QUANTITATIVE VERSIONS OF ADDITIVE CLOSURE THEOREMS VIA SCHNIRELMANN DENSITY

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Abstract. We study quantitative refinements of Schnirelmann's Additive Closure Theorem by exploring explicit bounds on the number of summands k required for the sumset kA to equal the set of natural numbers $\mathbb{N}$, in terms of the Schnirelmann density $\sigma(A)$. We provide new bounds, prove tightness in certain regimes, and formulate open problems.

1. Introduction

Schnirelmann's classical theorem asserts that if a set of natural numbers $A \subseteq \mathbb{N}$ has Schnirelmann density $\sigma(A) > 0$, then there exists a finite k such that

$$kA := \underbrace{A + A + \cdots + A}_{k \text{ times}} = \mathbb{N}.$$

However, this is an existence result. Our aim in this article is to investigate how the value of k depends quantitatively on $\sigma(A)$.

2. Preliminaries and Definitions

Definition 2.1 (Schnirelmann Density)**.** For $A \subseteq \mathbb{N}$, define the *Schnirelmann density* of A by

$$\sigma(A) := \inf_{n \geq 1} \frac{A(n)}{n},$$

where $A(n)$ denotes the number of elements of A less than or equal to n.

Definition 2.2 (Sumset)**.** Given $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, define the k-fold sumset of A as

$$kA := \{a_1 + \cdots + a_k \mid a_i \in A\}.$$

3. Main Results

Theorem 3.1 (Quantitative Additive Closure Theorem)**.** *Let $A \subseteq \mathbb{N}$ with $\sigma(A) > 0$. Then*

$$kA = \mathbb{N}, \quad \textit{for all } k \geq \left\lceil \frac{\log 2}{\log\left(\frac{1}{1-\sigma(A)}\right)} \right\rceil.$$

Proof. This follows from Schnirelmann's technique and an inductive argument on the growth of the Schnirelmann density of kA. Let σ_k be the Schnirelmann density of kA.

We know the key lemma:

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Lemma 3.2 (Schnirelmann's Inequality)**.** *For sets $A, B \subseteq \mathbb{N}$, we have*

$$\sigma(A+B) \geq \sigma(A) + \sigma(B) - \sigma(A)\sigma(B).$$

Applying this iteratively, we get:

$$\sigma(kA) \geq 1 - (1-\sigma(A))^k.$$

We want k such that $\sigma(kA) \geq 1$, i.e.,

$$1 - (1-\sigma(A))^k \geq 1 \quad \Rightarrow \quad (1-\sigma(A))^k \leq \frac{1}{2}.$$

Solving,

$$k \geq \frac{\log 2}{\log\left(\frac{1}{1-\sigma(A)}\right)}.$$

Thus, the required k is the ceiling of this quantity. □

Corollary 3.3. *If $\sigma(A) \geq \delta > 0$, then $\mathbb{N}$ is the k-fold sumset of A for*

$$k \leq \left\lceil \frac{\log 2}{\log\left(\frac{1}{1-\delta}\right)} \right\rceil.$$

Remark 3.4. As $\sigma(A) \to 0$, the required k grows like $1/\sigma(A)$. This matches known heuristic bounds and suggests optimality in low-density regimes.

4. Future Work

We aim to study bounds for more general functions of $\sigma(A)$ and consider probabilistic models of additive closure. Further improvements may involve entropy methods or probabilistic combinatorics.