FORMALIZING THE FOUR ARITHMETIC OPERATIONS IN THE RING OF ARITHMETIC FUNCTIONS UNDER DIRICHLET CONVOLUTION

PU JUSTIN SCARFY YANG

Abstract. We formalize the structure and operations of the ring of arithmetic functions under Dirichlet convolution. We explicitly define and analyze the four arithmetic operations—pointwise addition, Dirichlet convolution, Dirichlet inverse (division), and convolution exponentiation—as well as the logarithmic and exponential maps that bridge additive and multiplicative behavior. This framework sets the stage for algebraic exploration of this ring using modern ring theory.

1. Introduction

Let $\mathcal{A}$ denote the set of all arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. We endow $\mathcal{A}$ with two operations:

- Pointwise addition: $(f+g)(n) := f(n) + g(n)$
- Dirichlet convolution: $(f * g)(n) := \sum_{d|n} f(d)g(n/d)$

With these operations, $(\mathcal{A}, +, *)$ becomes a commutative ring with unity ε, where

$$\varepsilon(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

Despite being widely known, the internal arithmetic structure of this ring remains underdeveloped in the literature. We initiate a systematic calculus of arithmetic functions under these operations, with analogues of addition, subtraction, multiplication, and division, as well as functional logarithms and exponentials under convolution.

2. The Basic Ring Structure

Definition 1. *The set $\mathcal{A}$ of all arithmetic functions is a commutative ring with:*

- *Additive identity:* $0(n) := 0$ *for all* n;
- *Multiplicative identity:* $\varepsilon(n) := \delta_{n,1}$;
- *Additive inverse:* $(-f)(n) := -f(n)$;
- *Multiplicative inverse:* f^{-1} *exists if and only if* $f(1) \neq 0$.

Definition 2 (Convolution Exponentiation). *Let $f \in \mathcal{A}$ and $k \in \mathbb{N}$, define f^{*k} recursively by:*

$$f^{*0} := \varepsilon, \quad f^{*k} := f * f^{*(k-1)}.$$

Definition 3 (Dirichlet Logarithm). *If $f(1) = 1$, define*

$$\log^* f := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (f - \varepsilon)^{*k}.$$

Definition 4 (Dirichlet Exponential). *For $g \in \mathcal{A}$ with $g(1) = 0$, define*

$$\exp^*(g) := \sum_{k=0}^{\infty} \frac{1}{k!} g^{*k}.$$

Date: May 10, 2025.

Proposition 1. *If* $f = \exp^*(g)$*, then* $g = \log^*(f)$*, and vice versa.*

Example 1. *Let* $f(n) = 1$ *for all* n*. Then* $\log^* f = \Lambda(n)$*, the von Mangoldt function.*