Extended Research in Infinitary Algebraic Structures and Advanced Motives - Part III

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1 Advanced Infinitary Structures and Theorems

1.1 New Mathematical Notations

Infinitary Motive Category: The category of infinitary motives $\mathcal{M}_{\text{inf}}$ is equipped with the following notations:
- Mot_{inf}: The collection of infinitary motives.
- $\text{Hom}_{\text{inf}}(M_i, M_j)$: The space of morphisms between infinitary motives M_i and M_j, defined as:

$$\text{Hom}_{\text{inf}}(M_i, M_j) = \bigoplus_{k \in \mathbb{I}} \text{Hom}(M_{i,k}, M_{j,k})$$

Infinitary Cohomology Groups: For an infinitary variety X, the cohomology groups are given by:

$$H^n_{\text{inf}}(X) = \bigoplus_{i \in \mathbb{I}} H^n(X_i)$$

Infinitary L-functions: For a motive $\mathcal{M}$, the infinitary L-function is:

$$L_{\text{inf}}(s, \mathcal{M}) = \prod_{i \in \mathbb{I}} \frac{1}{\det(I - A_i s)}$$

where A_i are operators associated with the motive $\mathcal{M}_i$.

1.2 New Theorems and Proofs

Theorem 1: Infinitary Cohomology and K-Theory

Let $\mathcal{M}_{inf}$ be an infinitary motive category. If $\mathcal{M}_i$ are infinitary motives in $\mathcal{M}_{inf}$, then the infinitary K-theory group $K_0(\mathcal{M}_{inf})$ is given by:

$$K_0(\mathcal{M}_{\text{inf}}) = \bigoplus_{i \in \mathbb{I}} K_0(\mathcal{M}_i)$$

Proof: We will prove this by showing that $K_0(\mathcal{M}_{\text{inf}})$ is a direct sum of the K-groups of its components. By definition:

$$K_0(\mathcal{M}_{\text{inf}}) = \text{Grothendieck Group of Mot}_{\text{inf}}$$

The infinitary K-group can be decomposed as:

$$K_0(\mathcal{M}_{\text{inf}}) = \langle [M_i] \mid i \in \mathbb{I} \rangle$$

where $[M_i]$ denotes the K-theory class of the infinitary motive M_i. Since:

$$\text{Grothendieck Group of Mot}_{\text{inf}} = \bigoplus_{i \in \mathbb{I}} \text{Grothendieck Group of Mot}_i$$

it follows that:

$$K_0(\mathcal{M}_{\text{inf}}) = \bigoplus_{i \in \mathbb{I}} K_0(\mathcal{M}_i)$$

Theorem 2: Infinitary L-functions and Special Values

Let $L_{inf}(s, \mathcal{M})$ be the infinitary L-function for a motive $\mathcal{M}$. If s_0 is a special point in the domain of L_{inf}, then the value $L_{inf}(s_0, \mathcal{M})$ satisfies:

$$L_{\text{inf}}(s_0, \mathcal{M}) = \prod_{i \in \mathbb{I}} L(s_0, \mathcal{M}_i)$$

Proof: To prove this theorem, we use the definition of infinitary L-functions:

$$L_{\text{inf}}(s, \mathcal{M}) = \prod_{i \in \mathbb{I}} \frac{1}{\det(I - A_i s)}$$

At $s = s_0$, this becomes:

$$L_{\text{inf}}(s_0, \mathcal{M}) = \prod_{i \in \mathbb{I}} \frac{1}{\det(I - A_i s_0)}$$

By definition of $L(s_0, \mathcal{M}_i)$ as:

$$L(s_0, \mathcal{M}_i) = \frac{1}{\det(I - A_i s_0)}$$

it follows:

$$L_{\text{inf}}(s_0, \mathcal{M}) = \prod_{i \in \mathbb{I}} L(s_0, \mathcal{M}_i)$$

Theorem 3: Infinitary Moduli Spaces and Geometric Properties

Let $\mathcal{M}_{inf}$ be an infinitary moduli space. If X_i are infinitary varieties in $\mathcal{M}_{inf}$, then the infinitary moduli space $\mathcal{M}_{inf}$ can be decomposed as:

$$\mathcal{M}_{\text{inf}} = \left\langle \bigcup_{i \in \mathbb{I}} \mathcal{M}_i \right\rangle$$

Proof: The moduli space $\mathcal{M}_{\text{inf}}$ is defined as:

$$\mathcal{M}_{\text{inf}} = \text{Union of moduli spaces of Var}_{\text{inf}}$$

where:

$$\text{Var}_{\text{inf}}(X) = \bigcup_{i \in \mathbb{I}} \text{Var}(X_i)$$

Thus:

$$\mathcal{M}_{\text{inf}} = \left\langle \bigcup_{i \in \mathbb{I}} \mathcal{M}_i \right\rangle$$

showing that the infinitary moduli space is indeed a union of its components.

2 References

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