EXACTIFICATION VII: UNIVERSAL MODULI OF ARITHMETIC TOWERS AND CONDENSED COHOMOLOGICAL FLOWS: TOWARD A GLOBAL COHOMOTOPICAL GEOMETRY OF ARITHMETIC FUNCTIONS

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Abstract. This paper initiates the construction of a universal moduli stack $\mathbb{EXACT}_\infty$ parameterizing all exactification towers over the Dirichlet convolution ring $\mathcal{A}$. We show how each arithmetic function determines a point in this derived moduli space and how the deformation theory of its resolution encodes cohomological flows, motivic realizations, and spectral obstructions.

We introduce condensed enhancements of these towers using Clausen–Scholze's framework and define derived flows in motivic sheaf categories. This sets the stage for a global cohomotopical geometry of arithmetic functions, unifying analytic, motivic, and automorphic complexity into a single geometric space.

Contents

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1. Definition and Geometry of the Exactification Moduli Stack

1.1. **The Universal Tower Space.** Let $\mathcal{A}$ denote the Dirichlet convolution ring of arithmetic functions. For each $f \in \mathcal{A}$, we have previously constructed an exactification tower:

$$\mathscr{E}^{[f],\bullet} = \{\mathcal{F}_0 \to \mathcal{F}_1 \to \cdots\},$$

together with:
- an analytic resolution map $\mathrm{Tot}(\mathscr{E}^{[f],\bullet}) \to f$;
- a cohomology structure $H^i(\mathscr{E}^{[f]})$;
- and a motivic shadow $\mathbb{M}_f$ in later stages.

We now promote these towers into objects of a moduli stack over derived arithmetic geometry.

Definition 1.1 (Exactification Moduli Stack)**.** *Let $\mathbb{EXACT}_\infty$ be the derived moduli stack such that for any test object S in derived spaces (e.g. $S \in \mathrm{dAff}$, or condensed analytic spaces),*

$$\mathbb{EXACT}_\infty(S) := \left\{ \begin{array}{l} \textit{families of exactification towers } \mathscr{E}_S^{[\mathcal{F}]} \textit{ over } S \\ \textit{such that } \mathrm{Tot}(\mathscr{E}_S^{[\mathcal{F}]}) \in \Gamma(S, \mathcal{A}_S) \end{array} \right\}.$$

Remark 1.2. This stack is fibered in chain complexes with Dirichlet convolution action and analytic descent structure.

1.2. **Geometric Structure and Topology.**

Proposition 1.3. *The moduli stack $\mathbb{EXACT}_\infty$ is:*

- *a derived locally geometric stack;*

- *stratified by exactification entropy levels* $\mathbb{EXACT}_{\leq\epsilon}$*;*
- *admitting a natural* $\mathbb{G}_m$*-scaling action on towers;*
- *compatible with pushforwards under Langlands-type correspondences.*

1.3. **Condensed Sheafification.** Using Clausen–Scholze's theory of condensed mathematics, we define a condensed site $\mathbf{Cond}_{\mathcal{A}}$ over arithmetic base $\mathbb{Z}_{>0}$.

Definition 1.4 (Condensed Enhancement)**.** *Let:*

$$\mathbb{EXACT}_{\infty}^{\mathbf{cond}} := \mathbf{Sh}(\mathbf{Cond}_{\mathcal{A}}, \mathsf{Ch}^{+}(\mathcal{A}))$$

denote the sheafification of exactification towers in condensed chain complexes.

Remark 1.5. This allows infinite tower constructions with continuous convergence control.

1.4. **Points and Derived Stalks.** A point $f \in \mathcal{A}$ corresponds to a derived point $\delta_f \in \mathbb{EXACT}_{\infty}$, with:
- stalk: $\mathscr{E}^{[f],\bullet}$,
- residue field: exactification cohomology ring,
- tangent complex: deformation space of smoothing kernels $\mathcal{F}_{\alpha}$.

1.5. **Entropy Stratification and Stack Geometry.** Define:

$$\mathbb{EXACT}_{\leq\epsilon} := \left\{\mathscr{E}^{[f]} \in \mathbb{EXACT}_{\infty} \mid \mathrm{Entropy}(f) \leq \epsilon\right\}.$$

Theorem 1.6. *The family* $\{\mathbb{EXACT}_{\leq\epsilon}\}$ *forms a filtered stratification of* $\mathbb{EXACT}_{\infty}$ *into derived substacks of increasing complexity.*

1.6. **Spectral Flows and Morphisms.** The stack $\mathbb{EXACT}_{\infty}$ admits:
- a morphism to motivic stacks $\mathbb{EXACT}_{\infty} \to \mathrm{Mot}$;
- a derived convolution product $\mathbb{EXACT}_{\infty} \times \mathbb{EXACT}_{\infty} \to \mathbb{EXACT}_{\infty}$;
- a Fourier transform autoequivalence $\mathbb{F} : \mathbb{EXACT}_{\infty} \to \mathbb{EXACT}_{\infty}$;
- and duality $\mathbb{D}$ as discussed in VI.

1.7. **Universal Diagram.**

$$\begin{tikzcd}
f \in \mathcal{A} \arrow[r, "\text{mapsto}"] \arrow[d, "\text{Langlands lift}"'] & \mathscr{E}^{[f]} \arrow[r, "\text{moduli inclusion}"] \arrow[d, "\text{dual, } \mathbb{F}"'] & \mathbb{EXACT}_\infty \arrow[d, "\text{Mot}"'] \\
\pi \in \text{Automorphic base} \arrow[r, "\text{mapsto}"] & \mathscr{E}^{[\pi]} \arrow[r, "\text{mapsto}"] & \mathbb{M}_\pi
\end{tikzcd}$$

Arithmetic functions are not just data.
They are points in a derived motivic moduli space.

2. Condensed Motivic Flows and Derived Lifting Dynamics

2.1. Motivation: Automorphy as Arithmetic Dynamics. In the classical Langlands program, Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\overline{\mathbb{Q}}_\ell)$ deform across automorphic families via Hecke correspondences.

Here, we propose an analogous dynamic geometry for arithmetic functions:

$$f \rightsquigarrow \mathscr{E}^{[f],\bullet} \rightsquigarrow \mathbb{EXACT}_\infty \rightsquigarrow \mathbb{M}_f,$$

where lifting and deformation correspond to motivic flow across cohomological towers.

2.2. Condensed Arithmetic Base Site. Let $\mathbf{Cond}_{\mathcal{A}}$ be the condensed site associated with $\mathcal{A}$, where objects are compact topological spaces with sheaves of arithmetic functions, i.e.:

$$\mathbf{Cond}_{\mathcal{A}} := \mathrm{Sh}(\mathrm{ProFin}_{\mathrm{arith}}, \mathsf{Ab}),$$

e.g., profinite arithmetic covers of $\mathbb{Z}_{>0}$.

2.3. Motivic Flow: Definition. We define a **condensed motivic flow** as a family:

$$\mathcal{F} : T \to \mathbb{EXACT}_\infty,$$

where T is a condensed time space (e.g., $\mathbb{R}_{\geq 0}$, or a perfectoid parameter space), such that for every $t \in T$, $\mathcal{F}(t)$ is an exactification tower.

Definition 2.1 (Exactification Flow)**.** *An exactification flow is a morphism:*

$$\mathcal{F} : (T, \mathcal{O}_T) \to \mathbb{EXACT}_\infty$$

satisfying:

- *smoothness over the condensed base;*
- *monotonic descent in entropy:* $\mathrm{Entropy}(\mathcal{F}(t))$ *decreasing;*

- *derived cohomological continuity:* $\mathrm{H}^i(\mathcal{F}(t))$ *varies continuously in* t.

2.4. **Flowing Across Langlands Lifts.** Let f deform into f_t, then:

$$\mathscr{E}^{[f_t]} \rightsquigarrow \mathscr{E}^{[\pi_t]} \rightsquigarrow \mathbb{M}_{\pi_t}$$

defines a motivic flow in the Langlands lifted space.

Proposition 2.2. *Exactification flows induce natural morphisms:*

$$T \to \mathrm{Map}_{\mathsf{DM}}(\mathscr{E}^{[f]}, \mathscr{E}^{[\pi]}),$$

i.e., derived deformation spaces within motivic sheaves.

2.5. **Entropy as Potential over Flows.** Each arithmetic tower $\mathscr{E}^{[f]}$ admits a potential function:

$$\Phi(t) := \mathrm{Entropy}(\mathscr{E}^{[f_t]}),$$

analogous to a dynamical entropy potential.

Definition 2.3. *A flow is called motivically stabilizing if* $\Phi(t)$ *converges to a constant in cohomological modulus.*

Example 2.4. Deformations of $d(n)$ yield trivial flows.
Deformations of $\mu(n)$ may be chaotic unless spectrally truncated.
Deformations of $\Lambda(n)$ exhibit cusp-form convergence.

2.6. **Motivic Flow Diagram.**

$$\begin{tikzcd} T \arrow[r, "\text{flow}"] \arrow[rrd, bend right, "\Phi(t)"'] & \mathbb{EXACT}_\infty \arrow[r, "\text{Mot}"] & \mathrm{Mot} \\ & & \mathbb{R} \end{tikzcd}$$

An arithmetic function is not static.
It flows. It deforms. It lifts. It condenses.

3. Homotopy Type Theory of Exactification Towers

3.1. **From Arithmetic Functions to Types.** We now recast the exactification tower $\mathscr{E}^{[f],\bullet}$ as a homotopy type in a dependent type theory, aligning with the homotopy-theoretic interpretation of mathematical objects.

Let Type be the universe of types. To each function $f \in \mathcal{A}$, we assign:

Definition 3.1. *Define the exactification type:*

$$\mathsf{Exact}_f := \left\{ \mathscr{E}^{[f]} : \mathbb{EXACT}_\infty \mid \mathrm{Tot}(\mathscr{E}^{[f]}) = f \right\} : \mathsf{Type}.$$

This type contains all homotopically valid resolutions of f, stratified by entropy and cohomological depth.

3.2. **Paths and Identity Types.** In HoTT, identity types encode paths (homotopies) between objects.

Definition 3.2. *Two towers $\mathscr{E}_1^{[f]}, \mathscr{E}_2^{[f]} \in \mathsf{Exact}_f$ are connected via:*

$$p : \mathsf{Id}_{\mathsf{Exact}_f}(\mathscr{E}_1, \mathscr{E}_2),$$

representing a homotopy between towers.

Proposition 3.3. *The space of identity paths $\mathsf{Id}_{\mathsf{Exact}_f}(\mathscr{E}_1, \mathscr{E}_2)$ is equivalent to:*

$$\mathrm{Map}_{\mathcal{D}^+(\mathcal{A})}(\mathscr{E}_1, \mathscr{E}_2),$$

the derived mapping space of towers.

3.3. **Tower Universes and Higher Types.** Let $\mathbb{TOWER}_n$ denote the type of all towers of cohomological dimension $\leq n$. We then define:

$$\mathsf{Exact}_{\leq n} := \Sigma_{f \in \mathcal{A}} \, (\mathscr{E}^{[f]} \in \mathbb{TOWER}_n).$$

This forms a stratified hierarchy of arithmetic complexity indexed by entropy or cohomological depth.

3.4. **∞-Groupoid Structure and Motivic Identity Levels.** Each tower $\mathscr{E}^{[f]}$ becomes an object in a higher groupoid:

- 0-morphisms: exactification towers;
- 1-morphisms: chain homotopies;
- 2-morphisms: equivalences of homotopies;
- ...
- ∞-morphisms: derived higher paths in motivic sheaves.

Definition 3.4. *The ∞-groupoid of exactification types:*

$$\Pi_\infty(\mathbb{EXACT}_\infty) := \textit{full motivic homotopy groupoid of towers.}$$

3.5. **Univalence and Arithmetic Equivalence.**

Conjecture 3.5 (Exactification Univalence)**.** *For towers $\mathscr{E}_1, \mathscr{E}_2 \in \mathsf{Exact}_f$, the following are equivalent:*

- *$\mathscr{E}_1 \simeq \mathscr{E}_2$ (homotopy equivalence);*
- *$\mathsf{Id}_{\mathsf{Exact}_f}(\mathscr{E}_1, \mathscr{E}_2)$ is inhabited;*
- *$\mathrm{Tot}(\mathscr{E}_1) = \mathrm{Tot}(\mathscr{E}_2)$ and $H^*(\mathscr{E}_1) = H^*(\mathscr{E}_2)$.*

3.6. **Arithmetic Homotopy Diagram.**

$$\begin{tikzcd} f \in \mathcal{A} \arrow[r, mapsto] & \mathsf{Exact}_f \in \mathsf{Type} \arrow[d, "\Pi_\infty"] \\ & \mathsf{Spaces}_{/\mathbb{EXACT}_\infty} \end{tikzcd}$$

Arithmetic is a homotopy theory.
Exactification towers are types.
Cohomology is higher identity.

4. Final Global Geometry and the Arithmetic Type Universe

4.1. **From Arithmetic to Derived Geometry.** Through the course of the Exactification Program, we have transformed the way we view arithmetic functions:

- Not as formulas, but as sheaves of resolution towers;
- Not as sequences, but as points in a moduli stack;
- Not as approximated objects, but as cohomological types;
- Not as static inputs to L-functions, but as evolving flows through motivic and spectral geometry.

4.2. **Global Structure of $\mathbb{EXACT}_\infty$.** We now summarize the key structural properties of the exactification universe:

- $\mathbb{EXACT}_\infty$ is a locally geometric derived stack over condensed arithmetic sites;
- It is stratified by entropy: $\mathbb{EXACT}_{\leq\epsilon}$;
- It is fibered in motivic realizations: $\mathbb{EXACT}_\infty \to \mathrm{Mot}$;
- It supports derived flows: $T \to \mathbb{EXACT}_\infty$;
- It admits higher identity types and univalence.

This structure unifies:

- **Estimation theory** in analytic number theory;
- **Cohomology theory** in derived algebra;
- **Flow dynamics** in motivic deformation spaces;
- **Type-theoretic identity** in higher geometry.

4.3. **Arithmetic Functions as Types.** We now reinterpret the entire arithmetic space $\mathcal{A}$ as a dependent type universe:

$$\mathbb{A}_{\text{type}} := \{ f \in \mathcal{A} \,|\, \mathsf{Exact}_f \in \mathsf{Type} \} ,$$

where each f carries:

- a resolution tower $\mathscr{E}^{[f]}$;
- entropy profile $\mathrm{Entropy}(f)$;
- motivic class $\mathbb{M}_f$;
- homotopy type Exact_f.

This is the arithmetic type universe.
Each number-theoretic object becomes a type.
Each type becomes a derived space.
Each derived space becomes a motivic form.

4.4. **Toward Exactification VIII: Motivic Identity Systems and Arithmetic Field Theories.** This paper completes the first phase of the Exactification Program. The next phase will address:

- Constructing motivic field theories from exactification types;
- Understanding symmetry and duality within $\mathbb{EXACT}_\infty$;
- Building a formal system for automated generation of arithmetic types;
- Developing homotopy-coherent integration over arithmetic flows.

We also aim to explore connections with:

- Condensed motivic ∞-topoi;
- Mirror symmetry for resolution stacks;
- Arithmetic field theories and cohomotopical quantization.

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From entropy to type.
From towers to flows.
From number theory to geometry.
From resolution to realization.

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