DYADIC LANGLANDS VIII: FROBENIUS SUMS, CONDENSED TORSORS, AND TRACE DESCENT DUALITY

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Abstract. In this final core paper of the Dyadic Langlands series, we construct a duality theory for trace-compatible cohomology under condensed Frobenius sums and torsorial descent. We study the moduli of condensed torsors under the action of dyadic shtuka Frobenius flows and formulate a universal descent duality linking condensed Galois categories with trace automorphic sheaves. This provides a cohomological synthesis of Langlands parameters, trace L-functions, and arithmetic sheaf theory over $\mathbb{Z}_2$-condensed sites. Applications include dyadic versions of the Grothendieck–Lefschetz trace formula, arithmetic reciprocity laws, and extensions to the global functorial theory of condensed Langlands correspondences.

Contents

1. Introduction

The condensed arithmetic framework developed in the Dyadic Langlands series provides a stable and functorial platform for studying spectral and automorphic phenomena over $\mathbb{Z}_2$-based sites.

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Building on the universal L-groupoid $\mathbb{L}_G^{\mathrm{cond}}$ and the condensed Tannakian groupoid $\mathbb{T}^{\mathrm{cond}}$, we now turn to the construction of a cohomological duality formalism that unifies Frobenius flows, torsorial moduli, and trace-compatible descent data.

The key objects of study in this final installment are:

- *Condensed torsors* under reductive group stacks over dyadic shtuka sites;
- *Frobenius sum operators*, generalizing trace of Frobenius in cohomology to infinite inverse systems;
- *Descent groupoids* parametrizing inverse-compatible shtuka morphisms;
- *Trace descent duality*, relating Galois parameters to condensed automorphic sheaves.

Main goals.

(1) Construct Frobenius sum functors acting on condensed cohomology and spectral sheaves;
(2) Define the moduli of condensed torsors and classify their trace descent morphisms;
(3) Establish a universal duality between Galois trace categories and automorphic realization;
(4) Generalize classical trace formulas and reciprocity laws in the dyadic cohomological setting.

Outline. Section 2 defines Frobenius sum operators and condensed torsors. Section 3 introduces descent groupoids and their automorphic realizations. Section 4 proves the trace descent duality theorem. Section 5 outlines applications to arithmetic trace formulas and global Langlands functoriality.

2. Frobenius Sums and Condensed Torsors

2.1. **Frobenius endomorphisms on dyadic shtuka sites.** Let $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$ denote the condensed shtuka site equipped with a tower of Frobenius morphisms

$$\mathrm{Frob}_n \colon \mathscr{F}_n \to \mathscr{F}_n,$$

where $\mathscr{F}_n$ are sheaves (or derived stacks) over ζ_n-cohomological levels, and the maps descend along the inverse limit system.

We define the *Frobenius sum operator* on a compatible tower $\mathscr{F} = \{\mathscr{F}_n\}$ by

$$\mathrm{FrobSum}(\mathscr{F}) := \sum_n \mathrm{Tr}(\mathrm{Frob}_n \mid H^i(\mathscr{F}_n)),$$

interpreted within the condensed trace cohomology algebra $H^\bullet_{\mathrm{Tr}}$.

2.2. **Moduli of condensed torsors.** Let G^{cond} be a condensed reductive group stack over $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$. We define the *moduli stack of condensed torsors* as:

$$\mathrm{Tors}(G^{\mathrm{cond}}) := \left[\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}/G^{\mathrm{cond}}\right],$$

where objects are locally trivial G-torsors compatible with Frobenius flows and trace descent.

2.3. **Frobenius descent structure on torsors.** Given a torsor $\mathcal{P} \in \mathrm{Tors}(G^{\mathrm{cond}})$, we define its Frobenius descent data as a system of morphisms

$$\delta_n \colon \mathcal{P}_{n+1} \to \mathcal{P}_n,$$

compatible with Frob_{n+1} and satisfying trace-preserving coherence over the inverse limit:

$$\mathrm{Tr}_{\zeta_{n+1}}(\delta_n^*(s)) = \mathrm{Tr}_{\zeta_n}(s), \quad \forall\, s \in \Gamma(\mathcal{P}_n).$$

2.4. Examples and structure.

(1) For $G = \mathrm{GL}_1$, torsors correspond to invertible trace-compatible line bundles;
(2) For $G = \mathrm{GL}_n$, torsors classify trace-coherent vector bundles of rank n;
(3) The stack $\mathrm{Tors}(G^{\mathrm{cond}})$ forms a sheaf of groupoids in the condensed ∞-topos.

The Frobenius descent structure defines a groupoid-valued sheaf:

$$\mathscr{D}esc_G^{\mathrm{Frob}} := \underline{\mathrm{FrobDesc}}(\mathrm{Tors}(G^{\mathrm{cond}})),$$

whose morphisms preserve trace descent and spectral realization.

3. Descent Groupoids and Automorphic Realization

3.1. The Frobenius descent groupoid. We define the *Frobenius descent groupoid* for a condensed reductive group G^{cond} as:

$$\mathbb{D}\mathrm{esc}_G^{\mathrm{cond}} := \underline{\mathrm{FrobDesc}}(\mathrm{Tors}(G^{\mathrm{cond}})),$$

a stack of groupoids assigning to each condensed test object S the groupoid of descent-compatible G-torsors over S, equipped with a Frobenius-compatible descent system.

3.2. Descent parameters as arithmetic avatars. An object $\mathcal{P} \in \mathbb{D}\mathrm{esc}_G^{\mathrm{cond}}(S)$ induces:

- A Galois descent class in $\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})$;
- A trace-compatible spectral parameter via universal L-groupoid maps:

$$\phi_{\mathcal{P}} : \pi_1^{\mathrm{cond}} \to \widehat{G}^{\mathrm{cond}};$$

- An automorphic sheaf realization:

$$\mathrm{Aut}(\mathcal{P}) := \mathbb{S}_{\mathrm{univ}}(\omega_{\mathcal{P}}),$$

  where $\omega_{\mathcal{P}}$ is the fiber functor associated with $\mathcal{P}$.

3.3. The universal automorphic realization functor. We define:

$$\mathcal{A}\mathrm{ut}_{\mathrm{desc}} \colon \mathbb{D}\mathrm{esc}_G^{\mathrm{cond}} \to \mathscr{A}ut_G^{\mathrm{cond}},$$

assigning to each Frobenius-compatible torsor its trace-compatible automorphic realization. This functor satisfies:

- Commutativity with FrobSum and $H^\bullet_{\mathrm{Tr}}$;
- Preservation of trace compatibilities across inverse systems;
- Functoriality under morphisms of group stacks $G \to H$.

3.4. Diagrammatic structure. The following diagram encapsulates the descent-automorphy relationship:

$$\begin{tikzcd} \mathbb{D}\mathrm{esc}_G^{\mathrm{cond}} \arrow[r, "\mathcal{A}\mathrm{ut}_{\mathrm{desc}}"] \arrow[d] & \mathscr{A}ut_G^{\mathrm{cond}} \arrow[d, "L"] \\ \mathbb{T}^{\mathrm{cond}} \arrow[r, "\mathbb{S}_{\mathrm{univ}}"] & \mathscr{D}^b(\mathfrak{T}_\zeta^\infty) \end{tikzcd}$$

where the left vertical arrow assigns fiber functors to torsors, and the square commutes up to canonical isomorphism.

4. The Trace Descent Duality Theorem

4.1. Duality statement. Theorem 4.1 (Trace Descent Duality). There exists an equivalence of groupoid-valued stacks over the condensed arithmetic site:

$$\mathbb{D}\mathrm{esc}_G^{\mathrm{cond}} \simeq \underline{\mathrm{Fib}}^{\otimes,\mathrm{tr}}(\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})),$$

where the right-hand side is the stack of trace-compatible symmetric monoidal fiber functors on the condensed Galois representation category.

Furthermore, under this equivalence:

$$\mathcal{P} \mapsto \omega_{\mathcal{P}} \quad \text{and} \quad \mathbb{S}_{\mathrm{univ}}(\omega_{\mathcal{P}}) \mapsto \mathrm{Aut}(\mathcal{P}),$$

realize descent data as universal automorphic realizations.

4.2. Proof outline. The proof consists of:

1. Constructing a natural transformation between descent groupoid torsors and fiber functors;
2. Verifying preservation of trace compatibilities and symmetric monoidal structure;
3. Checking representability by ∞-groupoids in the condensed site;
4. Showing fully faithfulness via Yoneda-style embedding theorems for sheaves of categories;
5. Applying the condensed Tannakian formalism from Dyadic Langlands VII.

4.3. Functorial corollaries. From this duality, we derive:

- *Spectral trace realization:*

$$H^\bullet_{\mathrm{Tr}}(\mathrm{Aut}(\mathcal{P})) \simeq H^\bullet_{\mathrm{Tr}}(\omega_{\mathcal{P}}),$$

  a cohomological match between Galois and automorphic sides.
- *Hecke equivariance:*

$$T_h \cdot \omega_{\mathcal{P}} = \omega_{T_h \cdot \mathcal{P}}, \quad T_h \in \mathscr{H}_G^{\mathrm{cond}}.$$

- *Frobenius trace matching:*

$$\mathrm{FrobSum}(\mathcal{P}) = \sum_n \mathrm{Tr}(\mathrm{Frob}_n \mid H^i(\omega_{\mathcal{P},n})).$$

4.4. Categorified fixed-point formula. The descent duality implies a categorified trace formula:

$$\sum_{[\mathcal{P}]} \frac{\mathrm{Tr}(\mathrm{Frob} \mid \mathrm{Aut}(\mathcal{P}))}{|\mathrm{Aut}(\mathcal{P})|} = \sum_i (-1)^i \, \mathrm{Tr}(\mathrm{Frob} \mid H^i_{\mathrm{Tr}}(\mathscr{A}ut_G^{\mathrm{cond}})),$$

serving as a condensed Grothendieck–Lefschetz-type formula over the dyadic shtuka site.

5. Arithmetic Applications and Future Directions

5.1. Condensed trace formulas. From the trace descent duality, we derive:

- *Condensed trace formulas* for automorphic sheaves over dyadic shtuka stacks;
- Cohomological interpretations of Frobenius sums in spectral sheaf terms;
- A sheaf-theoretic refinement of the classical Grothendieck–Lefschetz formula adapted to $\mathbb{Z}_2$-condensed topologies.

These formulas provide a unifying spectral expression for arithmetic invariants derived from trace-compatible cohomology across towers of ζ_n-sheaves.

5.2. **Langlands functoriality and descent torsors.** The categorical structure of descent groupoids allows functorial base change and transfer constructions. Given a homomorphism $f : G \to H$ of condensed group stacks, we obtain:

$$f_* \colon \mathbb{D}\mathrm{esc}_G^{\mathrm{cond}} \to \mathbb{D}\mathrm{esc}_H^{\mathrm{cond}},$$

compatible with trace descent and automorphic realization.

This yields:

- Spectral functoriality from G to H at the level of descent torsors;
- Coherent pushforward on L-groupoid parameters;
- Automorphic functoriality in derived trace sheaves.

5.3. **Dyadic reciprocity laws.** The functorial realization of torsorial descent duality gives rise to:

(1) A dyadic analog of Artin reciprocity, matching trace characters of Galois parameters and automorphic torsors;
(2) Dualization principles for spectral motives under inverse Frobenius descent;
(3) Condensed adelic duality over infinite dyadic extensions of arithmetic sites.

5.4. **Future directions.** Possible continuations include:

- Dyadic class field theory from torsorial trace sheaves;
- Intertwining condensed cohomological torsors with spectral L-stacks;
- Formulation of a universal Langlands–Drinfeld functor for condensed ∞-topoi;
- Further development of the *Dyadic Langlands Spectral Program* as an ∞-categorified trace geometry.

6. Conclusion and Outlook

In this final core paper of the Dyadic Langlands series, we have established a geometric and cohomological framework that categorifies Frobenius descent, trace automorphy, and arithmetic torsors over the condensed dyadic shtuka site.

Main Achievements:

- Constructed the Frobenius sum operators on inverse shtuka towers;
- Defined the moduli of condensed G-torsors with trace-compatible descent data;
- Proved a trace descent duality theorem linking Galois parameters to automorphic realizations;
- Derived condensed trace formulas and cohomological analogs of Langlands reciprocity.

Outlook: The foundations developed here invite generalization in multiple directions:

- Integration into the global condensed Langlands program across spectral motives;
- Geometric class field theory for $\mathbb{Z}_2$-extensions and higher dyadic topologies;
- Universal L-groupoid structures over condensed arithmetic ∞-stacks;
- Applications to trace-compatible categories in condensed perfectoid and condensed motivic geometry.

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