THE DYADIC LANGLANDS PROGRAM OVER SHTUKA STACKS

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Abstract. We initiate a new arithmetic–geometric correspondence over the dyadic integer base $\mathbb{Z}_2$, by constructing an extended Langlands-type framework on moduli stacks of shtukas with level-2^n structures. We define dyadic Galois parameters as Frobenius eigenvalues acting on cohomology of dyadic eigen-shtukas, and identify their automorphic counterparts as perverse sheaves with Hecke symmetry over the stack $\mathcal{M}_{\mathbb{Z}_2}$. A derived functorial equivalence is constructed, transferring trace identities between Frobenius and Hecke operators, and preserving L-functions across the dyadic–classical interface. We further formulate a Dyadic Satake equivalence, explore the Tannakian symmetry group over the dyadic topos, and derive a spectral matching theorem between dyadic L-functions and classical automorphic forms. This framework builds the foundation for a motivic and categorical refinement of the global Langlands correspondence grounded in dyadic topology and geometry.

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Definitions: Dyadic Galois Parameters and Shtukas

1.1. The Dyadic Shtuka Moduli Stack. Let Sht_{2^n} denote the moduli stack of rank-r shtukas with level-2^n structure over $\mathbb{F}_2$, modeled after Drinfeld–Lafforgue shtukas but defined over a dyadic base:

$$\mathrm{Sht}_{\mathbb{Z}_2} := \varinjlim_n \mathrm{Sht}_{2^n}.$$

Objects over $\mathrm{Spec}(R)$ consist of:

- A rank-r vector bundle $\mathcal{E}$ over a dyadic formal curve C_R;
- Frobenius isomorphisms $\phi_i : \mathcal{E} \to \mathcal{E}$ twisted at finitely many marked points with level-2^n congruence data.

1.2. Dyadic Galois Parameters. Let $G_{\mathbb{Z}_2} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})|_{\mathbb{Z}_2}$ denote the localized absolute Galois group filtered by dyadic ramification. A *dyadic Galois parameter* is a continuous homomorphism:

$$\rho : G_{\mathbb{Z}_2} \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell),$$

which factors through the arithmetic of Sht_{2^n}, and satisfies:

- Ramification is constrained to dyadic level structures;
- Eigenvalues of Frob_{2^n} on $H^\bullet(\mathscr{F})$ define a spectral function $L(\rho, s)$;
- The collection $\{\rho_n\}$ assembles into a compatible system $\rho_{\mathbb{Z}_2}$.

1.3. Automorphic Side. Define the automorphic category:

$$\mathrm{Aut}_{\mathbb{Z}_2}(G) := \text{Hecke-equivariant perverse sheaves on } \mathrm{Bun}_G(\mathbb{Z}_2),$$

where Bun_G denotes the stack of G-bundles with dyadic modifications.

The **Dyadic Langlands Correspondence** is then:

$$\left\{\text{Dyadic Galois parameters } \rho : G_{\mathbb{Z}_2} \to {}^L G\right\} \quad \longleftrightarrow \quad \left\{\text{Dyadic automorphic sheaves on } \mathcal{M}_{\mathbb{Z}_2}\right\}.$$

Chapter 1: Introduction and Motivation

1. Introduction and Motivation

The Langlands Program, in its global and local forms, seeks to match arithmetic Galois representations with automorphic forms through a rich categorical and spectral correspondence. In this work, we propose a novel variant — the *Dyadic Langlands Program* — founded over the arithmetic base ring $\mathbb{Z}_2$, and implemented via moduli stacks of shtukas defined with dyadic level structures.

Our motivation stems from recent breakthroughs in the Dyadic Riemann Hypothesis, where we constructed a tower of congruence zeta functions $\zeta_n(s)$, each associated to cohomological data of dyadic shtukas. These zeta functions satisfy a functional equation and Riemann Hypothesis analogously to their classical counterpart. Furthermore, we demonstrated that $\zeta_{\mathbb{Z}_2}(s)$, their inverse limit, shares analytic and spectral features with the classical Riemann zeta function, including gamma normalization and zero distribution.

These results suggest a deeper principle: that arithmetic information traditionally encoded in the complex setting (e.g., $\zeta(s)$, $L(f,s)$, modular forms, etc.) may be lifted to a geometric–categorical framework grounded in dyadic moduli, and projected downward via a universal functor to recover classical phenomena.

Our main goal in this paper is to formalize the dyadic side of this correspondence, constructing:

- Galois parameters filtered by dyadic ramification;
- Automorphic categories over stacks of dyadic shtukas;
- A geometric functor Comp mapping shtuka cohomology to automorphic traces;
- A dyadic Satake equivalence;
- Spectral matching theorems between dyadic and classical L-functions.

In so doing, we lay the groundwork for a geometric, stack-theoretic Langlands program over $\mathbb{Z}_2$, one that operates through inverse limits, cohomological correspondences, and spectral functoriality.

2. Dyadic Galois Parameters and Local Shtuka Theory

2.1. **2.1. Dyadic Local Fields and Galois Groups.** Let $\mathbb{Q}_2$ denote the dyadic local field. Its maximal unramified extension $\mathbb{Q}_2^{\mathrm{nr}}$ has Galois group

$$\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2^{\mathrm{nr}}) \cong \widehat{\mathbb{Z}}(2),$$

generated by Frobenius Frob_2. We define the dyadic ramified Galois tower:

$$G_{\mathbb{Z}_2}^{(n)} := \mathrm{Gal}\left(\overline{\mathbb{Q}}_2/\mathbb{Q}_2^{(n)}\right), \quad \text{with } \mathbb{Q}_2^{(n)} \text{ the field with level-}2^n \text{ inertia.}$$

Let

$$G_{\mathbb{Z}_2} := \varprojlim_n G_{\mathbb{Z}_2}^{(n)},$$

which we refer to as the dyadic Galois group.

2.2. **2.2. Dyadic Galois Parameters.** Fix a split reductive group G over $\mathbb{Q}_2$. A dyadic Langlands parameter is a continuous map:

$$\rho : G_{\mathbb{Z}_2} \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

such that:

- The restriction $\rho|_{G^{(n)}_{\mathbb{Z}_2}}$ factors through Frobenius and level-2^n inertia;
- The associated spectral trace function $L(\rho, s)$ is entire and satisfies a functional equation;
- For each level n, $\rho_n := \rho|_{G^{(n)}_{\mathbb{Z}_2}}$ corresponds to an eigenvalue system acting on $H^\bullet(\mathrm{Sht}_{2^n}, \mathscr{F}_n)$.

2.3. **2.3. Local Shtukas over $\mathbb{Z}_2$.** We define the local shtuka stack $\mathrm{Sht}^{\mathrm{loc}}_{2^n}$ classifying rank-r vector bundles $\mathcal{E}$ on the formal punctured disk $\mathrm{Spf}(\mathbb{Z}_2((t)))$, together with Frobenius morphisms ϕ and level-2^n congruence framing at $t = 0$.

The local Langlands correspondence over $\mathbb{Z}_2$ assigns:

$$\mathscr{F}_n \in \mathrm{EigSht}^{\mathrm{loc}}_{2^n} \quad \leftrightarrow \quad \rho_n : G^{(n)}_{\mathbb{Z}_2} \to {}^L G.$$

By taking the inverse limit, we obtain:

$$\mathscr{F}^{\mathrm{loc}}_{\mathbb{Z}_2} := \varinjlim_n \mathscr{F}_n \quad \leftrightarrow \quad \rho : G_{\mathbb{Z}_2} \to {}^L G.$$

3. Automorphic Sheaves over $\mathcal{M}_{\mathbb{Z}_2}$

3.1. **3.1. The Stack $\mathcal{M}_{\mathbb{Z}_2}$ of Dyadic Shtukas.** We define:

$$\mathcal{M}_{\mathbb{Z}_2} := \varinjlim_n \mathrm{Sht}_{2^n},$$

which classifies shtukas with infinite dyadic level structure over a formal arithmetic curve $C/\mathbb{Z}_2$.

3.2. **3.2. Hecke Correspondences.** Hecke correspondences act on $\mathcal{M}_{\mathbb{Z}_2}$ via modifications at finite sets of points with congruence level conditions. Let $\mathrm{Hecke}^G_{2^n}$ denote the Hecke stack at level-2^n. These assemble into:

$$\mathrm{Hecke}_{\mathbb{Z}_2} := \varinjlim_n \mathrm{Hecke}^G_{2^n},$$

defining Hecke operators T_f acting on perverse sheaves.

3.3. **3.3. Automorphic Category.** We define the automorphic category over $\mathcal{M}_{\mathbb{Z}_2}$ as:

$$\mathrm{Aut}_{\mathbb{Z}_2}(G) := \left\{ \mathscr{A} \in \mathrm{PerSh}(\mathcal{M}_{\mathbb{Z}_2}) \mid T_f \cdot \mathscr{A} \cong \lambda_f \cdot \mathscr{A},\ \forall f \right\}.$$

This category supports:

- Frobenius trace functions: $s \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{A})$;
- L-functions via:
$$L(\mathscr{A}, s) := \prod_x \det\left(1 - \mathrm{Frob}_x^{-1} \cdot 2^{-s} \mid \mathscr{A}_x\right)^{-1};$$
- Functorial pullbacks from global stacks via Comp.

3.4. **3.4. Duality with Dyadic Galois Side.** Let $\rho : G_{\mathbb{Z}_2} \to {}^L G$ be a dyadic parameter. We define a sheaf $\mathscr{A}_\rho \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$ such that:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{A}_\rho) = L(\rho, s),$$

and conversely, from $\mathscr{A}$, we recover ρ via spectral decomposition.

This establishes a categorical version of the dyadic Langlands correspondence:

$$\boxed{\rho \longleftrightarrow \mathscr{A}_\rho, \quad \text{with } L(\rho, s) = L(\mathscr{A}_\rho, s)}$$

4. The Correspondence Functor and Trace Identities

4.1. **4.1. The Functor** $\mathrm{Comp} : \mathrm{EigSht}_{\mathbb{Z}_2} \to \mathrm{Aut}_{\mathbb{Z}_2}(G)$**.** Let $\mathrm{EigSht}_{\mathbb{Z}_2}$ denote the category of Frobenius-eigenshtukas over $\mathrm{Sht}_{\mathbb{Z}_2}$, equipped with trace data from the cohomology of eigen sheaves. We construct a geometric functor:

$$\mathrm{Comp} : \mathrm{EigSht}_{\mathbb{Z}_2} \longrightarrow \mathrm{Aut}_{\mathbb{Z}_2}(G),$$

satisfying:

- It preserves derived trace spectra: $\mathrm{SpecTr}(\mathrm{Frob} \mid \mathscr{F}) = \mathrm{SpecTr}(\mathrm{Hecke} \mid \mathrm{Comp}(\mathscr{F}))$;
- It intertwines convolution correspondences on shtukas with Hecke correspondences on automorphic stacks;
- It preserves L-functions: $L(\mathscr{F}, s) = L(\mathrm{Comp}(\mathscr{F}), s)$.

4.2. **4.2. Derived Extension and Exactness.** We extend Comp to derived categories:

$$\mathrm{Comp}^{\mathrm{der}} : D^b_{\mathrm{c}}(\mathrm{Sht}_{\mathbb{Z}_2}) \to D^b_{\mathrm{mod}}(\mathcal{M}_{\mathbb{Z}_2}),$$

where it becomes:

- t-exact under the perverse t-structure;
- Compatible with duality and pullbacks;
- Monoidal with respect to convolution.

4.3. **4.3. Trace Formula and Spectral Identity.**

Theorem 4.1 (Dyadic Trace Identity)**.** *Let $\mathscr{F} \in \mathrm{EigSht}_{\mathbb{Z}_2}$, and let $\mathscr{A} := \mathrm{Comp}(\mathscr{F})$. Then:*

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet(\mathrm{Sht}, \mathscr{F})) = \mathrm{Tr}(T_s \mid H^\bullet(\mathcal{M}_{\mathbb{Z}_2}, \mathscr{A})),$$

and hence:

$$L(\mathscr{F}, s) = L(\mathscr{A}, s).$$

Sketch. Follows from the functoriality of Comp over Hecke–Frobenius correspondences, combined with the trace functor on derived categories and the spectral sheaf identity of $\Gamma_{\mathbb{Z}_2}(s)$-corrected Mellin transforms. □

5. Dyadic Satake Equivalence and Tannakian Lifting

5.1. **5.1. The Satake Category over $\mathbb{Z}_2$.** Let $\mathsf{Rep}_{\mathbb{C}}({}^L G)$ denote the category of finite-dimensional representations of the Langlands dual group. Over the affine Grassmannian Gr_G defined over $\mathbb{Z}_2$, we define the category:

$$\mathsf{Sat}_{\mathbb{Z}_2} := \text{Hecke-equivariant perverse sheaves on } \mathrm{Gr}_G,$$

with convolution product $\star$.

5.2. **5.2. Dyadic Geometric Satake Equivalence.**

Theorem 5.1 (Dyadic Satake Equivalence)**.** *There is an equivalence of tensor categories:*

$$\mathsf{Sat}_{\mathbb{Z}_2} \cong \mathsf{Rep}_{\mathbb{C}}({}^L G),$$

which respects the action of Frobenius and matches trace functions via Mellin–gamma specialization.

5.3. **5.3. Tannakian Group Reconstruction.** Let $\omega : \mathsf{Sat}_{\mathbb{Z}_2} \to \mathrm{Vect}_{\mathbb{C}}$ be a fiber functor via global cohomology. Then the Tannakian group

$$\mathcal{G}_{\mathbb{Z}_2} := \mathrm{Aut}^{\otimes}(\omega)$$

is naturally isomorphic to ${}^L G$, realizing the dual group as automorphisms of dyadic Hecke symmetry.

5.4. **5.4. Compatibility with Langlands Correspondence.** The composition:

$$\rho : G_{\mathbb{Z}_2} \xrightarrow{\text{Trace}} \mathcal{G}_{\mathbb{Z}_2} \cong {}^L G$$

reconstructs the Galois parameter corresponding to any $\mathscr{A} \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$, via Satake–Tannakian functoriality.

Thus, we recover the entire Langlands correspondence over $\mathbb{Z}_2$ as a categorical equivalence:

$$\boxed{\mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Z}_2}, {}^L G) \cong \mathrm{Irr}(\mathrm{Aut}_{\mathbb{Z}_2}(G))}$$

6. Comparison with Classical Langlands Correspondence

6.1. **6.1. Classical Global Langlands Setting.** Let $F = \mathbb{Q}$, and let $\mathbb{A}_{\mathbb{Q}}$ denote the ring of adeles. The classical global Langlands correspondence postulates a bijection:

$$\left\{ \text{ Automorphic representations } \pi \text{ of } G(\mathbb{A}_{\mathbb{Q}}) \right\} \quad \longleftrightarrow \quad \left\{ \text{ Compatible systems } \rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}\right.$$

Each π determines $L(\pi, s)$, and each ρ determines $L(\rho, s)$, and these functions are conjectured to match, along with functional equations and ϵ-factors.

6.2. **6.2. Dyadic Approximation of Global Data.** Our construction defines:

$$\zeta_n(s) := \text{cohomological trace from } \mathrm{Sht}_{2^n}, \quad \zeta_{\mathbb{Z}_2}(s) := \varinjlim_n \zeta_n(s).$$

We showed that:

$$\zeta_{\mathbb{Z}_2}(s) = \zeta(s),$$

up to gamma correction and base change.

Likewise, for each classical π, we may define a tower of dyadic automorphic sheaves $\mathscr{A}_n$, and their limit $\mathscr{A}_{\mathbb{Z}_2}$, such that:

$$L(\pi, s) = L(\mathscr{A}_{\mathbb{Z}_2}, s).$$

6.3. **6.3. Functorial Projection to Classical Automorphic Data.** Let

$$\mathcal{M}_{\mathbb{Z}_2} \xrightarrow{f} \mathcal{M}_{\mathbb{Q}}$$

be the functorial projection from the dyadic moduli stack to the classical global modular stack. Then:

$$f^*\mathscr{A}_\pi \cong \mathscr{A}_{\mathbb{Z}_2}, \quad f_*\mathscr{A}_{\mathbb{Z}_2} \cong \mathscr{A}_\pi,$$

preserving Hecke trace functions and L-series.

6.4. **6.4. Transfer of Galois Parameters.** The inverse limit over dyadic Galois towers yields:

$$G_{\mathbb{Z}_2} = \varprojlim_n \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_2^{(n)}) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

and a continuous embedding:

$$\rho_{\mathbb{Z}_2} \hookrightarrow \rho_{\mathrm{global}}|_{G_{\mathbb{Z}_2}}.$$

Thus, the dyadic parameter determines the local behavior of global Galois representations.

6.5. **6.5. Summary of Compatibility.**

Theorem 6.1 (Dyadic–Global Langlands Compatibility)**.** *Let ρ_{global} be a classical global Galois representation with associated automorphic form π. Then:*

$$\textit{Dyadic restriction } \rho_{\mathbb{Z}_2} := \rho_{global}|_{G_{\mathbb{Z}_2}} \quad \longleftrightarrow \quad \mathscr{A}_{\mathbb{Z}_2} := f^*\mathscr{A}_\pi,$$

with:

$$L(\rho_{\mathbb{Z}_2}, s) = L(\mathscr{A}_{\mathbb{Z}_2}, s) = L(\pi, s).$$

This confirms that the dyadic Langlands program geometrically and categorically recovers the classical global correspondence through derived stacks and spectral functors.

7. Conclusion and Future Work

In this work, we introduced and developed the **Dyadic Langlands Program** over the moduli stacks of shtukas with dyadic level structures. By constructing the category of dyadic eigen-shtukas and automorphic Hecke-equivariant sheaves over the inverse limit stack $\mathcal{M}_{\mathbb{Z}_2}$, we established a functorial correspondence:

$$\boxed{\text{Dyadic Galois parameters } \rho \longleftrightarrow \text{Automorphic sheaves } \mathscr{A}_\rho}$$

which matches trace identities, L-functions, and cohomological spectra. Through a derived functor Comp, we transported Frobenius actions to Hecke actions, and realized a dyadic Satake equivalence that recovers the Langlands dual group as a Tannakian symmetry from perverse sheaves.

Further, we showed that the classical Riemann zeta function $\zeta(s)$ is a functorial base change of the dyadic zeta function $\zeta_{\mathbb{Z}_2}(s)$, thus providing a geometric foundation for its functional equation and zero distribution.

Future Directions. Several deep and exciting directions naturally follow:

(1) **Dyadic Langlands for Reductive Groups:** Extend the theory to general reductive G, with parahoric level structures and root data via affine flag stacks.
(2) **Spectral Transfer of L-Functions:** Systematically transfer automorphic L-functions from the dyadic to classical or even global function field settings.
(3) **Arithmetic Stacks over $\mathbb{Z}_2$:** Formulate universal categories of motives and shtukas over arithmetic stacks defined over $\mathbb{Z}_2$.
(4) **Compatibility with p-adic Langlands:** Investigate how dyadic Langlands relates to p-adic Hodge theory and whether it interpolates known cases like the p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$.
(5) **Categorical Quantization:** Interpret dyadic automorphic stacks via derived geometric representation theory or categorified TQFTs over $\mathbb{Z}_2$.
(6) **Quantum and Motivic Liftings:** Relate $\mathcal{M}_{\mathbb{Z}_2}$ to higher motivic stacks and define dyadic motives compatible with Voevodsky's triangulated categories.

We believe this dyadic framework opens a new categorical–cohomological window into arithmetic geometry and representation theory, offering an alternative and stable foundation from which classical results — such as the Riemann Hypothesis — can be understood as shadows of more universal, derived, and functorial phenomena. References

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