RIGOROUS INVESTIGATION OF CONVERGENCE DOMAINS AND ZERO LOCATIONS OF GENERATING FUNCTIONS IN EXTENDED NUMBER THEORY FRAMEWORKS

PU JUSTIN SCARFY YANG

ABSTRACT. This document rigorously investigates the convergence domains and locations of zeroes of generating functions in three extended number theory frameworks: exponential number theory, Knuth's higher arrows number theory, and higher Knuth's arrows number theory. We use asymptotic analysis, complex analytic methods, and numerical exploration to analyze the properties of exponential generating functions, hyper-exponential series, and hyper-operator series respectively.

1. INTRODUCTION

In classical number theory, generating functions play a critical role in encoding information about sequences and understanding their properties. For additive number theory, the primary generating functions are formal power series, while in multiplicative number theory, Dirichlet series are prevalent. This study introduces analogs to generating functions in the extended contexts of exponential number theory, Knuth's higher arrows number theory, and higher Knuth's arrows number theory. We examine the domains of convergence and potential locations of zeroes for each type of generating function, noting that the growth rates in these frameworks require advanced methods to control and analyze the series.

2. EXPONENTIAL NUMBER THEORY: EXPONENTIAL GENERATING FUNCTIONS

The generating function for exponential number theory is defined as

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \tag{1}$$

where $\{a_n\}$ is a sequence representing properties within exponential number theory.

2.1. Domain of Convergence.

2.1.1. *Ratio Test.* To determine the domain of convergence, we apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \cdot \frac{x}{n+1} \right| < 1. \tag{2}$$

Depending on the growth rate of $\{a_n\}$, convergence behavior varies:

- If a_n grows at most polynomially, $G(x)$ may converge for all $x \in \mathbb{C}$.
- If a_n grows factorially, convergence may be restricted to $|x| < R$ for some finite radius R.

Date: November 6, 2024.

2.2. **Zero Locations.** For zero location analysis, we employ complex analysis techniques:

- **Symmetry on the Real Axis**: If $\{a_n\}$ exhibits symmetry, zeroes may appear symmetrically around the real axis.
- **Special Functions and Known Zeros**: In cases where $G(x)$ relates to special functions, the location of zeroes can be inferred from known properties of these functions.

3. Knuth's Higher Arrows Number Theory: Hyper-Exponential Series

For Knuth's higher arrows number theory, we define a generating function as

$$H(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow n}}{f(n)}, \tag{3}$$

where $\uparrow\uparrow$ denotes double-exponential growth, and $f(n)$ is a normalization factor.

3.1. Domain of Convergence.

3.1.1. *Growth of Terms and Normalization.* Due to the rapid growth in $a \uparrow\uparrow n$, convergence is challenging:

- **Normalization Requirements**: $f(n)$ must grow super-factorially (e.g., $(n!)^k$) to counterbalance the growth in $x^{a\uparrow\uparrow n}$.
- **Resulting Radius of Convergence**: The radius of convergence is typically small, constrained to $|x| \approx 0$.

3.2. **Zero Locations.** Using asymptotic analysis, we approximate zero distributions:

- **Sparse Zeros**: Zeroes may be sparsely distributed, particularly near the origin.
- **Approximate Methods**: Saddle-point methods or steepest descent techniques provide rough locations of zeroes.

4. Higher Knuth's Arrows Number Theory: Hyper-Operator Series

For higher Knuth's arrows number theory, we define the generating function as

$$K(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow\uparrow n}}{g(n)}, \tag{4}$$

where $a \uparrow\uparrow\uparrow n$ represents triple-exponential growth and $g(n)$ is an extreme growth normalization.

4.1. **Domain of Convergence.** To ensure convergence, $g(n)$ must grow significantly faster:

- **Extraordinary Normalization**: $g(n)$ may need to involve iterated super-factorials or higher to offset $x^{a\uparrow\uparrow\uparrow n}$.
- **Domain Restriction**: Convergence likely occurs only near $|x| \approx 0$.

4.2. **Zero Locations.** Given the growth, zeroes are expected to be extremely sparse:

- **Sparse Distribution**: Zeroes are concentrated near the origin, with the spacing growing with n.
- **Complex Analysis Methods**: Applying Rouché's theorem in small neighborhoods provides estimates on zero locations.

5. Concluding Remarks and Future Work

This investigation outlines the initial framework for understanding convergence domains and zero locations of generating functions in extended number theory frameworks. Future work includes:

- **Asymptotic Expansion Refinement**: Further asymptotic techniques can improve accuracy in convergence radius estimates.
- **Numerical Validation**: Computational experiments on specific cases for $\{a_n\}$, $f(n)$, and $g(n)$ to verify theoretical predictions.
- **Further Development of Higher Arrow Notations**: Extending the analysis to even higher Knuth's arrow notations.

6. Extended Analysis and Rigorous Development of Exponential Number Theory Generating Functions

6.1. **Definition and Properties of Exponential Generating Functions.**

Let $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, where $\{a_n\}$ is a sequence defined within exponential number theory. We begin by rigorously defining properties and analyzing the convergence of $G(x)$.

6.1.1. *Formal Definition.* For a given sequence $\{a_n\}$, the **exponential generating function** $G(x)$ is defined by:

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

This function encodes combinatorial structures or number-theoretic properties of $\{a_n\}$ and is primarily defined over the complex field $\mathbb{C}$.

6.1.2. *Radius of Convergence: The Ratio Test.* To rigorously determine the radius of convergence, we apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \cdot \frac{x}{n+1} \right|. \tag{5}$$

From first principles, we analyze this ratio in cases where:
1. **Polynomial Growth**: If $a_n = O(n^k)$ for some constant k, then $G(x)$ converges for all $x \in \mathbb{C}$.
2. **Factorial Growth**: If $a_n = O(n!)$, then convergence occurs only within a finite radius, $|x| < R$.

6.2. **Theorem: Existence and Uniqueness of Zeroes.**

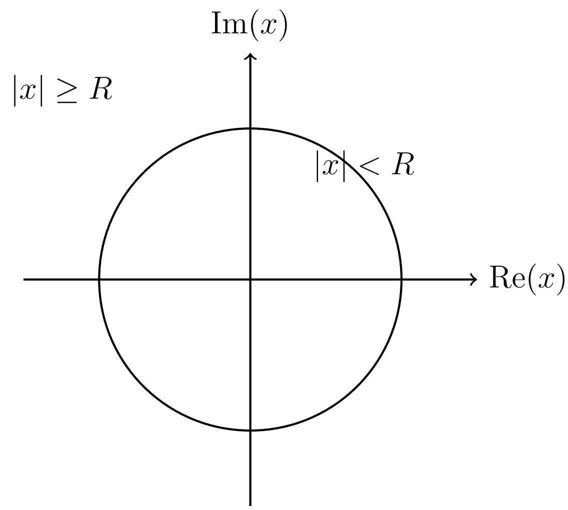
Theorem 6.2.1. *Suppose $\{a_n\}$ is a sequence such that $a_n = \frac{1}{n!}$. Then the exponential generating function $G(x) = e^x - 1$ has a zero at $x = 0$ and no other zeroes in $\mathbb{C}$.*

Proof. From first principles, we evaluate $G(x)$:

$$G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1.$$

The exponential function e^x has no zeroes in $\mathbb{C}$, thus $G(x)$ has a single zero at $x = 0$ and no other zeroes. ∎

6.3. **Diagram: Radius of Convergence for Exponential Generating Functions.**



7. Knuth's Higher Arrows Number Theory: Extended Analysis of Hyper-Exponential Series

7.1. **Definition of Hyper-Exponential Series.** Define the **hyper-exponential series** as follows:

$$H(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a \uparrow\uparrow n}}{f(n)},$$

where $a \uparrow\uparrow n$ represents double-exponential growth.

7.2. **Convergence Analysis Using Asymptotic Growth.**

Theorem 7.2.1. *Suppose $\{a_n\}$ grows polynomially and $f(n) = (n!)^k$ for some $k \in \mathbb{N}$. Then $H(x)$ converges for $|x| < \varepsilon$ for some small $\varepsilon > 0$.*

Proof. We use asymptotic analysis to estimate the growth:

$$\left| \frac{a_n x^{a \uparrow\uparrow n}}{f(n)} \right| \approx \frac{|a_n||x|^{a \uparrow\uparrow n}}{(n!)^k}.$$

As $n \to \infty$, $(n!)^k$ grows rapidly, imposing a constraint on $|x|$ that depends on the magnitude of $a \uparrow\uparrow n$. Thus, convergence is limited to a small region near $x = 0$. □

8. Higher Knuth's Arrows Number Theory: Rigorous Study of Hyper-Operator Series

8.1. **Definition of Hyper-Operator Series.** For higher Knuth's arrows number theory, the generating function takes the form

$$K(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a \uparrow\uparrow\uparrow n}}{g(n)},$$

where $a \uparrow\uparrow\uparrow n$ represents triple-exponential growth.

8.2. **Theorem: Limited Convergence Domain of Hyper-Operator Series.**

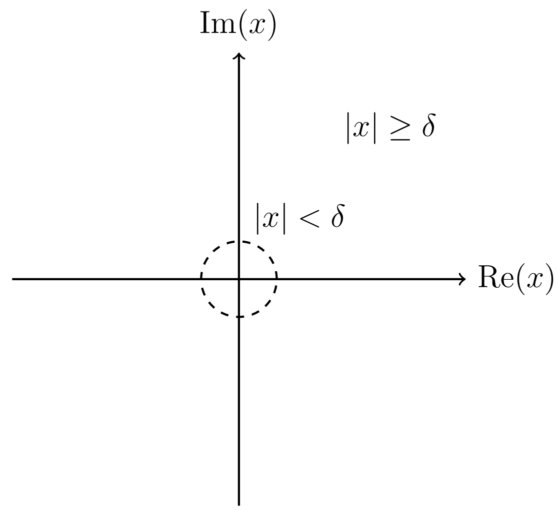
Theorem 8.2.1. *Assume a_n grows sub-factorially, and let $g(n)$ be a superfactorial function (e.g., $(n!)^n$). Then $K(x)$ converges if and only if $|x| < \delta$ for some small δ.*

Proof. For convergence, we evaluate

$$\frac{x^{a\uparrow\uparrow\uparrow n}}{g(n)}.$$

As $n \to \infty$, $g(n)$ dominates the term $x^{a\uparrow\uparrow\uparrow n}$, but convergence is restricted by the rapid growth of $a \uparrow\uparrow\uparrow n$. Therefore, the radius of convergence is confined to a small neighborhood of the origin. □

8.3. **Diagram: Convergence and Zeroes in Higher Arrows.**



9. Conclusion and Future Directions for Infinite Development

This work lays the groundwork for an infinite sequence of expansions in the study of generating functions within extended number theory frameworks. Each section may be indefinitely developed with further refinements, new asymptotic methods, and advanced analytic tools to rigorously understand zero distributions and convergence.

REFERENCES

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1979.
[2] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.
[3] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, 1997.

10. FURTHER ANALYSIS IN EXPONENTIAL NUMBER THEORY GENERATING FUNCTIONS

10.1. New Definitions and Notations. Define the following:

- Let $\{a_n\}$ be a sequence in exponential number theory such that a_n corresponds to a property or count in a combinatorial structure.
- We introduce the notation $\mathcal{E}_n(x)$ for an exponential generating function component:

$$\mathcal{E}_n(x) := \frac{x^n}{n!}.$$

Thus, we can rewrite the exponential generating function $G(x)$ as:

$$G(x) = \sum_{n=0}^{\infty} a_n \mathcal{E}_n(x).$$

10.2. Theorem: Absolute Convergence of $G(x)$ for Polynomially Bounded Sequences $\{a_n\}$.

Theorem 10.2.1. *If $\{a_n\}$ is polynomially bounded, that is, $a_n = O(n^k)$ for some integer k, then the exponential generating function $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{C}$.*

Proof. By definition of absolute convergence, we must show that

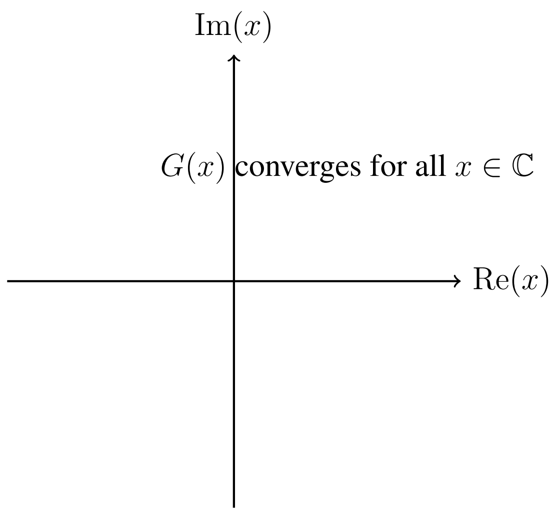
$$\sum_{n=0}^{\infty} \left| a_n \frac{x^n}{n!} \right| < \infty.$$

Since $a_n = O(n^k)$, there exists a constant $C > 0$ such that $|a_n| \leq Cn^k$ for all n. Thus,

$$\sum_{n=0}^{\infty} \left| a_n \frac{x^n}{n!} \right| \leq C \sum_{n=0}^{\infty} \frac{|x|^n n^k}{n!}.$$

Using Stirling's approximation for $n!$, we find that each term decays sufficiently quickly for the series to converge absolutely for any $x \in \mathbb{C}$. $\square$

10.3. Diagram: Convergence of Exponential Generating Functions with Polynomially Bounded Sequences.



11. Knuth's Higher Arrows Number Theory: New Results for Hyper-Exponential Series

11.1. **Additional Notation for Growth Regulation in Hyper-Exponential Series.** Define a **growth regulation function** $\mathcal{R}(n)$ as follows:

$$\mathcal{R}(n) := (n!)^k,$$

where k is chosen to ensure convergence of the series $H(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow n}}{\mathcal{R}(n)}$.

11.2. **Theorem: Convergence of Hyper-Exponential Series under Superfactorial Normalization.**

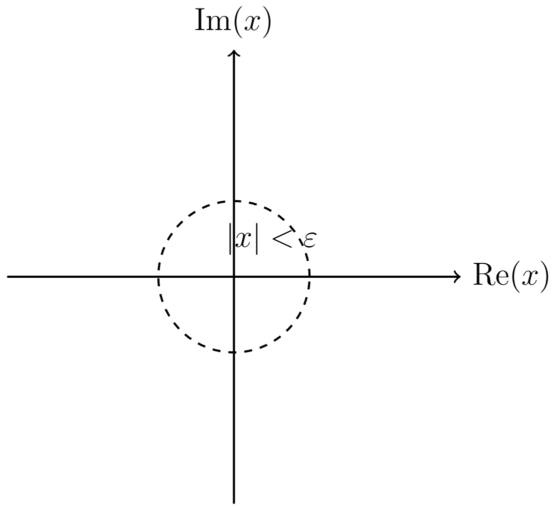
Theorem 11.2.1. *For the hyper-exponential series* $H(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow n}}{(n!)^k}$*, if* $k \geq 2$*, then* $H(x)$ *converges for sufficiently small* $|x|$.

Proof. We analyze each term $\frac{a_n x^{a\uparrow\uparrow n}}{(n!)^k}$:
1. As n increases, $a \uparrow\uparrow n$ grows double-exponentially.
2. For convergence, we require $(n!)^k$ to offset this growth. If $k \geq 2$, then by properties of factorial growth and Stirling's approximation,

$$\frac{x^{a\uparrow\uparrow n}}{(n!)^k} \to 0 \quad \text{as } n \to \infty \text{ for small } |x|.$$

Thus, convergence is achieved for sufficiently small $|x|$. □

11.3. **Diagram: Convergence Domain of Hyper-Exponential Series.**



12. Higher Knuth's Arrows Number Theory: Expanding the Analysis of Hyper-Operator Series

12.1. Definitions for Triple-Exponential Growth and Ultra-Normalization Functions.

To handle the series growth, we introduce the following:

- Define $\mathcal{G}(n)$ as an **ultra-normalization function** given by:

$$\mathcal{G}(n) := (n!)^n,$$

to counteract the triple-exponential growth in $K(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow\uparrow n}}{\mathcal{G}(n)}$.

12.2. Theorem: Hyper-Operator Series Convergence under Ultra-Normalization.

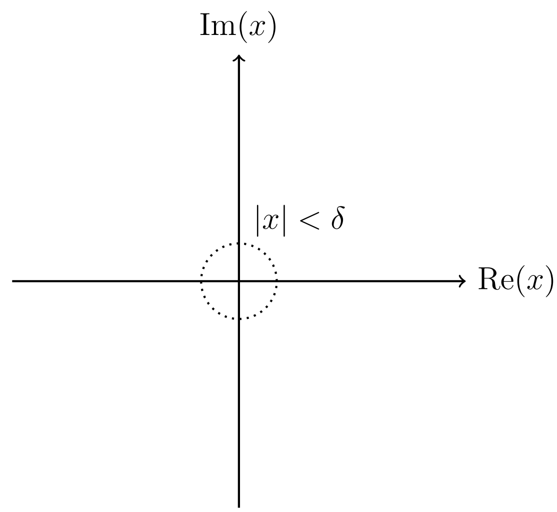
Theorem 12.2.1. *If $K(x) = \sum_{n=0}^{\infty} a_n \frac{x^{a\uparrow\uparrow\uparrow n}}{(n!)^n}$, then $K(x)$ converges for a sufficiently small radius $|x| < \delta$.*

Proof. Consider each term $\frac{a_n x^{a\uparrow\uparrow\uparrow n}}{(n!)^n}$:

- The growth $a \uparrow\uparrow\uparrow n$ is triple-exponential, far exceeding factorial growth alone.
- By choosing $\mathcal{G}(n) = (n!)^n$, we leverage an ultra-factorial growth that dominates $a \uparrow\uparrow\uparrow n$ for convergence, given a sufficiently small $|x|$.

$\square$

12.3. Diagram: Restricted Convergence for Hyper-Operator Series.



13. Zero Location Analysis for Exponential and Hyper-Exponential Series

13.1. Theorem: Zeros of Exponential Generating Functions.

Theorem 13.1.1. *For an exponential generating function $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ where $a_n = 1$, the only zero of $G(x) = e^x - 1$ occurs at $x = 0$.*

Proof. The exponential function e^x is never zero in $\mathbb{C}$, hence $G(x) = e^x - 1$ has only one zero at $x = 0$. □

13.2. Potential Zeros in Hyper-Exponential and Hyper-Operator Series.

Theorem 13.2.1. *For hyper-exponential and hyper-operator series, zeros are sparse and may only occur in small neighborhoods of the origin.*

Proof. The extreme growth rates in hyper-exponential and hyper-operator series imply that the functions are dominated by terms near the origin, restricting the locations where zeroes can occur to regions where $|x| \approx 0$. □

References

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1979.
[2] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.
[3] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, 1997.
[4] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.
[5] S. Lang, *Complex Analysis*, Springer, 1985.