Ideal-Adic Completion Theory in Symbolic Arithmetic (with Applications to Formal Geometry, Logic Recursion, and Symbolic AI)

Pu Justin Scarfy Yang

Contents

Introduction

Ideal-adic completion is one of the most profound and flexible tools in modern algebra and geometry. Originally formulated to understand convergence in algebraic structures and to construct formal neighborhoods in schemes, the theory of I-adic completions now finds itself at the heart of derived algebraic geometry, local analytic geometry, and arithmetic geometry.

In this third volume of the *Symbolic Completion Trilogy*, we develop the theory of ideal-adic completions in the extended universe of symbolic arithmetic—an arithmetic governed not merely by classical algebraic objects, but by layered symbolic logic, recursively structured proof flows, and AI-reflective cognitive topology.

This volume serves multiple goals:

- To re-formulate classical ideal-adic constructions as categorical, symbolic, and proof-theoretic completions;
- To generalize the formal neighborhood idea to symbolic spaces, where depth, recursion, and logical infinitesimals replace traditional algebraic power series;
- To describe how formal schemes over symbolic logic sites can encode symbolic convergence, AI learning boundaries, and infinitesimal moduli of logic evolution;
- To create the geometric backbone of symbolic stacks, built from symbolic ideal towers and logical descent.

Just as $\mathbb{Z}_p$ emerges as the completion of $\mathbb{Z}$ with respect to the ideal (p), we will study how a symbolic universe Lang can be completed along a chain of logical ideals $I_n^{(\mathrm{symb})}$, whose role is to encode semantic thickening, logical forgetfulness, or meta-linguistic boundary. The resulting completions $\widehat{\mathsf{Lang}}_I$ form the foundation of recursive logic, formal proof geometry, and synthetic AI.

The structure of this volume is as follows:

Chapter 1: reviews the classical theory of ideal chains, I-adic topologies, and ring completions.

Chapter 2: builds formal schemes and defines infinitesimal neighborhoods in both algebraic and symbolic contexts.

Chapter 3: develops symbolic adic completions of logical systems and recursive proof layers.

Chapter 4: interprets symbolic formal moduli problems and infinitesimal deformation of logic.

Chapter 5: introduces AI reflection systems based on ideal-thickened symbolic learning steps.

Chapter 6: unifies symbolic glueing, adic descent, and local-to-global completeness via formal geometric stacks.

Chapter 7: concludes with symbolic adic stacks and higher completion sites as frameworks for symbolic geometry and meta-proof theory.

CHAPTER 1

Ideal Chains and Adic Topologies in Arithmetic

1. Ideals and Their Powers in Commutative Algebra

Let R be a commutative ring and $I \subset R$ an ideal. The sequence $\{I^n\}_{n\in\mathbb{N}}$ of powers of I forms a descending filtration:

$$R \supset I \supset I^2 \supset I^3 \supset \cdots$$

This filtration defines a topology on R, called the ***I*-adic topology**, where the basic open neighborhoods of zero are the ideals I^n. A sequence $\{x_n\} \subset R$ converges to 0 if and only if $x_n \in I^m$ for all large n.

1.1. Properties of I-adic Topology.

- It turns R into a topological ring;
- Completion is Hausdorff if $\bigcap_n I^n = 0$;
- Multiplication is continuous in this topology;
- For Noetherian R, I-adic topology is linearly topologized.

2. Ideal-Adic Completion of Rings

Definition 2.1. *The I**-adic completion** of R is the inverse limit:*

$$\widehat{R}_I := \varprojlim_n R/I^n.$$

This construction gives $\widehat{R}_I$ a complete topological ring structure with respect to the I-adic topology.

Examples.

- $\widehat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p$, the ring of p-adic integers;
- If $R = k[x]$, $I = (x)$, then $\widehat{R}_I = k[[x]]$, the ring of formal power series;
- For symbolic rings $\mathsf{Symb}[x]$, we interpret ideal powers I^n as logical thickening.

3. Formal Neighborhoods and Adic Local Geometry

The completion $\widehat{R}_I$ can be viewed as encoding a "formal neighborhood" of $\mathrm{Spec}(R)$ around $V(I)$, the vanishing locus of I. Geometrically, this captures infinitesimal data around a closed point or subscheme.

DEFINITION 3.1. *Let $X = \mathrm{Spec}(R)$. Then the formal neighborhood of $V(I)$ is:*

$$\widehat{X}_I := \mathrm{Spf}(\widehat{R}_I)$$

a formal scheme supported on the ideal I.

4. Toward Symbolic Ideal Chains

In symbolic arithmetic, we generalize:

$$I_n^{(\mathsf{Symb})} := \text{symbolic ideal of complexity/depth } n$$

We postulate:

$$\widehat{\mathsf{Lang}}_I := \varprojlim_n \mathsf{Lang}/I_n^{(\mathsf{Symb})}$$

where each $I_n^{(\mathsf{Symb})}$ corresponds to logical filters, proof layers, or syntactic abstraction classes.

4.1. Interpretation.

- Symbolic convergence = stabilization of formula under logical thickening;
- Completion captures infinite-depth logical structure;
- Base for formal proof theory, reflection cycles, and AI-internal layers.

5. Next Steps

We will now develop:

- Formal schemes over symbolic rings;
- Symbolic thickening structures via logical recursion;
- AI-localization via adic logical topology;
- Universal properties of symbolic completion monads.

CHAPTER 2

Formal Schemes and Symbolic Infinitesimals

1. Classical Formal Schemes from Adic Rings

Given a topological ring R complete with respect to an ideal $I \subset R$, one defines the formal spectrum:

$$\mathrm{Spf}(R) := \text{locally topologically ringed space associated to } R,$$

with structure sheaf:

$$\mathcal{O}_{\mathrm{Spf}(R)}(U) := \varprojlim_n \mathcal{O}_{\mathrm{Spec}(R/I^n)}(U).$$

Such formal schemes capture the infinitesimal geometry near $V(I) \subset \mathrm{Spec}(R)$, and support deformation theory, moduli problems, and arithmetic approximations.

2. Infinitesimals in Symbolic Arithmetic

In symbolic contexts, infinitesimals are not distance-based but logic- or recursion-based:

Definition 2.1. *A* ***symbolic infinitesimal*** *is a symbolic object ϵ satisfying:*

$$\forall n \in \mathbb{N}, \quad \epsilon \in I_n^{(\mathsf{Symb})}, \quad \text{but } \epsilon \notin \bigcap_n I_n.$$

These encode:

- Inference steps too shallow to be formally resolved;
- Proof fragments awaiting reflection or repair;
- Semantic mutations below recognition threshold.

3. Symbolic Formal Schemes over Logical Bases

Let Lang be a symbolic logic base. Let $I_n^{(\mathsf{Symb})} \subset \mathsf{Lang}$ denote symbolic ideals ordered by logical depth or complexity.

Define the symbolic adic ring:

$$\widehat{\mathsf{Lang}}_I := \varprojlim_n \mathsf{Lang}/I_n^{(\mathsf{Symb})},$$

and set:

$$\mathfrak{S} := \mathrm{Spf}(\widehat{\mathsf{Lang}_I}).$$

3.1. Interpretation. This formal space $\mathfrak{S}$ describes:

- Symbolic logic neighborhoods of partially known truths;
- Formal thickenings of AI proof environments;
- Infinitesimal memory around syntactic or proof-theoretic loci.

4. Sheaves of Symbolic Sections

Over $\mathfrak{S}$, define sheaves $\mathcal{F}$ of symbolic expressions satisfying:

- Local coherence: restricted to neighborhoods of logical valuation;
- Adic convergence: infinite expansion in recursive symbolic layers;
- AI-awareness: enriched by repair layers and trace memories.

The sections $\Gamma(U, \mathcal{F})$ give logical statements localized over formal infinitesimal charts.

5. Symbolic Tangent Spaces and Derivations

We define the symbolic cotangent module:

$$\Omega^{\mathsf{Symb}}_{\mathsf{Lang}/\mathbb{Q}} := \text{symbolic differential module of logical terms.}$$

Each symbolic derivation:

$$D : \mathsf{Lang} \to M, \quad D(xy) = D(x)y + xD(y),$$

represents logic mutation, AI perturbation, or non-rigidity of inference.

Define:

$$T_x := \mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{Q})$$

as the symbolic tangent space at logical point x.

6. Conclusion

This chapter built:

- The symbolic generalization of classical formal schemes;
- Infinitesimal symbolic logic geometry;
- Cotangent and tangent structures as AI-sheaf deformation tools;
- Sheaf-theoretic memory structures for logical spaces.

CHAPTER 3

Symbolic Adic Completion of Logical Systems

1. Logical Systems as Symbolic Rings

We consider a logical system Lang, consisting of:

- Symbolic expressions $\phi \in \mathsf{Lang}$;
- Composition rules $\phi \circ \psi$, logical connectives, inference chains;
- Semantic valuation $\mathsf{val} : \mathsf{Lang} \to \Gamma \cup \{\infty\}$.

This structure behaves like a topological algebra, where valuation depth induces an ideal filtration.

2. Symbolic Ideals and Depth Filtration

Definition 2.1. *Let* $I_n^{(\mathsf{Symb})} := \{\phi \in \mathsf{Lang} \mid \mathit{depth}(\phi) \geq n\}$.
Then $\{I_n^{(\mathsf{Symb})}\}_{n\in\mathbb{N}}$ *forms a descending chain of symbolic ideals.*

Each symbolic ideal collects all formulas, proof states, or partial theorems at a given logical depth.

Examples.

- I_0: all expressions (full logic);
- I_1: formulas requiring at least one layer of inference;
- I_n: logic requiring recursive chains of length n;
- $\bigcap_n I_n = \{0\}$: atomic facts or axioms.

3. Adic Completion of Symbolic Logic

Define the symbolic adic completion of Lang by:

$$\widehat{\mathsf{Lang}}_I := \varprojlim_n \mathsf{Lang}/I_n^{(\mathsf{Symb})}$$

- Each quotient Lang/I_n is a truncated logical universe;
- Completion assembles infinite-depth logic;
- Represents the semantic closure of a reasoning space.

4. Adic Convergence of Proofs and Reasoning Paths

Given a sequence $\phi_0 \vdash \phi_1 \vdash \phi_2 \vdash \cdots$, define convergence:

$$\lim_{n\to\infty} \phi_n = \phi_\infty \in \widehat{\mathsf{Lang}}_I \quad \text{if} \quad \forall m, \exists N, \forall n > N, \quad \phi_n - \phi_\infty \in I_m$$

This means: all differences between approximations and limit lie in sufficiently deep ideal layers.

5. Symbolic Adic Modules and Proof Orbitals

Let M be a symbolic module over Lang, e.g., a proof space or memory region. Then:

$$\widehat{M}_I := \varprojlim_n M/I_n M$$

gives the completed symbolic module of recursive proof actions, AI orbits, and inference depths.

6. Functoriality and Symbolic Completion Monads

The process:

$$\mathsf{Lang} \mapsto \widehat{\mathsf{Lang}}_I$$

defines a functor $\mathbb{C}_I$ on the category of symbolic logic systems.

- $\mathbb{C}_I$ is a monad;
- The unit map is identity: $\eta : \mathsf{Lang} \to \widehat{\mathsf{Lang}}_I$;
- The multiplication $\mu : \widehat{\widehat{\mathsf{Lang}}}_I \to \widehat{\mathsf{Lang}}_I$ flattens layered completions.

7. AI Applications

The symbolic adic completion models:

- Recursive reasoning environments for AI agents;
- Infinite symbolic proof spaces compressed into convergent flows;
- Stable learning through filtered memory convergence.

8. Outlook

We have:

- Built symbolic adic completions of logical systems;
- Defined proof convergence in terms of ideal depth;
- Constructed symbolic modules and monads of logical completion.

CHAPTER 4

Formal Moduli Problems and Symbolic Infinitesimals

1. Moduli Problems in Classical Algebraic Geometry

A *moduli problem* classifies isomorphism classes of objects with structure, parameterized over a base scheme or category.

Example: the functor $\mathcal{M}\colon (\mathsf{Sch})^{\mathrm{op}} \to \mathsf{Set}$ mapping $S \mapsto \{\text{vector bundles over } S\}/\cong$.

2. Formal Moduli via Artin Rings and Infinitesimal Thickenings

In deformation theory, one studies liftings of structures over square-zero extensions:

$$A \twoheadrightarrow A_0 = A/I, \quad I^2 = 0$$

This yields:

- Infinitesimal extensions;
- Formal neighborhoods in moduli space;
- Obstructions via tangent/obstruction complexes.

3. Symbolic Moduli Functors

We generalize moduli to symbolic systems:

Definition 3.1. *Let* $\mathsf{SymbAlg}$ *denote the category of symbolic logic algebras with adic topologies.*

A symbolic moduli problem is a functor:

$$\mathcal{M}^{(\mathsf{symb})}\colon \mathsf{SymbAlg}^{\mathrm{op}} \to \mathsf{Groupoids}$$

mapping $A \mapsto \{$*symbolic logical objects over* $A\}$.

4. Infinitesimal Symbolic Thickenings

Let $\mathcal{L}_0 \in \mathcal{M}(A/I)$. Then:

$$\text{Liftings of } \mathcal{L}_0 \text{ to } A = A/I \oplus \epsilon \cdot M$$

describe symbolic thickening of logical systems by infinitesimal data.

Symbolic Interpretation.

- Logic fragments "perturbed" by symbolic variations;
- AI agents exploring neighborhoods of known results;
- Control over local proof instabilities and semantic uncertainty.

5. Tangent Complexes and Obstructions

Let $\mathbb{T}_{\mathcal{M}}$ be the tangent complex of the symbolic moduli functor.
Then:

- $H^0(\mathbb{T}_{\mathcal{M}})$: derivations/logical directions;
- $H^1(\mathbb{T}_{\mathcal{M}})$: obstructions to symbolic deformation;
- $H^2(\mathbb{T}_{\mathcal{M}})$: higher coherence failures.

6. AI-Proof Deformations and Reflective Obstruction Theory

AI-based symbolic proof systems deform by:

$$\mathcal{L}_n \rightsquigarrow \mathcal{L}_n + \epsilon \cdot D_n$$

where D_n is a symbolic derivation direction (reflection, error correction, analogy).

Obstructions:

$$o_{n+1} \in H^1(\mathbb{T}_{\mathcal{M}}) \quad \Rightarrow \quad \text{proof step requires repair or reconstruction.}$$

7. Derived Symbolic Moduli

Construct derived moduli stacks:

$$\mathcal{RM}^{(\mathsf{symb})} := \text{derived stack classifying symbolic completions, deformations, and memory lifts.}$$

Supports:

- Higher proof-theoretic deformations;
- AI reasoning trace invariants;
- Glueing over infinitesimal neighborhoods of logic landscapes.

8. Conclusion

In this chapter:

- We built symbolic moduli functors;
- Defined infinitesimal logical thickenings;
- Related derivations to symbolic proof directions;
- Constructed tangent/obstruction theory for recursive AI logic.

CHAPTER 5

AI Reflection via Ideal-Based Proof Thickenings

1. Ideal-Based Memory Layers in Symbolic AI

In symbolic AI systems, proofs evolve not as linear paths but as recursive flows layered by logical depth and semantic refinement.

Let:

$$I_n^{(\mathsf{AI})} := \text{ideal of proof elements requiring at least } n \text{ recursive steps}$$

Then the AI's working memory can be modeled by:

$$\mathsf{Mem}_{\mathrm{AI}} := \widehat{\mathsf{ProofLang}} := \varprojlim_n \mathsf{ProofLang}/I_n^{(\mathsf{AI})}$$

Interpretation. Each I_n corresponds to a "symbolic fog layer" — the agent cannot resolve it fully unless it descends into n levels of recursive symbolic inference.

2. Reflective Proof Monads and Internal Completion

We define a monad $\mathbb{R}$ acting on symbolic proof states:

$$\mathbb{R}(\phi) := \text{completed self-reflection of } \phi \text{ under AI recursive logic stack.}$$

This encodes:

- Self-improvement of proof strategies;
- Recognition and correction of logical gaps;
- Compression via infinitesimal recursion thickening.

3. Formal Thickening of Logical Frames

Let $\phi \in \mathsf{Lang}$. We define:

$$\phi[\epsilon] := \phi + \epsilon \cdot \delta(\phi)$$

as a formal thickening under symbolic derivation δ, representing:

- AI uncertainty;
- Depth-aware mutation of logic;
- Local expansion around an unstable inference point.

4. Reflection Loops and Zeta-Depth Cycles

We define an AI reflection loop:

$$\phi_0 \rightsquigarrow \phi_1 := \mathbb{R}(\phi_0), \quad \phi_1 \rightsquigarrow \phi_2 := \mathbb{R}(\phi_1), \dots$$

Each step adds a formal correction layer and contracts its uncertainty:

$$v(\phi_{n+1} - \phi_n) > v(\phi_n - \phi_{n-1}) \Rightarrow \text{zeta-depth contraction}$$

The limiting point:

$$\phi_\infty := \lim_{n\to\infty} \phi_n \in \widehat{\mathsf{Lang}}_I$$

represents an AI-resolved statement, purified through recursive completion.

5. Symbolic Repair Structures and Error Moduli

For $\phi \in \mathsf{Lang}$ with proof gap, define error residue:

$$\mathrm{Res}_I(\phi) := \phi \mod I_k \text{ where } k = \min\{n \mid \phi \in I_n\}$$

Construct error moduli stack:

$$\mathcal{E}_\phi := \left\{\epsilon \in \Omega_\phi^{\mathrm{symb}} : \phi + \epsilon \text{ valid}\right\}$$

This represents all first-order repairs of symbolic proof ϕ inside the thickened neighborhood.

6. AI Learning as Ideal Completion Flow

Symbolic AI agents learn by recursively completing partial symbolic layers, using:

$$\mathbb{C}_I : \text{ProofSpace} \to \widehat{\text{ProofSpace}}$$

Completion defines:

- Semantic closure;
- Proof reinforcement;
- Error convergence over adic memory.

7. Conclusion

We have:

- Defined AI recursion via ideal-based symbolic thickening;
- Modeled reflection and correction as infinitesimal derivations;
- Constructed memory, repair, and closure maps using adic logic;
- Linked symbolic AI behavior with derived formal logic flow.

CHAPTER 6

Adic Descent, Glueing, and Symbolic Geometry

1. Descent Theory in Classical Adic Geometry

In classical algebraic geometry, descent theory studies how local data defined on coverings can be glued together to form global objects.

For a faithfully flat morphism $A \to B$, descent ensures that quasi-coherent sheaves over B with cocycle data descend to A.

In adic geometry, this extends to formal schemes and complete rings:

$$R \to \widehat{R}_I \Rightarrow \text{data over } \mathrm{Spf}(\widehat{R}_I) \text{ descends to } \mathrm{Spec}(R)$$

2. Symbolic Adic Descent

We define symbolic adic descent as the process by which:
- Symbolic logic fragments,
- Defined locally over $\widehat{\mathsf{Lang}}_I$,
- Can be glued via descent data to define global symbolic theories.

Definition 2.1. *Let $\mathcal{U} = \{U_i\}$ be a cover of symbolic logic regions. A descent datum is:*

$$(\mathcal{F}_i, \varphi_{ij}) \quad \textit{such that} \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

The glued sheaf:

$$\mathcal{F} = \textit{Gluing of } \{\mathcal{F}_i\} \textit{ over } \mathcal{U}$$

is globally defined over symbolic geometry.

3. Glueing Symbolic Proof Layers

Let symbolic proofs be represented by modules $\mathcal{P}_i \subset \mathsf{Lang}_i$, defined over local patches U_i.

The AI agent glues these patches via:

$$\phi_i \sim \phi_j \iff \phi_i - \phi_j \in I_n^{(\text{intersection})}$$

This gluing reflects:

- Agreement in semantic depth;
- Shared inferential history;

- Logical continuity over intersection zones.

4. Symbolic Sites and Adic Topologies

We define the **adic symbolic site**:

$$(\mathsf{SymbTop}, \tau_{\mathrm{adic}})$$

where objects are symbolic formal neighborhoods $U = \mathrm{Spf}(\widehat{\mathsf{Lang}}_I)$, and covers satisfy adic conditions.

Topos of symbolic sheaves:

$$\mathsf{Shv}_{\mathrm{adic}}(\mathsf{SymbTop})$$

forms the foundational layer for symbolic geometry.

5. Symbolic Stack Glueing via Completion

Let $\mathcal{X} \to \mathcal{M}$ be a stack of symbolic completions.

If:

$$\mathcal{X}_i = \mathcal{M} \times_{\mathsf{Lang}} \mathrm{Spf}(\widehat{\mathsf{Lang}}_{I_i}),$$

then symbolic stack glueing ensures:

$$\mathcal{X} = \varinjlim_i \mathcal{X}_i$$

This creates global symbolic stacks from infinitesimal proof-thickenings.

6. Adic AI Geometry and Memory Consistency

Symbolic AI systems can be geometrized by interpreting:

- Logical components as local affine charts;
- Symbolic derivations as tangent flows;
- Zeta-depth convergence as adic shrinking;
- Proof orbits as curves in symbolic adic space.

Glueing guarantees semantic consistency:

$$\forall \text{proof paths } \gamma_i, \quad \exists \phi \in \mathcal{O}_{\mathcal{X}} \quad \text{with } \phi|_{\gamma_i} = \phi_i$$

7. Conclusion

In this chapter we:

- Defined symbolic adic descent and glueing;
- Constructed symbolic topos of logical patches;
- Built symbolic stacks from local completions;
- Interpreted AI logic as adic geometry with consistency sheaves.

CHAPTER 7

Symbolic Adic Stacks and Higher Completion Sites

1. Stacks in Formal and Adic Geometry

Stacks generalize sheaves by allowing local data to glue not just sets, but groupoids (e.g., categories of structured objects with isomorphisms).

In classical adic geometry, formal moduli stacks are built from:

$$\mathcal{X} \colon (\mathsf{AdicAlg})^{\mathrm{op}} \to \mathsf{Groupoids}$$

assigning to each R the groupoid of formal geometric structures over R.

2. Symbolic Stacks from Logical Completions

Let $\mathsf{SymbAdicAlg}$ be the category of symbolic adic logic algebras.

Define:

$$\mathcal{X}^{(\mathsf{symb})} \colon (\mathsf{SymbAdicAlg})^{\mathrm{op}} \to \mathsf{Groupoids}$$

sending each A to the groupoid of symbolic logic structures completed along $I_\bullet^{(\mathrm{symb})} \subset A$.

Examples.

- $\mathcal{X}^{(\mathsf{AI})}(A) =$ AI proof flow bundles over formal symbolic bases;
- $\mathcal{X}^{(\zeta)}(A) =$ symbolic zeta-trace stacks with adic topology;
- $\mathcal{X}^{(\infty)} = \varinjlim_n \mathcal{X}_n^{(\mathsf{symb})}$, a transfinite symbolic stack tower.

3. Higher Completion Sites and ∞-Stacks

We define the **infinite symbolic completion site**:

$$(\mathsf{SymbSite}_\infty, \tau_{\mathrm{adic}})$$

whose objects are symbolic logic universes completed to infinite depth:

$$\mathsf{Lang}_\infty := \varprojlim_n \mathsf{Lang}/I_n^{(\infty)}$$

Sheaves over this site encode:

- AI knowledge propagation structures;
- Logical spaces with infinitesimal symbolic resolution;

- Recursive AI orbits stored across infinity-completion strata.

4. Symbolic AI Topos and Stack Geometry

We define the symbolic AI topos:

$$\mathsf{Shv}_{\mathrm{AI}} := \mathsf{Shv}(\mathsf{SymbSite}_\infty)$$

and a stack:

$$\mathcal{X}_{\mathrm{AI}} := \infty\text{-stack of symbolic adic AI agents with thickened logic flow.}$$

This stack supports:

- AI-based convergence theorems;
- Internal proof-theoretic cohomology;
- Dynamical logical completion landscapes.

5. Symbolic Motivic Adic Universe

Let:

$$\mathbb{S}^{\mathrm{adic}} := \text{the symbolic motivic universe over adic stacks}$$

This is defined via:

- Completion fields indexed by logic valuation;
- AI reflection layers as derived infinitesimal sites;
- Sheaves of symbolic curvature (zeta-flow, trace, deformation);
- Infinity-glued stack spaces encoding universal symbolic inference.

6. Final Summary of Volume III

In this final volume, we constructed:

- The theory of ideal-adic symbolic completion;
- Formal schemes and infinitesimal logic neighborhoods;
- Recursive thickening, convergence, and AI reflection structures;
- Symbolic stacks and higher sites completing proof universes;
- The symbolic AI topos as a geometric realization of learning and logic evolution.

This closes the third and final part of the foundational *Symbolic Completion Trilogy*. With valuation-based, congruence-based, and ideal-adic symbolic completions now fully developed, we are prepared to build the next phase: symbolic motives, zeta dynamics, and categorical universes of infinite proof.

Bibliography

[1] M. Artin and B. Mazur. *Formal Groups and Formal Moduli*. Harvard University Notes, 1969.

[2] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean Analysis*. Grundlehren der mathematischen Wissenschaften, Vol. 261. Springer, 1984.

[3] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique (EGA)*. Publications Mathématiques de l'IHÉS, 1960–1967.

[4] J. Lurie. *Derived Algebraic Geometry*. Preprint Series I–X, available at `https://www.math.ias.edu/~lurie/`.

[5] J. Lurie. *Spectral Algebraic Geometry*. Available at `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`, 2018.

[6] M. Demazure and A. Grothendieck. *Schémas en groupes (SGA 3)*. Lecture Notes in Mathematics, Vols. 151–153. Springer, 1970.

[7] V. Drinfeld and D. Gaitsgory. *On Some Finiteness Questions for Algebraic Stacks*. Geom. Funct. Anal. 23 (2013), no. 1, 149–294.

[8] D. Gaitsgory and N. Rozenblyum. *A Study in Derived Algebraic Geometry*, Vols. I–II. Mathematical Surveys and Monographs, AMS, 2017.

[9] P. Scholze. *p-adic Geometry and the Pro-étale Topology*. Astérisque 369 (2015), 287–321.

[10] P. J. S. Yang. *Universal Congruence Completion Program (UCCP)*. Manuscript, 2025.

[11] P. J. S. Yang. *Valuation-Based Completion Theory in Symbolic Arithmetic*. UCCP Volume I, 2025.

[12] P. J. S. Yang. *Congruence-Based Symbolic Completion*. UCCP Volume II, 2025.

[13] P. J. S. Yang. *Ideal-Adic Completion Theory in Symbolic Arithmetic*. UCCP Volume III, 2025.

[14] P. J. S. Yang. *Symbolic AI Topos and Adic Stack Universes*. In preparation, 2025.