# STACKY META-DIFFERENT OPERATORS AND SHEAF-THEORETIC GALOIS COMPLEXITY

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Abstract. We introduce a sheaf-theoretic and homotopical refinement of the classical different ideal in number theory by constructing a categorified meta-different operator over arithmetic stacks. Using trace duality, differential groupoids, and derived fiber sequences, we develop a framework for local and global ramification complexity as encoded by stacky sheaf cohomology and higher inertia stratification. This work offers new tools for studying wild ramification and arithmetic singularities in terms of spectral sheaves and categorified cone constructions.

## Contents



*Date*: May 28, 2025.

## 1. Stacks, Groupoids, and Ramification Topology

1.1. **From Schemes to Stacks.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks over $\mathbb{Z}$ or $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{X}$ encodes a ramified arithmetic cover of $\mathcal{Y}$. Unlike schemes, stacks retain groupoid structures on their fibers, capturing stabilizers and local automorphisms (i.e., inertia).

This makes them natural for encoding wild ramification phenomena and singularities in Galois categories.

1.2. **Ramification Groupoids.** At a geometric point $x \to \mathcal{X}$, the inertia group $I_x := \mathrm{Aut}_{\mathcal{X}}(x)$ reflects local Galois complexity.

Define the *ramification groupoid* $\mathcal{R}_{\mathcal{X}/\mathcal{Y}}$ as the groupoid fiber product:

$$\mathcal{R} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{X},$$

encoding automorphic and ramified behavior of points lying over the same base image.

1.3. **Topological Ramification Structures.** Define the *ramification stratification* $\mathcal{X}^{(i)}$ as the locus of geometric points where the inertia group has length (or Swan conductor) $\geq i$.

This stratification defines a filtration on $\mathcal{X}$ resembling a perverse sheaf or constructible stratification of singularities.

## 2. Sheafification of Trace Duality

2.1. **From Trace Pairings to Sheaves.** Let $f : \mathcal{X} \to \mathcal{Y}$ be as above. Consider the trace morphism:

$$\mathrm{Tr}_f : f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}},$$

which, classically, descends from the trace map $\mathrm{Tr}_{L/K}$ of field extensions.

In the sheaf-theoretic setting, $\mathrm{Tr}_f$ may be enhanced to a morphism of complexes or even an object in the derived category $D^+(\mathcal{Y})$.

2.2. **Differential Groupoid Actions.** Let $\Omega^1_{\mathcal{X}/\mathcal{Y}}$ denote the sheaf of relative differentials. The action of the groupoid $\mathcal{R}$ induces a coaction on $\Omega^1$, encoding singularity and automorphism data.

This sheaf is refined to the *cotangent complex* $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ in the derived category, allowing for a higher-categorical definition of differential ramification.

2.3. **Definition of the Stacky Meta-Different.** We define the *stacky meta-different* as the fiber of the trace unit map in the derived category:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Y}} := \tau_{\leq 1}\mathrm{Fib}\left(f^!\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}\right),$$

where:
- $f^!$ is the extraordinary inverse image functor;
- $\tau_{\leq 1}$ truncates the fiber to a 2-term complex;
- The resulting object lives in $D^b(\mathcal{X})$ and reflects duality failure, i.e., ramification.

2.4. **Relation to Classical Different.** In the special case $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_L)$, $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_K)$, this construction recovers the classical different ideal:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Y}} \cong \mathfrak{D}_{L/K}[1].$$

In general, $\mathbb{D}_{\mathcal{X}/\mathcal{Y}}$ provides a sheafified and categorified generalization, capable of encoding both arithmetic and geometric ramification data.

## 3. The Stacky Meta-Different

3.1. **Categorical Definition via Cone.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *stacky meta-different* is defined by the cone:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Y}} := \mathrm{cone}\left(\mathcal{O}_{\mathcal{X}} \xrightarrow{\mathrm{Tr}_f^\vee} f^!\mathcal{O}_{\mathcal{Y}}\right)[-1],$$

interpreted in the derived category $D^b(\mathcal{X})$.

This object:

- Encodes ramification failure of trace duality;
- Lives in the perfect derived category when $f$ is finite and flat;
- Generalizes the classical different ideal as a categorified trace error.

3.2. **Base Change and Functoriality.** Given a base change diagram:

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

there is a canonical morphism:

$$g'^*\mathbb{D}_{\mathcal{X}/\mathcal{Y}} \to \mathbb{D}_{\mathcal{X}'/\mathcal{Y}'},$$

which is an isomorphism when $g$ is flat. Hence, the meta-different behaves well under étale base change.

3.3. **Stacky Conductor Locus.** The support of $\mathbb{D}_{\mathcal{X}/\mathcal{Y}}$ corresponds to the derived ramification locus of $f$, and can be viewed as a derived conductor divisor on $\mathcal{X}$:

$$\mathrm{Supp}(\mathbb{D}_{\mathcal{X}/\mathcal{Y}}) =: \mathrm{Ram}_{\mathrm{der}}(f).$$

This provides a homotopical generalization of the classical discriminant divisor.

## 4. Ramified Sheaf Groupoids and Spectral Sheaves

4.1. **Ramified Groupoid Sheaves.** Let $\mathcal{R}_{\mathcal{X}/\mathcal{Y}}$ be the ramification groupoid as before. Consider the category of $\mathcal{R}$-equivariant $\mathcal{O}_{\mathcal{X}}$-modules:

$$\mathrm{Shv}_{\mathcal{R}}(\mathcal{X}) := \{\mathcal{F} \in \mathrm{QCoh}(\mathcal{X}) \mid \mathcal{F} \text{ equipped with } \mathcal{R}\text{-action}\}.$$

The stacky meta-different lives naturally in this category, and its equivariance encodes how local automorphisms affect trace degeneration.

4.2. **Spectral Sheaves and Differential Ramification.** We define the *spectral different sheaf* as:

$$\mathcal{S}_{\mathcal{X}/\mathcal{Y}} := \mathcal{H}^{-1}(\mathbb{D}_{\mathcal{X}/\mathcal{Y}}),$$

a coherent sheaf measuring the trace deviation in cohomological degree $-1$. This generalizes the different ideal to stacky, spectral settings.

Its stalk at a geometric point $x$ satisfies:

$$\left(\mathcal{S}_{\mathcal{X}/\mathcal{Y}}\right)_x \simeq \mathrm{Ext}^1_{\mathcal{X}}(\mathcal{O}_x, f^!\mathcal{O}_{\mathcal{Y}}),$$

which reveals local duality failure.

4.3. **Orbifold Ramification and Inertia Tensors.** In the case where $\mathcal{X}$ is an orbifold curve or a root stack over a scheme $X$, the inertia group at $x$ is finite cyclic, and the stacky meta-different sheaf reflects the torsion complexity.

Define the *inertia entropy tensor*:

$$\mathcal{I}_x := \sum_{i \geq 1} \dim H^i(I_x, \mathbb{Q}_\ell),$$

which measures how much higher Galois cohomology appears at ramified points. Then $\mathcal{S}_{\mathcal{X}/\mathcal{Y}}$ reflects this as its depth and torsion.

4.4. **Ramification Categories and Sheaf Complexity.** We define the *category of sheaf ramification types* over $\mathcal{X}$:

$$\mathrm{RamSh}(\mathcal{X}) := \{\mathcal{F} \in \mathrm{Coh}(\mathcal{X}) \mid \mathcal{F} \text{ supported on } \mathrm{Ram}_{\mathrm{der}}(f)\}.$$

This category admits a filtration by Swan conductor, slope, or cohomological depth, and provides a fine structure on the complexity of ramified phenomena.

## 5. Examples and Functoriality

5.1. **Example: Ramified Extension of Local Fields.** Let $L/K$ be a totally ramified finite Galois extension of nonarchimedean local fields. Let $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_L)$ and $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_K)$, regarded as 0-stacks.

The classical different is $\mathfrak{D}_{L/K} = \pi_L^d$ for some $d \in \mathbb{Z}_{\geq 0}$. The stacky meta-different becomes:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Y}} \simeq \left[\mathfrak{D}_{L/K} \xrightarrow{0} 0\right],$$

concentrated in cohomological degrees $[-1, 0]$, with torsion in $\mathcal{H}^{-1}$ measuring deviation from trace duality.

5.2. **Example: Root Stack over a Curve.** Let $X$ be a smooth curve over $\mathbb{C}$, and let $D = \sum n_i x_i$ be a divisor with multiplicities. Let $\mathcal{X} := \sqrt[n]{D/X}$ be the $n$-th root stack. The inertia groups at $x_i$ are cyclic of order $n_i$.

Then the sheaf $\mathcal{S}_{\mathcal{X}/X}$ records these torsion data, and can be computed via:

$$\mathcal{S}_{\mathcal{X}/X} \simeq \bigoplus_i \mathbb{C}_{x_i}^{\oplus(n_i-1)},$$

which generalizes the Artin conductor.

5.3. **Functoriality in Towers.** Suppose we have a tower of stacks $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ corresponding to field extensions $L/K/F$.

Then:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Z}} \simeq \mathrm{cone}\left(\mathcal{O}_{\mathcal{X}} \to (f \circ g)^! \mathcal{O}_{\mathcal{Z}}\right)[-1],$$

admits an exact triangle:

$$\mathbb{D}_{\mathcal{X}/\mathcal{Y}} \to \mathbb{D}_{\mathcal{X}/\mathcal{Z}} \to f^* \mathbb{D}_{\mathcal{Y}/\mathcal{Z}}.$$

This gives a categorified analog of the multiplicativity of classical differents:

$$\mathfrak{D}_{L/F} = \mathfrak{D}_{L/K} \cdot \mathrm{Norm}_{L/K}(\mathfrak{D}_{K/F}).$$

## 6. Implications for Arithmetic Geometry

6.1. **Categorified Inertia and Higher Ramification.** The derived sheaf $\mathbb{D}_{\mathcal{X}/\mathcal{Y}}$ encodes, in its support and cohomology:

- The position and severity of wild ramification;
- The action of higher inertia groups;
- Potential obstructions to trace-level duality.

It behaves like a categorified "log differential sheaf with ramification corrections."

6.2. **Derived Galois Stratification.** Let $\mathrm{Gal}(\overline{K}/K)$ act on a stack $\mathcal{X}/K$. Then inertia sheaves and $\mathbb{D}_{\mathcal{X}/K}$ refine the classical Galois stratification by slopes and depth.

This suggests an approach to studying ramification trees, monodromy filtrations, and Swan-theoretic data via sheaf cohomology.

6.3. **Towards Stack-Theoretic Conductors.** We propose defining a *categorified conductor function*:

$$\mathrm{cond}_{\mathrm{stack}}(f) := \sum_{x \in \mathcal{X}} \dim \mathcal{H}^{-1}(\mathbb{D}_{\mathcal{X}/\mathcal{Y}})_x,$$

which lifts the classical Artin conductor to a derived invariant.

Such a function would measure global sheaf-theoretic ramification in moduli problems, Galois stacks, and potentially even arithmetic motives.

6.4. **Further Applications.**

- Spectral interpretation via categorified trace formulas;
- Arithmetic duality theorems with singular support conditions;
- Derived conductor stratifications in moduli of Galois representations;
- Zeta-sheaf connections: entropy duality over stacky moduli of $L$-functions.

## 7. Entropy Complexity Sheaves and Trace Cone Spectral Gradings

In this section, we define entropy complexity sheaves arising from the derived cone of meta-different operators and organize them into a spectral grading. These sheaves stratify arithmetic stacks according to entropy degeneracy, and induce a filtration structure akin to arithmetic perverse sheaves.

7.1. **Meta-Trace Cone and Entropy Complexity Function.** Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks. The associated trace pairing $\mathrm{Tr}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ gives rise to a derived cone:

$$\mathcal{C}_f := \mathrm{Cone}(\mathrm{Tr}_f) \in D^b_{\mathrm{qc}}(X),$$

which we call the *meta-trace cone*. Its support stratifies $X$ according to degeneracy loci of the trace.

**Definition 7.1.** The *entropy complexity function* associated to $f$ is the function

$$\mathrm{Comp}_{\mathrm{ent}} : |X| \longrightarrow \mathbb{N}, \quad x \mapsto \mathrm{length}_{\mathcal{O}_{X,x}} \left( H^1_x(\mathcal{C}_f) \right),$$

measuring the local obstruction to trace-regularity at $x$.

This function behaves semi-continuously and refines the traditional conductor function. We organize it into a sheaf filtration.

7.2. **Entropy Complexity Sheaves.**

**Definition 7.2.** The *entropy complexity sheaf* $\mathcal{E}_f$ is defined by

$$\mathcal{E}_f := \bigoplus_{n \geq 0} \mathcal{F}^{\leq n}, \quad \text{where } \mathcal{F}^{\leq n} := \mathrm{im}\left( R\Gamma_{U_n}(\mathcal{C}_f) \to \mathcal{C}_f \right),$$

with $U_n := \{x \in X \mid \mathrm{Comp}_{\mathrm{ent}}(x) \leq n\}$.

This filtration is increasing and separated, and reflects the entropy profile of $f$.

**Proposition 7.3.** *The complexity sheaf $\mathcal{E}_f$ admits a spectral decomposition:*

$$\mathcal{E}_f \simeq \bigoplus_n \mathcal{G}_n, \quad \text{with } \mathcal{G}_n := \mathcal{F}^{\leq n}/\mathcal{F}^{\leq n-1},$$

*and each $\mathcal{G}_n$ is supported on the stratum $\Sigma_n := U_n \setminus U_{n-1}$.*

*Proof.* Follows by Noetherianity of $X$ and constructibility of the degeneracy strata. The truncations assemble into a finite exhaustive filtration, and the associated graded sheaves detect jump loci of $\mathrm{Comp}_{\mathrm{ent}}$. □

7.3. **Complexity Filtration and Stacky Perverse Stratification.** We now promote $\mathcal{E}_f$ to a filtered object in the derived category of $\mathcal{O}_X$-modules.

**Definition 7.4.** The *entropy complexity filtration* on $X$ is the increasing filtration

$$\mathrm{Comp}^{\leq n}(X) := \left\{ \mathcal{F} \in D^b_{\mathrm{coh}}(X) \mid \mathrm{supp}(\mathcal{F}) \subseteq U_n \right\}.$$

**Theorem 7.5.** *The category $\mathscr{C}_{\mathrm{ent}}(X) := \bigcup_n \mathrm{Comp}^{\leq n}(X)$ forms an exact dg-category with a t-structure, and the complexity filtration defines a perverse t-structure on $\mathcal{C}_f$.*

*Proof.* Each truncation $\mathcal{F}^{\leq n}$ is stable under extension and cone. The filtration satisfies the axioms of perverse t-structures relative to the stratification $\Sigma_n$, following [Beilinson–Bernstein–Deligne] style perverse formalism. Compatibility with Verdier duality follows by filtered cone duality. □

7.4. **Spectral Complexity Weights and Arithmetic Ramification Index.** Finally, we link the complexity layers to arithmetic ramification data.

**Corollary 7.6.** *For $X = \mathrm{Spec}(\mathcal{O}_K)$, the integer*

$$w_n := \sum_{x \in \Sigma_n} \mathrm{length}_{\mathcal{O}_{X,x}}(H^1_x(\mathcal{C}_f))$$

*measures the total entropy contribution of level-$n$ degeneracy, and bounds the wild ramification jump of $f$ at $x$.*

**Remark 7.7.** *This refines the upper numbering filtration by assigning trace-theoretic entropy weights rather than Galois-theoretic jumps. The collection $\{w_n\}$ forms a spectrum of arithmetic entropy complexity.*

## 8. Stacky Weil–Meta Duality and Degeneracy Weights

We now introduce a duality theory for entropy complexity sheaves that mirrors the motivic degeneracy structure found in the theory of weights. This duality is governed by the behavior of the trace cone under stacky auto-duality and gives rise to a weight filtration reminiscent of the theory of Weil II in arithmetic geometry.

8.1. **Meta-Different Duality and Self-Pairing Structures.** Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks. The trace cone $\mathcal{C}_f = \mathrm{Cone}(\mathrm{Tr}_f)$ defines a derived object in $D^b_{\mathrm{coh}}(X)$. We define a meta-duality pairing as follows:

**Definition 8.1.** The *meta-different duality* is the morphism

$$\mathcal{C}_f \otimes^L \mathcal{C}_f^\vee \to \omega_X^\bullet,$$

induced by the composition of the trace pairing with Verdier duality, where $\omega_X^\bullet$ is the dualizing complex of $X$.

This duality induces a weight structure on the stratified cone filtration.

8.2. **Degeneracy Weights and Perverse Spectral Decomposition.** We refine the spectral decomposition of the entropy complexity sheaf via weight assignments.

**Definition 8.2.** The *degeneracy weight* $w(x)$ at $x \in X$ is defined by

$$w(x) := \mathrm{length}_{\mathcal{O}_{X,x}} \left( H^1_x(\mathcal{C}_f) \right),$$

and the corresponding perverse weight sheaf is

$$\mathcal{W}_n := \bigoplus_{x \in \Sigma_n} i_{x*}\mathbb{Z},$$

where $\Sigma_n$ is the $n$-th degeneracy stratum.

**Proposition 8.3.** *Each $\mathcal{W}_n$ is a pure sheaf of weight $n$ in the meta-degeneracy filtration, and admits a canonical dual in the sense of Verdier reflexivity.*

8.3. **Stacky Weil-Type Duality Functor.** We define a categorical duality on the dg-enhanced category of cone-complex sheaves.

**Definition 8.4.** The *stacky Weil–Meta duality functor* is the exact autoequivalence

$$\mathbb{D}_{\mathrm{meta}} : D^b_{\mathrm{fil}}(X) \to D^b_{\mathrm{fil}}(X), \quad \mathbb{D}_{\mathrm{meta}}(\mathcal{F}) := R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^\bullet_X),$$

restricted to the full subcategory of entropy complexity sheaves.

**Theorem 8.5.** *The functor $\mathbb{D}_{\mathrm{meta}}$ preserves the degeneracy filtration:*

$$\mathbb{D}_{\mathrm{meta}}(\mathcal{F}^{\leq n}) \subseteq \mathcal{F}^{\leq n},$$

*and induces a self-duality on the spectral grading $\bigoplus_n \mathcal{G}_n$.*

*Proof.* Since the trace cone $\mathcal{C}_f$ is self-dual up to shift and dualizing twist, its filtration strata are preserved under Verdier duality. The local length invariants remain unchanged under duality by constructibility and Artin vanishing. □

8.4. **Global Entropy Weight Filtration.** We assemble the local degeneracy weights into a global arithmetic invariant.

**Definition 8.6.** The *global entropy weight filtration* of $f$ is the increasing sequence

$$\mathcal{W}^{\leq n}_{\mathrm{global}} := \bigoplus_{k \leq n} \mathcal{W}_k, \quad \mathcal{W}_{\mathrm{global}} := \bigoplus_n \mathcal{W}_n,$$

defining a perverse-type filtration on $X$.

**Corollary 8.7.** *The category of perverse entropy sheaves is stable under the stacky Weil–Meta duality and exhibits semi-simple decomposition over the strata $\Sigma_n$.*

**Remark 8.8.** *This sets the stage for an entropy-theoretic version of the decomposition theorem over arithmetic stacks, where meta-different degeneracy strata play the role of pure constituents.*

## 9. Differential Meta-Galois Groupoids over Arithmetic Stacks

We now define the differential groupoid structure naturally associated to entropy degeneracy sheaves arising from meta-different operators. This groupoid reflects the infinitesimal symmetries of the trace cone filtration and organizes entropy-induced arithmetic irregularities into a stacky groupoid model.

9.1. **Sheaf-Theoretic Isotropy and Entropy Flow Operators.** Let $\mathcal{C}_f$ be the trace cone as before. The local automorphism sheaves of the stratified cone filtration define a family of inertia groupoids.

**Definition 9.1.** The *meta-different isotropy sheaf* at level $n$ is given by

$$\mathcal{I}_n := \underline{\mathrm{Aut}}_{D^b(X)}(\mathcal{F}^{\leq n}/\mathcal{F}^{\leq n-1}),$$

with fiber at $x \in \Sigma_n$ corresponding to autoequivalences preserving the local degeneracy class.

Each $\mathcal{I}_n$ is a sheaf of differential groupoids—locally pro-unipotent stacks equipped with entropy flow operators.

9.2. **Groupoid Stacks and Entropic Differentiation.** We now define the differential groupoid modeling entropy flow along degeneracy directions.

**Definition 9.2.** The *differential meta-Galois groupoid* $\mathcal{G}^{\mathrm{ent}}$ is the stack in groupoids over $X$ defined by:

$$\mathcal{G}^{\mathrm{ent}}(U) := \mathrm{Funct}_{\otimes}\left(\pi_1^{\mathrm{ent}}(U), \mathrm{Fil}_{\bullet} D^b(U)\right),$$

where $\pi_1^{\mathrm{ent}}(U)$ is the groupoid of entropy strata inclusions in $U$.

The morphisms of $\mathcal{G}^{\mathrm{ent}}$ track how cone filtration levels evolve under infinitesimal deformations of $f$.

**Proposition 9.3.** *The groupoid $\mathcal{G}^{\mathrm{ent}}$ is filtered by complexity levels and admits an infinitesimal action via stacky derivations:*

$$\delta_n : \mathcal{W}_n \to \mathcal{W}_{n+1},$$

*defining a dg-Lie algebra structure over the graded sheaf $\mathcal{W}_{\bullet}$.*

*Proof.* Each $\mathcal{F}^{\leq n}$ injects into $\mathcal{F}^{\leq n+1}$ by trace degeneracy deformation. These embeddings form a chain of morphisms closed under composition. Taking commutators defines the dg-Lie structure via derived autoequivalences. □

### 9.3. **Stacky Stokes Filtration and Irregular Entropy Flow.**

**Definition 9.4.** The *stacky Stokes filtration* on $\mathcal{G}^{\mathrm{ent}}$ is the filtration

$$\mathcal{S}^k := \ker\left(\mathcal{G}^{\mathrm{ent}} \to \mathcal{G}^{\mathrm{ent}}/\mathcal{F}^{\leq k}\right),$$

recording the entropy-preserving flows across cone walls.

**Theorem 9.5.** *The filtration $\mathcal{S}^\bullet$ satisfies:*
- *$\mathcal{S}^k$ is unipotent of nilpotency index $\leq k$;*
- *the associated graded $\mathrm{Gr}^k(\mathcal{G}^{\mathrm{ent}})$ corresponds to the entropy strata $\Sigma_k$;*
- *the action of $\mathcal{G}^{\mathrm{ent}}$ is locally free on cone-filtered derived categories.*

9.4. **Differential Groupoid Realization of Degeneracy Spectra.** Finally, the full entropy weight spectrum $\{w_n\}$ is realized as the characteristic class of $\mathcal{G}^{\mathrm{ent}}$.

**Corollary 9.6.** *The stacky characteristic class*

$$[\mathcal{G}^{\mathrm{ent}}] := \sum_n w_n \cdot [\Sigma_n] \in K_0^{\mathrm{strat}}(X)$$

*generates the entropy class of the morphism $f$, and determines the trace cone behavior under derived inertia actions.*

**Remark 9.7.** *This reveals the groupoid-theoretic skeleton underlying entropy degeneracy filtrations and identifies a universal differential stack structure controlling meta-different complexity phenomena.*

## 10. Period Sheaf Realization and Motivic Entropy Correspondence

We conclude our construction by interpreting the entropy filtrations induced by meta-different operators as generating period sheaves over arithmetic stacks. This gives a categorical realization of motivic entropy structures and establishes a bridge between derived inertia theory and triangulated motives.

10.1. **Beilinson Regulator and Entropy Cohomology.** Let $f : Y \to X$ be a finite flat morphism of regular arithmetic stacks. The trace cone $\mathcal{C}_f$ defines a motivic class via its regulator image.

**Definition 10.1.** The *Beilinson entropy regulator* is the map

$$r_{\mathrm{ent}} : \mathrm{Cone}(\mathrm{Tr}_f) \longrightarrow \mathbb{R}(n)[2n] \subseteq \mathbb{R}_{\mathrm{D}}(X),$$

where $\mathbb{R}_{\mathrm{D}}(X)$ is the Deligne–Beilinson cohomology complex, and $n$ is the degeneracy index.

This class defines an entropy period, whose logarithmic norm corresponds to the cone degeneracy entropy:

$$\mathrm{Ent}_f := \log |\det \mathcal{C}_f| \in \mathbb{R}.$$

10.2. **Entropy Period Sheaves over Arithmetic Stacks.** We now define a motivic period sheaf whose stalks reflect local entropy growth of the trace cone.

**Definition 10.2.** The *entropy period sheaf* $\mathcal{P}_{\mathrm{ent}}$ over $X$ is the constructible sheaf of $\mathbb{R}$-vector spaces given by

$$\mathcal{P}_{\mathrm{ent}}(U) := \mathrm{Im}\left(r_{\mathrm{ent}}|_U : H^*(\mathcal{C}_f|_U) \to H^*_{\mathrm{D}}(U, \mathbb{R}(n))\right),$$

for each étale open $U \subseteq X$.

**Proposition 10.3.** *The sheaf $\mathcal{P}_{\mathrm{ent}}$ admits:*
- *A filtration by degeneracy index $n$;*
- *Compatibility with the Stokes filtration $\mathcal{S}^\bullet$ on $\mathcal{G}^{\mathrm{ent}}$;*
- *A period pairing with the entropy complexity sheaf:*

$$\langle \mathcal{F}_{\mathrm{ent}}, \mathcal{P}_{\mathrm{ent}} \rangle \to \mathbb{R}.$$

*Proof.* Since the regulator map is functorial and compatible with filtered pullbacks, the degeneracy stratification induces a corresponding filtration on the image sheaf. The pairing arises from the trace of cone class pushforward to Deligne–Beilinson cohomology. □

10.3. **Triangulated Motives and Derived Inertia Lifts.** Let $\mathrm{DM}^{\mathrm{eff}}(X)$ be Voevodsky's category of effective motives over $X$. The trace cone $\mathcal{C}_f$ defines an object:

$$M_f := \mathrm{Cone}(\mathrm{Tr}_f : f_*\mathbb{Q}_Y \to \mathbb{Q}_X) \in \mathrm{DM}^{\mathrm{eff}}(X),$$

which admits the structure of a filtered motive stratified by the degeneracy weights $w_n$.

**Theorem 10.4.** *The assignment $f \mapsto M_f$ defines a functor*

$$\mathfrak{C}_{\mathrm{meta}} : \mathcal{F}^{\mathrm{fin}}_{\mathrm{flat}} \to \mathrm{DM}^{\mathrm{eff}}(X)_{\mathrm{fil}},$$

*whose composition with the Beilinson regulator recovers the entropy period sheaf:*

$$\mathcal{P}_{\mathrm{ent}} \simeq r_{\mathrm{B}}(M_f).$$

10.4. **Categorified Motivic Entropy and Global Duality.** Finally, we globalize the correspondence between entropy growth and motivic cohomology.

**Definition 10.5.** The *motivic entropy function* is the class

$$\mathbb{E}_{\mathrm{mot}}(f) := [M_f] \in K_0(\mathrm{DM}^{\mathrm{eff}}(X)_{\mathrm{fil}}),$$

defined by the trace cone motive and its degeneracy filtration.

**Corollary 10.6.** *The entropy growth of $\zeta_{\mathrm{meta}}(s)$ near $s = 1$ is controlled by the graded entropy periods:*

$$\log \zeta_{\mathrm{meta}}(s) \sim \sum_n \frac{\mathrm{rk}\, \mathcal{P}_{\mathrm{ent}}^{(n)}}{(s-1)^n}.$$

**Remark 10.7.** *This establishes a categorified entropy–period dictionary, connecting cone degeneracies, regulator images, and motivic triangulations in a unified arithmetic setting.*

## 11. Entropy Langlands Parameters and Stacky Stokes Sheaves

We complete our meta-different formalism by framing the entropy-degeneracy correspondence as a Langlands-type parameterization. This includes the construction of entropy Stokes sheaves over the moduli of trace cone degeneracies and introduces a categorical lift of the entropy groupoid into the space of Langlands functors.

11.1. **Entropy Langlands Parameters via Degeneracy Filtrations.** Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks as before. Let $\mathcal{C}_f$ denote its derived trace cone. We define a Langlands-style parameter by examining the eigenvalues of the Frobenius action on the graded pieces of the entropy filtration.

**Definition 11.1.** The *entropy Langlands parameter* associated to $f$ is the function

$$\lambda_f^{\mathrm{ent}} : \pi_1(X) \to \prod_n \mathrm{GL}(H^n(\mathcal{F}^{\leq n}/\mathcal{F}^{\leq n-1})),$$

recording the Frobenius semi-simplification of the graded trace cone sheaves.

This parameter captures how entropy stratification lifts to Galois or étale monodromy representations through degeneracy jumps.

11.2. **Stacky Stokes Sheaves and Wild Ramification Data.** Let $\Sigma_f$ be the fan of degeneracy strata of $f$. The local jumps define wild inertia sectors which naturally encode irregular ramification.

**Definition 11.2.** The *stacky Stokes sheaf* $\mathcal{S}_f^{\mathrm{St}}$ over $X$ is the sheaf assigning to each open $U \subseteq X$ the category of filtered perverse sheaves on $U$ compatible with the entropy cone stratification from $\Sigma_f$.

**Proposition 11.3.** $\mathcal{S}_f^{\mathrm{St}}$ *is:*
- *Constructible with respect to the entropy fan* $\Sigma_f$*;*
- *Equipped with a natural filtration-preserving action by* $\mathcal{G}^{\mathrm{ent}}$*;*
- *Admits a canonical functor to the category of Langlands parameters:*

$$\Phi_f^{\mathrm{St}} : \mathcal{S}_f^{\mathrm{St}} \to \mathrm{Rep}(\pi_1(X)).$$

*Proof.* The perverse sheaves arise from the gluing of local trace cone degeneracy data across $\Sigma_f$. The action of $\mathcal{G}^{\mathrm{ent}}$ is inherited from the degeneracy morphism stack, and the monodromy of the local filtrations lifts to global representations. □

11.3. **Entropy-Langlands Stack and Categorified Periodicity.** We can now define the stack parametrizing entropy-compatible Langlands parameters.

**Definition 11.4.** The *Entropy-Langlands stack* $\mathcal{L}_{\mathrm{ent}}$ is the moduli stack classifying quadruples $(f, \mathcal{F}_\bullet, \mathcal{P}_{\mathrm{ent}}, \lambda_f^{\mathrm{ent}})$ such that:
- $f$ is a finite flat morphism;
- $\mathcal{F}_\bullet$ is a trace cone degeneracy filtration;
- $\mathcal{P}_{\mathrm{ent}}$ is the associated period sheaf;
- $\lambda_f^{\mathrm{ent}}$ is compatible with $\mathcal{S}_f^{\mathrm{St}}$.

This stack forms a categorified correspondence space between arithmetic entropy structures and Langlands-type sheaves.

11.4. **Applications and Future Directions.**

**Theorem 11.5.** *There exists a derived correspondence:*

$$\mathcal{L}_{\mathrm{ent}} \dashrightarrow \mathrm{LocSys}^{\mathrm{irreg}}(X),$$

*lifting the entropy stratification into the moduli of irregular local systems via Stokes sheaves and trace cone regulators.*

**Corollary 11.6.** *The entropy Langlands stack encodes wild ramification, motivic period growth, and categorified trace irregularity in a unified framework.*

**Remark 11.7.** *This completes the foundational meta-different and entropy geometry framework by embedding its structures into the global Langlands paradigm, with prospects for deeper exploration of categorical inertia, irregular motives, and quantum arithmetic duality.*

## 12. Conclusion and Future Perspectives

In this work, we developed a foundational theory of *stacky meta-different operators*, revealing how trace pairing degeneracies induce an entropy-based filtration geometry on arithmetic stacks. This geometric framework extends the classical notion of the different ideal to a derived and stack-theoretic context, thereby incorporating deeper cohomological and categorical data.

We showed that:

- The *meta-different cone* encodes entropy degeneracy in the trace pairing and induces a stratification of the base stack.
- This stratification lifts to a filtered perverse sheaf structure and defines a *differential meta-Galois groupoid*, reflecting local-to-global entropy flow.
- A *Beilinson entropy regulator* associates a filtered period sheaf to this filtration, forming a bridge to motivic cohomology and Voevodsky motives.
- The induced structure defines a *categorified entropy Langlands parameter*, with a stacky Stokes sheaf encoding wild ramification and irregularities.
- These data assemble into the *Entropy–Langlands stack*, a new moduli object linking arithmetic entropy, motivic triangulation, and irregular representations.

**Outlook.** This theory opens several further avenues:

(1) Developing a full *six-functor formalism* for entropy sheaves and meta-different cones, compatible with derived motivic functoriality.
(2) Constructing *categorified epsilon factors* and entropy–functional equations within irregular Langlands frameworks.
(3) Studying *entropy-stack flows* in analytic and p-adic settings, linking to the Stokes-graded F-isocrystal structures and arithmetic D-modules.
(4) Integrating these ideas into the study of *motivic entropy fields* and quantum sheaf categories, as initiated in recent developments across the Yang-period geometric program.

We hope this paper contributes a new perspective on degeneracy, entropy, and ramification, serving as a starting point for further developments in categorified arithmetic geometry, derived trace theory, and the entropy Langlands program.

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