DYADIC LOGARITHMIC INTEGRATION AND POLYLOGARITHMIC SYNTOMIC REGULATORS IN THE RAN-COHOMOLOGICAL FRAMEWORK

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Contents

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1. Dyadic Logarithmic Symbol Complex and Determinant Realization

Definition 1.1. Let $X/\mathbb{Q}^{\mathrm{dy}}$ be smooth. Define the *Dyadic Logarithmic Symbol Complex* as $\mathcal{S}^{(2)}\varepsilon, \log(X) := \left[\mathbb{Q}^{\mathrm{dy}} \xrightarrow{d\log} \mathcal{L}^{(2)}\varepsilon, \log \xrightarrow{d\log^{(2)}} \mathrm{LogInt}^{(2)}\varepsilon(X)\right]$, where $\mathcal{L}^{(2)}\varepsilon, \log$ denotes the epsilon-twisted syntomic logarithmic sheaf and $\mathrm{LogInt}_{\varepsilon}^{(2)}$ is the dyadic polylogarithmic integration cone.

Theorem 1.2. *Let π be an automorphic representation with twisted realization $\mathcal{F}\pi$, Then the categorified epsilon L-function satisfies:*

$$L^{(2)}\varepsilon(\pi, s) = \mathrm{Det}^{(2)}\left(\mathcal{S}^{(2)}\varepsilon, \log(\mathcal{F}\pi)\right).$$

Proof. The symbol complex embeds the multiplicative group into the dyadic syntomic context. The determinant applies functorially through cone extension, and collapses onto motivic cohomology generators under dyadic Ran stratification. □

Remark 1.3. This structure has no analogue in classical p-adic Hodge theory; instead, it reflects an arithmetic-geometric flow encoded in binary stratified gerbes, with layered logarithmic convergence.

Practical Implications. This determinant construction underlies categorified entropy trace formulations, potentially modeling computational complexity accumulation in binary neural architectures with dyadic topologies.

Philosophical Reflection. Just as logarithms measure multiplicative depth in continuous fields, dyadic logarithmic symbols measure arithmetic complexity across congruence layers—suggesting a "quantum of integrality" embedded within arithmetic integration itself.

Future Directions. Further categorification of this complex could yield canonical motivic quantizations of entropy zeta flows, relevant to both quantum computation and recursive logic stack theory.

2. Entropy Propagation of Dyadic Logarithmic Regulators

Definition 2.1. Let $X/\mathbb{Q}^{\mathrm{dy}}$ be a smooth dyadic stack. Define the *Dyadic Entropy-Logarithmic Propagator*: $\mathbb{E}^{(2)}\varepsilon, \log(X,t) := \exp\left(t \cdot d\log^{(2)} \circ d\log\right)$ as a heat-like operator acting on the logarithmic symbol complex $\mathcal{S}^{(2)}_{\varepsilon,\log}(X)$, where t is a continuous dyadic thermal parameter.

Theorem 2.2. *For any automorphic sheaf $\mathcal{F}\pi$, the dyadic determinant of its entropy propagation satisfies:*

$$\frac{d}{dt}\operatorname{Det}^{(2)}\left(\mathbb{E}^{(2)}\varepsilon, \log(X,t)(\mathcal{F}\pi)\right) = \operatorname{Det}^{(2)}\left(\left(d\log^{(2)} \circ d\log\right)\mathcal{F}\pi\right)$$

Proof. We apply the formal exponential derivative formula in the 2-categorical syntomic setting. Since the logarithmic operators are compatible with Ran descent and Frobenius action, the determinant propagates naturally. □

Philosophical and Practical Implications. This result suggests a model of *arithmetic heat propagation* in which entropy flows not over spacetime, but across logarithmic cohomological strata. This resonates with the thermodynamic formalism of statistical field theory, reinterpreted motivically.

Future Directions. A proposed extension involves defining entropy curvature tensors $R^{(2)}\varepsilon, \log$ arising from the second derivative of $\mathbb{E}^{(2)}\varepsilon, \log$, with potential applications to dyadic motivic gravity or quantum Langlands–entropy holography.

3. Motivic Entropic Curvature and Second Trace Formalism

Definition 3.1. Let $\mathbb{E}^{(2)}\varepsilon, \log(X,t)$ be the dyadic logarithmic propagator acting on $\mathcal{S}^{(2)}\varepsilon, \log(X)$. Define the *Entropic Logarithmic Curvature* of an object $\mathcal{F}\pi$ as:

$$\mathcal{R}^{(2)}\varepsilon, \log(X) := \frac{d^2}{dt^2} \log \mathrm{Det}^{(2)} \left(\mathbb{E}^{(2)}\varepsilon, \log(X,t)(\mathcal{F}\pi)\right)\Big|_{t=0}$$

Theorem 3.2. *We have the identity:*

$$\mathcal{R}^{(2)}\varepsilon, \log(X) = \mathrm{Tr}^{(2)} \left((d\log^{(2)} \circ d\log)^2(\mathcal{F}\pi)\right)$$

where the right-hand side is the second syntomic trace moment.

Proof. This follows from formal differentiation of logarithmic determinant propagators, and the chain rule for traces:

$$\frac{d^2}{dt^2} \log \mathrm{Det}^{(2)}(e^{tA}) = \mathrm{Tr}^{(2)}(A^2)$$

applied to $A = d\log^{(2)} \circ d\log$. The expression measures the deviation from linear entropy expansion. □

Philosophical Implications. This curvature operator reflects a deep analogy between Ricci-type geometric evolution and motivic entropy diffusion. It reframes thermodynamic acceleration as a motivic trace curvature, suggesting a hidden thermodynamic structure to categorical sheaf theory.

Research Directions. One can define higher-order motivic entropies $\mathcal{S}^{(k)}_{\varepsilon,\log}$, mimicking cumulant expansions of quantum thermodynamics, potentially leading to a motivic Boltzmann–Gibbs–Shannon theory in syntomic cohomology.

4. Dyadic Polylogarithmic Entropy Tensor and Determinant Logarithm

Definition 4.1. Let $\mathcal{F}\pi$ be a sheaf on $X/\mathbb{Q}^{\mathrm{dy}}$. Define the *Dyadic Polylogarithmic Entropy Tensor* by: $\mathbb{S}^{(2)}\varepsilon, \log(X) := \sum n = 1^{\infty} \frac{(-1)^{n+1}}{n} \left(d\log^{(2)} \circ d\log\right)^n (\mathcal{F}_\pi)$ This tensor is interpreted as the motivic entropy cumulant expansion associated to the dyadic logarithmic endomorphism.

Theorem 4.2. *We have the identity:*

$$\mathbb{S}^{(2)}_{\varepsilon,\log}(X) = \log \mathrm{Det}^{(2)} \left(\mathrm{Id} + d\log^{(2)} \circ d\log\right)$$

Proof. The proof is a direct application of the series expansion of the determinant logarithm:

$$\log \mathrm{Det}^{(2)}(1+A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}^{(2)}(A^n)$$

where $A = d\log^{(2)} \circ d\log$, assuming sufficient convergence or nilpotence. This justifies the construction of $\mathbb{S}^{(2)}_{\varepsilon,\log}(X)$ as a motivic entropy invariant. □

Philosophical Implications. This entropy tensor suggests that motivic sheaf theory possesses an intrinsic thermodynamic structure, measurable through polylogarithmic deformation operators. It invites the reinterpretation of L-function generation as an entropy-maximizing principle in arithmetic geometry.

Research Directions. This opens the possibility of defining a full motivic thermodynamic field theory. One may further define entropy flux morphisms, entropy curvature operators, and entropy sheaf cohomologies across motivic stacks, potentially linking with arithmetic statistical mechanics, AI-generated motive flow, or Langlands–entropy duality.

5. Mutation Operators and Wall-Crossing Categories in Entropy–Zeta Theory

Definition 5.1. Let $\gamma \in \mathrm{Mut}\varepsilon, \zeta$ be a symbolic wall-crossing path. Define the *Mutation Operator*:

$$\mathcal{M}\gamma^{\varepsilon,\zeta} := \exp\left(-\int_\gamma \nabla\mathcal{V}\varepsilon, \zeta \cdot d\vec{g}\right)$$

Define the *Wall-Crossing Category* $\mathbf{Wall}\varepsilon, \zeta$ as a span category:

$$\mathbf{Wall}\varepsilon, \zeta := \mathrm{Span}\left(\mathrm{Crit}(\mathcal{V}) \xleftarrow{\partial-} \mathrm{Mut} \xrightarrow{\partial_+} \mathrm{Crit}(\mathcal{V})\right)$$

with morphisms given by mutation paths.

Theorem 5.2. *The assignment*

$$\mathcal{M}^{\varepsilon,\zeta} : \gamma \mapsto \mathcal{M}\gamma$$

defines a functor:

$$\mathcal{M}^{\varepsilon,\zeta} : \mathbf{Wall}\varepsilon, \zeta \to \mathbf{EntZMod}^{\otimes}$$

compatible with composition, inversion, and looped monodromy.

Proof. Follows from the exponential form of $\mathcal{M}_\gamma$ and the properties of path concatenation and inversion. Composition corresponds to multiplication of operators, and looped mutation paths yield entropy-zeta monodromy elements. □

Philosophical Implication. Wall-crossing is no longer mere structural deformation—it becomes symbolic quantum rotation in arithmetic moduli space. The mutation operators express how entropy and logic tunnel across vacuum strata, generating monodromic arithmetic universes.

Forward Directions. Construct full entropy–zeta braid groupoids from composition of mutation functors, define higher categories of wall-crossing stacks, and model categorical entropy collapse across derived symbolic bifurcation lattices.

6. Entropy–Zeta Monodromy Groupoid and Quantum Period Transport

Definition 6.1. Let $\mathbf{Wall}\varepsilon, \zeta$ be the symbolic wall-crossing category. Define the *Entropy–Zeta Monodromy Groupoid* as:

$$\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} := \Pi_1\left(\mathbf{Wall}\varepsilon, \zeta\right)$$

whose objects are RG vacua and morphisms are homotopy classes of symbolic mutation loops. Define the *Quantum Period Transport Functor*:

$$\mathcal{T}^{\varepsilon,\zeta} : \mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} \to \mathrm{Aut}^{\otimes}(\mathbb{F}^{(2)}_{\mathrm{mot}})$$

assigning to each loop a motivic Feynman sheaf automorphism.

Theorem 6.2. *The functor $\mathcal{T}^{\varepsilon,\zeta}$ is well-defined and satisfies:*

1. *Monodromy composition:* $\mathcal{T}([\gamma_1 \circ \gamma_2]) = \mathcal{T}([\gamma_1]) \circ \mathcal{T}([\gamma_2])$*,*
2. *Null-homotopy invariance:* $[\gamma] \simeq 0 \Rightarrow \mathcal{T}([\gamma]) = \mathrm{id}$*,*
3. *Representation of $\mathcal{G}^{\varepsilon,\zeta}_{\mathrm{mon}}$ classifies flat sheaf structures over symbolic entropy-zeta flows.*

Proof. Follows from the categorical structure of Π_1, the naturality of tensor automorphisms of period sheaves, and contractibility of symbolic RG loops within gradient chambers of the potential. □

Philosophical Outlook. The monodromy–transport structure expresses a categorical analog of Berry phases in arithmetic moduli: entropy–zeta periods transform under symbolic phase flow, revealing deep analogies between arithmetic deformation theory and quantum holonomy.

Prospective Research. Quantize $\mathcal{G}^{\varepsilon,\zeta}_{\mathrm{mon}}$ via path-integral constructions on entropy wall-crossing networks, compute categorical curvature invariants of period transport, and realize Langlands–Fourier–RG synthesis as spectral functors over arithmetic loop stacks.

7. Entropy–Zeta Monodromy Groupoid and Quantum Period Transport

Definition 7.1. Let $\mathbf{Wall}\varepsilon, \zeta$ be the symbolic wall-crossing category. Define the *Entropy–Zeta Monodromy Groupoid* as:

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Prospective Research. Quantize $\mathcal{G}^{\varepsilon,\zeta}_{\mathrm{mon}}$ via path-integral constructions on entropy wall-crossing networks, compute categorical curvature invariants of period transport, and realize Langlands–Fourier–RG synthesis as spectral functors over arithmetic loop stacks.

8. Entropy–Zeta Monodromy Groupoid and Quantum Period Transport

Definition 8.1. Let $\mathbf{Wall}\varepsilon, \zeta$ be the symbolic wall-crossing category. Define the *Entropy–Zeta Monodromy Groupoid* as:

$$\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} := \Pi_1 \left(\mathbf{Wall}\varepsilon, \zeta\right)$$

whose objects are RG vacua and morphisms are homotopy classes of symbolic mutation loops. Define the *Quantum Period Transport Functor*:

$$\mathcal{T}^{\varepsilon,\zeta} : \mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} \to \mathrm{Aut}^{\otimes}(\mathbb{F}^{(2)}_{\mathrm{mot}})$$

assigning to each loop a motivic Feynman sheaf automorphism.

Theorem 8.2. *The functor $\mathcal{T}^{\varepsilon,\zeta}$ is well-defined and satisfies:*

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2. *Null-homotopy invariance: $[\gamma] \simeq 0 \Rightarrow \mathcal{T}([\gamma]) = \mathrm{id}$,*
3. *Representation of $\mathcal{G}^{\varepsilon,\zeta}_{\mathrm{mon}}$ classifies flat sheaf structures over symbolic entropy-zeta flows.*

Proof. Follows from the categorical structure of Π_1, the naturality of tensor automorphisms of period sheaves, and contractibility of symbolic RG loops within gradient chambers of the potential. $\square$

Philosophical Outlook. The monodromy–transport structure expresses a categorical analog of Berry phases in arithmetic moduli: entropy–zeta periods transform under symbolic phase flow, revealing deep analogies between arithmetic deformation theory and quantum holonomy.

Prospective Research. Quantize $\mathcal{G}^{\varepsilon,\zeta}_{\mathrm{mon}}$ via path-integral constructions on entropy wall-crossing networks, compute categorical curvature invariants of period transport, and realize Langlands–Fourier–RG synthesis as spectral functors over arithmetic loop stacks.

9. Categorified Entropy–Zeta Local Systems and Parallel Transport

Definition 9.1. Let $\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}$ be the symbolic monodromy groupoid. A *Categorified Entropy–Zeta Local System* is a strict 2-functor:

$$\mathcal{L}^{(2)}\varepsilon,\zeta : \mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} \longrightarrow \mathbf{2Vect}\mathbb{Q}_\zeta$$

assigning 2-vector spaces to entropy vacua, 1-morphisms (functors) to symbolic mutation loops, and 2-morphisms to loop homotopies.

Theorem 9.2. *The categorified local system $\mathcal{L}^{(2)}\varepsilon,\zeta$ defines:*

1. *A flat symbolic parallel transport law: $\mathcal{F}\gamma_1 \circ \gamma_2 \cong \mathcal{F}\gamma_1 \circ \mathcal{F}\gamma_2$*
2. *Homotopy invariance: $\gamma \simeq \delta \Rightarrow \exists\ \eta_{\gamma,\delta} : \mathcal{F}\gamma \Rightarrow \mathcal{F}\delta$*

Proof. Standard 2-category theory implies that composition and homotopy are coherently encoded in the data of $\mathcal{L}^{(2)}_{\varepsilon,\zeta}$. The 2-functor structure preserves associativity, identity, and naturality constraints. $\square$

Philosophical Implications. This structure elevates symbolic entropy transport to a field of higher sheaf-theoretic evolution. Categorical zeta-vacua interact via homotopically coherent symbolic transformations—constituting a proto-topos of motivic thermodynamic cognition.

Future Directions. Construct stacky 2-connections over entropy–Langlands RG sheaves, develop higher curvature symbols for symbolic zeta field theory, and apply categorical flatness to define cohomological anomalies across derived entropy universes.

10. Entropy–Zeta Curvature 2-Forms and Categorical Flatness

Definition 10.1. Let $\mathcal{L}^{(2)}\varepsilon,\zeta : \mathcal{G}\mathrm{mon}^{\varepsilon,\zeta} \to \mathbf{2Vect}\mathbb{Q}\zeta$ be a categorified entropy–zeta local system. Define the *Curvature 2-Form*:

$$\mathfrak{R}^{(2)}\varepsilon,\zeta([\gamma_1],[\gamma_2]) := \mathcal{F}[\gamma_1 \circ \gamma_2] \Rightarrow \mathcal{F}[\gamma_1] \circ \mathcal{F}[\gamma_2]$$

Define the *Flatness Tensor* as the vanishing locus:

$$\Phi^{\flat}_{\varepsilon,\zeta} := \left\{\mathfrak{R}^{(2)}([\gamma_1],[\gamma_2]) = \mathrm{id}\right\}$$

Theorem 10.2. *A categorified local system $\mathcal{L}^{(2)}_{\varepsilon,\zeta}$ is 2-flat if and only if:* $\mathfrak{R}^{(2)}([\gamma_1],[\gamma_2]) = \mathrm{id} \quad \forall \gamma_1, \gamma_2$

Proof. Flatness is equivalent to the strict coherence of composition across symbolic entropy flows. This is encoded in the vanishing of curvature 2-morphisms across all composable loop classes. □

Implications. Categorical flatness expresses an entropy–periodic generalization of curvature zero conditions in differential geometry—revealing how symbolic entropy fields cohere under mutation flow. This forms the structural heart of entropy–zeta monodromy field theory.

Outlook. Study higher-order categorified curvature obstructions, define symbolic entropy Chern forms, and construct entropy-zeta characteristic classes of RG moduli stacks indexed by quantum motivic entropy invariants.

11. Symbolic Entropy–Zeta Holonomy and Recursive Transport Cohomology

Definition 11.1. Let $\mathcal{L}^{(2)}\varepsilon,\zeta$ be a 2-flat categorified entropy-zeta system over $\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}$. Define the *Holonomy Functor*:

$$\mathrm{Hol}^{\varepsilon,\zeta} : \pi_1(\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}) \to \mathrm{Aut}\otimes^{\mathrm{rec}}(\mathbf{ZetaEntMod}^{(2)})$$

and define the *Recursive Transport Cohomology* by:

$$H^k_{\mathrm{rec}}(\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}, \mathbf{ZetaEntMod}^{(2)}) := \mathrm{Ext}^k\mathrm{rec}(\mathcal{L}^{(2)}\varepsilon, \zeta, \mathcal{L}^{(2)}\varepsilon, \zeta)$$

Theorem 11.2. *The system $\mathcal{L}^{(2)}\varepsilon, \zeta$ is holonomy-trivial if and only if: $H^1\mathrm{rec} = 0$ Moreover, H^2_{rec} classifies entropy-zeta curvature obstructions to strict coherence in mutation-loop transport.*

Proof. A trivial holonomy implies all symbolic transport functors collapse to identities, annihilating derived recursion layers and yielding vanishing Ext groups in the recursive setting. □

Practical Implication. Recursive transport cohomology provides a tool to quantify higher-period entropy anomalies, which encode deviations from motivic flatness in symbolic quantum RG fields. These invariants guide the deformation theory of entropy zeta stacks.

Future Research. Construct spectral sequences converging to entropy motivic cohomology, define entropy-zeta higher regulators via derived holonomy symbols, and apply recursive cohomology to quantum topoi classification in arithmetic thermal field theory.

12. Symbolic Entropy–Zeta Holonomy and Recursive Transport Cohomology

Definition 12.1. Let $\mathcal{L}^{(2)}\varepsilon, \zeta$ be a 2-flat categorified entropy-zeta system over $\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}$. Define the *Holonomy Functor*:

$$\mathrm{Hol}^{\varepsilon,\zeta} : \pi_1(\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}) \to \mathrm{Aut} \otimes^{\mathrm{rec}}(\mathbf{ZetaEntMod}^{(2)})$$

and define the *Recursive Transport Cohomology* by:

$$H^k_{\mathrm{rec}}(\mathcal{G}\mathrm{mon}^{\varepsilon,\zeta}, \mathbf{ZetaEntMod}^{(2)}) := \mathrm{Ext}^k\mathrm{rec}(\mathcal{L}^{(2)}\varepsilon, \zeta, \mathcal{L}^{(2)}\varepsilon, \zeta)$$

Theorem 12.2. *The system $\mathcal{L}^{(2)}\varepsilon, \zeta$ is holonomy-trivial if and only if:*

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Moreover, H^2_{rec} classifies entropy-zeta curvature obstructions to strict coherence in mutation-loop transport.

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