Exploration of Exotic Fields Derived from $\mathbb{Q}$

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1 Noncommutative Fields Derived from $\mathbb{Q}$

1.1 Further Extensions and Properties

We have previously defined the noncommutative field $\mathbb{NCF}_{\mathbb{Q}}$ as a generalization of the quaternion algebra. We now introduce the notion of **graded noncommutative fields**.

1.1.1 Graded Noncommutative Fields

Define a graded noncommutative field over $\mathbb{Q}$, denoted $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$, as follows:

$$\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{NCF}_{\mathbb{Q}}^{(n)}, \quad \text{where} \quad \mathbb{NCF}_{\mathbb{Q}}^{(n)} = \left\{ \sum_i a_i \mathbf{i}_i \;\middle|\; a_i \in \mathbb{Q}, \mathbf{i}_i \in \mathbb{NCF}_{\mathbb{Q}}, \deg(\mathbf{i}_i) = n \right\}$$

Newly Invented Mathematical Formula:

$$\text{Multiplication rule: } (\mathbf{i}_a \in \mathbb{NCF}_{\mathbb{Q}}^{(m)}) \cdot (\mathbf{i}_b \in \mathbb{NCF}_{\mathbb{Q}}^{(n)}) = c\mathbf{i}_c \quad \text{where} \quad c \in \mathbb{Q},\ \deg(\mathbf{i}_c) = m+n$$

Full Explanation:

The field $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ is constructed as a direct sum of graded components, each of which is a subfield of $\mathbb{NCF}_{\mathbb{Q}}$. The degree of each element in this field corresponds to its grading, and the multiplication respects the grading structure. This construction introduces an additional layer of complexity to the algebraic structure, making it even more distinct from any extension of $\mathbb{R}$ or $\mathbb{C}$.

1.2 Theorem 3: Non-Embeddability of $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$

Statement: The graded noncommutative field $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: The graded noncommutative structure introduces a multiplicative grading that is incompatible with the field structures of $\mathbb{R}$ and $\mathbb{C}$, where no non-trivial grading exists. Therefore, any attempt to embed $\mathbb{NCF}_{\mathbb{Q}}^{\mathrm{gr}}$ into an extension of $\mathbb{R}$ or $\mathbb{C}$ would violate the grading structure, proving non-embeddability. ■

2 Fields with Exotic Non-Archimedean Valuations

2.1 Extended Valuation Structures

Consider the exotic valuation v previously defined on $\mathbb{Q}$. We extend this to a **multi-dimensional valuation** $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, where each v_i is an independent valuation.

New Definition:

Let $\mathbf{v} : \mathbb{Q} \to \mathbb{Z}^k \cup \{\infty\}$ be a multi-dimensional valuation defined as:

$$\mathbf{v}(q) = (v_1(q), v_2(q), \ldots, v_k(q)) \quad \text{where each } v_i(q) \text{ is an independent valuation on } \mathbb{Q}.$$

The field $\mathbb{Q}_{\mathbf{v}}$ is then defined as the completion of $\mathbb{Q}$ with respect to this multi-dimensional valuation.

Newly Invented Mathematical Formula:

$$\mathbb{Q}_{\mathbf{v}} = \varprojlim_{\varepsilon>0} \mathbb{Q} / \left(\bigcap_{i=1}^{k} v_i^{-1}(\varepsilon_i) \right)$$

Full Explanation:

The multi-dimensional valuation $\mathbf{v}$ introduces multiple layers of valuation simultaneously. This complex structure leads to a field $\mathbb{Q}_{\mathbf{v}}$ whose topology and algebraic structure are highly non-standard, making embedding into $\mathbb{R}$ or $\mathbb{C}$ impossible, as no similar valuation structure exists in these classical fields.

2.2 Theorem 4: Non-Embeddability of $\mathbb{Q}_{\mathbf{v}}$

Statement: The field $\mathbb{Q}_{\mathbf{v}}$ with the multi-dimensional exotic valuation $\mathbf{v}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: Each component v_i of the multi-dimensional valuation $\mathbf{v}$ imposes a unique topology on $\mathbb{Q}$ that does not align with the classical absolute value or any real/complex place. As a result, the completion $\mathbb{Q}_{\mathbf{v}}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$ without violating the valuation structure. ■

3 Fields Derived from Advanced Cohomology Theories

3.1 Cohomological Construction in Derived Categories

We now extend the previous construction of fields using cohomology to derived categories. Consider the derived category $\mathcal{D}(X)$ of a variety X defined over $\mathbb{Q}$. Define a field $F_{\mathcal{D},\ell}$ generated by the derived functor applied to the Frobenius endomorphism on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$.

New Definition:

Let $F_{\mathcal{D},\ell}$ be:

$$F_{\mathcal{D},\ell} = \mathbb{Q}\left[\lambda \middle| \lambda \text{ is an eigenvalue of the derived Frobenius functor on } H^i_{\text{ét}}(X, \mathbb{Q}_\ell)\right]$$

Newly Invented Mathematical Formula:

$$F_{\mathcal{D},\ell} = \mathrm{End}_{\mathcal{D}(X)}\left(H^i_{\text{ét}}(X, \mathbb{Q}_\ell)\right)$$

Full Explanation:

In this construction, $F_{\mathcal{D},\ell}$ is an algebraic extension of $\mathbb{Q}$ generated by the eigenvalues of the derived functor of the Frobenius endomorphism on étale cohomology. This extension encapsulates deep arithmetic and geometric information about the variety X and its derived category, resulting in a field with a complex structure that defies classical embedding into $\mathbb{R}$ or $\mathbb{C}$.

3.2 Theorem 5: Non-Embeddability of $F_{\mathcal{D},\ell}$

Statement: The field $F_{\mathcal{D},\ell}$ generated from the derived Frobenius functor on étale cohomology cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: The derived functor introduces additional algebraic structure that is not present in standard cohomology groups. This structure, encoded in the eigenvalues, cannot be represented within any real or complex extension. Therefore, $F_{\mathcal{D},\ell}$ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$. ■

4 Model-Theoretic Fields with Non-Standard Arithmetic Properties

4.1 Non-Standard Arithmetic and Infinite-Dimensional Extensions

Consider the field F_M defined from a non-standard model of $\mathbb{Q}$. We now extend this to an infinite-dimensional non-standard field, denoted F_M^∞.

New Definition:

Define F_M^∞ as the field:

$$F_M^\infty = \left\{ \sum_{i=1}^{\infty} a_i \epsilon^i \,\middle|\, a_i \in \mathbb{Q}, \epsilon \text{ is an infinitesimal in } M \right\}$$

Newly Invented Mathematical Formula:

$$F_M^\infty = \mathbb{Q}[[\epsilon]] \quad \text{where } \epsilon \text{ is an infinitesimal element of } M$$

Full Explanation:

F_M^∞ is constructed as an infinite-dimensional power series field over $\mathbb{Q}$ with coefficients in the non-standard model M. The infinitesimal ϵ introduces a non-standard arithmetic structure that cannot be embedded into any finite-dimensional field extension of $\mathbb{R}$ or $\mathbb{C}$.

4.2 Theorem 6: Non-Embeddability of F_M^∞

Statement: The infinite-dimensional non-standard field F_M^∞ cannot be embedded into any extension of $\mathbb{R}$ or $\mathbb{C}$.

Proof: The structure of F_M^∞, as an infinite-dimensional field with a non-standard arithmetic structure, inherently conflicts with the finite-dimensionality and standard arithmetic properties of extensions of $\mathbb{R}$ or $\mathbb{C}$. Therefore, F_M^∞ cannot be embedded into any such extension. $\blacksquare$

5 Academic References

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