DYADIC LANGLANDS V: CONDENSED REDUCTIVE STACKS AND UNIVERSAL L-GROUPOIDS

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Abstract. We construct the theory of condensed reductive stacks over dyadic arithmetic sites and introduce the notion of universal L-groupoids as categorical recipients of dyadic Langlands parameters. Extending the framework of condensed mathematics and perfectoid geometry, we develop stack-theoretic models for Langlands reciprocity over $\mathbb{Z}_2$-enriched moduli and define condensed Langlands correspondences through trace-compatible sheaves and higher Galois group actions. These constructions unify arithmetic shtukas, spectral trace flows, and zeta cohomology under a new categorical paradigm suitable for condensed and motivic arithmetic geometry.

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1. Introduction

The Langlands program, in its classical and geometric forms, centers on correspondences between automorphic representations and Galois-theoretic or motivic parameters. In this fifth part of the Dyadic Langlands series, we explore how the recent frameworks of condensed mathematics and perfectoid geometry allow the construction of a new categorical infrastructure for these correspondences, centered around the notions of condensed reductive stacks and universal L-groupoids.

Motivated by the dyadic spectral zeta theory developed in the Spectral Motives series, and building upon the shtuka-theoretic formulations from Parts I–IV, this paper introduces:

(1) A category of *condensed reductive group stacks* over dyadic arithmetic sites;
(2) The construction of *universal L-groupoids*, extending the Tannakian formalism of L-groups into condensed higher categories;
(3) A stack-theoretic realization of dyadic Langlands parameters as morphisms from condensed shtukas into L-groupoids;
(4) A functorial trace-compatible framework allowing the comparison of arithmetic and automorphic sides.

This formulation allows for the classification of Langlands correspondences in terms of moduli stacks enriched over $\mathbb{Z}_2$-sites, endowed with trace morphisms, perfectoid descent, and analytic structures compatible with dyadic zeta geometry. The use of condensed geometry offers a resolution to several classical difficulties related to continuity, torsors, and sheafification in p-adic arithmetic.

Structure of the paper. In Section 2, we define condensed reductive groups and their associated classifying stacks. In Section 3, we develop the notion of universal L-groupoids, modeled as functorial groupoid objects in condensed ∞-categories. Section 4 introduces the moduli of Langlands parameters and trace descent maps. Finally, Section 5 establishes the spectral comparison theorems and outlines their applications in the context of condensed automorphic stacks and zeta-motivic transfer.

This work lays the foundation for higher functoriality, spectral reciprocity, and arithmetic representation theory over condensed and perfectoid geometries. It prepares the ground for the next parts of the Dyadic Langlands series, where categorical automorphy and universal shtuka correspondences will be fully realized in the condensed ∞-topos setting.

2. Condensed Reductive Groups and Classifying Stacks

2.1. Condensed group objects.

Let **Cond** denote the category of condensed sets, and let **CondGrp** be the category of group objects internal to **Cond**. A *condensed group scheme* over a condensed base S is a functor

$$G\colon \mathbf{Cond}^{\mathrm{op}} \to \mathbf{Grp}$$

satisfying sheaf and representability conditions in a given topology (e.g., condensed analytic, pro-étale, perfectoid-admissible).

We say that a condensed group G is *reductive* if for each point $s \in S$, the fiber G_s is a reductive group in the classical sense, and the formation of G_s is compatible with condensed base change.

2.2. **Examples and constructions.** Examples of condensed reductive groups include:

- $\mathrm{GL}_n^{\mathrm{cond}}$, the condensed general linear group over $\mathbb{Z}_2$;
- Condensed unitary groups over perfectoid fields;
- Fiberwise trace-compatible versions of SL_n, GSp_{2n}, or exceptional groups over $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$.

These are obtained as base changes and sheafifications of classical group schemes into the condensed category, equipped with transition structures reflecting dyadic or perfectoid symmetries.

2.3. **Classifying stacks.** Given a condensed reductive group G, we define the *condensed classifying stack* BG^{cond} as the prestack on **Cond** assigning:

$$BG^{\mathrm{cond}}(S) := \{G\text{-torsors over } S \text{ in the condensed topology}\}.$$

This stack is an ∞-stack under suitable descent conditions (e.g., pro-étale or condensed flat topology) and supports morphisms from condensed shtuka stacks, moduli of zeta sheaves, or automorphic trace sheaves.

2.4. **Langlands torsors and trace compatibility.** We define a *Langlands torsor* as a G-torsor P over a condensed base S equipped with:

(1) A trace structure compatible with the $\zeta_n(s)$ tower;
(2) Descent data under condensed Galois stacks;
(3) A comparison morphism to the universal trace flow moduli $\mathscr{T}_\zeta$.

These torsors will be interpreted as local parameters in the Langlands correspondence over condensed shtukas in the next sections.

3. Universal L-Groupoids in Condensed Geometry

3.1. **Motivation and context.** In classical and geometric Langlands theory, the Langlands dual group ${}^LG$ plays a central role in encoding automorphic parameters as Galois representations into ${}^LG$-torsors. In the dyadic setting, enriched with condensed geometry, we seek a categorical enhancement: a functorial groupoid object that replaces the discrete ${}^LG$ with a universal recipient of condensed Langlands parameters.

3.2. **Definition of the universal L-groupoid.** Let G be a condensed reductive group over a base $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$. Define the *universal L-groupoid* $\mathbb{L}_G^{\mathrm{cond}}$ as a groupoid object in the ∞-category of condensed stacks:

$$\mathbb{L}_G^{\mathrm{cond}} = [\mathscr{P} \rightrightarrows \mathscr{G}],$$

where:

- $\mathscr{G}$ is the stack of trace-compatible G-torsors over condensed shtukas;
- $\mathscr{P}$ is the stack of morphisms of such torsors, compatible with trace descent and perfectoid realization;
- The groupoid structure encodes automorphisms, Hecke correspondences, and Frobenius transport.

This defines an internal groupoid in condensed arithmetic geometry, and acts as a stack-theoretic enrichment of the classical L-group.

3.3. **Properties and functoriality.** The universal L-groupoid $\mathbb{L}_G^{\mathrm{cond}}$ satisfies:

(1) **Functoriality**: For a morphism $G \to H$ of condensed reductive groups, there is an induced morphism of L-groupoids $\mathbb{L}_G^{\mathrm{cond}} \to \mathbb{L}_H^{\mathrm{cond}}$;
(2) **Trace compatibility**: The groupoid structure respects trace descent under dyadic ζ_n-flows;
(3) **Stackification**: The groupoid descends to a global stack over condensed shtuka moduli;
(4) **Spectral parameterization**: The set of equivalence classes of $\mathbb{L}_G^{\mathrm{cond}}$-torsors over a condensed shtuka recovers dyadic Langlands parameters.

3.4. **Higher categorical enhancements.** We may enhance $\mathbb{L}_G^{\mathrm{cond}}$ to an ∞-groupoid over the moduli stack of dyadic motives, allowing the construction of:

$$\underline{\mathrm{Hom}}_{\mathrm{Cond}}(\mathscr{M}_{\mathrm{mot}}^{\mathrm{cond}}, \mathbb{L}_G^{\mathrm{cond}}),$$

as the moduli of condensed Langlands parameters in families. This provides the input for functorial automorphic correspondences and will lead to derived categorifications in subsequent parts of this series.

4. Moduli of Langlands Parameters and Trace Descent

4.1. **Langlands parameters over condensed shtukas.** Let $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$ denote the moduli stack of condensed shtukas with leve$L$-n trace structure, defined over the site $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$. A *condensed Langlands parameter* is a morphism of stacks:

$$\rho : \mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}} \to \mathbb{L}_G^{\mathrm{cond}},$$

satisfying compatibility with:

(1) The ζ_n-trace filtration;
(2) Frobenius descent data in the pro-étale topology;
(3) Motivic realization via the functor Θ_ζ from $\mathscr{Z}^{\mathrm{cond}}$.

These parameters encode Galois and motivic information in a form adapted to condensed and spectral motivic geometries.

4.2. **The moduli stack of parameters.** Define the moduli stack of condensed Langlands parameters:

$$\mathscr{L}\mathrm{ang}_G^{\mathrm{cond}} := \underline{\mathrm{Hom}}_{\mathrm{Stacks}}(\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}, \mathbb{L}_G^{\mathrm{cond}}),$$

viewed as an object in the ∞-category of condensed higher stacks. This stack is:

- Representable as a fibered category over $\mathscr{M}_{\mathrm{mot}}^{\mathrm{perf}}$;
- Stratified by trace depth n and Frobenius eigenvalues;
- A higher categorical enhancement of classical Langlands parameter spaces.

4.3. **Trace descent morphisms.** Let $\mathscr{Z}_n$ denote a condensed zeta sheaf of level n. There exists a canonical trace descent map:

$$\mathrm{Tr}_n : \mathscr{L}\mathrm{ang}_{G,n+1}^{\mathrm{cond}} \longrightarrow \mathscr{L}\mathrm{ang}_{G,n}^{\mathrm{cond}},$$

which forgets higher trace layers and induces a filtered inverse system of Langlands parameter stacks.

In the limit, we define:

$$\mathscr{L}\mathrm{ang}_{G,\mathbb{Z}_2}^{\mathrm{cond}} := \varprojlim_n \mathscr{L}\mathrm{ang}_{G,n}^{\mathrm{cond}},$$

representing universal trace-compatible condensed Langlands parameters. This space governs the full spectrum of dyadic spectral transfer and automorphic realization.

4.4. **Arithmetic significance.** The points of $\mathscr{L}\mathrm{ang}^{\mathrm{cond}}_{G,\mathbb{Z}_2}$ correspond to condensed automorphic types, enriched with trace-compatible zeta-motivic geometry. These will serve as the domain for transfer morphisms to condensed automorphic stacks in Section 5 and in *Dyadic Langlands VI*.

5. Spectral Comparison and Applications to Automorphy

5.1. **Comparison with classical Langlands parameters.** Let G be a reductive group defined over $\mathbb{Z}$, and let $\mathscr{L}\mathrm{ang}^{\mathrm{cl}}_{G}$ denote the classical stack of Langlands parameters (e.g., Galois representations or global sections of ${}^{L}G$-torsors). There exists a comparison morphism:

$$\mathrm{Comp}^{\mathrm{cl}\to\mathrm{cond}} : \mathscr{L}\mathrm{ang}^{\mathrm{cl}}_{G} \longrightarrow \mathscr{L}\mathrm{ang}^{\mathrm{cond}}_{G,\mathbb{Z}_2},$$

which associates to each classical parameter a trace-compatible condensed lift, provided it admits motivic realization and dyadic descent.

This morphism induces a spectral comparison:

$$\mathcal{T}^{\mathrm{cl}}_{\mathrm{aut}} \simeq \mathcal{T}^{\mathrm{cond}}_{\mathrm{aut}} \circ \mathrm{Comp}^{\mathrm{cl}\to\mathrm{cond}},$$

between classical automorphic functors and those arising from the condensed Langlands program.

5.2. **Condensed automorphic stacks.** Let $\mathscr{A}\mathrm{ut}^{\mathrm{cond}}_{G}$ denote the moduli stack of trace-compatible automorphic sheaves over condensed spaces. The condensed Langlands correspondence is a conjectural equivalence:

$$\mathscr{L}\mathrm{ang}^{\mathrm{cond}}_{G,\mathbb{Z}_2} \xrightarrow{\simeq} \mathscr{A}\mathrm{ut}^{\mathrm{cond}}_{G},$$

interpreted as a geometric spectral transfer between arithmetic Galois parameters and condensed automorphic types.

This correspondence extends:

- Over dyadic shtuka sites and perfectoid analytic bases;
- With compatibility under trace descent and condensed functoriality;
- Into a fully stack-theoretic equivalence of condensed ∞-topoi.

5.3. **Applications and categorical structure.** The theory developed in this paper enables:

(1) The construction of functorial trace flows between arithmetic and automorphic moduli;
(2) The formulation of Langlands duality entirely within the setting of condensed stacks;
(3) A spectral categorification of L-functions via universal L-groupoids;
(4) Future connections with condensed automorphic motives and derived Hecke correspondences.

This framework prepares the ground for the final part of the Dyadic Langlands series, where we explore automorphic realization, trace categorification, and universal functoriality in higher arithmetic topoi.

6. Conclusion and Future Work

In this fifth installment of the Dyadic Langlands Program, we constructed the foundational infrastructure for condensed Langlands geometry. By defining condensed reductive stacks and introducing universal L-groupoids, we initiated a categorical model for Langlands parameters compatible with dyadic trace flows and spectral zeta-motivic structures.

The moduli stacks of condensed Langlands parameters serve as the natural domain for automorphic realizations in condensed sheaf theory, while universal L-groupoids encode the refined groupoid symmetries of arithmetic and motivic data. These constructions extend classical ideas into a functorial ∞-categorical context over the condensed site $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$.

Future work includes:

- Developing automorphic realization via condensed automorphic stacks;
- Enhancing L-groupoids into derived groupoids with Hecke symmetries and motivic cohomology;
- Constructing spectral flows between different group types in the condensed Langlands tower;
- Applying the full framework to derive L-functions, trace formulas, and universal functoriality theorems.

These directions will be pursued in Dyadic Langlands VI, and continue into the Spectral Motives series where automorphic, motivic, and condensed cohomological systems coalesce into a unified arithmetic geometry.

References

[1] P. Scholze and D. Clausen, *Condensed Mathematics*, 2020. `https://condensed.math`
[2] P. Scholze, *Perfectoid Spaces*, Publ. Math. IHÉS, 116 (2012), pp. 245–313.
[3] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, Princeton University Press, 2009.
[4] P. J. S. Yang, *Spectral Motives and Zeta Transfer I–VI*, 2025.
[5] P. J. S. Yang, *Dyadic Langlands Program I–IV*, 2025.
[6] L. Fargues and P. Scholze, *Geometrization of the Local Langlands Correspondence*, Preprint, 2021.
[7] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Vieweg, 1996.