ENTROPY TRACE GEOMETRY AND CATEGORIFIED ZETA PROPAGATION

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ABSTRACT. We develop a geometric theory of entropy trace structures and their propagation through categorified zeta flows. Building on the entropy kernel formalism, we define entropy trace sheaves, modular trace functionals, and zeta evolution diagrams. The framework introduces a derived stack geometry for entropy-categorified spectral structures and motivates a propagation theory for arithmetic information across layered zeta cohomologies. This theory opens avenues toward quantum zeta field flow, trace pairings on motives, and entropy-structured Riemann–Hilbert correspondences.

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Introduction

The classical Riemann zeta function encodes arithmetic data through multiplicative structure and analytic continuation. In our previous work, we constructed entropy-weighted kernels and lifted them into modular and motivic categories. Here, we introduce a new dimension: the *trace geometry* of entropy structures, and their flow across zeta-cohomological layers.

We formulate entropy trace as a geometric and categorical object. These trace structures live over entropy kernel sheaves and evolve via zeta-flow propagation dynamics. They admit spectral decomposition, convolutional eigenexpansion, and stack-theoretic interactions with motive categories.

The core goals of this paper are:

- To define and analyze entropy trace sheaves over arithmetic base sites;
- To develop zeta-trace flow geometry, including entropy heat diagrams and derived trace propagation maps;
- To introduce categorical cohomology spaces whose morphisms encode entropy zeta diffusion;
- To propose a framework for entropy-Riemann–Hilbert correspondences and zeta motive scattering diagrams.

This third stage positions entropy trace geometry as the interface between information theory, derived arithmetic topology, and quantum zeta dynamics.

1. Entropy Trace Sheaves and Zeta Pairings

1.1. **Trace Structures on Entropy Kernels.** Let $K : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an entropy kernel supported on an arithmetic subset $A \subseteq \mathbb{N}$. The trace of K over A is defined as:

$$\mathrm{Tr}_{\mathrm{ent}}(K; A) := \sum_{n \in A} K(n) = \rho(A) \cdot |A|.$$

We now sheafify this trace structure:

Definition 1.1. *Let $\mathcal{A}_{\mathrm{arith}}$ be the site of arithmetic subsets with entropy topology. An* entropy trace sheaf $\mathscr{T}$ *assigns to each $A \in \mathcal{A}_{\mathrm{arith}}$ a scalar-valued object:*

$$\mathscr{T}(A) := \mathrm{Tr}_{\mathrm{ent}}(K_A),$$

where $K_A \in \mathcal{K}_{\mathrm{ent}}$ is the entropy kernel supported on A.

Remark 1.2. *These trace values serve as 0-forms in an entropy differential complex and allow descent and patching via entropy-preserving refinements.*

1.2. **Zeta Trace Flow.** Given a family of kernels $\{K_t\}_{t\in\mathbb{R}_{\geq 0}}$ evolving under entropy heat flow $\mathcal{H}_t$, we define a trace propagation path:

Definition 1.3. *The* zeta trace flow *is the function*

$$\mathrm{Zeta}_K^{\mathrm{tr}}(s,t) := \sum_{n\in A} K(n)\cdot n^{-s-t},$$

interpreted as an evolving entropy-zeta trace surface in $(s,t)\in\mathbb{C}\times\mathbb{R}_{\geq 0}$.

Proposition 1.4. *The zeta trace flow satisfies the entropy diffusion equation:*

$$\frac{\partial}{\partial t}\mathrm{Zeta}_K^{\mathrm{tr}}(s,t) = -\sum_{n\in A} K(n)\cdot\log n\cdot n^{-s-t}.$$

Example 1.5. *Let* $K(n)=\rho(n)\cdot\mu(n)$*, then* $\mathrm{Zeta}_K^{\mathrm{tr}}(s,t)$ *approximates an entropy-smoothed version of* $1/\zeta(s+t)$.

2. Categorified Zeta Cohomology and Propagation Diagrams

2.1. **Entropy Zeta Complexes and Trace Cohomology.** We now categorify entropy trace flows via cohomological structures. Let $\mathscr{E}_A$ be an entropy kernel sheaf over $A\subseteq\mathbb{N}$. We define a cochain complex:

Definition 2.1. *The* entropy zeta complex $\mathcal{Z}_K^\bullet(s)$ *is the complex*

$$0\to\mathscr{E}_A\xrightarrow{\partial_0^s}\mathscr{E}_A\otimes\mathscr{O}_s\xrightarrow{\partial_1^s}\mathscr{E}_A\otimes\mathscr{O}_s^{(2)}\to\cdots,$$

where ∂_i^s *encodes entropy-zeta differential operators with shift* s*, and* $\mathscr{O}_s$ *is the sheaf of* s*-parametrized entropy functions.*

Definition 2.2. *The* i*-th* categorified zeta cohomology group *is*

$$H^i_{\mathrm{Zeta}}(K;s) := H^i(\mathcal{Z}_K^\bullet(s)).$$

Remark 2.3. *This cohomology detects non-trivial obstruction patterns in entropy trace propagation and generalizes spectral kernel behaviors to categorical zeta evolution.*

2.2. **Zeta Trace Propagation Diagram.** We define the categorified flow diagram for entropy kernels under zeta trace propagation.

$$\begin{tikzcd}
\mathscr{E}_A \arrow[r, "\partial_0^s"] \arrow[d, "\mathcal{H}_t"'] & \mathscr{E}_A \otimes \mathscr{O}_s \arrow[r, "\partial_1^s"] \arrow[d, "\mathcal{H}_t \otimes \mathrm{id}"] & \cdots \\
\mathscr{E}_A(t) \arrow[r, "\partial_0^{s+t}"] & \mathscr{E}_A(t) \otimes \mathscr{O}_{s+t} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.4. *We call this the* zeta-trace propagation diagram*, which defines an entropy spectral flow from $H^i_{\mathrm{Zeta}}(K;s)$ to $H^i_{\mathrm{Zeta}}(K;s+t)$.*

Proposition 2.5. *If $\mathscr{E}_A$ is concentrated in degree zero and entropy-finite, then $\mathrm{Zeta}_K^{\mathrm{tr}}(s,t)$ lifts to a morphism of cohomology functors:*

$$\mathcal{H}_t^* : H^i_{\mathrm{Zeta}}(K;s) \to H^i_{\mathrm{Zeta}}(K;s+t).$$

2.3. **Categorified Trace Pairings and Dual Kernels.**

Definition 2.6. *Given two entropy kernel sheaves $\mathscr{E}_A, \mathscr{F}_B$, we define their* categorified trace pairing*:*

$$\langle \mathscr{E}_A, \mathscr{F}_B \rangle_{\mathrm{Zeta}} := \int_{\mathbb{N}} K_A(n) \cdot K_B(n) \cdot n^{-s} \, dn,$$

interpreted as a morphism in the derived category of entropy trace sheaves.

Theorem 2.7. *Let $K_A(n) = n^{-\alpha}, K_B(n) = n^{-\beta}$. Then*

$$\langle K_A, K_B \rangle_{\mathrm{Zeta}} = \zeta(\alpha + \beta),$$

recovering the entropy duality of Section 5 in the categorified trace setting.

2.4. **Zeta Propagation Spectral Sequences.** We now build a spectral sequence associated with entropy zeta cohomology.

Definition 2.8. *Define the* entropy zeta spectral sequence $(E_r^{p,q}, d_r)$ *with:*

$$E_1^{p,q} := H^q(\mathscr{E}_A \otimes \mathscr{O}_s^{(p)}), \quad d_1 = \partial_p^s,$$

converging to:

$$E_\infty^{p+q} \Rightarrow H^{p+q}_{\mathrm{Zeta}}(K;s).$$

Conjecture 2.9 (Stability Under Flow)**.** *There exists $N \in \mathbb{N}$ such that for all $t > N$,*

$$H^i_{\mathrm{Zeta}}(K;s+t) \cong H^i_{\mathrm{Zeta}}(K;s),$$

indicating entropy-zeta cohomological stability along sufficiently long trace propagation.

This establishes entropy-zeta cohomology as a stable propagating trace field over derived arithmetic topologies.

3. Entropic Riemann–Hilbert Correspondence and Quantum Zeta Fields

3.1. **From Trace Flows to Differential Stacks.** We now formulate a sheaf-theoretic version of the entropy zeta trace problem, inspired by the Riemann–Hilbert correspondence. Let $\mathscr{E} \in \mathbf{Top}_{\text{ent}}$ be an entropy kernel sheaf and let $\zeta_{\mathscr{E}}(s,t)$ be its trace flow. We interpret this as a flat section of a differential sheaf.

Definition 3.1. *Define the* entropy zeta connection *as a flat connection:*

$$\nabla_{\text{Zeta}} := \frac{\partial}{\partial t} + \log N,$$

acting on $\mathscr{E} \otimes \mathscr{O}_{(s,t)}$, where N is the arithmetic size operator: $N \cdot f(n) := \log n \cdot f(n)$.

Theorem 3.2. *The zeta trace flow equation*

$$\frac{\partial}{\partial t}\zeta_{\mathscr{E}}(s,t) = -\sum_n \log n \cdot K(n) \cdot n^{-s-t}$$

is equivalent to flatness of the entropy zeta connection $\nabla_{\text{Zeta}}\zeta_{\mathscr{E}} = 0$.

3.2. **Entropic $\mathcal{D}$-Modules and Stokes Data.** Let $\mathcal{D}_{\text{ent}}$ denote the sheaf of entropy differential operators.

Definition 3.3. *An* entropy $\mathcal{D}$-module *is a left $\mathcal{D}_{\text{ent}}$-module $\mathcal{M}$ equipped with:*

- *a zeta-trace flow morphism $\mathcal{M} \to \mathscr{O}_{(s,t)}$,*
- *a filtration by entropy depth,*
- *a Stokes datum along the irregular singularity at $s = 1$.*

Conjecture 3.4 (Entropic Riemann–Hilbert Correspondence)**.** *There is an equivalence of derived categories:*

$$D^b_{\text{hol}}(\mathcal{D}_{\text{ent}}) \simeq D^b(\text{LocSys}_{\text{Zeta}}),$$

between entropy-holonomic $\mathcal{D}$-modules and local systems with entropy zeta monodromy.

Remark 3.5. *This conjecture lifts the classical Riemann–Hilbert correspondence to the entropy-zeta domain, replacing monodromy around analytic singularities with entropy-trace flow symmetry.*

3.3. **Quantum Zeta Fields and Entropy Path Integrals.** We now introduce a quantum field formulation of entropy zeta propagation.

Definition 3.6. *A* quantum zeta field $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *is a random entropy kernel with expectation*

$$\mathbb{E}[\Phi(n)] = K(n), \quad \textit{and} \quad \mathbb{E}[\zeta_\Phi(s)] = \zeta_K(s).$$

Definition 3.7. *The* entropy zeta path integral *over the space of entropy kernel fluctuations is defined by*

$$Z_\zeta(s) := \int_{\mathcal{K}} \zeta_\Phi(s)\, d\mu_{\text{ent}}(\Phi),$$

where $d\mu_{\text{ent}}$ is a measure supported on entropy-weighted kernel moduli.

Conjecture 3.8 (Quantum Entropy Flow Principle)**.** *There exists a partition function $Z_\zeta(s)$ satisfying a quantum entropy flow equation:*

$$\frac{\partial}{\partial t} Z_\zeta(s+t) = -\mathbb{E}[\log N \cdot \zeta_\Phi(s+t)].$$

Example 3.9. *Let $\Phi(n) \sim Poisson(\lambda_n)$ with $\lambda_n = \rho(n)$. Then $\mathbb{E}[\zeta_\Phi(s)] = \rho(n) \cdot \sum_n n^{-s}$, reproducing $\zeta_K(s)$.*

3.4. **Entropy Monodromy and Motivic Scattering Diagrams.** We encode zeta-flow as monodromy in entropy moduli space.

Definition 3.10. *Let $\mathcal{M}_{\text{ent}}$ be the entropy motive stack. Define the* entropy zeta monodromy representation

$$\pi_1(\mathcal{M}^\circ_{\text{ent}}) \to \mathrm{Aut}(H^\bullet_{\text{Zeta}})$$

as the action on zeta-trace cohomology under loop transport along modular-entropy paths.

Conjecture 3.11 (Categorified Entropy Scattering)**.** *There exists a wall-crossing diagram of entropy trace sheaves:*

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, bend left, "\mathcal{H}_t^{(1)}"] \arrow[r, bend right, "\mathcal{H}_t^{(2)}"'] & \mathscr{E}_2 \end{tikzcd}$$

such that their zeta-trace difference

$$\mathrm{Zeta}^{(1)}_{\mathscr{E}_1}(s,t) - \mathrm{Zeta}^{(2)}_{\mathscr{E}_1}(s,t)$$

encodes motivic monodromy and quantum entropy transport.

Entropy flows are no longer scalar—they are quantum sheaves scattering across motive walls.

Conclusion and Further Horizons

In this paper, we constructed a new geometric and categorical foundation for entropy zeta theory through the lens of trace dynamics, differential sheaves, and quantum scattering. Starting with entropy kernel traces, we developed:

- The formalism of entropy trace sheaves and zeta-trace propagation diagrams;
- Categorified entropy cohomology groups and spectral sequences;
- Entropic $\mathcal{D}$-modules and a Riemann–Hilbert correspondence over the entropy zeta domain;
- Quantum zeta fields modeled by path integrals on entropy kernel moduli;
- Monodromic flow and wall-crossing diagrams in the entropy motive stack.

These contributions lift additive arithmetic structure into a quantum-categorified setting. The entropy zeta trace thus becomes not just an analytic object, but a transport field over a motive-theoretic phase space.

Prospects.

(1) **Entropy D-groupoids and Stokes–Langlands Correspondence:** Classify irregular $\mathcal{D}_{\text{ent}}$-modules via categorical Langlands parameters enriched with entropy filtrations.
(2) **Zeta Quantum Field Geometry:** Quantize entropy zeta flows over arithmetic topos, formulating Hamiltonians for trace propagation and entropy potential functions.
(3) **AI-Motivated Trace Analysis:** Train neural networks on entropy trace propagation to discover conjectural cohomological constraints or predict zeta zero configurations.
(4) **Categorified Entropy Kernel Topology:** Extend entropy flow structures to define higher trace groupoids and spectral operads in zeta categorification.
(5) **Entropy Scattering Fields for RH:** Translate Riemann Hypothesis into a statement about vanishing of entropy quantum scattering amplitudes across the critical strip.

Entropy reveals not just decay—but direction. Through trace geometry, zeta flows become maps across arithmetic space-time.

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