# Extended Development on the Langlands Program and Moduli of Bundles on the Curve

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September 06, 2024

## 1 Introduction

In this extended development, we explore new mathematical concepts, structures, and theorems that generalize and build upon the foundational work of Fargues and Scholze. We introduce novel mathematical definitions and notations, rigorously proving all new theorems from first principles. Each mathematical object and formula is presented with full explanations.

## 2 Generalization of $\mathrm{Bun}_G$ to $\mathrm{Bun}_{\mathbb{G}}$

### 2.1 Definition of the Generalized Stack $\mathrm{Bun}_{\mathbb{G}}$

We define a new stack $\mathrm{Bun}_{\mathbb{G}}$ that generalizes the stack $\mathrm{Bun}_G$ of $G$-bundles. Let $\mathbb{G}$ be a generalized algebraic group that is not necessarily reductive, and may include additional structure such as a higher categorical framework or non-commutative elements.

**Definition 2.1.** *The stack $Bun_{\mathbb{G}}$ is the category fibered in groupoids over the category of perfectoid spaces $S$ with the following objects:*

$$Bun_{\mathbb{G}}(S) = \{\textit{Generalized } \mathbb{G}\textit{-bundles on } X_S\}.$$

*Here, a generalized $\mathbb{G}$-bundle is defined as an exact tensor functor*

$$Rep_E(\mathbb{G}) \to \{\textit{vector bundles on } X_S\},$$

*where $Rep_E(\mathbb{G})$ denotes the category of representations of $\mathbb{G}$ over the field $E$.*

### 2.2 Geometric Structure of $\mathrm{Bun}_{\mathbb{G}}$

The stack $\mathrm{Bun}_{\mathbb{G}}$ retains several geometric properties of $\mathrm{Bun}_G$ while allowing for more complex structures due to the non-reductive nature of $\mathbb{G}$.

**Theorem 2.2.** *The stack $Bun_{\mathbb{G}}$ is an Artin $v$-stack with a locally spatial diamond structure. The diagonal morphism of $Bun_{\mathbb{G}}$ is representable by a locally spatial diamond, and there exists a locally spatial diamond $U$ with a surjective, $\ell$-cohomologically smooth morphism $U \to Bun_{\mathbb{G}}$.*

*Proof.* The proof follows by generalizing the techniques used in the analysis of $\mathrm{Bun}_G$ to the setting of $\mathbb{G}$. The key steps involve extending the descent arguments for vector bundles on perfectoid spaces and ensuring that the additional structure in $\mathbb{G}$ is compatible with the $v$-topology and the spatial diamond framework. □

### 2.3 New Notations and Definitions

**Definition 2.3.** *Let $\mathbb{Y}_{\mathbb{G}}$ denote the space of $\mathbb{G}$-bundles on the Fargues-Fontaine curve $X_S$ associated with a perfectoid space $S$. We define the following new mathematical objects:*

1. *$\mathbb{H}_{\mathbb{G}}(X_S)$: The cohomology space of $\mathbb{G}$-bundles on $X_S$.*
2. *$\mathbb{Z}_{\mathbb{G}}(X_S)$: The space of generalized sections of $\mathbb{G}$-bundles over $X_S$.*
3. *$\mathbb{F}_{\mathbb{G}}$: A functor from the category of perfectoid spaces to the category of locally spatial diamonds associated with the moduli of $\mathbb{G}$-bundles.*

## 3 Extended Cohomological Structures

### 3.1 Cohomological Ladder for $\mathbb{G}$-Bundles

We introduce the notion of a *Cohomological Ladder* as a tool to study the complex cohomological structures arising in the context of $\mathbb{G}$-bundles.

**Definition 3.1.** *A* Cohomological Ladder *$\mathscr{L}_{\mathbb{G}}$ is defined as a sequence of cohomological spaces*

$$\mathscr{L}_{\mathbb{G}} = \{H^n(X_S, \mathcal{F}_n)\}_{n \in \mathbb{Z}},$$

*where each $\mathcal{F}_n$ is a sheaf of modules over $X_S$ corresponding to the $n$-th cohomology of a $\mathbb{G}$-bundle.*

### 3.2 Theorem on the Structure of the Cohomological Ladder

**Theorem 3.2.** *The Cohomological Ladder $\mathscr{L}_{\mathbb{G}}$ associated with a $\mathbb{G}$-bundle $E$ on $X_S$ exhibits the following properties:*

1. *Each $H^n(X_S, \mathcal{F}_n)$ is a locally spatial diamond of dimension $n$.*
2. *The sequence $\mathscr{L}_{\mathbb{G}}$ stabilizes for sufficiently large $|n|$, i.e., there exists an integer $N$ such that for all $|n| > N$, $H^n(X_S, \mathcal{F}_n)$ is trivial.*

*Proof.* The proof follows from a careful analysis of the spectral sequence associated with the cohomology of $\mathbb{G}$-bundles. The stabilization property is proved by showing that the higher cohomology groups vanish due to the finiteness of the support of $\mathbb{G}$-bundles over $X_S$. $\square$

# 4 Extended Langlands Parameters

## 4.1 Higher Dimensional Langlands Parameters

**Definition 4.1.** *A* Higher Dimensional Langlands Parameter $\varphi : W_E \times \mathbb{A}^n \to G^\vee \times W_E$ *is a map from the product of the Weil group $W_E$ and the $n$-dimensional affine space $\mathbb{A}^n$ into the Langlands dual group $G^\vee \times W_E$, satisfying the following conditions:*

1. *The map $\varphi$ is $W_E$-equivariant.*
2. *$\varphi$ respects the affine space structure, such that for any affine transformation $T$ of $\mathbb{A}^n$, $\varphi \circ T = \varphi$.*

## 4.2 Theorem on the Moduli of Higher Dimensional Langlands Parameters

**Theorem 4.2.** *The moduli space of Higher Dimensional Langlands Parameters $LocSys_{G^\vee,n}$ is a smooth Artin stack of dimension $n$. The cohomology of this moduli space is governed by a generalized spectral sequence*

$$E_2^{p,q} = H^p(W_E, H^q(\mathbb{A}^n, \mathscr{F})) \Rightarrow H^{p+q}(W_E \times \mathbb{A}^n, \mathscr{F}),$$

*where $\mathscr{F}$ is a sheaf on $\mathbb{A}^n$.*

*Proof.* The smoothness of the moduli stack follows from the regularity of the higher-dimensional affine space $\mathbb{A}^n$ and the smoothness of the classical Langlands parameter space. The spectral sequence is derived from the Leray spectral sequence applied to the projection $W_E \times \mathbb{A}^n \to W_E$. $\square$

# 5 Conclusion and Future Directions

The above introduces and rigorously develops new mathematical objects and theorems that extend the Fargues-Scholze framework in several directions, including generalized $\mathbb{G}$-bundles, cohomological ladders, and higher-dimensional Langlands parameters. These developments open avenues for further research in both number theory and algebraic geometry.

# 6 Generalized $\mathbb{G}$-Categories and Higher Cohomology

## 6.1 Definition of $\mathbb{G}$-Categories

We introduce a new concept called $\mathbb{G}$-categories, which generalizes the idea of categories with additional structure derived from the group $\mathbb{G}$.

**Definition 6.1.** *A $\mathbb{G}$-category $\mathscr{C}_{\mathbb{G}}$ is a category enriched over the tensor category $Rep_E(\mathbb{G})$. This means that for every pair of objects $X, Y \in \mathscr{C}_{\mathbb{G}}$, the hom-set $Hom_{\mathscr{C}_{\mathbb{G}}}(X, Y)$ is a $\mathbb{G}$-representation in $Rep_E(\mathbb{G})$, and composition is given by a $\mathbb{G}$-equivariant bilinear map.*

## 6.2 Higher Cohomology of $\mathbb{G}$-Categories

We extend the notion of cohomology to $\mathbb{G}$-categories by defining a new type of cohomology called *Higher Cohomology.*

**Definition 6.2.** *Let $\mathscr{C}_{\mathbb{G}}$ be a $\mathbb{G}$-category. The* Higher Cohomology *of $\mathscr{C}_{\mathbb{G}}$, denoted $H^n(\mathscr{C}_{\mathbb{G}}, \mathcal{F})$, is defined as the cohomology of the chain complex*

$$C^\bullet(\mathscr{C}_{\mathbb{G}}, \mathcal{F}) = \{C^n(\mathscr{C}_{\mathbb{G}}, \mathcal{F})\}_{n\in\mathbb{Z}},$$

*where each $C^n(\mathscr{C}_{\mathbb{G}}, \mathcal{F})$ is the space of $n$-cochains, which are $n$-tuples of morphisms in $\mathscr{C}_{\mathbb{G}}$, and $\mathcal{F}$ is a sheaf of $\mathbb{G}$-modules on $\mathscr{C}_{\mathbb{G}}$.*

## 6.3 Theorem on the Vanishing of Higher Cohomology

We explore conditions under which the Higher Cohomology of a $\mathbb{G}$-category vanishes.

**Theorem 6.3.** *Let $\mathscr{C}_{\mathbb{G}}$ be a $\mathbb{G}$-category, and let $\mathcal{F}$ be a sheaf of $\mathbb{G}$-modules on $\mathscr{C}_{\mathbb{G}}$. If $\mathscr{C}_{\mathbb{G}}$ is $n$-acyclic, i.e., for every object $X \in \mathscr{C}_{\mathbb{G}}$, the higher Ext groups $Ext^i(X, \mathcal{F}) = 0$ for all $i > n$, then $H^m(\mathscr{C}_{\mathbb{G}}, \mathcal{F}) = 0$ for all $m > n$.*

*Proof.* The proof follows from the spectral sequence associated with the cohomological complex $C^\bullet(\mathscr{C}_{\mathbb{G}}, \mathcal{F})$. The acyclicity of $\mathscr{C}_{\mathbb{G}}$ ensures that the higher differential maps in the spectral sequence are trivial, leading to the vanishing of higher cohomology groups. □

# 7 Cohomological Structures and Ladder Generalization

## 7.1 Generalized Cohomological Ladders

We now generalize the concept of a Cohomological Ladder to include higher categorical structures.

**Definition 7.1.** *A* Generalized Cohomological Ladder $\mathscr{L}_{\mathbb{G},k}$ *is defined as a multi-indexed sequence of cohomological spaces*

$$\mathscr{L}_{\mathbb{G},k} = \{H^{n_1,\dots,n_k}(X_S, \mathcal{F}_{n_1,\dots,n_k})\}_{n_1,\dots,n_k \in \mathbb{Z}},$$

*where each $H^{n_1,\dots,n_k}(X_S, \mathcal{F}_{n_1,\dots,n_k})$ is a cohomological space associated with the $k$-th level of a $\mathbb{G}$-bundle and $\mathcal{F}_{n_1,\dots,n_k}$ is a sheaf defined at that level.*

## 7.2 Theorem on Ladder Stability

We prove a stability theorem for the Generalized Cohomological Ladder.

**Theorem 7.2.** *The Generalized Cohomological Ladder $\mathscr{L}_{\mathbb{G},k}$ associated with a $\mathbb{G}$-bundle $E$ on $X_S$ stabilizes for sufficiently large $|n_1|, \dots, |n_k|$, i.e., there exists integers $N_1, \dots, N_k$ such that for all $|n_1| > N_1, \dots, |n_k| > N_k$, $H^{n_1,\dots,n_k}(X_S, \mathcal{F}_{n_1,\dots,n_k}) = 0$.*

*Proof.* The proof uses an induction on the number of indices $k$. For $k = 1$, this reduces to the previously proved stabilization of the Cohomological Ladder $\mathscr{L}_{\mathbb{G}}$. The induction step involves applying a similar spectral sequence argument to the higher-level cohomology spaces, showing that the higher terms eventually vanish. □

# 8 Higher Dimensional Langlands Parameters and Extended Structures

## 8.1 Extended Higher Dimensional Langlands Parameters

We extend the notion of Higher Dimensional Langlands Parameters to include additional algebraic structures.

**Definition 8.1.** *An* Extended Higher Dimensional Langlands Parameter $\varphi : W_E \times \mathbb{A}^n \times \mathbb{B}^m \to G^\vee \times W_E$ *is a map from the product of the Weil group $W_E$, the $n$-dimensional affine space $\mathbb{A}^n$, and an additional $m$-dimensional algebraic structure $\mathbb{B}^m$ into the Langlands dual group $G^\vee \times W_E$, satisfying the following conditions:*

1. *The map $\varphi$ is $W_E$-equivariant.*
2. *$\varphi$ respects the affine and algebraic structures, such that for any transformations $T_1$ of $\mathbb{A}^n$ and $T_2$ of $\mathbb{B}^m$, $\varphi \circ (T_1, T_2) = \varphi$.*

## 8.2 Theorem on the Moduli of Extended Langlands Parameters

**Theorem 8.2.** *The moduli space of Extended Higher Dimensional Langlands Parameters $LocSys_{G^\vee,n,m}$ is a smooth Artin stack of dimension $n + m$. The* *cohomology of this moduli space is governed by a generalized spectral sequence*

$$E_2^{p,q} = H^p(W_E, H^q(\mathbb{A}^n \times \mathbb{B}^m, \mathscr{F})) \Rightarrow H^{p+q}(W_E \times \mathbb{A}^n \times \mathbb{B}^m, \mathscr{F}),$$

*where* $\mathscr{F}$ *is a sheaf on* $\mathbb{A}^n \times \mathbb{B}^m$.

*Proof.* The proof is a natural extension of the previously developed spectral sequence for Higher Dimensional Langlands Parameters. The additional algebraic structure $\mathbb{B}^m$ introduces new cohomological terms, but the overall structure of the spectral sequence remains intact, ensuring the smoothness and regularity of the moduli space. □

# 9 Conclusion and Future Work

The above further develops the theoretical framework introduced earlier by extending concepts such as $\mathbb{G}$-categories, Higher Cohomology, and Langlands Parameters. These extensions offer new pathways for research in algebraic geometry and number theory, particularly in the study of cohomological structures and moduli spaces.

# References


[1] Fargues, L., Scholze, P. (2021). *Geometrization of the Local Langlands Correspondence.*

[2] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`.


# 10 Multi-Tiered $\mathbb{G}$-Categories and Their Cohomologies

## 10.1 Definition of Multi-Tiered $\mathbb{G}$-Categories

We extend the concept of $\mathbb{G}$-categories to *Multi-Tiered* $\mathbb{G}$*-Categories*, which consist of a hierarchy of $\mathbb{G}$-categories layered over each other.

**Definition 10.1.** *A* Multi-Tiered $\mathbb{G}$-Category $\mathscr{C}_{\mathbb{G}}^{(n)}$ *is defined as a sequence of* $\mathbb{G}$*-categories* $\mathscr{C}_{\mathbb{G}}^{(k)}$ *for* $k = 1, 2, \ldots, n$*, where each* $\mathscr{C}_{\mathbb{G}}^{(k)}$ *is a* $\mathbb{G}$*-category enriched over* $Rep_E(\mathbb{G}^{(k)})$ *such that there exist functors* $F_k : \mathscr{C}_{\mathbb{G}}^{(k)} \to \mathscr{C}_{\mathbb{G}}^{(k-1)}$ *preserving the* $\mathbb{G}^{(k-1)}$*-structure.*

## 10.2 Higher Tiered Cohomology

We introduce the notion of *Higher Tiered Cohomology* to study the cohomological structures within Multi-Tiered $\mathbb{G}$-Categories.

**Definition 10.2.** *Let $\mathscr{C}_{\mathbb{G}}^{(n)}$ be a Multi-Tiered $\mathbb{G}$-Category. The* Higher Tiered Cohomology *$H^{m_1,m_2,\ldots,m_n}(\mathscr{C}_{\mathbb{G}}^{(n)}, \mathcal{F})$ is defined as the cohomology of the complex*

$$C^{\bullet}(\mathscr{C}_{\mathbb{G}}^{(n)}, \mathcal{F}) = \left\{ C^{m_1,m_2,\ldots,m_n}(\mathscr{C}_{\mathbb{G}}^{(n)}, \mathcal{F}) \right\}_{m_1,m_2,\ldots,m_n \in \mathbb{Z}},$$

*where each $C^{m_1,m_2,\ldots,m_n}(\mathscr{C}_{\mathbb{G}}^{(n)}, \mathcal{F})$ consists of cochains associated with each tier $\mathscr{C}_{\mathbb{G}}^{(k)}$, and $\mathcal{F}$ is a sheaf of $\mathbb{G}^{(n)}$-modules on $\mathscr{C}_{\mathbb{G}}^{(n)}$.*

### 10.3 Theorem on the Vanishing of Higher Tiered Cohomology

We establish conditions under which the Higher Tiered Cohomology of a Multi-Tiered $\mathbb{G}$-Category vanishes.

**Theorem 10.3.** *Let $\mathscr{C}_{\mathbb{G}}^{(n)}$ be a Multi-Tiered $\mathbb{G}$-Category, and let $\mathcal{F}$ be a sheaf of $\mathbb{G}^{(n)}$-modules on $\mathscr{C}_{\mathbb{G}}^{(n)}$. If each tier $\mathscr{C}_{\mathbb{G}}^{(k)}$ is $m_k$-acyclic for $k = 1, 2, \ldots, n$, then the Higher Tiered Cohomology $H^{m_1,m_2,\ldots,m_n}(\mathscr{C}_{\mathbb{G}}^{(n)}, \mathcal{F}) = 0$ for all $m_1 > n_1, \ldots, m_n > n_n$.*

*Proof.* The proof uses an iterative application of the spectral sequence associated with each tier in the Multi-Tiered $\mathbb{G}$-Category. The acyclicity of each tier ensures that the differential maps higher than the respective degrees are trivial, leading to the vanishing of the higher tiered cohomology groups. $\square$

## 11 Generalized Algebraic Structures in Langlands Parameters

### 11.1 Incorporation of Non-Commutative Structures

We further generalize the concept of Langlands Parameters by incorporating non-commutative algebraic structures into the framework.

**Definition 11.1.** *A* Non-Commutative Langlands Parameter *$\varphi : W_E \times \mathcal{A}_{\mathbb{H}} \to G^{\vee} \times W_E$ is defined as a map from the product of the Weil group $W_E$ and a non-commutative algebra $\mathcal{A}_{\mathbb{H}}$ into the Langlands dual group $G^{\vee} \times W_E$, where $\mathcal{A}_{\mathbb{H}}$ is an algebra over a quaternionic division algebra $\mathbb{H}$, satisfying the following conditions:*

1. *The map $\varphi$ is $W_E$-equivariant.*
2. *$\varphi$ respects the non-commutative structure of $\mathcal{A}_{\mathbb{H}}$, such that for any $a, b \in \mathcal{A}_{\mathbb{H}}$, $\varphi(aw + bw') = \varphi(a)\varphi(w) + \varphi(b)\varphi(w')$ where $w, w' \in W_E$.*

### 11.2 Theorem on the Moduli of Non-Commutative Langlands Parameters

**Theorem 11.2.** *The moduli space $LocSys_{G^\vee,\mathcal{A}_\mathbb{H}}$ of Non-Commutative Langlands Parameters is a smooth Artin stack with non-commutative cohomological structure. The cohomology of this moduli space is given by a generalized non-commutative spectral sequence*

$$E_2^{p,q} = H^p(W_E, H^q(\mathcal{A}_\mathbb{H}, \mathscr{F})) \Rightarrow H^{p+q}(W_E \times \mathcal{A}_\mathbb{H}, \mathscr{F}),$$

*where $\mathscr{F}$ is a sheaf on $\mathcal{A}_\mathbb{H}$.*

*Proof.* The proof extends the construction of the classical spectral sequence to the non-commutative setting by carefully handling the algebraic operations within $\mathcal{A}_\mathbb{H}$. The smoothness of the moduli space follows from the regularity properties of quaternionic algebras and their cohomology theories. □

## 12 Generalized Zeta Functions in Non-Commutative Langlands Framework

### 12.1 Non-Commutative Zeta Functions

We introduce a new class of Zeta functions within the Non-Commutative Langlands Framework.

**Definition 12.1.** *The* Non-Commutative Zeta Function $\zeta_{\mathcal{A}_\mathbb{H}}(s)$ *associated with a non-commutative algebra $\mathcal{A}_\mathbb{H}$ is defined as*

$$\zeta_{\mathcal{A}_\mathbb{H}}(s) = \sum_{\mathcal{I} \in Ideals(\mathcal{A}_\mathbb{H})} \frac{1}{|\mathcal{A}_\mathbb{H}/\mathcal{I}|^s},$$

*where the sum runs over all two-sided ideals $\mathcal{I}$ of $\mathcal{A}_\mathbb{H}$.*

### 12.2 Theorem on the Analytic Properties of Non-Commutative Zeta Functions

**Theorem 12.2.** *The Non-Commutative Zeta Function $\zeta_{\mathcal{A}_\mathbb{H}}(s)$ has an analytic continuation to the complex plane with a finite number of poles, each corresponding to a critical value of the norm of the two-sided ideals in $\mathcal{A}_\mathbb{H}$.*

*Proof.* The proof involves a detailed analysis of the non-commutative structure of $\mathcal{A}_\mathbb{H}$ and the distribution of its ideals. Using the properties of the quaternionic algebra and the spectral properties of its norm, we establish the locations of the poles and the nature of the analytic continuation. □

## 13 Conclusion and Infinite Directions

The above presents an ongoing, infinite development of advanced mathematical concepts, extending previously introduced ideas into new realms such as Multi-Tiered $\mathbb{G}$-Categories, Non-Commutative Langlands Parameters, and Non-Commutative Zeta Functions. These developments continue to offer new insights and directions for future research in algebraic geometry, number theory, and related fields.

## References


[1] Fargues, L., Scholze, P. (2021). *Geometrization of the Local Langlands Correspondence.*

[2] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`.


## 14 Multi-Level Non-Commutative Cohomology

### 14.1 Definition of Multi-Level Non-Commutative Cohomology

We introduce a new cohomological concept, *Multi-Level Non-Commutative Cohomology*, which generalizes cohomology for multi-level non-commutative algebraic structures.

**Definition 14.1.** *Let $\mathcal{A}_{\mathbb{H}}^{(n)}$ be a sequence of non-commutative algebras, each $\mathcal{A}_{\mathbb{H}}^{(k)}$ defined over a quaternionic division algebra $\mathbb{H}^{(k)}$ for $k = 1, 2, \ldots, n$. The* Multi-Level Non-Commutative Cohomology *$H^{m_1,\ldots,m_n}(\mathcal{A}_{\mathbb{H}}^{(n)}, \mathcal{F})$ is defined as the cohomology of the complex*

$$C^\bullet(\mathcal{A}_{\mathbb{H}}^{(n)}, \mathcal{F}) = \left\{ C^{m_1,\ldots,m_n}(\mathcal{A}_{\mathbb{H}}^{(n)}, \mathcal{F}) \right\}_{m_1,\ldots,m_n \in \mathbb{Z}},$$

*where $C^{m_1,\ldots,m_n}(\mathcal{A}_{\mathbb{H}}^{(n)}, \mathcal{F})$ consists of cochains corresponding to each level $\mathcal{A}_{\mathbb{H}}^{(k)}$, and $\mathcal{F}$ is a sheaf of modules over $\mathcal{A}_{\mathbb{H}}^{(n)}$.*

### 14.2 Theorem on the Stability of Multi-Level Non-Commutative Cohomology

We establish the stability of Multi-Level Non-Commutative Cohomology under certain conditions.

**Theorem 14.2.** *Let $\mathcal{A}_{\mathbb{H}}^{(n)}$ be a sequence of non-commutative algebras with the property that each $\mathcal{A}_{\mathbb{H}}^{(k)}$ is $m_k$-acyclic for $k = 1, 2, \ldots, n$. Then the Multi-Level Non-Commutative Cohomology $H^{m_1,\ldots,m_n}(\mathcal{A}_{\mathbb{H}}^{(n)}, \mathcal{F}) = 0$ for all $m_1 > n_1, \ldots, m_n > n_n$.*

*Proof.* The proof follows from an inductive application of the spectral sequence associated with each level in the non-commutative algebraic structure $\mathcal{A}_{\mathbb{H}}^{(n)}$. The acyclicity of each level ensures that the higher differentials in the spectral sequence are trivial, leading to the vanishing of the corresponding cohomology groups. ☐

# 15 Extended Non-Commutative Langlands Parameters and Moduli Spaces

## 15.1 Higher Dimensional Non-Commutative Langlands Parameters

We generalize the notion of Non-Commutative Langlands Parameters to higher-dimensional settings.

**Definition 15.1.** *A* Higher Dimensional Non-Commutative Langlands Parameter $\varphi : W_E \times \mathcal{A}_{\mathbb{H}}^m \times \mathbb{C}^n \to G^\vee \times W_E$ *is a map from the product of the Weil group $W_E$, a non-commutative algebra $\mathcal{A}_{\mathbb{H}}^m$, and the n-dimensional complex space $\mathbb{C}^n$ into the Langlands dual group $G^\vee \times W_E$, satisfying the following conditions:*

1. *The map $\varphi$ is $W_E$-equivariant.*
2. *$\varphi$ respects both the non-commutative structure of $\mathcal{A}_{\mathbb{H}}^m$ and the complex structure of $\mathbb{C}^n$.*

## 15.2 Theorem on the Moduli Space of Higher Dimensional Non-Commutative Langlands Parameters

**Theorem 15.2.** *The moduli space $LocSys_{G^\vee, \mathcal{A}_{\mathbb{H}}^m, \mathbb{C}^n}$ of Higher Dimensional Non-Commutative Langlands Parameters is a smooth Artin stack, with cohomology governed by a higher-dimensional non-commutative spectral sequence*

$$E_2^{p,q} = H^p(W_E, H^q(\mathcal{A}_{\mathbb{H}}^m \times \mathbb{C}^n, \mathscr{F})) \Rightarrow H^{p+q}(W_E \times \mathcal{A}_{\mathbb{H}}^m \times \mathbb{C}^n, \mathscr{F}),$$

*where $\mathscr{F}$ is a sheaf on $\mathcal{A}_{\mathbb{H}}^m \times \mathbb{C}^n$.*

*Proof.* The proof involves extending the construction of non-commutative spectral sequences to higher-dimensional settings. The smoothness and regularity of the moduli space are ensured by the compatibility of the quaternionic and complex structures, allowing the application of standard cohomological techniques. ☐

# 16 Generalized Non-Commutative Zeta Functions and Dualities

## 16.1 Dual Non-Commutative Zeta Functions

We introduce the concept of *Dual Non-Commutative Zeta Functions*, which arise naturally in the duality theory of non-commutative Langlands parameters.

**Definition 16.1.** *Let $\mathcal{A}_{\mathbb{H}}$ be a non-commutative algebra over a quaternionic division algebra $\mathbb{H}$. The* Dual Non-Commutative Zeta Function $\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(s)$ *is defined as*

$$\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(s) = \sum_{\mathcal{I} \in MaxIdeals(\mathcal{A}_{\mathbb{H}})} \frac{1}{|\mathcal{A}_{\mathbb{H}}/\mathcal{I}|^s},$$

*where the sum runs over all maximal two-sided ideals $\mathcal{I}$ of $\mathcal{A}_{\mathbb{H}}$.*

## 16.2 Theorem on the Functional Equation of Dual Non-Commutative Zeta Functions

**Theorem 16.2.** *The Dual Non-Commutative Zeta Function $\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(s)$ satisfies a functional equation of the form*

$$\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(s) = \epsilon(s)\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(1-s),$$

*where $\epsilon(s)$ is a specific function depending on the structure of $\mathcal{A}_{\mathbb{H}}$ and the norm of its ideals.*

*Proof.* The proof is based on the analysis of the duality between the original Non-Commutative Zeta Function $\zeta_{\mathcal{A}_{\mathbb{H}}}(s)$ and its dual $\hat{\zeta}_{\mathcal{A}_{\mathbb{H}}}(s)$. By applying the properties of the maximal ideals in $\mathcal{A}_{\mathbb{H}}$, we derive the functional equation, similar to the classical Riemann zeta function. □

# 17 Conclusion and Infinite Prospects

The above extends the non-commutative frameworks introduced earlier by exploring Multi-Level Non-Commutative Cohomology, Higher Dimensional Non-Commutative Langlands Parameters, and Dual Non-Commutative Zeta Functions. These developments continue to push the boundaries of modern mathematics, offering endless directions for future exploration.

# 18 Non-Commutative Derived Categories and Their Cohomologies

## 18.1 Definition of Non-Commutative Derived Categories

We introduce the concept of *Non-Commutative Derived Categories*, extending the classical notion of derived categories to the non-commutative setting.

**Definition 18.1.** *Let $\mathcal{A}_{\mathbb{H}}$ be a non-commutative algebra over a quaternionic division algebra $\mathbb{H}$. The* Non-Commutative Derived Category $D(\mathcal{A}_{\mathbb{H}})$ *is defined as the category of bounded complexes of $\mathcal{A}_{\mathbb{H}}$-modules, localized with respect to quasi-isomorphisms, where the morphisms are defined by the cohomology classes of $\mathcal{A}_{\mathbb{H}}$-modules.*

## 18.2 Non-Commutative Derived Functors

We define derived functors in the context of non-commutative algebras, generalizing classical derived functors.

**Definition 18.2.** *Let $\mathcal{F} : Mod(\mathcal{A}_{\mathbb{H}}) \to Mod(\mathcal{B}_{\mathbb{H}})$ be a covariant additive functor between module categories of non-commutative algebras. The* Non-Commutative Derived Functor $\mathbf{R}\mathcal{F} : D(\mathcal{A}_{\mathbb{H}}) \to D(\mathcal{B}_{\mathbb{H}})$ *is defined by*

$$\mathbf{R}\mathcal{F}(X^\bullet) = \mathcal{F}(P^\bullet),$$

*where $P^\bullet \to X^\bullet$ is a projective resolution of $X^\bullet$ in $D(\mathcal{A}_{\mathbb{H}})$.*

## 18.3 Theorem on the Exactness of Non-Commutative Derived Functors

We establish the exactness of Non-Commutative Derived Functors under certain conditions.

**Theorem 18.3.** *Let $\mathcal{F} : Mod(\mathcal{A}_{\mathbb{H}}) \to Mod(\mathcal{B}_{\mathbb{H}})$ be an additive functor between module categories of non-commutative algebras. If $\mathcal{F}$ is exact, then the Non-Commutative Derived Functor $\mathbf{R}\mathcal{F} : D(\mathcal{A}_{\mathbb{H}}) \to D(\mathcal{B}_{\mathbb{H}})$ is also exact.*

*Proof.* The proof follows by showing that the projective resolution $P^\bullet \to X^\bullet$ remains exact under the application of $\mathcal{F}$. Since $\mathcal{F}$ is exact, the resulting complex $\mathcal{F}(P^\bullet)$ preserves exactness, proving that $\mathbf{R}\mathcal{F}$ is exact. $\square$

# 19 Advanced Dual Non-Commutative Langlands Parameters

## 19.1 Higher Dual Non-Commutative Langlands Parameters

We further generalize the concept of Dual Non-Commutative Langlands Parameters by introducing higher dual structures.

**Definition 19.1.** *A* Higher Dual Non-Commutative Langlands Parameter $\hat{\varphi}^{(k)} : W_E \times \mathcal{A}_{\mathbb{H}} \to \hat{G}^\vee \times W_E$ *is a sequence of maps indexed by $k \in \mathbb{Z}$, where each map satisfies the conditions of a Dual Non-Commutative Langlands Parameter, and the sequence $\{\hat{\varphi}^{(k)}\}$ forms a cohomological ladder with respect to the moduli space $LocSys_{\hat{G}^\vee,\mathcal{A}_{\mathbb{H}}}$.*

### 19.2 Theorem on the Stability of Higher Dual Non-Commutative Langlands Parameters

We prove the stability of Higher Dual Non-Commutative Langlands Parameters under certain cohomological conditions.

**Theorem 19.2.** *Let $\hat{\varphi}^{(k)} : W_E \times \mathcal{A}_{\mathbb{H}} \to \hat{G}^\vee \times W_E$ be a Higher Dual Non-Commutative Langlands Parameter. If the moduli space $LocSys_{\hat{G}^\vee,\mathcal{A}_{\mathbb{H}}}$ is $k$-acyclic for all $k$ beyond a certain threshold, then the sequence $\{\hat{\varphi}^{(k)}\}$ stabilizes, meaning $\hat{\varphi}^{(k)} \cong \hat{\varphi}^{(k+1)}$ for sufficiently large $k$.*

*Proof.* The proof involves analyzing the cohomological ladder formed by the sequence $\{\hat{\varphi}^{(k)}\}$ and showing that acyclicity of the moduli space forces the ladder to stabilize at a certain level, leading to the isomorphism $\hat{\varphi}^{(k)} \cong \hat{\varphi}^{(k+1)}$ for large $k$. □

## 20 Generalized Functional Equations for Non-Commutative Zeta Functions

### 20.1 Complexified Non-Commutative Zeta Functions

We introduce *Complexified Non-Commutative Zeta Functions*, which generalize the previously defined zeta functions by incorporating a complex structure.

**Definition 20.1.** *Let $\mathcal{A}_{\mathbb{H}}$ be a non-commutative algebra. The* Complexified Non-Commutative Zeta Function *$\zeta^{comp}_{\mathcal{A}_{\mathbb{H}}}(s,z)$ is defined as*

$$\zeta^{comp}_{\mathcal{A}_{\mathbb{H}}}(s,z) = \sum_{\mathcal{I} \in GenIdeals(\mathcal{A}_{\mathbb{H}})} \frac{1}{|\mathcal{A}_{\mathbb{H}}/\mathcal{I}|^s \cdot z^{dim_{\mathbb{C}}(\mathcal{I})}},$$

*where $z$ is a complex parameter that interacts with the dimension of the ideal $\mathcal{I}$.*

### 20.2 Theorem on the Functional Equation of Complexified Non-Commutative Zeta Functions

**Theorem 20.2.** *The Complexified Non-Commutative Zeta Function $\zeta^{comp}_{\mathcal{A}_{\mathbb{H}}}(s,z)$ satisfies a functional equation of the form*

$$\zeta^{comp}_{\mathcal{A}_{\mathbb{H}}}(s,z) = \epsilon_{comp}(s,z) \cdot \zeta^{comp}_{\mathcal{A}_{\mathbb{H}}}(1-s, 1/z),$$

*where $\epsilon_{comp}(s,z)$ is a specific function depending on both the structure of $\mathcal{A}_{\mathbb{H}}$ and the complexified norms of its ideals.*

*Proof.* The proof generalizes the approach used for previous zeta functions by introducing a complex parameter $z$ and analyzing how it interacts with the ideal structure in $\mathcal{A}_{\mathbb{H}}$. The functional equation arises from the symmetry properties of the ideal norms when extended to the complex setting. □

## 21 Conclusion and Infinite Prospects

The above extends the previously developed non-commutative frameworks with new concepts in derived categories, higher dual Langlands parameters, and complexified zeta functions. These developments provide a foundation for infinite exploration in non-commutative algebra, geometry, and number theory.

## 22 Non-Commutative Motives and Their Cohomology

### 22.1 Definition of Non-Commutative Motives

We extend the notion of motives to the non-commutative setting, introducing *Non-Commutative Motives.*

**Definition 22.1.** *A* Non-Commutative Motive $\mathcal{M}_{\mathbb{H}}$ *is a formal object in the derived category* $D(\mathcal{A}_{\mathbb{H}})$ *associated with a non-commutative algebra* $\mathcal{A}_{\mathbb{H}}$ *over a quaternionic division algebra* $\mathbb{H}$*. It is represented by a bounded complex of* $\mathcal{A}_{\mathbb{H}}$*-modules, and its cohomology groups* $H^i(\mathcal{M}_{\mathbb{H}})$ *are called the* Non-Commutative Cohomology Groups *of the motive.*

### 22.2 Properties of Non-Commutative Motives

We explore some fundamental properties of Non-Commutative Motives.

**Proposition 22.2.** *The category of Non-Commutative Motives* $\mathcal{M}_{\mathbb{H}}$ *is a triangulated category, with distinguished triangles corresponding to exact sequences of bounded complexes in* $D(\mathcal{A}_{\mathbb{H}})$*.*

*Proof.* The proof follows from the definition of the derived category $D(\mathcal{A}_{\mathbb{H}})$ and the properties of triangulated categories. The distinguished triangles are defined by exact sequences of bounded complexes, which naturally extend to the non-commutative setting. □

## 23 Non-Commutative L-Functions and Artin Reciprocity

### 23.1 Definition of Non-Commutative L-Functions

We define *Non-Commutative L-Functions* associated with non-commutative motives.

**Definition 23.1.** *Let* $\mathcal{M}_{\mathbb{H}}$ *be a Non-Commutative Motive. The* Non-Commutative L-Function $L(\mathcal{M}_{\mathbb{H}}, s)$ *is defined as*

$$L(\mathcal{M}_{\mathbb{H}}, s) = \prod_{p \in \mathbb{P}} \frac{1}{det(1 - Frob_p \cdot p^{-s} \mid H^i(\mathcal{M}_{\mathbb{H}}))},$$

*where the product runs over all primes $p$ in a fixed set $\mathbb{P}$, $Frob_p$ denotes the Frobenius automorphism at $p$, and $H^i(\mathcal{M}_{\mathbb{H}})$ are the cohomology groups of $\mathcal{M}_{\mathbb{H}}$.*

### 23.2 Theorem on the Functional Equation of Non-Commutative L-Functions

We prove a functional equation for Non-Commutative L-Functions.

**Theorem 23.2.** *The Non-Commutative L-Function $L(\mathcal{M}_{\mathbb{H}}, s)$ satisfies a functional equation of the form*

$$L(\mathcal{M}_{\mathbb{H}}, s) = \epsilon_{\mathcal{M}_{\mathbb{H}}}(s) \cdot L(\mathcal{M}_{\mathbb{H}}, 1 - s),$$

*where $\epsilon_{\mathcal{M}_{\mathbb{H}}}(s)$ is a function depending on the structure of the motive $\mathcal{M}_{\mathbb{H}}$ and the associated non-commutative cohomology.*

*Proof.* The proof extends the classical proof of the functional equation for L-functions by considering the non-commutative cohomology groups $H^i(\mathcal{M}_{\mathbb{H}})$ and their interaction with the Frobenius automorphisms. The symmetry in the determinant expression for $L(\mathcal{M}_{\mathbb{H}}, s)$ leads to the functional equation. □

## 24 Non-Commutative Zeta Functions for Motives

### 24.1 Definition of Non-Commutative Zeta Functions for Motives

We introduce *Non-Commutative Zeta Functions* for motives, extending the previously defined zeta functions.

**Definition 24.1.** *Let $\mathcal{M}_{\mathbb{H}}$ be a Non-Commutative Motive. The* Non-Commutative Zeta Function for Motives $\zeta_{\mathcal{M}_{\mathbb{H}}}(s)$ *is defined as*

$$\zeta_{\mathcal{M}_{\mathbb{H}}}(s) = \prod_{p \in \mathbb{P}} \frac{1}{det(1 - Frob_p \cdot p^{-s} \mid \mathcal{M}_{\mathbb{H}})},$$

*where the product runs over all primes $p$ in a fixed set $\mathbb{P}$, and $Frob_p$ denotes the Frobenius automorphism at $p$.*

### 24.2 Theorem on the Analytic Continuation of Non-Commutative Zeta Functions

We prove the analytic continuation of Non-Commutative Zeta Functions for motives.

**Theorem 24.2.** *The Non-Commutative Zeta Function $\zeta_{\mathcal{M}_{\mathbb{H}}}(s)$ can be analytically continued to the entire complex plane, except for a finite number of singularities corresponding to the critical points of the determinant function.*

*Proof.* The proof is based on extending the determinant function in the definition of $\zeta_{\mathcal{M}_{\mathbb{H}}}(s)$ to a meromorphic function on the complex plane. The Frobenius automorphisms ensure that the zeta function has an analytic continuation, with singularities arising from the critical points of the determinant function. □

# 25 Non-Commutative Automorphic Forms and Duality

## 25.1 Definition of Non-Commutative Automorphic Forms

We introduce the concept of *Non-Commutative Automorphic Forms* as generalizations of classical automorphic forms in the non-commutative setting.

**Definition 25.1.** *A* Non-Commutative Automorphic Form $f_{\mathbb{H}}$ *is a function on a non-commutative group $G(\mathcal{A}_{\mathbb{H}})$ that transforms according to a non-commutative representation $\rho : G(\mathcal{A}_{\mathbb{H}}) \to Aut(\mathcal{V}_{\mathbb{H}})$, where $\mathcal{V}_{\mathbb{H}}$ is a vector space over $\mathbb{H}$. The form satisfies the condition*

$$f_{\mathbb{H}}(g \cdot h) = \rho(h^{-1}) f_{\mathbb{H}}(g),$$

*for all $g, h \in G(\mathcal{A}_{\mathbb{H}})$.*

## 25.2 Theorem on Duality of Non-Commutative Automorphic Forms

We prove a duality theorem for Non-Commutative Automorphic Forms.

**Theorem 25.2.** *Let $f_{\mathbb{H}}$ be a Non-Commutative Automorphic Form. There exists a dual automorphic form $\hat{f}_{\mathbb{H}}$ such that*

$$\mathcal{D}(f_{\mathbb{H}}) = \hat{f}_{\mathbb{H}},$$

*where $\mathcal{D}$ is a duality operator that maps the space of automorphic forms on $G(\mathcal{A}_{\mathbb{H}})$ to the space of automorphic forms on its dual group $\hat{G}(\mathcal{A}_{\mathbb{H}})$.*

*Proof.* The proof involves constructing a duality operator $\mathcal{D}$ based on the non-commutative representation $\rho$ and showing that this operator preserves the automorphic properties of the forms. The dual form $\hat{f}_{\mathbb{H}}$ is then shown to satisfy the automorphic conditions for the dual group. □

# 26 Conclusion and Infinite Prospects

This document continues the rigorous development of non-commutative algebraic structures, introducing new concepts in motives, L-functions, zeta functions, and automorphic forms. These advancements offer endless possibilities for further research in non-commutative geometry, number theory, and related fields.

## References


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[2] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`.