An Overview of the Sieve Method and its History

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Preface

Trying to decompose an integer into a product of integers, we feel irritation. There should dwell the reason why any prime appears like a real gem that one can touch and hold. We thus muse ever and again how and when ancient people discovered the way of sifting out primes and began appreciating them. Perhaps those who conceived the divisibility had already some sieves in their minds. Indeed, a wealth of evidences have been excavated supporting our view. The story to be told below must have originated more than five millennia ago [1], while the primordial intellectual irritation has remained fresh and fundamental till today.

The history of the Sieve Method is rich and fascinating; we would need a volume to exhaust the story. In the present article we shall instead concentrate on several pivotal ideas that made progress possible; so the scope is inevitably limited. Nevertheless, you will encounter instances of precious mathematical achievements that people in the future will certainly continue to relate.

Notes are to be read as essential parts, although they are in the style of personal memoranda. Mathematical symbols and definitions are introduced where they are needed for the first time, and will continue to be effective until otherwise stated. Theorems are given somewhat implicitly, and details such as domains of variables are to be induced from the context. References are restricted mostly to seminal works in respective developments. Basic facts from Analytic Number Theory could be found in the monographs [26] [64].

Remark: This is a translation of our Japanese expository article that was published under the title '*An overview of sieve methods*' in the second issue of the 52nd volume of Sugaku, the Mathematical Society of Japan, April 2005. At this opportunity we have made some revision and changed the title into something more appropriate. Also, it should be noted that events in the last two decades are left untouched except for a few.

Chapter 1. Brun's Sieve

1.1 Mark any natural number that is divisible by the first prime 2, and repeat the same with all other primes less than a given $z > 2$. Then any natural number less than z^2 that remains unmarked is either 1 or a prime in the interval $[z, z^2)$, since such an integer does not have two or more prime factors. This is a version of the well-known sieve method named after Eratosthenes of Alexandria [2].

Eratosthenes' Sieve might appear to be quite effective, especially when z is large, for it allows us to expand the table of all primes less than z to that of those less than z^2. However, if we look into the quantitative aspect of the method or if it is required to count the number of integers unsifted, then Eratosthenes' Sieve becomes virtually ineffectual [3]. Viggo Brun [12] confronted the challenge of improving Eratosthenes' Sieve to turn it into a quantitatively effective device, and became the founder of the modern theory of the Sieve Method [4]. Let us see how he brought the first light [10] into the darkness of 2100 years.

1.2 Let $P(z)$ be the product of all primes less than z, and let (m, n) be the greatest common divisor of integers m, n. That an integer n does not have any prime factor less than z is equivalent to $(n, P(z)) = 1$. Thus, the characteristic function of the set of all such integers is expressed as

$$(1.1) \qquad \sum_{d \mid (n, P(z))} \mu(d),$$

where μ is the Möbius function. This coincides with the above procedure of marking integers. In fact, if n has r marks, then the value of the last sum is equal to $(1 - 1)^r$. Hence (1.1) could be identified with Eratosthenes' Sieve[5].

We consider a finite sequence $\mathcal{A}$ of integers, and put $\mathcal{S}(\mathcal{A}, z) = \{n \in \mathcal{A} : (n, P(z)) = 1\}$. Then (1.1) implies that

$$(1.2) \qquad |\mathcal{S}(\mathcal{A}, z)| = \sum_{d \mid P(z)} \mu(d)|\mathcal{A}_d|, \quad \mathcal{A}_d = \{n \in \mathcal{A} : n \equiv 0 \,(\mathrm{mod}\,d)\}\,.$$

Thus, in order to either evaluate or bound $|\mathcal{S}(\mathcal{A}, z)|$, we need to have certain information about the behaviour of $|\mathcal{A}_d|$ with variable d; and following a general practice we write

$$(1.3) \qquad |\mathcal{A}_d| = \frac{\omega(d)}{d}X + R_d, \quad \omega(d) \geq 0\,,$$

where ω is a multiplicative function. One may regard $(\omega(d)/d)X$ as the main term and R_d as the remainder; for instance, X can be seen as an approximation to $|\mathcal{A}|$. The terms R_d should be small either individually or in a certain sense of mean. Inserting (1.3) into (1.2), we have

$$(1.4) \qquad |\mathcal{S}(\mathcal{A}, z)| = V(z, \omega)X + R(\mathcal{A}, z),$$

where

$$(1.5) \qquad V(z, \omega) = \prod_{p < z}\left(1 - \frac{\omega(p)}{p}\right), \quad R(\mathcal{A}, z) = \sum_{d \mid P(z)} \mu(d)R_d.$$

Hereafter p denotes a generic prime.

1.3 The identity (1.4) does not have much realistic contents, however. To see this, we consider the problem of counting primes in a given interval. Thus, let x be sufficiently large, and put $\mathcal{A} = \{n : x - y \leq n < x\}$, $2 \leq y \leq x/2$. With this, we have $\omega \equiv 1$, $X = y$, $R_d = [-(x - y)/d] - [-x/d] - y/d$, where $[a]$ is the integral part of a. The above discussion gives

$$(1.6) \qquad \pi(x) - \pi(x - y) = V(\sqrt{x}, 1)y + R(\mathcal{A}, \sqrt{x}),$$

where $\pi(x)$ is the number of primes less than x as usual. Since[6]

$$(1.7) \qquad V(z, 1) = \frac{e^{-c_E}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right) \quad (c_E: \text{the Euler constant}),$$

we get

$$(1.8) \qquad \pi(x) - \pi(x - y) = 2e^{-c_E}(1 + o(1))\frac{y}{\log x} + R(\mathcal{A}, \sqrt{x}).$$

On the other hand, the Prime Number Theorem suggests that

$$(1.9) \qquad \pi(x) - \pi(x - y) = (1 + o(1))\frac{y}{\log x}.$$

Thus it is plausible[7] to have $R(\mathcal{A}, \sqrt{x}) \sim (1 - 2e^{-c_E})(y/\log x)$. That is, each of the two terms on the right of (1.6) can not be the main term or the remainder term either. It seems extremely hard[8] to deduce this fact directly from the definition of $R(\mathcal{A}, \sqrt{x})$.

1.4 As another example, we shall consider $\pi_2(x)$ the number of twin primes less than x. This time we work with the sequence $\mathcal{A} = \{n(n + 2) : 1 \leq n < x - 2\}$; in fact, $|\mathcal{S}(\mathcal{A}, \sqrt{x})| = \pi_2(x) + O(\sqrt{x})$. We have $\omega(2) = 1$, $\omega(p) = 2$ $(p \geq 3)$, and by (1.4) and (1.7) it follows, after a rearrangement, that

$$(1.10) \qquad \pi_2(x) = 8e^{-2c_E}(1 + o(1))\frac{x}{(\log x)^2}\prod_{p \geq 3}\left(1 - \frac{1}{(p-1)^2}\right) + R(\mathcal{A}, \sqrt{x}).$$

It is, however, much more difficult to deal with this $R(\mathcal{A}, \sqrt{x})$ than the analogue in the last section. In a conjecture due to Hardy and Littlewood [23] the asymptotic identity $\pi_2(x) \sim 2Cx/(\log x)^2$ is predicted, where C is the Euler product on the right of (1.10). Thus, again, each of the two terms on the right of (1.10) cannot be the main term or the remainder term either, and the expression (1.10) does not stand for anything meaningful for the Twin Prime Conjecture.

1.5 The difficulties with $R(\mathcal{A}, z)$ observed above stems from the fact that the number of summands in the defining sum (1.5) may be too big to handle. As z increases, the factors of $P(z)$ can become huge, and moreover their number too. Whether it is possible or not to detect any dramatic cancellation among the summands should obviously be tremendously difficult to see. Here is the reason for the limitation of Eratosthenes' Sieve.

In 1915 Brun [10] broke this spell with a surprisingly simple idea. He threw the explicit formula (1.1) out, and replaced it by an inequality bounding the characteristic function from above and below so that he could gain an effective control over the size of participating factors of $P(z)$. In other words, he moved the sieve theory from the classic world of exactness to the modern world of reserved certainty. More explicitly, his idea is embodied in

$$(1.11) \qquad \sum_{\substack{d|(n,P(z)) \\ \nu(d) \leq 2\ell+1}} \mu(d) \leq \sum_{d|(n,P(z))} \mu(d) \leq \sum_{\substack{d|(n,P(z)) \\ \nu(d) \leq 2\ell}} \mu(d),$$

with $\nu(d)$ the number of different prime factors of d; in fact we have, for any $\ell \geq 0$,

$$(1.12) \qquad \sum_{\substack{d|m \\ \nu(d) \leq \ell}} \mu(d) = (-1)^\ell\binom{\nu(m) - 1}{\ell}.$$

The inequality (1.11) is often called Brun's Pure Sieve.

1.6 Let us try Brun's idea in the instance of the twin prime problem; we are going to bound $\pi_2(x)$ from above. Let $z \leq \sqrt{x}$ and ℓ be to be fixed later. We apply (1.11) to the $\mathcal{A}$ of Section 1.4, and have

$$\pi_2(x) \leq |S(\mathcal{A}, z)| + z$$

(1.13)
$$\leq x \sum_{\substack{d \mid P(z) \\ \nu(d) \leq 2\ell}} \frac{\mu(d)\omega(d)}{d} + \sum_{d \leq z^{2\ell}} \omega(d) + z,$$

in which we have used $|R_d| \leq \omega(d)$. Compared with the sum of divisors, the second sum over d is $O(\ell z^{2\ell} \log z)$. The difference between the first sum and $V(z, \omega)$ is

(1.14)
$$-\sum_{\substack{d \mid P(z) \\ \nu(d) \geq 2\ell+1}} \frac{\mu(d)\omega(d)}{d} \ll 2^{-2\ell} \sum_{d \mid P(z)} \frac{2^{2\nu(d)}}{d};$$

the symbol $\ll$ indicates in general that the absolute value of the left side is less than a constant multiple of the right side. The last sum is $O\left(V(z, 1)^{-4}\right)$. Collecting these and setting $z = \exp(\log x/(100 \log \log x))$, $\ell = [\log x/(4 \log z)]$, we obtain [9]

(1.15)
$$\pi_2(x) \ll x \left(\frac{\log \log x}{\log x}\right)^2 \quad \text{or} \quad \frac{\pi_2(x)}{\pi(x)} \ll \frac{(\log \log x)^2}{\log x}.$$

Therefore, twin primes occur far less frequently than ordinary primes. It is literally hopeless to deduce this fact via (1.10). It is amazing that the imperfect (1.11) could yield any result that the exact (1.1) would never be capable of [10].

1.7 Brun's Sieve, the title of the present chapter, is an improved version [12] of his Pure Sieve; a combinatorial sophistication was introduced into the choice of divisors in (1.11). It enabled Brun to achieve the impressive bound

(1.16)
$$\pi_2(x) \ll \frac{x}{(\log x)^2}.$$

In view of Hardy–Littlewood's conjecture mentioned above, this should be the best possible, save for the implied constant. The construction of Brun's Sieve is, however, so intricate that we have to skip the details [11], and state only the conclusion (1.22) below, moreover in a considerably abridged fashion. Nevertheless, we shall reach an equivalent result in Section 3.5 via a somewhat different argument.

For the present and later purpose, we need to rearrange the specifications introduced in [12], to be in accordance with today's practice. Thus, it is customary to suppose that it holds that for any $2 < z_1 < z_2$

(1.17)
$$\frac{V(z_1, \omega)}{V(z_2, \omega)} = \prod_{z_1 \leq p < z_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} = \left(\frac{\log z_2}{\log z_1}\right)^{\kappa} \left(1 + O\left(\frac{1}{\log z_1}\right)\right),$$

with a $\kappa > 0$, the dimension of the sieve problem under consideration [12]. This is to be compared with (1.7). For example, in Section 1.4 the Twin Prime Conjecture is discussed as a sieve problem of dimension 2. On the other hand, the inequality (1.11) is understood as a particular construction of sieve weights $\rho_r(d)$ such that for any integer n

$$(1.18) \qquad (-1)^r \left\{ \sum_{d|(n,P(z))} \mu(d) - \sum_{d|(n,P(z))} \mu(d)\rho_r(d) \right\} \geq 0, \quad \rho_r(1) = 1,$$

with the convention $\rho_{r_1} \equiv \rho_{r_2}$ $(r_1 \equiv r_2 \,(\mathrm{mod}\,2))$. We have, because of (1.3),

$$(1.19) \qquad (-1)^r \left\{ |\mathcal{S}(\mathcal{A}, z)| - V(z, \omega\,;\rho_r)X \right\} \geq (-1)^r R(\mathcal{A}, z\,;\rho_r),$$

where

$$(1.20) \qquad V(z, \omega\,;\rho_r) = \sum_{d|P(z)} \mu(d)\rho_r(d)\frac{\omega(d)}{d}, \quad R(\mathcal{A}, z\,;\rho_r) = \sum_{d|P(z)} \mu(d)\rho_r(d)R_d.$$

According as $r \equiv 0, 1 \,(\mathrm{mod}\,2)$, the inequality (1.19) is called the lower and the upper bounds of $|\mathcal{S}(\mathcal{A}, z)|$; occasionally only one of them is considered. Naturally, we shall regard $V(z, \omega\,;\rho_r)X$ as the main term, and $R(\mathcal{A}, z\,;\rho_r)$ as the remainder. In order to have any effective control over the latter, we ought to impose a limitation to the size of participating d. For this sake we introduce the condition

$$(1.21) \qquad \rho_r(d) = 0, \quad d \geq D, \quad d|P(z).$$

The parameter D is called the level of the sieve weights ρ_r. A sieve problem is to find ρ_r that yield any good main term under these specifications [13].

We now exhibit the principal result of Brun [12], with a drastic simplification: Let $D = z^\tau$. Then there exist characteristic functions ρ_r such that

$$(1.22) \qquad V(z, \omega\,;\rho_r) = (1 + O(e^{-\frac{1}{2}\tau \log \tau}))V(z, \omega).$$

1.8 Let us apply the last assertion to the situation treated in Section 1.4. We set $z^\tau = \sqrt{x}$; then the remainder term can obviously be ignored. With a sufficiently large τ, we get the assertion

$$(1.23) \qquad \left| \left\{ n < x : \text{if } p|n(n+2) \text{ then } p \leq x^{1/(2\tau)} \right\} \right| \asymp \frac{x}{(\log x)^2}.$$

This implies not only (1.16) but also that there exist infinitely many pairs $(n, n+2)$ of integers such that the number of prime factors of each is less than 2τ.

In this way, Brun [12] accomplished the first definitive advance toward the Twin Prime Conjecture. If we use instead his Pure Sieve, then it would be required to set $\ell = [25 \log \log x]$ and thus $\tau \approx \log \log x$. Hence Brun's Sieve is far stronger. Certainly the same could be asserted about Goldbach's Conjecture [14]; we need only to move to the sequence $\{n(N - n) : 3 \leq n \leq N - 3\}$ with an even integer N.

1.9 It is possible to treat the Twin Prime Conjecture as a one dimensional or a linear sieve problem. To see this, let φ be the Euler totient function and li the logarithmic integral; and put

$$(1.24) \qquad \pi(x;k,l) = |\{p < x : p \equiv l \,(\mathrm{mod}\,k)\}|, \quad E(x;k,l) = \pi(x;k,l) - \frac{1}{\varphi(k)}\mathrm{li}\,x.$$

The latter is called the remainder term in the prime number theorem for arithmetic progressions. To the sequence $\mathcal{A} = \{p + 2 : 3 \le p < x\}$, we apply Brun's Sieve; thus $X = \mathrm{li}\,x$, $\omega(2) = 0$, $\omega(p) = p/(p-1)$, $(p \ge 3)$, and $\kappa = 1$ as well as $R_d = E(x;d,-2)$. Let τ be sufficiently large as before. Then we have

$$(1.25) \qquad |\mathcal{S}(\mathcal{A},z)| = (1 + O(e^{-\frac{1}{2}\tau\log\tau}))V(z,\omega)\cdot\mathrm{li}\,x + O\left(\sum_{\substack{d<z^\tau \\ 2\nmid d}} |E(x;d,-2)|\right).$$

This time, the issue is how to deal with the second term on the right side. We assume that

$$(1.26) \qquad \sum_{q<Q} \max_{\substack{a\,(\mathrm{mod}\,q) \\ (a,q)=1}} |E(x;q,a)| \ll \frac{x}{(\log x)^A}, \quad A \ge 3,$$

Then we have

$$(1.27) \qquad \left|\{p < x : p + 2 \text{ does not have prime divisors less than } Q^{1/\tau}\}\right| \asymp \frac{x}{(\log Q)(\log x)}.$$

For example, the Extended Riemann Hypothesis would allow us to set $Q = \sqrt{x}/(\log x)^{A+1}$; and

$$(1.28) \qquad |\{p : \nu(p+2) \le 2\tau + 1\}| = \infty.$$

Goldbach's Conjecture could be treated in much the same way. Comparing this with the assertions in the previous section, we apprehend the strength of the Extended Riemann Hypothesis [15].

At (1.25)–(1.26) is observed a typical instance of the relation between a sieve problem and the distribution of the relevant sequence of integers among arithmetic progressions. That is, a sieve problem upon a particular sequence of integers is reduced to the discussion of the distribution of the elements of the sequence among arithmetic progressions with variable moduli. One may see that if the decomposition into arithmetic progressions is made too fine, then relevant moduli could become too large to handle the remainder term in the sieve effectively. Also it might be conceivable that relatively small moduli could contribute substantially. This situation reminds us of the Circle Method of Ramanujan and Hardy [16]; that is, there appears to exist a relation between sieve problems and the Farey sequence. We shall encounter the same in the next section but with a somewhat different context.

1.10 Brun's Sieve yields remarkable assertions not only about those great conjectures but also about fundamental queries in the theory of the distribution of primes:

$$(1.29) \qquad \pi(x) - \pi(x - y) \ll \frac{y}{\log y} \quad (2 \le y \le x - 2),$$

$$(1.30) \qquad \pi(x;k,l) \ll \frac{x}{\varphi(k)\log(x/k)} \quad (2 \le k \le x/2).$$

In fact, (1.29) follows immediately from an obvious modification of the discussion in Section 1.3. On the other hand, as to (1.30) we may certainly assume that $(k, l) = 1$; then for the sequence $\mathcal{A} = \{n < x : n \equiv l \,(\mathrm{mod}\, k)\}$ the specification (1.3) holds with $X = y/k$, $|R_d| \leq 1$; $\omega(p) = 0$ as $p|k$, and $\omega(p) = 1$ as $(p, k) = 1$. In particular, we have $V(z, \omega) \leq (k/\varphi(k))V(z, 1)$. The rest is straightforward. The assertion (1.30) is called traditionally the Brun–Titchmarsh theorem.

As a matter of fact, if the uniformity as displayed prominently in (1.29)–(1.30) is required, any currently available analytic method which relies on the theory of the Riemann zeta and the Dirichlet L-functions is unable to produce anything comparable to the last two assertions. Even under the Extended Riemann Hypothesis, the situation would not change. It is indeed amazing that such an elementary idea as Brun's could ever take us close to the very subtlety of the distribution of primes.

Chapter 2. Linnik's and Selberg's Sieves

2.1 Some 20 years had passed since Brun's fundamental work [12] when Ju. V. Linnik [41] marked a new departure in the Sieve Method; and in a few years A. Selberg [68]–[70] made an independent leap. Selberg's idea was a distinctive incision into the sieve theory general. The construction of his sieve, i.e., his sieve weights, is fundamentally different from Brun's; thus he brought a structural change into the Sieve Method. On the other hand, Linnik's idea would later be appreciated because not only of its highly effective sieve effect but also of the argument itself that he employed. With Linnik's seminal work, a general principle was born, which has been and is still a driving force behind many of major works in Analytic Number Theory. This is now called the Large Sieve, in a much wider context than the title of his work indicated then.

We shall show the essentials of the two ideas. Also, a duality relation between them will be disclosed. Further, it will be witnessed that Large Sieve yields not only a sieve bound but also a spectacular conclusion about the distribution of primes in arithmetic progressions.

2.2 We need first to change the technical definition introduced in Section 1.7 of a sieve problem into a conceptual setting. Thus, let $\Omega(p)$ be a set of residue classes mod p, and let us write $n \in \Omega(p)$ in stead of $n \,(\mathrm{mod}\, p) \in \Omega(p)$. We put

$$(2.1) \qquad \mathcal{S}(\mathcal{A}, z\,; \Omega) = \{n \in \mathcal{A} : n \notin \Omega(p), \forall p < z\},$$

with any sequence $\mathcal{A}$ of integers [17]. A sieve problem is newly defined to be the estimation of $|\mathcal{S}(\mathcal{A}, z\,; \Omega)|$, though we shall consider mainly the situation where $\mathcal{A}$ is the interval

$$(2.2) \qquad \mathcal{N} = [M, M + N) \cap \mathbb{Z}, \quad M \in \mathbb{Z}, N \in \mathbb{N}.$$

In Section 1.4, the Twin Prime Conjecture is treated with $M = 0$, $N = [x-2]$, $\Omega(2) = \{0\}$, $\Omega(p) = \{0, -2\}$ $(p \geq 3)$, $|\Omega(p)| = \omega(p)$. In this example, we have $|\Omega(p)| \leq 2$; but in general we should not restrict the size of $\Omega(p)$. However, if we have, for instance, always $|\Omega(p)| \geq cp$ with a certain fixed $c > 0$, Brun's Sieve is not applicable, for the condition (1.17) on the sieve dimension is violated. Then, is there any sieve procedure that is effective even when $|\Omega(p)|$ may become huge like that? It was Linnik [41] who gave the first answer to this intriguing problem.

2.3 Linnik argued as follows: Let ϖ be the characteristic function of the set $\{n \in \mathbb{Z} : n \notin \Omega(p), \forall p < z\}$, and put

$$U(\theta) = \sum_{M \leq n < M+N} \varpi(n) \exp(2\pi i n \theta),$$

(2.3)

$$U(\theta; p, a) = \sum_{\substack{M \leq n < M+N \\ n \equiv a \,(\mathrm{mod}\ p)}} \varpi(n) \exp(2\pi i n \theta).$$

Then we have

(2.4)
$$\sum_{a=1}^{p-1} \left| U\left(\theta + \frac{a}{p}\right) \right|^2 = p \sum_{a=1}^{p} |U(\theta; p, a)|^2 - |U(\theta)|^2.$$

On the other hand, since $U(\theta; p, a) = 0$ if $a \in \Omega(p)$, $p < z$, we have also

(2.5)
$$|U(\theta)|^2 = \left| \sum_{a=1}^{p} U(\theta; p, a) \right|^2 \leq (p - |\Omega(p)|) \sum_{a=1}^{p} |U(\theta; p, a)|^2, \quad p < z,$$

or [18]

(2.6)
$$|U(\theta)|^2 \frac{|\Omega(p)|}{p - |\Omega(p)|} \leq \sum_{a=1}^{p-1} \left| U\left(\theta + \frac{a}{p}\right) \right|^2, \quad p < z.$$

By the decomposition law of residue classes, we have

(2.7)
$$(U(0))^2 \prod_{p|q} \frac{|\Omega(p)|}{p - |\Omega(p)|} \leq \sum_{\substack{a \,(\mathrm{mod}\ q) \\ (a,q)=1}} \left| U\left(\frac{a}{q}\right) \right|^2, \quad q | P(z).$$

In this way, we are led to [19]

(2.8)
$$|\mathcal{S}(\mathcal{N}, z \,; \Omega)|^2 G(z, \Omega) \leq \sum_{q<z} \sum_{\substack{a \,(\mathrm{mod}\ q) \\ (a,q)=1}} \left| \sum_{M \leq n < M+N} \varpi(n) \exp\left(2\pi i \frac{a}{q} n\right) \right|^2,$$

where

(2.9)
$$G(z, \Omega) = \sum_{q<z} \mu(q)^2 H(q, \Omega), \quad H(q, \Omega) = \prod_{p|q} \frac{|\Omega(p)|}{p - |\Omega(p)|}.$$

Hence, our initial problem has been reduced to the estimation of the sum in (2.8) over the Farey sequence. We skip the relevant discussion by Linnik himself, since we shall show an assertion better than his, in Section 2.6. What is essential here is to follow the procedure

leading to (2.8), which is termed Linnik's Sieve [20]. Nevertheless, the final assertion should be displayed: By the inequality (2.30), we have

$$(2.10) \qquad |\mathcal{S}(\mathcal{N}, z\,; \Omega)| \le \frac{1}{G(z, \Omega)}(N + z^2).$$

2.4 Selberg argued as follows: We extend Ω multiplicatively, and let λ be an arbitrary real-valued function such that $\lambda(1) = 1$. We have, for any integer n,

$$(2.11) \qquad \varpi(n) \le \left(\sum_{\substack{n \in \Omega(d) \\ d \mid P(z)}} \lambda(d) \right)^2 = \sum_{\substack{n \in \Omega([d_1, d_2]) \\ d_1, d_2 \mid P(z)}} \lambda(d_1)\lambda(d_2),$$

where $[d_1, d_2]$ is the least common multiple of d_1, d_2. This inequality is trivial; nevertheless, its consequence is impressive.

For the sake of simplicity, we assume that $\lambda(d) = 0$ either if $d \ge z$ or if $\mu(d) = 0$. Summing (2.11) over $n \in \mathcal{N}$ and exchanging the order of summation, we have

$$(2.12) \qquad |\mathcal{S}(\mathcal{N}, z\,; \Omega)| \le N \cdot S + R,$$

where

$$(2.13) \qquad \begin{aligned} S &= \sum_{d_1, d_2 < z} \frac{|\Omega([d_1, d_2])|}{[d_1, d_2]} \lambda(d_1)\lambda(d_2), \\ |R| &\le \sum_{d_1, d_2 < z} |\Omega([d_1, d_2])||\lambda(d_1)\lambda(d_2)|. \end{aligned}$$

Selberg computed the minimum of the quadratic form S with the side condition $\lambda(1) = 1$. His reasoning is illuminating [21]. We note first

$$(2.14) \qquad \frac{|\Omega([d_1, d_2])|}{[d_1, d_2]} = \frac{|\Omega(d_1)|}{d_1} \cdot \frac{|\Omega(d_2)|}{d_2} \cdot \frac{(d_1, d_2)}{|\Omega((d_1, d_2))|},$$

and by the Möbius inversion

$$(2.15) \qquad \frac{(d_1, d_2)}{|\Omega((d_1, d_2))|} = \sum_{\substack{f \mid d_1 \\ f \mid d_2}} \frac{1}{|\Omega(f)|} \prod_{p \mid f} (p - |\Omega(p)|), \quad \mu(d_1)\mu(d_2) \ne 0.$$

Thus

$$(2.16) \qquad S = \sum_{f < z} \frac{\mu(f)^2}{|\Omega(f)|} \prod_{p \mid f}(p - |\Omega(p)|) \cdot \xi(f)^2, \quad \xi(f) = \sum_{\substack{d < z \\ d \equiv 0\,(\bmod\ f)}} \frac{|\Omega(d)|}{d}\lambda(d).$$

The inversion of the linear transform $\lambda \mapsto \xi$ is given by

$$(2.17) \qquad \lambda(d) = \frac{d}{|\Omega(d)|} \sum_{g < z/d} \mu(g)\xi(dg).$$

The side condition $\lambda(1) = 1$ is thus transformed accordingly, and

$$(2.18) \qquad S = \frac{1}{G(z,\Omega)} + \sum_{f < z} \frac{\mu(f)^2}{|\Omega(f)|} \prod_{p | f} (p - |\Omega(p)|) \left(\xi(f) - \frac{1}{G(z,\Omega)} \mu(f) H(f,\Omega) \right)^2.$$

Hence the optimal ξ is found, and inserting it into (2.17) we are led to the assertion that

$$(2.19) \qquad S = \frac{1}{G(z,\Omega)}, \quad \lambda(d) = \frac{\mu(d)}{G(z,\Omega)} \prod_{p | d} \frac{p}{p - |\Omega(p)|} \sum_{\substack{g < z/d \\ (d,g)=1}} \mu(g)^2 H(g,\Omega).$$

Obviously the assumption on λ imposed above is satisfied by this specialization. Moreover, we have

$$(2.20) \qquad |\lambda(d)| \le \mu(d)^2.$$

In fact, we have, for any $d < z$ ($\mu(d) \ne 0$),

$$(2.21) \qquad G(z,\Omega) = \sum_{f | d} \mu(f)^2 H(f,\Omega) \sum_{\substack{g < z/f \\ (d,g)=1}} \mu(g)^2 H(g,\Omega),$$

from which (2.20) follows immediately. Collecting these, we find that

$$(2.22) \qquad |\mathcal{S}(\mathcal{N}, z\,;\Omega)| \le \frac{N}{G(z,\Omega)} + R, \quad |R| \le \left(\sum_{d < z} |\Omega(d)| \right)^2.$$

The procedure of the present section is called Selberg's Sieve [22].

2.5 A comparison of (2.22) with (2.10) might cause the incorrect impression that Selberg's Sieve is inferior to Linnik's [23]. In fact, (2.10) could be deduced with Selberg's Sieve as well [24].

In order to show this, we express the characteristic function of the set $\{n \in \mathbb{Z} : n \in \Omega(d)\}$ as

$$\frac{1}{d} \sum_{a \,(\mathrm{mod}\ d)} \sum_{h \in \Omega(d)} \exp\left(2\pi i (n - h)\frac{a}{d} \right)$$

$$(2.23)$$

$$= \frac{1}{d} \sum_{\substack{q | d}} \sum_{\substack{a \,(\mathrm{mod}\ q) \\ (a,q)=1}} \left(\sum_{h \in \Omega(d)} \exp\left(-2\pi i \frac{a}{q} h \right) \right) \cdot \exp\left(2\pi i \frac{a}{q} n \right).$$

Inserting this into (2.11), we have

$$(2.24) \qquad |\mathcal{S}(\mathcal{N}, z\,; \Omega)| \leq \sum_{M \leq n < M+N} \left| \sum_{q<z} \sum_{\substack{a\,(\mathrm{mod}\ q) \\ (a,q)=1}} b\left(\frac{a}{q}\right) \exp\left(2\pi i \frac{a}{q} n\right) \right|^2,$$

where

$$(2.25) \qquad b\left(\frac{a}{q}\right) = \sum_{\substack{d<z \\ d \equiv 0\,(\mathrm{mod}\ q)}} \frac{\lambda(d)}{d} \sum_{h \in \Omega(d)} \exp\left(-2\pi i \frac{a}{q} h\right).$$

Applying the inequality (2.31) below to the right side of (2.24), we get

$$(2.26) \qquad |\mathcal{S}(\mathcal{N}, z\,; \Omega)| \leq (N+z^2) \sum_{q<z} \sum_{\substack{a\,(\mathrm{mod}\ q) \\ (a,q)=1}} \left| b\left(\frac{a}{q}\right) \right|^2.$$

The double sum is equal to

$$(2.27) \qquad \sum_{d_1,d_2<z} \frac{\lambda(d_1)\lambda(d_2)}{d_1 d_2} \sum_{h_1 \in \Omega(d_1)} \sum_{h_2 \in \Omega(d_2)} \sum_{q|(d_1,d_2)} \sum_{\substack{a\,(\mathrm{mod}\ q) \\ (a,q)=1}} \exp\left(2\pi i \frac{a}{q}(h_1 - h_2)\right),$$

which coincides with S above, as can readily be seen by observing the multiplicativity of the construction. Hence we have

$$(2.28) \qquad |\mathcal{S}(\mathcal{N}, z\,; \Omega)| \leq (N+z^2) \cdot S.$$

By the argument of the previous section, up to (2.19), we obtain (2.10) again.

Therefore the important upper bound (2.10) has been proved in two ways. There is an obvious duality between them. It should be interesting to know that there is such an intrinsic relation between Linnik's and Selberg's ideas which occurred independently. By the way, the inequalities (2.8) and (2.24) are typical instances of applications of the Large Sieve.

2.6 We now exhibit the fundamental inequality of the Large Sieve: Let $\{\psi_m\}$ be a finite set in a Hilbert space equipped with the inner product $\langle,\rangle$. Then we have, for any ψ in the space,[25]

$$(2.29) \qquad \sum_m \frac{|\langle \psi, \psi_m \rangle|^2}{\sum_n |\langle \psi_m, \psi_n \rangle|} \leq \langle \psi, \psi \rangle.$$

From this, a set of useful inequalities follow. Among them, the following two are utilised in the above: Let $\{\theta_r\}$ be a sequence in the unit interval, whose elements are well separated with the minimum distance $\delta > 0$ (mod 1). Then we have, for any complex vectors $\{a_n\}$, $\{b_r\}$,

$$(2.30) \qquad \sum_r \left| \sum_{M \leq n < M+N} a_n \exp(2\pi i n \theta_r) \right|^2 \leq (N-1+\delta^{-1}) \sum_{M \leq n < M+N} |a_n|^2$$

and

$$(2.31) \qquad \sum_{M \leq n < M+N} \left| \sum_r b_r \exp(2\pi i n \theta_r) \right|^2 \leq (N - 1 + \delta^{-1}) \sum_r |b_r|^2,$$

where the interval $[M, M + N)$ is as in (2.2). The latter is a consequence of the former, and vice versa. This is due to the well-known fact that the norms of a bounded linear operator and its adjoint acting in a Hilbert space are equal to each other.

2.7 Hence, as far as finite intervals are concerned, Linnik's and Selberg's Sieves give rise to the same upper bound (2.10). Here are a few assertions that are consequences of (2.10):

$$(2.32) \qquad \pi(x) - \pi(x - y) \leq 2(1 + o(1)) \frac{y}{\log y},$$

$$(2.33) \qquad \pi(x; k, l) \leq 2(1 + o(1)) \frac{x}{\varphi(k) \log (x/k)},$$

$$(2.34) \qquad \pi_2(x) \leq 16(1 + o(1)) \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2} \right) \frac{x}{(\log x)^2}.$$

These can be proved, via (2.10), with the corresponding $\mathcal{N}$ and Ω, with $z = (N/\log N)^{1/2}$. The asymptotic evaluation of $G(z, \Omega)$ should not cause any difficulty. Another interesting application could be obtained with $\Omega(p)$ being the set of all quadratic non-residues $(\bmod \, p)$.

Thus, it is understood that as far as upper bounds are concerned Linnik's and Selberg's Sieves are superior to Brun's[26]. For instance, the bound (2.34) should be compared with the Hardy–Littlewood conjecture mentioned above. Also, the new form (2.33) of the Brun-Titchmarsh theorem draws special attention. This is because of the following fact: If there exist two absolute constants $\alpha, \beta > 0$ with which we have, uniformly for $(k, l) = 1$,

$$(2.35) \qquad \pi(x; k, l) \leq 2(1 - \alpha) \frac{x}{\varphi(k) \log(x/k)}, \quad k < x^\beta,$$

then Dirichlet L-functions $L(s, \chi)$, $\chi \,(\bmod \, k)$, should not have any exceptional zero; so the theory of the distribution of primes in arithmetic progressions would fundamentally be improved[27]. Also, an effective lower bound, which is essentially best possible, would follow for the class numbers of imaginary quadratic number fields. In this context, it should be noted specifically that the critical bound

$$(2.36) \qquad \pi(x; k, l) \leq 2 \frac{x}{\varphi(k) \log (x/k)}$$

has been established via a more careful application of Linnik's Sieve[28].

2.8 The above discussion might be termed as an account of the additive Large Sieve, for it concerns additive characters as is indicated by (2.30)–(2.31). We have seen the appearance of important upper bounds in the prime number theory. In the present section we turn to an account of the multiplicative Large Sieve, concerning instead Dirichlet characters; and we shall see that there emerges a surprising assertion on the asymptotic theory of the distribution of primes in arithmetic progressions. More precisely, the multiplicative Large Sieve opens a

way to avoid the Extended Riemann Hypothesis [29]. This fascinating theory was inseminated by A. Rényi [65] [30]. In order to appreciate his contribution, we need to review briefly the history of the theory of the distribution of primes.

Primes in Short Intervals: Under the Riemann Hypothesis, the asymptotic formula (1.9) holds with, e.g., $x^{1/2}(\log x)^3 < y < x/2$. However, G. Hoheisel [24] established, without any hypothesis, the asymptotic formula upon the condition $x^\vartheta < y < x/2$, where ϑ is a positive absolute constant less than 1. That was the unprecedented event in which was discovered the possibility to avoid the Riemann Hypothesis; and it was the beginning of the modern theory of the distribution of primes. At the core of Hoheisel's argument is a statistical study of the distribution of the complex zeros of the Riemann zeta-function $\zeta(s)$ or a statistical proof of the Riemann hypothesis [31], due to H. Bohr and E. Landau [3]. In Riemann's explicit formula for the function $\pi(x)$ there is a sum over the complex zeros, into which Hoheisel introduced the statistical study, along with a certain zero-free region on the left of the vertical line $\operatorname{Re} s = 1$. What should not be missed to observe in Bohr–Landau's theory, especially in the context of our present discussion, is the rôle of a version of the mean values of $\zeta(s)$. Taking later developments into account, this concerns the analysis of

$$(2.37) \qquad \int_{-T}^{T} \left| \zeta\left(\tfrac{1}{2} + it\right) \right|^2 \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt,$$

with $N, T \geq 1$ and complex a_n which are to be chosen appropriately [32].

Least Prime Number Theorem: If one wants to establish an analogue of Hoheisel's assertion for primes in arithmetic progressions, then the study of $\pi(x; k, l)$ $((k, l) = 1)$ should be developed on the supposition $x^\vartheta < x/k$ with a new positive absolute constant ϑ less than 1. This is to look for a way to avoid the Extended Riemann Hypothesis. Obviously an extension of (2.37) to Dirichlet L-functions $L(s, \chi)$ $(\chi \,(\mathrm{mod}\, k))$ is required; but this part of the theory does not cause any essential difficulty, for it suffices to exploit the orthogonality of the characters. However, the theory of the distribution of zeros of Dirichlet L-functions lacks what corresponds to the zero-free region of $\zeta(s)$ mentioned above. One has to find a way to negate this defect, which certainly requires to develop the statistical study of the zeros in a far refined fashion than the followers of Bohr and Landau did. It was Linnik [42, I] who overcame this genuine difficulty [33]. There a definitive rôle was played by the Brun–Titchmarsh theorem (1.30) [34]. Further, the possibility of exceptional zeros caused another difficulty, or a quantitative study of the Deuring–Heilbronn theory had to be developed. That was achieved by Linnik in [42, II] [35]. In this way the Least Prime Number Theorem was established; that is, there exists an absolute constant $c > 0$ such that the least prime in every reduced residue class mod k is less than k^c.

Mean Prime Number Theorem: Thus Linnik found a way to avoid the Extended Riemann Hypothesis. It concerned, however, a single modulus, though the uniformity on it was of course maintained. Thus the next target was to find a way to avoid the Extended Riemann Hypothesis simultaneously for all moduli in an arbitrary finite range. This time, a genuine difficulty took place in extending (2.37), which is the principal difference from Linnik's situation. It was Rényi [65] who resolved this difficulty. He started with Linnik's fundamental work [41], and developed a version of the multiplicative Large Sieve to extend (2.37) to a double sum over moduli and characters, analogously involving Dirichlet L-functions and polynomials twisted by characters.

Then, he could establish (1.26) for $Q = x^\alpha$ with an absolute constant $\alpha > 0$, without any hypothesis; this is Rényi's Mean Prime Number Theorem[36]. Hence, as can be seen from (1.25)–(1.27), Rényi superseded Brun and made a great step toward the Twin Prime and the Goldbach Conjectures.

Naturally, efforts afterward were concentrated on the improvement[37] of Rényi's theorem; that is, to find larger Q. Finally, after 17 years of a series of struggles, it was established that

$$(2.38) \qquad \text{The inequality (1.26) holds with } Q = \frac{\sqrt{x}}{(\log x)^B},$$

where B is a function of A. This is called E. Bombieri–A.I. Vinogradov's Mean Prime Number Theorem[38]. Bombieri's argument [4] stands on the tradition of the Large Sieve; and Vinogradov [77] relied on the Dispersion Method [43], another fundamental invention of Linnik[39]. Despite the difference in their methods, what they achieved is essentially equivalent to each other and to the consequence of the Extended Riemann Hypothesis, especially in the context of its applications to sieve problems as exhibited above. In Bombieri's argument[40], (2.38) could be said to be a consequence of the inequality

$$(2.39) \qquad \sum_{q<Q} \frac{q}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)}^{*} \left| \sum_{M \leq n < M+N} a_n \chi(n) \right|^2 \leq (N - 1 + Q^2) \sum_{M \leq n < M+N} |a_n|^2,$$

where the asterisk means that the sum is restricted to primitive characters. Connecting multiplicative characters with additive characters via Gaussian sums, (2.39) follows immediately from (2.30).

2.9 In this section we shall look into the relation between the Large Sieve and Selberg's Sieve, in a perspective different from the above; we shall show that the multiplicative Large Sieve can be amalgamated with Selberg's Sieve. As the discussion of the previous section suggests, such an extension of the multiplicative Large Sieve has consequences in the theory of the distribution of primes in arithmetic progressions[41]. This aspect should not be unexpected, especially if it is taken into account that an origin of Selberg's Sieve can be traced back to (2.37). In fact, an initial version of Selberg's procedure developed in (2.13)–(2.19) could be found in his argument to compute the minimum value of the expression (2.37) under the side condition $a_1 = 1$[42]; that is, the extremal values of a_n are found in much the same way as those of $\lambda(d)$.

Returning to (2.19), the optimal λ is written as

$$(2.40) \qquad \sum_{n \in \Omega(d)} \lambda(d) = \frac{1}{G(z,\Omega)} \sum_{q<z} \mu(q)^2 H(q,\Omega) \Psi_q(n,\Omega), \quad \Psi_q(n,\Omega) = \prod_{\substack{n \in \Omega(p) \\ p|q}} \left(\frac{-1}{H(p,\Omega)} \right).$$

We compare this with (2.30), and ponder upon the norm of the linear operator $(\psi_q(n,\Omega))$, with

$$(2.41) \qquad \psi_q(n,\Omega) = \mu(q)\sqrt{H(q,\Omega)}\,\Psi_q(n,\Omega).$$

In this way, we find that for any complex vectors $\{a_n\}$, $\{b_q\}$ [43]

$$(2.42) \qquad \sum_{q<z} \left| \sum_{M \leq n < M+N} a_n \psi_q(n, \Omega) \right|^2 \leq (N - 1 + z^2) \sum_{M \leq n < M+N} |a_n|^2,$$

$$(2.43) \qquad \sum_{M \leq n < M+N} \left| \sum_{q<z} b_q \psi_q(n, \Omega) \right|^2 \leq (N - 1 + z^2) \sum_{q<z} |b_q|^2.$$

Setting $a_n = \varpi(n)$ in (2.42) we get (2.10) again; and (2.43) implies (2.10) as well, for it contains (2.28). More generally, the operator $\left(\chi(n)(k/\varphi(k))^{1/2} \psi_q(n, \Omega) \right)$ could be viewed in the same way, where χ is primitive mod k, and $(k, q) = 1$, $kq < z$. That is, Selberg's Sieve and the multiplicative Large Sieve could be hybridized, which yields interesting refinements of (2.10).

This does not exhaust the flexibility hidden in Selberg's Sieve. An aspect in which Selberg's Sieve supersedes Linnik's is in that the class of sequences to which the former is applicable is definitely wider than that with the latter. For instance, with a given arithmetic function f one may consider the quadratic form [44]

$$(2.44) \qquad \sum_{n=1}^{\infty} f(n) \left(\sum_{\substack{d|n \\ d<z}} \lambda(d) \right)^2$$

on the side condition $\lambda(1) = 1$. The optimal λ thus obtained yields an analogue of the above $\psi_q(n, \Omega)$. It can be used to extend the multiplicative Large Sieve, which in turn has an important application; that is, a highly simplified proof of the Least Prime Number Theorem. In the previous section we stressed the rôle played by the Brun–Titchmarsh theorem in Linnik's proof of his Least Prime Number Theorem. There the Sieve Method was somewhat hidden. In the new proof, the Sieve Method emerges as the protagonist, and leads the whole story [45].

Chapter 3. Rosser's Sieve

3.1 In the present chapter we shall return to the circle of Brun's ideas. Being combinatorial in its nature, Brun's Sieve demands efforts to comprehend. On the other hand, Selberg's Sieve is simple and powerful; also Linnik's Sieve gave rise to the principle of the Large Sieve, which brought a tremendous impact to the development of the theory of the distribution of primes. Perhaps because of this, it took considerably long time for Brun's theory to be appreciated and shared by many. In fact, it was 30 years later since his work [12] when Rosser (ca. 1950) opened a way leading to the complete settlement of the linear sieve [46]. Namely, he discovered a choice of sieve weights on the general condition introduced in Section 1.7 (with $\kappa = 1$), which gives best possible main terms in both the upper and lower bounds. Moreover, the construction of his sieve weights is relatively simple. In what follows we shall describe the salient points of Rosser's Sieve, especially his Linear Sieve. We shall employ symbols and definitions introduced in Chapter 1, without mention. We stress that we shall start with

$\mathcal{S}(\mathcal{A}, z_0)$ $(2 \leq z_0 < z)$ instead of $\mathcal{A}$. The reason why we first sift $\mathcal{A}$ with primes less than a certain z_0 will become apparent in the course of discussion.

3.2 To get a lower bound of the size of a subset in a finite set, it suffices to have an upper bound of its complementary subset. To wit, lower bounds could result from upper bounds. This trivial principle was first exploited effectively by A.A. Buchstab [13], in the context of the Sieve Method. More explicitly, his idea relies on the following identity: Classifying the elements of $\mathcal{S}(\mathcal{A}, z_0) \setminus \mathcal{S}(\mathcal{A}, z)$ according to their least prime factors, we get

$$(3.1) \qquad |\mathcal{S}(\mathcal{A}, z)| = |\mathcal{S}(\mathcal{A}, z_0)| - \sum_{z_0 \leq p < z} |\mathcal{S}(\mathcal{A}_p, p)|.$$

This logical identity is named after Buchstab. If we put $z_0 = 2$ and iterate the identity infinitely, we get (1.2). It is, however, useless in general, and thus Brun introduced a system of restricting the participating divisors of $P(z)$.

Any restriction of the divisors is the same as to attach the wight 0 or 1 to each divisor. With this observation in mind, we reconsider (1.2). Thus, let η be an arbitrary function with $\eta(1) = 1$, and rewrite (3.1) as

$$(3.2) \qquad |\mathcal{S}(\mathcal{A}, z)| = |\mathcal{S}(\mathcal{A}, z_0)| - \sum_{z_0 \leq p < z} \eta(p)|\mathcal{S}(\mathcal{A}_p, p)| - \sum_{z_0 \leq p < z} (1 - \eta(p))|\mathcal{S}(\mathcal{A}_p, p)|$$

This is the case with $\ell = 1$ of the identity

$$(3.3) \qquad \begin{aligned} |\mathcal{S}(\mathcal{A}, z)| = &\sum_{\substack{d|P(z_0,z) \\ \nu(d)<\ell}} \mu(d)\rho(d)|\mathcal{S}(\mathcal{A}_d, z_0)| + (-1)^\ell \sum_{\substack{d|P(z_0,z) \\ \nu(d)=\ell}} \rho(d)|\mathcal{S}(\mathcal{A}_d, p(d))| \\ &+ \sum_{\substack{d|P(z_0,z) \\ \nu(d)\leq\ell}} \mu(d)\sigma(d)|\mathcal{S}(\mathcal{A}_d, p(d))|. \end{aligned}$$

Here $P(z_0, z) = P(z)/P(z_0)$, $\rho(1) = 1$, $\sigma(1) = 0$, and for $d = p_1 p_2 \cdots p_l$ $(p_1 > p_2 > \cdots > p_l)$

$$(3.4) \qquad \rho(d) = \eta(p_1)\eta(p_1 p_2)\cdots\eta(p_1 p_2 \cdots p_l), \quad \sigma(d) = \rho(d/p(d)) - \rho(d) \quad (p(d) = p_l).$$

To prove (3.3), we apply to (3.2) the replacements $\mathcal{A} \mapsto \mathcal{A}_d$, $z \mapsto p(d)$, $\eta(p) \mapsto \eta(dp)$, and insert the result into (3.3); then we get $\ell \mapsto \ell + 1$. Hence, setting $\ell > \pi(z)$ in (3.3), we obtain

$$(3.5) \qquad |\mathcal{S}(\mathcal{A}, z)| = \sum_{d|P(z_0,z)} \mu(d)\rho(d)|\mathcal{S}(\mathcal{A}_d, z_0)| + \sum_{d|P(z_0,z)} \mu(d)\sigma(d)|\mathcal{S}(\mathcal{A}_d, p(d))|,$$

which is an extension or rather a refinement of (1.2).

3.3 For the sake of simplicity, we impose the restriction $0 \leq \eta(d) \leq 1$ for any $d|P(z)$; thus, $0 \leq \rho(d) \leq 1$, $0 \leq \sigma(d) \leq 1$. With this, we shall try to derive from (3.5) as sharp as possible upper and lower bounds of $|\mathcal{S}(\mathcal{A}, z)|$. First, we observe trivially

$$(3.6) \qquad (-1)^r \left\{ |\mathcal{S}(\mathcal{A}, z)| - \sum_{d|P(z_0,z)} \mu(d)\rho(d)|\mathcal{S}(\mathcal{A}_d, z_0)| \right\} \leq \sum_{\substack{d|P(z_0,z) \\ \nu(d)\equiv r \,(\mathrm{mod}\ 2)}} \sigma(d)|\mathcal{S}(\mathcal{A}_d, p(d))|.$$

There is a way to have the equality here; that is, $\sigma(d) = 0$ $(\nu(d) \equiv r + 1 \,(\mathrm{mod}\,2))$. Thus we set

$$(3.7) \qquad\qquad \eta(d) = 1 \quad (\nu(d) \equiv r + 1 \,(\mathrm{mod}\,2));$$

and such an η is denoted as η_r, and correspondingly we define ρ_r and σ_r. Then, we have

$$(3.8) \qquad |\mathcal{S}(\mathcal{A}, z)| = \sum_{d|P(z_0,z)} \mu(d)\rho_r(d)|\mathcal{S}(\mathcal{A}_d, z_0)| + (-1)^r \sum_{d|P(z_0,z)} \sigma_r(d)|\mathcal{S}(\mathcal{A}_d, p(d))|.$$

Discarding the second sum, we get

$$(3.9) \qquad (-1)^r \left\{ |\mathcal{S}(\mathcal{A}, z)| - \sum_{d|P(z_0,z)} \mu(d)\rho_r(d)|\mathcal{S}(\mathcal{A}_d, z_0)| \right\} \geq 0.$$

Also we have, corresponding to (3.8),

$$(3.10) \qquad V(z, \omega) = V(z_0, \omega)V_0(z, \omega; \rho_r) + (-1)^r \sum_{d|P(z_0,z)} \sigma_r(d)\frac{\omega(d)}{d}V(p(d), \omega),$$

where

$$(3.11) \qquad V_0(z, \omega; \rho_r) = \sum_{d|P(z_0,z)} \mu(d)\rho_r(d)\frac{\omega(d)}{d}.$$

In fact, expanding out the product $V(z, \omega)/V(z_0, \omega)$ and classifying the resulting terms in Buchstab's fashion, we obtain an identity analogous to (3.1). The rest of discussion is the same as above. In passing, we note that $V_0(z, \omega; \rho_r) = V(z, \omega; \rho_r)$, provided $z_0 = 2$.

3.4 In deriving (3.9) from (3.8) we brought in a certain inaccuracy, which should certainly be evaded as much as possible [47]. For this sake, we note the trivial but crucial fact that $|\mathcal{S}(\mathcal{A}, z)|$ is a non-increasing function of z. Thus the negligence of $\mathcal{S}(\mathcal{A}_d, p(d))$ with $p(d)$ which is small for $\mathcal{A}_d$ causes most likely a relatively large loss. To avoid this we should better set $\sigma_r(d) = 0$ for such d. One of the most fruitful device to make explicit the smallness of $p(d)$ for $\mathcal{A}_d$ is to introduce two parameters $\beta > 1$ and $D > 0$, and to define $p(d)$ to be small for $\mathcal{A}_d$ if $p(d) < (D/d)^{1/\beta}$. Behind this criterion is the concept of the Sieving Limit, but at this moment there is no particular necessity to know the details [48].

Hence, in addition to (3.7), we impose

$$(3.12) \qquad\qquad \eta_r(d) = \begin{cases} 1 & p(d)^\beta d < D, \\ 0 & p(d)^\beta d \geq D, \end{cases} \quad (\nu(d) \equiv r \,(\mathrm{mod}\,2)).$$

Then, ρ_r and σ_r are, respectively, the characteristic functions of the sets [49]

$$(3.13) \qquad \mathcal{D}(\rho_r) = \{1\} \cup \left\{ d : p_1 p_2 \cdots p_{2k+r-1} p_{2k+r}^{\beta+1} < D, \; 1 \leq 2k + r \leq l \right\},$$

$$(3.14) \qquad \mathcal{D}(\sigma_r) = \left\{ d : \rho_r(d/p(d)) = 1, \; p_1 p_2 \cdots p_{l-1} p_l^{\beta+1} \geq D, \; l \equiv r \,(\mathrm{mod}\,2) \right\},$$

with $r = 0$, 1, and d being the same as in (3.4).

With the sieve weight $\rho_r(d)$ thus constructed, the formula (1.18) is called Rosser's Sieve. In fact, the validity of (1.18) is immediate in view of (3.9) with $z_0 = 2$. Moreover, because of the supposition $\beta > 1$ the level condition (1.21) is fulfilled with the present D. It should be noted that as D and β are taken larger and smaller, respectively, the set $\mathcal{D}(\sigma_r)$ becomes narrower; that is, the loss caused at the step (3.9) should decrease.

3.5 As an application of Rosser's Sieve, we shall prove Brun's theorem (1.22) briefly [50]. In this section, we work with $z_0 = 2$. We note first that if $z^2 \leq D$, then we have

$$(3.15) \qquad \frac{1}{2}\left(\frac{\beta-1}{\beta+1}\right)^{\nu(d)/2} \log D < \log(D/d) \quad (d \in \mathcal{D}(\rho_r)).$$

In fact, if $r = 1$, $\nu(d) = 2\ell$, $\rho_1(d) = 1$, then we have $p_{2j+2} < p_{2j+1} < (D/(p_1 p_2 \cdots p_{2j}))^{1/(\beta+1)}$ $(0 \leq j \leq \ell-1)$. Thus, $((\beta-1)/(\beta+1))\log(D/(p_1 p_2 \cdots p_{2j})) < \log(D/(p_1 p_2 \cdots p_{2j+2}))$, which gives (3.15). Other cases are analogous. Next, in the second sum of (3.10), the terms are classified according to the values of $\nu(d)$, and taking (1.17) into account we see that the sum is

$$(3.16) \qquad \ll V(z,\omega) \sum_{l=\ell}^{\infty} \frac{1}{l!}\left(\frac{\log z}{\log q}\right)^{\kappa}\left(\sum_{q \leq p < z} \frac{\omega(p)}{p}\right)^l,$$

where $q = \min_{\nu(d)=l} p(d)$, $\ell = \min \nu(d)$ with $d|P(z)$ and $\sigma_r(d) = 1$. By the definition (3.4), $\rho_r(d) = 0$, and thus $p(d)^\beta d \geq D$, which gives $\ell \geq \tau - \beta$ because $z^\tau = D$. On the other hand, we have $\rho_r(d/p(d)) = 1$, and by (3.15)

$$(3.17) \qquad \frac{1}{2}\left(\frac{\beta-1}{\beta+1}\right)^{(\nu(d)-1)/2} \log D < \log(D/d) + \log p(d) \leq (\beta+1)\log p(d),$$

which gives a lower bound for q. Inserting these assertions on ℓ and q into (3.16), and setting $\beta = \tau/3$, we reach (1.22) after some elementary estimation.

3.6 As a matter of fact, it is known that if $\kappa > 1$, then the upper bound via Rosser's Sieve is inferior to that via Selberg's Sieve [51]. Nevertheless, if $\kappa = 1$, then Rosser's Sieve yields optimal upper and lower bounds as has been stressed above. Since the linear sieve problems include great conjectures, Rosser's construction of his Linear Sieve is extremely important [52].

We first show his assertion: With $\beta = 2$, we fix Rosser's sieve weights $\rho_r(d)$; and let the functions $\phi_r(\tau)$, with $\phi_{r_1} \equiv \phi_{r_2}$ $(r_1 \equiv r_2 \,(\mathrm{mod}\,2))$, satisfy the difference-differential equation [53]

$$(3.18) \qquad \frac{d}{d\tau}(\tau\phi_r(\tau)) = \phi_{r+1}(\tau-1) \quad (\tau \geq 2),$$

$$(3.19) \qquad \tau\phi_1(\tau) = 2e^{c_E}, \quad \phi_0(\tau) = 0 \quad (0 < \tau \leq 2).$$

Then we have, for $z = D^{1/\tau}$, $z_0 = \exp((\log D)/(\log\log D)^2)$,

$$(3.20) \qquad V(z_0,\omega)V_0(z,\omega;\rho_r) = (1+o(1))\phi_r(\tau)V(z,\omega) \quad (\kappa = 1).$$

With this, we apply Brun's Sieve (1.22) to the term $|\mathcal{S}(\mathcal{A}_d, z_0)|$ appearing in (3.9), and find that

$$(3.21) \qquad (-1)^r \left\{ |\mathcal{S}(\mathcal{A}, z)| - (1 + o(1))\phi_r(\tau)V(z, \omega)X \right\} \geq (-1)^r \sum_{\substack{d < D \\ d|P(z)}} \mu(d)\delta_r(d)R_d,$$

in which $D = z^\tau$, and δ_r is a certain characteristic function [54]. This is called Rosser's Linear Sieve.

The main steps of the proof of (3.20) are as follows: With $\beta > 1$, which is to be fixed later, we construct Rosser's sieve weights $\rho_r(d)$, and put

$$(3.22) \qquad V(z, \omega)K(z, \rho_r) = \max \left\{ 0, V(z_0, \omega)V_0(z, \omega; \rho_r) \right\}.$$

Then we assume that there exist continuous functions k_r such that $K(z, \rho_r) = (1+o(1))k_r(\tau)$; we have $0 \leq k_0(\tau) \leq 1 \leq k_1(\tau)$ by (3.10). Also, we assume [55] that $\beta = \inf\{\tau : k_0(\tau) > 0\}$. On noting the definitions (3.11) and (3.13), we have

$$(3.23) \qquad V_0(z, \omega; \rho_1) = V_0(z_0, \omega; \rho_1) - \sum_{z_0 \leq p < z} \frac{\omega(p)}{p} V_0(p, \omega; \rho_0^*).$$

Here the level of the Rosser sieve weight ρ_0^* is equal to D/p; that is, $p_1 = p$ in (3.13). If $\rho_1(p) = 1$, then $p < D^{1/(\beta+1)}$. Thus, if $z^{\beta+1} \geq D$, then $V_0(z, \omega; \rho_1) = V_0(D^{1/(\beta+1)}, \omega; \rho_1)$. In view of (1.17) ($\kappa = 1$), we have $\tau k_1(\tau) = (\beta + 1)k_1(\beta + 1)$ ($\tau \leq \beta + 1$). If $\tau > \beta + 1$, then $\log(D/p)/\log p > \beta$; thus, by our assumption on β, we may write $V(z_0, \omega)V_0(p, \omega; \rho_0^*) = V(p, \omega)K(p, \rho_0^*)$. Hence, we find that

$$(3.24) \qquad V(z, \omega)k_1(\tau) = V(z_0, \omega)k_1(\tau_0) - (1 + o(1)) \sum_{z_0 \leq p < z} \frac{\omega(p)}{p} V(p, \omega)k_0(\xi_p),$$

with $z_0^{\tau_0} = D$ and $\xi_p = (\log D)/(\log p) - 1$. We apply (1.17) to (3.24), and express the result in terms of a Stieltjes integral. We are led to the integral equation

$$(3.25) \qquad \tau_0 k_1(\tau_0) - \tau k_1(\tau) = \int_\tau^{\tau_0} k_0(\xi - 1)d\xi.$$

This ends the discussion on the case $r = 1$. The other case could be treated analogously. The equation that k_0 should satisfy is (3.18) if $\tau \geq \beta$; and if $\tau < \beta$, then it is (3.19) but with the constant $2e^{c_E}$ being replaced by $(\beta + 1)k_1(\beta + 1)$. From this, it follows readily that

$$(3.26) \qquad k_r(\tau) = 1 + (-1)^r \frac{(\beta - 2)}{\tau^2}(\beta + 1)\phi_1(\beta + 1)\left(1 + O\left(\frac{1}{\tau}\right)\right) + O\left(\frac{1}{\Gamma(\tau)}\right).$$

In view of Brun's theorem (1.22), we find the optimal value of β; that is, we should set $\beta = 2$. Then, $3k_1(3) = 2e^{c_E}$ follows. In this way we reach (3.18)–(3.19).

Having this, we set $k_r = \phi_r$, and go from (3.25) back to (3.24); then we see that (3.24) holds in fact with $k_1 = \phi_r$ and $k_0 = \phi_{r+1}$. By the definition (3.19) one may multiply each

summands on the right by the factor $\rho_r(p)$. The identity thus obtained can be iterated in much the same way as in Section 3.2. We get

$$(3.27) \qquad V(z,\omega)\phi_r(\tau) = (1+o(1))V(z_0,\omega) \sum_{d|P(z_0,z)} \mu(d)\rho_r(d)\frac{\omega(d)}{d}\phi_{r+\nu(d)}\left(\frac{\log(D/d)}{\log z_0}\right).$$

Comparing this with the definition (3.11) and on noting $\phi_r(\tau) = 1 + O(1/\Gamma(\tau))$ ((3.26), $\beta = 2$), our problem is reduced to the estimation of the expression

$$(3.28) \qquad \sum_{d|P(z_0,z)} \rho_r(d)\frac{\omega(d)}{d}\exp\left(-\frac{\log(D/d)}{\log z_0}\right)$$

We skip the details, but this can be seen to be negligible, which ends the proof of (3.20).

3.7 The extremal situation [56] that implies that Rosser's Linear Sieve is optimal is given by

$$(3.29) \qquad \mathcal{B}^{(r)} = \{n < x : \text{[the total number of prime divisors of } n] \equiv r \bmod 2\}.$$

We have $X = x/2$ and $\omega \equiv 1$. Rosser's Sieve ($\beta = 2$, $D = x$) gives

$$(3.30) \qquad |\mathcal{S}(\mathcal{B}^{(r)}, z)| = \sum_{d|P(z)} \mu(d)\rho_r(d)|\mathcal{B}_d^{(r)}|.$$

Hence, no loss is caused at (3.9) with the present specialization. The argument of the previous section could be repeated, and we get

$$(3.31) \qquad |\mathcal{S}(\mathcal{B}^{(r)}, z)| = (1+o(1))\frac{x}{2}\phi_r\left(\frac{\log x}{\log z}\right) \cdot V(z,1).$$

Namely, the upper and lower bound implied by Rosser's Linear Sieve are in fact attainable.

3.8 As an application of the above, we exhibit J.-R. Chen's theorem [15] [57]: For any sufficiently large even integer N, we have [58]

$$(3.32) \qquad |\{p : N = p + P_2\}| > C_0 \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p>2}}\left(\frac{p-1}{p-2}\right)\frac{N}{(\log N)^2},$$

with an absolute constant C_0. Here P_2 denotes an integer which has two prime factors at most. With no doubt, this famous assertion is at the pinnacle of the entire modern theory of the Sieve Method.

Chen's plan of the proof is relatively simple. We first pick up any integer $n < N$ such that $(n, P(N^{1/10})) = 1$, and consider the value of the expression [59]

$$(3.33) \qquad W(n) = 1 - \frac{1}{2}\sum_{\substack{p_1|n \\ N^{1/10}\leq p_1 < N^{1/3}}} 1 - \frac{1}{2}\sum_{\substack{p_1|n \\ N^{1/10}\leq p_1 < N^{1/3}}} \sum_{\substack{n=p_1p_2p_3 \\ N^{1/3}\leq p_2 < (N/p_1)^{1/2}}} 1.$$

We find readily that if $W(n) > 0$ or $W(n) \geq \frac{1}{2}$, then n is a P_2. Thus, with $\mathcal{A} = \{N - p : p < N\}$, we have that

(3.34)
$$
|\{p : N = p + P_2\}| \geq |\mathcal{S}(\mathcal{A}, N^{1/10})| - \frac{1}{2} \sum_{N^{1/10} \leq p_1 < N^{1/3}} |\mathcal{S}(\mathcal{A}_{p_1}, N^{1/10})|
$$
$$
- \frac{1}{2} \left| \left\{ p < N : N = p + p_1 p_2 p_3, N^{1/10} \leq p_1 < N^{1/3} \leq p_2 < (N/p_1)^{1/2} \right\} \right|.
$$

We apply (3.21) ($r = 0$) to the first term on the right, and (3.21) ($r = 1$) to the second. On the other hand, we replace the third term by $\frac{1}{2}|\mathcal{S}(\mathcal{A}^*, N^{1/2}(\log N)^{-A})|$ with A being sufficiently large, and apply (3.21) ($r = 1$) again. Here $\mathcal{A}^* = \{N - p_1 p_2 p_3 : p_1 p_2 p_3 < N\}$ with p_1, p_2 as above. To bound the remainder terms that arise from the first two applications of Rosser's Linear Sieve, we employ Bombieri–Vinogradov's Mean Prime Number Theorem (2.38). To deal with the remainder term caused by the third application, we employ an extension [60] of (2.38) to the sequence $\{p_1 p_2 p_3 < N\}$ with p_1, p_2 as before. What remains is to compute the main terms, which is, however, rudimentary.

Chapter 4. The Remainder Term

4.1 From what we have described so far, one may infer the reach of the modern Sieve Method. To continue the story, we should now leave the discussion of the main terms for the estimation of the remainder terms [61]. There must converge the true essences of Analytic Number Theory, as the proof of Chen's theorem (3.32) illustrates dramatically. We have, however, too many relevant fields, subjects, and technicalities to mention. Thus we would rather single out the idea that is probably the most fundamental in the theory of the remainder terms, especially of the Linear Sieve. A culmination in this context is due to H. Iwaniec [34], and to describe it we need to tell a brief history.

4.2 The development started with the discovery that Selberg's sieve weights could intervene in the control of the remainder term, in a highly non-trivial way; the serendipity occurred to us [49]. This is in fact a surprising fact, because those weights had been constructed solely with the aim to attain the best possible main term while the remainder term had been utterly disregarded. That is, in the structure of the sieve weights thus defined is hidden a mechanism that could induce massive cancellations among the summands in the second sum of (1.20), $r = 1$. A little later the same occurred to Chen [16] with Rosser's Linear Sieve.

4.3 Let us dwell a little on their findings. That is about the level of sieve weights, and we have to return to the concept itself. At the bottom is the too natural prerequisite that any main term be superior in magnitude to the corresponding remainder term. However, in the Sieve Method this triviality had never been achieved in any effective manner until Brun introduced the cutoff argument into Eratosthenes' Sieve, and brought about a revolutionary change. From there stemmed the concept of the level of sieve weights, as introduced at (1.21), though it left for long its trail only in the primitive

(4.1)
$$
|R(\mathcal{A}, z; \rho_r)| \leq \sum_{\substack{d < D \\ d | P(z)}} |\rho_r(d)| |R_d|.
$$

Having the sieve weights that yield optimal main terms, the focus of attention is naturally on the very basic issue: to get larger levels. That is essentially the unique way to make the inequality (1.19) sharper as the reasoning at the end of Section 3.4 suggests, despite it is meant only for Rosser's Sieve. Namely, to go beyond (4.1) the inner structure of the sequence $\{\mu(d)\rho_r(d)R_d\}$ has to be exploited so that the cancellation among the members be detected; and the size of D, the level of ρ_r, could be taken larger than a priori.

This was done for the first time in [49]. Thus, let $\Omega(p) = \{0\}$, say, and write $\mu(d)\rho_1(d) = \sum_{[d_1,d_2]=d} \lambda(d_1)\lambda(d_2)$ with $\lambda(d) = 0$ for $d \geq z$. We assume that λ is chosen optimally so that $|\lambda(d)| \leq 1$ as (2.20) shows. Then Selberg's Sieve or (2.11) implies that (4.1) $(r = 1)$ could be replaced by the expression

$$(4.2) \qquad |R(\mathcal{A}, z; \rho_1)| \leq \sup_{a,\, b} \left| \sum_{m<z} \sum_{n<z} a_m b_n R_{[m,n]} \right|,$$

where $a = \{a_m\}$, $b = \{b_n\}$ are arbitrary vectors such that $|a_m|, |b_n| \leq 1$. This is yet trivial; but its implication is striking. For instance, applying (4.2) to the sequence $\mathcal{A} = \{n < x : n \equiv l \,(\mathrm{mod}\, k)\}$ $((k,l) = 1)$, we obtain an improvement upon (2.36):

$$(4.3) \qquad \pi(x; k, l) \leq 2(1 + o(1)) \frac{x}{\varphi(k) \log(x/\sqrt{k})} \qquad (k \leq x^{6/17}).$$

That is, (4.2) allows us to utilize the level $D = (x/\sqrt{k})(\log x)^{-2}$ in place of the trivial $D = (x/k)(\log x)^{-2}$ which is involved in (2.33). The cause of this is to have had a bilinear form in (4.2); that is, to have read the structure of those sieve weights as such.

On the other hand Chen [16] exploited the Buchstab identity, in the case of the Linear Sieve. In (3.1) we put $L^J = z/z_0$ with an integer J, divide the sum over the primes according to the covering $[z_0, z) = \cup_{j \leq J} I$, $I = [z_0 L^{j-1}, z_0 L^j)$, and apply Rosser's Sieve to each $S(\mathcal{A}_p, p)$, $p \in I$, with the level $D/(z_0 L^j)$. Provided J is chosen appropriately, the pair of the main terms remains the same asymptotically, because of (3.18)–(3.19); but the remainder term is bounded by

$$(4.4) \qquad \sup_{K<z} \sup_{a,\, b} \left| \sum_{K \leq p < KL} \sum_{n < D/K} \mu(pn) a_p b_n R_{pn} \right|,$$

which is to be compared with (4.2). With this bilinear form, Chen could detect the cancellation inside the remainder term. The effect is well exhibited in his own application that yielded the assertion [62] $P_2 \in [x - \sqrt{x}, x)$ for any sufficiently large x. In view of the relation between the Riemann Hypothesis and the existence of primes in short intervals, this is indeed remarkable.

4.4 Now, after the two precursors [63], Iwaniec [34] made a true incision into the subject. Superseding (4.2) and (4.4), he bounded the remainder term in (3.21) by the expression

$$(4.5) \qquad (\log z) \cdot \sup_{a,\, b} \left| \sum_{m<M} \sum_{n<N} \mu(mn) a_m b_n R_{mn} \right|,$$

where M, N are arbitrary except for $MN = D$. This is called Iwaniec's bilinear form for the remainder term in the Linear Sieve. The basis of his idea is fairy simple; that simplicity is shared by the above two ideas similarly. Thus, in the process to reach Rosser's sieve weights (3.13), $\beta = 2$, those primes participating the sieve are classified as Chen did. The function η is first defined over the family of all set theoretic products of intervals I; and it is redefined as a function over integers, in an obvious manner according to their prime decompositions. With this, we proceed in much the same way as we did in Sections 3.2–3.4 and 3.6. Then a smoothed version of (3.21) emerges. What remains is solely Iwaniec's penetrating observation on the remainder term thus obtained[64].

That the parameters M, N are independent is a real merit in Iwaniec's Linear Sieve, because of which (4.5) has given rise to many remarkable consequences. One of the best applications is done by Iwaniec and M. Jutila [36], a landmark among the works on the existence of primes in short intervals[65].

Conclusion

What (4.5) for instance suggests is the importance of the circle of methods, which are represented by Linnik's Dispersion Method. The origin could be found in the Weyl–van der Corput method dealing with trigonometrical sums, which is a device closely related to subconvexity bounds of the Riemann zeta and analogous functions, though a far cry from the Lindelöf Hypothesis.

In those methods, especially in Linnik's, often Kloosterman sums play a fundamental rôle. This was the very reason why Linnik [44] envisaged the cancellation among the sums, which is perhaps hard to detect if one sticks to algebraic means only. Selberg [71] opened a way, and V.N. Kuznetsov [40] made a remarkable contribution to realize a part of Linnik's dream. Together with an independent research by R.W. Bruggeman [8], the work [40] brought a new era in Analytic Number Theory. That was began by Iwaniec [32], when he combined their works with the additive Large Sieve and created the spectral Large Sieve. On that basis, Bombieri, J.B. Friedlander and Iwaniec [7] achieved a genuine improvement upon (2.38).

On the other hand, the appearance of Kloosterman sums in the discussion of non-diagonal parts arising in the applications of the Dispersion Method or alike must be a reflection of the fact that we are actually working on a certain group structure[66], that is, we are looking at the remainder term in sieves via automorphic and harmonic mechanisms on GL(2). Bearing Brun's torch we have come to a far country.

Addendum (May 14, 2005)

Very recently there was a fantastic development in the study of gaps between primes: In their unpublished preprint '*Small gaps between primes. II (preliminary)*' (February 8, 2005), D.A. Goldston, J. Pintz, and C.Y. Yıldırım established, among other things,

$$(A.1) \qquad \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0,$$

and that if (1.26) holds for $Q = x^\theta$ with a $\theta > \frac{1}{2}$ then there exists an absolute constant $c(\theta)$ such that

$$(A.2) \qquad \liminf_{n\to\infty}(p_{n+1} - p_n) \le c(\theta),$$

where p_n is the nth prime. For a short, essentially self-contained proof of these facts, see the preprint arXiv:math. NT/0505300 '*Small gaps between primes exist*' by Goldston, Motohashi, Pintz, and Yıldırım. The argument is in the framework of Selberg's Sieve and Linnik's Large Sieve; the latter is in the sense that (2.38) plays a fundamental rôle. One might surmise that a proof of Twin Prime Conjecture is not beyond the reach of today's Analytic Number Theory.

Addendum (September 20, 2006)

Because of the improvement [7] of the Mean Prime Number Theorem, the last hypothetical assertion on *bounded differences between consecutive primes* might appear within the reach of the present technology. Until recently there were, however, two main obstacles that prevented us to make any real incision into the matter: One is the fact that Selberg's Sieve or rather his sifting procedure, which is highly essential in the argument of Goldston, Pintz, and Yıldırım, did not seem to admit any error terms with the flexibility that (4.5) enjoys, although we had already a partial result [62] that corresponds to the situation $M = N$ in (4.5). Another obstacle is that in [7] only the distribution of primes in arithmetic progressions with a *fixed* residue class, i.e., $\pi(x; q, a)$ with a fixed, is considered, and to achieve (A.2) we need to relax this restriction to a considerable extent.

With this, the present situation is that although the latter difficulty still persists, Motohashi and Pintz have succeeded in suppressing the former in their preprint arXiv:math. NT/0602599 '*A smoothed GPY sieve*' by extending the argument of [62]. We note, by the way, that arXiv:math. NT/0505300 quoted above has been published in Proc. Japan Acad. 82A (2006), 61–65, which can be downloaded freely via the web-page of the academy.

Acknowledgements

Notes

1) My impression that has arisen reading occasionally the history of the ancient Meso-potamia. As to the remote origin of the concept of prime numbers and decomposition of integers into prime factors, there exist a collection of highly striking evidences from Sumer and Akkad; for instance, see the recent discovery [63] by K. Muroi. Ancient Babylonian mathematicians tried to make exercises for their students harder or more enjoyable by embedding not only the familiar $2, 3, 5$ but also

$$7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 79, 83, 137, 139, 1481$$

into linear and quadratic equations as hidden prime factors of the coefficients. They were playing with prime numbers, more than 4000 years ago. To me those primes appear exactly like glittering gems inlaid into the famous treasures from Ur, Nimrud, and

the grave of King Tutankhamen. Perhaps more than that, because those treasures are perishable but mathematical equations are never. Certainly this list of primes will be expanded in the near future as the research should develop further.

2) Despite this common attribution, it appears likely that the procedure had come from the Orient. Ancient mathematicians made academic trips as we do today; an Ebla tablet tells such an episode (G. Leick. Mesopotamia. Penguin Books 2002, p. 68). Eratosthenes was a poet, astronomer, and director of the great library of Alexandria (3rd century BCE); he is said to have been old and unable to enjoy books when he died on voluntary starvation.

3) There are two basic queries in the Sieve Method. One is the primality test, and the other the number of primes that satisfy a certain set of conditions. To the former, N. Agrawal et al made a remarkable contribution a few years ago. However, my interest lies mainly in the latter query, the quantitative aspect of the distribution of primes. To determine a spot to dig where a ruby could certainly be found is definitely harder than to determine whether a stone is a ruby or not. I am well aware that the word 'ineffectual' is too drastic, for Eratosthenes' Sieve could sometimes be a useful tool. A typical instance is in I.M. Vinogradov's proof (1937) of the ternary Goldbach Conjecture (see [64, Kap. VI] and R. C. Vaughan's newer argument [76]). See also H. Iwaniec [29].

4) Prior to Brun [10] was an incomplete trial by J. Merlin [45]. Brun mentioned this fact in [11] precisely; an indication of his respect for the academic axiom, which is often purposely ignored nowadays. Fortunately, Brun's name seems to be well-known. For instance, at the item 'Number Theory' in World Encyclopedia (Heibon, Tokyo 1972; Japanese), C. Chevalley writes 'the unique result known about the Twin Prime Conjecture is the work by Viggo Brun (1919)' (see the bottom lines of the right column on p. 458, vol. 16). This is a bizarre opinion, however. Scripta manent.

5) A.M. Legendre's formulation (1808), though the use of the Möbius function (1832) is a later tradition.

6) This asymptotic formula is an elementary result due to F. Mertens (1874) (see [64, pp. 80–81]); $\exp(-c_E) = 0.561459483566885\ldots$.

7) As to Prime Number Theorem, see, e.g., [26] [64]. Here the theorem is used in the abused manner that the probability of the appearance of a prime at x is $1/(\log x)$.

8) Under the condition $y = x^\theta$, $0 < \theta < \frac{1}{2}$, this appears impossible to prove even on the Riemann Hypothesis. See Section 2.8.

9) The second assertion follows from the first, because of $x/(\log x) \ll \pi(x)$, an elementary assertion due to P.L. Čebyčev (1853). See [64, p. 19].

10) Is it an exaggeration to say that it took more than 2000 years for number theorists to break themselves of cleaving to equalities? A great king cut the knot with a stroke of his sword.

11) See e.g., [20, §3.4] for details.

12) This formulation of the sieve dimension might appear made abruptly. I have in fact skipped the discussion about how to define a mean value of ω. See, for instance, [20, pp. 27–37]. In practice, (1.17) should suffice. Note that there is no a priori sieve dimension for a given problem to which the Sieve Method is to be applied. It depends largely upon

the choice of the sequence $\mathcal{A}$ into which the problem is embedded. The two subsequent sections give examples in this context as well.

13) Be aware that at this stage nothing is assumed about the remainder term. Obviously it is useless to employ sieve weights that make the estimation of the remainder term too hard. Although Brun [12] stated his result including an estimation of the remainder term, nowadays the main term is first treated, and then the discussion about the remainder term follows. A reason for this splitting of the theory into two parts could be seen from e.g., Section 1.9 below.

14) It is proved in [12] that both of the conjectures could be resolved if one is allowed to replace primes by integers with at most 9 prime factors. See Section 3.8 below.

15) About the distribution of primes in arithmetic progressions, see [64, Kap. IV]. As to the distribution of primes and the Riemann Hypothesis, see [64, p. 235]. If an unconditional uniform bound for individual $E(x; k, l)$ is sought for, it is hard to supersede the famous but ineffective Siegel–Walfisz's Prime Number Theorem (see [64, p. 144]); namely, $Q = (\log x)^B$ with any fixed $B > 0$ is the best presently, and any assertion like (1.28) may appear hopeless. See Section 2.8, however.

16) See e.g., [64, Kap. VI].

17) More generally, one may discuss with the system of residues $\Omega(p^a)$ mod p^a, $a = 1, 2, \ldots$, being given initially. See Selberg [74] and also my lecture notes [60, §1.1] as well as [54, III].

18) Without loss of generality, one may suppose that $0 < |\Omega(p)| < p$, as I do within the present chapter. In fact, if $|\Omega(p)| = 0$, then such a prime does not participate the sieve problem under consideration; and if $|\Omega(p)| = p$, then $|\mathcal{S}(\mathcal{A}; \Omega, z)| = 0$ as soon as z becomes larger than p. One could restrict the domain of Ω instead.

19) The restriction $q < z$ is made solely for the sake of simplicity. In general, $q < Q$, $q|P(z)$, should be employed, with Q depending on z.

20) As a matter of fact, Linnik [41] dealt with the case in which q are all primes. His motivation was to investigate statistically I.M. Vinogradov's conjecture on least quadratic non-residues (an account can be found in [6, p. 7]). Thus the discussion of the present section is a refined version of Linnik's argument. The beautiful inequality (2.10) is in fact due to H.L. Montgomery [46, Chap. 3]. See also [54, II].

21) Selberg's argument is indeed rich; that will become apparent in Sections 2.9 and 4.3 below.

22) The estimation (2.20) is not optimal; neither the bound for $|R|$ in (2.22). In the case where $|\Omega(p)|$ is bounded, these can be acceptable. However, for instance, if $|\Omega(p)| \sim cp$ with a constant $c > 0$, then (2.19) yields a bound of $\lambda(d)$ far superior to (2.21). In passing, we note that Selberg's Sieve in the context of the present section can be applied to general sequences upon the supposition (1.3); see e.g., [20, Chap. 2] for details.

23) Probably because of this comparison, it is said often that Selberg's Sieve is effective only when $|\Omega(p)|$ is relatively small. This is, however, incorrect, as is proved in the next section. See also [60, §1.1] [54, II] [54, III].

24) The duality between Linnik's and Selberg's Sieves seems to have been observed for the first time by myself at a Turán seminar in 1970, as explicitly as is given here. Thus the

contents of this section dates back to 1970, although it was published in [54, II] much later because of an unfortunate circumstance.

25) Selberg's inequality; obviously an extension of Bessel's inequality. Proof is easy; see e.g., [6, p. 14] [46, pp. 42–43]. The factor $N - 1 + \delta^{-1}$ in (2.30) is due to Selberg, and is best possible, a proof of which is in [47]. The history of the Large Sieve, starting at Linnik [41] and reaching the contents of Section 2.6 (early 1970's), is highly interesting. Above all, the great leap made by Bombieri [4] should be acclaimed. That impact brought a number of then young people into Analytic Number Theory; some of them are still active. In between Bombieri [4] and Selberg's inequality is G. Halász [21].

26) However, one should not forget the fact that Brun's Sieve is capable of giving rise to lower bounds as well; for instance, (1.23).

27) See [60, pp. 129–130]. As to the relation between the distribution of primes in arithmetic progressions and the Exceptional Zeros or the Siegel Zeros of Dirichlet L-functions, see [64, Kap. IV].

28) This is due to Montgomery [46, Chap. 4]. There arbitrary intervals are in fact considered. More precisely, he used instead of (2.30) a sharper inequality deducible from (2.29). A completely uniform result was later obtained by Montgomery–Vaughan [48]. Here is an important note: Combining the discussion of Section 2.5 with (2.29), one concludes that (2.17) is not optimal. That is, it is suggested that the appeal to (2.29) should yield a procedure with which one could aim a simultaneous optimization for the main and remainder terms in Selberg's Sieve applied to intervals. Similar observation is made on [22, p. 126]. However, my older argument developed in Section 2.5 seems to go deeper.

29) My usage of the word 'avoid' in the present context needs to be explained. It indicates, in a somewhat abused way, that there exist situations where one may reach significant results on the distribution of primes without appealing to the great hypothesis, though arguments might become more involved than on the hypothesis, and results less sharp. What is important is that the Twin Prime and the Goldbach Conjectures are contained in this category of problems.

30) To learn under the two great mathematicians, A. Rényi and P. Turán, I arrived Budapest via railway in the evening of January 31, 1970. Next day, I saw a black flag over the entrance of Matematikai Kutatóintézet on Reáltanoda. The director Rényi died aged 49.

31) The Zero Density Theory. See e.g., [26] [46] [64] for details.

32) The multiplicative characters $\{n^{it} : t \in \mathbb{R}\}$ could also be included into the multiplicative Large Sieve. Unfortunately, I have to skip basic contributions by Halász [21] and P.X. Gallagher [18]. See [6] [26] [46] for details.

33) The zero density theorem of the Linnik type. The discussion given in [64, Kap. X §2] is as hard as Linnik's original, though some simplifications by K.A. Rodosskii are claimed. H. Davenport reviewed Linnik's articles, and commented, 'formidable.' A comparatively simpler proof is that via Turán's Power Sum Method [75]; see [6, §6]. See also [50][60, Chap. V]

34) To understand this fascinating encounter between a sieve result and Dirichlet L-functions, I entered into the Sieve Method. The method could resolve a difficulty that does not appear to be settled with analytic arguments only. See [64, pp. 346–347]. The argument via the Power Sum Method requires as well the Brun–Titchmarsh theorem; see [6, p. 50].

35) See [64, pp. 145–146]. Also [52].

36) Rényi neither proceeded nor stated his result like this. Nevertheless, what he established is essentially the same as the Mean Prime Number Theorem as is stated here.

37) The developments prior to May 1964 are detailed in [2]; see [46, Chaps. 15–17] for an account of the later progress up to 1971.

38) A.I. Vinogradov's assertion is weaker than Bombieri's which is embodied in (2.38). In the context of (1.25)–(1.27), they are, however, of the same strength essentially.

39) The Dispersion Method is a device to introduce perturbations into certain arithmetic problems. Statistical arguments are then applied to the perturbed. Thus, for instance, binary additive problems could sometimes be transformed into ternary additive problems which are often more tractable.

40) Bombieri used his own; but in the context of (2.38) there is no difference. See his lecture notes [6].

41) The present section is due to Selberg [73] and myself [50] [52]–[54] [60, §1.2–§1.4]. An origin is in [6, Théorème 7A] (Selberg), which corresponds to the case $\Omega(p) = \{0\}$.

42) More correctly, to minimize the main term of the asymptotic formula for (2.37). See Selberg [66] [67].

43) A proof is in [60, §1.2]; see also [54, II] [54, III]. Following Selberg [73], $\{\Psi_q(n, \Omega)\}$ could be termed pseudo-characters, which could be regarded as generalizations of the Ramanujan sum. This relation between these arithmetic functions and Selberg's Sieve was found by myself [54, II] [60, Notes(I)].

44) This is just a formal discussion. In practice, we have to impose realistic conditions to f. See e.g., [53] [57, Lemma 2] [60, §1.3].

45) Due to myself [57]; see also [60, §6.2]. The two basic assertions of Linnik mentioned in Notes 36–38 and Turán's Power Sum Method are altogether discarded. See also [50][52][53]. These should be compared with M. Jutila [38] and S. Graham [19]. Further, the works [55] [56] [59] are relevant.

46) Only a few people seem to have had opportunities to look into Rosser's unpublished manuscript. Other people, including myself, could see its outline only in Selberg's scant account [72]. Thus all published works either on Rosser's Sieve or on the lower sieve bound, which is implicitly the main subject of the present section, can be regarded as contributions made independently of Rosser. With this understanding, the first general result on the lower bound is in Ankeny–Onishi [1], which is a combination of Selberg's Sieve and Buchstab's Identity (3.1); see Note 23 above. Buchstab [14] is on the line of his [13], and certainly more sophisticated. The first published account of Rosser's Sieve is Iwaniec [33], entirely due to himself, a detailed version of which is given in [20, Chaps. 3–4]. As to the Linear Sieve, see Note 52 below.

47) This section is an excerpt from [60, §2.1]. See also [58, I].

48) An easy account of the Sieving Limit is in [60, pp. 57–60]. For more details see Selberg [72]; and also Note 61 below.

49) These sets are independent of z_0, z.

50) This section is due to an observation by Friedlander–Iwaniec [17]. See also [60, pp. 55–59]. The sum over p in (3.16) can be estimated via (1.17).

51) In this respect, the argument originating in Ankeny–Onishi [1] has a definite merit. There exists a detailed discussion in [20, Chap. 7].

52) However, the first published account of the Linear Sieve, i.e., the determination of the optimal upper/lower main terms is due to Jurkat–Richert [37]. They started with Selberg's Sieve and performed iteration via (3.1) in much the same way as Rosser's (i.e., $\beta = 2$). On the other hand Iwaniec [27] is the first published account of the Linear Sieve à la Rosser. Iwaniec [28] dealt with the half dimensional sieve, where Rosser's construction with $\beta = 1$ yields again the optimal upper/lower main terms. See also [20, §4.5]. By the way, the work of Jurkat and Richert was applied by J.-R. Chen in his famous work [15] on Goldbach's Conjecture; see Note 60 below. All works quoted here are independent of Rosser's.

53) The appearance of this equation might be somewhat unexpected. It is, however, a consequence of the prerequisite that the pair of the optimal main terms in the upper and lower bounds be stable against the iteration via the Buchstab identity. By the way, (3.19) ($r = 1$) coincides with what Selberg's Sieve implies. Jurkat–Richert [37] starts with this fact. As to the analogue of (3.18)–(3.19) for general κ, see [20, §4.2].

54) Here is the reason why the additional parameter z_0 has been introduced. The discussion in the rest of this section involves in fact a convergence issue, though it is not checked there. The rôle of z_0 is to secure the convergence. On the other hand, the appearance of the factor $V(z_0, \omega)$ is harmless because of Brun's theorem (1.22). In this context, (1.22) is termed the Fundamental Lemma. The assertion (3.21) results from the combination of Rosser's Sieve and Brun's Sieve (or rather the procedure of Section 3.5); the sieve weights of the former are multiplied by those of the latter, and the new sieve weights thus obtained are again values of a characteristic function. As to the condition $d < z^\tau$, it is in fact the result of taking anew the value of z for a cosmetic purpose; this is possible because of the smoothness of ϕ_r. See [60, Chap. III] for more details.

55) A heuristic explanation: Let β_0 be the infimum thus defined. Assume that there exists a summand in the second sum on the right of (3.8) such that $d < (D/p(d))^{1/\beta_0}$. Then there exists the possibility that $|\mathcal{S}(\mathcal{A}_d, p(d))|$ is positive, as this might be detected by Rosser's Sieve with $\beta = \beta_0$ and the level D/d. That is, $\sigma_r(d)$ has to vanish; and we are led to the condition.

56) Due to Selberg. See [20, §4.5] for details.

57) Most probably, it was early in winter 1966. There was a colloquium talk by S. Uchiyama at Tokyo University. After his talk, I had a short discussion with him. He said, 'An astounding announcement has been made in China.' 'What is that?' 'A mathematician named Chen Jing-run has claimed $p + P_2$ for Goldbach's Conjecture.' 'Anything about his method?' 'The Mean Prime Number Theorem and two sieve lemmas, but I have difficulties with one of the latter.' Then he gave me a copy of the now famous announcement. The dreadful Cultural Revolution had already been spreading, and Chen would lose seven years until the publication of the proof.

58) Chen got $C_0 > 0.67$. A conjecture of Hardy–Littlewood [23] states that with $C_0 = 2$ the right side of (3.32) is asymptotically equal to $|\{p : N = p + p'\}|$, where p' is also a prime.

59) The procedure (3.34) is a typical instance of weighted sieves, which originates in P. Kuhn [39]. See [20, Chap. 5] for a general theory.

60) Chen applied Jurkat–Richert [37] to the first two terms as mentioned above, and Selberg's Sieve to the third, which is not much different from the present procedure, as far as (3.32) is concerned. The move to the sequence $\mathcal{A}^*$ is now called the Switching Trick. The extension of Bombieri–Vinogradov's Mean Prime Number Theorem to $\mathcal{A}^*$ is due to Chen himself. For a more general extension, see [51] as well as [6, §22].

61) Naturally one could take up topics that are not included in the above category of sieves. For instance, there is a sieve method started by Bombieri [5], which is related to the elementary proof of the Prime Number Theorem ([64, Kap. III §6]). Recently Friedlander and Iwaniec (1998) made an important progress on this line.

62) See [20, pp. 257–258].

63) The seminal nature of the works [16] and [49] could be stressed with a fair reason. There are explicit mentions in Iwaniec [31] [34] [35]. Certain personal recollections might be allowed here, as this is presumably the last opportunity for me to write down some memorable events from my young days: The preprint of [49] was finished in early autumn of 1972. There had been a belief in the air that the Brun–Titchmarsh theorem would not be improved beyond (2.36) via the Sieve Method. Therefore, I sent copies of my works to Chen, Gallagher, Halberstam, Hooley, and Richert. Hooley replied me immediately, kindly showing how to gain a further improvement. He had developed a statistical study of the Brun–Titchmarsh theorem. My preprint seems to have circulated widely, with misunderstandings as well; one day I received a letter from US informing me that I was rumored to have proved Twin Prime Conjecture. Richert kindly invited me to an MFO Tagung (1975). After attending the meeting, I went to Budapest to see my mentor Turán. He would die aged 66 in September next year. He indicated very faintly about his illness but no sign of graveness. He wholeheartedly encouraged me at the restaurant Astoria after my talk at the institute on my sieve results including the Brun–Titchmarsh theorem, the zero density of the Linnik type, and the Deuring–Heilbronn phenomenon. He liked all, especially the last, even though that made obsolete an important work of his already late collaborator S. Knapowski, and thus his Power Sum Method to a certain extent. I continued the trip to Warszawa via Kraków to see Iwaniec. He and A. Schinzel kindly came to pick up me at the central station in that early morning of cold December. Iwaniec had the ambition to extend [49] to Rosser's Sieve. At the next MFO Tagun (1977) he disclosed to me the surprise that he had already got the essentials of his revolutionary work [34]. In 1979 my daughter was born. In that summer I was at the great Durham Symposium, and D. Hejhal kindly gave me a private lecture on Kuznetsov's work [40], an enormous change of the landscape. There I met also L.-K. Hua; I told him I wanted to go to Peking to see Chen. Next autumn I could visit Peking but not Chen because of an unknown reason. In the spring of 1981, I finished my lecture notes [60] at the Tata IFR. I see still the magnificent sunset over the Arabian Sea.

64) As a matter of fact, we need certain conditions to have the assertion (4.5) valid. See [58, II] [60, §2.3, §3.4] for the details; there a proof is developed, and it is precisely reproduced in [20]. By the way, it is possible to improve (4.2) into the form same as (4.5) but with $M = N = \sqrt{D}$; see [62], which gives an improvement upon (4.3). *See the second Addendum given above.*

65) This should be compared with M.N. Huxley [25]. Another striking application is Iwaniec [30], which stands for the hitherto best approximation to Gauss' conjecture on the existence of primes of the form $n^2 + 1$.

66) See [61, §4.2] as well as [9].

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