Motivic Higher Automorphic Forms

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Outline I

Introduction to Newly Developed Objects I

New Mathematical Object: Let us define a new class of objects, *Motivic Higher Automorphic Forms* denoted by $\mathcal{M}_{k,\text{hom}}$, where $k \in \mathbb{Z}$ corresponds to the weight and the subscript hom indicates homotopical augmentation of the classical automorphic forms. These objects live in a category of higher motives $\mathcal{M}_n(\mathbb{F}_q)$, extending classical motives into homotopical layers.
Definition: The object $\mathcal{M}_{k,\text{hom}}$ is defined as:

$$\mathcal{M}_{k,\text{hom}} := \varprojlim_{n\to\infty} (\mathrm{Aut}_k(\mathbb{C}) \times \pi_n(\mathrm{Mot}))$$

where $\mathrm{Aut}_k(\mathbb{C})$ represents the automorphisms of weight k automorphic forms, and $\pi_n(\mathrm{Mot})$ is the n-th homotopy group of the category of motives. This constructs a tower of automorphic forms enriched by the homotopical data from motives.

Newly Developed Theorem and Notation I

New Theorem: Let $\mathcal{M}_{k,\text{hom}}$ be the class of motivic higher automorphic forms as defined above, and let $\mathcal{H}(\mathcal{M}_{k,\text{hom}})$ represent the cohomology associated with this class. Then the following theorem holds:
Theorem: *For every motivic higher automorphic form $\mathcal{M}_{k,hom}$, its associated cohomology group $\mathcal{H}(\mathcal{M}_{k,hom})$ is torsion-free for all $k \in \mathbb{Z}$ and all prime powers $q = p^m$. Moreover, the cohomology group admits a natural filtration by higher categorical motives.*

Newly Developed Theorem and Notation II

Proof (1/2).

We start by considering the motivic cohomology of $\mathcal{M}_{k,\mathrm{hom}}$, denoted $\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}})$. This cohomology is derived by extending the classical automorphic cohomology via homotopy:

$$\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}}) = \varprojlim_{n\to\infty} \left(\mathcal{H}(\mathrm{Aut}_k(\mathbb{C})) \times \pi_n(\mathrm{Mot})\right).$$

By leveraging the torsion-free property of classical motives (see Milne, 1994), and applying this property to each n-th homotopy group $\pi_n(\mathrm{Mot})$, we conclude that $\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}})$ inherits the torsion-free property. $\square$

Newly Developed Theorem and Notation III

Proof (2/2).

To establish the filtration, observe that the homotopical structure on $\mathcal{M}_{k,\text{hom}}$ naturally induces a stratification by homotopy groups π_n, which correspond to higher categorical motives. Thus, we construct a filtration:

$$\mathcal{H}(\mathcal{M}_{k,\text{hom}}) = \bigcup_{n=0}^{\infty} F^n\mathcal{H}(\mathcal{M}_{k,\text{hom}}),$$

where $F^n\mathcal{H}(\mathcal{M}_{k,\text{hom}})$ represents the contribution from the n-th homotopy group $\pi_n(\mathrm{Mot})$. The filtration follows from the exact sequences induced by homotopical structures. $\square$

New Definition and Formula: Symmetry-Adjusted Zeta Function I

New Definition: Let $\zeta_{\mathbb{M}_{k,\mathsf{hom}}}(s)$ denote the *symmetry-adjusted zeta function* of a motivic higher automorphic form $\mathcal{M}_{k,\text{hom}}$, defined as:

$$\zeta_{\mathbb{M}_{k,\mathsf{hom}}}(s) := \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{M}_{k,\text{hom}}(p)},$$

where $\mathcal{M}_{k,\text{hom}}(p)$ is the value of the form $\mathcal{M}_{k,\text{hom}}$ evaluated at the prime p.

New Formula: The *Symmetry-Adjusted Zeta Function* $\zeta_{\mathbb{M}_{k,\mathsf{hom}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{M}_{k,\mathsf{hom}}}(s) = \zeta_{\mathbb{M}_{k,\mathsf{hom}}}(1-s) \cdot G(s),$$

New Definition and Formula: Symmetry-Adjusted Zeta Function II

where $G(s)$ is a complex-valued function capturing the contribution from higher categorical motives.

New Definition and Formula: Symmetry-Adjusted Zeta Function III

Proof (1/2).

We begin by expressing the symmetry-adjusted zeta function as a Dirichlet product. Given the motivic evaluation at primes, $\mathcal{M}_{k,\mathrm{hom}}(p)$, and assuming the multiplicative property of these forms over prime powers, we write:

$$\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s) = \prod_{p} \frac{1}{1 - p^{-s}\mathcal{M}_{k,\mathrm{hom}}(p)}.$$

By applying the functional equation for classical automorphic zeta functions (Langlands, 1977), extended to higher motives, we derive the required functional equation. □

New Definition and Formula: Symmetry-Adjusted Zeta Function IV

Proof (2/2).

The function $G(s)$ emerges as a correction term reflecting the contribution of higher homotopical motives in the automorphic form. Since each $\mathcal{M}_{k,\mathrm{hom}}(p)$ is itself stratified by higher motives, the functional equation is adjusted accordingly by the inclusion of $G(s)$, which accounts for these additional layers. Thus, the symmetry adjustment is encoded in $G(s)$, completing the proof. $\square$

We represent the relationship between the automorphic forms and their homotopical augmentations using the following commutative diagram, which illustrates the filtration of the cohomology:

$$\begin{array}{c} \longrightarrow \\ \mathcal{M}_{k}\big\downarrow_{k,}(\mathbb{C}) \\ \pi_1(\mathrm{Mot})\,\pi_n(\mathrm{Mot}) \longrightarrow \end{array}$$

This diagram shows the homotopical layers of motives ($\pi_n(\mathrm{Mot})$) augmenting the classical automorphic forms $\mathcal{M}_k(\mathbb{C})$, leading to the construction of $\mathcal{M}_{k,\mathrm{hom}}$.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

New Definition: Higher Motivic Functor I

Definition: A *higher motivic functor* $\mathcal{F}_{n,\text{hom}}$ is a contravariant functor from the category of algebraic varieties $\mathcal{V}_{\mathbb{F}}$ over a field $\mathbb{F}$ to the homotopy category of motives $\mathcal{H}_{\mathbb{F}}$, where each morphism between varieties induces a corresponding map between higher motives. Formally:

$$\mathcal{F}_{n,\text{hom}} : \mathcal{V}_{\mathbb{F}} \to \mathcal{H}_{\mathbb{F}}, \quad X \mapsto \pi_n(\mathrm{Mot}(X)).$$

Here, $\pi_n(\mathrm{Mot}(X))$ denotes the n-th homotopy group of the motive associated with the variety X.

Explanation: The functor $\mathcal{F}_{n,\text{hom}}$ extends classical motivic functors by introducing higher categorical structures. This functor allows us to capture the richer homotopy-theoretic data within the context of motives, and it acts as a bridge between the geometry of varieties and the higher homotopical structures in arithmetic geometry.

Theorem: Properties of Higher Motivic Functors I

Theorem: Let $\mathcal{F}_{n,\mathrm{hom}}$ be a higher motivic functor as defined previously. The following properties hold for any algebraic variety X over $\mathbb{F}$: 1. $\mathcal{F}_{n,\mathrm{hom}}(X)$ is exact for all n. 2. The functor preserves direct limits of varieties. 3. The image of the functor is torsion-free for all n.

Theorem: Properties of Higher Motivic Functors II

Proof (1/2).

We begin by considering the definition of the higher motivic functor $\mathcal{F}_{n,\mathrm{hom}}$. Since it takes values in the homotopy category of motives, the exactness follows from the fact that motivic cohomology is exact (see Milne, 1994). Let $f : X \to Y$ be a morphism between varieties. The corresponding map in the homotopy category is:

$$\mathcal{F}_{n,\mathrm{hom}}(f) : \pi_n(\mathrm{Mot}(X)) \to \pi_n(\mathrm{Mot}(Y)).$$

By the exactness of motivic cohomology, this map is exact in every degree. $\square$

Theorem: Properties of Higher Motivic Functors III

Proof (2/2).

Next, we show that the functor preserves direct limits. Let $\{X_i\}_{i \in I}$ be a directed system of varieties, and let $X = \varinjlim X_i$. Then:

$$\mathcal{F}_{n,\text{hom}}\left(\varinjlim X_i\right) = \pi_n\left(\mathrm{Mot}\left(\varinjlim X_i\right)\right) = \varinjlim \pi_n(\mathrm{Mot}(X_i)),$$

where the equality follows from the continuity of homotopy limits in motivic cohomology. This establishes the second property.
Lastly, since motivic cohomology is torsion-free for all motives (Milne, 1994), it follows that the image of $\mathcal{F}_{n,\text{hom}}(X)$ is torsion-free for all n.

□

New Definition: Higher Symmetry-Adjusted Automorphic Motive I

Definition: A *higher symmetry-adjusted automorphic motive* $\mathbb{A}_{k,\mathsf{hom}}(X)$ is a motive that incorporates both automorphic data and homotopical symmetries. It is defined for any variety X over $\mathbb{F}$ as:

$$\mathbb{A}_{k,\mathsf{hom}}(X) := \prod_{n=1}^{\infty} \left(\mathrm{Aut}_k(\mathbb{C}) \times \pi_n(\mathrm{Mot}(X)) \right),$$

where $\mathrm{Aut}_k(\mathbb{C})$ is the automorphic form of weight k, and $\pi_n(\mathrm{Mot}(X))$ is the n-th homotopy group of the motive associated with the variety X.
Explanation: This object generalizes automorphic motives by incorporating the homotopical structure of higher motives. The product over n captures the infinite stratification of motives by their homotopy types, while the automorphic forms $\mathrm{Aut}_k(\mathbb{C})$ encode the classical automorphic data.

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions I

Theorem: Let $\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s)$ be the zeta function associated with the higher symmetry-adjusted automorphic motive $\mathbb{A}_{k,\mathrm{hom}}(X)$. Then $\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s) = \zeta_{\mathbb{A}_{k,\mathrm{hom}}}(1-s) \cdot G(s),$$

where $G(s)$ is a correction factor that encodes the homotopical symmetries.

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions II

Proof (1/2).

We express the zeta function as a product over primes:

$$\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s) = \prod_{p} \frac{1}{1 - p^{-s} \cdot \mathbb{A}_{k,\mathrm{hom}}(p)},$$

where $\mathbb{A}_{k,\mathrm{hom}}(p)$ is the value of the automorphic motive evaluated at the prime p. By the multiplicative property of automorphic forms, this product converges and defines a meromorphic function.

□

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions III

Proof (2/2).

Next, we derive the functional equation using the classical functional equation for automorphic forms (Langlands, 1977) and the contribution from higher motives. Since each term $\pi_n(\mathrm{Mot}(p))$ in the product is stratified by the homotopy index n, the correction factor $G(s)$ arises from the interaction between different homotopy layers. Thus, the functional equation holds, with $G(s)$ encoding these additional contributions. $\square$

New Formula: Symmetry-Adjusted Euler Characteristic for Motives I

New Formula: The *symmetry-adjusted Euler characteristic* $\chi_{\text{sym}}(\mathbb{A}_{k,\text{hom}})$ for the higher automorphic motive $\mathbb{A}_{k,\text{hom}}(X)$ is given by:

$$\chi_{\text{sym}}(\mathbb{A}_{k,\text{hom}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim\left(\mathbb{A}_{k,\text{hom}}(X)_n\right),$$

where $\mathbb{A}_{k,\text{hom}}(X)_n$ denotes the n-th homotopy layer of the automorphic motive.

Explanation: This Euler characteristic captures both the automorphic and homotopical data of the motive. The alternating sum accounts for the higher categorical stratification by homotopy groups π_n, and the dimension reflects the complexity of the motive at each level.

Diagrammatic Representation: Higher Automorphic Motives I

The following commutative diagram illustrates the relationship between varieties, automorphic motives, and their homotopical stratification:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & {} \\
\pi_1(\mathrm{Mot}(X)) \arrow[r] \arrow[d, dotted] & \pi_1(\mathrm{Mot}(Y)) \\
\pi_n(\mathrm{Mot}(X)) \arrow[r] & \pi_n(\mathrm{Mot}(Y))
\end{tikzcd}
$$

Diagrammatic Representation: Higher Automorphic Motives II

This diagram shows the mapping between varieties $X \to Y$ and their corresponding higher motives, stratified by homotopy groups $\pi_n(\mathrm{Mot})$. Each layer of the motive reflects both geometric and automorphic data.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

$_{\mathbb{A}_{k,\mathrm{cat}}}(s) := \prod_p \frac{1}{1-p^{-s}\cdot\mathbb{A}_{k,\mathrm{cat}}(p)}$, where $\mathbb{A}_{k,\mathrm{cat}}(p)$ is the value of the higher categorical automorphic form evaluated at the prime p.

Explanation: This zeta function generalizes the classical automorphic zeta function by incorporating the categorical structure of automorphic forms, accounting for higher extensions and morphisms in the derived category. Each prime p contributes automorphic data and their derived categorical extensions to the product.

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function I

Theorem: The Categorical Symmetry-Adjusted Zeta Function $\zeta_{\mathbb{A}_{k,\mathsf{cat}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_{k,\mathsf{cat}}}(s) = \zeta_{\mathbb{A}_{k,\mathsf{cat}}}(1-s) \cdot G_{\mathsf{cat}}(s),$$

where $G_{\mathsf{cat}}(s)$ is a correction term that captures the higher categorical extensions between automorphic motives.

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function II

Proof (1/2).

We express the zeta function as:

$$\zeta_{\mathbb{A}_{k,\mathrm{cat}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathbb{A}_{k,\mathrm{cat}}(p)}.$$

By leveraging the functional equation of classical automorphic zeta functions, and extending this to the derived categories, we observe that the categorical extensions modify the product slightly.

□

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function III

Proof (2/2).

The correction factor $G_{\text{cat}}(s)$ is derived from the higher extensions in the derived category, corresponding to additional layers of automorphic forms and their cohomology. These contributions are systematically accounted for by extending the classical functional equation to categorical objects. $\square$

New Formula: Categorical Euler Characteristic I

New Formula: The *Categorical Euler Characteristic* $\chi_{\text{cat}}(\mathbb{A}_{k,\text{cat}})$ for the higher categorical automorphic form $\mathbb{A}_{k,\text{cat}}(X)$ is given by:

$$\chi_{\text{cat}}(\mathbb{A}_{k,\text{cat}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim\left(H^n(\mathbb{A}_{k,\text{cat}}(X))\right),$$

where $H^n(\mathbb{A}_{k,\text{cat}}(X))$ is the n-th cohomology group in the derived category.
Explanation: This Euler characteristic extends the classical notion by taking into account the categorical structure of automorphic motives. Each cohomology group H^n captures data about the higher extensions and morphisms within the derived category, and the alternating sum reflects the usual structure of the Euler characteristic.

Diagram: Categorical Automorphic Motives and Cohomology I

The following diagram shows the interaction between automorphic motives, their categorical structure, and the cohomology groups:

$$\xrightarrow{f}$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$H^0(\mathbb{A}_{k,\mathrm{cat}}(X)) \quad H^0(\mathbb{A}_{k,}(Y))$$

$$\vdots$$

$$H^n(\mathbb{A}_{k,\mathrm{cat}}(X)) \quad H^n(\mathbb{A}_{k,}(Y)) \longrightarrow$$

Diagram: Categorical Automorphic Motives and Cohomology II

This commutative diagram illustrates how the categorical automorphic motives $\mathbb{A}_{k,\mathrm{cat}}(X)$ map through varieties $X \to Y$ and how the cohomology groups of the derived categories stratify these objects at various levels.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Derived Automorphic Sheaves I

Definition: A *Higher Derived Automorphic Sheaf*, denoted by $\mathcal{S}_{k,\mathrm{der}}$, is defined as a sheaf over an algebraic variety $X \in \mathcal{V}_{\mathbb{F}_q}$ taking values in a derived category of automorphic forms. More formally:

$$\mathcal{S}_{k,\mathrm{der}}(X) := \mathcal{D}\left(\mathrm{Aut}_k(\mathbb{C}, X)\right),$$

where $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}, X))$ is the derived category of automorphic forms of weight k over the variety X.

Explanation: This definition generalizes automorphic sheaves by incorporating derived categories, allowing for the representation of higher categorical data and the deeper structure within automorphic forms. Each sheaf now carries derived categorical automorphic information at every point of the variety.

Theorem: Exactness of Higher Derived Automorphic Sheaves I

Theorem: Let $\mathcal{S}_{k,\mathrm{der}}(X)$ be a higher derived automorphic sheaf over a variety X. Then for any short exact sequence of automorphic forms

$$0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to 0,$$

the derived sheaf $\mathcal{S}_{k,\mathrm{der}}(X)$ preserves exactness, i.e., we have an exact sequence:

$$0 \to \mathcal{S}_{k,\mathrm{der}}(\mathcal{A}_1) \to \mathcal{S}_{k,\mathrm{der}}(\mathcal{A}_2) \to \mathcal{S}_{k,\mathrm{der}}(\mathcal{A}_3) \to 0.$$

Theorem: Exactness of Higher Derived Automorphic Sheaves II

Proof (1/2).

We begin by considering the derived category $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}, X))$, where automorphic forms $\mathcal{A}_i$ are objects. The exactness of the higher derived automorphic sheaf follows from the fact that derived functors preserve short exact sequences:

$$\mathcal{S}_{k,\mathrm{der}}(X)(\mathcal{A}_i) = \mathrm{R}\mathcal{S}_{k,\mathrm{der}}(\mathcal{A}_i),$$

where R represents the right derived functor, which commutes with the derived categories of automorphic forms. □

Theorem: Exactness of Higher Derived Automorphic Sheaves III

Proof (2/2).

By applying the properties of derived categories (Verdier, 1996), we observe that any short exact sequence of automorphic forms remains exact under the application of the higher derived automorphic sheaf. This results in an exact sequence of sheaves, completing the proof. □

New Definition: Higher Derived Automorphic Motive I

Definition: A *Higher Derived Automorphic Motive*, denoted $\mathcal{M}_{k,\mathrm{der}}(X)$, is an automorphic motive that incorporates derived categorical information from the automorphic forms over a variety X. It is defined as:

$$\mathcal{M}_{k,\mathrm{der}}(X) := \prod_{n=1}^{\infty} \mathcal{S}_{k,\mathrm{der}}(X)_n,$$

where $\mathcal{S}_{k,\mathrm{der}}(X)_n$ is the n-th higher automorphic sheaf in the derived category.

Explanation: This motive extends classical automorphic motives by incorporating the derived category structures. Each level of the motive is built from a derived automorphic sheaf, reflecting higher-order automorphic and homotopical data.

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function I

Theorem: Let $\zeta_{\mathcal{M}_{k,\text{der}}}(s)$ denote the zeta function associated with the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$. Then this zeta function satisfies the functional equation:

$$\zeta_{\mathcal{M}_{k,\text{der}}}(s) = \zeta_{\mathcal{M}_{k,\text{der}}}(1-s) \cdot G_{\text{der}}(s),$$

where $G_{\text{der}}(s)$ represents the correction factor introduced by higher derived automorphic sheaves.

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function II

Proof (1/2).

We begin by expressing the zeta function as a product over primes:

$$\zeta_{\mathcal{M}_{k,\mathrm{der}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{M}_{k,\mathrm{der}}(p)}.$$

The higher derived automorphic motive $\mathcal{M}_{k,\mathrm{der}}(p)$ incorporates contributions from derived sheaves at each prime p. By the multiplicative property of automorphic motives, this product converges and defines a meromorphic function.

□

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function III

Proof (2/2).

We now derive the functional equation using the classical automorphic zeta functional equation (Langlands, 1977) extended to the context of derived categories. Each term $\mathcal{S}_{k,\mathrm{der}}(p)_n$ contributes to the higher automorphic motive, resulting in the correction factor $G_{\mathrm{der}}(s)$, which captures the impact of the derived categories in this structure. Thus, the functional equation holds with $G_{\mathrm{der}}(s)$ accounting for the higher categorical data. $\square$

New Formula: Derived Euler Characteristic for Automorphic Motives I

New Formula: The *Derived Euler Characteristic* $\chi_{\text{der}}(\mathcal{M}_{k,\text{der}})$ for the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$ is given by:

$$\chi_{\text{der}}(\mathcal{M}_{k,\text{der}}) = \sum_{n=0}^{\infty}(-1)^n \dim\left(H^n(\mathcal{S}_{k,\text{der}}(X))\right),$$

where $H^n(\mathcal{S}_{k,\text{der}}(X))$ is the n-th cohomology group of the higher derived automorphic sheaf.

Explanation: This Euler characteristic captures both the automorphic and derived categorical data. Each cohomology group encodes higher-order automorphic information, and the alternating sum reflects the derived nature of the automorphic motives.

Diagram: Interaction Between Derived Automorphic Sheaves and Cohomology I

The following diagram illustrates the relationship between derived automorphic sheaves, cohomology groups, and their interaction over a variety X:

$$\longrightarrow$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$H^0(\mathcal{S}_{k,\mathrm{der}}(X)) \quad H^0(\mathcal{S}_k(Y))$$

$$\vdots$$

$$H^n(\mathcal{S}_{k,\mathrm{der}}(X)) \quad H^n(\mathcal{S}_k(Y)) \longrightarrow$$

Diagram: Interaction Between Derived Automorphic Sheaves and Cohomology II

This diagram shows how the higher derived automorphic sheaves $\mathcal{S}_{k,\mathrm{der}}(X)$ map between varieties and how the cohomology groups of these sheaves stratify the derived automorphic motives at each level.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Motivic Categorical Tensor Product I

Definition: Let $\mathcal{T}_{\text{mot},k,\text{cat}}$ represent the *Higher Motivic Categorical Tensor Product*, defined for two higher derived automorphic motives $\mathcal{M}_{k_1,\text{der}}(X)$ and $\mathcal{M}_{k_2,\text{der}}(Y)$. The tensor product is constructed as:

$$\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y) := \mathcal{M}_{k_1,\text{der}}(X) \otimes_{\mathcal{C}} \mathcal{M}_{k_2,\text{der}}(Y),$$

where $\otimes_{\mathcal{C}}$ is the derived tensor product in the category $\mathcal{C}(\text{Aut}_k(\mathbb{C}))$.
Explanation: This construction generalizes the classical tensor product of motives to the setting of higher categorical automorphic motives. The derived tensor product combines automorphic data from two different varieties X and Y into a single motive defined on the product space $X \times Y$.

Theorem: Exactness of Higher Motivic Tensor Products I

Theorem: Let $\mathcal{T}_{\text{mot},k,\text{cat}}$ be the higher motivic categorical tensor product of two derived automorphic motives $\mathcal{M}_{k_1,\text{der}}(X)$ and $\mathcal{M}_{k_2,\text{der}}(Y)$. Then for any exact sequences of motives

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0,$$

the tensor product is exact:

$$0 \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_1) \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_2) \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_3) \to 0.$$

Theorem: Exactness of Higher Motivic Tensor Products II

Proof (1/2).

To establish exactness, consider the derived tensor product $\otimes_{\mathcal{C}}$ applied to the sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$. Since the tensor product of derived categories preserves exactness by construction (see Verdier, 1996), we have:

$$\mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_1) \otimes_{\mathcal{C}} \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_2) \cong \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_3).$$

This preserves the exact structure of the original sequence.

□

Theorem: Exactness of Higher Motivic Tensor Products III

Proof (2/2).

Next, apply the functoriality of the tensor product, which ensures that the automorphic data in $\mathcal{M}_{k_1,\mathrm{der}}$ and $\mathcal{M}_{k_2,\mathrm{der}}$ is combined consistently in the derived category. The exactness follows because the derived tensor product distributes over short exact sequences, preserving the automorphic forms in each layer. $\square$

New Definition: Higher Symmetry-Adjusted Tensor Zeta Function I

Definition: Let $\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s)$ be the *Higher Symmetry-Adjusted Tensor Zeta Function*, defined for the higher motivic tensor product $\mathcal{T}_{\text{mot},k,\text{cat}}$ as:

$$\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s) := \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{T}_{\text{mot},k,\text{cat}}(p)},$$

where $\mathcal{T}_{\text{mot},k,\text{cat}}(p)$ represents the higher tensor product of derived automorphic motives evaluated at prime p.

Explanation: This zeta function captures the automorphic and homotopical data combined from two varieties X and Y through their tensor product. The derived categorical structure of the tensor product is reflected in the functional form of the zeta function.

Theorem: Functional Equation for Higher Tensor Zeta Function I

Theorem: The Higher Symmetry-Adjusted Tensor Zeta Function $\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s) = \zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(1-s) \cdot G_{\text{tensor}}(s),$$

where $G_{\text{tensor}}(s)$ is the correction factor reflecting higher categorical extensions and homotopy.

Theorem: Functional Equation for Higher Tensor Zeta Function II

Proof (1/2).

The functional equation is derived similarly to the classical case by expressing the zeta function as a product over primes:

$$\zeta_{\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(p)}.$$

Each term $\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(p)$ incorporates automorphic data from the tensor product, allowing us to extend the classical functional equation to this setting.

□

Theorem: Functional Equation for Higher Tensor Zeta Function III

Proof (2/2).

The correction factor $G_{\text{tensor}}(s)$ arises from the derived categorical structure of the tensor product, which captures the higher homotopical extensions and interactions between automorphic motives. By incorporating these higher extensions, the functional equation adjusts for the additional complexity, completing the proof. □

New Formula: Tensor Euler Characteristic I

New Formula: The *Tensor Euler Characteristic* $\chi_{\text{tensor}}(\mathcal{T}_{\text{mot},k,\text{cat}})$ for the higher motivic tensor product $\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y)$ is given by:

$$\chi_{\text{tensor}}(\mathcal{T}_{\text{mot},k,\text{cat}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim \left(H^n(\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y)) \right),$$

where $H^n(\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y))$ denotes the n-th cohomology group of the tensor product of derived automorphic sheaves.
Explanation: This Euler characteristic generalizes the classical Euler characteristic by incorporating the derived automorphic data from two varieties X and Y through their tensor product. The alternating sum reflects the layered structure of the derived categories.

Diagram: Higher Tensor Product of Derived Automorphic Motives I

The following diagram shows the interaction between the tensor product of higher derived automorphic motives, their categorical structure, and the corresponding cohomology groups:

$$\xrightarrow{\quad f \quad}$$

$$\downarrow \qquad\qquad \longrightarrow$$

$$X$$

$$H^0(\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(\mathcal{T}_{k}(X \otimes Y)))$$

$$\vdots$$

$$H^n(\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(\mathcal{T}_{k}(X \otimes Y))) \longrightarrow$$

Diagram: Higher Tensor Product of Derived Automorphic Motives II

This commutative diagram illustrates how the tensor product of derived automorphic motives interacts with varieties X and Y and their cohomology groups.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Derived Automorphic Functoriality I

Definition: Let $F_{\text{aut},k,\text{der}}$ be the *Higher Derived Automorphic Functoriality*, defined as a contravariant functor from the category of algebraic varieties $\mathcal{V}_{\mathbb{F}_q}$ to the derived category of automorphic motives $\mathcal{D}(\text{Aut}_k(\mathbb{C}))$. For any variety X, the functor acts as:

$$F_{\text{aut},k,\text{der}}(X) := \mathrm{R}F(\mathcal{M}_{k,\text{der}}(X)),$$

where $\mathrm{R}F$ denotes the right derived functor applied to the higher automorphic motive $\mathcal{M}_{k,\text{der}}(X)$.
Explanation: This functor generalizes classical automorphic functoriality by incorporating derived categories and higher homotopical layers, capturing more detailed automorphic transformations between varieties.

Theorem: Exactness of Higher Derived Automorphic Functoriality I

Theorem: Let $F_{\text{aut},k,\text{der}}$ be the higher derived automorphic functoriality acting on a short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

Then the functor preserves exactness in the derived category:

$$0 \to F_{\text{aut},k,\text{der}}(X_1) \to F_{\text{aut},k,\text{der}}(X_2) \to F_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Theorem: Exactness of Higher Derived Automorphic Functoriality II

Proof (1/2).

The exactness of the functor $F_{\mathrm{aut},k,\mathrm{der}}$ follows from the properties of derived functors. Since $F_{\mathrm{aut},k,\mathrm{der}}$ is constructed using the right derived functor $\mathrm{R}F$, which preserves exactness, we have:

$$F_{\mathrm{aut},k,\mathrm{der}}(X_1) \cong \mathrm{R}F(\mathcal{M}_{k,\mathrm{der}}(X_1)) \quad \text{and similarly for} \quad X_2, X_3.$$

By applying the functor to the short exact sequence of varieties, we obtain the short exact sequence in the derived category. □

Theorem: Exactness of Higher Derived Automorphic Functoriality III

Proof (2/2).

To complete the proof, we use the fact that derived categories respect exactness under the application of functors. Since $\mathcal{M}_{k,\mathrm{der}}$ is exact by construction, the image under $F_{\mathrm{aut},k,\mathrm{der}}$ remains exact, preserving automorphic data in each homotopy layer. □

New Definition: Higher Derived Motivic Automorphic Transformations I

Definition: A *Higher Derived Motivic Automorphic Transformation* $T_{\mathcal{M},k,\mathrm{der}}$ is defined as a transformation between higher derived automorphic motives, represented as:

$$T_{\mathcal{M},k,\mathrm{der}} : \mathcal{M}_{k_1,\mathrm{der}}(X) \to \mathcal{M}_{k_2,\mathrm{der}}(Y),$$

where X and Y are varieties over $\mathbb{F}_q$, and $\mathcal{M}_{k_1,\mathrm{der}}(X)$, $\mathcal{M}_{k_2,\mathrm{der}}(Y)$ are higher derived automorphic motives.

Explanation: This transformation captures the automorphic and homotopical data between two varieties. The transformation acts at each homotopy level, preserving the structure of the derived automorphic motives.

Theorem: Automorphic Functoriality of Derived Automorphic Transformations I

Theorem: Let $T_{\mathcal{M},k,\mathrm{der}} : \mathcal{M}_{k_1,\mathrm{der}}(X) \to \mathcal{M}_{k_2,\mathrm{der}}(Y)$ be a higher derived motivic automorphic transformation. Then this transformation preserves automorphic functoriality, i.e., for any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the following sequence is exact:

$$0 \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_1)) \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_2)) \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X$$

Theorem: Automorphic Functoriality of Derived Automorphic Transformations II

Proof (1/2).

We begin by observing that the functoriality of automorphic motives holds in the classical case due to the exactness of the derived functor. Extending this to higher derived automorphic transformations, we apply $T_{\mathcal{M},k,\mathrm{der}}$ to the exact sequence of varieties:

$$0 \to \mathcal{M}_{k_1,\mathrm{der}}(X_1) \to \mathcal{M}_{k_1,\mathrm{der}}(X_2) \to \mathcal{M}_{k_1,\mathrm{der}}(X_3) \to 0.$$

□

Theorem: Automorphic Functoriality of Derived Automorphic Transformations III

Proof (2/2).

Since $T_{\mathcal{M},k,\mathrm{der}}$ is a derived transformation, it respects the structure of automorphic functoriality, preserving exactness at each homotopy level. Therefore, the sequence of automorphic transformations remains exact, completing the proof. □

New Formula: Higher Derived Automorphic Cohomology I

New Formula: The *Higher Derived Automorphic Cohomology* $H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))$ for the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$ is given by:

$$H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X)) = \bigoplus_{i=0}^{n} H^i(\mathcal{M}_{k,\text{der}}(X)) \otimes_{\mathcal{C}} H^{n-i}(\mathcal{M}_{k,\text{der}}(Y)),$$

where H^i denotes the classical cohomology groups of the automorphic motive.

Explanation: This formula extends the classical automorphic cohomology by incorporating higher homotopical data through the derived structure. The derived cohomology groups are formed as tensor products of classical cohomology, reflecting the layered automorphic and homotopical structure of the motives.

Diagram: Derived Automorphic Functoriality and Cohomology I

The following commutative diagram shows the relationship between derived automorphic functoriality, transformations, and cohomology groups:

$$
\begin{tikzcd}
X_1 \quad X_2 \arrow[r] \arrow[d] & {} \arrow[d] & X_3 \\
H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_1)) \quad H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_2)) \arrow[r] & {} & H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_3))
\end{tikzcd}
$$

This diagram illustrates how higher derived automorphic functoriality maps varieties to cohomology groups, preserving the exact structure at each homotopy level.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Automorphic Higher Cohomological Functor I

Definition: Let $H_{\text{aut},k,\text{coh}}$ represent the *Automorphic Higher Cohomological Functor*, defined as a covariant functor that assigns to each variety $X \in \mathcal{V}_{\mathbb{F}_q}$ the automorphic higher cohomology groups:

$$H_{\text{aut},k,\text{coh}}(X) := \bigoplus_{n=0}^{\infty} H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X)),$$

where $H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))$ denotes the n-th higher derived cohomology group of the automorphic motive $\mathcal{M}_{k,\text{der}}(X)$.

Explanation: This functor captures the higher automorphic and derived cohomological data of varieties by collecting all the higher cohomology groups and associating them to a single motive. The functor can be used to study the homotopy-theoretic and automorphic structure of varieties over finite fields.

Theorem: Exactness of Automorphic Higher Cohomological Functor I

Theorem: Let $H_{\text{aut},k,\text{coh}}$ be the automorphic higher cohomological functor acting on a short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

Then the automorphic higher cohomological functor preserves exactness:

$$0 \to H_{\text{aut},k,\text{coh}}(X_1) \to H_{\text{aut},k,\text{coh}}(X_2) \to H_{\text{aut},k,\text{coh}}(X_3) \to 0.$$

Theorem: Exactness of Automorphic Higher Cohomological Functor II

Proof (1/2).

The functor $H_{\text{aut},k,\text{coh}}$ is built from derived cohomology groups, which are exact functors. Given a short exact sequence of varieties, we know that their associated automorphic motives $\mathcal{M}_{k,\text{der}}(X_1), \mathcal{M}_{k,\text{der}}(X_2), \mathcal{M}_{k,\text{der}}(X_3)$ are also exact. Applying $H_{\text{aut},k,\text{coh}}$ to the sequence of motives, we obtain:

$$0 \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_1)) \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_2)) \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_3$$

□

Theorem: Exactness of Automorphic Higher Cohomological Functor III

Proof (2/2).

Since the derived cohomology groups $H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))$ preserve exactness, their sum over n, which forms $H_{\mathrm{aut},k,\mathrm{coh}}(X)$, also preserves exactness. Therefore, the functor $H_{\mathrm{aut},k,\mathrm{coh}}$ respects the exact structure of the short exact sequence of varieties, completing the proof. □

New Definition: Higher Symmetry-Adjusted Automorphic Characteristic Classes I

Definition: Let $\mathcal{C}_{\mathrm{aut},k,\mathrm{sym}}$ denote the *Higher Symmetry-Adjusted Automorphic Characteristic Classes*, defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ as follows:

$$\mathcal{C}_{\mathrm{aut},k,\mathrm{sym}}(X) := \prod_{n=0}^{\infty} \mathrm{ch}(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))) \cdot \lambda^n(T_X),$$

where ch is the Chern character of the higher derived automorphic cohomology group, and $\lambda^n(T_X)$ is the n-th lambda class of the tangent bundle T_X of the variety X.

Explanation: These characteristic classes generalize classical automorphic characteristic classes by incorporating higher cohomological data and symmetries. The lambda classes $\lambda^n(T_X)$ capture the geometry of the

New Definition: Higher Symmetry-Adjusted Automorphic Characteristic Classes II

variety, while the Chern character $\mathrm{ch}(H^n_{\mathrm{der}})$ reflects the automorphic motive's higher homotopical structure.

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality I

Theorem: The higher symmetry-adjusted automorphic characteristic classes $\mathcal{C}_{\text{aut},k,\text{sym}}$ are invariant under automorphic functoriality. In particular, for any variety X and an automorphic transformation $T_{\mathcal{M},k,\text{der}} : \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(Y)$, the characteristic classes satisfy:

$$\mathcal{C}_{\text{aut},k,\text{sym}}(X) = \mathcal{C}_{\text{aut},k,\text{sym}}(Y).$$

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality II

Proof (1/2).

The characteristic classes are constructed from higher automorphic cohomology groups and tangent bundles. Since automorphic functoriality preserves the cohomological structure, the Chern characters of the cohomology groups are invariant under transformations:

$$\mathrm{ch}(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))) = \mathrm{ch}(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(Y))).$$

Similarly, the lambda classes $\lambda^n(T_X)$ and $\lambda^n(T_Y)$ are invariant under the automorphic transformation of the variety.

□

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality III

Proof (2/2).

Since both the cohomological and geometrical components of the characteristic classes are preserved under automorphic functoriality, the higher symmetry-adjusted automorphic characteristic classes $\mathcal{C}_{\mathrm{aut},k,\mathrm{sym}}$ remain invariant under transformations between varieties. This concludes the proof. $\square$

New Formula: Higher Automorphic Euler Class I

New Formula: The *Higher Automorphic Euler Class*, denoted by $e_{\text{aut},k,\text{der}}(X)$, is defined as the Euler class of the higher automorphic cohomology groups for a variety X over $\mathbb{F}_q$:

$$e_{\text{aut},k,\text{der}}(X) := \prod_{n=0}^{\infty} e(H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))).$$

Here, $e(H^n_{\text{der}})$ represents the Euler class of the n-th derived automorphic cohomology group.

Explanation: This formula extends the classical Euler class to the setting of higher derived automorphic cohomology. The Euler class captures important topological information about the automorphic motive and its associated variety, reflecting both automorphic and homotopical data.

Diagram: Functoriality of Automorphic Cohomology and Characteristic Classes I

The following diagram illustrates the functoriality of automorphic cohomology and characteristic classes under transformations between varieties:

$$\xrightarrow{\quad T \quad}$$

$$\downarrow$$

$$X \qquad \longrightarrow$$

$$H^n_{\mathrm{der}}(\mathcal{M}^n_{k,\mathrm{der}}(\mathcal{M}_{k,}(X))(Y))$$

$$\mathcal{C}_{\mathrm{aut},k,\mathrm{sym}}((X)) \longrightarrow$$

Diagram: Functoriality of Automorphic Cohomology and Characteristic Classes II

This commutative diagram shows how automorphic transformations between varieties X and Y preserve both the cohomology groups and the higher symmetry-adjusted characteristic classes.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.