UNIFIED ARITHMETIC MOTIVE GEOMETRY: THE YANG–NISADELIC TOPOLOGIES AND THEIR MOTIVIC REALIZATIONS OVER $\mathbb{Q}$

PU JUSTIN SCARFY YANG

ABSTRACT. We construct a unified framework for arithmetic motives over $\mathbb{Q}$ by combining two new Grothendieck topologies: the Yang Arithmetic Topology and the Nisadelic Topology. The former captures intrinsic arithmetic structures based on Galois descent and homotopical semantics, while the latter refines the classical Nisnevich topology with adelic local refinements. We define two types of arithmetic motives: Yang-motives and Nisadelic motives, each supporting distinct but complementary cohomological invariants, regulator maps, and realization functors. We then formulate a unified motivic category, $\mathfrak{DM}_{\mathrm{arith}}(\mathbb{Q})$, with natural applications to special values of L-functions, Selmer groups, p-adic Hodge theory, and arithmetic Langlands duality.

CONTENTS

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1. Introduction

Motivic cohomology and $\mathbb{A}^1$-homotopy theory have revolutionized our understanding of schemes over fields. However, for arithmetic bases such as $\mathbb{Q}$ or $\mathbb{Q}_p$, existing topologies such as Nisnevich or étale fail to provide a natural and unified descent-theoretic framework. To resolve this, we propose two topological refinements:

(1) The **Yang Arithmetic Topology**, based on internal Galois-theoretic descent and type-theoretic homotopy motives.
(2) The **Nisadelic Topology**, which combines Nisnevich-style gluing with adelic local refinements at all places of $\mathbb{Q}$.

Each gives rise to a respective category of motives: $\mathrm{DM}_{\mathrm{Yang}}(\mathbb{Q})$ and $\mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q})$. These are then integrated into a larger system:

$$\mathfrak{DM}_{\mathrm{arith}}(\mathbb{Q}) := \mathrm{DM}_{\mathrm{Yang}}(\mathbb{Q}) \cup \mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q}).$$

2. The Yang Arithmetic Topology

2.1. **The Yang Arithmetic Topology: Definitions and Philosophy.** Let $\mathcal{C}_{\mathrm{arith}}$ be the category of finite-type arithmetic schemes over $\mathrm{Spec}(\mathbb{Q})$, possibly extended to pro-schemes via Galois towers.

Definition 2.1 (Yang Arithmetic Covering)**.** A family $\{U_i \to X\}$ is a Yang covering if:

- Each U_i admits a morphism to X which locally corresponds to a finite Galois cover of base fields;
- There exists a finite-index open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that fixes each U_i individually;

- These morphisms admit descent data compatible with the pro-Galois site structure.

Definition 2.2 (Yang Site)**.** The **Yang arithmetic site** is the site $(\mathcal{C}_{\mathrm{arith}}, \tau_{\mathrm{Yang}})$, where coverings are given by Yang arithmetic coverings.

From this, one defines presheaves and sheaves on the Yang site, as well as homotopy classes and descent cohomology. Motives over this site naturally contain arithmetic data, Galois homotopy types, and type-theoretic interpretations.

2.2. **The Nisadelic Topology: Geometry and Localization.** Let $\mathcal{C}_{\mathrm{sm}/\mathbb{Q}}$ be the category of smooth schemes over $\mathbb{Q}$.

Definition 2.3 (Nisadelic Covering)**.** A family $\{U_i \to X\}$ is a Nisadelic covering if:

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This defines a Grothendieck topology $\tau_{\mathrm{Nisadelic}}$ over $\mathcal{C}_{\mathrm{sm}/\mathbb{Q}}$. The associated derived category of motives $\mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q})$ supports:

- $\mathbb{A}^1$-invariance;
- Compatibility with local realization functors $\mathrm{DM}(\mathbb{Q}_v)$;
- A natural platform for Beilinson regulators and motivic period pairings.

2.3. **Cohomological Interactions and Fiber Realizations.** Each arithmetic motive M in $\mathfrak{DM}_{\mathrm{arith}}(\mathbb{Q})$ has two projections:

$$M \rightsquigarrow \begin{cases} M_{\mathrm{Yang}} \in \mathrm{DM}_{\mathrm{Yang}}(\mathbb{Q}) \\ M_{\mathrm{Nisadelic}} \in \mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q}) \end{cases}$$

One reflects internal Galois and syntactic structure, the other external adelic and topological geometry. Their comparison allows reconstruction of global arithmetic invariants.

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5. Iwasawa Motives and Λ-adic Structures

5.1. **Selmer–Nisadelic Motives.** For a motive $M \in \mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q})$, define the *Selmer–Nisadelic motive* as the sheaf $\mathcal{F}_M$ whose cohomology encodes local conditions across all v:

$$H^1_{\mathrm{Nisadelic}}(\mathbb{Q}, \mathcal{F}_M) \cong \mathrm{Sel}(M/\mathbb{Q}).$$

5.2. **Λ-adic Sheaves and Iwasawa Tower.** Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension. Define:

$$\mathcal{F}_\Lambda := \varprojlim_n \mathcal{F}_{M_n} \quad \text{where } M_n = M|_{\mathbb{Q}_n}.$$

This yields an object in a Λ-adic derived category of arithmetic sheaves.

6. p-adic Realizations and Comparison Functors

6.1. **Fontaine Realizations from Motives.** There exists a realization functor from Nisadelic motives to p-adic Hodge modules:

$$R_{\mathrm{pHd}} : \mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q}_p) \to \mathrm{Rep}^{\mathrm{adm}}_{B_{\mathrm{cris}}}.$$

6.2. **Comparison Motive Diagram.** The following square commutes for each M and each place v:

$$\begin{tikzcd}
H^i_{\mathrm{mot}}(M) \arrow[r] \arrow[d] & H^i_{\text{ét}}(M_{\overline{\mathbb{Q}}_v}, \mathbb{Q}_p) \arrow[d] \\
H^i_{\mathrm{dR}}(M) \arrow[r] & D_{\mathrm{dR}}(V)
\end{tikzcd}$$

This provides a cohomological description of p-adic periods via motive-theoretic paths.

7. Arithmetic Periods and Zeta-Motives

7.1. **Zeta-Motive of $\mathbb{Q}$.** Define the zeta-motive $M_\zeta(\mathbb{Q})$ such that:

$$\langle \text{regulator class}, \text{period class} \rangle = \zeta^{(n)}_{\mathbb{Q}}(0)$$

in terms of a period pairing over $\mathrm{DM}_{\mathrm{Nisadelic}}$.

7.2. **Beilinson Regulator Maps.** The Nisadelic site admits a natural Beilinson-style regulator:

$$\mathrm{reg}_B : K_{2n-1}(\mathbb{Q}) \to H^n_{\mathrm{Nisadelic}}(\mathbb{Q}, \mathbb{Q}(n)).$$

8. Future Directions

- Develop full ∞-categorical versions of $\mathrm{DM}_{\mathrm{Yang}}$ and $\mathrm{DM}_{\mathrm{Nisadelic}}$.
- Construct formal path objects and fundamental groups within both sites.
- Extend the unified theory to global function fields and the field with one element $\mathbb{F}_1$.
- Investigate implications for the Langlands program and categorical representations.

Appendix A. Notation Summary

- $\mathcal{T}_{\mathrm{Yang}}$: Yang Arithmetic Topology
- $\tau_{\mathrm{Nisadelic}}$: Nisadelic Topology
- $\mathrm{DM}_{\mathrm{Yang}}(\mathbb{Q})$: Motives over Yang site
- $\mathrm{DM}_{\mathrm{Nisadelic}}(\mathbb{Q})$: Motives over Nisadelic site
- $\mathfrak{DM}_{\mathrm{arith}}(\mathbb{Q})$: Unified category of arithmetic motives

Appendix B. Comparison Table

Aspect	Yang Topology	Nisadelic Topology
Source of Geometry	Galois descent	Adelic refinement
Cohomology Type	Descent, Galois-type	Period, local fiber
Motive Fiber	Pro-Galois tower	$\prod_v$-localized fibers
Homotopy Tool	Arithmetic HoTT	$\mathbb{A}^1$-homotopy
Period Structure	Implicit via type	Explicit via regulators

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