GENERALIZING SCHNIRELMANN-TYPE DENSITY TO TOPOLOGICAL AND MEASURABLE GROUP STRUCTURES

PU JUSTIN SCARFY YANG

Abstract. We develop analogues of Schnirelmann density for topological groups, locally compact groups, and groups equipped with Haar measure. This includes integration-based density, open neighborhood approximations, and applications to ergodic theory and measure-preserving systems.

1. Introduction

While Schnirelmann density has been studied extensively in discrete arithmetic, its extension to topological or measurable groups remains underdeveloped. We propose a framework to generalize Schnirelmann-type density to groups with topology and measure, using Haar measure, local neighborhoods, and invariant means.

2. Topological Density Concepts

Let G be a topological group with Borel σ-algebra and Haar measure μ.

Definition 2.1 (Haar Schnirelmann Density)**.** Let $A \subseteq G$ be measurable. Define

$$\sigma_\mu(A) := \inf_{K\subseteq G,\,\text{compact}} \frac{\mu(A\cap K)}{\mu(K)}.$$

Remark 2.2. This is a topological analogue of Schnirelmann density, based on relative inner measures over compact sets.

Definition 2.3 (Local Density in Neighborhoods)**.** Let $U \ni e$ be a symmetric open neighborhood. Define

$$\sigma_U(A) := \inf_{x\in G} \frac{\mu(A\cap xU)}{\mu(U)}.$$

Proposition 2.4. *If A is syndetic (i.e., $G = FA$ for finite F), then $\sigma_U(A) > 0$ for small U.*

3. Additive Closure and Ergodicity

Definition 3.1 (Haar Additive Closure)**.** Let $A \subseteq G$ be measurable. Define kA using group product. A is *k-Haar dense* if

$$\mu(kA) = \mu(G).$$

Theorem 3.2 (Ergodic Implication)**.** *Let $A \subseteq G$ be such that $\mu(A) > 0$ and G acts ergodically on itself by left translation. Then $kA = G$ for some k under convolution powers.*

Remark 3.3. This generalizes classical results of Følner sequences and applies to amenable groups.

Date: May 5, 2025.

4. Connections with Measure-Theoretic Entropy

Definition 4.1 (Entropy Density Estimate)**.** Define the additive entropy of A as

$$H(A) := -\int_G \log\left(\frac{d\mu_A}{d\mu}\right) d\mu_A,$$

where μ_A is the normalized restriction of μ to A.

Proposition 4.2. *Higher entropy correlates with more rapid growth of kA under convolution.*

5. Future Work

- Characterization of Schnirelmann-type density in locally compact non-abelian groups.
- Compact group analogues and torsion subgroup behaviors.
- Interactions with representation theory and harmonic analysis.
- Formalization in measure-theoretic ergodic systems and spectral decomposition.