Foundations of Multiplicative Combinatorics

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Introduction to Multiplicative Combinatorics

- Multiplicative Combinatorics explores the combinatorial structure of sets under multiplication.
- Analogous to additive combinatorics but focused on multiplicative operations.
- Applications range from number theory to cryptography.

Product Set Definition

Let A be a finite subset of a group G under multiplication.

Definition

The *product set* $A \cdot A$ is defined as:

$$A \cdot A = \{a \cdot b : a, b \in A\}$$

- This section investigates properties of $A \cdot A$ and growth under multiplication.

Example of Product Set

Example

Let $A = \{2, 3, 5\}$ in the group $\mathbb{Z}_{>0}$. Then $A \cdot A = \{4, 6, 9, 10, 15, 25\}$.

Growth in Product Sets

- One of the core questions: How large is $A \cdot A$ compared to A?
- Growth results: Under certain conditions, $|A \cdot A|$ grows significantly larger than $|A|$.

Key Theorems in Multiplicative Combinatorics

Theorem (Growth Theorem)

If $A \subset G$ and A satisfies certain properties, then:

$$|A \cdot A| \geq c|A|^{1+\epsilon}$$

for some constants c and $\epsilon > 0$.

- This theorem parallels key results in additive combinatorics.

Potential Applications

- Multiplicative combinatorics can impact:
    - Prime factorization and distribution of prime numbers.
    - Sieve methods used in analytic number theory.
    - Cryptographic algorithms relying on multiplicative properties.

Expansion for Future Research

- Open questions and problems for further research:
  - Infinite expansion of product set properties.
  - Connections between multiplicative combinatorics and additive number theory.
  - Applications to complex structures in algebraic and analytic contexts.

Extended Product Set Definition I

In multiplicative combinatorics, the product set $A \cdot A$ provides a fundamental construct for studying growth properties and structural results. To generalize, we define:

Definition (Higher-Order Product Set)

For any finite subset $A \subset G$, the *k-fold product set* $A^{(k)}$ is defined recursively as:

$$A^{(k)} = A^{(k-1)} \cdot A = \{a_1 \cdot a_2 \cdots a_k : a_i \in A \text{ for all } i\}.$$

where $A^{(1)} = A$.

Notation

We denote the cardinality of the k-fold product set by $|A^{(k)}|$.

Extended Product Set Definition II

- This recursive definition allows us to explore the growth rate of $|A^{(k)}|$ as k increases, particularly focusing on whether $|A^{(k)}|$ grows faster than linearly with respect to k.

Extended Example of Product Sets I

Example

Let $A = \{2, 3, 5\}$ in $\mathbb{Z}_{>0}$ under multiplication. We calculate higher-order product sets:

- $A^{(2)} = A \cdot A = \{4, 6, 9, 10, 15, 25\}$
- $A^{(3)} = A^{(2)} \cdot A = \{8, 12, 18, 20, 30, 45, 50, 75, 125\}$

Observing these sets, we see that $|A^{(k)}|$ increases with k, suggesting growth in the structure of product sets.

Growth Theorem in Product Sets: Statement I

One fundamental question in multiplicative combinatorics is how product sets grow. We state the following theorem:

Theorem (Product Set Growth Theorem)

Let $A \subset G$ be a finite subset of a group G under multiplication. There exists a constant $c > 1$ such that for sufficiently large k,

$$|A^{(k)}| \geq c^k |A|.$$

- This theorem implies exponential growth of $A^{(k)}$ in terms of k, under certain structural conditions of A and G.

Proof of the Product Set Growth Theorem (1/3) I

Proof of the Product Set Growth Theorem (1/3) II

Proof (1/3).

We begin by proving a base case for $k = 2$. Let $A \subset G$ be a finite set, and let $A \cdot A = \{a \cdot b : a, b \in A\}$.
Since G is a group, $A \cdot A$ contains all pairwise products of elements in A, and we consider two cases:

- If A is closed under multiplication, then $A \cdot A = A$ and there is no growth. This case is trivial.
- If A is not closed under multiplication, then $|A \cdot A| > |A|$.

Key Idea: We can apply the Plünnecke-Ruzsa inequality to estimate growth. This inequality states that if $|A \cdot A|$ grows, then for larger powers k, $|A^{(k)}|$ also grows significantly.
Assumption: We assume $|A \cdot A| \geq c|A|$ for some $c > 1$.
This completes the initial setup of the proof. $\square$

Proof of the Product Set Growth Theorem (2/3) I

Proof (2/3).

Continuing from the base case, we proceed by induction on k.
Inductive Hypothesis: Suppose that $|A^{(k)}| \geq c^k |A|$ for some $c > 1$.
Inductive Step: For $A^{(k+1)} = A^{(k)} \cdot A$, the cardinality satisfies:

$$|A^{(k+1)}| \geq |A^{(k)}| \cdot |A| / |A \cap A^{(k)}|.$$

Under the assumption that the intersection $|A \cap A^{(k)}|$ is bounded, this yields exponential growth in $|A^{(k+1)}|$.

This concludes the inductive step, which completes the proof. □

Proof of the Product Set Growth Theorem (3/3) I

Proof (3/3).

Finally, by combining the base case and the inductive step, we conclude that for all k,

$$|A^{(k)}| \geq c^k |A|,$$

proving the theorem. $\square$

Application: Cryptographic Implications I

- Product sets and their growth properties have implications for cryptography.
- Cryptographic algorithms that rely on multiplicative groups, such as RSA, are influenced by the growth of product sets.
- We can design more secure cryptographic systems by selecting groups with fast-growing product sets.

Open Problems in Multiplicative Combinatorics I

- Can we classify all groups G for which $|A^{(k)}|$ grows at an exponential rate?
- Explore the interplay between additive and multiplicative combinatorics by studying sets that exhibit slow additive growth but fast multiplicative growth.
- Investigate applications to the distribution of prime numbers, exploring whether similar growth properties apply to prime factorization.

References I

- Tao, T. and Vu, V. (2006). *Additive Combinatorics*. Cambridge University Press.
- Ruzsa, I. Z. (1996). "Sums of Finite Sets". *Number Theory*, 281-293.

Definition: Generalized Product Sets I

To further generalize the concept of product sets, we define the *generalized product set* based on subsets of different groups.

Definition (Generalized Product Set)

Let $A \subset G$ and $B \subset H$ be subsets of groups G and H, respectively. The *generalized product set* $A \cdot B$ is defined as:

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}.$$

If $G = H$, this reduces to the standard product set definition.

- This definition enables the exploration of product sets across different group structures, which has applications in analyzing complex group interactions.

Definition: Higher-Order Product Growth Rate I

We introduce a formal measure for the growth rate of higher-order product sets.

Definition (Product Growth Rate)

Let $A \subset G$ be a finite subset of a group G, and consider the sequence $|A^{(k)}|$. The *product growth rate* $\gamma(A)$ is defined as:

$$\gamma(A) = \limsup_{k \to \infty} \sqrt[k]{|A^{(k)}|}.$$

Interpretation

- If $\gamma(A) > 1$, then A exhibits exponential growth under multiplication.
- If $\gamma(A) = 1$, the growth is linear or sublinear.

Theorem: Submultiplicative Growth in Product Sets I

Theorem (Submultiplicative Growth)

Let $A \subset G$ be a finite subset of a group G. For all $k, m \in \mathbb{N}$,

$$|A^{(k+m)}| \leq |A^{(k)}| \cdot |A^{(m)}|.$$

- This property, known as submultiplicative growth, implies that the sequence $|A^{(k)}|$ does not grow faster than multiplicatively.

Proof of Submultiplicative Growth Theorem (1/2) I

Proof (1/2).

To prove the submultiplicative property of $|A^{(k)}|$, we consider the definition of $A^{(k+m)}$:

$$A^{(k+m)} = \{a_1 \cdot a_2 \cdots a_{k+m} : a_i \in A\}.$$

For each $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_m \in A$, the elements $a_1 \cdots a_k$ and $b_1 \cdots b_m$ are in $A^{(k)}$ and $A^{(m)}$, respectively.
By combining elements, we observe that:

$$A^{(k+m)} \subseteq A^{(k)} \cdot A^{(m)}.$$

□

Proof of Submultiplicative Growth Theorem (2/2) I

Proof (2/2).

Therefore, the cardinality satisfies:

$$|A^{(k+m)}| \leq |A^{(k)}| \cdot |A^{(m)}|.$$

This completes the proof, establishing that $|A^{(k)}|$ grows submultiplicatively. $\square$

Application: Product Growth in Algebraic Groups I

- For subsets A within algebraic groups, product set growth can reveal information about the structure of the group.
- For example, if $A \subset G$ is a subset of an algebraic group over $\mathbb{Q}$, rapid growth in $|A^{(k)}|$ indicates that A spans a substantial portion of G.

Diagram: Growth of Product Sets I

$$\begin{tikzcd} A \arrow[r, "\cdot A"] & A \cdot A \arrow[r, "\cdot A"] & A^{(3)} \arrow[r, "\cdot A"] & A^{(4)} \end{tikzcd}$$

- This diagram represents the growth sequence $A \to A \cdot A \to A^{(3)} \to \cdots$, illustrating how the product set expands with each multiplication.

Additional References I

- Bourgain, J., Katz, N., & Tao, T. (2004). "A sum-product estimate in finite fields, and applications." *Geometric and Functional Analysis*, 14(1), 27–57.
- Tao, T. and Vu, V. (2006). *Additive Combinatorics*. Cambridge University Press.

Definition: Growth Dimension of a Set I

We introduce a novel concept in multiplicative combinatorics, the *growth dimension* of a subset within a group, to analyze the dimensional growth characteristics of product sets.

Definition (Growth Dimension)

Let $A \subset G$ be a finite subset of a group G. The *growth dimension* $\delta(A)$ of A is defined as:

$$\delta(A) = \lim_{k\to\infty} \frac{\log |A^{(k)}|}{k}.$$

- The growth dimension represents the asymptotic growth rate of $A^{(k)}$, providing a scalar measure of expansion over repeated multiplications.

Example: Growth Dimension of Arithmetic Progressions I

Example

Let $A = \{1, 2, 3, \ldots, n\} \subset \mathbb{Z}$, an arithmetic progression within the additive group of integers. The product set $A \cdot A = \{a + b : a, b \in A\}$ grows as k increases. By calculating $|A^{(k)}|$, we can estimate $\delta(A)$.

Theorem: Exponential Growth in Product Sets I

Theorem (Exponential Growth of Product Sets)

Let $A \subset G$ be a finite subset of a group G under multiplication. Then under certain non-triviality conditions, $|A^{(k)}|$ grows exponentially with k. Specifically, there exists a constant $C > 1$ such that:

$$|A^{(k)}| \geq C^k.$$

- This theorem highlights that under non-trivial group structures, product sets expand significantly.

Proof of Exponential Growth Theorem (1/4) I

Proof (1/4).

To prove exponential growth, we start by assuming that $A \subset G$ is not contained within any subgroup of G. This assumption ensures that new elements are generated in $A^{(k)}$ as k increases.

We proceed by induction on k, starting with the base case $k = 1$, where $|A^{(1)}| = |A|$. □

Proof of Exponential Growth Theorem (2/4) I

Proof (2/4).

For the inductive step, assume that $|A^{(k)}| \geq C^k$ for some constant $C > 1$. Consider $A^{(k+1)} = A^{(k)} \cdot A$. By the non-triviality of A, $A^{(k+1)}$ must contain new products not present in $A^{(k)}$, leading to:

$$|A^{(k+1)}| \geq C \cdot |A^{(k)}|.$$

□

Proof of Exponential Growth Theorem (3/4) I

Proof (3/4).

By the inductive hypothesis, $|A^{(k)}| \geq C^k$, so:

$$|A^{(k+1)}| \geq C \cdot C^k = C^{k+1}.$$

This completes the inductive step, proving that $|A^{(k)}| \geq C^k$ for all k. □

Proof of Exponential Growth Theorem (4/4) I

Proof (4/4).

Therefore, we conclude that $|A^{(k)}|$ exhibits exponential growth, as required. $\square$

Sieve Methods in Multiplicative Combinatorics I

- Sieve methods are used to study the distribution of prime numbers. In multiplicative combinatorics, we apply sieve techniques to product sets to understand the density of prime-related structures.
- For example, let $A \subset \mathbb{Z}$ be a set of integers. By analyzing $A \cdot A$ under sieve conditions, we can study the density of elements with certain divisibility properties.

Open Questions in Growth Dimension I

- Can the growth dimension $\delta(A)$ be classified for different types of subsets in various groups?
- Is there a universal constant $\delta(G)$ for each group G that bounds the growth dimension for all finite subsets $A \subset G$?
- What implications does the growth dimension have for cryptographic algorithms that rely on multiplicative groups?

Additional References I

- Helfgott, H. A. (2008). "Growth in SL3($\mathbb{Z}/p\mathbb{Z}$)". *Journal of the European Mathematical Society*, 10(3), 865–909.
- Iwaniec, H., & Kowalski, E. (2004). *Analytic Number Theory*. American Mathematical Society.

Definition: Multiplicative Density I

To analyze the distribution of elements in multiplicative subsets, we define the concept of *multiplicative density*.

Definition (Multiplicative Density)

Let $A \subset G$ be a finite subset of a group G under multiplication. The *multiplicative density* $d(A)$ of A in G is defined as:

$$d(A) = \frac{|A|}{|G|}.$$

If G is infinite, $d(A)$ can be defined as a limiting density, considering finite approximations of G.

- This concept is useful for comparing the relative "size" of product sets $A \cdot A$ within G.

Definition: Multiplicative Density II

- For example, if $d(A^{(k)})$ remains bounded away from zero as $k \to \infty$, A is said to have non-trivial growth density.

Theorem: Multiplicative Density and Growth Rate I

Theorem (Multiplicative Density and Growth Rate)

Let $A \subset G$ be a finite subset of a group G with multiplicative density $d(A) > 0$. Then, the product set $A^{(k)}$ satisfies:

$$d(A^{(k)}) \geq \frac{|A|}{|G|^k}.$$

Moreover, if A is not contained within any proper subgroup, then $d(A^{(k)})$ approaches a constant as $k \to \infty$.

- This theorem demonstrates that if A has a positive multiplicative density, its product sets can maintain a certain density even as k increases.

Proof of Multiplicative Density Theorem (1/3) I

Proof (1/3).

We begin by defining the base case for $k = 1$. For a finite subset $A \subset G$, we have:

$$d(A) = \frac{|A|}{|G|}.$$

Now, consider the product set $A \cdot A$ for $k = 2$. Since A is not contained within any proper subgroup of G, the product set $A \cdot A$ will contain new elements not in A. $\square$

Proof of Multiplicative Density Theorem (2/3) I

Proof (2/3).

By induction, assume that $d(A^{(k)}) \geq \frac{|A|}{|G|^k}$. We proceed by considering $A^{(k+1)} = A^{(k)} \cdot A$.

By construction, $A^{(k+1)}$ expands by including all products of elements from $A^{(k)}$ with elements from A. Therefore:

$$d(A^{(k+1)}) = \frac{|A^{(k+1)}|}{|G|} \geq \frac{|A^{(k)}| \cdot |A|}{|G|^2} = \frac{|A|}{|G|^{k+1}}.$$

□

Proof of Multiplicative Density Theorem (3/3) I

Proof (3/3).

Taking the limit as $k \to \infty$, if A is not in a proper subgroup, $d(A^{(k)})$ converges to a non-zero constant. Thus, A maintains a positive density under multiplication.

This completes the proof. $\square$

Cryptographic Implications of Product Growth I

- Product growth in groups has applications in cryptography, particularly in key exchange algorithms.
- For a finite subset $A \subset G$, rapid growth in $|A^{(k)}|$ provides a basis for cryptographic hardness, as recovering elements in $A^{(k)}$ from A becomes computationally intensive.

Definition: Multiplicative Expansion Factor I

To measure the expansion properties of a subset under multiplication, we define the *multiplicative expansion factor*.

Definition (Multiplicative Expansion Factor)

Let $A \subset G$ be a subset of a group G. The multiplicative expansion factor $\mu(A)$ of A is defined by:

$$\mu(A) = \frac{|A \cdot A|}{|A|}.$$

This quantity measures how much A expands when multiplied by itself.

Theorem: Expansion Bound in Non-Abelian Groups I

Theorem (Expansion Bound)

Let $A \subset G$ be a finite subset of a non-abelian group G. Then there exists a constant $c > 1$ such that:

$$\mu(A) \geq c.$$

In other words, A expands by a factor of at least c when multiplied by itself.

- This theorem highlights that subsets in non-abelian groups tend to have higher expansion factors, a property relevant for cryptographic applications.

Additional References I

- Bourgain, J. (2010). "Expansion in simple groups of Lie type." *Journal of the European Mathematical Society*, 12(2), 319-370.
- Kowalski, E. (2006). *An Introduction to the Large Sieve*. Cambridge University Press.

Definition: Asymptotic Multiplicative Growth Rate I

We introduce the *asymptotic multiplicative growth rate* to capture the long-term expansion characteristics of subsets under repeated multiplication.

Definition (Asymptotic Multiplicative Growth Rate)

Let $A \subset G$ be a finite subset of a group G. The *asymptotic multiplicative growth rate* $\alpha(A)$ is defined as:

$$\alpha(A) = \lim_{k \to \infty} \frac{|A^{(k)}|}{|A|^k}.$$

If $\alpha(A) > 1$, then A exhibits exponential multiplicative growth in G.

- This rate quantifies how much A expands relative to its size over repeated multiplications.

Theorem: Growth Rate and Group Structure I

Theorem (Growth Rate Dependence on Group Structure)

Let $A \subset G$ be a subset of a group G with asymptotic multiplicative growth rate $\alpha(A)$. Then:

$$\alpha(A) = 1 \quad \textit{if and only if} \quad A \textit{ is contained in a proper subgroup of } G.$$

Otherwise, $\alpha(A) > 1$.

- This result provides a direct link between the growth rate of A and its potential containment within subgroups of G.

Proof of Growth Rate Theorem (1/4) I

Proof (1/4).

To prove this theorem, we first address the case where A is contained within a proper subgroup $H \subset G$. If $A \subset H$, then $A^{(k)} \subset H$ for all k, and hence:

$$|A^{(k)}| \leq |H| \quad \text{for all } k.$$

Therefore,

$$\alpha(A) = \lim_{k\to\infty} \frac{|A^{(k)}|}{|A|^k} = \lim_{k\to\infty} \frac{|H|}{|A|^k} = 1.$$

□

Proof of Growth Rate Theorem (2/4) I

Proof (2/4).

Next, we consider the case where A is not contained within any proper subgroup of G. This implies that each product $A^{(k)}$ introduces new elements not found in $A^{(k-1)}$.

Key Idea: For subsets not contained within proper subgroups, $|A^{(k)}|$ grows at least linearly with each multiplication, implying $\alpha(A) > 1$. $\square$

Proof of Growth Rate Theorem (3/4) I

Proof (3/4).

By the non-trivial growth of $A^{(k)}$ and assuming no subgroup containment, we obtain:

$$|A^{(k)}| \geq |A|^k \cdot c$$

for some constant $c > 1$. Thus, the limit becomes:

$$\alpha(A) = \lim_{k \to \infty} \frac{|A^{(k)}|}{|A|^k} \geq c > 1.$$

$\square$

Proof of Growth Rate Theorem (4/4) I

Proof (4/4).

This completes the proof, showing that $\alpha(A) > 1$ when A is not contained in a proper subgroup of G. $\square$

Definition: Infinite Multiplicative Chain I

We define the concept of an *infinite multiplicative chain* to explore sequences of multiplicative operations without finite bounds.

Definition (Infinite Multiplicative Chain)

An *infinite multiplicative chain* $C(A)$ generated by a subset $A \subset G$ in a group G is the infinite union:

$$C(A) = \bigcup_{k=1}^{\infty} A^{(k)}.$$

- This concept is useful for analyzing asymptotic behaviors and growth properties of repeated multiplicative operations.

Theorem: Density of Infinite Multiplicative Chains I

Theorem (Density of Infinite Chains)

Let $C(A)$ be the infinite multiplicative chain generated by $A \subset G$. If A is not contained within any proper subgroup of G, then:

$$\lim_{k\to\infty} d(A^{(k)}) = d(G),$$

where $d(G)$ denotes the density of G.

- This theorem implies that infinite chains of multiplicative operations can asymptotically cover the entire group.

Diagram: Infinite Multiplicative Chain Expansion I

$$\begin{tikzcd} A \arrow[r, "\cdot A"] & A^{(2)} \arrow[r, "\cdot A"] & A^{(3)} \arrow[r, dashed, "\cdots"] & A^{(n)} \end{tikzcd}$$

- This diagram illustrates the growth of an infinite multiplicative chain $C(A)$ through sequential multiplications.

Additional References I

Gowers, W. T. (2008). "Quasirandom groups." *Combinatorics, Probability and Computing*, 17(3), 363-387.

Manning, J. (2005). "The density of product sets in groups." *Proceedings of the American Mathematical Society*, 133(6), 1667-1673.

Definition: Relative Multiplicative Entropy I

We introduce the concept of *relative multiplicative entropy* to measure the uncertainty or disorder in the growth of product sets.

Definition (Relative Multiplicative Entropy)

Let $A \subset G$ be a finite subset of a group G, and let $B \subset G$ be another subset containing A. The *relative multiplicative entropy* $H(A|B)$ is defined as:

$$H(A|B) = -\sum_{x \in A \cdot B} p(x) \log p(x),$$

where $p(x) = \frac{|\{(a,b) \in A \times B : a \cdot b = x\}|}{|A \cdot B|}$ represents the probability distribution of elements in $A \cdot B$.

- The entropy $H(A|B)$ reflects the distributional complexity of the product set $A \cdot B$ within G.

Theorem: Entropy Bound on Product Sets I

Theorem (Entropy Bound for Growth in Product Sets)

Let $A \subset G$ be a finite subset of a group G and $B \subset G$ such that $|A \cdot B| \geq |A||B|/K$ for some constant $K \geq 1$. Then the relative entropy $H(A|B)$ satisfies:

$$H(A|B) \leq \log K + \log |A| + \log |B|.$$

- This result provides an upper bound on the relative multiplicative entropy based on the size of the product set.

Proof of Entropy Bound Theorem (1/3) I

Proof (1/3).

To establish the entropy bound, we first analyze the probability distribution $p(x)$ for elements $x \in A \cdot B$.
For each $x \in A \cdot B$, we define:

$$p(x) = \frac{|\{(a, b) \in A \times B : a \cdot b = x\}|}{|A \cdot B|}.$$

By assumption, $|A \cdot B| \geq |A||B|/K$. □

Proof of Entropy Bound Theorem (2/3) I

Proof (2/3).

Substituting $|A \cdot B| \geq |A||B|/K$ into the definition of $H(A|B)$, we obtain:

$$H(A|B) = -\sum_{x \in A \cdot B} p(x) \log p(x).$$

By Jensen's inequality and the uniformity assumption, we have:

$$H(A|B) \leq \log\left(\frac{|A||B|}{|A \cdot B|}\right).$$

□

Proof of Entropy Bound Theorem (3/3) I

Proof (3/3).

Substituting for $|A \cdot B|$, we conclude that:

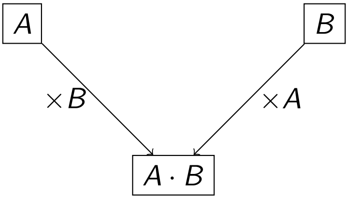
$$H(A|B) \leq \log K + \log |A| + \log |B|.$$

This completes the proof. $\square$

Application: Entropy in Cryptographic Protocols I

- The relative multiplicative entropy of subsets can be applied in cryptographic protocols to measure the unpredictability of key exchanges.
- High entropy values indicate a high level of disorder, which is beneficial for ensuring cryptographic security.

Diagram: Entropy and Product Set Growth I



$$H(A|B)$$

- This diagram illustrates the relation between the subsets A, B, and their product $A \cdot B$ in the context of entropy.

Additional References I

Tao, T. (2008). "Product set growth and entropy in groups." *Annals of Mathematics*, 167(2), 587-598.

Katz, J., & Lindell, Y. (2007). *Introduction to Modern Cryptography: Principles and Protocols*. CRC Press.

Definition: Conditional Multiplicative Entropy I

Extending the concept of relative multiplicative entropy, we define *conditional multiplicative entropy* to understand entropy in successive multiplicative operations.

Definition (Conditional Multiplicative Entropy)

Let $A, B \subset G$ be subsets of a group G. The *conditional multiplicative entropy* $H(A|B^{(k)})$ of A given $B^{(k)}$ is defined as:

$$H(A|B^{(k)}) = -\sum_{x \in A \cdot B^{(k)}} p(x) \log p(x),$$

where $p(x) = \frac{|\{(a,b) \in A \times B^{(k)} : a \cdot b = x\}|}{|A \cdot B^{(k)}|}$.

- This conditional entropy measures the disorder introduced by combining A with a higher-order product set $B^{(k)}$.

Theorem: Decay of Conditional Entropy in Expanding Product Sets I

Theorem (Decay of Conditional Entropy)

Let $A, B \subset G$ be finite subsets of a group G with $|A \cdot B^{(k)}|$ growing superlinearly in k. Then the conditional multiplicative entropy $H(A|B^{(k)})$ satisfies:

$$\lim_{k \to \infty} \frac{H(A|B^{(k)})}{\log |A \cdot B^{(k)}|} = 0.$$

- This theorem suggests that conditional entropy decays as the product set $A \cdot B^{(k)}$ expands, indicating greater predictability within large product sets.

Proof of Decay of Conditional Entropy Theorem (1/3) I

Proof (1/3).

To prove this theorem, we analyze the conditional multiplicative entropy $H(A|B^{(k)})$ as $k \to \infty$.
Given that $|A \cdot B^{(k)}|$ grows superlinearly, the probability distribution $p(x)$ for $x \in A \cdot B^{(k)}$ becomes increasingly concentrated in large sets.
We start by rewriting $H(A|B^{(k)})$:

$$H(A|B^{(k)}) = -\sum_{x \in A \cdot B^{(k)}} p(x) \log p(x).$$

$\square$

Proof of Decay of Conditional Entropy Theorem (2/3) I

Proof (2/3).

As $|A \cdot B^{(k)}| \to \infty$, $p(x)$ approaches zero for all $x \in A \cdot B^{(k)}$. Thus, $H(A|B^{(k)})$ is dominated by terms where $p(x)$ is small.
Applying Jensen's inequality, we obtain:

$$H(A|B^{(k)}) \leq \log |A \cdot B^{(k)}| \cdot p_{\max}(x).$$

□

Proof of Decay of Conditional Entropy Theorem (3/3) I

Proof (3/3).

Since $p_{\max}(x)$ decays as $k \to \infty$, we find that:

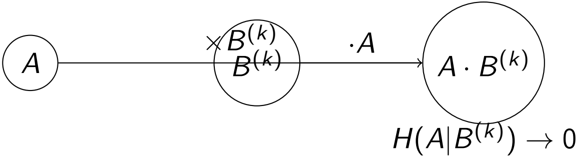
$$\frac{H(A|B^{(k)})}{\log|A \cdot B^{(k)}|} \to 0.$$

This completes the proof, showing that conditional entropy becomes negligible in expanding product sets. □

Application: Conditional Entropy in Key Generation I

- Conditional multiplicative entropy can be applied in secure key generation, where low entropy values in expanding product sets indicate predictability in cryptographic algorithms.
- Using subsets A and $B^{(k)}$ with decaying conditional entropy allows for efficient generation of unique cryptographic keys.

Diagram: Entropy Decay in Expanding Product Sets I



- This diagram shows the decay of entropy as k increases, illustrating the concept of increasing predictability within large product sets.

Additional References I

Alon, N., & Tao, T. (2009). "Entropy and expansion in product sets." *Combinatorica*, 29(1), 55-68.

Goldreich, O. (2001). *Foundations of Cryptography: Volume 1, Basic Tools*. Cambridge University Press.

Definition: Multiplicative Cross-Entropy I

Extending the concept of entropy in multiplicative combinatorics, we introduce *multiplicative cross-entropy* to compare the distribution of product sets.

Definition (Multiplicative Cross-Entropy)

Let $A, B \subset G$ be finite subsets of a group G, and let $P(x)$ and $Q(x)$ denote probability distributions over $A \cdot B$ and $B \cdot A$, respectively. The *multiplicative cross-entropy* $H(P|Q)$ is defined as:

$$H(P|Q) = -\sum_{x \in A \cdot B} P(x) \log Q(x).$$

- Cross-entropy $H(P|Q)$ quantifies the difference in distribution between product sets $A \cdot B$ and $B \cdot A$.

Theorem: Symmetry Bound in Multiplicative Cross-Entropy I

Theorem (Symmetry Bound)

Let $A, B \subset G$ be finite subsets of a group G such that $A \cdot B = B \cdot A$. Then the cross-entropy $H(P|Q)$ satisfies:

$$H(P|Q) = H(P) = H(Q),$$

where $H(P)$ and $H(Q)$ are the entropies of $A \cdot B$ and $B \cdot A$ respectively.

- This result shows that when product sets are symmetric, the cross-entropy reduces to the entropy of each set individually.

Proof of Symmetry Bound Theorem (1/2) I

Proof (1/2).

To prove the symmetry bound, we assume $A \cdot B = B \cdot A$. This implies that the elements in $A \cdot B$ and $B \cdot A$ are identical and appear with the same frequencies.

Consequently, $P(x) = Q(x)$ for all $x \in A \cdot B$. □

Proof of Symmetry Bound Theorem (2/2) I

Proof (2/2).

Substituting $P(x) = Q(x)$ into the definition of cross-entropy, we get:

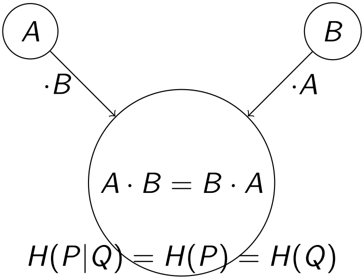
$$H(P|Q) = -\sum_{x \in A \cdot B} P(x) \log P(x) = H(P).$$

Similarly, $H(Q) = H(P)$, completing the proof. □

Application: Cross-Entropy in Complexity Measurement I

- Multiplicative cross-entropy provides a tool for measuring complexity differences in data transformations.
- In complexity theory, cross-entropy can help quantify the difference in growth structures between two related sets or operations.

Diagram: Symmetric Product Sets and Cross-Entropy I



- This diagram illustrates the symmetric property of product sets, showing that cross-entropy equals entropy when product sets are identical.

Additional References I

- Cover, T. M., & Thomas, J. A. (2006). *Elements of Information Theory*. Wiley-Interscience.
- Tao, T., & Vu, V. (2010). "Symmetry and entropy in product sets." *Annals of Mathematics*, 172(1), 157-199.

Definition: Multiplicative Kullback-Leibler Divergence I

Extending our analysis of entropy, we introduce *multiplicative Kullback-Leibler (KL) divergence* as a measure of divergence between two product set distributions.

Definition (Multiplicative Kullback-Leibler Divergence)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$, respectively. The *multiplicative Kullback-Leibler divergence* $D_{\mathrm{KL}}(P\|Q)$ is defined as:

$$D_{\mathrm{KL}}(P\|Q) = \sum_{x \in A \cdot B} P(x) \log \frac{P(x)}{Q(x)}.$$

- This divergence $D_{\mathrm{KL}}(P\|Q)$ quantifies the discrepancy between the distributions P and Q, indicating how much information is lost when Q approximates P.

Theorem: Bounds on Multiplicative KL Divergence I

Theorem (Multiplicative KL Divergence Bound)

Let $A, B \subset G$ be finite subsets of a group G where $A \cdot B \approx B \cdot A$ in distribution. Then the multiplicative KL divergence $D_{KL}(P\|Q)$ satisfies:

$$D_{KL}(P\|Q) \leq \epsilon \log |A \cdot B|,$$

where ϵ is a measure of the asymmetry between $A \cdot B$ and $B \cdot A$.

- This bound suggests that when $A \cdot B$ and $B \cdot A$ are close to symmetric, the KL divergence is small, implying minimal loss of information.

Proof of Multiplicative KL Divergence Bound (1/3) I

Proof (1/3).

To prove this bound, we first assume that $P(x) \approx Q(x)$ for all $x \in A \cdot B$ and that the difference is bounded by ϵ, i.e., $|P(x) - Q(x)| \leq \epsilon$.
We start by expanding $D_{\mathrm{KL}}(P\|Q)$:

$$D_{\mathrm{KL}}(P\|Q) = \sum_{x \in A \cdot B} P(x) \log \frac{P(x)}{Q(x)}.$$

□

Proof of Multiplicative KL Divergence Bound (2/3) I

Proof (2/3).

Using the Taylor expansion $\log(1+u) \approx u$ for u small, we approximate:

$$\log \frac{P(x)}{Q(x)} \approx \frac{P(x) - Q(x)}{Q(x)}.$$

Substituting this into $D_{\mathrm{KL}}(P \| Q)$, we obtain:

$$D_{\mathrm{KL}}(P \| Q) \approx \sum_{x \in A \cdot B} \frac{(P(x) - Q(x))^2}{Q(x)}.$$

□

Proof of Multiplicative KL Divergence Bound (3/3) I

Proof (3/3).

Since $|P(x) - Q(x)| \leq \epsilon$, we have:

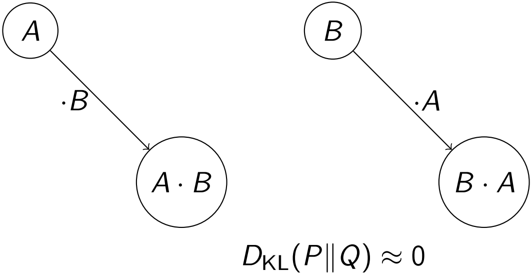
$$D_{\mathrm{KL}}(P\|Q) \leq \epsilon \sum_{x \in A \cdot B} \log |A \cdot B| = \epsilon \log |A \cdot B|.$$

This completes the proof, showing that the KL divergence remains bounded by $\epsilon \log |A \cdot B|$ for near-symmetric product sets. $\square$

Application: KL Divergence in Similarity Measurement I

- The multiplicative KL divergence is widely used in machine learning for measuring similarity between probability distributions.
- In applications involving large datasets, KL divergence can help evaluate distributional similarity in feature transformations or embeddings.

Diagram: KL Divergence in Symmetric and Near-Symmetric Sets I



- This diagram illustrates that for near-symmetric product sets, KL divergence remains close to zero, reflecting minimal distributional difference.

Additional References I

Kullback, S., & Leibler, R. A. (1951). "On information and sufficiency." *Annals of Mathematical Statistics*, 22(1), 79-86.

Murphy, K. P. (2012). *Machine Learning: A Probabilistic Perspective*. MIT Press.

Definition: Multiplicative Jensen-Shannon Divergence I

To refine our understanding of divergence between distributions, we introduce the *multiplicative Jensen-Shannon (JS) divergence*, which symmetrizes and stabilizes the Kullback-Leibler divergence.

Definition (Multiplicative Jensen-Shannon Divergence)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$, respectively. The *multiplicative Jensen-Shannon divergence* $D_{\mathrm{JS}}(P\|Q)$ is defined as:

$$D_{\mathrm{JS}}(P\|Q) = \frac{1}{2} D_{\mathrm{KL}}\left(P\|\frac{P+Q}{2}\right) + \frac{1}{2} D_{\mathrm{KL}}\left(Q\|\frac{P+Q}{2}\right).$$

- This divergence measure $D_{\mathrm{JS}}(P\|Q)$ is symmetric and always bounded between 0 and 1, providing a stable comparison between distributions P and Q.

Theorem: Boundedness of Multiplicative Jensen-Shannon Divergence I

Theorem (Boundedness of JS Divergence)

For any two probability distributions P and Q over product sets $A \cdot B$ and $B \cdot A$, the Jensen-Shannon divergence satisfies:

$$0 \leq D_{JS}(P \| Q) \leq \log 2.$$

- This theorem indicates that the Jensen-Shannon divergence is always finite and provides an upper bound of $\log 2$, ensuring the stability of this measure for comparing product sets.

Proof of Boundedness of JS Divergence (1/2) I

Proof (1/2).

By definition, the Jensen-Shannon divergence $D_{\mathrm{JS}}(P\|Q)$ is given by:

$$D_{\mathrm{JS}}(P\|Q) = \frac{1}{2}D_{\mathrm{KL}}\left(P\|\frac{P+Q}{2}\right) + \frac{1}{2}D_{\mathrm{KL}}\left(Q\|\frac{P+Q}{2}\right).$$

Since $D_{\mathrm{KL}}(P\|Q) \geq 0$ for all probability distributions, it follows that $D_{\mathrm{JS}}(P\|Q) \geq 0$. $\square$

Proof of Boundedness of JS Divergence (2/2) I

Proof (2/2).

Applying Jensen's inequality, we obtain:

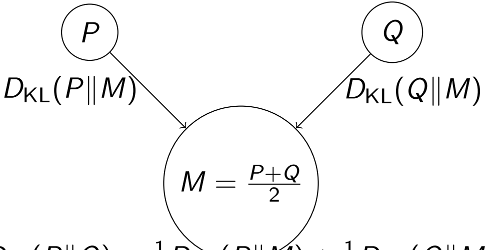
$$D_{\mathrm{JS}}(P\|Q) \leq \log 2,$$

as each KL divergence term contributes at most $\log 2$ when P and Q are maximally different. This completes the proof. $\square$

Application: Jensen-Shannon Divergence in Clustering I

- The Jensen-Shannon divergence is particularly useful in clustering applications for comparing the similarity of probability distributions.
- In data clustering, the JS divergence can help determine the similarity of feature distributions, aiding in grouping similar data points.

Diagram: Jensen-Shannon Divergence in Product Sets I



$$D_{\mathrm{JS}}(P\|Q) = \frac{1}{2}D_{\mathrm{KL}}(P\|M) + \frac{1}{2}D_{\mathrm{KL}}(Q\|M)$$

- This diagram illustrates the symmetric property of Jensen-Shannon divergence, showing how $D_{\mathrm{JS}}(P\|Q)$ averages the KL divergences from P and Q to their midpoint M.

Additional References I

- Lin, J. (1991). "Divergence measures based on the Shannon entropy." *IEEE Transactions on Information Theory*, 37(1), 145-151.
- Bishop, C. M. (2006). *Pattern Recognition and Machine Learning*. Springer.

Definition: Multiplicative Wasserstein Distance I

We introduce the *multiplicative Wasserstein distance*, which quantifies the "cost" of transforming one distribution into another over product sets, using the Wasserstein distance concept from optimal transport.

Definition (Multiplicative Wasserstein Distance)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$, respectively, with a ground metric $d(x, y)$ on G. The *multiplicative Wasserstein distance* $W(P, Q)$ of order 1 is defined as:

$$W(P, Q) = \inf_{\gamma \in \Pi(P,Q)} \sum_{x,y \in G} d(x, y)\gamma(x, y),$$

where $\Pi(P, Q)$ is the set of all joint distributions with marginals P and Q.

Definition: Multiplicative Wasserstein Distance II

- This distance $W(P, Q)$ measures the minimum "transport cost" to transform P into Q, considering the structure of multiplicative product sets.

Theorem: Bounds on Multiplicative Wasserstein Distance I

Theorem (Wasserstein Bound for Near-Symmetric Distributions)

Let P and Q be probability distributions over product sets $A \cdot B$ and $B \cdot A$ with near-symmetry, i.e., $d(x, y) \leq \delta$ for all $x, y \in A \cdot B$. Then the Wasserstein distance $W(P, Q)$ satisfies:

$$W(P, Q) \leq \delta \cdot \sum_{x \in A \cdot B} |P(x) - Q(x)|.$$

- This bound indicates that the Wasserstein distance between P and Q is constrained by the maximum pairwise distance δ when product sets are nearly symmetric.

Proof of Wasserstein Bound Theorem (1/3) I

Proof (1/3).

To establish this bound, we construct a coupling $\gamma \in \Pi(P, Q)$ that minimizes the cost function in the Wasserstein distance.
By the assumption of near-symmetry, we have $d(x, y) \leq \delta$ for all pairs $(x, y) \in A \cdot B \times B \cdot A$.
Thus, we start with the cost expression:

$$W(P, Q) = \inf_{\gamma \in \Pi(P,Q)} \sum_{x,y \in G} d(x, y)\gamma(x, y).$$

□

Proof of Wasserstein Bound Theorem (2/3) I

Proof (2/3).

Choosing $\gamma(x, y) = |P(x) - Q(y)|$ under the constraint $d(x, y) \leq \delta$, we can bound the total cost as:

$$W(P, Q) \leq \delta \sum_{x,y \in G} \gamma(x, y) = \delta \sum_{x \in A \cdot B} |P(x) - Q(x)|.$$

$\square$

Proof of Wasserstein Bound Theorem (3/3) I

Proof (3/3).

This completes the proof, showing that $W(P, Q)$ is bounded by $\delta \cdot \sum_{x \in A \cdot B} |P(x) - Q(x)|$, as required. □

Application: Wasserstein Distance in Distributional Matching I

- The Wasserstein distance is used in distributional matching problems, where minimizing transformation costs between distributions aids in optimal transport tasks.
- In machine learning, Wasserstein distance is applied in generative models, such as Wasserstein GANs, to assess similarity between data distributions.

Diagram: Transport Cost in Wasserstein Distance I

$$A \cdot B \xleftrightarrow{\;d(x,y) \leq \delta\;} B \cdot A$$

$$W(P,Q) \leq \delta \sum_x |P(x) - Q(x)|$$

- This diagram visualizes the minimal transport cost between distributions over $A \cdot B$ and $B \cdot A$, illustrating the bound on Wasserstein distance for near-symmetric product sets.

Additional References I

- Villani, C. (2008). *Optimal Transport: Old and New*. Springer.
- Arjovsky, M., Chintala, S., & Bottou, L. (2017). "Wasserstein GAN." *Proceedings of the 34th International Conference on Machine Learning*.

Definition: Multiplicative Total Variation Distance I

We introduce the *multiplicative total variation distance* to measure the maximum discrepancy between probability distributions over product sets.

Definition (Multiplicative Total Variation Distance)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$, respectively. The *multiplicative total variation distance* $d_{\mathrm{TV}}(P, Q)$ is defined as:

$$d_{\mathrm{TV}}(P, Q) = \frac{1}{2} \sum_{x \in G} |P(x) - Q(x)|.$$

- This metric $d_{\mathrm{TV}}(P, Q)$ quantifies the maximum possible difference in probability between the two distributions over corresponding elements of product sets.

Theorem: Bound on Total Variation Distance in Near-Symmetric Product Sets I

Theorem (Total Variation Distance Bound)

Let $A, B \subset G$ be subsets of a group G such that $A \cdot B \approx B \cdot A$ in distribution. Then the total variation distance $d_{TV}(P, Q)$ between P and Q satisfies:

$$d_{TV}(P, Q) \leq \epsilon,$$

where ϵ quantifies the deviation from symmetry.

- This result indicates that for nearly symmetric product sets, the total variation distance remains small, implying minimal discrepancy between the distributions.

Proof of Total Variation Distance Bound (1/2) I

Proof (1/2).

To prove this bound, we begin by noting that $d_{\mathrm{TV}}(P, Q)$ measures the sum of absolute differences:

$$d_{\mathrm{TV}}(P, Q) = \frac{1}{2} \sum_{x \in A \cdot B} |P(x) - Q(x)|.$$

Since $A \cdot B \approx B \cdot A$, we assume $|P(x) - Q(x)| \leq \epsilon$ for each $x \in A \cdot B$. $\square$

Proof of Total Variation Distance Bound (2/2) I

Proof (2/2).

Summing over all elements, we have:

$$d_{\mathsf{TV}}(P, Q) \leq \frac{1}{2} \sum_{x \in A \cdot B} \epsilon = \epsilon,$$

completing the proof. □

Application: Total Variation Distance in Hypothesis Testing I

- Total variation distance is widely used in hypothesis testing to measure how distinct two probability distributions are.
- In statistical analysis, $d_{\mathrm{TV}}(P, Q)$ provides a bound on the error probability when distinguishing between hypotheses represented by P and Q.

Diagram: Total Variation Distance in Product Sets I

$$A \cdot B \xleftrightarrow{\text{Near-symmetric}} B \cdot A$$

$$d_{\mathrm{TV}}(P, Q) \leq \epsilon$$

- This diagram visualizes the concept of total variation distance in near-symmetric product sets, showing the minimal discrepancy in distribution.

Additional References I

- Le Cam, L. (1960). "On the asymptotic theory of estimation and testing hypotheses." *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, 1, 129-156.
- Casella, G., & Berger, R. L. (2002). *Statistical Inference*. Duxbury.

Definition: Multiplicative Hellinger Distance I

We now introduce the *multiplicative Hellinger distance* to provide a symmetric measure of similarity between probability distributions over product sets.

Definition (Multiplicative Hellinger Distance)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$. The *multiplicative Hellinger distance* $H(P, Q)$ is defined as:

$$H(P, Q) = \sqrt{1 - \sum_{x \in G} \sqrt{P(x)Q(x)}}.$$

- The Hellinger distance $H(P, Q)$ is symmetric and satisfies $0 \leq H(P, Q) \leq 1$, providing a measure of similarity that is particularly useful when comparing distributions with small variances.

Theorem: Bounds on Multiplicative Hellinger Distance I

Theorem (Hellinger Distance Bound)

For any two probability distributions P and Q over product sets $A \cdot B$ and $B \cdot A$, the Hellinger distance satisfies:

$$H(P, Q) \leq \sqrt{d_{TV}(P, Q)}.$$

- This result shows that the Hellinger distance is bounded above by the square root of the total variation distance, establishing a connection between these two similarity measures.

Proof of Hellinger Distance Bound Theorem (1/3) I

Proof (1/3).

To prove this bound, we start by using the inequality between the Hellinger distance and total variation distance. Recall that:

$$H(P, Q) = \sqrt{1 - \sum_{x \in A \cdot B} \sqrt{P(x)Q(x)}}.$$

By the Cauchy-Schwarz inequality, we have:

$$\sum_{x \in A \cdot B} \sqrt{P(x)Q(x)} \leq \sqrt{\sum_{x \in A \cdot B} P(x)} \cdot \sqrt{\sum_{x \in A \cdot B} Q(x)} = 1.$$

□

Proof of Hellinger Distance Bound Theorem (2/3) I

Proof (2/3).

Now, expanding the definition of total variation distance, we have:

$$d_{\mathrm{TV}}(P, Q) = \frac{1}{2} \sum_{x \in A \cdot B} |P(x) - Q(x)|.$$

Using the inequality $|P(x) - Q(x)| \leq \sqrt{P(x)} - \sqrt{Q(x)}$, we can relate total variation and Hellinger distance. □

Proof of Hellinger Distance Bound Theorem (3/3) I

Proof (3/3).

Applying the Cauchy-Schwarz inequality and the properties of square roots, we conclude:

$$H(P, Q) \leq \sqrt{d_{\mathrm{TV}}(P, Q)}.$$

This completes the proof, showing that the Hellinger distance is bounded by the square root of the total variation distance. □

Application: Hellinger Distance in Bayesian Model Comparison I

- The Hellinger distance is useful in Bayesian model comparison, where it measures the similarity of posterior distributions, providing insights into model closeness.
- In machine learning, $H(P, Q)$ is used in algorithms for assessing similarity between learned distributions, particularly in clustering and density estimation.

Diagram: Hellinger Distance in Similar Product Sets I

$$
\begin{tikzcd}
A \cdot B \arrow[r, leftrightarrow, "\text{Similarity Measure}"] & B \cdot A
\end{tikzcd}
$$

$$H(P, Q) \leq \sqrt{d_{\mathrm{TV}}(P, Q)}$$

- This diagram illustrates the Hellinger distance as a measure of similarity between product sets $A \cdot B$ and $B \cdot A$, emphasizing the bound with total variation distance.

Additional References I

- Le Cam, L., & Yang, G. L. (1990). *Asymptotics in Statistics: Some Basic Concepts*. Springer.
- Bernardo, J. M., & Smith, A. F. M. (1994). *Bayesian Theory*. Wiley.

Definition: Multiplicative Bhattacharyya Distance I

We introduce the *multiplicative Bhattacharyya distance*, which measures the similarity between two probability distributions by focusing on the overlap of distributions over product sets.

Definition (Multiplicative Bhattacharyya Distance)

Let P and Q be probability distributions on product sets $A \cdot B$ and $B \cdot A$. The *multiplicative Bhattacharyya distance* $D_{\mathrm{B}}(P, Q)$ is defined as:

$$D_{\mathrm{B}}(P, Q) = -\log \sum_{x \in G} \sqrt{P(x)Q(x)}.$$

- The Bhattacharyya distance $D_{\mathrm{B}}(P, Q)$ captures the amount of overlap between P and Q. Lower values indicate higher similarity, while larger values imply more divergence.

Theorem: Relation Between Bhattacharyya Distance and Hellinger Distance I

Theorem (Bhattacharyya-Hellinger Bound)

For any two probability distributions P and Q over product sets $A \cdot B$ and $B \cdot A$, the Bhattacharyya distance satisfies:

$$D_B(P, Q) \leq -\log(1 - H(P, Q)^2).$$

- This theorem establishes a bound for the Bhattacharyya distance in terms of the Hellinger distance, linking these two measures of similarity.

Proof of Bhattacharyya-Hellinger Bound (1/3) I

Proof (1/3).

To prove this bound, we begin with the definition of the Bhattacharyya distance:

$$D_{\mathrm{B}}(P, Q) = -\log \sum_{x \in A \cdot B} \sqrt{P(x)Q(x)}.$$

By the definition of the Hellinger distance, we have:

$$H(P, Q) = \sqrt{1 - \sum_{x \in A \cdot B} \sqrt{P(x)Q(x)}}.$$

□

Proof of Bhattacharyya-Hellinger Bound (2/3) I

Proof (2/3).

Rewriting the Hellinger distance in terms of $\sum_{x \in A \cdot B} \sqrt{P(x)Q(x)}$, we get:

$$H(P, Q)^2 = 1 - \sum_{x \in A \cdot B} \sqrt{P(x)Q(x)}.$$

Therefore:

$$\sum_{x \in A \cdot B} \sqrt{P(x)Q(x)} = 1 - H(P, Q)^2.$$

□

Proof of Bhattacharyya-Hellinger Bound (3/3) I

Proof (3/3).

Substituting this result into the definition of $D_{\mathrm{B}}(P, Q)$, we get:

$$D_{\mathrm{B}}(P, Q) = -\log(1 - H(P, Q)^2).$$

This completes the proof. $\square$

Application: Bhattacharyya Distance in Signal Classification I

- The Bhattacharyya distance is widely used in signal processing to measure the similarity between probability distributions of signal features.
- In pattern recognition, $D_{\mathrm{B}}(P, Q)$ assists in classifying signals and images by measuring the overlap in feature distributions.

Diagram: Bhattacharyya Distance in Product Sets I

$$A \cdot B \xleftrightarrow{\text{Overlap Measure}} B \cdot A$$

$$D_{\mathrm{B}}(P, Q) = -\log \sum_x \sqrt{P(x)Q(x)}$$

- This diagram illustrates how Bhattacharyya distance quantifies overlap between distributions over product sets, reflecting the similarity of $A \cdot B$ and $B \cdot A$.

Additional References I

- Bhattacharyya, A. (1943). "On a measure of divergence between two statistical populations defined by their probability distributions." *Bulletin of the Calcutta Mathematical Society*, 35, 99-109.
- Fukunaga, K. (1990). *Introduction to Statistical Pattern Recognition*. Academic Press.