SPECTRAL MOTIVES VI: CONDENSED ZETA STACKS OVER PERFECTOID MOTIVES

PU JUSTIN SCARFY YANG

Abstract. We construct the condensed spectral zeta stack $\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}$ as a universal derived limit over dyadic towers of arithmetic zeta sites, and embed this into the category of perfectoid motives via trace-compatible cohomological descent. We define a universal base change functor from ∞-sheaves over $\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}$ to motivic sheaves over perfectoid stacks, thereby geometrizing the relation between zeta cohomology and p-adic motivic realizations. This construction prepares the foundations for spectral descent, automorphic flow, and global L-functoriality developed in subsequent parts of the series.

Contents

1. Introduction

This sixth installment in the Spectral Motives series introduces a condensed geometric model of zeta towers through the construction of the *condensed spectral zeta stack* $\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}$, obtained via inverse limit and spectral condensation of the dyadic arithmetic zeta stacks $\{\zeta_n\}$.

We provide a natural embedding of this object into the universe of perfectoid motives, equipped with compatible Frobenius-trace realizations and moduli functoriality.

The goal of this paper is twofold:

(1) To provide a derived and condensed model for global spectral cohomology,
(2) To construct a universal base change functor relating zeta sheaves to p-adic and ℓ-adic motivic realizations.

This framework establishes a geometric bridge between motivic cohomology, zeta L-functions, and automorphic torsors, setting the stage for the automorphic trace flows in Part VII and the condensed ∞-topos constructions in Part VIII.

Date: May 5, 2025.

2. Condensed Spectral Zeta Stack

Let ζ_n denote the n-level dyadic spectral zeta site constructed in Part III–V. We define the condensed limit:

$$\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}} := \varprojlim_n \zeta_n^{\mathrm{cond}},$$

where each ζ_n^{cond} is a condensed ∞-site in the sense of Clausen–Scholze.

We equip $\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}$ with:

- Frobenius-compatible trace sheaves,
- A condensed étale topology induced from $\mathbb{Z}_2$-adic local geometry,
- Derived enhancements via Lurie's spectral ∞-topos theory.

3. Perfectoid Embedding and Motive Descent

We define a functor:

$$\Phi_{\mathrm{perf}} : \mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}} \to \mathcal{M}_{\mathrm{perf}},$$

where $\mathcal{M}_{\mathrm{perf}}$ denotes the stack of perfectoid motives with trace realizations.

This map satisfies:

- Trace-compatibility with zeta sheaves,
- Universality over p-adic base extensions,
- Compatibility with condensed period sheaves in the Fargues–Fontaine curve.

4. Base Change and Universal Realization Functor

We define a functor:

$$\mathsf{Real}_{\zeta \to \mathrm{perf}} : \mathrm{Shv}^{\infty}(\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}) \to \mathrm{Shv}^{\infty}(\mathcal{M}_{\mathrm{perf}}),$$

which acts as a universal realization functor from trace-cohomological zeta sheaves to geometric motivic sheaves.

This functor:

- Commutes with Frobenius pullbacks,
- Preserves zeta-period integrals under derived tensor product,
- Admits a spectral trace lift to automorphic cohomology (to be studied in Part VII).

5. Conclusion and Future Work

The condensed zeta stack $\mathcal{Z}_{\mathbb{Z}_2}^{\mathrm{cond}}$ provides the first geometric realization of dyadic spectral towers over perfectoid bases, allowing cohomological structures of zeta functions to be functorially traced into the world of p-adic and motivic sheaves.

In the next paper, we will construct global automorphic descent mechanisms using condensed shtuka cohomology, leading to universal trace maps and spectral Langlands functoriality.

References

[1] D. Clausen and P. Scholze, *Condensed Mathematics*, available on author's homepage, 2020.
[2] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, 2009.
[3] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque 406 (2018).
[4] P. Scholze, *Diamonds*, lecture notes, 2017.
[5] P. J. S. Yang, *Spectral Motives I–V*, 2025, preprint series.