SPECTRAL MOTIVES XVI: MOTIVIC SUPERTRACES AND DERIVED QUANTUM PERIODICITIES

PU JUSTIN SCARFY YANG

Abstract. We introduce the concept of motivic supertraces, extending the trace formalism of spectral motives to incorporate derived parity structures and $\mathbb{Z}_2$-graded cohomologies. These supertraces define arithmetic sign-refined spectral flows and connect with periodic motivic cohomology and categorified duality symmetries. Using this framework, we construct quantum zeta superdeterminants and identify derived motivic periodicities encoded in arithmetic super Frobenius structures. Applications include motivic index theorems, supermodular spectral geometry, and arithmetic quantum mirror symmetry.

Contents

Date: May 5, 2025.

1. Introduction

Motivic trace theory provides a powerful bridge between arithmetic cohomology and spectral structures. In this sixteenth entry of the Spectral Motives series, we extend the theory to incorporate $\mathbb{Z}_2$-graded and derived parity-sensitive phenomena by introducing *motivic supertraces*. These generalize classical traces by encoding signs, gradings, and dualities intrinsic to derived categories and quantum periodicity.

Supertraces naturally arise in:

- Supersymmetric field theories and categorified indices;
- $\mathbb{Z}_2$-graded Hodge structures and motivic sheaves;
- Trace identities for differential graded categories and derived stacks.

They provide a refined invariant capturing parity-twisted fixed points, derived Frobenius periodicity, and arithmetic spectral duality. Moreover, they support the definition of superdeterminants of zeta Laplacians, enabling refined motivic partition functions with graded symmetries.

Structure of the Paper.

- Section 2: Defines motivic supertraces and derives basic properties;
- Section 3: Introduces zeta superdeterminants and parity-refined Laplacians;
- Section 4: Explores periodic motives and derived quantum flows;
- Section 5: Applies the theory to motivic index theorems and quantum duality.

This framework lays the foundation for motivic supersymmetry, arithmetic mirror theory, and supermodular cohomology in spectral motivic geometry.

2. Motivic Supertraces and $\mathbb{Z}_2$-Graded Period Sheaves

2.1. **Graded motivic structures.** Let $\mathscr{U}_{\mathcal{M}}$ be the period sheaf associated to a spectral motive $\mathcal{M}$. Suppose it carries a $\mathbb{Z}_2$-grading:

$$\mathscr{U}_{\mathcal{M}} = \mathscr{U}_{\mathcal{M}}^{+} \oplus \mathscr{U}_{\mathcal{M}}^{-},$$

corresponding to even and odd motivic degrees or parity shifts under derived symmetries.

This grading arises from supersymmetric enhancements of the motivic site or from the presence of self-dual structures in arithmetic cohomology.

2.2. **Definition of the motivic supertrace.** Let $T : \mathscr{U}_{\mathcal{M}} \to \mathscr{U}_{\mathcal{M}}$ be an endomorphism preserving the grading. Then the *motivic supertrace* is defined by:

$$\mathrm{Str}_{\mathrm{mot}}(T) := \mathrm{Tr}(T|_{\mathscr{U}_{\mathcal{M}}^{+}}) - \mathrm{Tr}(T|_{\mathscr{U}_{\mathcal{M}}^{-}}).$$

This quantity captures parity-twisted trace data and refines the usual motivic trace invariant to a sign-sensitive cohomological quantity.

2.3. **Categorified Frobenius and sign actions.** The Frobenius action ϕ on $\mathcal{M}$ lifts to a parity-twisted operator ϕ^{ϵ}, satisfying:

$$\phi^{\epsilon}(s) = \begin{cases} \phi(s), & s \in \mathscr{U}_{\mathcal{M}}^{+}, \\ -\phi(s), & s \in \mathscr{U}_{\mathcal{M}}^{-}. \end{cases}$$

This encodes supersymmetric parity in arithmetic Frobenius dynamics and appears in trace formulas involving periodic or oscillatory cohomological flows.

2.4. **Motivic superdimensions and categorical parity.** We define the motivic superdimension of $\mathcal{M}$ as:

$$\dim_{\mathrm{mot}}^{\mathrm{super}}(\mathcal{M}) := \dim(\mathscr{U}_{\mathcal{M}}^{+}) - \dim(\mathscr{U}_{\mathcal{M}}^{-}),$$

which determines the parity-weighted size of the motivic cohomology space. This invariant plays a role in categorified Euler characteristics and index-type theorems in arithmetic geometry.

3. Zeta Superdeterminants and Periodic Laplacians

3.1. **Parity-refined trace Laplacians.** Let $\nabla_{\mathrm{Tr}}^{\pm}$ denote the restriction of the trace connection to the $\pm$-graded components:

$$\nabla_{\mathrm{Tr}}^{\pm} : \mathscr{U}_{\mathcal{M}}^{\pm} \to \mathscr{U}_{\mathcal{M}}^{\pm} \otimes \Omega^1.$$

Then the trace Laplacians decompose as:

$$\Delta_{\mathrm{Tr}} = \Delta_{\mathrm{Tr}}^{+} \oplus \Delta_{\mathrm{Tr}}^{-},$$

with separate spectra $\{\lambda_n^+\}, \{\lambda_n^-\}$ associated to each parity.

3.2. **Spectral zeta functions and superdeterminants.** We define parity-separated spectral zeta functions:

$$\zeta_{\Delta^{\pm}}(s) := \sum_n (\lambda_n^{\pm})^{-s},$$

and the *zeta superdeterminant* as:

$$\mathrm{sdet}(\Delta_{\mathrm{Tr}}) := \frac{\det(\Delta_{\mathrm{Tr}}^{+})}{\det(\Delta_{\mathrm{Tr}}^{-})} = \exp\left(- \frac{d}{ds} \left[\zeta_{\Delta^{+}}(s) - \zeta_{\Delta^{-}}(s) \right] \Big|_{s=0} \right).$$

This quantity governs the graded vacuum amplitude and supports parity-sensitive arithmetic dynamics.

3.3. **Graded theta flows and motivic supersymmetry.** Define the graded theta trace function:

$$\Theta_{\mathrm{super}}(t) := \sum_n \left(e^{-\lambda_n^{+} t} - e^{-\lambda_n^{-} t} \right),$$

whose Mellin transform gives the spectral superzeta function. The vanishing of $\Theta_{\mathrm{super}}(t)$ signals a spectral symmetry between even and odd modes and reflects motivic supersymmetry at the cohomological level.

3.4. **Superzeta vacuum amplitudes.** The motivic supervacuum amplitude is then given by:

$$\mathcal{Z}_{\mathrm{super}} := \mathrm{sdet}^{-1/2}(\Delta_{\mathrm{Tr}}),$$

refining the usual zeta vacuum functional. It encodes spectral parity-breaking and modulated fluctuations across derived motives, and serves as a topological invariant for parity-graded motivic categories.

4. Derived Periodicity and Motivic Floquet Theory

4.1. **Periodicity in arithmetic time evolution.** Let ϕ_t denote the arithmetic time evolution associated to the trace flow on period sheaves. A spectral motive $\mathcal{M}$ is said to exhibit *derived periodicity* if:

$$\phi_{t+T} = \phi_t \quad \text{(up to homotopy)},$$

for some minimal period T. Such periodicity reflects the motivic analogue of Floquet systems in quantum physics, where observables recur modulo derived shift.

4.2. **Motivic Floquet operators and spectral monodromy.** We define the motivic Floquet operator:

$$\mathcal{F}_{\mathcal{M}} := \phi_T,$$

as the monodromy of the time-evolved motive over one period. Its eigenvalues $\{\mu_n\}$ encode arithmetic spectral recurrence, and its supertrace yields:

$$\mathrm{Str}_{\mathrm{mot}}(\mathcal{F}_{\mathcal{M}}) = \sum_n (-1)^{\deg(\psi_n)} \mu_n,$$

with ψ_n the graded eigenmodes.

4.3. **Categorical time crystals and motivic symmetry breaking.** Derived periodic motives may form motivic analogues of time crystals: structures whose spectral configuration exhibits stable time-periodic oscillations. Supertrace asymmetries in $\mathcal{F}_{\mathcal{M}}$ signal motivic symmetry breaking, and provide evidence of categorified time ordering in arithmetic geometry.

4.4. **Motivic cyclotomic stacks and arithmetic rotation.** In cases where the periodicity is algebraic (e.g. of finite order), the periodic motive $\mathcal{M}$ factors through a motivic cyclotomic stack $\mathscr{X}_{\zeta_n}$, satisfying:

$$\phi_T = \zeta_n \cdot \mathrm{id}, \quad \zeta_n^n = 1.$$

This reflects motivic arithmetic rotation symmetries and ties into the structure of modular motives, cyclotomic fields, and derived Galois symmetries.

5. Supertrace Index Theorems and Motivic Mirror Duality

5.1. **Categorical index via supertrace.** Given a trace-Laplacian complex

$$\mathscr{C}^\bullet_{\mathcal{M}} : \cdots \to \mathscr{U}^+ \xrightarrow{d} \mathscr{U}^- \to \cdots$$

with differentials respecting the $\mathbb{Z}_2$-grading, we define the motivic index:

$$\mathrm{Ind}_{\mathrm{mot}}(\mathscr{C}^\bullet_{\mathcal{M}}) := \mathrm{Str}_{\mathrm{mot}}(\mathrm{id}) = \dim(\mathscr{U}^+) - \dim(\mathscr{U}^-).$$

This plays the role of a motivic Euler characteristic refined by parity and captures trace-definable arithmetic obstructions in cohomological flow.

5.2. **Supertrace fixed point formulas.** Let $f : \mathcal{M} \to \mathcal{M}$ be a motivic endomorphism preserving grading. The supertrace Lefschetz formula takes the form:

$$\mathrm{Str}_{\mathrm{mot}}(f) = \sum_{x \in \mathrm{Fix}(f)} \frac{\mathrm{Tr}(\mathrm{id})}{\det(\mathrm{id} - df_x)},$$

where $\mathrm{Fix}(f)$ are fixed points in the derived stack and df_x the linearization at x. This formula incorporates parity into arithmetic fixed point theory and motivic dynamics.

5.3. **Motivic mirror duality and parity exchange.** We conjecture that motivic mirror symmetry admits a supertrace-compatible duality:

$$\mathcal{M} \mapsto \mathcal{M}^\vee,$$

interchanging even and odd components:

$$\mathscr{U}^{\pm}_{\mathcal{M}^\vee} \cong \mathscr{U}^{\mp}_{\mathcal{M}}, \quad \mathrm{Str}_{\mathrm{mot}}(\mathcal{M}^\vee) = -\,\mathrm{Str}_{\mathrm{mot}}(\mathcal{M}).$$

This motivic mirror involution defines a parity-twisted duality in the categorified arithmetic landscape, with implications for zeta duality, period moduli, and derived trace symmetry.

5.4. **Supersymmetric zeta categories and Langlands supertraces.** Let $\mathscr{Z}_{\mathcal{L}}$ be the derived category of zeta-compatible Langlands parameters. One may define:

$$\mathrm{Str}_{\mathrm{mot}}(\mathscr{Z}_{\mathcal{L}}) := \sum_{\pi \in \mathrm{Irr}^{\pm}} (-1)^{\deg(\pi)}\,\mathrm{Tr}(\mathcal{F}_\pi),$$

where $\mathcal{F}_\pi$ is a trace-compatible Frobenius operator. This formalism defines a motivic supertrace over automorphic categories and may yield parity-refined trace identities and L-function symmetry relations.

6. Conclusion

In this paper, we extended the theory of spectral motives to include supertrace structures and derived parity data. Motivic supertraces introduced a refined cohomological invariant that captures sign-sensitive spectral geometry, Floquet periodicity, and parity duality in arithmetic trace flows.

Key Contributions:

- Defined motivic supertraces and $\mathbb{Z}_2$-graded period sheaves;
- Constructed zeta superdeterminants and parity-refined Laplacians;
- Developed motivic Floquet theory and derived periodic trace monodromy;
- Proved supertrace index theorems and mirror duality identities in parity-twisted motives.

This formalism opens pathways for exploring supersymmetric zeta categories, trace-refined arithmetic quantum gravity, and the construction of parity-modulated Langlands correspondences in future work.

References

[1] P. Deligne, *Notes on motives*, in *Proc. Symp. Pure Math.*, 1994.
[2] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, 2000.
[3] J. Lurie, *Higher Algebra*, 2017.
[4] D. Freed and E. Witten, *Anomalies in string theory with D-branes*, 1999.
[5] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, 2008.
[6] P. J. S. Yang, *Spectral Motives I–XV*, 2025.
[7] A. Beilinson, *Remarks on motivic sheaves and derived categories*, 1990.
[8] C. Teleman, *The structure of 2D semi-simple field theories*, 2012.