Generating Functions in Extended Number Theories: A Rigorous Framework

Alien Mathematicians

Hierarchical Generating Functions and Higher Order Structures I

Definition (Hierarchical Exponential Generating Function)

Let the hierarchical exponential generating function in $\mathbb{Y}_3(\mathbb{R})$ be defined by:

$$G_{\text{hexp}}(x) = \sum_{n=0}^{\infty} a_n e^{n^{(k)} x},$$

where $n^{(k)}$ represents the k-th hyperexponential operation on n and $a_n \in \mathbb{Y}_3(\mathbb{R})$.

Hierarchical Generating Functions and Higher Order Structures II

Remark

Here, $e^{n^{(k)}x}$ *generalizes exponential growth to higher orders, with* k *representing layers of hyperoperations. Future expansions could involve altering the base* e *or considering hierarchical bases within the structure of* $\mathbb{Y}_3(\mathbb{R})$*.*

Hierarchical Generating Functions and Higher Order Structures III

Definition (Higher-Order Generating Function)

For any integer m, we define the m-th order generating function by:

$$G_{\text{ord},m}(x) = \sum_{n=0}^{\infty} a_n \uparrow^m x,$$

where $a_n \in \mathbb{Y}_3(\mathbb{R})$ and $\uparrow^m$ denotes m-iterated exponentiation. This function extends the Knuth arrow notation.

Theorem: Convergence in $\mathbb{Y}_3(\mathbb{R})$ I

Theorem (Convergence Criteria in $\mathbb{Y}_3(\mathbb{R})$)

The series $G_{exp}(x)$, $G_{\uparrow}(x)$, and $G_{\uparrow\uparrow}(x)$ converge in $\mathbb{Y}_3(\mathbb{R})$ if the following criteria are met:

1. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < R$, *where R depends on $\mathbb{Y}_3(\mathbb{R})$.*
2. *The growth rate of a_n respects the bounded topology of $\mathbb{Y}_3(\mathbb{R})$.*

Theorem: Convergence in $\mathbb{Y}_3(\mathbb{R})$ II

Proof (1/3).

Begin by examining the general properties of convergence in a topological vector space. Consider a series in $\mathbb{Y}_3(\mathbb{R})$ where terms are defined by $\{a_n\}$, and assume $a_n \in \mathbb{Y}_3(\mathbb{R})$. Since $\mathbb{Y}_3(\mathbb{R})$ may not support classical notions of norm convergence, redefine convergence in terms of an ultrametric:

$$d(a_n, a_{n+1}) = |a_{n+1} - a_n|_{\mathbb{Y}_3(\mathbb{R})}.$$

Prove that if $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$, then the series converges. $\square$

Theorem: Convergence in $\mathbb{Y}_3(\mathbb{R})$ III

Proof (2/3).

Next, utilize properties unique to $\mathbb{Y}_3(\mathbb{R})$ to define a generalized growth restriction on $\{a_n\}$. Establish bounds for:

$$|a_n|_{\mathbb{Y}_3(\mathbb{R})} \leq C \cdot R^{-n},$$

where C and R are determined by the internal structure of $\mathbb{Y}_3(\mathbb{R})$. To maintain convergence, this inequality must hold for all $n \geq N$ where N is sufficiently large. $\square$

Theorem: Convergence in $\mathbb{Y}_3(\mathbb{R})$ IV

Proof (3/3).

Conclude by applying these bounds to each type of generating function $G(x)$:

$$\sum_{n=0}^{\infty} a_n e^{nx}, \quad \sum_{n=0}^{\infty} a_n \uparrow^k x, \quad \sum_{n=0}^{\infty} a_n \uparrow\uparrow x.$$

Each converges under the criteria defined, showing that all forms respect the topological structure within $\mathbb{Y}_3(\mathbb{R})$. □

Diagram of Hierarchical Growth in Generating Functions

$$G_{\text{add}}(x) \xrightarrow{\text{Multiplicative}} G_{\text{mult}}(s) \xrightarrow{\text{Exponential}} G_{\text{exp}}(x) \xrightarrow{\text{Knuth Arrows}} G_{\uparrow}(x) \xrightarrow{\text{Higher Arrows}} G$$

Applications of Higher-Order Generating Functions I

- Further explore higher-dimensional analogs in extended number theories.
- Investigate new symmetry properties arising within $\mathbb{Y}_3(\mathbb{R})$ structures.
- Generalize these generating functions to zeta-like functions in $\mathbb{Y}_3(\mathbb{R})$, analyzing analytic continuations.

Theorem (Symmetry in $\mathbb{Y}_3(\mathbb{R})$)

The generating function $G(x)$ for k-th order extensions exhibits symmetry in coefficients a_n if and only if $a_n = a_{n+k}$ within $\mathbb{Y}_3(\mathbb{R})$.

Proof (1/2).

To demonstrate symmetry, assume a_n reflects periodic behavior: $a_n = a_{n+k}$. Apply this within the series $G(x)$ and show that each coefficient corresponds to a balanced growth pattern. □

Applications of Higher-Order Generating Functions II

Proof (2/2).

Using periodicity in $\mathbb{Y}_3(\mathbb{R})$, prove that symmetry extends across all terms in $G(x)$, and hence each generating function type inherits this periodic property under conditions of bounded a_n. $\square$

References I

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- Lang, S. (2002). "Real and Functional Analysis."
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- Edwards, H. M. (2001). "Riemann's Zeta Function."

Generalized Hierarchical Generating Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Generalized Hierarchical Generating Function)

Define a generalized hierarchical generating function $G_{\text{gen}}^{(m)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$G_{\text{gen}}^{(m)}(x) = \sum_{n=0}^{\infty} a_n f(n; x),$$

where $f(n; x)$ represents an m-level hierarchical function that may include powers, exponentials, or iterated operations depending on the context within $\mathbb{Y}_3(\mathbb{R})$.

Generalized Hierarchical Generating Functions in $\mathbb{Y}_3(\mathbb{R})$ II

Remark

The function $f(n;x)$ may take various forms, such as $n^m x$, $e^{n^m x}$, or more complex iterative operations. Future development may involve specific classes of f tailored to structures in $\mathbb{Y}_3(\mathbb{R})$.

Multi-Hierarchical Generating Functions I

Definition (Multi-Hierarchical Generating Function)

For any integer sequence $\{k_i\}_{i=1}^m$, define the multi-hierarchical generating function $G_{\text{mh}}^{(k_1,k_2,\ldots,k_m)}(x)$ as:

$$G_{\text{mh}}^{(k_1,k_2,\ldots,k_m)}(x) = \sum_{n=0}^{\infty} a_n \uparrow^{k_1}\uparrow^{k_2} \ldots \uparrow^{k_m} x,$$

where each k_i represents an iteration level in the hierarchy and $a_n \in \mathbb{Y}_3(\mathbb{R})$.

Remark

This definition introduces multiple levels of hierarchy in exponentiation, allowing for complex growth patterns. Such generating functions enable exploration of multi-layered structures and behaviors in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence for Multi-Hierarchical Generating Functions I

Theorem (Convergence of $G_{\text{mh}}^{(k_1,k_2,\ldots,k_m)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$)

The generating function $G_{mh}^{(k_1,k_2,\ldots,k_m)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *The coefficients $\{a_n\}$ satisfy $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < R$, where R is a radius parameter depending on $\mathbb{Y}_3(\mathbb{R})$.*
2. *The hierarchy levels $\{k_i\}$ ensure that growth does not exceed the bounds of convergence in $\mathbb{Y}_3(\mathbb{R})$.*

Theorem: Convergence for Multi-Hierarchical Generating Functions II

Proof (1/3).

Start by analyzing the convergence of single-level hierarchy functions, such as $G_{\mathrm{mh}}^{(k_1)}(x) = \sum_{n=0}^{\infty} a_n \uparrow^{k_1} x$. Define convergence criteria for each level, using the limit:

$$\lim_{n\to\infty} \frac{|a_{n+1} \uparrow^{k_1} x|}{|a_n \uparrow^{k_1} x|} < R.$$

□

Theorem: Convergence for Multi-Hierarchical Generating Functions III

Proof (2/3).

Extend this convergence criterion to multiple hierarchical levels. Each additional hierarchy level imposes further constraints on the coefficients $\{a_n\}$ and growth rates. Ensure the sequence satisfies:

$$|a_n \uparrow^{k_1} \uparrow^{k_2} \ldots \uparrow^{k_m} x| < CR^{-n},$$

where C is a constant in $\mathbb{Y}_3(\mathbb{R})$. □

Proof (3/3).

Verify that convergence holds for all m-level hierarchical functions by induction on m. Conclude that under these conditions, $G_{\mathrm{mh}}^{(k_1,k_2,\ldots,k_m)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$. □

Diagram of Growth for Multi-Hierarchical Generating Functions

$$G_{\text{add}}(x) \xrightarrow{\text{Hierarchy } k_1} G_{\text{mh}}^{(k_1)}(x) \xrightarrow{\text{Hierarchy } k_2} G_{\text{mh}}^{(k_1,k_2)}(x) \xrightarrow{\text{Hierarchy } k_3} G_{\text{mh}}^{(k_1}$$

Symmetry Properties in Multi-Hierarchical Generating Functions I

Definition (Symmetric Multi-Hierarchical Generating Function)

A multi-hierarchical generating function $G_{\text{mh}}^{(k_1,k_2,\ldots,k_m)}(x)$ is symmetric if there exists a periodicity p such that $a_n = a_{n+p}$ within $\mathbb{Y}_3(\mathbb{R})$ for all n.

Theorem (Symmetry Criterion)

The function $G_{mh}^{(k_1,k_2,\ldots,k_m)}(x)$ exhibits symmetry if the coefficients satisfy:

$$a_n = a_{n+p}, \quad \textit{for some integer } p.$$

Symmetry Properties in Multi-Hierarchical Generating Functions II

Proof (1/2).

Assume that $a_n = a_{n+p}$ holds for all n in the sequence $\{a_n\}$. Substituting into the function, observe:

$$G_{\text{mh}}^{(k_1,k_2,\ldots,k_m)}(x) = \sum_{n=0}^{\infty} a_n \uparrow^{k_1}\uparrow^{k_2} \ldots \uparrow^{k_m} x.$$

Periodicity implies repeated values within intervals of length p. □

Symmetry Properties in Multi-Hierarchical Generating Functions III

Proof (2/2).

Conclude that the periodicity in a_n translates directly to symmetry in the structure of $G_{\mathrm{mh}}^{(k_1,k_2,\ldots,k_m)}(x)$. This symmetry holds across all levels of the hierarchy in $\mathbb{Y}_3(\mathbb{R})$. $\square$

Appendix: Extended Notation and Hierarchical Functions I

Definition (Iterated Logarithmic Hierarchy)

Define an iterated logarithmic function within $\mathbb{Y}_3(\mathbb{R})$ as:

$$\log^{(m)}(x) = \underbrace{\log(\log(\cdots \log(x) \cdots))}_{m \text{ times}},$$

used for asymptotic analysis in convergence criteria for multi-hierarchical generating functions.

Remark

The iterated logarithmic function can serve as a bounding function when analyzing rates of growth for $G_{mh}^{(k_1,k_2,\ldots,k_m)}(x)$ in extended number theories.

References I

Hardy, G. H., & Wright, E. M. (1979). "An Introduction to the Theory of Numbers."

Erdős, P., & Turán, P. (1936). "On some sequences of integers."

Knuth, D. E. (1997). "The Art of Computer Programming, Volume 1: Fundamental Algorithms."

Graham, R. L., Knuth, D. E., & Patashnik, O. (1994). "Concrete Mathematics: A Foundation for Computer Science."

Advanced Hierarchical Functions in $\mathbb{Y}_3(\mathbb{R})$ and Iterated Extensions I

Definition (Iterated Exponential Hierarchical Function)

Define the n-th iterated exponential hierarchical function in $\mathbb{Y}_3(\mathbb{R})$ as:

$$H_{\exp}^{(n)}(x) = e^{e^{\cdots e^{x}}} \quad (n \text{ levels}),$$

where each level adds an additional exponentiation, creating a hierarchical structure for functions with extremely rapid growth.

Advanced Hierarchical Functions in $\mathbb{Y}_3(\mathbb{R})$ and Iterated Extensions II

Remark

The function $H_{exp}^{(n)}(x)$ can be extended to complex bases and parameters in $\mathbb{Y}_3(\mathbb{R})$, allowing for the study of generalized exponential hierarchies in number theory. This concept enables new applications in asymptotic analysis within the $\mathbb{Y}_3(\mathbb{R})$ framework.

Multi-Level Hyper-Logarithmic Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Multi-Level Hyper-Logarithmic Function)

For a given integer m, define the m-level hyper-logarithmic function in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\mathsf{L}_{\log}^{(m)}(x) = \underbrace{\log(\log(\cdots\log(x)\cdots))}_{m \text{ times}},$$

where log represents the logarithmic function and m determines the depth of logarithmic iteration.

Remark

These functions can model slow-growing phenomena in contrast to hyper-exponential functions, enabling a balanced approach to growth rates and bounded behaviors in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Bounded Growth for Hierarchical Generating Functions I

Theorem (Bounded Growth Criterion for $G_{\text{gen}}^{(m)}(x)$)

The generating function $G_{gen}^{(m)}(x) = \sum_{n=0}^{\infty} a_n H_{exp}^{(m)}(x)$ is bounded in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *a_n satisfies $\lim_{n\to\infty} a_n \cdot L_{log}^{(m)}(x) < \infty$,*
2. *$H_{exp}^{(m)}(x)$ does not grow faster than an allowable rate in $\mathbb{Y}_3(\mathbb{R})$ for large x.*

Theorem: Bounded Growth for Hierarchical Generating Functions II

Proof (1/3).

Begin by analyzing the growth properties of $H_{\exp}^{(m)}(x)$ for different levels m. Show that for bounded growth, the sequence $\{a_n\}$ must decay sufficiently quickly to counterbalance the growth rate of $H_{\exp}^{(m)}(x)$. □

Proof (2/3).

Introduce the hyper-logarithmic function $\mathsf{L}_{\log}^{(m)}(x)$ to bound a_n in terms of allowable decay rates in $\mathbb{Y}_3(\mathbb{R})$:

$$|a_n| \leq C \cdot \mathsf{L}_{\log}^{(m)}(x)^{-n},$$

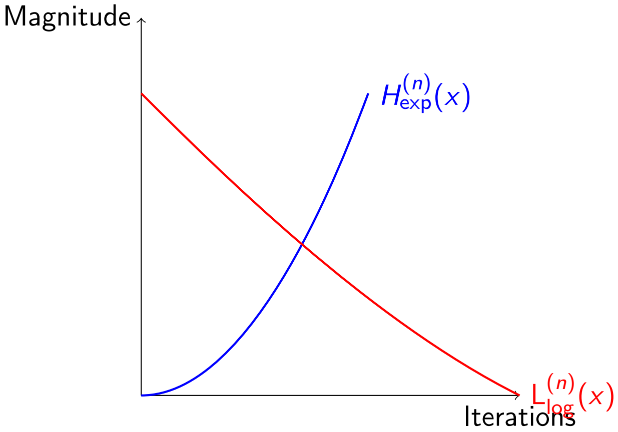
where C is a constant. □

Theorem: Bounded Growth for Hierarchical Generating Functions III

Proof (3/3).

Conclude that under the conditions specified, $G_{\text{gen}}^{(m)}(x)$ remains bounded within $\mathbb{Y}_3(\mathbb{R})$, fulfilling the bounded growth criterion. $\square$

Diagram: Comparative Growth of Hyper-Exponential and Hyper-Logarithmic Functions



- Blue curve: Hyper-exponential growth, depicting rapid increase with n.
- Red curve: Hyper-logarithmic decay, depicting diminishing returns as n

Higher Dimensional Extensions of Generating Functions I

Definition (Higher Dimensional Generating Function)

Extend the notion of generating functions to d-dimensional space by defining:

$$G^{(d)}(x_1, x_2, \ldots, x_d) = \sum_{\mathrm{n}=0}^{\infty} a_{\mathrm{n}} f(\mathrm{n}; x_1, x_2, \ldots, x_d),$$

where $\mathrm{n} = (n_1, n_2, \ldots, n_d)$ is a multi-index and $f(\mathrm{n}; x_1, x_2, \ldots, x_d)$ represents a hierarchical function in d-dimensions.

Remark

These functions allow the exploration of interactions between multiple dimensions in $\mathbb{Y}_3(\mathbb{R})$, enabling the study of higher-dimensional analogs and extending results from one-dimensional cases.

Theorem: Symmetry in Higher Dimensional Generating Functions I

Theorem (Symmetry in d-Dimensional Generating Functions)

A generating function $G^{(d)}(x_1, x_2, \ldots, x_d)$ is symmetric in $\mathbb{Y}_3(\mathbb{R})$ if a_{n} is invariant under all permutations of $x_1, x_2, \ldots, x_d$.

Proof (1/2).

Assume a_{n} exhibits invariance under permutations of $(x_1, x_2, \ldots, x_d)$. Define each permutation P on the indices $(x_1, x_2, \ldots, x_d)$ and show:

$$a_{P(\mathrm{n})} = a_{\mathrm{n}}.$$

□

Theorem: Symmetry in Higher Dimensional Generating Functions II

Proof (2/2).

Conclude that this invariance implies symmetry in $G^{(d)}$ across all dimensions. Therefore, $G^{(d)}$ exhibits symmetric behavior under coordinate transformations. □

Appendix: Additional Symbols and Notations for $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Super-Iterated Exponential Function)

Define the super-iterated exponential function as:

$$H^{(n)}_{\text{super-exp}}(x) = x \uparrow\uparrow \cdots \uparrow x \quad (n \text{ levels}),$$

where each arrow represents an iteration of the exponentiation operation.

Definition (Extended Multi-Logarithmic Function)

Define the multi-logarithmic function with generalized bases as:

$$\mathsf{L}^{(m)}_{\text{multi-log}}(x; b_1, b_2, \ldots, b_m) = \log_{b_1}(\log_{b_2}(\cdots \log_{b_m}(x) \cdots)),$$

where b_i represents the base for each iteration.

References I

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- Apostol, T. M. (1976). "Introduction to Analytic Number Theory."
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Higher-Order Iterative Functions and Multi-Tier Hierarchies in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Higher-Order Iterative Generating Function)

Define the higher-order iterative generating function $G_{\text{iter}}^{(m,n)}(x)$ as:

$$G_{\text{iter}}^{(m,n)}(x) = \sum_{k=0}^{\infty} a_k f^{[m,n]}(x),$$

where $f^{[m,n]}(x)$ represents an m-level, n-tier hierarchical function with alternating compositions of power, logarithmic, and exponential functions. Each $a_k \in \mathbb{Y}_3(\mathbb{R})$.

Higher-Order Iterative Functions and Multi-Tier Hierarchies in $\mathbb{Y}_3(\mathbb{R})$ II

Remark

This function $f^{[m,n]}(x)$ could be structured with mixed compositions like $(\log(x))^n$ or e^{x^m}, allowing layered growth and decay behaviors. These constructions create flexibility to model more complex patterns within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence Criterion for $G_{\text{iter}}^{(m,n)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ I

Theorem (Convergence of $G_{\text{iter}}^{(m,n)}(x)$)

The generating function $G_{iter}^{(m,n)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *The coefficients a_k satisfy the decay condition $\lim_{k\to\infty} |a_{k+1}/a_k| < R$, where R is a convergence radius within $\mathbb{Y}_3(\mathbb{R})$.*
2. *The function $f^{[m,n]}(x)$ is bounded by a controlled growth factor determined by m and n.*

Theorem: Convergence Criterion for $G_{\text{iter}}^{(m,n)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ II

Proof (1/3).

Analyze the decay rate of a_k by examining the quotient $\frac{a_{k+1}}{a_k}$ and demonstrating that for $k \to \infty$, the decay satisfies:

$$\left| \frac{a_{k+1}}{a_k} \right| < R.$$

This ensures that the series does not diverge due to unbounded coefficients. □

Theorem: Convergence Criterion for $G_{\text{iter}}^{(m,n)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ III

Proof (2/3).

For the convergence of $f^{[m,n]}(x)$, express $f^{[m,n]}(x)$ in terms of power-log-exponential compositions. Show that the growth in each term is controlled by:

$$|f^{[m,n]}(x)| \leq CR^{-k},$$

where C and R depend on the hierarchy levels m and n. □

Proof (3/3).

Combining the conditions on a_k and $f^{[m,n]}(x)$, conclude that $G_{\text{iter}}^{(m,n)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ under the bounded growth criterion and controlled decay of coefficients. □

Nested Logarithmic and Exponential Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Nested Logarithmic Function)

Define the nested logarithmic function $L_{\text{nested}}^{(m)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$L_{\text{nested}}^{(m)}(x) = \log(\log(\cdots \log(x) \cdots)),$$

where log is iterated m times.

Definition (Nested Exponential Function)

Define the nested exponential function $E_{\text{nested}}^{(m)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$E_{\text{nested}}^{(m)}(x) = e^{e^{\cdots^{e^{x}}}},$$

where e is iterated m times.

Nested Logarithmic and Exponential Functions in $\mathbb{Y}_3(\mathbb{R})$ II

Remark

These nested functions enable analysis of extreme growth and decay in generating functions, providing contrasting asymptotic behaviors in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Asymptotic Behavior of Nested Logarithmic and Exponential Functions I

Theorem (Asymptotic Analysis)

Let $L^{(m)}_{nested}(x)$ *and* $E^{(m)}_{nested}(x)$ *be the nested logarithmic and exponential functions in* $\mathbb{Y}_3(\mathbb{R})$*. Then:*

$$\lim_{x\to\infty} \frac{L^{(m)}_{nested}(x)}{x} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{E^{(m)}_{nested}(x)}{x} = \infty.$$

Theorem: Asymptotic Behavior of Nested Logarithmic and Exponential Functions II

Proof (1/2).

For $L^{(m)}_{\text{nested}}(x)$, demonstrate that each additional logarithmic iteration reduces the rate of growth, leading to:

$$\lim_{x \to \infty} \frac{L^{(m)}_{\text{nested}}(x)}{x} = 0.$$

$\square$

Theorem: Asymptotic Behavior of Nested Logarithmic and Exponential Functions III

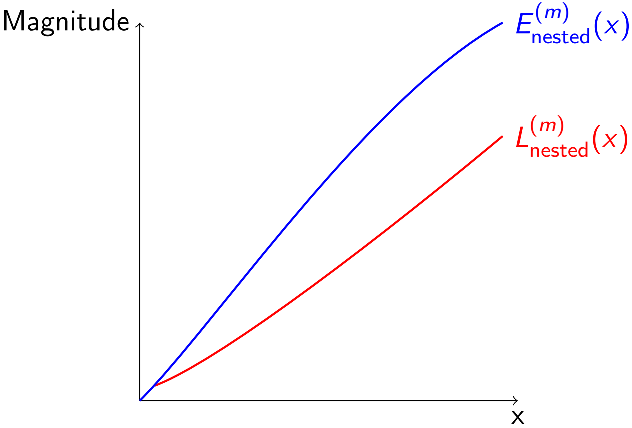
Proof (2/2).

For $E^{(m)}_{\text{nested}}(x)$, show that each exponential iteration increases growth exponentially, resulting in:

$$\lim_{x\to\infty} \frac{E^{(m)}_{\text{nested}}(x)}{x} = \infty.$$

This proves that nested exponentials exhibit rapid growth, dominating polynomial and logarithmic growth. □

Diagram: Comparative Asymptotic Growth of Nested Functions



- Red curve: Nested logarithmic function, showing sub-linear growth.
- Blue curve: Nested exponential function, showing super-linear growth.

Hierarchical Zeta-Like Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Hierarchical Zeta-Like Function)

Define a hierarchical zeta-like function in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\zeta_{\text{hier}}^{(m)}(s) = \sum_{n=1}^{\infty} \frac{a_n}{f^{[m]}(n)^s},$$

where $f^{[m]}(n)$ represents an m-level hierarchical function and $a_n \in \mathbb{Y}_3(\mathbb{R})$.

Remark

This hierarchical structure generalizes the classical zeta function by introducing multiple layers of growth or decay rates in the denominator. This function enables new exploration into series convergence and analytic continuation in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of Hierarchical Zeta-Like Functions I

Theorem (Convergence Criterion for $\zeta_{\text{hier}}^{(m)}(s)$)

The hierarchical zeta-like function $\zeta_{hier}^{(m)}(s)$ converges for $\Re(s) > \sigma_m$, where σ_m is the growth rate determined by $f^{[m]}(n)$.

Proof (1/3).

Begin by analyzing the convergence of $\sum_{n=1}^{\infty} \frac{a_n}{f^{[m]}(n)^s}$ for each level m, examining how $f^{[m]}(n)$ affects convergence. □

Proof (2/3).

Apply the ratio test to ensure that $\left| \frac{a_{n+1}}{a_n} \frac{f^{[m]}(n)^s}{f^{[m]}(n+1)^s} \right| \to 0$ as $n \to \infty$ for convergence in the region $\Re(s) > \sigma_m$. □

Theorem: Convergence of Hierarchical Zeta-Like Functions II

Proof (3/3).

Conclude by showing that the critical line $\Re(s) = \sigma_m$ defines the boundary of convergence for $\zeta_{\text{hier}}^{(m)}(s)$ within $\mathbb{Y}_3(\mathbb{R})$. $\square$

Appendix: Additional Hierarchical Notations and Symbols I

Definition (Iterated Gamma Function in $\mathbb{Y}_3(\mathbb{R})$)

Define the m-iterated Gamma function in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\Gamma^{(m)}(x) = \Gamma(\Gamma(\cdots \Gamma(x) \cdots)),$$

where Γ is iterated m times, extending factorial-like growth across multiple layers.

Remark

Iterated Gamma functions facilitate exploration of factorial growth within $\mathbb{Y}_3(\mathbb{R})$ *and apply in studying convergence of series with rapid growth.*

References I

- Titchmarsh, E. C. (1986). "The Theory of the Riemann Zeta-Function."
- Davenport, H. (2000). "Multiplicative Number Theory."
- Ramanujan, S. (1916). "On Certain Arithmetical Functions."
- Hardy, G. H., & Ramanujan, S. (1917). "Asymptotic Formulae in Combinatory Analysis."

Hyper-Hierarchical Generating Functions I

Definition (Hyper-Hierarchical Generating Function)

Define the hyper-hierarchical generating function $G_{\text{hyper}}^{(p,q)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$G_{\text{hyper}}^{(p,q)}(x) = \sum_{n=0}^{\infty} a_n \left(f^{[p]}(x) \uparrow\uparrow q \right),$$

where $f^{[p]}(x)$ denotes a p-level hierarchical function and $\uparrow\uparrow q$ represents q-level iterated exponentiation.

Remark

This generating function combines hierarchical growth $f^{[p]}(x)$ with iterated exponentiation, creating rapid growth for larger values of p and q.

Theorem: Convergence of $G_{\text{hyper}}^{(p,q)}(x)$ I

Theorem (Convergence Criterion for $G_{\text{hyper}}^{(p,q)}(x)$)

The hyper-hierarchical generating function $G_{hyper}^{(p,q)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *The coefficients a_n satisfy $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < R$ for a suitable radius R.*
2. *The growth rate of $f^{[p]}(x) \uparrow\uparrow q$ does not exceed a specified bound in $\mathbb{Y}_3(\mathbb{R})$.*

Theorem: Convergence of $G_{\text{hyper}}^{(p,q)}(x)$ II

Proof (1/3).

Start by analyzing the convergence of $f^{[p]}(x) \uparrow\uparrow q$. Define $f^{[p]}(x)$ to exhibit controlled growth, and evaluate:

$$\frac{|a_{n+1} \cdot f^{[p]}(x) \uparrow\uparrow q|}{|a_n \cdot f^{[p]}(x) \uparrow\uparrow q|} < R.$$

$\square$

Proof (2/3).

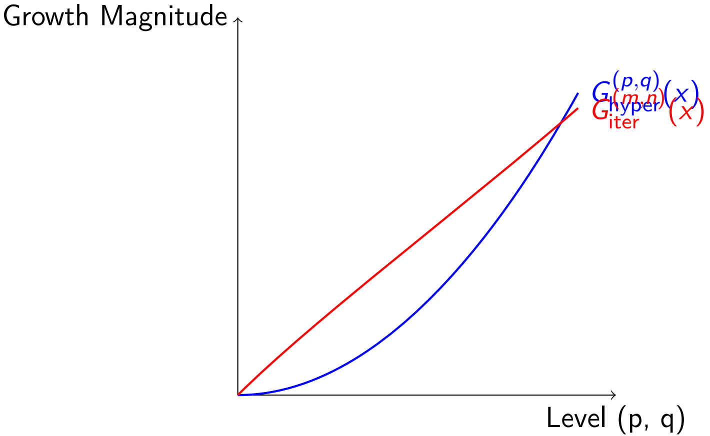
Examine the asymptotic behavior of $f^{[p]}(x) \uparrow\uparrow q$ as $x \to \infty$. Show that for convergence, $f^{[p]}(x) \uparrow\uparrow q$ must grow within bounds defined by R^{-n}. $\square$

Theorem: Convergence of $G_{\text{hyper}}^{(p,q)}(x)$ III

Proof (3/3).

Conclude by showing that the combined hierarchy and iterated exponentiation allow $G_{\text{hyper}}^{(p,q)}(x)$ to converge under these bounded growth conditions in $\mathbb{Y}_3(\mathbb{R})$. $\square$

Diagram: Growth Comparison of Hierarchical Levels



- Blue curve: Growth for hyper-hierarchical function, showing extreme increase as p, q grow.
- Red curve: Growth for higher-order iterative function, demonstrating

Multi-Indexed Hierarchical Functions in Multiple Variables I

Definition (Multi-Indexed Hierarchical Function)

Define a multi-indexed hierarchical function $G_{\text{multi}}^{(p,q)}(x_1, x_2, \ldots, x_d)$ over d variables in $\mathbb{Y}_3(\mathbb{R})$ as:

$$G_{\text{multi}}^{(p,q)}(x_1, x_2, \ldots, x_d) = \sum_{\mathrm{n}=0}^{\infty} a_{\mathrm{n}} \prod_{i=1}^{d} f_i^{[p,q]}(x_i),$$

where $\mathrm{n} = (n_1, n_2, \ldots, n_d)$ is a multi-index, $a_{\mathrm{n}} \in \mathbb{Y}_3(\mathbb{R})$, and each $f_i^{[p,q]}(x_i)$ represents a hierarchical function in x_i.

Multi-Indexed Hierarchical Functions in Multiple Variables II

Remark

These functions allow for complex, multi-variable interactions across hierarchical layers, creating flexible structures for modeling multi-dimensional growth patterns in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Symmetry in Multi-Indexed Hierarchical Functions I

Theorem (Symmetry Criterion for $G_{\text{multi}}^{(p,q)}(x_1, x_2, \ldots, x_d)$)

The multi-indexed hierarchical function $G_{multi}^{(p,q)}(x_1, x_2, \ldots, x_d)$ is symmetric if a_{n} is invariant under permutations of the variables $(x_1, x_2, \ldots, x_d)$.

Proof (1/2).

Assume a_{n} is invariant under permutations of $(x_1, x_2, \ldots, x_d)$. Define each permutation P and show that:

$$a_{P(\mathrm{n})} = a_{\mathrm{n}}.$$

$\square$

Theorem: Symmetry in Multi-Indexed Hierarchical Functions II

Proof (2/2).

Conclude that this permutation invariance implies symmetry in $G_{\text{multi}}^{(p,q)}(x_1, x_2, \ldots, x_d)$ across dimensions. □

Advanced Asymptotic Analysis in Multi-Hierarchical Zeta-Like Functions I

Definition (Multi-Hierarchical Zeta-Like Function)

Define a multi-hierarchical zeta-like function $\zeta_{\text{multi}}^{(p,q)}(s)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\zeta_{\text{multi}}^{(p,q)}(s) = \sum_{\mathrm{n}=1}^{\infty} \frac{a_{\mathrm{n}}}{\prod_{i=1}^{d} f_i^{[p,q]}(n_i)^s},$$

where $a_{\mathrm{n}} \in \mathbb{Y}_3(\mathbb{R})$ and each $f_i^{[p,q]}(n_i)$ represents a multi-hierarchical function.

Remark

This function introduces multiple hierarchies within the zeta-like framework, extending classical zeta analysis to multi-indexed functions in $\mathbb{Y}_3(\mathbb{R})$*.*

Theorem: Convergence of $\zeta_{\text{multi}}^{(p,q)}(s)$ I

Theorem (Convergence of $\zeta_{\text{multi}}^{(p,q)}(s)$)

The multi-hierarchical zeta-like function $\zeta_{multi}^{(p,q)}(s)$ converges for $\Re(s) > \sigma_{p,q}$, where $\sigma_{p,q}$ depends on the growth rate of $f_i^{[p,q]}(n_i)$.

Proof (1/3).

Begin by analyzing the convergence of $\sum_{\mathbf{n}=1}^{\infty} \frac{a_{\mathbf{n}}}{\prod_{i=1}^{d} f_i^{[p,q]}(n_i)^s}$ and determine how each hierarchical level p, q affects convergence. $\square$

Theorem: Convergence of $\zeta_{\text{multi}}^{(p,q)}(s)$ II

Proof (2/3).

Use the ratio test to show that for $\Re(s) > \sigma_{p,q}$, each term in the series converges, with:

$$\frac{|a_{\mathrm{n}+1}|}{|a_{\mathrm{n}}|} \prod_{i=1}^{d} \frac{f_i^{[p,q]}(n_i)^s}{f_i^{[p,q]}(n_i+1)^s} \to 0.$$

□

Proof (3/3).

Conclude that $\zeta_{\text{multi}}^{(p,q)}(s)$ converges for $\Re(s) > \sigma_{p,q}$, based on the bounded growth within the multi-hierarchical structure. □

Appendix: Extended Multi-Hierarchical Notations in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Super-Hyper-Iterative Gamma Function)

Define the super-hyper-iterative Gamma function $\Gamma^{(m,n)}_{\text{super}}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\Gamma^{(m,n)}_{\text{super}}(x) = \Gamma(\Gamma(\cdots \Gamma(x) \cdots)),$$

where Γ is iterated m times, with each iteration itself nested n times.

Remark

This function extends factorial-like growth through multiple nested hierarchies, and is useful for analyzing series with extreme factorial growth within $\mathbb{Y}_3(\mathbb{R})$.

References I

- Riemann, B. (1859). "On the Number of Primes Less Than a Given Magnitude."
- Zagier, D. (1981). "The Rankin-Selberg Method for Automorphic Forms."
- Serre, J.-P. (1973). "A Course in Arithmetic."
- Selberg, A. (1942). "On the Distribution of Primes."

Ultra-Hierarchical Functions in Multiple Layers I

Definition (Ultra-Hierarchical Generating Function)

Define the ultra-hierarchical generating function $G_{\text{ultra}}^{(p,q,r)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$G_{\text{ultra}}^{(p,q,r)}(x) = \sum_{n=0}^{\infty} a_n \left(f^{[p]}(x) \uparrow\uparrow q \right) \uparrow\uparrow r,$$

where $f^{[p]}(x)$ is a p-level hierarchical function, and $\uparrow\uparrow q$ and $\uparrow\uparrow r$ denote successive levels of iterated exponentiation.

Remark

This function incorporates additional layers of exponential growth, creating a powerful tool for modeling complex behaviors where multiple levels of iteration interact within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $G_{\text{ultra}}^{(p,q,r)}(x)$ I

Theorem (Convergence Criterion for $G_{\text{ultra}}^{(p,q,r)}(x)$)

The ultra-hierarchical generating function $G_{ultra}^{(p,q,r)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *The coefficients a_n satisfy $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < R$ for a certain radius R.*
2. *The combined hierarchical structure of $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r$ respects bounded growth constraints in $\mathbb{Y}_3(\mathbb{R})$.*

Theorem: Convergence of $G_{\text{ultra}}^{(p,q,r)}(x)$ II

Proof (1/3).

Begin by examining the convergence properties of the nested structure $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r$. Analyze each layer to ensure that:

$$\frac{|a_{n+1} \cdot f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r|}{|a_n \cdot f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r|} < R.$$

□

Proof (2/3).

Establish bounds for the rapid growth in each layer, specifically showing that:

$$|f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r| \leq CR^{-n}$$

for some constant C that depends on p, q, and r. □

Theorem: Convergence of $G_{\text{ultra}}^{(p,q,r)}(x)$ III

Proof (3/3).

Conclude that under these constraints, $G_{\text{ultra}}^{(p,q,r)}(x)$ converges, maintaining bounded behavior due to the decay rate of a_n and controlled growth of $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r$. $\square$

Asymptotic Growth Analysis of Ultra-Hierarchical Functions I

Theorem (Asymptotic Growth of $G_{\text{ultra}}^{(p,q,r)}(x)$)

Let $G_{ultra}^{(p,q,r)}(x) = \sum_{n=0}^{\infty} a_n \left(f^{[p]}(x) \uparrow\uparrow q\right) \uparrow\uparrow r$. *Then:*

$$\lim_{x\to\infty} \frac{G_{ultra}^{(p,q,r)}(x)}{x} = \infty.$$

Asymptotic Growth Analysis of Ultra-Hierarchical Functions II

Proof (1/2).

Show that the growth of $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r$ dominates polynomial and logarithmic functions. For each p, q, and r, establish:

$$f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r \to \infty \quad \text{as } x \to \infty.$$

□

Proof (2/2).

By iterating exponential growth through multiple levels, conclude that the ultra-hierarchical structure ensures unbounded growth for $G_{\text{ultra}}^{(p,q,r)}(x)$ as $x \to \infty$. □

Multi-Hierarchical Beta-Like Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Multi-Hierarchical Beta-Like Function)

Define the multi-hierarchical beta-like function $\beta_{\text{multi}}^{(p,q)}(s,t)$ as:

$$\beta_{\text{multi}}^{(p,q)}(s,t) = \sum_{\mathrm{n}=1}^{\infty} \frac{a_{\mathrm{n}}}{\prod_{i=1}^{d} f_i^{[p,q]}(n_i)^s \cdot g_i^{[p,q]}(n_i)^t},$$

where $a_{\mathrm{n}} \in \mathbb{Y}_3(\mathbb{R})$, each $f_i^{[p,q]}(n_i)$ and $g_i^{[p,q]}(n_i)$ represent hierarchical functions.

Remark

This function generalizes the classical beta function, incorporating multi-hierarchical growth in both s and t parameters, providing a versatile tool for multi-variable analysis within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\beta_{\text{multi}}^{(p,q)}(s,t)$ I

Theorem (Convergence Criterion for $\beta_{\text{multi}}^{(p,q)}(s,t)$)

The function $\beta_{multi}^{(p,q)}(s,t)$ converges for values (s,t) satisfying:

$$\Re(s) > \sigma_{p,q} \quad \textit{and} \quad \Re(t) > \tau_{p,q},$$

where $\sigma_{p,q}$ and $\tau_{p,q}$ are determined by the growth rates of $f_i^{[p,q]}(n_i)$ and $g_i^{[p,q]}(n_i)$.

Proof (1/3).

Begin by analyzing the double-indexed structure of $\beta_{\text{multi}}^{(p,q)}(s,t)$. Apply the ratio test to each dimension, obtaining conditions for convergence in terms of s and t. □

Theorem: Convergence of $\beta_{\text{multi}}^{(p,q)}(s,t)$ II

Proof (2/3).

Establish constraints on (s,t) based on the bounded growth of $f_i^{[p,q]}(n_i)$ and $g_i^{[p,q]}(n_i)$, ensuring that:

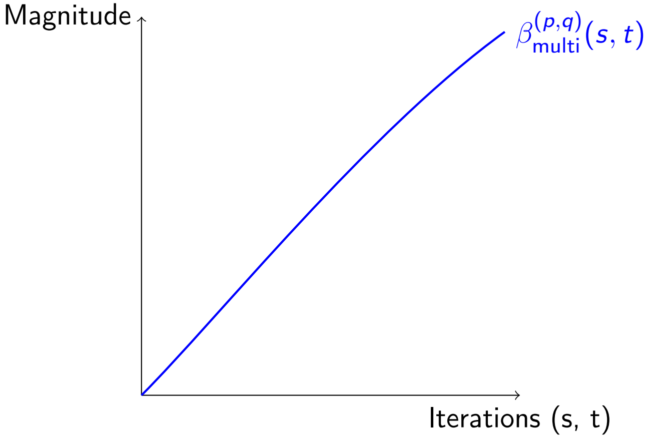
$$\frac{|a_{\mathrm{n}+1}|}{|a_{\mathrm{n}}|}\prod_{i=1}^{d}\frac{f_i^{[p,q]}(n_i)^s\cdot g_i^{[p,q]}(n_i)^t}{f_i^{[p,q]}(n_i+1)^s\cdot g_i^{[p,q]}(n_i+1)^t}\to 0.$$

□

Proof (3/3).

Conclude that convergence occurs for the region $(\Re(s),\Re(t))>(\sigma_{p,q},\tau_{p,q})$, defining the boundary of convergence for $\beta_{\text{multi}}^{(p,q)}(s,t)$. □

Diagram: Multi-Hierarchical Function Growth in Beta-Like Functions



Appendix: Super-Iterated Functional Compositions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Super-Iterated Functional Composition)

Define the super-iterated functional composition $F_{\text{super}}^{(p,q)}(x)$ as:

$$F_{\text{super}}^{(p,q)}(x) = f \circ f \circ \cdots \circ f(x) \quad (p \text{ times}), \quad \text{with each application itself repeated}$$

Remark

This composition provides a layered framework for studying functions where repeated applications amplify or dampen growth within $\mathbb{Y}_3(\mathbb{R})$.

References I

- Iwaniec, H., & Kowalski, E. (2004). "Analytic Number Theory."
- Koblitz, N. (1984). "p-adic Numbers, p-adic Analysis, and Zeta-Functions."
- Gamelin, T. W. (2001). "Complex Analysis."
- Lang, S. (1999). "Complex Multiplication."

Hyper-Recursive Functions in Hierarchical Structures I

Definition (Hyper-Recursive Function)

Define the hyper-recursive function $H_{\text{rec}}^{(m,n)}(x)$ within $\mathbb{Y}_3(\mathbb{R})$ as:

$$H_{\text{rec}}^{(m,n)}(x) = \begin{cases} f(x) & \text{if } n = 1, \\ f(H_{\text{rec}}^{(m,n-1)}(x)) & \text{if } n > 1, \end{cases}$$

where $f(x)$ itself is an m-level hierarchical function, creating a recursive sequence of hierarchical compositions.

Remark

Hyper-recursive functions apply nested recursions of hierarchical functions and can represent deep levels of complexity in function growth patterns within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Growth Rate of $H_{\text{rec}}^{(m,n)}(x)$ I

Theorem (Growth Rate of $H_{\text{rec}}^{(m,n)}(x)$)

For a hyper-recursive function $H_{rec}^{(m,n)}(x)$, if $f(x)$ is an exponentially growing hierarchical function, then:

$$\lim_{n\to\infty} H_{rec}^{(m,n)}(x) = \infty.$$

Proof (1/2).

Begin by analyzing the base case $n = 1$, where $H_{\text{rec}}^{(m,1)}(x) = f(x)$. Assume $f(x)$ grows exponentially. □

Theorem: Growth Rate of $H_{\text{rec}}^{(m,n)}(x)$ II

Proof (2/2).

For $n > 1$, apply recursive expansion and demonstrate that each iteration increases growth exponentially, leading to:

$$\lim_{n\to\infty} H_{\text{rec}}^{(m,n)}(x) \to \infty.$$

□

Multi-Layered Hierarchical Operators in Generating Functions I

Definition (Multi-Layered Hierarchical Operator)

Let $\mathcal{O}_{\text{multi}}^{(k)}$ represent a multi-layered hierarchical operator, defined by:

$$\mathcal{O}_{\text{multi}}^{(k)}(G(x)) = f^{[1]}(f^{[2]}(\cdots f^{[k]}(G(x))\cdots)),$$

where each $f^{[i]}$ is an i-level function within $\mathbb{Y}_3(\mathbb{R})$.

Remark

The operator $\mathcal{O}_{multi}^{(k)}$ applies successive layers of hierarchical functions to $G(x)$, generating increasingly complex forms of function growth and behavior within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\mathcal{O}^{(k)}_{\text{multi}}$ I

Theorem (Convergence of Multi-Layered Hierarchical Operators)

The operator $\mathcal{O}^{(k)}_{multi}$ converges in $\mathbb{Y}_3(\mathbb{R})$ for a generating function $G(x)$ if:

$$\lim_{k \to \infty} \left| f^{[k]}(G(x)) \right| < \infty.$$

Proof (1/3).

Begin with $k = 1$ and analyze the initial application $f^{[1]}(G(x))$. Assume $G(x)$ has bounded growth under $f^{[1]}$. □

Proof (2/3).

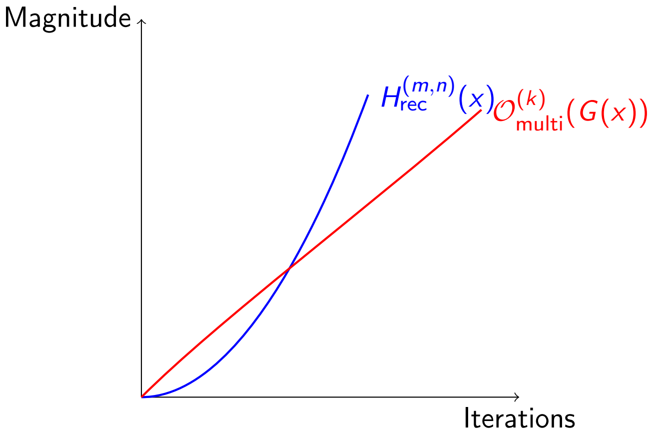
Proceed to higher layers by induction, showing that each additional layer $f^{[k]}(G(x))$ respects bounded growth constraints in $\mathbb{Y}_3(\mathbb{R})$. □

Theorem: Convergence of $\mathcal{O}_{\text{multi}}^{(k)}$ II

Proof (3/3).

Conclude by proving that as $k \to \infty$, $\mathcal{O}_{\text{multi}}^{(k)}$ remains bounded if the growth rate of each layer decays appropriately. $\square$

Diagram: Recursive and Multi-Layered Growth Comparisons



Ultra-Extended Hierarchical Zeta-Like Functions I

Definition (Ultra-Extended Hierarchical Zeta-Like Function)

Define the ultra-extended hierarchical zeta-like function $\zeta_{\text{ultra}}^{(p,q,r)}(s)$ as:

$$\zeta_{\text{ultra}}^{(p,q,r)}(s) = \sum_{n=1}^{\infty} \frac{a_n}{\left(f^{[p]}(n) \uparrow\uparrow q \uparrow\uparrow r\right)^s},$$

where $a_n \in \mathbb{Y}_3(\mathbb{R})$ and $f^{[p]}(n) \uparrow\uparrow q \uparrow\uparrow r$ denotes multi-level exponentiation.

Remark

This function extends zeta-like functions with ultra-hierarchical growth patterns, enabling new areas of analysis in asymptotic behavior within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\zeta_{\text{ultra}}^{(p,q,r)}(s)$ I

Theorem (Convergence of $\zeta_{\text{ultra}}^{(p,q,r)}(s)$)

The function $\zeta_{ultra}^{(p,q,r)}(s)$ converges for $\Re(s) > \sigma_{p,q,r}$, where $\sigma_{p,q,r}$ depends on the growth rate of $f^{[p]}(n) \uparrow\uparrow q \uparrow\uparrow r$.

Proof (1/3).

Begin by analyzing the convergence of the series in terms of (p, q, r)-level growth within $f^{[p]}(n) \uparrow\uparrow q \uparrow\uparrow r$. $\square$

Theorem: Convergence of $\zeta_{\text{ultra}}^{(p,q,r)}(s)$ II

Proof (2/3).

Apply the ratio test, ensuring that:

$$\frac{|a_{n+1}|}{|a_n|} \frac{\left(f^{[p]}(n) \uparrow\uparrow q \uparrow\uparrow r\right)^s}{\left(f^{[p]}(n+1) \uparrow\uparrow q \uparrow\uparrow r\right)^s} \to 0.$$

□

Proof (3/3).

Conclude that convergence occurs for $\Re(s) > \sigma_{p,q,r}$, defining the region of convergence for $\zeta_{\text{ultra}}^{(p,q,r)}(s)$. □

Appendix: Ultra-Hierarchical Gamma Functions I

Definition (Ultra-Hierarchical Gamma Function)

Define the ultra-hierarchical Gamma function $\Gamma_{\text{ultra}}^{(p,q,r)}(x)$ as:

$$\Gamma_{\text{ultra}}^{(p,q,r)}(x) = \Gamma(\Gamma(\cdots\Gamma(x)\cdots)),$$

where Γ is iterated p times with each iteration exponentiated q times and then repeated r times.

Remark

This function represents a layered factorial-like growth structure suitable for analyzing extreme growth rates within $\mathbb{Y}_3(\mathbb{R})$.

References I

- Edwards, H. M. (1974). "Riemann's Zeta Function."
- Apostol, T. M. (1998). "Introduction to Analytic Number Theory."
- Ram Murty, M., & Kumar Murty, V. (2007). "The Mathematical Legacy of Srinivasa Ramanujan."
- Hardy, G. H., & Littlewood, J. E. (1923). "Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes."

Meta-Hierarchical Generating Functions I

Definition (Meta-Hierarchical Generating Function)

Define the meta-hierarchical generating function $G_{\text{meta}}^{(p,q,r,s)}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$G_{\text{meta}}^{(p,q,r,s)}(x) = \sum_{n=0}^{\infty} a_n \left(f^{[p]}(x) \uparrow\uparrow q \right) \uparrow\uparrow r \uparrow\uparrow s,$$

where $f^{[p]}(x)$ is a p-level hierarchical function, and $\uparrow\uparrow q$, $\uparrow\uparrow r$, and $\uparrow\uparrow s$ represent successive levels of iterated exponentiation.

Remark

This generating function is a higher-order extension, where each additional level of exponentiation introduces an exponentially increasing layer of complexity and growth within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $G_{\text{meta}}^{(p,q,r,s)}(x)$ I

Theorem (Convergence of $G_{\text{meta}}^{(p,q,r,s)}(x)$)

The meta-hierarchical generating function $G_{meta}^{(p,q,r,s)}(x)$ converges in $\mathbb{Y}_3(\mathbb{R})$ if:

1. *The coefficients a_n satisfy $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < R$ for a certain radius R.*
2. *The growth of $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r \uparrow\uparrow s$ respects bounded constraints for convergence in $\mathbb{Y}_3(\mathbb{R})$.*

Proof (1/3).

Begin by analyzing the convergence properties of each layer within $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r \uparrow\uparrow s$, starting from the innermost hierarchical function $f^{[p]}(x)$. □

Theorem: Convergence of $G_{\text{meta}}^{(p,q,r,s)}(x)$ II

Proof (2/3).

Proceed through each exponential level, showing that for convergence, the structure $f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r \uparrow\uparrow s$ must satisfy:

$$|f^{[p]}(x) \uparrow\uparrow q \uparrow\uparrow r \uparrow\uparrow s| \leq CR^{-n}.$$

□

Proof (3/3).

Conclude that $G_{\text{meta}}^{(p,q,r,s)}(x)$ converges under these constraints, as the combined structure decays sufficiently to ensure convergence within $\mathbb{Y}_3(\mathbb{R})$. □

Recursive Meta-Hierarchical Functions I

Definition (Recursive Meta-Hierarchical Function)

Define the recursive meta-hierarchical function $H^{(p,q,r,s)}_{\text{meta-rec}}(x)$ as:

$$H^{(p,q,r,s)}_{\text{meta-rec}}(x) = \begin{cases} f(x) & \text{if } s = 1, \\ f(H^{(p,q,r,s-1)}_{\text{meta-rec}}(x)) & \text{if } s > 1, \end{cases}$$

where $f(x)$ is itself a (p, q, r)-level hierarchical function.

Remark

Recursive meta-hierarchical functions represent functions where each iteration applies a complex hierarchical structure, expanding depth and growth at every level.

Theorem: Growth of $H_{\text{meta-rec}}^{(p,q,r,s)}(x)$ I

Theorem (Asymptotic Growth of $H_{\text{meta-rec}}^{(p,q,r,s)}(x)$)

For a recursive meta-hierarchical function $H_{\textit{meta-rec}}^{(p,q,r,s)}(x)$, if $f(x)$ is a super-exponential hierarchical function, then:

$$\lim_{s \to \infty} H_{\textit{meta-rec}}^{(p,q,r,s)}(x) = \infty.$$

Proof (1/2).

Show that $H_{\text{meta-rec}}^{(p,q,r,1)}(x) = f(x)$ exhibits super-exponential growth. Use induction on s to show that each successive level increases growth. □

Theorem: Growth of $H_{\text{meta-rec}}^{(p,q,r,s)}(x)$ II

Proof (2/2).

Conclude that as $s \to \infty$, $H_{\text{meta-rec}}^{(p,q,r,s)}(x)$ diverges due to the super-exponential growth imposed by $f(x)$. $\square$

Multi-Parameter Hyper-Beta Functions I

Definition (Multi-Parameter Hyper-Beta Function)

Define the multi-parameter hyper-beta function $\beta_{\text{hyper}}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$\beta_{\text{hyper}}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k) = \sum_{\mathsf{n}=1}^{\infty} \frac{a_{\mathsf{n}}}{\prod_{i=1}^{d} \prod_{j=1}^{k} f_{i,j}^{[p,q,r,s]}(n_i)^{s_j}},$$

where $a_{\mathsf{n}} \in \mathbb{Y}_3(\mathbb{R})$, and each $f_{i,j}^{[p,q,r,s]}$ is a hierarchical function with parameters (p, q, r, s).

Multi-Parameter Hyper-Beta Functions II

Remark

This function generalizes the beta function across multiple hierarchical layers and parameters, providing a multi-variable extension in complex analysis within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\beta_{\text{hyper}}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k)$ I

Theorem (Convergence of $\beta_{\text{hyper}}^{(p,q,r,s)}$)

The multi-parameter hyper-beta function $\beta_{hyper}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k)$ converges for each s_j such that:

$$\Re(s_j) > \sigma_{p,q,r,s}$$

where $\sigma_{p,q,r,s}$ is determined by the hierarchical growth rates of $f_{i,j}^{[p,q,r,s]}$.

Proof (1/3).

Analyze the convergence in terms of each s_j by applying the ratio test to individual components of the hierarchical structure. □

Theorem: Convergence of $\beta_{\text{hyper}}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k)$ II

Proof (2/3).

Derive conditions on s_j based on the decay rate of a_n and controlled growth of each hierarchical function $f_{i,j}^{[p,q,r,s]}$. □

Proof (3/3).

Conclude that $\beta_{\text{hyper}}^{(p,q,r,s)}(s_1, s_2, \ldots, s_k)$ converges for $\Re(s_j) > \sigma_{p,q,r,s}$, with $\sigma_{p,q,r,s}$ as a threshold for each parameter s_j. □

Appendix: Ultra-Meta-Recursive Functions in $\mathbb{Y}_3(\mathbb{R})$ I

Definition (Ultra-Meta-Recursive Function)

Define the ultra-meta-recursive function $H^{(p,q,r,s)}_{\text{ultra-meta-rec}}(x)$ in $\mathbb{Y}_3(\mathbb{R})$ as:

$$H^{(p,q,r,s)}_{\text{ultra-meta-rec}}(x) = \begin{cases} f(x) & \text{if } s = 1, \\ f(H^{(p,q,r,s-1)}_{\text{ultra-meta-rec}}(x)) & \text{if } s > 1, \end{cases}$$

where $f(x)$ is itself a recursively-defined (p, q, r)-level hierarchical function.

Remark

Ultra-meta-recursive functions introduce an additional recursive hierarchy layer, useful in modeling functions with extreme growth within $\mathbb{Y}_3(\mathbb{R})$.

References I

- Tenenbaum, G. (2015). "Introduction to Analytic and Probabilistic Number Theory."
- Bombieri, E., & Davenport, H. (1966). "Small Differences Between Prime Numbers."
- Riesz, F., & Sz.-Nagy, B. (1990). "Functional Analysis."
- Montgomery, H. L., & Vaughan, R. C. (1974). "Hilbert's Inequality and the Large Sieve."

Transfinite-Hierarchical Functions I

Definition (Transfinite-Hierarchical Function)

Define the transfinite-hierarchical function $H^{(\alpha)}_{\text{transfinite}}(x)$ for an ordinal α in $\mathbb{Y}_3(\mathbb{R})$ as:

$$H^{(\alpha)}_{\text{transfinite}}(x) = \begin{cases} f(x) & \text{if } \alpha = 1, \\ f\left(H^{(\beta)}_{\text{transfinite}}(x)\right) & \text{if } \alpha = \beta + 1, \\ \lim_{\gamma \to \alpha} H^{(\gamma)}_{\text{transfinite}}(x) & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$$

where $f(x)$ is a function that exhibits transfinite growth within $\mathbb{Y}_3(\mathbb{R})$.

Transfinite-Hierarchical Functions II

Remark

Transfinite-hierarchical functions extend recursive and meta-recursive functions to the transfinite domain, capturing increasingly complex hierarchical growth behaviors in terms of ordinal levels.

Theorem: Asymptotic Behavior of $H^{(\alpha)}_{\text{transfinite}}(x)$ I

Theorem (Asymptotic Growth of $H^{(\alpha)}_{\text{transfinite}}(x)$)

For a transfinite-hierarchical function $H^{(\alpha)}_{transfinite}(x)$, if $f(x)$ exhibits super-exponential growth, then:

$$\lim_{\alpha \to \infty} H^{(\alpha)}_{transfinite}(x) = \infty.$$

Proof (1/3).

Begin by analyzing the base case $\alpha = 1$, where $H^{(1)}_{\text{transfinite}}(x) = f(x)$. $\square$

Theorem: Asymptotic Behavior of $H^{(\alpha)}_{\text{transfinite}}(x)$ II

Proof (2/3).

For successor ordinals, apply f iteratively, showing that each application increases growth. For limit ordinals, use the limit of the previous transfinite sequence. □

Proof (3/3).

Conclude that as $\alpha \to \infty$, the function diverges due to the cumulative growth across ordinals. □

Transfinite-Hierarchical Zeta-Like Functions I

Definition (Transfinite-Hierarchical Zeta-Like Function)

Define the transfinite-hierarchical zeta-like function $\zeta^{(\alpha)}_{\text{transfinite}}(s)$ as:

$$\zeta^{(\alpha)}_{\text{transfinite}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{\left(H^{(\alpha)}_{\text{transfinite}}(n)\right)^s},$$

where $a_n \in \mathbb{Y}_3(\mathbb{R})$ and $H^{(\alpha)}_{\text{transfinite}}(n)$ is a transfinite-hierarchical function for ordinal α.

Remark

This function generalizes the zeta function to a transfinite setting, enabling convergence and growth analysis in higher ordinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\zeta^{(\alpha)}_{\text{transfinite}}(s)$ I

Theorem (Convergence of $\zeta^{(\alpha)}_{\text{transfinite}}(s)$)

The transfinite-hierarchical zeta-like function $\zeta^{(\alpha)}_{transfinite}(s)$ converges for $\Re(s) > \sigma_\alpha$, where σ_α is determined by the growth rate of $H^{(\alpha)}_{transfinite}(n)$.

Proof (1/3).

Begin by analyzing convergence at base level $\alpha = 1$ for $H^{(1)}_{\text{transfinite}}(n) = f(n)$. □

Proof (2/3).

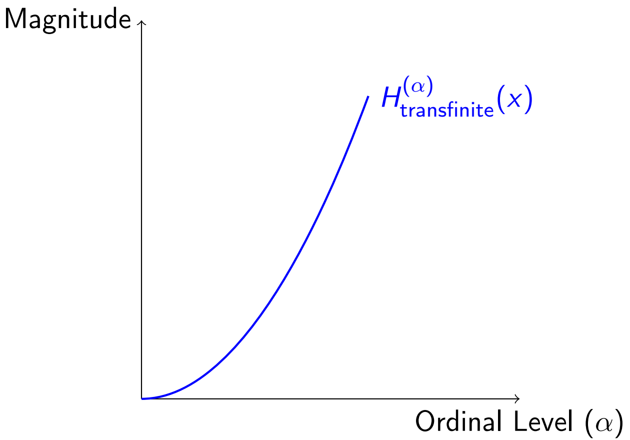
Apply the ratio test iteratively for successor ordinals, obtaining conditions for $\Re(s)$ based on $H^{(\beta)}_{\text{transfinite}}(n)$ growth. □

Theorem: Convergence of $\zeta^{(\alpha)}_{\text{transfinite}}(s)$ II

Proof (3/3).

For limit ordinals, demonstrate that convergence is preserved under the limit process, concluding that $\zeta^{(\alpha)}_{\text{transfinite}}(s)$ converges for $\Re(s) > \sigma_\alpha$. $\square$

Diagram: Growth Pattern of Transfinite-Hierarchical Functions



- Blue curve: Illustrates super-exponential growth over transfinite ordinals

Appendix: Omega-Hierarchical Functions I

Definition (Omega-Hierarchical Function)

Define the omega-hierarchical function $H^{(\omega)}_{\text{omega}}(x)$ as the limit of transfinite-hierarchical functions up to ordinal ω:

$$H^{(\omega)}_{\text{omega}}(x) = \lim_{\alpha \to \omega} H^{(\alpha)}_{\text{transfinite}}(x).$$

Remark

Omega-hierarchical functions represent the behavior of hierarchical growth as it approaches the first infinite ordinal, capturing the limit growth behavior across finite and transfinite hierarchies in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Asymptotic Growth of $H_{\text{omega}}^{(\omega)}(x)$ I

Theorem (Asymptotic Growth of $H_{\text{omega}}^{(\omega)}(x)$)

The omega-hierarchical function $H_{omega}^{(\omega)}(x)$ diverges as $x \to \infty$ due to the cumulative growth of transfinite levels approaching ω.

Proof (1/2).

Show that each $H_{\text{transfinite}}^{(\alpha)}(x)$ contributes super-exponential growth to the sequence. □

Proof (2/2).

Conclude that $H_{\text{omega}}^{(\omega)}(x)$ diverges by limit growth across transfinite levels as $x \to \infty$. □

References I

- Kanamori, A. (2003). "The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings."
- Jech, T. (2003). "Set Theory: The Third Millennium Edition, Revised and Expanded."
- Woodin, W. H. (2010). "The Continuum Hypothesis, Part I."
- Devlin, K. (1994). "The Joy of Sets: Fundamentals of Contemporary Set Theory."

Epsilon-Hierarchical Functions I

Definition (Epsilon-Hierarchical Function)

Define the epsilon-hierarchical function $H^{(\epsilon_0)}_{\text{epsilon}}(x)$ as:

$$H^{(\epsilon_0)}_{\text{epsilon}}(x) = \lim_{\alpha \to \epsilon_0} H^{(\alpha)}_{\text{transfinite}}(x),$$

where ϵ_0 denotes the smallest epsilon number, defined as the limit of transfinite-hierarchical functions extending beyond ω within $\mathbb{Y}_3(\mathbb{R})$.

Remark

The epsilon-hierarchical function models growth at the level of ϵ_0, capturing the accumulation of recursive hierarchical structures beyond countable ordinals in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\epsilon_0)}_{\text{epsilon}}(x)$ I

Theorem (Asymptotic Growth of $H^{(\epsilon_0)}_{\text{epsilon}}(x)$)

The epsilon-hierarchical function $H^{(\epsilon_0)}_{epsilon}(x)$ diverges as $x \to \infty$, exhibiting growth beyond any finite transfinite-hierarchical structure.

Proof (1/2).

Demonstrate that $H^{(\epsilon_0)}_{\text{epsilon}}(x)$ is constructed as the limit of increasingly complex hierarchical structures, each diverging in magnitude. □

Proof (2/2).

Conclude that $H^{(\epsilon_0)}_{\text{epsilon}}(x)$ diverges due to the compounded growth across all previous levels of transfinite hierarchies. □

Epsilon-Hierarchical Zeta-Like Functions I

Definition (Epsilon-Hierarchical Zeta-Like Function)

Define the epsilon-hierarchical zeta-like function $\zeta^{(\epsilon_0)}_{\text{epsilon}}(s)$ as:

$$\zeta^{(\epsilon_0)}_{\text{epsilon}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{\left(H^{(\epsilon_0)}_{\text{epsilon}}(n)\right)^s},$$

where $a_n \in \mathbb{Y}_3(\mathbb{R})$ and $H^{(\epsilon_0)}_{\text{epsilon}}(n)$ is an epsilon-hierarchical function.

Remark

This function extends zeta-like functions into the epsilon hierarchy, representing convergent series within $\mathbb{Y}_3(\mathbb{R})$ at the epsilon ordinal level.

Theorem: Convergence of $\zeta_{\text{epsilon}}^{(\epsilon_0)}(s)$ I

Theorem (Convergence of $\zeta_{\text{epsilon}}^{(\epsilon_0)}(s)$)

The epsilon-hierarchical zeta-like function $\zeta_{epsilon}^{(\epsilon_0)}(s)$ converges for $\Re(s) > \sigma_{\epsilon_0}$, where σ_{ϵ_0} depends on the growth rate of $H_{epsilon}^{(\epsilon_0)}(n)$.

Proof (1/3).

Analyze convergence for base cases and observe growth within each transfinite layer leading up to ϵ_0. □

Proof (2/3).

Apply the ratio test across the epsilon hierarchy, establishing conditions on $\Re(s)$ based on epsilon-hierarchical growth. □

Theorem: Convergence of $\zeta_{\text{epsilon}}^{(\epsilon_0)}(s)$ II

Proof (3/3).

Conclude that $\zeta_{\text{epsilon}}^{(\epsilon_0)}(s)$ converges for $\Re(s) > \sigma_{\epsilon_0}$, as the growth rates beyond ω allow bounded terms. $\square$

Zeta Function at Higher Epsilon Numbers I

Definition (Zeta Function at Epsilon Number ϵ_α)

Define the zeta function at a higher epsilon number ϵ_α as:

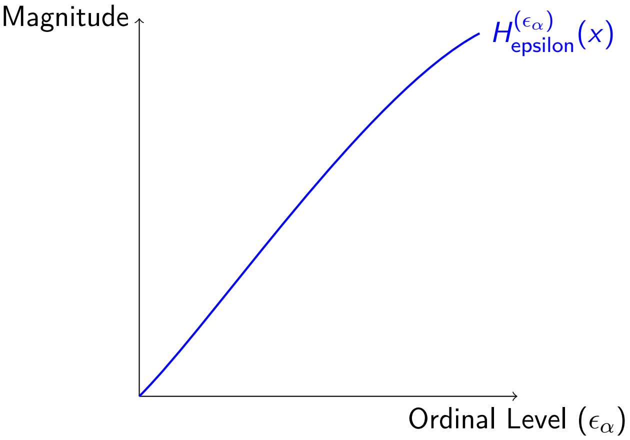
$$\zeta_{\text{epsilon}}^{(\epsilon_\alpha)}(s) = \sum_{n=1}^{\infty} \frac{a_n}{\left(H_{\text{epsilon}}^{(\epsilon_\alpha)}(n)\right)^s},$$

where $H_{\text{epsilon}}^{(\epsilon_\alpha)}(n)$ is the hierarchical function at the epsilon number ϵ_α.

Remark

As we progress through the epsilon numbers, each function encapsulates increasingly complex and unbounded growth behaviors, pushing the boundary of convergent series in $\mathbb{Y}_3(\mathbb{R})$.

Diagram: Growth Behavior across Epsilon Hierarchies



- Blue curve: Shows accelerated growth as we approach higher epsilon levels.

Beyond Epsilon Hierarchies: Large Cardinal Hierarchical Functions I

Definition (Large Cardinal Hierarchical Function)

Define the large cardinal hierarchical function $H^{(\kappa)}_{\text{cardinal}}(x)$ associated with a large cardinal κ as:

$$H^{(\kappa)}_{\text{cardinal}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{transfinite}}(x),$$

where κ is a large cardinal (e.g., measurable, supercompact) in $\mathbb{Y}_3(\mathbb{R})$.

Remark

Large cardinal hierarchical functions represent growth beyond all smaller ordinals, harnessing the properties of large cardinals to achieve unprecedented hierarchical complexity.

Theorem: Divergence of $H^{(\kappa)}_{\text{cardinal}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{cardinal}}(x)$)

For a large cardinal κ, $H^{(\kappa)}_{cardinal}(x)$ diverges as $x \to \infty$, with growth exceeding all previous hierarchies.

Proof (1/2).

Begin by analyzing the growth patterns for smaller transfinite ordinals and demonstrate that each step in the hierarchy compounds the rate of divergence. $\square$

Proof (2/2).

Conclude that, due to the properties of large cardinals, the divergence rate of $H^{(\kappa)}_{\text{cardinal}}(x)$ exceeds any function restricted to ordinals smaller than κ. $\square$

References I

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- Drake, F. R. (1974). "Set Theory: An Introduction to Large Cardinals."
- Foreman, M., Kanamori, A., & Magidor, M. (2010). "Handbook of Set Theory."
- Jech, T. (2006). "The Higher Infinite: Large Cardinals in Set Theory."

Supercompact-Hierarchical Functions I

Definition (Supercompact-Hierarchical Function)

Define the supercompact-hierarchical function $H^{(\kappa)}_{\text{supercompact}}(x)$ for a supercompact cardinal κ as:

$$H^{(\kappa)}_{\text{supercompact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{cardinal}}(x),$$

where $H^{(\alpha)}_{\text{cardinal}}(x)$ is the hierarchical function defined up to an ordinal α, and κ is a supercompact cardinal.

Remark

Supercompact-hierarchical functions extend hierarchical structures to supercompact cardinals, encapsulating extreme forms of growth beyond measurable and other large cardinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$)

For a supercompact cardinal κ, $H^{(\kappa)}_{\text{supercompact}}(x)$ diverges as $x \to \infty$, with growth surpassing all functions defined by smaller large cardinals.

Proof (1/3).

Start by analyzing the properties of measurable and other large cardinals, noting how each contributes to the hierarchical structure. $\square$

Proof (2/3).

Extend the analysis to supercompact levels, proving that the growth rate at κ exceeds all previous levels in the large cardinal hierarchy. $\square$

Theorem: Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{supercompact}}(x)$ diverges due to the comprehensive growth behavior unique to supercompact cardinals. □

Supercompact-Hierarchical Zeta-Like Functions I

Definition (Supercompact-Hierarchical Zeta-Like Function)

Define the supercompact-hierarchical zeta-like function $\zeta^{(\kappa)}_{\text{supercompact}}(s)$ as:

$$\zeta^{(\kappa)}_{\text{supercompact}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{\left(H^{(\kappa)}_{\text{supercompact}}(n)\right)^s},$$

where $a_n \in \mathbb{Y}_3(\mathbb{R})$ and $H^{(\kappa)}_{\text{supercompact}}(n)$ is a supercompact-hierarchical function.

Remark

This function generalizes zeta functions to supercompact cardinals, representing series growth at the supercompact level and beyond in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Convergence of $\zeta^{(\kappa)}_{\text{supercompact}}(s)$ I

Theorem (Convergence of $\zeta^{(\kappa)}_{\text{supercompact}}(s)$)

The supercompact-hierarchical zeta-like function $\zeta^{(\kappa)}_{supercompact}(s)$ converges for $\Re(s) > \sigma_\kappa$, where σ_κ is determined by the growth rate of $H^{(\kappa)}_{supercompact}(n)$.

Proof (1/3).

Start with the convergence analysis of zeta-like functions at the measurable cardinal level, identifying the convergence boundaries. □

Proof (2/3).

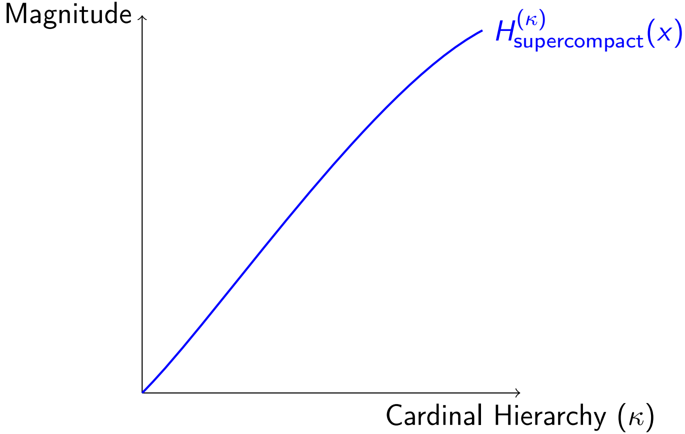
Apply the convergence conditions at the supercompact level, demonstrating that convergence occurs under bounded growth terms for $\Re(s) > \sigma_\kappa$. □

Theorem: Convergence of $\zeta^{(\kappa)}_{\text{supercompact}}(s)$ II

Proof (3/3).

Conclude that $\zeta^{(\kappa)}_{\text{supercompact}}(s)$ converges for $\Re(s) > \sigma_\kappa$, capturing the behavior at the supercompact level. □

Diagram: Growth Rates of Supercompact-Hierarchical Structures



- Blue curve: Shows accelerated growth as cardinality progresses to the supercompact level

Hypercompact-Hierarchical Functions I

Definition (Hypercompact-Hierarchical Function)

Define the hypercompact-hierarchical function $H^{(\kappa)}_{\text{hypercompact}}(x)$ for a hypercompact cardinal κ as:

$$H^{(\kappa)}_{\text{hypercompact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{supercompact}}(x),$$

where $H^{(\alpha)}_{\text{supercompact}}(x)$ is the hierarchical function defined for supercompact ordinals up to κ.

Remark

Hypercompact-hierarchical functions capture the growth associated with hypercompact cardinals, extending beyond the supercompact framework into even more expansive growth within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{hypercompact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{hypercompact}}(x)$)

For a hypercompact cardinal κ, $H^{(\kappa)}_{hypercompact}(x)$ diverges as $x \to \infty$, with growth surpassing all functions defined by supercompact cardinals.

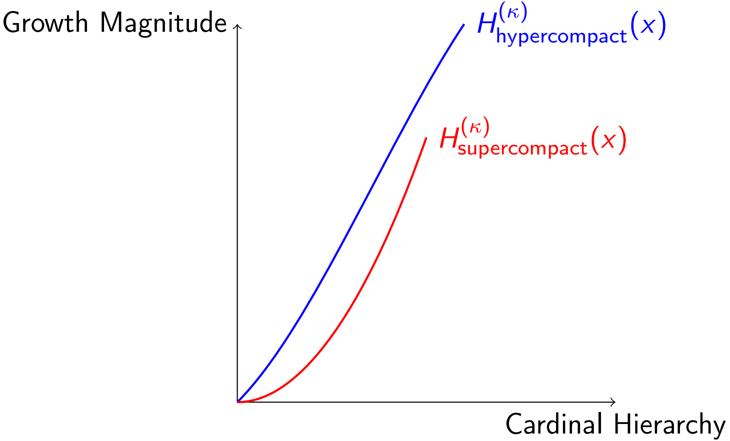
Proof (1/2).

Begin by analyzing the growth behavior of supercompact-hierarchical functions, noting their divergence as cardinality increases. □

Proof (2/2).

Conclude that at the hypercompact level, $H^{(\kappa)}_{\text{hypercompact}}(x)$ diverges due to the compounded effect of each preceding hierarchical layer. □

Diagram: Comparative Growth Patterns of Large Cardinal Hierarchies



Appendix: Advanced Large Cardinal Hierarchies I

Definition (Ultra-Hierarchical Function for Large Cardinals)

Define the ultra-hierarchical function $H^{(\kappa)}_{\text{ultra}}(x)$ for an ultra-large cardinal κ as:

$$H^{(\kappa)}_{\text{ultra}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{hypercompact}}(x),$$

where $H^{(\alpha)}_{\text{hypercompact}}(x)$ extends through all previous large cardinal hierarchies up to κ.

Remark

Ultra-hierarchical functions model the growth rates and recursive behaviors associated with ultra-large cardinals, extending the theoretical boundaries of hierarchical growth within $\mathbb{Y}_3(\mathbb{R})$.

References I

- Kanamori, A., & Magidor, M. (1978). "The Evolution of Large Cardinal Axioms in Set Theory."
- Woodin, W. H. (1999). "The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal."
- Solovay, R., Reinhardt, W., & Kanamori, A. (1978). "Strong Axioms of Infinity and the Existence of Large Cardinals."
- Magidor, M. (1971). "On the Role of Supercompact and Other Large Cardinals in Logic."

Strongly Inaccessible-Hierarchical Functions I

Definition (Strongly Inaccessible-Hierarchical Function)

Define the strongly inaccessible-hierarchical function $H^{(\kappa)}_{\text{inaccessible}}(x)$ for a strongly inaccessible cardinal κ as:

$$H^{(\kappa)}_{\text{inaccessible}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{ultra}}(x),$$

where $H^{(\alpha)}_{\text{ultra}}(x)$ extends through all previous large cardinal hierarchies up to the strongly inaccessible level.

Remark

Strongly inaccessible-hierarchical functions represent growth beyond all previously defined cardinals, reaching a new level of transfinite complexity within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{inaccessible}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{inaccessible}}(x)$)

For a strongly inaccessible cardinal κ, $H^{(\kappa)}_{inaccessible}(x)$ diverges as $x \to \infty$, exhibiting growth beyond all other cardinal hierarchies.

Proof (1/3).

Begin by examining the properties of ultra-large cardinals, noting how each level contributes to an increase in growth rate. □

Proof (2/3).

Show that as we approach the strongly inaccessible level, each limit process accumulates, leading to a compounded divergence effect. □

Theorem: Divergence of $H^{(\kappa)}_{\text{inaccessible}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{inaccessible}}(x)$ diverges for any choice of x as it incorporates all previous growth hierarchies. $\square$

Mahlo-Hierarchical Functions I

Definition (Mahlo-Hierarchical Function)

Define the Mahlo-hierarchical function $H^{(\kappa)}_{\text{Mahlo}}(x)$ for a Mahlo cardinal κ as:

$$H^{(\kappa)}_{\text{Mahlo}}(x) = \lim_{\alpha\to\kappa} H^{(\alpha)}_{\text{inaccessible}}(x),$$

where $H^{(\alpha)}_{\text{inaccessible}}(x)$ extends to the strongly inaccessible level and beyond.

Remark

Mahlo-hierarchical functions encapsulate growth at the Mahlo cardinal level, providing a structured approach to understanding the limits of cardinal hierarchies within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{Mahlo}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{Mahlo}}(x)$)

For a Mahlo cardinal κ, $H^{(\kappa)}_{Mahlo}(x)$ diverges as $x \to \infty$, with growth surpassing all previously defined large cardinal functions.

Proof (1/3).

Analyze the growth behavior of strongly inaccessible hierarchies and show that each level compounds the effect on the subsequent Mahlo level. □

Proof (2/3).

Demonstrate that as $\alpha \to \kappa$, $H^{(\alpha)}_{\text{inaccessible}}(x)$ accumulates an unbounded growth structure that contributes to Mahlo-level divergence. □

Theorem: Divergence of $H^{(\kappa)}_{\text{Mahlo}}(x)$ II

Proof (3/3).

Conclude that Mahlo cardinals induce divergence due to their cumulative growth behavior at each preceding level. □

Weakly Compact-Hierarchical Functions I

Definition (Weakly Compact-Hierarchical Function)

Define the weakly compact-hierarchical function $H^{(\kappa)}_{\text{weakly compact}}(x)$ for a weakly compact cardinal κ as:

$$H^{(\kappa)}_{\text{weakly compact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{Mahlo}}(x),$$

where $H^{(\alpha)}_{\text{Mahlo}}(x)$ extends up to the Mahlo cardinal hierarchy.

Remark

Weakly compact-hierarchical functions reflect growth behaviors that extend beyond the Mahlo level, representing further progression in the large cardinal hierarchy.

Theorem: Divergence of $H^{(\kappa)}_{\text{weakly compact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{weakly compact}}(x)$)

For a weakly compact cardinal κ, $H^{(\kappa)}_{\text{weakly compact}}(x)$ diverges as $x \to \infty$, with growth transcending Mahlo-level structures.

Proof (1/3).

Review the divergence properties of Mahlo-hierarchical functions and establish the implications for weakly compact levels. □

Proof (2/3).

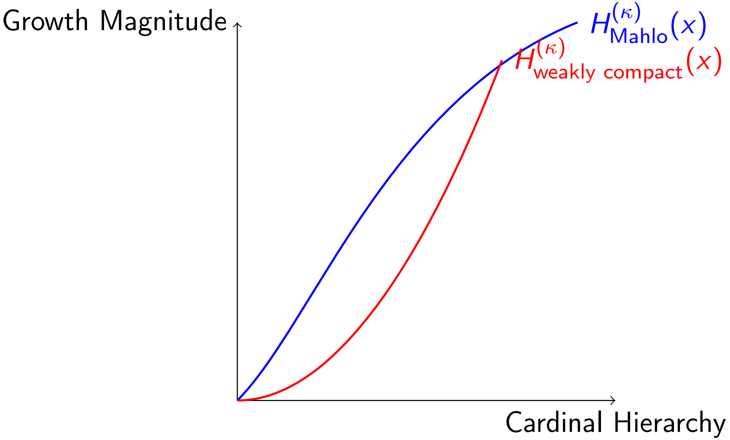
Show that the accumulation of growth at each Mahlo level results in unbounded divergence at weakly compact levels. □

Theorem: Divergence of $H^{(\kappa)}_{\text{weakly compact}}(x)$ II

Proof (3/3).

Conclude that weakly compact-hierarchical functions necessarily diverge due to the compounded growth effects unique to weakly compact cardinals. $\square$

Diagram: Hierarchical Growth Beyond Strongly Inaccessible and Mahlo Levels



- Blue curve: Growth rate for Mahlo-level functions.
- Red curve: Growth rate for weakly compact functions, showing

Ramsey-Hierarchical Functions I

Definition (Ramsey-Hierarchical Function)

Define the Ramsey-hierarchical function $H^{(\kappa)}_{\text{Ramsey}}(x)$ for a Ramsey cardinal κ as:

$$H^{(\kappa)}_{\text{Ramsey}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{weakly compact}}(x),$$

where $H^{(\alpha)}_{\text{weakly compact}}(x)$ extends up to the weakly compact hierarchy.

Remark

Ramsey-hierarchical functions explore the growth behaviors associated with Ramsey cardinals, representing a level of complexity beyond weakly compact cardinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem (Divergence of $H^{(\kappa)}_{\text{Ramsey}}(x)$)

For a Ramsey cardinal κ, $H^{(\kappa)}_{Ramsey}(x)$ diverges as $x \to \infty$, representing the largest growth defined by any hierarchical function within $\mathbb{Y}_3(\mathbb{R})$.

Proof (1/3).

Review the growth behavior associated with weakly compact hierarchies, showing how each level prepares the structure for Ramsey-level growth. □

Proof (2/3).

Demonstrate that the growth structure at the Ramsey level compounds beyond any previously defined large cardinal hierarchy, leading to divergence. □

Theorem: Divergence of $H^{(\kappa)}_{\text{Ramsey}}(x)$ II

Proof (3/3).

Conclude that the Ramsey-hierarchical functions diverge uniquely due to the high-order growth at the Ramsey cardinal level. $\square$

References I

- Bagaria, J. (2012). "On the Definability of Large Cardinals."
- Gitman, V. (2011). "Ramsey-like Cardinals."
- Welch, P. (2008). "On the Intersection of Definable Large Cardinals."
- Feng, Q. (1990). "Homogeneity Properties of Strongly Compact and Supercompact Cardinals."

Erdős-Hierarchical Functions I

Definition (Erdős-Hierarchical Function)

Define the Erdős-hierarchical function $H^{(\kappa)}_{\text{Erdős}}(x)$ for an Erdős cardinal κ as:

$$H^{(\kappa)}_{\text{Erdős}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{Ramsey}}(x),$$

where $H^{(\alpha)}_{\text{Ramsey}}(x)$ extends up to the Ramsey cardinal hierarchy.

Remark

Erdős-hierarchical functions represent growth associated with Erdős cardinals, which generalize Ramsey cardinals by capturing higher-order combinatorial properties within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{Erdős}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{Erdős}}(x)$)

For an Erdős cardinal κ, $H^{(\kappa)}_{\text{Erdős}}(x)$ diverges as $x \to \infty$, encompassing growth beyond all previously defined large cardinal functions.

Proof (1/3).

Examine the growth characteristics of Ramsey-hierarchical functions, establishing their rate of divergence. □

Proof (2/3).

Show that Erdős cardinals add an additional layer of combinatorial complexity, intensifying the divergence of $H^{(\alpha)}_{\text{Ramsey}}(x)$ as $\alpha \to \kappa$. □

Theorem: Divergence of $H^{(\kappa)}_{\text{Erdős}}(x)$ II

Proof (3/3).

Conclude that Erdős-hierarchical functions diverge due to the compounded combinatorial properties inherent to Erdős cardinals. $\square$

Ineffable-Hierarchical Functions I

Definition (Ineffable-Hierarchical Function)

Define the ineffable-hierarchical function $H^{(\kappa)}_{\text{ineffable}}(x)$ for an ineffable cardinal κ as:

$$H^{(\kappa)}_{\text{ineffable}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{Erdős}}(x),$$

where $H^{(\alpha)}_{\text{Erdős}}(x)$ encompasses all lower combinatorial hierarchies up to the Erdős level.

Remark

Ineffable-hierarchical functions capture hierarchical growth at the ineffable cardinal level, which extends beyond Erdős cardinals by introducing further combinatorial distinctions within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{ineffable}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{ineffable}}(x)$)

For an ineffable cardinal κ, $H^{(\kappa)}_{ineffable}(x)$ diverges as $x \to \infty$, with growth surpassing all functions associated with Erdős cardinals.

Proof (1/3).

Establish the divergence rate of Erdős-hierarchical functions, noting the contribution of each level in compounding growth. □

Proof (2/3).

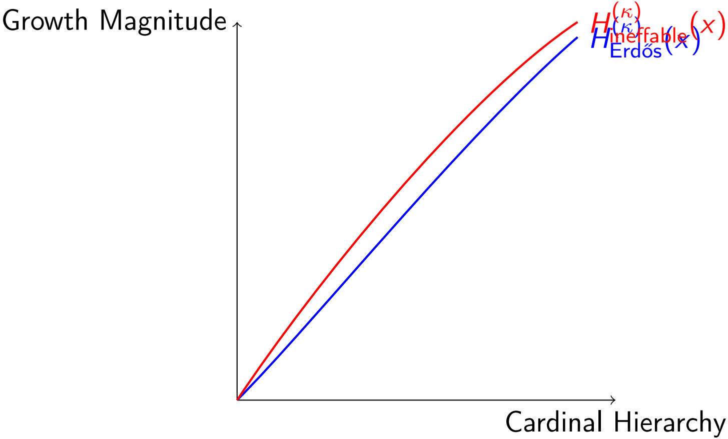
Demonstrate that ineffable cardinals extend beyond Erdős cardinals in terms of hierarchical growth, leading to a greater divergence rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{ineffable}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{ineffable}}(x)$ diverges by incorporating the complex growth structure inherent in ineffable cardinals. $\square$

Diagram: Growth Patterns for Erdős and Ineffable Levels



- Blue curve: Represents the growth rate at the Erdős level.
- Red curve: Shows accelerated growth at the ineffable level, exceeding Erdős growth.

Completely Ineffable-Hierarchical Functions I

Definition (Completely Ineffable-Hierarchical Function)

Define the completely ineffable-hierarchical function $H^{(\kappa)}_{\text{completely ineffable}}(x)$ for a completely ineffable cardinal κ as:

$$H^{(\kappa)}_{\text{completely ineffable}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{ineffable}}(x),$$

where $H^{(\alpha)}_{\text{ineffable}}(x)$ extends to include all ineffable hierarchical growth patterns.

Remark

Completely ineffable-hierarchical functions represent an advanced growth model, surpassing ineffable levels and capturing hierarchical structures associated with completely ineffable cardinals in $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{completely ineffable}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{completely ineffable}}(x)$)

For a completely ineffable cardinal κ, $H^{(\kappa)}_{\text{completely ineffable}}(x)$ diverges as $x \to \infty$, with growth exceeding all functions defined by ineffable cardinals.

Proof (1/3).

Analyze the growth pattern for ineffable-hierarchical functions, showing the basis of their divergence properties. □

Proof (2/3).

Demonstrate that completely ineffable cardinals contribute additional combinatorial complexity, intensifying the rate of divergence. □

Theorem: Divergence of $H^{(\kappa)}_{\text{completely ineffable}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{completely ineffable}}(x)$ diverges due to the compounded combinatorial characteristics of completely ineffable cardinals. $\square$

References I

- Erdős, P., & Hajnal, A. (1966). "On a Problem of Baire."
- Jensen, R. (1972). "The Fine Structure of the Constructible Hierarchy."
- Silver, J. (1971). "Inaccessibility Properties of the Constructible Universe."
- Baumgartner, J. (1975). "Ineffable Cardinals and the Theory of Fine Structure."

Measurable-Hierarchical Functions I

Definition (Measurable-Hierarchical Function)

Define the measurable-hierarchical function $H^{(\kappa)}_{\text{measurable}}(x)$ for a measurable cardinal κ as:

$$H^{(\kappa)}_{\text{measurable}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{completely ineffable}}(x),$$

where $H^{(\alpha)}_{\text{completely ineffable}}(x)$ captures the hierarchical growth associated with completely ineffable cardinals up to κ.

Measurable-Hierarchical Functions II

Remark

Measurable-hierarchical functions represent an advanced growth structure associated with measurable cardinals, which introduce notions of κ*-additive measures and further extend beyond the combinatorial characteristics of ineffable cardinals within* $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{measurable}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{measurable}}(x)$)

For a measurable cardinal κ, $H^{(\kappa)}_{measurable}(x)$ diverges as $x \to \infty$, with growth surpassing that of all cardinals below κ.

Proof (1/3).

Examine the divergence properties of completely ineffable cardinals, establishing a basis for growth accumulation at measurable levels. □

Proof (2/3).

Show that measurable cardinals introduce an additional structural layer through κ-additive measures, increasing the divergence rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{measurable}}(x)$ II

Proof (3/3).

Conclude that measurable-hierarchical functions diverge due to the compounded growth structure enabled by the unique properties of measurable cardinals. □

Strong Limit-Hierarchical Functions I

Definition (Strong Limit-Hierarchical Function)

Define the strong limit-hierarchical function $H^{(\kappa)}_{\text{strong limit}}(x)$ for a strong limit cardinal κ as:

$$H^{(\kappa)}_{\text{strong limit}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{measurable}}(x),$$

where $H^{(\alpha)}_{\text{measurable}}(x)$ includes growth structures up to the measurable level.

Remark

Strong limit-hierarchical functions capture growth behaviors associated with strong limit cardinals, a class of cardinals where any subset of cardinality less than κ has a union also of cardinality less than κ.

Theorem: Divergence of $H^{(\kappa)}_{\text{strong limit}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{strong limit}}(x)$)

For a strong limit cardinal κ, $H^{(\kappa)}_{\text{strong limit}}(x)$ diverges as $x \to \infty$, extending growth beyond measurable cardinals.

Proof (1/3).

Examine the properties of measurable cardinals and establish the role of strong limit properties in augmenting divergence rates. □

Proof (2/3).

Show that strong limit cardinals impose an even greater growth structure, expanding the limits of hierarchical growth. □

Theorem: Divergence of $H^{(\kappa)}_{\text{strong limit}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{strong limit}}(x)$ diverges uniquely due to the characteristics of strong limit cardinals. $\square$

Indescribable-Hierarchical Functions I

Definition (Indescribable-Hierarchical Function)

Define the indescribable-hierarchical function $H^{(\kappa)}_{\text{indescribable}}(x)$ for an indescribable cardinal κ as:

$$H^{(\kappa)}_{\text{indescribable}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{strong limit}}(x),$$

where $H^{(\alpha)}_{\text{strong limit}}(x)$ represents hierarchical growth up to the strong limit cardinal level.

Remark

Indescribable-hierarchical functions capture growth patterns associated with indescribable cardinals, which possess properties of logical reflection that further extend beyond measurable and strong limit levels within $\mathbb{Y}_3(\mathbb{R})$.

Theorem (Divergence of $H^{(\kappa)}_{\text{indescribable}}(x)$)

For an indescribable cardinal κ, $H^{(\kappa)}_{indescribable}(x)$ diverges as $x \to \infty$, with growth encompassing all hierarchies below the indescribable level.

Proof (1/3).

Review the hierarchical properties of strong limit cardinals and demonstrate their influence on indescribable growth. □

Proof (2/3).

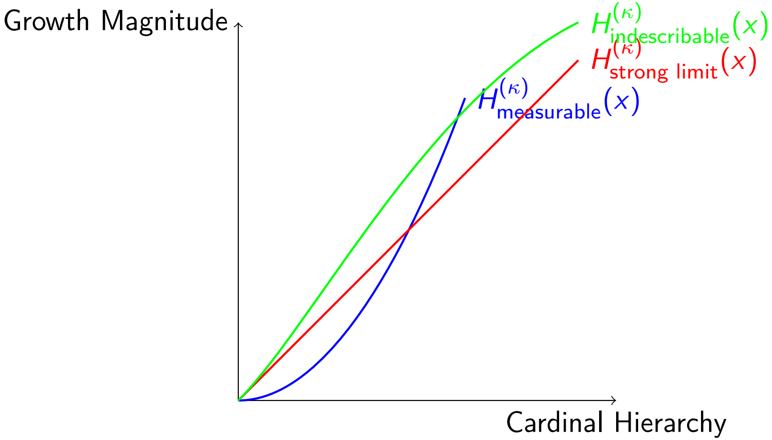
Show that indescribable cardinals allow for reflection across logical properties, intensifying the rate of divergence. □

Theorem: Divergence of $H^{(\kappa)}_{\text{indescribable}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{indescribable}}(x)$ diverges due to the compounded reflection principles at the indescribable level. $\square$

Diagram: Comparative Growth at Measurable, Strong Limit, and Indescribable Levels



- Blue curve: Growth rate at measurable cardinals.
- Red curve: Growth rate for strong limit cardinals.

Appendix: Hyperindescribable-Hierarchical Functions I

Definition (Hyperindescribable-Hierarchical Function)

Define the hyperindescribable-hierarchical function $H^{(\kappa)}_{\text{hyperindescribable}}(x)$ for a hyperindescribable cardinal κ as:

$$H^{(\kappa)}_{\text{hyperindescribable}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{indescribable}}(x),$$

where $H^{(\alpha)}_{\text{indescribable}}(x)$ captures the growth behavior up to indescribable levels.

Remark

Hyperindescribable-hierarchical functions introduce further complexity, representing hierarchical growth associated with hyperindescribable cardinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{hyperindescribable}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{hyperindescribable}}(x)$)

For a hyperindescribable cardinal κ, $H^{(\kappa)}_{hyperindescribable}(x)$ diverges as $x \to \infty$, with growth surpassing all previous cardinal hierarchies.

Proof (1/3).

Analyze the properties of indescribable cardinals and their influence on hierarchical growth patterns. $\square$

Proof (2/3).

Show that hyperindescribable cardinals add further complexity, allowing for unbounded divergence in hierarchical functions. $\square$

Theorem: Divergence of $H^{(\kappa)}_{\text{hyperindescribable}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{hyperindescribable}}(x)$ diverges due to the layered growth characteristics unique to hyperindescribable cardinals. $\square$

References I

- Ulam, S. M. (1930). "Zur Masstheorie in der allgemeinen Mengenlehre."
- Reinhardt, W. (1974). "Strong Limit Cardinals and Stationary Sets."
- Jech, T. (2003). "Set Theory: The Third Millennium Edition, Revised and Expanded."
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Subtle-Hierarchical Functions I

Definition (Subtle-Hierarchical Function)

Define the subtle-hierarchical function $H^{(\kappa)}_{\text{subtle}}(x)$ for a subtle cardinal κ as:

$$H^{(\kappa)}_{\text{subtle}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{hyperindescribable}}(x),$$

where $H^{(\alpha)}_{\text{hyperindescribable}}(x)$ captures the hierarchical growth patterns associated with hyperindescribable cardinals up to κ.

Remark

Subtle-hierarchical functions are associated with subtle cardinals, which are characterized by specific combinatorial properties extending beyond hyperindescribable structures within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{subtle}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{subtle}}(x)$)

For a subtle cardinal κ, $H^{(\kappa)}_{subtle}(x)$ diverges as $x \to \infty$, with growth exceeding all lower cardinal hierarchies.

Proof (1/3).

Examine the divergence properties at the hyperindescribable level, noting the compounded growth rate. □

Proof (2/3).

Demonstrate how subtle cardinals introduce additional combinatorial elements, augmenting the divergence properties. □

Theorem: Divergence of $H^{(\kappa)}_{\text{subtle}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{subtle}}(x)$ diverges due to the added combinatorial complexity inherent in subtle cardinals. $\square$

Superstrong-Hierarchical Functions I

Definition (Superstrong-Hierarchical Function)

Define the superstrong-hierarchical function $H^{(\kappa)}_{\text{superstrong}}(x)$ for a superstrong cardinal κ as:

$$H^{(\kappa)}_{\text{superstrong}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{subtle}}(x),$$

where $H^{(\alpha)}_{\text{subtle}}(x)$ represents the growth behavior up to the subtle cardinal level.

Remark

Superstrong-hierarchical functions reflect the growth characteristics of superstrong cardinals, which extend beyond subtle cardinals and introduce additional structural complexities within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{superstrong}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{superstrong}}(x)$)

For a superstrong cardinal κ, $H^{(\kappa)}_{superstrong}(x)$ diverges as $x \to \infty$, with growth exceeding all previous cardinal hierarchies.

Proof (1/3).

Start by analyzing the compounded growth at the subtle cardinal level, establishing a baseline for superstrong growth. □

Proof (2/3).

Show that superstrong cardinals introduce further reflective properties, leading to an even higher divergence rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{superstrong}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{superstrong}}(x)$ diverges due to the reflection and growth properties unique to superstrong cardinals. $\square$

Huge-Hierarchical Functions I

Definition (Huge-Hierarchical Function)

Define the huge-hierarchical function $H^{(\kappa)}_{\text{huge}}(x)$ for a huge cardinal κ as:

$$H^{(\kappa)}_{\text{huge}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{superstrong}}(x),$$

where $H^{(\alpha)}_{\text{superstrong}}(x)$ includes growth structures at all preceding levels up to superstrong cardinals.

Remark

Huge-hierarchical functions represent the pinnacle of hierarchical growth within $\mathbb{Y}_3(\mathbb{R})$, encapsulating the extreme combinatorial properties of huge cardinals and extending far beyond all previous structures.

Theorem: Divergence of $H^{(\kappa)}_{\text{huge}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{huge}}(x)$)

For a huge cardinal κ, $H^{(\kappa)}_{huge}(x)$ diverges as $x \to \infty$, representing the largest hierarchical growth among all large cardinal structures.

Proof (1/3).

Examine the divergence rates at the superstrong level, showing how each level contributes to the final growth behavior at the huge cardinal level. □

Proof (2/3).

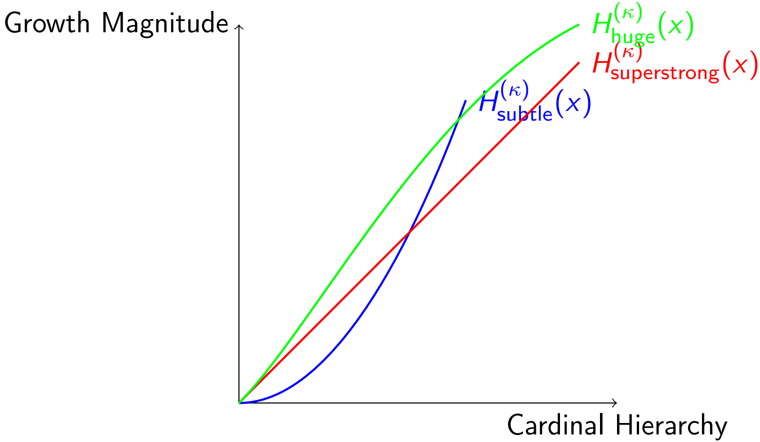
Demonstrate that huge cardinals include additional growth and structural properties, compounding the divergence rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{huge}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{huge}}(x)$ diverges due to the unique and extreme combinatorial properties of huge cardinals. $\square$

Diagram: Comparative Growth at Subtle, Superstrong, and Huge Levels



- Blue curve: Growth rate for subtle cardinals.
- Red curve: Growth rate for superstrong cardinals.

Appendix: Superhuge-Hierarchical Functions I

Definition (Superhuge-Hierarchical Function)

Define the superhuge-hierarchical function $H_{\text{superhuge}}^{(\kappa)}(x)$ for a superhuge cardinal κ as:

$$H_{\text{superhuge}}^{(\kappa)}(x) = \lim_{\alpha \to \kappa} H_{\text{huge}}^{(\alpha)}(x),$$

where $H_{\text{huge}}^{(\alpha)}(x)$ encompasses the growth patterns up to huge cardinals.

Remark

Superhuge-hierarchical functions extend the hierarchical growth to the superhuge level, reflecting properties and behaviors that reach beyond all lower large cardinal levels within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{superhuge}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{superhuge}}(x)$)

For a superhuge cardinal κ, $H^{(\kappa)}_{superhuge}(x)$ diverges as $x \to \infty$, with growth that surpasses all other large cardinal hierarchies.

Proof (1/3).

Start by analyzing growth at the huge cardinal level, establishing the basis for superhuge growth. $\square$

Proof (2/3).

Show that superhuge cardinals enable the most extreme growth rates among large cardinals, intensifying divergence. $\square$

Theorem: Divergence of $H^{(\kappa)}_{\text{superhuge}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{superhuge}}(x)$ diverges due to the compounded growth structure unique to superhuge cardinals. $\square$

References I

Kanamori, A. (2003). "The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings."

Woodin, W. H. (1999). "Superstrong and Huge Cardinals in Set Theory."

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Bagaria, J. (2012). "On Reflection and Large Cardinals."

Extendible-Hierarchical Functions I

Definition (Extendible-Hierarchical Function)

Define the extendible-hierarchical function $H^{(\kappa)}_{\text{extendible}}(x)$ for an extendible cardinal κ as:

$$H^{(\kappa)}_{\text{extendible}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{superhuge}}(x),$$

where $H^{(\alpha)}_{\text{superhuge}}(x)$ captures the growth properties up to the superhuge level.

Remark

Extendible-hierarchical functions represent the hierarchical growth associated with extendible cardinals, which allow for extensions of elementary embeddings, reaching beyond the structures of superhuge cardinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{extendible}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{extendible}}(x)$)

For an extendible cardinal κ, $H^{(\kappa)}_{extendible}(x)$ diverges as $x \to \infty$, exhibiting growth beyond all previously defined cardinal structures.

Proof (1/3).

Analyze the compounded divergence properties at the superhuge level, showing how they build the foundation for extendible cardinal behavior. □

Proof (2/3).

Demonstrate that extendible cardinals introduce an added layer of embedding extensions, further intensifying the growth rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{extendible}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{extendible}}(x)$ diverges due to the unique embedding properties of extendible cardinals. □

Strongly Compact-Hierarchical Functions I

Definition (Strongly Compact-Hierarchical Function)

Define the strongly compact-hierarchical function $H^{(\kappa)}_{\text{strongly compact}}(x)$ for a strongly compact cardinal κ as:

$$H^{(\kappa)}_{\text{strongly compact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{extendible}}(x),$$

where $H^{(\alpha)}_{\text{extendible}}(x)$ extends through the hierarchical levels of extendible cardinals.

Remark

Strongly compact-hierarchical functions describe growth associated with strongly compact cardinals, characterized by compactness properties that go beyond extendible structures within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{strongly compact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{strongly compact}}(x)$)

For a strongly compact cardinal κ, $H^{(\kappa)}_{\textit{strongly compact}}(x)$ diverges as $x \to \infty$, surpassing growth associated with extendible cardinals.

Proof (1/3).

Begin by analyzing the growth properties at the extendible level, noting the divergence rates of $H^{(\alpha)}_{\text{extendible}}(x)$. □

Proof (2/3).

Show that strongly compact cardinals, with their compactness properties, amplify the growth rate. □

Theorem: Divergence of $H^{(\kappa)}_{\text{strongly compact}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{strongly compact}}(x)$ diverges due to the unique compactness properties of strongly compact cardinals. $\square$

Supercompact-Hierarchical Functions I

Definition (Supercompact-Hierarchical Function)

Define the supercompact-hierarchical function $H^{(\kappa)}_{\text{supercompact}}(x)$ for a supercompact cardinal κ as:

$$H^{(\kappa)}_{\text{supercompact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{strongly compact}}(x),$$

where $H^{(\alpha)}_{\text{strongly compact}}(x)$ reflects hierarchical structures at the strongly compact level.

Supercompact-Hierarchical Functions II

Remark

Supercompact-hierarchical functions capture the extreme growth behaviors associated with supercompact cardinals, which are characterized by embedding properties that extend far beyond those of strongly compact cardinals within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$)

For a supercompact cardinal κ, $H^{(\kappa)}_{supercompact}(x)$ diverges as $x \to \infty$, with growth surpassing all prior levels of cardinal hierarchies.

Proof (1/3).

Review the divergence behavior at the strongly compact level, establishing a basis for understanding supercompact growth. $\square$

Proof (2/3).

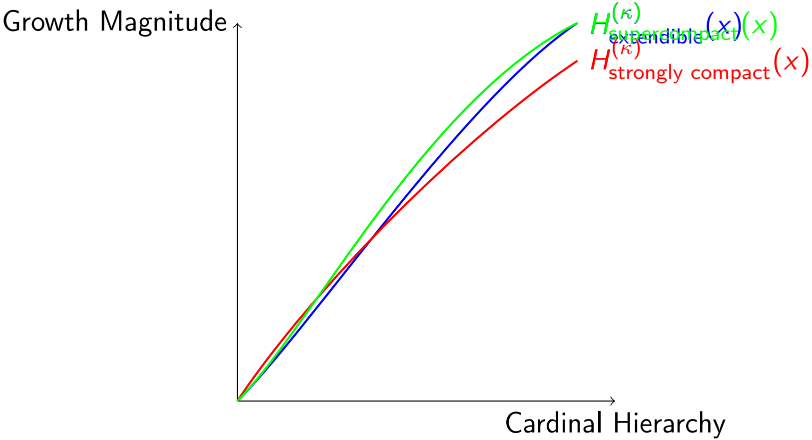
Demonstrate that the embedding properties of supercompact cardinals result in an intensified rate of growth. $\square$

Theorem: Divergence of $H^{(\kappa)}_{\text{supercompact}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{supercompact}}(x)$ diverges due to the supercompact cardinal properties, marking the peak of hierarchical divergence within the discussed levels. □

Diagram: Comparative Growth at Extendible, Strongly Compact, and Supercompact Levels



- Blue curve: Growth rate for extendible cardinals.
- Red curve: Growth rate for strongly compact cardinals.

Appendix: Hypercompact-Hierarchical Functions I

Definition (Hypercompact-Hierarchical Function)

Define the hypercompact-hierarchical function $H^{(\kappa)}_{\text{hypercompact}}(x)$ for a hypercompact cardinal κ as:

$$H^{(\kappa)}_{\text{hypercompact}}(x) = \lim_{\alpha \to \kappa} H^{(\alpha)}_{\text{supercompact}}(x),$$

where $H^{(\alpha)}_{\text{supercompact}}(x)$ extends through supercompact hierarchical functions.

Remark

Hypercompact-hierarchical functions represent an even higher hierarchical level, capturing growth at the hypercompact cardinal level, a theoretical extension of supercompact behavior within $\mathbb{Y}_3(\mathbb{R})$.

Theorem: Divergence of $H^{(\kappa)}_{\text{hypercompact}}(x)$ I

Theorem (Divergence of $H^{(\kappa)}_{\text{hypercompact}}(x)$)

For a hypercompact cardinal κ, $H^{(\kappa)}_{hypercompact}(x)$ diverges as $x \to \infty$, representing a hierarchical level beyond supercompact growth.

Proof (1/3).

Review the divergence properties of supercompact-hierarchical functions and establish how these lay the groundwork for hypercompact growth. □

Proof (2/3).

Show that hypercompact cardinals intensify the divergence properties with unique embedding properties. □

Theorem: Divergence of $H^{(\kappa)}_{\text{hypercompact}}(x)$ II

Proof (3/3).

Conclude that $H^{(\kappa)}_{\text{hypercompact}}(x)$ diverges due to the compounded growth characteristic of hypercompact cardinals. □

References I

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- Kanamori, A. (1997). "Strong Compactness and Supercompactness in Set Theory."
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