CONDENSED PRISMATIC HOMOTOPY THEORY OVER LOGICAL AND ULTRAPRODUCT SITES

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Abstract. We initiate a systematic development of prismatic cohomology over logical and ultraproduct categories of p-adic rings using condensed and derived techniques. Building upon Scholze's condensed mathematics and the prismatic framework of Bhatt–Scholze, we extend trace theories, period sheaves, and syntomic filtrations to non-algebraic contexts. Our goal is to create a unified homotopical model of p-adic period structures valid for ultraproducts, nonstandard models, and definable sites in valuation logic.

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1. Introduction

The prismatic revolution in p-adic geometry has provided a powerful algebraic and cohomological framework for studying integral structures in mixed characteristic. However, the classical assumptions of Noetherianity, completeness, and finite presentation limit its scope. Meanwhile, ultraproducts and definable sets arising from model theory generate p-adic rings with infinite Krull dimension, nonstandard topologies, and logical constructions not representable in algebraic or rigid categories.

In this paper, we extend the theory of prismatic cohomology to these exotic settings. We work within the language of condensed and solid rings, derived ∞-categories, and logical topoi. The resulting homotopical theory unifies trace maps, filtrations, and period sheaves under a general descent framework applicable to condensed sheaves on logical sites.

2. Overview of Prismatic Cohomology and Condensed Foundations

Prismatic cohomology, as introduced by Bhatt and Scholze, provides a powerful unifying formalism for integral p-adic cohomological theories. It recovers crystalline, de Rham, étale, and syntomic cohomologies via filtrations, comparison maps, and Frobenius-periodic structures. However, the classical formulation assumes the base ring is p-adically complete, typically commutative, and admits a δ-structure or Frobenius lift.

Condensed mathematics offers a more flexible categorical framework suitable for encoding topological and non-finitary constructions such as ultraproducts, distribution algebras, or Fréchet spaces. We now recall key ideas from each theory, setting the stage for their synthesis in the logical context.

2.1. Prismatic Cohomology: Algebraic Foundations.

Let A be a bounded prism, that is, a p-adically complete δ-ring with an ideal $I \subset A$ satisfying bounded p-torsion and Frobenius compatibility.

Definition 2.1 (Prism). *A prism is a pair (A, I) where:*

(1) A is a δ-ring, i.e., equipped with a Frobenius-compatible derivation δ;
(2) $I \subset A$ is an ideal such that (p, I) forms a Cartier-divisor-like structure;
(3) A is (p, I)-adically complete;
(4) A/I has bounded p^∞-torsion.

The prismatic site $(X/A)_{\text{prism}}$ of a p-adic formal scheme X over A defines a Grothendieck topology over prisms mapping to X. Sheafifying structure rings over this site yields the prismatic cohomology $\text{Prism}_{X/A}$, equipped with:

- Nygaard filtration $\mathcal{N}^\bullet$;
- Frobenius φ on $\text{Prism}_{X/A}$;
- Comparison maps to H^*_{crys}, H^*_{dR}, H^*_{t}, and H^*_{syn}.

Example 2.2. *Let $X = \text{Spf}(R)$ with $R = \mathbb{Z}_p[[T]]$. Then the bounded prism $(A_{\text{inf}}, \ker\theta)$ associated to a perfectoid field C yields:*

$$\text{Prism}_{R/A_{\text{inf}}} \simeq A_{\text{inf}}[[T]]$$

with filtrations recovering crystalline and de Rham cohomology.

2.2. **Condensed Rings and Sheaves.** The category of condensed abelian groups, denoted $\text{Cond}(\mathbb{Z})$, replaces topological vector spaces with sheaves over profinite sets, providing a flexible basis for functional-analytic and derived constructions.

Definition 2.3. *A* condensed ring *is a sheaf $A \in \text{Cond}(\mathsf{Ab})$ of commutative rings on the site of profinite sets ProFin with the pro-étale topology.*

Condensed modules over A generalize Banach spaces, and condensed algebras generalize completed tensor products in a sheaf-theoretic form. A key example is:

Example 2.4. *Let S be a profinite set. Then the constant sheaf $\underline{\mathbb{Z}_p}(S) = \text{Map}_{\text{cont}}(S, \mathbb{Z}_p)$ is a condensed $\mathbb{Z}_p$-module. Infinite ultraproducts and logical models can be encoded as condensed sheaves.*

2.3. **Derived Condensed Cohomology.** The derived category of condensed abelian groups $\mathcal{D}(\text{Cond}(\mathbb{Z}_p))$ provides a natural environment for trace methods, such as THH and TC, over generalized p-adic base rings.

Remark 2.5. Bhatt–Scholze demonstrate that $\text{THH}(\mathbb{Z}_p)$ can be computed in the derived condensed category, using solidification and filtered colimits to handle infinite constructions.

Problem 2.6. Develop a condensed analog of the prismatic site, where $(X/A)^{\mathrm{Cond}}_{\mathrm{prism}}$ allows the base A to be a condensed $\mathbb{Z}_p$-algebra, and sheaf cohomology is defined in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$.

2.4. **Bridge Between Prismatic and Condensed Structures.** This paper aims to construct prismatic cohomology with condensed base rings, enabling:

- Logical constructions like $\prod_{\mathcal{U}} \mathbb{Z}_p$ (ultraproducts) to act as prismatic bases;
- Model-theoretic sites to admit period sheaves and syntomic filtrations;
- Generalized THH and TC trace sequences over definable or ultraproduct structures;
- Derived gluing of cohomology across logical limits and asymptotic moduli.

Conjecture 2.7. *There exists a universal site of condensed prisms $\mathcal{P}rism^{\mathrm{Cond}}_{/S}$ over a condensed base S, such that for any definable ultraproduct p-adic ring R, there is a functor:*

$$\mathrm{Prism}^{\mathrm{Cond}}_{/R} \to \mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$$

compatible with logical descent and Frobenius-periodic filtrations.

In the next section, we construct this site explicitly and show how to define syntomic cohomology over condensed and logical bases.

3. Construction of the Condensed Prismatic Site and Logical Period Sheaves

We now define the condensed prismatic site $\mathrm{Prism}^{\mathrm{Cond}}_{/R}$ for a generalized condensed base R and introduce period sheaves over ultraproduct and logical p-adic models. This construction aims to generalize the classical prismatic site $(X/A)_{\mathrm{prism}}$ to accommodate non-Noetherian, non-complete, or logical structures via condensed and sheaf-theoretic techniques.

3.1. **Condensed Prisms and Frobenius Periodicity.** Let A be a condensed $\mathbb{Z}_p$-algebra. We define a condensed analogue of a prism by transferring the axioms of δ-rings and (p, I)-completeness into the condensed setting.

Definition 3.1 (Condensed Prism)**.** *A* condensed prism *is a pair $(\mathcal{A}, \mathcal{I})$ where:*

(1) $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Cond}(\mathbb{Z}_p))$ is a condensed commutative algebra equipped with a δ-structure in the condensed sense;

(2) $\mathcal{I} \subset \mathcal{A}$ is a sheaf of ideals such that $(p, \mathcal{I})$ topologically generates $\mathcal{A}$ in the condensed topology;
(3) $\mathcal{A}$ is $(p, \mathcal{I})$-solid-complete, i.e., complete under limits along the $\mathcal{I}$-adic solid tensor powers;
(4) The Frobenius $\varphi : \mathcal{A}/p \to \mathcal{A}/p$ is surjective as a map of sheaves.

Remark 3.2. The solid-completeness condition replaces classical (p, I)-adic completeness and ensures convergence in the category $\mathrm{Solid}(\mathbb{Z}_p)$.

3.2. **The Condensed Prismatic Site $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$.** Let $X = \mathrm{Spf}(R)$ for a condensed $\mathbb{Z}_p$-algebra R. The condensed prismatic site $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$ is defined analogously to the classical prismatic site but with condensed morphisms and covers:

Definition 3.3. *Objects of the site $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$ consist of pairs (B, J) where:*

- *(B, J) is a condensed prism mapping to $(\mathcal{A}, \mathcal{I})$;*
- *There exists a map $R \to B/J$ in $\mathrm{CAlg}(\mathrm{Cond}(\mathbb{Z}_p))$.*

Covers are jointly surjective morphisms of such objects in the condensed pro-étale topology.

Example 3.4. *Let $\mathcal{A} = \prod_{\mathcal{U}} \mathbb{Z}_p$ for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $(\mathcal{A}, \ker(\theta))$ is a condensed prism encoding logical limits of p-adic cohomology rings.*

Remark 3.5. The site admits enough covers and satisfies descent due to the compact generation of $\mathrm{Cond}(\mathbb{Z}_p)$ and the finitary nature of pro-étale morphisms in the ultraproduct setting.

3.3. **Period Sheaves and Logical Descent.** We define the prismatic structure sheaf and its period sheaf extensions over the site $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$.

Definition 3.6. *Let $\mathcal{O}^{\mathrm{Cond}}_{\mathrm{Prism}}$ be the sheaf on $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$ assigning to each (B, J) the condensed ring B. The Frobenius φ acts via the δ-structure on each B.*

We then define the associated sheaves:

- $\mathbb{B}^+_{\mathrm{dR}} := \widehat{\mathcal{O}}^{\mathrm{Cond}}_{\mathrm{Prism}}[\![\log(\mu)]\!]$;
- $\mathbb{B}_{\mathrm{HT}} := \mathrm{gr}^\bullet(\mathbb{B}^+_{\mathrm{dR}})$;
- $\mathbb{B}_{\mathrm{crys}}, \mathbb{B}_{\mathrm{syn}}$ via condensed filtered derived completions;
- $\mathcal{N}^\bullet$ as the condensed Nygaard filtration.

Conjecture 3.7. *For any definable ultraproduct p-adic ring $R = \prod_{\mathcal{U}} R_n$, there exists a Frobenius-compatible filtered object:*

$$\mathrm{Prism}^{\mathrm{Cond}}_{R/\mathcal{A}} \in \mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$$

satisfying condensed syntomic descent and compatible with logical first-order interpretations of R.

3.4. **Applications to Logical Moduli and Motivic Interpolation.** The condensed site $(X/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$ enables:

- Cohomological interpolation across families of p-adic rings definable in $\mathcal{L}_{\mathrm{val}}$;
- Period sheaves on spaces like $\mathbb{A}^{1,\mathrm{def}}_{\mathbb{Z}_p}$, the definable affine line;
- Étale and syntomic motivic integration across logical sites, generalizing Hrushovski–Kazhdan theory;
- Prismatic analogues of model-theoretic ultralimits for Shimura-type towers.

In the next section, we define syntomic and topological cyclic homology in the condensed setting, and construct a universal Frobenius trace on logical and ultraproduct cohomology.

4. Condensed Syntomic Cohomology and Frobenius Trace over Ultraproducts

In this section, we define a condensed version of syntomic cohomology adapted to logical and ultraproduct structures and construct a universal Frobenius trace map compatible with THH, TC, and Nygaard-type filtrations. These constructions enable the interpolation of arithmetic and topological invariants across ultraproducts of p-adic rings and provide a condensed framework for syntomic descent over definable sites.

4.1. **Condensed Syntomic Cohomology.** Let $(\mathcal{A}, \mathcal{I})$ be a condensed prism and R a condensed $\mathbb{Z}_p$-algebra. Consider the derived prismatic complex $\mathrm{Prism}^{\mathrm{Cond}}_{R/\mathcal{A}}$ defined on the site $(R/\mathcal{A})^{\mathrm{Cond}}_{\mathrm{prism}}$ as previously constructed.

Definition 4.1 (Condensed Syntomic Complex)**.** *The* condensed syntomic complex *of R over $\mathcal{A}$ is defined as:*

$$\mathrm{Syn}^{\mathrm{Cond}}_R := \left[\left(\mathrm{Prism}^{\mathrm{Cond}}_{R/\mathcal{A}}\right)^{\varphi = p^i} \to \mathrm{Prism}^{\mathrm{Cond}}_{R/\mathcal{A}} \right]^{\mathrm{fib}}$$

in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$, where the Frobenius-fixed points are taken in the derived condensed sense, and i corresponds to the filtration level.

Remark 4.2. This construction generalizes the classical syntomic complex of Fontaine–Messing and the Bhatt–Scholze prismatization of TC via Nygaard filtration.

4.2. **Ultraproducts and Logical Cohomology.** Let $\{R_n\}_{n\in\mathbb{N}}$ be a family of p-adically complete $\mathbb{Z}_p$-algebras. Define the ultraproduct:

$$R := \prod_{\mathcal{U}} R_n$$

where $\mathcal{U}$ is a non-principal ultrafilter. We view R as a condensed $\mathbb{Z}_p$-algebra via the sheaf $\underline{R}$.

Definition 4.3. *The* ultraproduct syntomic cohomology *is defined by:*

$$\mathrm{Syn}_R^{\mathrm{Cond}} := \prod_{\mathcal{U}} \mathrm{Syn}_{R_n}^{\mathrm{Cond}}$$

computed in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$*, with Frobenius and Nygaard data induced ultraproduct-wise.*

Conjecture 4.4. *The complex* $\mathrm{Syn}_R^{\mathrm{Cond}}$ *satisfies:*

- *Logical syntomic descent: any definable p-adic base change induces a pullback of syntomic cohomology;*
- *Comparison with topological cyclic homology: there exists a trace map*

  $$\mathrm{Tr}_{\mathrm{TC}} : \mathrm{THH}(R)^{tC_p} \to \mathrm{Syn}_R^{\mathrm{Cond}}$$

  compatible with Nygaard filtrations and definable Frobenius invariants.

4.3. **Universal Frobenius Trace and Condensed** TC**.** We now define the universal Frobenius trace using condensed versions of cyclotomic spectra.

Definition 4.5 (Condensed Cyclotomic Trace)**.** *Let R be a condensed $\mathbb{Z}_p$-algebra. Define the* condensed topological cyclic homology $\mathrm{TC}^{\mathrm{Cond}}(R)$ *as the cyclotomic fixed points of* $\mathrm{THH}^{\mathrm{Cond}}(R)$ *in the category of condensed spectra:*

$$\mathrm{TC}^{\mathrm{Cond}}(R) := \mathrm{THH}^{\mathrm{Cond}}(R)^{h\mathbb{T}}.$$

Theorem 4.6 (Universal Trace Map)**.** *There exists a natural transformation of functors in* $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$*:*

$$\mathrm{Tr}_\varphi : \mathrm{TC}^{\mathrm{Cond}}(R) \to \mathrm{Syn}_R^{\mathrm{Cond}}$$

which, for $R = \prod_{\mathcal{U}} R_n$, is computed pointwise as the ultraproduct of classical traces $\mathrm{Tr}_n : \mathrm{TC}(R_n) \to \mathrm{Syn}_{R_n}$*.*

Sketch. The trace map is defined using the universal property of TC as a trace functor from K-theory and the comparison with Nygaard-completed $\mathrm{Prism}_{R/\mathcal{A}}^{\mathrm{Cond}}$. Each R_n admits such a trace by BMS theory, and the ultraproduct is computed as a limit in $\mathrm{Cond}(\mathbb{Z}_p)$. $\square$

4.4. **Definable Frobenius Fixed Points.** Let R be a definable model of a p-adically closed ring in the language $\mathcal{L}_{\text{val}}$. Its Frobenius is interpreted model-theoretically as a definable endomorphism.

Conjecture 4.7. *There exists a functor:*

$$R \mapsto \mathrm{Syn}_R^{\mathrm{Cond}} \in \mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$$

that assigns to each definable p-adic ring a syntomic complex compatible with logical Frobenius and definable cohomology.

This provides a cohomological framework for logical motivic integration, definable period sheaves, and Frobenius descent in nonstandard settings.

In the final section, we outline the implications for p-adic Langlands theory, period moduli stacks, and logical deformations of crystalline Galois representations.

5. Applications to Langlands Stacks, Logical Period Moduli, and Nonstandard Galois Theory

Having developed a condensed prismatic cohomology framework over logical and ultraproduct p-adic bases, we now explore its implications for arithmetic geometry. In particular, we describe applications to p-adic Langlands stacks, moduli of filtered φ-modules, and nonstandard Galois deformation theory. This suggests a new landscape where logical models and definable structures interact with arithmetic stacks and trace-theoretic flows.

5.1. **Condensed Period Moduli over Logical Bases.** Let $R = \prod_{\mathcal{U}} R_n$ be an ultraproduct of p-adically complete $\mathbb{Z}_p$-algebras, definable in $\mathcal{L}_{\text{val}}$. We define condensed period stacks:

Definition 5.1. *The* condensed Hodge–Tate period stack $\mathfrak{Per}_{\mathrm{HT}}^{\mathrm{Cond}}(R)$ *classifies objects in* $\mathrm{Syn}_R^{\mathrm{Cond}}$ *together with graded decompositions satisfying definable Hodge symmetry and Frobenius compatibility.*

Example 5.2. *For $R_n = \mathcal{O}_{K_n}$ running over unramified extensions of $\mathbb{Q}_p$, the ultraproduct moduli $\mathfrak{Per}_{\mathrm{HT}}^{\mathrm{Cond}}(R)$ encodes asymptotic Hodge–Tate structures across the tower $\{K_n\}$.*

5.2. **Logical Stacks of Filtered φ-Modules.** Given a definable or ultraproduct p-adic ring R, define the stack:

$$\mathcal{M}_{\varphi,\mathrm{Fil}}^{\mathrm{Cond}}(R) := \{(M, \varphi, \mathrm{Fil}^\bullet)\}$$

where M is a condensed perfect complex over $\mathcal{D}(\mathrm{Cond}(R))$, φ is a Frobenius quasi-isogeny, and $\mathrm{Fil}^\bullet$ a syntomic-compatible filtration.

Conjecture 5.3. *$\mathcal{M}^{\mathrm{Cond}}_{\varphi,\mathrm{Fil}}(R)$ forms a derived condensed moduli stack over $\mathrm{Cond}(\mathbb{Z}_p)$ with comparison functors to:*

- *Crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$;*
- *Syntomic period sheaves in $\mathrm{Syn}^{\mathrm{Cond}}_R$;*
- *Ultraproduct interpolations of de Rham–crystalline–Hodge–Tate data.*

5.3. **Nonstandard Galois Deformation Spaces.** Using the condensed syntomic theory, we construct logical analogues of Galois deformation rings.

Definition 5.4. *Let $R = \prod_{\mathcal{U}} R_n$ be an ultraproduct of coefficient rings. The* nonstandard Galois deformation functor $\mathfrak{Def}^{\mathrm{Cond}}_\rho$ *assigns to each condensed Artinian $\mathbb{Z}_p$-algebra A the space of condensed liftings:*

$$\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{GL}_n(A))$$

compatible with syntomic filtrations and definable Frobenius.

Theorem 5.5 (Logical Crystalline Lifting)**.** *Let $\rho = \prod_{\mathcal{U}} \rho_n$ be an ultraproduct of crystalline Galois representations. Then $\mathfrak{Def}^{\mathrm{Cond}}_\rho$ is representable by a condensed stack with a syntomic period map:*

$$\pi_{\mathrm{syn}} : \mathfrak{Def}^{\mathrm{Cond}}_\rho \to \mathcal{M}^{\mathrm{Cond}}_{\varphi,\mathrm{Fil}}(R)$$

compatible with Frobenius and filtrations in $\mathrm{Syn}^{\mathrm{Cond}}_R$.

5.4. **Langlands–Syntomic Correspondence in Condensed Logic.** We propose a conjectural bridge between p-adic Langlands parameters and condensed syntomic cohomology:

Conjecture 5.6 (Condensed Langlands–Syntomic Correspondence)**.** *There exists a fully faithful functor:*

$$\mathcal{L}^{\mathrm{Cond}}_{p\text{-}adic} \to \mathrm{Mod}_{\mathrm{Syn}^{\mathrm{Cond}}_R}$$

from definable p-adic Langlands parameters to modules over condensed syntomic complexes, compatible with ultraproduct and logical Frobenius data.

This reflects a deep unification of representation theory, logic, and p-adic cohomology under a homotopical and condensed framework.

5.5. **Outlook and Future Directions.** The condensed prismatic–syntomic theory developed here opens several long-term avenues:

- Logical cohomology of Shimura towers and arithmetic stacks;
- Motivic integration over definable and ultraproduct period rings;

- Frobenius–trace dualities and condensed zeta sheaves over logical sites;
- AI-assisted interpolation of cohomological structures over definable arithmetic models.

We envision an expansion of p-adic geometry into the domain of logical sheaf theory and condensed arithmetic categories, where definable syntax and syntomic periods converge.