LOCAL ARITHMETIC FUNCTIONS: A PRIMEWISE FRAMEWORK

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Abstract. This work introduces a new framework for arithmetic functions by localizing classical global arithmetic data into prime-indexed components. Termed *local arithmetic functions*, these objects isolate the contribution of each prime p via functions $f_p : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, assembled into global functions by multiplicative or additive aggregation. The formalism naturally extends to define local zeta transforms, primewise convolution rings, and sheaf-theoretic interpretations over $\mathrm{Spec}(\mathbb{Z})$. This approach unifies concepts from analytic number theory, algebraic geometry, and arithmetic cohomology, and offers generalizations to motivic, perfectoid, and p-adic Hodge theoretic settings. Connections to the Yang-number systems provide a potential foundation for future frameworks in arithmetic geometry and motivic analysis.

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1. Local Arithmetic Functions: Definitions and Foundational Framework

1.1. Motivation. In classical analytic number theory, arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ typically reflect global data of the integer n, such as its complete prime factorization. In this section, we initiate the development of a theory of *local arithmetic functions* that isolate and encode information at each individual prime p, forming a primewise analytical structure.

1.2. Primewise Decomposition. Let $\mathbb{P}$ denote the set of all prime numbers. For any $n \in \mathbb{N}$, its prime factorization can be uniquely expressed as:

$$n = \prod_{p \in \mathbb{P}} p^{v_p(n)}$$

where $v_p(n) \in \mathbb{Z}_{\geq 0}$ is the p-adic valuation of n, and only finitely many $v_p(n)$ are non-zero.

1.3. Definition of Local Arithmetic Function.

Definition 1 (Local Arithmetic Function):

A **local arithmetic function** at a prime p is a function

$$f_p : \mathbb{Z}_{\geq 0} \to \mathbb{C}$$

which depends only on the exponent $v_p(n)$ of p in the factorization of n. The collection $\{f_p\}_{p \in \mathbb{P}}$ defines a local arithmetic system.

Definition 2 (Global Function Induced by Local Family):

Given a local system $\{f_p\}$, define the corresponding global arithmetic function $F : \mathbb{N} \to \mathbb{C}$ by either

$$F(n) := \prod_{p^\alpha \| n} f_p(\alpha), \quad \text{or} \quad F(n) := \sum_{p^\alpha \| n} f_p(\alpha)$$

depending on whether a multiplicative or additive aggregate is desired.

1.4. Examples.

Example 1. Local Degree Function: $f_p(k) = k$ leads to $F(n) = \sum_{p|n} v_p(n)$.

Example 2. Local Exponential Function: $f_p(k) = p^k$ recovers the original n, i.e., $F(n) = n$.

Example 3. Local Möbius-Like Function:

$$f_p(k) = \begin{cases} -1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \\ 1 & \text{if } k = 0 \end{cases} \quad \Rightarrow \quad F(n) = \mu(n)$$

1.5. Local Multiplicativity.

Definition 3 (Local Multiplicativity):

A local system $\{f_p\}$ is said to be **locally multiplicative** if the induced global function F satisfies

$$F(mn) = F(m)F(n) \quad \text{whenever } \gcd(m,n) = 1$$

This holds automatically when the f_p depend only on $v_p(n)$ and interact multiplicatively under disjoint support.

1.6. Future Directions. This framework permits generalization to local zeta transforms, convolution algebras on prime-indexed spaces, and interactions with p-adic, adelic, or motivic arithmetic settings. Extensions to new arithmetic structures (e.g., Yang-number systems) may further arise from this foundation.

2. Local Zeta Transforms and Euler Product Structure

Given a local arithmetic system $\{f_p\}_{p\in\mathbb{P}}$, we define its local zeta transform at each prime p as:

$$\zeta_{f_p}(s) := \sum_{k=0}^{\infty} \frac{f_p(k)}{p^{ks}}, \quad \text{for } \Re(s) > \sigma_0$$

where the abscissa of convergence σ_0 depends on the growth of $f_p(k)$.

Definition 4 (Global Zeta Function from Local System):

The global zeta function associated to the local arithmetic system is given by the Euler product:

$$\zeta_{\{f_p\}}(s) := \prod_{p\in\mathbb{P}} \zeta_{f_p}(s)$$

This framework generalizes classical zeta functions such as:

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{corresponding to } f_p(k) = 1$$

By tuning each $f_p(k)$, one may encode twisting, local ramification data, or even deformation parameters.

3. Local Convolution and Ring Structures

3.1. Primewise Convolution. Let $\{f_p\}, \{g_p\}$ be two local arithmetic systems. We define their *local convolution* at prime p as:

$$(f_p * g_p)(k) := \sum_{i=0}^{k} f_p(i) g_p(k-i)$$

The corresponding global function becomes:

$$(F * G)(n) := \prod_{p^\alpha \| n} (f_p * g_p)(\alpha)$$

3.2. Algebraic Structure. The set of all finitely supported local systems forms a commutative ring under:

- Pointwise addition: $(f_p + g_p)(k) := f_p(k) + g_p(k)$
- Primewise convolution product as above

This ring structure reflects a graded algebra over $\mathbb{P} \times \mathbb{Z}_{\geq 0}$.

3.3. Connections to Analytic, Algebraic, and Geometric Structures.

- **Analytic**: The zeta transforms define meromorphic functions whose poles/zeros may reflect classical or generalized L-functions.
- **Algebraic**: The convolution ring may be studied as a prime-indexed module with multiplicative grading, interpretable via formal group laws.
- **Geometric**: Local systems can be viewed as stalks of sheaves over $\mathrm{Spec}(\mathbb{Z})$, yielding arithmetic sheaves.
- **Cohomological**: Taking derived functors or cohomology of such sheaves leads to novel arithmetic invariants and obstructions.

4. Sheaf-Theoretic Interpretation over $\mathrm{Spec}(\mathbb{Z})$

4.1. Arithmetic Sheaves via Local Systems. We reinterpret the local arithmetic system $\{f_p\}_{p\in\mathbb{P}}$ as defining a presheaf of arithmetic data over $\mathrm{Spec}(\mathbb{Z})$, where each point $p \in \mathbb{P}$ corresponds to a closed point $\mathfrak{p} \subset \mathrm{Spec}(\mathbb{Z})$.

For each p, define a stalk:

$$\mathscr{F}_p := \{f_p(k)\}_{k\geq 0}$$

This defines a sheaf $\mathscr{F}$ over the set of primes, i.e.,

$$\mathscr{F} : \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Ab}$$

with $\mathscr{F}(U) = \prod_{p\in U} \mathscr{F}_p$ for $U \subset \mathbb{P}$.

4.2. Structure Sheaf Interpretation. Let $\mathscr{O}_{\mathrm{arith}}$ be the sheaf of local arithmetic functions over $\mathrm{Spec}(\mathbb{Z})$, assigning to each open U a set of local systems supported on U.

This allows us to view local arithmetic functions as global sections:

$$\Gamma(\mathrm{Spec}(\mathbb{Z}), \mathscr{O}_{\mathrm{arith}}) = \{\{f_p\}_{p\in\mathbb{P}}\}$$

4.3. Derived Functors and Cohomological Invariants. Define the derived functors:

$$\mathbb{R}^i\Gamma(\mathrm{Spec}(\mathbb{Z}), \mathscr{F})$$

which measure arithmetic obstructions and global compatibility among local data. These cohomology groups potentially encode:

- Compatibility of local factors in zeta product expansions;
- Non-trivial interactions or extensions among local systems;
- Hidden global constraints not apparent from primewise values alone.

4.4. Toward Arithmetic Motives. If we equip each stalk $\mathscr{F}_p$ with further structure (e.g., ℓ-adic representations, Frobenius traces, etc.), then $\mathscr{F}$ may be enhanced to a motivic sheaf, yielding arithmetic motivic cohomology:

$$H^i_{\mathrm{mot}}(\mathrm{Spec}(\mathbb{Z}), \mathscr{F})$$

This aligns with the vision of a motivic interpretation of arithmetic functions via their local avatars, possibly extending into the realm of the Langlands program, perfectoid cohomology, or categorical arithmetic frameworks.

5. Extensions: Toward Yang Number Systems, Motivic Cohomology, and Perfectoid Geometry

5.1. Yang$_n(F)$-Local Arithmetic Structures. Let F be a base field, and consider a Yang-type number system $\mathbb{Y}_n(F)$ with arithmetic-like operations. Define a local arithmetic system:

$$\{f_{p,\mathbb{Y}_n} : \mathbb{Z}_{\geq 0} \to \mathbb{Y}_n(F)\}$$

with induced global function:

$$F_{\mathbb{Y}_n}(n) := \prod_{p^\alpha \| n} f_{p,\mathbb{Y}_n}(\alpha)$$

This can be viewed as a sheaf:

$$\mathscr{F}_{\mathbb{Y}_n} : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{Y}_n(F)\text{-Mod}$$

whose sections encode Yang-arithmetic flows. This opens the path toward defining $\mathbb{Y}_n$-valued arithmetic cohomology and Yang-motivic zeta structures.

5.2. Motivic Cohomology and Functorial Lifts. Enhance each local stalk f_p by attaching geometric or categorical data:

$$f_p(k) \rightsquigarrow (f_p(k), \rho_p^{(k)}, \mathrm{Frob}_p^{(k)})$$

where ρ_p denotes a Galois or étale representation and Frob_p the Frobenius element.

This yields a motivic sheaf $\mathscr{F}_{\mathrm{mot}}$ and cohomology:

$$H^i_{\mathrm{mot}}(\mathrm{Spec}(\mathbb{Z}), \mathscr{F}_{\mathrm{mot}})$$

capturing not only numeric values but deeper categorical structures. Such cohomology groups may encode hidden functional equations, special values, and generalizations of L-function symmetries.

5.3. Perfectoid Extensions and Tilting. Let $\mathbb{F}_p$ be a perfect field and consider $\mathbb{Z}_p^\flat$, the tilt of the perfectoid ring $\mathbb{Z}_p$.

We define a perfectoid local arithmetic function:

$$f_p^\flat(k) \in \mathbb{Z}_p^\flat, \quad \text{inducing } F^\flat(n) := \prod_{p^\alpha \| n} f_p^\flat(\alpha)$$

and the corresponding sheaf $\mathscr{F}^\flat$ over the perfectoid site. This permits construction of:

$$H^i_{\text{pro-ét}}(\mathrm{Spec}(\mathbb{Z}_p), \mathscr{F}^\flat)$$

integrating the theory of diamonds and perfectoid towers.

5.4. p-adic Hodge Theoretic Cohomology. Each $f_p(k)$ may be lifted to a p-adic Hodge-theoretic object:

$$f_p(k) \rightsquigarrow D_{\mathrm{HT}}(f_p(k)), D_{\mathrm{dR}}(f_p(k)), D_{\mathrm{crys}}(f_p(k))$$

allowing us to define cohomology theories:

$$H^i_{\mathrm{HT}}, \quad H^i_{\mathrm{dR}}, \quad H^i_{\mathrm{crys}}$$

for local arithmetic sheaves, thereby unifying arithmetic functions with Fontaine's p-adic cohomological worlds.

These constructions not only connect analytic number theory with arithmetic geometry, but also offer a blueprint for future arithmetic invariants in the spirit of the Langlands program and beyond.

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