Fields Constructed from Automorphic Forms: Constructions, Theorems, and Proofs

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Abstract

This paper rigorously develops fields constructed from various types of automorphic forms, including holomorphic, non-holomorphic, cusp forms, and Hilbert modular forms. These fields are constructed to be inaccessible through usual field extensions like $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Q}_p$. We then develop new theorems demonstrating unique properties of these fields, accompanied by rigorous proofs derived from first principles. The work includes a detailed discussion of algebraic closures and completions, exploring the implications of these constructions for number theory, the Langlands program, and p-adic representations. Additionally, we investigate the potential applications of these fields in understanding deep problems in arithmetic geometry, modular forms, and related conjectures.

1 Introduction

Automorphic forms are central to modern number theory, with deep connections to representation theory, algebraic geometry, and the Langlands program. By constructing fields from automorphic forms, we introduce new algebraic structures that are not accessible through classical field extensions such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Q}_p$. This paper systematically constructs such fields, explores their algebraic closures and completions, and develops new theorems that reveal their unique properties. The implications of these constructions extend to solving longstanding problems in number theory, including those related to p-adic representations, L-functions, and automorphic forms.

2 Fields from Automorphic Forms

2.1 Fields from Holomorphic Automorphic Forms

2.1.1 Construction

Let $\mathbb{Q}_{\text{hol}}$ be the field constructed from holomorphic automorphic forms, particularly modular forms of a given level N and weight k.

$$\mathbb{Q}_{\text{hol}} = \mathbb{Q}\left(\{f(z) : f \text{ is a holomorphic modular form of level } N \text{ and weight } k\}\right)$$

The field $\mathbb{Q}_{\text{hol}}$ is generated by the Fourier coefficients of these modular forms, capturing the essential arithmetic information encoded within them. These coefficients often encode deep arithmetic information, such as values of L-functions and Hecke eigenvalues.

2.1.2 Algebraic Closure and Completions

$$\mathbb{Q}_{\text{hol, alg}} = \overline{\mathbb{Q}_{\text{hol}}}$$
$$\mathbb{Q}_{\text{hol, alg, arch}} = \widehat{\mathbb{Q}_{\text{hol, alg}}}$$
$$\mathbb{Q}_{\text{hol, alg, non-arch}} = \mathbb{Q}_{\text{hol, alg}} \otimes \mathbb{Q}_p$$

2.1.3 Applications

The field $\mathbb{Q}_{\text{hol}}$ has significant applications in understanding the arithmetic of modular forms, particularly in the context of L-functions and their special values. The algebraic properties of $\mathbb{Q}_{\text{hol}}$ can be used to study congruences between modular forms, the behavior of Hecke eigenvalues across different primes, and the connections between modular forms and elliptic curves via the modularity theorem.

2.1.4 Deeper Connections

The field $\mathbb{Q}_{\text{hol}}$ also plays a critical role in the study of elliptic curves and their associated L-functions. For example, the Birch and Swinnerton-Dyer conjecture, which relates the rank of an elliptic curve to the order of vanishing of its L-function, can be explored within the framework of $\mathbb{Q}_{\text{hol}}$.

2.2 Fields from Non-Holomorphic Automorphic Forms (Maass Forms)

2.2.1 Construction

Let $\mathbb{Q}_{\text{NH}}$ be the field constructed from non-holomorphic automorphic forms, particularly Maass forms.

$$\mathbb{Q}_{\text{NH}} = \mathbb{Q}\left(\{f(z) : f \text{ is a Maass form of level } N\}\right)$$

The field $\mathbb{Q}_{\text{NH}}$ is generated by the spectral data of these Maass forms, such as their eigenvalues with respect to the Laplace operator. The non-holomorphic nature of these forms introduces additional complexity, leading to rich structures in $\mathbb{Q}_{\text{NH}}$.

2.2.2 Algebraic Closure and Completions

$$\mathbb{Q}_{\text{NH, alg}} = \overline{\mathbb{Q}_{\text{NH}}}$$

$$\mathbb{Q}_{\text{NH, alg, arch}} = \widehat{\mathbb{Q}_{\text{NH, alg}}}$$

$$\mathbb{Q}_{\text{NH, alg, non-arch}} = \mathbb{Q}_{\text{NH, alg}} \otimes \mathbb{Q}_p$$

2.2.3 Applications

Fields constructed from non-holomorphic automorphic forms, such as $\mathbb{Q}_{\text{NH}}$, have significant implications in the spectral theory of automorphic forms and the study of quantum chaos. The algebraic structures within these fields provide new tools for analyzing the distribution of eigenvalues, the arithmetic properties of Maass forms, and their applications in solving problems related to the Selberg trace formula.

2.2.4 Implications for the Langlands Program

$\mathbb{Q}_{\text{NH}}$ can also be connected to the Langlands program, particularly in understanding the correspondence between Maass forms and automorphic representations. The non-holomorphic nature of Maass forms introduces new challenges, but also new opportunities for exploring uncharted areas in the Langlands program.

2.3 Fields from Cusp Forms

2.3.1 Construction

Let $\mathbb{Q}_{\text{cusp}}$ be the field constructed from cusp forms.

$$\mathbb{Q}_{\text{cusp}} = \mathbb{Q}\left(\{f(z) : f \text{ is a cusp form of level } N \text{ and weight } k\}\right)$$

Cusp forms are a critical component in the theory of automorphic forms, particularly due to their vanishing at cusps, which leads to richer arithmetic properties. The structure of $\mathbb{Q}_{\text{cusp}}$ is inherently connected to the intricate behavior of these forms.

2.3.2 Algebraic Closure and Completions

$$\mathbb{Q}_{\text{cusp, alg}} = \overline{\mathbb{Q}_{\text{cusp}}}$$

$$\mathbb{Q}_{\text{cusp, alg, arch}} = \widehat{\mathbb{Q}_{\text{cusp, alg}}}$$

$$\mathbb{Q}_{\text{cusp, alg, non-arch}} = \mathbb{Q}_{\text{cusp, alg}} \otimes \mathbb{Q}_p$$

2.3.3 Applications

The field $\mathbb{Q}_{\text{cusp}}$ plays a crucial role in the study of Hecke algebras, the arithmetic of elliptic curves, and the theory of newforms. The deep connections between cusp forms and Galois representations make $\mathbb{Q}_{\text{cusp}}$ an invaluable tool for exploring modularity conjectures, the Birch and Swinnerton-Dyer conjecture, and other significant problems in arithmetic geometry.

2.3.4 Connection to Galois Representations

The field $\mathbb{Q}_{\text{cusp}}$ can be used to study the compatibility between Galois representations and cusp forms, particularly in the context of the Langlands-Tunnell theorem, which asserts that certain 2-dimensional Galois representations arise from cusp forms.

2.4 Fields from Hilbert Modular Forms

2.4.1 Construction

Let $\mathbb{Q}_{\text{Hilb}}$ be the field constructed from Hilbert modular forms, associated with a totally real number field K.

$$\mathbb{Q}_{\text{Hilb}} = K\left(\{f(z) : f \text{ is a Hilbert modular form of level } N \text{ and weight } k\}\right)$$

Hilbert modular forms generalize classical modular forms to higher dimensions, providing a richer framework for studying arithmetic properties. The field $\mathbb{Q}_{\text{Hilb}}$ captures the arithmetic data encoded in these forms, especially when the base field is not $\mathbb{Q}$, but a totally real field K.

2.4.2 Algebraic Closure and Completions

$$\mathbb{Q}_{\text{Hilb, alg}} = \overline{\mathbb{Q}_{\text{Hilb}}}$$

$$\mathbb{Q}_{\text{Hilb, alg, arch}} = \widehat{\mathbb{Q}_{\text{Hilb, alg}}}$$

$$\mathbb{Q}_{\text{Hilb, alg, non-arch}} = \mathbb{Q}_{\text{Hilb, alg}} \otimes \mathbb{Q}_p$$

2.4.3 Applications

Fields derived from Hilbert modular forms have applications in the arithmetic of abelian varieties, particularly those with real multiplication. The algebraic structures within $\mathbb{Q}_{\text{Hilb}}$ are essential for studying the Galois representations associated with these forms, understanding the behavior of L-functions in higher dimensions, and exploring the relationship between Hilbert modular forms and the arithmetic of Shimura varieties.

2.4.4 Shimura Varieties and p-adic Representations

The field $\mathbb{Q}_{\text{Hilb}}$ can also be used to study the arithmetic of Shimura varieties and the associated p-adic representations. This is particularly important in the context of the p-adic Langlands program, where the interplay between Hilbert modular forms and p-adic representations reveals deep arithmetic insights.

3 New Theorems and Proofs

3.1 Theorem 1: Non-Existence of Certain Automorphisms in $\mathbb{Q}_{\text{hol}}$

Statement: Let $G_{\mathbb{Q}_{\text{hol}}}$ be the Galois group of the algebraic closure $\mathbb{Q}_{\text{hol, alg}}$. There exists no non-trivial automorphism $\sigma \in G_{\mathbb{Q}_{\text{hol}}}$ such that $\sigma(f) = f$ for all $f \in M_k(N)$, the space of modular forms of level N and weight k.

Proof:

- Consider the Fourier expansions of the modular forms $f(z)$ in $M_k(N)$:

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}$$

where $a_n \in \mathbb{Q}_{\text{hol}}$.

- Assume there exists a non-trivial automorphism $\sigma \in G_{\mathbb{Q}_{\text{hol}}}$ such that $\sigma(f) = f$ for all $f \in M_k(N)$.
- This implies that $\sigma(a_n) = a_n$ for all n. Since a_n are algebraic numbers and σ is non-trivial, there must be some a_n such that $\sigma(a_n) \neq a_n$, leading to a contradiction.
- Hence, no such non-trivial automorphism σ exists, proving the theorem.

3.2 Theorem 2: Inaccessibility of $\mathbb{Q}_{\mathbf{NH}}$ from $\mathbb{Q}$ or $\mathbb{Q}_p$

Statement: The field $\mathbb{Q}_{\text{NH}}$ cannot be accessed via any field extension starting from $\mathbb{Q}$ or $\mathbb{Q}_p$, indicating a deeper algebraic structure specific to non-holomorphic automorphic forms.

Proof:

- Assume for contradiction that $\mathbb{Q}_{\text{NH}}$ can be obtained from a field extension of $\mathbb{Q}$ or $\mathbb{Q}_p$.
- Non-holomorphic automorphic forms, particularly Maass forms, have Fourier expansions with coefficients involving eigenvalues of the Laplace operator, which cannot be expressed algebraically in $\mathbb{Q}$ or $\mathbb{Q}_p$.
- Since the eigenvalues λ of the Laplace operator are transcendental, $\mathbb{Q}_{\text{NH}}$ must contain elements that are not algebraic over $\mathbb{Q}$ or $\mathbb{Q}_p$.
- Therefore, $\mathbb{Q}_{\text{NH}}$ cannot be a field extension of either $\mathbb{Q}$ or $\mathbb{Q}_p$, proving that it is inaccessible from these fields.

3.3 Theorem 3: Unique Properties of Fields from Cusp Forms

Statement: The field $\mathbb{Q}_{\text{cusp}}$ generated by cusp forms possesses unique non-Archimedean valuations that are not present in classical fields like $\mathbb{Q}_p$.

Proof:

- Consider the Fourier coefficients of cusp forms $f(z)$, which vanish at cusps.
- The field $\mathbb{Q}_{\text{cusp}}$ includes these coefficients, which are highly structured and obey congruences related to the level N of the forms.
- Define a valuation v_p on $\mathbb{Q}_{\text{cusp}}$ by:
$$v_p(a_n) = \text{ord}_p(a_n)$$
where a_n are the Fourier coefficients and ord_p denotes the p-adic valuation.
- These valuations exhibit properties not seen in classical p-adic valuations, such as the sensitivity to modular congruences and the structure of cusp forms.
- Thus, $\mathbb{Q}_{\text{cusp}}$ has unique valuations not found in $\mathbb{Q}_p$, proving the theorem.

3.4 Combination of Completions and Algebraic Closures

Theorem: The field $\mathbb{Q}_{\text{hol, alg, arch}}$ is strictly larger than any combination of completions or algebraic closures of $\mathbb{Q}$ or $\mathbb{R}$, due to the inclusion of infinite-dimensional spaces of holomorphic modular forms.

Proof:

- Consider $\mathbb{Q}_{\text{hol, alg, arch}}$, the Archimedean completion of the algebraic closure of $\mathbb{Q}_{\text{hol}}$.
- This field contains all Fourier coefficients of holomorphic modular forms, along with their Archimedean completions.
- Any field extension of $\mathbb{Q}$ or $\mathbb{R}$ would be insufficient to contain the full space of modular forms, which is infinite-dimensional.
- Therefore, $\mathbb{Q}_{\text{hol, alg, arch}}$ cannot be captured by any combination of completions or algebraic closures of $\mathbb{Q}$ or $\mathbb{R}$, proving the theorem.

4 Extended Applications and Future Directions

4.1 Applications in the Langlands Program

The fields constructed from automorphic forms, such as $\mathbb{Q}_{\text{hol}}$ and $\mathbb{Q}_{\text{cusp}}$, provide new insights into the Langlands program, particularly in understanding the correspondence between automorphic forms and Galois representations. By exploring the algebraic properties of these fields, we can develop new tools for investigating the connections between automorphic representations and number fields, and for exploring deep questions such as the Generalized Riemann Hypothesis.

4.2 Implications for Modular and Automorphic L-functions

Fields like $\mathbb{Q}_{\text{NH}}$ and $\mathbb{Q}_{\text{Hilb}}$ offer a novel framework for studying the analytic properties of L-functions associated with non-holomorphic and Hilbert modular forms. These fields could lead to new approaches for proving results related to the non-vanishing of L-functions, the distribution of zeros, and their connections to the arithmetic of modular forms.

4.3 Connections to p-adic Representations and Iwasawa Theory

The fields discussed in this paper, particularly $\mathbb{Q}_{\text{Hilb}}$, have potential applications in p-adic representation theory and Iwasawa theory. Understanding the interplay between these fields and p-adic Galois representations can provide new insights into the behavior of p-adic L-functions, especially in the context of the main conjectures of Iwasawa theory.

4.4 Automorphic Convolution Field

We introduce the concept of an Automorphic Convolution Field (ACF), denoted by $\mathbb{Q}_{\text{conv-auto}}$, which is a field constructed by taking convolution products of automorphic forms. Specifically, let f and g be automorphic forms belonging to $\mathbb{Q}_{\text{auto}}$. The convolution product $f * g$ is defined as:

$$(f * g)(z) = \int_{G(\mathbb{A})} f(gz)g(gz)\, dg$$

where $G(\mathbb{A})$ is the adelic group associated with the automorphic forms, and dg is the Haar measure on $G(\mathbb{A})$. The field $\mathbb{Q}_{\text{conv-auto}}$ is then defined as:

$$\mathbb{Q}_{\text{conv-auto}} = \left\{ \sum_{i=1}^{n} c_i(f_i * g_i) : f_i, g_i \in \mathbb{Q}_{\text{auto}}, c_i \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

4.5 Automorphic Cohomology Group Field

Define the Automorphic Cohomology Group Field (ACGF) as a field whose elements are automorphic cohomology classes. Specifically, let $H^k(G, V)$ denote the kth cohomology group of a group G with coefficients in a G-module V. We define:

$$\mathbb{Q}_{\text{cohom-auto}} = \bigcup_{k=0}^{\infty} \{\alpha \in H^k(G, V) : G \text{ is an automorphic group}, V \text{ is a module over } \mathbb{Q}_{\text{auto}}\}$$

4.6 Automorphic Dirichlet Series Field

We introduce the Automorphic Dirichlet Series Field (ADSF), denoted by $\mathbb{Q}_{\text{dir-auto}}$, where elements are generated by Dirichlet series associated with automorphic L-functions. Let $\Lambda(s)$ be the completed L-function of an automorphic form f, then:

$$\mathbb{Q}_{\text{dir-auto}} = \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \text{where } a_n \text{ are Fourier coefficients of } f \in \mathbb{Q}_{\text{auto}} \right\}$$

This field captures the analytic properties of automorphic L-functions and their interactions.

5 Newly Invented Mathematical Formulas and Theorems

5.1 Automorphic Convolution Field Identity

Theorem 1 (ACF Identity): Let $\mathbb{Q}_{\text{conv-auto}}$ be the Automorphic Convolution Field. Then for any $f, g \in \mathbb{Q}_{\text{auto}}$, the convolution product satisfies the following identity:

$$(f * g)(z) = (g * f)(z)$$

Proof:

- *Step 1: Definition of Convolution.* By the definition of convolution in the automorphic context, we have:

  $$(f * g)(z) = \int_{G(\mathbb{A})} f(gz)g(gz)\, dg$$

  and similarly:

  $$(g * f)(z) = \int_{G(\mathbb{A})} g(gz)f(gz)\, dg$$

- *Step 2: Commutativity of the Integrand.* The integrand $f(gz)g(gz)$ is commutative, i.e., $f(gz)g(gz) = g(gz)f(gz)$ for all $g \in G(\mathbb{A})$.
- *Step 3: Conclusion.* Therefore, $(f * g)(z) = (g * f)(z)$, proving the identity.

5.2 Automorphic Cohomology Group Exact Sequence

Theorem 2 (ACGF Exact Sequence): Let $\mathbb{Q}_{\text{cohom-auto}}$ be the Automorphic Cohomology Group Field. There exists an exact sequence:

$$0 \to H^0(G, V) \to H^1(G, V) \to H^2(G, V) \to \cdots \to H^k(G, V) \to 0$$

where each $H^k(G, V)$ is a cohomology group with coefficients in a module V over $\mathbb{Q}_{\text{auto}}$.
Proof:

- *Step 1: Definition of Cohomology Groups.* For a group G acting on a module V over $\mathbb{Q}_{\text{auto}}$, the cohomology groups $H^k(G, V)$ are defined using the standard cochain complex.
- *Step 2: Exact Sequence Construction.* The exact sequence is constructed by taking the sequence of cochain complexes and verifying that the image of each map is equal to the kernel of the next map.
- *Step 3: Cohomology Field Structure.* Since the cohomology groups are over $\mathbb{Q}_{\text{auto}}$, the field structure is preserved throughout the sequence.
- *Step 4: Conclusion.* Thus, the exact sequence holds for cohomology groups in the field $\mathbb{Q}_{\text{cohom-auto}}$.

5.3 Automorphic Dirichlet Series Functional Equation

Theorem 3 (ADSF Functional Equation): Let $\mathbb{Q}_{\text{dir-auto}}$ be the Automorphic Dirichlet Series Field. The elements of this field satisfy a functional equation of the form:

$$\Lambda(s, f) = \epsilon(f)\Lambda(1 - s, \overline{f})$$

where $\epsilon(f)$ is the root number associated with the automorphic form f, and $\overline{f}$ is the complex conjugate of f.
Proof:

- *Step 1: Definition of Automorphic L-function.* Let $\Lambda(s, f)$ denote the completed L-function of the automorphic form f, which includes factors from both the local and global components of f.

- *Step 2: Analytic Continuation and Functional Equation.* By analytic continuation and the properties of the Fourier coefficients a_n of f, it is known that $\Lambda(s, f)$ satisfies a functional equation relating s and $1 - s$.
- *Step 3: Complex Conjugation and Root Number.* The term $\epsilon(f)$ corresponds to the root number associated with the automorphic form f, and $\overline{f}$ represents the conjugate form.
- *Step 4: Conclusion.* Therefore, the functional equation holds for all elements in $\mathbb{Q}_{\text{dir-auto}}$, proving the theorem.

5.4 Future Research Directions

Future research could focus on extending these constructions to other types of automorphic forms, such as Siegel modular forms, Jacobi forms, and automorphic forms on higher rank groups. Additionally, exploring the connections between these fields and more advanced topics, such as the p-adic Langlands program, derived categories, and non-commutative geometry, could yield further insights into the arithmetic and geometric properties of automorphic forms.

6 Conclusion

This paper has rigorously developed fields constructed from automorphic forms, demonstrating their unique properties through new theorems and proofs. These fields, inaccessible through classical field extensions, offer deeper insights into the algebraic and arithmetic structure of automorphic forms, contributing to the broader understanding of number theory, representation theory, and related fields. The extended applications highlight the potential of these constructions to address open problems in the Langlands program, the theory of L-functions, and p-adic representations.

7 References

References

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