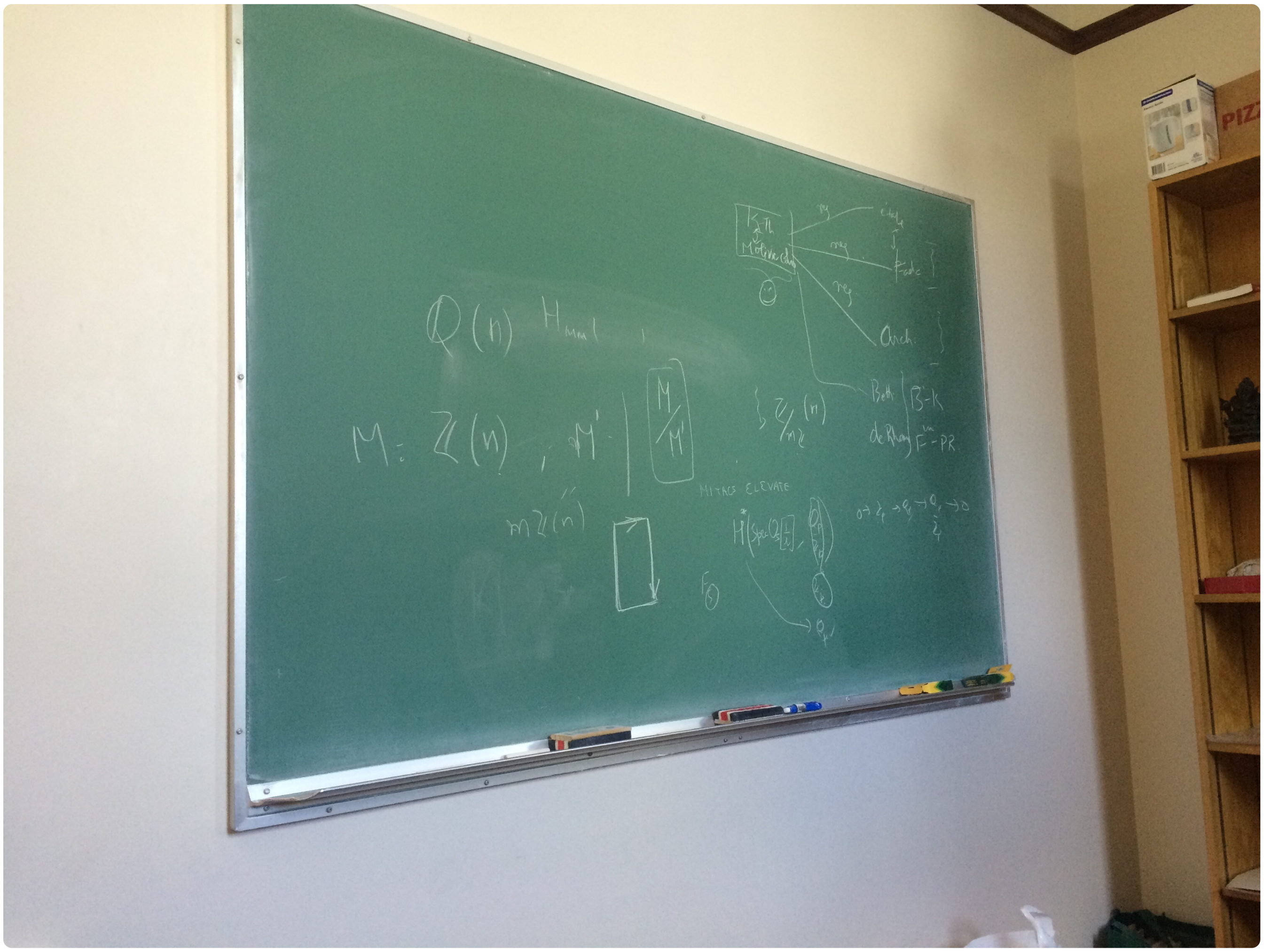
2016-04-10 5

Bloch-Kato w/ Sujatha



↑

$\boxed{2016-04-04}$

Bloch-Kato.

Defin

4

$Ш_{BK}(\quad)$ $\quad Ш_{PT}(\quad)$

Tate-S group

Galois cohomology

1

2016-03-18

"Bloch-Kato Conjecture is invariant under isogeny"

(Original article of B-K).

L-fns for ell. curves for CM.

Guido Kings → reln. with Iwasawa theory

MathSciNet

→ Motives coming from (CM) ell. curves

→ Motives — " — ab. varieties

Also look at Otmar Venjakob.

Diamond? L. Guo — Motives coming from modular forms.
Kato

Project: Known (B-K) that B-K conj. is invariant under isogeny.

One component of B-K → K-theory.

Q: What does this "invariance under isogeny" translate to when viewed from the K-theory optic?

3

2016-03-28

Easter

Google

"Gyanome Ramanujan and more"

on Youtube and watch the interview

(Google Hangout) with Ken Ono that we

did on Ramanujan's birthday last year.

x

→ Dialysis machines

→ Eye Care

Forus Health Care

IC - IMPACTS www.ic-impacts.org

MITACS www.mitacs.ca

$\boxed{4}$ $X/\mathbb{Q}$ smooth variety.

2016-04-04

Motivic coho of X (bigraded family of rational vector spaces)

$$H^i_\mu(X, j) = (K_{2j-i}(X) \otimes \mathbb{Q})^{(j)}$$

$\mathcal{X}/\mathrm{Spec}\,\mathbb{Z}$

Superscript (j) denotes the generalised simultaneous eigenspace of all Adams operators $\psi^k, k \geq 1$, belonging to the eigenvalues k^j.

$X = \mathrm{Spec}\,\mathcal{O}_S\left[\frac{1}{\ell}\right]$

$K_*(X)$: Quillen K-groups.

p. 175 CUP Book.

$\mathbb{Q}(n)$

$\mathbb{Z}(n)$, $\mathbb{Z}_\ell(n)$

$$X \hookrightarrow \mathcal{X}$$

5

- Understand Ш$_{BK}$ and Ш$_{PT}$ for {Tate motive, ell. curves}
- For the Tamagawa number conjecture, understand how L-fns come in. (L-fn for E-C & Riemann-zeta fn)

Just say what TNC is. (B-K ...)

Justin:

- Understand how L-fns side change for isogeny } Greg Martin
- Understand how TNC is invariant under motives

M

M'

M/M'

$T_\ell E$

$T_\ell E'$

Broad picture Sketch. via K-theory + mot. coho.
(Regulators)

Stanford lectures

$T = \mathbb{Z}_\ell(n)$ $\quad$ T' : $m\mathbb{Z}_\ell(n)$

TNC (Gealy)

$\boxed{2016\text{-}04\text{-}10}$

BK paper

Define Tamagawa #

Example w/ $SL_n(\mathbb{Q})$

$\mathrm{Tam}(SL_n(\mathbb{Q})) = 1 = \mathrm{vol}(SL_n(\mathbb{Q}) \backslash \prod'_{p\le\infty} SL_n(\mathbb{Q}_p) = SL_n(\mathbb{Q}) \backslash SL_n(\mathbb{A}_\mathbb{Q})$

w.r.t. μ on $SL_n(\mathbb{A}_\mathbb{Q})$

choose iso det = high end power

$\boxed{0.1}$ $\det_{\mathbb{Q}}(sl_n(\mathbb{Q})) \cong \mathbb{Q}$.

Define Haar me μ_p on $SL_n(\mathbb{Q}_p)$ $\forall p \le \infty$

& μ on $SL_n(\mathbb{A}_\mathbb{Q})$

product formula implies μ independent of ch in $\boxed{0.1}$

Note that strong approx $\Rightarrow$

Recall: from [Platonov & Rep pg 14]

$V^{\mathbb{Q}}$: set of all valuations on $\mathbb{Q}$

$= \{\infty, 2, 3, \dots\}$

$S \subset V^{\mathbb{Q}}$ finite set

dlf $S \neq \emptyset \Rightarrow$ the image of $\mathbb{Q}$ under the diagonal emb is dense in $\mathbb{A}_S$ = finite adele

$\mu(SL_n(\mathbb{Q}) \backslash SL_n(\mathbb{A}_\mathbb{Q})) = \mu_\infty(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})) \prod_{p<\infty} \mu_p(SL_n(\mathbb{Z}_p))$

If (0.1) chosen w/ $\det_{\mathbb{Z}}(sl_n(\mathbb{Z})) \cong \mathbb{Z}$,

$$\mu_p(SL_n(\mathbb{Z}_p)) = \prod_{i=2}^{n}(1-p^{-i})$$

& $\mathrm{Tam}(SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A}))=1$ implies

$$\mu_\infty(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \left(\prod_{p<\infty}\mu_p(SL_n(\mathbb{Z}_p))^{-1} = \left(\prod_{p<\infty}\prod_{i=2}^{n}(1-p^{-i})\right)^{-1}\right.$$

$$= \prod_{i=2}^{n}\left(\prod_{p<\infty}(1-p^{-i})^{-1}\right)$$

$$= \prod_{i=2}^{n}\zeta(i)$$

---

motif M (pure) a direct factor of $H^r(X, \mathbb{Z}(n))$ $\exists$ universal

cohom. theory

X = complete, smooth variety

→ no singular pts

→ $\forall$ variety Y, the map $X\times Y \to Y$ closed

assume $n \geq (r+1)/2$ $\Leftrightarrow$ M weight ≤ -1

ASSUME M weight -2.

Intermediate jacobian of Griffiths.

$$A(\mathbb{C}) := \frac{H^r(X(\mathbb{C}),\mathbb{C})}{\{H^r(X,\mathbb{Z}(n)) + F^n H^r(X(\mathbb{C}),\mathbb{C})\}}$$

recall F^n = Hodge filtration $F^n H^r = \bigoplus_{i\geq n} H^{i,r-i}$

$A(\mathbb{C})$ has real structure given by invariants of conjugation acting

simultan on $X(\mathbb{C})$ as a topological space and on coeffic $\mathbb{C}$.

Moreover, the canonical identification

$$H^r(X(\mathbb{C}), \mathbb{C}) \cong H^r_{DR}(X/\mathbb{Q}) \otimes \mathbb{C}$$

def a $\mathbb{Q}$-struct on the tangent space $H^r(X(\mathbb{C}), \mathbb{C}) / F^n(X(\mathbb{C}), \mathbb{C})$
$\longrightarrow A(\mathbb{C})$

The choice of an iso

(5.2) $\quad \det(H^r_{dR}(X/\mathbb{Q}) / F^n H^r_{dR}(X/\mathbb{Q})) \xrightarrow{\sim} \mathbb{Q}$

determine a Haar meas w_∞ on $A(\mathbb{C})$

Deligne consid $A(\mathbb{R})$ compact, and his conject read

$$L(M, w) \in w_\infty(A(\mathbb{R})) \cdot \mathbb{Q}$$

$\Rightarrow$ $L(M,w)$ a $\mathbb{Q}$-multiple of volume of $A(\mathbb{R})$.

When $A(\mathbb{R})$ not compact, Delig point out Beilinson conj

§1. B_{cris} & B_{dR}

OK from seminar last year ✓

2. C-W homo & F-M theory

motif $\mathbb{Z}(r)$

Let K = unramified finite extens of $\mathbb{Q}_p$

$$\begin{tikzcd}[row sep=small]
 & \overline{K} \arrow[d, dash] & \\
 & K(\zeta_{p^\infty}) = \bigcup_{n\geq 1} K(\zeta_{p^n}) \arrow[d, dash] & \\
 & K(\zeta_{p^\infty})^{ab} \arrow[d, dash] & \\
 & K \arrow[dl, dash] \arrow[d, dash] & \\
\mathbb{Q}_p \arrow[d, dash] & \mathcal{O}_K & \\
\mathbb{Z}_p \arrow[ur, dash] & &
\end{tikzcd}$$

$\mathbb{Z}_p \cong G \left\{ \right.$ (between $K(\zeta_{p^\infty})$ and K), $\left.\right\} P$ (between $K(\zeta_{p^\infty})^{ab}$ and K)

$G = \mathrm{Gal}(K(\zeta_{p^\infty})/K)$

$P = \mathrm{Gal}(K(\zeta_{p^\infty})^{ab}/K)$

Let $U = \varprojlim_n (\mathcal{O}_K[\zeta_{p^n}]^\times)$ = inertia subgroup of P by LCFT

recall inertia subgrp of P, $I_p \triangleleft D_p = \int$

Recall C-W homo a canonical cont $G = \mathrm{Gal}(K(\zeta_{p^\infty})/K)$ -homo

$$\phi_{CW}^r : U \longrightarrow K(r) = K \otimes \mathbb{Q}(r) \quad \text{defined for } r \geq 1$$

Next denote the cont Gal-coh $H^q(\mathrm{Gal}(\overline{K}/K), -)$

by $H^q(K, -)$

Since $P/U \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$

w/ trivial G-action

also since $H^q(G, \mathbb{Q}_p(r)) = (0) \quad \forall q, r \geq 1,$

we have $\forall r \geq 1$,

$$H^1(K, K(r)) = H^1(K, K \otimes \mathbb{Q}(r))$$

$$\cong H^0(G, H^1(K(\zeta_{p^\infty}), \overbrace{K \otimes \mathbb{Q}(r)}^{K(r)}))$$

$$\cong \mathrm{Hom}_G(P, K(r))$$

$$\cong \mathrm{Hom}_G(U, K(r))$$

regard $\phi^r_{CW} \in H^1(K, K(r))$ by above

Thm 2.1

$K/\mathbb{Q}_p$ finite unramified ext, p = odd prime, $r \geq 1$

$$\partial^r : K = H^0(K, B^+_{\mathrm{crys}}) \to H^1(K, \mathbb{Q}_p(r))$$

satisfies $\partial^r(a) = T(a \cdot \phi^r_{CW})/(r-1)!$

where $T : H^1(K, K(r)) \to H^1(K, \mathbb{Q}_p(r))$ the trace for $K/\mathbb{Q}_p$.

In particular, if $K = \mathbb{Q}_p$, we have

$$\phi^r_{CW} = (r-1)! \cdot \partial^r(1)$$

motif $\mathbb{Z}(r)$ given in Thm 2.6

classical explicit recip law as the explicit recip law for $\mathbb{Z}(1)$ OR for $\mathbb{G}_m$.

We expect $\exists$ exp rec la for EC w/ CM

Recall Coleman def of C-W hom

Let $R = \mathcal{O}_K[[T]]$; $Z = 1+T$

R = formal mult. group $\hat{\mathbb{G}}_m / \mathcal{O}_K$ w/ can formal $Z := 1+T$

Let $a \in \mathbb{Z}_p$, $\sigma_a / f / \varphi : R \to R$ be the identit / frob of $\mathcal{O}_K$ /

$$\sigma_a(Z) = Z^a \,/\, f(Z) = Z^p \,/\, \varphi(Z) = Z$$

where $Z^a = \sum_{n=0}^{\infty} (n!)^{-1} \left(\prod_{i=0}^{n-1} (a-i) \right) (Z-1)^n \in R$

$\Rightarrow \sigma_a$ comes from the a-th power map

σ_a com from $\hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$

Since σ_p & f are finite flat map of degree p,

$\exists$ $N_p / N_f : R \to R$ (norm)

$T_p / T_f : R \to R$ (trace)

$N_p \circ \sigma_p(x) = x^p$; $T_p \circ \sigma_p(x) = px$

$N_f \circ f(x) = x^p$; $T_f \circ f(x) = px$ $\quad \forall x \in R$.

Define the action of $G = Gal(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ on R as follows

§3. H^1 of local Gal rep

$K/\mathbb{Q}_p$ finite as above.

M/K motif

Example: A: ab variety

$A(K) \hookrightarrow H^1(K,T)$
$\shortparallel$
K-points of A

T = Tate module of $A = \varprojlim_n {}_nA(\bar{K})$

from

$0 \to {}_nA(\bar{K}) \xrightarrow{n} A(\bar{K}) \to A(\bar{K}) \to 0$

$A(K)$ motivic as the group exten. $/K$ of the dual abelian variety $A^\vee = \mathrm{Hom}(A, \mathbb{G}_m)$

mot. interpr. of T by etale coh. generally easily ☺

$\forall\ \hat{\mathbb{Z}}$-module T of finite rank, w/ cont act of $\mathrm{Gal}(\bar{K}/K)$

$$H^1_e(K,T) \subset H^1_f(K,T) \subset H^1_g(K,T) \subset H^1(K,T) \quad \boxed{3.1}$$

$\uparrow$ exponential $\quad \uparrow$ finite $\quad \uparrow$ gen part

In partic.

$$H^1_g(K,T) = \mathrm{Ker}(H^1(K,T) \to H^1(K, B_{dR} \otimes_{\hat{\mathbb{Z}}} T)) \quad \boxed{3.2}$$

The action of G on $\mathbb{Z}_p(1)$ def an iso $\kappa : G \cong \mathbb{Z}_p^\times$

action of $\kappa^{-1}(a)$ on R is by σ_a.

<u>Theorem 2.2</u> (Coleman)

$\exists$ an iso of G-module

$$U \cong \{ a \in R^\times \mid N_f(a) = a \}, \quad u = \varprojlim u_n \mapsto g_u$$

where g_u is ch as the $\exists!$ element of $R^\times$ s.t.

$$\rho_n(\varphi^{-n}(g_u)) = u_n \quad \forall n \geq 1$$

Write $t = \log(Z) = \sum_{i=1}^{\infty} (-1)^{i-1} T^i/i \ ; \ T = Z - 1$

and conside $R \subseteq K[[t]] \cong K[[T]]$

Def the homo $\Phi^r : U \to K$ $(r \geq 0)$ by

$$\log(g_u) = \sum_{n=0}^{\infty}$$

When T = Tate module of abel variety A.

$$H^1_e(K,T) = H^1_f(K,T) = H^1_g(K,T) = \text{image of } A(K) \hookrightarrow H^1(K,T)$$

$\Rightarrow$ $\forall$ ab varieties A,

$$\boxed{7.3} \quad \underbrace{H^1(K, A(\bar{K}))}_{} \cong \frac{H^1(K, T\otimes \mathbb{Q}/\mathbb{Z})}{H^1_*(K,T)\otimes\mathbb{Q}/\mathbb{Z}} \qquad (* = e, f, g)$$

needed to def. $\underline{Ш}(A)$

When T the Tate-module of general motif.

general

$$H^1_*(K,T) \quad \& \quad H^1(K, M(\bar{K})) = \frac{H^1(K, T\otimes\mathbb{Q}/\mathbb{Z})}{H^1_*(K,T)\otimes\mathbb{Q}/\mathbb{Z}} \qquad (*: e, f, g)$$

analogues

Duality result

$A(K) \cong$ Pontryagin dual of $H^1(K, A^\vee(\bar{K}))$

Let $k = K/m$ = residue field & K_0 = quotient of the Witt vectors $W(k)$

$K_0 \subset K$ max unramified subfield of K.

for V = f.d. $\mathbb{Q}_p$-v.s. endowed w/ cont $G = \mathrm{Gal}(\bar{K}/K)$ action, Fontaine def

$$\boxed{3.5}\begin{cases} Cys(V) := H^0(K, B_{cys} \otimes V) \\ DR(V) := H^0(K, B_{dR} \otimes V) \end{cases}$$

Prime # ℓ & a $\mathbb{Q}_p$-v.s V of finite dim endowed w/ contin. action of $Gal(\bar{K}/K)$, we def $\mathbb{Q}_\ell$-subsp.

$$H^1_e(K,V) \subset H^1_f(K,V) \subset H^1_g(K,V) \subset H^1(K,V) \qquad \boxed{3.7}$$

If $\ell \neq p$

$$\boxed{3.7.1}\begin{cases} H^1_e(K,V) = 0 \\ H^1_f(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K_{nr},V)) \\ H^1_g(K,V) = H^1(K,V) \end{cases}$$

K_{nr} = max unramif

If $\ell = p$

$$\boxed{3.7.2}\begin{cases} H^1_e(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B^{f=1}_{cys} \otimes V)) \\ H^1_f(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B_{cys} \otimes V)) \\ H^1_g(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B_{dR} \otimes V)) \end{cases}$$

Prime l w/ free $\mathbb{Z}_l$-module T of finite rank, endowed w/ continu $\mathrm{Gal}(\bar{K}/K)$-action, we defi

$$H^1_*(K,T) := \iota^{-1}(H^1_*(K, T\otimes\mathbb{Q}))$$

where $\iota : H^1(K,T) \to H^1(K, T\otimes\mathbb{Q})$

$(* = e, f, g)$

$\Rightarrow H^1_*(K,T)$ always contain the tors subg of $H^1(K,T)$.

– If T a free $\hat{\mathbb{Z}}$-module of finit rank w/ cont

$G = \mathrm{Gal}(\bar{K}/K)$-action, let $T_l = T\otimes_{\hat{\mathbb{Z}}} \mathbb{Z}_l$.

Defi

$\boxed{3.7.4}$ $\quad H^1_*(K,T) := \prod H^1_*(K, T_l)$, $* = e, f, g$

Let V be above, $\alpha \in H^1(K,V)$

α correspond to

$0 \to V \to E \to \mathbb{Q}_l \to 0$ an extension

Assume

$l \neq p$	V unramified, then	E unra		$\alpha \in H^1_f(K,V)$
$l = p$	V cris	E cris	iff	$\alpha \in H^1_f(K,V)$
$l = p$	V de Rham	E de Rham		$\alpha \in H^1_g(K,V)$

Prop 3.8

ℓ = prime, V = f.d. $\mathbb{Q}_p$ v.s. } w/ cont $G = \mathrm{Gal}(\bar{K}/K)$ action

T = free $\hat{\mathbb{Z}}$-module of finite rank

§4 values & L-function, local situation

$$
\begin{array}{ll}
 & \bar{K} \\
 & | \\
\mathrm{Gal}(\bar{K}/K) = G & K_{nr} = \text{max unramif ext of } K. \\
 & | \\
 & K \\
\text{finite} & K_0 = \text{max unramif subfld of } K/\mathbb{Q}_p \\
 & \mathbb{Q}_p
\end{array}
$$

l = prim, V = f.d. $\mathbb{Q}_l$-v.s. w/ cont. action of $\mathrm{Gal}(\bar{K}/K)$

Let

$$
P(V,u) = \begin{cases} \det_{\mathbb{Q}_l}(1 - f_K u : H^0(K_{nr}, V)) \in \mathbb{Q}_l[u] & \text{if } l \neq p \\ \det_{K_0}(1 - f_K u : D_{cris}(V)) \in K_0[u] & \text{if } l = p. \end{cases}
$$

When $l \neq p$, f_K = action of an elem of $G = \mathrm{Gal}(\bar{K}/K)$ on $\mathbb{Z}_l(-1)$ by $p^{[K:\mathbb{Q}_p]}$

$l = p$, f_K = the K_0-lin map $f^{[K:\mathbb{Q}_p]}$

We call $P(V,u)^{-1}$ the local L-function attached to V.

Thm 4.1

Let l = prime & V = f.d. $\mathbb{Q}_l$-v.s. w/ cont. action of $\mathrm{Gal}(\bar{K}/K)$

assume $P(V,1) \neq 0$

i) Assume $l \neq p$.

Then (a) $= H^1_e(K,V) = H^1_f(K,V)$

If V = unramified & T is $\mathrm{Gal}(\bar{K}/K)$-stable $\mathbb{Z}_\ell$-lattice in V, then

$$\# H^1_f(K,T) = |P(V,1)|_\ell^{-1}$$

where $|\cdot|_\ell$ the normalized absolute value on $\mathbb{Q}_\ell$.

ii) Assume $\ell = p$ and V a de Rham rep.

Then $DR(V)/DR(V)^0 \xrightarrow[\cong]{\exp} H^1_e(K,V) = H^1_f(K,V)$

iii) Assume $\ell = p$, $K/\mathbb{Q}_p$ unramified, V = crys rep. and

(☆) $\exists\, i \le 0$ & $j \ge 1$ w/ $j - i < p$ s.t. $DR(V)^i = DR(V)$ and $DR(V)^j = (0)$

filtration

Let $D \subset Cry_s(V) = DR(V)$ be a strongly div lattice,

(finitely generated $\mathcal{O}_K$-submodule of $DR(V)$ s.t.

$$D = \sum p^{-i} f(D^i) \text{ w/ } D^i = D \cap DR(V)^i \;)$$

Let $T \subset V$ be a Gal stable sublattice

Theorem 4.1

Let l and V be as above, and assume $P(V,1) \neq 0$

i) Assume $l \neq p$.

Then $(0) = H^1_e(K,V) = H^1_f($

If V

ii) assume $l = p$ and V is a de Rham representation.

Then $DR(V)/DR(V)^0 \longrightarrow H^1_e(K,V) = H^1_f(K,V)$

iii) assume $l = p$, $K/\mathbb{Q}_p$ unramified

V = crystalline rep. and $\circledast$ holds

$\circledast$ $\exists$ $i \leq 0$ & $j \geq 1$ w/ $j - i < p$ s.t. $DR(V)^i = DR(V)$

Theorem 4.2

Let K be an unramified finite extension of $\mathbb{Q}_p$, p = odd prime, and let $r \geq 2$.

The Haar measure μ on $H^1(K, \hat{\mathbb{Z}}(r))$ induced by Haar measure on K

$$\exp : K \xrightarrow{\sim} H^1(K, \hat{\mathbb{Z}}(r)) \otimes \mathbb{Q}$$

Then

$$\mu(H^1(K, \hat{\mathbb{Z}}(r))) = (1 - q^{-r}) |(r-1)!|_K \cdot \# H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(1-r)).$$

where q the order of the residue field of K and $|\cdot|_K$ the normalized absolute value of K.

A filtered Dieudonne module / O_K is an O_K-module D of finite type endowed with:

a) $(D^i)_{i \in \mathbb{Z}}$ where the D^i are direct fact of D

b) a family of frob lin map $f_i : D^i \to D$

$D^i = D \quad \forall i \ll 0$

5. Global conj

$\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ = finite adel of $\mathbb{Q}$

$$\begin{array}{ccc} \mathcal{O}_K & - & K \\ | & & | \\ \mathbb{Z} & - & \mathbb{Q} \end{array}$$

v = ple of K & K_v = completion

Def 5.1

Let $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell, \hat{\mathbb{Z}}$ OR $\mathbb{A}_f$

$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$

T: free Λ-module of finite rank endowed w/ cont Λ-linear act of $\mathrm{Gal}(\bar{K}/K)$

$\forall\ \emptyset \neq U \subset \mathrm{Spec}(\mathcal{O}_K)$, define

$$H^1_{f,U}(K,T) \subset H^1(K,T)$$

$H^1_{f,U}(K,T)$ = set of cohomolog class whose imag in $H^1_f(K_v, T)$ $\forall$ finite place $v \in U$
$H^1_g(K_v, T)$ $\forall$ finite place $v \notin U$.

Define $H^1_g(K,T) = \varinjlim_U H^1_{f,U}(K,T)$

- If $\Lambda = \mathbb{Z}_\ell$, and if T unramified on U & $\ell \notin U$,

we regard $H^1_{f,U}(K,T)$ as a sub-$\mathbb{Z}_\ell$-module of

$$H^1(U, T) = \varprojlim H^1(U_{et}, T/\ell^n T)$$

Conjecture relating

$H^1_{f.u}$, H^1_g & K-theory.

X = smooth, proj scheme

Let $m, r \in \mathbb{Z}$ fixed

$$V = \begin{cases} gr^r(K_{2r-m-1}(X) \otimes \mathbb{Q}) & \text{if } m \neq 2r-1 \\ (CH^r(X) \otimes \mathbb{Q})_{hom \sim 0} & \text{if } m = 2r-1 \end{cases}$$

w/ gr taken w.r.t the γ-filtration

Def 5.5

A motivic pair (V, D) a pair of f.d. $\mathbb{Q}$-v.s. w/ i) – iii) below

i) $V \otimes \mathbb{A}_f$ has cont $\mathbb{A}_f$-linear Gal action s.t. $V \subset V \otimes \mathbb{A}_f$ stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) < \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

ii) D has decreasing filtration $(D^i)_{i \in \mathbb{Z}}$ by $\mathbb{Q}$-subspaces s.t.

$D^i = (0) \ \forall i \gg 0$

$D^i = D \ \forall i \ll 0$

iii) $\forall p < \infty$ we are given an iso of $\mathbb{Q}_p$-v.s.

$\Theta_p : D_p \cong DR(V_p)$ preserving filtration

$D_p = D \otimes \mathbb{Q}_p$ $\quad$ $V_p = V \otimes \mathbb{Q}_p$ & $DR(-)$ defined w.r.t. the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

When $p = \infty$,

$\Theta_\infty : D_\infty \cong (V_\infty \otimes_{\mathbb{R}} \mathbb{C})^+$

$D_\infty = D \otimes \mathbb{R}$ $\quad$ $V_\infty = V \otimes \mathbb{R}$ $\quad$ $(-)^+ = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-fixed part

action of $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $V_\infty \otimes \mathbb{C}$ is $\sigma \otimes \sigma$

Subject to the following axioms:

P1) $\exists\, \emptyset \neq U \subset \mathrm{Spec}(\mathbb{Z})$ open s.t. $\forall p \in U$,

$V_\ell = V \otimes \mathbb{Q}_\ell$ unramified at p $\forall \ell \neq p$

$V_p = V \otimes \mathbb{Q}_p$ crystalline

P2] M a $\mathbb{Z}$-lattice in V

L a $\mathbb{Z}$-lattice in D

Then $\exists$ finite set $S \subset \mathrm{Spec}(\mathbb{Z})$, [bad prime]

$\infty \in S$ and that $\forall p \notin S$, V_p = crystalline rep of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

P3] Let $p < \infty$, let $P_p(V_\ell, u)$ be the polynomial $P(V_\ell, u)$ of §4.

define for the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module V_ℓ.

Then $P_p(V_\ell, u) \in \mathbb{Q}[u]$ $\forall \ell$ and these polynom are independent of ℓ

P4] If $p < \infty$, $\exists$ $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-stable $\mathbb{Z}$-lattice

$T \subset V \otimes \mathbb{A}_f$ s.t. $H^0 | \mathbb{Q}$

Definition 5.5.1

A moli pair (V, D) weight $\leq w$ if $\forall p < \infty$, the polynomial $P_p(V, u)$ ha the form $\prod(1-\alpha_i u)$

Suppose (V, D) h weight $\leq w$, and S is a finite set of pl of $\mathbb{Q}$ cont ∞,

The L-function $L_S(V, s)$ defined by

$$L_S(V,s) = \prod_{p \notin S} P_p(V, p^{-s})^{-1}$$

Product $< \infty$ absolutely $Re(s) > w/2 + 1$.

Let (V, D) be a mo p w/ weig ≤ -1.

Fix a $\mathbb{Z}$-lattice $M \subseteq V$ s.t. $M \otimes \hat{\mathbb{Z}}$ is $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-stable in $V \otimes \mathbb{A}_f$.

(Note $\mathbb{Z}$-latt in V are in one-t-one correspondence w/ $\hat{\mathbb{Z}}$-lattice in $V \otimes \mathbb{A}_f$

We define groups $A(\mathbb{Q}_p)$ $\forall p \leq \infty$ associated t M,

analogue to group of $\mathbb{Q}_p$-rational pai of a commutative alg group

A = only notation

When (V,D) has weights ≤ -3, we define the Tamagawa m on

$$\prod_{p\le\infty} A(\mathbb{Q}_p)$$

Let

$\boxed{5.6}$ $$A(\mathbb{Q}_p) = \begin{cases} H^1_f(\mathbb{Q}_p, M\otimes\hat{\mathbb{Z}}) & \text{if } p<\infty \\ ((D_\infty\otimes_{\mathbb{R}}\mathbb{C})/((D^0_\infty\otimes_{\mathbb{R}}\mathbb{C})+M))^+ & \text{if } p=\infty. \end{cases}$$

(The inclusion $M \hookrightarrow D_\infty\otimes_{\mathbb{R}}\mathbb{C}$ giv by the identification $D_\infty\otimes_{\mathbb{R}}\mathbb{C} = V_\infty\otimes\mathbb{C}$)

We regard $A(\mathbb{Q}_p)$ f $p<\infty$ as a compact group w/ the natural topolo

$A(\mathbb{R})$ = locally compact group.

For $p\le\infty$, we have

$\exp: D_p/D_p^0 \dashrightarrow A(\mathbb{Q}_p)$ local iso defined on a neigh of

Indeed, $p<\infty$, hypoth $w\le -1 \Rightarrow P_p(V,1)\ne 0$, so

$\boxed{5.7}$ $\exp: D_p/D_p^0 \cong H^1_f(\mathbb{Q}_p, V_p)$

From (P3) & (P4), we easily

$$A(\mathbb{Q}_p)/H^1_f(\mathbb{Q}_p, M\otimes\mathbb{Z}_p) \cong \prod_{l\ne p} H^1(\mathbb{Q}_p, M\otimes\mathbb{Q}_l/\mathbb{Z}_l) \quad \text{finite,}$$

So $\boxed{5.7}$ a local iso.

$p=\infty$, def exp map

Assume weights ≤ -3.

Define Tamagawa # on $\prod A(\mathbb{Q}_p)$ as foll

Fix an iso

$$\omega : \det_{\mathbb{Q}}(D/D^0) \cong \mathbb{Q}$$

$\forall p \leq \infty \Rightarrow$

$$\det_{\mathbb{Q}}(D_p/D_p^0) \cong \mathbb{Q}_p$$

Trivialization of determ $\Rightarrow$ a Haar measure on th

By [4.1], suff large set S of pl of $\mathbb{Q}$, $\infty \in S$, we have for $p \neq S$

$$\mu_{p,\omega}(A(\mathbb{Q}_p)) = P_p(V,1).$$

Since the weights are ≤ -3, the product

$$L_S(V,0)^{-1} = \prod_{p \notin S} \mu_{p,\omega}(A(\mathbb{Q}_p)) < \infty$$

So the product meas $\mu = \prod_{p \leq \infty} \mu_{p,\omega}$ on $\prod_{p \leq \infty} A(\mathbb{Q}_p)$ is def

Def 5.7

μ is the Tamag meo mod (D, V)

If we only assume the weight of (D,V) are ≤ -1, we can def the

Tamagawa meas μ on $\prod A(\mathbb{Q}_p)$ if we assume $L_S(V,s)$ can be analytically continued to $\mathrm{Re}(s) > -\varepsilon$ $\exists \varepsilon > 0$

Indeed, let $r = \mathrm{ord}_{s=0} L_S(V,s)$ and define

$$\mu = |\lim s^r L_S(V,s)^{-1}| \cdot \prod_{p \in S} \mu_{p,w} \cdot \prod_{s \notin S} (P_p(V,1)^{-1} \mu_{p,w})$$

Hope that a motif $/\mathbb{Q} \Rightarrow$ motivic pair

Consid the pure motif $H^m(X)(r)$, X = smooth, prop. scheme $/\mathbb{Q}$

Define

$V = H^n(X(\mathbb{C}), \mathbb{Q}(2\pi i)^r)$; $D = H^m_{DR}(X/\mathbb{Q})$

$V \otimes \mathbb{A}_f \cong H^m_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{A}_f)(r)$ where the gal action

Filtration on D is deduced from the Hodge filtration on H_{DR} by

$D^i = \mathrm{Fil}^{r+i} H^m_{dR}(X/\mathbb{Q})$

The iso $\theta_\infty : D_\infty \cong (\mathbb{C} \otimes_{\mathbb{R}} V_\infty)^+$ is standard

De Rham coh of Fal scheme $X_{\mathbb{Q}_p}$, $p<\infty$ proved by Faltings

$\exists$ canonical iso $\Theta_p : D_p \cong DR(V_\infty)$

Work of Fontaine & Messing show (V, D) has the property that

(P1) & (P2). (P3) $\overset{\text{hold}}{\vee}$ almost $\forall p$

(P4) holds if $m=1$.

<u>Global points</u>

$A(\mathbb{Q})$. Let (V, D) be a mod pr of weight ≤ -3, given a finite dim $\mathbb{Q}$-v.s. Φ endowed w/ an iso of $\mathbb{R}$-v.s

$$R_\infty : \Phi \otimes \mathbb{R} \cong D_\infty / (D_\infty^{\circ} + V_\infty^{+})$$

and an iso of $\mathbb{A}_f$-module

$$R_{Gal} : \Phi \otimes \mathbb{A}_f \cong H^1_{f, \mathrm{Spec}(\mathbb{Z})}(\mathbb{Q}, V \otimes \mathbb{A}_f)$$

Fix a lattice M in V s.t. $M \otimes \hat{\mathbb{Z}}$ is gal stable in $V \otimes \mathbb{A}_f$.

Define $A(\mathbb{Q}) \subset H^1_{f, \mathrm{Spec}(\mathbb{Z})}(\mathbb{Q}, M \otimes \hat{\mathbb{Z}})$

$$\parallel$$

$$R_{gal}^{-1}(R_{gal}(\Phi))$$

Note that $A(\mathbb{Q})$ is a finitely generated ab group

$A(\mathbb{Q}) \otimes \hat{\mathbb{Z}} = H^1_{f, \mathrm{Spec}(\mathbb{Z})}(\mathbb{Q}, M \otimes \hat{\mathbb{Z}})$ and $A(\mathbb{Q}) \otimes \mathbb{Q} = \Phi$

Lem

i) $A(\mathbb{Q})$

ii) $A(\mathbb{Q}_p)_t \cong H^0(\mathbb{Q}_p, M\otimes \mathbb{Q}/\mathbb{Z})$ $p\le\infty$

Proof Exe.

$\exists$ natural homo $A(\mathbb{Q}) \longrightarrow A(\mathbb{Q}_p)$ $p<\infty$, as well as $A(\mathbb{Q}) \to A(\mathbb{R})/A(\mathbb{R})_{cpt} = D_\infty/(D_\infty^0 + V_\infty^+)$

where $A(\mathbb{R})_{cpt}$ = max compact sub of $A(\mathbb{R})$.

ii) above f $p=\infty$ $\Rightarrow$ $A(\mathbb{R})_{tor} \cong H^0(\mathbb{R}, M\otimes \mathbb{Q}/\mathbb{Z})$

$\Rightarrow$ $A(\mathbb{Q})_t \subset A(\mathbb{R})_{t}$

So we can choose $h: A(\mathbb{Q}) \longrightarrow A(\mathbb{R})$ lift ab map.

$\Rightarrow$ If (V,D,Φ) come from a motif, a canonical lift

Define $\quad T_{am}(M) = \mu(\prod A(\mathbb{Q}_p)/A(\mathbb{Q}))$

hyp $\Rightarrow$ that the im of $A(\mathbb{Q})$ in $A(\mathbb{R})/A(\mathbb{R})_{cpt}$ discrete &

cocompact, $\Rightarrow T_{am}(M)$ defined

When (V,D) associal

$\Phi = \mathrm{Image}(g_r^W(K_{2r-m-1}(\mathcal{X}\otimes\mathbb{Q}) \longrightarrow$

Let (V, D, Φ, M)

map $\alpha_M : \dfrac{H^1(\mathbb{Q}, M\otimes\mathbb{Q}/\mathbb{Z})}{A(\mathbb{Q})\otimes\mathbb{Q}/\mathbb{Z}} \longrightarrow \bigoplus \dfrac{H^1(\mathbb{Q}_p, M\otimes\mathbb{Q}/\mathbb{Z})}{}$

Define $Ш(M) = \mathrm{Ker}(\alpha_M)$

Prop 5.14

i) $\forall$ prime number l, the l-p part $Ш(M)[l]$ finite

ii) $\mathrm{Cok}(\alpha_M)$ finite and

iii) Assume $Ш(M)$ finite, and def $\chi(M) \in \mathbb{R}^\times$ by

$$\chi(M) = \mathrm{Tam}(M) \cdot \#(Ш(M)) \cdot \#(H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))^{-1}.$$

then $\forall \mathbb{Z}$-lattice $M' \subset V$ s.t. $M'\otimes\hat{\mathbb{Z}}$ is gal stable in $V\otimes\mathbb{A}_f$, we have $Ш(M')$ al

Conjecture
(Tam # conjecture)

Assume triple (V, D, Φ) come f motif.

Let M be a $\mathbb{Z}$-latt in V s.t. $M\otimes\hat{\mathbb{Z}}$ is gal stable in $V\otimes\mathbb{A}_f$

Then Ш(M) finite, and

$$\mathrm{Tam}(M) = \frac{\#(H^0(\mathbb{Q}, M^\vee \otimes \mathbb{Q}/\mathbb{Z}(1))}{\#(Ш(M))}$$

By (5.14), validity of conj is indeped of the ch of M.

If (V, D, Φ) com from $H^m(X)(r)$, we can take

$M = H^m(X(\mathbb{C}), \mathbb{Z}(2\pi i)^r)/t$

However, when the motive is only the image of a project,
∃ frequently NO canonical choice f M so it is nice to have conj
"isog invariant"

This conj can be rewritten to emph

$$L_S(V, 0) = \frac{\#(Ш(M))}{\#(H^0(\mathbb{Q}, M^\vee \otimes \mathbb{Q}/\mathbb{Z}(1)) \mu_{\infty,\omega}(A(\mathbb{R})/A(\mathbb{Q})) \cdot \prod_{p \in S - \infty} \mu_{p,\omega}(A(\mathbb{Q}_p))}$$

w/ S sufficiently large (dep on ω as well as (V, D, Φ, M))

<u>Remark 5.15.2</u>

Suppose (V, D, Φ, M) corresponds to $H^m(X)(r)$ w/ $m - 2r \leq -3$ and X smooth and prop /$\mathbb{Q}$.

Problem of even def

Finite of Ш(M) & cocom (P4)

$$\Phi \otimes \mathbb{R} \cong D_\infty / (D^0_\infty + V^+_\infty);$$

$$\Phi \otimes \mathbb{Q}_l \cong \mathrm{Ker}\left(H^1(\mathbb{Q}, V_l) \to H^1(\mathbb{Q}_l, V_l) / H^1_f(\mathbb{Q}_l, V_l) \oplus \prod_{p \neq l} H^1(\mathbb{Q}_p, V_l)\right).$$

Note $P_l(V_l, 1) \neq 0$ holds at least if X is proj & has good reduction at l.

Define the l-Tamagawa #

$$\mathrm{Tam}^{(l)}(M) \in \mathbb{R}^\times / \mathbb{Z}^\times_{(l)}$$

we use the groups

$$A^{(l)}(\mathbb{Q}_p) \overset{\mathrm{def}}{=} \begin{cases} H^1_f(\mathbb{Q}_l, M \otimes \mathbb{Z}_l) & \text{if } p = l \\ H^1(\mathbb{Q}_l, M \otimes \mathbb{Z}_l)_{L} & \text{if } p \neq l, p < \infty \end{cases}$$

One can show finit of

$$Ш^{(l)}(M) \overset{\mathrm{def}}{=} \mathrm{Ker}\left(\frac{H^1(\mathbb{Q}, M \otimes \mathbb{Q}_l/\mathbb{Z}_l)}{\mathrm{image}(\Phi)} \to \frac{H^1(\mathbb{Q}_l, M \otimes \mathbb{Q}_l/\mathbb{Z}_l)}{H^1_f(\mathbb{Q}_l, M \otimes \mathbb{Z}_l) \otimes \mathbb{Q}/\mathbb{Z}} \bigoplus_{p \neq l} H^1(\mathbb{Q}_p, M \otimes \mathbb{Q}_l/\mathbb{Z}_l)\right)$$

and we can actually ask if

$$\mathrm{Tam}^{(l)}(M) \cdot \# Ш^{(l)}(M) \cdot \# H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}_l/\mathbb{Z}_l(1)) = 1 \quad \text{in } \mathbb{R}^\times/\mathbb{Z}^\times_{(l)}.$$

no proof of (5.14)

finit of $Ш\{l\}$ & $\mathrm{Coker}(\alpha_M)\{l\}$ foll

<u>Lem 5.16</u>

Let l = prime, U = non-empty open set of $\mathrm{Spec}(\mathbb{Z})$ NOT cont l,

T = free $\mathbb{Z}_\ell$-mod of finite rank w/ cont. action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and $V = T \otimes \mathbb{Q}_\ell$

Assume (a)–(d) below hold:

a) V is unramified on U

b) V a de Rham rep of $Gal(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$

c) $P_p(V,1) \neq 0 \ \forall p \notin U,\ p \neq \infty$.

d) $P_p(V(-1), 1) \neq 0 \ \forall p \in U$.

$$(5.16.1) \qquad \alpha_T : \frac{H^1(\mathbb{Q}, T\otimes \mathbb{Q}/\mathbb{Z})}{H^1_{f,\,Spec(\mathbb{Z})}(\mathbb{Q}, T)\otimes \mathbb{Q}/\mathbb{Z}} \longrightarrow \bigoplus_{p\leq\infty} \frac{H^1(\mathbb{Q}_p, T\otimes \mathbb{Q}/\mathbb{Z})}{H^1_f(\mathbb{Q}_p, T)\otimes \mathbb{Q}/\mathbb{Z}}$$

$$\beta_T : H^2(\ ,\ T\otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{p\leq\infty} H^2(\mathbb{Q}_p, T\otimes \mathbb{Q}/\mathbb{Z})$$

When $p=\infty$, $H^1_f(\mathbb{Q}_p, T)\otimes \mathbb{Q}/\mathbb{Z} = 0$

Then $Ker(\alpha_T)$ is finite, β_T is surjective,

$Coker(\alpha_T)$ and $Ker(\beta_T)$ are of co-finite type

$Coker(\alpha_T) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus (\text{finite})$

$Ker(\beta_T) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus$ finite

$H^1_{f,\,Spec(\mathbb{Z})}(\mathbb{Q}, V) = \dim(V) - \dim(DR(V$

<u>Proof</u> of 5.16

Let $\alpha_{T,U} : \dfrac{H^1(U, T\otimes\mathbb{Q}/\mathbb{Z})}{} \longrightarrow \bigoplus_{p\notin U} \dfrac{H^1(\mathbb{Q}_p, T\otimes\mathbb{Q}/\mathbb{Z})}{}$

$$\beta_{T,U} : H^2(U, T \otimes \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{p \notin U} H^2(\mathbb{Q}_p, T \otimes \mathbb{Q}/\mathbb{Z})$$

$$\operatorname{Ker}(\alpha_{T,U}) \cong \operatorname{Ker}(\alpha_T)$$

$$0 \to \operatorname{Coker}(\alpha_{T,U}) \to \operatorname{Coker}(\alpha_T) \to \operatorname{Ker}(\beta_{T,U}) \to K(\beta_T) \to 0$$

$$\operatorname{corank}(\operatorname{Coker}(\alpha_T)) + \operatorname{corank}(\operatorname{Ker}(\beta_T)) = \operatorname{corank}(\operatorname{Coker}(\alpha_{T,U})) + \operatorname{corank}(\operatorname{Ker}(\beta_{T,U}))$$

$$= \sum_{i=0}^{2} (-1)^i \operatorname{corank} H^i(U, T \otimes \mathbb{Q}/\mathbb{Z}) - \sum$$

We have

$$\sum (-1)^i \operatorname{corank} H^i(U, T \otimes \mathbb{Q}/\mathbb{Z}) = \dim(V^+) - \dim(V)$$

Conj 6.2

The image of $c_r(\alpha)$ & $n^{1-r}c_{r,n}(\alpha)$ coincide up to sign in $H^1(\mathbb{Q}(\alpha), A_f(r))$

Lemma 6.3

Assume $r > 2$

i) $$\mathrm{Tam}(\mathbb{Z}(r)) = \pm \frac{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))}{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))} \cdot \frac{2}{\zeta(1-r)}$$

(ii) If r is odd, $$\mathrm{Tam}(\mathbb{Z}(r)) = \frac{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))}{\chi(A(\mathbb{Q}) : \mathbb{Z} \cdot c_r(r))}$$

Lemma 6.3

Assume $r \geq 2$

i) $\mathrm{Tam}(\mathbb{Z}(r)) = \pm \dfrac{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))}{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))} \cdot \dfrac{2}{\zeta(1-r)}$

ii) If r odd, $\mathrm{Tam}(\mathbb{Z}(r)) = \pm \dfrac{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))}{\chi(A(\mathbb{Q}) : \mathbb{Z} \cdot c_r(1))}$

Here $\mathbb{Z} \cdot c_r(1) \subset A(\mathbb{Q}) \otimes \mathbb{Q}$ the free abe group generated by $c_r(1)$

and by def

$\chi(A(\mathbb{Q}) : \mathbb{Z} \cdot c_r(1)) = [A(\mathbb{Q}) : L] / [\mathbb{Z} \cdot c_r(1) : L]$

$\forall$ free subg L of $A(\mathbb{Q})$ of finite index wh.

Proof

Take $1 \in \mathbb{Q} = H^0_{dR}(\mathrm{Spec}(\mathbb{Q}))$ canonical base.

and let μ_p be the correspond measure on $A(\mathbb{Q}_p)$ $\forall p \leq \infty$

If r = even, $A(\mathbb{R}) = \mathbb{R}/(2\pi)^r \mathbb{Z}$ w/ Leb mea μ_∞, so we have

$\mathrm{Tam}(\mathbb{Z}(r)) = (\# A(\mathbb{Q}))^{-1} \prod_{p \leq \infty} \mu_p(A(\mathbb{Q}_p))$

$= \dfrac{1}{\# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))} \cdot (2\pi)^r \cdot \#$

If r odd, $A(\mathbb{R}) = A(\mathbb{C})^+ = (\mathbb{C}/(2\pi i)^r\mathbb{Z})^+ \cong \mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z}$

$$A(\mathbb{Q}) \hookrightarrow A(\mathbb{R})$$

and the calculat reads

$$T_{am}(\mathbb{Z}(r)) = \mu\Big(\prod_{p\le\infty} A(\mathbb{Q}_p)/A(\mathbb{Q})\Big)$$

$$= \mu_\infty(A(\mathbb{R})/A(\mathbb{Q}))\ \#H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))\ ((r-1)!\,\zeta(r))^{-1}$$

$$= \#H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))\ \chi(A(\mathbb{Q}) : \mathbb{Z}\cdot c_r(1))^{-1}$$

Note that we did NOT use conj 6.1.

WTS Thm 6.1, we must show

(6.5) $\zeta(1-r) = 2\cdot\#(Ш)\cdot\#H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))^{-1}$ r even

(6.6) $\#(Ш) = \chi(A(\mathbb{Q}) : \mathbb{Z}\cdot c_r(1))$ r odd

6. Rieman ζ-function

Theorem 6.1

Let $r \geq 2$ be given

i) If r = even, then TNC is true modulo a pow of 2 f the motif $\mathbb{Q}(r)$

ii) r = odd, then TNC true modulo a po of 2 f the motif $\mathbb{Q}(r)$ if the conj (6.2) below true in the case $\alpha = 1$.

Unfortunate power of 2 ambiguit

Compatib.

Conj 6.2

Lem 6.7

If $p \neq 2$, the p-pr part of Ш is iso to $H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(r))$

If r = even, p-p part of Ш also is to $H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))$

Proof Consid the CD of E.S.

Ш{p} Ш{p}

$$
\begin{tikzcd}
 & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] & & \\
0 \arrow[r] & \dfrac{H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))}{A(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \arrow[r] \arrow[d, "s"] & \dfrac{H^1(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p(r))}{A(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \arrow[r] \arrow[d, "t"] & \bigoplus_{l \neq p} H^0(\mathbb{F}_l, \mathbb{Q}_p/\mathbb{Z}_p(r-1)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{H^1(\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(r))}{A(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \arrow[r] & S \arrow[r] & \bigoplus_{l \neq p} H^0(\mathbb{F}_l, \mathbb{Q}_p/\mathbb{Z}_p(r-1)) \arrow[r] & 0
\end{tikzcd}
$$

where

$$S = \bigoplus_{l} \frac{H^1(\mathbb{Q}_l, \mathbb{Q}_p/\mathbb{Z}_p(r))}{A(\mathbb{Q}_l) \otimes \mathbb{Q}_p/\mathbb{Z}_p} = \frac{H^1(\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(r))}{A(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \oplus \bigoplus_{l \neq p} H^0(\mathbb{F}_l, \mathbb{Q}_p/\mathbb{Z}_p(r-1))$$

We have

$$Ш\{p\} \cong \operatorname{Ker}(H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(r)) \longrightarrow H^2(\mathbb{Q}_p, \mathbb{Z}_p(r)) \longrightarrow H^2(\mathbb{Q}_p, \mathbb{Z}_p(r))$$

map on the

$$H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(r)) \longrightarrow H^2(\mathbb{Q}_p, \mathbb{Z}_p(r)) \longrightarrow H^0(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(1-r))^* \longrightarrow 0$$

Proof of (6.5) $\zeta(1-r) = 2 \cdot \#(Ш) \cdot \# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))^{-1}$ $\quad r$ even

follows from Mazur-Wiles [MW]

$$\zeta(1-r) = \pm \prod_{p<\infty} \frac{\# H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))}{\# H^0(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))} \qquad (r \geq 2 \text{ even})$$

Next consider (6.6) $\#(Ш) = \chi(A(\mathbb{Q}) :: \mathbb{Z} \cdot c_r(1))$ $\quad r$ odd

Let

$G = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$;

$P = \mathrm{Gal}(\mathbb{Q}_p(\zeta_p)^{ab}/\mathbb{Q}_p(\zeta_{p^\infty}))$

$X = \pi_1(\mathrm{Spec}(\mathbb{Z}[1/p][\zeta_{p^\infty}]))^{a}$

$C =$ the $\hat{\mathbb{Z}}[G]$-submodule of P generated by

$(1-\zeta_{p^n})_n \in \varprojlim \mathbb{Q}_p(\zeta_{p^n})^\times \subset P$.

$\Rightarrow$ we have a canonical G-hom $P/C \to X$.

Consider the foll CD

$$\begin{tikzcd}
0 \arrow[d] & \\
H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(r))^- \arrow[d] & 0 \arrow[d] \\
H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \arrow[r, "a"] \arrow[d] & H^0(G, (P/C)^-(1-r)) \arrow[d] \\
H^1(\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \arrow[r, "b"] \arrow[d] & H^0(G, P^-(1-r)) \arrow[d] \\
{} & {}
\end{tikzcd}$$

$$\begin{array}{ccc} H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(1)) & \xrightarrow{c} & H^0(G, C^*(1-r)) \\ \downarrow & & \\ H^2(\mathbb{Z}[1/p]) & & \end{array}$$

* dual dual $= \mathrm{Hom}_{cont}(\ , \mathbb{Q}_p/\mathbb{Z}_p)$

So $P^* = \mathrm{Hom}_{cont}(P, \mathbb{Q}_p/\mathbb{Z}_p)$

$= H^1(\mathbb{Q}_p(\zeta_{p^\infty}), \mathbb{Q}_p/\mathbb{Z}_p)$

$\Rightarrow$ the map a fact as $a = a_1 \circ a_2$

$$H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \overset{a_1}{\cong} H^1(G, X^*(1-r)) \xrightarrow{a_2} H^1(G, (P/C)^*(1-r))$$

w/ a_1 bijective

note $X^* = \mathrm{Hom}_{cont}(\pi_1(\mathrm{Spec}(\mathbb{Z}[1/p][\zeta_{p^\infty}]))^{ab}, \mathbb{Q}_p/\mathbb{Z}_p)$

$= H^1(\mathbb{Z}[1/p][\zeta_{p^\infty}], \mathbb{Q}_p/\mathbb{Z}_p)$

b also bijective, and c the surjection induced by

$$\varprojlim \mathbb{Z}[1/p][\zeta_{p^n}]^\times \longrightarrow \varprojlim H^1(\mathbb{Z}[1/p][\zeta_{p^n}], \mathbb{Z}_p(1))(1-r)$$

$$\xrightarrow{Norm} H^1(\mathbb{Z}[1/p][\zeta_p], \mathbb{Z}_p(1)(1-r))$$

Finally, d fact as

$$H^2(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \overset{d_1}{\cong} H^1(G, X^*(1-r)) \xrightarrow{d_2} H^1(G, (P/C)^*(1-r))$$

w/ d_2 bijective

By Mazur-Wiles [MW], the kernel & cokernel of

$$(P/C)^{+}(p) \longrightarrow X^{+}(p)$$

§7. CM EC

For an EC$/\mathbb{Q}$ w/ CM, we reduce the l-pr part of our Tama # conj f $H^1(E)(2)$

i.e. f $L(H^1(E),2)$ where E such an EC. to the question

$$\dim gr^2(K_2(E)\otimes\mathbb{Q}) \overset{?}{=} 1$$ to a certa. prob on the Gal coho of the

7.1 we rev the "cyclo elements" in K_2 of ECs

k = field and E an EC$/k$.

For $a, n \in \mathbb{Z}\setminus\{0\}$ s.t. $(a,n)=1$ and all points in ${}_aE \cup {}_nE$ are k-rational

here ${}_bE = \mathrm{Ker}(b : E \to E)$ $\Rightarrow$ ${}_aE \cup {}_nE = \mathrm{Ker}(a : E \to E) \cup \mathrm{Ker}(b : E \to E)$

$\forall \beta \in {}_nE - \{0\} = \mathrm{Ker}(n : E \to E) - \{0\}$,

define $C_n^a(\beta) \in \Gamma(E_{\mathrm{Zar}}, K_2)/K_2(k)$

as foll

Take g, s, t_γ ($\gamma \in \mathrm{Ker}(a : E \to E)\setminus\{0\}$) on E s.t.

$\mathrm{div}(g) = a^2(0) - {}_aE$,

$\mathrm{div}(s) = N(\beta) - n(0)$

$\mathrm{div}(t_\gamma) = a(\gamma) - a(0)$,

Then,

$$C_n^a(\beta) \overset{\mathrm{def}}{=} a\{g(\beta)^{-1}g, s\} - \sum_{\substack{a\gamma=0 \\ \gamma\neq 0}} \{s(\gamma).t_\gamma\} \in K_2(k(E))/K_2(k)$$

easy check

tame symbol map

$$K_2(k(E))/K_2(k) \longrightarrow \bigoplus_{\substack{x \in E \\ \text{closed}}} k(x)^\times$$

$$\mathrm{Tam}(\mathbb{Z}(r)) = \mu\Big(\prod_{p \le \infty} A(\mathbb{Q}_p)/A(\mathbb{Q})\Big)$$

$$= \mu_\infty(A(\mathbb{R})/A(\mathbb{Q})) \ \# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r)) \ ((r-1)!\, \zeta(r))^{-1}$$

$$= \# H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r)) \ \chi(A(\mathbb{Q}) \cdot \mathbb{Z} \cdot c_0(r))^{-1}$$

Recall $\dim V = 2$

$$H^1(G, S) = \frac{\ker(}{\quad}$$

1. Special val of ζ-function.

The zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ $\sigma > 1$

where $s = \sigma + it$ $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$

1. (Euler product)

$\sigma > 1$ we have

$\zeta(s) = \prod_p (1 - 1/p^s)^{-1}$.

product over all p.

2. (Analy cont)

$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \gamma_k (s-1)^k$, γ = Euler constant.

3. (Functi

$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$

where $\Gamma(s)$ the usual Γ-function.

Let $\zeta_\infty(s) = \pi^{-s/2} \Gamma(s/2)$. and def the completed zeta funct by

$\Lambda(s) := \zeta_\infty(s) \zeta(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$

Critical val of $\zeta(s)$

Bernoulli #s are def by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

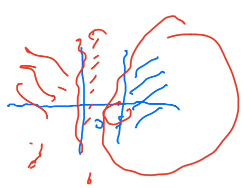
$B_0 = 1$

$B_1 = -1/2$

$B_2 = 1/6$

$B_3 = 0$

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$



2. K-theo Backgrou

Let G = top group

cl. space BG of G = the

Theorem 2.1.1

Let (X, x) be a path connected space, N a perfect n

$\phi : (X, x) \to (X^+, x^+)$

a) $\exists$ an exact seq

$0 \to N \to \pi_1(X, x) \to$

b) $\forall$ local coeff system L on X^+, the induced homo

$\phi_* : H_n(X, \phi^* L) \to H_n(X^+, L)$

an iso $\forall\, n \geq 0$.

c) If $g : (X, x) \to (Y, y)$ a cts map s.t. $N \subset \mathrm{Ker}(g_*)$.

R = ring

$GL_n(R)$ = group of $(n \times n)$ mat $/R$ that are invertible

$\forall$ matrix g is naturally viewed as an $(n+1) \times (n+1)$ matrix g by $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$

$GL_n(R) \hookrightarrow GL_{n+1}(R)$

Group $GL(R)$ defined

$\bigcup_n GL_n(R)$ called the infinite linear group $/R$.

1.2.1

A motive / $\mathbb{Q}$ w/ coeff al in $\mathbb{Q}$ is a five tuple of data

$$M = (M_B, M_{dR}, M_f, I_\infty, I_f)$$

where M_B and M_{dR} are finite-dim $\mathbb{Q}$-v.s. both of dim d,

and M_f is an $\mathbb{A}_f$-module of rank d.

$$I_\infty : M_B \otimes_{\mathbb{Q}} \mathbb{C} \to M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$$

$$I_f : M_B \otimes_{\mathbb{Q}} \mathbb{A}_f \to M_f$$

Conjecture 1.2.1

at $s=0$,

$$L(0,M) \sim c^+(M)$$

where $\sim$ means up to an element of $\mathbb{Q}$.

If $m \in \mathbb{Z}$ a critical integ $L(s,M)$

then

$$L_f(m,M) \sim (2\pi i)^{md^\pm} c^\pm(M), \qquad \pm = (-1)^m.$$

Motive for $\zeta(s)$

$n \geq 1$ def the motive

$$\mathbb{Q}(-n) := H^{2n}(\mathbb{P}^n)$$

- Betti realization

$$\mathbb{Q}(-n)_B = H^{2n}_B(\mathbb{P}^n(\mathbb{C}), \mathbb{Q})$$

Tate motive $\mathbb{Q}(1)$

(V, D) of finite-dim $\mathbb{Q}$-

2. K-theory background

G: top group, recall BG the cl space of G is the quotient of a weakly contractible (all of whose homotopy groups are triv) EG by a free action of G.

Principal G-bundle

$$p: EG \to BG$$

which is universal in the sense that if X is of the homotopy type of a CW-complex, and $P \to X$ a principal G-bundle, then $\exists$ cont map $f: X \to BG$ (defined up to homotopy) s.t. P occ as the pullback f^*EG. The spa BG and EG are defined up to homotopy equival, and are connected.

The topology of G is cru when cons the cl space.

Understand the $\mathbb{Z}_p(1)$ case.

Recall the 4 term exact sequence as in my Cyclotomic Fields book.

$H^1_{Iw}(T_pE/F^{cyc}) \longrightarrow$ Elliptic Curve

For $\mathbb{Z}_p(1)$, can similarly describe

$H^1_{Iw}(\mathbb{Z}_p(1)/F^{cyc})$. In here, you have the special element coming from the Euler System. Lets call this special element z, $\langle z \rangle$ is the Iwasawa module generated by this element. $F = \mathbb{Q}$

$$
\begin{tikzcd}
0 \arrow[r] & H^1_{Iw}(\mathbb{Z}_p(1)/\mathbb{Q}^{cyc}) \arrow[r] & H^1_{Iw}(\mathbb{Z}_p(1)/\mathbb{Q}_p^{cyc})/\langle z \rangle \arrow[d, "\varphi"] \\
 & & X_\infty \arrow[d] \\
 & & Y_\infty \arrow[d] \\
 & & 0
\end{tikzcd}
$$

IMC $\Leftrightarrow$ char. ideals of the first & last terms are equal

$\Leftrightarrow$ char. ideals of the middle 2 terms are equal. Factor φ as:

$$
\begin{aligned}
H^1_{Iw}(\mathbb{Z}_p(1)/\mathbb{Q}_p^{cyc}) &\xrightarrow{\mathcal{E}xp} \Lambda \\
z &\longmapsto \mathcal{L}_p
\end{aligned}
$$

Write $\dfrac{H^1_{Iw}(\mathbb{Z}_p(1))}{\mathbb{Q}_p^{cyc}} \xleftrightarrow{\exp} \Lambda/\mathcal{L}_p$

Char. ideal of 2nd term

$$\frac{H^1_{Iw}(\mathbb{Z}_p(1))}{\mathbb{Q}_p^{cyc}} = \mathcal{L}_p$$

IMC $\longleftrightarrow$ $\mathcal{L}_p$ = char. ideal of X_∞

Relation with TNC:

$$H^1_{Iw}(T_pE/\mathbb{Q}^{cyc}) \supseteq \{z\} ; \langle z \rangle$$

$$\left[H^1_{Iw}(T_pE/\mathbb{Q}^{cyc}) / \langle z \rangle \right]$$ torsion as

a $\Lambda(\Gamma)$-module

$$H^1_{Iw}(T_pE/\mathbb{Q}^{cyc})_\Gamma \longrightarrow H^1_{Iw}(T_pE/\mathbb{Q})$$

$$\| $$

$$H^1(F_S/\mathbb{Q}, T_pE)$$

$$\langle z \rangle \longmapsto z_\Gamma$$

$$\chi_\Gamma\left(H^1_{Iw}(T_pE/\mathbb{Q}^{cyc})\right) = \frac{\# H^0_\Gamma}{\# H^1_\Gamma}$$

$H^1_\Gamma = H^2(F_{S/F}, T_pE)$, conjectured to be finite.

$$\# H^0_\Gamma = [H^1(F_S/\mathbb{Q}, T_pE) : z_\Gamma] < \infty$$

TNC $\Longleftrightarrow$ $\#[H^1(F_S/F, T_pE : z_\Gamma]$

$$\|$$

$$\# H^2(F_S/F, T_pE).$$

Guido Kings; 7. Bars

CM case ✓

$$\begin{tikzcd} K \arrow[d, dash] & \mathfrak{p} \arrow[dr, dash] & & \bar{\mathfrak{p}} \arrow[dl, dash] \\ \mathbb{Q} & & p & \end{tikzcd}$$

TNC holds.