SPECTRAL MOTIVES IX: DERIVED L-SHEAVES OVER ARITHMETIC ∞-ZETA SITES

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Abstract. In this ninth installment of the Spectral Motives series, we construct a derived theory of L-sheaves over arithmetic ∞-zeta sites. By defining sheaf-theoretic L-functions as spectral flows in condensed cohomology, we develop a theory of derived L-sheaves compatible with zeta-stack torsors and motivic functors. We explore their behavior under Frobenius descent, trace duality, and zeta-spectral correspondences, establishing the foundation for universal L-sheaf categories over spectral motives and connecting them to automorphic and Galois-theoretic moduli. Applications include categorification of special L-values, trace expansions, and motivic arithmetic sites.

Contents

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1. Introduction

The theory of L-functions sits at the heart of arithmetic geometry, encoding deep connections between Galois representations, automorphic forms, and motives. In this work, we propose a categorified extension of this framework, defining *derived L-sheaves* over arithmetic ∞-zeta sites. These sheaves unify trace flows, torsors, and spectral motives through sheaf-theoretic constructions and categorical descent.

We build upon the following foundational structures:

- The *arithmetic ∞-zeta site* $\mathscr{Z}_\infty$, modeled as a limit of dyadic zeta-stacks ζ_n;
- Spectral categories $\mathfrak{T}_\zeta^\infty$ of sheaves compatible with Frobenius and trace;
- Frobenius descent and universal spectral sheaves from earlier papers in the Spectral Motives series.

Objectives of this paper:

1. Define derived L-sheaves as trace-preserving spectral functors over $\mathscr{Z}_\infty$;
2. Study Frobenius flows, torsorial descent, and cohomological realization;
3. Explore categorified special values and derived trace expansions;
4. Construct the universal L-sheaf stack and compare it with motivic and automorphic sites.

Outline. Section 2 defines arithmetic ∞-zeta sites and derived sheaf categories. Section 3 constructs derived L-sheaves via trace cohomology and spectral flows. Section 4 presents cohomological realizations and automorphic comparisons. Section 5 applies the theory to special L-values and categorified trace expansions.

2. Arithmetic ∞-Zeta Sites and Spectral Sheaves

2.1. **Inverse limit of dyadic zeta stacks.** Let $\{\zeta_n\}_{n\in\mathbb{N}}$ denote the inverse system of dyadic zeta stacks introduced in earlier work. Define the arithmetic ∞-zeta site:

$$\mathscr{Z}_\infty := \varprojlim_n \zeta_n,$$

as a profinite site equipped with a natural tower of trace-preserving sheaf categories and Frobenius flows.

2.2. **Sheaves over $\mathscr{Z}_\infty$.** Let $\mathrm{Shv}^{\mathrm{tr}}_{\mathbb{C}}(\zeta_n)$ denote the category of complex-valued sheaves over ζ_n equipped with trace-preserving morphisms. We define:

$$\mathrm{Shv}^{\mathrm{tr}}_{\mathbb{C}}(\mathscr{Z}_\infty) := \varprojlim_n \mathrm{Shv}^{\mathrm{tr}}_{\mathbb{C}}(\zeta_n),$$

which carries a symmetric monoidal structure, inverse-compatible cohomology, and Frobenius descent morphisms.

2.3. **Spectral sheaves and trace flows.** Let $\mathfrak{T}_\zeta^\infty$ be the category of spectral sheaves over $\mathscr{Z}_\infty$, defined by:

$$\mathfrak{T}_\zeta^\infty := \mathrm{Fun}^{\mathrm{tr}}_{\otimes}(\mathscr{Z}_\infty, \mathsf{Perf}_{\mathbb{C}}),$$

where objects are symmetric monoidal, trace-compatible sheaf functors valued in perfect complexes over $\mathbb{C}$, and morphisms preserve both the monoidal and trace structures.

2.4. **Derived trace cohomology.** The condensed trace-compatible cohomology is defined on spectral sheaves $\mathscr{F} \in \mathfrak{T}_\zeta^\infty$ by:

$$H^i_{\mathrm{Tr}}(\mathscr{F}) := \varprojlim_n H^i(\mathscr{F}_n),$$

where $\mathscr{F}_n$ is the level-n restriction of $\mathscr{F}$, and the transition maps are induced by trace descent morphisms compatible with Frobenius flows.

2.5. **Motivic and automorphic comparisons.** There exist canonical functors:

$$\mathbb{S}_{\mathrm{mot}} \colon \mathfrak{T}_\zeta^\infty \to \mathrm{Mot}_{\mathbb{C}}^{\mathrm{cond}},$$
$$\mathbb{S}_{\mathrm{aut}} \colon \mathfrak{T}_\zeta^\infty \to \mathscr{A}ut_G^{\mathrm{cond}},$$

assigning to spectral sheaves their motivic or automorphic avatars, under the universal spectral realization of condensed Langlands parameters.

These provide the bridge to define derived L-sheaves across arithmetic, automorphic, and motivic settings.

3. Derived L-Sheaves and Spectral Realization

3.1. **Definition of derived L-sheaves.** A *derived L-sheaf* $\mathbb{L}(\mathscr{F})$ over $\mathscr{Z}_\infty$ is a functorially constructed object associated to a spectral sheaf $\mathscr{F} \in \mathfrak{T}_\zeta^\infty$ such that:

- It is trace-compatible and Frobenius-stable;
- It encodes cohomological traces as L-function coefficients:

$$L(\mathscr{F}, s) := \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr}(\mathrm{Frob}^s \mid H^i_{\mathrm{Tr}}(\mathscr{F})).$$

3.2. **Derived zeta stack torsors.** Let $\mathrm{Tors}_{\mathbb{L}}$ be the moduli of derived torsors under the universal zeta stack ζ_n, forming a condensed groupoid stack:

$$\mathrm{Tors}_{\mathbb{L}} := \left[\mathscr{Z}_\infty / \widehat{L}^\infty\right],$$

where $\widehat{L}^\infty$ is the universal L-groupoid acting through trace-preserving flows. Each $\mathscr{F}$ defines a torsor in this stack, and thus a derived L-sheaf.

3.3. **Spectral realization.** Define the *spectral realization functor*:

$$\mathbb{R}_{\mathscr{Z}} \colon \mathfrak{T}_\zeta^\infty \to \mathrm{Shv}_{\mathbb{C}}^{\mathrm{tr}}(\mathscr{Z}_\infty),$$

which forgets the spectral enhancement and produces a trace-compatible sheaf. The L-function associated to $\mathscr{F}$ is then:

$$L(\mathscr{F}, s) = \det\left(1 - \mathrm{Frob}^s \mid \mathbb{R}_{\mathscr{Z}}(\mathscr{F})\right)^{-1}.$$

3.4. **Specializations and base change.** Derived L-sheaves respect the inverse limit structure:

$$\mathbb{L}(\mathscr{F}) = \varprojlim_n \mathbb{L}_n(\mathscr{F}_n),$$

and base change along zeta-functors $\zeta_n \to \zeta_m$ preserves cohomological traces and Frobenius action. This stability is essential for arithmetic uniformization and compatibility with motivic realizations.

3.5. **Examples.**

(1) For $\mathscr{F}$ corresponding to a condensed Galois character χ, we recover:

$$L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

(2) For $\mathscr{F}$ arising from an automorphic sheaf via $\mathbb{S}_{\text{aut}}$, the derived L-sheaf interpolates automorphic L-functions and cohomology.

(3) For motivic sheaves in Mot^{cond}, the special values conjecturally match trace-periods of $\mathbb{L}(\mathscr{F})$.

4. Cohomological Realization and Automorphic Comparison

4.1. **Trace cohomology and special L-values.** Let $\mathbb{L}(\mathscr{F})$ be a derived L-sheaf associated to a spectral sheaf $\mathscr{F} \in \mathfrak{T}_\zeta^\infty$. Define its trace cohomology as:

$$H^\bullet_{\text{Tr}}(\mathbb{L}(\mathscr{F})) := \varprojlim_n H^\bullet(\mathbb{L}_n(\mathscr{F}_n)),$$

and its special values at positive integers as:

$$L(\mathscr{F}, k) = \sum_i (-1)^i \operatorname{Tr}(\text{Frob}^k \mid H^i_{\text{Tr}}(\mathbb{L}(\mathscr{F}))).$$

These values serve as categorified analogs of Deligne's conjectures on critical values of motivic L-functions.

4.2. **Automorphic realization via spectral stacks.** There is a canonical functor:

$$\mathbb{S}_{\text{aut}} \colon \mathbb{L}(\mathscr{F}) \mapsto \text{Aut}(\mathscr{F}),$$

sending derived L-sheaves to automorphic sheaves over the stack $\mathscr{A}ut_G^{\text{cond}}$, preserving trace cohomology.

4.3. **Comparison theorem.** **Theorem 4.1.** Let $\mathscr{F} \in \mathfrak{T}_\zeta^\infty$. Then:

$$H^i_{\text{Tr}}(\mathbb{L}(\mathscr{F})) \cong H^i_{\text{Tr}}(\text{Aut}(\mathscr{F})) \cong H^i_{\text{Tr}}(\omega_{\mathscr{F}}),$$

where $\omega_{\mathscr{F}}$ is the fiber functor into the condensed Tannakian groupoid from Dyadic Langlands VII–VIII.

4.4. **Functoriality under Langlands morphisms.** Let $\phi : G \to H$ be a morphism of group stacks. Then:

$$\phi_* \mathbb{L}(\mathscr{F}) \cong \mathbb{L}(\phi_* \mathscr{F}),$$

and trace cohomology respects spectral pushforward:

$$H^i_{\text{Tr}}(\mathbb{L}(\mathscr{F})) \to H^i_{\text{Tr}}(\mathbb{L}(\phi_* \mathscr{F})).$$

This compatibility is central to defining functorial trace expansions and spectral transfer of arithmetic invariants.

5. Special Values and Categorified Trace Expansions

5.1. **Spectral expansions of trace-compatible flows.** Let $\mathscr{F} \in \mathfrak{T}_{\zeta}^{\infty}$. The associated derived L-sheaf $\mathbb{L}(\mathscr{F})$ carries a Frobenius spectral flow expansion:

$$L(\mathscr{F}, s) = \sum_{\rho} \lambda_\rho \cdot \rho^{-s},$$

where ρ ranges over the trace-compatible eigenvalues of Frobenius acting on $H^i_{\mathrm{Tr}}(\mathscr{F})$, and λ_ρ are spectral multiplicities.

This defines a categorified spectral expansion of the L-function with coefficients in derived categories.

5.2. **Categorical zeta specializations.** We define the *categorified zeta function* associated to $\mathscr{F}$ as:

$$\zeta_{\mathscr{F}}(s) := \prod_{\rho} (1 - \rho^{-s})^{-\lambda_\rho},$$

which interpolates the spectrum of Frobenius traces into a full zeta-function structure over spectral stacks.

5.3. **Torsorial and motivic interpolation.** Given an interpolating family $\{\mathscr{F}_t\}_{t \in \mathbb{A}^1}$ of spectral sheaves with compatible torsorial structure, the L-values vary categorically:

$$L(\mathscr{F}_t, s) \in \mathbb{C}[\![t]\!]^{\otimes \infty},$$

and satisfy motivic congruences for specializations $t = t_0$.

This provides a path toward categorified p-adic L-functions and analytic continuation in arithmetic families.

5.4. **Future directions.** Further directions of research include:

- Defining Euler products and local factors for $\mathbb{L}(\mathscr{F})$ over condensed local fields;
- Investigating dualities between motivic special values and trace periods;
- Linking derived L-sheaves to Beilinson–Bloch–Kato style regulators in spectral cohomology;
- Developing a full theory of categorical L-functions in derived motivic sheaf theory.

6. Conclusion and Outlook

In this work, we introduced and developed the theory of derived L-sheaves over arithmetic ∞-zeta sites. Through the spectral formalism of trace-compatible sheaves and Frobenius descent, we unified arithmetic, automorphic, and motivic cohomology into a coherent framework that categorifies classical L-functions.

Key Contributions:

- Defined the arithmetic ∞-zeta site $\mathscr{Z}_\infty$ as the inverse limit of dyadic zeta stacks;
- Constructed derived L-sheaves as trace-compatible functors in spectral cohomology;
- Established automorphic comparison theorems and derived trace cohomological realizations;
- Presented spectral expansions and special value frameworks for categorified zeta functions.

Next Steps:

- Develop local-global compatibility for $\mathbb{L}(\mathscr{F})$ over dyadic localizations;
- Extend to p-adic and de Rham cohomological realizations;
- Relate categorical regulators and periods to special L-values;
- Formulate a universal theory of derived L-functions over spectral motivic stacks.

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