Iterative Construction of Algebraic Structures Between Group-Like Objects

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September 5, 2024

Abstract

In this paper, we systematically develop an infinite sequence of algebraic structures between traditional group-like objects. By iteratively introducing and refining properties such as associativity, identity, and invertibility, we construct new mathematical objects that serve as operators on groups. This approach is inspired by similar developments between vector spaces and fields. We present the first five iterations of this process, including rigorous descriptions and diagrams illustrating the relationships between these structures.

1 Motivation

The motivation behind this work is to explore the algebraic landscape between well-known group-like structures such as Magma, Quasigroup, Semigroup, Loop, Monoid, and Group. Similar to how the space between vector spaces and fields was systematically expanded by introducing intermediate structures, we aim to develop an infinite series of new algebraic structures. These structures are built by iteratively refining properties like associativity, identity, and invertibility, thus expanding the traditional algebraic framework and providing new tools for advanced mathematical applications.

2 Introduction

Group theory and its related algebraic structures form a cornerstone of modern mathematics, with applications ranging from abstract algebra to cryptography and quantum computing. However, the space between these traditional structures has not been fully explored. This paper presents a systematic approach to construct new algebraic objects by iteratively refining and combining properties such as associativity, identity, and invertibility. These objects are treated as operators on groups, analogous to the method used to explore spaces between vector spaces and fields.

3 The Iterative Process

The iterative process we perform is as follows:

- First iteration: Introduce basic intermediate structures by relaxing or strengthening properties such as associativity, identity, and invertibility.
- Second iteration: Further refine these structures by adding additional layers or combining properties, creating multi-level or hybrid structures.
- Third iteration: Introduce more complexity by layering these properties further and exploring their interactions, leading to highly specialized structures.
- Subsequent iterations: Continue this process, systematically increasing the complexity and specificity of the structures.

4 First Iteration

In the first iteration, we introduce basic intermediate structures between the traditional group-like objects.

4.1 Definitions and Potential Uses

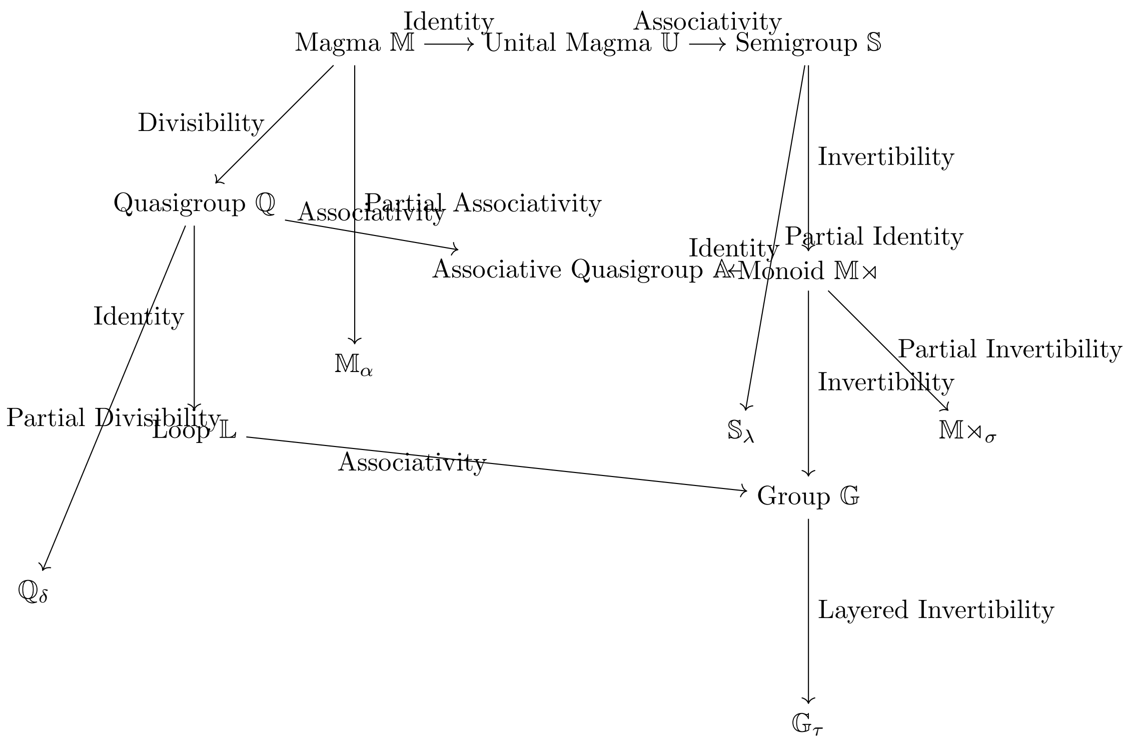
- $\mathbb{M}_\alpha$**: Partially Associative Magma**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a partially associative magma $\mathbb{M}_\alpha$ if there exists a subset $\alpha \subseteq G \times G \times G$ such that for all $(a, b, c) \in \alpha$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
  Potential Uses: Could be used in modeling systems where associativity only holds for certain operations or within certain contexts, such as in computer science for partial parallel computations or in physics for localized interactions.
- $\mathbb{Q}_\delta$**: Partially Divisible Quasigroup**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a partially divisible quasigroup $\mathbb{Q}_\delta$ if there exists a subset $\delta \subseteq G \times G$ such that for all $(a, b) \in \delta$, there exist unique $x, y \in G$ satisfying $a \cdot x = b$ and $y \cdot a = b$.
  Potential Uses: Applicable in cryptography, where partial invertibility could be useful in designing secure encryption algorithms that require certain operations to be reversible under specific conditions.
- $\mathbb{S}_\lambda$**: Semigroup with Partial Identity**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a semigroup with partial identity $\mathbb{S}_\lambda$ if there exists a subset $\lambda \subseteq G$ such that for all $e \in \lambda$ and $a \in G$, $e \cdot a = a \cdot e = a$.
  Potential Uses: Could be used in algebraic coding theory, where certain

elements act as identity elements within specific subsets, allowing for more flexible and efficient encoding schemes.

- $\mathbb{M}\rtimes_\sigma$: **Monoid with Partial Invertibility**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a monoid with partial invertibility $\mathbb{M}\rtimes_\sigma$ if there exists a subset $\sigma \subseteq G$ such that for all $a \in \sigma$, there exists $a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$.
  Potential Uses: Useful in systems where some elements are invertible while others are not, such as in fault-tolerant computing or in designing algorithms with partially reversible steps.

- $\mathbb{G}_\tau$: **Group with Layered Invertibility**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a group with layered invertibility $\mathbb{G}_\tau$ if G can be partitioned into subsets G_τ such that each G_τ is closed under inversion, and for each $a \in G_\tau$, there exists $a^{-1} \in G_\tau$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$.
  Potential Uses: Could be applied in complex systems where different layers or levels of invertibility are needed, such as in hierarchical encryption schemes or multi-level decision-making processes.

4.2 Diagram



5 Second Iteration

In the second iteration, we refine the previous structures by adding additional layers or combining properties.

5.1 Definitions and Potential Uses

- **$\mathbb{M}_\alpha^{(n)}$: n-level Partially Associative Magma**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is an n-level partially associative magma $\mathbb{M}_\alpha^{(n)}$ if there exist n subsets $\alpha_1, \alpha_2, \ldots, \alpha_n \subseteq G \times G \times G$ such that for all $(a, b, c) \in \alpha_i$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for each $i = 1, 2, \ldots, n$.
  Potential Uses: Useful in hierarchical data structures or systems where multiple levels of associativity are needed, such as multi-level parallel computing or layered network protocols.

- **$\mathbb{Q}_{\xi,\rho}$: Quasi-Invertible Quasigroup**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quasi-invertible quasigroup $\mathbb{Q}_{\xi,\rho}$ if there exist subsets $\xi, \rho \subseteq G \times G$ such that for all $(a, b) \in \xi$, there exist $x, y \in G$ with $a \cdot x = b$ and $y \cdot a = b$, and for all $(a, b) \in \rho$, there exist $x', y' \in G$ such that $x' \cdot a = b$

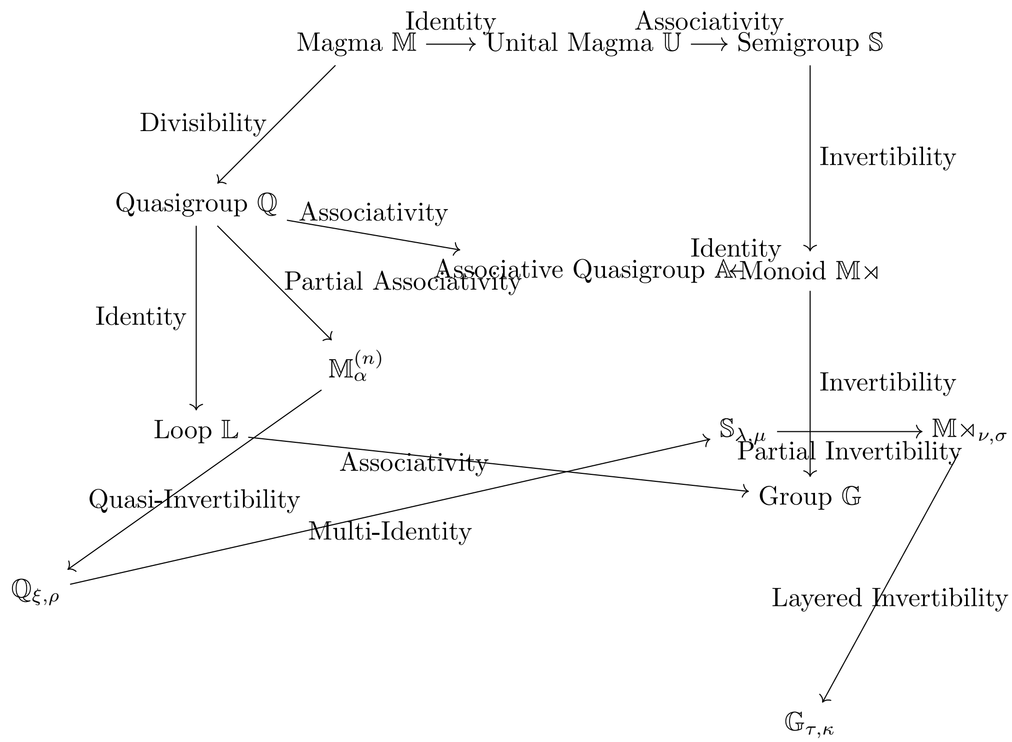
and $a \cdot y' = b$.
Potential Uses: Applicable in cryptographic systems where operations need partial invertibility, providing flexibility in security protocols or reversible computations.

- $\mathbb{S}_{\lambda,\mu}$**: Multi-Identity Semigroup**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a multi-identity semigroup $\mathbb{S}_{\lambda,\mu}$ if there exist subsets $\lambda, \mu \subseteq G$ such that for all $e_1 \in \lambda$, $e_2 \in \mu$, and $a \in G$, $e_1 \cdot a = a \cdot e_2 = a$.
  Potential Uses: Could be used in complex algebraic coding or modular systems where different identities are needed depending on the operation or context.

- $\mathbb{M}\rtimes_{\nu,\sigma}$**: Multi-Identity Monoid with Partial Invertibility**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a multi-identity monoid $\mathbb{M}\rtimes_{\nu,\sigma}$ if there exist subsets $\nu, \sigma \subseteq G$ such that for all $a \in \nu$, a has an inverse $a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$, and for all $b \in \sigma$, $b \cdot b^{-1} = b^{-1} \cdot b = e'$, where $e' \in G$ is another identity.
  Potential Uses: Useful in modular cryptography, where multiple identities and partial inverses are necessary, or in designing complex modular arithmetic systems.

- $\mathbb{G}_{\tau,\kappa}$**: Layered Invertible Group**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a layered invertible group $\mathbb{G}_{\tau,\kappa}$ if G can be partitioned into subsets G_τ and G_κ such that for each $a \in G_\tau$ and $b \in G_\kappa$, there exist $a^{-1} \in G_\tau$ and $b^{-1} \in G_\kappa$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$, and $b \cdot b^{-1} = b^{-1} \cdot b = e$.
  Potential Uses: Relevant in hierarchical cryptographic systems or multi-layered decision-making processes where layered operations and inverses are necessary.

5.2 Diagram



6 Third Iteration

In the third iteration, we introduce additional layers of complexity, leading to even more specialized structures.

6.1 Definitions and Potential Uses

- $\mathbb{M}_\alpha^{(n,m)}$**: Double-Layer Partially Associative Magma**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a double-layer partially associative magma $\mathbb{M}_\alpha^{(n,m)}$ if there exist subsets $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m \subseteq G \times G \times G$ such that for all $(a, b, c) \in \alpha_i$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for each $i = 1, 2, \ldots, n$, and similarly for β_j, $j = 1, 2, \ldots, m$.
  Potential Uses: Applicable in complex hierarchical systems or networks where multiple, layered associativity operations are needed, such as in layered protocol stacks or hierarchical data processing.

- $\mathbb{Q}_{\xi,\rho,\sigma}$**: Triple Quasi-Invertible Quasigroup**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a triple quasi-invertible quasigroup $\mathbb{Q}_{\xi,\rho,\sigma}$ if there exist subsets $\xi, \rho, \sigma \subseteq G \times G$ such that for all $(a, b) \in \xi$, ρ, or σ, there exist

$x, y \in G$ with $a \cdot x = b$ and $y \cdot a = b$, and $x', y' \in G$ such that $x' \cdot a = b$ and $a \cdot y' = b$.
Potential Uses: Useful in advanced cryptographic systems where multiple layers of partial invertibility are needed, ensuring different security levels for different operations.

- $\mathbb{S}_{\lambda,\mu,\nu}$: **Triple-Identity Semigroup**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a triple-identity semigroup $\mathbb{S}_{\lambda,\mu,\nu}$ if there exist subsets $\lambda, \mu, \nu \subseteq G$ such that for all $e_1 \in \lambda$, $e_2 \in \mu$, $e_3 \in \nu$, and $a \in G$, $e_1 \cdot a = a \cdot e_2 = a \cdot e_3 = a$.
Potential Uses: Could be applied in complex modular systems or coding theory where different identities are necessary depending on the operation.

- $\mathbb{M}\rtimes_{\nu,\sigma,\tau}$: **Multi-Identity Monoid with Layered Invertibility**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a multi-identity monoid $\mathbb{M}\rtimes_{\nu,\sigma,\tau}$ if there exist subsets $\nu, \sigma, \tau \subseteq G$ such that for all $a \in \nu$, a has an inverse $a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$, and for all $b \in \sigma$ and $c \in \tau$, $b \cdot b^{-1} = b^{-1} \cdot b = e'$, and $c \cdot c^{-1} = c^{-1} \cdot c = e''$, where $e', e'' \in G$ are different identities.
Potential Uses: Useful in advanced modular arithmetic systems or cryptography, where layered inverses and multiple identities are required.

- $\mathbb{G}_{\tau,\kappa,\lambda}$: **Triple-Layer Invertible Group**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a triple-layer invertible group $\mathbb{G}_{\tau,\kappa,\lambda}$ if G can be partitioned into subsets G_τ, G_κ, and G_λ such that for each $a \in G_\tau$, $b \in G_\kappa$, and $c \in G_\lambda$, there exist $a^{-1} \in G_\tau$, $b^{-1} \in G_\kappa$, and $c^{-1} \in G_\lambda$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$, $b \cdot b^{-1} = b^{-1} \cdot b = e$, and $c \cdot c^{-1} = c^{-1} \cdot c = e$.
Potential Uses: Applicable in quantum computing or control systems, where multiple layers of inversion are needed, each with its own distinct operation.

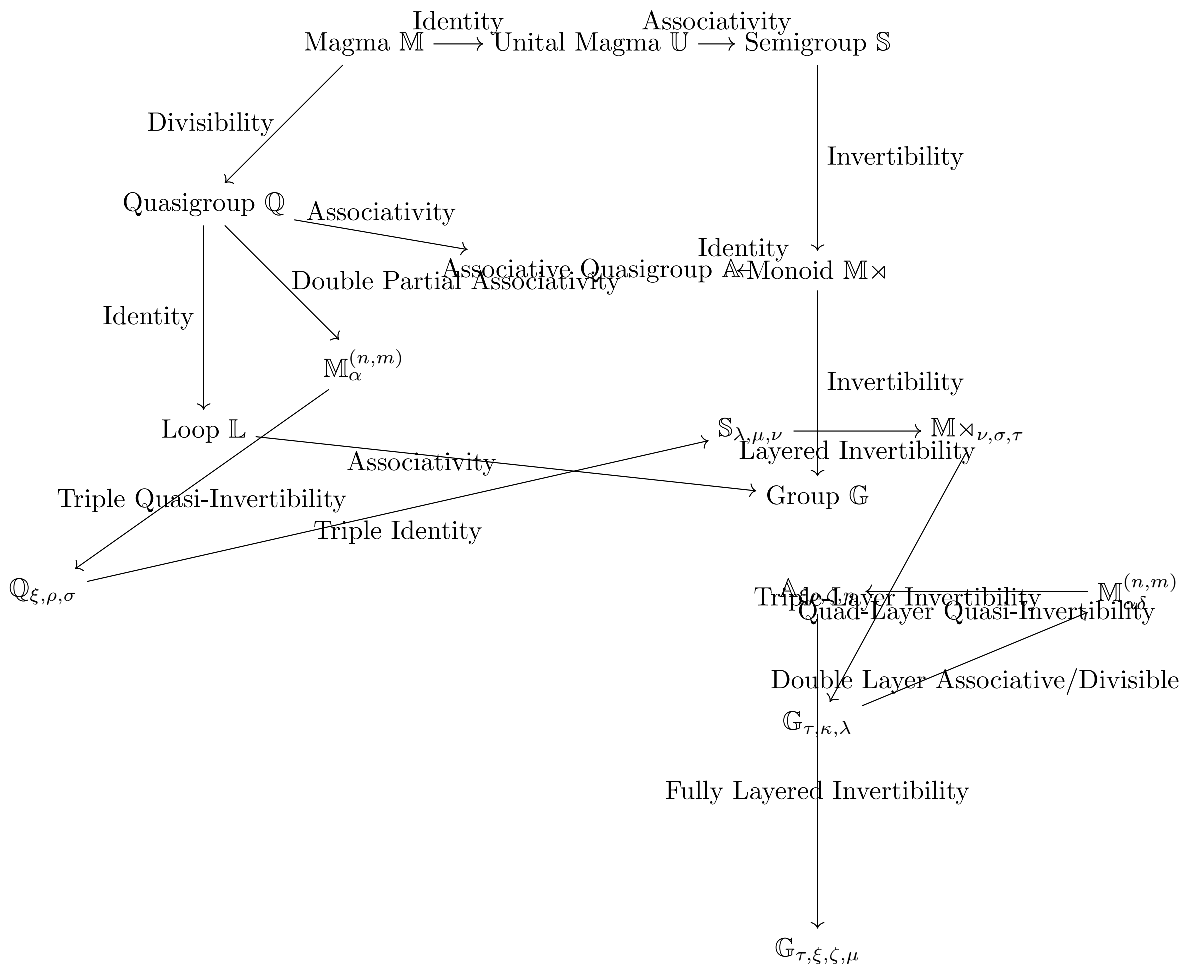
- $\mathbb{M}_{\alpha\delta}^{(n,m)}$: **Double-Layer Partially Associative and Divisible Magma**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a double-layer partially associative and divisible magma $\mathbb{M}_{\alpha\delta}^{(n,m)}$ if there exist subsets $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\delta_1, \delta_2, \ldots, \delta_m \subseteq G \times G$ such that for all $(a, b, c) \in \alpha_i$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for each $i = 1, 2, \ldots, n$, and for all $(a, b) \in \delta_j$, there exist $x, y \in G$ satisfying $a \cdot x = b$ and $y \cdot a = b$.
Potential Uses: Could be used in layered network protocols or hierarchical data processing systems that require multiple, layered operations.

- $\mathbb{A}_{\xi\rho,\zeta,\eta}$: **Quad-Layer Quasi-Invertible Associative Quasigroup**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quad-layer quasi-invertible associative quasigroup

$\mathbb{A}_{\xi\rho,\zeta,\eta}$ if there exist subsets $\xi, \rho, \zeta, \eta \subseteq G \times G$ such that for all $(a, b) \in \xi$, ρ, ζ, or η, there exist $x, y \in G$ with $a \cdot x = b$ and $y \cdot a = b$, and for all $(a, b, c) \in G \times G \times G$, associativity holds: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Potential Uses: Useful in advanced encryption algorithms or error-correcting codes where layered, partial invertibility and associativity are necessary.

- $\mathbb{G}_{\tau,\xi,\zeta,\mu}$: **Fully Layered Invertible Group**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a fully layered invertible group $\mathbb{G}_{\tau,\xi,\zeta,\mu}$ if G can be partitioned into subsets G_τ, G_ξ, G_ζ, and G_μ such that for each $a \in G_\tau$, $b \in G_\xi$, $c \in G_\zeta$, and $d \in G_\mu$, there exist $a^{-1} \in G_\tau$, $b^{-1} \in G_\xi$, $c^{-1} \in G_\zeta$, and $d^{-1} \in G_\mu$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$, $b \cdot b^{-1} = b^{-1} \cdot b = e$, $c \cdot c^{-1} = c^{-1} \cdot c = e$, and $d \cdot d^{-1} = d^{-1} \cdot d = e$.
Potential Uses: Could be applied in multi-dimensional data structures, quantum algorithms, or control theory, where deeply nested or layered inverses and operations are needed.

6.2 Diagram



7 Fourth Iteration

7.1 Definitions and Potential Uses

- $\mathbb{M}\alpha^{(n,m,p)}$**: Triple-Layer Partially Associative Magma** *Definition*: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a triple-layer partially associative magma $\mathbb{M}\alpha^{(n,m,p)}$ if there exist subsets $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_m$, and $\gamma_1, \gamma_2, \ldots, \gamma_p \subseteq G \times G \times G$ such that for all $(a, b, c) \in \alpha_i$, β_j, or γ_k, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$. *Potential Uses*: Useful in modeling extremely complex hierarchical systems or networks that require multiple, nested associativity conditions, such as in advanced layered computing architectures or multi-tier data processing frameworks.

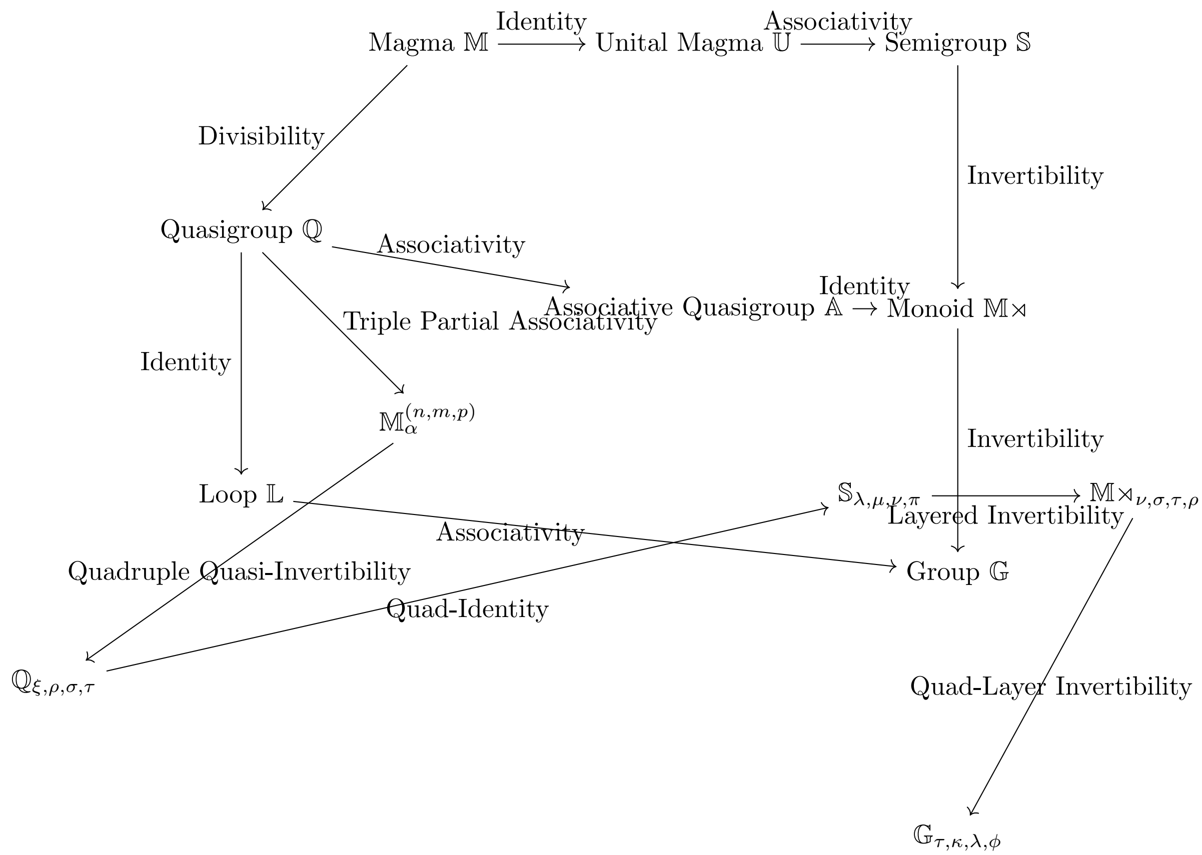
- $\mathbb{Q}_{\xi,\rho,\sigma,\tau}$: **Quadruple Quasi-Invertible Quasigroup**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quadruple quasi-invertible quasigroup $\mathbb{Q}_{\xi,\rho,\sigma,\tau}$ if there exist subsets $\xi, \rho, \sigma, \tau \subseteq G \times G$ such that for all $(a, b) \in \xi$, ρ, σ, or τ, there exist $x, y \in G$ with $a \cdot x = b$ and $y \cdot a = b$, and $x', y' \in G$ such that $x' \cdot a = b$ and $a \cdot y' = b$.
Potential Uses: Applicable in cryptographic systems where high levels of security and complexity are required, particularly in scenarios where operations need to be reversible under different conditions, ensuring multiple layers of encryption or decryption.

- $\mathbb{S}_{\lambda,\mu,\nu,\pi}$: **Quad-Identity Semigroup**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quad-identity semigroup $\mathbb{S}_{\lambda,\mu,\nu,\pi}$ if there exist subsets $\lambda, \mu, \nu, \pi \subseteq G$ such that for all $e_1 \in \lambda$, $e_2 \in \mu$, $e_3 \in \nu$, $e_4 \in \pi$, and $a \in G$, $e_1 \cdot a = a \cdot e_2 = a \cdot e_3 = a \cdot e_4 = a$.
Potential Uses: Could be applied in advanced algebraic systems where multiple identities are necessary depending on the operation or in scenarios requiring highly complex modular arithmetic or coding systems.

- $\mathbb{M}\rtimes_{\nu,\sigma,\tau,\rho}$: **Multi-Identity Monoid with Quad-Layer Invertibility**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a multi-identity monoid $\mathbb{M}\rtimes_{\nu,\sigma,\tau,\rho}$ if there exist subsets $\nu, \sigma, \tau, \rho \subseteq G$ such that for all $a \in \nu$, $b \in \sigma$, $c \in \tau$, $d \in \rho$, there exist $a^{-1}, b^{-1}, c^{-1}, d^{-1} \in G$ with $a \cdot a^{-1} = e$, $b \cdot b^{-1} = e'$, $c \cdot c^{-1} = e''$, and $d \cdot d^{-1} = e'''$, where e', e'', e''' are different identities in G.
Potential Uses: Useful in highly complex modular systems or cryptography, where multiple identities and layered inverses are required for secure operations and computations.

- $\mathbb{G}_{\tau,\kappa,\lambda,\phi}$: **Quad-Layer Invertible Group**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a quad-layer invertible group $\mathbb{G}_{\tau,\kappa,\lambda,\phi}$ if G can be partitioned into subsets G_τ, G_κ, G_λ, and G_ϕ such that for each $a \in G_\tau$, $b \in G_\kappa$, $c \in G_\lambda$, and $d \in G_\phi$, there exist $a^{-1}, b^{-1}, c^{-1}, d^{-1} \in G$ with $a \cdot a^{-1} = e$, $b \cdot b^{-1} = e'$, $c \cdot c^{-1} = e''$, and $d \cdot d^{-1} = e'''$.
Potential Uses: Relevant in advanced quantum computing, multi-dimensional control systems, or any context requiring deeply nested layers of invertibility and operations.

7.2 Diagram



8 Fifth Iteration

8.1 Definitions and Potential Uses

- **$\mathbb{M}\alpha^{(n,m,p,q)}$: Quadruple-Layer Partially Associative Magma**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quadruple-layer partially associative magma $\mathbb{M}\alpha^{(n,m,p,q)}$ if there exist subsets $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_m$, $\gamma_1, \gamma_2, \ldots, \gamma_p$, and $\delta_1, \delta_2, \ldots, \delta_q \subseteq G \times G \times G$ such that for all $(a, b, c) \in \alpha_i$, β_j, γ_k, or δ_ℓ, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$, and $\ell = 1, 2, \ldots, q$.
  Potential Uses: Applicable in ultra-complex systems or computational models where multiple, deeply nested layers of associativity are necessary, such as in the most advanced hierarchical network protocols or deep learning architectures.

- **$\mathbb{Q}_{\xi,\rho,\sigma,\tau,\omega}$: Quintuple Quasi-Invertible Quasigroup**
  Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The

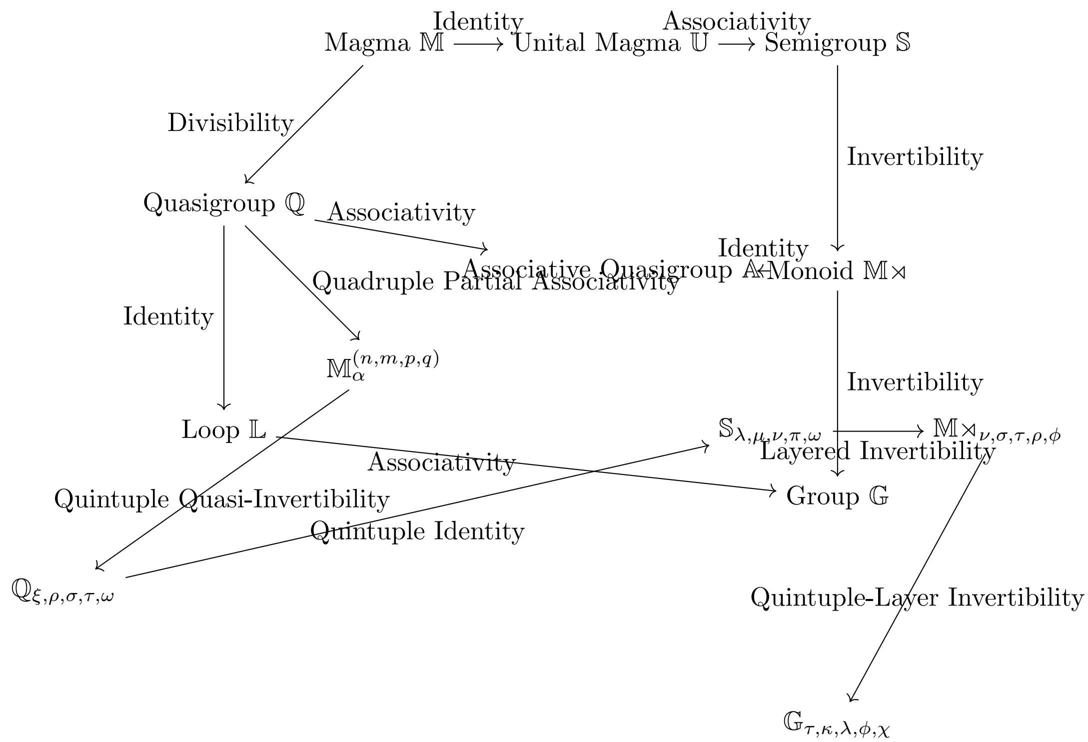
structure $(G, \cdot)$ is a quintuple quasi-invertible quasigroup $\mathbb{Q}_{\xi,\rho,\sigma,\tau,\omega}$ if there exist subsets $\xi, \rho, \sigma, \tau, \omega \subseteq G \times G$ such that for all $(a, b) \in \xi$, ρ, σ, τ, or ω, there exist $x, y \in G$ with $a \cdot x = b$ and $y \cdot a = b$, and $x', y' \in G$ such that $x' \cdot a = b$ and $a \cdot y' = b$.
Potential Uses: Useful in the most sophisticated cryptographic systems or secure communications protocols where multiple, nested layers of reversibility are essential for securing data and operations.

- $\mathbb{S}_{\lambda,\mu,\nu,\pi,\omega}$**: Quintuple-Identity Semigroup**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$. The structure $(G, \cdot)$ is a quintuple-identity semigroup $\mathbb{S}_{\lambda,\mu,\nu,\pi,\omega}$ if there exist subsets $\lambda, \mu, \nu, \pi, \omega \subseteq G$ such that for all $e_1 \in \lambda$, $e_2 \in \mu$, $e_3 \in \nu$, $e_4 \in \pi$, $e_5 \in \omega$, and $a \in G$, $e_1 \cdot a = a \cdot e_2 = a \cdot e_3 = a \cdot e_4 = a \cdot e_5 = a$.
Potential Uses: Could be applied in highly advanced modular systems or coding theories where extremely complex operations require multiple identity elements.

- $\mathbb{M}\rtimes_{\nu,\sigma,\tau,\rho,\phi}$**: Multi-Identity Monoid with Quintuple-Layer Invertibility**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a multi-identity monoid $\mathbb{M}\rtimes_{\nu,\sigma,\tau,\rho,\phi}$ if there exist subsets $\nu, \sigma, \tau, \rho, \phi \subseteq G$ such that for all $a \in \nu$, $b \in \sigma$, $c \in \tau$, $d \in \rho$, and $e \in \phi$, there exist $a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1} \in G$ with $a \cdot a^{-1} = e$, $b \cdot b^{-1} = e'$, $c \cdot c^{-1} = e''$, $d \cdot d^{-1} = e'''$, and $e \cdot e^{-1} = e''''$, where e', e'', e''', e'''' are different identities in G.
Potential Uses: Useful in the most complex modular arithmetic systems, quantum computing, or multi-layered cryptography, where multiple identities and layered inverses are essential for secure and efficient operations.

- $\mathbb{G}_{\tau,\kappa,\lambda,\phi,\chi}$**: Quintuple-Layer Invertible Group**
Definition: Let G be a set with a binary operation $\cdot : G \times G \to G$ and an identity element $e \in G$. The structure $(G, \cdot, e)$ is a quintuple-layer invertible group $\mathbb{G}_{\tau,\kappa,\lambda,\phi,\chi}$ if G can be partitioned into subsets G_τ, G_κ, G_λ, G_ϕ, and G_χ such that for each $a \in G_\tau$, $b \in G_\kappa$, $c \in G_\lambda$, $d \in G_\phi$, and $e \in G_\chi$, there exist $a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1} \in G$ with $a \cdot a^{-1} = e$, $b \cdot b^{-1} = e'$, $c \cdot c^{-1} = e''$, $d \cdot d^{-1} = e'''$, and $e \cdot e^{-1} = e''''$.
Potential Uses: Relevant in the most advanced areas of quantum computing, complex control theory, and multi-dimensional data structures where deeply nested layers of invertibility are necessary for the system's integrity and functionality.

8.2 Diagram



9 Conclusion and Future Work

This above part introduces a systematic approach to developing new algebraic structures between group-like objects. We explored the first five iterations in detail, revealing a rich landscape of algebraic possibilities. The iterative process outlined here can be extended indefinitely, leading to increasingly complex and specialized structures. Future work may involve exploring specific applications of these structures in cryptography, quantum computing, and advanced control systems.

Cardinality and Construction of Intermediate Algebraic Structures Between Magmas and Groups

Abstract

This below investigates the cardinality of the set of intermediate algebraic structures that exist between well-defined algebraic systems such as magmas, monoids, semigroups, and groups. Specifically, we analyze the cardinality of the continuum, $\mathfrak{c}$, which arises when considering the number of possible structures in each transition from magmas to monoids, monoids to semigroups, and semigroups to groups, particularly when the cardinality of the underlying set n becomes infinite. We also provide explicit methods for constructing these intermediate structures, demonstrating the rich variety of possible algebraic systems. Furthermore, we explore a vast array of adjectives and ideas that can be applied to generalize and extend algebraic structures, leading to an infinite landscape of possible mathematical objects.

10 Introduction

Algebraic structures such as magmas, monoids, semigroups, and groups are fundamental in mathematics, each defined by specific properties related to associativity, identity elements, and inverses. The study of intermediate structures between these well-known systems provides insights into the complexity and richness of algebraic operations. This paper examines the number of such intermediate structures and demonstrates that, in the case of infinite sets, the cardinality of these structures is $\mathfrak{c}$, the cardinality of the continuum. Furthermore, we provide explicit methods for constructing these intermediate structures, highlighting the diversity of possible configurations. Additionally, we explore an infinite number of adjectives and ideas that can be used to generalize algebraic structures further, leading to a boundless exploration of mathematical possibilities.

11 Cardinality of Structures Between Magmas and Groups

To understand the full scope of intermediate structures, we consider the transitions from magmas to monoids, monoids to semigroups, and semigroups to groups. At each step, we introduce or relax certain properties such as associativity, the existence of an identity element, and the existence of inverses.

11.1 Magmas to Monoids

Magmas are sets M equipped with a binary operation $\cdot$ but with no additional properties such as associativity or identity elements.

Monoids are magmas with the additional property of having an identity element e and an associative operation.

The transition from magmas to monoids involves introducing associativity and identity elements, which can be defined partially, conditionally, or in layers. For an infinite set M, the number of ways to define these properties is vast, and the total number of possible intermediate structures between magmas and monoids is $\mathfrak{c}$.

11.2 Monoids to Semigroups

Monoids possess an identity element and associative operation.

Semigroups are defined by associativity alone, without necessarily having an identity element.

In this case, we may consider the introduction of partial identities or the removal of identities in certain subsets of the monoid. For an infinite set M, the number of configurations for introducing these properties leads to a space of structures with cardinality $\mathfrak{c}$.

11.3 Semigroups to Groups

Semigroups have associative operations but do not require the existence of identity elements or inverses.

Groups are semigroups where every element has both an identity element and an inverse.

Transitioning from semigroups to groups involves introducing identity elements (if not already present) and inverses for each element. The introduction of partial or layered inverses contributes to a continuum of possible structures, resulting in a cardinality of $\mathfrak{c}$ when n is infinite.

12 Explicit Construction of Intermediate Structures

In this section, we detail methods for explicitly constructing the intermediate algebraic structures between magmas, monoids, semigroups, and groups.

12.1 Constructing Intermediate Structures Between Magmas and Monoids

To construct intermediate structures between magmas and monoids, we focus on introducing associativity and identity elements in a gradual and controlled manner:

1. **Partial Associativity:** Define a subset $A \subseteq M \times M \times M$ where associativity holds. For example, define $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $(a, b, c) \in A$, but not necessarily for all triples in $M \times M \times M$.

2. **Layered Associativity:** Partition M into disjoint subsets $M_1, M_2, \ldots, M_k$, and define associativity within each layer M_i. Associativity might hold in M_1 but not in M_2, or it may vary between layers.

3. **Conditional Associativity:** Define associativity based on an external condition or parameter. For instance, associativity could hold only when a certain condition $C(a, b, c)$ is satisfied.

4. **Partial Identity:** Introduce a partial identity by defining a subset $I \subseteq M$ such that $e \cdot a = a \cdot e = a$ for all $e \in I$ and $a \in M$. This identity might only act as an identity for a subset of the elements in M.

12.2 Constructing Intermediate Structures Between Monoids and Semigroups

To construct structures between monoids and semigroups, we explore variations in the identity element:

1. **Partial Identity Removal:** In a monoid $(M, \cdot, e)$, gradually remove the identity property for certain elements. Define a subset $S \subseteq M$ where the identity element e does not act as an identity, i.e., $e \cdot s \neq s$ for some $s \in S$.

2. **Layered Identity:** Define different identity elements within different layers of M. For example, e_1 might be the identity for layer M_1, while e_2 is the identity for layer M_2. The structure as a whole may not have a global identity element.

3. **Conditional Identity:** Introduce an identity element that acts as such only under specific conditions. For instance, e could be an identity only when a condition $C(e, a)$ holds true.

12.3 Constructing Intermediate Structures Between Semigroups and Groups

Finally, to construct intermediate structures between semigroups and groups, we introduce partial or conditional inverses:

1. **Partial Inverses:** In a semigroup $(S, \cdot)$, define a subset $I \subseteq S$ where for each $a \in I$, there exists an element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, where e is the identity element. Not every element in S needs to have an inverse.

2. **Layered Inverses:** Partition S into layers, and define inverses within specific layers. For example, elements in S_1 might have inverses, while those in S_2 do not.

3. **Conditional Inverses:** Define inverses that only exist under certain conditions. For example, an element $a \in S$ might have an inverse a^{-1} only if a condition $C(a)$ is satisfied.

13 Exploring an Infinite Landscape of Adjectives and Ideas

In addition to "partial," "layered," and "conditional," there are infinitely many other adjectives and ideas that can be applied to generalize and create new algebraic structures. Here are some additional examples:

13.1 Additional Adjectives and Concepts

1. **Fractal:** Introduces self-similar or recursive properties within the structure. This can be used to study structures where operations are recursively defined or where similar patterns recur within substructures.

2. **Stochastic:** Operations or properties are defined probabilistically rather than deterministically, which is useful in modeling systems with inherent randomness or uncertainty.

3. **Fuzzy:** Elements and operations do not have sharply defined characteristics but instead possess degrees of membership or participation, relevant in fuzzy logic systems.

4. **Discontinuous:** Operations or properties hold only in certain parts of the structure or under specific conditions, leading to discontinuities in behavior.

5. **Hybrid:** Combines properties from different algebraic structures within the same framework, leading to a "hybrid" structure that inherits features from both.

6. **Adaptive:** The structure evolves over time or in response to external stimuli, relevant in dynamic systems and control theory.

7. **Hierarchical:** The structure is organized in layers or levels, with different properties or operations applying at each level, useful in multiscale modeling and network theory.

8. **Contextual:** Operations and properties depend on the context or environment in which the structure is considered, relevant in parametric systems.

9. **Modular:** The structure is divided into modules or components, each with its own internal algebraic properties, relevant in modular arithmetic and component-based design.

10. **Quantum:** Operations might involve quantum properties such as superposition or entanglement, important in quantum computing.

11. **Approximate:** Properties or operations are approximately defined rather than exact, useful in numerical methods and optimization.

12. **Fractional:** Elements or operations might involve fractional or non-integer values, relevant in fractional calculus and scaling systems.

13. **Non-commutative:** Operations are not commutative, leading to structures with richer or more complex behaviors, important in quantum mechanics.

14. **Tropical:** Operations follow the rules of tropical algebra, where usual addition and multiplication are replaced by operations based on maximum (or minimum) and addition.

15. **Non-associative:** Operations are not necessarily associative, leading to structures where the grouping of operations affects the outcome, relevant in the study of Lie algebras.

16. **Dual:** The structure is defined in terms of its dual, where the roles of elements and operations are reversed or otherwise transformed, important in algebraic geometry.

17. **Graded:** The structure is decomposed into "grades" or "levels," each of which might have different algebraic properties, relevant in homological algebra.

18. **Multivalued:** Operations might produce multiple outputs rather than a single result, leading to a multivalued algebraic structure, useful in multi-functions and uncertainty modeling.

13.2 Infinite Possibilities

The list above provides just a glimpse into the vast array of adjectives and ideas that can be used to explore different algebraic structures. Each adjective corresponds to a different way of modifying or generalizing existing structures, leading to an infinite number of possible structures and areas of study.

Many of these adjectives can be combined to create even more specialized and complex structures. For example, "partial quantum layered non-commutative structures" could describe an algebraic system that integrates several different concepts simultaneously. Beyond the traditional adjectives, researchers can invent new adjectives or concepts tailored to specific problems or applications, further expanding the landscape of algebraic structures.

14 Conclusion

The cardinality of the continuum $\mathfrak{c}$ emerges naturally when analyzing the number of intermediate algebraic structures between magmas and groups, particularly when the underlying set M is infinite. Each intermediate set of structures (magmas to monoids, monoids to semigroups, and semigroups to groups) contributes to this continuum number, reflecting the rich and complex landscape of algebraic systems. The introduction and variation of properties such as associativity, identity, and inverses across infinite sets lead to uncountably many configurations, affirming that the cardinality of the continuum is intrinsic to the study of these intermediate structures.

Moreover, the exploration of additional adjectives and ideas, such as fractal, stochastic, fuzzy, and others, opens up an infinite landscape of possible algebraic structures. This richness allows for the development of entirely new mathematical theories and applications, pushing the boundaries of what is possible in algebra and beyond.

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