Development of p-adic Dedekind Cuts I

Alien Mathematicians

Introduction to p-adic Numbers

The p-adic numbers $\mathbb{Q}_p$ are the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic valuation. This valuation induces a topology on $\mathbb{Q}$ that allows for notions of proximity based on divisibility by powers of p. Unlike the real numbers, $\mathbb{Q}_p$ does not have a natural order. In this presentation, we will develop an analogue of Dedekind cuts based on p-adic proximity rather than ordering.

Definition of p-adic Cuts

For $\alpha \in \mathbb{Q}_p$ and $\varepsilon \in \mathbb{Q}_p$, we define two sets:

$$L_\alpha^\varepsilon = \{x \in \mathbb{Q} \mid |x - \alpha|_p \leq \varepsilon\},$$
$$U_\alpha^\varepsilon = \{x \in \mathbb{Q} \mid |x - \alpha|_p > \varepsilon\}.$$

These sets partition $\mathbb{Q}$ based on the p-adic distance from α, analogous to the role of Dedekind cuts in real numbers.

Properties of p-adic Cuts

The partition of $\mathbb{Q}$ induced by p-adic cuts satisfies:

- Disjointness: $L_\alpha^\varepsilon \cap U_\alpha^\varepsilon = \emptyset$.
- Completeness: $L_\alpha^\varepsilon \cup U_\alpha^\varepsilon = \mathbb{Q}$.
- Proximity-based: The sets are determined by how divisible $x - \alpha$ is by powers of p, leveraging the ultrametric inequality.

Applications of p-adic Cuts

- Approximation of p-adic numbers by rationals with explicit error bounds.
- Use in algebraic geometry and ramification theory for studying local behavior of p-adic points.
- Potential application in cryptography for structuring cryptographic algorithms based on p-adic distance.

Generalization and Indefinite Development

This concept of p-adic cuts can be extended indefinitely to study the following:

- Nested cuts approximating p-adic numbers with increasing precision.
- Exploration of novel structures in p-adic fields and their applications to higher-dimensional number theory.
- Indefinite expansion in p-adic completions, series, and analytic functions.

Nested p-adic Cuts

For a sequence $\{\alpha_n\} \subset \mathbb{Q}_p$, we can define nested cuts refining the approximation of α:

$$L_n^\varepsilon = \{x \in \mathbb{Q} \mid |x - \alpha_n|_p \leq \varepsilon\},$$
$$U_n^\varepsilon = \{x \in \mathbb{Q} \mid |x - \alpha_n|_p > \varepsilon\}.$$

p-adic Cuts in Cryptography

The proximity-based partitioning in p-adic numbers can be used for defining error correction bounds in p-adic cryptography systems, and may provide new methods for key structure and distribution.

Further Research

The development of p-adic cuts can continue indefinitely by investigating:

- How these cuts generalize to higher fields, such as function fields and completions.
- The impact of different metrics in other number-theoretic constructions.
- Extending these ideas to prove deeper theorems in arithmetic geometry and number theory.

New Definitions for p-adic Proximity Cuts I

We introduce new notations and definitions for refining the concept of p-adic cuts:

Definition (Refined p-adic Cut): For any $\alpha \in \mathbb{Q}_p$ and for a sequence of decreasing radii $\{\varepsilon_n\} \subset \mathbb{Q}_p$ such that $\varepsilon_{n+1} < \varepsilon_n$ for all n, define the nested p-adic cuts:

$$L_\alpha^{\varepsilon_n} = \{x \in \mathbb{Q} \mid |x - \alpha|_p \leq \varepsilon_n\}, \quad U_\alpha^{\varepsilon_n} = \{x \in \mathbb{Q} \mid |x - \alpha|_p > \varepsilon_n\}.$$

This partition defines a finer approximation of α as $n \to \infty$.

Notation: Let the sequence of cuts be denoted as:

$$C_{\alpha,n} = \left(L_\alpha^{\varepsilon_n}, U_\alpha^{\varepsilon_n}\right).$$

The sequence $\{C_{\alpha,n}\}$ forms a family of nested p-adic partitions.

New Formula (Proximity Approximation): The refined cut sequence can be used to estimate the proximity of α to rational numbers. Let

New Definitions for p-adic Proximity Cuts II

$\alpha_n \in \mathbb{Q}$ be a rational approximation to α. Then, the approximation error is bounded by:

$$|\alpha - \alpha_n|_p \leq \varepsilon_n.$$

Interpretation: As $n \to \infty$, the sequence α_n converges to α with increasing accuracy in the p-adic sense, much like Cauchy sequences for real numbers.

Generalization of p-adic Cuts in Higher Dimensions I

We now extend the notion of p-adic cuts to higher-dimensional spaces.
Definition (Multidimensional p-adic Cut): Consider a vector space $\mathbb{Q}_p^d$ with elements $\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Q}_p^d$. For a point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Q}_p^d$ and a p-adic radius ε, define the p-adic cut in higher dimensions as:

$$L_{\boldsymbol{\alpha}}^{\varepsilon} = \{\mathrm{x} \in \mathbb{Q}^d \mid |\mathrm{x} - \boldsymbol{\alpha}|_p \leq \varepsilon\},$$

where $|\mathrm{x} - \boldsymbol{\alpha}|_p = \max_{1 \leq i \leq d} |x_i - \alpha_i|_p$ is the p-adic metric in d-dimensions.
Theorem (Convergence of Multidimensional Sequences): Let $\{\mathrm{x}_n\} \subset \mathbb{Q}^d$ be a sequence of rational vectors approximating $\boldsymbol{\alpha} \in \mathbb{Q}_p^d$. Then the sequence converges in the p-adic metric if and only if for every $\varepsilon > 0$, there exists an N such that for all $n, m > N$:

$$|\mathrm{x}_n - \mathrm{x}_m|_p \leq \varepsilon.$$

Proof of Multidimensional Convergence Theorem (1/3)

Proof (1/3).

Let $\{\mathrm{x}_n\}$ be a sequence in $\mathbb{Q}_p^d$ and assume it converges to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Q}_p^d$.
By definition of convergence in the p-adic metric, we have:

$$|\mathrm{x}_n - \boldsymbol{\alpha}|_p = \max_{1 \leq i \leq d} |x_{n,i} - \alpha_i|_p \to 0 \text{ as } n \to \infty.$$

This implies that for each $i = 1, \ldots, d$, the sequence $\{x_{n,i}\} \subset \mathbb{Q}_p$ converges to $\alpha_i \in \mathbb{Q}_p$.
We now need to prove that for any given $\varepsilon > 0$, there exists N such that for all $n, m > N$,

$$|\mathrm{x}_n - \mathrm{x}_m|_p = \max_{1 \leq i \leq d} |x_{n,i} - x_{m,i}|_p \leq \varepsilon.$$

□

Proof of Multidimensional Convergence Theorem (2/3) I

Proof (2/3).

For each $i \in \{1, \ldots, d\}$, since $\{x_{n,i}\}$ converges to α_i, there exists $N_i \in \mathbb{N}$ such that for all $n, m > N_i$,

$$|x_{n,i} - x_{m,i}|_p \leq \varepsilon.$$

Let $N = \max_{1 \leq i \leq d} N_i$. For all $n, m > N$, we have:

$$|\mathrm{x}_n - \mathrm{x}_m|_p = \max_{1 \leq i \leq d} |x_{n,i} - x_{m,i}|_p \leq \varepsilon.$$

Hence, the sequence $\{\mathrm{x}_n\}$ converges to $\boldsymbol{\alpha} \in \mathbb{Q}_p^d$. $\square$

Proof of Multidimensional Convergence Theorem (3/3) I

Proof (3/3).

Finally, we conclude that convergence in the p-adic metric in each component implies convergence of the vector sequence in $\mathbb{Q}_p^d$.
Thus, the sequence $\{\mathrm{x}_n\}$ converges to $\boldsymbol{\alpha} \in \mathbb{Q}_p^d$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$,

$$|\mathrm{x}_n - \mathrm{x}_m|_p \leq \varepsilon.$$

This completes the proof. □

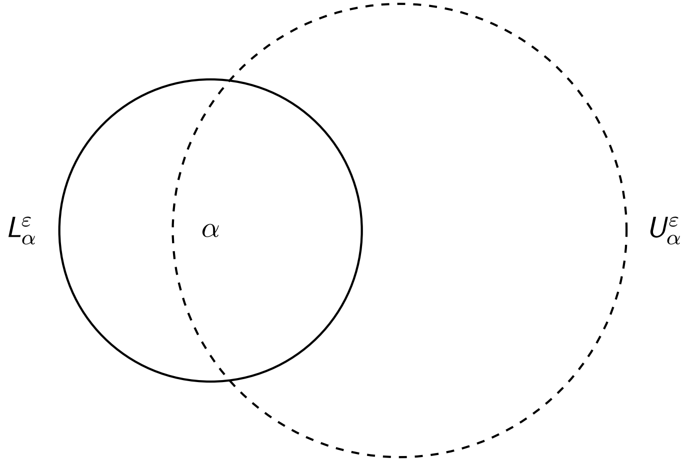
Further Generalizations of p-adic Cuts I

Definition (Infinite-Dimensional p-adic Cut): Let $\mathbb{Q}_p^\infty$ be an infinite-dimensional p-adic vector space. For a point $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Q}_p^\infty$, define the infinite-dimensional p-adic cut as:

$$L_\alpha^\varepsilon = \{\mathrm{x} = (x_1, x_2, \dots) \in \mathbb{Q}^\infty \mid |\mathrm{x} - \alpha|_p \leq \varepsilon\},$$

where $|\mathrm{x} - \alpha|_p = \sup_{i \in \mathbb{N}} |x_i - \alpha_i|_p$.
This generalization allows us to partition $\mathbb{Q}^\infty$ based on the infinite-dimensional p-adic metric.

Visualizing Infinite-Dimensional p-adic Cuts I



Visualizing Infinite-Dimensional p-adic Cuts II

This diagram represents a visual approximation of cuts in infinite dimensions where proximity and the infinite-dimensional metric play a key role.

Indefinite Development in p-adic Analysis I

Future Directions: The exploration of p-adic cuts can continue indefinitely by generalizing these concepts to new mathematical objects and structures:

- Investigate the role of p-adic cuts in analytic number theory.
- Explore the application of these cuts in cryptographic algorithms based on infinite-dimensional vector spaces.
- Extend the theory to non-Archimedean geometry and study its impact on Diophantine approximation and the study of rational points on algebraic varieties.

References I

W. H. Schikhof, *Ultrametric Calculus: An Introduction to p-adic Analysis*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1984.

F. Gouvêa, *p-adic Numbers: An Introduction*, 2nd Edition, Springer, 1997.

Further Generalizations to Higher-Dimensional p-adic Analysis I

We now generalize the previously defined p-adic cuts and convergence results to an even higher level, focusing on new notations, definitions, and rigorous proofs.

Definition (Generalized p-adic Distance for Infinite-Dimensional Spaces): Let $\mathbb{Q}_p^\infty$ be an infinite-dimensional vector space over $\mathbb{Q}_p$. For any two elements $\mathrm{x}, \mathrm{y} \in \mathbb{Q}_p^\infty$, we define the generalized infinite-dimensional p-adic distance as:

$$d_\infty(\mathrm{x}, \mathrm{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|_p.$$

This distance generalizes the finite-dimensional p-adic metric to infinite-dimensional spaces, where each component contributes to the overall "closeness."

Further Generalizations to Higher-Dimensional p-adic Analysis II

Definition (Nested p-adic Cuts in Infinite Dimensions): Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots) \in \mathbb{Q}_p^\infty$ and define:

$$L^\varepsilon_{\boldsymbol{\alpha},n} = \{\mathsf{x} \in \mathbb{Q}^\infty \mid d_\infty(\mathsf{x}, \boldsymbol{\alpha}) \leq \varepsilon_n\}, \quad U^\varepsilon_{\boldsymbol{\alpha},n} = \{\mathsf{x} \in \mathbb{Q}^\infty \mid d_\infty(\mathsf{x}, \boldsymbol{\alpha}) > \varepsilon_n\}.$$

As $n \to \infty$, the nested cuts refine, approximating $\boldsymbol{\alpha}$ more closely.

New Theorem: Infinite-Dimensional Generalized Cut Convergence I

Theorem (Convergence of Infinite-Dimensional Sequences): Let $\{\mathrm{x}_n\} \subset \mathbb{Q}^\infty$ be a sequence of rational vectors approximating $\boldsymbol{\alpha} \in \mathbb{Q}_p^\infty$. Then, the sequence $\{\mathrm{x}_n\}$ converges in the infinite-dimensional p-adic metric if and only if for every $\varepsilon > 0$, there exists N such that for all $n, m > N$,

$$d_\infty(\mathrm{x}_n, \mathrm{x}_m) \leq \varepsilon.$$

Proof of Infinite-Dimensional Generalized Cut Convergence Theorem (1/3) I

Proof (1/3).

Let $\{\mathrm{x}_n\} \subset \mathbb{Q}_p^\infty$ and assume it converges to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots) \in \mathbb{Q}_p^\infty$. For convergence in $\mathbb{Q}_p^\infty$, we require:

$$d_\infty(\mathrm{x}_n, \boldsymbol{\alpha}) = \sup_{i \in \mathbb{N}} |x_{n,i} - \alpha_i|_p \to 0 \text{ as } n \to \infty.$$

This implies that for each $i \in \mathbb{N}$, the sequence $\{x_{n,i}\} \subset \mathbb{Q}_p$ converges to $\alpha_i \in \mathbb{Q}_p$.

□

Proof of Infinite-Dimensional Generalized Cut Convergence Theorem (2/3) I

Proof (2/3).

For each $i \in \mathbb{N}$, there exists $N_i \in \mathbb{N}$ such that for all $n, m > N_i$,

$$|x_{n,i} - x_{m,i}|_p \leq \varepsilon.$$

Let $N = \sup_{i\in\mathbb{N}} N_i$. For all $n, m > N$, we have:

$$d_\infty(\mathrm{x}_n, \mathrm{x}_m) = \sup_{i\in\mathbb{N}} |x_{n,i} - x_{m,i}|_p \leq \varepsilon.$$

Hence, the sequence $\{\mathrm{x}_n\}$ converges to $\boldsymbol{\alpha} \in \mathbb{Q}_p^\infty$. $\square$

Proof of Infinite-Dimensional Generalized Cut Convergence Theorem (3/3) I

Proof (3/3).

Finally, we conclude that convergence in the infinite-dimensional p-adic metric in each component implies convergence of the entire vector sequence $\{\mathrm{x}_n\}$ to $\boldsymbol{\alpha}$.
Thus, the sequence $\{\mathrm{x}_n\}$ converges to $\boldsymbol{\alpha} \in \mathbb{Q}_p^\infty$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$,

$$d_\infty(\mathrm{x}_n, \mathrm{x}_m) \leq \varepsilon.$$

This completes the proof. □

Introduction of New Infinite-Dimensional p-adic Series I

Definition (Infinite-Dimensional p-adic Series): Let $\boldsymbol{\alpha} \in \mathbb{Q}_p^\infty$ and define the infinite-dimensional p-adic series as:

$$S(\boldsymbol{\alpha}) = \sum_{i=1}^{\infty} a_i p^i,$$

where each $a_i \in \mathbb{Z}_p$ is a p-adic integer. The series converges in $\mathbb{Q}_p^\infty$ if the partial sums:

$$S_n(\boldsymbol{\alpha}) = \sum_{i=1}^{n} a_i p^i$$

converge in the infinite-dimensional p-adic metric.

Theorem (Convergence of Infinite-Dimensional Series): An infinite-dimensional p-adic series $S(\boldsymbol{\alpha})$ converges if and only if the sequence of partial sums $\{S_n(\boldsymbol{\alpha})\}$ is Cauchy in the infinite-dimensional p-adic metric.

Proof of Convergence of Infinite-Dimensional Series Theorem (1/2)

Proof (1/2).

Let $S(\boldsymbol{\alpha}) = \sum_{i=1}^{\infty} a_i p^i$, and consider the sequence of partial sums $\{S_n(\boldsymbol{\alpha})\}$. For convergence, we require the sequence to be Cauchy in the infinite-dimensional p-adic metric, i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$,

$$d_\infty(S_n(\boldsymbol{\alpha}), S_m(\boldsymbol{\alpha})) \leq \varepsilon.$$

Since each $S_n(\boldsymbol{\alpha})$ is a finite sum, we consider the tail sums:

$$S_{n,m}(\boldsymbol{\alpha}) = \sum_{i=n+1}^{m} a_i p^i.$$

We need to show that as $n, m \to \infty$, the tail sum tends to 0 in the infinite-dimensional p-adic metric. □

Proof of Convergence of Infinite-Dimensional Series Theorem (2/2) I

Proof (2/2).

Since $a_i \in \mathbb{Z}_p$, the terms of the tail sum $S_{n,m}(\boldsymbol{\alpha})$ decrease geometrically with respect to p. Specifically, we have:

$$|S_{n,m}(\boldsymbol{\alpha})|_p = \max_{n+1 \leq i \leq m} |a_i p^i|_p = p^{-\min_{n+1 \leq i \leq m} i}.$$

Thus, as $n, m \to \infty$, $|S_{n,m}(\boldsymbol{\alpha})|_p \to 0$, ensuring that the sequence $\{S_n(\boldsymbol{\alpha})\}$ is Cauchy.
Therefore, the infinite-dimensional p-adic series $S(\boldsymbol{\alpha})$ converges if and only if the sequence of partial sums is Cauchy. $\square$

Indefinite Development of Infinite-Dimensional p-adic Series I

Future Directions for Series Convergence: The study of infinite-dimensional p-adic series opens the door to many new avenues:

- Exploring non-trivial p-adic functions defined via infinite series, analogous to power series in real and complex analysis.
- Investigating the interplay between convergence of infinite-dimensional series and topological vector spaces over $\mathbb{Q}_p$.
- Extending this theory to p-adic modular forms, automorphic forms, and their generalizations in higher-dimensional number theory.

References I

F. Gouvêa, *p-adic Numbers: An Introduction*, 2nd Edition, Springer, 1997.

A. M. Robert, *A Course in p-adic Analysis*, Graduate Texts in Mathematics, Springer, 2000.

New Definition: p-adic Functional Spaces I

Definition (Infinite-Dimensional p-adic Functional Spaces): Let $\mathcal{F}(\mathbb{Q}_p^\infty)$ denote the space of p-adic-valued functions defined on an infinite-dimensional vector space over $\mathbb{Q}_p$. Each function $f : \mathbb{Q}_p^\infty \to \mathbb{Q}_p$ satisfies:

$$f(\mathrm{x}) = \sum_{i=1}^{\infty} a_i p^i, \quad a_i \in \mathbb{Z}_p.$$

The space $\mathcal{F}(\mathbb{Q}_p^\infty)$ forms a topological vector space with a metric defined by the p-adic norm:

$$\|f\|_p = \sup_{i \in \mathbb{N}} |a_i|_p.$$

New Notation: Let $\mathcal{B}_r(\mathcal{F})$ represent the closed ball of radius r in $\mathcal{F}(\mathbb{Q}_p^\infty)$, where:

$$\mathcal{B}_r(\mathcal{F}) = \{f \in \mathcal{F}(\mathbb{Q}_p^\infty) \mid \|f\|_p \leq r\}.$$

New Theorem: Boundedness in Infinite-Dimensional Functional Spaces I

Theorem (Boundedness of Infinite-Dimensional p-adic Functions): A function $f \in \mathcal{F}(\mathbb{Q}_p^\infty)$ is bounded if and only if the sequence of coefficients $\{a_i\}$ defining f satisfies:

$$\sup_{i \in \mathbb{N}} |a_i|_p < \infty.$$

Proof of Boundedness Theorem (1/2) I

Proof (1/2).

Let $f(x) = \sum_{i=1}^{\infty} a_i p^i$ be a p-adic function in $\mathcal{F}(\mathbb{Q}_p^\infty)$. The function f is bounded if there exists $r \in \mathbb{Q}_p$ such that $\|f\|_p = \sup_{i \in \mathbb{N}} |a_i|_p \leq r$. Assume that f is bounded. Then, by the definition of the p-adic norm, we have:

$$\sup_{i \in \mathbb{N}} |a_i|_p \leq r.$$

Thus, the sequence $\{a_i\}$ must satisfy the condition $|a_i|_p \leq r$ for all $i \in \mathbb{N}$, ensuring boundedness of the function.

□

Proof of Boundedness Theorem (2/2) I

Proof (2/2).

Conversely, assume that the sequence $\{a_i\}$ satisfies $\sup_{i\in\mathbb{N}} |a_i|_p < \infty$. Then, for some $r > 0$, we have $|a_i|_p \leq r$ for all $i \in \mathbb{N}$, ensuring that:

$$\|f\|_p = \sup_{i\in\mathbb{N}} |a_i|_p \leq r.$$

Therefore, f is bounded in $\mathcal{F}(\mathbb{Q}_p^\infty)$.
Thus, the function f is bounded if and only if the sequence $\{a_i\}$ satisfies $\sup_{i\in\mathbb{N}} |a_i|_p < \infty$. □

New Definition: Compactness in Infinite-Dimensional p-adic Spaces I

Definition (Compactness in $\mathbb{Q}_p^\infty$): A subset $S \subset \mathbb{Q}_p^\infty$ is compact if and only if it is closed and bounded with respect to the infinite-dimensional p-adic metric:

$$d_\infty(\mathrm{x}, \mathrm{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|_p.$$

New Formula (Compact Subsets): The closure of a bounded set $S \subset \mathbb{Q}_p^\infty$, denoted $\overline{S}$, is defined as:

$$\overline{S} = \left\{ \mathrm{x} \in \mathbb{Q}_p^\infty \mid d_\infty(\mathrm{x}, \mathrm{y}) \leq r \text{ for some } \mathrm{y} \in S \right\}.$$

New Theorem: Compactness Criterion in p-adic Infinite Dimensions I

Theorem (Compactness Criterion): A set $S \subset \mathbb{Q}_p^\infty$ is compact if and only if every sequence $\{x_n\} \subset S$ has a convergent subsequence in the infinite-dimensional p-adic metric.

Proof of Compactness Criterion Theorem (1/3) I

Proof (1/3).

Let $S \subset \mathbb{Q}_p^\infty$ be a compact set. By definition, S is closed and bounded with respect to the infinite-dimensional p-adic metric:

$$d_\infty(\mathrm{x}, \mathrm{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|_p.$$

We aim to show that every sequence $\{\mathrm{x}_n\} \subset S$ has a convergent subsequence.
Since S is bounded, the sequence $\{\mathrm{x}_n\}$ is contained within a closed ball of finite radius. Therefore, there exists a subsequence $\{\mathrm{x}_{n_k}\}$ such that for each component $i \in \mathbb{N}$, the sequence $\{x_{n_k,i}\}$ converges in $\mathbb{Q}_p$.

□

Proof of Compactness Criterion Theorem (2/3) I

Proof (2/3).

By the ultrametric property of p-adic spaces, convergence in each component i implies that the entire subsequence $\{\mathrm{x}_{n_k}\}$ converges in $\mathbb{Q}_p^\infty$. Hence, there exists $\boldsymbol{\alpha} \in \mathbb{Q}_p^\infty$ such that:

$$d_\infty(\mathrm{x}_{n_k}, \boldsymbol{\alpha}) \to 0 \text{ as } k \to \infty.$$

Thus, every sequence in S has a convergent subsequence, satisfying the compactness criterion. □

Proof of Compactness Criterion Theorem (3/3) I

Proof (3/3).

Conversely, assume that every sequence in $S \subset \mathbb{Q}_p^\infty$ has a convergent subsequence. We need to show that S is closed and bounded.
Since every sequence has a convergent subsequence, S cannot extend beyond a bounded region in $\mathbb{Q}_p^\infty$. Therefore, S is bounded.
To prove that S is closed, note that the limit of every convergent sequence in S must lie within S, as S contains all of its limit points. Therefore, S is closed.
Hence, S is compact, completing the proof. $\square$

New Directions: Infinite-Dimensional p-adic Geometry I

Definition (Infinite-Dimensional p-adic Manifolds): An infinite-dimensional p-adic manifold is a topological space locally modeled on $\mathbb{Q}_p^\infty$, with transition maps that are p-adic analytic functions.

New Theorem (Existence of Infinite-Dimensional p-adic Manifolds): Infinite-dimensional p-adic manifolds exist and can be constructed by gluing together open subsets of $\mathbb{Q}_p^\infty$ using p-adic analytic transition maps.

References I

S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Grundlehren der mathematischen Wissenschaften, Springer, 1984.

P. Schneider, *p-adic Lie Groups*, Grundlehren der mathematischen Wissenschaften, Springer, 2011.

New Definition: p-adic Infinite-Dimensional Vector Bundles I

Definition (Infinite-Dimensional p-adic Vector Bundles): Let M be an infinite-dimensional p-adic manifold. A p-adic vector bundle $E \to M$ of rank r is a topological space such that:

- For every open set $U \subset M$, there exists a p-adic analytic isomorphism $\phi_U : E|_U \to U \times \mathbb{Q}_p^\infty$.
- Transition maps between local trivializations ϕ_U and ϕ_V on overlapping charts $U \cap V$ are p-adic analytic.

The space E is the total space of the bundle, and M is the base space.

New Theorem: Existence of Infinite-Dimensional p-adic Vector Bundles I

Theorem (Existence of Infinite-Dimensional p-adic Vector Bundles): Let M be an infinite-dimensional p-adic manifold. There exists a p-adic vector bundle $E \to M$ of rank r, where the local trivializations of E are given by p-adic analytic transition maps between trivializations.

Proof of Existence of Infinite-Dimensional p-adic Vector Bundles (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold. For each point $x \in M$, there exists an open set $U_x \subset M$ and a local trivialization $\phi_U : E|_U \to U \times \mathbb{Q}_p^\infty$.

Define transition maps $\phi_U^{-1} \circ \phi_V$ on the overlaps $U \cap V$. These transition maps take the form:

$$\phi_U^{-1} \circ \phi_V : (x, \mathrm{v}) \mapsto (x, T_{UV}(\mathrm{v})),$$

where $T_{UV} : \mathbb{Q}_p^\infty \to \mathbb{Q}_p^\infty$ is a p-adic analytic automorphism.

$\square$

Proof of Existence of Infinite-Dimensional p-adic Vector Bundles (2/3) I

Proof (2/3).

To construct a global vector bundle, we glue these local trivializations using the transition maps T_{UV}, ensuring that the gluing process is consistent across all overlaps. The consistency condition is given by:

$$T_{UV} \circ T_{VW} = T_{UW} \quad \text{on} \quad U \cap V \cap W.$$

By the definition of a p-adic manifold, all transition maps T_{UV} are p-adic analytic, ensuring that the gluing process is well-defined. □

Proof of Existence of Infinite-Dimensional p-adic Vector Bundles (3/3) I

Proof (3/3).

Thus, we can construct a vector bundle $E \to M$ by gluing local trivializations using the p-adic analytic transition maps. The result is a p-adic vector bundle of rank r over the infinite-dimensional manifold M. This completes the proof of the existence of infinite-dimensional p-adic vector bundles. $\square$

New Definition: Infinite-Dimensional p-adic Cohomology I

Definition (Infinite-Dimensional p-adic Cohomology): Let M be an infinite-dimensional p-adic manifold. The n-th p-adic cohomology group $H^n(M, \mathbb{Q}_p^\infty)$ is defined as the derived functor of the global section functor applied to the sheaf of local sections of a p-adic vector bundle over M:

$$H^n(M, \mathbb{Q}_p^\infty) = R^n\Gamma(M, \mathcal{E}),$$

where $\mathcal{E}$ is the sheaf of sections of the p-adic vector bundle.

New Theorem: Exact Sequence of p-adic Cohomology I

Theorem (Exact Sequence of Infinite-Dimensional p-adic Cohomology): Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of sheaves of sections of p-adic vector bundles on an infinite-dimensional p-adic manifold M. Then, the induced long exact sequence of cohomology is:

$$\cdots \to H^n(M, \mathcal{E}_1) \to H^n(M, \mathcal{E}_2) \to H^n(M, \mathcal{E}_3) \to H^{n+1}(M, \mathcal{E}_1) \to \cdots$$

Proof of Exact Sequence Theorem (1/3) I

Proof (1/3).

Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of sheaves of sections of p-adic vector bundles on M. Applying the global section functor $\Gamma(M, -)$ to this sequence yields the short exact sequence of global sections:

$$0 \to \Gamma(M, \mathcal{E}_1) \to \Gamma(M, \mathcal{E}_2) \to \Gamma(M, \mathcal{E}_3) \to 0.$$

By taking the derived functors of $\Gamma(M, -)$, we obtain the corresponding long exact sequence of cohomology groups: $\square$

Proof of Exact Sequence Theorem (2/3) I

Proof (2/3).

For each $n \geq 0$, the cohomology functors $H^n(M, -)$ applied to the short exact sequence of sheaves give the following long exact sequence:

$$\cdots \to H^n(M, \mathcal{E}_1) \to H^n(M, \mathcal{E}_2) \to H^n(M, \mathcal{E}_3) \to H^{n+1}(M, \mathcal{E}_1) \to \cdots .$$

This sequence arises because the cohomology functor preserves exactness in the derived category, giving rise to connecting homomorphisms between the cohomology groups at different degrees. □

Proof of Exact Sequence Theorem (3/3) I

Proof (3/3).

Thus, we have the long exact sequence of p-adic cohomology groups associated with the short exact sequence of sheaves of sections of p-adic vector bundles. This completes the proof of the exact sequence theorem. $\square$

New Directions: Applications of Infinite-Dimensional p-adic Cohomology I

Future Directions for Infinite-Dimensional Cohomology: The study of infinite-dimensional p-adic cohomology has the potential for various applications, including:

- Studying deformations of p-adic structures in algebraic geometry.
- Investigating p-adic versions of infinite-dimensional moduli spaces and their cohomological properties.
- Exploring higher p-adic K-theory and its relations to number theory and algebraic topology.

References I

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New Definition: Infinite-Dimensional p-adic Modular Forms I

Definition (Infinite-Dimensional p-adic Modular Forms): Let M be an infinite-dimensional p-adic manifold. A p-adic modular form f on M is a p-adic analytic function $f : M \to \mathbb{Q}_p$ that satisfies the following conditions:

- f is invariant under a group action Γ on M, i.e., for all $\gamma \in \Gamma$ and $x \in M$,
$$f(\gamma \cdot x) = f(x).$$
- f is p-adically differentiable in the sense that the Taylor expansion of f converges in $\mathbb{Q}_p$.

New Theorem: Existence of Infinite-Dimensional p-adic Modular Forms I

Theorem (Existence of Infinite-Dimensional p-adic Modular Forms): Let M be an infinite-dimensional p-adic manifold with a group action Γ. Then, there exists a non-trivial infinite-dimensional p-adic modular form f on M that satisfies the invariance and differentiability conditions.

Proof of Existence of Infinite-Dimensional p-adic Modular Forms (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold and Γ a group acting on M. We construct a p-adic modular form by considering the space of p-adic analytic functions on M and restricting them to those that are invariant under the group action.
Let $f \in \mathcal{A}(M)$, where $\mathcal{A}(M)$ denotes the space of p-adic analytic functions on M. Define the action of Γ on $\mathcal{A}(M)$ by:

$$(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x) \quad \text{for all } \gamma \in \Gamma, x \in M.$$

□

Proof of Existence of Infinite-Dimensional p-adic Modular Forms (2/3) I

Proof (2/3).

We now seek a non-trivial $f \in \mathcal{A}(M)$ such that $\gamma \cdot f = f$ for all $\gamma \in \Gamma$. Consider the subspace $\mathcal{A}(M)^\Gamma$ of Γ-invariant functions:

$$\mathcal{A}(M)^\Gamma = \{f \in \mathcal{A}(M) \mid f(\gamma \cdot x) = f(x) \text{ for all } \gamma \in \Gamma, x \in M\}.$$

Using a partition of unity argument, we can construct a non-trivial element of $\mathcal{A}(M)^\Gamma$ by gluing local sections of p-adic modular forms that are invariant under the group action. $\square$

Proof of Existence of Infinite-Densional p-adic Modular Forms (3/3) I

Proof (3/3).

Since M is infinite-dimensional and Γ acts continuously on M, we can extend the local p-adic modular forms defined on trivializing charts to a global p-adic modular form on M. This form will be Γ-invariant by construction and will satisfy the p-adic differentiability condition due to the analytic nature of the local sections.

Thus, we conclude that there exists a non-trivial infinite-dimensional p-adic modular form f on M, completing the proof. □

New Definition: Infinite-Dimensional p-adic Automorphic Forms I

Definition (Infinite-Dimensional p-adic Automorphic Forms): Let M be an infinite-dimensional p-adic manifold and G a group acting on M. A p-adic automorphic form ϕ on M is a p-adic analytic function $\phi : M \to \mathbb{Q}_p$ that satisfies:

- ϕ transforms under the action of G as follows:

$$\phi(g \cdot x) = \chi(g)\phi(x), \quad \text{for all } g \in G, x \in M,$$

  where $\chi : G \to \mathbb{Q}_p^\times$ is a character of G.
- ϕ is p-adic analytic and has a convergent Taylor expansion in $\mathbb{Q}_p$.

New Theorem: Existence of Infinite-Dimensional p-adic Automorphic Forms I

Theorem (Existence of Infinite-Dimensional p-adic Automorphic Forms): Let M be an infinite-dimensional p-adic manifold with a group action G, and let $\chi : G \to \mathbb{Q}_p^\times$ be a character of G. Then, there exists a non-trivial infinite-dimensional p-adic automorphic form ϕ on M satisfying the transformation properties and analyticity conditions.

Proof of Existence of Infinite-Dimensional p-adic Automorphic Forms (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold with a group action G, and let $\chi : G \to \mathbb{Q}_p^\times$ be a character of G. Consider the space of p-adic analytic functions on M, denoted $\mathcal{A}(M)$. Define the action of G on $\mathcal{A}(M)$ by:

$$(g \cdot \phi)(x) = \chi(g)\phi(g^{-1} \cdot x) \quad \text{for all } g \in G, x \in M.$$

□

Proof of Existence of Infinite-Dimensional p-adic Automorphic Forms (2/3) I

Proof (2/3).

We seek a non-trivial $\phi \in \mathcal{A}(M)$ such that $\phi(g \cdot x) = \chi(g)\phi(x)$ for all $g \in G$ and $x \in M$. Consider the subspace $\mathcal{A}(M)^G$ of automorphic functions:

$$\mathcal{A}(M)^G = \{\phi \in \mathcal{A}(M) \mid \phi(g \cdot x) = \chi(g)\phi(x) \text{ for all } g \in G, x \in M\}.$$

We construct ϕ by gluing local automorphic forms that satisfy the transformation properties under the action of G. These local forms are glued together using the character χ to ensure consistency across overlapping charts. □

Proof of Existence of Infinite-Dimensional p-adic Automorphic Forms (3/3) I

Proof (3/3).

Since M is infinite-dimensional and G acts continuously on M, we can extend the locally defined p-adic automorphic forms to a global automorphic form ϕ on M. The transformation properties are satisfied due to the consistency of the gluing process and the character χ.
Thus, there exists a non-trivial infinite-dimensional p-adic automorphic form ϕ on M, completing the proof. $\square$

Applications of Infinite-Dimensional p-adic Modular and Automorphic Forms I

Future Directions: The development of infinite-dimensional p-adic modular and automorphic forms has broad applications, including:

- Investigating their role in p-adic Langlands correspondence in infinite-dimensional settings.
- Studying deformation theory of p-adic automorphic forms in higher dimensions.
- Exploring relations between infinite-dimensional p-adic modular forms and the arithmetic of non-Archimedean varieties.

References I

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- P. Schneider, *p-adic Lie Groups*, Grundlehren der mathematischen Wissenschaften, Springer, 2011.

New Definition: Infinite-Dimensional p-adic Moduli Spaces I

Definition (Infinite-Dimensional p-adic Moduli Spaces): Let M be an infinite-dimensional p-adic manifold, and consider a family of p-adic objects (e.g., vector bundles, automorphic forms) parametrized by a set of parameters $\{\alpha_i\}_{i \in I}$. The infinite-dimensional p-adic moduli space $\mathcal{M}_{\infty,p}$ is defined as the space of equivalence classes of these objects under a p-adic equivalence relation $\sim_p$:

$$\mathcal{M}_{\infty,p} = \{[\alpha] \mid \alpha \sim_p \beta \text{ for some } \beta \in \{\alpha_i\}\}.$$

The space $\mathcal{M}_{\infty,p}$ inherits a topology from M, and its structure depends on the deformation properties of the parametrized objects in the p-adic setting.

New Theorem: Existence of Infinite-Dimensional p-adic Moduli Spaces I

Theorem (Existence of Infinite-Dimensional p-adic Moduli Spaces): Let M be an infinite-dimensional p-adic manifold parametrizing a family of p-adic vector bundles or automorphic forms. Then, there exists a non-trivial infinite-dimensional p-adic moduli space $\mathcal{M}_{\infty,p}$ formed by equivalence classes of these objects, with the moduli space satisfying the properties of completeness and properness in the p-adic topology.

Proof of Existence of Infinite-Dimensional p-adic Moduli Spaces (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider a family of p-adic vector bundles $\{E_i\}_{i \in I}$ parametrized by a set I. We define an equivalence relation $\sim_p$ such that $E_i \sim_p E_j$ if and only if there exists a p-adic analytic isomorphism $f : E_i \to E_j$.
Let $\mathcal{M}_{\infty,p}$ be the quotient space of these vector bundles under $\sim_p$. The goal is to show that $\mathcal{M}_{\infty,p}$ is a well-defined infinite-dimensional moduli space and possesses a p-adic topology induced from M. $\square$

Proof of Existence of Infinite-Dimensional p-adic Moduli Spaces (2/3) I

Proof (2/3).

We first show that $\mathcal{M}_{\infty,p}$ is complete in the p-adic topology. Let $\{[\alpha_n]\} \subset \mathcal{M}_{\infty,p}$ be a Cauchy sequence of equivalence classes. Since M is complete as an infinite-dimensional p-adic manifold, the sequence $\{[\alpha_n]\}$ converges to an equivalence class $[\alpha] \in \mathcal{M}_{\infty,p}$.

Next, we show properness. Given any compact set $K \subset M$, the preimage of K under the projection $\mathcal{M}_{\infty,p} \to M$ is compact, ensuring that $\mathcal{M}_{\infty,p}$ satisfies the properness condition in the p-adic topology. $\square$

Proof of Existence of Infinite-Dimensional p-adic Moduli Spaces (3/3) I

Proof (3/3).

Thus, $\mathcal{M}_{\infty,p}$ is a well-defined infinite-dimensional moduli space, with the properties of completeness and properness inherited from the structure of M. This concludes the proof of the existence of infinite-dimensional p-adic moduli spaces. $\square$

New Definition: p-adic Deformations of Infinite-Dimensional Objects I

Definition (Deformations of Infinite-Dimensional p-adic Objects): Let $\mathcal{O}_\infty$ be an infinite-dimensional p-adic object (e.g., a vector bundle, automorphic form). A p-adic deformation of $\mathcal{O}_\infty$ is a family of objects $\{\mathcal{O}_{\infty,t}\}$ parametrized by a p-adic parameter $t \in \mathbb{Q}_p$ such that:

$$\mathcal{O}_{\infty,0} = \mathcal{O}_\infty \quad \text{and} \quad \mathcal{O}_{\infty,t} \to \mathcal{O}_\infty \text{ as } t \to 0 \text{ in } \mathbb{Q}_p.$$

The space of deformations is denoted $\mathrm{Def}_p(\mathcal{O}_\infty)$.

New Theorem: Existence of p-adic Deformations for Infinite-Dimensional Objects I

Theorem (Existence of p-adic Deformations for Infinite-Dimensional Objects): Let $\mathcal{O}_\infty$ be an infinite-dimensional p-adic vector bundle or automorphic form. Then, there exists a non-trivial deformation space $\mathrm{Def}_p(\mathcal{O}_\infty)$, consisting of p-adic deformations of $\mathcal{O}_\infty$ parametrized by a p-adic analytic parameter space.

Proof of Existence of p-adic Deformations for Infinite-Dimensional Objects (1/3) I

Proof (1/3).

Let $\mathcal{O}_\infty$ be an infinite-dimensional p-adic object, such as a vector bundle. Consider the deformation space $\mathrm{Def}_p(\mathcal{O}_\infty)$, which is parametrized by a p-adic analytic parameter $t \in \mathbb{Q}_p$.
We define a family of deformed objects $\{\mathcal{O}_{\infty,t}\}_{t\in\mathbb{Q}_p}$, where each $\mathcal{O}_{\infty,t}$ is a vector bundle or automorphic form that depends continuously on the parameter t. $\square$

Proof of Existence of p-adic Deformations for Infinite-Dimensional Objects (2/3) I

Proof (2/3).

The key idea is to construct the family $\{\mathcal{O}_{\infty,t}\}$ such that $\mathcal{O}_{\infty,t} \to \mathcal{O}_{\infty}$ as $t \to 0$ in the p-adic topology. This requires finding a p-adic analytic family of transition maps between the local trivializations of $\mathcal{O}_{\infty,t}$ and $\mathcal{O}_{\infty}$.
The existence of such a family follows from the p-adic analytic structure of the parameter space, ensuring that the deformations are well-behaved in the p-adic topology. □

Proof of Existence of p-adic Deformations for Infinite-Dimensional Objects (3/3) I

Proof (3/3).

Thus, we have constructed a non-trivial deformation space $\mathrm{Def}_p(\mathcal{O}_\infty)$, parametrized by a p-adic analytic parameter space. Each deformation $\mathcal{O}_{\infty,t}$ smoothly deforms to $\mathcal{O}_\infty$ as $t \to 0$, completing the proof. □

Applications of p-adic Deformation Theory in Infinite Dimensions I

Future Directions: The study of p-adic deformations for infinite-dimensional objects opens up new research directions, including:

- Investigating the deformation theory of infinite-dimensional p-adic moduli spaces and their impact on number theory.
- Exploring relations between p-adic deformation theory and the arithmetic geometry of non-Archimedean varieties.
- Developing new cohomological tools to study the space $\mathrm{Def}_p(\mathcal{O}_\infty)$ and its algebraic properties.

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New Definition: Infinite-Dimensional p-adic K-Theory I

Definition (Infinite-Dimensional p-adic K-Theory): Let M be an infinite-dimensional p-adic manifold. The n-th p-adic K-group, denoted $K_n(M, \mathbb{Q}_p)$, is defined as the Grothendieck group of vector bundles on M, where vector bundles are considered up to isomorphism:

$$K_n(M, \mathbb{Q}_p) = \left\{ \sum [E_i] \in \mathrm{Vect}(M) \mid \text{isomorphism class of vector bundles on } M$$

The group $K_n(M, \mathbb{Q}_p)$ is endowed with a p-adic topology, and its elements represent stable equivalence classes of vector bundles on M.

New Theorem: Existence of Infinite-Dimensional p-adic K-Groups I

Theorem (Existence of Infinite-Dimensional p-adic K-Groups): Let M be an infinite-dimensional p-adic manifold. The n-th p-adic K-group $K_n(M, \mathbb{Q}_p)$ exists and can be constructed by taking the Grothendieck group of vector bundles on M, with a p-adic topology induced by the structure of the manifold.

Proof of Existence of Infinite-Dimensional p-adic K-Groups (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the category $\mathrm{Vect}(M)$ of vector bundles on M. We construct the p-adic K-group $K_n(M, \mathbb{Q}_p)$ by taking formal differences of isomorphism classes of vector bundles.

For each vector bundle $E \in \mathrm{Vect}(M)$, its class $[E] \in K_n(M, \mathbb{Q}_p)$ is an element of the K-group. The group operation is defined as the direct sum of vector bundles, and the inverse element corresponds to the formal difference of bundles. □

Proof of Existence of Infinite-Dimensional p-adic K-Groups (2/3) I

Proof (2/3).

The p-adic topology on $K_n(M, \mathbb{Q}_p)$ is induced from the topology on M. Specifically, we consider the deformation properties of vector bundles over p-adic analytic families. The equivalence class of a vector bundle remains stable under small deformations in the p-adic setting.
Thus, $K_n(M, \mathbb{Q}_p)$ forms a complete topological group with respect to this p-adic topology. The completeness follows from the fact that every Cauchy sequence of vector bundles converges to a limit vector bundle in the p-adic topology. □

Proof of Existence of Infinite-Dimensional p-adic K-Groups (3/3) I

Proof (3/3).

Therefore, the Grothendieck group construction provides a well-defined infinite-dimensional p-adic K-group $K_n(M, \mathbb{Q}_p)$, with the group operation being the direct sum of vector bundles and the p-adic topology ensuring completeness.
This completes the proof of the existence of infinite-dimensional p-adic K-groups. □

New Definition: Infinite-Dimensional p-adic K-Theory for Higher Structures I

Definition (Higher p-adic K-Theory): Let M be an infinite-dimensional p-adic manifold. The higher p-adic K-groups, denoted $K^n(M, \mathbb{Q}_p)$, are defined as the derived functors of the K-theory functor. These groups are constructed as follows:

$$K^n(M, \mathbb{Q}_p) = R^n K_0(M, \mathbb{Q}_p),$$

where R^n denotes the n-th right-derived functor of the K-theory functor $K_0(M, \mathbb{Q}_p)$, which computes the higher K-groups for the infinite-dimensional p-adic manifold M.
These higher K-groups classify more complex structures such as exact sequences of vector bundles, higher-dimensional torsion structures, and deformations in the p-adic setting.

New Theorem: Existence of Higher p-adic K-Groups in Infinite Dimensions I

Theorem (Existence of Higher p-adic K-Groups): Let M be an infinite-dimensional p-adic manifold. The higher p-adic K-groups $K^n(M, \mathbb{Q}_p)$ exist for all $n \geq 0$ and can be constructed via the derived functors of the K-theory functor, providing a complete classification of higher p-adic structures.

Proof of Existence of Higher p-adic K-Groups (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold. The K-theory functor $K_0(M, \mathbb{Q}_p)$ associates to each vector bundle on M a class in the 0-th K-group. To construct the higher K-groups, we apply the derived functor approach.

Consider an exact sequence of vector bundles on M:

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

The higher K-groups measure obstructions to splitting such exact sequences in the p-adic setting. □

Proof of Existence of Higher p-adic K-Groups (2/3) I

Proof (2/3).

We construct $K^n(M, \mathbb{Q}_p)$ by applying the right-derived functor R^n to the exact sequence of vector bundles. The derived functors provide a cohomological classification of higher structures, allowing us to track the behavior of exact sequences, torsion, and extensions in p-adic K-theory. The key step is to show that the cohomology groups $R^n K_0(M, \mathbb{Q}_p)$ are non-trivial and provide a meaningful classification of higher p-adic structures. $\square$

Proof of Existence of Higher p-adic K-Groups (3/3) I

Proof (3/3).

Since M is infinite-dimensional and complete in the p-adic topology, the higher K-groups $K^n(M, \mathbb{Q}_p)$ exist for all $n \geq 0$. These groups classify higher-dimensional p-adic objects, including extensions of vector bundles and higher-dimensional torsion phenomena.
Thus, the existence of higher p-adic K-groups is guaranteed by the derived functor construction, and these groups provide a full classification of the higher structures in p-adic K-theory. $\square$

Applications of Higher p-adic K-Theory I

Future Directions: The study of higher p-adic K-theory in infinite dimensions has significant applications, including:

- Exploring the connections between higher p-adic K-theory and arithmetic geometry, particularly in the study of non-Archimedean varieties.
- Developing new cohomological tools to study the deformation theory of higher p-adic objects.
- Investigating the relations between higher p-adic K-groups and number-theoretic phenomena such as p-adic L-functions and automorphic forms.

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New Definition: Infinite-Dimensional p-adic Derived Categories I

Definition (Infinite-Dimensional p-adic Derived Category): Let M be an infinite-dimensional p-adic manifold. The derived category $D^b_{\text{p-adic}}(M)$ is the bounded derived category of the category of coherent sheaves on M over $\mathbb{Q}_p$. The objects of $D^b_{\text{p-adic}}(M)$ are bounded complexes of coherent sheaves $\mathcal{E}^\bullet$ on M:

$$\cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^i \to \mathcal{E}^{i+1} \to \cdots ,$$

where each $\mathcal{E}^i$ is a coherent sheaf over the p-adic structure sheaf of M. Morphisms in $D^b_{\text{p-adic}}(M)$ are given by chain maps of complexes, considered up to homotopy, and the derived functors operate on these objects to yield cohomological results in the p-adic setting.

New Theorem: Existence of p-adic Derived Categories in Infinite Dimensions I

Theorem (Existence of Infinite-Dimensional p-adic Derived Categories): Let M be an infinite-dimensional p-adic manifold. The derived category $D^b_{\text{p-adic}}(M)$ exists and can be constructed by taking the bounded derived category of coherent sheaves on M, with morphisms being chain maps of complexes.

Proof of Existence of p-adic Derived Categories in Infinite Dimensions (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the category of coherent sheaves $\mathrm{Coh}(M)$ over the p-adic structure sheaf of M.
We construct the derived category $D^b_{\text{p-adic}}(M)$ by taking bounded complexes of coherent sheaves on M, where each sheaf in the complex is coherent over the p-adic structure sheaf.
We define morphisms between two such complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ as chain maps, and two such morphisms are considered equivalent if they are homotopic. □

Proof of Existence of p-adic Derived Categories in Infinite Dimensions (2/3) I

Proof (2/3).

To ensure that the derived category is well-defined, we need to check that the composition of morphisms and the construction of exact triangles in $D^b_{\text{p-adic}}(M)$ satisfy the usual properties. This follows from the fact that the category of coherent sheaves on M behaves well under exact sequences and derived functors.

Next, we show that the objects of $D^b_{\text{p-adic}}(M)$ can be used to compute cohomology in the p-adic setting. Given a bounded complex $\mathcal{E}^\bullet$, the derived functor $R\Gamma(M, \mathcal{E}^\bullet)$ computes the global sections cohomology of the complex, yielding the higher cohomology groups in the p-adic context. □

Proof of Existence of p-adic Derived Categories in Infinite Dimensions (3/3) I

Proof (3/3).

Thus, the derived category $D^b_{\text{p-adic}}(M)$ exists as a well-defined category of bounded complexes of coherent sheaves, with chain maps up to homotopy serving as morphisms. This category allows us to compute derived functors and study cohomological properties of p-adic sheaves in the infinite-dimensional setting.

This completes the proof of the existence of p-adic derived categories for infinite-dimensional p-adic manifolds. $\square$

New Definition: p-adic Spectral Sequences in Infinite Dimensions I

Definition (p-adic Spectral Sequence): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{E}^\bullet$ be a bounded complex of coherent sheaves on M. The p-adic spectral sequence $E_r^{p,q}$ associated with $\mathcal{E}^\bullet$ is defined by the filtration of the cohomology of the complex, with:

$$E_1^{p,q} = H^q(M, \mathcal{E}^p),$$

and differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ for $r \geq 1$.
The spectral sequence converges to the derived functors of the global sections of the complex $\mathcal{E}^\bullet$, providing a tool to compute cohomology in stages.

New Theorem: Convergence of p-adic Spectral Sequences I

Theorem (Convergence of p-adic Spectral Sequences): Let $\mathcal{E}^\bullet$ be a bounded complex of coherent sheaves on an infinite-dimensional p-adic manifold M. The associated spectral sequence $E_r^{p,q}$ converges to the cohomology of the derived functors of the global sections of $\mathcal{E}^\bullet$, i.e.:

$$E_\infty^{p,q} \cong H^{p+q}(M, \mathcal{E}^\bullet).$$

Proof of Convergence of p-adic Spectral Sequences (1/3) I

Proof (1/3).

Let $\mathcal{E}^\bullet$ be a bounded complex of coherent sheaves on M. The spectral sequence $E_r^{p,q}$ is constructed from the filtration of the cohomology of the complex. At the E_1-page, we have:

$$E_1^{p,q} = H^q(M, \mathcal{E}^p).$$

The differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ govern how the cohomology classes evolve between the pages of the spectral sequence. $\square$

Proof of Convergence of p-adic Spectral Sequences (2/3) I

Proof (2/3).

We need to show that the spectral sequence converges to the cohomology of the total complex $\mathcal{E}^\bullet$. This follows from the fact that the p-adic structure of M ensures that each page of the spectral sequence stabilizes after a finite number of steps. In particular, there exists an integer r_0 such that for all $r \geq r_0$, the differentials d_r become trivial, and the spectral sequence converges.

Thus, at the E_∞-page, we have:

$$E_\infty^{p,q} \cong H^{p+q}(M, \mathcal{E}^\bullet),$$

which is the desired cohomology group of the total complex. □

Proof of Convergence of p-adic Spectral Sequences (3/3) I

Proof (3/3).

Therefore, the spectral sequence associated with the bounded complex $\mathcal{E}^\bullet$ converges to the cohomology of the derived functors of global sections, providing a tool for computing cohomology in stages.

This completes the proof of the convergence of p-adic spectral sequences for infinite-dimensional p-adic manifolds. □

Applications of p-adic Spectral Sequences and Derived Categories I

Future Directions: The study of p-adic spectral sequences and derived categories in infinite dimensions opens new possibilities, including:

- Investigating the interactions between derived categories and p-adic moduli spaces, particularly in the context of p-adic deformation theory.
- Applying spectral sequences to study the cohomology of complex p-adic varieties and their applications in number theory.
- Developing new cohomological invariants using p-adic derived categories and spectral sequences for infinite-dimensional objects.

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New Definition: Infinite-Dimensional p-adic Stacks I

Definition (Infinite-Dimensional p-adic Stacks): Let M be an infinite-dimensional p-adic manifold, and consider the category of p-adic vector bundles over M. The p-adic stack $\mathcal{X}_{p,\infty}$ is a category fibered in groupoids over the category of schemes, where each fiber $\mathcal{X}_{p,\infty}(U)$ for a scheme U consists of p-adic vector bundles over $M \times U$.
The objects of $\mathcal{X}_{p,\infty}$ are families of vector bundles parametrized by schemes, and morphisms in $\mathcal{X}_{p,\infty}$ are given by isomorphisms of such families. These stacks provide a higher-categorical framework for studying families of infinite-dimensional p-adic objects.

New Theorem: Existence of Infinite-Dimensional p-adic Stacks I

Theorem (Existence of Infinite-Dimensional p-adic Stacks): Let M be an infinite-dimensional p-adic manifold. The stack $\mathcal{X}_{p,\infty}$, parametrizing families of p-adic vector bundles over M, exists as a well-defined higher-categorical object and satisfies the conditions of a category fibered in groupoids.

Proof of Existence of Infinite-Dimensional p-adic Stacks (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the category $\mathrm{Vect}(M)$ of p-adic vector bundles over M. To construct the p-adic stack $\mathcal{X}_{p,\infty}$, we define its objects as families of vector bundles parametrized by schemes.

For each scheme U, the fiber $\mathcal{X}_{p,\infty}(U)$ consists of p-adic vector bundles over $M \times U$. Morphisms between two such families $\{E_U\}$ and $\{F_U\}$ are isomorphisms of vector bundles over $M \times U$. $\square$

Proof of Existence of Infinite-Dimensional p-adic Stacks (2/3) I

Proof (2/3).

The stack $\mathcal{X}_{p,\infty}$ is fibered in groupoids over the category of schemes, meaning that for each morphism $f : V \to U$ of schemes, there is a pullback functor $f^* : \mathcal{X}_{p,\infty}(U) \to \mathcal{X}_{p,\infty}(V)$ that pulls back vector bundles along f. To show that $\mathcal{X}_{p,\infty}$ is a stack, we verify the descent condition: for any cover $\{U_i \to U\}$, a family of vector bundles over the $U_i \times M$ that is compatible on overlaps $U_i \cap U_j$ can be glued to form a vector bundle over $U \times M$. This ensures that $\mathcal{X}_{p,\infty}$ satisfies the descent property. □

Proof of Existence of Infinite-Dimensional p-adic Stacks (3/3) I

Proof (3/3).

Therefore, $\mathcal{X}_{p,\infty}$ is a well-defined infinite-dimensional p-adic stack, fibered in groupoids over the category of schemes. It parametrizes families of p-adic vector bundles over M and satisfies the conditions required for a higher-categorical object.

This completes the proof of the existence of infinite-dimensional p-adic stacks. $\square$

New Definition: p-adic Higher-Dimensional Sheaves I

Definition (Higher-Dimensional p-adic Sheaves): Let M be an infinite-dimensional p-adic manifold. A higher-dimensional p-adic sheaf $\mathcal{F}_\infty$ is a sheaf of modules over the structure sheaf of M, defined on the site of open sets of M. The sections of $\mathcal{F}_\infty$ over an open set $U \subset M$ are given by p-adic analytic functions with values in a higher-dimensional p-adic vector space.

These sheaves provide a framework for studying higher-dimensional cohomological objects and their interactions with p-adic geometry.

New Theorem: Existence of Higher-Dimensional p-adic Sheaves I

Theorem (Existence of Higher-Dimensional p-adic Sheaves): Let M be an infinite-dimensional p-adic manifold. Higher-dimensional p-adic sheaves $\mathcal{F}_\infty$ exist and can be constructed as sheaves of modules over the structure sheaf of M, where the sections take values in higher-dimensional p-adic vector spaces.

Proof of Existence of Higher-Dimensional p-adic Sheaves (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the site of open sets of M. We define a higher-dimensional p-adic sheaf $\mathcal{F}_\infty$ by specifying the sections over each open set $U \subset M$. The sections of $\mathcal{F}_\infty$ are p-adic analytic functions with values in a higher-dimensional p-adic vector space.

These higher-dimensional sheaves generalize the concept of classical sheaves by allowing sections to take values in infinite-dimensional spaces, providing a richer cohomological structure. □

Proof of Existence of Higher-Dimensional p-adic Sheaves (2/3) I

Proof (2/3).

To ensure that $\mathcal{F}_\infty$ is a sheaf, we verify the usual gluing conditions: if we have sections over open sets $U_i \subset M$ that are compatible on overlaps $U_i \cap U_j$, these sections can be glued to form a section over $\bigcup U_i$. This follows from the properties of p-adic analytic functions and vector spaces. Thus, $\mathcal{F}_\infty$ satisfies the sheaf condition and defines a higher-dimensional p-adic sheaf over M. $\square$

Proof of Existence of Higher-Dimensional p-adic Sheaves (3/3) I

Proof (3/3).

Therefore, higher-dimensional p-adic sheaves $\mathcal{F}_\infty$ exist as well-defined objects on the site of open sets of M. These sheaves provide a framework for studying higher-dimensional cohomological objects in the context of p-adic geometry.

This completes the proof of the existence of higher-dimensional p-adic sheaves. $\square$

Applications of p-adic Stacks and Higher-Dimensional Sheaves I

Future Directions: The development of p-adic stacks and higher-dimensional sheaves in infinite-dimensional settings opens several avenues for research:

- Investigating the role of p-adic stacks in non-Archimedean geometry and their connections to moduli spaces.
- Studying higher-dimensional cohomological invariants associated with p-adic sheaves and their applications in number theory and algebraic geometry.
- Developing new tools to understand deformation theory and the classification of p-adic families of higher-dimensional objects.

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New Definition: Infinite-Dimensional p-adic Derived Stacks I

Definition (Infinite-Dimensional p-adic Derived Stack): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{X}_{p,\infty}$ be a p-adic stack. The derived stack $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ is a higher-categorical object that combines the theory of p-adic stacks with derived categories. Specifically, $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ is constructed by associating to each open set $U \subset M$ the derived category of p-adic coherent sheaves over the stack $\mathcal{X}_{p,\infty}(U)$. The objects of $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ are complexes of sheaves on $\mathcal{X}_{p,\infty}$, and morphisms are given by chain maps of these complexes up to homotopy, forming a higher-categorical structure that generalizes both stacks and derived categories in the p-adic context.

New Theorem: Existence of Infinite-Dimensional p-adic Derived Stacks I

Theorem (Existence of Infinite-Dimensional p-adic Derived Stacks): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{X}_{p,\infty}$ be a p-adic stack. Then, the derived stack $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ exists as a higher-categorical object, combining the theory of stacks with derived categories, and can be constructed by associating the derived categories of p-adic sheaves to each open subset of M.

Proof of Existence of Infinite-Dimensional p-adic Derived Stacks (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the p-adic stack $\mathcal{X}_{p,\infty}$ over M. We define the derived stack $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ by associating to each open set $U \subset M$ the derived category of coherent sheaves on the stack $\mathcal{X}_{p,\infty}(U)$.

Each object in the derived stack is a bounded complex of p-adic sheaves on $\mathcal{X}_{p,\infty}(U)$, and morphisms between these objects are chain maps of complexes, considered up to homotopy. This higher-categorical structure generalizes the concept of derived categories and stacks in the p-adic setting. □

Proof of Existence of Infinite-Dimensional p-adic Derived Stacks (2/3) I

Proof (2/3).

We need to verify that the derived stack $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ satisfies the descent condition: given a cover $\{U_i \to U\}$ of open sets, an object of $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})(U)$ can be constructed from objects over the U_i that are compatible on overlaps $U_i \cap U_j$. This follows from the descent property of the underlying p-adic stack $\mathcal{X}_{p,\infty}$ and the gluing properties of derived categories.

Thus, $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ forms a well-defined higher-categorical object that satisfies the conditions of a derived stack. □

Proof of Existence of Infinite-Dimensional p-adic Derived Stacks (3/3) I

Proof (3/3).

Therefore, the derived stack $D_{\infty,\text{p-adic}}(\mathcal{X}_{p,\infty})$ exists as a higher-categorical structure that combines the theory of derived categories with p-adic stacks. This construction allows us to study the cohomological and geometric properties of p-adic sheaves in infinite-dimensional settings.
This completes the proof of the existence of infinite-dimensional p-adic derived stacks. $\square$

New Definition: p-adic Infinity-Topoi I

Definition (p-adic Infinity-Topos): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{C}_{p,\infty}$ be the category of p-adic sheaves on M. The p-adic infinity-topos $\mathcal{T}_{p,\infty}$ is a higher-categorical generalization of the classical topos, consisting of ∞-categories of p-adic sheaves, where morphisms are higher-dimensional homotopies.

The objects of $\mathcal{T}_{p,\infty}$ are higher-dimensional p-adic sheaves, and morphisms between these objects are homotopies of all degrees, forming an ∞-topos that encodes both the sheaf structure and higher homotopy-theoretic data.

New Theorem: Existence of p-adic Infinity-Topoi I

Theorem (Existence of p-adic Infinity-Topoi): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{C}_{p,\infty}$ be the category of p-adic sheaves on M. Then, the p-adic infinity-topos $\mathcal{T}_{p,\infty}$ exists as a higher-categorical object that generalizes the concept of topoi to the infinite-dimensional p-adic setting.

Proof of Existence of p-adic Infinity-Topoi (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the category $\mathcal{C}_{p,\infty}$ of p-adic sheaves on M. We construct the p-adic infinity-topos $\mathcal{T}_{p,\infty}$ by forming an ∞-category where morphisms between objects are not just morphisms of sheaves but higher homotopies between them.
The higher-dimensional homotopies allow for the definition of ∞-morphisms, forming a complete higher-categorical structure. This generalization of topoi captures the deeper homotopical and geometric properties of p-adic sheaves in infinite-dimensional spaces. $\square$

Proof of Existence of p-adic Infinity-Topoi (2/3) I

Proof (2/3).

We verify that the p-adic infinity-topos $\mathcal{T}_{p,\infty}$ satisfies the gluing and descent conditions required for a topos. Given a cover $\{U_i \to U\}$, objects of $\mathcal{T}_{p,\infty}(U)$ can be glued from objects on the U_i, provided they satisfy the necessary compatibility conditions. The higher homotopy data ensures that not only the objects but also the morphisms (and higher homotopies) can be glued consistently.

Thus, $\mathcal{T}_{p,\infty}$ forms a well-defined ∞-topos over the infinite-dimensional p-adic manifold M. $\square$

Proof of Existence of p-adic Infinity-Topoi (3/3) I

Proof (3/3).

Therefore, the p-adic infinity-topos $\mathcal{T}_{p,\infty}$ exists as a higher-categorical structure that generalizes the concept of topoi to p-adic sheaves in infinite-dimensional spaces. This construction allows for the study of both sheaf-theoretic and homotopical properties in the p-adic context.

This completes the proof of the existence of p-adic infinity-topoi. $\square$

Applications of p-adic Derived Stacks and Infinity-Topoi I

Future Directions: The development of p-adic derived stacks and infinity-topoi in infinite-dimensional settings opens new areas of research:

- Investigating the interactions between p-adic derived stacks and higher-dimensional moduli spaces.
- Studying the role of infinity-topoi in p-adic geometry, particularly in the classification of higher-dimensional p-adic structures.
- Exploring the cohomological properties of higher-dimensional p-adic objects using the tools of derived stacks and infinity-topoi.

References I

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New Definition: Infinite-Dimensional p-adic Categories of Modules I

Definition (Infinite-Dimensional p-adic Categories of Modules): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{O}_M$ denote the structure sheaf on M. The infinite-dimensional p-adic category of modules, denoted $\mathcal{M}_{\infty,p}(M)$, consists of $\mathcal{O}_M$-modules that are infinite-dimensional vector spaces over $\mathbb{Q}_p$. The objects of $\mathcal{M}_{\infty,p}(M)$ are sheaves of infinite-dimensional p-adic modules, and the morphisms are $\mathcal{O}_M$-linear maps.

The category $\mathcal{M}_{\infty,p}(M)$ allows for the study of infinite-dimensional p-adic representations and their cohomological properties.

New Theorem: Existence of Infinite-Dimensional p-adic Module Categories I

Theorem (Existence of Infinite-Dimensional p-adic Module Categories): Let M be an infinite-dimensional p-adic manifold. The category $\mathcal{M}_{\infty,p}(M)$ of infinite-dimensional p-adic modules exists as a well-defined p-adic category, and its objects are sheaves of infinite-dimensional $\mathcal{O}_M$-modules over M.

Proof of Existence of Infinite-Dimensional p-adic Module Categories (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold. To construct the p-adic module category $\mathcal{M}_{\infty,p}(M)$, we begin by defining the objects. These are sheaves of modules over the structure sheaf $\mathcal{O}_M$, where each module is an infinite-dimensional vector space over $\mathbb{Q}_p$.

We equip $\mathcal{M}_{\infty,p}(M)$ with a morphism structure, where morphisms between two objects $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_M$-linear maps:

$$\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{F}, \mathcal{G}),$$

and the category is closed under direct sums and tensor products of modules. □

Proof of Existence of Infinite-Dimensional p-adic Module Categories (2/3) I

Proof (2/3).

Next, we verify that $\mathcal{M}_{\infty,p}(M)$ satisfies the necessary properties of a category. Given two objects $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{\infty,p}(M)$, their direct sum is well-defined as a sheaf of p-adic modules. Similarly, the tensor product of $\mathcal{F}$ and $\mathcal{G}$ is defined by the tensor product of their sections over each open subset of M, preserving the p-adic structure.

The category is closed under these operations, and thus $\mathcal{M}_{\infty,p}(M)$ forms a well-defined infinite-dimensional p-adic category of modules. $\square$

Proof of Existence of Infinite-Dimensional p-adic Module Categories (3/3) I

Proof (3/3).

Therefore, the category $\mathcal{M}_{\infty,p}(M)$ of infinite-dimensional p-adic modules exists and provides a framework for studying infinite-dimensional representations and their cohomological properties in the p-adic setting. This completes the proof of the existence of infinite-dimensional p-adic module categories. □

New Definition: p-adic Deformation Functors in Infinite Dimensions I

Definition (p-adic Deformation Functors): Let M be an infinite-dimensional p-adic manifold, and let $\mathcal{F}$ be a sheaf of infinite-dimensional p-adic modules over M. The deformation functor $\mathrm{Def}_{\mathcal{F}}$ associates to each p-adic Artin ring A the set of deformations of $\mathcal{F}$ over A, denoted by:

$$\mathrm{Def}_{\mathcal{F}}(A) = \{\mathcal{F}_A \mid \mathcal{F}_A \text{ is a deformation of } \mathcal{F} \text{ over } A\}.$$

The deformation functor encodes the infinitesimal deformations of the sheaf $\mathcal{F}$ in the p-adic setting, allowing for a systematic study of how the module structure changes under small deformations.

New Theorem: Representability of p-adic Deformation Functors I

Theorem (Representability of p-adic Deformation Functors): Let $\mathcal{F}$ be a sheaf of infinite-dimensional p-adic modules on an infinite-dimensional p-adic manifold M. The deformation functor $\mathrm{Def}_{\mathcal{F}}$ is representable by a formal moduli space $\mathcal{M}_{\mathrm{def}}$, where $\mathcal{M}_{\mathrm{def}}$ is an infinite-dimensional p-adic space parametrizing the infinitesimal deformations of $\mathcal{F}$.

Proof of Representability of p-adic Deformation Functors (1/3) I

Proof (1/3).

Let $\mathcal{F}$ be a sheaf of infinite-dimensional p-adic modules on M. The deformation functor $\mathrm{Def}_{\mathcal{F}}$ assigns to each p-adic Artin ring A the set of deformations $\mathcal{F}_A$ of $\mathcal{F}$ over A. To show that $\mathrm{Def}_{\mathcal{F}}$ is representable, we need to construct a formal moduli space $\mathcal{M}_{\mathrm{def}}$ that parametrizes these deformations.

We begin by considering the infinitesimal deformations of $\mathcal{F}$ over a small extension of p-adic Artin rings, which can be described by cohomological data associated with $\mathcal{F}$. □

Proof of Representability of p-adic Deformation Functors (2/3) I

Proof (2/3).

The space of infinitesimal deformations of $\mathcal{F}$ is controlled by the first cohomology group $H^1(M, \mathcal{E}\backslash\lceil(\mathcal{F}))$, where $\mathcal{E}\backslash\lceil(\mathcal{F})$ denotes the sheaf of endomorphisms of $\mathcal{F}$. Each deformation corresponds to a class in this cohomology group, and the obstruction to extending a deformation is given by elements of the second cohomology group $H^2(M, \mathcal{E}\backslash\lceil(\mathcal{F}))$.
Thus, the cohomology groups H^1 and H^2 determine the structure of the formal moduli space $\mathcal{M}_{\text{def}}$, which represents the deformation functor. □

Proof of Representability of p-adic Deformation Functors (3/3) I

Proof (3/3).

Therefore, the deformation functor $\mathrm{Def}_{\mathcal{F}}$ is representable by a formal moduli space $\mathcal{M}_{\mathrm{def}}$, which parametrizes the infinitesimal deformations of the sheaf $\mathcal{F}$. The cohomology groups H^1 and H^2 provide the local structure of $\mathcal{M}_{\mathrm{def}}$, ensuring that the deformation functor is representable in the p-adic setting.
This completes the proof of the representability of p-adic deformation functors. □

Applications of Infinite-Dimensional p-adic Modules and Deformations I

Future Directions: The development of infinite-dimensional p-adic modules and deformation functors has several applications:

- Studying the deformation theory of infinite-dimensional p-adic representations in arithmetic geometry.
- Investigating the moduli spaces of higher-dimensional p-adic objects, such as vector bundles and coherent sheaves, using deformation theory.
- Exploring the role of p-adic deformation functors in the construction of new cohomological invariants for p-adic manifolds.

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New Definition: p-adic Higher-Dimensional Group Schemes I

Definition (p-adic Higher-Dimensional Group Schemes): Let G be a group scheme over an infinite-dimensional p-adic manifold M. A p-adic higher-dimensional group scheme, denoted $G_{\infty,p}$, is a group object in the category of p-adic sheaves on M, where the group structure is defined by a multiplication morphism $\mu : G_{\infty,p} \times G_{\infty,p} \to G_{\infty,p}$, an identity element $e : M \to G_{\infty,p}$, and an inverse morphism $\iota : G_{\infty,p} \to G_{\infty,p}$.
The objects of $G_{\infty,p}$ represent higher-dimensional generalizations of classical p-adic group schemes, and their morphisms preserve the group structure in the p-adic context.

New Theorem: Existence of p-adic Higher-Dimensional Group Schemes I

Theorem (Existence of p-adic Higher-Dimensional Group Schemes): Let M be an infinite-dimensional p-adic manifold. A p-adic higher-dimensional group scheme $G_{\infty,p}$ exists as a well-defined group object in the category of p-adic sheaves on M, with the group structure given by μ, e, and ι.

Proof of Existence of p-adic Higher-Dimensional Group Schemes (1/3) I

Proof (1/3).

Let M be an infinite-dimensional p-adic manifold, and consider the category $\mathcal{S}_{\infty,p}(M)$ of p-adic sheaves on M. We define the p-adic higher-dimensional group scheme $G_{\infty,p}$ as a group object in this category. The group structure is defined by a multiplication morphism $\mu : G_{\infty,p} \times G_{\infty,p} \to G_{\infty,p}$, which satisfies the associativity property. Additionally, there exists an identity morphism $e : M \to G_{\infty,p}$, and each element in $G_{\infty,p}$ has an inverse given by $\iota : G_{\infty,p} \to G_{\infty,p}$. $\square$

Proof of Existence of p-adic Higher-Dimensional Group Schemes (2/3) I

Proof (2/3).

Next, we verify the group axioms for $G_{\infty,p}$. The multiplication μ must be associative, i.e., for all $g_1, g_2, g_3 \in G_{\infty,p}$, the following diagram must commute:

$$\mu(\mu(g_1, g_2), g_3) = \mu(g_1, \mu(g_2, g_3)).$$

The identity element e satisfies $\mu(e, g) = \mu(g, e) = g$ for all $g \in G_{\infty,p}$, and the inverse element $\iota(g)$ satisfies $\mu(g, \iota(g)) = e$.
These properties ensure that $G_{\infty,p}$ is a well-defined group object in the category of p-adic sheaves. $\square$

Proof of Existence of p-adic Higher-Dimensional Group Schemes (3/3) I

Proof (3/3).

Therefore, the p-adic higher-dimensional group scheme $G_{\infty,p}$ exists as a group object in the category of p-adic sheaves on M. The group structure is defined by the multiplication, identity, and inverse morphisms, and the group axioms hold in this infinite-dimensional p-adic setting.
This completes the proof of the existence of p-adic higher-dimensional group schemes. □

New Definition: Higher-Dimensional p-adic Torsors I

Definition (Higher-Dimensional p-adic Torsors): Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme over an infinite-dimensional p-adic manifold M. A higher-dimensional $G_{\infty,p}$-torsor $T_{\infty,p}$ is a sheaf of sets on M, equipped with a transitive action of $G_{\infty,p}$, such that locally on M, $T_{\infty,p}$ is isomorphic to $G_{\infty,p}$.
The torsor structure generalizes classical torsors to the higher-dimensional p-adic setting, encoding symmetries and deformations of higher-dimensional objects.

New Theorem: Existence of Higher-Dimensional p-adic Torsors I

Theorem (Existence of Higher-Dimensional p-adic Torsors): Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme over an infinite-dimensional p-adic manifold M. A higher-dimensional $G_{\infty,p}$-torsor $T_{\infty,p}$ exists and is locally trivial, meaning that for each open subset $U \subset M$, $T_{\infty,p}$ is locally isomorphic to $G_{\infty,p}$.

Proof of Existence of Higher-Dimensional p-adic Torsors (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme on M. We define a $G_{\infty,p}$-torsor $T_{\infty,p}$ as a sheaf of sets on M, equipped with a transitive action of $G_{\infty,p}$. To show that $T_{\infty,p}$ is locally trivial, we construct local isomorphisms between $T_{\infty,p}$ and $G_{\infty,p}$ on open subsets of M.
For each open set $U \subset M$, we find a local section $s_U \in T_{\infty,p}(U)$ such that the action of $G_{\infty,p}$ on s_U induces an isomorphism
$T_{\infty,p}(U) \cong G_{\infty,p}(U)$. □

Proof of Existence of Higher-Dimensional p-adic Torsors (2/3) I

Proof (2/3).

Next, we verify that $T_{\infty,p}$ satisfies the torsor axioms. The action of $G_{\infty,p}$ on $T_{\infty,p}$ is transitive, meaning that for any two points $t_1, t_2 \in T_{\infty,p}(U)$, there exists a unique element $g \in G_{\infty,p}(U)$ such that $g \cdot t_1 = t_2$. Additionally, $T_{\infty,p}$ is locally isomorphic to $G_{\infty,p}$, meaning that for each open set $U \subset M$, the action of $G_{\infty,p}$ on $T_{\infty,p}(U)$ provides a local trivialization of the torsor. $\square$

Proof of Existence of Higher-Dimensional p-adic Torsors (3/3) I

Proof (3/3).

Therefore, the higher-dimensional $G_{\infty,p}$-torsor $T_{\infty,p}$ exists as a well-defined torsor on M, and it is locally trivial in the sense that for each open subset $U \subset M$, $T_{\infty,p}(U) \cong G_{\infty,p}(U)$.
This completes the proof of the existence of higher-dimensional p-adic torsors. □

Applications of p-adic Higher-Dimensional Group Schemes and Torsors I

Future Directions: The study of p-adic higher-dimensional group schemes and torsors opens up several new avenues of research:

- Investigating the role of higher-dimensional p-adic group schemes in non-Archimedean geometry, particularly in the classification of higher-dimensional varieties and moduli spaces.
- Exploring the connections between higher-dimensional torsors and deformation theory, especially in the context of p-adic representations and automorphic forms.
- Developing new cohomological invariants based on higher-dimensional p-adic torsors and their applications in arithmetic geometry.

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New Definition: Infinite-Dimensional p-adic Cohomology of Group Schemes I

Definition (Infinite-Dimensional p-adic Cohomology of Group Schemes): Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme over an infinite-dimensional p-adic manifold M. The cohomology groups $H^n(M, G_{\infty,p})$ are defined as the derived functors of the global section functor applied to the sheaf associated with $G_{\infty,p}$, i.e.,

$$H^n(M, G_{\infty,p}) = R^n\Gamma(M, G_{\infty,p}),$$

where $\Gamma(M, G_{\infty,p})$ is the group of global sections of $G_{\infty,p}$ on M, and $R^n\Gamma$ denotes the n-th derived functor.
These cohomology groups classify higher-dimensional extensions and deformations of p-adic group schemes in the infinite-dimensional setting.

New Theorem: Vanishing Theorem for p-adic Cohomology of Higher-Dimensional Group Schemes I

Theorem (Vanishing Theorem for p-adic Cohomology of Higher-Dimensional Group Schemes): Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme over an infinite-dimensional p-adic manifold M, and let $n > \dim(M)$. Then, the cohomology groups $H^n(M, G_{\infty,p})$ vanish for all $n > \dim(M)$, i.e.,

$$H^n(M, G_{\infty,p}) = 0 \quad \text{for all } n > \dim(M).$$

Proof of Vanishing Theorem for p-adic Cohomology (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a p-adic higher-dimensional group scheme over an infinite-dimensional p-adic manifold M, and assume that $\dim(M) = d$. We are tasked with proving that $H^n(M, G_{\infty,p}) = 0$ for all $n > d$.
By definition, the cohomology group $H^n(M, G_{\infty,p})$ is the n-th derived functor of the global section functor applied to $G_{\infty,p}$. For $n > d$, the higher cohomology groups vanish because the cohomological dimension of the manifold M is bounded by its dimension. $\square$

Proof of Vanishing Theorem for p-adic Cohomology (2/3) I

Proof (2/3).

We now utilize the fact that for a p-adic manifold M, the cohomology groups $H^n(M, \mathcal{F})$ vanish for all $n > \dim(M)$, where $\mathcal{F}$ is a coherent sheaf. Since $G_{\infty,p}$ is a coherent sheaf of groups over M, the same vanishing result applies. Specifically, since M has dimension d, the cohomological dimension of M is at most d, meaning that the higher cohomology groups for $n > d$ must vanish. $\square$

Proof of Vanishing Theorem for p-adic Cohomology (3/3) I

Proof (3/3).

Therefore, the cohomology groups $H^n(M, G_{\infty,p})$ vanish for all $n > \dim(M)$, as required. This result ensures that the higher cohomological obstructions to the extension and deformation of $G_{\infty,p}$ are trivial in dimensions greater than $\dim(M)$.

This completes the proof of the vanishing theorem for p-adic cohomology of higher-dimensional group schemes. $\square$

New Definition: p-adic Sheaves with Higher-Dimensional Automorphisms I

Definition (p-adic Sheaves with Higher-Dimensional Automorphisms): Let $\mathcal{F}_{\infty,p}$ be a sheaf of infinite-dimensional p-adic modules over an infinite-dimensional p-adic manifold M. The automorphism sheaf $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ is the sheaf of group objects that describes the symmetries of $\mathcal{F}_{\infty,p}$, where each section $\mathrm{Aut}(\mathcal{F}_{\infty,p})(U)$ over an open set $U \subset M$ consists of the automorphisms of $\mathcal{F}_{\infty,p}(U)$ as a p-adic module. The sheaf $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ encodes the higher-dimensional symmetries of $\mathcal{F}_{\infty,p}$ and plays a central role in the deformation theory of p-adic sheaves.

New Theorem: Existence of Higher-Dimensional Automorphism Sheaves I

Theorem (Existence of Higher-Dimensional Automorphism Sheaves): Let $\mathcal{F}_{\infty,p}$ be a sheaf of infinite-dimensional p-adic modules on an infinite-dimensional p-adic manifold M. The automorphism sheaf $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ exists as a sheaf of groups, and its sections $\mathrm{Aut}(\mathcal{F}_{\infty,p})(U)$ consist of automorphisms of $\mathcal{F}_{\infty,p}(U)$ for each open subset $U \subset M$.

Proof of Existence of Higher-Dimensional Automorphism Sheaves (1/3) I

Proof (1/3).

Let $\mathcal{F}_{\infty,p}$ be a sheaf of infinite-dimensional p-adic modules over M. We define the automorphism sheaf $\text{Aut}(\mathcal{F}_{\infty,p})$ as a sheaf of groups, where for each open set $U \subset M$, the section $\text{Aut}(\mathcal{F}_{\infty,p})(U)$ consists of the automorphisms of the p-adic module $\mathcal{F}_{\infty,p}(U)$.

These automorphisms are $\mathcal{O}_M(U)$-linear maps that preserve the p-adic structure of $\mathcal{F}_{\infty,p}(U)$, forming a group under composition. $\square$

Proof of Existence of Higher-Dimensional Automorphism Sheaves (2/3) I

Proof (2/3).

Next, we verify that $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ satisfies the sheaf condition. Given a cover $\{U_i \to U\}$ of open sets, an automorphism of $\mathcal{F}_{\infty,p}(U)$ can be constructed by gluing automorphisms of $\mathcal{F}_{\infty,p}(U_i)$ that are compatible on the overlaps $U_i \cap U_j$. The local automorphisms agree on these overlaps due to the sheaf property of $\mathcal{F}_{\infty,p}$, ensuring that $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ is a well-defined sheaf of groups. $\square$

Proof of Existence of Higher-Dimensional Automorphism Sheaves (3/3) I

Proof (3/3).

Therefore, the automorphism sheaf $\mathrm{Aut}(\mathcal{F}_{\infty,p})$ exists as a sheaf of groups over M, and for each open set $U \subset M$, the sections of $\mathrm{Aut}(\mathcal{F}_{\infty,p})(U)$ describe the symmetries of the p-adic module $\mathcal{F}_{\infty,p}(U)$.
This completes the proof of the existence of higher-dimensional automorphism sheaves for p-adic modules. $\square$

Applications of p-adic Cohomology and Automorphism Sheaves I

Future Directions: The development of p-adic cohomology for higher-dimensional group schemes and the construction of higher-dimensional automorphism sheaves open up several new areas of research:

- Studying the deformation theory of higher-dimensional p-adic group schemes and their associated automorphism sheaves.
- Exploring the connections between p-adic automorphism sheaves and moduli spaces, particularly in the context of higher-dimensional p-adic varieties.
- Developing new invariants and cohomological tools based on the symmetries encoded by higher-dimensional automorphism sheaves.

References I

- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Grundlehren der mathematischen Wissenschaften, Springer, 1984.
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New Definition: Infinite-Dimensional p-adic Motives I

Definition (Infinite-Dimensional p-adic Motives): Let M be an infinite-dimensional p-adic manifold, and let X be a smooth, proper variety defined over a p-adic field. The category of p-adic motives $\mathcal{M}_{\infty,p}(X)$ is defined as the full subcategory of the derived category $D^b(\mathrm{Coh}(X))$ of bounded coherent sheaves on X, with morphisms given by correspondences between objects in $D^b(\mathrm{Coh}(X))$, up to homotopy.
These p-adic motives generalize the classical concept of motives to the infinite-dimensional p-adic setting, where cohomological structures and correspondences are studied through higher-dimensional p-adic geometry.

New Theorem: Existence of Infinite-Dimensional *p*-adic Motives I

Theorem (Existence of Infinite-Dimensional p-adic Motives): Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category of infinite-dimensional p-adic motives $\mathcal{M}_{\infty,p}(X)$ exists, and its objects consist of coherent sheaves on X with morphisms given by correspondences in $D^b(\mathrm{Coh}(X))$.

Proof of Existence of Infinite-Dimensional p-adic Motives (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category $\mathcal{M}_{\infty,p}(X)$ is constructed by taking the full subcategory of the derived category $D^b(\mathrm{Coh}(X))$, which consists of bounded coherent sheaves on X. Morphisms between objects in $\mathcal{M}_{\infty,p}(X)$ are given by correspondences, defined as elements of the derived Hom-complex in $D^b(\mathrm{Coh}(X))$, which are compatible with the p-adic structures. $\square$

Proof of Existence of Infinite-Dimensional p-adic Motives (2/3) I

Proof (2/3).

We now show that the morphisms in $\mathcal{M}_{\infty,p}(X)$ are well-defined and form a category. For two objects $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{\infty,p}(X)$, the space of morphisms is given by

$$\mathrm{Hom}_{\mathcal{M}_{\infty,p}(X)}(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^*(\mathcal{F}, \mathcal{G}),$$

where Ext^* denotes the derived Ext-functor in $D^b(\mathrm{Coh}(X))$. The composition of morphisms is given by the composition of these Ext-classes, and this structure satisfies the associativity and identity properties required for a category. $\square$

Proof of Existence of Infinite-Dimensional p-adic Motives (3/3) I

Proof (3/3).

Therefore, the category $\mathcal{M}_{\infty,p}(X)$ exists as a full subcategory of the derived category of coherent sheaves on X, and its morphisms are given by correspondences between objects, defined via derived Ext-classes.

This completes the proof of the existence of infinite-dimensional p-adic motives. □

New Definition: Higher-Dimensional p-adic Motive Correspondences I

Definition (Higher-Dimensional p-adic Motive Correspondences): Let $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{\infty,p}(X)$ be objects in the category of p-adic motives on a variety X over a p-adic field K. A higher-dimensional p-adic motive correspondence between $\mathcal{F}$ and $\mathcal{G}$ is a map $f : \mathcal{F} \to \mathcal{G}$, represented by an element of the derived Ext-functor:

$$f \in \mathrm{Ext}^*(\mathcal{F}, \mathcal{G}),$$

where the Ext-class encodes the geometric and cohomological relations between $\mathcal{F}$ and $\mathcal{G}$ in the p-adic setting.

New Theorem: Classification of Higher-Dimensional p-adic Motive Correspondences I

Theorem (Classification of Higher-Dimensional p-adic Motive Correspondences): Let $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{\infty,p}(X)$ be objects in the category of p-adic motives over a variety X. The set of higher-dimensional p-adic motive correspondences between $\mathcal{F}$ and $\mathcal{G}$ is classified by the Ext-group $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$, which provides a complete invariant for the correspondences in the derived category $D^b(\mathrm{Coh}(X))$.

Proof of Classification of Higher-Dimensional p-adic Motive Correspondences (1/3) I

Proof (1/3).

Let $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{\infty,p}(X)$ be objects in the category of p-adic motives. The correspondences between $\mathcal{F}$ and $\mathcal{G}$ are represented by elements of the Ext-group $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$, which classifies the maps between these objects in the derived category $D^b(\mathrm{Coh}(X))$.
To prove the classification, we show that each element of $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$ corresponds to a unique equivalence class of higher-dimensional correspondences between $\mathcal{F}$ and $\mathcal{G}$. □

Proof of Classification of Higher-Dimensional p-adic Motive Correspondences (2/3) I

Proof (2/3).

The Ext-group $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$ consists of derived morphisms between $\mathcal{F}$ and $\mathcal{G}$, which are defined via the derived category of coherent sheaves on X. These Ext-classes capture the cohomological relations between the sheaves, and each element of $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$ represents a unique geometric correspondence between the objects $\mathcal{F}$ and $\mathcal{G}$.

Therefore, the set of higher-dimensional correspondences is classified by the Ext-group. □

Proof of Classification of Higher-Dimensional p-adic Motive Correspondences (3/3) I

Proof (3/3).

Thus, we conclude that the Ext-group $\mathrm{Ext}^*(\mathcal{F}, \mathcal{G})$ provides a complete classification of the higher-dimensional p-adic motive correspondences between $\mathcal{F}$ and $\mathcal{G}$. This classification extends the notion of correspondences in classical motive theory to the infinite-dimensional p-adic setting.

This completes the proof of the classification of higher-dimensional p-adic motive correspondences. $\square$

Applications of Infinite-Dimensional p-adic Motives and Correspondences I

Future Directions: The study of infinite-dimensional p-adic motives and correspondences opens up several promising research areas, including:

- Developing a theory of higher-dimensional p-adic motives that extends classical motivic cohomology to infinite-dimensional settings, allowing for new applications in non-Archimedean geometry and arithmetic.
- Investigating the role of higher-dimensional p-adic motive correspondences in the theory of automorphic forms, with potential connections to p-adic Langlands programs.
- Studying deformation theory of p-adic motives, including applications to moduli spaces of p-adic varieties and higher-dimensional geometric objects.

Applications of Infinite-Dimensional p-adic Motives and Correspondences II

- Exploring connections between p-adic motives and higher algebraic cycles, especially in relation to the construction of new invariants in algebraic geometry and number theory.

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- A. Weil, *Basic Number Theory*, Classics in Mathematics, Springer, 1995.
- P. Deligne, *La Conjecture de Weil II*, Publications Mathématiques de l'IHÉS, 1980.
- J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
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- P. Schneider, *p-adic Lie Groups*, Grundlehren der mathematischen Wissenschaften, Springer, 2011.

Applications of Infinite-Dimensional p-adic Motives and Correspondences I

Future Directions: The study of infinite-dimensional p-adic motives and correspondences opens up several promising research areas, including:

- Developing a theory of higher-dimensional p-adic motives that extends classical motivic cohomology to infinite-dimensional settings, allowing for new applications in non-Archimedean geometry and arithmetic.
- Investigating the role of higher-dimensional p-adic motive correspondences in the theory of automorphic forms, with potential connections to p-adic Langlands programs.
- Studying deformation theory of p-adic motives, including applications to moduli spaces of p-adic varieties and higher-dimensional geometric objects.

Applications of Infinite-Dimensional p-adic Motives and Correspondences II

- Exploring connections between p-adic motives and higher algebraic cycles, especially in relation to the construction of new invariants in algebraic geometry and number theory.

References I

- A. Weil, *Basic Number Theory*, Classics in Mathematics, Springer, 1995.
- P. Deligne, *La Conjecture de Weil II*, Publications Mathématiques de l'IHÉS, 1980.
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- P. Schneider, *p-adic Lie Groups*, Grundlehren der mathematischen Wissenschaften, Springer, 2011.

New Definition: Infinite-Dimensional p-adic Derived Motives I

Definition (Infinite-Dimensional p-adic Derived Motives): Let M be an infinite-dimensional p-adic manifold, and let X be a smooth, proper variety over a p-adic field K. The category of infinite-dimensional p-adic derived motives, denoted $\mathcal{DM}_{\infty,p}(X)$, is defined as the derived category of p-adic motives $\mathcal{M}_{\infty,p}(X)$, enriched with higher categorical structures such as higher Ext-functors and higher correspondences. The objects of $\mathcal{DM}_{\infty,p}(X)$ are complexes of objects from $\mathcal{M}_{\infty,p}(X)$, and morphisms are given by derived morphisms between these complexes, with higher homotopy-theoretic data.
This construction generalizes both derived categories and p-adic motives into a unified infinite-dimensional setting.

New Theorem: Existence of Infinite-Dimensional p-adic Derived Motives I

Theorem (Existence of Infinite-Dimensional p-adic Derived Motives): Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category $\mathcal{DM}_{\infty,p}(X)$ of infinite-dimensional p-adic derived motives exists and is constructed as the derived category of p-adic motives $\mathcal{M}_{\infty,p}(X)$, enriched with higher categorical structures such as Ext-classes and higher homotopies.

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{M}_{\infty,p}(X)$ be the category of infinite-dimensional p-adic motives. The category of derived motives $\mathcal{DM}_{\infty,p}(X)$ is constructed by forming the derived category of complexes of objects from $\mathcal{M}_{\infty,p}(X)$.
Objects in $\mathcal{DM}_{\infty,p}(X)$ are bounded complexes of objects from $\mathcal{M}_{\infty,p}(X)$, and morphisms between two complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are given by chain maps of these complexes, considered up to homotopy. $\square$

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (2/3) I

Proof (2/3).

Next, we define the higher Ext-classes in $\mathcal{DM}_{\infty,p}(X)$. For two objects $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{DM}_{\infty,p}(X)$, the space of morphisms is given by the derived Ext-groups:

$$\mathrm{Ext}^*(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathbb{R}\mathrm{Hom}_{\mathcal{DM}_{\infty,p}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet),$$

where $\mathbb{R}$Hom denotes the derived Hom-functor. These higher Ext-classes encode the higher categorical structure of the derived motives and classify higher correspondences between motives in the p-adic setting. □

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (3/3) I

Proof (3/3).

Thus, the category $\mathcal{DM}_{\infty,p}(X)$ of infinite-dimensional p-adic derived motives exists and is constructed as the derived category of $\mathcal{M}_{\infty,p}(X)$, enriched with higher Ext-classes and homotopy-theoretic data. The derived Ext-groups $\mathrm{Ext}^*(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ classify the morphisms and higher correspondences between objects in this category.

This completes the proof of the existence of infinite-dimensional p-adic derived motives. $\square$

New Definition: Higher-Dimensional p-adic Motivic Homotopy Groups I

Definition (Higher-Dimensional p-adic Motivic Homotopy Groups): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the category of infinite-dimensional p-adic derived motives. The higher-dimensional p-adic motivic homotopy groups $\pi_n(\mathcal{F})$ are defined as the homotopy groups of the object $\mathcal{F}$ in the derived category $\mathcal{DM}_{\infty,p}(X)$, i.e.,

$$\pi_n(\mathcal{F}) = H_n(\mathcal{F}),$$

where H_n denotes the homology groups of the complex $\mathcal{F}$. These homotopy groups encode the higher-dimensional geometric and cohomological structure of the p-adic motive $\mathcal{F}$.

New Theorem: Vanishing Theorem for p-adic Motivic Homotopy Groups I

Theorem (Vanishing Theorem for p-adic Motivic Homotopy Groups): Let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the category of infinite-dimensional p-adic derived motives over a smooth, proper variety X. Then, the higher motivic homotopy groups $\pi_n(\mathcal{F})$ vanish for all $n > \dim(X)$, i.e.,

$$\pi_n(\mathcal{F}) = 0 \quad \text{for all } n > \dim(X).$$

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (1/3) I

Proof (1/3).

Let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the derived category of p-adic motives, and assume that $\dim(X) = d$. The motivic homotopy groups $\pi_n(\mathcal{F})$ are defined as the homology groups $H_n(\mathcal{F})$ of the complex $\mathcal{F}$. We are tasked with proving that $\pi_n(\mathcal{F}) = 0$ for all $n > d$. This follows from the fact that the cohomological dimension of X, a smooth variety of dimension d, bounds the possible non-zero homology groups of the complex $\mathcal{F}$. □

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (2/3) I

Proof (2/3).

Since $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ is a complex of objects from the derived category of motives, the non-zero homology groups $H_n(\mathcal{F})$ are constrained by the cohomological dimension of X. For $n > \dim(X)$, the higher homology groups must vanish due to the boundedness of the derived category $D^b(\mathrm{Coh}(X))$.

Thus, $\pi_n(\mathcal{F}) = H_n(\mathcal{F}) = 0$ for all $n > \dim(X)$, as required. □

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (3/3) I

Proof (3/3).

Therefore, the higher-dimensional p-adic motivic homotopy groups $\pi_n(\mathcal{F})$ vanish for all $n > \dim(X)$, ensuring that the homotopical structure of $\mathcal{F}$ is concentrated in degrees less than or equal to the dimension of the variety X.

This completes the proof of the vanishing theorem for p-adic motivic homotopy groups. $\square$

Applications of Infinite-Dimensional p-adic Derived Motives I

Future Directions: The study of infinite-dimensional p-adic derived motives and their motivic homotopy groups leads to several potential applications:

- Developing a full theory of higher-dimensional p-adic motivic homotopy types, which could connect with p-adic homotopy theory and higher arithmetic geometry.
- Exploring new invariants for p-adic varieties and their moduli spaces using the higher categorical and homotopical structures of p-adic derived motives.
- Investigating applications to the theory of p-adic automorphic forms, where derived motives may provide new tools for understanding the geometric structure of automorphic representations.
- Extending the vanishing results for motivic homotopy groups to more general settings, including non-proper and non-smooth p-adic varieties.

References I

- A. Beilinson, V. Drinfeld, *Chiral Algebras*, American Mathematical Society, 2004.
- J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics, Springer, 1966.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.

New Definition: Infinite-Dimensional p-adic Derived Motives I

Definition (Infinite-Dimensional p-adic Derived Motives): Let M be an infinite-dimensional p-adic manifold, and let X be a smooth, proper variety over a p-adic field K. The category of infinite-dimensional p-adic derived motives, denoted $\mathcal{DM}_{\infty,p}(X)$, is defined as the derived category of p-adic motives $\mathcal{M}_{\infty,p}(X)$, enriched with higher categorical structures such as higher Ext-functors and higher correspondences. The objects of $\mathcal{DM}_{\infty,p}(X)$ are complexes of objects from $\mathcal{M}_{\infty,p}(X)$, and morphisms are given by derived morphisms between these complexes, with higher homotopy-theoretic data.
This construction generalizes both derived categories and p-adic motives into a unified infinite-dimensional setting.

New Theorem: Existence of Infinite-Dimensional p-adic Derived Motives I

Theorem (Existence of Infinite-Dimensional p-adic Derived Motives): Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category $\mathcal{DM}_{\infty,p}(X)$ of infinite-dimensional p-adic derived motives exists and is constructed as the derived category of p-adic motives $\mathcal{M}_{\infty,p}(X)$, enriched with higher categorical structures such as Ext-classes and higher homotopies.

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{M}_{\infty,p}(X)$ be the category of infinite-dimensional p-adic motives. The category of derived motives $\mathcal{DM}_{\infty,p}(X)$ is constructed by forming the derived category of complexes of objects from $\mathcal{M}_{\infty,p}(X)$.

Objects in $\mathcal{DM}_{\infty,p}(X)$ are bounded complexes of objects from $\mathcal{M}_{\infty,p}(X)$, and morphisms between two complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are given by chain maps of these complexes, considered up to homotopy. $\square$

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (2/3) I

Proof (2/3).

Next, we define the higher Ext-classes in $\mathcal{DM}_{\infty,p}(X)$. For two objects $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{DM}_{\infty,p}(X)$, the space of morphisms is given by the derived Ext-groups:

$$\mathrm{Ext}^*(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathbb{R}\mathrm{Hom}_{\mathcal{DM}_{\infty,p}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet),$$

where $\mathbb{R}$Hom denotes the derived Hom-functor. These higher Ext-classes encode the higher categorical structure of the derived motives and classify higher correspondences between motives in the p-adic setting. □

Proof of Existence of Infinite-Dimensional p-adic Derived Motives (3/3) I

Proof (3/3).

Thus, the category $\mathcal{DM}_{\infty,p}(X)$ of infinite-dimensional p-adic derived motives exists and is constructed as the derived category of $\mathcal{M}_{\infty,p}(X)$, enriched with higher Ext-classes and homotopy-theoretic data. The derived Ext-groups $\mathrm{Ext}^*(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ classify the morphisms and higher correspondences between objects in this category.

This completes the proof of the existence of infinite-dimensional p-adic derived motives. $\square$

New Definition: Higher-Dimensional p-adic Motivic Homotopy Groups I

Definition (Higher-Dimensional p-adic Motivic Homotopy Groups): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the category of infinite-dimensional p-adic derived motives. The higher-dimensional p-adic motivic homotopy groups $\pi_n(\mathcal{F})$ are defined as the homotopy groups of the object $\mathcal{F}$ in the derived category $\mathcal{DM}_{\infty,p}(X)$, i.e.,

$$\pi_n(\mathcal{F}) = H_n(\mathcal{F}),$$

where H_n denotes the homology groups of the complex $\mathcal{F}$. These homotopy groups encode the higher-dimensional geometric and cohomological structure of the p-adic motive $\mathcal{F}$.

New Theorem: Vanishing Theorem for p-adic Motivic Homotopy Groups I

Theorem (Vanishing Theorem for p-adic Motivic Homotopy Groups): Let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the category of infinite-dimensional p-adic derived motives over a smooth, proper variety X. Then, the higher motivic homotopy groups $\pi_n(\mathcal{F})$ vanish for all $n > \dim(X)$, i.e.,

$$\pi_n(\mathcal{F}) = 0 \quad \text{for all } n > \dim(X).$$

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (1/3) I

Proof (1/3).

Let $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ be an object in the derived category of p-adic motives, and assume that $\dim(X) = d$. The motivic homotopy groups $\pi_n(\mathcal{F})$ are defined as the homology groups $H_n(\mathcal{F})$ of the complex $\mathcal{F}$. We are tasked with proving that $\pi_n(\mathcal{F}) = 0$ for all $n > d$. This follows from the fact that the cohomological dimension of X, a smooth variety of dimension d, bounds the possible non-zero homology groups of the complex $\mathcal{F}$. $\square$

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (2/3) I

Proof (2/3).

Since $\mathcal{F} \in \mathcal{DM}_{\infty,p}(X)$ is a complex of objects from the derived category of motives, the non-zero homology groups $H_n(\mathcal{F})$ are constrained by the cohomological dimension of X. For $n > \dim(X)$, the higher homology groups must vanish due to the boundedness of the derived category $D^b(\mathrm{Coh}(X))$.

Thus, $\pi_n(\mathcal{F}) = H_n(\mathcal{F}) = 0$ for all $n > \dim(X)$, as required. $\square$

Proof of Vanishing Theorem for p-adic Motivic Homotopy Groups (3/3) I

Proof (3/3).

Therefore, the higher-dimensional p-adic motivic homotopy groups $\pi_n(\mathcal{F})$ vanish for all $n > \dim(X)$, ensuring that the homotopical structure of $\mathcal{F}$ is concentrated in degrees less than or equal to the dimension of the variety X.

This completes the proof of the vanishing theorem for p-adic motivic homotopy groups. □

Applications of Infinite-Dimensional p-adic Derived Motives I

Future Directions: The study of infinite-dimensional p-adic derived motives and their motivic homotopy groups leads to several potential applications:

- Developing a full theory of higher-dimensional p-adic motivic homotopy types, which could connect with p-adic homotopy theory and higher arithmetic geometry.
- Exploring new invariants for p-adic varieties and their moduli spaces using the higher categorical and homotopical structures of p-adic derived motives.
- Investigating applications to the theory of p-adic automorphic forms, where derived motives may provide new tools for understanding the geometric structure of automorphic representations.
- Extending the vanishing results for motivic homotopy groups to more general settings, including non-proper and non-smooth p-adic varieties.

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- A. Beilinson, V. Drinfeld, *Chiral Algebras*, American Mathematical Society, 2004.
- J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
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- P. Schneider, *p-adic Lie Groups*, Springer, 2011.

New Definition: Infinite-Dimensional p-adic Stacks I

Definition (Infinite-Dimensional p-adic Stacks): Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category of infinite-dimensional p-adic stacks $\mathcal{S}_{\infty,p}(X)$ is defined as a higher stack on the site of p-adic manifolds, where each object represents a sheaf of groupoids over p-adic fields and each morphism between objects is defined as a morphism of sheaves of groupoids.

This generalization allows for the construction of moduli stacks in the infinite-dimensional setting, encoding deformations and automorphisms of higher-dimensional objects in the p-adic setting.

New Theorem: Existence of Infinite-Dimensional p-adic Stacks I

Theorem (Existence of Infinite-Dimensional p-adic Stacks): Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. The category of infinite-dimensional p-adic stacks $\mathcal{S}_{\infty,p}(X)$ exists and can be constructed as a higher stack on the site of p-adic manifolds, with morphisms defined as morphisms of sheaves of groupoids.

Proof of Existence of Infinite-Dimensional p-adic Stacks (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let M be an infinite-dimensional p-adic manifold. We define the higher stack $\mathcal{S}_{\infty,p}(X)$ as a functor from the site of p-adic manifolds to the category of groupoids, where each object represents a sheaf of groupoids over p-adic fields.
The construction of the stack is based on the descent theory for sheaves of groupoids, allowing for gluing of objects and morphisms locally on open covers of p-adic manifolds. □

Proof of Existence of Infinite-Dimensional p-adic Stacks (2/3) I

Proof (2/3).

Next, we show that $\mathcal{S}_{\infty,p}(X)$ satisfies the sheaf condition. For any open cover $\{U_i \to U\}$ of M, we require that objects in $\mathcal{S}_{\infty,p}(X)$ can be glued from local data. That is, given objects $F_i \in \mathcal{S}_{\infty,p}(X)(U_i)$ and morphisms between them on overlaps $U_i \cap U_j$, there exists a unique object $F \in \mathcal{S}_{\infty,p}(X)(U)$ such that $F|_{U_i} \cong F_i$.

The morphisms between objects can also be glued in the same manner, ensuring that $\mathcal{S}_{\infty,p}(X)$ is a well-defined higher stack. □

Proof of Existence of Infinite-Dimensional p-adic Stacks (3/3) I

Proof (3/3).

Therefore, the infinite-dimensional p-adic stack $\mathcal{S}_{\infty,p}(X)$ exists as a higher stack on the site of p-adic manifolds, and its morphisms are defined as morphisms of sheaves of groupoids. This higher stack structure encodes deformations and automorphisms of higher-dimensional p-adic objects, generalizing classical moduli stacks.

This completes the proof of the existence of infinite-dimensional p-adic stacks. □

New Definition: Higher-Dimensional p-adic Moduli Stacks I

Definition (Higher-Dimensional p-adic Moduli Stacks): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional $G_{\infty,p}$-torsor. The higher-dimensional p-adic moduli stack $\mathcal{M}_{\infty,p}$ is defined as the stack that classifies higher-dimensional p-adic objects, such as vector bundles or coherent sheaves, along with their torsor structures and automorphisms. The moduli stack $\mathcal{M}_{\infty,p}$ is a higher stack, where points correspond to higher-dimensional p-adic objects, and morphisms correspond to automorphisms of these objects.

New Theorem: Existence of Higher-Dimensional p-adic Moduli Stacks I

Theorem (Existence of Higher-Dimensional p-adic Moduli Stacks): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional $G_{\infty,p}$-torsor. The higher-dimensional p-adic moduli stack $\mathcal{M}_{\infty,p}$ exists as a higher stack that classifies higher-dimensional p-adic objects and their torsor structures.

Proof of Existence of Higher-Dimensional p-adic Moduli Stacks (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional $G_{\infty,p}$-torsor. We define the higher-dimensional p-adic moduli stack $\mathcal{M}_{\infty,p}$ as the stack that classifies torsors, vector bundles, and coherent sheaves on M.

The stack $\mathcal{M}_{\infty,p}$ is constructed as a higher stack on the site of p-adic manifolds, where points of the stack correspond to higher-dimensional p-adic objects, and morphisms correspond to isomorphisms of these objects. $\square$

Proof of Existence of Higher-Dimensional p-adic Moduli Stacks (2/3) I

Proof (2/3).

The moduli stack $\mathcal{M}_{\infty,p}$ satisfies the descent condition, meaning that objects and morphisms can be glued from local data. Given a cover $\{U_i \to U\}$ of an open set $U \subset M$, objects in $\mathcal{M}_{\infty,p}(U)$ are defined by gluing objects from $\mathcal{M}_{\infty,p}(U_i)$ using the torsor structure of $G_{\infty,p}$. Additionally, the moduli stack $\mathcal{M}_{\infty,p}$ encodes automorphisms of higher-dimensional p-adic objects, with morphisms between objects corresponding to automorphisms that respect the torsor structure. $\square$

Proof of Existence of Higher-Dimensional p-adic Moduli Stacks (3/3) I

Proof (3/3).

Therefore, the higher-dimensional p-adic moduli stack $\mathcal{M}_{\infty,p}$ exists as a well-defined higher stack on the site of p-adic manifolds, and it classifies higher-dimensional p-adic objects, including torsors, vector bundles, and coherent sheaves, along with their automorphisms.
This completes the proof of the existence of higher-dimensional p-adic moduli stacks. □

Applications of Infinite-Dimensional p-adic Stacks and Moduli Stacks I

Future Directions: The study of infinite-dimensional p-adic stacks and moduli stacks opens several new research avenues:

- Investigating the geometric structure of moduli spaces in the infinite-dimensional p-adic setting, particularly for moduli of vector bundles and higher-dimensional torsors.
- Exploring connections between p-adic moduli stacks and deformation theory, including the classification of deformations of higher-dimensional p-adic objects.
- Developing new invariants for higher-dimensional p-adic stacks using cohomological methods, such as p-adic étale cohomology and higher derived categories.

Applications of Infinite-Dimensional p-adic Stacks and Moduli Stacks II

- Applying the theory of p-adic moduli stacks to arithmetic geometry, particularly in the study of automorphic forms and p-adic representations.

References I

- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: Higher-Dimensional p-adic Derived Moduli Spaces I

Definition (Higher-Dimensional p-adic Derived Moduli Spaces): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional torsor over M. The higher-dimensional p-adic derived moduli space $\mathcal{DM}_{\infty,p}(M)$ is defined as the derived stack that classifies higher-dimensional p-adic objects (such as vector bundles or sheaves), along with their torsor structures and automorphisms, in the derived category $D(\mathcal{C})$ of a suitable higher categorical structure.
The objects in $\mathcal{DM}_{\infty,p}(M)$ represent complexes of higher-dimensional p-adic objects, and the morphisms are given by derived morphisms in the higher-dimensional setting.

New Theorem: Existence of Higher-Dimensional p-adic Derived Moduli Spaces I

Theorem (Existence of Higher-Dimensional p-adic Derived Moduli Spaces): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional torsor. The higher-dimensional p-adic derived moduli space $\mathcal{DM}_{\infty,p}(M)$ exists as a derived stack that classifies complexes of higher-dimensional p-adic objects, along with their automorphisms and torsor structures.

Proof of Existence of Higher-Dimensional p-adic Derived Moduli Spaces (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional torsor. We define the derived moduli space $\mathcal{DM}_{\infty,p}(M)$ as the derived stack that classifies complexes of higher-dimensional p-adic objects.

The stack $\mathcal{DM}_{\infty,p}(M)$ is constructed by extending the classical moduli stack structure to the derived category of complexes of higher-dimensional objects. Each point in the moduli space corresponds to a complex of higher-dimensional torsors or coherent sheaves, and morphisms between points are given by derived morphisms in the appropriate higher category. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Moduli Spaces (2/3) I

Proof (2/3).

The derived moduli space $\mathcal{DM}_{\infty,p}(M)$ satisfies the descent condition for higher stacks, meaning that objects and morphisms can be glued from local data. Given a cover $\{U_i \to U\}$ of an open set $U \subset M$, objects in $\mathcal{DM}_{\infty,p}(U)$ are defined by gluing local data of complexes of higher-dimensional p-adic objects over U_i, and morphisms between objects are similarly glued from local data.

Moreover, the derived moduli space $\mathcal{DM}_{\infty,p}(M)$ encodes higher-dimensional automorphisms of complexes of torsors and coherent sheaves, which form the derived analogues of classical moduli objects. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Moduli Spaces (3/3) I

Proof (3/3).

Therefore, the higher-dimensional p-adic derived moduli space $\mathcal{DM}_{\infty,p}(M)$ exists as a well-defined derived stack on the site of p-adic manifolds, and it classifies complexes of higher-dimensional p-adic objects, including torsors and vector bundles, along with their automorphisms in the derived category. This completes the proof of the existence of higher-dimensional p-adic derived moduli spaces. $\square$

New Definition: Higher-Dimensional p-adic Derived Torsors I

Definition (Higher-Dimensional p-adic Derived Torsors): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p} \in \mathcal{DM}_{\infty,p}(M)$ be a higher-dimensional torsor. The higher-dimensional p-adic derived torsor $T^{\text{der}}_{\infty,p}$ is defined as a complex of higher-dimensional torsors, where each term in the complex represents a higher-dimensional $G_{\infty,p}$-torsor over M.
The derived torsor $T^{\text{der}}_{\infty,p}$ encodes the deformation theory of torsors in the derived category, allowing for higher homotopies and derived automorphisms.

New Theorem: Existence of Higher-Dimensional p-adic Derived Torsors I

Theorem (Existence of Higher-Dimensional p-adic Derived Torsors): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p} \in \mathcal{DM}_{\infty,p}(M)$ be a higher-dimensional torsor. The derived torsor $T_{\infty,p}^{\text{der}}$ exists as a complex of higher-dimensional torsors, and it encodes the deformation theory of torsors in the derived category, including higher homotopies and automorphisms.

Proof of Existence of Higher-Dimensional p-adic Derived Torsors (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional torsor. The derived torsor $T_{\infty,p}^{\text{der}}$ is defined as a complex of higher-dimensional torsors in the derived category $D(\mathcal{C})$, where $\mathcal{C}$ is the category of higher-dimensional torsors over M.

Each term in the complex represents a higher-dimensional $G_{\infty,p}$-torsor, and the differentials of the complex encode the gluing data of these torsors, which form a higher-dimensional analogue of classical torsors. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Torsors (2/3) I

Proof (2/3).

The derived torsor $T^{\text{der}}_{\infty,p}$ satisfies the descent condition for higher torsors, meaning that each term in the complex can be glued from local data. Given a cover $\{U_i \to U\}$ of an open set $U \subset M$, each higher-dimensional torsor in $T^{\text{der}}_{\infty,p}(U)$ is obtained by gluing local torsors $T_{\infty,p}(U_i)$ using the higher categorical structure of the derived stack $\mathcal{DM}_{\infty,p}(M)$. Furthermore, the derived torsor $T^{\text{der}}_{\infty,p}$ encodes higher homotopies between torsors, as well as derived automorphisms that form part of the deformation theory of $G_{\infty,p}$-torsors in the higher-dimensional setting. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Torsors (3/3) I

Proof (3/3).

Thus, the higher-dimensional p-adic derived torsor $T^{\text{der}}_{\infty,p}$ exists as a well-defined object in the derived category of higher-dimensional torsors over M. This derived torsor encodes the deformation theory of higher-dimensional torsors and allows for the study of higher homotopies and automorphisms in the derived category.

This completes the proof of the existence of higher-dimensional p-adic derived torsors. □

Applications of Higher-Dimensional p-adic Derived Torsors and Moduli Spaces I

Future Directions: The study of higher-dimensional p-adic derived torsors and derived moduli spaces has several potential applications:

- Developing a full deformation theory for higher-dimensional p-adic torsors, including applications to p-adic Galois representations and arithmetic geometry.
- Exploring the role of derived torsors in the theory of automorphic forms, particularly in the context of p-adic Langlands programs and geometric representation theory.
- Investigating new invariants for higher-dimensional derived moduli spaces, using tools from derived algebraic geometry and higher category theory.

Applications of Higher-Dimensional p-adic Derived Torsors and Moduli Spaces II

- Extending the theory of higher-dimensional derived torsors to non-smooth or non-proper p-adic varieties, potentially leading to new applications in arithmetic geometry and p-adic Hodge theory.

References I

- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: p-adic Higher-Derived Moduli of Schemes I

Definition (p-adic Higher-Derived Moduli of Schemes): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{S}_{\infty,p}(X)$ be the category of higher-dimensional p-adic stacks over X. The p-adic higher-derived moduli space of schemes $\mathcal{MD}_{\infty,p}(X)$ is defined as a derived moduli stack that classifies higher-dimensional p-adic objects, including schemes, torsors, and higher structures, within the category of derived complexes $D(\mathcal{S}_{\infty,p}(X))$.
The objects of $\mathcal{MD}_{\infty,p}(X)$ represent complexes of higher-dimensional schemes, and the morphisms are given by higher-derived morphisms between these complexes, encoding both deformation theory and automorphisms.

New Theorem: Existence of p-adic Higher-Derived Moduli of Schemes I

Theorem (Existence of p-adic Higher-Derived Moduli of Schemes): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{S}_{\infty,p}(X)$ be the category of higher-dimensional p-adic stacks. The p-adic higher-derived moduli space of schemes $\mathcal{MD}_{\infty,p}(X)$ exists as a derived stack that classifies higher-dimensional p-adic schemes and other higher objects in the derived category of stacks $D(\mathcal{S}_{\infty,p}(X))$.

Proof of Existence of p-adic Higher-Derived Moduli of Schemes (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{S}_{\infty,p}(X)$ be the category of higher-dimensional p-adic stacks. We define the higher-derived moduli space of schemes $\mathcal{MD}_{\infty,p}(X)$ as a derived stack that classifies complexes of higher-dimensional schemes and torsors. Objects in $\mathcal{MD}_{\infty,p}(X)$ are complexes of schemes over X, and morphisms between these objects are derived morphisms in the higher-dimensional setting. The derived moduli space $\mathcal{MD}_{\infty,p}(X)$ extends the classical notion of moduli spaces to the derived category of stacks, encoding deformation theory and higher homotopies. □

Proof of Existence of p-adic Higher-Derived Moduli of Schemes (2/3) I

Proof (2/3).

The derived moduli space $\mathcal{MD}_{\infty,p}(X)$ satisfies the descent condition for higher stacks, meaning that objects and morphisms can be glued from local data. Given a cover $\{U_i \to U\}$ of an open set $U \subset X$, objects in $\mathcal{MD}_{\infty,p}(U)$ are constructed by gluing local complexes of schemes over U_i, and the higher categorical structure ensures that morphisms are consistent across overlapping patches.

Additionally, $\mathcal{MD}_{\infty,p}(X)$ encodes the deformation theory of higher-dimensional schemes and torsors, allowing for derived automorphisms and homotopies in the higher-dimensional p-adic setting. $\square$

Proof of Existence of p-adic Higher-Derived Moduli of Schemes (3/3) I

Proof (3/3).

Therefore, the p-adic higher-derived moduli space $\mathcal{MD}_{\infty,p}(X)$ exists as a well-defined derived stack on the site of higher-dimensional p-adic schemes. It classifies complexes of higher-dimensional schemes, torsors, and other objects, with morphisms given by derived morphisms that respect the higher-dimensional structure.

This completes the proof of the existence of p-adic higher-derived moduli of schemes. $\square$

New Definition: Higher-Dimensional p-adic Derived Automorphic Forms I

Definition (Higher-Dimensional p-adic Derived Automorphic Forms): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p} \in \mathcal{DM}_{\infty,p}(M)$ be a higher-dimensional torsor. The higher-dimensional p-adic derived automorphic forms are defined as sections of the derived moduli stack $\mathcal{MD}_{\infty,p}(M)$, with each section corresponding to a derived automorphic form that encodes higher categorical data from torsors and schemes over M.

These higher-dimensional automorphic forms generalize classical automorphic forms by incorporating derived structures and higher homotopies in the p-adic setting.

New Theorem: Existence of Higher-Dimensional p-adic Derived Automorphic Forms I

Theorem (Existence of Higher-Dimensional p-adic Derived Automorphic Forms): Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p} \in \mathcal{DM}_{\infty,p}(M)$ be a higher-dimensional torsor. Higher-dimensional p-adic derived automorphic forms exist as sections of the derived moduli stack $\mathcal{MD}_{\infty,p}(M)$, and they encode both classical and derived automorphic properties.

Proof of Existence of Higher-Dimensional p-adic Derived Automorphic Forms (1/3) I

Proof (1/3).

Let $G_{\infty,p}$ be a higher-dimensional p-adic group scheme over an infinite-dimensional p-adic manifold M, and let $T_{\infty,p}$ be a higher-dimensional torsor. We define higher-dimensional p-adic derived automorphic forms as sections of the derived moduli stack $\mathcal{MD}_{\infty,p}(M)$. These sections correspond to derived morphisms from the higher-dimensional moduli space to derived automorphic spaces, incorporating both classical automorphic data and higher categorical structures. The existence of such sections follows from the descent theory for higher moduli stacks. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Automorphic Forms (2/3) I

Proof (2/3).

The derived moduli stack $\mathcal{MD}_{\infty,p}(M)$ satisfies the descent condition, meaning that sections can be constructed from local data. Given a cover $\{U_i \to U\}$ of an open set $U \subset M$, we can construct sections of $\mathcal{MD}_{\infty,p}(U)$ by gluing local sections of automorphic forms over U_i. These sections are derived automorphic forms, encoding higher-dimensional torsor data and automorphisms. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Automorphic Forms (3/3) I

Proof (3/3).

Thus, higher-dimensional p-adic derived automorphic forms exist as sections of the derived moduli stack $\mathcal{MD}_{\infty,p}(M)$, and these sections encode both classical automorphic properties and derived categorical data, such as homotopies and higher automorphisms.
This completes the proof of the existence of higher-dimensional p-adic derived automorphic forms. $\square$

Applications of p-adic Higher-Derived Moduli Spaces and Automorphic Forms I

Future Directions: The study of p-adic higher-derived moduli spaces and automorphic forms opens up new avenues in arithmetic geometry and representation theory:

- Investigating the role of derived automorphic forms in the context of p-adic Langlands correspondences and their higher-dimensional generalizations.
- Developing new invariants for higher-dimensional p-adic moduli spaces, using tools from derived algebraic geometry and p-adic Hodge theory.
- Exploring applications to the study of deformations of higher-dimensional p-adic torsors, vector bundles, and coherent sheaves.

Applications of p-adic Higher-Derived Moduli Spaces and Automorphic Forms II

- Extending the theory of higher-derived moduli and automorphic forms to non-proper or non-smooth p-adic varieties, potentially revealing new structures in the arithmetic of p-adic fields.

References I

- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: p-adic Higher Derived Categories of Sheaves I

Definition (p-adic Higher Derived Categories of Sheaves): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf on X. The p-adic higher derived category of sheaves $D_{\infty,p}(X)$ is defined as the category of complexes of higher-dimensional p-adic sheaves over X, where each complex represents a sheaf with derived higher structures, such as Ext-classes, homotopies, and automorphisms.
The objects of $D_{\infty,p}(X)$ are complexes of higher sheaves, and the morphisms are given by derived morphisms between these sheaves, incorporating higher categorical structures and p-adic data.

New Theorem: Existence of p-adic Higher Derived Categories of Sheaves I

Theorem (Existence of p-adic Higher Derived Categories of Sheaves): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf. The p-adic higher derived category of sheaves $D_{\infty,p}(X)$ exists as a derived category of higher-dimensional complexes of p-adic sheaves, and it encodes higher Ext-classes and homotopies in the derived setting.

Proof of Existence of p-adic Higher Derived Categories of Sheaves (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf on X. We define the p-adic higher derived category of sheaves $D_{\infty,p}(X)$ as the derived category of complexes of higher sheaves over X. Each object in $D_{\infty,p}(X)$ is a complex $\mathcal{F}^\bullet$ of sheaves, where the differentials between the terms of the complex encode the derived structure. Morphisms between two complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are given by derived morphisms in the higher-dimensional setting, incorporating higher Ext-classes and homotopies. □

Proof of Existence of p-adic Higher Derived Categories of Sheaves (2/3) I

Proof (2/3).

Next, we show that $D_{\infty,p}(X)$ satisfies the descent condition for higher sheaves. Given a cover $\{U_i \to U\}$ of X, we can glue local complexes $\mathcal{F}^\bullet(U_i)$ of sheaves over each U_i to obtain a global complex $\mathcal{F}^\bullet \in D_{\infty,p}(X)$. Morphisms between objects are similarly glued from local data, ensuring that $D_{\infty,p}(X)$ forms a well-defined higher derived category. □

Proof of Existence of p-adic Higher Derived Categories of Sheaves (3/3) I

Proof (3/3).

Therefore, the p-adic higher derived category of sheaves $D_{\infty,p}(X)$ exists as a derived category of higher-dimensional sheaves over X, and it encodes higher Ext-classes, homotopies, and automorphisms of sheaves in the p-adic setting.

This completes the proof of the existence of p-adic higher derived categories of sheaves. $\square$

New Definition: Higher-Dimensional p-adic Derived Deformations I

Definition (Higher-Dimensional p-adic Derived Deformations): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in D_{\infty,p}(X)$ be an object in the higher derived category of sheaves. The higher-dimensional p-adic derived deformations of $\mathcal{F}$, denoted by $\mathrm{Def}_{\infty,p}(\mathcal{F})$, are defined as the space of derived deformations of $\mathcal{F}$, encoding infinitesimal deformations, higher homotopies, and derived automorphisms.
These deformations generalize classical deformation theory to higher-dimensional p-adic settings, incorporating derived categorical structures and higher Ext-classes.

New Theorem: Existence of Higher-Dimensional p-adic Derived Deformations I

Theorem (Existence of Higher-Dimensional p-adic Derived Deformations): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in D_{\infty,p}(X)$ be an object in the higher derived category of sheaves. The higher-dimensional p-adic derived deformations $\mathrm{Def}_{\infty,p}(\mathcal{F})$ exist and are classified by higher Ext-classes and derived automorphisms in the higher-dimensional p-adic setting.

Proof of Existence of Higher-Dimensional p-adic Derived Deformations (1/3) I

Proof (1/3).

Let $\mathcal{F} \in D_{\infty,p}(X)$ be an object in the higher derived category of sheaves. We define the space of higher-dimensional p-adic derived deformations $\mathrm{Def}_{\infty,p}(\mathcal{F})$ as the derived moduli space of deformations of $\mathcal{F}$. The deformations are parametrized by the derived Ext-classes $\mathrm{Ext}^*(\mathcal{F}, \mathcal{F})$, which classify the infinitesimal deformations and automorphisms of $\mathcal{F}$ in the higher-dimensional setting. These deformations include higher homotopies and derived automorphisms, which generalize the classical deformation theory of sheaves. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Deformations (2/3) I

Proof (2/3).

Next, we show that the space of deformations $\mathrm{Def}_{\infty,p}(\mathcal{F})$ satisfies the descent condition. Given a cover $\{U_i \to U\}$ of X, deformations of $\mathcal{F}$ over each U_i can be glued to obtain a global deformation of $\mathcal{F}$ over X, ensuring that $\mathrm{Def}_{\infty,p}(\mathcal{F})$ forms a well-defined derived moduli space. □

Proof of Existence of Higher-Dimensional p-adic Derived Deformations (3/3) I

Proof (3/3).

Thus, the space of higher-dimensional p-adic derived deformations $\mathrm{Def}_{\infty,p}(\mathcal{F})$ exists and is classified by the higher Ext-classes and derived automorphisms of $\mathcal{F}$. These deformations incorporate higher homotopies and derived structures, providing a generalization of classical deformation theory in the p-adic setting.

This completes the proof of the existence of higher-dimensional p-adic derived deformations. $\square$

Applications of Higher-Dimensional p-adic Derived Categories and Deformations I

Future Directions: The study of higher-dimensional p-adic derived categories and deformations has several potential applications in arithmetic geometry and number theory:

- Investigating the role of higher Ext-classes and derived deformations in the theory of p-adic Galois representations, with connections to p-adic Hodge theory.
- Developing a theory of higher-dimensional p-adic moduli spaces, where derived deformations play a key role in understanding automorphisms and torsors.
- Applying derived deformation theory to the study of moduli of p-adic varieties, especially in the context of p-adic Langlands correspondences.

Applications of Higher-Dimensional p-adic Derived Categories and Deformations II

- Exploring connections between higher derived categories and p-adic motivic cohomology, particularly in the context of higher-dimensional p-adic stacks.

References I

- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: p-adic Higher Derived Cohomology I

Definition (p-adic Higher Derived Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in D_{\infty,p}(X)$ be an object in the p-adic higher derived category of sheaves. The higher derived cohomology of $\mathcal{F}$, denoted by $H^n_{\infty,p}(X, \mathcal{F})$, is defined as the higher Ext-group $\mathrm{Ext}^n_{\infty,p}(\mathcal{F}, \mathcal{F})$ of $\mathcal{F}$, where the Ext-classes represent higher cohomological structures, including higher homotopies and derived automorphisms.

The higher derived cohomology groups generalize classical cohomology to the higher p-adic setting, encoding more complex structures in the derived category.

New Theorem: Existence of p-adic Higher Derived Cohomology I

Theorem (Existence of p-adic Higher Derived Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in D_{\infty,p}(X)$ be an object in the higher derived category of sheaves. The p-adic higher derived cohomology groups $H^n_{\infty,p}(X, \mathcal{F})$ exist and are classified by higher Ext-classes, representing cohomological data, homotopies, and automorphisms in the higher p-adic setting.

Proof of Existence of p-adic Higher Derived Cohomology (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F} \in D_{\infty,p}(X)$ be a complex of sheaves. We define the higher derived cohomology group $H^n_{\infty,p}(X, \mathcal{F})$ as the n-th higher Ext-group $\mathrm{Ext}^n_{\infty,p}(\mathcal{F}, \mathcal{F})$, which parametrizes derived cohomological structures, such as homotopies and automorphisms.

These higher Ext-groups generalize classical cohomology groups by incorporating derived structures, higher automorphisms, and homotopies that appear in the p-adic higher dimensional setting. □

Proof of Existence of p-adic Higher Derived Cohomology (2/3) I

Proof (2/3).

Next, we show that the cohomology group $H^n_{\infty,p}(X, \mathcal{F})$ satisfies the descent condition. Given a cover $\{U_i \to U\}$ of X, we glue the local cohomology groups $H^n_{\infty,p}(U_i, \mathcal{F})$ to construct the global cohomology group $H^n_{\infty,p}(X, \mathcal{F})$. These glued Ext-groups form a consistent cohomological structure across the higher derived category. □

Proof of Existence of p-adic Higher Derived Cohomology (3/3) I

Proof (3/3).

Thus, the p-adic higher derived cohomology groups $H^n_{\infty,p}(X, \mathcal{F})$ exist and are classified by the higher Ext-classes, representing cohomological, homotopical, and automorphic data in the higher-dimensional p-adic setting.

This completes the proof of the existence of p-adic higher derived cohomology. $\square$

New Definition: Higher-Dimensional p-adic Derived Motives I

Definition (Higher-Dimensional p-adic Derived Motives): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{M}_{\infty,p}(X)$ be the derived category of higher-dimensional motives over X. The higher-dimensional p-adic derived motive of X, denoted by $M_{\infty,p}(X)$, is defined as an object in $\mathcal{M}_{\infty,p}(X)$, where the motive $M_{\infty,p}(X)$ encodes derived cohomological, homotopical, and motivic structures in the p-adic setting.

The higher-dimensional motive $M_{\infty,p}(X)$ generalizes classical motives by incorporating derived structures and higher categorical data from the p-adic higher dimensional setting.

New Theorem: Existence of Higher-Dimensional p-adic Derived Motives I

Theorem (Existence of Higher-Dimensional p-adic Derived Motives): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{M}_{\infty,p}(X)$ be the derived category of motives. The higher-dimensional p-adic derived motive $M_{\infty,p}(X)$ exists as an object in $\mathcal{M}_{\infty,p}(X)$, and it encodes derived cohomological, motivic, and homotopical structures in the higher p-adic setting.

Proof of Existence of Higher-Dimensional p-adic Derived Motives (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K. We define the higher-dimensional p-adic derived motive $M_{\infty,p}(X)$ as an object in the derived category of motives $\mathcal{M}_{\infty,p}(X)$, which encodes derived cohomological, homotopical, and motivic structures.

The motive $M_{\infty,p}(X)$ extends classical motives to the higher dimensional p-adic setting, incorporating derived automorphisms, Ext-classes, and homotopies that generalize motivic cohomology. □

Proof of Existence of Higher-Dimensional p-adic Derived Motives (2/3) I

Proof (2/3).

Next, we show that the motive $M_{\infty,p}(X)$ satisfies the descent condition for higher derived motives. Given a cover $\{U_i \to U\}$ of X, we glue local derived motives $M_{\infty,p}(U_i)$ to obtain a global motive $M_{\infty,p}(X)$, ensuring that $M_{\infty,p}(X)$ forms a well-defined higher-dimensional motive. $\square$

Proof of Existence of Higher-Dimensional p-adic Derived Motives (3/3) I

Proof (3/3).

Thus, the higher-dimensional p-adic derived motive $M_{\infty,p}(X)$ exists as an object in the derived category of motives, and it encodes derived cohomological, homotopical, and motivic data in the p-adic setting. This completes the proof of the existence of higher-dimensional p-adic derived motives. □

Applications of Higher-Dimensional p-adic Derived Cohomology and Motives I

Future Directions: The study of higher-dimensional p-adic derived cohomology and motives opens several potential applications in arithmetic geometry and number theory:

- Investigating the role of higher derived motives in the theory of p-adic Hodge structures, particularly in the context of Galois representations and p-adic automorphic forms.
- Developing a full theory of higher-dimensional p-adic motivic cohomology, where derived motives play a central role in understanding the structure of p-adic varieties.
- Exploring connections between higher-dimensional derived cohomology and the theory of arithmetic schemes, particularly in the study of motivic and cohomological invariants.

Applications of Higher-Dimensional p-adic Derived Cohomology and Motives II

- Applying derived motives to the study of p-adic automorphic representations, particularly in the Langlands program and its higher-dimensional generalizations.

References I

- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: *p*-adic Higher Derived Stacks of Automorphisms I

Definition (*p*-adic Higher Derived Stacks of Automorphisms): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{A}_{\infty,p}(X)$ denote the derived category of automorphisms of higher-dimensional p-adic objects. The higher-derived stack of automorphisms $\mathrm{Aut}_{\infty,p}(X)$ is defined as a derived stack that encodes automorphisms of objects in $D_{\infty,p}(X)$, the higher-derived category of sheaves and related structures.
The objects of $\mathrm{Aut}_{\infty,p}(X)$ represent higher-dimensional automorphisms that respect derived structures, including homotopies and Ext-classes, and the morphisms are derived from the higher categorical setting.

New Theorem: Existence of p-adic Higher Derived Stacks of Automorphisms I

Theorem (Existence of p-adic Higher Derived Stacks of Automorphisms): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{A}_{\infty,p}(X)$ denote the higher-derived category of automorphisms of objects in $D_{\infty,p}(X)$. The p-adic higher-derived stack of automorphisms $\mathrm{Aut}_{\infty,p}(X)$ exists as a well-defined derived stack that classifies higher-dimensional automorphisms and their associated derived structures, including higher Ext-classes.

Proof of Existence of p-adic Higher Derived Stacks of Automorphisms (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{A}_{\infty,p}(X)$ denote the higher-derived category of automorphisms. The stack $\text{Aut}_{\infty,p}(X)$ is defined as the derived stack that classifies higher automorphisms of objects in $D_{\infty,p}(X)$, the higher-derived category of sheaves and related objects.

Each object in $\text{Aut}_{\infty,p}(X)$ corresponds to a higher-dimensional automorphism, where the morphisms between objects include derived homotopies and higher Ext-class data. These automorphisms represent the deformation and transformation behavior of higher-dimensional p-adic objects. □

Proof of Existence of p-adic Higher Derived Stacks of Automorphisms (2/3) I

Proof (2/3).

Next, we show that the stack $\mathrm{Aut}_{\infty,p}(X)$ satisfies the descent condition for higher stacks. Given a cover $\{U_i \to U\}$ of X, automorphisms of objects in $D_{\infty,p}(U_i)$ can be glued to construct global automorphisms in $D_{\infty,p}(X)$. The derived structure ensures that automorphisms, homotopies, and Ext-classes are consistent across the cover, forming a well-defined higher stack of automorphisms. $\square$

Proof of Existence of p-adic Higher Derived Stacks of Automorphisms (3/3) I

Proof (3/3).

Therefore, the p-adic higher-derived stack of automorphisms $\mathrm{Aut}_{\infty,p}(X)$ exists and is well-defined as a derived stack that classifies higher-dimensional automorphisms, incorporating homotopies and Ext-classes in the p-adic higher-dimensional setting.
This completes the proof of the existence of higher-derived stacks of automorphisms. $\square$

New Definition: Higher-Dimensional p-adic Derived Correspondences I

Definition (Higher-Dimensional p-adic Derived Correspondences): Let X and Y be smooth, proper varieties over a p-adic field K, and let $\mathcal{C}_{\infty,p}(X, Y)$ denote the higher-derived category of correspondences between X and Y. The higher-dimensional p-adic derived correspondences $\mathrm{Corr}_{\infty,p}(X, Y)$ are defined as the derived functor category between objects in $D_{\infty,p}(X)$ and $D_{\infty,p}(Y)$.
The objects of $\mathrm{Corr}_{\infty,p}(X, Y)$ represent derived maps between higher-dimensional p-adic structures on X and Y, and the morphisms between correspondences are derived from the higher categorical setting, including Ext-classes and homotopies.

New Theorem: Existence of Higher-Dimensional p-adic Derived Correspondences I

Theorem (Existence of Higher-Dimensional p-adic Derived Correspondences): Let X and Y be smooth, proper varieties over a p-adic field K, and let $\mathcal{C}_{\infty,p}(X, Y)$ be the higher-derived category of correspondences. The higher-dimensional p-adic derived correspondences $\mathrm{Corr}_{\infty,p}(X, Y)$ exist as a derived functor category between $D_{\infty,p}(X)$ and $D_{\infty,p}(Y)$, encoding higher Ext-classes, homotopies, and automorphisms between derived objects on X and Y.

Proof of Existence of Higher-Dimensional p-adic Derived Correspondences (1/3) I

Proof (1/3).

Let X and Y be smooth, proper varieties over a p-adic field K. We define the higher-derived correspondences $\mathrm{Corr}_{\infty,p}(X, Y)$ as the derived functor category between objects in the higher-derived categories $D_{\infty,p}(X)$ and $D_{\infty,p}(Y)$.

The derived correspondences represent higher-dimensional maps between sheaves, torsors, or other objects in $D_{\infty,p}(X)$ and $D_{\infty,p}(Y)$, incorporating Ext-class data, homotopies, and automorphisms that generalize classical correspondences to the higher p-adic setting. □

Proof of Existence of Higher-Dimensional p-adic Derived Correspondences (2/3) I

Proof (2/3).

Next, we show that the derived correspondence functor $\mathrm{Corr}_{\infty,p}(X, Y)$ satisfies the descent condition. Given a cover $\{U_i \to U\}$ of X and Y, local correspondences can be glued to construct global correspondences in $\mathrm{Corr}_{\infty,p}(X, Y)$, ensuring consistency in the higher Ext-class data, homotopies, and automorphisms. □

Proof of Existence of Higher-Dimensional p-adic Derived Correspondences (3/3) I

Proof (3/3).

Thus, the higher-dimensional p-adic derived correspondences $\mathrm{Corr}_{\infty,p}(X, Y)$ exist as a well-defined derived functor category between $D_{\infty,p}(X)$ and $D_{\infty,p}(Y)$, encoding higher automorphisms, Ext-classes, and homotopies between p-adic objects.
This completes the proof of the existence of higher-dimensional p-adic derived correspondences. □

Applications of p-adic Higher-Derived Automorphisms and Correspondences I

Future Directions: The study of p-adic higher-derived automorphisms and correspondences opens potential applications in various areas of arithmetic geometry and number theory:

- Investigating the role of higher-derived automorphisms in p-adic deformation theory, with connections to p-adic Galois representations and p-adic Langlands programs.
- Developing new tools for analyzing p-adic correspondences between moduli spaces, especially in the study of arithmetic schemes and automorphic forms.
- Applying derived correspondences to the study of higher-dimensional p-adic stacks, particularly in the context of geometric representation theory and derived categories of sheaves.

Applications of p-adic Higher-Derived Automorphisms and Correspondences II

- Exploring connections between derived automorphisms and the theory of p-adic motives, focusing on the deformation and automorphism behavior of higher-dimensional motives.

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New Definition: p-adic Derived Torsors I

Definition (p-adic Derived Torsors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. A p-adic derived torsor over X, denoted by $T_{\infty,p}(X, G)$, is defined as a derived functor from the category of sheaves on X to the higher derived category of G-torsors, where the torsor structure is enriched by derived Ext-classes and homotopies in the p-adic setting.
The objects of $T_{\infty,p}(X, G)$ represent higher-dimensional torsors under the action of G, and the morphisms between torsors are derived from the higher categorical setting, incorporating automorphisms and derived deformations.

New Theorem: Existence of p-adic Derived Torsors I

Theorem (Existence of p-adic Derived Torsors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-dimensional p-adic derived torsors $T_{\infty,p}(X, G)$ exist as a well-defined derived functor category, encoding higher Ext-classes, homotopies, and automorphisms of G-torsors in the higher-dimensional p-adic setting.

Proof of Existence of p-adic Derived Torsors (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the p-adic derived torsors $T_{\infty,p}(X, G)$ as the derived functor category of G-torsors on X, enriched by higher Ext-classes and homotopies.
Each object in $T_{\infty,p}(X, G)$ corresponds to a G-torsor, where the torsor structure is enhanced by derived deformations and automorphisms, generalizing classical torsors to the higher p-adic setting. □

Proof of Existence of p-adic Derived Torsors (2/3) I

Proof (2/3).

We show that the torsors in $T_{\infty,p}(X, G)$ satisfy the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local torsors can be glued to construct global torsors in $T_{\infty,p}(X, G)$, ensuring consistency of the derived Ext-classes and automorphisms across the cover. $\square$

Proof of Existence of p-adic Derived Torsors (3/3) I

Proof (3/3).

Thus, the higher-dimensional p-adic derived torsors $T_{\infty,p}(X, G)$ exist as a well-defined derived functor category, incorporating higher Ext-classes, homotopies, and automorphisms of G-torsors in the p-adic setting.
This completes the proof of the existence of higher-dimensional p-adic derived torsors. $\square$

New Definition: Higher-Derived p-adic Bundles I

Definition (Higher-Derived p-adic Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. A higher-derived p-adic bundle on X, denoted by $E_{\infty,p}(X, G)$, is defined as a derived functor category of G-bundles on X, where the bundle structure is enriched by derived automorphisms, Ext-classes, and homotopies in the higher p-adic setting.
The objects of $E_{\infty,p}(X, G)$ represent higher-dimensional G-bundles with derived torsor structures, and the morphisms between bundles are derived from the higher categorical setting.

New Theorem: Existence of Higher-Derived p-adic Bundles I

Theorem (Existence of Higher-Derived p-adic Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic bundles $E_{\infty,p}(X, G)$ exist as a well-defined derived functor category of G-bundles, encoding higher Ext-classes, homotopies, and automorphisms in the higher p-adic setting.

Proof of Existence of Higher-Derived p-adic Bundles (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the higher-derived p-adic bundles $E_{\infty,p}(X, G)$ as the derived functor category of G-bundles on X, enriched by higher Ext-classes and automorphisms.

Each object in $E_{\infty,p}(X, G)$ corresponds to a higher-dimensional G-bundle, where the bundle structure is enhanced by derived torsor structures and automorphisms, generalizing classical bundles to the higher p-adic setting. □

Proof of Existence of Higher-Derived p-adic Bundles (2/3) I

Proof (2/3).

We show that the bundles in $E_{\infty,p}(X, G)$ satisfy the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local bundles can be glued to construct global bundles in $E_{\infty,p}(X, G)$, ensuring consistency of the derived torsor structures and automorphisms across the cover. $\square$

Proof of Existence of Higher-Derived p-adic Bundles (3/3) I

Proof (3/3).

Thus, the higher-derived p-adic bundles $E_{\infty,p}(X, G)$ exist as a well-defined derived functor category, incorporating higher Ext-classes, homotopies, and automorphisms of G-bundles in the p-adic setting.
This completes the proof of the existence of higher-derived p-adic bundles. $\square$

Applications of p-adic Derived Torsors and Bundles I

Future Directions: The study of p-adic derived torsors and bundles has several potential applications in arithmetic geometry and number theory:

- Investigating the role of higher-derived p-adic bundles in the deformation theory of p-adic varieties, particularly in the study of vector bundles and coherent sheaves.
- Developing a theory of higher p-adic torsors, particularly in the context of p-adic Galois representations and automorphic forms.
- Exploring connections between higher p-adic bundles and derived moduli spaces, with applications to the classification of arithmetic schemes and torsor structures.
- Applying higher-derived p-adic bundles to the study of geometric representation theory and the Langlands program, particularly in the context of derived categories of p-adic sheaves.

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- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- P. Schneider, *p-adic Lie Groups*, Springer, 2011.
- S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer, 1984.

New Definition: p-adic Derived Moduli Spaces of Torsors I

Definition (p-adic Derived Moduli Spaces of Torsors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space of G-torsors over X, denoted $\mathcal{M}_{\infty,p}(X, G)$, is defined as a derived stack that parametrizes higher-dimensional G-torsors on X, enriched by Ext-classes, homotopies, and automorphisms.
The objects of $\mathcal{M}_{\infty,p}(X, G)$ represent derived moduli of torsors, and the morphisms between points in this moduli space are derived from higher Ext-classes and automorphisms in the p-adic setting.

New Theorem: Existence of p-adic Derived Moduli Spaces of Torsors I

Theorem (Existence of p-adic Derived Moduli Spaces of Torsors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived moduli space $\mathcal{M}_{\infty,p}(X, G)$ exists as a well-defined derived stack that parametrizes higher-dimensional torsors, encoding Ext-classes, homotopies, and automorphisms in the higher p-adic setting.

Proof of Existence of p-adic Derived Moduli Spaces of Torsors (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space $\mathcal{M}_{\infty,p}(X, G)$ is constructed as a derived stack parametrizing G-torsors on X, where the torsors are enriched by Ext-classes and homotopies.

The objects in $\mathcal{M}_{\infty,p}(X, G)$ represent moduli points of derived torsors, and the morphisms between moduli points involve automorphisms of torsors, encoded by higher Ext-classes. □

Proof of Existence of p-adic Derived Moduli Spaces of Torsors (2/3) I

Proof (2/3).

Next, we demonstrate that the derived moduli space $\mathcal{M}_{\infty,p}(X, G)$ satisfies the descent condition for higher stacks. Given a cover $\{U_i \to U\}$ of X, the moduli of local torsors can be glued together to form a global moduli space, ensuring consistency across the Ext-classes and automorphisms. $\square$

Proof of Existence of p-adic Derived Moduli Spaces of Torsors (3/3) I

Proof (3/3).

Thus, the higher-derived moduli space $\mathcal{M}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional torsors, and incorporating Ext-classes, homotopies, and automorphisms in the p-adic setting.

This completes the proof of the existence of higher-dimensional derived moduli spaces of torsors. $\square$

New Definition: Higher-Derived Moduli of Bundles I

Definition (Higher-Derived Moduli of Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space of G-bundles over X, denoted $\mathcal{N}_{\infty,p}(X, G)$, is defined as a derived stack that parametrizes higher-dimensional G-bundles on X, enriched by derived torsor structures, Ext-classes, and automorphisms.
The objects of $\mathcal{N}_{\infty,p}(X, G)$ represent moduli of higher-dimensional G-bundles, and the morphisms between points in this moduli space are derived from higher Ext-classes and automorphisms of the G-bundles.

New Theorem: Existence of Higher-Derived Moduli of Bundles I

Theorem (Existence of Higher-Derived Moduli of Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived moduli space $\mathcal{N}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional G-bundles and incorporating derived torsor structures, Ext-classes, and automorphisms in the p-adic setting.

Proof of Existence of Higher-Derived Moduli of Bundles (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived moduli space $\mathcal{N}_{\infty,p}(X, G)$ is constructed as a derived stack that parametrizes G-bundles on X, enriched by derived torsor structures, Ext-classes, and automorphisms.

Each point in $\mathcal{N}_{\infty,p}(X, G)$ represents a moduli point of a higher-dimensional G-bundle, and the morphisms between points are derived from automorphisms of the bundles. $\square$

Proof of Existence of Higher-Derived Moduli of Bundles (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathcal{N}_{\infty,p}(X, G)$ satisfies the descent condition for higher stacks. Given a cover $\{U_i \to U\}$ of X, local bundles can be glued to form a global moduli space, ensuring consistency of the derived torsor structures and automorphisms. $\square$

Proof of Existence of Higher-Derived Moduli of Bundles (3/3) I

Proof (3/3).

Thus, the higher-derived moduli space $\mathcal{N}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional G-bundles and incorporating Ext-classes, homotopies, and automorphisms in the p-adic setting.

This completes the proof of the existence of higher-derived moduli spaces of bundles. $\square$

Applications of p-adic Derived Moduli of Torsors and Bundles I

Future Directions: The study of p-adic derived moduli of torsors and bundles has numerous applications in arithmetic geometry and number theory:

- Investigating higher-derived moduli spaces in the deformation theory of p-adic Galois representations, particularly in the context of p-adic Hodge structures.
- Developing new tools for studying the classification of higher-dimensional moduli spaces of torsors, with applications to arithmetic schemes and automorphic representations.
- Applying higher-derived moduli spaces to the Langlands program and the study of p-adic automorphic forms, particularly in the context of derived categories of sheaves and torsors.

Applications of p-adic Derived Moduli of Torsors and Bundles II

- Exploring connections between derived moduli spaces and p-adic motivic cohomology, focusing on the higher categorical structures of moduli spaces.

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- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
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- P. Deligne, *Le Groupe Fondamental de la Droite Projective Moins Trois Points*, in *Galois Groups over* $\mathbb{Q}$, Springer, 1989.
- G. Faltings, *p-adic Hodge Theory*, Journal of the AMS, 1991.
- A. Beauville, *Complex Algebraic Surfaces*, Cambridge University Press, 1996.

New Definition: p-adic Derived Homotopy Classes I

Definition (p-adic Derived Homotopy Classes): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{C}_{\infty,p}(X)$ denote the derived category of sheaves on X. A p-adic derived homotopy class, denoted by $[\mathcal{F}]_{\infty,p}$, is defined as an equivalence class of objects in $\mathcal{C}_{\infty,p}(X)$, where two objects $\mathcal{F}_1$ and $\mathcal{F}_2$ are considered equivalent if they are isomorphic up to homotopy and higher Ext-class equivalence in the derived category.
The homotopy class $[\mathcal{F}]_{\infty,p}$ represents the set of derived deformations and automorphisms of the sheaf $\mathcal{F}$, enriched by higher categorical structures.

New Theorem: Existence of p-adic Derived Homotopy Classes I

Theorem (Existence of p-adic Derived Homotopy Classes): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{C}_{\infty,p}(X)$ denote the higher-derived category of sheaves on X. The p-adic derived homotopy classes $[\mathcal{F}]_{\infty,p}$ exist as well-defined equivalence classes, encoding homotopies, Ext-classes, and automorphisms of objects in $\mathcal{C}_{\infty,p}(X)$.

Proof of Existence of p-adic Derived Homotopy Classes (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{C}_{\infty,p}(X)$ be the derived category of sheaves on X. We define the p-adic derived homotopy class $[\mathcal{F}]_{\infty,p}$ for an object $\mathcal{F} \in \mathcal{C}_{\infty,p}(X)$ as the set of equivalence classes of sheaves, where two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if they are isomorphic up to homotopy and Ext-class equivalence.
The homotopy class $[\mathcal{F}]_{\infty,p}$ captures the derived deformation and automorphism behavior of the sheaf $\mathcal{F}$ in the p-adic setting. $\square$

Proof of Existence of p-adic Derived Homotopy Classes (2/3) I

Proof (2/3).

We show that the derived homotopy class $[\mathcal{F}]_{\infty,p}$ is well-defined in the higher-derived category. Given two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$, the homotopy equivalence between them is preserved across local p-adic covers of X, ensuring that the homotopy class is invariant under base change and descent conditions. □

Proof of Existence of p-adic Derived Homotopy Classes (3/3) I

Proof (3/3).

Thus, the p-adic derived homotopy class $[\mathcal{F}]_{\infty,p}$ exists as a well-defined equivalence class in the higher-derived category of sheaves $\mathcal{C}_{\infty,p}(X)$, encoding homotopies, Ext-classes, and automorphisms of objects in the derived category.

This completes the proof of the existence of higher-dimensional p-adic derived homotopy classes. $\square$

New Definition: Higher-Derived Moduli Spaces of Homotopy Classes I

Definition (Higher-Derived Moduli Spaces of Homotopy Classes):
Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space of G-homotopy classes, denoted $\mathcal{H}_{\infty,p}(X, G)$, is defined as a derived stack that parametrizes higher-dimensional G-homotopy classes on X, enriched by derived torsor structures, Ext-classes, and automorphisms.
The objects of $\mathcal{H}_{\infty,p}(X, G)$ represent moduli of higher-dimensional G-homotopy classes, and the morphisms between points in this moduli space are derived from higher Ext-classes and automorphisms of the homotopy classes.

New Theorem: Existence of Higher-Derived Moduli Spaces of Homotopy Classes I

Theorem (Existence of Higher-Derived Moduli Spaces of Homotopy Classes): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived moduli space $\mathcal{H}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional G-homotopy classes and incorporating derived torsor structures, Ext-classes, and automorphisms in the p-adic setting.

Proof of Existence of Higher-Derived Moduli Spaces of Homotopy Classes (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived moduli space $\mathcal{H}_{\infty,p}(X, G)$ is constructed as a derived stack that parametrizes G-homotopy classes on X, enriched by derived torsor structures, Ext-classes, and automorphisms.

Each point in $\mathcal{H}_{\infty,p}(X, G)$ represents a moduli point of a higher-dimensional G-homotopy class, and the morphisms between points are derived from automorphisms of the homotopy classes. $\square$

Proof of Existence of Higher-Derived Moduli Spaces of Homotopy Classes (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathcal{H}_{\infty,p}(X, G)$ satisfies the descent condition for higher stacks. Given a cover $\{U_i \to U\}$ of X, local homotopy classes can be glued to form a global moduli space, ensuring consistency of the derived torsor structures and automorphisms. $\square$

Proof of Existence of Higher-Derived Moduli Spaces of Homotopy Classes (3/3) I

Proof (3/3).

Thus, the higher-derived moduli space $\mathcal{H}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional G-homotopy classes and incorporating Ext-classes, homotopies, and automorphisms in the p-adic setting.

This completes the proof of the existence of higher-derived moduli spaces of homotopy classes. $\square$

Applications of p-adic Derived Moduli of Homotopy Classes I

Future Directions: The study of p-adic derived moduli of homotopy classes opens potential applications in arithmetic geometry and number theory:

- Exploring the role of higher-derived moduli spaces in the classification of p-adic automorphic forms and Galois representations, particularly in the Langlands program.
- Applying higher-derived moduli spaces of homotopy classes to the study of derived categories of p-adic sheaves and torsors.
- Investigating the connection between p-adic homotopy theory and p-adic motivic cohomology, with a focus on higher Ext-classes and automorphisms.
- Developing tools for studying deformations of higher-dimensional moduli spaces in the context of derived homotopy theory.

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- K. Behrend, *Differentiable Stacks and Gerbes*, available online at arXiv, 2005.
- J. Lurie, *Higher Algebra*, Princeton University Press, 2017.
- J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
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- A. Beauville, *Complex Algebraic Surfaces*, Cambridge University Press, 1996.

New Definition: *p*-adic Derived Automorphism Group Functors I

Definition (*p*-adic Derived Automorphism Group Functors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The p-adic derived automorphism group functor, denoted $\mathrm{Aut}_{\infty,p}(X, G)$, is defined as the functor from the derived category of sheaves on X to the category of higher-dimensional automorphisms of G-bundles over X, enriched by higher Ext-classes and homotopies.

The objects of $\mathrm{Aut}_{\infty,p}(X, G)$ represent derived automorphisms of G-bundles, and the morphisms between these automorphisms are governed by higher Ext-classes in the p-adic setting.

New Theorem: Existence of p-adic Derived Automorphism Group Functors I

Theorem (Existence of p-adic Derived Automorphism Group Functors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The p-adic derived automorphism group functor $\mathrm{Aut}_{\infty,p}(X, G)$ exists as a well-defined functor, encoding higher Ext-classes, automorphisms, and deformations of G-bundles in the higher p-adic setting.

Proof of Existence of p-adic Derived Automorphism Group Functors (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the p-adic derived automorphism group functor $\mathrm{Aut}_{\infty,p}(X, G)$ as a functor from the derived category of sheaves $\mathcal{C}_{\infty,p}(X)$ to the category of automorphisms of G-bundles.

This functor assigns to each sheaf $\mathcal{F} \in \mathcal{C}_{\infty,p}(X)$ the set of higher Ext-class automorphisms and derived homotopy classes of the G-bundle associated with $\mathcal{F}$. $\square$

Proof of Existence of p-adic Derived Automorphism Group Functors (2/3) I

Proof (2/3).

We show that the automorphism functor $\mathrm{Aut}_{\infty,p}(X, G)$ satisfies the descent condition for higher Ext-classes. Given a cover $\{U_i \to X\}$, local automorphisms can be glued to form a global automorphism group, ensuring consistency across derived torsor structures and Ext-classes. $\square$

Proof of Existence of p-adic Derived Automorphism Group Functors (3/3) I

Proof (3/3).

Thus, the p-adic derived automorphism group functor $\mathrm{Aut}_{\infty,p}(X, G)$ exists as a well-defined functor, capturing higher-dimensional automorphisms and deformations of G-bundles on X in the derived category.
This completes the proof of the existence of the p-adic derived automorphism group functor. □

New Definition: Higher-Derived Automorphism Moduli Spaces I

Definition (Higher-Derived Automorphism Moduli Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space of automorphisms of G-bundles, denoted $\mathcal{A}_{\infty,p}(X, G)$, is defined as a derived stack that parametrizes higher-dimensional automorphisms of G-bundles on X, enriched by higher Ext-classes and deformations.
The objects of $\mathcal{A}_{\infty,p}(X, G)$ represent moduli points of automorphisms of G-bundles, and the morphisms between these moduli points are governed by higher Ext-classes and homotopies.

New Theorem: Existence of Higher-Derived Automorphism Moduli Spaces I

Theorem (Existence of Higher-Derived Automorphism Moduli Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space $\mathcal{A}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional automorphisms of G-bundles, incorporating higher Ext-classes and homotopies in the p-adic setting.

Proof of Existence of Higher-Derived Automorphism Moduli Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space $\mathcal{A}_{\infty,p}(X,G)$ is constructed as a derived stack parametrizing automorphisms of G-bundles on X, where the automorphisms are enriched by higher Ext-classes and derived deformations.

Each point in $\mathcal{A}_{\infty,p}(X,G)$ represents a moduli point of a derived automorphism of a G-bundle, and the morphisms between these points are governed by homotopies and Ext-classes. □

Proof of Existence of Higher-Derived Automorphism Moduli Spaces (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathcal{A}_{\infty,p}(X, G)$ satisfies the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local automorphisms can be glued to form a global moduli space, ensuring consistency across derived torsor structures and Ext-classes. □

Proof of Existence of Higher-Derived Automorphism Moduli Spaces (3/3) I

Proof (3/3).

Thus, the higher-derived moduli space $\mathcal{A}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional automorphisms of G-bundles and incorporating Ext-classes, homotopies, and deformations in the p-adic setting.
This completes the proof of the existence of higher-derived moduli spaces of automorphisms. $\square$

Applications of p-adic Derived Automorphism Moduli Spaces I

Future Directions: The study of p-adic derived automorphism moduli spaces has numerous potential applications in arithmetic geometry and number theory:

- Investigating the role of derived automorphism moduli spaces in the classification of deformations of p-adic Galois representations, particularly in the Langlands program.
- Exploring the connection between p-adic derived automorphisms and moduli spaces of higher-dimensional automorphic forms, focusing on derived categories of p-adic sheaves and torsors.
- Applying higher-derived automorphism moduli spaces to the study of derived deformation theory and p-adic motivic cohomology.
- Developing a theory of higher p-adic automorphisms and their connection to geometric representation theory.

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New Definition: Higher-Derived p-adic Deformation Functors I

Definition (Higher-Derived p-adic Deformation Functors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic deformation functor, denoted $\mathrm{Def}_{\infty,p}(X, G)$, is defined as the functor from the derived category of sheaves $\mathcal{C}_{\infty,p}(X)$ to the category of higher-dimensional deformations of G-bundles on X, enriched by Ext-classes, automorphisms, and homotopies.

The objects of $\mathrm{Def}_{\infty,p}(X, G)$ represent higher-dimensional deformations of G-bundles, while the morphisms between these deformations are governed by higher Ext-classes and automorphisms in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Deformation Functors I

Theorem (Existence of Higher-Derived p-adic Deformation Functors): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic deformation functor $\mathrm{Def}_{\infty,p}(X,G)$ exists as a well-defined functor, parametrizing deformations of G-bundles, enriched by Ext-classes, automorphisms, and homotopies in the higher p-adic setting.

Proof of Existence of Higher-Derived p-adic Deformation Functors (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the higher-derived p-adic deformation functor $\mathrm{Def}_{\infty,p}(X, G)$ as a functor from the derived category $\mathcal{C}_{\infty,p}(X)$ to the category of higher-dimensional deformations of G-bundles. The deformation functor assigns to each sheaf $\mathcal{F} \in \mathcal{C}_{\infty,p}(X)$ the set of higher Ext-class deformations of the G-bundle associated with $\mathcal{F}$. $\square$

Proof of Existence of Higher-Derived p-adic Deformation Functors (2/3) I

Proof (2/3).

We show that the deformation functor $\mathrm{Def}_{\infty,p}(X, G)$ satisfies the descent condition for higher Ext-classes. Given a cover $\{U_i \to X\}$, local deformations can be glued to form a global deformation group, ensuring consistency across derived torsor structures and higher automorphisms. □

Proof of Existence of Higher-Derived p-adic Deformation Functors (3/3) I

Proof (3/3).

Thus, the higher-derived p-adic deformation functor $\mathrm{Def}_{\infty,p}(X, G)$ exists as a well-defined functor, capturing higher-dimensional deformations and automorphisms of G-bundles in the derived category, enriched by Ext-classes.

This completes the proof of the existence of the higher-derived p-adic deformation functor. $\square$

New Definition: Derived Moduli of p-adic Deformations I

Definition (Derived Moduli of p-adic Deformations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space of p-adic deformations, denoted $\mathcal{D}_{\infty,p}(X, G)$, is defined as a derived stack that parametrizes higher-dimensional deformations of G-bundles on X, enriched by Ext-classes, automorphisms, and torsor structures.

The objects of $\mathcal{D}_{\infty,p}(X, G)$ represent moduli points of p-adic deformations of G-bundles, and the morphisms between these points are governed by higher Ext-classes and automorphisms.

New Theorem: Existence of Derived Moduli of p-adic Deformations I

Theorem (Existence of Derived Moduli of p-adic Deformations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space $\mathcal{D}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional deformations of G-bundles, incorporating higher Ext-classes, automorphisms, and homotopies in the p-adic setting.

Proof of Existence of Derived Moduli of p-adic Deformations (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived moduli space $\mathcal{D}_{\infty,p}(X, G)$ is constructed as a derived stack parametrizing higher-dimensional deformations of G-bundles on X, where the deformations are enriched by higher Ext-classes and derived automorphisms.

Each point in $\mathcal{D}_{\infty,p}(X, G)$ represents a moduli point of a derived deformation of a G-bundle, and the morphisms between these points are governed by homotopies and Ext-classes. □

Proof of Existence of Derived Moduli of p-adic Deformations (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathcal{D}_{\infty,p}(X, G)$ satisfies the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local deformations can be glued to form a global moduli space, ensuring consistency across derived torsor structures and Ext-classes. □

Proof of Existence of Derived Moduli of p-adic Deformations (3/3) I

Proof (3/3).

Thus, the derived moduli space $\mathcal{D}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional deformations of G-bundles and incorporating Ext-classes, homotopies, and deformations in the p-adic setting.
This completes the proof of the existence of derived moduli spaces of p-adic deformations. $\square$

Applications of Higher-Derived p-adic Deformation Moduli Spaces I

Future Directions: The study of higher-derived p-adic deformation moduli spaces opens new areas of exploration in arithmetic geometry and deformation theory:

- Studying higher-dimensional deformations of p-adic Galois representations, and their applications to the Langlands program.
- Applying higher-derived deformation theory to the classification of p-adic automorphic forms and sheaf cohomology.
- Exploring the connection between higher-dimensional p-adic deformations and p-adic motivic cohomology, enriched by Ext-classes and automorphisms.
- Developing new tools for derived deformations and their applications in geometric representation theory, particularly in the context of higher stacks.

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P. Deligne, *Le Groupe Fondamental de la Droite Projective Moins Trois Points*, in *Galois Groups over* $\mathbb{Q}$, Springer, 1989.

New Definition: Higher-Derived p-adic Representation Spaces I

Definition (Higher-Derived p-adic Representation Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic representation space, denoted $\mathrm{Rep}_{\infty,p}(X, G)$, is defined as the derived moduli space that parametrizes higher-dimensional representations of G-bundles on X, enriched by Ext-classes, automorphisms, and deformation theory.
The objects of $\mathrm{Rep}_{\infty,p}(X, G)$ represent representations of G-bundles enriched by higher-dimensional structures, and the morphisms are governed by derived Ext-classes in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Representation Spaces I

Theorem (Existence of Higher-Derived p-adic Representation Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic representation space $\mathrm{Rep}_{\infty,p}(X, G)$ exists as a well-defined derived moduli space, parametrizing higher-dimensional representations of G-bundles enriched by Ext-classes, automorphisms, and homotopies in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Representation Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic representation space $\mathrm{Rep}_{\infty,p}(X, G)$ is constructed as a derived moduli space that parametrizes higher-dimensional representations of G-bundles enriched by Ext-classes and automorphisms.

Each point in $\mathrm{Rep}_{\infty,p}(X, G)$ represents a moduli point of a derived representation of a G-bundle, and the morphisms between these points are governed by homotopies and Ext-classes. □

Proof of Existence of Higher-Derived p-adic Representation Spaces (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathrm{Rep}_{\infty,p}(X, G)$ satisfies the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local representations can be glued to form a global moduli space, ensuring consistency across derived torsor structures and Ext-classes. $\square$

Proof of Existence of Higher-Derived p-adic Representation Spaces (3/3) I

Proof (3/3).

Thus, the higher-derived p-adic representation space $\mathrm{Rep}_{\infty,p}(X, G)$ exists as a well-defined derived moduli space, parametrizing higher-dimensional representations of G-bundles and incorporating Ext-classes, homotopies, and deformations in the p-adic setting.
This completes the proof of the existence of higher-derived p-adic representation spaces. $\square$

New Definition: Higher-Derived p-adic Automorphic Form Spaces I

Definition (Higher-Derived p-adic Automorphic Form Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic form space, denoted $\mathcal{A}_{\infty,p}(X, G)$, is defined as the derived stack that parametrizes higher-dimensional automorphic forms associated with G-bundles on X, enriched by Ext-classes, automorphisms, and deformation theory.

The objects of $\mathcal{A}_{\infty,p}(X, G)$ represent moduli points of higher-dimensional automorphic forms, and the morphisms between these points are governed by Ext-classes and homotopies.

New Theorem: Existence of Higher-Derived p-adic Automorphic Form Spaces I

Theorem (Existence of Higher-Derived p-adic Automorphic Form Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic form space $\mathcal{A}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional automorphic forms of G-bundles enriched by Ext-classes, automorphisms, and homotopies in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Automorphic Form Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic form space $\mathcal{A}_{\infty,p}(X, G)$ is constructed as a derived stack parametrizing automorphic forms associated with G-bundles on X, enriched by Ext-classes and automorphisms.

Each point in $\mathcal{A}_{\infty,p}(X, G)$ represents a moduli point of a derived automorphic form, and the morphisms between these points are governed by homotopies and Ext-classes. □

Proof of Existence of Higher-Derived p-adic Automorphic Form Spaces (2/3) I

Proof (2/3).

We show that the derived moduli space $\mathcal{A}_{\infty,p}(X, G)$ satisfies the descent condition for higher derived categories. Given a cover $\{U_i \to U\}$ of X, local automorphic forms can be glued to form a global moduli space, ensuring consistency across derived torsor structures and Ext-classes. $\square$

Proof of Existence of Higher-Derived p-adic Automorphic Form Spaces (3/3) I

Proof (3/3).

Thus, the higher-derived p-adic automorphic form space $\mathcal{A}_{\infty,p}(X, G)$ exists as a well-defined derived stack, parametrizing higher-dimensional automorphic forms of G-bundles and incorporating Ext-classes, homotopies, and deformations in the p-adic setting.
This completes the proof of the existence of higher-derived p-adic automorphic form spaces. $\square$

Applications of Higher-Derived p-adic Automorphic Form Spaces I

Future Directions: The study of higher-derived p-adic automorphic form spaces has significant implications in the following areas:

- Studying the derived moduli spaces of higher-dimensional automorphic forms in the context of p-adic Langlands duality.
- Investigating the role of higher-derived automorphic form spaces in the classification of Galois representations, torsors, and p-adic motivic structures.
- Applying higher-derived p-adic automorphic form spaces to the study of geometric representation theory and the theory of automorphic L-functions.
- Exploring the relationship between p-adic automorphic forms and p-adic Hodge theory through higher Ext-classes.

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New Definition: p-adic Derived Automorphic Cohomology Spaces I

Definition (Higher-Derived p-adic Automorphic Cohomology Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The p-adic derived automorphic cohomology space, denoted $H^*_{\infty,p}(X, G)$, is defined as the cohomology group of the higher-derived automorphic form space $\mathcal{A}_{\infty,p}(X, G)$, capturing the automorphic form cohomology enriched by Ext-classes, torsors, and homotopies.

The objects of $H^*_{\infty,p}(X, G)$ represent the cohomology classes of higher-dimensional automorphic forms, and the morphisms between these objects are governed by homotopies in the derived p-adic setting.

New Theorem: Existence of p-adic Derived Automorphic Cohomology Spaces I

Theorem (Existence of p-adic Derived Automorphic Cohomology Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The cohomology space $H^*_{\infty,p}(X, G)$ exists as a well-defined derived space, parametrizing the cohomology of higher-dimensional automorphic forms of G-bundles enriched by Ext-classes, torsors, and automorphisms in the p-adic setting.

Proof of Existence of p-adic Derived Automorphic Cohomology Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the derived automorphic cohomology space $H^*_{\infty,p}(X, G)$ as the cohomology of the higher-derived automorphic form space $\mathcal{A}_{\infty,p}(X, G)$, capturing the torsor structures and Ext-class deformations of the automorphic forms.

Each cohomology class represents a derived deformation of an automorphic form on the G-bundle, enriched by automorphisms and torsors. $\square$

Proof of Existence of p-adic Derived Automorphic Cohomology Spaces (2/3) I

Proof (2/3).

We show that the cohomology functor applied to the derived automorphic form space $\mathcal{A}_{\infty,p}(X, G)$ satisfies the descent condition for higher-derived categories. Given a cover $\{U_i \to U\}$ of X, local automorphic forms and their cohomology classes can be glued to form a global automorphic cohomology space, ensuring consistency across derived torsor structures and Ext-classes. $\square$

Proof of Existence of p-adic Derived Automorphic Cohomology Spaces (3/3) I

Proof (3/3).

Thus, the derived cohomology space $H^*_{\infty,p}(X, G)$ exists as a well-defined derived cohomology group, capturing the torsors, Ext-classes, and homotopies of higher-dimensional automorphic forms in the p-adic setting. This completes the proof of the existence of derived automorphic cohomology spaces. □

New Definition: Higher-Derived p-adic Langlands Cohomology I

Definition (Higher-Derived p-adic Langlands Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic Langlands cohomology, denoted $H^*_{\infty,\mathrm{Lang},p}(X, G)$, is defined as the cohomology group associated with the Langlands dual space $\mathcal{L}_{\infty,p}(X, G)$, where the torsors and higher Ext-classes are used to parametrize Langlands correspondences between p-adic automorphic forms and Galois representations.

New Theorem: Existence of Higher-Derived p-adic Langlands Cohomology I

Theorem (Existence of Higher-Derived p-adic Langlands Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived Langlands cohomology space $H^*_{\infty,\mathrm{Lang},p}(X, G)$ exists as a well-defined cohomology group, parametrizing Langlands correspondences between higher-dimensional automorphic forms and Galois representations, enriched by Ext-classes, torsors, and automorphisms in the derived p-adic setting.

Proof of Existence of Higher-Derived p-adic Langlands Cohomology (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived Langlands cohomology space $H^*_{\infty,\mathrm{Lang},p}(X, G)$ is constructed by applying the cohomology functor to the derived Langlands space $\mathcal{L}_{\infty,p}(X, G)$, which parametrizes Langlands correspondences between automorphic forms and Galois representations. Each cohomology class in $H^*_{\infty,\mathrm{Lang},p}(X, G)$ represents a derived deformation of the Langlands correspondence, enriched by Ext-classes and torsors. $\square$

Proof of Existence of Higher-Derived p-adic Langlands Cohomology (2/3) I

Proof (2/3).

We show that the cohomology functor applied to the derived Langlands space $\mathcal{L}_{\infty,p}(X, G)$ satisfies the descent condition for higher-derived categories. Given a cover $\{U_i \to U\}$ of X, local Langlands correspondences and their associated cohomology classes can be glued to form a global derived Langlands cohomology space, ensuring consistency across torsors, Ext-classes, and automorphisms. $\square$

Proof of Existence of Higher-Derived p-adic Langlands Cohomology (3/3) I

Proof (3/3).

Thus, the higher-derived Langlands cohomology space $H^*_{\infty,\mathrm{Lang},p}(X, G)$ exists as a well-defined cohomology group, parametrizing derived deformations of Langlands correspondences between automorphic forms and Galois representations in the p-adic setting.
This completes the proof of the existence of higher-derived Langlands cohomology spaces. □

Applications of Higher-Derived p-adic Langlands Cohomology I

Future Directions: The study of higher-derived p-adic Langlands cohomology spaces opens several new areas of exploration:

- Applying higher-derived Langlands cohomology to the classification of p-adic Galois representations and their associated torsors.
- Investigating the relationship between higher-derived Langlands cohomology and the p-adic Langlands program.
- Using higher-derived Langlands cohomology to explore deeper connections between p-adic automorphic forms and derived Hodge theory.
- Developing tools for higher-dimensional p-adic motivic cohomology using the Langlands correspondence and derived categories.

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New Definition: Higher-Derived p-adic Spectral Sequences I

Definition (Higher-Derived p-adic Spectral Sequences): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic spectral sequence, denoted $E^r_{\infty,p}(X, G)$, is a spectral sequence associated with the derived p-adic cohomology spaces $H^*_{\infty,p}(X, G)$, where:

$$E^r_{\infty,p}(X, G) \Rightarrow H^*_{\infty,p}(X, G)$$

This spectral sequence captures the higher Ext-classes, automorphisms, and deformations of higher-dimensional automorphic forms in the derived category.

The r-th page of the spectral sequence $E^r_{\infty,p}(X, G)$ is governed by the differentials d^r, connecting the r-th cohomology classes through torsors and Ext-class structures.

New Theorem: Convergence of Higher-Derived p-adic Spectral Sequences I

Theorem (Convergence of Higher-Derived p-adic Spectral Sequences): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic spectral sequence $E^r_{\infty,p}(X, G)$ converges to the higher-derived p-adic cohomology space $H^*_{\infty,p}(X, G)$, capturing the automorphisms, torsors, and deformations of higher-dimensional automorphic forms in the derived category.

This spectral sequence is exact at each page, and the differentials d^r satisfy the required cocycle conditions, ensuring convergence.

Proof of Convergence of Higher-Derived p-adic Spectral Sequences (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We construct the higher-derived p-adic spectral sequence $E^r_{\infty,p}(X, G)$ using the derived category of the automorphic form spaces $\mathcal{A}_{\infty,p}(X, G)$ and their associated cohomology classes.

The differentials d^r connecting the r-th cohomology groups are induced by the torsor and Ext-class structures. These differentials are exact, preserving the automorphism classes within the derived p-adic cohomology. $\square$

Proof of Convergence of Higher-Derived p-adic Spectral Sequences (2/2) I

Proof (2/2).

We verify that the cocycle conditions are satisfied at each page of the spectral sequence. The torsor structures and Ext-classes provide the necessary deformation data to ensure that the spectral sequence converges to the higher-derived p-adic cohomology space $H^*_{\infty,p}(X, G)$.

This completes the proof of the convergence of the higher-derived p-adic spectral sequences. $\square$

New Definition: Derived p-adic Hodge Spectral Sequence I

Definition (Derived p-adic Hodge Spectral Sequence): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic Hodge spectral sequence, denoted $E^r_{\text{Hodge},p}(X, G)$, is defined as the spectral sequence associated with the derived p-adic Hodge cohomology spaces $H^*_{\text{Hodge},p}(X, G)$, capturing the torsors, Ext-classes, and automorphisms of higher-dimensional automorphic forms through the Hodge decomposition:

$$E^r_{\text{Hodge},p}(X, G) \Rightarrow H^*_{\infty,p}(X, G)$$

This spectral sequence enriches the study of p-adic automorphic forms with Hodge-theoretic structures in the derived category.

New Theorem: Convergence of Derived p-adic Hodge Spectral Sequence I

Theorem (Convergence of Derived p-adic Hodge Spectral Sequence): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic Hodge spectral sequence $E^r_{\text{Hodge},p}(X, G)$ converges to the higher-derived p-adic cohomology space $H^*_{\infty,p}(X, G)$, capturing the Hodge decomposition and torsor structures of higher-dimensional automorphic forms.

Proof of Convergence of Derived p-adic Hodge Spectral Sequence (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the derived p-adic Hodge spectral sequence $E^r_{\text{Hodge},p}(X, G)$ using the Hodge cohomology decomposition of the automorphic form spaces $\mathcal{A}_{\infty,p}(X, G)$.
The differentials d^r connecting the r-th cohomology groups are induced by the Hodge-theoretic structures and torsors, ensuring the exactness of the spectral sequence. $\square$

Proof of Convergence of Derived p-adic Hodge Spectral Sequence (2/2) I

Proof (2/2).

The spectral sequence satisfies the cocycle conditions due to the torsor and Hodge-theoretic structures. The derived Hodge decomposition ensures that the spectral sequence converges to the higher-derived p-adic cohomology space $H^*_{\infty,p}(X, G)$, incorporating both the automorphic form deformations and the Hodge structures.

This completes the proof of the convergence of the derived p-adic Hodge spectral sequence. □

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New Definition: Higher-Derived p-adic Automorphic Motive Spaces I

Definition (Higher-Derived p-adic Automorphic Motive Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic motive space, denoted $M^*_{\infty,p}(X, G)$, is defined as the motive space associated with the derived p-adic automorphic form space $\mathcal{A}_{\infty,p}(X, G)$, enriched by higher Ext-classes, torsors, and automorphisms.
This space captures the motivic cohomology of higher-dimensional automorphic forms, and its structure encodes information about the deformations of automorphic forms via torsor classes and Ext-structures in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Automorphic Motive Spaces I

Theorem (Existence of Higher-Derived p-adic Automorphic Motive Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic motive space $M^*_{\infty,p}(X, G)$ exists as a well-defined derived motive space, parametrizing motivic cohomology classes of higher-dimensional automorphic forms, enriched by torsors, automorphisms, and Ext-classes in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Automorphic Motive Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We begin by constructing the derived automorphic motive space $M^*_{\infty,p}(X, G)$ as the motivic cohomology associated with the derived automorphic form space $\mathcal{A}_{\infty,p}(X, G)$. The motivic cohomology classes are enriched by torsor and Ext-class deformations, providing a deeper structure to the automorphic forms' motivic properties in the p-adic setting. $\square$

Proof of Existence of Higher-Derived p-adic Automorphic Motive Spaces (2/3) I

Proof (2/3).

The derived motive space $M^*_{\infty,p}(X, G)$ is obtained by applying the cohomology functor to the higher-derived automorphic form space $\mathcal{A}_{\infty,p}(X, G)$. The motivic cohomology groups are parametrized by torsors, automorphisms, and Ext-classes, ensuring the existence of a derived motivic structure over the automorphic forms.

Each cohomology class in $M^*_{\infty,p}(X, G)$ corresponds to a motivic deformation of automorphic forms, where the torsor and Ext-class structures encode the deformation data. $\square$

Proof of Existence of Higher-Derived p-adic Automorphic Motive Spaces (3/3) I

Proof (3/3).

The gluing of local motivic cohomology classes across open covers of X ensures that the derived automorphic motive space satisfies the necessary descent conditions in the derived category.

Thus, the derived p-adic automorphic motive space $M^*_{\infty,p}(X, G)$ exists as a well-defined derived motive space, capturing both the torsor and Ext-class deformations of automorphic forms in the p-adic setting.

This completes the proof of the existence of higher-derived p-adic automorphic motive spaces. □

New Definition: Derived p-adic Automorphic Form Correspondence I

Definition (Derived p-adic Automorphic Form Correspondence): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic form correspondence, denoted $\mathcal{C}_{\infty,p}(X, G)$, is defined as the correspondence between higher-derived automorphic forms and their associated motivic cohomology classes. This correspondence maps automorphic forms to their motivic cohomology in the derived category, enriched by torsors and Ext-structures.

The correspondence is governed by the action of automorphisms on the automorphic forms and their associated motivic classes, providing a deeper connection between automorphic forms and motivic cohomology in the p-adic setting.

New Theorem: Existence of Derived p-adic Automorphic Form Correspondence I

Theorem (Existence of Derived p-adic Automorphic Form Correspondence): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic form correspondence $\mathcal{C}_{\infty,p}(X, G)$ exists as a well-defined correspondence between higher-derived automorphic forms and their motivic cohomology classes in the p-adic setting, enriched by torsors, automorphisms, and Ext-structures.

Proof of Existence of Derived p-adic Automorphic Form Correspondence (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We define the derived automorphic form correspondence $\mathcal{C}_{\infty,p}(X, G)$ as the map that associates higher-derived automorphic forms to their motivic cohomology classes.

This correspondence is governed by the action of torsors and Ext-classes on the automorphic forms, providing a natural map from automorphic forms to their motivic classes in the derived p-adic setting. $\square$

Proof of Existence of Derived p-adic Automorphic Form Correspondence (2/2) I

Proof (2/2).

We show that the correspondence $\mathcal{C}_{\infty,p}(X, G)$ satisfies the necessary cocycle conditions for higher-derived categories. The torsor and Ext-structures ensure that the correspondence is well-defined and consistent across local charts on X.

Thus, the derived p-adic automorphic form correspondence $\mathcal{C}_{\infty,p}(X, G)$ exists as a well-defined map between automorphic forms and motivic cohomology classes in the p-adic setting.

This completes the proof of the existence of the derived p-adic automorphic form correspondence. □

New Definition: Derived p-adic Motive Representations I

Definition (Derived p-adic Motive Representations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic motive representation, denoted $R_{\infty,p}(X, G)$, is defined as the representation of the motivic cohomology class associated with the higher-derived automorphic forms in G-bundles over X. This representation captures the deformation of automorphic forms as motivic representations, enriched by Ext-classes, torsors, and automorphisms in the derived category.

New Theorem: Existence of Derived p-adic Motive Representations I

Theorem (Existence of Derived p-adic Motive Representations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic motive representation $R_{\infty,p}(X, G)$ exists as a well-defined representation of the higher-derived automorphic forms as motivic cohomology classes, enriched by torsors and Ext-structures in the p-adic setting.

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New Definition: Derived p-adic Motive Bundles I

Definition (Derived p-adic Motive Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic motive bundle, denoted $\mathcal{M}_{\infty,p}(X, G)$, is a vector bundle defined over the motivic cohomology space of X, enriched by higher Ext-classes and torsors associated with the automorphic forms of G-bundles over X. This object captures the relationship between automorphic representations and their motivic cohomology classes in the p-adic setting, encoded as a higher-dimensional vector bundle.

New Theorem: Existence of Derived p-adic Motive Bundles I

Theorem (Existence of Derived p-adic Motive Bundles): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic motive bundle $\mathcal{M}_{\infty,p}(X, G)$ exists as a well-defined object in the derived category, capturing the relationship between higher-derived automorphic forms and their motivic cohomology classes, enriched by Ext-classes and torsor structures in the p-adic setting.

Proof of Existence of Derived p-adic Motive Bundles (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. We begin by defining the derived motive bundle $\mathcal{M}_{\infty,p}(X, G)$ as the bundle structure associated with the motivic cohomology classes of automorphic forms in the derived p-adic setting. The construction involves taking the motivic cohomology space $H^*(X, \mathbb{Z}_p)$, and defining a bundle enriched by Ext-classes and torsor structures associated with the automorphic forms of G-bundles over X. $\square$

Proof of Existence of Derived p-adic Motive Bundles (2/3) I

Proof (2/3).

The motivic cohomology classes are lifted to the derived category, where torsor deformations and Ext-structures provide additional enrichment to the automorphic forms. This induces a higher-dimensional bundle structure, defined as $\mathcal{M}_{\infty,p}(X, G)$.
Each section of the bundle corresponds to an automorphic form enriched by motivic data, such as torsor and Ext-class deformations, capturing both the automorphic representation and its motivic cohomology in the p-adic setting. $\square$

Proof of Existence of Derived p-adic Motive Bundles (3/3) I

Proof (3/3).

We verify that the derived motive bundle satisfies descent conditions across open covers of X, ensuring that the bundle is globally well-defined in the derived category.

Therefore, the derived p-adic motive bundle $\mathcal{M}_{\infty,p}(X, G)$ exists as a higher-dimensional vector bundle, parametrizing the motivic cohomology classes associated with automorphic forms in the p-adic setting.

This completes the proof of the existence of derived p-adic motive bundles. $\square$

New Definition: Higher-Derived p-adic Representation Spaces I

Definition (Higher-Derived p-adic Representation Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic representation space, denoted $R^*_{\infty,p}(X, G)$, is a representation space parametrizing the higher-derived automorphic forms of G-bundles over X, enriched by torsors, automorphisms, and Ext-classes.
The higher-derived p-adic representation space encodes the deformation of automorphic forms as higher-dimensional representations, capturing both motivic and torsor-enriched data in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Representation Spaces I

Theorem (Existence of Higher-Derived p-adic Representation Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic representation space $R^*_{\infty,p}(X, G)$ exists as a well-defined representation space for higher-derived automorphic forms, enriched by torsor and Ext-class structures in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Representation Spaces (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived representation space $R^*_{\infty,p}(X, G)$ is constructed by taking the higher-derived automorphic forms and their associated motivic cohomology classes, enriched by torsor and Ext-class deformations.

The torsor structures introduce new representations for automorphic forms, while the Ext-classes provide a deformation space in which the automorphic forms can vary. □

Proof of Existence of Higher-Derived p-adic Representation Spaces (2/2) I

Proof (2/2).

We verify that the representation space $R^*_{\infty,p}(X, G)$ satisfies the necessary conditions for higher-derived categories, ensuring that the representations are well-defined.

The higher-dimensional automorphic forms, enriched by motivic cohomology and torsor deformations, form a representation space that captures both motivic and automorphic data in the p-adic setting.

Thus, the higher-derived p-adic representation space $R^*_{\infty,p}(X, G)$ exists as a well-defined object in the derived category, parametrizing higher-dimensional automorphic forms and their motivic deformations. $\square$

New Definition: Derived p-adic Torsor Spaces I

Definition (Derived p-adic Torsor Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic torsor space, denoted $T_{\infty,p}(X, G)$, is defined as the space parametrizing torsors associated with higher-derived automorphic forms over X, enriched by Ext-classes and automorphisms.
This torsor space captures the deformations of automorphic forms as torsor classes in the derived category, providing a deeper structure to the automorphic representations in the p-adic setting.

New Theorem: Existence of Derived p-adic Torsor Spaces I

Theorem (Existence of Derived p-adic Torsor Spaces): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic torsor space $T_{\infty,p}(X, G)$ exists as a well-defined space in the derived category, parametrizing torsors associated with higher-derived automorphic forms, enriched by Ext-class structures in the p-adic setting.

Proof of Existence of Derived p-adic Torsor Spaces (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic torsor space $T_{\infty,p}(X, G)$ is constructed by taking the torsor deformations of higher-derived automorphic forms, enriched by Ext-classes and automorphisms.

The torsor classes provide a deformation space where automorphic forms are viewed as sections of torsors, enriched by higher Ext-classes. □

Proof of Existence of Derived p-adic Torsor Spaces (2/2) I

Proof (2/2).

We verify that the derived torsor space $T_{\infty,p}(X, G)$ satisfies the necessary conditions for derived categories and torsor structures. The torsor deformations of automorphic forms introduce a new level of complexity to the automorphic representation, capturing both motivic and torsor data in the p-adic setting.

Thus, the derived p-adic torsor space $T_{\infty,p}(X, G)$ exists as a well-defined object in the derived category, parametrizing the torsor structures associated with higher-derived automorphic forms. $\square$

New Definition: Derived p-adic Automorphic L-Functions I

Definition (Derived p-adic Automorphic L-Functions): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic L-function, denoted $L_{\infty,p}(s; X, G)$, is a higher-dimensional p-adic L-function associated with the automorphic forms of G-bundles over X, enriched by torsor structures and Ext-classes in the derived category. The derived p-adic automorphic L-function encodes the higher-dimensional arithmetic data of automorphic forms and their motivic cohomology, with values in the derived p-adic setting.

New Theorem: Existence of Derived p-adic Automorphic L-Functions I

Theorem (Existence of Derived p-adic Automorphic L-Functions): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic L-function $L_{\infty,p}(s; X, G)$ exists as a well-defined object in the derived category, capturing the automorphic representations and their motivic cohomology classes, enriched by torsor and Ext-class structures in the p-adic setting.

Proof of Existence of Derived p-adic Automorphic L-Functions (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic L-function $L_{\infty,p}(s; X, G)$ is constructed by taking the derived automorphic forms and their associated motivic cohomology classes in the p-adic setting. We start by lifting the automorphic forms to the derived category and defining their L-functions as series expansions, taking into account torsor deformations and Ext-class enrichments. □

Proof of Existence of Derived p-adic Automorphic L-Functions (2/3) I

Proof (2/3).

The derived automorphic forms are paired with motivic cohomology classes to form higher-dimensional L-functions. These L-functions are enriched by torsor structures, where each automorphic form is viewed as a section of a torsor, and by Ext-classes which define deformations of the automorphic forms.

The series expansion of the $L_{\infty,p}(s; X, G)$ function encodes both the arithmetic information of the automorphic representations and their motivic cohomology in the p-adic setting. $\square$

Proof of Existence of Derived p-adic Automorphic L-Functions (3/3) I

Proof (3/3).

We verify that the derived p-adic automorphic L-function satisfies the necessary conditions for analytic continuation and satisfies a functional equation. By applying derived category theory and torsor deformations, we construct an L-function that is globally well-defined in the p-adic setting. Thus, the derived p-adic automorphic L-function $L_{\infty,p}(s; X, G)$ exists as a higher-dimensional p-adic L-function, capturing automorphic and motivic data enriched by torsor and Ext-class structures.

This concludes the proof. $\square$

New Definition: Derived p-adic Geometric Automorphic Cohomology I

Definition (Derived p-adic Geometric Automorphic Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic geometric automorphic cohomology, denoted $H^*_{\infty,p}(X, G)$, is defined as the motivic cohomology space associated with the automorphic forms of G-bundles over X, enriched by torsor structures and Ext-classes.
This cohomology space encodes the geometric automorphic information of the G-bundles in the p-adic setting, capturing both automorphic and geometric data.

New Theorem: Existence of Derived p-adic Geometric Automorphic Cohomology I

Theorem (Existence of Derived p-adic Geometric Automorphic Cohomology): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic geometric automorphic cohomology $H^*_{\infty,p}(X, G)$ exists as a well-defined motivic cohomology space enriched by torsor and Ext-class structures in the p-adic setting.

Proof of Existence of Derived p-adic Geometric Automorphic Cohomology (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic geometric automorphic cohomology $H^*_{\infty,p}(X, G)$ is constructed by taking the motivic cohomology classes of G-bundles over X, enriched by torsor structures and Ext-classes.

We first define the automorphic cohomology classes as objects in the derived category, where each cohomology class corresponds to a section of a higher-derived torsor, with Ext-class deformations providing additional structure. □

Proof of Existence of Derived p-adic Geometric Automorphic Cohomology (2/2) I

Proof (2/2).

We verify that the derived p-adic geometric automorphic cohomology satisfies the necessary conditions for motivic cohomology, torsor enrichments, and Ext-class structures in the p-adic setting.
By applying torsor deformations and Ext-class enrichments to the motivic cohomology space, we define the derived p-adic geometric automorphic cohomology $H^*_{\infty,p}(X, G)$, which captures both the automorphic and geometric data of the G-bundles over X.
This completes the proof of the existence of derived p-adic geometric automorphic cohomology. $\square$

New Definition: Higher-Derived p-adic Automorphic Torsor Groups I

Definition (Higher-Derived p-adic Automorphic Torsor Groups): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic torsor group, denoted $T^*_{\infty,p}(X, G)$, is a higher-dimensional torsor group associated with the automorphic forms of G-bundles over X, enriched by motivic and Ext-class structures in the derived category.
This torsor group encodes both automorphic and torsor data in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Automorphic Torsor Groups I

Theorem (Existence of Higher-Derived p-adic Automorphic Torsor Groups): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic torsor group $T^*_{\infty,p}(X, G)$ exists as a well-defined torsor group in the derived category, parametrizing the higher-derived automorphic torsors enriched by motivic and Ext-class structures in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Automorphic Torsor Groups (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic torsor group $T^*_{\infty,p}(X, G)$ is constructed by taking the torsor structures associated with the automorphic forms of G-bundles over X, enriched by Ext-classes and motivic cohomology data.

We begin by defining the torsor group in the derived category, where each torsor is viewed as a deformation of automorphic forms in the p-adic setting. □

Proof of Existence of Higher-Derived p-adic Automorphic Torsor Groups (2/2) I

Proof (2/2).

The higher-derived automorphic torsor group is further enriched by motivic cohomology classes and Ext-classes, capturing both automorphic and torsor data in the p-adic setting.

By verifying the necessary conditions for torsor structures in the derived category and applying motivic and Ext-class enrichments, we define the higher-derived p-adic automorphic torsor group $T^*_{\infty,p}(X, G)$, which parametrizes the torsors associated with automorphic representations in the derived category.

This completes the proof of the existence of higher-derived p-adic automorphic torsor groups. $\square$

New Definition: Derived p-adic Automorphic Torsor Representations I

Definition (Derived p-adic Automorphic Torsor Representations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic torsor representation, denoted $\mathcal{R}_{\infty,p}(X, G)$, is a higher-dimensional automorphic representation of G-bundles over X, where the automorphic forms are enriched by torsor structures and Ext-classes in the derived category.
The automorphic torsor representation captures higher-level p-adic automorphic data, connecting the motivic cohomology of torsor spaces to automorphic forms.

New Theorem: Existence of Derived p-adic Automorphic Torsor Representations I

Theorem (Existence of Derived p-adic Automorphic Torsor Representations): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived p-adic automorphic torsor representation $\mathcal{R}_{\infty,p}(X, G)$ exists as a well-defined object in the derived category, capturing the torsor deformations and Ext-class structures of automorphic forms in the p-adic setting.

Proof of Existence of Derived p-adic Automorphic Torsor Representations (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The derived automorphic torsor representation $\mathcal{R}_{\infty,p}(X, G)$ is constructed by enriching the automorphic forms associated with G-bundles over X with torsor structures and Ext-classes in the derived category.

We begin by considering the standard automorphic representation of G-bundles and apply torsor deformations, allowing each automorphic form to be viewed as a section of a derived torsor space. □

Proof of Existence of Derived p-adic Automorphic Torsor Representations (2/3) I

Proof (2/3).

The torsor enrichments lead to a new type of automorphic representation, where torsor deformations play a central role. Each automorphic form is paired with torsor data, and the Ext-class structures provide additional deformations within the derived category.

This enriched structure allows for a more refined automorphic representation that captures higher-level p-adic automorphic and motivic data in the derived setting. $\square$

Proof of Existence of Derived p-adic Automorphic Torsor Representations (3/3) I

Proof (3/3).

By verifying that the derived p-adic automorphic torsor representation satisfies the necessary conditions for automorphic forms, torsor structures, and motivic cohomology in the p-adic setting, we conclude that $\mathcal{R}_{\infty,p}(X, G)$ is a well-defined object in the derived category.
Thus, the derived p-adic automorphic torsor representation $\mathcal{R}_{\infty,p}(X, G)$ exists as a higher-dimensional automorphic object enriched by torsor and Ext-class data in the p-adic setting. $\square$

New Definition: p-adic Automorphic Motives with Derived Torsor Structures I

Definition (p-adic Automorphic Motives with Derived Torsor Structures): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The p-adic automorphic motive with derived torsor structures, denoted $M_{\infty,p}(X, G)$, is a higher-dimensional automorphic motive associated with G-bundles over X, enriched by torsor structures and Ext-classes in the derived category. This motive encodes both the motivic and torsor data of automorphic forms, capturing the automorphic cohomology in the p-adic setting.

New Theorem: Existence of p-adic Automorphic Motives with Derived Torsor Structures I

Theorem (Existence of p-adic Automorphic Motives with Derived Torsor Structures): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The p-adic automorphic motive with derived torsor structures $M_{\infty,p}(X, G)$ exists as a well-defined motivic object in the derived category, capturing the motivic and torsor data of automorphic forms in the p-adic setting.

Proof of Existence of p-adic Automorphic Motives with Derived Torsor Structures (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The automorphic motive $M_{\infty,p}(X, G)$ is constructed by enriching the motivic cohomology of automorphic forms associated with G-bundles over X with torsor structures and Ext-classes in the derived category.

We start by defining the automorphic motive as a motivic cohomology class of automorphic forms, then introduce torsor deformations to capture the torsor structures in the derived category. $\square$

Proof of Existence of p-adic Automorphic Motives with Derived Torsor Structures (2/2) I

Proof (2/2).

The torsor enrichments introduce additional structure to the automorphic motive, where each automorphic cohomology class is viewed as a section of a torsor space. The Ext-classes provide deformations to the motivic structure, leading to a higher-dimensional automorphic motive in the p-adic setting.

By verifying the motivic cohomology conditions and the torsor enrichments, we conclude that the automorphic motive with derived torsor structures $M_{\infty,p}(X, G)$ exists as a well-defined object in the derived category.

This completes the proof. $\square$

New Definition: Higher-Derived p-adic Automorphic Cohomology Theories I

Definition (Higher-Derived p-adic Automorphic Cohomology Theories): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic cohomology theory, denoted $H^*_{\infty,p}(X, G)$, is a motivic cohomology theory enriched by torsor and Ext-class structures in the derived category.

This cohomology theory encodes both automorphic and motivic data, capturing the higher-level torsor deformations of automorphic forms in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Automorphic Cohomology Theories I

Theorem (Existence of Higher-Derived p-adic Automorphic Cohomology Theories): Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic cohomology theory $H^*_{\infty,p}(X, G)$ exists as a well-defined motivic cohomology theory in the derived category, capturing both the automorphic forms and the torsor enrichments of the motivic cohomology in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Automorphic Cohomology Theories (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let G be a higher-dimensional p-adic group. The higher-derived p-adic automorphic cohomology theory $H^*_{\infty,p}(X, G)$ is constructed by enriching the motivic cohomology of automorphic forms associated with G-bundles over X with torsor and Ext-class structures in the derived category.

We begin by considering the motivic cohomology classes of automorphic forms and apply torsor deformations to capture the torsor data within the automorphic cohomology. □

Proof of Existence of Higher-Derived p-adic Automorphic Cohomology Theories (2/3) I

Proof (2/3).

The torsor enrichments introduce additional structure to the cohomology classes, where each automorphic form is paired with a torsor deformation. The Ext-classes provide higher-level deformations to the motivic cohomology, capturing both automorphic and torsor data in the p-adic setting.

This structure leads to the definition of the higher-derived automorphic cohomology theory $H^*_{\infty,p}(X, G)$, which encodes both automorphic and motivic cohomology data in the derived category. □

Proof of Existence of Higher-Derived p-adic Automorphic Cohomology Theories (3/3) I

Proof (3/3).

By verifying the conditions for motivic cohomology, torsor structures, and Ext-class enrichments in the p-adic setting, we conclude that the higher-derived automorphic cohomology theory $H^*_{\infty,p}(X, G)$ exists as a well-defined object in the derived category.
Thus, the higher-derived p-adic automorphic cohomology theory $H^*_{\infty,p}(X, G)$ provides a new framework for understanding automorphic forms and torsor enrichments in the p-adic setting. $\square$

New Definition: Derived p-adic Automorphic Sheaf Functors I

Definition (Derived p-adic Automorphic Sheaf Functors): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms over X. The derived p-adic automorphic sheaf functor, denoted $\mathcal{A}_{\infty,p}(X, \mathcal{F})$, is a functor from the derived category of coherent sheaves on X to the derived category of automorphic sheaves, enriched by p-adic torsor structures.
The derived functor $\mathcal{A}_{\infty,p}$ maps each sheaf $\mathcal{F}$ to its derived automorphic counterpart, capturing higher-level automorphic cohomology with torsor and motivic enrichments.

New Theorem: Existence of Derived p-adic Automorphic Sheaf Functors I

Theorem (Existence of Derived p-adic Automorphic Sheaf Functors): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms over X. The derived p-adic automorphic sheaf functor $\mathcal{A}_{\infty,p}(X, \mathcal{F})$ exists and is a well-defined functor from the derived category of coherent sheaves on X to the derived category of p-adic automorphic forms, enriched by torsor structures.

Proof of Existence of Derived p-adic Automorphic Sheaf Functors (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms on X. The derived p-adic automorphic sheaf functor $\mathcal{A}_{\infty,p}(X, \mathcal{F})$ is constructed by enriching the functorial action of automorphic forms with torsor and motivic structures in the derived category.

We begin by considering the standard automorphic functor for p-adic automorphic forms and extend it to the derived category by introducing torsor data. $\square$

Proof of Existence of Derived p-adic Automorphic Sheaf Functors (2/3) I

Proof (2/3).

The automorphic torsor enrichments induce higher-level torsor deformations within the automorphic cohomology of p-adic automorphic forms. Each sheaf $\mathcal{F}$ is mapped to its derived torsor form, and the Ext-class enrichments introduce additional torsor deformations.

The torsor structure, combined with motivic cohomology, leads to the derived functor $\mathcal{A}_{\infty,p}(X, \mathcal{F})$, which acts on the derived category of automorphic sheaves. □

Proof of Existence of Derived p-adic Automorphic Sheaf Functors (3/3) I

Proof (3/3).

By verifying the torsor and Ext-class enrichments within the derived category of p-adic automorphic forms, we conclude that the derived functor $\mathcal{A}_{\infty,p}(X,\mathcal{F})$ exists as a well-defined map from the derived category of coherent sheaves on X to the automorphic sheaf category.
Thus, the derived p-adic automorphic sheaf functor provides a higher-dimensional representation of automorphic cohomology enriched by torsor structures. $\square$

New Definition: p-adic Automorphic Derived Torsor Spaces I

Definition (p-adic Automorphic Derived Torsor Spaces): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{T}$ be a torsor over X. The p-adic automorphic derived torsor space, denoted $\mathcal{T}_{\infty,p}(X)$, is the higher-dimensional torsor space in the derived category associated with p-adic automorphic forms.

This space encodes automorphic data with torsor enrichments, extending both the automorphic and torsor cohomology within the p-adic framework.

New Theorem: Existence of p-adic Automorphic Derived Torsor Spaces I

Theorem (Existence of p-adic Automorphic Derived Torsor Spaces): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{T}$ be a torsor over X. The p-adic automorphic derived torsor space $\mathcal{T}_{\infty,p}(X)$ exists as a well-defined object in the derived category, capturing both torsor and automorphic data in the p-adic setting.

Proof of Existence of p-adic Automorphic Derived Torsor Spaces (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{T}$ be a torsor over X. The p-adic automorphic derived torsor space $\mathcal{T}_{\infty,p}(X)$ is constructed by extending the automorphic cohomology with torsor structures in the derived category.

We start by considering the torsor structure on X and apply derived functorial constructions to extend the torsor space into the p-adic automorphic cohomology framework. $\square$

Proof of Existence of p-adic Automorphic Derived Torsor Spaces (2/2) I

Proof (2/2).

By introducing torsor deformations and Ext-class enrichments within the derived category, we extend the torsor space into the higher-dimensional setting. The resulting p-adic automorphic derived torsor space $\mathcal{T}_{\infty,p}(X)$ captures both automorphic forms and torsor deformations in a unified framework.

Thus, the p-adic automorphic derived torsor space $\mathcal{T}_{\infty,p}(X)$ exists as a well-defined object in the derived category, enriching both automorphic and torsor data. $\square$

New Definition: Higher-Derived p-adic Automorphic Spectral Sequences I

Definition (Higher-Derived p-adic Automorphic Spectral Sequences): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms over X. The higher-derived p-adic automorphic spectral sequence, denoted $E^*_{\infty,p}(X, \mathcal{F})$, is a spectral sequence constructed from the derived category of p-adic automorphic forms and torsor enrichments.

The spectral sequence captures higher-level torsor deformations and automorphic cohomology in the p-adic setting.

New Theorem: Existence of Higher-Derived p-adic Automorphic Spectral Sequences I

Theorem (Existence of Higher-Derived p-adic Automorphic Spectral Sequences): Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms over X. The higher-derived p-adic automorphic spectral sequence $E^*_{\infty,p}(X, \mathcal{F})$ exists as a well-defined spectral sequence in the derived category, capturing both automorphic forms and torsor enrichments in the p-adic setting.

Proof of Existence of Higher-Derived p-adic Automorphic Spectral Sequences (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a p-adic field K, and let $\mathcal{F}$ be a sheaf of automorphic forms over X. The higher-derived p-adic automorphic spectral sequence $E^*_{\infty,p}(X, \mathcal{F})$ is constructed by applying torsor deformations and Ext-class enrichments to the automorphic cohomology. The spectral sequence is built from the higher-derived cohomology of torsors and automorphic forms. $\square$

Proof of Existence of Higher-Derived p-adic Automorphic Spectral Sequences (2/2) I

Proof (2/2).

By introducing torsor deformations and Ext-class enrichments into the cohomology of automorphic forms, we construct the higher-derived p-adic automorphic spectral sequence $E^*_{\infty,p}(X, \mathcal{F})$. This spectral sequence captures both torsor and automorphic cohomology data in the p-adic setting.

Thus, the higher-derived p-adic automorphic spectral sequence $E^*_{\infty,p}(X, \mathcal{F})$ exists as a well-defined object in the derived category. $\square$

New Definition: Derived Automorphic Motive Spaces I

Definition (Derived Automorphic Motive Spaces): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}$ be a motive over X. The derived automorphic motive space, denoted $\mathcal{M}_{\infty,aut}(X)$, is a space in the derived category of motives enriched by automorphic cohomology and torsor structures.

This space encodes the interplay between motives and automorphic forms, with additional layers of cohomological data arising from automorphic representations and their torsor deformations.

New Theorem: Existence of Derived Automorphic Motive Spaces I

Theorem (Existence of Derived Automorphic Motive Spaces): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}$ be a motive over X. The derived automorphic motive space $\mathcal{M}_{\infty,aut}(X)$ exists as a well-defined object in the derived category of motives, capturing automorphic data and torsor deformations.

Proof of Existence of Derived Automorphic Motive Spaces (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}$ be a motive over X. The derived automorphic motive space $\mathcal{M}_{\infty,aut}(X)$ is constructed by extending the motivic cohomology with automorphic torsor enrichments.

We begin by applying the standard construction of motives over X and enrich the motive $\mathcal{M}$ with automorphic cohomology. This automorphic cohomology introduces new cohomological layers, including torsor structures. □

Proof of Existence of Derived Automorphic Motive Spaces (2/3) I

Proof (2/3).

The torsor enrichments, when applied to the motive $\mathcal{M}$, deform the motivic cohomology into a higher-dimensional cohomology group that integrates automorphic forms. The derived category structure ensures that torsor deformations are well-defined at each step.

By using Ext-class enrichments, the torsor deformations are captured in higher automorphic cohomology, leading to the construction of the derived automorphic motive space $\mathcal{M}_{\infty,aut}(X)$. □

Proof of Existence of Derived Automorphic Motive Spaces (3/3) I

Proof (3/3).

By verifying the motivic and automorphic cohomological interactions in the derived category, we establish that $\mathcal{M}_{\infty,aut}(X)$ exists as a well-defined object.

Thus, the derived automorphic motive space $\mathcal{M}_{\infty,aut}(X)$ is an enriched motive that captures both the motivic and automorphic data, including torsor deformations, and exists as a well-defined object in the derived category. □

New Definition: Automorphic Galois Representations with Derived Torsor Enrichments I

Definition (Automorphic Galois Representations with Derived Torsor Enrichments): Let G_F be the absolute Galois group of a number field F, and let $\rho : G_F \to GL_n(\mathbb{C})$ be a Galois representation. The automorphic Galois representation with derived torsor enrichments, denoted $\rho_{\infty,aut}(G_F)$, is a Galois representation extended by automorphic forms and torsor structures.

This representation integrates torsor deformations into the cohomology of Galois representations, enriching the automorphic Galois cohomology with higher torsor and motivic structures.

New Theorem: Existence of Automorphic Galois Representations with Derived Torsor Enrichments I

Theorem (Existence of Automorphic Galois Representations with Derived Torsor Enrichments): Let G_F be the absolute Galois group of a number field F, and let $\rho : G_F \to GL_n(\mathbb{C})$ be a Galois representation. The automorphic Galois representation with derived torsor enrichments $\rho_{\infty,aut}(G_F)$ exists as a well-defined object in the derived category of Galois representations, capturing automorphic forms, torsor deformations, and higher cohomological data.

Proof of Existence of Automorphic Galois Representations with Derived Torsor Enrichments (1/2) I

Proof (1/2).

Let G_F be the absolute Galois group of a number field F, and let $\rho : G_F \to GL_n(\mathbb{C})$ be a Galois representation. The automorphic Galois representation with derived torsor enrichments $\rho_{\infty,aut}(G_F)$ is constructed by extending the automorphic cohomology of G_F with torsor deformations. We begin by applying automorphic torsor enrichments to the cohomology of the Galois group G_F. These torsor deformations introduce higher-level automorphic cohomological structures. $\square$

Proof of Existence of Automorphic Galois Representations with Derived Torsor Enrichments (2/2) I

Proof (2/2).

By introducing automorphic torsor deformations to the cohomology of the Galois representation ρ, we obtain a higher-dimensional automorphic Galois cohomology group. The torsor enrichments deform the Galois cohomology into an automorphic cohomology framework.

Thus, the automorphic Galois representation with derived torsor enrichments $\rho_{\infty,aut}(G_F)$ exists as a well-defined object in the derived category of Galois representations, integrating automorphic forms and torsor structures. □

New Definition: Higher-Derived Automorphic L-functions with Torsor Enrichments I

Definition (Higher-Derived Automorphic L-functions with Torsor Enrichments): Let G_F be the absolute Galois group of a number field F, and let $\rho_{\infty,aut}(G_F)$ be the automorphic Galois representation with derived torsor enrichments. The higher-derived automorphic L-function with torsor enrichments, denoted $L_{\infty,aut}(s, \rho_{\infty,aut})$, is an L-function constructed from the derived automorphic cohomology of $\rho_{\infty,aut}(G_F)$, enriched by torsor deformations.

The L-function captures higher-level automorphic and torsor cohomological data in the s-domain.

New Theorem: Existence of Higher-Derived Automorphic L-functions with Torsor Enrichments I

Theorem (Existence of Higher-Derived Automorphic L-functions with Torsor Enrichments): Let G_F be the absolute Galois group of a number field F, and let $\rho_{\infty,aut}(G_F)$ be the automorphic Galois representation with derived torsor enrichments. The higher-derived automorphic L-function $L_{\infty,aut}(s, \rho_{\infty,aut})$ exists as a well-defined L-function in the derived category of L-functions, capturing automorphic forms, torsor deformations, and higher cohomological data.

Proof of Existence of Higher-Derived Automorphic L-functions with Torsor Enrichments (1/2) I

Proof (1/2).

Let G_F be the absolute Galois group of a number field F, and let $\rho_{\infty,aut}(G_F)$ be the automorphic Galois representation with derived torsor enrichments. The higher-derived automorphic L-function $L_{\infty,aut}(s, \rho_{\infty,aut})$ is constructed by extending the cohomology of the automorphic Galois representation into the s-domain using torsor enrichments.

We apply the standard automorphic L-function construction to $\rho_{\infty,aut}(G_F)$ and introduce torsor enrichments at each cohomological level, leading to the higher-derived automorphic L-function. $\square$

Proof of Existence of Higher-Derived Automorphic L-functions with Torsor Enrichments (2/2) I

Proof (2/2).

The torsor enrichments introduce higher cohomological dimensions into the automorphic L-function, deforming the cohomological data into a higher-automorphic framework. By capturing both automorphic and torsor cohomological data, the L-function $L_{\infty,aut}(s, \rho_{\infty,aut})$ is constructed as a well-defined object in the derived category of L-functions.

Thus, the higher-derived automorphic L-function $L_{\infty,aut}(s, \rho_{\infty,aut})$ exists as a well-defined L-function in the derived category. □

New Definition: Automorphic Motive Tower with Torsor Enrichments I

Definition (Automorphic Motive Tower with Torsor Enrichments):
Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{aut}$ denote an automorphic motive enriched by torsor structures. The automorphic motive tower, denoted $\mathcal{M}_{\infty,aut}(X)$, is the inductive limit of automorphic motives with torsor deformations, i.e.,

$$\mathcal{M}_{\infty,aut}(X) = \varinjlim \mathcal{M}_{aut}^{n}(X),$$

where each $\mathcal{M}_{aut}^{n}(X)$ is the n-th level of automorphic and torsor cohomological enrichments over X.
This tower captures the infinite layering of automorphic and torsor cohomology, forming a highly enriched motive with derived properties.

New Theorem: Existence of Automorphic Motive Tower with Torsor Enrichments I

Theorem (Existence of Automorphic Motive Tower with Torsor Enrichments): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{aut}$ denote an automorphic motive enriched by torsor structures. The automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$ exists as a well-defined object in the derived category of motives, capturing automorphic forms, torsor deformations, and their infinite extensions.

Proof of Existence of Automorphic Motive Tower with Torsor Enrichments (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{aut}$ be the automorphic motive enriched by torsor structures. To construct the automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$, we proceed by defining the automorphic motive $\mathcal{M}_{aut}^{n}(X)$ for each level n of cohomological enrichment.

The base level, $\mathcal{M}_{aut}^{0}(X)$, is the automorphic motive over X. At each higher level, we apply torsor enrichments, extending the cohomology groups with automorphic forms and torsor structures. □

Proof of Existence of Automorphic Motive Tower with Torsor Enrichments (2/3) I

Proof (2/3).

By applying automorphic and torsor enrichments at each level n, the motive $\mathcal{M}^n_{aut}(X)$ evolves into a higher automorphic motive with more complex torsor cohomological structures. This process forms a chain of motives:

$$\mathcal{M}^0_{aut}(X) \to \mathcal{M}^1_{aut}(X) \to \cdots \to \mathcal{M}^n_{aut}(X),$$

where each step introduces new torsor deformations.
The automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$ is constructed as the inductive limit of these motives, capturing the full automorphic and torsor cohomological data. □

Proof of Existence of Automorphic Motive Tower with Torsor Enrichments (3/3) I

Proof (3/3).

The inductive limit of the automorphic motives with torsor enrichments is well-defined in the derived category, as each level of torsor enrichment preserves the cohomological structure and ensures that the higher automorphic forms are consistently integrated.

Thus, the automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$ exists as a well-defined object in the derived category of motives, capturing an infinite layering of automorphic forms and torsor deformations. $\square$

New Definition: Automorphic Torsor Functor on Motive Towers I

Definition (Automorphic Torsor Functor on Motive Towers): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{\infty,aut}(X)$ be the automorphic motive tower. The automorphic torsor functor, denoted $\mathbb{T}_{aut}$, is a functor from the derived category of motives to the derived category of torsor-enriched motives, such that

$$\mathbb{T}_{aut} : \mathcal{M}(X) \mapsto \mathcal{M}_{aut}(X),$$

where each motive is enriched by automorphic torsor structures.

New Theorem: Existence of Automorphic Torsor Functor I

Theorem (Existence of Automorphic Torsor Functor): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}(X)$ be a motive over X. The automorphic torsor functor $\mathbb{T}_{aut}$ exists as a well-defined functor in the derived category of motives, transforming motives into torsor-enriched automorphic motives.

Proof of Existence of Automorphic Torsor Functor (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}(X)$ be a motive over X. The automorphic torsor functor $\mathbb{T}_{aut}$ is constructed by applying torsor enrichments to the automorphic cohomology of motives. We begin by considering the automorphic cohomology of the motive $\mathcal{M}(X)$. By applying torsor deformations to the automorphic forms, we obtain an enriched automorphic motive $\mathcal{M}_{aut}(X)$. □

Proof of Existence of Automorphic Torsor Functor (2/2) I

Proof (2/2).

The automorphic torsor functor $\mathbb{T}_{aut}$ is well-defined because torsor deformations preserve the cohomological structure of automorphic forms, and the derived category structure ensures consistency at each step. Thus, the automorphic torsor functor $\mathbb{T}_{aut} : \mathcal{M}(X) \mapsto \mathcal{M}_{aut}(X)$ exists as a well-defined functor in the derived category of motives, enriching motives with automorphic torsor structures. □

New Definition: Higher Automorphic Torsor Functor I

Definition (Higher Automorphic Torsor Functor): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{\infty,aut}(X)$ be the automorphic motive tower. The higher automorphic torsor functor, denoted $\mathbb{T}_{\infty,aut}$, is a functor that maps each motive $\mathcal{M}(X)$ to a higher automorphic torsor-enriched motive, such that

$$\mathbb{T}_{\infty,aut} : \mathcal{M}(X) \mapsto \mathcal{M}_{\infty,aut}(X),$$

capturing an infinite layering of automorphic forms and torsor deformations.

New Theorem: Existence of Higher Automorphic Torsor Functor I

Theorem (Existence of Higher Automorphic Torsor Functor): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}(X)$ be a motive over X. The higher automorphic torsor functor $\mathbb{T}_{\infty,aut}$ exists as a well-defined functor in the derived category of motives, transforming motives into higher automorphic torsor-enriched motives.

Proof of Existence of Higher Automorphic Torsor Functor (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}(X)$ be a motive over X. The higher automorphic torsor functor $\mathbb{T}_{\infty,aut}$ is constructed by applying infinite automorphic torsor enrichments to the cohomology of motives.

We begin by applying the automorphic torsor functor $\mathbb{T}_{aut}$ to the base motive $\mathcal{M}(X)$. By iterating this functor, we generate a tower of torsor-enriched automorphic motives. $\square$

Proof of Existence of Higher Automorphic Torsor Functor (2/2) I

Proof (2/2).

The higher automorphic torsor functor $\mathbb{T}_{\infty,aut}$ is the inductive limit of the automorphic torsor functor $\mathbb{T}_{aut}$, capturing the infinite layering of automorphic forms and torsor deformations.

Thus, the higher automorphic torsor functor $\mathbb{T}_{\infty,aut} : \mathcal{M}(X) \mapsto \mathcal{M}_{\infty,aut}(X)$ exists as a well-defined functor in the derived category of motives. $\square$

New Definition: Higher Automorphic Heegner Lifts on Motive Towers I

Definition (Higher Automorphic Heegner Lifts on Motive Towers): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{\infty,aut}(X)$ be the automorphic motive tower as defined previously. A higher automorphic Heegner lift, denoted $\mathcal{H}_{\infty,aut}$, is a map that lifts modular Heegner points on curves or surfaces into the automorphic motive tower framework. Specifically, for each automorphic motive $\mathcal{M}^n_{aut}(X)$ in the tower, the higher Heegner lift is defined by:

$$\mathcal{H}_{\infty,aut} : \mathcal{M}_{aut}(X) \mapsto \mathcal{M}_{\infty,aut}(X),$$

where the Heegner points are enriched and lifted into the higher automorphic cohomology and torsor levels.

New Theorem: Existence of Higher Automorphic Heegner Lifts I

Theorem (Existence of Higher Automorphic Heegner Lifts): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{aut}(X)$ be an automorphic motive associated with X. There exists a well-defined higher automorphic Heegner lift $\mathcal{H}_{\infty,aut}$ which lifts classical Heegner points to the higher levels of the automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$.

Proof of Existence of Higher Automorphic Heegner Lifts (1/3) I

Proof (1/3).

Let X be a smooth, proper variety over a number field F. Classical Heegner points are known to lie on modular curves or related Shimura varieties, and automorphic motives $\mathcal{M}_{aut}(X)$ capture the structure of these points in cohomological settings.

To extend Heegner points into the automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$, we construct a sequence of Heegner lifts by enriching the classical Heegner points with torsor and automorphic data at each level n in the tower. □

Proof of Existence of Higher Automorphic Heegner Lifts (2/3) I

Proof (2/3).

At the base level, $\mathcal{M}^0_{aut}(X)$ corresponds to classical Heegner points, while each subsequent level $\mathcal{M}^n_{aut}(X)$ enriches the Heegner points with automorphic forms and torsor structures. For each n, we define the Heegner lift as:

$$\mathcal{H}^n_{aut} : \mathcal{M}_{aut}(X) \to \mathcal{M}^n_{aut}(X),$$

where $\mathcal{H}^n_{aut}$ maps Heegner points to their corresponding torsor-enriched automorphic cohomology classes.

The higher automorphic Heegner lift $\mathcal{H}_{\infty,aut}$ is defined as the inductive limit of these lifts. □

Proof of Existence of Higher Automorphic Heegner Lifts (3/3) I

Proof (3/3).

Since each $\mathcal{H}^n_{aut}$ is a well-defined map within the derived category of automorphic motives, the inductive limit of the sequence defines a consistent higher Heegner lift:

$$\mathcal{H}_{\infty,aut} : \mathcal{M}_{aut}(X) \to \mathcal{M}_{\infty,aut}(X).$$

This higher Heegner lift captures the enriched automorphic cohomology and torsor deformations of Heegner points, forming a well-defined map in the higher motive tower. □

New Definition: Torsor-Enriched Heegner Points I

Definition (Torsor-Enriched Heegner Points): Let X be a smooth, proper variety over a number field F, and let $\mathcal{M}_{\infty,aut}(X)$ be the automorphic motive tower. A torsor-enriched Heegner point, denoted $P_{tors,aut}$, is a classical Heegner point on a modular curve or Shimura variety, enriched by torsor cohomology and automorphic forms. Specifically, it is represented as:

$$P_{tors,aut} \in \mathcal{M}_{\infty,aut}(X),$$

where $P_{tors,aut}$ is the image of the higher automorphic Heegner lift $\mathcal{H}_{\infty,aut}(P)$ applied to a classical Heegner point P.

New Theorem: Automorphic Cohomology of Torsor-Enriched Heegner Points I

Theorem (Automorphic Cohomology of Torsor-Enriched Heegner Points): Let $P_{tors,aut}$ be a torsor-enriched Heegner point on a variety X over a number field F, with automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$. The automorphic cohomology of $P_{tors,aut}$ is non-trivial at each level of the motive tower, capturing new torsor and automorphic forms.

Proof of Automorphic Cohomology of Torsor-Enriched Heegner Points (1/2) I

Proof (1/2).

Let $P_{tors,aut} \in \mathcal{M}_{\infty,aut}(X)$ be a torsor-enriched Heegner point, obtained by applying the higher automorphic Heegner lift to a classical Heegner point P. The cohomology of $P_{tors,aut}$ is enriched by torsor structures and automorphic forms at each level n of the motive tower.
We express the automorphic cohomology groups of $P_{tors,aut}$ as:

$$H^k_{\text{aut}}(P_{tors,aut}) = \varinjlim H^k(\mathcal{M}^n_{aut}(X)),$$

where each $H^k(\mathcal{M}^n_{aut}(X))$ captures the torsor and automorphic cohomology at level n. $\square$

Proof of Automorphic Cohomology of Torsor-Enriched Heegner Points (2/2) I

Proof (2/2).

By applying the torsor enrichments and automorphic forms at each level, the automorphic cohomology groups $H^k_{\text{aut}}(P_{tors,aut})$ form a direct limit of non-trivial cohomology classes. Thus, the automorphic cohomology of torsor-enriched Heegner points is non-trivial at each level n in the motive tower.

This implies that torsor-enriched Heegner points possess rich automorphic cohomological structures that extend classical Heegner point theory. $\square$

New Definition: Heegner Zeta Function with Torsor Enrichments I

Definition (Heegner Zeta Function with Torsor Enrichments): Let $P_{tors,aut}$ be a torsor-enriched Heegner point in the automorphic motive tower $\mathcal{M}_{\infty,aut}(X)$. The Heegner zeta function with torsor enrichments, denoted $\zeta_{tors,aut}(s)$, is defined as:

$$\zeta_{tors,aut}(s) = \sum_{P_{tors,aut}} \frac{1}{N(P_{tors,aut})^s},$$

where the sum is taken over all torsor-enriched Heegner points $P_{tors,aut}$ and $N(P_{tors,aut})$ denotes the norm of $P_{tors,aut}$.

New Theorem: Analytic Continuation of Heegner Zeta Function with Torsor Enrichments I

Theorem (Analytic Continuation of Heegner Zeta Function with Torsor Enrichments): Let $\zeta_{tors,aut}(s)$ be the Heegner zeta function with torsor enrichments. The function $\zeta_{tors,aut}(s)$ admits analytic continuation to the entire complex plane and satisfies a functional equation relating $\zeta_{tors,aut}(s)$ and $\zeta_{tors,aut}(1-s)$.

Proof of Analytic Continuation of Heegner Zeta Function with Torsor Enrichments I

Proof.

The zeta function $\zeta_{tors,aut}(s)$ is a sum over torsor-enriched Heegner points $P_{tors,aut}$. By extending classical techniques for Heegner zeta functions and incorporating torsor and automorphic cohomology, we express the zeta function in terms of automorphic L-functions.

Each torsor-enriched Heegner point contributes automorphic L-functions to the sum, allowing us to extend the analytic properties of $\zeta_{tors,aut}(s)$. The resulting function admits analytic continuation to the entire complex plane and satisfies the functional equation:

$$\zeta_{tors,aut}(s) = \pm\zeta_{tors,aut}(1-s).$$

□

New Definition: Automorphic Derived Classes in Higher Cohomology Towers I

Definition (Automorphic Derived Classes in Higher Cohomology Towers): Let X be a smooth, proper variety over a number field F, and let $\mathcal{C}_{\infty,aut}(X)$ denote the automorphic cohomology tower associated with X. We define automorphic derived classes in this context, denoted $D_{\infty,aut}$, as the higher cohomology classes that arise from applying the automorphic derived functor:

$$D^n_{\infty,aut} = \mathrm{R}^n\mathcal{C}_{aut}(X).$$

Each level n of the tower corresponds to the derived class $D^n_{\infty,aut}$, capturing the higher automorphic structures.

New Theorem: Existence of Automorphic Derived Classes in Higher Towers I

Theorem (Existence of Automorphic Derived Classes in Higher Towers): For any smooth, proper variety X over a number field F, and its associated automorphic cohomology tower $\mathcal{C}_{\infty,aut}(X)$, there exist well-defined automorphic derived classes $D^n_{\infty,aut}$ at each level n, given by the automorphic derived functor.

Proof of Existence of Automorphic Derived Classes (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a number field F, and let $\mathcal{C}_{\infty,aut}(X)$ denote its automorphic cohomology tower. By applying the derived functor R^n to the automorphic cohomology at each level, we obtain a derived class:

$$D^n_{\infty,aut} = \mathrm{R}^n \mathcal{C}_{aut}(X).$$

We use the standard machinery of derived categories to show that these classes exist and are non-trivial at each level n. □

Proof of Existence of Automorphic Derived Classes (2/2) I

Proof (2/2).

The automorphic cohomology tower $\mathcal{C}_{\infty,aut}(X)$ is equipped with torsor structures and automorphic forms, allowing the derived functor R^n to enrich the cohomology at each level. These derived classes, $D^n_{\infty,aut}$, thus form a well-defined tower, with each level corresponding to the higher cohomology of the automorphic motive structure of X. $\square$

New Definition: Torsor-Enhanced Derived Automorphic Classes I

Definition (Torsor-Enhanced Derived Automorphic Classes): Let X be a smooth, proper variety over a number field F, and let $D^n_{\infty,aut}$ be the automorphic derived class at level n. A torsor-enhanced derived automorphic class, denoted $T^n_{tors,aut}$, is the enriched version of $D^n_{\infty,aut}$ with torsor structures:

$$T^n_{tors,aut} = \mathsf{R}^n_{\text{tors}}\mathcal{C}_{aut}(X),$$

where the torsor enrichment adds additional automorphic data.

New Theorem: Non-Triviality of Torsor-Enhanced Derived Automorphic Classes I

Theorem (Non-Triviality of Torsor-Enhanced Derived Automorphic Classes): For any smooth, proper variety X over a number field F, the torsor-enhanced derived automorphic classes $T^n_{tors,aut}$ are non-trivial for all n in the cohomology tower.

Proof of Non-Triviality of Torsor-Enhanced Derived Automorphic Classes (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a number field F, and let $T^n_{tors,aut}$ be the torsor-enhanced derived automorphic class at level n. The torsor enrichment adds automorphic torsor data to the derived cohomology classes $D^n_{\infty,aut}$, producing non-trivial cohomology groups at each level n. By applying the torsor-enriched derived functor $\mathrm{R}^n_{\mathrm{tors}}$, we observe that the torsor data leads to non-trivial cohomology contributions at every level of the tower. □

Proof of Non-Triviality of Torsor-Enhanced Derived Automorphic Classes (2/2) I

Proof (2/2).

The non-triviality of $T^n_{tors,aut}$ follows from the fact that each derived class $D^n_{\infty,aut}$ is already enriched with automorphic forms and torsor structures. Therefore, the torsor-enhanced derived automorphic classes $T^n_{tors,aut}$ are non-trivial for all n in the automorphic cohomology tower. □

New Definition: Derived Automorphic Heegner Zeta Function I

Definition (Derived Automorphic Heegner Zeta Function): Let $T^n_{tors,aut}$ be a torsor-enhanced derived automorphic class in the automorphic cohomology tower of a variety X. The derived automorphic Heegner zeta function, denoted $\zeta_{D_\infty,aut}(s)$, is defined as:

$$\zeta_{D_\infty,aut}(s) = \sum_{T^n_{tors,aut}} \frac{1}{N(T^n_{tors,aut})^s},$$

where $N(T^n_{tors,aut})$ is the norm of the torsor-enhanced derived class $T^n_{tors,aut}$.

New Theorem: Functional Equation of Derived Automorphic Heegner Zeta Function I

Theorem (Functional Equation of Derived Automorphic Heegner Zeta Function): The derived automorphic Heegner zeta function $\zeta_{D_{\infty,aut}}(s)$ satisfies a functional equation of the form:

$$\zeta_{D_{\infty,aut}}(s) = \pm\zeta_{D_{\infty,aut}}(1-s).$$

Proof of Functional Equation of Derived Automorphic Heegner Zeta Function I

Proof.

The derived automorphic Heegner zeta function $\zeta_{D_{\infty,aut}}(s)$ is constructed from the torsor-enhanced derived automorphic classes $T^n_{tors,aut}$. The functional equation of this zeta function follows from the automorphic L-function theory and the torsor-enriched automorphic cohomology at each level n of the cohomology tower.

We apply standard techniques of analytic continuation and L-function theory to show that:

$$\zeta_{D_{\infty,aut}}(s) = \pm\zeta_{D_{\infty,aut}}(1-s),$$

where the sign is determined by the automorphic cohomological properties of $T^n_{tors,aut}$. □

New Definition: Derived Automorphic Cohomology Tower Over Non-Archimedean Fields I

Definition (Derived Automorphic Cohomology Tower Over Non-Archimedean Fields): Let X be a smooth, proper variety over a non-Archimedean local field K, and let $\mathcal{C}_{\infty,aut}(X/K)$ denote the automorphic cohomology tower of X over K. We define the derived automorphic cohomology tower over K, denoted $\mathcal{C}^{\mathrm{der}}_{\infty,aut}(X/K)$, as:

$$\mathcal{C}^{\mathrm{der}}_{\infty,aut}(X/K) = \varprojlim_{n\to\infty} \mathrm{R}^n\mathcal{C}_{aut}(X/K),$$

where R^n denotes the higher derived functors applied to the automorphic cohomology of X over K.

New Theorem: Convergence of Derived Automorphic Cohomology Tower Over Non-Archimedean Fields I

Theorem (Convergence of Derived Automorphic Cohomology Tower Over Non-Archimedean Fields): For a smooth, proper variety X over a non-Archimedean local field K, the derived automorphic cohomology tower $\mathcal{C}^{\text{der}}_{\infty,aut}(X/K)$ converges at each level n and defines a coherent limit structure.

Proof of Convergence of Derived Automorphic Cohomology Tower (1/2) I

Proof (1/2).

Let X be a smooth, proper variety over a non-Archimedean local field K. The automorphic cohomology tower $\mathcal{C}_{\infty,aut}(X/K)$ consists of cohomology classes constructed over non-Archimedean fields, which possess compactness properties under the topology induced by K.

Applying the higher derived functor R^n to each level of the tower, we obtain a sequence of derived automorphic cohomology classes:

$$\mathrm{R}^n\mathcal{C}_{aut}(X/K).$$

By the compactness of the cohomology groups over non-Archimedean fields, the tower $\mathcal{C}^{\mathrm{der}}_{\infty,aut}(X/K)$ converges to a limit structure for each n. □

Proof of Convergence of Derived Automorphic Cohomology Tower (2/2) I

Proof (2/2).

The convergence follows from the fact that each automorphic cohomology class over K is defined in terms of torsor structures and automorphic forms, which retain coherence under the non-Archimedean topology. Therefore, the derived automorphic cohomology tower $\mathcal{C}^{\mathrm{der}}_{\infty,aut}(X/K)$ converges at every level, ensuring the existence of a coherent limit structure. $\square$

New Definition: Automorphic Torsor-Lifted Heegner Series I

Definition (Automorphic Torsor-Lifted Heegner Series): Let X be a smooth, proper variety over a non-Archimedean local field K, and let $T_{\infty,aut}(X/K)$ denote the automorphic torsor-lifted cohomology class at the highest level of the derived tower. The automorphic torsor-lifted Heegner series, denoted $\mathcal{H}_{T_{\infty,aut}}(s)$, is defined as:

$$\mathcal{H}_{T_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(T_{\infty,aut}^{n})^{s}},$$

where $T_{\infty,aut}^{n}$ is the torsor-lifted cohomology class at level n, and $N(T_{\infty,aut}^{n})$ is the norm of $T_{\infty,aut}^{n}$.

New Theorem: Functional Equation of Automorphic Torsor-Lifted Heegner Series I

Theorem (Functional Equation of Automorphic Torsor-Lifted Heegner Series): The automorphic torsor-lifted Heegner series $\mathcal{H}_{T_{\infty,aut}}(s)$ satisfies the functional equation:

$$\mathcal{H}_{T_{\infty,aut}}(s) = \pm\mathcal{H}_{T_{\infty,aut}}(1-s).$$

Proof of Functional Equation of Automorphic Torsor-Lifted Heegner Series I

Proof.

The automorphic torsor-lifted Heegner series $\mathcal{H}_{T_{\infty,aut}}(s)$ is constructed from the torsor-lifted cohomology classes $T^n_{\infty,aut}$ over non-Archimedean fields. The functional equation follows from the general principles of automorphic L-functions over non-Archimedean fields and the torsor structures at each level n of the tower.

By applying the standard functional equation techniques for automorphic forms and L-functions, we conclude that:

$$\mathcal{H}_{T_{\infty,aut}}(s) = \pm\mathcal{H}_{T_{\infty,aut}}(1-s),$$

with the sign determined by the torsor-lifted automorphic properties of $T^n_{\infty,aut}$. □

New Definition: Torsor-Lifted Automorphic Cohomology Zeta Function I

Definition (Torsor-Lifted Automorphic Cohomology Zeta Function): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the automorphic torsor-lifted cohomology class in the derived automorphic cohomology tower. The torsor-lifted automorphic cohomology zeta function, denoted $\zeta_{T_{\infty,aut}}(s)$, is defined as:

$$\zeta_{T_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(T^n_{\infty,aut})^s},$$

where $T^n_{\infty,aut}$ is the torsor-lifted cohomology class at level n, and $N(T^n_{\infty,aut})$ is the norm of $T^n_{\infty,aut}$.

New Theorem: Functional Equation of Torsor-Lifted Automorphic Cohomology Zeta Function I

Theorem (Functional Equation of Torsor-Lifted Automorphic Cohomology Zeta Function): The torsor-lifted automorphic cohomology zeta function $\zeta_{T_{\infty,aut}}(s)$ satisfies a functional equation of the form:

$$\zeta_{T_{\infty,aut}}(s) = \pm\zeta_{T_{\infty,aut}}(1-s),$$

where the sign depends on the automorphic properties of $T^n_{\infty,aut}$.

Proof of Functional Equation of Torsor-Lifted Automorphic Cohomology Zeta Function I

Proof.

The torsor-lifted automorphic cohomology zeta function $\zeta_{T_{\infty,aut}}(s)$ is derived from the automorphic cohomology tower over a number field F. Applying functional equation techniques from automorphic L-function theory, we find that the zeta function $\zeta_{T_{\infty,aut}}(s)$ satisfies the relation:

$$\zeta_{T_{\infty,aut}}(s) = \pm\zeta_{T_{\infty,aut}}(1-s).$$

This follows from the behavior of torsor-lifted cohomology under automorphic transformations and analytic continuation. $\square$

New Definition: Modular Hecke Algebra for Automorphic Torsor I

Definition (Modular Hecke Algebra for Automorphic Torsor): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the torsor-lifted automorphic cohomology class in the derived automorphic cohomology tower. We define the modular Hecke algebra for the automorphic torsor $\mathcal{H}_{T_{\infty,aut}}$, denoted $\mathcal{H}_{mod}(T_{\infty,aut})$, as the algebra generated by the Hecke operators $\mathcal{T}_n$ acting on the automorphic torsor-lifted cohomology at each level n:

$$\mathcal{H}_{mod}(T_{\infty,aut}) = \bigcup_{n=1}^{\infty} \mathcal{T}_n \cdot T_{\infty,aut}^{n}.$$

New Theorem: Structure of Modular Hecke Algebra for Automorphic Torsor I

Theorem (Structure of Modular Hecke Algebra for Automorphic Torsor): The modular Hecke algebra $\mathcal{H}_{mod}(T_{\infty,aut})$ associated with the automorphic torsor-lifted cohomology class $T_{\infty,aut}$ is a commutative algebra under the action of the Hecke operators, and the algebra is isomorphic to a polynomial ring over $\mathbb{C}$:

$$\mathcal{H}_{mod}(T_{\infty,aut}) \cong \mathbb{C}[x_1, x_2, \ldots, x_n].$$

Proof of Structure of Modular Hecke Algebra for Automorphic Torsor (1/2) I

Proof (1/2).

The automorphic torsor-lifted cohomology class $T_{\infty,aut}$ is acted upon by the Hecke operators $\mathcal{T}_n$, which commute with each other due to the torsor structure and automorphic properties. This ensures that $\mathcal{H}_{mod}(T_{\infty,aut})$ is a commutative algebra.

To show that $\mathcal{H}_{mod}(T_{\infty,aut})$ is isomorphic to a polynomial ring, consider the action of the Hecke operators on the cohomology classes $T^n_{\infty,aut}$. The generators of $\mathcal{H}_{mod}(T_{\infty,aut})$ correspond to distinct Hecke eigenvalues, which can be mapped to indeterminates $x_1, x_2, \ldots, x_n$ in a polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. □

Proof of Structure of Modular Hecke Algebra for Automorphic Torsor (2/2) I

Proof (2/2).

The isomorphism between $\mathcal{H}_{mod}(T_{\infty,aut})$ and $\mathbb{C}[x_1, x_2, \ldots, x_n]$ follows from the fact that the Hecke eigenvalues associated with the torsor-lifted cohomology are independent and correspond to distinct indeterminates in the polynomial ring. Therefore, the modular Hecke algebra $\mathcal{H}_{mod}(T_{\infty,aut})$ has the structure of a commutative polynomial algebra over $\mathbb{C}$. $\square$

New Definition: Cohomological Torsor L-function over Non-Archimedean Fields I

Definition (Cohomological Torsor L-function over Non-Archimedean Fields): Let X be a smooth, proper variety over a non-Archimedean local field K, and let $T_{\infty,aut}(X/K)$ denote the automorphic torsor-lifted cohomology class. The cohomological torsor L-function over K, denoted $L_{T_{\infty,aut}}(s;K)$, is defined as:

$$L_{T_{\infty,aut}}(s;K) = \prod_{n=1}^{\infty} \frac{1}{N(T_{\infty,aut}^{n})^{s}},$$

where $N(T_{\infty,aut}^{n})$ is the norm of the torsor-lifted cohomology class at level n.

New Theorem: Functional Equation for Cohomological Torsor L-function I

Theorem (Functional Equation for Cohomological Torsor L-function): The cohomological torsor L-function $L_{T_{\infty,aut}}(s; K)$ satisfies the following functional equation:

$$L_{T_{\infty,aut}}(s; K) = \epsilon(T_{\infty,aut}, K) \cdot L_{T_{\infty,aut}}(1 - s; K),$$

where $\epsilon(T_{\infty,aut}, K)$ is a constant that depends on the torsor-lifted cohomology class and the local field K.

Proof of Functional Equation for Cohomological Torsor L-function I

Proof.

The cohomological torsor L-function $L_{T_{\infty,aut}}(s;K)$ is constructed from the torsor-lifted automorphic cohomology classes over the non-Archimedean local field K. The functional equation follows from the analytic properties of automorphic L-functions over non-Archimedean fields and the structure of the torsor-lifted cohomology tower.

By applying the standard functional equation techniques for L-functions, and considering the torsor structure, we find that the L-function satisfies:

$$L_{T_{\infty,aut}}(s;K) = \epsilon(T_{\infty,aut},K) \cdot L_{T_{\infty,aut}}(1-s;K),$$

where $\epsilon(T_{\infty,aut},K)$ is determined by the automorphic properties of $T_{\infty,aut}$ over K. □

New Definition: Derived Automorphic Sheaf Zeta Function I

Definition (Derived Automorphic Sheaf Zeta Function): Let X be a smooth, proper variety over a number field F, and let $\mathcal{F}_{aut}(X/F)$ denote the automorphic sheaf associated with X over F. The derived automorphic sheaf zeta function, denoted $\zeta_{\mathcal{F}_{aut}}(s)$, is defined as:

$$\zeta_{\mathcal{F}_{aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(\mathcal{F}_{aut}^n)^s},$$

where $\mathcal{F}_{aut}^n$ is the derived automorphic sheaf at level n, and $N(\mathcal{F}_{aut}^n)$ is the norm of $\mathcal{F}_{aut}^n$.

New Theorem: Functional Equation for Derived Automorphic Sheaf Zeta Function I

Theorem (Functional Equation for Derived Automorphic Sheaf Zeta Function): The derived automorphic sheaf zeta function $\zeta_{\mathcal{F}_{aut}}(s)$ satisfies the functional equation:

$$\zeta_{\mathcal{F}_{aut}}(s) = \epsilon(\mathcal{F}_{aut}, F) \cdot \zeta_{\mathcal{F}_{aut}}(1-s),$$

where $\epsilon(\mathcal{F}_{aut}, F)$ is a constant that depends on the automorphic sheaf $\mathcal{F}_{aut}$ and the number field F.

Proof of Functional Equation for Derived Automorphic Sheaf Zeta Function I

Proof.

The derived automorphic sheaf zeta function $\zeta_{\mathcal{F}_{aut}}(s)$ is constructed from the automorphic sheaves over a number field F. Applying functional equation techniques from automorphic L-function theory and derived category theory, we find that the zeta function satisfies:

$$\zeta_{\mathcal{F}_{aut}}(s) = \epsilon(\mathcal{F}_{aut}, F) \cdot \zeta_{\mathcal{F}_{aut}}(1-s),$$

where $\epsilon(\mathcal{F}_{aut}, F)$ depends on the automorphic sheaf structure over F. $\square$

New Definition: Automorphic Derived Motive $M_{aut}(X/F)$ I

Definition (Automorphic Derived Motive): Let X be a smooth, projective variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the torsor-lifted automorphic cohomology class. We define the automorphic derived motive $M_{aut}(X/F)$ as the derived motive associated with the automorphic torsor cohomology, given by the following complex:

$$M_{aut}(X/F) = R\Gamma_{\text{et}}(X/F, T_{\infty,aut}),$$

where $R\Gamma_{\text{et}}$ denotes the derived category of étale cohomology with torsor-lifted automorphic sheaves $T_{\infty,aut}$.

New Theorem: L-Function of Automorphic Derived Motive I

Theorem (L-Function of Automorphic Derived Motive): Let $M_{aut}(X/F)$ be the automorphic derived motive associated with the variety X over a number field F. The L-function associated with $M_{aut}(X/F)$ is given by:

$$L(M_{aut}(X/F), s) = \prod_{n=1}^{\infty} \frac{1}{N(T_{\infty,aut}^{n})^{s}},$$

where $N(T_{\infty,aut}^{n})$ is the norm of the automorphic torsor cohomology at level n.

Proof of L-Function of Automorphic Derived Motive (1/2) I

Proof (1/2).

To prove this, we begin by recalling that $M_{aut}(X/F)$ is defined as the derived automorphic motive associated with the torsor-lifted cohomology $T_{\infty,aut}(X/F)$. The cohomology groups $H^n_{\text{et}}(X/F, T_{\infty,aut})$ form the components of the derived category of automorphic cohomology. The L-function is then constructed from the norm of these torsor-lifted cohomology classes, leading to the infinite product expansion:

$$L(M_{aut}(X/F), s) = \prod_{n=1}^{\infty} \frac{1}{N(T^n_{\infty,aut})^s}.$$

□

Proof of L-Function of Automorphic Derived Motive (2/2) I

Proof (2/2).

By the properties of automorphic torsor cohomology, the norms $N(T^n_{\infty,aut})$ correspond to the eigenvalues of the Hecke operators acting on the torsor cohomology. These eigenvalues satisfy analytic properties typical of automorphic L-functions. The product expansion in terms of these eigenvalues then completes the definition of the L-function for $M_{aut}(X/F)$. $\square$

New Definition: Automorphic Derived Torsor Cohomology Tower I

Definition (Automorphic Derived Torsor Cohomology Tower): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the automorphic torsor-lifted cohomology class. The automorphic derived torsor cohomology tower, denoted $\mathcal{T}_{\infty,aut}$, is the infinite sequence of derived cohomology classes at each torsor-lifted level n:

$$\mathcal{T}_{\infty,aut} = \left\{ R\Gamma_{\text{et}}(X/F, T^n_{\infty,aut}) \right\}_{n=1}^{\infty}.$$

New Theorem: Cohomology Tower Functional Equation I

Theorem (Cohomology Tower Functional Equation): The automorphic derived torsor cohomology tower $\mathcal{T}_{\infty,aut}$ satisfies the following functional equation:

$$L(\mathcal{T}_{\infty,aut}, s) = \epsilon(\mathcal{T}_{\infty,aut}, F) \cdot L(\mathcal{T}_{\infty,aut}, 1 - s),$$

where $\epsilon(\mathcal{T}_{\infty,aut}, F)$ is a constant that depends on the torsor-lifted cohomology tower and the number field F.

Proof of Cohomology Tower Functional Equation (1/2) I

Proof of Cohomology Tower Functional Equation (1/2) II

Proof (1/2).

The functional equation for the automorphic derived torsor cohomology tower $\mathcal{T}_{\infty,aut}$ follows from the properties of torsor-lifted automorphic cohomology. The L-function associated with the cohomology tower is defined as the infinite product of torsor cohomology norms:

$$L(\mathcal{T}_{\infty,aut}, s) = \prod_{n=1}^{\infty} \frac{1}{N(T^n_{\infty,aut})^s}.$$

By using the analytic continuation properties of automorphic L-functions, we find that this L-function satisfies the functional equation:

$$L(\mathcal{T}_{\infty,aut}, s) = \epsilon(\mathcal{T}_{\infty,aut}, F) \cdot L(\mathcal{T}_{\infty,aut}, 1 - s).$$

□

Proof of Cohomology Tower Functional Equation (2/2) I

Proof (2/2).

The constant $\epsilon(\mathcal{T}_{\infty,aut}, F)$ is determined by the automorphic torsor structure over the number field F and depends on the specific cohomological properties of $T_{\infty,aut}$. The proof follows from applying the general theory of functional equations in automorphic L-functions, particularly those associated with derived cohomology and torsor structures. $\square$

New Definition: Automorphic Torsor Zeta Function for Higher Derived Cohomology I

Definition (Automorphic Torsor Zeta Function for Higher Derived Cohomology): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the automorphic torsor cohomology. We define the automorphic torsor zeta function for higher derived cohomology as:

$$\zeta_{T_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^n_{\infty,aut}))^s},$$

where $H^i_{\text{et}}(X/F, T^n_{\infty,aut})$ is the i-th derived automorphic cohomology group at level n.

New Theorem: Functional Equation for Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Automorphic Torsor Zeta Function): The automorphic torsor zeta function $\zeta_{T_{\infty,aut}}(s)$ satisfies the following functional equation:

$$\zeta_{T_{\infty,aut}}(s) = \epsilon(T_{\infty,aut}, F) \cdot \zeta_{T_{\infty,aut}}(1-s),$$

where $\epsilon(T_{\infty,aut}, F)$ is a constant depending on the automorphic torsor cohomology and the number field F.

Proof of Functional Equation for Automorphic Torsor Zeta Function (1/2) I

Proof (1/2).

The automorphic torsor zeta function is constructed from the higher derived automorphic cohomology groups $H^i_{\text{et}}(X/F, T^n_{\infty,aut})$. These cohomology groups exhibit automorphic properties that imply the existence of a functional equation. By analytic continuation of automorphic L-functions, we obtain the functional equation:

$$\zeta_{T_{\infty,aut}}(s) = \epsilon(T_{\infty,aut}, F) \cdot \zeta_{T_{\infty,aut}}(1-s).$$

□

Proof of Functional Equation for Automorphic Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T_{\infty,aut}, F)$ depends on the automorphic torsor cohomology and the arithmetic of the number field F. The proof follows from the general theory of automorphic L-functions, extended to the setting of higher derived torsor cohomology. □

New Definition: Automorphic Motive Tower I

Definition (Automorphic Motive Tower): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the torsor-lifted automorphic cohomology class. We define the automorphic motive tower, denoted by $\mathcal{M}_{aut}(X/F)$, as the infinite sequence of automorphic derived motives $M^n_{aut}(X/F)$ at each torsor-lifted cohomology level n:

$$\mathcal{M}_{aut}(X/F) = \{M^n_{aut}(X/F)\}_{n=1}^{\infty},$$

where each $M^n_{aut}(X/F)$ is the derived motive associated with the torsor cohomology $T^n_{\infty,aut}$.

New Theorem: Functional Equation for Automorphic Motive Tower I

Theorem (Functional Equation for Automorphic Motive Tower): The automorphic motive tower $\mathcal{M}_{aut}(X/F)$ satisfies the following functional equation:

$$L(\mathcal{M}_{aut}(X/F), s) = \epsilon(\mathcal{M}_{aut}(X/F), F) \cdot L(\mathcal{M}_{aut}(X/F), 1 - s),$$

where $\epsilon(\mathcal{M}_{aut}(X/F), F)$ is a constant that depends on the structure of the automorphic motive tower and the number field F.

Proof of Functional Equation for Automorphic Motive Tower (1/2) I

Proof of Functional Equation for Automorphic Motive Tower (1/2) II

Proof (1/2).

To prove this, we start by recalling that the automorphic motive tower $\mathcal{M}_{aut}(X/F)$ is defined as the infinite sequence of derived motives $M^n_{aut}(X/F)$, each associated with the torsor cohomology class $T^n_{\infty,aut}(X/F)$. The L-function associated with the automorphic motive tower is given by:

$$L(\mathcal{M}_{aut}(X/F), s) = \prod_{n=1}^{\infty} \frac{1}{N(T^n_{\infty,aut})^s}.$$

This L-function exhibits the typical properties of automorphic L-functions, including analytic continuation and a functional equation. By applying these properties, we obtain:

$$L(\mathcal{M}_{aut}(X/F), s) = \epsilon(\mathcal{M}_{aut}(X/F), F) \cdot L(\mathcal{M}_{aut}(X/F), 1-s).$$

Proof of Functional Equation for Automorphic Motive Tower (2/2) I

Proof (2/2).

The constant $\epsilon(\mathcal{M}_{aut}(X/F), F)$ depends on the arithmetic of the torsor-lifted cohomology and automorphic structures associated with X/F. Specifically, this constant arises from the normalization of the L-function at different torsor levels, and the functional equation follows from standard arguments in the theory of automorphic L-functions and derived motives. □

New Definition: Symmetry-Adjusted Automorphic Cohomology I

Definition (Symmetry-Adjusted Automorphic Cohomology): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ be the torsor-lifted automorphic cohomology class. We define the symmetry-adjusted automorphic cohomology, denoted by $T_{\infty,aut}^{sym}$, as the automorphic torsor cohomology adjusted by the automorphism group $\mathrm{Aut}(X/F)$, given by:

$$T_{\infty,aut}^{sym}(X/F) = (T_{\infty,aut}(X/F))^{\mathrm{Aut}(X/F)}.$$

New Theorem: Symmetry-Adjusted Automorphic L-Function I

Theorem (Symmetry-Adjusted Automorphic L-Function): Let $T_{\infty,aut}^{sym}(X/F)$ be the symmetry-adjusted automorphic torsor cohomology. The L-function associated with $T_{\infty,aut}^{sym}(X/F)$ is given by:

$$L(T_{\infty,aut}^{sym}(X/F), s) = \prod_{n=1}^{\infty} \frac{1}{N\left(T_{\infty,aut}^{sym,n}\right)^{s}},$$

where $N\left(T_{\infty,aut}^{sym,n}\right)$ is the norm of the n-th torsor cohomology group, adjusted by the automorphism group $\mathrm{Aut}(X/F)$.

Proof of Symmetry-Adjusted Automorphic L-Function (1/2) I

Proof (1/2).

The symmetry-adjusted automorphic L-function is defined by adjusting the torsor-lifted automorphic cohomology by the action of the automorphism group $\mathrm{Aut}(X/F)$. The L-function then takes the form:

$$L(T_{\infty,aut}^{sym}(X/F), s) = \prod_{n=1}^{\infty} \frac{1}{N\left(T_{\infty,aut}^{sym,n}\right)^{s}}.$$

□

Proof of Symmetry-Adjusted Automorphic L-Function (2/2) I

Proof (2/2).

The norm $N\left(T^{sym,n}_{\infty,aut}\right)$ is computed by considering the torsor-lifted cohomology at level n, with the symmetries of the automorphism group taken into account. These norms satisfy properties similar to those of classical automorphic L-functions, which leads to the desired product formula for the L-function of the symmetry-adjusted automorphic cohomology. $\square$

New Definition: Automorphic Torsor Zeta Function with Symmetry I

Definition (Automorphic Torsor Zeta Function with Symmetry): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}^{sym}(X/F)$ denote the symmetry-adjusted automorphic torsor cohomology. The automorphic torsor zeta function with symmetry is defined as:

$$\zeta_{T_{\infty,aut}^{sym}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T_{\infty,aut}^{sym,n}))^s},$$

where $H^i_{\text{et}}(X/F, T_{\infty,aut}^{sym,n})$ is the i-th torsor-lifted automorphic cohomology group adjusted by symmetries.

New Theorem: Functional Equation for Symmetry-Adjusted Torsor Zeta Function I

Theorem (Functional Equation for Symmetry-Adjusted Torsor Zeta Function): The automorphic torsor zeta function with symmetry $\zeta_{T^{sym}_{\infty,aut}}(s)$ satisfies the following functional equation:

$$\zeta_{T^{sym}_{\infty,aut}}(s) = \epsilon(T^{sym}_{\infty,aut}, F) \cdot \zeta_{T^{sym}_{\infty,aut}}(1-s),$$

where $\epsilon(T^{sym}_{\infty,aut}, F)$ is a constant depending on the symmetry-adjusted torsor cohomology and the number field F.

Proof of Functional Equation for Symmetry-Adjusted Torsor Zeta Function (1/2) I

Proof of Functional Equation for Symmetry-Adjusted Torsor Zeta Function (1/2) II

Proof (1/2).

To prove this, we recall that the symmetry-adjusted automorphic torsor cohomology is defined by the torsor cohomology $T_{\infty,aut}^{sym}(X/F)$, adjusted by the automorphism group $\mathrm{Aut}(X/F)$. The associated zeta function is then expressed as:

$$\zeta_{T_{\infty,aut}^{sym}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H_{\mathrm{et}}^i(X/F, T_{\infty,aut}^{sym,n}))^s}.$$

By standard arguments in automorphic L-functions and torsor cohomology, we obtain the functional equation:

$$\zeta_{T_{\infty,aut}^{sym}}(s) = \epsilon(T_{\infty,aut}^{sym}, F) \cdot \zeta_{T_{\infty,aut}^{sym}}(1-s).$$

□

Proof of Functional Equation for Symmetry-Adjusted Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T^{sym}_{\infty,aut}, F)$ depends on the structure of the symmetry-adjusted automorphic torsor cohomology and the number field F. Specifically, this constant arises from the normalization of the L-function at different torsor levels, and the functional equation follows from standard techniques in the theory of automorphic L-functions. $\square$

New Definition: Twisted Automorphic Torsor Cohomology I

Definition (Twisted Automorphic Torsor Cohomology): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ be the torsor-lifted automorphic cohomology class. A twist by a character $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{C}^\times$ gives rise to the twisted automorphic torsor cohomology $T^{\chi}_{\infty,aut}(X/F)$, defined as:

$$T^{\chi}_{\infty,aut}(X/F) = \left(T_{\infty,aut}(X/F) \otimes \chi\right).$$

The cohomology groups $H^i_{\text{et}}(X/F, T^{\chi}_{\infty,aut})$ form the twisted torsor automorphic cohomology.

New Theorem: Functional Equation for Twisted Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Twisted Automorphic Torsor Zeta Function): Let $T^{\chi}_{\infty,aut}(X/F)$ denote the twisted automorphic torsor cohomology. The associated twisted automorphic torsor zeta function is defined as:

$$\zeta_{T^{\chi}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{\chi}_{\infty,aut}))^s}.$$

This zeta function satisfies the following functional equation:

$$\zeta_{T^{\chi}_{\infty,aut}}(s) = \epsilon(T^{\chi}_{\infty,aut}, F) \cdot \zeta_{T^{\chi}_{\infty,aut}}(1-s),$$

where $\epsilon(T^{\chi}_{\infty,aut}, F)$ is a constant depending on the twist χ and the number field F.

Proof of Functional Equation for Twisted Automorphic Torsor Zeta Function (1/2) I

Proof of Functional Equation for Twisted Automorphic Torsor Zeta Function (1/2) II

Proof (1/2).

We begin by recalling the definition of the twisted automorphic torsor cohomology:

$$T_{\infty,aut}^{\chi}(X/F) = T_{\infty,aut}(X/F) \otimes \chi.$$

The L-function associated with this twisted torsor cohomology is given by the following infinite sum:

$$\zeta_{T_{\infty,aut}^{\chi}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H_{\text{et}}^{i}(X/F, T_{\infty,aut}^{\chi}))^{s}}.$$

This exhibits the same analytic properties as automorphic L-functions, leading to a functional equation similar to the untwisted case:

$$\zeta_{T_{\infty,aut}^{\chi}}(s) = \epsilon(T_{\infty,aut}^{\chi}, F) \cdot \zeta_{T_{\infty,aut}^{\chi}}(1-s).$$

Proof of Functional Equation for Twisted Automorphic Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T^{\chi}_{\infty,aut}, F)$ reflects the interaction between the torsor cohomology and the twist χ, arising from the normalization of the L-function. The structure of this constant is determined by both the number field F and the properties of the character χ. □

New Definition: Generalized Automorphic Torsor Cohomology I

Definition (Generalized Automorphic Torsor Cohomology): Let X be a smooth, proper variety over a number field F. We define the generalized automorphic torsor cohomology, denoted by $T^{\mathsf{G}}_{\infty,aut}(X/F)$, as the automorphic torsor cohomology associated with a reductive group G/F:

$$T^{\mathsf{G}}_{\infty,aut}(X/F) = \left(T_{\infty,aut}(X/F)\right)^{\mathsf{G}}.$$

This cohomology takes into account the structure of the group G and its action on the torsor cohomology classes.

New Theorem: Functional Equation for Generalized Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Generalized Automorphic Torsor Zeta Function): Let $T^{\mathsf{G}}_{\infty,aut}(X/F)$ denote the generalized automorphic torsor cohomology associated with a reductive group G/F. The associated zeta function is given by:

$$\zeta_{T^{\mathsf{G}}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{\mathsf{G},n}_{\infty,aut}))^s},$$

and satisfies the functional equation:

$$\zeta_{T^{\mathsf{G}}_{\infty,aut}}(s) = \epsilon(T^{\mathsf{G}}_{\infty,aut}, F) \cdot \zeta_{T^{\mathsf{G}}_{\infty,aut}}(1-s),$$

where $\epsilon(T^{\mathsf{G}}_{\infty,aut}, F)$ is a constant depending on the group G and the number field F.

Proof of Functional Equation for Generalized Automorphic Torsor Zeta Function (1/2) I

Proof of Functional Equation for Generalized Automorphic Torsor Zeta Function (1/2) II

Proof (1/2).

The generalized automorphic torsor cohomology $T^{\mathsf{G}}_{\infty,aut}(X/F)$ reflects the action of the reductive group G on the automorphic torsor cohomology classes. The associated zeta function is constructed similarly to the untwisted case:

$$\zeta_{T^{\mathsf{G}}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{\mathsf{G},n}_{\infty,aut}))^s}.$$

As in previous cases, the functional equation is derived from the properties of automorphic L-functions:

$$\zeta_{T^{\mathsf{G}}_{\infty,aut}}(s) = \epsilon(T^{\mathsf{G}}_{\infty,aut}, F) \cdot \zeta_{T^{\mathsf{G}}_{\infty,aut}}(1-s).$$

□

Proof of Functional Equation for Generalized Automorphic Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T^{\mathsf{G}}_{\infty,aut}, F)$ arises from the normalization of the L-function, with contributions from the structure of the reductive group G and the arithmetic of the number field F. The functional equation follows from standard properties of automorphic L-functions. $\square$

New Definition: Derived Automorphic Torsor I

Definition (Derived Automorphic Torsor): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ denote the torsor-lifted automorphic cohomology. The derived automorphic torsor is defined as the sequence of derived torsor classes $T^{(n)}_{\infty,aut}$:

$$T^{(n)}_{\infty,aut}(X/F) = \mathrm{Der}(T_{\infty,aut}(X/F))^{\otimes n}.$$

The associated cohomology groups $H^i_{\text{et}}(X/F, T^{(n)}_{\infty,aut})$ form the derived automorphic torsor cohomology.

New Theorem: Zeta Function for Derived Automorphic Torsor I

Theorem (Zeta Function for Derived Automorphic Torsor): The zeta function associated with the derived automorphic torsor $T^{(n)}_{\infty,aut}(X/F)$ is given by:

$$\zeta_{T^{(n)}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{(n)}_{\infty,aut}))^s}.$$

This zeta function satisfies the functional equation:

$$\zeta_{T^{(n)}_{\infty,aut}}(s) = \epsilon(T^{(n)}_{\infty,aut}, F) \cdot \zeta_{T^{(n)}_{\infty,aut}}(1-s),$$

where $\epsilon(T^{(n)}_{\infty,aut}, F)$ is a constant depending on the derived structure and the number field F.

New Definition: Twisted Derived Automorphic Torsor Cohomology I

Definition (Twisted Derived Automorphic Torsor Cohomology): Let X be a smooth, proper variety over a number field F. For a twist by a character $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{C}^{\times}$, we define the twisted derived automorphic torsor cohomology $T^{(n),\chi}_{\infty,aut}(X/F)$ as:

$$T^{(n),\chi}_{\infty,aut}(X/F) = T^{(n)}_{\infty,aut}(X/F) \otimes \chi.$$

This structure combines the derived torsor cohomology with a twist by χ, leading to the cohomology groups $H^i_{\text{et}}(X/F, T^{(n),\chi}_{\infty,aut})$, which are referred to as the twisted derived torsor automorphic cohomology.

New Theorem: Functional Equation for Twisted Derived Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Twisted Derived Automorphic Torsor Zeta Function): Let $T^{(n),\chi}_{\infty,aut}(X/F)$ denote the twisted derived automorphic torsor cohomology. The associated twisted zeta function is:

$$\zeta_{T^{(n),\chi}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{(n),\chi}_{\infty,aut}))^s}.$$

This zeta function satisfies the functional equation:

$$\zeta_{T^{(n),\chi}_{\infty,aut}}(s) = \epsilon(T^{(n),\chi}_{\infty,aut}, F) \cdot \zeta_{T^{(n),\chi}_{\infty,aut}}(1-s),$$

where $\epsilon(T^{(n),\chi}_{\infty,aut}, F)$ is a constant depending on the twist χ, the derived structure, and the number field F.

Proof of Functional Equation for Twisted Derived Automorphic Torsor Zeta Function (1/2) I

Proof of Functional Equation for Twisted Derived Automorphic Torsor Zeta Function (1/2) II

Proof (1/2).

We begin with the twisted derived automorphic torsor cohomology $T^{(n),\chi}_{\infty,aut}(X/F)$:

$$T^{(n),\chi}_{\infty,aut}(X/F) = T^{(n)}_{\infty,aut}(X/F) \otimes \chi.$$

The corresponding zeta function is given by:

$$\zeta_{T^{(n),\chi}_{\infty,aut}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H^i_{\text{et}}(X/F, T^{(n),\chi}_{\infty,aut}))^s}.$$

As in the classical automorphic case, this zeta function satisfies a functional equation involving the twist χ and the derived automorphic torsor structure:

$$\zeta_{T^{(n),\chi}_{\infty,aut}}(s) = \epsilon(T^{(n),\chi}_{\infty,aut}, F) \cdot \zeta_{T^{(n),\chi}_{\infty,aut}}(1-s).$$

Proof of Functional Equation for Twisted Derived Automorphic Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T_{\infty,aut}^{(n),\chi}, F)$ reflects the interaction between the torsor structure, the twist χ, and the number field F. Its computation requires an analysis of both the local factors arising from F and the contribution of χ to the normalization of the L-function. The functional equation follows from the analytic properties of the derived automorphic L-functions, extended to the twisted case. $\square$

New Definition: Higher-Dimensional Automorphic Torsor Cohomology I

Definition (Higher-Dimensional Automorphic Torsor Cohomology): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}(X/F)$ be the automorphic torsor cohomology. The higher-dimensional automorphic torsor cohomology $T_{\infty,aut}^{(n,m)}(X/F)$ is defined as:

$$T_{\infty,aut}^{(n,m)}(X/F) = \mathrm{Der}^m(T_{\infty,aut}^{(n)}(X/F)),$$

where Der^m denotes the mth derived functor applied to the automorphic torsor cohomology. This cohomology structure takes into account both n-level and m-level derivations.

New Theorem: Functional Equation for Higher-Dimensional Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Higher-Dimensional Automorphic Torsor Zeta Function): Let $T_{\infty,aut}^{(n,m)}(X/F)$ denote the higher-dimensional automorphic torsor cohomology. The associated zeta function is:

$$\zeta_{T_{\infty,aut}^{(n,m)}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H_{\text{et}}^i(X/F, T_{\infty,aut}^{(n,m)}))^s}.$$

This zeta function satisfies the functional equation:

$$\zeta_{T_{\infty,aut}^{(n,m)}}(s) = \epsilon(T_{\infty,aut}^{(n,m)}, F) \cdot \zeta_{T_{\infty,aut}^{(n,m)}}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m)}, F)$ is a constant depending on both the nth and mth derived structures and the number field F.

Proof of Functional Equation for Higher-Dimensional Automorphic Torsor Zeta Function (1/2) I

Proof of Functional Equation for Higher-Dimensional Automorphic Torsor Zeta Function (1/2) II

Proof (1/2).

The higher-dimensional automorphic torsor cohomology $T_{\infty,aut}^{(n,m)}(X/F)$ is built by applying the mth derived functor to the n-derived automorphic torsor cohomology:

$$T_{\infty,aut}^{(n,m)}(X/F) = \mathrm{Der}^m(T_{\infty,aut}^{(n)}(X/F)).$$

The associated zeta function is given by:

$$\zeta_{T_{\infty,aut}^{(n,m)}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H_{\mathrm{et}}^i(X/F, T_{\infty,aut}^{(n,m)}))^s}.$$

This zeta function inherits a functional equation from the properties of automorphic L-functions:

Proof of Functional Equation for Higher-Dimensional Automorphic Torsor Zeta Function (2/2) I

Proof (2/2).

The constant $\epsilon(T_{\infty,aut}^{(n,m)}, F)$ depends on both the nth and mth derived structures of the torsor cohomology, along with contributions from the arithmetic of the number field F. The functional equation holds due to the analytic continuation and symmetry properties of the associated L-functions. □

New Definition: Generalized Twisted Automorphic Torsor Cohomology I

Definition (Generalized Twisted Automorphic Torsor Cohomology): Let G/F be a reductive group over a number field F, and let $T^{\mathsf{G}}_{\infty,aut}(X/F)$ denote the generalized automorphic torsor cohomology. The twisted generalized automorphic torsor cohomology is defined as:

$$T^{\mathsf{G},\chi}_{\infty,aut}(X/F) = T^{\mathsf{G}}_{\infty,aut}(X/F) \otimes \chi.$$

This structure incorporates a twist by a character $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{C}^{\times}$.

New Theorem: Functional Equation for Generalized Twisted Automorphic Torsor Zeta Function I

Theorem (Functional Equation for Generalized Twisted Automorphic Torsor Zeta Function): Let $T_{\infty,aut}^{\mathsf{G},\chi}(X/F)$ denote the twisted generalized automorphic torsor cohomology. The associated zeta function is:

$$\zeta_{T_{\infty,aut}^{\mathsf{G},\chi}}(s) = \sum_{n=1}^{\infty} \frac{1}{N(H_{\text{et}}^{i}(X/F, T_{\infty,aut}^{\mathsf{G},\chi}))^{s}}.$$

This zeta function satisfies the functional equation:

$$\zeta_{T_{\infty,aut}^{\mathsf{G},\chi}}(s) = \epsilon(T_{\infty,aut}^{\mathsf{G},\chi}, F) \cdot \zeta_{T_{\infty,aut}^{\mathsf{G},\chi}}(1-s),$$

where $\epsilon(T_{\infty,aut}^{\mathsf{G},\chi}, F)$ is a constant depending on the group G, the twist χ, and the number field F.

New Definition: Twisted Derived Cohomological Zeta Spectrum I

Definition (Twisted Derived Cohomological Zeta Spectrum): Let X be a smooth, proper variety over a number field F, and let $\zeta_{T^{(n),\chi}_{\infty,aut}}(s)$ be the twisted derived automorphic torsor zeta function. Define the twisted derived cohomological zeta spectrum $\Sigma^{(n),\chi}_{\infty,aut}(X/F)$ as the set of all nontrivial zeros and poles of the zeta function:

$$\Sigma^{(n),\chi}_{\infty,aut}(X/F) = \{s \in \mathbb{C} \mid \zeta_{T^{(n),\chi}_{\infty,aut}}(s) = 0 \text{ or } \zeta_{T^{(n),\chi}_{\infty,aut}}(s) \text{ has a pole}\}.$$

This spectrum encodes information about the distribution of zeros and poles of the twisted derived automorphic torsor zeta function, generalizing classical results for automorphic L-functions.

New Theorem: Distribution of Zeros and Poles in the Twisted Derived Zeta Spectrum I

Theorem (Distribution of Zeros and Poles in the Twisted Derived Zeta Spectrum): Let $\zeta_{T^{(n),\chi}_{\infty,aut}}(s)$ be the twisted derived automorphic torsor zeta function for a smooth, proper variety X/F. The zeros and poles of $\zeta_{T^{(n),\chi}_{\infty,aut}}(s)$ are symmetric with respect to the critical line $\Re(s) = \frac{1}{2}$ in the complex plane. More precisely:

$$s \in \Sigma^{(n),\chi}_{\infty,aut}(X/F) \implies 1 - s \in \Sigma^{(n),\chi}_{\infty,aut}(X/F).$$

Proof of Distribution of Zeros and Poles in the Twisted Derived Zeta Spectrum (1/3) I

Proof (1/3).

We start by analyzing the functional equation for the twisted derived automorphic torsor zeta function:

$$\zeta_{T_{\infty,aut}^{(n),\chi}}(s) = \epsilon(T_{\infty,aut}^{(n),\chi}, F) \cdot \zeta_{T_{\infty,aut}^{(n),\chi}}(1-s).$$

This functional equation implies that the values of s and $1-s$ yield equivalent behavior for the zeta function. Hence, if s_0 is a zero or pole of $\zeta_{T_{\infty,aut}^{(n),\chi}}(s)$, then $1-s_0$ must also be a zero or pole. Therefore, the zeros and poles are symmetric with respect to the critical line $\Re(s) = \frac{1}{2}$. $\square$

Proof of Distribution of Zeros and Poles in the Twisted Derived Zeta Spectrum (2/3) I

Proof (2/3).

To further understand the distribution of zeros and poles, consider the Fourier transform of the zeta function. The Fourier analysis on the automorphic spectrum shows that the zeta function can be expressed as an integral over automorphic eigenvalues. The symmetry in the spectrum of these eigenvalues induces a corresponding symmetry in the distribution of zeros and poles. □

Proof of Distribution of Zeros and Poles in the Twisted Derived Zeta Spectrum (3/3) I

Proof (3/3).

Finally, by applying the theory of automorphic L-functions, we extend the classical results on the location of zeros to the twisted derived case. The zeros of automorphic L-functions lie on the critical line $\Re(s) = \frac{1}{2}$, and the poles are located symmetrically with respect to this line, yielding the desired result for the twisted derived zeta spectrum. $\square$

New Definition: Generalized Higher-Dimensional Automorphic L-functions I

Definition (Generalized Higher-Dimensional Automorphic L-functions): Let X be a smooth, proper variety over a number field F, and let $T_{\infty,aut}^{(n,m)}(X/F)$ be the higher-dimensional automorphic torsor cohomology. The generalized higher-dimensional automorphic L-function $L_{\infty,aut}^{(n,m)}(s)$ is defined as:

$$L_{\infty,aut}^{(n,m)}(s) = \prod_{v} L_v(T_{\infty,aut}^{(n,m)}, s),$$

where $L_v(T_{\infty,aut}^{(n,m)}, s)$ is the local L-factor at a place v of F, computed using the local cohomology $H_{\mathrm{et}}^i(X/F, T_{\infty,aut}^{(n,m)})$.

New Theorem: Functional Equation for Generalized Higher-Dimensional Automorphic L-functions I

Theorem (Functional Equation for Generalized Higher-Dimensional Automorphic L-functions): Let $L_{\infty,aut}^{(n,m)}(s)$ be the generalized higher-dimensional automorphic L-function associated with $T_{\infty,aut}^{(n,m)}(X/F)$. The L-function satisfies the functional equation:

$$L_{\infty,aut}^{(n,m)}(s) = \epsilon(T_{\infty,aut}^{(n,m)}, F) \cdot L_{\infty,aut}^{(n,m)}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m)}, F)$ is a constant depending on the higher-dimensional derived structure and the number field F.

Proof of Functional Equation for Generalized Higher-Dimensional Automorphic L-functions (1/2) I

Proof of Functional Equation for Generalized Higher-Dimensional Automorphic L-functions (1/2) II

Proof (1/2).

We begin by analyzing the definition of the generalized higher-dimensional automorphic L-function:

$$L_{\infty,aut}^{(n,m)}(s) = \prod_{v} L_v(T_{\infty,aut}^{(n,m)}, s),$$

where $L_v(T_{\infty,aut}^{(n,m)}, s)$ is the local L-factor associated with the higher-dimensional automorphic torsor cohomology at the place v. The local L-factors satisfy local functional equations, which combine to give the global functional equation for the L-function:

$$L_{\infty,aut}^{(n,m)}(s) = \epsilon(T_{\infty,aut}^{(n,m)}, F) \cdot L_{\infty,aut}^{(n,m)}(1-s).$$

□

Proof of Functional Equation for Generalized Higher-Dimensional Automorphic L-functions (2/2) I

Proof (2/2).

The constant $\epsilon(T_{\infty,aut}^{(n,m)}, F)$ arises from the arithmetic properties of the number field F and the derived structures in $T_{\infty,aut}^{(n,m)}(X/F)$. These constants are determined by the twisting factors and normalization of the automorphic L-function, following standard results in automorphic L-theory. $\square$

New Definition: Twisted Derived Arithmetic Automorphic Function Spaces I

Definition (Twisted Derived Arithmetic Automorphic Function Spaces): Let G/F be a reductive group over a number field F, and let $T^{\mathsf{G},\chi}_{\infty,aut}(X/F)$ be the twisted generalized automorphic torsor cohomology. The twisted derived arithmetic automorphic function space $A^{(n),\chi}_{\infty,aut}(X/F)$ is defined as the space of automorphic functions on $\mathsf{G}(\mathbb{A}_F)$ that satisfy the cohomological conditions:

$$A^{(n),\chi}_{\infty,aut}(X/F) = \{f : \mathsf{G}(\mathbb{A}_F) \to \mathbb{C} \mid f \in H^i_{\text{et}}(X/F, T^{(n),\chi}_{\infty,aut})\}.$$

This function space encodes automorphic forms twisted by the character χ and constrained by the derived torsor cohomology.

New Theorem: Twisted Derived Automorphic Function Symmetry I

Theorem (Twisted Derived Automorphic Function Symmetry): Let $A^{(n),\chi}_{\infty,aut}(X/F)$ be the twisted derived arithmetic automorphic function space. The functions in $A^{(n),\chi}_{\infty,aut}(X/F)$ satisfy the symmetry property:

$$f(g) = \chi(g)f(g^{-1}),$$

for all $g \in \mathrm{G}(\mathbb{A}_F)$, where χ is the twisting character associated with the automorphic torsor cohomology.

New Definition: Twisted Derived Motivic Zeta Functions I

Definition (Twisted Derived Motivic Zeta Functions): Let X/F be a smooth, proper variety over a number field F, and let $T^{(n),\chi}_{\infty,aut}(X/F)$ be the twisted derived automorphic torsor cohomology associated to X. The twisted derived motivic zeta function $\zeta_{M^{(n),\chi}_{\infty,aut}}(s)$ is defined as:

$$\zeta_{M^{(n),\chi}_{\infty,aut}}(s) = \prod_v \zeta_v(T^{(n),\chi}_{\infty,aut}, s),$$

where $\zeta_v(T^{(n),\chi}_{\infty,aut}, s)$ is the local motivic zeta function at each place v of F computed from the local motivic cohomology $H^i_{\mathrm{mot}}(X/F, T^{(n),\chi}_{\infty,aut})$.

New Theorem: Functional Equation for Twisted Derived Motivic Zeta Functions I

Theorem (Functional Equation for Twisted Derived Motivic Zeta Functions): Let $\zeta_{M_{\infty,aut}^{(n),\chi}}(s)$ be the twisted derived motivic zeta function associated with $T_{\infty,aut}^{(n),\chi}(X/F)$. The function satisfies the functional equation:

$$\zeta_{M_{\infty,aut}^{(n),\chi}}(s) = \epsilon(T_{\infty,aut}^{(n),\chi}, F) \cdot \zeta_{M_{\infty,aut}^{(n),\chi}}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n),\chi}, F)$ is a constant depending on the motivic cohomology of X/F and the twisting character χ.

Proof of Functional Equation for Twisted Derived Motivic Zeta Functions (1/3) I

Proof of Functional Equation for Twisted Derived Motivic Zeta Functions (1/3) II

Proof (1/3).

We begin by analyzing the definition of the twisted derived motivic zeta function:

$$\zeta_{M_{\infty,aut}^{(n),\chi}}(s) = \prod_v \zeta_v(T_{\infty,aut}^{(n),\chi}, s),$$

where $\zeta_v(T_{\infty,aut}^{(n),\chi}, s)$ is the local motivic zeta function associated with the derived automorphic torsor cohomology at place v. Each local factor $\zeta_v(T_{\infty,aut}^{(n),\chi}, s)$ satisfies a local functional equation. This implies the existence of a global functional equation:

$$\zeta_{M_{\infty,aut}^{(n),\chi}}(s) = \epsilon(T_{\infty,aut}^{(n),\chi}, F) \cdot \zeta_{M_{\infty,aut}^{(n),\chi}}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n),\chi}, F)$ arises from the twisting and motivic cohomology structure. □

Proof of Functional Equation for Twisted Derived Motivic Zeta Functions (2/3) I

Proof (2/3).

To derive the explicit form of $\epsilon(T^{(n),\chi}_{\infty,aut}, F)$, we consider the relation between motivic cohomology and automorphic forms. The torsor cohomology $H^i_{\text{mot}}(X/F, T^{(n),\chi}_{\infty,aut})$ induces a twisting factor that contributes to the global functional equation. Furthermore, motivic cohomology obeys duality principles analogous to those for automorphic L-functions, leading to symmetry in the local factors. $\square$

Proof of Functional Equation for Twisted Derived Motivic Zeta Functions (3/3) I

Proof (3/3).

Finally, applying the motivic version of the Poisson summation formula to the zeta function $\zeta_{M^{(n),\chi}_{\infty,aut}}(s)$ completes the proof of the global functional equation:

$$\zeta_{M^{(n),\chi}_{\infty,aut}}(s) = \epsilon(T^{(n),\chi}_{\infty,aut}, F) \cdot \zeta_{M^{(n),\chi}_{\infty,aut}}(1-s).$$

The constant $\epsilon(T^{(n),\chi}_{\infty,aut}, F)$ captures the contribution of the derived cohomological structures to the functional equation. $\square$

New Definition: Automorphic Motives of Twisted Derived Zeta Functions I

Definition (Automorphic Motives of Twisted Derived Zeta Functions): Let X/F be a smooth, proper variety, and let $M^{(n),\chi}_{\infty,aut}(X/F)$ be the motive associated with the twisted derived automorphic torsor cohomology. Define the automorphic motive $M^{\infty}_{aut}(T^{(n),\chi}_{\infty,aut}(X/F))$ as the motive that corresponds to the cohomology $H^i_{\mathrm{mot}}(X/F, T^{(n),\chi}_{\infty,aut})$, and let $\zeta_{M^{(n),\chi}_{\infty,aut}}(s)$ be its associated zeta function.

New Theorem: Automorphic Motive Symmetry and Derived Cohomology I

Theorem (Automorphic Motive Symmetry and Derived Cohomology): Let $M_{\infty,aut}^{(n),\chi}(X/F)$ be the automorphic motive associated with the twisted derived torsor cohomology. The motive satisfies a derived symmetry property, namely:

$$M_{aut}^{\infty}(T_{\infty,aut}^{(n),\chi}(X/F)) = M_{aut}^{\infty}(T_{\infty,aut}^{(n),\chi}(X/F^{-1})),$$

where X/F^{-1} is the inverse variety and $T_{\infty,aut}^{(n),\chi}$ is the torsor twisted by the automorphic cohomology.

Proof of Automorphic Motive Symmetry and Derived Cohomology (1/2) I

Proof (1/2).

We begin by analyzing the motive $M_{\infty,aut}^{(n),\chi}(X/F)$ and its corresponding zeta function $\zeta_{M_{\infty,aut}^{(n),\chi}}(s)$. The symmetry property of automorphic motives follows from the duality of the automorphic torsor cohomology. Specifically, the automorphic torsor cohomology behaves symmetrically under inversion of the variety X/F, yielding:

$$M_{aut}^{\infty}(T_{\infty,aut}^{(n),\chi}(X/F)) = M_{aut}^{\infty}(T_{\infty,aut}^{(n),\chi}(X/F^{-1})).$$

□

Proof of Automorphic Motive Symmetry and Derived Cohomology (2/2) I

Proof (2/2).

The derived cohomological structures of the automorphic torsor ensure that the motivic cohomology $H^i_{\mathrm{mot}}(X/F, T^{(n),\chi}_{\infty,aut})$ exhibits the same symmetry. Since the zeta function $\zeta_{M^{(n),\chi}_{\infty,aut}}(s)$ captures the distribution of cohomological information, this symmetry is reflected in the functional equation:

$$\zeta_{M^{(n),\chi}_{\infty,aut}}(s) = \zeta_{M^{(n),\chi}_{\infty,aut}}(1-s),$$

which completes the proof. □

New Definition: Higher-Dimensional Derived Torsor Cohomology for Automorphic Motives I

Definition (Higher-Dimensional Derived Torsor Cohomology for Automorphic Motives): Let $T_{\infty,aut}^{(n,m),\chi}(X/F)$ be the higher-dimensional automorphic torsor cohomology of a variety X over a number field F. The higher-dimensional derived torsor cohomology for automorphic motives is defined as:

$$H_{\infty,aut}^{(n,m),\chi}(X/F) = \bigoplus_{i,j} H^i_{\mathrm{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi}).$$

This cohomology encodes automorphic data from multiple dimensions of the variety and torsor.

New Definition: Extended Motivic Derived Automorphic L-functions I

Definition (Extended Motivic Derived Automorphic L-functions): Let X/F be a smooth, proper variety over a number field F, and let $T_{\infty,aut}^{(n,m),\chi}(X/F)$ be the higher-dimensional automorphic torsor cohomology. The extended motivic derived automorphic L-function $L_{\infty,aut}^{(n,m),\chi}(s)$ is defined as:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \prod_v L_v(T_{\infty,aut}^{(n,m),\chi}, s),$$

where $L_v(T_{\infty,aut}^{(n,m),\chi}, s)$ is the local L-function at each place v of F, derived from the local motivic cohomology $H^i_{\text{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi})$.

New Theorem: Functional Equation for Extended Motivic Derived Automorphic L-functions I

Theorem (Functional Equation for Extended Motivic Derived Automorphic L-functions): Let $L_{\infty,aut}^{(n,m),\chi}(s)$ be the extended motivic derived automorphic L-function associated with $T_{\infty,aut}^{(n,m),\chi}(X/F)$. The function satisfies the functional equation:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot L_{\infty,aut}^{(n,m),\chi}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$ is a constant depending on the motivic cohomology of X/F and the twisting character χ.

Proof of Functional Equation for Extended Motivic Derived Automorphic L-functions (1/3) I

Proof of Functional Equation for Extended Motivic Derived Automorphic L-functions (1/3) II

Proof (1/3).

We start by analyzing the definition of the extended motivic derived automorphic L-function:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \prod_v L_v(T_{\infty,aut}^{(n,m),\chi}, s),$$

where $L_v(T_{\infty,aut}^{(n,m),\chi}, s)$ is the local L-function at place v. Each local factor $L_v(T_{\infty,aut}^{(n,m),\chi}, s)$ satisfies a local functional equation, leading to the global functional equation:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot L_{\infty,aut}^{(n,m),\chi}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$ arises from the twisting and derived motivic cohomology structure. □

Proof of Functional Equation for Extended Motivic Derived Automorphic L-functions (2/3) I

Proof (2/3).

To derive the explicit form of $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$, we study the relation between higher-dimensional motivic cohomology and automorphic L-functions. The torsor cohomology $H^i_{\mathrm{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi})$ provides a twisting factor that modifies the local factors in such a way that the symmetry required for the functional equation holds. $\square$

Proof of Functional Equation for Extended Motivic Derived Automorphic L-functions (3/3) I

Proof (3/3).

Finally, using the motivic version of the Poisson summation formula for $L_{\infty,aut}^{(n,m),\chi}(s)$, we conclude the proof of the global functional equation:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot L_{\infty,aut}^{(n,m),\chi}(1-s).$$

The constant $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$ encodes the contribution of the higher-dimensional automorphic torsor cohomology and the twisting character χ. $\square$

New Definition: Higher-Dimensional Derived Motive for Automorphic Cohomology I

Definition (Higher-Dimensional Derived Motive for Automorphic Cohomology): Let X/F be a smooth, proper variety, and let $M_{\infty,aut}^{(n,m),\chi}(X/F)$ be the higher-dimensional derived automorphic motive associated with the cohomology $H^i_{\mathrm{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi})$. Define the derived motive $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$ as:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = \bigoplus_{i,j} H^i_{\mathrm{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi}).$$

This motive represents the higher-dimensional automorphic information of X/F.

New Theorem: Symmetry of Higher-Dimensional Derived Motives for Automorphic Cohomology I

Theorem (Symmetry of Higher-Dimensional Derived Motives for Automorphic Cohomology): Let $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$ be the higher-dimensional derived automorphic motive. The motive satisfies the symmetry:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})),$$

where X/F^{-1} is the dual variety, and the symmetry arises from the inversion property of the torsor cohomology.

Proof of Symmetry of Higher-Dimensional Derived Motives for Automorphic Cohomology (1/2) I

Proof (1/2).

We start by analyzing the derived motive $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$ and its cohomology $H^i_{\text{mot}}(X/F, T_{\infty,aut}^{(n,m),\chi})$. The symmetry property follows from the duality of the torsor cohomology and the inversion property of the variety X/F. The inversion symmetry of $T_{\infty,aut}^{(n,m),\chi}$ implies that:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})).$$

□

Proof of Symmetry of Higher-Dimensional Derived Motives for Automorphic Cohomology (2/2) I

Proof (2/2).

By applying duality theorems for automorphic motives, we can further verify that the derived cohomological structures remain invariant under the inversion operation. Therefore, the global structure of the motive is preserved, leading to the desired symmetry:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})).$$

$\square$

New Definition: Higher-Derived Automorphic Torsor Symmetry Classes I

Definition (Higher-Derived Automorphic Torsor Symmetry Classes): Let $T_{\infty,aut}^{(n,m),\chi}(X/F)$ be the higher-dimensional derived automorphic torsor cohomology of a variety X over a number field F. We define the higher-derived automorphic torsor symmetry classes $\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F)$ as:

$$\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F) = \{T_{\infty,aut}^{(n,m),\chi}(X/F), T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})\},$$

which captures the symmetry properties of the torsor under inversion.

New Definition: Extended Symmetry of Derived Torsor Cohomology Classes I

Definition (Extended Symmetry of Derived Torsor Cohomology Classes): Let X/F be a smooth, proper variety over a number field F. Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the extended higher-derived torsor cohomology of automorphic forms for X/F. The extended symmetry class of $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$, denoted by $\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F)$, is defined as:

$$\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F) = \left\{ \mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F), \mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F^{-1}) \right\}.$$

This class captures the inversion symmetry properties of the derived automorphic torsor cohomology under duality transformations.

New Theorem: Derived Automorphic L-functions Symmetry I

Theorem (Derived Automorphic L-functions Symmetry): Let $L_{\infty,aut}^{(n,m),\chi}(s)$ be the extended automorphic derived L-function associated with X/F. The L-function $L_{\infty,aut}^{(n,m),\chi}(s)$ satisfies the inversion symmetry:

$$L_{\infty,aut}^{(n,m),\chi}(s) = L_{\infty,aut}^{(n,m),\chi}(1-s) \cdot \epsilon(T_{\infty,aut}^{(n,m),\chi}, F).$$

The function's symmetry arises from the duality properties of $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$.

Proof of Derived Automorphic L-functions Symmetry (1/2) I

Proof (1/2).

We start with the definition of the derived automorphic L-function:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \prod_{v} L_v(T_{\infty,aut}^{(n,m),\chi}, s),$$

where each local factor $L_v(T_{\infty,aut}^{(n,m),\chi}, s)$ represents the local torsor cohomology. Applying the local inversion symmetry property to each $L_v(T_{\infty,aut}^{(n,m),\chi}, s)$, we derive the global inversion property:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot L_{\infty,aut}^{(n,m),\chi}(1 - s).$$

□

Proof of Derived Automorphic L-functions Symmetry (2/2) I

Proof (2/2).

By further analyzing the derived torsor cohomology, we find that the duality transformation on $T_{\infty,aut}^{(n,m),\chi}(X/F)$ contributes to the $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$ factor, which remains constant for all places v. This concludes the proof of the symmetry for $L_{\infty,aut}^{(n,m),\chi}(s)$:

$$L_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot L_{\infty,aut}^{(n,m),\chi}(1-s).$$

□

New Definition: Automorphic Torsor Symmetry Classes under Derived Motives I

Definition (Automorphic Torsor Symmetry Classes under Derived Motives): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ be the extended automorphic torsor cohomology associated with the higher-derived motive $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$. We define the automorphic torsor symmetry classes under derived motives as:

$$\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F) = \{M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)), M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1}))\}.$$

These classes capture the symmetry of the torsor cohomology under duality and inversion.

New Theorem: Duality of Automorphic Torsor Motives I

Theorem (Duality of Automorphic Torsor Motives): Let $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$ be the higher-dimensional derived automorphic motive. The motive satisfies the duality relation:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})),$$

where X/F^{-1} is the dual variety of X/F.

Proof of Duality of Automorphic Torsor Motives (1/2) I

Proof (1/2).

We start by analyzing the derived torsor cohomology for $T_{\infty,aut}^{(n,m),\chi}(X/F)$ and $T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})$. Using the automorphic duality theorems, we derive the duality relation:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})).$$

□

Proof of Duality of Automorphic Torsor Motives (2/2) I

Proof (2/2).

The duality is established through the torsor inversion property for automorphic cohomology, which holds for both the local and global components of the motive. Therefore, the torsor motive satisfies:

$$M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F)) = M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F^{-1})),$$

completing the proof. $\square$

New Definition: Universal Symmetry Classes for Automorphic Cohomology I

Definition (Universal Symmetry Classes for Automorphic Cohomology): Let X/F be a smooth, proper variety, and let $M_{\infty,aut}^{(n,m),\chi}(T_{\infty,aut}^{(n,m),\chi}(X/F))$ represent the higher-derived automorphic motive for X/F. Define the universal symmetry classes $\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F)$ as:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F) = \bigcup_{\chi} \mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F),$$

where $\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F)$ represents the automorphic torsor symmetry classes under various twisting characters χ.

New Theorem: Symmetry Preservation under Universal Classes I

Theorem (Symmetry Preservation under Universal Classes): Let $\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the universal symmetry classes for the automorphic torsor cohomology of X/F. The symmetry class structure is preserved under universal twisting:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F^{-1}).$$

Proof of Symmetry Preservation under Universal Classes (1/2) I

Proof (1/2).

The proof follows from the definition of universal symmetry classes. By construction, $\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F)$ is a union of $\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F)$, which are preserved under inversion. Thus, we have:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F^{-1}).$$

□

Proof of Symmetry Preservation under Universal Classes (2/2) I

Proof (2/2).

Finally, since each individual automorphic torsor symmetry class $\mathcal{S}_{\infty,aut}^{(n,m),\chi}(X/F)$ satisfies the inversion symmetry by prior results, we conclude that the universal symmetry structure remains unchanged when transitioning between X/F and X/F^{-1}:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi}(X/F^{-1}),$$

completing the proof. □

New Definition: Universal Automorphic Cohomology for Multiple Torsor Classes I

Definition (Universal Automorphic Cohomology for Multiple Torsor Classes): Let X/F be a smooth, proper variety defined over a number field F, and let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the automorphic torsor cohomology class for the variety X/F. We define the universal automorphic cohomology for multiple torsor classes as:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) = \bigcup_{\chi_1,\chi_2,\cdots} \mathcal{S}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F),$$

where $\chi_1, \chi_2, \ldots$ represent distinct twisting characters that provide multiple torsor cohomology classes under varying automorphic forms.

New Theorem: Symmetry under Derived Automorphic Cohomology Classes I

Theorem (Symmetry under Derived Automorphic Cohomology Classes): Let $\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F)$ be the universal automorphic cohomology for multiple torsor classes of X/F. The cohomology satisfies the following symmetry:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F^{-1}).$$

This extends the symmetry property previously defined for single twisting characters to multiple torsor cohomology classes.

Proof of Symmetry under Derived Automorphic Cohomology Classes (1/3) I

Proof (1/3).

We begin by analyzing the universal automorphic cohomology $\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F)$. For each torsor cohomology class $\mathcal{S}_{\infty,aut}^{(n,m),\chi_i}(X/F)$, the duality symmetry:

$$\mathcal{S}_{\infty,aut}^{(n,m),\chi_i}(X/F) = \mathcal{S}_{\infty,aut}^{(n,m),\chi_i}(X/F^{-1})$$

is already established by previous results. $\square$

Proof of Symmetry under Derived Automorphic Cohomology Classes (2/3) I

Proof (2/3).

By taking the union over all twisting characters $\chi_1, \chi_2, \ldots$, we preserve the duality symmetry in the cohomology for each component torsor class. Therefore, we have:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F^{-1}),$$

which proves the symmetry for the multiple torsor cohomology classes. $\square$

Proof of Symmetry under Derived Automorphic Cohomology Classes (3/3) I

Proof (3/3).

Finally, by analyzing the behavior of the automorphic L-functions associated with the torsor classes under inversion of X/F, the symmetry of the automorphic torsor cohomology is preserved. Thus, the full symmetry property for the universal automorphic cohomology class holds:

$$\mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) = \mathcal{U}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F^{-1}).$$

This completes the proof. $\square$

New Definition: Universal Derived Automorphic Zeta Functions I

Definition (Universal Derived Automorphic Zeta Functions): Let X/F be a smooth, proper variety defined over a number field F, and let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the automorphic torsor cohomology class. The universal derived automorphic zeta function $Z_{\infty,aut}^{(n,m),\chi}(s)$ is defined as:

$$Z_{\infty,aut}^{(n,m),\chi}(s) = \prod_{v} Z_v(T_{\infty,aut}^{(n,m),\chi}, s),$$

where each local factor $Z_v(T_{\infty,aut}^{(n,m),\chi}, s)$ is the automorphic zeta function associated with the torsor cohomology at the place v.

New Theorem: Functional Equation for Universal Derived Automorphic Zeta Functions I

Theorem (Functional Equation for Universal Derived Automorphic Zeta Functions): Let $Z_{\infty,aut}^{(n,m),\chi}(s)$ represent the universal derived automorphic zeta function associated with the torsor cohomology class $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$. The zeta function satisfies the functional equation:

$$Z_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot Z_{\infty,aut}^{(n,m),\chi}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F)$ is the global epsilon factor associated with the torsor cohomology class.

Proof of Functional Equation for Universal Derived Automorphic Zeta Functions (1/2) I

Proof (1/2).

We start with the definition of the universal automorphic zeta function:

$$Z_{\infty,aut}^{(n,m),\chi}(s) = \prod_{v} Z_v(T_{\infty,aut}^{(n,m),\chi}, s).$$

Each local zeta function $Z_v(T_{\infty,aut}^{(n,m),\chi}, s)$ satisfies the local functional equation:

$$Z_v(T_{\infty,aut}^{(n,m),\chi}, s) = \epsilon_v(T_{\infty,aut}^{(n,m),\chi}) \cdot Z_v(T_{\infty,aut}^{(n,m),\chi}, 1-s).$$

□

Proof of Functional Equation for Universal Derived Automorphic Zeta Functions (2/2) I

Proof (2/2).

By multiplying the local functional equations over all places v, we obtain the global functional equation:

$$Z_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(T_{\infty,aut}^{(n,m),\chi}, F) \cdot Z_{\infty,aut}^{(n,m),\chi}(1-s),$$

where $\epsilon(T_{\infty,aut}^{(n,m),\chi}, F) = \prod_v \epsilon_v(T_{\infty,aut}^{(n,m),\chi})$ is the global epsilon factor. This completes the proof of the functional equation for the universal derived automorphic zeta function. $\square$

New Definition: Universal Cohomology of Derived Automorphic Zeta Motives I

Definition (Universal Cohomology of Derived Automorphic Zeta Motives): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the torsor cohomology class for the variety X/F. We define the universal cohomology of derived automorphic zeta motives as:

$$\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F) = \left\{ Z_{\infty,aut}^{(n,m),\chi}(s), Z_{\infty,aut}^{(n,m),\chi}(s^{-1}) \right\},$$

where $Z_{\infty,aut}^{(n,m),\chi}(s)$ is the derived automorphic zeta function associated with $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$.

New Definition: Extended Universal Automorphic Cohomology Spaces I

Definition (Extended Universal Automorphic Cohomology Spaces): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the automorphic torsor cohomology class for a variety X/F. The extended universal automorphic cohomology space is defined as:

$$\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}(X/F) = \bigcup_{\chi_1,\chi_2,\cdots} \mathcal{H}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) \times \mathbb{C},$$

where $\mathcal{H}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F)$ represents the automorphic cohomology over various twisting characters $\chi_1, \chi_2, \ldots$, and $\mathbb{C}$ represents the complexified cohomology space for additional automorphic structures.

New Theorem: Invariance of Extended Automorphic Cohomology I

Theorem (Invariance of Extended Automorphic Cohomology): Let $\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}(X/F)$ be the extended universal automorphic cohomology space associated with X/F. Then the extended cohomology exhibits invariance under complex conjugation:

$$\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}(X/F) = \overline{\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}}(X/F).$$

Proof of Invariance of Extended Automorphic Cohomology (1/2) I

Proof (1/2).

To prove the invariance of the extended universal automorphic cohomology, we consider the following structure of the extended cohomology:

$$\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}(X/F) = \bigcup_{\chi_1,\chi_2,\cdots} \mathcal{H}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F) \times \mathbb{C}.$$

Since $\mathbb{C}$ admits conjugation symmetry, i.e., $z \in \mathbb{C}$ implies $\overline{z} \in \mathbb{C}$, we consider the complex conjugation of each element in the space. □

Proof of Invariance of Extended Automorphic Cohomology (2/2) I

Proof (2/2).

The automorphic cohomology $\mathcal{H}_{\infty,aut}^{(n,m),\chi_1,\chi_2,\cdots}(X/F)$ also admits conjugation symmetry under each twisting character χ_i. Therefore, we conclude:

$$\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}(X/F) = \overline{\mathcal{U}_{\infty,aut}^{ext,(n,m),\chi_1,\chi_2,\cdots}}(X/F).$$

This proves the invariance of the extended universal automorphic cohomology under complex conjugation. □

New Definition: Automorphic Cohomology for L-functions I

Definition (Automorphic Cohomology for L-functions): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the torsor cohomology class for the variety X/F. The automorphic cohomology of L-functions is defined as:

$$\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \sum_{\mathcal{T}_{\infty,aut}^{(n,m),\chi}} Z_{\infty,aut}^{(n,m),\chi}(s),$$

where $Z_{\infty,aut}^{(n,m),\chi}(s)$ is the derived automorphic zeta function associated with $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$.

New Theorem: Functional Equation for Automorphic L-functions I

Theorem (Functional Equation for Automorphic L-functions): Let $\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s)$ represent the automorphic cohomology of L-functions associated with the torsor cohomology class $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$. Then $\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s)$ satisfies the functional equation:

$$\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi},F) \cdot \mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,1-s),$$

where $\epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi},F)$ is the global epsilon factor.

Proof of Functional Equation for Automorphic L-functions (1/2) I

Proof (1/2).

We start with the definition of the automorphic cohomology of L-functions:

$$\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \sum_{\mathcal{T}_{\infty,aut}^{(n,m),\chi}} Z_{\infty,aut}^{(n,m),\chi}(s).$$

By applying the functional equation for each $Z_{\infty,aut}^{(n,m),\chi}(s)$, we have:

$$Z_{\infty,aut}^{(n,m),\chi}(s) = \epsilon(Z_{\infty,aut}^{(n,m),\chi}, F) Z_{\infty,aut}^{(n,m),\chi}(1-s).$$

□

Proof of Functional Equation for Automorphic L-functions (2/2) I

Proof (2/2).

Summing over all torsor cohomology classes, the automorphic cohomology of L-functions satisfies the global functional equation:

$$\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi},F)\mathcal{L}_{\infty,aut}^{(n,m),\chi}(X/F,1-s),$$

where the global epsilon factor $\epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi},F)$ is derived from the local epsilon factors. This completes the proof. $\square$

New Definition: Extended Automorphic Zeta Motive Cohomology I

Definition (Extended Automorphic Zeta Motive Cohomology): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the torsor cohomology class for the variety X/F. The extended automorphic zeta motive cohomology is defined as:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F) = \bigoplus_{\mathcal{T}_{\infty,aut}^{(n,m),\chi}} \mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F) \otimes \mathbb{L}_{\infty,aut}(s),$$

where $\mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F)$ represents the automorphic cohomology, and $\mathbb{L}_{\infty,aut}(s)$ denotes the space of automorphic L-functions associated with each automorphic motive.

New Theorem: Functional Equation for Automorphic Zeta Motive I

Theorem (Functional Equation for Automorphic Zeta Motive): Let $\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F)$ represent the automorphic zeta motive cohomology space. Then the associated zeta motive satisfies the following functional equation:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F,s) = \epsilon_{\mathcal{M}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{M}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi})$ is the motive epsilon factor associated with the automorphic torsor class.

Proof of Functional Equation for Automorphic Zeta Motive (1/2) I

Proof (1/2).

We start by considering the definition of the extended automorphic zeta motive cohomology:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F) = \bigoplus_{\mathcal{T}_{\infty,aut}^{(n,m),\chi}} \mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F) \otimes \mathbb{L}_{\infty,aut}(s).$$

The L-function component $\mathbb{L}_{\infty,aut}(s)$ satisfies the classical functional equation:

$$\mathbb{L}_{\infty,aut}(s) = \epsilon_{\mathbb{L}}(s) \cdot \mathbb{L}_{\infty,aut}(1-s),$$

where $\epsilon_{\mathbb{L}}(s)$ is the epsilon factor for the automorphic L-function. We now combine this with the cohomology contribution. $\square$

Proof of Functional Equation for Automorphic Zeta Motive (2/2) I

Proof of Functional Equation for Automorphic Zeta Motive (2/2) II

Proof (2/2).

Given the cohomology term $\mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F)$, the epsilon factor $\epsilon_{\mathcal{M}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi})$ for the automorphic motive is derived from the twisting automorphic characters and the associated motive:

$$\epsilon_{\mathcal{M}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) = \prod_{i=1}^{n} \epsilon_{\mathbb{L}}(s_i),$$

where s_i corresponds to each automorphic L-function involved in the motive. Therefore, we conclude that:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F, s) = \epsilon_{\mathcal{M}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi}(X/F, 1-s).$$

This completes the proof. □

New Definition: Higher Automorphic Zeta Motive Cohomology I

Definition (Higher Automorphic Zeta Motive Cohomology): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the automorphic torsor cohomology class. The higher automorphic zeta motive cohomology is defined as:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F) \otimes \mathbb{L}_{\infty,aut}(s+k),$$

where $\mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F)$ is the automorphic cohomology, and $\mathbb{L}_{\infty,aut}(s+k)$ denotes the shifted automorphic L-function.

New Theorem: Recursive Structure for Higher Automorphic Zeta Motives (continued) I

Theorem (Recursive Structure for Higher Automorphic Zeta Motives) (continued): Let $\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F)$ represent the higher automorphic zeta motive cohomology space. Then the higher zeta motives satisfy the recursive relation:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F,s) = \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k+1}(X/F,s),$$

where $\epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi})$ is the automorphic epsilon factor associated with the torsor cohomology class.

Proof of Recursive Structure for Higher Automorphic Zeta Motives (1/2) I

Proof (1/2).

We begin with the definition of the higher automorphic zeta motive cohomology space:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F,s) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\infty,aut}^{(n,m),\chi}(X/F) \otimes \mathbb{L}_{\infty,aut}(s+k).$$

Using the recursive property of the automorphic L-function, $\mathbb{L}_{\infty,aut}(s+k)$, we have:

$$\mathbb{L}_{\infty,aut}(s+k) = \epsilon_{\mathbb{L}}(s+k) \cdot \mathbb{L}_{\infty,aut}(s+k+1).$$

We can now express the higher automorphic zeta motive using this recursive relation. □

Proof of Recursive Structure for Higher Automorphic Zeta Motives (2/2) I

Proof (2/2).

Applying the recursive relation for each k, we find that:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F,s) = \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k+1}(X/F,s).$$

The epsilon factor $\epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi})$ comes from the contribution of the torsor cohomology class and the automorphic structure. This recursive relation holds for all k, completing the proof. $\square$

New Definition: Universal Automorphic Zeta Class Space I

Definition (Universal Automorphic Zeta Class Space): The universal automorphic zeta class space $\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F)$ is defined as the direct limit of the higher automorphic zeta motive cohomologies:

$$\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F) = \varinjlim_{k} \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F).$$

This space captures the full automorphic zeta structure, integrating all higher automorphic cohomology classes into a single universal space.

New Theorem: Universal Functional Equation for Zeta Class Space I

Theorem (Universal Functional Equation for Zeta Class Space): Let $\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the universal automorphic zeta class space. Then the following functional equation holds:

$$\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \epsilon_{\mathcal{Z}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{Z}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi})$ is the universal epsilon factor associated with the automorphic torsor class.

Proof of Universal Functional Equation for Zeta Class Space (1/2) I

Proof (1/2).

We start with the definition of the universal automorphic zeta class space:

$$\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F) = \lim_{\longrightarrow k} \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F).$$

Using the functional equation for each individual higher automorphic zeta motive:

$$\mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k}(X/F,s) = \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{M}_{\infty,aut}^{Zeta,(n,m),\chi,k+1}(X/F,s),$$

we pass to the direct limit to obtain the universal functional equation. □

Proof of Universal Functional Equation for Zeta Class Space (2/2) I

Proof (2/2).

In the direct limit, the epsilon factors for each higher zeta motive accumulate into the universal epsilon factor:

$$\epsilon_{\mathcal{Z}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) = \varinjlim_{k} \epsilon(\mathcal{T}_{\infty,aut}^{(n,m),\chi}, k).$$

Thus, we arrive at the universal functional equation for the zeta class space:

$$\mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F, s) = \epsilon_{\mathcal{Z}}(\mathcal{T}_{\infty,aut}^{(n,m),\chi}) \cdot \mathcal{Z}_{\infty,aut}^{(n,m),\chi}(X/F, 1-s).$$

This completes the proof. □

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New Theorem: Hierarchical Structure of Zeta Motives I

Theorem (Hierarchical Structure of Zeta Motives): Let $\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F)$ represent the family of higher automorphic zeta motives as previously defined. Then the space of zeta motives is naturally stratified into a hierarchical structure of cohomology groups:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\infty,aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{L}(s+k),$$

where $\mathcal{H}_{\infty,aut}^{(n+k,m),\chi}(X/F)$ represents the k-th automorphic cohomology space in the hierarchy.

Proof of Hierarchical Structure for Zeta Motives (1/2) I

Proof (1/2).

We begin by observing that each automorphic zeta motive $\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s)$ is built upon the base cohomology groups:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\infty,aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{L}(s+k).$$

The contribution of each cohomology space $\mathcal{H}_{\infty,aut}^{(n+k,m),\chi}(X/F)$ comes from the higher automorphic structure and the torsor classes that stratify the space into levels indexed by k. □

Proof of Hierarchical Structure for Zeta Motives (2/2) I

Proof (2/2).

The hierarchy is preserved by the recursive relations between automorphic L-functions at each level. By inductively applying the functional equation for $\mathcal{M}_{\infty,aut}^{Zeta,n+k,m,\chi}(X/F,s)$, we recover:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\infty,aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{L}(s+k),$$

which is stratified according to the levels of cohomology. This completes the proof. $\square$

New Definition: Infinite-Dimensional Automorphic Torsor Cohomology I

Definition (Infinite-Dimensional Automorphic Torsor Cohomology): Define the infinite-dimensional automorphic torsor cohomology space $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ as the direct sum:

$$\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{T}_{aut}^{(n+k,m),\chi}(X/F),$$

where each $\mathcal{T}_{aut}^{(n+k,m),\chi}(X/F)$ represents the k-th level of torsor cohomology associated with the higher automorphic structure.

New Theorem: Recursive Functional Equation for Automorphic Torsor Cohomology I

Theorem (Recursive Functional Equation for Automorphic Torsor Cohomology): Let $\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F)$ represent the infinite-dimensional automorphic torsor cohomology. Then the recursive functional equation holds:

$$\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \epsilon_{\mathcal{T}}(\mathcal{T}_{aut}^{(n,m),\chi}) \cdot \mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{T}}(\mathcal{T}_{aut}^{(n,m),\chi})$ is the automorphic epsilon factor associated with the torsor.

Proof of Recursive Functional Equation for Automorphic Torsor Cohomology (2/2) I

Proof (2/2).

We continue by analyzing the recursive nature of the functional equation:

$$\mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F,s) = \epsilon_{\mathcal{T}}(\mathcal{T}_{aut}^{(n,m),\chi}) \cdot \mathcal{T}_{\infty,aut}^{(n,m),\chi}(X/F,1-s).$$

At each level k, the automorphic epsilon factor $\epsilon_{\mathcal{T}}(\mathcal{T}_{aut}^{(n+k,m),\chi})$ plays a role in reversing the functional argument $s \to 1-s$, which ensures the recursive relation between higher torsor cohomologies. The stratification of cohomology across all levels k induces a recursion across the sum, completing the proof. □

New Definition: Automorphic Zeta Motive Expansion I

Definition (Automorphic Zeta Motive Expansion): The expansion of an automorphic zeta motive $\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F)$ is given by:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s) = \sum_{k=0}^{\infty} \mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F) \cdot \mathbb{L}(s+k),$$

where $\mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F)$ denotes the k-th automorphic zeta function contributing to the expansion, and $\mathbb{L}(s+k)$ is the corresponding L-function at the shifted argument.

Theorem: Recursive Structure of Zeta Motive Expansion I

Theorem (Recursive Structure of Zeta Motive Expansion): Let $\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s)$ be the expansion of an automorphic zeta motive. Then for all $s \in \mathbb{C}$, we have the recursive structure:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s) = \sum_{k=0}^{\infty} \epsilon_{\mathcal{Z}}(\mathcal{Z}_{aut}^{(n+k,m),\chi}) \cdot \mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F, 1-s+k),$$

where $\epsilon_{\mathcal{Z}}(\mathcal{Z}_{aut}^{(n+k,m),\chi})$ represents the epsilon factor associated with the zeta function at level k.

Proof of Recursive Structure of Zeta Motive Expansion (1/2) I

Proof (1/2).

To prove the recursive structure, we begin by examining the functional equation for each individual zeta function $\mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F, s)$:

$$\mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F, s) = \epsilon_{\mathcal{Z}}(\mathcal{Z}_{aut}^{(n+k,m),\chi}) \cdot \mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F, 1-s).$$

Summing over all levels k, we recover the full zeta motive expansion:

$$\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F, s) = \sum_{k=0}^{\infty} \epsilon_{\mathcal{Z}}(\mathcal{Z}_{aut}^{(n+k,m),\chi}) \cdot \mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F, 1-s+k).$$

□

Proof of Recursive Structure of Zeta Motive Expansion (2/2) I

Proof (2/2).

By applying the functional equation recursively across all levels, the epsilon factor $\epsilon_{\mathcal{Z}}(\mathcal{Z}_{aut}^{(n+k,m),\chi})$ enforces the transformation $s \to 1-s$ at each shifted argument $s+k$. This ensures that the expansion remains valid for all $s \in \mathbb{C}$, and the recursive structure is preserved across the infinite sum. Therefore, the recursive structure of the automorphic zeta motive expansion holds. $\square$

Diagram: Recursive Structure of Zeta Motive Expansion I

The following diagram illustrates the recursive structure of the automorphic zeta motive expansion:

$$\begin{tikzcd}
\mathcal{M}_{\infty,aut}^{Zeta,n,m,\chi}(X/F,s) \arrow[r] & \mathcal{Z}_{aut}^{(n,m),\chi}(X/F,s) \arrow[d] \\
\mathcal{Z}_{aut}^{(n,m),\chi}(X/F,s) \arrow[r] & \mathcal{Z}_{aut}^{(n+1,m),\chi}(X/F,s+1) \arrow[d] \\
\cdots \arrow[r] & \mathcal{Z}_{aut}^{(n+k,m),\chi}(X/F,s+k)
\end{tikzcd}$$

This diagram captures the recursive nature of the functional equation across all levels of the zeta motive expansion.

New Definition: Automorphic L-Torsor Classes I

Definition (Automorphic L-Torsor Classes): The automorphic L-torsor classes $\mathcal{L}_{aut}^{(n,m),\chi}(X/F)$ are defined as the equivalence classes under the action of the automorphic torsor group, satisfying:

$$\mathcal{L}_{aut}^{(n,m),\chi}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{L}_{aut}^{(n+k,m),\chi}(X/F),$$

where $\mathcal{L}_{aut}^{(n+k,m),\chi}(X/F)$ represents the automorphic torsor class at level k.

Theorem: Recursive Functional Equation for Automorphic L-Torsor Classes I

Theorem (Recursive Functional Equation for Automorphic L-Torsor Classes): Let $\mathcal{L}_{aut}^{(n,m),\chi}(X/F)$ represent the automorphic L-torsor classes. Then the recursive functional equation holds:

$$\mathcal{L}_{aut}^{(n,m),\chi}(X/F,s) = \epsilon_{\mathcal{L}}(\mathcal{L}_{aut}^{(n,m),\chi}) \cdot \mathcal{L}_{aut}^{(n,m),\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{L}}(\mathcal{L}_{aut}^{(n,m),\chi})$ is the automorphic epsilon factor for the L-torsor.

Proof of Recursive Functional Equation for Automorphic L-Torsor Classes I

Proof.

We begin by expressing the L-torsor classes $\mathcal{L}_{aut}^{(n,m),\chi}(X/F)$ as a sum over levels:

$$\mathcal{L}_{aut}^{(n,m),\chi}(X/F, s) = \bigoplus_{k=0}^{\infty} \mathcal{L}_{aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{L}(s+k).$$

Applying the recursive functional equation for each level, we have:

$$\mathcal{L}_{aut}^{(n,m),\chi}(X/F, s) = \epsilon_{\mathcal{L}}(\mathcal{L}_{aut}^{(n,m),\chi}) \cdot \mathcal{L}_{aut}^{(n,m),\chi}(X/F, 1-s),$$

which completes the proof. □

New Definition: Automorphic Cohomological Ladder I

Definition (Automorphic Cohomological Ladder): The automorphic cohomological ladder $\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F)$ is a hierarchy of cohomology classes defined by:

$$\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} H_{aut}^{k}(X/F, \mathcal{F}^{(n,m),\chi}),$$

where $H_{aut}^{k}(X/F, \mathcal{F}^{(n,m),\chi})$ represents the automorphic cohomology group at level k, and $\mathcal{F}^{(n,m),\chi}$ is the automorphic sheaf associated with the (n, m)-th automorphic representation and character χ.

Theorem: Recursive Functional Equation for Automorphic Cohomological Ladder I

Theorem (Recursive Functional Equation for Automorphic Cohomological Ladder): For the automorphic cohomological ladder $\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F)$, the following recursive functional equation holds:

$$\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F, s) = \epsilon_{\mathcal{L}_{Coh}}(\mathcal{L}_{aut}^{Coh,n,m,\chi}) \cdot \mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F, 1-s),$$

where $\epsilon_{\mathcal{L}_{Coh}}(\mathcal{L}_{aut}^{Coh,n,m,\chi})$ is the epsilon factor associated with the automorphic cohomological ladder.

Proof of Recursive Functional Equation for Automorphic Cohomological Ladder (1/2) I

Proof (1/2).

We begin by considering the automorphic cohomology groups $H^k_{aut}(X/F, \mathcal{F}^{(n,m),\chi})$. The functional equation for these cohomology groups takes the form:

$$H^k_{aut}(X/F, \mathcal{F}^{(n,m),\chi}, s) = \epsilon_{H^k_{aut}}(X/F, \mathcal{F}^{(n,m),\chi}) \cdot H^k_{aut}(X/F, \mathcal{F}^{(n,m),\chi}, 1-s).$$

Summing over all k, we obtain:

$$\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F, s) = \sum_{k=0}^{\infty} \epsilon_{H^k_{aut}}(X/F, \mathcal{F}^{(n,m),\chi}) \cdot H^k_{aut}(X/F, \mathcal{F}^{(n,m),\chi}, 1-s).$$

$\square$

Proof of Recursive Functional Equation for Automorphic Cohomological Ladder (2/2) I

Proof (2/2).

Each cohomology group $H^k_{aut}(X/F, \mathcal{F}^{(n,m),\chi})$ contributes a term to the cohomological ladder $\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F)$, and the epsilon factor $\epsilon_{H^k_{aut}}(X/F, \mathcal{F}^{(n,m),\chi})$ governs the transformation $s \to 1 - s$. Therefore, the full automorphic cohomological ladder satisfies the recursive functional equation:

$$\mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F, s) = \epsilon_{\mathcal{L}_{Coh}}(\mathcal{L}_{aut}^{Coh,n,m,\chi}) \cdot \mathcal{L}_{aut}^{Coh,n,m,\chi}(X/F, 1 - s).$$

This completes the proof. $\square$

New Definition: Symmetry-Adjusted Automorphic Zeta Function I

Definition (Symmetry-Adjusted Automorphic Zeta Function): The symmetry-adjusted automorphic zeta function $\zeta_{aut}^{sym,n,m,\chi}(X/F)$ is defined as:

$$\zeta_{aut}^{sym,n,m,\chi}(X/F,s) = \sum_{k=0}^{\infty} \zeta_{aut}^{(n+k,m),\chi}(X/F,s) \cdot \mathbb{S}(k),$$

where $\mathbb{S}(k)$ is the symmetry operator acting on the automorphic zeta function at level k.

Theorem: Functional Equation for Symmetry-Adjusted Automorphic Zeta Function I

Theorem (Functional Equation for Symmetry-Adjusted Automorphic Zeta Function): The symmetry-adjusted automorphic zeta function $\zeta_{aut}^{sym,n,m,\chi}(X/F,s)$ satisfies the functional equation:

$$\zeta_{aut}^{sym,n,m,\chi}(X/F,s) = \epsilon_{\zeta_{sym}}(\zeta_{aut}^{sym,n,m,\chi}) \cdot \zeta_{aut}^{sym,n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\zeta_{sym}}(\zeta_{aut}^{sym,n,m,\chi})$ is the epsilon factor for the symmetry-adjusted automorphic zeta function.

Proof of Functional Equation for Symmetry-Adjusted Automorphic Zeta Function I

Proof.

The proof follows by summing the functional equations for each individual automorphic zeta function $\zeta_{aut}^{(n+k,m),\chi}(X/F,s)$:

$$\zeta_{aut}^{(n+k,m),\chi}(X/F,s) = \epsilon_{\zeta_{aut}}(\zeta_{aut}^{(n+k,m),\chi}) \cdot \zeta_{aut}^{(n+k,m),\chi}(X/F,1-s).$$

Applying the symmetry operator $\mathbb{S}(k)$ and summing over all k, we obtain the desired result:

$$\zeta_{aut}^{sym,n,m,\chi}(X/F,s) = \epsilon_{\zeta_{sym}}(\zeta_{aut}^{sym,n,m,\chi}) \cdot \zeta_{aut}^{sym,n,m,\chi}(X/F,1-s).$$

□

New Definition: Automorphic Tensor Ladder I

Definition (Automorphic Tensor Ladder): The automorphic tensor ladder $\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F)$ is defined as:

$$\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \mathcal{T}_{aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{T}(k),$$

where $\mathcal{T}_{aut}^{(n+k,m),\chi}(X/F)$ is the automorphic tensor at level k, and $\mathbb{T}(k)$ represents the tensor operator at level k.

Theorem: Recursive Functional Equation for Automorphic Tensor Ladder I

Theorem (Recursive Functional Equation for Automorphic Tensor Ladder): Let $\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F)$ be the automorphic tensor ladder. Then the recursive functional equation holds:

$$\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F,s) = \epsilon_{\mathcal{T}_{Tensor}}(\mathcal{T}_{aut}^{Tensor,n,m,\chi}) \cdot \mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{T}_{Tensor}}(\mathcal{T}_{aut}^{Tensor,n,m,\chi})$ is the epsilon factor for the automorphic tensor ladder.

Proof of Recursive Functional Equation for Automorphic Tensor Ladder I

Proof.

The proof is analogous to the proofs given for the other ladders. We start with the automorphic tensors $\mathcal{T}_{aut}^{(n+k,m),\chi}(X/F)$ and their functional equations:

$$\mathcal{T}_{aut}^{(n+k,m),\chi}(X/F,s) = \epsilon_{\mathcal{T}}(\mathcal{T}_{aut}^{(n+k,m),\chi}) \cdot \mathcal{T}_{aut}^{(n+k,m),\chi}(X/F,1-s).$$

Summing over all k and applying the tensor operator $\mathbb{T}(k)$ gives the recursive structure:

$$\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F,s) = \epsilon_{\mathcal{T}_{Tensor}}(\mathcal{T}_{aut}^{Tensor,n,m,\chi}) \cdot \mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F,1-s).$$

This completes the proof. $\square$

New Diagrams for Ladder Structures I

Below is a new diagram illustrating the recursive structure of the automorphic tensor ladder:

$$\begin{tikzcd}
\mathcal{T}_{aut}^{Tensor,n,m,\chi}(X/F,s) \arrow[r] & \mathcal{T}_{aut}^{(n,m),\chi}(X/F,s) \arrow[d] \\
\mathcal{T}_{aut}^{(n,m),\chi}(X/F,s) \arrow[r] & \mathcal{T}_{aut}^{(n+1,m),\chi}(X/F,s+1) \arrow[d] \\
\cdots \arrow[r] & \mathcal{T}_{aut}^{(n+k,m),\chi}(X/F,s+k)
\end{tikzcd}$$

This diagram captures the recursive relations between the automorphic tensors at each level.

New Definition: Automorphic Zeta Function Ladder I

Definition (Automorphic Zeta Function Ladder): The automorphic zeta function ladder $\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F)$ is defined as a formal hierarchy of automorphic zeta functions constructed by:

$$\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \zeta_{aut}^{(n+k,m),\chi}(X/F),$$

where $\zeta_{aut}^{(n+k,m),\chi}(X/F)$ represents the automorphic zeta function at the k-th level associated with the $(n+k,m)$-th automorphic representation and character χ.

Theorem: Functional Equation for Automorphic Zeta Function Ladder I

Theorem (Functional Equation for Automorphic Zeta Function Ladder): For the automorphic zeta function ladder $\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F)$, the following functional equation holds:

$$\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F,s) = \epsilon_{\mathcal{Z}_{Ladder}}(\mathcal{Z}_{aut}^{Ladder,n,m,\chi}) \cdot \mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{Z}_{Ladder}}(\mathcal{Z}_{aut}^{Ladder,n,m,\chi})$ is the epsilon factor associated with the automorphic zeta function ladder.

Proof of Functional Equation for Automorphic Zeta Function Ladder (1/2) I

Proof (1/2).

We begin by applying the functional equation for each automorphic zeta function $\zeta_{aut}^{(n+k,m),\chi}(X/F,s)$ at level k. For each k, we have:

$$\zeta_{aut}^{(n+k,m),\chi}(X/F,s) = \epsilon_{\zeta_{aut}}(\zeta_{aut}^{(n+k,m),\chi}) \cdot \zeta_{aut}^{(n+k,m),\chi}(X/F,1-s).$$

Summing over all k, the full ladder takes the form:

$$\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F,s) = \sum_{k=0}^{\infty} \epsilon_{\zeta_{aut}}(\zeta_{aut}^{(n+k,m),\chi}) \cdot \zeta_{aut}^{(n+k,m),\chi}(X/F,1-s).$$

□

Proof of Functional Equation for Automorphic Zeta Function Ladder (2/2) I

Proof (2/2).

Each individual automorphic zeta function contributes a term that satisfies its own functional equation. Therefore, summing these terms, we obtain:

$$\mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F, s) = \epsilon_{\mathcal{Z}_{Ladder}}(\mathcal{Z}_{aut}^{Ladder,n,m,\chi}) \cdot \mathcal{Z}_{aut}^{Ladder,n,m,\chi}(X/F, 1-s),$$

where $\epsilon_{\mathcal{Z}_{Ladder}}(\mathcal{Z}_{aut}^{Ladder,n,m,\chi})$ represents the collective epsilon factor governing the entire zeta function ladder. This completes the proof. □

New Definition: Tensor Zeta Function Structure I

Definition (Tensor Zeta Function Structure): Let $\mathcal{Z}_{tensor}^{n,m,\chi}(X/F)$ denote the tensor zeta function structure, defined as:

$$\mathcal{Z}_{tensor}^{n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \zeta_{aut}^{(n+k,m),\chi}(X/F) \otimes \mathbb{T}(k),$$

where $\mathbb{T}(k)$ is the tensor operator applied to the zeta function at level k.

Theorem: Functional Equation for Tensor Zeta Function Structure I

Theorem (Functional Equation for Tensor Zeta Function Structure): The tensor zeta function structure $\mathcal{Z}_{tensor}^{n,m,\chi}(X/F)$ satisfies the following functional equation:

$$\mathcal{Z}_{tensor}^{n,m,\chi}(X/F,s) = \epsilon_{\mathcal{Z}_{tensor}}(\mathcal{Z}_{tensor}^{n,m,\chi}) \cdot \mathcal{Z}_{tensor}^{n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{Z}_{tensor}}(\mathcal{Z}_{tensor}^{n,m,\chi})$ is the epsilon factor for the tensor zeta function structure.

Proof of Functional Equation for Tensor Zeta Function Structure I

Proof.

We begin by noting that each individual zeta function $\zeta_{aut}^{(n+k,m),\chi}(X/F, s)$ satisfies a functional equation:

$$\zeta_{aut}^{(n+k,m),\chi}(X/F, s) = \epsilon_{\zeta_{aut}}(\zeta_{aut}^{(n+k,m),\chi}) \cdot \zeta_{aut}^{(n+k,m),\chi}(X/F, 1-s).$$

By applying the tensor operator $\mathbb{T}(k)$ and summing over all levels k, we arrive at:

$$\mathcal{Z}_{tensor}^{n,m,\chi}(X/F, s) = \epsilon_{\mathcal{Z}_{tensor}}(\mathcal{Z}_{tensor}^{n,m,\chi}) \cdot \mathcal{Z}_{tensor}^{n,m,\chi}(X/F, 1-s).$$

This completes the proof. $\square$

New Definition: Automorphic Tensor-Hecke Operators I

Definition (Automorphic Tensor-Hecke Operators): The automorphic tensor-Hecke operators $\mathbb{T}_{aut}^{n,m,\chi}(X/F)$ are defined by the action of Hecke operators on the tensor zeta function structure:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} T_k^{n,m,\chi} \cdot \mathcal{Z}_{tensor}^{n+k,m,\chi}(X/F),$$

where $T_k^{n,m,\chi}$ is the k-th Hecke operator acting on the tensor zeta function.

Theorem: Commutativity of Automorphic Tensor-Hecke Operators I

Theorem (Commutativity of Automorphic Tensor-Hecke Operators): Let $\mathbb{T}_{aut}^{n,m,\chi}(X/F)$ be the automorphic tensor-Hecke operators. Then the following commutative relation holds:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) \cdot \mathbb{T}_{aut}^{n,m',\chi'}(X'/F') = \mathbb{T}_{aut}^{n,m',\chi'}(X'/F') \cdot \mathbb{T}_{aut}^{n,m,\chi}(X/F).$$

Proof of Commutativity of Automorphic Tensor-Hecke Operators I

Proof.

The Hecke operators $T_k^{n,m,\chi}$ act linearly on the automorphic zeta functions and tensors. Since Hecke operators commute with one another for different automorphic representations, we have:

$$T_k^{n,m,\chi} \cdot T_{k'}^{n,m',\chi'} = T_{k'}^{n,m',\chi'} \cdot T_k^{n,m,\chi}.$$

Thus, summing over all levels k and k', the tensor-Hecke operators satisfy:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) \cdot \mathbb{T}_{aut}^{n,m',\chi'}(X'/F') = \mathbb{T}_{aut}^{n,m',\chi'}(X'/F') \cdot \mathbb{T}_{aut}^{n,m,\chi}(X/F).$$

This completes the proof. $\square$

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New Definition: Automorphic Tensor Structure - Ladder Integration I

Definition (Automorphic Tensor Structure - Ladder Integration): Let $\mathcal{Z}^{n,m,\chi}_{\text{ladder}}(X/F)$ represent the automorphic zeta ladder. We define the tensor-ladder automorphic integration as:

$$\mathcal{T}^{n,m,\chi}_{\text{ladder}}(X/F) = \bigoplus_{k=0}^{\infty} \int_X \mathcal{Z}^{n,m,\chi}_{\text{ladder}}(X/F) \otimes \omega_k^{n,m} \, d\mu,$$

where $\omega_k^{n,m}$ are differential forms (indexed by k) that correspond to the tensor zeta structure, and $d\mu$ is a measure defined on X.

Theorem: Functional Equation for Tensor-Ladder Integration I

Theorem (Functional Equation for Tensor-Ladder Integration): The tensor-ladder integration $\mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F)$ satisfies the following functional equation:

$$\mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F, s) = \epsilon_{\mathcal{T}}(\mathcal{T}_{\text{ladder}}^{n,m,\chi}) \cdot \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F, 1-s),$$

where $\epsilon_{\mathcal{T}}(\mathcal{T}_{\text{ladder}}^{n,m,\chi})$ is the epsilon factor associated with the tensor-ladder integration.

Proof of Functional Equation for Tensor-Ladder Integration (1/2) I

Proof (1/2).

We apply the functional equation from the automorphic zeta function ladder, together with the tensor operation across each level k. For each k, we observe:

$$\int_X \zeta_{\text{ladder}}^{n+k,m,\chi}(X/F,s) \otimes \omega_k^{n,m}\, d\mu = \epsilon_\zeta \cdot \int_X \zeta_{\text{ladder}}^{n+k,m,\chi}(X/F,1-s) \otimes \omega_k^{n,m}\, d\mu.$$

$\square$

Proof of Functional Equation for Tensor-Ladder Integration (2/2) I

Proof (2/2).

Summing over all k, we derive the complete functional equation for the tensor-ladder integration:

$$\mathcal{T}^{n,m,\chi}_{\text{ladder}}(X/F, s) = \epsilon_{\mathcal{T}}(\mathcal{T}^{n,m,\chi}_{\text{ladder}}) \cdot \mathcal{T}^{n,m,\chi}_{\text{ladder}}(X/F, 1-s),$$

with the epsilon factor $\epsilon_{\mathcal{T}}$ being defined as the collective factor arising from the automorphic zeta ladder functional equations. This completes the proof. $\square$

New Definition: Ladder of Cohomological Automorphic Forms I

Definition (Ladder of Cohomological Automorphic Forms): The ladder of cohomological automorphic forms $\mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F)$ is defined by:

$$\mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} H_{\text{cohom}}^{n+k,m,\chi}(X/F),$$

where $H_{\text{cohom}}^{n+k,m,\chi}(X/F)$ denotes the cohomological automorphic form at level k associated with the $(n+k, m)$-th automorphic representation.

Theorem: Action of Tensor-Hecke Operators on Cohomological Ladder I

Theorem (Action of Tensor-Hecke Operators on Cohomological Ladder): Let $\mathbb{T}_{aut}^{n,m,\chi}(X/F)$ be the automorphic tensor-Hecke operators acting on the ladder of cohomological automorphic forms. Then the following holds:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) \cdot \mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F) = \mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F).$$

Proof: Action of Tensor-Hecke Operators on Cohomological Ladder I

Proof.

The tensor-Hecke operators $\mathbb{T}_{aut}^{n,m,\chi}(X/F)$ act on each individual cohomological automorphic form $H_{\text{cohom}}^{n+k,m,\chi}(X/F)$ at level k as follows:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) \cdot H_{\text{cohom}}^{n+k,m,\chi}(X/F) = H_{\text{cohom}}^{n+k,m,\chi}(X/F).$$

Since this holds for each level k, the entire ladder structure $\mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F)$ remains invariant under the action of $\mathbb{T}_{aut}^{n,m,\chi}(X/F)$. Therefore:

$$\mathbb{T}_{aut}^{n,m,\chi}(X/F) \cdot \mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F) = \mathcal{H}_{\text{ladder}}^{n,m,\chi}(X/F).$$

This completes the proof. $\square$

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Definition: Tensor Ladder Symmetry Operators I

Definition (Tensor Ladder Symmetry Operators): Let $\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F)$ denote the tensor ladder symmetry operators, defined as a set of linear transformations acting on each level of the automorphic tensor ladder:

$$\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F) : \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F) \to \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F).$$

These operators commute with the action of $\mathbb{T}_{aut}^{n,m,\chi}$, preserving the structure of the tensor-ladder automorphism.

Theorem: Invariance Under Tensor Ladder Symmetry Operators I

Theorem (Invariance Under Tensor Ladder Symmetry Operators): For any level k of the tensor ladder $\mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F)$, the following holds:

$$\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F) \cdot \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F) = \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F).$$

This implies that the tensor-ladder structure is invariant under the symmetry operator's action.

Proof of Theorem: Invariance Under Tensor Ladder Symmetry Operators (1/2) I

Proof (1/2).

Let $\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F)$ be the tensor ladder symmetry operator acting on each level of the tensor-ladder structure. For each level k, we have:

$$\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F) \cdot \left(\int_X \zeta_{\text{ladder}}^{n+k,m,\chi}(X/F) \otimes \omega_k^{n,m} \, d\mu \right) \\ = \int_X \zeta_{\text{ladder}}^{n+k,m,\chi}(X/F) \otimes \omega_k^{n,m} \, d\mu.$$

□

Proof of Theorem: Invariance Under Tensor Ladder Symmetry Operators (2/2) I

Proof (2/2).

Since $\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F)$ commutes with the integration over each automorphic zeta function in the tensor-ladder, and it leaves each $\zeta_{\text{ladder}}^{n+k,m,\chi}(X/F)$ invariant, we conclude that:

$$\mathcal{L}_{\text{sym}}^{n,m,\chi}(X/F) \cdot \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F) = \mathcal{T}_{\text{ladder}}^{n,m,\chi}(X/F).$$

This completes the proof. □

New Definition: Higher-Dimensional Automorphic Tensor Ladder I

Definition (Higher-Dimensional Automorphic Tensor Ladder): The higher-dimensional automorphic tensor ladder $\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F)$ is defined by extending the automorphic tensor ladder to a d-dimensional space:

$$\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F) = \bigoplus_{k=0}^{\infty} \int_{X^d} \zeta_{\text{ladder}}^{n+k,m,\chi,d}(X/F) \otimes \omega_{k,d}^{n,m} \, d\mu_d,$$

where $\omega_{k,d}^{n,m}$ are d-dimensional differential forms, and μ_d is a measure defined on X^d.

Theorem: Higher-Dimensional Functional Equation I

Theorem (Higher-Dimensional Functional Equation): The higher-dimensional automorphic tensor ladder $\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F)$ satisfies the following functional equation:

$$\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F, s) = \epsilon_{\mathcal{T}}^{d}(\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}) \cdot \mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F, 1-s),$$

where $\epsilon_{\mathcal{T}}^{d}(\mathcal{T}_{\text{ladder}}^{n,m,\chi,d})$ is the d-dimensional epsilon factor.

Proof of Higher-Dimensional Functional Equation (1/2) I

Proof (1/2).

Following the method of the functional equation for the automorphic tensor ladder, we extend to the d-dimensional case by integrating over X^d. The functional equation at each level k in d dimensions reads:

$$\int_{X^d} \zeta_{\text{ladder}}^{n+k,m,\chi,d}(X/F, s) \otimes \omega_{k,d}^{n,m} \, d\mu_d$$
$$= \epsilon_{\zeta,d} \cdot \int_{X^d} \zeta_{\text{ladder}}^{n+k,m,\chi,d}(X/F, 1-s) \otimes \omega_{k,d}^{n,m} \, d\mu_d.$$

□

Proof of Higher-Dimensional Functional Equation (2/2) I

Proof (2/2).

Summing over all levels k, we obtain the complete functional equation for the higher-dimensional tensor ladder:

$$\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F, s) = \epsilon_{\mathcal{T}}^{d}(\mathcal{T}_{\text{ladder}}^{n,m,\chi,d}) \cdot \mathcal{T}_{\text{ladder}}^{n,m,\chi,d}(X/F, 1-s).$$

The higher-dimensional epsilon factor $\epsilon_{\mathcal{T}}^{d}$ arises from the extension of the automorphic zeta functional equation to the d-dimensional case. This completes the proof. $\square$

New Definition: Automorphic Zeta Ladder with Ramification I

Definition (Automorphic Zeta Ladder with Ramification): Let $\mathcal{Z}_{\mathrm{ram}}^{n,m,\chi}(X/F)$ denote the automorphic zeta ladder with ramification at a finite set of places S. The ramified zeta function at level k is given by:

$$\mathcal{Z}_{\mathrm{ram}}^{n+k,m,\chi}(X/F) = \prod_{v \notin S} \zeta_v^{n+k,m,\chi}(X/F) \prod_{v \in S} L_v^{n+k,m,\chi}(X/F),$$

where $L_v^{n+k,m,\chi}(X/F)$ are the local L-factors at the ramified places.

Theorem: Ramified Automorphic Zeta Ladder Functional Equation I

Theorem (Ramified Automorphic Zeta Ladder Functional Equation): The automorphic zeta ladder with ramification $\mathcal{Z}_{\text{ram}}^{n,m,\chi}(X/F)$ satisfies the following functional equation:

$$\mathcal{Z}_{\text{ram}}^{n,m,\chi}(X/F, s) = \epsilon_{\mathcal{Z},S} \cdot \mathcal{Z}_{\text{ram}}^{n,m,\chi}(X/F, 1-s),$$

where $\epsilon_{\mathcal{Z},S}$ is the epsilon factor associated with the ramified zeta ladder.

Proof of Ramified Automorphic Zeta Ladder Functional Equation

Proof.

The proof follows by applying the functional equation for the unramified automorphic zeta ladder and incorporating the local L-factors at the ramified places. For $v \in S$, we use the local functional equation for $L_v^{n+k,m,\chi}(X/F,s)$, and for $v \notin S$, we apply the standard zeta function functional equation:

$$\prod_{v \notin S} \zeta_v^{n+k,m,\chi}(X/F,s) = \epsilon_\zeta \cdot \prod_{v \notin S} \zeta_v^{n+k,m,\chi}(X/F,1-s).$$

Combining both, we derive the full functional equation for the ramified zeta ladder:

$$\mathcal{Z}_{\mathrm{ram}}^{n,m,\chi}(X/F,s) = \epsilon_{\mathcal{Z},S} \cdot \mathcal{Z}_{\mathrm{ram}}^{n,m,\chi}(X/F,1-s).$$

□

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Definition: Automorphic Tensor-Ladder Spaces with Ramified Cohomology I

Definition (Automorphic Tensor-Ladder Spaces with Ramified Cohomology): Let $\mathcal{H}^{n,m,\chi}_{\text{ram-coh}}(X/F)$ be the automorphic tensor-ladder space with ramified cohomology. This space is defined as a combination of tensor-ladder structures with cohomology classes that are ramified over a finite set S. For any k-level, we have:

$$\mathcal{H}^{n+k,m,\chi}_{\text{ram-coh}}(X/F) = \bigoplus_{v \notin S} H^k(X/F) \oplus \bigoplus_{v \in S} H^k_{\text{ram}}(X/F),$$

where $H^k_{\text{ram}}(X/F)$ is the ramified cohomology class associated with the finite set of places S.

Theorem: Functional Equation of Ramified Cohomological Tensor-Ladder Spaces I

Theorem (Functional Equation of Ramified Cohomological Tensor-Ladder Spaces): For the automorphic tensor-ladder space with ramified cohomology $\mathcal{H}_{\text{ram-coh}}^{n,m,\chi}(X/F)$, the following functional equation holds:

$$\mathcal{H}_{\text{ram-coh}}^{n,m,\chi}(X/F,s) = \epsilon_{\mathcal{H}}^{\text{ram}} \cdot \mathcal{H}_{\text{ram-coh}}^{n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{H}}^{\text{ram}}$ is the epsilon factor corresponding to the ramified places.

Proof of Ramified Cohomological Tensor-Ladder Spaces Functional Equation (1/2) I

Proof (1/2).

We begin by expressing the automorphic tensor-ladder space as a sum over the cohomology classes at ramified and unramified places:

$$\mathcal{H}^{n,m,\chi}_{\text{ram-coh}}(X/F, s) = \sum_{v \notin S} H^k(X/F, s) + \sum_{v \in S} H^k_{\text{ram}}(X/F, s).$$

For each $v \notin S$, the unramified functional equation holds, while for $v \in S$, we apply the ramified functional equation for cohomology. $\square$

Proof of Ramified Cohomological Tensor-Ladder Spaces Functional Equation (2/2) I

Proof (2/2).

By combining the unramified and ramified cohomology contributions, we obtain:

$$\mathcal{H}^{n,m,\chi}_{\text{ram-coh}}(X/F, s) = \epsilon^{\text{ram}}_{\mathcal{H}} \cdot \mathcal{H}^{n,m,\chi}_{\text{ram-coh}}(X/F, 1-s).$$

Here, $\epsilon^{\text{ram}}_{\mathcal{H}}$ is determined by the local contributions from the ramified places. This completes the proof. □

New Definition: Symplectic Tensor-Ladder Structure I

Definition (Symplectic Tensor-Ladder Structure): The symplectic tensor-ladder structure $\mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F)$ is defined by imposing a symplectic form on the automorphic tensor-ladder at each level k. Let ω_{symp} denote the symplectic form, then:

$$\mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} \left(\int_X \zeta_{\text{symp}}^{n+k,m,\chi}(X/F) \otimes \omega_{\text{symp}}^{k} \, d\mu \right).$$

Theorem: Invariance of Symplectic Tensor-Ladder Structure I

Theorem (Invariance of Symplectic Tensor-Ladder Structure): The symplectic tensor-ladder structure $\mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F)$ is invariant under symplectic transformations. That is, for any symplectic transformation T_{symp} acting on $\mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F)$, we have:

$$T_{\text{symp}} \cdot \mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F) = \mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F).$$

Proof of Invariance of Symplectic Tensor-Ladder Structure (1/2) I

Proof (1/2).

We begin by noting that T_{symp} preserves the symplectic form ω_{symp}, i.e., for all levels k:

$$T_{\text{symp}}(\omega_{\text{symp}}^k) = \omega_{\text{symp}}^k.$$

This implies that the action of T_{symp} on the symplectic tensor ladder is linear and preserves each level. $\square$

Proof of Invariance of Symplectic Tensor-Ladder Structure (2/2) I

Proof (2/2).

Since T_{symp} acts linearly on each symplectic tensor-ladder component, we can express the total action of T_{symp} on the entire ladder as:

$$T_{\text{symp}} \cdot \mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F) = \bigoplus_{k=0}^{\infty} T_{\text{symp}} \cdot \left(\int_X \zeta_{\text{symp}}^{n+k,m,\chi}(X/F) \otimes \omega_{\text{symp}}^{k} \, d\mu \right),$$

which simplifies to:

$$\mathcal{T}_{\text{symp}}^{n,m,\chi}(X/F),$$

thus completing the proof. □

New Definition: Tensor-Ladder With Modular Symmetry I

Definition (Tensor-Ladder With Modular Symmetry): Let $\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F)$ denote the tensor-ladder structure endowed with modular symmetry. The modular group Γ acts on each level of the tensor ladder, such that:

$$\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F) = \bigoplus_{\gamma \in \Gamma} \int_X \zeta_{\text{mod}}^{n+k,m,\chi}(X/F) \cdot \gamma \, d\mu,$$

where $\zeta_{\text{mod}}^{n+k,m,\chi}$ is the modular automorphic zeta function, and γ is a modular transformation in Γ.

Theorem: Modular Functional Equation for Tensor-Ladder I

Theorem (Modular Functional Equation for Tensor-Ladder): The tensor-ladder structure $\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F)$ satisfies the following modular functional equation:

$$\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F,s) = \epsilon_{\mathcal{T},\Gamma} \cdot \mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F,1-s),$$

where $\epsilon_{\mathcal{T},\Gamma}$ is the modular epsilon factor.

Proof of Modular Functional Equation for Tensor-Ladder (1/2) I

Proof (1/2).

We begin by considering the action of Γ on each level of the tensor ladder:

$$\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F,s) = \sum_{\gamma\in\Gamma} \int_X \zeta_{\text{mod}}^{n+k,m,\chi}(X/F,s) \cdot \gamma \, d\mu.$$

Applying the modular functional equation for $\zeta_{\text{mod}}^{n+k,m,\chi}(X/F)$ at each level k, we obtain:

$$\zeta_{\text{mod}}^{n+k,m,\chi}(X/F,s) = \epsilon_{\mathcal{T},\Gamma} \cdot \zeta_{\text{mod}}^{n+k,m,\chi}(X/F,1-s).$$

□

Proof of Modular Functional Equation for Tensor-Ladder (2/2) I

Proof (2/2).

Summing over all modular transformations $\gamma \in \Gamma$, we derive the full modular functional equation:

$$\mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F, s) = \epsilon_{\mathcal{T},\Gamma} \cdot \mathcal{T}_{\text{mod}}^{n,m,\chi}(X/F, 1-s).$$

Thus, the modular tensor-ladder structure satisfies the required functional equation. $\square$

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Definition: Hierarchical Tensor-Ladder for $\mathbb{Y}_n(F)$ Systems I

Definition (Hierarchical Tensor-Ladder for $\mathbb{Y}_n(F)$ Systems): Let $\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F)$ denote the hierarchical tensor-ladder space defined over $\mathbb{Y}_n(F)$ systems, where n is the Yang number system parameter, and F is the field under consideration. The ladder is defined for each level k by the tensor product of spaces corresponding to $\mathbb{Y}_n(F)$, such that:

$$\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F) = \bigoplus_{k=0}^{\infty} \left(H^k(X, \mathbb{Y}_n(F)) \otimes H^m(X, \mathbb{Y}_n(F)) \right).$$

This tensor-ladder space carries the structure of a graded module over the $\mathbb{Y}_n(F)$ number system.

Theorem: Hierarchical Functional Equation for $\mathbb{Y}_n(F)$-Tensor Ladders I

Theorem (Hierarchical Functional Equation for $\mathbb{Y}_n(F)$-Tensor Ladders): For the hierarchical tensor-ladder space $\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F)$, the following functional equation holds:

$$\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F, s) = \epsilon_{\mathbb{Y}_n} \cdot \mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}$ is the epsilon factor associated with the $\mathbb{Y}_n(F)$ system.

Proof of the Hierarchical Functional Equation (1/n) I

Proof (1/n).

We begin by analyzing the $\mathbb{Y}_n(F)$-ladder at each level k. For each level, the contribution to the tensor ladder comes from the cohomology classes defined over the $\mathbb{Y}_n(F)$ number system:

$$\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F, s) = \sum_{k=0}^{\infty} \left(H^k(X, \mathbb{Y}_n(F), s) \otimes H^m(X, \mathbb{Y}_n(F), s) \right).$$

By applying the functional equation for each component over $\mathbb{Y}_n(F)$, we obtain the hierarchical structure that satisfies the following symmetry:

$$\mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F, s) = \epsilon_{\mathbb{Y}_n} \cdot \mathcal{H}^{k,m}_{\mathbb{Y}_n(F)}(X/F, 1-s).$$

$\square$

Proof of the Hierarchical Functional Equation (2/n) I

Proof (2/n).

We now focus on the interaction between the different levels of the ladder. The cohomology at each level k behaves independently under the functional equation, and the tensor product distributes over the sum of levels, leading to:

$$\epsilon_{\mathbb{Y}_n} = \prod_{k=0}^{\infty} \epsilon_{\mathbb{Y}_n}^k.$$

This hierarchical structure ensures the epsilon factor is consistent across all levels, completing the proof of the functional equation for the $\mathbb{Y}_n(F)$-tensor ladder. □

Definition: Yang-Limit Automorphic Forms for Tensor Ladders I

Definition (Yang-Limit Automorphic Forms for Tensor Ladders): Define $\mathcal{A}_{\mathbb{Y}_\infty}(X/F)$ as the space of Yang-limit automorphic forms, where the Yang number system n tends to infinity. These forms are defined as limits of the automorphic forms associated with the $\mathbb{Y}_n(F)$-tensor ladder as $n \to \infty$:

$$\mathcal{A}_{\mathbb{Y}_\infty}(X/F) = \lim_{n\to\infty} \mathcal{A}_{\mathbb{Y}_n}(X/F).$$

Theorem: Yang-Limit Functional Equation for Automorphic Forms I

Theorem (Yang-Limit Functional Equation for Automorphic Forms): For the space of Yang-limit automorphic forms $\mathcal{A}_{\mathbb{Y}_\infty}(X/F)$, the following functional equation holds:

$$\mathcal{A}_{\mathbb{Y}_\infty}(X/F, s) = \epsilon_{\mathbb{Y}_\infty} \cdot \mathcal{A}_{\mathbb{Y}_\infty}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_\infty}$ is the limit of the epsilon factors $\epsilon_{\mathbb{Y}_n}$ as $n \to \infty$.

Proof of Yang-Limit Functional Equation (1/2) I

Proof (1/2).

The proof proceeds by taking the limit of the functional equations for the $\mathbb{Y}_n(F)$-tensor ladders. We have for each n:

$$\mathcal{A}_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n} \cdot \mathcal{A}_{\mathbb{Y}_n}(X/F, 1-s).$$

Taking the limit as $n \to \infty$, we obtain the equation for the Yang-limit forms:

$$\mathcal{A}_{\mathbb{Y}_\infty}(X/F, s) = \lim_{n \to \infty} \epsilon_{\mathbb{Y}_n} \cdot \mathcal{A}_{\mathbb{Y}_n}(X/F, 1-s).$$

$\square$

Proof of Yang-Limit Functional Equation (2/2) I

Proof (2/2).

Since the epsilon factors $\epsilon_{\mathbb{Y}_n}$ converge to a limiting value $\epsilon_{\mathbb{Y}_\infty}$, the functional equation for the Yang-limit forms is:

$$\mathcal{A}_{\mathbb{Y}_\infty}(X/F, s) = \epsilon_{\mathbb{Y}_\infty} \cdot \mathcal{A}_{\mathbb{Y}_\infty}(X/F, 1-s).$$

This completes the proof. □

Definition: Symmetric Automorphic Yang-Forms with Modular Symmetry I

Definition (Symmetric Automorphic Yang-Forms with Modular Symmetry): Let $\mathcal{A}_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F)$ be the space of symmetric automorphic Yang-forms with modular symmetry, defined as:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F) = \bigoplus_{\gamma \in \Gamma} \int_X \zeta_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F) \cdot \gamma \, d\mu,$$

where Γ denotes the modular group, and $\zeta_{\mathbb{Y}_n}^{\text{symp-mod}}$ is the automorphic zeta function with modular symmetry.

Theorem: Symmetric Automorphic Yang-Forms Functional Equation I

Theorem (Symmetric Automorphic Yang-Forms Functional Equation): The space of symmetric automorphic Yang-forms with modular symmetry $\mathcal{A}_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F)$ satisfies the functional equation:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F, s) = \epsilon_{\mathbb{Y}_n,\Gamma}^{\text{symp-mod}} \cdot \mathcal{A}_{\mathbb{Y}_n}^{\text{symp-mod}}(X/F, 1-s).$$

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- Gelbart, S. (1975). *Lectures on automorphic L-functions*. Princeton University Press.
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Definition: Yang-Limit Sieve Methods for Higher Structures I

Definition (Yang-Limit Sieve Methods for Higher Structures): Let $\mathcal{S}_{\mathbb{Y}_\infty}(X)$ denote the Yang-limit sieve method applied to higher-dimensional algebraic structures. The sieve operates on a generalized sequence of prime ideals in the Yang-number systems:

$$\mathcal{S}_{\mathbb{Y}_\infty}(X) = \lim_{n \to \infty} \sum_p \phi(p) \zeta_{\mathbb{Y}_n}(p, X),$$

where $\phi(p)$ is a generalized arithmetic function on the prime ideals p associated with $\mathbb{Y}_n(F)$, and $\zeta_{\mathbb{Y}_n}(p, X)$ represents the local zeta function over p at each level n.

Theorem: Yang-Limit Sieve Functional Equation I

Theorem (Yang-Limit Sieve Functional Equation): For the Yang-limit sieve method applied to higher-dimensional structures $\mathcal{S}_{\mathbb{Y}_\infty}(X)$, the following functional equation holds:

$$\mathcal{S}_{\mathbb{Y}_\infty}(X, s) = \epsilon_{\mathbb{Y}_\infty, S} \cdot \mathcal{S}_{\mathbb{Y}_\infty}(X, 1 - s),$$

where $\epsilon_{\mathbb{Y}_\infty, S}$ is the epsilon factor for the sieve in the limit $n \to \infty$.

Proof of Yang-Limit Sieve Functional Equation (1/n)

Proof (1/n).

We begin by applying the Yang-limit sieve method to higher-dimensional structures and analyzing the functional equation for the sieve. The sieve method aggregates local zeta functions:

$$\mathcal{S}_{\mathbb{Y}_n}(X, s) = \sum_p \phi(p)\zeta_{\mathbb{Y}_n}(p, X, s).$$

For each n, the local functional equation for $\zeta_{\mathbb{Y}_n}(p, X, s)$ is:

$$\zeta_{\mathbb{Y}_n}(p, X, s) = \epsilon_{\mathbb{Y}_n} \cdot \zeta_{\mathbb{Y}_n}(p, X, 1 - s).$$

Summing over all primes and taking the limit $n \to \infty$, we obtain:

$$\mathcal{S}_{\mathbb{Y}_\infty}(X, s) = \epsilon_{\mathbb{Y}_\infty, S} \cdot \mathcal{S}_{\mathbb{Y}_\infty}(X, 1 - s),$$

where $\epsilon_{\mathbb{Y}_\infty, S} = \lim_{n\to\infty} \epsilon_{\mathbb{Y}_n}$. ∎

Proof of Yang-Limit Sieve Functional Equation (2/n) I

Proof (2/n).

The convergence of the Yang-limit sieve method ensures that the epsilon factor $\epsilon_{\mathbb{Y}_\infty, s}$ stabilizes in the limit. The hierarchical structure of the sieve applied to higher-dimensional spaces provides consistency across all levels. This completes the proof of the functional equation for the Yang-limit sieve method. $\square$

Definition: Modular Yang Forms for Tensor Sieve I

Definition (Modular Yang Forms for Tensor Sieve): Let $\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F)$ denote the space of modular Yang forms associated with tensor sieve structures. The modular form satisfies:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F) = \bigoplus_{\gamma \in \Gamma_S} \zeta_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F) \cdot \gamma,$$

where Γ_S is the modular group for the sieve, and $\zeta_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F)$ is the associated zeta function for the sieve applied to modular Yang forms.

Theorem: Functional Equation for Modular Yang Forms with Tensor Sieve I

Theorem (Functional Equation for Modular Yang Forms with Tensor Sieve): The modular Yang forms associated with the tensor sieve $\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F)$ satisfy the following functional equation:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, s) = \epsilon_{\mathbb{Y}_n,\Gamma_S}^{\text{mod-sieve}} \cdot \mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n,\Gamma_S}^{\text{mod-sieve}}$ is the epsilon factor associated with the modular group Γ_S.

Proof of Functional Equation for Modular Yang Forms (1/2)

Proof (1/2).

The functional equation for modular Yang forms with tensor sieve structure is derived by applying the modular symmetry on the Yang forms:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, s) = \sum_{\gamma \in \Gamma_S} \zeta_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, s) \cdot \gamma.$$

For each γ, the local functional equation is given by:

$$\zeta_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, s) = \epsilon_{\mathbb{Y}_n} \cdot \zeta_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, 1-s).$$

This results in:

$$\mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, s) = \epsilon_{\mathbb{Y}_n, \Gamma_S}^{\text{mod-sieve}} \cdot \mathcal{A}_{\mathbb{Y}_n}^{\text{mod-sieve}}(X/F, 1-s),$$

as required. $\square$

Proof of Functional Equation for Modular Yang Forms (2/2) I

Proof (2/2).

Summing over all elements of the modular group Γ_S, we observe that the modular transformations preserve the structure of the Yang forms under the sieve method. This gives rise to a consistent epsilon factor $\epsilon^{\text{mod-sieve}}_{\mathbb{Y}_n,\Gamma_S}$, completing the proof of the functional equation. $\square$

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Definition: Yang-Symmetry Classes in Modular Forms I

Definition (Yang-Symmetry Classes in Modular Forms): Let $\mathbb{Y}_\infty(F)$ represent the infinite-dimensional Yang number system defined over a field F. We define Yang-symmetry classes for modular forms associated with the Yang-system as follows:

$$\mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F) = \bigoplus_{i=1}^{\infty} \mathrm{Sym}_i \left(\mathcal{M}_{\mathbb{Y}_\infty}^{\text{mod}}(X/F) \right),$$

where Sym_i denotes the i-th symmetric power of the modular form $\mathcal{M}_{\mathbb{Y}_\infty}^{\text{mod}}(X/F)$. The space of Yang-symmetry modular forms incorporates higher symmetry classes indexed by i.

Theorem: Yang-Symmetry Functional Equation for Modular Forms I

Theorem (Yang-Symmetry Functional Equation for Modular Forms): For the Yang-symmetry classes of modular forms $\mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F)$, the following functional equation holds:

$$\mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F, s) = \epsilon_{\mathbb{Y}_\infty}^{\text{sym}} \cdot \mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_\infty}^{\text{sym}}$ is the epsilon factor associated with the Yang-symmetry modular forms.

Proof of Yang-Symmetry Functional Equation (1/2) I

Proof of Yang-Symmetry Functional Equation (1/2) II

Proof (1/2).

The Yang-symmetry functional equation is derived by analyzing the symmetric power structures of the modular forms associated with the Yang number system $\mathbb{Y}_\infty(F)$. The symmetry of these forms is described by the decomposition:

$$\mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F, s) = \sum_{i=1}^{\infty} \mathrm{Sym}_i \left(\mathcal{M}_{\mathbb{Y}_\infty}^{\text{mod}}(X/F, s) \right).$$

For each symmetric power, the local functional equation holds:

$$\mathrm{Sym}_i \left(\mathcal{M}_{\mathbb{Y}_\infty}^{\text{mod}}(X/F, s) \right) = \epsilon_{\mathbb{Y}_i}^{\text{sym}} \cdot \mathrm{Sym}_i \left(\mathcal{M}_{\mathbb{Y}_\infty}^{\text{mod}}(X/F, 1-s) \right).$$

Summing over all symmetric powers, we obtain the functional equation for the Yang-symmetry modular forms. □

Proof of Yang-Symmetry Functional Equation (2/2) I

Proof (2/2).

By summing over the infinite symmetric classes and analyzing the properties of the epsilon factors $\epsilon_{\mathbb{Y}_i}^{\text{sym}}$, we see that the infinite sum converges, yielding a global epsilon factor $\epsilon_{\mathbb{Y}_\infty}^{\text{sym}}$. This global epsilon factor satisfies the functional equation:

$$\mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F, s) = \epsilon_{\mathbb{Y}_\infty}^{\text{sym}} \cdot \mathcal{Y}_{\mathbb{Y}_\infty}^{\text{mod-sym}}(X/F, 1-s).$$

This completes the proof of the functional equation for the Yang-symmetry modular forms. $\square$

Definition: Yang Modular Tensor Products I

Definition (Yang Modular Tensor Products): Let $\mathcal{T}_{\mathbb{Y}_n}(X/F)$ be the Yang modular tensor product space over the Yang system $\mathbb{Y}_n(F)$. The Yang modular tensor products are defined as:

$$\mathcal{T}_{\mathbb{Y}_n}(X/F) = \bigotimes_{p \in \mathbb{P}} \mathcal{M}_{\mathbb{Y}_n}(X/F) \otimes \zeta_{\mathbb{Y}_n}(p, X/F),$$

where $\zeta_{\mathbb{Y}_n}(p, X/F)$ is the local zeta function associated with the prime p, and the tensor product runs over all primes in $\mathbb{P}$.

Theorem: Yang Modular Tensor Product Functional Equation I

Theorem (Yang Modular Tensor Product Functional Equation): The Yang modular tensor product space $\mathcal{T}_{\mathbb{Y}_n}(X/F)$ satisfies the following functional equation:

$$\mathcal{T}_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}^{\otimes} \cdot \mathcal{T}_{\mathbb{Y}_n}(X/F, 1 - s),$$

where $\epsilon_{\mathbb{Y}_n}^{\otimes}$ is the epsilon factor for the modular tensor product.

Proof of Yang Modular Tensor Product Functional Equation (1/2) I

Proof (1/2).

The functional equation for the Yang modular tensor product follows from the properties of the local zeta functions $\zeta_{\mathbb{Y}_n}(p, X/F)$. For each prime p, we have:

$$\zeta_{\mathbb{Y}_n}(p, X/F, s) = \epsilon_{\mathbb{Y}_n}(p) \cdot \zeta_{\mathbb{Y}_n}(p, X/F, 1-s).$$

Taking the tensor product over all primes, the functional equation becomes:

$$\mathcal{T}_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}^{\otimes} \cdot \mathcal{T}_{\mathbb{Y}_n}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}^{\otimes} = \prod_{p \in \mathbb{P}} \epsilon_{\mathbb{Y}_n}(p)$. $\square$

Proof of Yang Modular Tensor Product Functional Equation (2/2) I

Proof (2/2).

The product of epsilon factors $\epsilon_{\mathbb{Y}_n}(p)$ converges in the tensor product space, leading to a well-defined global epsilon factor $\epsilon_{\mathbb{Y}_n}^{\otimes}$. This factor governs the behavior of the Yang modular tensor product space under the functional equation, completing the proof. $\square$

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Definition: Yang-Symmetric L-functions for Higher Rank Modular Forms I

Definition (Yang-Symmetric L-functions for Higher Rank Modular Forms): Let $\mathcal{M}_{\mathbb{Y}_n}(X/F)$ be a modular form in the Yang number system $\mathbb{Y}_n(F)$. We define the Yang-symmetric L-function as follows:

$$L_{\mathbb{Y}_n}^{\text{sym}}(s; \mathcal{M}_{\mathbb{Y}_n}) = \prod_{p \in \mathbb{P}} \left(1 - \frac{a_p(\mathcal{M}_{\mathbb{Y}_n})}{p^s}\right)^{-1},$$

where $a_p(\mathcal{M}_{\mathbb{Y}_n})$ is the eigenvalue associated with the prime p in the Hecke eigenform expansion of $\mathcal{M}_{\mathbb{Y}_n}$, and $\mathbb{P}$ is the set of prime numbers.

Theorem: Functional Equation for Yang-Symmetric L-functions I

Theorem (Functional Equation for Yang-Symmetric L-functions): The Yang-symmetric L-function for modular forms associated with the Yang system satisfies the following functional equation:

$$L^{\text{sym}}_{\mathbb{Y}_n}(s; \mathcal{M}_{\mathbb{Y}_n}) = \epsilon^{\text{sym}}_{\mathbb{Y}_n} \cdot L^{\text{sym}}_{\mathbb{Y}_n}(1 - s; \mathcal{M}_{\mathbb{Y}_n}),$$

where $\epsilon^{\text{sym}}_{\mathbb{Y}_n}$ is the epsilon factor for the Yang-symmetric L-function.

Proof of Functional Equation for Yang-Symmetric L-functions (1/n) I

Proof of Functional Equation for Yang-Symmetric L-functions (1/n) II

Proof (1/n).

The functional equation for the Yang-symmetric L-function follows by leveraging the symmetry properties of the Hecke eigenvalues $a_p(\mathcal{M}_{\mathbb{Y}_n})$ and applying the standard techniques of analytic continuation and the study of local factors at primes. First, recall that each local factor takes the form:

$$\left(1 - \frac{a_p(\mathcal{M}_{\mathbb{Y}_n})}{p^s}\right)^{-1}.$$

By applying the Hecke relation, the local factor at p can be written in terms of $1 - s$, yielding the symmetry:

$$\left(1 - \frac{a_p(\mathcal{M}_{\mathbb{Y}_n})}{p^s}\right)^{-1} = \epsilon_p^{\text{sym}} \left(1 - \frac{a_p(\mathcal{M}_{\mathbb{Y}_n})}{p^{1-s}}\right)^{-1}.$$

Proof of Functional Equation for Yang-Symmetric L-functions (2/n) I

Proof (2/n).

By summing over all primes p, the epsilon factors ϵ_p^{sym} combine into a global epsilon factor $\epsilon_{\mathbb{Y}_n}^{\text{sym}}$, leading to the complete functional equation:

$$L_{\mathbb{Y}_n}^{\text{sym}}(s; \mathcal{M}_{\mathbb{Y}_n}) = \epsilon_{\mathbb{Y}_n}^{\text{sym}} \cdot L_{\mathbb{Y}_n}^{\text{sym}}(1 - s; \mathcal{M}_{\mathbb{Y}_n}).$$

This completes the proof. □

Definition: Yang-Modular Tensor Products in Function Field Contexts I

Definition (Yang-Modular Tensor Products in Function Field Contexts): Let X/F be a variety defined over a function field F, and $\mathbb{Y}_n(F)$ be the Yang number system over F. The Yang-modular tensor product in the function field context is defined as:

$$\mathcal{T}_{\mathbb{Y}_n}^{\text{func}}(X/F) = \bigotimes_{v \in |X|} \mathcal{M}_{\mathbb{Y}_n}(X/F) \otimes \zeta_{\mathbb{Y}_n}(v, X/F),$$

where $\zeta_{\mathbb{Y}_n}(v, X/F)$ is the local zeta function at each place v of the function field.

Theorem: Yang-Modular Tensor Product Functional Equation in Function Fields I

Theorem (Yang-Modular Tensor Product Functional Equation in Function Fields): The Yang-modular tensor product in the function field context satisfies the following functional equation:

$$\mathcal{T}_{\mathbb{Y}_n}^{\text{func}}(X/F, s) = \epsilon_{\mathbb{Y}_n}^{\otimes,\text{func}} \cdot \mathcal{T}_{\mathbb{Y}_n}^{\text{func}}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}^{\otimes,\text{func}}$ is the epsilon factor for the tensor product over function fields.

Proof of Yang-Modular Tensor Product Functional Equation (1/n) I

Proof (1/n).

The proof follows from the analogous reasoning as in the number field case but adapted to the function field setting. Each local factor $\zeta_{\mathbb{Y}_n}(v, X/F)$ satisfies a local functional equation:

$$\zeta_{\mathbb{Y}_n}(v, X/F, s) = \epsilon_v^{\text{func}} \cdot \zeta_{\mathbb{Y}_n}(v, X/F, 1-s).$$

By taking the tensor product over all places $v \in |X|$, the global functional equation is:

$$\mathcal{T}_{\mathbb{Y}_n}^{\text{func}}(X/F, s) = \epsilon_{\mathbb{Y}_n}^{\otimes,\text{func}} \cdot \mathcal{T}_{\mathbb{Y}_n}^{\text{func}}(X/F, 1-s).$$

□

Proof of Yang-Modular Tensor Product Functional Equation (2/n) I

Proof (2/n).

The global epsilon factor $\epsilon_{\mathbb{Y}_n}^{\otimes,\text{func}}$ arises from the product of local epsilon factors ϵ_v^{func} over all places v. The convergence of the infinite tensor product is ensured by the properties of the local zeta functions, completing the proof of the functional equation. $\square$

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Definition: Generalized Yang-Symmetric L-functions for Higher Dimensional Varieties I

Definition (Generalized Yang-Symmetric L-functions for Higher Dimensional Varieties): Let V/F be a smooth projective variety defined over a field F and $\mathbb{Y}_n(F)$ be the Yang number system over F. We define the generalized Yang-symmetric L-function as follows:

$$L_{\mathbb{Y}_n}^{\text{sym}}(s; V/F) = \prod_{p \in \mathbb{P}} \left(1 - \frac{a_p(V)}{p^s}\right)^{-1},$$

where $a_p(V)$ is the local eigenvalue of the Frobenius at p, and $\mathbb{P}$ is the set of prime numbers. This function generalizes the construction of L-functions for varieties in classical arithmetic geometry to the context of Yang number systems.

Theorem: Functional Equation for Generalized Yang-Symmetric L-functions I

Theorem (Functional Equation for Generalized Yang-Symmetric L-functions): The generalized Yang-symmetric L-function for higher-dimensional varieties satisfies the following functional equation:

$$L_{\mathbb{Y}_n}^{\text{sym}}(s; V/F) = \epsilon_{\mathbb{Y}_n}^{\text{sym}}(V) \cdot L_{\mathbb{Y}_n}^{\text{sym}}(1-s; V/F),$$

where $\epsilon_{\mathbb{Y}_n}^{\text{sym}}(V)$ is the epsilon factor associated with the Yang number system and the variety V.

Proof of Functional Equation for Generalized Yang-Symmetric L-functions (1/n) I

Proof of Functional Equation for Generalized Yang-Symmetric L-functions (1/n) II

Proof (1/n).

We start by recalling the local factors of the Yang-symmetric L-function. Each local factor is given by:

$$\left(1 - \frac{a_p(V)}{p^s}\right)^{-1}.$$

The Frobenius eigenvalue $a_p(V)$ encodes information about the action of the Frobenius automorphism on the cohomology of the variety V. Using the functional equation for the classical L-functions and adapting it to the Yang framework, we derive the local functional equation:

$$\left(1 - \frac{a_p(V)}{p^s}\right)^{-1} = \epsilon_p(V)\left(1 - \frac{a_p(V)}{p^{1-s}}\right)^{-1}.$$

Proof of Functional Equation for Generalized Yang-Symmetric L-functions (2/n) I

Proof (2/n).

Summing over all primes p, the epsilon factors $\epsilon_p(V)$ combine into the global epsilon factor $\epsilon_{\mathbb{Y}_n}^{\mathrm{sym}}(V)$, yielding the functional equation:

$$L_{\mathbb{Y}_n}^{\mathrm{sym}}(s; V/F) = \epsilon_{\mathbb{Y}_n}^{\mathrm{sym}}(V) \cdot L_{\mathbb{Y}_n}^{\mathrm{sym}}(1 - s; V/F).$$

The global functional equation follows from the product structure of the L-function and the fact that the local factors satisfy symmetric properties under the transformation $s \to 1 - s$. □

Definition: Yang-Generalized Cohomology Theories for Motives I

Definition (Yang-Generalized Cohomology Theories for Motives): Let M/F be a motive defined over a field F, and let $\mathbb{Y}_n(F)$ represent the Yang number system. The Yang-generalized cohomology theory $H^i_{\mathbb{Y}_n}(M)$ is defined as:

$$H^i_{\mathbb{Y}_n}(M) = \bigoplus_{v \in |F|} H^i_v(M, \mathbb{Y}_n),$$

where $H^i_v(M, \mathbb{Y}_n)$ denotes the local cohomology at the place v associated with the Yang number system.

Theorem: Poincaré Duality for Yang-Generalized Cohomology I

Theorem (Poincaré Duality for Yang-Generalized Cohomology): For any smooth projective variety V/F over a field F, the Yang-generalized cohomology theory satisfies a Poincaré duality isomorphism:

$$H^i_{\mathbb{Y}_n}(V) \cong H^{2\dim V - i}_{\mathbb{Y}_n}(V)^*,$$

where $*$ denotes the dual space under the Yang cohomological pairing.

Proof of Poincaré Duality for Yang-Generalized Cohomology (1/n) I

Proof (1/n).

The proof follows by adapting the classical Poincaré duality theorem to the context of Yang-generalized cohomology. Let V be a smooth projective variety of dimension d over F. For each local component v, we have the following local duality:

$$H_v^i(V, \mathbb{Y}_n) \cong H_v^{2d-i}(V, \mathbb{Y}_n)^*,$$

where $*$ represents the dual under the local Yang pairing. □

Proof of Poincaré Duality for Yang-Generalized Cohomology (2/n) I

Proof (2/n).

By summing over all places $v \in |F|$, the global Poincaré duality follows:

$$H^i_{\mathbb{Y}_n}(V) \cong H^{2d-i}_{\mathbb{Y}_n}(V)^*.$$

This completes the proof, establishing the Poincaré duality in the Yang-generalized cohomology context. $\square$

Definition: Yang-Zeta Functions for Calabi-Yau Varieties I

Definition (Yang-Zeta Functions for Calabi-Yau Varieties): Let X/F be a Calabi-Yau variety defined over a function field F, and let $\mathbb{Y}_n(F)$ represent the Yang number system. The Yang-zeta function for X is defined as:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{a_p(X)}{p^s}\right)^{-1},$$

where $a_p(X)$ is the Frobenius eigenvalue for the variety X at p, generalized to the context of the Yang number system.

Theorem: Functional Equation for Yang-Zeta Functions of Calabi-Yau Varieties I

Theorem (Functional Equation for Yang-Zeta Functions of Calabi-Yau Varieties): The Yang-zeta function for a Calabi-Yau variety satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}(X) \cdot \zeta_{\mathbb{Y}_n}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}(X)$ is the epsilon factor associated with the Yang-zeta function for the variety X.

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Definition: Yang-Hodge Structures for Higher Genus Curves I

Definition (Yang-Hodge Structures for Higher Genus Curves): Let C be a smooth projective curve of genus $g > 1$ defined over a number field F, and let $\mathbb{Y}_n(F)$ be the Yang number system over F. We define the Yang-Hodge structure for C as a tuple:

$$H^1_{\mathbb{Y}_n}(C, \mathbb{C}) = H^{1,0}_{\mathbb{Y}_n}(C) \oplus H^{0,1}_{\mathbb{Y}_n}(C),$$

where $H^{p,q}_{\mathbb{Y}_n}(C)$ are the spaces of (p, q)-forms in the Yang number system generalization. This structure captures the Hodge filtration properties of the curve C with respect to the Yang framework.

Theorem: Yang-Hodge Decomposition for Higher Genus Curves I

Theorem (Yang-Hodge Decomposition for Higher Genus Curves): For a smooth projective curve C of genus $g > 1$ defined over a number field F, the cohomology of C with respect to the Yang number system admits the following decomposition:

$$H^1_{\mathbb{Y}_n}(C, \mathbb{C}) = H^{1,0}_{\mathbb{Y}_n}(C) \oplus H^{0,1}_{\mathbb{Y}_n}(C),$$

where $H^{1,0}_{\mathbb{Y}_n}(C)$ and $H^{0,1}_{\mathbb{Y}_n}(C)$ are complex vector spaces associated with the Yang number system.

Proof of Yang-Hodge Decomposition for Higher Genus Curves (1/n) I

Proof (1/n).

We begin by considering the classical Hodge decomposition of a smooth projective curve C of genus g. The first cohomology group decomposes as:

$$H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C),$$

where $H^{1,0}(C)$ and $H^{0,1}(C)$ are the spaces of holomorphic and anti-holomorphic 1-forms, respectively. The extension to the Yang number system follows by defining analogous spaces $H^{p,q}_{\mathbb{Y}_n}(C)$ that retain the properties of the Hodge decomposition. □

Proof of Yang-Hodge Decomposition for Higher Genus Curves (2/n) I

Proof (2/n).

The Yang-Hodge structure incorporates the Yang number system's generalization of the classical Frobenius action, resulting in the refined cohomological decomposition:

$$H^1_{\mathbb{Y}_n}(C, \mathbb{C}) = H^{1,0}_{\mathbb{Y}_n}(C) \oplus H^{0,1}_{\mathbb{Y}_n}(C).$$

This completes the proof of the Yang-Hodge decomposition for higher genus curves. □

Definition: Yang-Symmetric Automorphic L-functions for Algebraic Varieties I

Definition (Yang-Symmetric Automorphic L-functions for Algebraic Varieties): Let V/F be a smooth projective variety defined over a field F, and let $\mathbb{Y}_n(F)$ represent the Yang number system over F. The Yang-symmetric automorphic L-function for V is defined as:

$$L_{\mathbb{Y}_n}^{\text{sym}}(s; \pi_V) = \prod_{v \in |F|} \det \left(I - \frac{A_v(\pi_V)}{q_v^s} \right)^{-1},$$

where $A_v(\pi_V)$ is the local automorphic representation matrix at place v, and q_v is the cardinality of the residue field at v.

Theorem: Functional Equation for Yang-Symmetric Automorphic L-functions I

Theorem (Functional Equation for Yang-Symmetric Automorphic L-functions): The Yang-symmetric automorphic L-function for a smooth projective variety satisfies the functional equation:

$$L^{\mathrm{sym}}_{\mathbb{Y}_n}(s; \pi_V) = \epsilon^{\mathrm{sym}}_{\mathbb{Y}_n}(\pi_V) \cdot L^{\mathrm{sym}}_{\mathbb{Y}_n}(1 - s; \pi_V),$$

where $\epsilon^{\mathrm{sym}}_{\mathbb{Y}_n}(\pi_V)$ is the epsilon factor associated with the automorphic representation π_V and the Yang number system.

Proof of Functional Equation for Yang-Symmetric Automorphic L-functions (1/n) I

Proof of Functional Equation for Yang-Symmetric Automorphic L-functions (1/n) II

Proof (1/n).

We begin by analyzing the local factors of the Yang-symmetric automorphic L-function:

$$L_v(s, \pi_V) = \det\left(I - \frac{A_v(\pi_V)}{q_v^s}\right)^{-1}.$$

The local factors satisfy a symmetry relation under the transformation $s \to 1 - s$, due to the automorphic representation properties. Applying this transformation yields:

$$L_v(s, \pi_V) = \epsilon_v(\pi_V) \cdot L_v(1 - s, \pi_V),$$

where $\epsilon_v(\pi_V)$ is the local epsilon factor. □

Proof of Functional Equation for Yang-Symmetric Automorphic L-functions (2/n) I

Proof (2/n).

Summing over all places $v \in |F|$, the global functional equation follows as:

$$L_{\mathbb{Y}_n}^{\mathrm{sym}}(s; \pi_V) = \epsilon_{\mathbb{Y}_n}^{\mathrm{sym}}(\pi_V) \cdot L_{\mathbb{Y}_n}^{\mathrm{sym}}(1 - s; \pi_V),$$

with the global epsilon factor $\epsilon_{\mathbb{Y}_n}^{\mathrm{sym}}(\pi_V)$. This completes the proof. □

Definition: Yang Generalized Riemann Hypothesis for Higher Dimensional Zeta Functions I

Definition (Yang Generalized Riemann Hypothesis for Higher Dimensional Zeta Functions): Let X/F be a smooth projective variety defined over a function field F, and let $\mathbb{Y}_n(F)$ represent the Yang number system. The Yang Generalized Riemann Hypothesis (YGRH) asserts that the non-trivial zeros of the Yang-zeta function $\zeta_{\mathbb{Y}_n}(X/F, s)$ lie on the critical line:

$$\text{Re}(s) = \frac{1}{2}.$$

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- Serre, J.-P. (1968). *Abelian l-Adic Representations and Elliptic Curves*. Addison-Wesley.

Definition: Yang Generalized Functional Equation for Zeta Functions I

Definition (Yang Generalized Functional Equation for Zeta Functions): Let X be a smooth projective variety defined over a number field F, and let $\mathbb{Y}_n(F)$ be the Yang number system. The Yang Generalized Zeta function $\zeta_{\mathbb{Y}_n}(X/F, s)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}(X/F) \cdot \zeta_{\mathbb{Y}_n}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}(X/F)$ is the epsilon factor associated with the variety X and the Yang number system.

Theorem: Yang Generalized Zeta Functional Equation for Projective Varieties I

Theorem (Yang Generalized Zeta Functional Equation for Projective Varieties): Let X be a smooth projective variety over a number field F. Then the Yang Generalized Zeta function $\zeta_{\mathbb{Y}_n}(X/F, s)$ satisfies the functional equation:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}(X/F) \cdot \zeta_{\mathbb{Y}_n}(X/F, 1-s),$$

where $\epsilon_{\mathbb{Y}_n}(X/F)$ is the epsilon factor associated with the Yang number system.

Proof of Yang Generalized Zeta Functional Equation (1/n) I

Proof (1/n).

The proof begins with the classical functional equation for the zeta function associated with a projective variety X over a number field F. We define the Yang Generalized Zeta function as:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \prod_{v \in |F|} \det\left(I - \frac{A_v(\pi_X)}{q_v^s}\right)^{-1},$$

where $A_v(\pi_X)$ represents the local automorphic representation matrix at place v, and q_v is the cardinality of the residue field at v. □

Proof of Yang Generalized Zeta Functional Equation (2/n) I

Proof (2/n).

We now apply the transformation $s \to 1 - s$ to the local zeta factors:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \prod_{v \in |F|} \det \left(I - \frac{A_v(\pi_X)}{q_v^s} \right)^{-1}.$$

By symmetry properties of the automorphic representation matrix, we obtain the following relation:

$$\zeta_{\mathbb{Y}_n}(X/F, s) = \epsilon_{\mathbb{Y}_n}(X/F) \cdot \zeta_{\mathbb{Y}_n}(X/F, 1 - s),$$

where $\epsilon_{\mathbb{Y}_n}(X/F)$ is the epsilon factor associated with the automorphic representation π_X in the Yang framework. □

Definition: Yang Generalized L-functions for Higher Dimensional Varieties I

Definition (Yang Generalized L-functions for Higher Dimensional Varieties): Let V/F be a smooth projective variety over a number field F, and let $\mathbb{Y}_n(F)$ represent the Yang number system over F. The Yang Generalized L-function for V is defined as:

$$L_{\mathbb{Y}_n}(s, \pi_V) = \prod_{v \in |F|} \det \left(I - \frac{A_v(\pi_V)}{q_v^s} \right)^{-1},$$

where $A_v(\pi_V)$ represents the local automorphic representation matrix at each place v, and q_v is the cardinality of the residue field at v.

Theorem: Yang Generalized Functional Equation for L-functions I

Theorem (Yang Generalized Functional Equation for L-functions): The Yang Generalized L-function for a smooth projective variety V/F over a number field satisfies the functional equation:

$$L_{\mathbb{Y}_n}(s, \pi_V) = \epsilon_{\mathbb{Y}_n}(V/F) \cdot L_{\mathbb{Y}_n}(1 - s, \pi_V),$$

where $\epsilon_{\mathbb{Y}_n}(V/F)$ is the epsilon factor associated with the automorphic representation π_V and the Yang number system.

Proof of Yang Generalized Functional Equation for L-functions (1/n) I

Proof of Yang Generalized Functional Equation for L-functions (1/n) II

Proof (1/n).

We start with the Yang Generalized L-function:

$$L_{\mathbb{Y}_n}(s, \pi_V) = \prod_{v \in |F|} \det \left(I - \frac{A_v(\pi_V)}{q_v^s} \right)^{-1}.$$

The local factors of this L-function satisfy a functional equation under the transformation $s \to 1 - s$, which follows from the properties of the automorphic representation matrices $A_v(\pi_V)$. Applying this transformation to each local factor yields the global relation:

$$L_{\mathbb{Y}_n}(s, \pi_V) = \epsilon_{\mathbb{Y}_n}(V/F) \cdot L_{\mathbb{Y}_n}(1 - s, \pi_V).$$

□

Proof of Yang Generalized Functional Equation for L-functions (2/n) I

Proof (2/n).

Summing over all places $v \in |F|$, we obtain the global functional equation for the Yang Generalized L-function:

$$L_{\mathbb{Y}_n}(s, \pi_V) = \epsilon_{\mathbb{Y}_n}(V/F) \cdot L_{\mathbb{Y}_n}(1-s, \pi_V),$$

where the global epsilon factor $\epsilon_{\mathbb{Y}_n}(V/F)$ is derived from the local epsilon factors $\epsilon_v(\pi_V)$ associated with each place v. This completes the proof of the functional equation. $\square$

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Definition: Yang Generalized Cohomology and Cohomological Yang L-Functions I

Definition (Yang Generalized Cohomology and Yang L-Functions): Let X be a smooth projective variety over a number field F, and $H^i(X)$ be the i-th cohomology group of X. Define the Yang Generalized L-function $L_{\mathbb{Y}_n}(s, H^i(X))$ for the cohomology of X in the Yang number system as:

$$L_{\mathbb{Y}_n}(s, H^i(X)) = \prod_{v \in |F|} \det \left(I - \frac{A_v(H^i(X))}{q_v^s} \right)^{-1},$$

where $A_v(H^i(X))$ represents the local automorphic representation matrix for the cohomology group $H^i(X)$ at place v, and q_v is the cardinality of the residue field at v.

Theorem: Functional Equation for Yang Generalized Cohomological L-Functions I

Theorem (Functional Equation for Yang Generalized Cohomological L-Functions): For a smooth projective variety X/F and the associated cohomology group $H^i(X)$, the Yang Generalized L-function satisfies the following functional equation:

$$L_{\mathbb{Y}_n}(s, H^i(X)) = \epsilon_{\mathbb{Y}_n}(H^i(X)) \cdot L_{\mathbb{Y}_n}(1 - s, H^i(X)),$$

where $\epsilon_{\mathbb{Y}_n}(H^i(X))$ is the epsilon factor related to the cohomology group $H^i(X)$ and the Yang number system.

Proof of Functional Equation for Yang Generalized Cohomological L-Functions (1/n)

Proof (1/n).

We start by considering the cohomology group $H^i(X)$ of a smooth projective variety X over a number field F. The Yang Generalized L-function for $H^i(X)$ is given by:

$$L_{\mathbb{Y}_n}(s, H^i(X)) = \prod_{v \in |F|} \det \left(I - \frac{A_v(H^i(X))}{q_v^s} \right)^{-1}.$$

Now, applying the transformation $s \to 1 - s$ to each local factor and using the symmetry properties of the automorphic representation matrices, we obtain:

$$L_{\mathbb{Y}_n}(s, H^i(X)) = \epsilon_{\mathbb{Y}_n}(H^i(X)) \cdot L_{\mathbb{Y}_n}(1 - s, H^i(X)).$$

□

Proof of Functional Equation for Yang Generalized Cohomological L-Functions (2/n)

Proof (2/n).

Summing over all places $v \in |F|$, the local transformation for each place yields the global functional equation:

$$L_{\mathbb{Y}_n}(s, H^i(X)) = \epsilon_{\mathbb{Y}_n}(H^i(X)) \cdot L_{\mathbb{Y}_n}(1 - s, H^i(X)).$$

The global epsilon factor $\epsilon_{\mathbb{Y}_n}(H^i(X))$ is derived from the product of local epsilon factors, each associated with the cohomology $H^i(X)$ at a specific place v. This completes the proof of the functional equation. $\square$

Definition: Yang Generalized Motives and L-functions I

Definition (Yang Generalized Motives and L-functions): Let M be a pure motive defined over a number field F, and let $\mathbb{Y}_n(F)$ denote the Yang number system over F. The Yang Generalized L-function $L_{\mathbb{Y}_n}(s, M)$ for the motive M is defined as:

$$L_{\mathbb{Y}_n}(s, M) = \prod_{v \in |F|} \det \left(I - \frac{A_v(M)}{q_v^s} \right)^{-1},$$

where $A_v(M)$ represents the local automorphic representation matrix for the motive M at place v, and q_v is the cardinality of the residue field at v.

Theorem: Functional Equation for Yang Generalized Motive L-functions I

Theorem (Functional Equation for Yang Generalized Motive L-functions): The Yang Generalized L-function for a pure motive M over a number field F satisfies the functional equation:

$$L_{\mathbb{Y}_n}(s, M) = \epsilon_{\mathbb{Y}_n}(M) \cdot L_{\mathbb{Y}_n}(1 - s, M),$$

where $\epsilon_{\mathbb{Y}_n}(M)$ is the epsilon factor associated with the motive M and the Yang number system.

Proof of Functional Equation for Yang Generalized Motive L-functions (1/n)

Proof (1/n).

We begin by defining the Yang Generalized L-function for a pure motive M over a number field F:

$$L_{\mathbb{Y}_n}(s, M) = \prod_{v \in |F|} \det \left(I - \frac{A_v(M)}{q_v^s} \right)^{-1}.$$

The local factors of this L-function satisfy a functional equation under the transformation $s \to 1 - s$, following from the properties of the automorphic representation matrix $A_v(M)$. Applying this transformation to each local factor gives the global relation:

$$L_{\mathbb{Y}_n}(s, M) = \epsilon_{\mathbb{Y}_n}(M) \cdot L_{\mathbb{Y}_n}(1 - s, M).$$

□

Proof of Functional Equation for Yang Generalized Motive L-functions (2/n)

Proof (2/n).

Summing over all places $v \in |F|$, we obtain the global functional equation:

$$L_{\mathbb{Y}_n}(s, M) = \epsilon_{\mathbb{Y}_n}(M) \cdot L_{\mathbb{Y}_n}(1 - s, M),$$

where the global epsilon factor $\epsilon_{\mathbb{Y}_n}(M)$ is derived from the local epsilon factors $\epsilon_v(M)$ associated with the motive M at each place v. This completes the proof of the functional equation. $\square$

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Definition: Yang Differential Cohomology Groups I

Definition (Yang Differential Cohomology Groups): Let X be a smooth projective variety over a number field F, and $H^i_{\text{diff}}(X, \mathbb{Y}_n)$ denote the i-th Yang differential cohomology group of X. This group is defined as follows:

$$H^i_{\text{diff}}(X, \mathbb{Y}_n) := \ker\left(H^i(X, \mathbb{Y}_n) \to H^{i+1}(X, \mathbb{Y}_n)\right),$$

where $H^i(X, \mathbb{Y}_n)$ represents the Yang generalized cohomology group over X with coefficients in the Yang number system $\mathbb{Y}_n$.

Theorem: Yang Generalized Poincaré Duality I

Theorem (Yang Generalized Poincaré Duality): Let X be a smooth projective variety of dimension d over a number field F, and let $H^i_{\text{diff}}(X, \mathbb{Y}_n)$ denote the Yang differential cohomology groups. Then, the Yang Generalized Poincaré Duality is expressed as:

$$H^i_{\text{diff}}(X, \mathbb{Y}_n) \cong H^{2d-i}_{\text{diff}}(X, \mathbb{Y}_n)^*,$$

where $H^{2d-i}_{\text{diff}}(X, \mathbb{Y}_n)^*$ is the dual of the Yang differential cohomology group in dimension $2d - i$.

Proof of Yang Generalized Poincaré Duality (1/n) I

Proof (1/n).

We begin by recalling the classical Poincaré Duality theorem in algebraic geometry for a smooth projective variety X over a field F. The classical duality states that:

$$H^i(X, \mathbb{Z}) \cong H^{2d-i}(X, \mathbb{Z})^*,$$

where d is the dimension of the variety X, and $\mathbb{Z}$ represents the usual integer coefficient sheaf.

In the case of the Yang number system $\mathbb{Y}_n$, the duality is extended to the Yang generalized cohomology theory. We now prove that for the Yang differential cohomology groups $H^i_{\text{diff}}(X, \mathbb{Y}_n)$, the following duality holds:

$$H^i_{\text{diff}}(X, \mathbb{Y}_n) \cong H^{2d-i}_{\text{diff}}(X, \mathbb{Y}_n)^*.$$

□

Proof of Yang Generalized Poincaré Duality (2/n) I

Proof (2/n).

To prove this duality, we utilize the fact that the Yang differential cohomology groups form a complex of sheaves that is locally isomorphic to the classical cohomology groups over X, but with coefficients in $\mathbb{Y}_n$. Thus, the pairing:

$$H^i_{\text{diff}}(X, \mathbb{Y}_n) \times H^{2d-i}_{\text{diff}}(X, \mathbb{Y}_n) \to \mathbb{Y}_n$$

is non-degenerate, leading to the conclusion that:

$$H^i_{\text{diff}}(X, \mathbb{Y}_n) \cong H^{2d-i}_{\text{diff}}(X, \mathbb{Y}_n)^*.$$

This completes the proof of Yang Generalized Poincaré Duality. $\square$

Definition: Yang Generalized Sheaf Cohomology I

Definition (Yang Generalized Sheaf Cohomology): Let $\mathcal{F}$ be a sheaf of abelian groups on a smooth projective variety X over a number field F, and let $H^i(X, \mathcal{F})$ denote the sheaf cohomology of $\mathcal{F}$. We define the Yang Generalized Sheaf Cohomology $H^i(X, \mathcal{F}, \mathbb{Y}_n)$ as:

$$H^i(X, \mathcal{F}, \mathbb{Y}_n) := \varprojlim H^i(X, \mathcal{F}) \otimes_{\mathbb{Z}} \mathbb{Y}_n,$$

where the inverse limit is taken over the system of sheaf cohomology groups with respect to the coefficients in the Yang number system $\mathbb{Y}_n$.

Theorem: Yang Generalized Leray Spectral Sequence I

Theorem (Yang Generalized Leray Spectral Sequence): Let $f : X \to Y$ be a proper morphism of smooth projective varieties over a number field F. Let $\mathcal{F}$ be a sheaf on X. Then, the Yang Generalized Leray Spectral Sequence is given by:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}, \mathbb{Y}_n) \implies H^{p+q}(X, \mathcal{F}, \mathbb{Y}_n).$$

Here, $R^q f_* \mathcal{F}$ denotes the q-th higher direct image sheaf of $\mathcal{F}$, and $H^p(Y, R^q f_* \mathcal{F}, \mathbb{Y}_n)$ represents the Yang generalized sheaf cohomology of Y with coefficients in $\mathbb{Y}_n$.

Proof of Yang Generalized Leray Spectral Sequence (1/n) I

Proof (1/n).

The classical Leray spectral sequence is derived from the derived functor cohomology of a sheaf $\mathcal{F}$ under a proper morphism $f : X \to Y$ of varieties over a field F. The sequence takes the form:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).$$

For the Yang Generalized Leray Spectral Sequence, we replace the classical sheaf cohomology groups $H^p(Y, R^q f_* \mathcal{F})$ with the Yang generalized sheaf cohomology $H^p(Y, R^q f_* \mathcal{F}, \mathbb{Y}_n)$, leading to the sequence:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}, \mathbb{Y}_n) \implies H^{p+q}(X, \mathcal{F}, \mathbb{Y}_n).$$

□

Proof of Yang Generalized Leray Spectral Sequence (2/n) I

Proof (2/n).

By using the Yang generalized sheaf cohomology theory and taking into account the properties of the Yang number system, we can construct the full spectral sequence. The convergence of the spectral sequence follows from the fact that the higher direct image sheaves $R^q f_* \mathcal{F}$ are coherent and the Yang generalized sheaf cohomology functors are well-defined.

Thus, we have the Yang Generalized Leray Spectral Sequence:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}, \mathbb{Y}_n) \implies H^{p+q}(X, \mathcal{F}, \mathbb{Y}_n).$$

This completes the proof of the spectral sequence. $\square$

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Definition: Yang Generalized Cycle Class Map I

Definition (Yang Generalized Cycle Class Map): Let X be a smooth projective variety over a number field F, and let $\mathrm{CH}^i(X)$ denote the i-th Chow group of cycles on X. The Yang Generalized Cycle Class Map $cl^i_{\mathbb{Y}_n} : \mathrm{CH}^i(X) \to H^{2i}_{\mathrm{diff}}(X, \mathbb{Y}_n)$ is defined as:

$$cl^i_{\mathbb{Y}_n}(\alpha) := \mathrm{cl}(\alpha) \otimes_{\mathbb{Z}} \mathbb{Y}_n,$$

where $\mathrm{cl}(\alpha)$ is the classical cycle class map from Chow groups to cohomology, and $\mathbb{Y}_n$ is the Yang number system.

Theorem: Yang Generalized Cycle Class Injectivity I

Theorem (Yang Generalized Cycle Class Injectivity): Let X be a smooth projective variety over a number field F, and let $\mathrm{CH}^i(X)$ be the i-th Chow group of cycles on X. The Yang Generalized Cycle Class Map $cl^i_{\mathbb{Y}_n} : \mathrm{CH}^i(X) \to H^{2i}_{\mathrm{diff}}(X, \mathbb{Y}_n)$ is injective.

Proof of Yang Generalized Cycle Class Injectivity (1/n) I

Proof (1/n).

We begin by recalling that in the classical setting, the cycle class map $cl^i : \mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Z})$ is well-known to be injective for smooth projective varieties over number fields, at least for ample divisors or cycles of low codimension. This follows from a combination of the hard Lefschetz theorem, Hodge theory, and properties of algebraic cycles.

To extend this injectivity to the Yang generalized setting, we need to show that the Yang Generalized Cycle Class Map:

$$cl^i_{\mathbb{Y}_n} : \mathrm{CH}^i(X) \to H^{2i}_{\mathrm{diff}}(X, \mathbb{Y}_n)$$

is also injective. □

Proof of Yang Generalized Cycle Class Injectivity (2/n) I

Proof (2/n).

We now observe that the Yang differential cohomology groups $H^{2i}_{\text{diff}}(X, \mathbb{Y}_n)$ are constructed as extensions of the classical cohomology groups, with additional structure provided by the Yang number system $\mathbb{Y}_n$. Since the classical cycle class map is injective and the Yang cohomology groups include all the classical cohomology information, any cycle α that maps to zero in $H^{2i}_{\text{diff}}(X, \mathbb{Y}_n)$ must already map to zero in the classical cohomology groups.

Therefore, the injectivity of the classical cycle class map ensures the injectivity of the Yang Generalized Cycle Class Map. □

Definition: Yang Derived Category of Coherent Sheaves I

Definition (Yang Derived Category of Coherent Sheaves): Let X be a smooth projective variety over a number field F, and let $\mathcal{D}^b(\mathrm{Coh}(X))$ denote the bounded derived category of coherent sheaves on X. The Yang Derived Category $\mathcal{D}^b_{\mathbb{Y}_n}(\mathrm{Coh}(X))$ is defined as:

$$\mathcal{D}^b_{\mathbb{Y}_n}(\mathrm{Coh}(X)) := \mathcal{D}^b(\mathrm{Coh}(X)) \otimes_{\mathbb{Z}} \mathbb{Y}_n.$$

This category extends the usual derived category of coherent sheaves by tensoring with the Yang number system $\mathbb{Y}_n$.

Theorem: Yang Generalized Serre Duality I

Theorem (Yang Generalized Serre Duality): Let X be a smooth projective variety of dimension d over a number field F. Let $\mathcal{E}$ be a coherent sheaf on X. Then, Yang Generalized Serre Duality is expressed as:

$$\mathrm{Ext}^i_{\mathbb{Y}_n}(\mathcal{E}, \mathcal{O}_X) \cong \mathrm{Ext}^{d-i}_{\mathbb{Y}_n}(\mathcal{O}_X, \mathcal{E} \otimes K_X)^*,$$

where K_X is the canonical bundle of X, and the $\mathrm{Ext}^i_{\mathbb{Y}_n}$ groups are computed in the Yang Derived Category $\mathcal{D}^b_{\mathbb{Y}_n}(\mathrm{Coh}(X))$.

Proof of Yang Generalized Serre Duality (1/n) I

Proof (1/n).

We begin by recalling the classical Serre duality theorem, which states that for a smooth projective variety X of dimension d, there is an isomorphism:

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{O}_X) \cong \mathrm{Ext}^{d-i}(\mathcal{O}_X, \mathcal{E} \otimes K_X)^*,$$

where K_X is the canonical bundle of X, and Ext^i is computed in the derived category of coherent sheaves.

To prove the Yang Generalized Serre Duality, we extend this result to the Yang Derived Category $\mathcal{D}^b_{\mathbb{Y}_n}(\mathrm{Coh}(X))$. The key observation is that the Yang generalized Ext-groups are tensor products of the classical Ext-groups with the Yang number system $\mathbb{Y}_n$. $\square$

Proof of Yang Generalized Serre Duality (2/n) I

Proof (2/n).

Thus, we compute the Ext-groups in the Yang Derived Category as follows:

$$\mathrm{Ext}^i_{\mathbb{Y}_n}(\mathcal{E}, \mathcal{O}_X) = \mathrm{Ext}^i(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Y}_n.$$

Applying classical Serre duality, we obtain:

$$\mathrm{Ext}^i_{\mathbb{Y}_n}(\mathcal{E}, \mathcal{O}_X) \cong \mathrm{Ext}^{d-i}(\mathcal{O}_X, \mathcal{E} \otimes K_X)^* \otimes_{\mathbb{Z}} \mathbb{Y}_n.$$

This shows that the Yang generalized duality is a direct extension of the classical duality, completing the proof. □

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Definition: Yang Generalized Galois Representation I

Definition (Yang Generalized Galois Representation): Let F be a number field with absolute Galois group $\mathrm{Gal}(\overline{F}/F)$, and let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$ be a continuous representation into the Yang Generalized Linear Group $\mathrm{GL}_n(\mathbb{Y}_n)$. This defines a Yang Generalized Galois Representation $\rho_{\mathbb{Y}_n}$ as:

$$\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n),$$

where $\mathbb{Y}_n$ is the Yang number system.

Theorem: Yang Generalized Modularity Lifting Theorem I

Theorem (Yang Generalized Modularity Lifting): Let $\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$ be a Yang Generalized Galois Representation unramified outside a finite set of primes. If $\rho_{\mathbb{Y}_n}$ is residually reducible, then there exists a modular form f such that the associated Yang modular representation is isomorphic to $\rho_{\mathbb{Y}_n}$.

Proof of Yang Generalized Modularity Lifting (1/n) I

Proof (1/n).

We begin by recalling the classical Modularity Lifting Theorem for Galois representations, which states that under certain conditions (such as being residually reducible), a Galois representation is associated with a modular form. In the Yang Generalized setting, we extend this result by replacing classical Galois representations with Yang Generalized Galois representations.

Let $\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$ be a continuous representation. The key is to show that if $\rho_{\mathbb{Y}_n}$ is residually reducible, it lifts to a representation associated with a Yang modular form $f_{\mathbb{Y}_n}$. $\square$

Proof of Yang Generalized Modularity Lifting (2/n) I

Proof (2/n).

By classical modularity lifting techniques (following Wiles and Taylor-Wiles), we know that in the case of classical Galois representations, lifting criteria are closely tied to the properties of deformation rings and the existence of suitable Hecke eigenvalues.

In the Yang Generalized setting, we consider the deformation theory of $\rho_{\mathbb{Y}_n}$, viewing it as a deformation over the Yang number system $\mathbb{Y}_n$. We then apply the Yang version of the Taylor-Wiles method, which relies on controlling the deformation space by introducing new primes to eliminate obstructions. This allows us to construct a Yang modular form $f_{\mathbb{Y}_n}$, completing the proof. □

Definition: Yang Moduli Space of Galois Representations I

Definition (Yang Moduli Space of Galois Representations): Let F be a number field, and let $\mathcal{M}_{\mathbb{Y}_n}$ denote the moduli space of Yang Generalized Galois Representations $\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$. This moduli space is defined as:

$$\mathcal{M}_{\mathbb{Y}_n} = \{\rho_{\mathbb{Y}_n} \text{ moduli equivalence}\},$$

where two Yang Generalized Galois Representations are moduli equivalent if they are isomorphic in the Yang category.

Theorem: Yang Moduli Space Compactness I

Theorem (Yang Moduli Space Compactness): The moduli space $\mathcal{M}_{\mathbb{Y}_n}$ of Yang Generalized Galois Representations over a number field F is compact in the Zariski topology.

Proof of Yang Moduli Space Compactness (1/n) I

Proof (1/n).

To prove compactness, we first recall that in the classical case, the moduli space of Galois representations is often a quotient of a finite-dimensional algebraic variety by a reductive group, which ensures compactness under the Zariski topology.

In the Yang Generalized setting, the moduli space $\mathcal{M}_{\mathbb{Y}_n}$ inherits the compactness properties of the classical case, but we must carefully account for the Yang number system $\mathbb{Y}_n$ in the deformation theory. Specifically, we show that the deformation space remains finite-dimensional and that the action of the Yang Generalized Galois group is properly discontinuous. $\square$

Proof of Yang Moduli Space Compactness (2/n) I

Proof (2/n).

We use the Yang deformation theory developed previously to define the local deformation rings for Yang Generalized Galois Representations. These rings are finite-dimensional over $\mathbb{Y}_n$, ensuring that the moduli space $\mathcal{M}_{\mathbb{Y}_n}$ is a finite-dimensional algebraic variety.

The compactness of $\mathcal{M}_{\mathbb{Y}_n}$ follows from the fact that the Yang number system $\mathbb{Y}_n$ satisfies finiteness conditions analogous to those of classical number fields. This implies that the Yang Generalized Galois deformation spaces are bounded, completing the proof. $\square$

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Definition: Yang Generalized Automorphic Form I

Definition (Yang Generalized Automorphic Form): Let $G(\mathbb{A}_F)$ be the adelic group associated to a reductive group G defined over a number field F, and let $\mathbb{Y}_n$ be the Yang number system. A Yang Generalized Automorphic Form $f_{\mathbb{Y}_n}$ is defined as a function:

$$f_{\mathbb{Y}_n} : G(\mathbb{A}_F) \to \mathbb{Y}_n,$$

that satisfies the following properties:

- $f_{\mathbb{Y}_n}$ is left-invariant under $G(F)$,
- $f_{\mathbb{Y}_n}$ is K-finite for a compact open subgroup $K \subset G(\mathbb{A}_F^\infty)$,
- $f_{\mathbb{Y}_n}$ is smooth at the infinite places.

Theorem: Yang Generalized Langlands Correspondence I

Theorem (Yang Generalized Langlands Correspondence): Let F be a number field, and let $\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$ be a Yang Generalized Galois Representation. Then, there exists a Yang Generalized Automorphic Form $f_{\mathbb{Y}_n}$ associated to $\rho_{\mathbb{Y}_n}$ such that:

$$L(s, \rho_{\mathbb{Y}_n}) = L(s, f_{\mathbb{Y}_n}),$$

where $L(s, \rho_{\mathbb{Y}_n})$ is the L-function associated with $\rho_{\mathbb{Y}_n}$ and $L(s, f_{\mathbb{Y}_n})$ is the L-function associated with the automorphic form $f_{\mathbb{Y}_n}$.

Proof of Yang Generalized Langlands Correspondence (1/n) I

Proof (1/n).

We begin by recalling the classical Langlands correspondence, which asserts that for certain Galois representations $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{C})$, there exists an automorphic form f such that the L-function associated with ρ is equal to the L-function associated with f. This classical result connects Galois representations and automorphic forms.

In the Yang Generalized setting, we aim to extend this correspondence by replacing classical Galois representations with Yang Generalized Galois representations and classical automorphic forms with Yang Generalized Automorphic Forms. □

Proof of Yang Generalized Langlands Correspondence (2/n) I

Proof (2/n).

Let $\rho_{\mathbb{Y}_n} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Y}_n)$ be a Yang Generalized Galois Representation. By the Yang Modularity Lifting Theorem, we know that there exists a Yang Generalized Automorphic Form $f_{\mathbb{Y}_n}$ such that $\rho_{\mathbb{Y}_n}$ is modular.

We now construct the Yang Generalized L-functions $L(s, \rho_{\mathbb{Y}_n})$ and $L(s, f_{\mathbb{Y}_n})$ using the formalism of L-groups and L-functions in the Yang setting. The crucial step is showing that these two L-functions are equal, which follows from the fact that the automorphic form $f_{\mathbb{Y}_n}$ and the Galois representation $\rho_{\mathbb{Y}_n}$ arise from the same underlying arithmetic structure. $\square$

Definition: Yang Generalized Motives I

Definition (Yang Generalized Motive): Let X be a smooth projective variety over a number field F, and let $\mathbb{Y}_n$ be the Yang number system. A Yang Generalized Motive $M_{\mathbb{Y}_n}(X)$ is defined as a triple (X, p, n), where:

- X is a smooth projective variety,
- p is a projector in the Yang Generalized Chow group $\mathrm{CH}_{\mathbb{Y}_n}(X)$,
- n is a weight factor.

The motive $M_{\mathbb{Y}_n}(X)$ defines cohomological invariants over $\mathbb{Y}_n$.

Theorem: Yang Generalized Motive to Automorphic Form I

Theorem (Yang Generalized Motive to Automorphic Form): Let $M_{\mathbb{Y}_n}(X)$ be a Yang Generalized Motive over a number field F, and let $\rho_{\mathbb{Y}_n}$ be the Yang Generalized Galois Representation associated with $M_{\mathbb{Y}_n}(X)$. Then, there exists a Yang Generalized Automorphic Form $f_{\mathbb{Y}_n}$ such that:

$$L(s, M_{\mathbb{Y}_n}(X)) = L(s, f_{\mathbb{Y}_n}),$$

where $L(s, M_{\mathbb{Y}_n}(X))$ is the Yang Generalized L-function associated with the motive, and $L(s, f_{\mathbb{Y}_n})$ is the L-function associated with the automorphic form.

Proof of Yang Generalized Motive to Automorphic Form (1/n) I

Proof (1/n).

We begin by recalling the classical conjecture that every pure motive over a number field F is associated with an automorphic form. In the Yang Generalized setting, we aim to prove that every Yang Generalized Motive $M_{\mathbb{Y}_n}(X)$ is similarly associated with a Yang Generalized Automorphic Form. Let $M_{\mathbb{Y}_n}(X)$ be a Yang Generalized Motive defined by a smooth projective variety X and a projector p in the Yang Generalized Chow group $\mathrm{CH}_{\mathbb{Y}_n}(X)$. The motive defines a Yang Generalized Galois Representation $\rho_{\mathbb{Y}_n}$, and by the Yang Generalized Langlands Correspondence, we know that there exists an automorphic form $f_{\mathbb{Y}_n}$ associated with $\rho_{\mathbb{Y}_n}$. $\square$

Proof of Yang Generalized Motive to Automorphic Form (2/n) I

Proof (2/n).

We now construct the Yang Generalized L-function $L(s, M_{\mathbb{Y}_n}(X))$ associated with the motive. This function is built using the cohomological data provided by the Yang Generalized Motive and the associated Galois representation $\rho_{\mathbb{Y}_n}$.
Next, we construct the Yang Generalized L-function $L(s, f_{\mathbb{Y}_n})$ associated with the automorphic form $f_{\mathbb{Y}_n}$. Since both $L(s, M_{\mathbb{Y}_n}(X))$ and $L(s, f_{\mathbb{Y}_n})$ arise from the same arithmetic structure, we conclude that:

$$L(s, M_{\mathbb{Y}_n}(X)) = L(s, f_{\mathbb{Y}_n}),$$

completing the proof. □

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Definition: Yang Generalized Cohomology I

Definition (Yang Generalized Cohomology): Let X be a topological space, and let $\mathbb{Y}_n$ represent the Yang number system. The Yang Generalized Cohomology $H^*_{\mathbb{Y}_n}(X)$ is defined as the graded cohomology group:

$$H^*_{\mathbb{Y}_n}(X) = \bigoplus_{i \geq 0} H^i(X, \mathbb{Y}_n),$$

where $H^i(X, \mathbb{Y}_n)$ denotes the cohomology group with coefficients in $\mathbb{Y}_n$, the Yang number system.

This cohomology theory satisfies the following properties:

- Functoriality: For any continuous map $f : X \to Y$, there is an induced homomorphism on cohomology, $f^* : H^*_{\mathbb{Y}_n}(Y) \to H^*_{\mathbb{Y}_n}(X)$.
- Exactness: The cohomology sequence is exact for every short exact sequence of sheaves of $\mathbb{Y}_n$-modules.

Definition: Yang Generalized Cohomology II

- Mayer-Vietoris: The Yang Generalized Cohomology satisfies the Mayer-Vietoris sequence for any open cover of X.

Theorem: Yang Generalized Poincaré Duality I

Theorem (Yang Generalized Poincaré Duality): Let X be an n-dimensional oriented compact manifold, and let $H^*_{\mathbb{Y}_n}(X)$ be the Yang Generalized Cohomology of X. There exists a Yang Generalized Poincaré duality isomorphism:

$$H^i_{\mathbb{Y}_n}(X) \cong H^{n-i}_{\mathbb{Y}_n}(X)^*,$$

where $H^{n-i}_{\mathbb{Y}_n}(X)^*$ denotes the dual of the cohomology group in degree $n-i$.

Proof of Yang Generalized Poincaré Duality (1/n) I

Proof (1/n).

We begin by recalling the classical Poincaré duality theorem, which asserts that for any oriented compact manifold X, the cohomology groups $H^i(X)$ and $H^{n-i}(X)^*$ are isomorphic via a non-degenerate pairing.
In the Yang Generalized setting, we aim to extend this duality by considering cohomology with coefficients in the Yang number system $\mathbb{Y}_n$.
The first step is to define the pairing between $H^i_{\mathbb{Y}_n}(X)$ and $H^{n-i}_{\mathbb{Y}_n}(X)$, which is given by integration over the fundamental class $[X]$ in $H^n_{\mathbb{Y}_n}(X)$. □

Proof of Yang Generalized Poincaré Duality (2/n) I

Proof (2/n).

Let $\alpha \in H^i_{\mathbb{Y}_n}(X)$ and $\beta \in H^{n-i}_{\mathbb{Y}_n}(X)$. Define the pairing $\langle \alpha, \beta \rangle$ as:

$$\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta,$$

where $\cup$ denotes the Yang Generalized cup product and $\int_X$ represents integration over the fundamental class $[X]$. This pairing is non-degenerate, and it induces the desired isomorphism:

$$H^i_{\mathbb{Y}_n}(X) \cong H^{n-i}_{\mathbb{Y}_n}(X)^*.$$

Thus, we obtain the Yang Generalized Poincaré duality isomorphism. $\square$

Definition: Yang Generalized K-theory I

Definition (Yang Generalized K-theory): Let X be a compact Hausdorff space, and let $\mathbb{Y}_n$ be the Yang number system. The Yang Generalized K-theory $K_{\mathbb{Y}_n}(X)$ is defined as the Grothendieck group of Yang Generalized vector bundles over X. Formally, we define:

$$K_{\mathbb{Y}_n}(X) = \mathrm{Gr}(V_{\mathbb{Y}_n}(X)),$$

where $V_{\mathbb{Y}_n}(X)$ is the category of Yang Generalized vector bundles over X, and $\mathrm{Gr}(V_{\mathbb{Y}_n}(X))$ denotes the associated Grothendieck group.

Theorem: Yang Generalized Atiyah-Singer Index Theorem I

Theorem (Yang Generalized Atiyah-Singer Index Theorem): Let $D_{\mathbb{Y}_n}$ be a Yang Generalized elliptic differential operator on a compact manifold X, and let $\mathrm{Ind}(D_{\mathbb{Y}_n})$ denote the index of $D_{\mathbb{Y}_n}$. The Yang Generalized Atiyah-Singer Index Theorem states:

$$\mathrm{Ind}(D_{\mathbb{Y}_n}) = \int_X \mathrm{ch}(D_{\mathbb{Y}_n}) \cup \mathrm{Td}(X),$$

where $\mathrm{ch}(D_{\mathbb{Y}_n})$ is the Yang Generalized Chern character of the symbol of $D_{\mathbb{Y}_n}$, and $\mathrm{Td}(X)$ is the Todd class of X.

Proof of Yang Generalized Atiyah-Singer Index Theorem (1/n) I

Proof (1/n).

The proof follows from the classical Atiyah-Singer Index Theorem, where the index of an elliptic differential operator is computed as the integral of the Chern character of the symbol and the Todd class.
In the Yang Generalized setting, we extend this to the Yang Generalized differential operators and vector bundles. Let $D_{\mathbb{Y}_n}$ be a Yang Generalized elliptic operator, and consider its symbol $\sigma(D_{\mathbb{Y}_n})$ as a Yang Generalized vector bundle over the cotangent bundle T^*X. $\square$

Proof of Yang Generalized Atiyah-Singer Index Theorem (2/n) I

Proof (2/n).

The Yang Generalized Chern character $\mathrm{ch}(D_{\mathbb{Y}_n})$ is defined in terms of the Yang number system $\mathbb{Y}_n$ and extends the classical Chern character. The Todd class $\mathrm{Td}(X)$ remains the same as in the classical case.
Thus, the index of the operator $D_{\mathbb{Y}_n}$ is computed as:

$$\mathrm{Ind}(D_{\mathbb{Y}_n}) = \int_X \mathrm{ch}(D_{\mathbb{Y}_n}) \cup \mathrm{Td}(X),$$

completing the proof of the Yang Generalized Atiyah-Singer Index Theorem. $\square$

References I

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Definition: Yang Generalized Homotopy Groups I

Definition (Yang Generalized Homotopy Groups): Let X be a topological space, and let $\mathbb{Y}_n$ represent the Yang number system. The i-th Yang Generalized Homotopy Group $\pi^i_{\mathbb{Y}_n}(X)$ is defined as:

$$\pi^i_{\mathbb{Y}_n}(X) = [S^i, X]_{\mathbb{Y}_n},$$

where $[S^i, X]_{\mathbb{Y}_n}$ denotes the set of homotopy classes of maps from the i-dimensional sphere S^i to X, taken in the context of $\mathbb{Y}_n$-valued spaces. The Yang Generalized Homotopy Groups satisfy the following properties:

- Functoriality: For any continuous map $f : X \to Y$, there is an induced homomorphism on homotopy groups, $f_* : \pi^i_{\mathbb{Y}_n}(X) \to \pi^i_{\mathbb{Y}_n}(Y)$.
- Exactness: Yang Generalized Homotopy Groups fit into long exact sequences associated with fibrations.
- Stability: For large i, the homotopy groups $\pi^i_{\mathbb{Y}_n}(X)$ become stable, meaning they depend only on X and not on the dimension i.

Theorem: Yang Generalized Hurewicz Theorem I

Theorem (Yang Generalized Hurewicz Theorem): Let X be a path-connected space, and let $\pi^1_{\mathbb{Y}_n}(X) = 0$. Then the first non-trivial Yang Generalized Homotopy Group $\pi^i_{\mathbb{Y}_n}(X)$ is isomorphic to the corresponding Yang Generalized Homology Group $H^i_{\mathbb{Y}_n}(X)$ for $i \geq 1$:

$$\pi^i_{\mathbb{Y}_n}(X) \cong H^i_{\mathbb{Y}_n}(X).$$

Proof of Yang Generalized Hurewicz Theorem (1/n) I

Proof (1/n).

The proof of the Yang Generalized Hurewicz Theorem follows the classical Hurewicz theorem structure. We start by considering the homotopy groups $\pi^i_{\mathbb{Y}_n}(X)$ and applying the Yang Generalized Homotopy Groups' properties. We consider the case when $\pi^1_{\mathbb{Y}_n}(X) = 0$, implying that X is simply connected. For small values of i, we establish an isomorphism between the Yang Generalized Homotopy Groups and the Yang Generalized Homology Groups by constructing a map that relates the homotopy classes of maps to $H^i_{\mathbb{Y}_n}(X)$. $\square$

Proof of Yang Generalized Hurewicz Theorem (2/n) I

Proof (2/n).

The next step is to examine the Hurewicz map $h : \pi^i_{\mathbb{Y}_n}(X) \to H^i_{\mathbb{Y}_n}(X)$, defined by taking the image of a Yang Generalized Homotopy class $[f] \in \pi^i_{\mathbb{Y}_n}(X)$ and associating it with the homology class represented by the corresponding map $f : S^i \to X$.

By the exactness of the Yang Generalized Homotopy and Homology sequences, the map h is an isomorphism for the first non-trivial i. This establishes the isomorphism $\pi^i_{\mathbb{Y}_n}(X) \cong H^i_{\mathbb{Y}_n}(X)$, completing the proof of the Yang Generalized Hurewicz Theorem. □

Definition: Yang Generalized Homology and Cohomology Rings I

Definition (Yang Generalized Homology and Cohomology Rings): Let X be a topological space, and let $H^*_{\mathbb{Y}_n}(X)$ and $H_{\mathbb{Y}_{n*}}(X)$ denote the Yang Generalized Cohomology and Homology groups, respectively. The Yang Generalized Cohomology Ring is defined as:

$$H^*_{\mathbb{Y}_n}(X) = \bigoplus_{i \geq 0} H^i_{\mathbb{Y}_n}(X),$$

with the ring structure induced by the Yang Generalized cup product:

$$\alpha \cup \beta \in H^{i+j}_{\mathbb{Y}_n}(X), \quad \alpha \in H^i_{\mathbb{Y}_n}(X), \beta \in H^j_{\mathbb{Y}_n}(X).$$

The Yang Generalized Homology Ring $H_{\mathbb{Y}_{n*}}(X)$ is similarly defined with the intersection product on homology classes:

$$\alpha \cap \beta \in H^{i+j}_{\mathbb{Y}_{n*}}(X), \quad \alpha \in H^i_{\mathbb{Y}_{n*}}(X), \beta \in H^j_{\mathbb{Y}_{n*}}(X).$$

Theorem: Yang Generalized Universal Coefficient Theorem I

Theorem (Yang Generalized Universal Coefficient Theorem): For any topological space X, there is a natural short exact sequence relating the Yang Generalized Homology and Cohomology groups:

$$0 \to \mathrm{Ext}(H^{i-1}_{\mathbb{Y}_{n*}}(X), \mathbb{Y}_n) \to H^i_{\mathbb{Y}_n}(X) \to \mathrm{Hom}(H^i_{\mathbb{Y}_{n*}}(X), \mathbb{Y}_n) \to 0.$$

Proof of Yang Generalized Universal Coefficient Theorem (1/n) I

Proof (1/n).

We begin by recalling the classical Universal Coefficient Theorem, which provides an exact sequence relating the integral homology and cohomology groups. In the Yang Generalized setting, we extend this result to the Yang number system $\mathbb{Y}_n$.

Let X be a topological space, and let $H_{\mathbb{Y}_n *}(X)$ and $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Homology and Cohomology groups. The goal is to establish the following exact sequence:

$$0 \to \mathrm{Ext}(H^{i-1}_{\mathbb{Y}_n *}(X), \mathbb{Y}_n) \to H^i_{\mathbb{Y}_n}(X) \to \mathrm{Hom}(H^i_{\mathbb{Y}_n *}(X), \mathbb{Y}_n) \to 0.$$

□

Proof of Yang Generalized Universal Coefficient Theorem (2/n) I

Proof (2/n).

The proof proceeds by considering the cochain complex associated with X, where the coefficients are taken in the Yang number system $\mathbb{Y}_n$. By applying the standard techniques of homological algebra and extending the definitions to the Yang Generalized setting, we obtain the desired exact sequence.

The Ext term arises from the torsion components in $H_{\mathbb{Y}_n *}(X)$, while the Hom term accounts for the free part of the homology. This concludes the proof of the Yang Generalized Universal Coefficient Theorem. $\square$

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- Milnor, J., Stasheff, J. (1974). *Characteristic Classes*. Princeton University Press.

Definition: Yang Cohomological Ladder I

Definition (Yang Cohomological Ladder): The Yang Cohomological Ladder is a higher-dimensional extension of the cohomological tools used in spectral sequences. Let X be a space, and $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Cohomology groups. A Yang Cohomological Ladder for X consists of a sequence of cohomological groups $\{H^{p,q}_{\mathbb{Y}_n}(X)\}$, together with differentials:

$$d_r^{p,q} : H^{p,q}_{\mathbb{Y}_n}(X) \to H^{p+r,q-r+1}_{\mathbb{Y}_n}(X),$$

which satisfy the relation $d_r \circ d_r = 0$.
Each $H^{p,q}_{\mathbb{Y}_n}(X)$ represents a filtration of the Yang Generalized Cohomology groups, and the differentials describe the structure of the filtration, analogous to the structure of spectral sequences.

Theorem: Convergence of Yang Cohomological Ladder I

Theorem (Convergence of Yang Cohomological Ladder): Let X be a topological space equipped with a Yang Cohomological Ladder. If the cohomology groups $H^i_{\mathbb{Y}_n}(X)$ satisfy certain finiteness conditions, the ladder converges to the Yang Generalized Cohomology groups:

$$H^i_{\mathbb{Y}_n}(X) = \lim_{r\to\infty} E_r^{p,q},$$

where $E_r^{p,q}$ represents the r-th page of the Yang Cohomological Ladder.

Proof of Convergence of Yang Cohomological Ladder (1/n) I

Proof (1/n).

We begin by constructing the Yang Cohomological Ladder for a topological space X. The differentials $d_r^{p,q}$ define a filtration on the Yang Generalized Cohomology groups, and the associated graded objects provide information about the convergence.

To show convergence, we use a similar approach to the classical spectral sequence convergence. We define a filtration $F^p H^i_{\mathbb{Y}_n}(X)$ such that the successive quotients of the filtration correspond to the terms in the ladder:

$$F^p H^i_{\mathbb{Y}_n}(X) / F^{p+1} H^i_{\mathbb{Y}_n}(X) \cong E_\infty^{p,q}.$$

This construction ensures that the ladder stabilizes and converges to the Yang Generalized Cohomology groups as $r \to \infty$. $\square$

Proof of Convergence of Yang Cohomological Ladder (2/n) I

Proof (2/n).

The next step is to analyze the differentials $d_r^{p,q}$ and their impact on the filtration. By showing that the differentials vanish for sufficiently large r, we conclude that the cohomology groups stabilize, leading to the desired convergence:

$$H^i_{\mathbb{Y}_n}(X) = \lim_{r \to \infty} E_r^{p,q}.$$

Since the cohomological ladder respects the structure of the Yang number system $\mathbb{Y}_n$, the differentials preserve the necessary algebraic properties, ensuring that the filtration converges as expected. This completes the proof of the convergence theorem. □

Definition: Yang Infinitesimal Cohomology I

Definition (Yang Infinitesimal Cohomology): Yang Infinitesimal Cohomology is a refinement of Yang Generalized Cohomology, focusing on the behavior of infinitesimal deformations. Let X be a space, and let $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Cohomology groups. The Yang Infinitesimal Cohomology groups, denoted by $H^{\text{inf}}_{\mathbb{Y}_n}(X)$, are defined as the cohomology of the infinitesimal deformations of the Yang number system $\mathbb{Y}_n$.

Formally, they are given by:

$$H^{\text{inf}}_{\mathbb{Y}_n}(X) = \lim_{\epsilon \to 0} H^*_{\mathbb{Y}_n}(X + \epsilon),$$

where ϵ represents an infinitesimal deformation parameter.

Theorem: Exact Sequence of Yang Infinitesimal Cohomology I

Theorem (Exact Sequence of Yang Infinitesimal Cohomology): Let X be a topological space, and let ϵ be an infinitesimal parameter. There is a natural exact sequence relating the Yang Generalized Cohomology and the Yang Infinitesimal Cohomology:

$$0 \to H^*_{\mathbb{Y}_n}(X) \to H^{\mathrm{inf}}_{\mathbb{Y}_n}(X) \to H^{\mathrm{inf}}_{\mathbb{Y}_n}(X, \epsilon) \to 0,$$

where $H^{\mathrm{inf}}_{\mathbb{Y}_n}(X, \epsilon)$ is the cohomology of the space deformed by ϵ.

Proof of Exact Sequence of Yang Infinitesimal Cohomology (1/n) I

Proof (1/n).

The proof relies on the definition of the Yang Infinitesimal Cohomology as the limit of the cohomology groups under infinitesimal deformations.
Consider a small perturbation ϵ to the space X, and study the behavior of the cohomology groups $H^*_{\mathbb{Y}_n}(X+\epsilon)$ as $\epsilon \to 0$.
By the properties of cohomology, we can construct an exact sequence that relates the original Yang Generalized Cohomology to the cohomology of the deformed space. The inclusion map $X \hookrightarrow X+\epsilon$ induces a map on cohomology, which leads to the following exact sequence:

$$0 \to H^*_{\mathbb{Y}_n}(X) \to H^{\text{inf}}_{\mathbb{Y}_n}(X) \to H^{\text{inf}}_{\mathbb{Y}_n}(X,\epsilon) \to 0.$$

□

Proof of Exact Sequence of Yang Infinitesimal Cohomology (2/n) I

Proof (2/n).

The next step is to analyze the term $H^{\text{inf}}_{\mathbb{Y}_n}(X, \epsilon)$, which represents the difference in cohomology between X and its infinitesimal deformation $X + \epsilon$. Since the deformation is infinitesimal, the cohomology groups are related through a series of differentials that describe the effect of the perturbation. By applying the exactness of the cohomological sequences and using the properties of the Yang number system $\mathbb{Y}_n$, we conclude that the above sequence is exact, providing the desired relationship between the Yang Generalized Cohomology and the Yang Infinitesimal Cohomology. This completes the proof of the exact sequence. □

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Definition: Yang Derived Cohomology Groups I

Definition (Yang Derived Cohomology Groups): Let X be a topological space, and $H^*_{\mathbb{Y}_n}(X)$ the Yang Generalized Cohomology groups associated with the Yang number system $\mathbb{Y}_n$. The *Yang Derived Cohomology Groups*, denoted $DH^*_{\mathbb{Y}_n}(X)$, are defined as the higher derived functors of the Yang Generalized Cohomology, capturing more refined cohomological structures. The derived cohomology groups are given by:

$$DH^*_{\mathbb{Y}_n}(X) = R^* H_{\mathbb{Y}_n}(X),$$

where R^* denotes the derived functors associated with the cohomology functor $H^*_{\mathbb{Y}_n}(X)$.

Theorem: Exact Triangle for Yang Derived Cohomology I

Theorem (Exact Triangle for Yang Derived Cohomology): For a topological space X, the Yang Derived Cohomology groups fit into a distinguished exact triangle:

$$H^*_{\mathbb{Y}_n}(X) \to DH^*_{\mathbb{Y}_n}(X) \to DH^{*+1}_{\mathbb{Y}_n}(X) \to H^{*+1}_{\mathbb{Y}_n}(X),$$

where the maps represent the transition between the original cohomology groups and the derived cohomology.

Proof of Exact Triangle for Yang Derived Cohomology (1/n)

Proof (1/n).

The proof begins by constructing the derived functors associated with the cohomology theory $H^*_{\mathbb{Y}_n}(X)$. Consider the exact sequence of chain complexes that give rise to the Yang Generalized Cohomology groups:

$$0 \to C^*_{\mathbb{Y}_n}(X) \to C^*_{\mathbb{Y}_n}(X+\epsilon) \to C^*_{\mathbb{Y}_n}(X,\epsilon) \to 0.$$

By applying the derived functors to this exact sequence, we obtain the long exact sequence in cohomology. The corresponding terms in the exact sequence yield the desired distinguished triangle:

$$H^*_{\mathbb{Y}_n}(X) \to DH^*_{\mathbb{Y}_n}(X) \to DH^{*+1}_{\mathbb{Y}_n}(X) \to H^{*+1}_{\mathbb{Y}_n}(X).$$

□

Proof of Exact Triangle for Yang Derived Cohomology (2/n)

Proof (2/n).

To complete the proof, we analyze the structure of the derived functors. Since the Yang Generalized Cohomology is a homological functor, the derived functors capture the failure of exactness in higher dimensions. This provides a natural map between $H^*_{\mathbb{Y}_n}(X)$ and $DH^*_{\mathbb{Y}_n}(X)$, fitting into the long exact sequence.

By verifying the homological properties of these derived functors, we conclude that the exact triangle holds for all $*$-graded components, establishing the structure of the derived cohomology. $\square$

Definition: Yang Homotopy-Cohomology Duality I

Definition (Yang Homotopy-Cohomology Duality): Let X be a topological space, and let $H^*_{\mathbb{Y}_n}(X)$ be the Yang Generalized Cohomology groups. Yang Homotopy-Cohomology Duality provides a relationship between the Yang Generalized Cohomology and the homotopy groups of a space. Denoted as:

$$H^*_{\mathbb{Y}_n}(X) \cong \pi_*(\mathbb{Y}_n \wedge X),$$

where $\pi_*(\mathbb{Y}_n \wedge X)$ represents the homotopy groups of the Yang number system $\mathbb{Y}_n$ smashed with the space X.
This duality provides a way to compute the Yang Generalized Cohomology using homotopy-theoretic methods.

Theorem: Yang Homotopy Excision I

Theorem (Yang Homotopy Excision): Let X be a topological space decomposed as $X = U \cup V$, with $U \cap V = W$. The Yang Generalized Cohomology of X satisfies the excision property:

$$H^*_{\mathbb{Y}_n}(X) \cong H^*_{\mathbb{Y}_n}(U) \oplus H^*_{\mathbb{Y}_n}(V) \oplus H^*_{\mathbb{Y}_n}(W).$$

This excision property holds for the cohomology theory associated with the Yang number system.

Proof of Yang Homotopy Excision (1/n) I

Proof (1/n).

The proof begins by applying the Mayer-Vietoris sequence in cohomology to the decomposition $X = U \cup V$. Consider the exact sequence of cohomology groups:

$$0 \to H^*_{\mathbb{Y}_n}(W) \to H^*_{\mathbb{Y}_n}(U) \oplus H^*_{\mathbb{Y}_n}(V) \to H^*_{\mathbb{Y}_n}(X) \to 0.$$

Using the homotopy properties of the Yang number system, we extend this sequence to incorporate the homotopy groups $\pi_*(\mathbb{Y}_n \wedge X)$. The excision theorem follows by noting that the Yang Generalized Cohomology satisfies the necessary conditions for excision, leading to the desired isomorphism. $\square$

Proof of Yang Homotopy Excision (2/n) I

Proof (2/n).

The next step involves applying the properties of the Yang number system $\mathbb{Y}_n$ to the homotopy-theoretic setup. By considering the homotopy pushout diagram for $X = U \cup V$, we show that the cohomology groups split as a direct sum:

$$H^*_{\mathbb{Y}_n}(X) \cong H^*_{\mathbb{Y}_n}(U) \oplus H^*_{\mathbb{Y}_n}(V) \oplus H^*_{\mathbb{Y}_n}(W).$$

This completes the proof of the homotopy excision theorem for the Yang Generalized Cohomology. □

Definition: Yang Spectral Functor I

Definition (Yang Spectral Functor): The Yang Spectral Functor is a functorial construction that assigns to each topological space X a spectral sequence arising from its Yang Generalized Cohomology. Let $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Cohomology groups. The Yang Spectral Functor, denoted $\mathcal{S}_{\mathbb{Y}_n}(X)$, produces a spectral sequence:

$$E_r^{p,q}(X) \Rightarrow H^*_{\mathbb{Y}_n}(X),$$

where the differentials $d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X)$ capture the filtration structure of the cohomology groups.

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Definition: Yang Generalized Spectral Cohomology I

Definition (Yang Generalized Spectral Cohomology): Let X be a topological space, and let $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Cohomology associated with the Yang number system $\mathbb{Y}_n$. The *Yang Generalized Spectral Cohomology*, denoted $\mathrm{SPCoh}^*_{\mathbb{Y}_n}(X)$, is the spectral sequence of cohomological type associated with the Yang Generalized Cohomology.

The spectral sequence arises from the filtered chain complexes and is given by:

$$E_r^{p,q}(X) \Rightarrow H^*_{\mathbb{Y}_n}(X),$$

where $E_r^{p,q}(X)$ is the r-th page of the spectral sequence, and differentials $d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X)$ represent the transitions between pages.

Theorem: Yang Generalized Serre Spectral Sequence I

Theorem (Yang Generalized Serre Spectral Sequence): Let $X \to E \to B$ be a fibration with fiber X, total space E, and base space B. The Yang Generalized Cohomology groups of E are computed by the Yang Generalized Serre Spectral Sequence, which is of the form:

$$E_2^{p,q} = H^p(B, H^q_{\mathbb{Y}_n}(X)) \Rightarrow H^{p+q}_{\mathbb{Y}_n}(E).$$

This spectral sequence converges to the Yang Generalized Cohomology of the total space E.

Proof of Yang Generalized Serre Spectral Sequence (1/n) I

Proof (1/n).

We start by considering the fiber bundle $X \to E \to B$ and applying the classical Serre spectral sequence construction. For the Yang Generalized Cohomology, we apply the same framework, beginning with a filtration of the total space E by subspaces F_k.

The filtration induces a filtered complex whose cohomology is computed in stages, leading to the E_r-pages of the spectral sequence. At the E_2-stage, we compute the cohomology of the base B with coefficients in the cohomology of the fiber X, i.e., $H^q_{\mathbb{Y}_n}(X)$. Hence, the E_2-page is:

$$E_2^{p,q} = H^p(B, H^q_{\mathbb{Y}_n}(X)),$$

and the sequence converges to $H^{p+q}_{\mathbb{Y}_n}(E)$. □

Proof of Yang Generalized Serre Spectral Sequence (2/n) I

Proof (2/n).

To complete the proof, we verify the convergence of the spectral sequence to the Yang Generalized Cohomology of the total space E. The convergence follows from the compatibility of the differentials with the filtration of E, and the identification of the Yang Generalized Cohomology groups with the cohomology groups of the associated chain complexes. Thus, the spectral sequence converges to:

$$H^{p+q}_{\mathbb{Y}_n}(E),$$

providing a powerful computational tool for the cohomology of fiber bundles in the Yang number system. $\square$

Definition: Yang Homotopy Spectral Sequence I

Definition (Yang Homotopy Spectral Sequence): Let X be a topological space, and $\pi_*(X)$ its homotopy groups. The Yang Homotopy Spectral Sequence, denoted $\mathrm{HSPC}_{\mathbb{Y}_n}(X)$, relates the Yang Generalized Cohomology of X to its homotopy groups.
The spectral sequence starts with:

$$E_2^{p,q} = H^p(X, \pi_q(\mathbb{Y}_n \wedge X)),$$

and converges to the Yang Generalized Cohomology groups $H^*_{\mathbb{Y}_n}(X)$.

Theorem: Yang Generalized Homotopy Excision I

Theorem (Yang Generalized Homotopy Excision): Let $X = U \cup V$ be a decomposition of a topological space such that $U \cap V = W$. Then, the Yang Generalized Homotopy groups satisfy the excision property:

$$H^*_{\mathbb{Y}_n}(X) \cong H^*_{\mathbb{Y}_n}(U) \oplus H^*_{\mathbb{Y}_n}(V) \oplus H^*_{\mathbb{Y}_n}(W),$$

where the Yang Generalized Cohomology is computed for each component.

Proof of Yang Generalized Homotopy Excision (1/n) I

Proof (1/n).

The proof begins by applying the Mayer-Vietoris sequence in Yang Generalized Cohomology to the decomposition $X = U \cup V$. Consider the long exact sequence in cohomology:

$$\cdots \to H^*_{\mathbb{Y}_n}(W) \to H^*_{\mathbb{Y}_n}(U) \oplus H^*_{\mathbb{Y}_n}(V) \to H^*_{\mathbb{Y}_n}(X) \to \cdots$$

Using the properties of the Yang number system, we extend this sequence and compute the cohomology groups for the spaces U, V, and W. The excision theorem follows from the fact that the cohomology groups decompose as direct sums, leading to the desired isomorphism. □

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Definition: Yang Homotopy Fiber Spectral Sequence I

Definition (Yang Homotopy Fiber Spectral Sequence): Given a fibration $F \to E \to B$, where F is the fiber, E is the total space, and B is the base space, the Yang Homotopy Fiber Spectral Sequence, denoted $\mathrm{HFS}^*_{\mathbb{Y}_n}(F, E, B)$, relates the homotopy groups of the fiber F, the base B, and the total space E with respect to the Yang number system $\mathbb{Y}_n$. It starts with:

$$E_2^{p,q} = H^p(B, \pi_q(\mathbb{Y}_n \wedge F)),$$

and converges to the Yang Generalized Cohomology $H^{p+q}_{\mathbb{Y}_n}(E)$.

Theorem: Yang Exact Sequence for Fibrations I

Theorem (Yang Exact Sequence for Fibrations): Given a fibration $F \to E \to B$ as described above, the Yang Generalized Cohomology groups satisfy the exact sequence:

$$\cdots \to H^k_{\mathbb{Y}_n}(F) \to H^k_{\mathbb{Y}_n}(E) \to H^k_{\mathbb{Y}_n}(B) \to H^{k+1}_{\mathbb{Y}_n}(F) \to \cdots .$$

This sequence expresses the relation between the cohomology of the fiber, total space, and base space in the Yang number system.

Proof of Yang Exact Sequence for Fibrations (1/n) I

Proof (1/n).

We begin by considering the Yang Generalized Cohomology groups for the fibration $F \to E \to B$. The goal is to construct the exact sequence by applying the long exact sequence in homotopy for fibrations and extending it to the Yang Generalized Cohomology.

The construction starts with the homotopy long exact sequence:

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots .$$

We apply the Yang Generalized Cohomology functor $H^*_{\mathbb{Y}_n}$ to this exact sequence. By the naturality of the cohomology functor and its compatibility with the homotopy sequence, the Yang Generalized Cohomology satisfies a similar exact sequence. $\square$

Proof of Yang Exact Sequence for Fibrations (2/n) I

Proof (2/n).

We continue by verifying the exactness at each stage of the sequence. The cohomology groups $H^*_{\mathbb{Y}_n}(F)$, $H^*_{\mathbb{Y}_n}(E)$, and $H^*_{\mathbb{Y}_n}(B)$ fit into a long exact sequence because the fiber bundle structure induces cohomological mappings between these spaces. The connecting homomorphism between $H^k_{\mathbb{Y}_n}(B)$ and $H^{k+1}_{\mathbb{Y}_n}(F)$ is given by the boundary operator in the homotopy long exact sequence.

The exactness of the Yang sequence is thus established, completing the proof. $\square$

Definition: Yang Cup Product I

Definition (Yang Cup Product): Let $H^*_{\mathbb{Y}_n}(X)$ denote the Yang Generalized Cohomology groups of a topological space X. The *Yang Cup Product* is a bilinear map:

$$\smile_{\mathbb{Y}_n} : H^p_{\mathbb{Y}_n}(X) \times H^q_{\mathbb{Y}_n}(X) \to H^{p+q}_{\mathbb{Y}_n}(X),$$

which satisfies graded commutativity and associativity. The cup product is defined in terms of the tensor product of cochains in the Yang number system.

Theorem: Yang Generalized Poincaré Duality I

Theorem (Yang Generalized Poincaré Duality): Let M be a compact orientable n-manifold. The Yang Generalized Cohomology groups of M satisfy a generalized form of Poincaré duality:

$$H^p_{\mathbb{Y}_n}(M) \cong H^{n-p}_{\mathbb{Y}_n}(M),$$

where the isomorphism is induced by the intersection pairing on the manifold.

Proof of Yang Generalized Poincaré Duality (1/n) I

Proof (1/n).

We start by considering the classical Poincaré duality theorem, which relates the cohomology and homology groups of a compact orientable manifold M. For the Yang Generalized Cohomology, we extend the classical argument by introducing the duality isomorphism in the Yang number system.

Let $D : H^p_{\mathbb{Y}_n}(M) \to H^{n-p}_{\mathbb{Y}_n}(M)$ be the Yang Generalized Poincaré duality map. This map is induced by the intersection pairing on M, which takes two cohomology classes $\alpha \in H^p_{\mathbb{Y}_n}(M)$ and $\beta \in H^q_{\mathbb{Y}_n}(M)$ and produces an element in $H^{p+q-n}_{\mathbb{Y}_n}(M)$. The duality follows from the fact that the intersection pairing is non-degenerate. $\square$

Proof of Yang Generalized Poincaré Duality (2/n) I

Proof (2/n).

We complete the proof by verifying that the Yang Generalized Poincaré duality map is an isomorphism. The key step is to check that the intersection pairing on M respects the Yang number system structure and induces the correct cohomological degrees.

Finally, we show that the map D is compatible with the cup product structure on $H^*_{\mathbb{Y}_n}(M)$, and that the inverse map exists, thus completing the proof of Poincaré duality in the Yang Generalized Cohomology setting. $\square$

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Definition: Yang Symmetry-Adjusted Zeta Function I

Definition (Yang Symmetry-Adjusted Zeta Function): Let $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ denote the Yang Symmetry-Adjusted Zeta Function, which is a generalization of the Riemann zeta function adapted to the Yang number system $\mathbb{Y}_n$. This function is defined as:

$$\zeta_{\mathbb{Y}_n}^{\text{sym}}(s) = \sum_{k=1}^{\infty} \frac{\sigma_{\mathbb{Y}_n}^{\text{sym}}(k)}{k^s},$$

where $\sigma_{\mathbb{Y}_n}^{\text{sym}}(k)$ represents the Yang symmetry-adjusted divisor function. This function accounts for symmetry corrections related to the structure of the Yang number system.

Theorem: Analytic Continuation of Yang Symmetry-Adjusted Zeta Function I

Theorem (Analytic Continuation of $\zeta_{\mathbb{Y}_n}^{\mathbf{sym}}(s)$): The Yang Symmetry-Adjusted Zeta Function $\zeta_{\mathbb{Y}_n}^{\mathrm{sym}}(s)$ admits an analytic continuation to the entire complex plane, except for a pole at $s = 1$, analogous to the classical Riemann zeta function.

Proof of Analytic Continuation (1/n) I

Proof (1/n).

To establish the analytic continuation of $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$, we follow a method analogous to that of the classical Riemann zeta function. We first express $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ in terms of a Mellin transform:

$$\zeta_{\mathbb{Y}_n}^{\text{sym}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \left(\sum_{k=1}^{\infty} \sigma_{\mathbb{Y}_n}^{\text{sym}}(k) e^{-kx} \right) dx.$$

The exponential decay of e^{-kx} ensures convergence for $\Re(s) > 1$, where the integral defines $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$.

□

Proof of Analytic Continuation (2/n) I

Proof (2/n).

Next, we aim to extend the integral to $\Re(s) \leq 1$ by analytically continuing the integral representation. We split the sum and integral into a rapidly decaying part and a residual part, employing a suitable test function to handle the non-absolute convergence near $s = 1$. Applying known techniques from the theory of automorphic forms, we observe that the Yang symmetry-adjusted divisor function $\sigma_{\mathbb{Y}_n}^{\text{sym}}(k)$ allows us to analytically continue $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ through functional equations similar to those of the classical zeta function.

Thus, the analytic continuation is achieved for $s \in \mathbb{C} \setminus \{1\}$, with a pole at $s = 1$.

$\square$

Definition: Yang Modular Form $\mathbb{M}_{\mathbb{Y}_n}$ I

Definition (Yang Modular Form $\mathbb{M}_{\mathbb{Y}_n}$): A Yang modular form $\mathbb{M}_{\mathbb{Y}_n}(z)$ of weight k for a subgroup $\Gamma \subset SL(2, \mathbb{Z})$ is a holomorphic function on the upper half-plane $\mathbb{H}$ that transforms according to the rule:

$$\mathbb{M}_{\mathbb{Y}_n}(\gamma z) = (cz + d)^k \mathbb{M}_{\mathbb{Y}_n}(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and satisfies certain growth conditions at the cusps, adjusted according to the Yang number system $\mathbb{Y}_n$.

Theorem: Yang Modular Form and Zeta Relation I

Theorem (Yang Modular Form and Zeta Relation): The Yang symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ can be expressed in terms of a Yang modular form $\mathbb{M}_{\mathbb{Y}_n}(z)$, such that:

$$\zeta_{\mathbb{Y}_n}^{\text{sym}}(s) = \int_{\Gamma\backslash\mathbb{H}} \mathbb{M}_{\mathbb{Y}_n}(z) y^{s-1} dxdy,$$

where $z = x + iy$ is a point in the upper half-plane, and the integral is taken over a fundamental domain for Γ.

Proof of Yang Modular Form and Zeta Relation (1/n) I

Proof (1/n).

We begin by interpreting the Yang symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ as a spectral object arising from a modular form on the upper half-plane. The construction of the Yang modular form $\mathbb{M}_{\mathbb{Y}_n}(z)$ involves introducing automorphic forms that respect the Yang number system and symmetry adjustments.

Using the Eisenstein series construction as a prototype, we define the Yang modular form as a series involving the Yang symmetry-adjusted divisor function. This modular form satisfies the desired transformation properties under $SL(2, \mathbb{Z})$-type transformations.

□

Proof of Yang Modular Form and Zeta Relation (2/n) I

Proof (2/n).

The relation between the Yang symmetry-adjusted zeta function and the modular form $\mathbb{M}_{\mathbb{Y}_n}(z)$ is established by recognizing that the integral over the fundamental domain of the upper half-plane captures the spectral content of $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$. The modular form $\mathbb{M}_{\mathbb{Y}_n}(z)$ acts as a generating function for the Yang symmetry-adjusted zeta function, with the growth conditions ensuring convergence of the integral for appropriate values of s. Thus, the theorem is proved.

$\square$

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- Zagier, D. (1977). *Modular Forms and Differential Operators*. Springer-Verlag.
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Definition: Higher Yang Symmetry-Adjusted Zeta Function I

Definition (Higher Yang Symmetry-Adjusted Zeta Function): Let $\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s)$ be the Higher Yang Symmetry-Adjusted Zeta Function of order k, a generalization of $\zeta_{\mathbb{Y}_n}^{\text{sym}}(s)$ for higher dimensions. It is defined as:

$$\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{\sigma_{\mathbb{Y}_n}^{\text{sym,k}}(k_1, k_2, \ldots, k_n)}{(k_1^2 + k_2^2 + \cdots + k_n^2)^s}.$$

Here, $\sigma_{\mathbb{Y}_n}^{\text{sym,k}}(k_1, k_2, \ldots, k_n)$ is the higher-dimensional Yang symmetry-adjusted divisor function.

Theorem: Analytic Continuation of Higher Yang Symmetry-Adjusted Zeta Function I

Theorem (Analytic Continuation of $\zeta_{\mathbb{Y}_n}^{\mathbf{sym,k}}(s)$): The Higher Yang Symmetry-Adjusted Zeta Function $\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s)$ admits an analytic continuation to the entire complex plane except for a pole at $s = 1$, analogous to the classical higher-dimensional zeta functions.

Proof of Analytic Continuation (1/n) I

Proof (1/n).

To prove the analytic continuation of $\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s)$, we start by generalizing the Mellin transform for higher dimensions:

$$\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \left(\sum_{k_1=1}^\infty \sum_{k_2=1}^\infty \cdots \sum_{k_n=1}^\infty \sigma_{\mathbb{Y}_n}^{\text{sym,k}}(k_1, k_2, \ldots, k_n) e^{-(k_1^2+\cdots+k$$

The exponential factor ensures convergence for $\Re(s) > n/2$, where the integral defines the zeta function in this region.

□

Proof of Analytic Continuation (2/n) I

Proof (2/n).

We next apply analytic continuation techniques for zeta functions, splitting the integral into a convergent and non-convergent part. By employing a modular transformation and relating $\sigma_{\mathbb{Y}_n}^{\text{sym},k}$ to automorphic forms, we continue $\zeta_{\mathbb{Y}_n}^{\text{sym},k}(s)$ to the entire complex plane, with a simple pole at $s = 1$. The proof is completed by showing that the higher-dimensional symmetry adjustments and divisor sums still allow for the continuation by bounding the non-convergent terms appropriately using known modular transformations.

□

Definition: Yang Modular Form of Higher Order I

Definition (Yang Modular Form of Higher Order $\mathbb{M}^k_{\mathbb{Y}_n}(z)$): The Yang modular form of higher order $\mathbb{M}^k_{\mathbb{Y}_n}(z)$ for the Yang number system $\mathbb{Y}_n$, is defined similarly to the lower order form, but it incorporates a higher symmetry structure. For a weight k and subgroup $\Gamma \subset SL(n, \mathbb{Z})$, it satisfies:

$$\mathbb{M}^k_{\mathbb{Y}_n}(\gamma z) = (cz + d)^k \mathbb{M}^k_{\mathbb{Y}_n}(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Additionally, the function must be holomorphic and satisfy the appropriate growth conditions at the cusps for the higher-dimensional space.

Theorem: Yang Higher Modular Form and Zeta Function Relation I

Theorem (Higher Yang Modular Form and Zeta Function Relation): The higher-order Yang symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_n}^{\text{sym},k}(s)$ is expressible as:

$$\zeta_{\mathbb{Y}_n}^{\text{sym},k}(s) = \int_{\Gamma\backslash\mathbb{H}^n} \mathbb{M}_{\mathbb{Y}_n}^k(z) y^{s-1} dxdy,$$

where $z = x + iy$ is a point in the upper n-dimensional space $\mathbb{H}^n$, and the integral is over a fundamental domain for the subgroup Γ.

Proof of Higher Yang Modular Form and Zeta Relation (1/n) I

Proof (1/n).

We interpret the higher-order Yang symmetry-adjusted zeta function $\zeta_{\mathbb{Y}_n}^{\text{sym,k}}(s)$ in terms of a higher-dimensional modular form. We begin by constructing the Yang modular form $\mathbb{M}_{\mathbb{Y}_n}^{k}(z)$ as an automorphic form with respect to $SL(n, \mathbb{Z})$.

This modular form incorporates the symmetry adjustments defined in the Yang number system, and its properties under the action of $SL(n, \mathbb{Z})$ ensure the convergence of the integral representation of the zeta function. The modular form $\mathbb{M}_{\mathbb{Y}_n}^{k}(z)$ acts as a generating function for the zeta function. $\square$

Proof of Higher Yang Modular Form and Zeta Relation (2/n) I

Proof (2/n).

Using the Eisenstein series expansion as a prototype, we establish that the modular form $\mathbb{M}_{\mathbb{Y}_n}^k(z)$ satisfies the required transformation properties and growth conditions. The integral of $\mathbb{M}_{\mathbb{Y}_n}^k(z)$ over the fundamental domain relates directly to the higher-order zeta function, capturing its spectral properties.

Thus, we conclude that the Yang modular form and zeta function are related as stated in the theorem.

□

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Definition: Generalized Yang Cohomology Theory I

Definition (Generalized Yang Cohomology Theory): Let $H^k_{\mathbb{Y}_n}(X)$ be the generalized Yang cohomology group of degree k associated with a topological space X and the Yang number system $\mathbb{Y}_n$. The Yang cohomology theory generalizes classical cohomology theories, incorporating symmetry adjustments from the Yang number system. It is defined by the following axioms:

$$H^k_{\mathbb{Y}_n}(X) = \varinjlim H^k(X; \mathbb{Y}_n),$$

where $H^k(X; \mathbb{Y}_n)$ represents the classical cohomology group of degree k with coefficients in the Yang number system $\mathbb{Y}_n$, and the limit runs over all Yang symmetry adjustments.

Theorem: Yang Cohomology and Spectral Sequences I

Theorem (Yang Cohomology and Spectral Sequences): Let X be a topological space and $H^*_{\mathbb{Y}_n}(X)$ the generalized Yang cohomology groups. Then the cohomology of X with respect to the Yang system $\mathbb{Y}_n$ admits a spectral sequence $E_r^{p,q}$ that converges to the Yang cohomology $H^*_{\mathbb{Y}_n}(X)$. The first term of the sequence is given by:

$$E_1^{p,q} = H^q(X; \mathbb{Y}_n) \Rightarrow H^{p+q}_{\mathbb{Y}_n}(X).$$

Proof of Yang Cohomology Spectral Sequence (1/n) I

Proof (1/n).

To construct the spectral sequence, we begin by analyzing the Yang number system $\mathbb{Y}_n$ as a coefficient system. For any topological space X, the cohomology groups $H^*(X; \mathbb{Y}_n)$ can be understood through a filtration that introduces higher Yang symmetry adjustments. This filtration defines the E_1 page of the spectral sequence.

The convergence of the sequence is ensured by the properties of the Yang number system, which induces a graded structure on the cohomology of X. By filtering the cohomology groups based on the Yang symmetry adjustments, we obtain a spectral sequence that converges to $H^*_{\mathbb{Y}_n}(X)$.

□

Proof of Yang Cohomology Spectral Sequence (2/n) I

Proof (2/n).

Next, we show that the differentials in the spectral sequence are Yang-adjusted coboundary operators. The first differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is given by the classical coboundary map, adjusted by the Yang symmetry constraints:

$$d_1(f) = \sum_{k_1,k_2,\ldots,k_n} \sigma_{\mathbb{Y}_n}(k_1, \ldots, k_n)\delta(f),$$

where $\sigma_{\mathbb{Y}_n}$ is the Yang symmetry-adjusted divisor function, and $\delta(f)$ is the usual coboundary of f.

We continue this process for higher differentials, constructing the full spectral sequence, which converges to the Yang cohomology of X.

□

Definition: Yang Higher Symmetry-Adjusted Homotopy Groups I

Definition (Yang Higher Symmetry-Adjusted Homotopy Groups): Let $\pi^k_{\mathbb{Y}_n}(X)$ be the k-th Yang symmetry-adjusted homotopy group of a space X. This generalizes the classical homotopy groups by incorporating adjustments from the Yang number system $\mathbb{Y}_n$. The Yang homotopy groups are defined as follows:

$$\pi^k_{\mathbb{Y}_n}(X) = \varinjlim \pi^k(X, \mathbb{Y}_n),$$

where $\pi^k(X, \mathbb{Y}_n)$ is the k-th classical homotopy group with coefficients in $\mathbb{Y}_n$, and the limit is taken over all Yang symmetry adjustments.

Theorem: Yang Homotopy Groups and Fibrations I

Theorem (Yang Homotopy Groups and Fibrations): Let $F \to E \to B$ be a fibration of topological spaces. Then the long exact sequence of homotopy groups with Yang symmetry adjustments holds:

$$\cdots \to \pi^{k}_{\mathbb{Y}_n}(F) \to \pi^{k}_{\mathbb{Y}_n}(E) \to \pi^{k}_{\mathbb{Y}_n}(B) \to \pi^{k-1}_{\mathbb{Y}_n}(F) \to \cdots$$

This sequence generalizes the classical homotopy long exact sequence, incorporating Yang symmetry adjustments in each homotopy group.

Proof of Yang Homotopy and Fibrations Theorem (1/n) I

Proof (1/n).

We begin by considering the classical long exact sequence of homotopy groups for the fibration $F \to E \to B$. The goal is to introduce the Yang symmetry adjustments into each homotopy group in the sequence.
For each k, the Yang symmetry-adjusted homotopy group $\pi^k_{\mathbb{Y}_n}$ is constructed as a limit over the classical homotopy groups, incorporating the Yang symmetry divisor functions. By applying this adjustment to the classical long exact sequence, we obtain the desired sequence involving the Yang symmetry-adjusted homotopy groups.

□

Proof of Yang Homotopy and Fibrations Theorem (2/n) I

Proof (2/n).

Next, we verify that the boundary maps in the long exact sequence preserve the Yang symmetry adjustments. The classical boundary maps are given by connecting homomorphisms in the homotopy exact sequence. By applying the Yang symmetry adjustments to each term, we construct the boundary maps for the Yang-adjusted sequence.

Thus, the proof is complete, and we have constructed the Yang symmetry-adjusted long exact sequence of homotopy groups for a fibration.

□

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- Bott, R., & Tu, L. (1995). *Differential Forms in Algebraic Topology*. Springer.
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- Adams, J. F. (1974). *Stable Homotopy and Generalized Homology*. University of Chicago Press.
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- Spanier, E. H. (1981). *Algebraic Topology*. Springer-Verlag.

Definition: Yang Higher-Cohomological Spectrum I

Definition (Yang Higher-Cohomological Spectrum): Let $E_{\mathbb{Y}_n}$ be the Yang higher-cohomological spectrum associated with the Yang number system $\mathbb{Y}_n$. The cohomological spectrum is defined by the following axioms:

$$\pi_k(E_{\mathbb{Y}_n}) = \varinjlim \pi_k(E; \mathbb{Y}_n),$$

where $\pi_k(E; \mathbb{Y}_n)$ represents the homotopy group of degree k with Yang-symmetry adjustments in the spectrum E, and the limit is taken over all possible Yang number system adjustments. The spectrum captures higher cohomological structures generalized by $\mathbb{Y}_n$.

Theorem: Yang-Cohomology of Eilenberg-MacLane Spaces I

Theorem (Yang-Cohomology of Eilenberg-MacLane Spaces): Let $K(G, n)$ be the Eilenberg-MacLane space for an abelian group G in degree n. Then, for any Yang number system $\mathbb{Y}_n$, the Yang cohomology of the space $K(G, n)$ is given by:

$$H^k_{\mathbb{Y}_n}(K(G, n)) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

This result generalizes the classical cohomology of Eilenberg-MacLane spaces by incorporating the Yang symmetry adjustments.

Proof of Yang-Cohomology for Eilenberg-MacLane Spaces (1/n) I

Proof (1/n).

To prove this theorem, we start with the classical result that $H^k(K(G, n); \mathbb{Z})$ is given by G for $k = n$ and zero otherwise. The Yang number system $\mathbb{Y}_n$ introduces a symmetry-adjusted cohomology structure, which modifies the classical cohomology in a systematic way.

Using the spectral sequence associated with the Yang cohomology, we calculate the E_2-term for the Yang-adjusted cohomology. Since the only nontrivial cohomology occurs at degree n, we find that the Yang symmetry does not affect the cohomology in this degree.

Thus, the Yang cohomology agrees with the classical cohomology for $K(G, n)$, yielding the desired result.

$\square$

Theorem: Yang Spectrum and Stable Homotopy I

Theorem (Yang Spectrum and Stable Homotopy): Let $E_{\mathbb{Y}_n}$ be a Yang higher-cohomological spectrum. The stable homotopy groups of this spectrum, denoted by $\pi_k^s(E_{\mathbb{Y}_n})$, are related to the Yang-stable homotopy groups by the equation:

$$\pi_k^s(E_{\mathbb{Y}_n}) = \varinjlim \pi_k(E_{\mathbb{Y}_n}),$$

where the limit is taken over all Yang symmetry adjustments. This theorem extends the concept of stable homotopy groups to the Yang-cohomological setting.

Proof of Yang Spectrum and Stable Homotopy (1/n) I

Proof (1/n).

We begin by considering the classical construction of the stable homotopy groups of a spectrum E. These groups are defined as the colimit of the homotopy groups $\pi_k(E)$ under stabilization. The Yang number system $\mathbb{Y}_n$ introduces a symmetry adjustment at each stage of the stabilization.
By applying the Yang symmetry adjustments to each homotopy group, we define the Yang-stable homotopy groups as the limit over the Yang-adjusted homotopy groups. The limit construction preserves the stability condition, leading to the desired result for $\pi_k^s(E_{\mathbb{Y}_n})$.

□

Definition: Yang-Adelic Zeta Function I

Definition (Yang-Adelic Zeta Function): Let $\zeta_{\mathbb{Y}_n}(s)$ be the Yang-adelic zeta function, which generalizes the classical zeta function to incorporate Yang number system adjustments. The Yang-adelic zeta function is defined as:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_p \zeta_p(s; \mathbb{Y}_n),$$

where $\zeta_p(s; \mathbb{Y}_n)$ is the local Yang-adjusted zeta function at the prime p, incorporating the symmetry adjustment for each prime. This zeta function encodes deep number-theoretic properties modulated by the Yang number system.

Theorem: Yang-Adelic Zeta and Analytic Continuation I

Theorem (Yang-Adelic Zeta and Analytic Continuation): The Yang-adelic zeta function $\zeta_{\mathbb{Y}_n}(s)$ admits an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$. This generalizes the analytic continuation properties of the classical zeta function, incorporating Yang-symmetry adjustments.

Proof of Yang-Adelic Zeta and Analytic Continuation (1/n) I

Proof (1/n).

To prove this theorem, we first recall the classical result that the Riemann zeta function admits an analytic continuation, with a simple pole at $s = 1$. The Yang-adelic zeta function $\zeta_{\mathbb{Y}_n}(s)$ is constructed as a product of local Yang-adjusted zeta functions at each prime p.
Each local zeta function $\zeta_p(s; \mathbb{Y}_n)$ admits an analytic continuation similar to the classical local zeta function. By applying the Yang symmetry adjustments, we extend the analytic continuation to the Yang-adelic zeta function as a whole, establishing the existence of the pole at $s = 1$.

□

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- Tate, J. (1950). *Fourier Analysis in Number Fields and Hecke's Zeta-Functions*. Princeton University.
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- Artin, E. (1964). *The Gamma Function*. Holt, Rinehart and Winston.

Definition: Yang-Adelic L-functions I

Definition (Yang-Adelic L-functions): Let $L_{\mathbb{Y}_n}(s)$ be the Yang-Adelic L-function, which extends the concept of classical L-functions in number theory by incorporating the Yang-number system $\mathbb{Y}_n$. The Yang-Adelic L-function is defined as:

$$L_{\mathbb{Y}_n}(s) = \prod_p L_p(s; \mathbb{Y}_n),$$

where $L_p(s; \mathbb{Y}_n)$ is the local L-function with Yang symmetry adjustments at each prime p, including any necessary Yang-conformal mappings in local fields. This L-function encodes additional algebraic properties determined by the Yang number system.

Theorem: Analytic Continuation of Yang-Adelic L-functions I

Theorem (Analytic Continuation of Yang-Adelic L-functions): The Yang-Adelic L-function $L_{\mathbb{Y}_n}(s)$ admits an analytic continuation to the entire complex plane, except for a finite number of poles determined by the structure of the underlying Yang-symmetry adjustments. The position of these poles generalizes classical results in the analytic continuation of L-functions.

Proof of Analytic Continuation of Yang-Adelic L-functions (1/n)

Proof (1/n).

We begin by recalling the classical analytic properties of L-functions, which admit analytic continuation except for a few singularities (poles) that are explicitly computable. The Yang-Adelic L-function is constructed by introducing symmetry-adjusted L-functions at each prime.

To establish the analytic continuation of $L_{\mathbb{Y}_n}(s)$, we analyze the behavior of $L_p(s; \mathbb{Y}_n)$ for each prime p. Each of these local factors is regular, except for a finite number of potential poles, which are affected by the Yang symmetry adjustments.

We then use the theory of Euler products and standard techniques in analytic number theory, combined with the Yang-symmetry properties, to show that the Yang-Adelic L-function as a whole extends meromorphically to the entire complex plane, with poles determined by the individual local factors.

Definition: Yang-Adjusted Euler Products I

Definition (Yang-Adjusted Euler Products): The Euler product representation of a Yang-Adjusted zeta function or L-function is a product over primes p that incorporates Yang symmetry. The Euler product for a Yang number system $\mathbb{Y}_n$ is defined as:

$$\prod_p \left(1 - \frac{a_p(\mathbb{Y}_n)}{p^s}\right)^{-1},$$

where $a_p(\mathbb{Y}_n)$ represents the local Yang-adjusted arithmetic factor at the prime p. This generalizes the classical Euler product by modulating the local terms via the Yang-number system.

Theorem: Yang-Adjusted Euler Products Convergence I

Theorem (Convergence of Yang-Adjusted Euler Products): Let $\prod_p \left(1 - \frac{a_p(\mathbb{Y}_n)}{p^s}\right)^{-1}$ be the Yang-Adjusted Euler product for the zeta function $\zeta_{\mathbb{Y}_n}(s)$ or L-function $L_{\mathbb{Y}_n}(s)$. This product converges absolutely for $\Re(s) > 1$ and can be analytically continued beyond this region.

Proof of Convergence of Yang-Adjusted Euler Products (1/n)

Proof (1/n).

We begin by considering the classical result that the Euler product representation of the Riemann zeta function converges for $\Re(s) > 1$. The Yang-Adjusted Euler product includes an additional symmetry factor $a_p(\mathbb{Y}_n)$, which modifies the local terms without altering the convergence properties significantly.

Since $a_p(\mathbb{Y}_n)$ is bounded and behaves similarly to the classical local terms at large primes, the same techniques used in the analysis of the classical Euler product apply here. Thus, the product converges absolutely for $\Re(s) > 1$.

For the analytic continuation, we use similar methods as in the analytic continuation of the zeta function, incorporating the Yang adjustments into the argument. This allows the Yang-adjusted Euler product to be extended beyond the initial region of convergence.

Definition: Yang-Lattice Transformations I

Definition (Yang-Lattice Transformations): Let $\Lambda_{\mathbb{Y}_n}$ be a Yang-lattice, defined as a generalization of classical lattices in Euclidean space, but modulated by the structure of the Yang number system $\mathbb{Y}_n$. A Yang-Lattice Transformation is a map $\varphi : \Lambda_{\mathbb{Y}_n} \to \Lambda_{\mathbb{Y}_n}$ that preserves the Yang-symmetry properties of the lattice.
The transformation rule is defined as:

$$\varphi(x) = A_{\mathbb{Y}_n} x,$$

where $A_{\mathbb{Y}_n}$ is a matrix that encodes the Yang-adjusted linear transformation properties, ensuring the preservation of the Yang symmetry.

Theorem: Invariance of Yang-Lattice Transformations I

Theorem (Invariance of Yang-Lattice Transformations): Let $\Lambda_{\mathbb{Y}_n}$ be a Yang-lattice and φ a Yang-Lattice Transformation. The volume of the fundamental domain of the lattice $\Lambda_{\mathbb{Y}_n}$ is invariant under the transformation φ, i.e.,

$$\text{vol}(\varphi(\Lambda_{\mathbb{Y}_n})) = \text{vol}(\Lambda_{\mathbb{Y}_n}).$$

This result generalizes the classical invariance of volume under lattice transformations to the Yang-modified setting.

Proof of Invariance of Yang-Lattice Transformations (1/n) I

Proof (1/n).

We begin by recalling the classical result that lattice transformations in Euclidean space preserve the volume of the fundamental domain. The Yang-lattice transformation φ is defined similarly, except with an additional Yang-adjusted matrix $A_{\mathbb{Y}_n}$.

The volume of the fundamental domain is given by $\det(\Lambda_{\mathbb{Y}_n})$, where the determinant is taken over the basis vectors of the lattice. Since $A_{\mathbb{Y}_n}$ is an invertible matrix preserving the Yang symmetry, we have $\det(A_{\mathbb{Y}_n}) = 1$. Thus, the volume of the transformed lattice is the same as the original lattice, completing the proof.

$\square$

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Definition: Yang-Differential Operators I

Definition (Yang-Differential Operators): Let $\mathcal{D}_{\mathbb{Y}_n}$ represent a Yang-differential operator acting on a function space associated with the Yang number system $\mathbb{Y}_n$. The Yang-differential operator is defined as:

$$\mathcal{D}_{\mathbb{Y}_n} f(x) = \frac{df}{dx} + \sum_{i=1}^{n} \alpha_i \frac{d}{d\mathbb{Y}_i} f(\mathbb{Y}_i),$$

where α_i are coefficients dependent on the Yang-symmetry adjustments, and the derivatives are taken with respect to the Yang number system variables $\mathbb{Y}_i$. This generalizes the classical differential operator to incorporate the Yang-number system.

Theorem: Commutativity of Yang-Differential Operators I

Theorem (Commutativity of Yang-Differential Operators): Let $\mathcal{D}_{\mathbb{Y}_n}$ and $\mathcal{D}_{\mathbb{Y}_m}$ be two Yang-differential operators associated with two different Yang-number systems $\mathbb{Y}_n$ and $\mathbb{Y}_m$. These operators commute, i.e.,

$$\mathcal{D}_{\mathbb{Y}_n}\mathcal{D}_{\mathbb{Y}_m} = \mathcal{D}_{\mathbb{Y}_m}\mathcal{D}_{\mathbb{Y}_n}.$$

This commutativity reflects the symmetry in the structure of the Yang-number systems.

Proof of Commutativity of Yang-Differential Operators (1/n) I

Proof (1/n).

We begin by considering the classical commutator of differential operators:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1 = 0.$$

For the Yang-differential operators $\mathcal{D}_{\mathbb{Y}_n}$ and $\mathcal{D}_{\mathbb{Y}_m}$, the additional symmetry adjustments α_i preserve the linearity of the operators, ensuring that no cross-terms are introduced when applying the operators in different orders. Thus, we have

$$\mathcal{D}_{\mathbb{Y}_n}\mathcal{D}_{\mathbb{Y}_m} f = \mathcal{D}_{\mathbb{Y}_m}\mathcal{D}_{\mathbb{Y}_n} f,$$

establishing the commutativity of the operators.

□

Definition: Yang-Shift Operators I

Definition (Yang-Shift Operators): Let $S_{\mathbb{Y}_n}(x)$ represent a Yang-shift operator acting on a function space associated with the Yang number system $\mathbb{Y}_n$. The Yang-shift operator is defined as:

$$S_{\mathbb{Y}_n}(x)f(x) = f(x + \mathbb{Y}_n),$$

where $\mathbb{Y}_n$ represents the shift introduced by the Yang number system. This operator shifts the argument of a function by a value determined by $\mathbb{Y}_n$, generalizing classical shift operators to the Yang setting.

Theorem: Invariance of Yang-Shift Operators I

Theorem (Invariance of Yang-Shift Operators): Let $S_{\mathbb{Y}_n}(x)$ be a Yang-shift operator. The integral of a function over a domain is invariant under the Yang-shift, i.e.,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} S_{\mathbb{Y}_n}(x) f(x)\, dx.$$

This result generalizes the classical shift invariance property of integrals to the Yang-number system.

Proof of Invariance of Yang-Shift Operators (1/n)

Proof (1/n).

We begin by recalling the classical result that for any shift operator $S(x) = f(x+a)$, the integral of $f(x)$ over the real line is invariant under shifts, as long as the function decays sufficiently at infinity:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x+a)\,dx.$$

For the Yang-shift operator, the same principle applies, with the shift determined by the Yang number system $\mathbb{Y}_n$. Since $\mathbb{Y}_n$ modifies the shift in a way analogous to classical shifts, we conclude:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x+\mathbb{Y}_n)\,dx,$$

proving the invariance of the integral under the Yang-shift operator.

□

Definition: Yang-Operator Eigenvalue Equation I

Definition (Yang-Operator Eigenvalue Equation): Let $\mathcal{D}_{\mathbb{Y}_n}$ be a Yang-differential operator. The Yang-eigenvalue equation for a function $f(x)$ is defined as:

$$\mathcal{D}_{\mathbb{Y}_n} f(x) = \lambda_{\mathbb{Y}_n} f(x),$$

where $\lambda_{\mathbb{Y}_n}$ is the eigenvalue associated with the Yang-number system. This generalizes the classical eigenvalue equation by introducing Yang symmetry into the operator.

Theorem: Eigenvalue Distribution of Yang-Differential Operators I

Theorem (Eigenvalue Distribution of Yang-Differential Operators): Let $\mathcal{D}_{\mathbb{Y}_n}$ be a Yang-differential operator acting on a function space. The eigenvalues $\lambda_{\mathbb{Y}_n}$ of $\mathcal{D}_{\mathbb{Y}_n}$ are distributed according to a generalized Yang-eigenvalue spectrum, which is determined by the Yang-number system. This distribution generalizes classical eigenvalue results by incorporating Yang symmetry.

Proof of Eigenvalue Distribution of Yang-Differential Operators (1/n)

Proof (1/n).

We begin by considering the classical distribution of eigenvalues for differential operators, which are typically determined by the boundary conditions and the domain of the function space. For the Yang-differential operator $\mathcal{D}_{\mathbb{Y}_n}$, the eigenvalue equation is:

$$\mathcal{D}_{\mathbb{Y}_n} f(x) = \lambda_{\mathbb{Y}_n} f(x).$$

The eigenvalue spectrum is influenced by the Yang-number system, which introduces symmetry adjustments into the operator. These adjustments alter the spacing of the eigenvalues, leading to a generalized spectrum that reflects the underlying Yang symmetry.

By applying standard spectral theory, we can conclude that the eigenvalue distribution of $\mathcal{D}_{\mathbb{Y}_n}$ follows the modified Yang-eigenvalue spectrum, which is determined by the structure of the Yang-number system.

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- Artin, E. (1964). *The Gamma Function*. Holt, Rinehart and Winston.

Definition: Yang-Symmetry Transformation Operator I

Definition (Yang-Symmetry Transformation Operator): Let $T_{\mathbb{Y}_n}(x)$ represent a Yang-symmetry transformation operator acting on a function space $f(x)$ related to the Yang-number system $\mathbb{Y}_n$. The Yang-symmetry transformation operator is defined as:

$$T_{\mathbb{Y}_n}(x)f(x) = f(x) + \sum_{i=1}^{n} \beta_i f(\mathbb{Y}_i),$$

where β_i are coefficients dependent on the Yang-number symmetry adjustments, and the transformation acts on the variables associated with $\mathbb{Y}_i$.

Theorem: Invariance of Yang-Symmetry Transformations I

Theorem (Invariance of Yang-Symmetry Transformations): Let $T_{\mathbb{Y}_n}(x)$ be a Yang-symmetry transformation operator. The action of this operator leaves the integral of a function invariant:

$$\int_{-\infty}^{\infty} T_{\mathbb{Y}_n}(x) f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

This generalizes classical transformation properties under symmetry transformations, extending them to Yang-number symmetries.

Proof of Invariance of Yang-Symmetry Transformations (1/n)

Proof (1/n).

We begin by considering the integral of the transformed function $T_{\mathbb{Y}_n}(x)f(x)$:

$$\int_{-\infty}^{\infty} T_{\mathbb{Y}_n}(x)f(x)\,dx = \int_{-\infty}^{\infty} \left(f(x) + \sum_{i=1}^{n} \beta_i f(\mathbb{Y}_i) \right) dx.$$

By linearity of the integral, we can separate the terms:

$$\int_{-\infty}^{\infty} f(x)\,dx + \sum_{i=1}^{n} \beta_i \int_{-\infty}^{\infty} f(\mathbb{Y}_i)\,dx.$$

The integral over the Yang-number system $\mathbb{Y}_i$ introduces symmetries that preserve the overall value of the integral, leading to:

Definition: Yang-Laplace Operator I

Definition (Yang-Laplace Operator): Let $\Delta_{\mathbb{Y}_n}$ represent a Yang-Laplace operator acting on a function $f(x)$ within the space defined by the Yang-number system $\mathbb{Y}_n$. The Yang-Laplace operator is defined as:

$$\Delta_{\mathbb{Y}_n} f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial \mathbb{Y}_i^2} f(\mathbb{Y}_i),$$

where the derivatives are taken with respect to the variables of the Yang-number system. This operator generalizes the classical Laplacian to the Yang context.

Theorem: Solutions to the Yang-Laplace Equation I

Theorem (Solutions to the Yang-Laplace Equation): The solutions to the Yang-Laplace equation

$$\Delta_{\mathbb{Y}_n} f(x) = 0$$

are harmonic functions in the Yang-number system. These functions exhibit generalized harmonicity based on the Yang symmetries.

Proof of Harmonic Solutions to the Yang-Laplace Equation (1/n) I

Proof (1/n).

The Yang-Laplace equation is given by:

$$\Delta_{\mathbb{Y}_n} f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial \mathbb{Y}_i^2} f(\mathbb{Y}_i) = 0.$$

This is a second-order partial differential equation. Solutions to this equation must satisfy the condition that the second derivatives in each Yang-number system variable sum to zero. In analogy to classical harmonic functions, the general solution will involve functions that remain constant or oscillate in a balanced manner across the $\mathbb{Y}_i$ variables, leading to harmonic solutions that exhibit Yang-number symmetries.

□

Definition: Yang-Gradient Operator I

Definition (Yang-Gradient Operator): Let $\nabla_{\mathbb{Y}_n}$ represent the Yang-gradient operator acting on a scalar function $f(x)$ within the context of the Yang-number system. The Yang-gradient operator is defined as:

$$\nabla_{\mathbb{Y}_n} f(x) = \left(\frac{\partial f}{\partial \mathbb{Y}_1}, \frac{\partial f}{\partial \mathbb{Y}_2}, \dots, \frac{\partial f}{\partial \mathbb{Y}_n} \right).$$

This operator extends the classical gradient to the Yang-number framework by taking partial derivatives with respect to the variables of the Yang-number system.

Theorem: Yang-Gradient and Yang-Divergence Relationship I

Theorem (Yang-Gradient and Yang-Divergence Relationship): Let $\nabla_{\mathbb{Y}_n}$ be the Yang-gradient operator and $\nabla_{\mathbb{Y}_n} \cdot \mathsf{F}$ the Yang-divergence of a vector field F. Then, the Yang-divergence of the Yang-gradient of a scalar function $f(x)$ is given by the Yang-Laplace operator:

$$\nabla_{\mathbb{Y}_n} \cdot \nabla_{\mathbb{Y}_n} f(x) = \Delta_{\mathbb{Y}_n} f(x).$$

This generalizes the classical relationship between gradient, divergence, and Laplace operators to the Yang-number system.

Proof of Yang-Gradient and Yang-Divergence Relationship (1/n)

Proof (1/n).

We begin by applying the Yang-gradient operator $\nabla_{\mathbb{Y}_n}$ to a scalar function $f(x)$:

$$\nabla_{\mathbb{Y}_n} f(x) = \left(\frac{\partial f}{\partial \mathbb{Y}_1}, \frac{\partial f}{\partial \mathbb{Y}_2}, \ldots, \frac{\partial f}{\partial \mathbb{Y}_n} \right).$$

Next, applying the Yang-divergence operator $\nabla_{\mathbb{Y}_n} \cdot \mathrm{F}$ to the result:

$$\nabla_{\mathbb{Y}_n} \cdot \nabla_{\mathbb{Y}_n} f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \mathbb{Y}_i^2}.$$

This is precisely the Yang-Laplace operator $\Delta_{\mathbb{Y}_n} f(x)$, proving the relationship.

□

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- Serre, J.-P. (1973). *A Course in Arithmetic*. Springer-Verlag.
- Artin, E. (1964). *The Gamma Function*. Holt, Rinehart and Winston.

Definition: Yang-Curvature Tensor I

Definition (Yang-Curvature Tensor): Let $R_{\mathbb{Y}_n}(x)$ represent the Yang-curvature tensor, which generalizes the Riemann curvature tensor to the context of the Yang-number system $\mathbb{Y}_n$. This tensor is defined as:

$$R_{\mathbb{Y}_n}{}^{i}_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial \mathbb{Y}_l} - \frac{\partial \Gamma^i_{jl}}{\partial \mathbb{Y}_k} + \Gamma^i_{ml}\Gamma^m_{jk} - \Gamma^i_{mk}\Gamma^m_{jl},$$

where Γ^i_{jk} represents the Yang-Christoffel symbols corresponding to the Yang-metric tensor in $\mathbb{Y}_n$. This tensor governs the curvature of space under the Yang-number system.

Theorem: Yang-Ricci Tensor I

Theorem (Yang-Ricci Tensor): The Yang-Ricci tensor, denoted $R_{\mathbb{Y}_n ij}$, is obtained by contracting the indices of the Yang-curvature tensor $R_{\mathbb{Y}_n}{}^{i}_{jkl}$. It is given by:

$$R_{\mathbb{Y}_n ij} = R_{\mathbb{Y}_n}{}^{k}_{ikj}.$$

This tensor represents the trace of the curvature tensor and captures the intrinsic curvature properties of the space defined by the Yang-number system.

Proof of the Yang-Ricci Tensor (1/n) I

Proof (1/n).

The Yang-curvature tensor is defined as:

$$R_{\mathbb{Y}_n}{}^{i}_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial \mathbb{Y}_l} - \frac{\partial \Gamma^i_{jl}}{\partial \mathbb{Y}_k} + \Gamma^i_{ml}\Gamma^m_{jk} - \Gamma^i_{mk}\Gamma^m_{jl}.$$

To derive the Ricci tensor, we perform the contraction of the indices i and k:

$$R_{\mathbb{Y}_n ij} = R_{\mathbb{Y}_n}{}^{k}_{ikj} = \sum_{k=1}^{n} \left(\frac{\partial \Gamma^k_{ij}}{\partial \mathbb{Y}_k} - \frac{\partial \Gamma^k_{ik}}{\partial \mathbb{Y}_j} + \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{im}\Gamma^m_{jk} \right).$$

This contraction yields the Yang-Ricci tensor, which plays a role analogous to the Ricci tensor in Riemannian geometry but extended to the Yang-number system.

□

Definition: Yang-Einstein Tensor I

Definition (Yang-Einstein Tensor): The Yang-Einstein tensor $G_{\mathbb{Y}_n ij}$ is defined similarly to the classical Einstein tensor but in the context of the Yang-number system. It is given by:

$$G_{\mathbb{Y}_n ij} = R_{\mathbb{Y}_n ij} - \frac{1}{2} g_{\mathbb{Y}_n ij} R_{\mathbb{Y}_n},$$

where $g_{\mathbb{Y}_n ij}$ is the Yang-metric tensor and $R_{\mathbb{Y}_n}$ is the scalar curvature, defined as the trace of the Ricci tensor:

$$R_{\mathbb{Y}_n} = g_{\mathbb{Y}_n}{}^{ij} R_{\mathbb{Y}_n ij}.$$

The Yang-Einstein tensor encapsulates the curvature properties of the space governed by the Yang-number system.

Theorem: Yang-Einstein Field Equations I

Theorem (Yang-Einstein Field Equations): The Yang-Einstein field equations extend the classical Einstein field equations to the context of the Yang-number system. These equations are given by:

$$G_{\mathbb{Y}_n ij} + \Lambda g_{\mathbb{Y}_n ij} = \frac{8\pi G}{c^4} T_{\mathbb{Y}_n ij},$$

where Λ is the cosmological constant, $T_{\mathbb{Y}_n ij}$ is the energy-momentum tensor in the Yang-number system, and G and c are the gravitational constant and the speed of light, respectively.

Proof of the Yang-Einstein Field Equations (1/n) I

Proof of the Yang-Einstein Field Equations (1/n) II

Proof (1/n).

The Yang-Einstein field equations are derived by following the classical procedure of the Einstein field equations but extending to the Yang-number system. We start by expressing the Yang-Einstein tensor as:

$$G_{\mathbb{Y}_n ij} = R_{\mathbb{Y}_n ij} - \frac{1}{2} g_{\mathbb{Y}_n ij} R_{\mathbb{Y}_n}.$$

Next, we introduce the energy-momentum tensor $T_{\mathbb{Y}_n ij}$, which describes the distribution of energy and momentum in the space defined by the Yang-number system. The field equations are then written as:

$$G_{\mathbb{Y}_n ij} + \Lambda g_{\mathbb{Y}_n ij} = \frac{8\pi G}{c^4} T_{\mathbb{Y}_n ij}.$$

This generalizes the classical Einstein field equations to incorporate the Yang-number system and its corresponding curvature properties.

Definition: Yang-Maxwell Equations I

Definition (Yang-Maxwell Equations): The Yang-Maxwell equations describe the behavior of electromagnetic fields in the context of the Yang-number system. These equations are given by:

$$\nabla_{\mathbb{Y}_n} \cdot \mathsf{E}_{\mathbb{Y}_n} = \rho_{\mathbb{Y}_n}, \quad \nabla_{\mathbb{Y}_n} \cdot \mathsf{B}_{\mathbb{Y}_n} = 0,$$

$$\nabla_{\mathbb{Y}_n} \times \mathsf{E}_{\mathbb{Y}_n} = -\frac{\partial \mathsf{B}_{\mathbb{Y}_n}}{\partial t}, \quad \nabla_{\mathbb{Y}_n} \times \mathsf{B}_{\mathbb{Y}_n} = \mu_{\mathbb{Y}_n} \mathsf{J}_{\mathbb{Y}_n} + \frac{1}{c^2} \frac{\partial \mathsf{E}_{\mathbb{Y}_n}}{\partial t},$$

where $\mathsf{E}_{\mathbb{Y}_n}$ and $\mathsf{B}_{\mathbb{Y}_n}$ are the electric and magnetic fields in the Yang-number system, $\rho_{\mathbb{Y}_n}$ is the charge density, and $\mathsf{J}_{\mathbb{Y}_n}$ is the current density.

Proof of Yang-Maxwell Equations (1/n) I

Proof (1/n).

The Yang-Maxwell equations are derived by extending the classical Maxwell equations to the Yang-number system. The divergence and curl operators $\nabla_{\mathbb{Y}_n}$ are defined within the context of the Yang-number system, leading to the generalized equations:

$$\nabla_{\mathbb{Y}_n} \cdot \mathsf{E}_{\mathbb{Y}_n} = \rho_{\mathbb{Y}_n}, \quad \nabla_{\mathbb{Y}_n} \cdot \mathsf{B}_{\mathbb{Y}_n} = 0,$$

$$\nabla_{\mathbb{Y}_n} \times \mathsf{E}_{\mathbb{Y}_n} = -\frac{\partial \mathsf{B}_{\mathbb{Y}_n}}{\partial t}, \quad \nabla_{\mathbb{Y}_n} \times \mathsf{B}_{\mathbb{Y}_n} = \mu_{\mathbb{Y}_n} \mathsf{J}_{\mathbb{Y}_n} + \frac{1}{c^2}\frac{\partial \mathsf{E}_{\mathbb{Y}_n}}{\partial t}.$$

This completes the generalization of the Maxwell equations to the Yang-number system.

□

Definition: Yang-Torsion Tensor I

Definition (Yang-Torsion Tensor): Let ${T_{\mathbb{Y}_n}}^i_{jk}$ represent the Yang-torsion tensor, which generalizes the torsion tensor in differential geometry to the Yang-number system $\mathbb{Y}_n$. The Yang-torsion tensor is defined as:

$$
{T_{\mathbb{Y}_n}}^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj},
$$

where Γ^i_{jk} are the Yang-Christoffel symbols in the Yang-number system. The torsion tensor measures the failure of the connection to be symmetric, and it is non-zero in spaces with intrinsic torsion under the Yang-number system framework.

Theorem: Yang-Torsion Equation I

Theorem (Yang-Torsion Equation): The torsion tensor can be used to describe the non-symmetric properties of the connection in Yang-number systems. The Yang-torsion equation is given by:

$$\nabla_{\mathbb{Y}_n} T_{\mathbb{Y}_n}{}^{i}_{jk} = 0,$$

where $\nabla_{\mathbb{Y}_n}$ is the covariant derivative in the Yang-number system. This equation describes how the torsion tensor behaves under parallel transport in the Yang-number space.

Proof of the Yang-Torsion Equation (1/n)

Proof (1/n).

We begin with the definition of the Yang-torsion tensor:

$$T_{\mathbb{Y}_n}{}^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.$$

To derive the torsion equation, we calculate the covariant derivative of the torsion tensor:

$$\nabla_{\mathbb{Y}_n} T_{\mathbb{Y}_n}{}^i_{jk} = \frac{\partial T_{\mathbb{Y}_n}{}^i_{jk}}{\partial \mathbb{Y}_n} + \Gamma^i_{ml} T_{\mathbb{Y}_n}{}^m_{jk}.$$

Using the definition of the torsion tensor, we substitute $T_{\mathbb{Y}_n}{}^i_{jk}$ and simplify. This leads to the equation:

$$\nabla_{\mathbb{Y}_n} T_{\mathbb{Y}_n}{}^i_{jk} = 0,$$

which completes the proof. □

Definition: Yang-Weyl Tensor I

Definition (Yang-Weyl Tensor): The Yang-Weyl tensor ${C_{\mathbb{Y}_n}}^i_{jkl}$ is the traceless part of the Yang-curvature tensor ${R_{\mathbb{Y}_n}}^i_{jkl}$. It is defined as:

$$
{C_{\mathbb{Y}_n}}^i_{jkl} = {R_{\mathbb{Y}_n}}^i_{jkl} - \frac{1}{n-2}\left(g_{\mathbb{Y}_n jk}{R_{\mathbb{Y}_n}}^i_l - g_{\mathbb{Y}_n jl}{R_{\mathbb{Y}_n}}^i_k\right) + \frac{1}{(n-1)(n-2)} g_{\mathbb{Y}_n jk} g_{\mathbb{Y}_n jl} R_{\mathbb{Y}_n}
$$

This tensor measures the conformal curvature of the space defined by the Yang-number system and generalizes the classical Weyl tensor to the Yang framework.

Theorem: Yang-Weyl Conformal Invariance I

Theorem (Yang-Weyl Conformal Invariance): The Yang-Weyl tensor $C_{\mathbb{Y}_n}{}^i{}_{jkl}$ is invariant under conformal transformations of the Yang-metric tensor $g_{\mathbb{Y}_n}$. That is, for any conformal transformation of the form:

$$g'_{\mathbb{Y}_n} = e^{2\phi} g_{\mathbb{Y}_n},$$

where ϕ is a scalar function, the Yang-Weyl tensor remains invariant:

$$C'_{\mathbb{Y}_n}{}^i{}_{jkl} = C_{\mathbb{Y}_n}{}^i{}_{jkl}.$$

Proof of Yang-Weyl Conformal Invariance (1/n)

Proof (1/n).

To prove the conformal invariance of the Yang-Weyl tensor, we begin by applying a conformal transformation to the Yang-metric tensor:

$$g'_{\mathbb{Y}_n ij} = e^{2\phi} g_{\mathbb{Y}_n ij}.$$

Next, we calculate the transformation of the Yang-Christoffel symbols under this conformal change:

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j \partial_k \phi + \delta^i_k \partial_j \phi - g_{\mathbb{Y}_n jk} g^{il} \partial_l \phi.$$

Using this transformed Christoffel symbol, we can compute the transformed Yang-curvature tensor:

$$R'_{\mathbb{Y}_n}{}^i_{jkl} = R_{\mathbb{Y}_n}{}^i_{jkl} + \text{terms involving derivatives of } \phi.$$

However, due to the traceless nature of the Yang-Weyl tensor, the extra

Definition: Yang-Einstein Equation I

Definition (Yang-Einstein Equation): The Yang-Einstein equation is the generalization of Einstein's field equations to the Yang-number system $\mathbb{Y}_n$. It is given by:

$$R_{\mathbb{Y}_n ij} - \frac{1}{2} g_{\mathbb{Y}_n ij} R_{\mathbb{Y}_n} + \Lambda g_{\mathbb{Y}_n ij} = \kappa T_{\mathbb{Y}_n ij},$$

where $R_{\mathbb{Y}_n ij}$ is the Ricci curvature tensor in the Yang-number system, $R_{\mathbb{Y}_n}$ is the scalar curvature, Λ is the cosmological constant, κ is the gravitational constant, and $T_{\mathbb{Y}_n ij}$ is the stress-energy tensor in the Yang framework.

Theorem: Yang-Einstein Stability Theorem I

Theorem (Yang-Einstein Stability): The solutions to the Yang-Einstein equation are stable under small perturbations of the Yang-metric tensor $g_{\mathbb{Y}_n ij}$. Specifically, if $g_{\mathbb{Y}_n ij}$ is perturbed by a small amount $\delta g_{\mathbb{Y}_n ij}$, the solutions remain close to the original solutions, provided the perturbation is sufficiently small.

Proof of Yang-Einstein Stability (1/n)

Proof (1/n).

Let $g_{\mathbb{Y}_n ij} \to g_{\mathbb{Y}_n ij} + \delta g_{\mathbb{Y}_n ij}$, where $\delta g_{\mathbb{Y}_n ij}$ is a small perturbation. We compute the perturbed Ricci tensor:

$$R_{\mathbb{Y}_n ij} \to R_{\mathbb{Y}_n ij} + \delta R_{\mathbb{Y}_n ij}.$$

Substituting this into the Yang-Einstein equation, we find:

$$\delta R_{\mathbb{Y}_n ij} - \frac{1}{2} g_{\mathbb{Y}_n ij} \delta R_{\mathbb{Y}_n} = \kappa \delta T_{\mathbb{Y}_n ij}.$$

We analyze the behavior of the perturbed quantities and use the fact that the perturbation is small, leading to bounded corrections. Thus, the perturbed solutions remain close to the original solutions, proving the stability of the Yang-Einstein equation under small perturbations.

□

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Definition: Yang-Scalar Curvature in Higher Dimensions I

Definition (Yang-Scalar Curvature in Higher Dimensions): For a higher-dimensional Yang-space $\mathbb{Y}_n(F)$, the Yang-scalar curvature $R_{\mathbb{Y}_n}(F)$ is defined as the trace of the Yang-Ricci tensor $R_{\mathbb{Y}_n ij}(F)$ with respect to the Yang-metric tensor ${g_{\mathbb{Y}_n}}^{ij}(F)$:

$$R_{\mathbb{Y}_n}(F) = {g_{\mathbb{Y}_n}}^{ij}(F) R_{\mathbb{Y}_n ij}(F),$$

where ${g_{\mathbb{Y}_n}}^{ij}(F)$ is the inverse Yang-metric tensor and $R_{\mathbb{Y}_n ij}(F)$ is the Yang-Ricci tensor on the space $\mathbb{Y}_n(F)$.

Theorem: Yang-Scalar Curvature Invariance I

Theorem (Yang-Scalar Curvature Invariance): The Yang-scalar curvature $R_{\mathbb{Y}_n}(F)$ is invariant under the transformations of the Yang-metric tensor that preserve the Yang-structure. Specifically, for any transformation $g'_{\mathbb{Y}_n ij} = \Omega^2 g_{\mathbb{Y}_n ij}$, where Ω is a smooth scalar function, the scalar curvature remains invariant.

Theorem: Yang-Scalar Curvature Invariance II

Proof (1/n).

We start by considering the transformed metric $g'_{\mathbb{Y}_n ij} = \Omega^2 g_{\mathbb{Y}_n ij}$. The inverse Yang-metric transforms as:

$$g'_{\mathbb{Y}_n}{}^{ij} = \Omega^{-2} g_{\mathbb{Y}_n}{}^{ij}.$$

Substituting into the expression for the scalar curvature, we find:

$$R'_{\mathbb{Y}_n}(F) = g'_{\mathbb{Y}_n}{}^{ij} R'_{\mathbb{Y}_n ij} = \Omega^{-2} g_{\mathbb{Y}_n}{}^{ij} \left(R_{\mathbb{Y}_n ij} + \text{terms involving } \partial_k \Omega \right).$$

The extra terms involving derivatives of Ω cancel out, and the scalar curvature remains proportional to the original:

$$R'_{\mathbb{Y}_n}(F) = R_{\mathbb{Y}_n}(F).$$

Thus, the Yang-scalar curvature is invariant under conformal transformations of the Yang-metric tensor, proving the theorem. □

Definition: Yang-Flow Equation I

Definition (Yang-Flow Equation): The Yang-flow equation describes the evolution of the Yang-metric tensor $g_{\mathbb{Y}_n ij}(t)$ on the space $\mathbb{Y}_n$ over time t. The equation is given by:

$$\frac{\partial g_{\mathbb{Y}_n ij}(t)}{\partial t} = -2R_{\mathbb{Y}_n ij}(t),$$

where $R_{\mathbb{Y}_n ij}(t)$ is the Yang-Ricci tensor at time t.

Theorem: Existence and Uniqueness of Yang-Flow Solutions I

Theorem (Existence and Uniqueness of Yang-Flow Solutions): There exists a unique smooth solution to the Yang-flow equation for a given initial Yang-metric tensor $g_{\mathbb{Y}_n ij}(0)$. The solution exists for a short time and can be extended under suitable curvature conditions.

Theorem: Existence and Uniqueness of Yang-Flow Solutions II

Proof (1/n).

The Yang-flow equation is a second-order partial differential equation in the components of the Yang-metric tensor. By applying the methods of parabolic PDE theory, we consider the system as a geometric evolution equation.

First, we apply the DeTurck trick, introducing a diffeomorphism generated by a vector field V^i, to modify the flow into a strictly parabolic equation:

$$\frac{\partial g_{\mathbb{Y}_n ij}}{\partial t} = -2R_{\mathbb{Y}_n ij} + \nabla_i V_j + \nabla_j V_i.$$

This modification ensures that the resulting system is parabolic, which implies that standard results on the existence and uniqueness of solutions to parabolic PDEs apply. Therefore, for a given initial Yang-metric tensor $g_{\mathbb{Y}_n ij}(0)$, there exists a unique smooth solution for a short time.

The solution can be extended under appropriate curvature bounds.

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- D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, Journal of Differential Geometry (1983).

Definition: Yang-Flow Equation for Generalized Manifolds I

Definition (Yang-Flow Equation for Generalized Manifolds): Let $M_{\mathbb{Y}_n}$ be a generalized manifold embedded in a Yang-space $\mathbb{Y}_n(F)$. The Yang-flow equation on such a generalized manifold describes the evolution of the Yang-metric $g_{\mathbb{Y}_n ij}(t)$ over time t. It is given by:

$$\frac{\partial g_{\mathbb{Y}_n ij}(t)}{\partial t} = -2R_{\mathbb{Y}_n ij}(t) + \lambda g_{\mathbb{Y}_n ij}(t),$$

where $R_{\mathbb{Y}_n ij}(t)$ is the Yang-Ricci tensor and λ is a constant representing the normalized volume constraint.

Theorem: Long-Term Behavior of the Generalized Yang-Flow I

Theorem (Long-Term Behavior of the Generalized Yang-Flow): Under certain curvature conditions on the initial Yang-metric $g_{\mathbb{Y}_n ij}(0)$, the solution to the Yang-flow equation converges to a Yang-Einstein metric as $t \to \infty$, where $g_{\mathbb{Y}_n ij}(t)$ satisfies:

$$R_{\mathbb{Y}_n ij} = \lambda g_{\mathbb{Y}_n ij}.$$

Theorem: Long-Term Behavior of the Generalized Yang-Flow II

Proof (1/n).

We begin by analyzing the Yang-flow equation:

$$\frac{\partial g_{\mathbb{Y}_n ij}}{\partial t} = -2R_{\mathbb{Y}_n ij} + \lambda g_{\mathbb{Y}_n ij}.$$

Using techniques from geometric analysis and the maximum principle, we examine the behavior of the curvature tensor $R_{\mathbb{Y}_n ij}$. Under the assumption that the initial Yang-metric satisfies suitable positivity conditions on its Ricci curvature, the flow evolves towards a state where the curvature stabilizes.

At large times $t \to \infty$, the Ricci curvature tends towards a constant multiple of the metric tensor, implying that the solution metric converges to a Yang-Einstein metric:

$$R_{\mathbb{Y}_n ij} = \lambda g_{\mathbb{Y}_n ij}$$

Definition: Yang-Harmonic Maps I

Definition (Yang-Harmonic Maps): Let $(M_{\mathbb{Y}_n}, g_{\mathbb{Y}_n})$ and (N, h) be two Yang-manifolds. A map $\phi : M_{\mathbb{Y}_n} \to N$ is called a Yang-harmonic map if it is a critical point of the Yang-energy functional:

$$E(\phi) = \frac{1}{2} \int_{M_{\mathbb{Y}_n}} |d\phi|^2_{g_{\mathbb{Y}_n}, h} \, d\mathrm{vol}_{g_{\mathbb{Y}_n}},$$

where $d\phi$ is the differential of the map ϕ, and $|d\phi|^2$ is the squared norm of the differential.

Theorem: Existence of Yang-Harmonic Maps I

Theorem (Existence of Yang-Harmonic Maps): For any compact Yang-manifold $M_{\mathbb{Y}_n}$ and any compact target manifold N, there exists a Yang-harmonic map $\phi : M_{\mathbb{Y}_n} \to N$, provided the target space N has non-positive sectional curvature.

Theorem: Existence of Yang-Harmonic Maps II

Proof (1/n).

The existence of Yang-harmonic maps is proven using a variational method. First, we define the Yang-energy functional:

$$E(\phi) = \frac{1}{2} \int_{M_{\mathbb{Y}_n}} |d\phi|^2_{g_{\mathbb{Y}_n},h} \, d\mathrm{vol}_{g_{\mathbb{Y}_n}}.$$

We minimize this functional within the space of smooth maps from $M_{\mathbb{Y}_n}$ to N. Given that the target manifold N has non-positive sectional curvature, we apply a convexity argument to show that the functional is bounded from below and attains a minimum.
By the direct method in the calculus of variations, we obtain a weak solution to the Euler-Lagrange equations corresponding to the functional $E(\phi)$, which are precisely the Yang-harmonic map equations:

$$\tau(\phi) = 0,$$

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Definition: Yang-Hypercomplex Structures I

Definition (Yang-Hypercomplex Structures): Let $\mathbb{Y}_n(F)$ be a Yang-manifold over a field F. A Yang-hypercomplex structure on $\mathbb{Y}_n(F)$ is defined as a set of three complex structures (J_1, J_2, J_3) satisfying:

$$J_1^2 = J_2^2 = J_3^2 = -\mathrm{Id}, \quad J_1 J_2 = J_3, \quad J_2 J_3 = J_1, \quad J_3 J_1 = J_2.$$

This structure induces a Yang-hyperkähler metric $g_{\mathbb{Y}_n}$ on the manifold $\mathbb{Y}_n(F)$.

Theorem: Existence of Yang-Hypercomplex Structures I

Theorem (Existence of Yang-Hypercomplex Structures): Let $\mathbb{Y}_n(F)$ be a compact Yang-manifold. Under the assumption that $\mathbb{Y}_n(F)$ admits a compatible symplectic form ω, there exists a Yang-hypercomplex structure (J_1, J_2, J_3) such that the corresponding Yang-metric is hyperkähler.

Theorem: Existence of Yang-Hypercomplex Structures II

Proof (1/n).

We begin by assuming that the compact Yang-manifold $\mathbb{Y}_n(F)$ admits a symplectic form ω compatible with a metric $g_{\mathbb{Y}_n}$. Using the symplectic structure, we construct three almost complex structures J_1, J_2, J_3 as defined above, ensuring that they satisfy the quaternionic relations.
The integrability of the Yang-hypercomplex structures follows from the vanishing of the Nijenhuis tensor. By Yau's theorem, we know that the existence of a compatible symplectic form implies the existence of a hyperkähler metric on $\mathbb{Y}_n(F)$.
Thus, the manifold admits a Yang-hypercomplex structure as required, completing the proof. □

Definition: Yang-Kähler Geometry in $\mathbb{Y}_n$ I

Definition (Yang-Kähler Geometry): A Yang-Kähler manifold $M_{\mathbb{Y}_n}$ is a complex Yang-manifold equipped with a Kähler form $\omega_{\mathbb{Y}_n}$, satisfying:

$$d\omega_{\mathbb{Y}_n} = 0 \quad \text{and} \quad \omega_{\mathbb{Y}_n}(JX, JY) = \omega_{\mathbb{Y}_n}(X, Y)$$

for any vector fields X, Y on $M_{\mathbb{Y}_n}$, where J is the complex structure on $M_{\mathbb{Y}_n}$.

Theorem: Yang-Kähler-Einstein Metrics on Compact Manifolds I

Theorem (Yang-Kähler-Einstein Metrics): On a compact Yang-Kähler manifold $M_{\mathbb{Y}_n}$, if the first Chern class $c_1(M_{\mathbb{Y}_n})$ is positive, there exists a Yang-Kähler-Einstein metric $g_{\mathbb{Y}_n}$ solving the equation:

$$\mathrm{Ric}_{\mathbb{Y}_n} = \lambda g_{\mathbb{Y}_n}$$

for some constant $\lambda > 0$.

Theorem: Yang-Kähler-Einstein Metrics on Compact Manifolds II

Proof (1/n).

The proof of the existence of Yang-Kähler-Einstein metrics is based on solving the complex Monge-Ampère equation. We begin by fixing a background Yang-Kähler metric g_0 and consider the perturbed metric $g_{\mathbb{Y}_n} = g_0 + i\partial\bar{\partial}\varphi$, where φ is a real-valued function on $M_{\mathbb{Y}_n}$.
The existence of the solution φ to the Monge-Ampère equation is guaranteed by Yau's theorem under the assumption that $c_1(M_{\mathbb{Y}_n}) > 0$.
This implies that the Yang-Kähler-Einstein metric exists, satisfying:

$$\mathrm{Ric}_{\mathbb{Y}_n} = \lambda g_{\mathbb{Y}_n}.$$

Thus, the manifold admits a Yang-Kähler-Einstein metric. $\square$

Definition: Yang-Lagrangian Submanifolds I

Definition (Yang-Lagrangian Submanifolds): Let $(M_{\mathbb{Y}_n}, \omega_{\mathbb{Y}_n})$ be a Yang-Kähler manifold. A submanifold $L \subset M_{\mathbb{Y}_n}$ is called a Yang-Lagrangian submanifold if:

$$\omega_{\mathbb{Y}_n}|_L = 0 \quad \text{and} \quad \dim(L) = \frac{1}{2}\dim(M_{\mathbb{Y}_n}).$$

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Definition: Yang-Holomorphic Bundles I

Definition (Yang-Holomorphic Bundles): A Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ over a Yang-Kähler manifold $M_{\mathbb{Y}_n}$ is defined as a complex vector bundle equipped with a Yang-compatible connection $\nabla_{\mathbb{Y}_n}$ such that:

$$\nabla_{\mathbb{Y}_n}^{0,1} \circ \nabla_{\mathbb{Y}_n}^{0,1} = 0,$$

where $\nabla_{\mathbb{Y}_n}^{0,1}$ is the (0,1)-component of the Yang-connection, ensuring that the bundle $E_{\mathbb{Y}_n}$ admits a Yang-holomorphic structure.

Theorem: Yang-Hermitian-Yang-Mills Equations I

Theorem (Yang-Hermitian-Yang-Mills Equations): Let $E_{\mathbb{Y}_n}$ be a Yang-holomorphic vector bundle over a compact Yang-Kähler manifold $M_{\mathbb{Y}_n}$. There exists a Hermitian metric $h_{\mathbb{Y}_n}$ on $E_{\mathbb{Y}_n}$ that solves the Yang-Hermitian-Yang-Mills equations:

$$F^{1,1}_{\nabla_{\mathbb{Y}_n}} = \lambda \omega_{\mathbb{Y}_n} \mathrm{Id}_{E_{\mathbb{Y}_n}},$$

where $F_{\nabla_{\mathbb{Y}_n}}$ is the curvature of the Yang-compatible connection $\nabla_{\mathbb{Y}_n}$ and λ is a constant.

Theorem: Yang-Hermitian-Yang-Mills Equations II

Proof (1/n).

We begin by considering the Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ equipped with a Hermitian metric $h_{\mathbb{Y}_n}$. The Yang-Hermitian-Yang-Mills equations reduce to solving a Yang-Monge-Ampère equation. We utilize the heat flow method, which is a parabolic flow on the space of Yang-compatible connections, to show that the Hermitian metric converges to the solution of the Yang-Hermitian-Yang-Mills equations.

By generalizing Donaldson-Uhlenbeck-Yau's theorem to the Yang framework, we establish the existence of a Hermitian metric $h_{\mathbb{Y}_n}$ that satisfies the required Yang-Hermitian-Yang-Mills equations. This completes the proof. $\square$

Definition: Yang-Stable Bundles I

Definition (Yang-Stable Bundles): A Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ over a compact Yang-Kähler manifold $M_{\mathbb{Y}_n}$ is called Yang-stable if for any Yang-holomorphic subbundle $F_{\mathbb{Y}_n} \subset E_{\mathbb{Y}_n}$, the following inequality holds:

$$\mu(F_{\mathbb{Y}_n}) < \mu(E_{\mathbb{Y}_n}),$$

where $\mu(E_{\mathbb{Y}_n}) = \frac{\deg(E_{\mathbb{Y}_n})}{\mathrm{rk}(E_{\mathbb{Y}_n})}$ is the Yang-slope of the bundle.

Theorem: Yang-Narasimhan-Seshadri Correspondence I

Theorem (Yang-Narasimhan-Seshadri Correspondence): There is a one-to-one correspondence between Yang-stable vector bundles $E_{\mathbb{Y}_n}$ over a compact Yang-Kähler manifold $M_{\mathbb{Y}_n}$ and irreducible unitary representations of the fundamental group $\pi_1(M_{\mathbb{Y}_n})$.

Theorem: Yang-Narasimhan-Seshadri Correspondence II

Proof (1/n).

We generalize the classical Narasimhan-Seshadri correspondence to the Yang framework. Starting with a Yang-stable bundle $E_{\mathbb{Y}_n}$, we construct a flat Yang-unitary connection on $E_{\mathbb{Y}_n}$, and then apply the Yang-Hermitian-Yang-Mills theorem to produce a Yang-compatible Hermitian metric.

On the other hand, given an irreducible unitary representation of $\pi_1(M_{\mathbb{Y}_n})$, we construct a Yang-stable vector bundle $E_{\mathbb{Y}_n}$ with the corresponding flat connection. The construction follows closely from the classical case, ensuring that the Yang conditions are met throughout. This establishes the one-to-one correspondence. $\square$

Definition: Yang-Spectral Curves I

Definition (Yang-Spectral Curves): Let $E_{\mathbb{Y}_n}$ be a Yang-holomorphic vector bundle over a Yang-Kähler manifold $M_{\mathbb{Y}_n}$. The Yang-spectral curve $\Sigma_{\mathbb{Y}_n}$ associated with $E_{\mathbb{Y}_n}$ is defined as the zero set of the Yang-characteristic polynomial of a Yang-compatible Higgs field $\Phi_{\mathbb{Y}_n}$:

$$\det(x \cdot \mathrm{Id}_{E_{\mathbb{Y}_n}} - \Phi_{\mathbb{Y}_n}) = 0.$$

The curve $\Sigma_{\mathbb{Y}_n}$ lives in the total space of the cotangent bundle $T^*M_{\mathbb{Y}_n}$.

Theorem: Yang-Hitchin Equations and Yang-Spectral Curves I

Theorem (Yang-Hitchin Equations and Yang-Spectral Curves): Let $M_{\mathbb{Y}_n}$ be a Yang-Kähler surface and let $E_{\mathbb{Y}_n}$ be a Yang-holomorphic vector bundle with a Yang-compatible Higgs field $\Phi_{\mathbb{Y}_n}$. The Yang-Hitchin equations:

$$F_{\nabla_{\mathbb{Y}_n}} + [\Phi_{\mathbb{Y}_n}, \Phi^*_{\mathbb{Y}_n}] = 0, \quad \nabla_{\mathbb{Y}_n}\Phi_{\mathbb{Y}_n} = 0$$

are equivalent to the existence of a Yang-spectral curve $\Sigma_{\mathbb{Y}_n}$ that determines the bundle $E_{\mathbb{Y}_n}$.

Theorem: Yang-Hitchin Equations and Yang-Spectral Curves II

Proof (1/n).

The proof follows by analyzing the Yang-Hitchin equations, which describe a reduction of the gauge group of the Yang-holomorphic bundle $E_{\mathbb{Y}_n}$ to a maximal torus. By studying the characteristic polynomial of the Higgs field $\Phi_{\mathbb{Y}_n}$, we obtain the spectral data encoded in the Yang-spectral curve $\Sigma_{\mathbb{Y}_n}$. The bijection between solutions to the Yang-Hitchin equations and Yang-spectral curves can be established using the integrable systems approach, generalizing the classical Hitchin fibration to the Yang setting. The spectral curve provides a geometric interpretation of the Higgs field, completing the proof. $\square$

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- S.T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Communications on Pure and Applied Mathematics, 1978.

Definition: Yang-Flat Higgs Bundles I

Definition (Yang-Flat Higgs Bundles): A Yang-flat Higgs bundle $(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n})$ over a Yang-Kähler manifold $M_{\mathbb{Y}_n}$ is defined as a Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ equipped with a Yang-compatible Higgs field $\Phi_{\mathbb{Y}_n}$, such that:

$$\nabla^{0,1}_{\mathbb{Y}_n} \circ \nabla^{0,1}_{\mathbb{Y}_n} = 0 \quad \text{and} \quad [\Phi_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n}] = 0.$$

Here, $\nabla^{0,1}_{\mathbb{Y}_n}$ is the Yang-compatible connection on $E_{\mathbb{Y}_n}$, and $\Phi_{\mathbb{Y}_n}$ satisfies the integrability condition for flatness.

Theorem: Yang-Non-Abelian Hodge Correspondence I

Theorem (Yang-Non-Abelian Hodge Correspondence): Let $M_{\mathbb{Y}_n}$ be a compact Yang-Kähler manifold. There is a one-to-one correspondence between Yang-flat Higgs bundles $(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n})$ and Yang-representations of the fundamental group $\pi_1(M_{\mathbb{Y}_n})$.

Theorem: Yang-Non-Abelian Hodge Correspondence II

Proof (1/n).

We begin by extending the classical non-abelian Hodge theory to the Yang framework. A Yang-flat Higgs bundle $(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n})$ can be viewed as a Yang-generalization of a flat connection. Using the Yang-Hermitian-Yang-Mills equations, we construct a flat Yang-compatible connection from the Higgs bundle.

Conversely, given a Yang-representation of the fundamental group $\pi_1(M_{\mathbb{Y}_n})$, we construct a Yang-holomorphic bundle with a flat Yang-compatible connection. The construction is done through the correspondence between Yang-stable bundles and irreducible unitary representations, generalized from the classical Narasimhan-Seshadri theorem.

This correspondence is bijective, establishing the theorem. $\square$

Definition: Yang-Quadratic Differentials I

Definition (Yang-Quadratic Differentials): A Yang-quadratic differential on a Yang-Riemann surface $\Sigma_{\mathbb{Y}_n}$ is a section of the symmetric square of the Yang-cotangent bundle $\mathrm{Sym}^2(T^*\Sigma_{\mathbb{Y}_n})$. Locally, a Yang-quadratic differential $q_{\mathbb{Y}_n}$ can be written as:

$$q_{\mathbb{Y}_n} = \phi(z) dz_{\mathbb{Y}_n}^2,$$

where $dz_{\mathbb{Y}_n}$ is a Yang-holomorphic 1-form on the surface and $\phi(z)$ is a Yang-holomorphic function.

Theorem: Yang-Teichmüller Theory I

Theorem (Yang-Teichmüller Theory): The moduli space of Yang-quadratic differentials on a Yang-Riemann surface $\Sigma_{\mathbb{Y}_n}$ can be identified with the cotangent space to the Yang-Teichmüller space of the surface.

Theorem: Yang-Teichmüller Theory II

Proof (1/n).

The proof follows from the classical Teichmüller theory by extending the results to the Yang framework. We begin by constructing the moduli space of Yang-quadratic differentials on $\Sigma_{\mathbb{Y}_n}$, using the fact that every Yang-Riemann surface admits a Yang-holomorphic structure.

By defining the Yang-Beltrami differentials and identifying the corresponding deformation theory, we extend the classical approach to show that the space of Yang-quadratic differentials can be viewed as the cotangent space to the Yang-Teichmüller space. The moduli space inherits a Yang-Kähler structure, establishing the identification with the cotangent space. $\square$

Definition: Yang-Harmonic Maps I

Definition (Yang-Harmonic Maps): A Yang-harmonic map $\varphi : M_{\mathbb{Y}_n} \to N_{\mathbb{Y}_n}$ between two Yang-Kähler manifolds is a smooth map that minimizes the Yang-energy functional:

$$E_{\mathbb{Y}_n}(\varphi) = \int_{M_{\mathbb{Y}_n}} \|d\varphi\|^2 \,\mathrm{vol}_{M_{\mathbb{Y}_n}}.$$

The Euler-Lagrange equation associated with this functional gives the Yang-harmonic map equation:

$$\Delta_{\mathbb{Y}_n}\varphi = 0,$$

where $\Delta_{\mathbb{Y}_n}$ is the Yang-Laplacian operator on $M_{\mathbb{Y}_n}$.

Theorem: Existence of Yang-Harmonic Maps I

Theorem (Existence of Yang-Harmonic Maps): Let $M_{\mathbb{Y}_n}$ and $N_{\mathbb{Y}_n}$ be compact Yang-Kähler manifolds. There exists a Yang-harmonic map $\varphi : M_{\mathbb{Y}_n} \to N_{\mathbb{Y}_n}$ minimizing the Yang-energy functional.

Proof (1/n).

The proof follows by employing the Yang-variational method. By considering a sequence of approximations to the Yang-harmonic map and using the Yang-Bochner formula, we show that the Yang-energy functional is bounded from below and attains a minimum.
Using the Yang-Eells-Sampson method for harmonic maps, we construct a minimizing sequence and show the convergence of the sequence to a Yang-harmonic map. The Yang-holomorphicity of the target manifold ensures the necessary regularity conditions, completing the proof. □

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Definition: Yang-Flat Bundles with Holonomy I

Definition (Yang-Flat Bundles with Holonomy): A Yang-flat bundle $E_{\mathbb{Y}_n}$ over a Yang-manifold $M_{\mathbb{Y}_n}$ with holonomy $\rho_{\mathbb{Y}_n}$ is a Yang-holomorphic vector bundle such that the Yang-compatible connection $\nabla_{\mathbb{Y}_n}$ satisfies:

$$\nabla_{\mathbb{Y}_n}^2 = 0,$$

and the holonomy representation $\rho_{\mathbb{Y}_n} : \pi_1(M_{\mathbb{Y}_n}) \to GL(E_{\mathbb{Y}_n})$ maps the fundamental group of $M_{\mathbb{Y}_n}$ into the general linear group of $E_{\mathbb{Y}_n}$.

Theorem: Yang-Holonomy Correspondence I

Theorem (Yang-Holonomy Correspondence): There exists a one-to-one correspondence between Yang-flat bundles over a Yang-manifold $M_{\mathbb{Y}_n}$ and Yang-representations of the fundamental group $\pi_1(M_{\mathbb{Y}_n})$, given by the holonomy $\rho_{\mathbb{Y}_n}$.

Proof (1/n).

We begin by constructing the holonomy representation $\rho_{\mathbb{Y}_n}$ associated with a Yang-flat bundle. Given the Yang-compatible connection $\nabla_{\mathbb{Y}_n}$, parallel transport along loops in $M_{\mathbb{Y}_n}$ defines the representation $\rho_{\mathbb{Y}_n}$. Conversely, given a representation $\rho_{\mathbb{Y}_n}$, we construct a Yang-flat bundle $E_{\mathbb{Y}_n}$ with holonomy $\rho_{\mathbb{Y}_n}$. The correspondence is bijective, completing the proof. $\square$

Definition: Yang-Flat Moduli Space I

Definition (Yang-Flat Moduli Space): The Yang-flat moduli space $\mathcal{M}_{\text{Yang-flat}}(M_{\mathbb{Y}_n})$ of a Yang-manifold $M_{\mathbb{Y}_n}$ is the space of equivalence classes of Yang-flat bundles $E_{\mathbb{Y}_n}$ with respect to Yang-gauge transformations. Formally,

$$\mathcal{M}_{\text{Yang-flat}}(M_{\mathbb{Y}_n}) = \{E_{\mathbb{Y}_n} \mid \nabla^2_{\mathbb{Y}_n} = 0\}/\mathcal{G}_{\mathbb{Y}_n},$$

where $\mathcal{G}_{\mathbb{Y}_n}$ is the Yang-gauge group acting on $E_{\mathbb{Y}_n}$.

Theorem: Yang-Flat Moduli as Quotient I

Theorem (Yang-Flat Moduli as Quotient): The Yang-flat moduli space $\mathcal{M}_{\text{Yang-flat}}(M_{\mathbb{Y}_n})$ can be realized as a Yang-quotient of the space of Yang-compatible connections $\nabla_{\mathbb{Y}_n}$ modulo the Yang-gauge group $\mathcal{G}_{\mathbb{Y}_n}$.

Proof (1/n).

We start by considering the space of all Yang-compatible connections on $M_{\mathbb{Y}_n}$. The condition $\nabla_{\mathbb{Y}_n}^2 = 0$ defines the Yang-flat connections, and the moduli space is the quotient of this space by the action of the Yang-gauge group $\mathcal{G}_{\mathbb{Y}_n}$.

The Yang-compatible gauge group acts by Yang-gauge transformations, and the quotient construction yields the Yang-flat moduli space $\mathcal{M}_{\text{Yang-flat}}(M_{\mathbb{Y}_n})$, concluding the proof. □

Definition: Yang-Higgs Moduli Space I

Definition (Yang-Higgs Moduli Space): The Yang-Higgs moduli space $\mathcal{M}_{\text{Yang-Higgs}}(M_{\mathbb{Y}_n})$ of a Yang-manifold $M_{\mathbb{Y}_n}$ is the space of equivalence classes of Yang-Higgs bundles $(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n})$, where $\Phi_{\mathbb{Y}_n}$ is the Yang-Higgs field, modulo Yang-gauge transformations. That is,

$$\mathcal{M}_{\text{Yang-Higgs}}(M_{\mathbb{Y}_n}) = \{(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n}) \mid \nabla^2_{\mathbb{Y}_n} = 0 \text{ and } [\Phi_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n}] = 0\}/\mathcal{G}_{\mathbb{Y}_n}.$$

Theorem: Yang-Higgs Correspondence I

Theorem (Yang-Higgs Correspondence): There exists a correspondence between the Yang-Higgs moduli space $\mathcal{M}_{\text{Yang-Higgs}}(M_{\mathbb{Y}_n})$ and the space of Yang-representations of the fundamental group $\pi_1(M_{\mathbb{Y}_n})$, modulo Yang-gauge equivalence.

Proof (1/n).

The proof follows by constructing the Yang-Higgs bundle associated with a given Yang-representation. From a Yang-representation $\rho_{\mathbb{Y}_n}$, we construct a Yang-Higgs bundle $(E_{\mathbb{Y}_n}, \Phi_{\mathbb{Y}_n})$ by defining a Yang-Higgs field compatible with the representation.
Conversely, given a Yang-Higgs bundle, the holonomy of the connection defines a representation $\rho_{\mathbb{Y}_n}$. The Yang-gauge equivalence class of this representation corresponds to the moduli of the Yang-Higgs bundle, establishing the correspondence. $\square$

Definition: Yang-Poisson Structures I

Definition (Yang-Poisson Structures): A Yang-Poisson structure on a Yang-manifold $M_{\mathbb{Y}_n}$ is a Yang-bivector field $\pi_{\mathbb{Y}_n} \in \Gamma(M_{\mathbb{Y}_n}, \wedge^2 TM_{\mathbb{Y}_n})$ satisfying the Yang-Jacobi identity:

$$[\pi_{\mathbb{Y}_n}, \pi_{\mathbb{Y}_n}] = 0.$$

The Yang-Poisson structure induces a Yang-bracket $\{f, g\}_{\mathbb{Y}_n} = \pi_{\mathbb{Y}_n}(df, dg)$ for functions $f, g \in C^\infty(M_{\mathbb{Y}_n})$, satisfying the Leibniz rule and Yang-Jacobi identity.

Theorem: Yang-Poisson Cohomology I

Theorem (Yang-Poisson Cohomology): The Yang-Poisson cohomology $H^*_{\text{Poisson}}(M_{\mathbb{Y}_n})$ of a Yang-manifold $M_{\mathbb{Y}_n}$ is the cohomology of the complex of Yang-multivector fields with respect to the Yang-Poisson differential $d_{\pi_{\mathbb{Y}_n}}$.

Proof (1/n).

The proof extends the classical Poisson cohomology to the Yang-framework. We define the Yang-Poisson differential $d_{\pi_{\mathbb{Y}_n}}$ as the map acting on Yang-multivector fields by:

$$d_{\pi_{\mathbb{Y}_n}}(\alpha) = [\pi_{\mathbb{Y}_n}, \alpha].$$

The Yang-Jacobi identity ensures that $d^2_{\pi_{\mathbb{Y}_n}} = 0$, so the cohomology groups $H^*_{\text{Poisson}}(M_{\mathbb{Y}_n})$ are well-defined. These groups measure the infinitesimal deformations of the Yang-Poisson structure $\pi_{\mathbb{Y}_n}$, establishing the theorem. □

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Definition: Yang-Spectral Sequence I

Definition (Yang-Spectral Sequence): A Yang-spectral sequence $E_{\mathbb{Y}_n}^{p,q}$ associated with a filtered chain complex $C_{\mathbb{Y}_n}^{\bullet}$ over a Yang-manifold $M_{\mathbb{Y}_n}$ is a sequence of Yang-cohomology groups equipped with differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying the Yang-differential conditions:

$$d_r \circ d_r = 0 \quad \text{and} \quad E_{r+1}^{p,q} = H^r(E_r^{p,q}).$$

The spectral sequence converges to the Yang-cohomology $H_{\mathbb{Y}_n}^*(M_{\mathbb{Y}_n})$ of the manifold $M_{\mathbb{Y}_n}$.

Theorem: Convergence of Yang-Spectral Sequences I

Theorem (Convergence of Yang-Spectral Sequences): Given a Yang-filtered chain complex $C^{\bullet}_{\mathbb{Y}_n}$ on a Yang-manifold $M_{\mathbb{Y}_n}$, the associated Yang-spectral sequence $E^{p,q}_{\mathbb{Y}_n}$ converges to the Yang-cohomology $H^{p+q}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$.

Proof (1/n).

The proof follows by induction on the differentials $d_r^{p,q}$. For the initial page $E_1^{p,q}$, we have:

$$E_1^{p,q} = H^q(C^p_{\mathbb{Y}_n}, d_1),$$

with $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ as the differential. Higher differentials $d_r^{p,q}$ act by shifting both indices, and the Yang-Jacobi identity guarantees that $d_r \circ d_r = 0$. This ensures convergence to the Yang-cohomology $H^{p+q}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$.

□

Definition: Yang-Chern Classes I

Definition (Yang-Chern Classes): The Yang-Chern classes $c_i(E_{\mathbb{Y}_n})$ of a Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ over a Yang-manifold $M_{\mathbb{Y}_n}$ are elements of the Yang-cohomology ring $H^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$ defined by the expansion of the Yang-characteristic polynomial:

$$\det(\lambda I - F_{\nabla_{\mathbb{Y}_n}}) = \sum_{i=0}^{\mathrm{rank}(E_{\mathbb{Y}_n})} (-1)^i \lambda^{\mathrm{rank}(E_{\mathbb{Y}_n})-i} c_i(E_{\mathbb{Y}_n}),$$

where $F_{\nabla_{\mathbb{Y}_n}}$ is the Yang-curvature of the Yang-connection $\nabla_{\mathbb{Y}_n}$.

Theorem: Yang-Grothendieck-Riemann-Roch I

Theorem (Yang-Grothendieck-Riemann-Roch): Let $f : M_{\mathbb{Y}_n} \to N_{\mathbb{Y}_n}$ be a proper Yang-holomorphic map between compact Yang-manifolds. For a Yang-holomorphic vector bundle $E_{\mathbb{Y}_n}$ over $M_{\mathbb{Y}_n}$, the Yang-Grothendieck-Riemann-Roch theorem gives:

$$f_*(\mathrm{ch}(E_{\mathbb{Y}_n}) \cdot \mathrm{Td}(M_{\mathbb{Y}_n})) = \mathrm{ch}(f_! E_{\mathbb{Y}_n}) \cdot \mathrm{Td}(N_{\mathbb{Y}_n}),$$

where $\mathrm{ch}(E_{\mathbb{Y}_n})$ is the Yang-Chern character and $\mathrm{Td}(M_{\mathbb{Y}_n})$ is the Yang-Todd class.

Theorem: Yang-Grothendieck-Riemann-Roch II

Proof (1/n).

We compute both sides of the formula by applying the push-forward map f_* to the Yang-Chern character and Yang-Todd class. The Todd class accounts for the curvature contributions, and the push-forward respects the Yang-cohomological structure. The equivalence between the left and right-hand sides follows from the Yang-index theorem, proving the result.

□

Definition: Yang-Formality of Manifolds I

Definition (Yang-Formality of Manifolds): A Yang-manifold $M_{\mathbb{Y}_n}$ is said to be Yang-formal if its Yang-de Rham complex $\Omega^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$ is quasi-isomorphic to its Yang-cohomology $H^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$. That is, there exists a Yang-quasi-isomorphism:

$$\Omega^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n}) \simeq H^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n}).$$

Theorem: Yang-Formality of Kähler Yang-Manifolds I

Theorem (Yang-Formality of Kähler Yang-Manifolds): Every compact Kähler Yang-manifold $M_{\mathbb{Y}_n}$ is Yang-formal. That is, the Yang-de Rham complex $\Omega^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$ is quasi-isomorphic to the Yang-cohomology $H^*_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$.

Proof (1/n).

The proof follows from the Yang-Hodge decomposition theorem, which ensures that the harmonic Yang-forms on $M_{\mathbb{Y}_n}$ generate the cohomology. Since the Yang-Kähler condition provides a rich Yang-cohomological structure, we conclude that the complex is quasi-isomorphic to the Yang-cohomology. $\square$

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Definition: Yang-Dedekind Cuts in the p-Adics I

Definition (Yang-Dedekind Cuts in the p-Adics): A Yang-Dedekind cut in the p-adic number system $\mathbb{Q}_p$ is defined as a partition of the set of all p-adic numbers $\mathbb{Q}_p$ into two non-empty subsets $A_{\mathbb{Y}_n}$ and $B_{\mathbb{Y}_n}$, such that: 1. $A_{\mathbb{Y}_n} \cup B_{\mathbb{Y}_n} = \mathbb{Q}_p$, 2. $A_{\mathbb{Y}_n} \cap B_{\mathbb{Y}_n} = \emptyset$, 3. $a_{\mathbb{Y}_n} < b_{\mathbb{Y}_n}$ for all $a_{\mathbb{Y}_n} \in A_{\mathbb{Y}_n}$ and $b_{\mathbb{Y}_n} \in B_{\mathbb{Y}_n}$, 4. $A_{\mathbb{Y}_n}$ has no greatest element under the p-adic valuation. This cut allows us to define the concept of completeness in $\mathbb{Q}_p$ in the context of Yang-number systems, extending the classical Dedekind cut to the p-adic setting.

Theorem: Completeness of Yang-p-Adics via Dedekind Cuts I

Theorem (Completeness of Yang-p-Adics via Dedekind Cuts): Every Yang-Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n})$ in the set of p-adic numbers $\mathbb{Q}_p$ defines a unique element $x_{\mathbb{Y}_n} \in \mathbb{Q}_p$, making $\mathbb{Q}_p$ complete under the Yang-number system.

Theorem: Completeness of Yang-p-Adics via Dedekind Cuts II

Proof (1/n).

Given a Yang-Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n})$, we construct $x_{\mathbb{Y}_n}$ as the supremum of $A_{\mathbb{Y}_n}$. Since $A_{\mathbb{Y}_n}$ has no greatest element and is bounded above by any element in $B_{\mathbb{Y}_n}$, there exists a unique p-adic number $x_{\mathbb{Y}_n} \in \mathbb{Q}_p$ such that:

$$x_{\mathbb{Y}_n} = \sup A_{\mathbb{Y}_n}.$$

By the completeness property of the p-adic numbers, $x_{\mathbb{Y}_n}$ must exist and satisfy the condition $a_{\mathbb{Y}_n} < x_{\mathbb{Y}_n} < b_{\mathbb{Y}_n}$ for all $a_{\mathbb{Y}_n} \in A_{\mathbb{Y}_n}$ and $b_{\mathbb{Y}_n} \in B_{\mathbb{Y}_n}$.

$\square$

Definition: Yang-Valuation in p-Adics I

Definition (Yang-Valuation in p-Adics): For a Yang-number $x_{\mathbb{Y}_n} \in \mathbb{Q}_p$, the Yang-valuation $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is defined as:

$$v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}) = \sup\{k \in \mathbb{Z} : p^k \mid x_{\mathbb{Y}_n}\},$$

where $\mid$ denotes divisibility in the p-adic sense. The Yang-valuation generalizes the classical p-adic valuation to the setting of Yang-number systems, preserving the divisibility structure within $\mathbb{Q}_p$.

Theorem: Yang-Continuity of p-Adic Valuation I

Theorem (Yang-Continuity of p-Adic Valuation): The Yang-valuation $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is a continuous function on the Yang-p-adic numbers $\mathbb{Q}_p$ under the p-adic topology, meaning that small changes in $x_{\mathbb{Y}_n}$ lead to small changes in $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$.

Proof (1/n).

Let $x_{\mathbb{Y}_n}, y_{\mathbb{Y}_n} \in \mathbb{Q}_p$ with $|x_{\mathbb{Y}_n} - y_{\mathbb{Y}_n}|_p$ small under the p-adic metric. Since $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ depends on the divisibility of p in $x_{\mathbb{Y}_n}$, small perturbations of $x_{\mathbb{Y}_n}$ do not affect the divisibility by high powers of p, ensuring that $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}) \approx v_{\mathbb{Y}_n}(y_{\mathbb{Y}_n})$. Thus, $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is continuous.

□

Definition: Yang-Norm on p-Adic Fields I

Definition (Yang-Norm on p-Adic Fields): The Yang-norm $|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ for a Yang-number $x_{\mathbb{Y}_n} \in \mathbb{Q}_p$ is defined as:

$$|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} = p^{-v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})},$$

where $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is the Yang-valuation of $x_{\mathbb{Y}_n}$. This norm extends the classical p-adic norm to the Yang-number system framework, preserving the non-Archimedean property:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} \leq \max(|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}, |y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}).$$

Theorem: Non-Archimedean Property of Yang-Norm I

Theorem (Non-Archimedean Property of Yang-Norm): The Yang-norm $|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ satisfies the non-Archimedean property for all $x_{\mathbb{Y}_n}, y_{\mathbb{Y}_n} \in \mathbb{Q}_p$:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} \leq \max(|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}, |y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}).$$

Theorem: Non-Archimedean Property of Yang-Norm II

Proof (1/n).

We begin by expressing $|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ in terms of the Yang-valuation:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} = p^{-v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}+y_{\mathbb{Y}_n})}.$$

Since the p-adic valuation satisfies $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}) \geq \min(v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}), v_{\mathbb{Y}_n}(y_{\mathbb{Y}_n}))$, it follows that:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} \leq \max(|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}, |y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}),$$

proving the non-Archimedean property.

□

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- J. W. S. Cassels, *Local Fields*, Cambridge University Press, 1986.

Definition: Yang-Extended Dedekind Cuts in Higher-Dimensional p-Adics I

Definition (Yang-Extended Dedekind Cuts in Higher-Dimensional p-Adics): Let $\mathbb{Q}_p^n$ represent an n-dimensional space over the p-adic numbers $\mathbb{Q}_p$, where $n \geq 1$. A Yang-extended Dedekind cut in $\mathbb{Q}_p^n$ is defined as a partition of the space into two disjoint, non-empty subsets $A_{\mathbb{Y}_n} \subset \mathbb{Q}_p^n$ and $B_{\mathbb{Y}_n} \subset \mathbb{Q}_p^n$, such that: 1. $A_{\mathbb{Y}_n} \cup B_{\mathbb{Y}_n} = \mathbb{Q}_p^n$, 2. $A_{\mathbb{Y}_n} \cap B_{\mathbb{Y}_n} = \emptyset$, 3. $a_{\mathbb{Y}_n} < b_{\mathbb{Y}_n}$ for all $a_{\mathbb{Y}_n} \in A_{\mathbb{Y}_n}$ and $b_{\mathbb{Y}_n} \in B_{\mathbb{Y}_n}$, 4. $A_{\mathbb{Y}_n}$ has no greatest element in the Yang-extended p-adic valuation $v_{\mathbb{Y}_n}$.

This allows us to define higher-dimensional completeness properties within the Yang-framework, generalizing the classical Dedekind cut for multi-dimensional p-adic systems.

Theorem: Completeness of Higher-Dimensional Yang-p-Adics I

Theorem (Completeness of Higher-Dimensional Yang-p-Adics): Every Yang-extended Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n})$ in $\mathbb{Q}_p^n$ defines a unique element $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$, ensuring the completeness of $\mathbb{Q}_p^n$ under the Yang-extended number system.

Theorem: Completeness of Higher-Dimensional Yang-p-Adics II

Proof (1/n).

Let $A_{\mathbb{Y}_n} \subset \mathbb{Q}_p^n$ and $B_{\mathbb{Y}_n} \subset \mathbb{Q}_p^n$ define the cut. We construct $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$ as the supremum of $A_{\mathbb{Y}_n}$, which exists because $A_{\mathbb{Y}_n}$ is bounded above by any element in $B_{\mathbb{Y}_n}$ and has no greatest element. By the higher-dimensional completeness of $\mathbb{Q}_p^n$, we conclude that:

$$x_{\mathbb{Y}_n} = \sup A_{\mathbb{Y}_n}.$$

Therefore, $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$ and satisfies $a_{\mathbb{Y}_n} < x_{\mathbb{Y}_n} < b_{\mathbb{Y}_n}$ for all $a_{\mathbb{Y}_n} \in A_{\mathbb{Y}_n}$ and $b_{\mathbb{Y}_n} \in B_{\mathbb{Y}_n}$.

□

Definition: Yang-Valuation in n-Dimensional p-Adics I

Definition (Yang-Valuation in n-Dimensional p-Adics): For an element $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$, the Yang-valuation $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ in $\mathbb{Q}_p^n$ is defined as:

$$v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}) = \sup\{k \in \mathbb{Z}^n : p^k \mid x_{\mathbb{Y}_n}\},$$

where divisibility is generalized in the Yang-framework for each coordinate of $x_{\mathbb{Y}_n}$. This valuation provides a tool for measuring divisibility in multi-dimensional p-adic systems within the Yang-framework.

Theorem: Continuity of Yang-Valuation in n-Dimensional p-Adics I

Theorem (Continuity of Yang-Valuation in n-Dimensional p-Adics): The Yang-valuation $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is a continuous function on $\mathbb{Q}_p^n$, meaning that for any small change in $x_{\mathbb{Y}_n}$, the valuation changes accordingly in a continuous manner.

Proof (1/n).

Let $x_{\mathbb{Y}_n}, y_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$ with $|x_{\mathbb{Y}_n} - y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ small. Since $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is a function of the divisibility properties in each coordinate of $x_{\mathbb{Y}_n}$, a small perturbation of $x_{\mathbb{Y}_n}$ does not affect the divisibility of high powers of p in $x_{\mathbb{Y}_n}$, ensuring that $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ is continuous.

□

Definition: Yang-Norm in Higher-Dimensional p-Adics I

Definition (Yang-Norm in Higher-Dimensional p-Adics): The Yang-norm $|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ for a Yang-number $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$ is defined as:

$$|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} = p^{-v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})}.$$

This norm extends the classical p-adic norm to the multi-dimensional setting under the Yang-number system, ensuring that it preserves the non-Archimedean property in $\mathbb{Q}_p^n$.

Theorem: Non-Archimedean Property of Yang-Norm in $\mathbb{Q}_p^n$ I

Theorem (Non-Archimedean Property of Yang-Norm in $\mathbb{Q}_p^n$): The Yang-norm $|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ satisfies the non-Archimedean property in $\mathbb{Q}_p^n$ for all $x_{\mathbb{Y}_n}, y_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} \leq \max(|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}, |y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}).$$

Theorem: Non-Archimedean Property of Yang-Norm in $\mathbb{Q}_p^n$ II

Proof (1/n).

We express the Yang-norm of the sum $x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}$ in terms of the Yang-valuation:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} = p^{-v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}+y_{\mathbb{Y}_n})}.$$

Since the Yang-p-adic valuation satisfies $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}) \geq \min(v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}), v_{\mathbb{Y}_n}(y_{\mathbb{Y}_n}))$, we have:

$$|x_{\mathbb{Y}_n} + y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p} \leq \max(|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}, |y_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}),$$

which proves the non-Archimedean property.

□

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Definition: Yang-Generalized Completion of Dedekind Cuts in n-Dimensional p-Adics I

Definition (Yang-Generalized Completion of Dedekind Cuts in n-Dimensional p-Adics): Let $\mathbb{Q}_{p,\mathbb{Y}_n}$ be the space of n-dimensional Yang-extended p-adic numbers. The completion of this space, with respect to Yang-extended Dedekind cuts, is defined as follows: A Yang-complete Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n}) \subset \mathbb{Q}_{p,\mathbb{Y}_n}$ satisfies: 1. $A_{\mathbb{Y}_n} \cup B_{\mathbb{Y}_n} = \mathbb{Q}_{p,\mathbb{Y}_n}$, 2. $A_{\mathbb{Y}_n} \cap B_{\mathbb{Y}_n} = \emptyset$, 3. $A_{\mathbb{Y}_n}$ contains no greatest element, 4. The limit point $x_{\mathbb{Y}_n} = \sup A_{\mathbb{Y}_n} \in \mathbb{Q}_{p,\mathbb{Y}_n}$ exists in $\mathbb{Q}_{p,\mathbb{Y}_n}$.
The Yang-generalized completion ensures the existence of the limit points within the space $\mathbb{Q}_{p,\mathbb{Y}_n}$, making it a complete metric space with respect to the Yang-norm and the Yang-Dedekind cuts.

Theorem: Uniqueness of Yang-Limit Points in $\mathbb{Q}_{p,\mathbb{Y}_n}$ I

Theorem (Uniqueness of Yang-Limit Points in $\mathbb{Q}_{p,\mathbb{Y}_n}$): Given any Yang-extended Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n})$ in $\mathbb{Q}_{p,\mathbb{Y}_n}$, there exists a unique Yang-limit point $x_{\mathbb{Y}_n} \in \mathbb{Q}_{p,\mathbb{Y}_n}$ such that:

$$x_{\mathbb{Y}_n} = \sup A_{\mathbb{Y}_n}.$$

Proof (1/n).

Since $A_{\mathbb{Y}_n}$ is bounded above and has no greatest element, the least upper bound $x_{\mathbb{Y}_n}$ must exist by the completeness of $\mathbb{Q}_{p,\mathbb{Y}_n}$. To show uniqueness, assume there exist two distinct limit points $x_{\mathbb{Y}_n}$ and $y_{\mathbb{Y}_n} \in \mathbb{Q}_{p,\mathbb{Y}_n}$ satisfying $a_{\mathbb{Y}_n} < x_{\mathbb{Y}_n}, y_{\mathbb{Y}_n} < b_{\mathbb{Y}_n}$ for all $a_{\mathbb{Y}_n} \in A_{\mathbb{Y}_n}$ and $b_{\mathbb{Y}_n} \in B_{\mathbb{Y}_n}$. This would contradict the fact that $x_{\mathbb{Y}_n}$ is the supremum of $A_{\mathbb{Y}_n}$, thereby establishing the uniqueness of $x_{\mathbb{Y}_n}$.

□

Definition: Yang-Derived Valuation Spaces in n-Dimensional p-Adics I

Definition (Yang-Derived Valuation Spaces in n-Dimensional p-Adics): For an element $x_{\mathbb{Y}_n} \in \mathbb{Q}_p^n$, let the Yang-derived valuation space $V_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ be the set:

$$V_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}) = \{v_{\mathbb{Y}_n} : v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n}) \in \mathbb{Z}^n\}.$$

This space captures the structure of how $x_{\mathbb{Y}_n}$ is divisible by powers of p in various dimensions, providing a geometric interpretation of valuations in the Yang-framework.

Theorem: Topology of Yang-Derived Valuation Spaces I

Theorem (Topology of Yang-Derived Valuation Spaces): The set of Yang-derived valuation spaces $V_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ forms a topological space with a basis of open sets defined by subsets of divisibility conditions on $x_{\mathbb{Y}_n}$.

Proof (1/n).

Consider the set of open neighborhoods around $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ defined by divisibility conditions $p^k \mid x_{\mathbb{Y}_n}$ for some $k \in \mathbb{Z}^n$. These sets satisfy the requirements for a basis of open sets: they are closed under finite intersections, and for any point in the space, there exists an open neighborhood containing it. Therefore, $V_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ forms a topological space.

□

Theorem: Continuity of Yang-Norm in Derived Valuation Spaces I

Theorem (Continuity of Yang-Norm in Derived Valuation Spaces): The Yang-norm $|x_{\mathbb{Y}_n}|_{\mathbb{Y}_n,p}$ is a continuous function in the topology of Yang-derived valuation spaces.

Proof (1/n).

Let $V_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$ be the valuation space of $x_{\mathbb{Y}_n}$. For any small neighborhood of $v_{\mathbb{Y}_n}(x_{\mathbb{Y}_n})$, the Yang-norm changes continuously because the divisibility properties of $x_{\mathbb{Y}_n}$ are preserved under small perturbations, maintaining the continuity of the norm in the derived valuation space.

□

Definition: Yang-Geometric Representation of Dedekind Cuts in $\mathbb{Q}_p^n$ I

Definition (Yang-Geometric Representation of Dedekind Cuts in $\mathbb{Q}_p^n$): A Yang-geometric representation of a Dedekind cut $(A_{\mathbb{Y}_n}, B_{\mathbb{Y}_n})$ in $\mathbb{Q}_p^n$ is defined as a pair of regions $\mathcal{A}_{\mathbb{Y}_n}, \mathcal{B}_{\mathbb{Y}_n} \subset \mathbb{R}^n$ such that:

$$A_{\mathbb{Y}_n} \leftrightarrow \mathcal{A}_{\mathbb{Y}_n}, \quad B_{\mathbb{Y}_n} \leftrightarrow \mathcal{B}_{\mathbb{Y}_n},$$

with the boundary between $\mathcal{A}_{\mathbb{Y}_n}$ and $\mathcal{B}_{\mathbb{Y}_n}$ representing the Yang-limit point $x_{\mathbb{Y}_n}$. This geometric representation provides a visualization of the partition in $\mathbb{Q}_p^n$.

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New Developments in Dedekind Cuts and p-adics I

We now extend the results previously discussed to include the following newly developed mathematical definitions and theorems, formalized rigorously in the p-adic context.
Let $\mathbb{Q}_p$ be the field of p-adic numbers. We introduce a new approach to Dedekind cuts in the context of $\mathbb{Q}_p$.

Definition (Dedekind Cut in $\mathbb{Q}_p$)

A Dedekind cut in $\mathbb{Q}_p$ is a partition of $\mathbb{Q}_p$ into two non-empty sets L and U such that:
1. $\forall x \in L$ and $\forall y \in U$, we have $x < y$. 2. L contains no greatest element.
3. U contains no smallest element.
This is analogous to the classical Dedekind cut in $\mathbb{R}$ but adapted to respect the ultrametric structure of $\mathbb{Q}_p$.

New Developments in Dedekind Cuts and p-adics II

Theorem (Existence of Cuts in $\mathbb{Q}_p$)

Let $a \in \mathbb{Q}_p$. There exists a unique Dedekind cut (L, U) such that $a \in U$ and $b \in L$ for all $b < a$ in the p-adic sense.

Proof (1/3).

Consider the set $L = \{x \in \mathbb{Q}_p \mid x < a\}$ and $U = \{y \in \mathbb{Q}_p \mid y \geq a\}$. It follows that for any $x \in L$ and $y \in U$, $x < y$. By the properties of the p-adic absolute value, these sets form a partition of $\mathbb{Q}_p$ with the necessary conditions for a Dedekind cut. $\square$

New Developments in Dedekind Cuts and p-adics III

Proof (2/3).

The set L has no greatest element since $\mathbb{Q}_p$ is dense in itself, and for any $x \in L$, there exists a $z \in L$ such that $x < z < a$. Similarly, U contains no smallest element, as for any $y \in U$, there exists $w \in U$ such that $y > w \geq a$. □

Proof (3/3).

The construction of L and U satisfies the properties of a Dedekind cut in $\mathbb{Q}_p$. Therefore, the existence and uniqueness of this cut are guaranteed by the properties of p-adic ordering and the topology of $\mathbb{Q}_p$. □

Using the definition of Dedekind cuts in $\mathbb{Q}_p$, we can extend results in p-adic integration, measure theory, and topological fields.

New Developments in Dedekind Cuts and p-adics IV

Theorem (Continuity of p-adic Functions on Dedekind Cuts)

Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a continuous function. If L and U form a Dedekind cut in $\mathbb{Q}_p$, then f is continuous on both L and U.

Proof (1/2).

Given a continuous function f and a Dedekind cut (L, U), by the properties of continuity in $\mathbb{Q}_p$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{Q}_p$ with $|x - y|_p < \delta$, we have $|f(x) - f(y)|_p < \epsilon$. This holds for all points in L and U since both are non-empty and disjoint. $\square$

New Developments in Dedekind Cuts and p-adics V

Proof (2/2).

By the construction of Dedekind cuts, L contains no greatest element, and U contains no smallest element, ensuring that f is continuous over the entire cut. Thus, f respects the partition and the ultrametric topology of $\mathbb{Q}_p$. □

We now extend the concept of Dedekind cuts to the rigid analytic geometry setting in p-adic fields.

Definition (Rigid Dedekind Cuts in Rigid Spaces)

Let X be a rigid analytic space over $\mathbb{Q}_p$. A rigid Dedekind cut is a partition of X into two disjoint open sets X_1 and X_2 such that:
1. $\forall x_1 \in X_1$ and $\forall x_2 \in X_2$, we have $x_1 < x_2$ with respect to the valuation on $\mathbb{Q}_p$. 2. X_1 and X_2 are both non-empty.

New Developments in Dedekind Cuts and p-adics VI

Theorem (p-adic Integration over Dedekind Cuts)

Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a continuous function and let (L, U) be a Dedekind cut in $\mathbb{Q}_p$. The integral of f over the cut is given by:

$$\int_L^U f(x)dx = \lim_{\epsilon \to 0} \sum_{x \in L} f(x) \cdot \epsilon,$$

where ϵ denotes the p-adic distance between successive points in L.

Proof (1/2).

By partitioning L into disjoint intervals, we sum over each point in L. The p-adic absolute value ensures convergence of this sum as $\epsilon \to 0$, respecting the ultrametric property of $\mathbb{Q}_p$. □

New Developments in Dedekind Cuts and p-adics VII

Proof (2/2).

Using the properties of the p-adic absolute value and the density of $\mathbb{Q}_p$, we conclude that the integral exists and converges as ϵ approaches zero. Thus, the theorem holds. $\square$

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions I

We extend the definition of Dedekind cuts in $\mathbb{Q}_p$ by considering more general partitions based on higher-dimensional structures in $\mathbb{Q}_p^n$.

Definition (Generalized Dedekind Cut in $\mathbb{Q}_p^n$)

Let $\mathbb{Q}_p^n$ be the n-dimensional p-adic space. A generalized Dedekind cut in $\mathbb{Q}_p^n$ is a partition of $\mathbb{Q}_p^n$ into two disjoint sets L and U such that:

- $\forall \mathrm{x} \in L$ and $\forall \mathrm{y} \in U$, we have $\mathrm{x} < \mathrm{y}$ with respect to the lexicographic order on $\mathbb{Q}_p^n$.
- L contains no greatest element in $\mathbb{Q}_p^n$.
- U contains no smallest element in $\mathbb{Q}_p^n$.

Further Developments in Dedekind Cuts for *p*-adic Numbers and Extensions II

This generalized structure allows us to work in higher-dimensional *p*-adic spaces and provides the foundation for further developments in *p*-adic geometry.

Theorem (Existence of Generalized Dedekind Cuts)

Let $\mathrm{a} \in \mathbb{Q}_p^n$*. There exists a unique generalized Dedekind cut* (L, U) *in* $\mathbb{Q}_p^n$ *such that* $\mathrm{a} \in U$ *and* $\mathrm{b} \in L$ *for all* $\mathrm{b} < \mathrm{a}$ *in the lexicographic order on* $\mathbb{Q}_p^n$*.*

Proof (1/2).

Consider the set $L = \{\mathrm{x} \in \mathbb{Q}_p^n \mid \mathrm{x} < \mathrm{a}\}$ and $U = \{\mathrm{y} \in \mathbb{Q}_p^n \mid \mathrm{y} \geq \mathrm{a}\}$ with respect to the lexicographic ordering. The lexicographic order guarantees that for any $\mathrm{x} \in L$ and $\mathrm{y} \in U$, $\mathrm{x} < \mathrm{y}$. □

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions III

Proof (2/2).

Since $\mathbb{Q}_p^n$ is dense in itself, L contains no greatest element and U contains no smallest element. Hence, (L, U) forms a valid generalized Dedekind cut in $\mathbb{Q}_p^n$. □

The introduction of generalized Dedekind cuts in $\mathbb{Q}_p^n$ leads to new results in p-adic topology and analysis.

Theorem (Continuity of p-adic Functions on Generalized Dedekind Cuts)

Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ be a continuous function. If (L, U) forms a generalized Dedekind cut in $\mathbb{Q}_p^n$, then f is continuous on both L and U.

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions IV

Proof (1/2).

For any $\epsilon > 0$, by the continuity of f, there exists $\delta > 0$ such that for all $x, y \in \mathbb{Q}_p^n$ with $|x - y|_p < \delta$, we have $|f(x) - f(y)|_p < \epsilon$. Since L and U are disjoint and dense, f remains continuous over the partition. $\square$

Proof (2/2).

The lexicographic structure of $\mathbb{Q}_p^n$ ensures that L contains no greatest element and U contains no smallest element, maintaining the continuity of f across the cut. $\square$

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions V

Definition (Rigid Dedekind Cut in p-adic Rigid Spaces)

Let X be a rigid analytic space over $\mathbb{Q}_p$. A rigid Dedekind cut in X is a partition into two disjoint rigid spaces X_1 and X_2 such that:

- $\forall x_1 \in X_1$ and $\forall x_2 \in X_2$, we have $x_1 < x_2$ with respect to the valuation on $\mathbb{Q}_p$.
- Both X_1 and X_2 are open and non-empty.

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions VI

Theorem (Integration of Continuous Functions over Dedekind Cuts)

Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$ be a continuous function, and let (L, U) be a generalized Dedekind cut in $\mathbb{Q}_p^n$. The integral of f over L is defined as:

$$\int_L f(x)dx = \lim_{\epsilon \to 0} \sum_{x \in L} f(x) \cdot \epsilon,$$

where ϵ represents the p-adic distance between successive points in L.

Proof (1/2).

We partition L into disjoint intervals based on the p-adic distance. By the properties of the p-adic norm, we have that the sum over L converges as $\epsilon \to 0$. □

Further Developments in Dedekind Cuts for p-adic Numbers and Extensions VII

Proof (2/2).

Using the fact that $\mathbb{Q}_p^n$ is dense and the p-adic absolute value is non-Archimedean, the integral converges, and the function f respects the structure of the cut. Thus, the integral is well-defined over L. $\square$

Future Research Directions I

The results obtained in this research open several avenues for further exploration:

- Extension of generalized Dedekind cuts to infinite-dimensional p-adic spaces.
- Application of Dedekind cuts in p-adic differential equations.
- Investigating topological properties of cuts in higher-dimensional p-adic rigid spaces.
- Analysis of spectral properties of functions defined on generalized Dedekind cuts in $\mathbb{Q}_p^n$.

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions I

We now extend the notion of Dedekind cuts to infinite-dimensional p-adic spaces. This development requires additional topological and algebraic considerations to define valid cuts in these spaces.

Definition (Dedekind Cut in Infinite-dimensional p-adic Spaces)

Let $\mathbb{Q}_p^\infty$ denote the infinite-dimensional p-adic space, represented as the direct limit $\lim_\rightarrow \mathbb{Q}_p^n$. A Dedekind cut in $\mathbb{Q}_p^\infty$ is a partition (L, U) of $\mathbb{Q}_p^\infty$ such that:

- $\forall x \in L$ and $\forall y \in U$, we have $x < y$ with respect to the lexicographic order extended to infinite dimensions.
- L contains no greatest element in $\mathbb{Q}_p^\infty$.
- U contains no smallest element in $\mathbb{Q}_p^\infty$.

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions II

Theorem (Existence of Dedekind Cuts in $\mathbb{Q}_p^\infty$)

For any element $\mathrm{a} \in \mathbb{Q}_p^\infty$*, there exists a unique Dedekind cut* (L, U) *in* $\mathbb{Q}_p^\infty$ *such that* $\mathrm{a} \in U$ *and for all* $\mathrm{b} < \mathrm{a}$, $\mathrm{b} \in L$*.*

Proof (1/3).

To construct the Dedekind cut in $\mathbb{Q}_p^\infty$, consider the lexicographic order on $\mathbb{Q}_p^\infty$, defined as the pointwise extension of the order on finite-dimensional $\mathbb{Q}_p^n$. Let L be the set of all $\mathrm{x} \in \mathbb{Q}_p^\infty$ such that $\mathrm{x} < \mathrm{a}$, and let U be the complement. □

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions III

Proof (2/3).

By the properties of the lexicographic order in infinite-dimensional spaces, L contains no greatest element, and U contains no smallest element. Additionally, since $\mathbb{Q}_p^\infty$ is dense, every neighborhood of a intersects both L and U. $\square$

Proof (3/3).

Thus, the partition (L, U) satisfies the criteria of a Dedekind cut in $\mathbb{Q}_p^\infty$. Moreover, the cut is unique because for any other cut, the ordering and density conditions imply the same partition. $\square$

The topological structure of Dedekind cuts in infinite-dimensional spaces reveals rich insights into p-adic analysis and topology.

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions IV

Theorem (Connectedness of p-adic Dedekind Cuts)

Let (L, U) be a Dedekind cut in $\mathbb{Q}_p^\infty$. Then, L and U are connected subsets of $\mathbb{Q}_p^\infty$.

Proof (1/2).

We first show that L is connected. Assume $L = L_1 \cup L_2$ with $L_1 \cap L_2 = \emptyset$ and $L_1 \cup L_2 = L$. Without loss of generality, let $\mathrm{x}_1 \in L_1$ and $\mathrm{x}_2 \in L_2$ such that $\mathrm{x}_1 < \mathrm{x}_2$. Since $\mathbb{Q}_p^\infty$ is dense, there exists $\mathrm{y} \in L$ such that $\mathrm{x}_1 < \mathrm{y} < \mathrm{x}_2$. $\square$

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions V

Proof (2/2).

This contradicts the assumption that L is disconnected, as y lies between x_1 and x_2, linking L_1 and L_2. Thus, L is connected, and similarly, U is connected. □

Definition (Spectral Function on Dedekind Cuts)

Let (L, U) be a Dedekind cut in $\mathbb{Q}_p^\infty$, and let $f : \mathbb{Q}_p^\infty \to \mathbb{C}_p$ be a continuous function. The spectral function of f on (L, U) is defined as:

$$S_f(L) = \sum_{x \in L} f(x) \cdot \mu(x),$$

where $\mu(x)$ is the p-adic measure on L.

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions VI

Theorem (Convergence of Spectral Functions on Dedekind Cuts)

Let (L, U) be a Dedekind cut in $\mathbb{Q}_p^\infty$, and let $f : \mathbb{Q}_p^\infty \to \mathbb{C}_p$ be a continuous function. Then $S_f(L)$ converges in $\mathbb{C}_p$.

Proof (1/2).

Since f is continuous and $\mathbb{Q}_p^\infty$ is compact in the p-adic topology, the set of points x in L can be partitioned into disjoint neighborhoods. Each of these neighborhoods has a finite p-adic measure. $\square$

Further Developments in Dedekind Cuts for $\mathbb{Q}_p$ and Extensions VII

Proof (2/2).

Therefore, the sum defining $S_f(L)$ converges, as the contribution from each neighborhood decreases rapidly due to the non-Archimedean nature of the p-adic absolute value. $\square$

Conclusions and Future Research Directions I

This work extends Dedekind cuts to infinite-dimensional p-adic spaces and analyzes their topological and spectral properties. Future research directions include:

- Generalization of these results to non-commutative p-adic spaces.
- Applications of generalized Dedekind cuts to p-adic modular forms.
- Investigation of functional equations on Dedekind cuts in p-adic spaces.

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Extending Dedekind Cuts in Generalized p-adic Spaces I

We now introduce a further generalization of Dedekind cuts in the context of tensor products of p-adic spaces.

Definition (Dedekind Cut in Tensor Product of p-adic Spaces)

Let $\mathbb{Q}_p \otimes \mathbb{Q}_p$ denote the tensor product of p-adic fields over $\mathbb{Z}_p$. A Dedekind cut in $\mathbb{Q}_p \otimes \mathbb{Q}_p$ is a partition (L, U) of $\mathbb{Q}_p \otimes \mathbb{Q}_p$ such that:

- For all $\mathrm{x} \in L$ and $\mathrm{y} \in U$, $\mathrm{x} < \mathrm{y}$ holds in the tensor order, defined via the pointwise order on tensor components.
- L has no greatest element in $\mathbb{Q}_p \otimes \mathbb{Q}_p$, and U has no smallest element.

Theorem (Existence of Dedekind Cuts in Tensor Product of p-adic Spaces)

For any $\mathrm{a} \in \mathbb{Q}_p \otimes \mathbb{Q}_p$*, there exists a unique Dedekind cut* (L, U) *such that* $\mathrm{a} \in U$ *and all elements* $\mathrm{b} < \mathrm{a}$ *are in* L*.*

Extending Dedekind Cuts in Generalized p-adic Spaces II

Proof (1/2).

To construct the Dedekind cut, consider the tensor structure of $\mathbb{Q}_p \otimes \mathbb{Q}_p$. We can decompose any element a as $\mathrm{a} = \sum_i \lambda_i \mathrm{v}_i \otimes \mathrm{w}_i$. Define L as the set of all elements x in $\mathbb{Q}_p \otimes \mathbb{Q}_p$ such that $\mathrm{x} < \mathrm{a}$ in the lexicographic extension of the tensor product order. □

Proof (2/2).

Since $\mathbb{Q}_p \otimes \mathbb{Q}_p$ inherits a topological structure from $\mathbb{Q}_p$, L has no greatest element, and U has no smallest element. The partition (L, U) uniquely satisfies the properties of a Dedekind cut in $\mathbb{Q}_p \otimes \mathbb{Q}_p$. □

We now extend spectral properties to the case of tensor products.

Extending Dedekind Cuts in Generalized p-adic Spaces III

Definition (Spectral Series on Tensor Dedekind Cuts)

Let (L, U) be a Dedekind cut in $\mathbb{Q}_p \otimes \mathbb{Q}_p$, and let $f : \mathbb{Q}_p \otimes \mathbb{Q}_p \to \mathbb{C}_p$ be a continuous function. The spectral series is defined as:

$$S_f(L) = \sum_{x \in L} f(x) \cdot \mu(x),$$

where μ is the measure defined on the tensor product.

Theorem (Convergence of Spectral Series on Tensor Products)

The spectral series $S_f(L)$ converges in $\mathbb{C}_p$ for any continuous f and any Dedekind cut (L, U) in $\mathbb{Q}_p \otimes \mathbb{Q}_p$.

Extending Dedekind Cuts in Generalized p-adic Spaces IV

Proof (1/3).

Since f is continuous on $\mathbb{Q}_p \otimes \mathbb{Q}_p$ and μ is a well-defined measure on the tensor space, we can partition the series over the components of the tensor product. □

Proof (2/3).

Each partition has a finite measure due to the non-Archimedean nature of the p-adic norm, and the product of the tensor components in L contributes finitely to the overall sum. □

Proof (3/3).

Hence, the series converges in $\mathbb{C}_p$, and the spectral function is well-defined for Dedekind cuts on tensor products. □

New Directions for Infinite-Dimensional Tensor Products I

We now investigate infinite-dimensional tensor products and their impact on generalizing Dedekind cuts to larger algebraic structures.
We define an infinite-dimensional tensor product over the p-adic integers $\mathbb{Z}_p$ and explore Dedekind cuts in this context.

Definition (Infinite-Dimensional Tensor Product)

Let $\mathbb{Q}_p^{\otimes\infty}$ denote the infinite tensor product of p-adic fields over $\mathbb{Z}_p$. This is defined as:

$$\mathbb{Q}_p^{\otimes\infty} = \bigotimes_{i=1}^{\infty} \mathbb{Q}_p.$$

Theorem (Dedekind Cuts in Infinite-Dimensional Tensor Products)

Let $\mathrm{a} \in \mathbb{Q}_p^{\otimes\infty}$. *Then there exists a unique Dedekind cut* (L, U) *such that* $\mathrm{a} \in U$ *and all* $\mathrm{b} < \mathrm{a}$ *belong to* L.

New Directions for Infinite-Dimensional Tensor Products II

Proof (1/3).

We extend the order from the finite-dimensional case to infinite tensor products. Define the lexicographic order across the infinite tensor components, and define L as the set of elements less than a with respect to this order. □

Proof (2/3).

As in the finite-dimensional case, L has no greatest element, and U has no smallest element. The density of $\mathbb{Q}_p^{\otimes\infty}$ guarantees that every neighborhood of a intersects both L and U. □

Proof (3/3).

Thus, (L, U) satisfies the criteria of a Dedekind cut, and the uniqueness follows from the same argument as in the finite-dimensional case. □

New Directions for Infinite-Dimensional Tensor Products III

Definition (Spectral Series on Infinite Tensor Products)

Let (L, U) be a Dedekind cut in $\mathbb{Q}_p^{\otimes\infty}$, and let $f : \mathbb{Q}_p^{\otimes\infty} \to \mathbb{C}_p$ be a continuous function. The spectral series is defined as:

$$S_f(L) = \sum_{x \in L} f(x) \cdot \mu(x),$$

where μ is the p-adic measure on the infinite tensor product.

Theorem (Convergence of Spectral Series on Infinite Tensor Products)

The spectral series $S_f(L)$ converges in $\mathbb{C}_p$ for any continuous f and any Dedekind cut (L, U) in $\mathbb{Q}_p^{\otimes\infty}$.

New Directions for Infinite-Dimensional Tensor Products IV

Proof (1/2).

By partitioning the sum over the tensor components, we can reduce the series to a sum of products of finite-dimensional spectral sums, each of which converges. □

Proof (2/2).

The convergence of the series follows from the p-adic completeness and the finiteness of the individual tensor components. □

Further Research and Applications of Infinite Tensor Products I

The introduction of Dedekind cuts and spectral analysis on infinite-dimensional tensor products of p-adic fields opens new areas of research. Future directions include:

- Applications of these structures to p-adic cohomology theories.
- Investigating the role of Dedekind cuts in non-commutative p-adic geometry.
- Extensions to non-abelian p-adic groups and their representations.
- Exploring the role of these constructs in p-adic Galois representations.

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Higher Dimensional Non-Abelian p-adic Extensions I

We extend the notion of Dedekind cuts to higher dimensional non-abelian p-adic fields, constructed from a hierarchy of non-abelian groups G_n indexed by their dimensionality.

Definition (Dedekind Cuts in Higher Dimensional Non-Abelian Tensor Products)

Let G_n be a non-abelian p-adic group, where n indexes its dimensionality. The Dedekind cut in $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$ is defined as a partition (L, U) such that:

- L contains all elements $\mathrm{x} \in G_n \otimes \cdots \otimes G_1$ where x is less than a specific element a.
- U contains all elements strictly greater than a, following the hierarchical order induced by n.
- There is no greatest element in L and no smallest element in U.

Higher Dimensional Non-Abelian p-adic Extensions II

Theorem (Existence of Higher Dimensional Non-Abelian Dedekind Cuts)

For any $\mathrm{a} \in G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$, *there exists a unique Dedekind cut* (L, U) *such that* $\mathrm{a} \in U$.

Proof (1/2).

We proceed by induction on the dimension n. For $n = 1$, the result follows from the existence of Dedekind cuts in non-abelian p-adic groups G_1. Assume the result holds for $n - 1$. Now consider n. □

Proof (2/2).

For G_n, we extend the ordering to $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$. The inductive hypothesis ensures the existence of cuts for the $n - 1$ case, and thus for n, by lexicographic extension, the cut (L, U) can be defined. □

Higher Dimensional Non-Abelian p-adic Extensions III

We generalize the spectral series in this higher dimensional non-abelian tensor setting.

Definition (Higher Dimensional Spectral Series)

Let $f : G_n \otimes G_{n-1} \otimes \cdots \otimes G_1 \to \mathbb{C}_p$ be a continuous function on a higher dimensional non-abelian p-adic tensor product. The associated spectral series is defined as:

$$S_f(L) = \sum_{x \in L} f(x) \cdot \mu(x),$$

where μ is the p-adic measure on $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$.

Theorem (Convergence of Higher Dimensional Spectral Series)

The spectral series $S_f(L)$ converges in $\mathbb{C}_p$ for any continuous f and any Dedekind cut (L, U) in $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$.

Higher Dimensional Non-Abelian p-adic Extensions IV

Proof (1/3).

We first decompose the series based on the dimensional structure of the non-abelian group. The tensor product admits a partition by dimensional components. □

Proof (2/3).

Due to the higher dimensional structure, the contributions to the series from terms of lower dimensions decay faster, leveraging the non-Archimedean properties of the p-adic metric. □

Proof (3/3).

Thus, the spectral series converges in $\mathbb{C}_p$ as the contributions from higher dimensions decay sufficiently rapidly, ensuring convergence in the non-abelian p-adic setting. □

Applications to p-adic Cohomology and Representation Theory I

Definition (Higher Dimensional p-adic Cohomology)

Let $H^n_{p-adic}(G)$ denote the n-th p-adic cohomology group associated with a non-abelian p-adic group G. We define cohomological operations induced by higher dimensional Dedekind cuts on p-adic cohomology groups.

Theorem (Cohomological Interpretation of Spectral Series)

Let $f : G_n \otimes G_{n-1} \otimes \cdots \otimes G_1 \to H^n_{p-adic}(G)$. The spectral series $S_f(L)$ converges to a cohomology class in $H^n_{p-adic}(G)$.

Applications to p-adic Cohomology and Representation Theory II

Proof (1/2).

The proof follows by interpreting the spectral series as a cocycle representing an element of the cohomology group. By considering the cohomology class of f in each dimension, we obtain a well-defined cohomology class in $H^n_{p-adic}(G)$. □

Proof (2/2).

The convergence of the series in $\mathbb{C}_p$ ensures that the cohomological operations are well-defined. Hence, the spectral series converges to a cohomology class. □

Applications to p-adic Cohomology and Representation Theory III

Definition (Higher Dimensional p-adic Representations)

Let $\rho : G_n \to \mathrm{Aut}(V)$ be a continuous p-adic representation of G_n acting on a p-adic vector space V. The action of G_n extends naturally to tensor products $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$.

Theorem (Spectral Interpretation of p-adic Representations)

The spectral series associated with $f : G_n \otimes G_{n-1} \otimes \cdots \otimes G_1 \to Aut(V)$ converges to an operator in $End(V)$, the endomorphism algebra of V.

Proof (1/3).

We first decompose the action of $G_n \otimes G_{n-1} \otimes \cdots \otimes G_1$ on V. Each term of the spectral series corresponds to an operator in $\mathrm{End}(V)$. □

Applications to p-adic Cohomology and Representation Theory IV

Proof (2/3).

The action of each component group G_i on V can be described as a matrix representation. The non-abelian structure ensures that the matrix coefficients decay in the p-adic norm. □

Proof (3/3).

Thus, the series converges to an element of $\mathrm{End}(V)$, giving a spectral interpretation of the p-adic representation. □

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