Yang}}}$$

for a surface Σ with moduli space $\mathcal{M}_{\mathbb{Y}_n}$.

35.2. **Gauge Theory and Yang-Langlands Duality.** We propose an explicit formulation of gauge-theoretic duality:

$$\mathcal{D}_{\mathbb{Y}_n}(G) \simeq \mathcal{D}_{\mathbb{Y}_n^{\vee}}(G^{\vee}).$$

36. Concluding Remarks and Next Steps

- Extend the Yang-Langlands framework to motivic homotopy theory.
- Investigate explicit geometric applications of Yang-Galois functoriality.
- Develop explicit examples in global fields and function fields.
- Establish p-adic Hodge theoretic formulations of Yang-Langlands representations.

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37. Yang-Langlands and Higher-Dimensional Motives

37.1. **Motivic Yang-Galois Group.** We define the motivic Yang-Galois group associated with $\mathbb{Y}_n(F)$ as:

$$G_{\mathbb{Y}_n}^{\mathrm{mot}} = \mathrm{Aut}(\mathcal{T}_{\mathbb{Y}_n}),$$

where $\mathcal{T}_{\mathbb{Y}_n}$ is the Tannakian category of mixed motives over $\mathbb{Y}_n(F)$. This establishes a bridge between motivic homotopy theory and the Yang-Langlands correspondence.

37.2. **Motivic L-Functions in Yang-Langlands.** We extend the notion of L-functions to a motivic setting:

$$L_{\mathbb{Y}_n}^{\mathrm{mot}}(s) = \prod_v \det(1 - \mathrm{Frob}_v q_v^{-s} | H^*_{\mathrm{mot}}(X_{\mathbb{Y}_n})),$$

where $H^*_{\mathrm{mot}}(X_{\mathbb{Y}_n})$ is the motivic cohomology of a variety $X_{\mathbb{Y}_n}$ over $\mathbb{Y}_n(F)$. This defines an arithmetic L-function arising from the motivic structure.

38. Quantum Yang-Langlands and Derived Geometric Langlands

38.1. **Derived Geometric Yang-Langlands Correspondence.** We introduce an ∞-categorical refinement:

$$D^b_{\mathrm{QCoh}}(\mathcal{B}G_{\mathbb{Y}_n}) \simeq D^b_{\text{Ind-Rep}}(G^\vee_{\mathbb{Y}_n}),$$

where $\mathcal{B}G_{\mathbb{Y}_n}$ is the classifying stack of $G_{\mathbb{Y}_n}$ and $D^b_{\text{Ind-Rep}}(G^\vee_{\mathbb{Y}_n})$ is the derived category of ind-coherent representations.

38.2. **Quantum Geometric Yang-Langlands.** We introduce a deformation quantization of the moduli space:

$$\mathcal{D}_\hbar \, \mathrm{Bun}_{G_{\mathbb{Y}_n}} \simeq \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee_{\mathbb{Y}_n}}),$$

where $\mathcal{D}_\hbar$ denotes the deformation quantization at parameter $\hbar$. This describes the quantum structure of the geometric Yang-Langlands program.

39. Higher-Dimensional Functoriality in Yang-Langlands

39.1. **Functoriality for Higher-Dimensional Galois Representations.** For an extension of number fields $\mathbb{Y}_m(F) \subseteq \mathbb{Y}_n(F)$, functoriality dictates a natural transformation:

$$\mathrm{Hom}(G_{\mathbb{Y}_m}, A_{\mathbb{Y}_m}) \to \mathrm{Hom}(G_{\mathbb{Y}_n}, A_{\mathbb{Y}_n}).$$

This provides a higher-dimensional generalization of Langlands functoriality.

39.2. **Langlands Parametrization in the Yang Framework.** We propose a refined Langlands parameter map:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \to {}^L G_{\mathbb{Y}_n}(\mathbb{A}_F),$$

where ${}^L G_{\mathbb{Y}_n}(\mathbb{A}_F)$ is the Langlands dual group in the adelic setting. This generalizes the classical local-global compatibility in the Langlands program.

40. Topological Field Theory and Higher Yang-Langlands Structures

40.1. **2D and 4D TQFT Realization.** The partition function of a higher-dimensional Yang-Langlands topological quantum field theory is given by:

$$Z_{\mathbb{Y}_n}(\Sigma_4) = \int_{\mathcal{M}_{\mathbb{Y}_n}} e^{iS_{\mathrm{Yang}}},$$

where $\mathcal{M}_{\mathbb{Y}_n}$ is the moduli space of gauge connections on a four-manifold Σ_4.

40.2. **Higher-Dimensional Gauge Theory and Langlands Duality.** We introduce a higher-dimensional gauge-theoretic interpretation:

$$\mathcal{D}_{\mathbb{Y}_n}(G) \simeq \mathcal{D}_{\mathbb{Y}_n^\vee}(G^\vee).$$

This extends the S-duality conjecture to the Yang framework.

41. Conclusions and Future Research Directions

- Establish explicit motivic cohomology computations in the Yang-Langlands framework.
- Develop an explicit categorification of functoriality principles in the higher-dimensional setting.
- Investigate p-adic and geometric realizations of the motivic L-functions.
- Explore connections between Yang-Langlands and string theory compactifications.

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42. Yang-Langlands and Non-Abelian Iwasawa Theory

42.1. **Iwasawa Deformations of Yang-Galois Representations.** We define a non-abelian Iwasawa deformation ring associated with Yang-Galois representations:

$$\mathcal{R}_{\mathbb{Y}_n} = \varprojlim \mathrm{Rep}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, \mathbb{Z}/p^k\mathbb{Z}).$$

This controls the deformation theory of p-adic Yang-Galois representations and their Iwasawa theoretic properties.

42.2. **Yang-Langlands Iwasawa Main Conjecture.** We propose a non-abelian extension of the classical Iwasawa main conjecture:

$$\zeta^{\mathrm{Iw}}_{\mathbb{Y}_n}(s) \stackrel{?}{=} \det(1 - \mathrm{Frob}_p\, p^{-s} | H^*_{\mathrm{Iw}}(X_{\mathbb{Y}_n}, \mathbb{Z}_p)).$$

This establishes a direct link between Yang-Iwasawa theory and the p-adic Yang-Langlands program.

43. Higher Reciprocity Laws in the Yang-Langlands Framework

43.1. **Non-Abelian Artin Reciprocity.** We propose an extension of Artin reciprocity to the non-abelian Yang-Langlands setting:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \twoheadrightarrow \pi_1^{\mathrm{arith}}(\mathbb{Y}_n(F)) \longrightarrow \mathrm{Aut}(\mathbb{A}_{\mathbb{Y}_n}^{\times}).$$

This non-abelian reciprocity map connects Yang-Galois groups with the adelic structure of automorphic representations.

43.2. **Langlands-Kummer Reciprocity in the Yang Framework.** We generalize Kummer theory within the Yang-Langlands program:

$$\mathrm{Gal}(\mathbb{Y}_n(\mu_{p^\infty})/\mathbb{Y}_n(F)) \longleftrightarrow H^1_{\mathrm{et}}(\mathbb{Y}_n(F), \mathbb{Z}_p(1)).$$

This provides a reciprocity map between the Yang-Galois group of cyclotomic extensions and p-adic étale cohomology.

44. Noncommutative and Derived Yang-Langlands Program

44.1. **Yang-Langlands for Noncommutative Groups.** For a noncommutative algebraic group G over $\mathbb{Y}_n(F)$, we propose a Langlands-type correspondence:

$$\mathrm{Gal}(\mathbb{Y}_n(\overline{F})/\mathbb{Y}_n(F)) \longleftrightarrow {}^L G^{\mathrm{nc}}_{\mathbb{Y}_n}.$$

Here, ${}^L G^{\mathrm{nc}}_{\mathbb{Y}_n}$ is the noncommutative Langlands dual group.

44.2. **Derived Langlands Correspondence over Yang-Fargues-Fontaine Space.** We extend the Fargues-Fontaine curve construction to the derived category:

$$D^b_{\mathrm{coh}}(\mathcal{X}_{\mathbb{Y}_n}) \simeq D^b_{\text{Ind-Rep}}(G^\vee_{\mathbb{Y}_n}).$$

This establishes a higher-dimensional geometric Langlands duality.

45. Yang-Langlands and Topological Field Theories

45.1. **Extended 4D TQFTs for Yang-Langlands.** The partition function for a 4D Yang-Langlands TQFT is given by:

$$Z_{\mathbb{Y}_n}(\Sigma_4) = \int_{\mathcal{M}_{\mathbb{Y}_n}} e^{iS_{\mathrm{Yang}}},$$

where $\mathcal{M}_{\mathbb{Y}_n}$ is the moduli space of connections on a four-manifold Σ_4.

45.2. **Yang-Langlands Duality and Higher-Dimensional Gauge Theory.** We propose a higher-dimensional S-duality:

$$\mathcal{D}_{\mathbb{Y}_n}(G) \simeq \mathcal{D}_{\mathbb{Y}_n^\vee}(G^\vee).$$

This extends gauge-theoretic duality principles to the Yang framework.

46. Yang-Langlands and Arithmetic String Theory

46.1. **Automorphic String Compactifications.** We propose a connection between Yang-Langlands L-functions and string compactifications:

$$Z_{\mathrm{string}}(\mathbb{Y}_n) = \int_{\mathbb{M}_{\mathbb{Y}_n}} e^{-S_{\mathrm{eff}}}$$

where $\mathbb{M}_{\mathbb{Y}_n}$ is the moduli space of string vacua over $\mathbb{Y}_n(F)$.

46.2. **Quantum Modularity in Yang-Langlands.** We introduce a quantum modularity conjecture for the Yang-Langlands program:

$$\mathcal{Z}_{\mathbb{Y}_n}(q) = \sum_{\lambda} a_\lambda q^{h_\lambda} \overset{?}{\sim} \sum_{\mu} b_\mu q^{g_\mu}.$$

This encodes the arithmetic properties of modularity transformations in the quantum Yang-Langlands framework.

47. Future Research Directions

- Develop an explicit arithmetic application of the Yang-Langlands functoriality principle.
- Investigate explicit p-adic and perfectoid structures underlying the Yang framework.
- Explore connections between noncommutative motives and Yang-Langlands duality.
- Establish links between automorphic representations and higher categorical TQFTs.

48. Geometric Extensions

48.1. **Perfectoid Spaces and the Role of New Valuations in Yang-Langlands.** A cornerstone of modern Langlands is the use of **perfectoid spaces** (developed by Scholze) to bridge continuous and discrete arithmetic worlds. Perfectoid spaces are highly non-Archimedean analytic spaces where the Frobenius map is surjective (in characteristic p). In the Yang-Langlands Program, one considers generalized valuation fields $Q_{v\alpha}$ that extend beyond classical p-adic completions. For example, for a prime p, one can introduce:

- Maximally completed fields $Q_p^{\max}$ and its perfectoid closure Q_p^{perf}, enabling us to apply geometry to p-adic number theory. In particular, Q_p^{perf} is a perfectoid field, enabling new tools from p-adic Hodge theory to study the correspondence.
- Generalized completions $Q_{v\alpha}$ for new types of valuations v_α yielding new absolute Galois groups $G_{Q_{v\alpha}} = \text{Gal}(Q_{v\alpha}^{\text{alg}}/Q_{v\alpha})$.

We propose that these fields allow the study of automorphic representations in new settings where the fundamental group takes on a richer structure.

48.2. **Higher Stacks and Derived Structures for Moduli Spaces.** Classical Langlands often requires considering moduli stacks. The Yang-Langlands Program pushes this further by incorporating higher stacks and derived algebraic geometry to handle new "moduli of moduli" that arise. In particular, higher Galois theories suggest that instead of a single fundamental group, we should study higher homotopy invariants of arithmetic schemes. For instance, define an **arithmetic homotopy type** $\Pi_n(Q_{v\alpha})$ as a higher-dimensional analogue of the fundamental group of $\operatorname{Spec} Q_{v\alpha}$.

- Higher π_n (homotopy groups) correspond to new extension classes that capture higher cohomological invariants.
- Define $\Pi_\infty(Q_{v\alpha})$ as an ∞-groupoid whose π_1 is the usual Galois group $G_{Q_{v\alpha}}$ and whose higher π_n groups correspond to higher extension data.

The moduli of local systems for such higher Galois groups can be studied using derived categories of sheaves, and the Yang-Langlands Program aims to extend the classical correspondence to these new structures.

48.3. **Arithmetic Fundamental Groups and Automorphic Forms.** The classical Langlands correspondence can be viewed as a nonabelian generalization of class field theory, linking the fundamental group of arithmetic schemes to automorphic forms. In the Yang-Langlands Program, we extend this idea by introducing a hierarchy of arithmetic fundamental groups:

- $\pi_1(\operatorname{Spec} F) \cong \operatorname{Gal}(F^{\mathrm{alg}}/F)$ for a field F.
- Higher $\pi_n(\operatorname{Spec} F)$ for $n \geq 2$, defined via an arithmetic Postnikov tower, capturing higher extension classes.

These higher fundamental groups will correspond to automorphic forms in higher categories, where higher Hecke algebras act on the cohomology of moduli spaces of higher local systems.

49. Explicit Examples

49.1. **Function Fields Correspondence in Yang-Langlands.** In the Yang-Langlands Program, one extends the classical Langlands correspondence to generalized function fields. For example, consider the function field $K = \mathbb{F}_q(C)$, where C is a curve over a finite field. The Langlands correspondence provides a bijection between:

- Irreducible cuspidal automorphic representations of $GL_n(\mathbf{A}_K)$ and
- n-dimensional ℓ-adic Galois representations $\rho : \operatorname{Gal}(\overline{K}/K) \to GL_n(\mathbb{Q}_\ell)$.

For the Yang-Langlands extension, we conjecture a similar correspondence for new fields $Q_{v\alpha}$:

$$\left\{ \begin{array}{c} \text{Automorphic forms over } Q_{v\alpha} \\ \text{(e.g. on } GL_n) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} n\text{-dimensional representations of} \\ \operatorname{Gal}(\overline{Q}_{v\alpha}/Q_{v\alpha}) \end{array} \right\}.$$

49.2. **Modular Forms and Yang-Langlands.** The Yang-Langlands Program suggests that modular forms are a special case of automorphic forms. The classical correspondence between elliptic curves and modular forms is extended by considering automorphic forms on generalized groups over new fields $Q_{v\alpha}$. The local factors in the L-function for these new fields correspond to Hecke operators, which act on the higher Galois representations. We propose that the Yang-Langlands correspondence will generalize the classical modularity result to these new automorphic objects.

50. Connections with Physics

50.1. **Yang-Langlands and Supersymmetric Gauge Theories.** In geometric Langlands, S-duality of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions leads to a correspondence between D-modules and local systems. The Yang-Langlands Program extends this idea by considering new fields $Q_{v\alpha}$ and new types of gauge theories. We propose that these new gauge theories, when compactified on a suitable space, correspond to new classes of automorphic representations on fields like $Q_{v\alpha}$.

50.2. **AdS/CFT Correspondences and String-Theoretic Implications.** The AdS/CFT correspondence in string theory, which posits an equivalence between a gravitational theory on an AdS space and a conformal field theory on its boundary, has direct implications for the Yang-Langlands Program. We speculate that an analogue of this correspondence exists in number theory, where the boundary CFT corresponds to automorphic forms and the bulk theory corresponds to new types of Galois groups, such as $G_{Q_{v\alpha}}$ or $G_\infty(F_\infty)$.

51. Category-Theoretic Formalization

51.1. **Yang-Langlands Structures in Infinity-Categories.** The Yang-Langlands correspondence can be formulated in the language of ∞-categories. We define an equivalence between the ∞-category of Galois representations and the ∞-category of automorphic forms:

$$\mathcal{F} : \mathcal{G}(F) \xrightarrow{\sim} \mathcal{A}(F),$$

where $\mathcal{F}$ is a functor that relates objects in the Galois category $\mathcal{G}(F)$ to objects in the automorphic category $\mathcal{A}(F)$. This equivalence is expected to extend to new types of Galois groups, such as those corresponding to the fields $Q_{v\alpha}$.

51.2. **Extended Categorical Functoriality.** The functoriality in the Yang-Langlands Program is expected to be extended to new fields and higher-dimensional representations. For example, given a homomorphism ${}^LH \to {}^LG$, the automorphic and Galois categories must respect this homomorphism, extending the classical Langlands functoriality to new contexts.

51.3. **Derived Algebraic Geometry and Tannakian Formalism.** In the Yang-Langlands Program, many of the categories involved are not ordinary Tannakian categories but derived versions. These derived categories can be used to study new types of Galois representations and automorphic forms, especially in the context of new fields like $Q_{v\alpha}$. This involves understanding the higher categories that govern these objects and ensuring that the equivalences between them are consistent with the structure of derived algebraic geometry.

52. Advanced Structures in the Yang-Langlands Program

52.1. **Beyond Classical Automorphic Forms: Introducing Yang-Automorphic Forms.** The Yang-Langlands extension of the program proposes a novel category of *Yang-Automorphic Forms* defined over fields of type $\mathbb{Y}_n(F)$. These forms are generalizations of classical automorphic forms, where instead of acting on classical groups like GL_n, the actions are extended to groups corresponding to the Yang-Galois group structures and their associated Galois representations. In particular, we introduce:

$$\mathcal{A}_n^{\mathbb{Y}_n}(F) \quad \text{to denote the space of Yang-Automorphic forms over the field } \mathbb{Y}_n(F).$$

These forms can be viewed as representations of the Yang-Galois groups defined earlier, such as $Gal(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))$, with the added complication of higher local systems and structure extending to new algebraic and analytic domains.

52.2. **Extended Galois Representations and Yang-Galois Groups.** Let us consider the extended Galois representations defined on the fields $\mathbb{Y}_n(F)$ and their associated objects:

$$\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)) \to GL_n(\mathbb{Q}_\ell),$$

where $\mathbb{Q}_\ell$ denotes the ℓ-adic numbers. These representations are defined not only for classical fields like $\mathbb{Q}$ or finite extensions of $\mathbb{Q}$ but also for the fields $\mathbb{Y}_n(F)$ and the Yang-Galois groups. We then propose the study of the Yang-Langlands correspondence in this broader context as:

$$\text{Yang-Langlands Correspondence:} \quad \mathcal{A}_n^{\mathbb{Y}_n}(F) \longleftrightarrow \rho_{\mathbb{Y}_n}.$$

This correspondence links Yang-Automorphic forms and the extended Galois representations over new fields of the form $\mathbb{Y}_n(F)$, serving as a bridge between non-Archimedean structures, higher representations, and classical number-theoretic objects.

52.3. **Higher Galois Groups and the Study of Hecke Algebras.** The Hecke algebra plays a central role in the classical Langlands correspondence. In the extended framework, we introduce higher Hecke algebras associated with the Yang-Galois groups. The Hecke algebra is now defined over $\mathbb{Y}_n(F)$ and is modeled as a noncommutative algebra:

$$\mathcal{H}_n^{\mathbb{Y}_n}(F) = \text{Hecke algebra associated with the Yang-Galois group.}$$

The higher Hecke operators act on the space of Yang-Automorphic forms $\mathcal{A}_n^{\mathbb{Y}_n}(F)$, extending the action of classical Hecke operators in a manner consistent with the higher dimensional structure of the Yang-Langlands correspondence.

52.4. **The Yang-Langlands Correspondence for Function Fields.** Given a function field K of a smooth projective curve, we study the extended Langlands correspondence in the context of higher-dimensional function fields. The classical Langlands correspondence for function fields links the automorphic forms on GL_n over a function field $K = \mathbb{F}_q(C)$ (where C is a curve) to the corresponding Galois representations. We extend this to new fields $\mathbb{Y}_n(F)$ and propose the following generalized correspondence:

$$\mathcal{A}_n^{\mathbb{Y}_n}(F) \longleftrightarrow \rho_{\mathbb{Y}_n}(\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))).$$

This extension applies to function fields over finite fields $\mathbb{F}_q$, bringing new insights into the application of Langlands to higher-dimensional and non-Archimedean fields.

52.5. **The Role of Modular Forms in the Yang-Langlands Framework.** Modular forms play an important role in the classical Langlands correspondence, particularly in their connection with elliptic curves. In the extended Yang-Langlands framework, modular forms are extended to the Yang-Automorphic forms over $\mathbb{Y}_n(F)$. We hypothesize that these new forms preserve many properties of classical modular forms while also incorporating additional structure due to the new Galois and Yang-Galois groups.

Let us define the space of Yang-Modular forms over the field $\mathbb{Y}_n(F)$ as:

$$M_n^{\mathbb{Y}_n}(F) = \text{Space of Yang-Modular forms over } \mathbb{Y}_n(F).$$

The Yang-Modular forms are expected to be the bridge between higher-dimensional number theory and classical modularity, extending the classical connection between modular forms and elliptic curves to a new level.

52.6. **Categorical Structures and Functors in the Yang-Langlands Framework.** The study of categories and functors plays a pivotal role in the extended Langlands correspondence. In particular, we focus on the extension of Tannakian categories and their use in understanding automorphic forms and Galois representations. We introduce the following new categories:

$$\mathcal{C}_n^{\mathbb{Y}_n}(F) \quad \text{(category of representations over } \mathbb{Y}_n(F) \text{ with higher functors).}$$

These categories serve as the foundation for studying the deeper relationships between Yang-Galois representations, automorphic forms, and their associated structures. Functors between these categories preserve the essential correspondences of the Yang-Langlands program, extending the classical framework by allowing higher-dimensional representations and extensions.

52.7. **Admissible Categories for Yang-Automorphic Forms.** To further extend the theory, we define the notion of *admissible categories* for Yang-Automorphic forms over the field $\mathbb{Y}_n(F)$. These categories will generalize the classical notion of admissible representations by incorporating higher-dimensional structures. We hypothesize the following construction:

$$\mathcal{A}_n^{\mathrm{adm}}(\mathbb{Y}_n(F)) \quad \text{(admissible categories of Yang-Automorphic forms).}$$

These admissible categories are expected to allow a finer study of the properties of Yang-Automorphic forms, extending the classical framework of admissibility in the theory of automorphic forms.

53. Further Conjectures and Open Problems

53.1. **On the Structure of Higher Galois Groups.** Given the rich structure of higher Galois groups in the context of $\mathbb{Y}_n(F)$ fields, several conjectures remain open. In particular, we conjecture that the higher Galois groups associated with Yang-Galois representations satisfy certain structural properties akin to those seen in the classical setting. Specifically, we propose the following conjecture regarding the Galois groups of the fields $\mathbb{Y}_n(F)$:

Conjecture 1. *For any Yang-Galois representation $\rho_{\mathbb{Y}_n}$ over a field $\mathbb{Y}_n(F)$, the associated Galois group is a well-behaved object with respect to the extension of the base field. That is,* $\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))$ *exhibits properties similar to classical Galois groups over number fields, but with extensions into higher dimensions.*

53.2. **Generalized Langlands Correspondence for Higher Dimensions.** The primary open problem in this area is the generalization of the Langlands correspondence to higher-dimensional schemes, especially in the context of non-Archimedean fields like $\mathbb{Y}_n(F)$. While the classical theory applies primarily to one-dimensional schemes, we expect the extended correspondence to hold in higher dimensions as well. This would require the development of new techniques in algebraic geometry and the theory of higher categories.

54. Extended Yang-Langlands Duality and Higher Dimensional Correspondences

54.1. **Yang-Langlands Duality for Extended Reductive Groups.** In classical Langlands, duality plays a crucial role in connecting automorphic forms with Galois representations. We extend this notion by defining a new duality for reductive groups over $\mathbb{Y}_n(F)$:

$$^LG_{\mathbb{Y}_n(F)} = \widehat{G}_{\mathbb{Y}_n(F)} \rtimes W_{\mathbb{Y}_n(F)},$$

where $W_{\mathbb{Y}_n(F)}$ is the Weil group of $\mathbb{Y}_n(F)$ and $\widehat{G}_{\mathbb{Y}_n(F)}$ is the Langlands dual group defined over the extended field. The key new property of this duality is that it incorporates additional structure induced by the higher Galois symmetries of $\mathbb{Y}_n(F)$.

54.2. **Higher-Dimensional Automorphic Sheaves and Geometric Realization.** In the geometric Langlands program, automorphic sheaves provide a categorical interpretation of the Langlands correspondence. In the extended Yang-Langlands setting, we introduce a novel class of *higher-dimensional automorphic sheaves* over $\mathbb{Y}_n(F)$:

$$\mathcal{A}^{\mathbb{Y}_n}_{\text{geom}}(F) = D_{\text{coh}}(\mathbb{Y}_n(F), G_{\mathbb{Y}_n(F)}),$$

where D_{coh} denotes the category of coherent sheaves on the moduli space of $\mathbb{Y}_n(F)$-representations. These sheaves encode the Yang-Automorphic forms in a geometric framework, allowing us to extend geometric Langlands techniques to new arithmetic settings.

54.3. **Higher Ramification Theory and Non-Abelian Class Field Theory.** Ramification plays a central role in understanding local Galois representations. In the Yang-Langlands setting, we extend the classical ramification theory to *higher ramification groups* in $\mathbb{Y}_n(F)$:

$$\mathcal{R}^i_{\mathbb{Y}_n(F)} = \text{Higher ramification filtration of } \operatorname{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)).$$

This structure provides a non-abelian class field theory, generalizing the classical Langlands program to incorporate more refined ramification structures.

54.4. **Yang-Langlands Functoriality in Higher Dimensions.** A major component of the classical Langlands program is functoriality, which relates automorphic representations on different groups. We propose the following conjecture in the Yang-Langlands setting:

Conjecture 2. *For any homomorphism of extended Langlands dual groups:*

$$^LH_{\mathbb{Y}_n(F)} \to {}^LG_{\mathbb{Y}_n(F)},$$

there exists a transfer of automorphic representations:

$$\Pi(H_{\mathbb{Y}_n(F)}) \longrightarrow \Pi(G_{\mathbb{Y}_n(F)}),$$

preserving the extended structures and compatibility with the higher-dimensional Galois representations.

This provides a refined understanding of functoriality in the non-Archimedean setting of $\mathbb{Y}_n(F)$.

54.5. Higher-Dimensional Motives and the Yang-Langlands Conjecture. The theory of motives plays a crucial role in modern number theory. In the Yang-Langlands program, we propose a novel category of *higher-dimensional motives* over the fields $\mathbb{Y}_n(F)$:

$$\mathcal{M}^{\mathbb{Y}_n}(F) = \text{Category of higher motives over } \mathbb{Y}_n(F).$$

We conjecture that these motives correspond to the new Yang-Galois representations:

Conjecture 3. *There exists a natural equivalence:*

$$\mathcal{M}^{\mathbb{Y}_n}(F) \cong Rep_{\mathbb{Y}_n(F)}(\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))),$$

where $Rep_{\mathbb{Y}_n(F)}$ denotes the category of continuous representations of the extended Galois group.

54.6. Categorification of the Yang-Langlands Correspondence. One of the most exciting developments in higher representation theory is the categorification of the Langlands correspondence. In the context of $\mathbb{Y}_n(F)$, we propose a new categorical correspondence:

$$D^b_{\mathbb{Y}_n}(F) \simeq D^b_{\text{moduli}}(G_{\mathbb{Y}_n(F)}),$$

where $D^b_{\mathbb{Y}_n}(F)$ is the derived category of the space of Yang-Galois representations, and $D^b_{\text{moduli}}(G_{\mathbb{Y}_n(F)})$ is the derived category of automorphic sheaves on the moduli space of $G_{\mathbb{Y}_n(F)}$. This provides a deeper categorical framework for understanding the Yang-Langlands program.

54.7. Future Directions and Open Problems. The extension of the Langlands program to $\mathbb{Y}_n(F)$ fields presents numerous challenges and open questions:

- How do the extended Galois groups $\mathrm{Gal}(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F))$ interact with the classical absolute Galois groups of $\mathbb{Q}$?
- What are the implications of Yang-Langlands for the global Langlands correspondence over number fields?
- Can we define a Yang-Langlands spectral decomposition that refines the trace formula?
- What role does derived algebraic geometry play in the classification of automorphic representations over $\mathbb{Y}_n(F)$?

We propose that these questions guide further research into the Yang-Langlands program, bridging new number-theoretic frameworks with higher representation theory and non-Archimedean geometry.

55. Noncommutative Geometry and the Yang-Langlands Program

55.1. Extended Noncommutative Geometric Interpretation. Classical Langlands theory has deep connections with noncommutative geometry, particularly in the study of C*-algebras and spectral triples. In the Yang-Langlands setting, we propose an extended noncommutative geometric framework:

$$\mathcal{A}^{\mathbb{Y}_n}(F) = C^*(G_{\mathbb{Y}_n(F)})$$

where $C^*(G_{\mathbb{Y}_n(F)})$ denotes the C*-algebra of the Yang-Galois group. The spectral properties of these algebras are expected to encode the higher ramification structure of $\mathbb{Y}_n(F)$, extending the classical work of Connes on spectral triples.

55.2. **Higher Hecke Operators and Noncommutative Fourier Transform.** In the classical setting, Hecke operators provide an algebraic realization of Langlands functoriality. In the Yang-Langlands program, we introduce a *noncommutative Fourier transform* defined over the space of automorphic functions:

$$\mathcal{F}_{\mathbb{Y}_n} : \mathcal{H}_n^{\mathbb{Y}_n}(F) \longrightarrow \widehat{\mathcal{H}_n^{\mathbb{Y}_n}(F)}$$

where $\mathcal{H}_n^{\mathbb{Y}_n}(F)$ is the noncommutative Hecke algebra associated with the Yang-Galois group. This Fourier transform allows for a spectral decomposition of automorphic forms in a noncommutative setting.

56. Topological Quantum Field Theory and the Yang-Langlands Program

56.1. **Extended TQFTs and Langlands Duality.** The geometric Langlands program has well-known connections with supersymmetric gauge theories and topological quantum field theory (TQFT). In the Yang-Langlands setting, we introduce a new class of *extended TQFTs* defined over $\mathbb{Y}_n(F)$. These TQFTs take the form:

$$Z_{\mathbb{Y}_n}(M) = \int_{\mathbb{Y}_n(F)} e^{-S_{\mathbb{Y}_n}(A)} dA$$

where $S_{\mathbb{Y}_n}(A)$ is an action functional defined on gauge fields associated with the Yang-Galois representations.

56.2. **Quantum Groups and Extended Langlands Duality.** The quantum group $U_q(\mathfrak{g})$ associated with the Langlands dual group plays a central role in categorified representation theory. In the Yang-Langlands program, we define an *extended quantum group*:

$$U_{\mathbb{Y}_n}(q) = \text{Deformation of } U_q(\mathfrak{g}) \text{ over } \mathbb{Y}_n(F).$$

This quantum group is expected to provide new insights into the category of automorphic sheaves over the moduli space of Galois representations.

57. Extended Riemann Hypothesis and the Yang-Zeta Function

57.1. **Generalized Zeta Functions for $\mathbb{Y}_n(F)$.** In the classical setting, the Riemann hypothesis predicts the distribution of nontrivial zeros of the Riemann zeta function. We introduce a new *Yang-Zeta function* defined over $\mathbb{Y}_n(F)$:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Y}_n(F))} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

This function generalizes the Dedekind zeta function and incorporates the higher-dimensional number-theoretic properties of $\mathbb{Y}_n(F)$.

57.2. **Yang-Riemann Hypothesis.** We conjecture that the nontrivial zeros of $\zeta_{\mathbb{Y}_n}(s)$ lie on a critical line analogous to the classical Riemann hypothesis:

Conjecture 4. *All nontrivial zeros of $\zeta_{\mathbb{Y}_n}(s)$ satisfy:*

$$\Re(s) = \frac{1}{2}.$$

This extends classical techniques in analytic number theory to the new number systems introduced in the Yang-Langlands program.

58. Higher Category Theory and Extended Tannakian Formalism

58.1. **Infinity-Categories and Higher Stacks.** In the classical setting, Tannakian formalism provides an equivalence between categories of representations and fundamental groups. In the Yang-Langlands setting, we extend this to the realm of ∞-categories:

$$\mathcal{C}^{\mathbb{Y}_n}(F) \simeq \mathrm{Fun}(\mathbb{Y}_n(F), \mathrm{Spectra}).$$

This equivalence allows for a deeper understanding of the categorical structures underlying automorphic representations.

58.2. **Derived Categories of Automorphic Sheaves.** We propose that the category of automorphic sheaves over $\mathbb{Y}_n(F)$ should be viewed as a derived category:

$$D^b_{\mathrm{Aut}}(\mathbb{Y}_n(F)) \simeq D^b_{\mathrm{Galois}}(\mathbb{Y}_n(F)).$$

This provides a powerful tool for studying functoriality and higher categorical correspondences in the Yang-Langlands program.

59. Open Problems and Future Directions

- Can we establish a full spectral decomposition of automorphic forms in the setting of $\mathbb{Y}_n(F)$?
- How do the higher Hecke algebras interact with the extended Langlands dual groups?
- What are the physical implications of the Yang-Langlands correspondence in the context of gauge theories and quantum gravity?
- Can we extend the theory of ℓ-adic cohomology to higher-dimensional representations over $\mathbb{Y}_n(F)$?
- What is the role of motivic integration in the Yang-Langlands framework?

60. Yang-Langlands and Noncommutative Motives

60.1. **Noncommutative Motives over $\mathbb{Y}_n(F)$.** The classical theory of motives provides a universal cohomological framework for algebraic varieties. In the Yang-Langlands setting, we introduce *noncommutative motives* over $\mathbb{Y}_n(F)$, extending the notion of derived categories of motives:

$$\mathcal{M}^{\mathbb{Y}_n}_{\mathrm{nc}}(F) = D^b_{\mathrm{nc}}(\mathbb{Y}_n(F))$$

where $D^b_{\mathrm{nc}}(\mathbb{Y}_n(F))$ is the derived category of noncommutative spaces associated with $\mathbb{Y}_n(F)$. We conjecture that these noncommutative motives encode the structure of the Yang-Galois representations.

60.2. **Higher Motivic Galois Groups and Tannakian Formalism.** Motivic Galois groups govern the structure of periods and special values of L-functions. In the extended Yang-Langlands setting, we define *higher motivic Galois groups*:

$$G_{\mathcal{M}^{\mathbb{Y}_n}(F)} = \mathrm{Aut}^{\otimes}(\mathcal{M}^{\mathbb{Y}_n}(F))$$

where $\mathrm{Aut}^{\otimes}(\mathcal{M}^{\mathbb{Y}_n}(F))$ is the Tannakian fundamental group of the category of Yang-motives. This provides a link between higher-dimensional motives and the representation theory of noncommutative Galois groups.

61. Extended Adelic and Automorphic Theory

61.1. **Adelic Langlands Correspondence over $\mathbb{Y}_n(F)$.** In classical Langlands theory, adelic methods play a crucial role in understanding automorphic representations. We extend this perspective by defining *adelic structures* over $\mathbb{Y}_n(F)$:

$$\mathbb{A}_{\mathbb{Y}_n(F)} = \prod_v{}' \mathbb{Y}_n(F_v),$$

where the restricted product is taken over all places v of $\mathbb{Y}_n(F)$. The automorphic representations in this setting take the form:

$$\pi : \mathbb{A}^{\times}_{\mathbb{Y}_n(F)} \longrightarrow \mathbb{C}^{\times}.$$

We conjecture that these adelic automorphic representations correspond to new Galois representations in the extended Langlands setting.

61.2. **Fourier Analysis on Noncommutative Adeles.** A central tool in classical automorphic theory is the Fourier expansion of automorphic forms. In the Yang-Langlands setting, we propose a *noncommutative Fourier transform* on the adelic space:

$$\mathcal{F}_{\mathbb{Y}_n(F)} : L^2(\mathbb{A}_{\mathbb{Y}_n(F)}) \longrightarrow \widehat{L^2}(\mathbb{A}_{\mathbb{Y}_n(F)}).$$

This transform allows the study of spectral decompositions of Yang-Automorphic forms and their relation to Galois representations.

62. Higher Stacks and Derived Categories in the Yang-Langlands Program

62.1. **Higher Moduli Stacks of Local Systems.** The moduli of Galois representations and automorphic forms can be studied using higher stacks. We define the moduli stack of Yang-Galois representations as a derived stack:

$$\mathcal{M}^{\mathbb{Y}_n}_{\mathrm{loc}}(F) = \left[\mathrm{Spec}\ \mathbb{Y}_n(F)/G_{\mathbb{Y}_n(F)}\right]_{\infty}.$$

This higher stack encodes the deformation theory of local systems and their connection to automorphic representations.

62.2. **Derived Categories of Automorphic Sheaves.** We propose an extended geometric Langlands correspondence by considering derived categories of automorphic sheaves:

$$D^b_{\text{aut}}(\mathbb{Y}_n(F)) \simeq D^b_{\text{Gal}}(G_{\mathbb{Y}_n(F)}),$$

where $D^b_{\text{aut}}(\mathbb{Y}_n(F))$ is the derived category of automorphic sheaves and $D^b_{\text{Gal}}(G_{\mathbb{Y}_n(F)})$ is the category of sheaves on the moduli space of Yang-Galois representations.

63. Topological Field Theory and Higher Langlands Functoriality

63.1. **Yang-Langlands and Extended Chern-Simons Theory.** The geometric Langlands program is closely related to Chern-Simons theory in 3D gauge theory. In the extended Yang-Langlands program, we define an *extended Chern-Simons action*:

$$S_{\mathbb{Y}_n} = \int_M \text{Tr}(A \wedge dA + A \wedge A \wedge A)$$

where A is a gauge field associated with the Yang-Galois representations. We conjecture that this extended theory provides a physical realization of the functoriality in higher dimensions.

63.2. **Higher TQFTs and the Yang-Langlands Spectrum.** In the classical setting, TQFTs provide invariants of 3-manifolds. We extend this by defining *higher TQFTs* over $\mathbb{Y}_n(F)$, whose partition function takes the form:

$$Z_{\mathbb{Y}_n}(M) = \sum_{\pi} e^{-S_{\mathbb{Y}_n}(\pi)},$$

where the sum is over automorphic representations π contributing to the spectrum of the Yang-Langlands program.

64. Future Directions and Open Conjectures

64.1. **Non-Abelian Reciprocity Laws in Higher Dimensions.** We conjecture an extension of class field theory to the setting of non-abelian reciprocity laws for the extended fields $\mathbb{Y}_n(F)$:

Conjecture 5. *There exists a natural correspondence:*

$$Gal(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)) \longleftrightarrow \text{Cohomology of higher stacks on } \mathbb{Y}_n(F).$$

This correspondence extends class field theory and establishes a bridge between non-abelian Galois representations and the structure of automorphic forms in the extended Yang-Langlands setting.

64.2. **Higher L-Functions and New Spectral Invariants.** The theory of L-functions plays a central role in Langlands theory. In the extended setting, we propose the study of *higher L-functions* associated with the fields $\mathbb{Y}_n(F)$:

$$L(s, \pi_{\mathbb{Y}_n}) = \prod_v L_v(s, \pi_{\mathbb{Y}_n}),$$

where each local factor encodes the representation theory of the Yang-Galois group at the place v. We conjecture that these higher L-functions satisfy new functional equations and encode spectral invariants of the noncommutative Yang-Langlands program.

64.3. **Yang-Langlands and the Arithmetic of Stacks.** Stacks have become central to modern algebraic geometry. We propose the following conjecture regarding the interaction of stacks and the Yang-Langlands correspondence:

Conjecture 6. *The moduli stack of Yang-Galois representations $\mathcal{M}^{\mathbb{Y}_n}(F)$ governs the spectral decomposition of automorphic forms in the extended Yang-Langlands program.*

This suggests that the arithmetic geometry of stacks plays a foundational role in extending Langlands correspondences to new mathematical objects.

65. Noncommutative Geometry and the Yang-Langlands Program

65.1. **Extended Noncommutative Geometric Interpretation.** Classical Langlands theory has deep connections with noncommutative geometry, particularly in the study of C*-algebras and spectral triples. In the Yang-Langlands setting, we propose an extended noncommutative geometric framework:

$$\mathcal{A}^{\mathbb{Y}_n}(F) = C^*(G_{\mathbb{Y}_n(F)})$$

where $C^*(G_{\mathbb{Y}_n(F)})$ denotes the C*-algebra of the Yang-Galois group. The spectral properties of these algebras are expected to encode the higher ramification structure of $\mathbb{Y}_n(F)$, extending the classical work of Connes on spectral triples.

65.2. **Higher Hecke Operators and Noncommutative Fourier Transform.** In the classical setting, Hecke operators provide an algebraic realization of Langlands functoriality. In the Yang-Langlands program, we introduce a *noncommutative Fourier transform* defined over the space of automorphic functions:

$$\mathcal{F}_{\mathbb{Y}_n} : \mathcal{H}_n^{\mathbb{Y}_n}(F) \longrightarrow \widehat{\mathcal{H}_n^{\mathbb{Y}_n}(F)}$$

where $\mathcal{H}_n^{\mathbb{Y}_n}(F)$ is the noncommutative Hecke algebra associated with the Yang-Galois group. This Fourier transform allows for a spectral decomposition of automorphic forms in a noncommutative setting.

66. Topological Quantum Field Theory and the Yang-Langlands Program

66.1. **Extended TQFTs and Langlands Duality.** The geometric Langlands program has well-known connections with supersymmetric gauge theories and topological quantum field theory (TQFT). In the Yang-Langlands setting, we introduce a new class of *extended TQFTs* defined over $\mathbb{Y}_n(F)$. These TQFTs take the form:

$$Z_{\mathbb{Y}_n}(M) = \int_{\mathbb{Y}_n(F)} e^{-S_{\mathbb{Y}_n}(A)} dA$$

where $S_{\mathbb{Y}_n}(A)$ is an action functional defined on gauge fields associated with the Yang-Galois representations.

66.2. **Quantum Groups and Extended Langlands Duality.** The quantum group $U_q(\mathfrak{g})$ associated with the Langlands dual group plays a central role in categorified representation theory. In the Yang-Langlands program, we define an *extended quantum group*:

$$U_{\mathbb{Y}_n}(q) = \text{Deformation of } U_q(\mathfrak{g}) \text{ over } \mathbb{Y}_n(F).$$

This quantum group is expected to provide new insights into the category

$$U_{\mathbb{Y}_n}(q) = \text{Deformation of } U_q(\mathfrak{g}) \text{ over } \mathbb{Y}_n(F).$$

This quantum group is expected to provide new insights into the category of Yang-Galois representations, allowing for a categorification of the Yang-Langlands correspondence. We conjecture that there exists a natural equivalence:

$$\text{Rep}(G_{\mathbb{Y}_n(F)}) \simeq D^b(U_{\mathbb{Y}_n}(q)\text{-mod}),$$

where $D^b(U_{\mathbb{Y}_n}(q)\text{-mod})$ is the derived category of modules over the quantum group $U_{\mathbb{Y}_n}(q)$. This equivalence extends the classical geometric Langlands duality to the noncommutative and higher-dimensional setting of $\mathbb{Y}_n(F)$.

67. Categorification and Higher Yang-Langlands Structures

67.1. **Extended Infinity-Categories and Higher Tannakian Formalism.** To fully realize the Yang-Langlands correspondence in higher dimensions, we propose the study of ∞-categories of Yang-Galois representations and automorphic forms. Define:

$$\mathcal{C}_{\mathbb{Y}_n}(F) = \text{Fun}(\mathbb{Y}_n(F), \text{Spectra})$$

as the ∞-category of Yang-Galois representations, where Spectra denotes the stable homotopy category. The functoriality conjecture in the Yang-Langlands program now takes the form:

$$D^b_{\text{Aut}}(\mathbb{Y}_n(F)) \simeq \mathcal{C}_{\mathbb{Y}_n}(F),$$

extending the classical Tannakian formalism to a derived, homotopical framework.

67.2. **Higher Stacks and Derived Moduli of Yang-Galois Representations.** We propose the study of the moduli space of higher-dimensional Yang-Galois representations as a derived stack:

$$\mathcal{M}^{\mathbb{Y}_n}_{\text{Yang}}(F) = \left[\text{Spec}(\mathbb{Y}_n(F))/G_{\mathbb{Y}_n(F)}\right]_{\infty}.$$

This stack encodes higher deformation theory and the structure of noncommutative motives in the extended Yang-Langlands setting.

68. Concluding Remarks and Open Questions

- What is the role of extended quantum groups in categorifying the Yang-Langlands correspondence?
- How does noncommutative geometry affect the representation theory of automorphic forms over $\mathbb{Y}_n(F)$?
- Can we explicitly compute L-functions in the noncommutative and higher-dimensional setting of Yang-Langlands?
- What are the physical implications of the extended TQFT models for Yang-Galois representations?

The Yang-Langlands program opens new avenues in number theory, representation theory, algebraic geometry, and mathematical physics, extending the classical Langlands program into an enriched, noncommutative, and higher-dimensional framework.

69. Extended Yang-Langlands Reciprocity and Higher Functoriality

69.1. Generalized Reciprocity Laws in the Yang-Langlands Program. One of the fundamental aspects of the Langlands program is its non-abelian generalization of class field theory. In the extended Yang-Langlands setting, we propose a new class of reciprocity laws governing higher-dimensional fields $\mathbb{Y}_n(F)$:

Conjecture 7. *There exists a natural functoriality map between the extended Yang-Galois groups and automorphic forms:*

$$Gal(\overline{\mathbb{Y}_n(F)}/\mathbb{Y}_n(F)) \longleftrightarrow H^i_{Aut}(\mathbb{Y}_n(F), G_{\mathbb{Y}_n(F)}),$$

where $H^i_{Aut}(\mathbb{Y}_n(F), G_{\mathbb{Y}_n(F)})$ denotes the higher cohomology of automorphic forms with respect to the moduli space of Yang-Galois representations.

This reciprocity extends classical non-abelian class field theory and provides a framework for studying the higher structure of the Yang-Langlands correspondence.

69.2. Functoriality of Automorphic Representations over $\mathbb{Y}_n(F)$. A key component of the classical Langlands program is the transfer of automorphic representations between groups. In the extended setting, we introduce a new functoriality principle:

Conjecture 8. *For any homomorphism of extended Langlands dual groups:*

$$^L H_{\mathbb{Y}_n(F)} \to {}^L G_{\mathbb{Y}_n(F)},$$

there exists an induced map of automorphic representations:

$$\Pi(H_{\mathbb{Y}_n(F)}) \longrightarrow \Pi(G_{\mathbb{Y}_n(F)}),$$

that preserves the spectral and arithmetic structures of the representations.

This provides an extended functoriality framework that incorporates higher category theory, noncommutative geometry, and topological quantum field theory.

70. Higher Yang-Langlands Correspondence for Derived Stacks

70.1. Extended Derived Moduli of Yang-Galois Representations. We refine the moduli space of Yang-Galois representations to a *derived stack*, incorporating homotopical and derived deformation theory:

$$\mathcal{M}^{\mathbb{Y}_n}_{\text{Yang}}(F) = \left[\text{Spec}(\mathbb{Y}_n(F))/G_{\mathbb{Y}_n(F)}\right]_\infty .$$

This stack parametrizes higher Yang-Galois representations, extending the classical moduli space of ℓ-adic representations.

70.2. Categorical Langlands and Higher Stacks. The Yang-Langlands correspondence can be formulated categorically using higher stacks and derived categories:

$$D^b_{\text{Aut}}(\mathbb{Y}_n(F)) \simeq D^b_{\text{Gal}}(G_{\mathbb{Y}_n(F)}),$$

where $D^b_{\text{Aut}}(\mathbb{Y}_n(F))$ is the derived category of automorphic sheaves, and $D^b_{\text{Gal}}(G_{\mathbb{Y}_n(F)})$ is the derived category of Galois representations.

71. Extended Langlands Duality and the Yang-Langlands Spectral Theory

71.1. Spectral Decomposition of Automorphic Forms over $\mathbb{Y}_n(F)$. The classical Langlands spectral decomposition expresses automorphic forms in terms of eigenvalues of Hecke operators. In the Yang-Langlands setting, we introduce a *noncommutative spectral decomposition*:

$$L^2_{\text{Aut}}(\mathbb{Y}_n(F)) = \bigoplus_\pi m_\pi \pi,$$

where π ranges over Yang-automorphic representations and m_π denotes their multiplicities. The spectral theory of these automorphic forms is governed by the extended noncommutative Hecke algebra.

71.2. Higher-Order Hecke Operators and Langlands Functoriality. We introduce higher Hecke operators acting on the space of Yang-automorphic forms:

$$\mathcal{H}^{\mathbb{Y}_n}_{\text{Yang}}(F) = \text{Noncommutative Hecke algebra over } \mathbb{Y}_n(F).$$

These operators encode the action of Yang-Galois groups on automorphic functions, extending classical harmonic analysis in the Langlands setting.

72. Future Directions and Open Problems

- Can the Yang-Langlands functoriality principle be explicitly realized in terms of higher categorical correspondences?
- What is the precise role of derived stacks and homotopical methods in the classification of Yang-Galois representations?
- How do the higher Hecke algebras relate to the structure of automorphic L-functions in the extended setting?
- Can the extended Langlands duality be formulated in terms of noncommutative motives and TQFT?

These open problems will guide future developments in the Yang-Langlands program, bridging arithmetic geometry, representation theory, higher category theory, and quantum field theory.

73. Yang-Langlands Correspondence in Non-Archimedean and Perfectoid Settings

73.1. Perfectoid Spaces and Higher Ramification in $\mathbb{Y}_n(F)$. A crucial aspect of the modern Langlands program is its formulation in perfectoid geometry, introduced by Scholze. In the Yang-Langlands program, we define an extended perfectoid structure associated with $\mathbb{Y}_n(F)$:

$$\mathbb{Y}_n(F)^{\text{perf}} = \varprojlim_{\varphi} \mathbb{Y}_n(F)/p^n.$$

This allows us to study the ramification theory of Yang-Galois representations in an infinitely deep tower of field extensions. We conjecture that there exists a higher ramification filtration:

$$G^{(i)}_{\mathbb{Y}_n(F)} = \{\sigma \in G_{\mathbb{Y}_n(F)} \mid \sigma \equiv \text{id} \mod p^i\},$$

which generalizes the classical ramification theory to infinite-dimensional settings.

73.2. P-Adic Hodge Theory and the $\mathbb{Y}_n(F)$-Analogue of (φ, Γ)-Modules. Classical p-adic Hodge theory relates Galois representations to semi-linear algebraic structures called (φ, Γ)-modules. In the Yang-Langlands program, we propose a new class of $(\varphi, \Gamma_{\mathbb{Y}_n})$-modules:

$$D^{\varphi,\Gamma}_{\mathbb{Y}_n}(V) = \text{Filtered module associated with } V \in \text{Rep}_{\mathbb{Y}_n(F)}(G_{\mathbb{Y}_n(F)}).$$

We conjecture that these modules serve as an intermediary between automorphic forms and Galois representations, extending classical p-adic Hodge theory.

74. Yang-Langlands Duality and Derived Categories of L-Functions

74.1. Derived Langlands Duality and Spectral Stacks. Classical Langlands duality relates reductive groups to their Langlands duals. In the Yang-Langlands program, we extend this notion to derived stacks:

$${}^L G_{\mathbb{Y}_n(F)} = \widehat{G}_{\mathbb{Y}_n(F)} \rtimes W_{\mathbb{Y}_n(F)},$$

where $\widehat{G}_{\mathbb{Y}_n(F)}$ is the Langlands dual group, and $W_{\mathbb{Y}_n(F)}$ is the Weil group of the higher-dimensional field $\mathbb{Y}_n(F)$. This allows us to define spectral stacks parameterizing automorphic L-functions.

74.2. Higher-Dimensional L-Functions and Yang-Langlands Zeta Functions. A key feature of the classical Langlands program is its relation to L-functions. In the Yang-Langlands framework, we introduce an extended class of Yang-Langlands zeta functions:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathbb{Y}_n(F))} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

These zeta functions are expected to satisfy functional equations governed by the noncommutative Fourier transform of higher Hecke operators.

75. Noncommutative TQFT and Yang-Langlands Gauge Theory

75.1. **Extended Chern-Simons Theory and the Yang-Galois Action.** The classical geometric Langlands program has deep connections with Chern-Simons theory. In the Yang-Langlands setting, we introduce a new gauge-theoretic formulation:

$$S_{\mathbb{Y}_n} = \int_M \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A)$$

where A is a gauge field associated with a Yang-Galois representation. The path integral over these gauge fields defines a noncommutative TQFT that governs the extended Langlands duality.

75.2. **AdS/CFT Correspondences and the Yang-Langlands Duality.** The AdS/CFT correspondence in physics posits a duality between gravity in an anti-de Sitter (AdS) space and a conformal field theory (CFT) on its boundary. We conjecture an analogous correspondence in number theory:

$$\text{Yang-Galois Theory on } \mathbb{Y}_n(F) \longleftrightarrow \text{Automorphic Theory on the Boundary of } \mathbb{Y}_n(F).$$

This provides a bridge between quantum field theory and higher-dimensional automorphic forms.

76. Future Research Directions

- Can we explicitly construct $(\varphi, \Gamma_{\mathbb{Y}_n})$-modules in the setting of higher-dimensional p-adic representations?
- What are the implications of higher noncommutative geometry for L-functions in the Yang-Langlands framework?
- How does the categorification of Hecke algebras contribute to the functoriality principle in noncommutative Langlands duality?
- Can we define an explicit geometric model for the AdS/CFT correspondence in number theory?

The Yang-Langlands program extends the scope of classical Langlands theory, integrating number theory, representation theory, algebraic geometry, and quantum physics into a unified mathematical framework. We propose a new class of (φ, Γ)-*modules* adapted to the structure of $\mathbb{Y}_n(F)$:

$$D^{\varphi,\Gamma}_{\mathbb{Y}_n(F)} = \text{Category of } (\varphi, \Gamma)\text{-modules over } \mathbb{Y}_n(F).$$

These modules are expected to encode the p-adic representations of the Yang-Galois groups and provide a bridge between arithmetic geometry and p-adic analysis in the extended Langlands setting.

77. Higher L-Functions and Extended Functional Equations

77.1. Yang-Langlands L-Functions and Higher Euler Products. The classical Langlands program associates L-functions to automorphic representations and Galois representations. We extend this by defining the *higher Yang-Langlands L-function*:

$$L(s, \pi_{\mathbb{Y}_n}) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Y}_n(F))} (1 - \lambda_{\mathfrak{p}} N(\mathfrak{p})^{-s})^{-1},$$

where $\lambda_{\mathfrak{p}}$ are the Hecke eigenvalues associated with the automorphic representation $\pi_{\mathbb{Y}_n}$.

77.2. Functional Equation of the Yang-Zeta Function. We conjecture that the Yang-Zeta function satisfies a functional equation generalizing the classical Riemann zeta function:

$$\zeta_{\mathbb{Y}_n}(s) = \Gamma_{\mathbb{Y}_n}(s) L(s, \pi_{\mathbb{Y}_n}),$$

where $\Gamma_{\mathbb{Y}_n}(s)$ is an explicitly computable gamma factor depending on the structure of $\mathbb{Y}_n(F)$.

78. Topological Field Theories and the Physics of Yang-Langlands

78.1. Yang-Langlands TQFT and Extended Geometric Duality. We introduce a new class of *Yang-Langlands topological quantum field theories (TQFTs)* that provide a physical realization of the extended Langlands duality. The partition function of such a theory takes the form:

$$Z_{\mathbb{Y}_n}(M) = \sum_{\pi} e^{-S_{\mathbb{Y}_n}(\pi)},$$

where $S_{\mathbb{Y}_n}(\pi)$ is the action functional associated with an automorphic representation.

78.2. Yang-Magnetic Monopoles and Non-Abelian Gauge Symmetries. Inspired by S-duality in supersymmetric gauge theories, we propose a new interpretation of automorphic forms in terms of *Yang-magnetic monopoles*:

$$F_{\mathbb{Y}_n} = dA + A \wedge A,$$

where A is a gauge field associated with the Yang-Galois group. We conjecture that these gauge-theoretic objects play a fundamental role in the extended Langlands program.

79. Future Directions and Research Problems

- Can we explicitly construct (φ, Γ)-modules over $\mathbb{Y}_n(F)$ and relate them to p-adic representations?
- How do the noncommutative Hecke algebras interact with the spectral decomposition of Yang-Langlands automorphic forms?
- What is the precise role of geometric Langlands duality in the setting of extended quantum groups?
- Can we derive a trace formula for Yang-Langlands representations in non-Archimedean settings?

- How do higher TQFTs provide a physical interpretation of automorphic forms over $\mathbb{Y}_n(F)$?

80. From Langlands Duality to Langlands m-ality

80.1. **Classical Langlands Duality.** Langlands duality is the central principle of the Langlands program, establishing a deep correspondence between:

- Galois representations $\rho : \mathrm{Gal}(\overline{F}/F) \to {}^L G(\mathbb{C})$, where ${}^L G$ is the Langlands dual group of G, and
- Automorphic representations π of $G(\mathbb{A}_F)$.

This duality operates on a one-to-one level: each automorphic representation is expected to correspond to a Galois representation under this framework.

80.2. **Generalizing to Langlands m-ality.** Langlands m-ality extends beyond classical duality by introducing multi-layered correspondences rather than simple one-to-one mappings. In this framework:

- Instead of a single Langlands dual group ${}^L G$, we consider a hierarchy of Langlands dual groups ${}^L G_n$, each capturing additional structure.
- Instead of a single correspondence, we explore higher-categorical functorial maps between multiple automorphic representations and Galois representations.

We define Langlands m-ality as an extension of Langlands duality where the standard Langlands group ${}^L G$ (which acts as an intermediary between Galois representations and automorphic forms) is replaced by a tower of groups:

$$\cdots \to {}^L G_n \to {}^L G_{n-1} \to \cdots \to {}^L G_1 \to {}^L G.$$

Each ${}^L G_n$ represents an extension that incorporates additional arithmetic, geometric, or categorical structure.

80.3. **Higher Langlands Correspondences in m-ality.**

- Classical Langlands duality establishes a bijection:
$$\mathrm{Gal}(\overline{F}/F) \longleftrightarrow \mathrm{Aut}(G(\mathbb{A}_F)).$$
- In Langlands m-ality, this is extended to a hierarchy of multi-level mappings:

$$\cdots \to \mathrm{Gal}_n(\overline{F}/F) \longleftrightarrow \mathrm{Aut}_n(G(\mathbb{A}_F)) \to \cdots \to \mathrm{Gal}_1(\overline{F}/F) \longleftrightarrow \mathrm{Aut}_1(G(\mathbb{A}_F)).$$

Each level captures additional hidden symmetries, analogous to how higher algebraic structures encode finer invariants in homotopy theory.

80.4. **Geometric and Categorical Interpretations of Langlands m-ality.**

80.4.1. *Geometric Langlands m-ality.* The classical geometric Langlands program establishes a duality between D-modules on Bun_G (the moduli of principal G-bundles) and automorphic sheaves. In the extended setting:

- Langlands m-ality introduces derived stacks and higher moduli spaces, yielding a multi-level geometric Langlands correspondence:
$$D^b_{\mathrm{Aut}_n}(G) \simeq D^b_{\mathrm{Gal}_n}(G),$$
where D^b denotes the bounded derived category of automorphic sheaves and Galois representations at level n.

80.4.2. *Categorified Hecke Operators.* In classical Langlands theory, the Hecke algebra $\mathcal{H}$ acts on automorphic forms. In Langlands m-ality, we introduce a hierarchy of higher Hecke algebras:

$$\mathcal{H}_n = \bigoplus_{i=0}^{n} \mathcal{H}_i.$$

Each $\mathcal{H}_i$ encodes a deeper functoriality layer in the Langlands m-ality correspondence.

80.5. **Relation to Higher Langlands Functoriality.** Langlands functoriality conjectures predict that if ${}^L H$ and ${}^L G$ are related by a homomorphism of L-groups, then their automorphic representations should also be related:

$$\Pi(H) \longrightarrow \Pi(G).$$

Langlands m-ality enhances this principle by introducing higher functorial transformations:

$$\Pi_n(H) \longrightarrow \Pi_n(G) \to \Pi_{n-1}(G) \to \cdots \to \Pi_1(G).$$

This allows for a multi-step functorial transformation, capturing additional symmetries beyond the classical setting.

80.6. **Application to Noncommutative and Perfectoid Extensions.** Since Langlands m-ality introduces additional layers of structure, it naturally extends to:

- **Noncommutative Langlands m-ality**, where Hecke operators and automorphic representations are defined in a noncommutative setting.
- **Perfectoid Langlands m-ality**, where each level in the hierarchy corresponds to a perfectoid space structure:
$$\mathbb{Y}_n(F)^{\mathrm{perf}} = \varprojlim_{\varphi} \mathbb{Y}_n(F)/p^n.$$

80.7. **Conclusion.** By leveraging m-ality, we systematically generalize Langlands duality into Langlands m-ality, capturing deeper arithmetic symmetries, higher functorial correspondences, and hierarchical automorphic representation structures. This framework is particularly useful for:

- Noncommutative generalizations of the Langlands program,
- Higher categorical formulations of automorphic and Galois representations,
- Geometric and derived stack-theoretic approaches to Langlands correspondences.

These insights pave the way for further developments in the Langlands program, extending it into a multi-layered, hierarchical framework with broad applications in arithmetic geometry, representation theory, and mathematical physics.

81. From Langlands Duality to Langlands m-ality

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This duality operates on a one-to-one level: each automorphic representation is expected to correspond to a Galois representation under this framework.

81.2. **Generalizing to Langlands m-ality.** Langlands m-ality extends beyond classical duality by introducing multi-layered correspondences rather than simple one-to-one mappings. In this framework:

- Instead of a single Langlands dual group ${}^LG$, we consider a hierarchy of Langlands dual groups ${}^LG_m$, each capturing additional structure.
- Instead of a single correspondence, we explore higher-categorical functorial maps between multiple automorphic representations and Galois representations.

We define Langlands m-ality as an extension of Langlands duality where the standard Langlands group ${}^LG$ (which acts as an intermediary between Galois representations and automorphic forms) is replaced by a tower of groups:

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Each $\mathcal{H}_i$ encodes a deeper functoriality layer in the Langlands m-ality correspondence.

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$$\Pi(H) \longrightarrow \Pi(G).$$

Langlands m-ality enhances this principle by introducing higher functorial transformations:

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- Geometric and derived stack-theoretic approaches to Langlands correspondences.

These insights pave the way for further developments in the Langlands program, extending it into a multi-layered, hierarchical framework with broad applications in arithmetic geometry, representation theory, and mathematical physics.

83. Refined Functoriality and Extended Langlands m-ality Structures

83.1. **Tower of Langlands Dual Groups and Higher Functorial Transformations.** To refine Langlands m-ality, we propose a systematic study of the hierarchy of Langlands dual groups:

$$\cdots \to {}^L G_m \to {}^L G_{m-1} \to \cdots \to {}^L G_1 \to {}^L G.$$

Each group ${}^L G_m$ is constructed to incorporate additional structures, such as:

- **Geometric Data:** Moduli spaces of higher principal bundles,
- **Categorical Data:** Higher categories of representations,
- **Noncommutative Deformations:** Quantum and p-adic extensions.

For a given functoriality map between L-groups:

$$^{L}H_m \to {}^{L}G_m,$$

we conjecture that there exists an induced hierarchy of automorphic representations:

$$\Pi_m(H) \longrightarrow \Pi_m(G) \to \Pi_{m-1}(G) \to \cdots \to \Pi_1(G).$$

This multi-step functorial transformation extends the classical Langlands functoriality principle by encoding refined structural symmetries.

83.2. **Extended Moduli of Yang-Galois Representations.** We introduce a refined moduli space for higher Galois representations associated with the fields $\mathbb{Y}_n(F)$:

$$\mathcal{M}_{\mathrm{Gal}}^{\mathbb{Y}_n,m}(F) = \left[\mathrm{Spec}(\mathbb{Y}_n(F))/G_m\right]_{\infty}.$$

This derived stack extends classical moduli spaces by incorporating:

- Non-abelian cohomological deformations,
- Higher categorical structure of automorphic representations,
- Interaction with perfectoid and noncommutative frameworks.

83.3. **Higher Noncommutative Hecke Algebras and Spectral Theory.** To describe the action of Hecke operators in Langlands m-ality, we define the *higher noncommutative Hecke algebra*:

$$\mathcal{H}_m = \bigoplus_{i=0}^{m} \mathcal{H}_i,$$

where each $\mathcal{H}_i$ corresponds to an additional functorial level. The spectral decomposition of automorphic forms over $\mathbb{Y}_n(F)$ in this setting is conjectured to satisfy:

$$L^2_{\mathrm{Aut}_m}(\mathbb{Y}_n(F)) = \bigoplus_{\pi_m} m_{\pi_m}\pi_m.$$

The operators from $\mathcal{H}_m$ act on this space, allowing for a refined spectral analysis.

83.4. **Extended L-Functions and Multi-Layered Zeta Functions.** The classical Langlands program associates L-functions to automorphic and Galois representations. In Langlands m-ality, we introduce *multi-layered L-functions*:

$$L_m(s, \pi_{\mathbb{Y}_n}) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Y}_n(F))} (1 - \lambda_{\mathfrak{p},m} N(\mathfrak{p})^{-s})^{-1}.$$

Each level of this function captures additional spectral and arithmetic invariants.

We conjecture that the associated Yang-Zeta function satisfies a functional equation:

$$\zeta^{(m)}_{\mathbb{Y}_n}(s) = \Gamma^{(m)}_{\mathbb{Y}_n}(s) L_m(s, \pi_{\mathbb{Y}_n}),$$

where $\Gamma^{(m)}_{\mathbb{Y}_n}(s)$ is a multi-layered gamma factor depending on Langlands m-ality.

83.5. **Langlands m-ality and Higher Categorical TQFTs.** A fundamental connection between the Langlands program and physics arises from topological quantum field theory (TQFT). In Langlands m-ality, we propose an extension to *higher categorical TQFTs*, defined via:

$$Z_{\mathbb{Y}_n}^{(m)}(M) = \sum_{\pi_m} e^{-S_{\mathbb{Y}_n}^{(m)}(\pi_m)},$$

where $S_{\mathbb{Y}_n}^{(m)}(\pi_m)$ is the action functional associated with the multi-layered automorphic representations.

83.6. **Future Directions.**

- Can we explicitly compute higher Hecke algebra actions in the setting of noncommutative Langlands m-ality?
- How does the interaction between moduli stacks of Yang-Galois representations and perfectoid spaces extend the arithmetic of Langlands m-ality?
- What is the role of derived categories and higher TQFTs in refining functoriality principles?
- Can we establish a trace formula in Langlands m-ality that incorporates higher categorical functoriality?

84. Higher-Dimensional Motives and Langlands m-ality

84.1. **Noncommutative Motives and Derived Categories.** In classical arithmetic geometry, the category of motives provides a universal cohomological framework for varieties. In Langlands m-ality, we introduce the notion of *higher-dimensional noncommutative motives*:

$$\mathcal{M}^{\mathbb{Y}_n,m}(F) = D^b_{\mathrm{nc}}(\mathbb{Y}_n(F)),$$

where $D^b_{\mathrm{nc}}(\mathbb{Y}_n(F))$ denotes the bounded derived category of noncommutative spaces associated with $\mathbb{Y}_n(F)$ at level m. These motives serve as the bridge between automorphic representations and higher Langlands functoriality.

84.2. **Motivic Galois Groups and Higher Tannakian Formalism.** Motivic Galois groups govern the structure of periods and special values of L-functions. In Langlands m-ality, we define the *hierarchy of motivic Galois groups*:

$$G_{\mathcal{M}^{\mathbb{Y}_n,m}(F)} = \mathrm{Aut}^{\otimes}(\mathcal{M}^{\mathbb{Y}_n,m}(F)),$$

where $\mathrm{Aut}^{\otimes}$ represents the Tannakian fundamental group of the category of noncommutative Yang-motives. We conjecture that these groups satisfy a refined functoriality principle:

$$G_{\mathcal{M}^{\mathbb{Y}_n,m}(F)} \longrightarrow {}^L G_m.$$

This provides an explicit link between motivic structures and Langlands m-ality.

84.3. **Adelic Langlands m-ality and Higher Local-Global Principles.**

84.3.1. *Adelic Automorphic Representations.* We refine the notion of adelic structures in Langlands m-ality by defining:

$$\mathbb{A}_{\mathbb{Y}_n,m}(F) = \prod_v{}' \mathbb{Y}_n(F_v)^{(m)},$$

where $\mathbb{Y}_n(F_v)^{(m)}$ represents the local field $\mathbb{Y}_n(F_v)$ extended to level m. The space of automorphic representations in this framework is:

$$\Pi_m(\mathbb{A}_{\mathbb{Y}_n}(F)) = L^2(\mathbb{A}_{\mathbb{Y}_n,m}(F)).$$

84.3.2. *Higher Local-Global Correspondence.* We propose an extension of the classical local-global principle in Langlands theory by introducing a stratified hierarchy of local Langlands correspondences:

$$\Pi_m(G(F_v)) \longrightarrow \Pi_{m-1}(G(F_v)) \to \cdots \to \Pi_1(G(F_v)).$$

This structure encodes deeper functoriality across local and global automorphic representations.

84.4. **Topological Higher Langlands Duality.**

84.4.1. *Yang-Langlands Duality and Higher Homotopy Theory.* The geometric Langlands program has been linked to topological field theory and higher category theory. In the setting of Langlands m-ality, we propose an *extended homotopy-theoretic Langlands correspondence*:

$$\mathcal{C}_{\mathbb{Y}_n,m}(F) \simeq \mathrm{Fun}(\mathbb{Y}_n(F), \mathrm{Spectra}).$$

This equivalence allows for a homotopical interpretation of automorphic representations and Galois groups.

84.4.2. *Yang-TQFT and Extended Functoriality.* We define a *hierarchy of topological quantum field theories* (TQFTs) associated with Langlands m-ality:

$$Z_{\mathbb{Y}_n,m}(M) = \int_{\mathbb{Y}_n(F)} e^{-S_{\mathbb{Y}_n,m}(A)} dA.$$

This partition function encodes automorphic spectral data through a gauge-theoretic perspective.

84.5. **Future Research Questions and Open Problems.**

- Can we develop an explicit spectral decomposition of automorphic forms using the hierarchy of Hecke algebras in Langlands m-ality?
- How do noncommutative motives contribute to higher functoriality in Galois representations?
- What is the role of higher categorical TQFTs in extending trace formulas for automorphic L-functions?
- Can we establish a cohomological realization of Langlands m-ality using topological spectral stacks?

85. Langlands m-ality and Derived Arithmetic Geometry

85.1. Higher-Dimensional Derived Stacks and Moduli of Langlands Representations. In classical Langlands theory, moduli spaces of Galois representations and automorphic forms play a central role. In the framework of Langlands m-ality, we introduce the notion of *higher derived stacks*:

$$\mathcal{M}^{\mathbb{Y}_n,m}_{\text{Langlands}}(F) = [\text{Spec}(\mathbb{Y}_n(F))/G_m]_\infty .$$

This derived moduli stack parametrizes higher Yang-Galois representations and their interactions with automorphic forms in Langlands m-ality.

85.2. Categorified Langlands Correspondences and Infinity-Topoi. We propose an extension of the Langlands correspondence to the setting of higher category theory:

$$\mathcal{D}^b_{\text{Aut}_m}(\mathbb{Y}_n(F)) \simeq \mathcal{D}^b_{\text{Gal}_m}(\mathbb{Y}_n(F)),$$

where $\mathcal{D}^b$ denotes the derived category of automorphic sheaves and Galois representations at level m. These equivalences are conjectured to be realized as natural transformations in the setting of ∞-topoi.

85.3. Perfectoid Langlands m-ality and Higher Ramification Structures. We extend Scholze's perfectoid geometry to the setting of Langlands m-ality by considering the following tower of perfectoid spaces:

$$\mathbb{Y}_n(F)^{\text{perf},m} = \varprojlim_{\varphi} \mathbb{Y}_n(F)/p^m.$$

The associated ramification theory introduces *higher ramification filtrations*:

$$G^{(m,i)}_{\mathbb{Y}_n(F)} = \{\sigma \in G_{\mathbb{Y}_n(F)} \mid \sigma \equiv \text{id} \mod p^{m,i}\}.$$

We conjecture that this refined filtration allows for an extended reciprocity law in the framework of Langlands m-ality.

85.4. Extended Hecke Operators and Langlands m-ality.

85.4.1. *Higher Hecke Algebra Representations.* The action of Hecke operators on automorphic representations is fundamental to classical Langlands theory. In Langlands m-ality, we define an extended hierarchy of Hecke algebras:

$$\mathcal{H}_{m,n} = \bigoplus_{i=0}^{m} \mathcal{H}_{i,n},$$

where each $\mathcal{H}_{i,n}$ corresponds to an additional functorial structure that interacts with the hierarchy of Langlands dual groups.

85.4.2. *Functoriality of Hecke Operators in Noncommutative Settings.* We introduce an extended functoriality principle for the action of Hecke operators:

$$\text{Ind}^{G_m}_{H_m}\mathcal{H}_m \to \text{Ind}^{G_{m-1}}_{H_{m-1}}\mathcal{H}_{m-1} \to \cdots \to \text{Ind}^{G_1}_{H_1}\mathcal{H}_1.$$

This functorial transformation captures the hierarchical structure of Hecke actions in Langlands m-ality.

85.5. Higher Adelic Structures and Extended Automorphic L-Functions.

85.5.1. *Multi-Layered Adelic Decompositions.* We refine the notion of adelic structures in Langlands m-ality by defining:

$$\mathbb{A}_{\mathbb{Y}_n,m}(F) = \prod_v{}' \mathbb{Y}_n(F_v)^{(m)}.$$

The space of automorphic representations in this framework is given by:

$$\Pi_m(\mathbb{A}_{\mathbb{Y}_n}(F)) = L^2(\mathbb{A}_{\mathbb{Y}_n,m}(F)).$$

85.5.2. *Higher Langlands L-Functions and Spectral Analysis.* The classical Langlands program associates L-functions to automorphic and Galois representations. In Langlands m-ality, we introduce *multi-layered Langlands L-functions*:

$$L_{m,n}(s, \pi_{\mathbb{Y}_n}) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Y}_n(F))} (1 - \lambda_{\mathfrak{p},m,n} N(\mathfrak{p})^{-s})^{-1}.$$

These functions are conjectured to satisfy an extended functional equation:

$$\zeta_{\mathbb{Y}_n}^{(m,n)}(s) = \Gamma_{\mathbb{Y}_n}^{(m,n)}(s) L_{m,n}(s, \pi_{\mathbb{Y}_n}),$$

where $\Gamma_{\mathbb{Y}_n}^{(m,n)}(s)$ is a spectral gamma factor that encodes automorphic symmetries in Langlands m-ality.

85.6. Yang-Langlands Topological Quantum Field Theories (TQFTs).

85.6.1. *Extended Langlands Duality and Higher Gauge Theories.* Inspired by the interaction between the geometric Langlands program and topological field theories, we propose a higher-categorical formulation of Langlands duality in the setting of gauge theory:

$$Z_{\mathbb{Y}_n,m}(M) = \sum_{\pi_{m,n}} e^{-S_{\mathbb{Y}_n,m,n}(\pi_{m,n})}.$$

The path integral formulation allows for a topological realization of automorphic forms in Langlands m-ality.

85.6.2. *Higher Chern-Simons Theories and Langlands Duality.* We propose an *extended Chern-Simons action* associated with Langlands m-ality:

$$S_{\mathbb{Y}_n,m} = \int_M \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A),$$

where A is a gauge field associated with the Langlands dual hierarchy. The partition function:

$$Z_{\mathbb{Y}_n,m}(M) = \int \mathcal{D}A \; e^{-S_{\mathbb{Y}_n,m}(A)}$$

encodes the spectral decomposition of automorphic representations in a topological framework.

85.7. **Open Problems and Future Research Directions.**

- Can we establish a full functorial hierarchy for Hecke algebras in Langlands m-ality?
- How do higher motivic structures contribute to the reciprocity laws in Langlands m-ality?
- What are the implications of Langlands m-ality for higher categorical representation theory?
- Can we develop explicit formulas for automorphic L-functions in the setting of multi-layered adelic analysis?
- How does the interaction of noncommutative geometry and TQFT refine the spectral theory of automorphic forms?

We introduce an extended noncommutative Hecke algebra:

$$\mathcal{H}_m = \bigoplus_{i=0}^{m} \mathcal{H}_i,$$

where each $\mathcal{H}_i$ corresponds to a refined functorial level of automorphic representations. The spectral decomposition of automorphic forms in this setting is conjectured to satisfy:

$$L^2_{\mathrm{Aut}_m}(\mathbb{Y}_n(F)) = \bigoplus_{\pi_m} m_{\pi_m} \pi_m.$$

The operators from $\mathcal{H}_m$ act on this space, capturing new functorial symmetries in Langlands m-ality.

85.7.1. *Multi-Layered Hecke Operators and Functoriality.* In classical Langlands, Hecke operators serve as a bridge between automorphic forms and Galois representations. In Langlands m-ality, we introduce a *multi-layered Hecke functor*:

$$\mathcal{H}_m \colon \Pi_m(G) \to \Pi_{m-1}(G) \to \cdots \to \Pi_1(G).$$

This construction refines the functorial correspondence between automorphic and Galois representations in an extended categorical framework.

86. Langlands m-ality and the Structure of Higher L-Functions

86.1. **Hierarchical L-Functions and Refined Functional Equations.** In the classical Langlands program, L-functions encode deep arithmetic and spectral properties of automorphic forms. In Langlands m-ality, we propose an extended structure of *hierarchical L-functions*:

$$L_m(s, \pi_{\mathbb{Y}_n}) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Y}_n(F))} (1 - \lambda_{\mathfrak{p},m} N(\mathfrak{p})^{-s})^{-1}.$$

Each level of this function encodes refined spectral and arithmetic invariants. The associated *hierarchical Langlands zeta function* satisfies the extended functional equation:

$$\zeta^{(m)}_{\mathbb{Y}_n}(s) = \Gamma^{(m)}_{\mathbb{Y}_n}(s) L_m(s, \pi_{\mathbb{Y}_n}),$$

where $\Gamma^{(m)}_{\mathbb{Y}_n}(s)$ is a stratified gamma factor capturing hidden symmetries in the noncommutative Langlands setting.

87. Quantum Field Theory and the Topological Realization of Langlands m-ality

87.1. Higher-Categorical TQFTs and Extended Automorphic Functoriality. Topological quantum field theories (TQFTs) have been deeply linked to the geometric Langlands program. In the framework of Langlands m-ality, we propose an *extended TQFT* encoding the spectral decomposition of automorphic forms:

$$Z_{\mathbb{Y}_n,m}(M) = \sum_{\pi_m} e^{-S_{\mathbb{Y}_n,m}(\pi_m)},$$

where $S_{\mathbb{Y}_n,m}(\pi_m)$ is the action functional associated with an automorphic representation at level m.

87.2. AdS/CFT Duality and the Arithmetic of Langlands m-ality. Inspired by the AdS/CFT correspondence in physics, we conjecture an analogous *Langlands m-ality duality*:

$$\text{Higher Yang-Galois Theory on } \mathbb{Y}_n(F) \longleftrightarrow \text{Automorphic Theory on the Boundary of } \mathbb{Y}_n(F).$$

This perspective suggests a deep connection between arithmetic geometry, representation theory, and quantum field theory.

88. Future Research Directions

- Can we explicitly classify automorphic sheaves in the setting of higher categorical Langlands m-ality?
- What is the precise role of perfectoid geometry in refining functoriality principles in Langlands m-ality?
- How do derived stacks of Yang-Galois representations interact with noncommutative motives?
- Can we establish a trace formula in Langlands m-ality that captures higher categorical functoriality?
- How do quantum field theoretic models refine spectral decomposition in noncommutative automorphic theory?