THE CATEGORICAL DISCRIMINANT AS A DERIVED INERTIA MEASUREMENT OVER ARITHMETIC STACKS

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Abstract. We propose a new definition of the discriminant of arithmetic and geometric morphisms using categorical and derived structures. The failure of étaleness and smoothness is interpreted via inertia stacks and the failure of full faithfulness in the derived category. We develop a notion of *categorical discriminant sheaves* that measure singularity, ramification, and arithmetic irregularity, generalizing classical invariants such as the discriminant ideal and Artin conductors.

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Date: May 29, 2025.

Abstract. We define a categorical and motivic refinement of the classical discriminant of a finite morphism of arithmetic stacks, based on the derived cone of the trace pairing. This *meta-discriminant* arises as a determinant line in the derived category and measures the failure of the trace to be perfect, reflecting ramification complexity. We introduce the *entropy zeta function* associated to this structure, interpret it geometrically via an *entropy–motivic stack*, and establish a functional equation analogous to the motivic trace formula. The theory yields a categorified Galois groupoid, a ramification gerbe structure, and a sheaf-theoretic Stokes stratification on the cone. These constructions suggest a new theory of entropy motives and provide a framework for interpreting L-functions and trace anomalies through derived and stack-theoretic tools.

1. Introduction

Discriminants occupy a central role in number theory and algebraic geometry, measuring how far an algebraic object is from being regular or unramified. Traditionally defined through the vanishing of the determinant of a trace pairing, the discriminant encodes the loci of ramification and singularity. Yet this classical notion is fundamentally numerical and lacks a categorical or homotopical interpretation.

In this paper, we develop a refined theory of the discriminant—what we term the *categorical discriminant* or *meta-discriminant*—arising from the failure of the trace pairing to be perfect, realized as a *derived cone* in the sense of homological algebra à la Weibel. This allows us to access the discriminant's internal structure via triangulated categories and derived geometry.

The derived cone becomes a powerful invariant: it lifts the degeneracy locus into the derived category, permits spectral decomposition, and admits motivic interpretation through determinant lines and entropy growth. We associate to this structure an *entropy zeta function*, recording degeneracy stratification across arithmetic strata, and show that it satisfies a duality functional equation.

This motivates the definition of the *entropy–motivic stack*, a derived moduli object encoding trace pairing flows, and the construction of a categorified Galois groupoid $\mathcal{G}_f^{\mathrm{ent}}$ classifying trace cone automorphisms. These tools unify notions from ramification theory, motivic cohomology, and categorical sheaf theory into a single framework.

From these perspectives emerge several foundational results:

- The entropy zeta function lifts to motivic cohomology via the Beilinson regulator.
- The trace cone admits a perverse sheaf stratification whose Stokes data corresponds to ramification jumps.
- The categorical discriminant induces a motivic Fourier duality and governs spectral trace flows.
- A new interpretation of local epsilon factors emerges from the monodromy gerbes of cone degeneracy.

We conclude by speculating on the emergence of a theory of *entropy motives*, where motivic filtrations arise dynamically from trace degeneracies, and on applications to categorified class field theory and derived ramification structures.

2. Revisiting the Discriminant through Stacks

2.1. **Classical Discriminant and Étale Failure.** Given a finite extension L/K of number fields, the classical discriminant $\Delta_{L/K}$ quantifies the non-smoothness of $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. It measures:

- Ramification: primes where $\mathcal{O}_L$ is not étale over $\mathcal{O}_K$;
- Failure of basis orthogonality in the trace form;
- Degeneracy of the determinant of the trace matrix.

2.2. **Inertia and Branch Loci in Stacks.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks. Define the *inertia stack*:

$$\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X},$$

encoding automorphisms of objects in the stack.

The *branch locus* $\mathrm{Br}(f)$ is the support of the cone of the unit map:

$$\eta_f : \mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}.$$

Its derived fiber detects where f fails to be fully faithful (i.e., not étale).

2.3. **Categorical Discriminant Definition.** We define the *categorical discriminant sheaf* of f as:

$$\Delta_f^{\mathrm{cat}} := \mathrm{Supp}\left(\mathrm{cone}\left(\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}\right)\right),$$

enhanced by higher Ext-sheaves:

$$\mathcal{D}_f^i := \mathcal{E}xt^i_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}}, f_*\mathcal{O}_{\mathcal{X}}),$$

for $i > 0$, capturing higher-dimensional non-étaleness.

2.4. **Failure of Full Faithfulness and Derived Obstructions.** The map f is étale iff the adjunction morphism $f^*f_* \to \mathrm{id}$ is an isomorphism.

The derived discriminant sheaf can thus be interpreted as the obstruction to full faithfulness:

$$\mathrm{cone}(\mathcal{O}_{\mathcal{X}} \xrightarrow{f^*f_*} \mathcal{O}_{\mathcal{X}}),$$

whose support defines the derived non-smooth locus.

3. Discriminant Sheaf and Inertia Action

3.1. **Discriminant as Support of Non-Étaleness.** For a morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$, we define the *categorical discriminant locus* as:

$$\mathrm{Disc}^{\mathrm{cat}}(f) := \mathrm{Supp}\left(\mathrm{cone}\left(\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}\right)\right),$$

which generalizes the classical discriminant ideal to a coherent support sheaf, sensitive to derived and stack-theoretic features.

In the case where f is representable and finite, this recovers the classical discriminant divisor.

3.2. **Inertia Stack and Monodromy.** The inertia stack $\mathcal{I}_{\mathcal{X}}$ carries information about local automorphisms and ramification data:

$$\mathcal{I}_{\mathcal{X}}(T) = \left\{(x, \alpha) \mid x \in \mathcal{X}(T), \alpha \in \mathrm{Aut}_{\mathcal{X}}(x)\right\}.$$

The fixed point loci of inertia under base change yield stratified ramification layers. The action of inertia on $f_*\mathcal{O}_{\mathcal{X}}$ creates higher cohomology terms.

3.3. **Categorical Monodromy Tensor.** We define the *categorical monodromy tensor* at a geometric point $\bar{y} \in \mathcal{Y}$ as:

$$\mathbb{M}_f(\bar{y}) := \mathrm{cone}\left(\mathbb{Q}_\ell \xrightarrow{\mathrm{id}} \mathrm{R}\Gamma(\mathcal{X}_{\bar{y}}, \mathbb{Q}_\ell)\right),$$

which detects the failure of full fiber triviality and reflects higher monodromy in the derived category.

This tensor measures "how non-trivial" the stacky structure is over the point $\bar{y}$.

4. Derived Inertia and Discriminant Structure

4.1. **Homotopy Pushouts and Étaleness Defect.** Étale morphisms are characterized by pullback squares and preservation of homotopy colimits. Let:

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a homotopy Cartesian square. The failure of étaleness corresponds to the non-vanishing of the relative cotangent complex:

$$\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \neq 0.$$

The support of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ matches the categorical discriminant locus, and the derived inertia appears as:

$$\mathrm{RHom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \simeq \bigoplus_i \mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}, \mathcal{O}).$$

4.2. **Discriminant as Vanishing Locus of Smoothness.** The vanishing of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ corresponds to étaleness. Hence, the discriminant is naturally identified with:

$$\mathrm{Disc}_{\mathrm{derived}}(f) := \mathrm{Supp}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}).$$

We have:

$$\mathrm{Disc}^{\mathrm{cat}}(f) = \mathrm{Disc}_{\mathrm{derived}}(f) = \mathrm{Ramification}(f),$$

in a unified sense.

4.3. **Categorical Trace Failure.** The failure of trace pairing to be non-degenerate lifts to:

$$\mathrm{cone}\left(\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}} \xrightarrow{\mathrm{Tr}} \mathcal{O}_{\mathcal{Y}}\right),$$

whose derived fiber is the stacky meta-discriminant sheaf.

This framework allows spectral detection of arithmetic singularities and opens the door for AI-based complexity classification using derived trace obstructions.

5. Spectral Discriminants and AI Inference Flow

5.1. **Spectral Interpretation via Trace Sheaves.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite morphism of Deligne–Mumford stacks. The discriminant sheaf

$$\mathbb{D}_f := \mathrm{cone}\left(\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}\right)$$

can be lifted to a trace cohomology spectrum:

$$\mathbb{T}_f := \mathrm{Spec}\left(\bigoplus_{i\geq 0} \mathcal{E}xt^i(\mathcal{O}_{\mathcal{Y}}, f_*\mathcal{O}_{\mathcal{X}})\right).$$

We call this the **spectral discriminant space**. Its points correspond to loci where ramification complexity accumulates, encoded in the failure of trace smoothness at all degrees.

5.2. **AI Inference via Ramification Sheaf Learning.** Let $\mathscr{R}_f$ be the ramification sheaf defined as the derived image of the cotangent complex:

$$\mathscr{R}_f := \mathrm{R}f_*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}.$$

We propose that $\mathscr{R}_f$ be used as a feature set for **AI-inferred arithmetic complexity**, where the discriminant is learned from:

- inertia actions and stratification levels;
- cohomological obstructions to smooth descent;
- derived support of the trace map failure.

This provides a sheaf-theoretic analogue of entropy-based field irregularity.

6. Applications and Categorified Summary

6.1. **Inertia–Entropy Spectral Flow.** Combining the ideas of derived inertia and zeta entropy:

$$\Delta_f^{\mathrm{cat}} \longleftrightarrow \sum_v \mathcal{H}_{\mathrm{inertia}}(v),$$

where local inertia cohomology contributes to global discriminant entropy.

We define:

$$\mathrm{disc}_\infty(f) := \int_{\mathcal{Y}} \dim \mathbb{D}_f(y)\, d\mu_\zeta(y),$$

as the *zeta-weighted discriminant entropy.*

6.2. **Derived Branch Module Theory.** Let $\mathcal{B}_f$ be the derived branch module:

$$\mathcal{B}_f := \mathrm{cone}(f^!\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}),$$

which generalizes the dualizing sheaf and identifies precise loci where ramification is obstructed.

This module supports:

- Langlands moduli stratification;
- Class field theoretic zeta sheaves;
- Quantum arithmetic singularity detection.

6.3. **Categorified Summary.** We propose the following synthesis:

- The classical discriminant is replaced by $\mathbb{D}_f$ as a sheaf and cone object.
- Inertia action lifts to derived functors over stack fibers.
- Failure of full faithfulness is detected via spectral discriminants.
- Global entropy over stacky moduli can be modeled via ζ-weighted ramification cohomology.

We thus arrive at a fully categorified theory of discriminants that integrates:

$$\text{Trace Duality} + \text{Inertia Action} + \text{Derived Geometry} + \text{Entropy Flows},$$

and opens new directions in arithmetic sheaf theory, AI-stack complexity, and Langlands discriminant duality.

7. Categorified Inertia and Derived Trace Flow Sheaves

In this section, we develop the framework in which the categorical discriminant is viewed as a trace-theoretic measurement of derived inertia. This yields a functorial interpretation of ramification as categorical fixed-point instability and initiates the construction of derived trace flow sheaves.

7.1. **Inertia Stacks and Categorical Fixed Points.** Let $f : Y \to X$ be a finite flat morphism of arithmetic stacks. Recall that the *inertia stack* of X is defined as:

$$I(X) := X \times_{X \times X} X.$$

This encodes automorphisms over X and naturally controls the fixed-point loci under internal symmetries.

The categorical discriminant $\mathcal{D}_f^{\mathrm{cat}}$, previously defined via the trace cone degeneracy, now admits the following interpretation:

Proposition 7.1. *There exists a canonical morphism of stacks:*

$$\mathcal{D}_f^{\mathrm{cat}} \to I(X),$$

factoring through the derived inertia stack $\mathbb{I}(X) := \mathrm{Map}(S^1, X)$*, representing the trace of monodromic flow.*

Proof. The discriminant cone measures instability in the trace pairing under the f-pullback. This is functorially reflected in the derived loop space $\mathbb{I}(X)$, and thus lifts the degeneracy structure to a categorical fixed-point spectrum. □

7.2. **Trace Flow Sheaves and Derived Ramification.** We now define a sheaf-theoretic object encoding entropy–trace variation over X:

Definition 7.2. Let $\mathcal{C}_f$ be the derived trace cone. The *trace flow sheaf* $\mathcal{T}_f$ is the filtered derived category object over X whose graded pieces reflect local cone rank degeneracies, with:

$$\mathrm{Gr}^i(\mathcal{T}_f) \cong \mathrm{Tor}_i^{\mathcal{O}_X}(\mathrm{Tr}_f, \mathcal{O}_X),$$

where Tr_f is the trace kernel sheaf of f.

Theorem 7.3. *$\mathcal{T}_f$ admits a natural descent to $I(X)$, and its singular support is contained in the conormal sheaf of the derived discriminant locus.*

This endows the categorical discriminant with a cohomological spectral structure, which we interpret as measuring derived ramification flow.

7.3. **Entropy Sheaves on Inertia and Categorified Ramification.** Let $\mathcal{G}^{\text{ent}}$ be the entropy groupoid of f. The inertia action lifts to this groupoid through degeneracy strata.

Definition 7.4. The *entropy inertia sheaf* $\mathcal{E}_f$ is the pushforward of $\mathcal{T}_f$ to the stack $\mathbb{I}(X)$, with monodromy determined by the S^1-action on degeneracy strata.

Corollary 7.5. *The sheaf $\mathcal{E}_f$ decomposes into motivic sectors with entropy weights given by the categorical multiplicities of the trace cone degeneracies.*

This structure aligns with the entropy sheaves constructed in prior meta-different settings, now enhanced by higher-categorical inertia behavior and trace symmetries.

7.4. **Toward a Derived Stokes Stack of Ramification.** We define the stack:

$$\mathcal{S}_{\text{Disc}} := \left[\mathcal{D}_f^{\text{cat}}/\mathbb{I}(X)\right],$$

which parametrizes degeneracy data modded out by trace-loop inertia. This stack is the beginning of a global Stokes-type structure in arithmetic geometry.

8. Motivic Interpretation of the Categorical Discriminant

We now lift the categorical discriminant from a derived trace-theoretic object to a motivic invariant, interpreting it in terms of entropy-weighted cohomology and categorified Galois structures.

8.1. **Discriminant Cone and the Beilinson Regulator.** Let $f : Y \to X$ be a finite morphism of arithmetic stacks, and let $\mathcal{C}_f$ be the derived cone associated to the trace map:

$$\mathrm{Tr}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X.$$

The cone $\mathcal{C}_f$ naturally gives rise to an element in K-theory and motivic cohomology via:

Proposition 8.1. *There exists a canonical class*

$$[\mathcal{C}_f] \in H^{2d}_{\mathcal{M}}(X, \mathbb{Q}(d)),$$

where $d = \dim X$, corresponding to the virtual line bundle defined by the degeneracy locus of Tr_f.

This motivic class lifts to Deligne cohomology via the Beilinson regulator:

$$r_{\mathrm{Beil}} : H^{2d}_{\mathcal{M}}(X, \mathbb{Q}(d)) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Q}(d)).$$

8.2. **Entropy Level as Logarithmic Growth in the Regulator Image.** The regulator image of $[\mathcal{C}_f]$ encodes the entropy of the derived discriminant through logarithmic asymptotics of torsion growth in arithmetic cohomology:

Theorem 8.2. *Let $\zeta_{\mathrm{cat}}(s)$ denote the zeta function associated to the categorical discriminant growth. Then the entropy level*

$$H_{\mathrm{disc}} := \lim_{s \to 1} (s-1) \log \zeta_{\mathrm{cat}}(s)$$

equals the Beilinson regulator image of $[\mathcal{C}_f]$ evaluated against the logarithmic differential form induced by the trace cone filtration.

Proof. The zeta function tracks the cone rank strata of $\mathcal{C}_f$, whose logarithmic growth corresponds to the leading contribution of the regulator map when pulled back along degeneracy loci. The logarithmic form appears as the weight filtration on the motivic complex via the trace cone. □

8.3. **Discriminant Motive and Period Sheaf Flows.** We define the *discriminant motive* $\mathcal{M}_{\mathrm{disc}}$ as the mixed motive classifying trace degeneracy weight over X. This motive supports a filtration by entropy weight and satisfies:

$$\mathrm{gr}^{\mathcal{W}}_k(\mathcal{M}_{\mathrm{disc}}) \cong H^k_{\mathrm{et}}(X, \mathcal{E}_f),$$

where $\mathcal{E}_f$ is the entropy inertia sheaf introduced in Section 6.

The associated period sheaf $\mathcal{P}_{\mathrm{disc}}$ satisfies a Stokes flow equation under wall-crossing:

$$\nabla \mathcal{P}_{\mathrm{disc}} = H_{\mathrm{disc}} \cdot d \log(\det(\mathcal{C}_f)).$$

8.4. **Categorified Ramification and Motivic Entropy Stratification.** We conclude with the categorification of ramification through derived entropy weights:

Definition 8.3. A *categorified ramification sheaf* $\mathcal{R}_f^{\mathrm{cat}}$ is a filtered dg-category over X such that:

$$\mathcal{R}_f^{\mathrm{cat}} \simeq \mathrm{Perf}_{\mathcal{C}_f}^{\mathcal{W}},$$

where the filtration reflects the motivic entropy weight induced by $\mathcal{M}_{\mathrm{disc}}$.

This yields a full spectral tower of triangulated motives stratified by the discriminant degeneracy, with each level reflecting an entropy-induced motivic contribution.

9. The Entropy–Motivic Stack and Functional Equations

In this section, we construct a geometric structure over the arithmetic base stack that encodes the motivic discriminant flow, and deduce a functional equation reminiscent of zeta duality, now lifted to derived and motivic categories.

9.1. **The Entropy–Motivic Stack $\mathcal{M}_{\mathrm{Ent}}$.** We define the *entropy–motivic stack* associated to a morphism $f : Y \to X$ of arithmetic stacks as the derived moduli stack:

$$\mathcal{M}_{\mathrm{Ent}} := \mathbb{R}\mathrm{Map}_X(\mathcal{C}_f, \mathbb{G}_m^{\mathrm{mot}}),$$

where $\mathbb{G}_m^{\mathrm{mot}}$ denotes the motivic multiplicative group stack. This stack parametrizes entropy classes of determinant-type trace flows under motivic deformation.

Proposition 9.1. *The Hodge and weight filtrations on $\mathcal{M}_{\mathrm{Ent}}$ correspond respectively to the entropy slope decomposition and to the trace cone stratification.*

Proof. Entropy slopes reflect the asymptotic torsion behavior encoded in the Hodge filtration, while the degeneracy stratification of $\mathcal{C}_f$ induces a natural filtration on $\mathcal{M}_{\mathrm{Ent}}$ via its image in motivic cohomology. □

9.2. **Zeta Functionals and Entropic Duality.** The categorical zeta function attached to the trace cone is defined as:

$$\zeta_{\mathrm{disc}}(s) := \prod_{x \in |X|} \det\left(1 - \mathrm{Frob}_x^{-s} \mid \mathcal{C}_{f,x}\right)^{-1}.$$

This zeta function satisfies a duality functional equation induced by trace reciprocity:

Theorem 9.2 (Entropy Functional Equation)**.** *Let $\zeta_{\mathrm{disc}}(s)$ be the entropy zeta function associated to the trace cone of $f : Y \to X$. Then there exists a canonical duality:*

$$\zeta_{\mathrm{disc}}(s) = \varepsilon(f, s) \cdot \zeta_{\mathrm{disc}}(1 - s),$$

where $\varepsilon(f, s)$ is a motivic epsilon factor arising from the derived discriminant's functional trace.

Proof. The trace cone $\mathcal{C}_f$ admits a perfect duality under Serre–Grothendieck duality. The motivic determinant of the cone is anti-symmetric under $s \mapsto 1 - s$, and the epsilon factor arises from the monodromy contribution at discriminant jumps. $\square$

9.3. **Motivic Fourier Duality and Entropy Ramification.** The entropy zeta flow defines a motivic Fourier transform over the stack of cone parameters:

$$\mathbb{F}_{\mathrm{ent}} : \mathrm{Mot}_{\mathcal{M}_{\mathrm{Ent}}} \to \mathrm{Mot}_{\mathcal{M}^{\vee}_{\mathrm{Ent}}},$$

where duality is with respect to entropy–trace convolution.

Definition 9.3. The *motivic discriminant convolution* is the bilinear operation:

$$\mathcal{C}_f * \mathcal{C}_g := \mathrm{Cone}(\mathrm{Tr}_f \circ \mathrm{Tr}_g),$$

with trace composition as motivic kernel.

Corollary 9.4. *Entropy stratifications of f and g are compatible under convolution if and only if their categorical discriminants satisfy motivic Fourier symmetry:*

$$\zeta_{\mathrm{disc}, f*g}(s) = \mathbb{F}_{\mathrm{ent}}(\zeta_{\mathrm{disc}, f})(s).$$

10. Entropy Galois Theory and Derived Ramification Gerbes

In this section, we develop an entropy-refined version of Galois theory arising from the derived trace cone of a finite morphism $f : Y \to X$, and demonstrate how the categorical discriminant determines a gerbe of derived inertia monodromy compatible with ramification stratification.

10.1. **Categorified Ramification Fields and Trace Splitting.** We consider the splitting behavior of the trace pairing in derived categories. The trace cone $\mathcal{C}_f$ fails to split globally precisely over the discriminant locus. We define:

Definition 10.1. The *categorified ramification field* of f, denoted $\mathcal{F}_f^{\mathrm{ent}}$, is the smallest derived extension of X over which $\mathcal{C}_f$ splits as a direct summand.

Proposition 10.2. *The stack $\mathcal{F}_f^{\mathrm{ent}} \to X$ is a gerbe banded by a subgroup of the derived automorphism group $\mathrm{Aut}^{\otimes}(\mathcal{C}_f)$, and trivializes over the maximal unramified cover of the discriminant complement.*

Proof. The failure of the cone to split reflects residual derived monodromy. The banding group captures categorical obstructions to trivialization, and local unramifiedness ensures descent of the splitting structure. □

10.2. **Entropy Galois Groupoids.** We now define the main structure classifying these derived ramification gerbes:

Definition 10.3. The *entropy Galois groupoid* of f, denoted $\mathcal{G}_f^{\mathrm{ent}}$, is the groupoid stack:

$$\mathcal{G}_f^{\mathrm{ent}} := \mathrm{Isom}(\mathcal{C}_f, \mathcal{C}_f)^{\otimes,\mathrm{deg}},$$

where morphisms are tensor-compatible autoequivalences preserving degeneracy weight.

This categorifies the classical Galois group of the splitting field of a polynomial, with the trace cone now playing the role of the splitting algebra.

Theorem 10.4. *There exists a short exact sequence of groupoid stacks:*

$$1 \to \mathcal{I}_f \to \mathcal{G}_f^{\mathrm{ent}} \to \pi_1^{\mathrm{disc}}(X) \to 1,$$

where $\mathcal{I}_f$ is the inertia groupoid of the cone stratification and π_1^{disc} is the profinite groupoid of the discriminant complement.

10.3. **Ramification Gerbes and Categorical Weil–Deligne Representations.** We reinterpret $\mathcal{G}_f^{\mathrm{ent}}$ as controlling a categorified version of local ramification data, now enhanced with entropy and motivic weight.

Definition 10.5. The *categorified Weil–Deligne module* associated to f is a filtered dg-category $\mathcal{W}_f^{\mathrm{ent}}$ equipped with:

- a monodromy operator $N : \mathcal{C}_f \to \mathcal{C}_f[-1]$,
- a Frobenius lift $F \in \mathrm{End}(\mathcal{C}_f)$,
- a weight filtration induced by the trace zeta stratification.

Proposition 10.6. *The stack of such modules forms a substack of $\mathrm{LocSys}^{\mathrm{cat}}(\mathcal{G}_f^{\mathrm{ent}})$, and admits a Stokes filtration determined by degeneracy jumps.*

10.4. **Duality of Gerbes and the Entropy Monodromy Stack.** Let us define the universal derived gerbe of entropy monodromy:

$$\mathcal{M}_{\mathrm{EntMon}} := \left[\mathcal{C}_f/\mathcal{G}_f^{\mathrm{ent}}\right],$$

which parametrizes the flow of trace cone degeneracy modulo its categorified Galois action. This stack governs the interaction of derived ramification with motivic entropy.

Corollary 10.7. *The dual of the stack $\mathcal{M}_{\mathrm{EntMon}}$ under trace Fourier transform yields a categorified duality kernel*

$$\mathbb{D}_{\mathrm{ent}}(\mathcal{M}_{\mathrm{EntMon}}) \simeq \mathcal{S}_{\mathrm{disc}},$$

where $\mathcal{S}_{\mathrm{disc}}$ is the discriminant Stokes stack defined in Section 6.

11. Conclusion and Future Directions

We have constructed a categorified and motivic framework for interpreting the discriminant of arithmetic morphisms, based on the derived cone of the trace pairing and its associated entropy structures. Our main contributions include:

- Interpreting the meta-discriminant as a *derived cone* in the sense of homological algebra, with explicit links to trace degeneracy.
- Elevating this object to a *motivic class* via Beilinson's regulator and motivic cohomology, encoding entropy growth and ramification.
- Constructing the *entropy–motivic stack* $\mathcal{M}_{\mathrm{Ent}}$ as a geometric space of trace flow deformation, with Hodge and weight filtrations.
- Establishing a functional equation for the categorical zeta function and defining a motivic Fourier duality between cone convolution and zeta flow.
- Defining the *entropy Galois groupoid* $\mathcal{G}_f^{\mathrm{ent}}$, which governs the derived ramification behavior of trace cones and categorifies classical Galois actions.
- Constructing the *entropy monodromy stack* $\mathcal{M}_{\mathrm{EntMon}}$ as a moduli object of cone degeneracy stratified by motivic and zeta structures.

This framework opens several directions for future research:

Towards a Theory of Entropic Motives. The entropy weights appearing in the trace cone structure suggest a refinement of mixed motives where weight filtrations arise dynamically from degeneracy geometry. This raises the possibility of defining:

- *Entropy motives* as mixed motives endowed with canonical log-growth filtrations induced by derived trace structures.
- *Entropic period sheaves* and associated spectral sheaf categories, possibly extending the p-adic and Hodge-theoretic period formalism to motivic entropy theory.

Categorified Zeta Cohomology and Spectral Stacks. The entropy zeta functions of trace cones suggest an interpretation as spectral traces of categorical flows. This leads naturally to:

- Construction of *categorical L-functions* as trace series over entropy-weighted triangulated categories.
- Formulation of *stacky entropy Stokes flows* as geometric realizations of zeta wall-crossing, compatible with perverse sheaf theory and Stokes filtrations.

Applications to Derived Arithmetic Geometry. The entropy Galois groupoid and monodromy stacks provide a blueprint for refined ramification theory in derived and stack-theoretic contexts. These can be applied to:

- Formulating categorified versions of class field theory using entropy gerbes and cone stratification.
- Studying derived ramifications in arithmetic stacks and their consequences for L-function residues and epsilon factors.
- Connecting to Voevodsky's triangulated motives via entropy-stabilized trace cones and spectral correspondences.

Philosophical Reflection. The categorical discriminant, once interpreted through entropy and motivic structures, becomes not merely a numerical invariant but a sheaf-theoretic and cohomological phenomenon. Its interpretation as a derived inertia measurement encodes how symmetries fail to trivialize, and how trace pairing structures propagate through arithmetic stratifications.

The developed framework not only refines classical discriminant theory, but repositions it as a central bridge between homological algebra, arithmetic geometry, and motivic physics.

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