META-DIFFERENT AND ENTROPY GEOMETRY OVER ARITHMETIC STACKS

PU JUSTIN SCARFY YANG

Abstract. We investigate the geometric and stack-theoretic realization of the meta-different complex, defining entropy sheaves over arithmetic stacks as categorified measures of arithmetic curvature and ramification energy. By interpreting the cone of the derived trace pairing as an entropy flow field, we construct a theory of entropy cohomology, entropy divisors, and curvature sheaves over stacks of Galois representations and arithmetic sites. This framework enriches classical ramification theory with homological and moduli-theoretic structure.

Contents

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1. Introduction: Entropy as Arithmetic Curvature

The classical different $\mathfrak{D}_{L/K}$ and discriminant $\Delta_{L/K}$ of a finite extension L/K of number fields serve as global and local invariants of ramification. In Articles 1 and 2, we introduced the meta-different $\mathbb{D}^{\mathrm{meta}}_{L/K}$ as the cone of the derived trace pairing, and its determinant, the meta-discriminant $\Delta^{\mathrm{meta}}_{L/K}$.

This article develops a geometric interpretation of $\mathbb{D}^{\mathrm{meta}}$ in terms of arithmetic entropy and derived curvature. Inspired by analogies with energy–momentum tensors in physics and trace stress in topology, we define entropy sheaves as categorified ramification flows.

Over arithmetic stacks, such as moduli of Galois representations, these sheaves encode localized arithmetic fluctuation and obstruction to étaleness. We construct entropy cohomology groups, entropy divisors, and entropy curvature fields that extend classical ramification loci.

2. Meta-Different Revisited: Stacky View and Trace Cone

2.1. Definition Recap.

Given a finite extension L/K of number fields, let $\mathcal{O}_L$, $\mathcal{O}_K$ be their rings of integers. The trace pairing:

$$\mathrm{Tr}_{L/K} : \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \longrightarrow \mathcal{O}_K$$

extends to a morphism in the derived category:

$$\mathrm{Tr}^{\bullet} : \mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_K.$$

We define:

$$\mathbb{D}^{\mathrm{meta}}_{L/K} := \mathrm{cone}(\mathrm{Tr}^{\bullet})[-1].$$

This object lives in $D^{\mathrm{perf}}(\mathcal{O}_K)$ and reflects the derived failure of symmetry in L/K. It measures, in particular, higher ramification data beyond the classical different ideal.

2.2. **Moduli Stacks of Extensions.** Let $\mathcal{M}_{\mathrm{Ext}}$ denote the moduli stack of rank-n finite locally free $\mathcal{O}_K$-algebras. Over this stack, there exists a universal extension:

$$\mathcal{A} \to \mathcal{R},$$

and we may define a universal trace morphism:

$$\mathrm{Tr}^\bullet : \mathcal{A} \overset{L}{\otimes}_{\mathcal{R}} \mathcal{A} \to \mathcal{R}.$$

Definition 2.1. The *universal meta-different complex* on $\mathcal{M}_{\mathrm{Ext}}$ is:

$$\mathbb{D}^{\mathrm{meta}} := \mathrm{cone}(\mathrm{Tr}^\bullet)[-1] \in D^{\mathrm{perf}}(\mathcal{M}_{\mathrm{Ext}}).$$

This construction is functorial in families and can be pulled back to base schemes, arithmetic moduli, or stacks of Galois representations.

2.3. **Arithmetic Ramification via Derived Support.** We define the derived ramification locus of an extension as the support of the cone complex:

$$\mathrm{Ram}^{\mathrm{meta}}_{L/K} := \mathrm{Supp}\left(\mathbb{D}^{\mathrm{meta}}_{L/K}\right).$$

This subscheme (or substack in moduli setting) refines the classical ramification divisor and detects depth of wildness, residual torsion, and cohomological obstruction.

Example 2.2. Let $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_p)$ for p prime. Then $\mathbb{D}^{\mathrm{meta}}_{L/K}$ detects nontrivial higher-order terms in the ramification filtration at p.

3. Entropy Sheaves: Categorified Ramification Fields

3.1. **Definition of Entropy Sheaves.** We define the *entropy sheaf* associated to a finite extension L/K as the derived object:

$$\mathcal{E}_{L/K} := \mathbb{D}^{\mathrm{meta}}_{L/K} = \mathrm{cone}(\mathrm{Tr}^\bullet)[-1] \in D^{\mathrm{perf}}(\mathcal{O}_K).$$

Definition 3.1. The entropy sheaf $\mathcal{E}_{L/K}$ is a categorified flow field encoding the homological resistance of $\mathcal{O}_L$ to the symmetry imposed by the trace pairing. It generalizes the notion of "ramification pressure" across $\mathrm{Spec}(\mathcal{O}_K)$.

3.2. **Properties and Functoriality.**

- **Locality**: $\mathcal{E}_{L/K}$ is compatible with localization at primes $\mathfrak{p}$ of $\mathcal{O}_K$.
- **Base Change**: For $K \to K'$ flat, $\mathcal{E}_{L/K} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \simeq \mathcal{E}_{L\otimes_K K'/K'}$.
- **Determinant**: $\det(\mathcal{E}_{L/K}) = \Delta^{\mathrm{meta}}_{L/K}$, recovering the meta-discriminant.

3.3. **Interpretation as Ramification Energy Field.** In analogy with stress tensors in differential geometry and entropy gradients in thermodynamics, we interpret $\mathcal{E}_{L/K}$ as a curvature-density complex describing how ramification "curves" the arithmetic geometry of $\mathcal{O}_L$ over $\mathcal{O}_K$.

$$\boxed{\text{Entropy} = \text{Ramification Curvature} = \partial^2(\text{Arithmetic Deviation})}$$

Example 3.2. For wildly ramified L/K, $\mathcal{E}_{L/K}$ contains higher cohomology and torsion, while for unramified extensions, it is acyclic.

3.4. **Entropy Covariant Structure.** Let $f : X \to Y$ be a morphism of arithmetic stacks or schemes representing an extension. Then:

$$f^*\mathcal{E}_Y \longrightarrow \mathcal{E}_X$$

defines an *entropy covariant flow*, measuring curvature propagation across morphisms.

This structure induces a tensorial category of entropy sheaves with:

- Pullback: entropy curvature transport,
- Pushforward: entropy compaction,
- Tensor: entropy interference fields.

4. Entropy Flows, Divisors, and Derived Curvature

4.1. **Entropy Flow Fields.** We define an entropy flow field $\mathscr{F}_{L/K}$ as the map:

$$\mathscr{F}_{L/K} := \nabla_{\mathrm{Tr}} := d\,\mathrm{Tr}^\bullet$$

interpreted as a "covariant differential" of the trace morphism.

Then:

$$\mathrm{Ker}(\mathscr{F}_{L/K}) \subseteq \text{symmetrizable directions}, \quad \mathrm{Coker}(\mathscr{F}_{L/K}) \simeq \mathcal{E}_{L/K}.$$

4.2. **Entropy Divisors.** We define the *entropy divisor* $\mathrm{Div}^{\mathrm{ent}}_{L/K}$ as the derived support:

$$\mathrm{Div}^{\mathrm{ent}}_{L/K} := \mathrm{Supp}(\mathcal{E}_{L/K}).$$

This refines the classical discriminant divisor and controls where ramification causes arithmetic divergence from symmetry.

4.3. **Entropy Curvature Tensor.** We define the entropy curvature $\mathcal{R}_{L/K}$ formally as:

$$\mathcal{R}_{L/K} := d^2 \mathcal{E}_{L/K}$$

interpreted in the dg-category of complexes over $\mathcal{O}_K$ or over an arithmetic stack.

This curvature controls:
- Failure of linear propagation of ramification;
- Entropic concentration at wildly ramified points;
- Derived deviation from classical étale topology.

4.4. **Entropy Lorentz Structure (Optional).** In derived arithmetic sites with period sheaf enhancement, we conjecture the existence of an entropy pseudo-metric:

$$g^{\mathrm{ent}} : \mathcal{E}_{L/K} \otimes \mathcal{E}_{L/K} \to \mathcal{O}_K[-1],$$

making $\mathcal{E}_{L/K}$ into a sheaf-theoretic analog of a Lorentzian tangent bundle.

5. Entropy Cohomology and Duality Theories

5.1. **Entropy Cohomology Groups.** Given an entropy sheaf $\mathcal{E}_{L/K}$ over $\mathrm{Spec}(\mathcal{O}_K)$ or over a more general base arithmetic stack $\mathcal{X}$, we define the entropy cohomology groups as:

$$\mathbb{H}^i_{\mathrm{ent}}(K, L) := \mathbb{H}^i(\mathcal{X}, \mathcal{E}_{L/K}).$$

These groups quantify obstruction to trace-symmetry at different cohomological levels:

- $\mathbb{H}^0$: classical trace-symmetry deviations (different);
- $\mathbb{H}^1$: wild ramification and local torsion;
- $\mathbb{H}^{>1}$: derived residual entropy and stacky effects.

5.2. **Entropy Duality.** Let $f : \mathcal{O}_L \to \mathcal{O}_K$ be a finite flat extension, and $\mathcal{E}_{L/K}$ its entropy sheaf.

We define the entropy duality pairing:

$$\langle -, - \rangle_{\mathrm{ent}} : \mathcal{E}_{L/K} \otimes \mathcal{E}_{L/K}^{\vee} \to \mathbb{Z}[-1],$$

and obtain the entropy Serre duality:

$$\mathbb{H}^i_{\mathrm{ent}}(K, L) \cong \mathbb{H}^{1-i}_{\mathrm{ent}}(K, L)^{\vee}.$$

5.3. **Entropy Conductors and Ramification Indices.** We define the entropy conductor as:

$$\mathfrak{f}_{\text{ent}}(L/K) := \text{ord}_{\mathfrak{p}}\left(\det(\mathcal{E}_{L/K})\right),$$

which refines the Artin conductor and measures derived ramification depth.

6. Future Directions

6.1. **Entropy Motives and Categorified Artin Maps.** We propose the existence of an entropy motive sheaf $\mathcal{M}^{\text{ent}}_{L/K}$ such that:

$$\mathcal{E}_{L/K} \simeq \mathcal{H}^1(\mathcal{M}^{\text{ent}}_{L/K}),$$

with motivic realization in a suitable category of derived Galois sheaves.

This leads to the conjectural entropy Artin reciprocity:

$$\text{Gal}^{\text{ab}}(K) \longrightarrow \text{Pic}^{\text{ent}}(\mathcal{X}_K),$$

mapping abelian Galois data to entropy line bundles.

6.2. **Entropy Langlands Program.** We expect entropy sheaves to appear naturally in categorified Langlands settings, especially in:
- Moduli of local systems with wild ramification;
- Stacks of flat bundles over arithmetic curves;
- Fourier-entropy transforms of automorphic sheaves.

6.3. **Entropy Period Sheaves and Curvature Quantization.** Let $\mathcal{X}$ be a stack over $\text{Spec}(\mathbb{Z})$. We define the entropy period sheaf:

$$\mathscr{P}_{\text{ent}} := R\Gamma(\mathcal{X}, \mathcal{E}_{L/K}),$$

which plays the role of a curvature-quantized period ring.

This suggests an arithmetic analog of the Riemann–Hilbert correspondence:

$$\text{Entropy Representations} \longleftrightarrow \text{Entropy Sheaves}.$$

6.4. **Entropy Gerbes and Global Ramification Networks.** We propose a categorified global conductor theory using entropy gerbes:

$$\mathcal{G}^{\text{ent}}_{L/K} \in \mathsf{Gerbes}_{\mathbb{D}^{\text{meta}}}$$

with curvature defined by the trace failure field, and banded by motivic period groups.

Such objects could unify:
- Class field theory;

- Motives and L-functions;
- Derived automorphic recursion.

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