Higher Sumsets and Energies in Additive Combinatorics and Number Theory

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Introduction

Let $\mathbf{G} = (\mathbf{G}, +)$ be an (abelian) group.

The *sumset* and of A, $B \subseteq \mathbf{G}$ is

$$A + B = \{a + b \ : \ a \in A, \, b \in B\} \, .$$

In a similar way we define the *difference sets* and the *k–fold sumset* of A, e.g., $2A - A$ is $A + A - A$.

If $\mathcal{R} = (\mathcal{R}, +, *)$ is a ring, then the *product* set of A, $B \subseteq \mathcal{R}$ is

$$A * B = \{a * b \ : \ a \in A, \, b \in B\} \, .$$

The study of the structure of sumsets is a fundamental problem in classical additive combinatorics.

Higher difference sets

We have the following characterization of difference sets

$$A - A := \{a - b \ : \ a, b \in A\} = \{x \in \mathbf{G} \ : \ A \cap (A + x) \neq \emptyset\} .$$

A natural generalization of the difference set $A - A$ is the set

$$\{(x_1, \dots, x_k) \in \mathbf{G}^k \ : \ A \cap (A + x_1) \cap \dots \cap (A + x_k) \neq \emptyset\} = A^k - \Delta_k(A) ,$$

which is called the *higher difference set*. Here

$$\Delta_k(A) := \{(a, a, \dots, a) \ : \ a \in A\} \subseteq \mathbf{G}^k .$$

In $A^k - \Delta_k(A)$ the sets A^k and $\Delta_k(A)$ have completely different natures.

The higher differences appear naturally in AC. The classical Ruzsa triangle inequality:

$$|X||Y-Z| \leq |Y \times Z - \Delta_2(X)| \leq |Y-X||Z-X|\,.$$

Lemma (generalized triangle inequality)

Let k be a positive integer, $Y \subseteq \mathbf{G}^k$ and $X, Z \subseteq \mathbf{G}$. Then

$$|X||Y - \Delta_k(Z)| \leq |Y \times Z - \Delta_{k+1}(X)|\,,$$

and for any $k \geq 2$ and sets $A_1, \dots, A_k \subseteq \mathbf{G}$ one has ($|\mathbf{G}| < \infty$)

$$|A_1 \times \dots \times A_k - \Delta_k(\mathbf{G})| = |\mathbf{G}||A_1 \times \dots \times A_{k-1} - \Delta_{k-1}(A_k)|\,.$$

One can think about the higher difference sets $A^k - \Delta_k(A)$ as projections of the Cartesian product A^k along the lines

$$\begin{cases} x_1 = t + c_1 \\ \dots\dots\dots\dots \\ x_k = t + c_k \end{cases}$$

The difference of two set of different nature in formula

$$\{(x_1,\dots,x_k) \in \mathbf{G}^k \ : \ A\cap(A+x_1)\cap\dots\cap(A+x_k) \neq \emptyset\} = A^k - \Delta_k(A)$$

provides an interesting combinatorics of the higher difference sets. Further, the family of sets

$$A_{\vec{x}} = A \cap (A+x_1) \cap \dots \cap (A+x_k)$$

has some specific properties and is very far from being random. Indeed, let $k=1$ and thus we consider the family

$$A_s = A \cap (A-s) \subseteq A, \qquad s \in A-A\,.$$

Usually the following holds

$$|A^2 - \Delta_2(A)| = o(|A-A|^2)\,.$$

Thus for a.e. pairs $s,t \in A-A$, namely, for $(s,t) \notin A^2 - \Delta_2(A)$ one has

$$A_s \cap A_t = \emptyset$$

and therefore the family $\{A_s\}_{s\in A-A}$ is absolutely non–random.

It is very interesting to give a full description of this family of sets.

One can study the family of sets $\{A_s\}_{s\in A-A}$ using the usual matrix methods. For example, one can consider the symmetric matrix $M(s,t) = |A_s \cap A_t|$ and try to compute the spectrum

$$\mathrm{Spec}\,(M) = \{\mu_1 \geq \mu_2 \geq \dots \geq \mu_{|A|} \geq 0\}$$

of M and it brings us to the *eigenvalues method*.
It turns out that the family $\{A_s\}_{s\in A-A}$ is connected with submatrices of the classical Cayley graphs: for $A \subseteq \mathbf{G}$ the Cayley graph $\mathrm{Cay}(\mathbf{G}, A)$ has the vertex set $\mathbf{G}$ and $(x,y) \in \mathbf{G}\times\mathbf{G}$ belongs the edge set iff $y - x \in A$. Then the system $\{A_s\}_{s\in A-A}$ corresponds to the *submatrix* $\mathrm{T}(x,y)$ of the adjacency matrix of $\mathrm{Cay}(\mathbf{G}, A)$, having the same spectrum as M. Moreover, one can compute

$$\mu_1|A| \geq \langle \mathrm{T}A, A\rangle := \sum_{x,y\in A} \mathrm{T}(x,y)$$

$$= |\{(a,b,c,d)\in A^4 \ : \ a+b = c+d\}| := \mathrm{E}(A)$$

and the last quantity is called the *additive energy* of our set A.

The energy $\mathrm{E}(A)$ plays a huge role in AC and NT. On the other hand,

$$\mathrm{E}(A) = \sum_{s \in A-A} |A_s|^2$$

and it leads us to the *higher energies* method

$$\mathrm{E}_\alpha(A) := \sum_{s \in A-A} |A_s|^\alpha \,,$$

where $\alpha \geq 0$ be any real number.

In particular, $\mathrm{E}(A) = \mathrm{E}_2(A)$ and

$$\sum_{j=1}^{|A|} \mu_j^2 = \mathrm{E}_3(A) = \sum_{s \in A-A} |A_s|^3 \,.$$

We know that the higher difference sets $A^k - \Delta_k(A)$ are projections of the Cartesian product A^k along the lines

$$\begin{cases} x_1 = t + c_1 \\ \dots\dots\dots\dots \\ x_k = t + c_k \end{cases}$$

On the other hand, projections along hyperspaces $x_1 + \dots + x_k = 0$ are associated with another type of sumsets, namely kA or $kA - kA$, and also with another type of energies $\mathrm{T}_k(A)$

$$\mathrm{T}_k(A) = |\{(a_1, \dots, a_k, a'_1, \dots, a'_k) \in A^{2k} \;:\; a_1 + \dots + a_k = a'_1 + \dots + a'_k\}|$$

and this dual object (L_{2k}–norm of the Fourier transform of the characteristic function of the set A) is a classical tool in NT and AC. One has (if $|\mathbf{G}| < \infty$)

$$\mathrm{T}_k(|\widehat{A}|^2) = |\mathbf{G}|^{2k-1}\mathrm{E}_{2k}(A)\,,$$

and there is even an uncertainty principle

$$\mathrm{E}_k(A)\mathrm{T}_k(A) \geq \left(\frac{\mathrm{E}_{3/2}(A)}{|A|}\right)^{2k} .$$

There are plenty of non–trivial relations between the quantities E_α and E_β for different α and β, e.g., under some mild conditions on a set A for any $s \in [1,2]$ one has

$$\mathrm{E}_s(A) \gg |A|^{1-s/2}\mathrm{E}^{s/2}(A)\,.$$

This estimate cannot be obtained using standard Hölder–type inequalities.

Another important property of the higher energies is their *duality*, which in a very special case can be expressed by an identity connecting different intersections from above, namely, for any positive integers k and l one has

$$\sum_{s_1,\dots,s_k} |(A-s_1)\cap(A-s_2)\cap\dots\cap(A-s_k)|^l$$

$$= \sum_{x_1,\dots,x_l} |(A-x_1)\cap(A-x_2)\cap\dots\cap(A-x_l)|^k\,.$$

Applications: classical additive combinatorics

The classical Ruzsa triangle inequality:

$$|C||A-B| \le |A-C||B-C|\,.$$

The higher difference appears naturally in the proof

$$|C||A-B| \le |A\times B - \Delta_2(C)| \le |A-C||B-C|\,.$$

The lower bound follows from the formula

$$|A\times B - \Delta_2(C)| = \sum_{s\in A-B} |C - A\cap(B+s)| \ge |C||A-B|\,.$$

Of course, the identity above contains much more information.

Theorem (Schoen–S., 2016, symmetric case)

Let $A\subset\mathbb{R}^n$ be a compact convex set. Then

$$\mu(A-A) \ll \frac{\mu(A+A)^2}{\sqrt{n}\mu(A)}\,.$$

Theorem (Croot–Sisask, 2010)

Let $\epsilon \in (0,1)$, $K \geq 1$ be real numbers, p be a positive integer, $A, B \subseteq \mathbf{G}$ s.t. $|A+B| \leq K|A|$ and let f be a function on $\mathbf{G}$. Then there are $b \in B$ and a set $T \subseteq B$, $|T| \geq |B|(2K)^{-O(\epsilon^{-2}p)}$ s.t. for all $t \in T - b$ one has

$$\|(f*A)(x+t) - (f*A)(x)\|_{L_p(\mathbf{G},x)} \leq \epsilon|A|\|f\|_{L_p(\mathbf{G})}$$

One can easily check that the set of almost periods T is, actually, a large subset of the intersections

$$A_{\vec{s}} = A \cap (A+s_1) \cap \dots \cap (A+s_k)\,.$$

So, any new information about structure of the collection of $A_{\vec{s}}$, $\vec{s} \in A^k - \Delta_k(A)$ can help to understand these central additive-combinatorial questions more deeply.

The classical Balog–Szemerédi–Gowers gives us a connection between the additive energy $\mathrm{E}(A) = \mathrm{E}_2(A)$ and the trivial energy $\mathrm{E}_0(A) = |A|^2$.

Theorem (Balog–Szemerédi–Gowers, 1994, 1998)

Let $A \subseteq \mathbf{G}$ be a set. Then we have two equivalent conditions

$$\mathrm{E}_0^{3/2}(A) \ge \mathrm{E}(A) \gg \frac{|A|^3}{K^{C_1}} = \frac{\mathrm{E}_0^{3/2}(A)}{K^{C_1}}\,,$$

and

$$\exists A_* \subseteq A,\ |A_* + A_*| \ll K^{C_2}|A_*| \quad \text{and} \quad |A_*| \gg |A|/K^{C_3}\,.$$

Are there any connections between other pairs of energies $\mathrm{E}_\alpha/\mathrm{E}_\beta$?

The answer is positive and we formulate here the most useful result (so–called E_2–E_3 theorem).

By the Hölder inequality, we have

$$\mathrm{E}_2^2(A) \le \mathrm{E}_3(A)|A|^2 \,. \tag{1}$$

It turns out that it is possible to describe all sets A such that the bound (1) is M–sharp in the sense that the following inequality holds: $\mathrm{E}_3(A)|A|^2 \le M\mathrm{E}_2^2(A)$? The answer is "yes":

$$\mathrm{E}_2^2(A) \sim_M \mathrm{E}_3(A)|A|^2 \Leftrightarrow |(A_* - A_*) + (A_* - A_*)| \ll_M |A_* - A_*| \,.$$

Theorem (S., 2013)

Let $\mathbf{G}$ be an abelian group, $A \subseteq \mathbf{G}$, $\mathrm{E}(A) \ge |A|^3/K$, and $\mathrm{E}_3(A) = M|A|^4/K^2$. Then there is a set $A_* \subseteq A$, $|A_*| \gg_M |A|$ such that for any positive integers n, m the following holds

$$|nA_* - mA_*| \ll M^{O(n+m)} K |A_*| \,.$$

Non–commutative structural E_2–E_3 result was used to obtain some new results for the sum–product phenomenon.

There is a connection of the higher energies with Gowers norms, additive dimension, universality and so on.

Definition

Let $|\mathbf{G}| < \infty$, and let $A \subseteq \mathbf{G}$ be a set. We write

$$\mathrm{cov}(A) = \min\{|X| \;:\; X \subseteq \mathbf{G},\, A + X = \mathbf{G}\}$$

and the quantity $\mathrm{cov}(A)$ is called the *covering number* of A.

For any set $A \subseteq \mathbf{G}$ one has

$$\frac{|\mathbf{G}|}{|A|} \le \mathrm{cov}(A) \ll \frac{|\mathbf{G}|}{|A|} \log |\mathbf{G}| \,.$$

Theorem (S., 2024)

Let $|\mathbf{G}| < \infty$, $A, B \subseteq \mathbf{G}$, $|A| = \alpha|\mathbf{G}|$, and $|B| = \beta|\mathbf{G}|$. Then

$$\mathrm{cov}(A+B) \le \frac{1}{\alpha} \log \frac{1}{\beta} + 1 \,.$$

The higher energies are connected with the family of sets $A_s = A \cap (A - s)$. We have

$$\sum_{s \in A-A} |A_s| = |A|^2$$

and hence the typical size of A_s is $\frac{|A|^2}{|A-A|}$. We showed (Lev–S., 2023) that under some mild conditions on an arbitrary subset $A \subset \mathbb{Z}$ (or $\mathbb{F}_p$) there is $s \neq 0$ such that

$$|A_s| \geq (2 - o(1)) \cdot \frac{|A|^2}{|A - A|} \qquad (2)$$

and the last bound is tight. Moreover, it was showed that the collection of s such that bound (2) takes place forms a set of *measure zero*. This fact once again emphasizes that higher sumsets methods work on small sets of measure zero.

This approach also allows us to make progress on the classical Freiman $3k - 4$ theorem in the prime fields (Lev–S., 2020 and Lev–Serra, 2023).

Applications: the sum–product phenomenon and incidence geometry

Conjecture (Erdős–Szemerédi, 1983)

Let $A \subset \mathbb{Z}$, $|A| < \infty$. Then

$$\max\{|A+A|, |AA|\} \gg_\varepsilon |A|^{2-\varepsilon}\,, \qquad |A| \to \infty\,,$$

where $\varepsilon > 0$ is any number.

Theorem (Erdős–Szemerédi, 1983)

Let $A \subset \mathbb{Z}$, $|A| < \infty$. Then

$$\max\{|A+A|, |AA|\} \gg |A|^{1+c}\,,$$

where $c > 0$ is an absolute constant.

Applications: NT, AC, DS, CS, cryptography, ...

Theorem (Konyagin–S., 2016, ..., Rudnev–Stevens, 2022)

Let $A \subset \mathbb{R}$. Then

$$\max\{|A+A|, |A\cdot A|\} \gg |A|^{4/3+c}\,, \qquad |A| \to \infty\,,$$

where $c < \frac{2}{1167}$ is an absolute constant.

The existence of c above is important for further results.

It also turns out that a similar approach works in the case $\mathcal{R} = \mathbb{F}_p$.

Theorem (Rudnev–Shakan–S., 2020; Mohammadi–Stevens, 2023)

Let $A \subset \mathbb{F}_p$, $|A| < p^{1/2}$. Then

$$\max\{|A+A|, |A\cdot A|\} \gg |A|^{1+1/4+o(1)}\,.$$

On a question of Klurman–Pohoata

$S \subseteq \mathbf{G}$ is a *Sidon set* iff for all $a, b, c, d \in S$

$$a + b = c + d \implies \{a, b\} = \{c, d\}$$

or, equivalently,

$$0 \neq a - b = c - d \implies a = c,\ b = d\,. \tag{3}$$

We write $S \in \mathrm{Sid}_1$. If (3) has at most g solutions, then let $S \in \mathrm{Sid}_g$. For $A \subseteq \mathbf{G}$ we write

$$\mathrm{Sid}(A) = \mathrm{Sid}_1(A) = \max\{|S| \ :\ S \subseteq A,\, S \text{ is a Sidon set}\}\,,$$

and similarly for $\mathrm{Sid}_g(A)$.

Theorem (Komlós–Sulyok–Szemerédi, 1975 + Semchankau, 2017)

Let $\mathbf{G}$ be an abelian group. Then for any $A \subseteq \mathbf{G}$ one has

$$\mathrm{Sid}(A) \geq c\sqrt{|A|}\,.$$

Question (Klurman–Pohoata)

For any $A \subset \mathbb{R}$ one has

$$\max\{\mathrm{Sid}^{+}(A), \mathrm{Sid}^{\times}(A)\} \gg |A|^{1/2+c},$$

where $c > 0$ is an absolute constant.

This is a sum-product–type question.

Upper bounds for the maximum: $\exists A$ s.t.

$$\max\{\mathrm{Sid}^{+}(A), \mathrm{Sid}^{\times}(A)\} \ll |A|^{2/3}$$

(independently, Green–Peluse, Roche-Newton–Warren, S.).

Theorem (S., 2021)

Let $A \subseteq \mathbb{F}$ be a set, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{F}_p$ and $|A| < \sqrt{p}$, say. Then there are some absolute constants $c > 0$, $K \geq 1$ such that

$$\max\{\mathrm{Sid}^{+}_{K}(A), \mathrm{Sid}^{\times}_{K}(A)\} \gg |A|^{1/2+c}\,.$$

On the other hand, for any integer $k \geq 1$ there is $A \subseteq \mathbb{F}$ with

$$\max\{\mathrm{Sid}^{+}_{k}(A), \mathrm{Sid}^{\times}_{k}(A)\} \ll k^{1/2}|A|^{2/3}\,.$$

Improvements: Mudgal, Jing–Mudgal.

Theorem (S., 2014)

Let $A \subseteq \mathbf{G}$ be a set, $\delta, \varepsilon \in (0,1]$ be parameters. Then there is $k = k(\delta, \varepsilon)$ such that either

$$\mathrm{E}_k(A) \le |A|^{k+\delta}$$

or there is $H \subseteq \mathbf{G}$ and $\Lambda \subseteq \mathbf{G}$ such that $|H| \gg |A|^{\delta(1-\varepsilon)}$ and

$$|H+H| \ll |H||A|^{\varepsilon} \quad \text{and} \quad |A|^{1-\varepsilon} \ll |\Lambda||H| \ll |A|^{1+\varepsilon},$$

$$|(H \dotplus \Lambda) \cap A| \gg |A|^{1-\varepsilon}.$$

In other words, for any set $A \subseteq \mathbf{G}$ there is $k = O(1)$ s.t. either $\mathrm{E}_k(A) \sim |A|^{k+o(1)}$ or

$$A \approx H \dotplus \Lambda, \qquad \text{where} \qquad |H+H| \ll |H|^{1+o(1)}.$$

It is a *criterion*.

The improvements to the Erdős–Szemerédi conjecture are interesting in themselves, but what is even more exciting is that our applications of the higher energies to the sum–product–type questions lead to fundamentally new results and effects. We give just two examples.

I) A new necessary condition for a set of reals to be a sumset = any sumset must grow under multiplication.

Theorem (S.–Zhelezov, 2018)

Let $B, C \subset \mathbb{R}$ and $|B|, |C| > 1$. Then

$$|(B+C)(B+C)| \gg |B+C|^{1+c_*},$$

where $c_* > 0$ is an absolute constant.

In $\mathbb{F}_p$ a similar result is known just in the case $(A-A)(A-A)$ (Murphy–Petridis–Roche-Newton–Rudnev–S., 2019).

II) *Additive decompositions* of sets.

The classic question of Ostmann asks about the possibility of representing the set of prime numbers $\mathcal{P}$ as the sum of two sets B and C such that $|B|, |C| > 1$, i.e. is it possible to find B and C with $|B|, |C| > 1$ such that

$$|\mathcal{P} \triangle (B + C)| < \infty\,?$$

Another classical question of Sárközy asks the same question about the set of quadratic residues.

Theorem (S., 2020)

Let $\Gamma \le \mathbb{F}_p$ and $|\Gamma| \le p^{2/3-\varepsilon}$, where $\varepsilon > 0$. Then

$$\Gamma \neq B + C, \qquad \forall B, C, \quad |B|, |C| > 1\,.$$

Let us underline one more time that the results above cannot be obtained without proof of existence of our new sum–product constant $c > 0$.

Applications: number theory

Let p be a prime number. Thanks to Fermat's little theorem we know that for any $n \in \mathbb{F}_p$ the ratio

$$q(n) := \frac{n^p - n}{p}$$

is an integer. What can be said about the distribution of the *Fermat quotients* $q(n)$? This is a classic question and the distribution is controlled by the *Heilbronn exponential sum*

$$S(a) = \sum_{n=1}^{p} e^{2\pi i \cdot \frac{an^p}{p^2}}\,.$$

D.R. Heath–Brown, 1996 (also, Konyagin, V'yugin, Solodkova, S.)

Theorem (S., 2014)

Let p be a prime, and $a \neq 0 \pmod p$. Then

$$|S(a)| \ll p^{\frac{5}{6}} \log^{\frac{1}{6}} p\,.$$

It is easy to see that the Heilbronn sum is taken over the multiplicative subgroup

$$\Gamma = \{m^p \ : \ m \in \mathbb{Z}/(p^2\mathbb{Z})\,, m \neq 0\} \subseteq \mathbb{Z}/(p^2\mathbb{Z})$$

and these methods work for exponential sums over subgroups, Gauss–type sums, for example, trilinear and trinomial exponential sums.

It turns out that a similar method (Hanson, Volostnov, Schoen, S.) is applicable to the other classical sums of the from

$$\sum_{a\in A,\, b\in B} \chi(a+b)\,,$$

where A, B are arbitrary subsets of the field $\mathbb{F}_p$ and χ is a non–trivial multiplicative character.

Non–trivial bounds are unknown for $|A| \sim |B| = o(\sqrt{p})$ even for A, B of special form. The well–known *Paley graph conjecture* predicts exponential savings for any $|A|, |B| > p^{\varepsilon}$.

Moreover, even at the combinatorial level, nothing is known. Namely,

The product–sum conjecture

Let $A \subseteq \mathbb{F}_p$ and $|A| > p^\varepsilon$. Then there is $n = n(\varepsilon)$ such that

$$\frac{(nA-nA)^n}{(nA-nA)^n} = \mathbb{F}_p\,.$$

With Schoen we broke the square–root barrier and we obtained an affirmative answer to a question of A. Balog, 2012.

Theorem (Schoen–S., 2022)

Let p be a prime number, and $A \subseteq \mathbb{F}_p$ be a set of size $|A| \ge \sqrt{p} \cdot \exp(-O(\log^{1/5} p))$. Then

$$\frac{(2A-2A)^3}{(2A-2A)^2} = \mathbb{F}_p\,.$$

Applications: Fourier analysis

Given an abelian group $\mathbf{G}$ define by $\widehat{\mathbf{G}}$ its dual group. For any function $f : \mathbf{G} \to \mathbb{C}$ and $\xi \in \widehat{\mathbf{G}}$ define the Fourier transform

$$\widehat{f}(\xi) = \sum_{g \in \mathbf{G}} f(g)\overline{\xi(g)}\,.$$

A number of uncertainty principles on $\mathbf{G}$ assert, roughly, that a function on $\mathbf{G}$ and its Fourier transform cannot be simultaneously highly concentrated, e.g.,

$$|\mathrm{supp}\, f| \cdot |\mathrm{supp}\, \widehat{f}| \ge |\mathbf{G}|\,,$$

and the classical Heisenberg inequality for $\mathbf{G} = \mathbb{R}$ states that for any $a, b \in \mathbb{R}$ one has

$$\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, dx \cdot \int_{\mathbb{R}} (\xi - b)^2 |\widehat{f}(\xi)|^2 \, d\xi \ge \frac{\|f\|_2^4}{16\pi^2}\,.$$

Theorem (S., 2024)

Let $\mathbf{G}$ be a finite abelian group, $A \subseteq \mathbf{G}$, $|A| = \delta|\mathbf{G}|$, and let $|A-A| = K|A|$. Suppose that $|A| \to \infty$, $\log^2 K = o(\log|A|)$ and

$$K^2\delta = o(1)\,.$$

Then

$$M(A)\rho(A) \ge \frac{|A|^3}{K} \cdot (1-o(1))\,,$$

where

$$\rho(A) := \max_{x\neq 0} |A \cap (A+x)|\,, \qquad \text{and} \qquad M(A) := \max_{\xi \neq 0} |\widehat{1}_A(\xi)|^2\,.$$

This result is tight.

Modern results and future work

Theorem (Kelley–Meka, 2023, also see Bloom–Sisask, 2023)

Let $A \subseteq \{1, \dots, N\}$ has no arithmetic progressions of length three. Then

$$|A| \ll N \exp(-\Omega(\log^c N)) \,.$$

The proof uses the higher energy (now k is a growing parameter depending on the density $|A|/N$):

$$\mathrm{E}_k(f) = \sum_x \left(\sum_y f(x)f(x+y) \right)^k$$

of the balanced function of the set A, i.e.

$$f(x) = 1_A(x) - \frac{|A|}{N} \cdot 1_{[N]}(x)$$

to control the number of arithmetic progressions in A.

Recently, S. generalized this method for the higher energies $\mathrm{E}_{k,l}$. It would be very interesting to develop this topic and understand the limitations of this approach. For example, it is unclear whether any regularity result can be obtained using this method. The main difficulty here is that, as we have seen above, the higher energies act on sets of measure zero, and this may become an obstacle to obtaining such analytic results.

Problem

Let $k > 2$ be an integer, $A \subseteq \{1, \dots, N\}$ be a set, $|A| = \delta N$ and $\varepsilon, \omega \in (0,1)$ be some parameters. Is there a partition of $\{1, \dots, N\}$ into arithmetic progressions $P_1, \dots, P_s$ and a set Ω such that

- $|\Omega| \le \omega N$,
- $s = O_{\delta,\varepsilon,\omega}(1)$,
- $\mathrm{E}_k(f_{A \cap P_j}) \le \varepsilon N^{k+1}$ for all $j \in [s]$?

Theorem (Gowers–Green–Manners–Tao, 2024)

Let $\mathbf{G}$ be an abelian group with torsion m. Suppose that $A \subseteq \mathbf{G}$ s.t. $|A+A| \le K|A|$. Then A is covered by at most $(2K)^{O(m^3)}$ cosets of $H \le \mathbf{G}$ and $|H| \le |A|$.

One of the main steps of the proof was the so–called fibring lemma, which establishes a connection between the *entropy* distances of the sets $A, A-A$ and A_s. The first result in this direction (at the energy level) was obtained by Katz–Koester. One of the modern possible forms (Schoen–S.)

$$\mathrm{E}_0^4(A) = |A|^8 \le \mathrm{E}_4(A)\mathrm{E}(A-A) = \sum_{s,t} \mathrm{E}(A_s, A_t) \cdot \mathrm{E}(A-A)\,.$$

The entropy itself can be thought of as the energy $\mathrm{E}_{1+\varepsilon}(A)$, where $\varepsilon > 0$ is an arbitrary small number. It would be very interesting to obtain some deeper connections between $A, A-A$ and A_s (and similar quantities) at the level of energies and/or sumsets.

Thank you for your attention!

Using the higher energies method, we succeeded in obtaining some new bounds in the spirit of Katz–Koester, connecting $A, A - A$ and A_s, for example

$$\sum_{x+y=z+w} |A_x|^2 |A_y|^2 |A_z|^2 |A_w|^2 \geq \frac{\mathrm{E}^6(A)}{|A|^7} \,.$$

It would be very interesting to obtain a connection between similar quantities at the level of energies and/or sumsets.