ZERO MODULUS FIELD AND VARIATIONAL STRUCTURE OF DEFORMED EULER ZETA FAMILIES

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ABSTRACT. We extend the study of the deformed Dirichlet family

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}$$

by analyzing the modulus-squared field $\mathcal{F}_t(s) := \log |L_t(s)|^2$ over the complex plane. We interpret this as an energy or pressure field and investigate its variational structure. This provides a new route toward understanding the localization and focusing behavior of potential zeros under deformation.

CONTENTS

1. Zero Modulus Field: Definition

We define:

$$\mathcal{F}_t(s) := \log |L_t(s)|^2 = 2\Re \left[\sum_p \sum_{k=1}^{\infty} \frac{t}{k} \cdot \frac{1}{p^{ks}} \right].$$

This is a real-valued function on $s = \sigma + i\tau \in \mathbb{C}$ and is harmonic away from singularities.

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2. Gradient and Variational Structure

Let $s = \sigma + i\tau$, then:

$$\frac{\partial \mathcal{F}_t}{\partial \sigma} = -2t \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{k\sigma}} \cos(k\tau \log p),$$

$$\frac{\partial \mathcal{F}_t}{\partial \tau} = 2t \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{k\sigma}} \sin(k\tau \log p).$$

The zero set of $\nabla \mathcal{F}_t$ corresponds to local extrema of $|L_t(s)|$.
We define critical points s_* of $\mathcal{F}_t$ by:

$$\nabla \mathcal{F}_t(s_*) = 0.$$

These are candidate locations for modulus valleys, i.e., potential zero precursors.

3. Second Variation and Stability

We define the second directional derivatives:

$$\frac{\partial^2 \mathcal{F}_t}{\partial \sigma^2} = 2t \sum_p \sum_{k=1}^{\infty} \frac{(\log p)^2 k}{p^{k\sigma}} \cos(k\tau \log p),$$

$$\frac{\partial^2 \mathcal{F}_t}{\partial \tau^2} = -2t \sum_p \sum_{k=1}^{\infty} \frac{(\log p)^2 k}{p^{k\sigma}} \cos(k\tau \log p).$$

We interpret these as curvature in the modulus field. Local minima satisfy:

$$\frac{\partial^2 \mathcal{F}_t}{\partial \sigma^2} > 0, \quad \text{and} \quad \mathrm{Hessian}(\mathcal{F}_t) \succ 0.$$

4. Variational Principle

Define the action:

$$\mathcal{S}_t[\gamma] := \int_\gamma \|\nabla \mathcal{F}_t(s)\|^2 \, ds.$$

Then zero focusing paths can be thought of as extremals minimizing $\mathcal{S}_t[\gamma]$ in the deformation limit $t \to 1^-$.

5. Outlook

This framework sets the stage for defining zero-flow dynamics under the modulus pressure field and offers a new lens for understanding the attractor nature of $\Re(s) = \frac{1}{2}$.