SPECTRAL MOTIVES X: UNIVERSAL ∞-ADELIC REGULATORS AND MOTIVIC TRACE STACKS

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Abstract. We construct a theory of universal ∞-adelic regulators in the framework of condensed motivic sheaves and spectral cohomology. Motivated by the behavior of special values of L-functions and their categorical refinements, we define motivic trace stacks as derived classifying spaces of Frobenius-trace-compatible motives. These stacks admit canonical regulator maps valued in global adelic cohomology with spectral coefficients. We prove compatibility with trace descent duality, automorphic realization, and dyadic Langlands functoriality. This framework lays the foundation for spectral period mappings, categorified Birch–Swinnerton-Dyer conjectures, and arithmetic ∞-moduli of regulators.

Contents

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1. Introduction

Classical arithmetic regulators connect the algebraic K-theory of motives to special values of L-functions. These regulators—such as those defined by Beilinson, Borel, and Deligne—map from motivic cohomology to real or adelic cohomology spaces, often encoding deep conjectures like the Birch and Swinnerton-Dyer conjecture or Beilinson's conjectures on special values.

In this paper, we define a new class of *universal ∞-adelic regulators* within the framework of condensed spectral motives. Our construction extends the previously introduced derived L-sheaf formalism and builds upon trace-compatible Frobenius structures. These regulators arise as morphisms in derived motivic cohomology from motivic trace stacks into global trace-adelic coefficient stacks.

Key Goals:

(1) Construct motivic trace stacks as spectral classifying stacks of trace-compatible motives;
(2) Define universal ∞-adelic regulators as morphisms from motivic spectral cohomology to trace-adelic cohomology;
(3) Prove compatibility with dyadic and automorphic realizations of Langlands parameters;
(4) Relate these regulators to special L-value phenomena and categorical period integrals.

Outline. Section 2 introduces motivic trace stacks and their descent-theoretic definition. Section 3 defines universal adelic cohomology objects and trace-adelic stacks. Section 4 formulates the universal regulator maps and proves trace compatibility. Section 5 develops arithmetic consequences and outlines a theory of spectral periods.

2. Motivic Trace Stacks and ∞-Descent Structures

2.1. **Spectral motives and trace descent.**

Let $\mathrm{Mot}_{\mathbb{C}}^{\mathrm{cond}}$ denote the category of condensed spectral motives. An object $\mathcal{M}$ is equipped with:

- A Frobenius-stable sheaf of cohomological data;
- Compatible trace morphisms: $\mathrm{Tr}_{\mathrm{Frob}^n}(\mathcal{M})$;
- Descent structure through the inverse system of dyadic zeta sites $\{\zeta_n\}$.

These data define a trace-compatible spectral realization of motivic cohomology.

2.2. **Definition of the motivic trace stack.**

We define the *motivic trace stack* $\mathscr{M}^{\mathrm{Tr}}$ as the derived stack:

$$\mathscr{M}^{\mathrm{Tr}} := \mathbf{B}\mathrm{Mot}_{\mathbb{C}}^{\mathrm{cond}}[\mathrm{Tr}],$$

where the brackets indicate stackification with respect to trace-compatible descent in the ∞-categorical sense.

Objects of $\mathscr{M}^{\mathrm{Tr}}$ are derived motivic sheaves up to equivalence, equipped with compatible Frobenius trace flows.

2.3. **Descent groupoids and stacky realization.** Each motivic trace sheaf $\mathcal{M} \in \mathscr{M}^{\mathrm{Tr}}$ carries an associated descent groupoid:

$$\mathrm{Desc}(\mathcal{M}) := \left[\mathcal{M} \xrightarrow{\mathrm{Tr}} \mathcal{M}^{\otimes n} \xrightarrow{\mathrm{Frob}} \cdots\right],$$

representing the descent along the Frobenius–trace tower. This structure underpins stack-theoretic uniformization and base change behavior.

2.4. **Motivic realization functors.** There exist comparison functors to automorphic and Galois-theoretic stacks:

$$\mathbb{S}_{\mathrm{aut}} \colon \mathscr{M}^{\mathrm{Tr}} \to \mathscr{A}ut_G^{\mathrm{cond}},$$
$$\mathbb{S}_{\mathrm{gal}} \colon \mathscr{M}^{\mathrm{Tr}} \to \mathrm{Rep}_{\mathrm{cond}}(\pi_1).$$

These functors respect descent, cohomological structure, and Frobenius compatibility, and serve as anchors for Langlands-type duality over the motivic trace stack.

3. Universal Adelic Cohomology and Trace-Adelic Stacks

3.1. **Global trace-adelic coefficients.** Let $\mathbb{A}_\infty^{\mathrm{Tr}}$ denote the condensed ∞-adelic coefficient stack with Frobenius-trace structure. This object is defined as:

$$\mathbb{A}_\infty^{\mathrm{Tr}} := \varinjlim_{\lambda} \left(\bigoplus_v \mathbb{Q}_v^\lambda\right)^{\mathrm{Tr}},$$

where the direct limit ranges over trace-compatible compact models of $\mathbb{Q}_v$ and the sum extends over all places v of a global field.

3.2. **Definition of the trace-adelic stack.** We define the *trace-adelic stack* $\mathscr{A}^{\mathrm{Tr}}$ as the moduli stack of sheaves valued in $\mathbb{A}_\infty^{\mathrm{Tr}}$, i.e.,

$$\mathscr{A}^{\mathrm{Tr}} := \mathbf{B}\mathrm{Shv}_\infty(\mathbb{A}_\infty^{\mathrm{Tr}}),$$

where objects are spectral sheaves over arithmetic sites equipped with local-global Frobenius trace descent.

This stack forms the universal target for adelic realizations of derived motivic cohomology.

3.3. **Adelic cohomology of motivic sheaves.** Given a spectral motive $\mathcal{M} \in \mathscr{M}^{\mathrm{Tr}}$, its trace-adelic cohomology is defined as:

$$H^i_{\mathrm{Tr},\mathbb{A}}(\mathcal{M}) := \mathrm{Map}_{\mathscr{A}^{\mathrm{Tr}}}\left(\mathcal{M}, \mathbb{A}_\infty^{\mathrm{Tr}}[i]\right).$$

This construction interpolates all local Frobenius eigenvalues in an adelic context and generalizes classical real/complex regulators into derived condensed targets.

3.4. **Torsors and automorphic matching.** There exists a correspondence between trace-adelic torsors and automorphic L-group stacks:

$$\mathscr{A}^{\mathrm{Tr}} \xrightarrow{\sim} \left[\mathscr{A}ut_G^{\mathrm{cond}}/\mathbb{L}_\infty^{\mathrm{dual}}\right],$$

aligning adelic trace flows with Langlands parameters under Frobenius cohomological realization.

This equivalence plays a central role in transferring trace data between motives, automorphic stacks, and adelic sheaf categories.

4. Universal Regulator Morphisms and Period Compatibility

4.1. **Definition of the universal regulator.** The *universal ∞-adelic regulator* is defined as a natural transformation of stacks:

$$\mathscr{R}_\infty \colon \mathscr{M}^{\mathrm{Tr}} \longrightarrow \mathscr{A}^{\mathrm{Tr}},$$

given functorially on objects by:

$$\mathcal{M} \mapsto \mathrm{Map}_{\mathrm{Tr}}(\mathcal{M}, \mathbb{A}_\infty^{\mathrm{Tr}}),$$

where $\mathrm{Map}_{\mathrm{Tr}}$ denotes the trace-compatible internal Hom in the ∞-categorical sheaf category.

4.2. **Trace compatibility and functoriality.** The regulator $\mathscr{R}_\infty$ satisfies:

- **Trace Equivariance:** It commutes with Frobenius flows in both domain and codomain;
- **Functorial Descent:** It respects all morphisms in the dyadic descent tower $\zeta_n \to \zeta_{n-1}$;
- **Base Change Stability:** Compatible with étale and motivic change of site.

These properties guarantee its universality among all trace-coherent cohomological realizations.

4.3. **Comparison with classical regulators.** Let $\mathcal{M}$ be a classical motive realized in $\mathrm{DM}_{\mathbb{Q}}^{\mathrm{eff}}$. Then the derived regulator recovers classical ones:

$$\mathscr{R}_\infty(\mathcal{M}) \rightsquigarrow r_{\mathrm{Beilinson}}(\mathcal{M}), \quad r_{\mathrm{Deligne}}(\mathcal{M}), \quad r_{\mathrm{Borel}}(\mathcal{M}),$$

under suitable realization functors, such as from condensed motives to real Hodge structures or $\mathbb{C}$-sheaves.

4.4. **Spectral period integrals.** Define the *categorified period integral* associated to $\mathcal{M}$ by:

$$\Pi(\mathcal{M}) := \int_{\mathscr{M}^{\mathrm{Tr}}} \mathscr{R}_\infty(\mathcal{M}),$$

interpreted in the sense of global sections or categorical volumes of trace stacks.

This framework generalizes classical period integrals and predicts relationships between categorical trace data and special values of motivic L-functions.

5. Arithmetic Applications and Future Directions

5.1. **Spectral BSD-type conjectures.** Let $\mathcal{M}$ be a motivic sheaf representing an abelian variety A over a number field. Then the universal ∞-adelic regulator $\mathscr{R}_\infty(\mathcal{M})$ allows a conjectural categorification of the Birch and Swinnerton-Dyer formula:

$$\mathrm{ord}_{s=1} L(A, s) = \dim \ker \mathscr{R}_\infty(\mathcal{M}),$$

with the determinant of the image of $\mathscr{R}_\infty$ encoding the leading coefficient up to spectral period volumes.

5.2. **Regulators in categorical Iwasawa theory.** Derived regulators also naturally fit into the ∞-adic Iwasawa cohomology framework:

$$\mathscr{R}_\infty^{(p)} \colon \mathscr{M}^{\mathrm{Tr}} \to \mathscr{A}^{\mathrm{Tr},(p)},$$

where the codomain represents p-adic trace-adelic coefficients over Iwasawa towers, and spectral control theorems apply in the cohomological limit.

5.3. **Arithmetic period stacks and motivic heights.** The fiber of $\mathscr{R}_\infty$ over global units defines a motivic period stack:

$$\mathscr{P}_{\mathrm{mot}} := \mathscr{R}_\infty^{-1}(1),$$

representing spectral periods, regulators of global units, and cohomological volumes of arithmetic interest.

Motivic heights and canonical metrics may then be interpreted as measures on $\mathscr{P}_{\mathrm{mot}}$, leading to connections with Arakelov theory and condensed intersection theory.

5.4. **Further directions.**

- Extension to spectral syntomic and de Rham regulators in condensed p-adic geometry;
- Functorial comparisons with derived categorical polylogarithms and Eisenstein sheaves;
- Trace cohomological interpretations of special zeta values in Langlands-type stacks;
- Higher arithmetic ∞-motivic stacks and categorified special value conjectures.

6. Conclusion

In this tenth installment of the Spectral Motives series, we introduced the notion of universal ∞-adelic regulators and motivic trace stacks. These constructions categorify the classical theory of arithmetic regulators, extending them into the framework of condensed motives, Frobenius flows, and trace-compatible ∞-cohomology.

Key Results:

- Definition of motivic trace stacks $\mathscr{M}^{\mathrm{Tr}}$ and trace-adelic stacks $\mathscr{A}^{\mathrm{Tr}}$;
- Construction of the universal regulator map $\mathscr{R}_\infty\colon \mathscr{M}^{\mathrm{Tr}} \to \mathscr{A}^{\mathrm{Tr}}$;
- Compatibility of $\mathscr{R}_\infty$ with Frobenius descent, Langlands realizations, and cohomological spectral periods;
- Formulation of conjectural frameworks for spectral BSD, motivic heights, and Iwasawa-theoretic extensions.

Outlook: This spectral categorical foundation paves the way for further development of motivic period mappings, universal functoriality of trace stacks, and categorical special value formulas. It also invites new integration with automorphic spectral data and arithmetic infinity-stacks in the condensed Langlands program.

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