On the isogeny invariance of the Bloch-Kato's Tamagawa Numbers conjecture:

a K-theoretic point of view

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF GRADUATE STUDIES
(Mathematics)

The University Of British Columbia
(Vancouver)

December 2016

Abstract

The arithmetic properties of various interesting objects (e.g. number fields, varieties, or even a motive) are all encoded in their respective L-functions, in which the Riemann zeta function is the simplest example. The understanding of these L-functions undergoes three phases [Kat1993a]:

1. analytic properties and the rationality of values,
2. algebraic properties shed light by the p-adic properties of values,
3. arith-geometric significance of values when viewed as an Euler system.

To date most of these understanding has been achieved for the Riemann zeta function [CRSS2015] (except the locations of the zeros, or the zero free region). In their article published in the Grothendieck Festschrift [BK1990], Spencer Bloch and Kazuya Kato removed the $\mathbb{Q}^\times$ ambiguity given in the Deligne and Beilinson conjectures [Be1985], [De1979] to formulate a more concise conjecture on the values at integers points of L-functions associated to motives, given by the Tamagawa numbers and showed it is isogeny invariant. Kato even proved the conjecture for the Riemann zeta function and elliptic curves with complex multiplication in [BK1990], and the book [CRSS2015] exploits this isogeny invariance condition in the optic of K-theory for the Riemann zeta function.

This thesis provides a K-theoretic point of view of the isogeny invariance of the Bloch-Kato's Tamagawa conjectures for elliptic curves, modular forms, and abelian varieties with complex multiplication.

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Acknowledgments

First and foremost, I am deeply indebted to Professor Sujatha Ramdorai for providing me with extraordinary amounts of time and patience (since my first year undergraduate), sharing her seemingly endless bounty of mathematical knowledge, and challenging me with this exciting and multi-faceted problem. I have learned a great deal from our many conversations and feel privileged to have studied under her supervision. I am also very grateful for the funding I have received from NSERC, the department, and Professor Sujatha.

In addition, I am extremely grateful to Professor Greg Martin for his patience and suffering since my first year undergraduate studies in guiding me through the world of analytic number theory, enabling me to understand the many recent exciting breakthroughs in the field, especially in understanding the notorious Riemann Hypothesis, and I want to thank Professor Lior Silberman for enabling me to understand its automorphic version. I also want to thank Professors Patrick Brosnan and Bruno Kahn on pointing me to the latest advances in the Hodge conjecture, to Professor Tai-Peng Tsai for his inspiring lectures on the latest results on the Navier-Stokes existence and smoothness problem, to Professor Izabella Łaba for her lectures with the latest results on the Kakeya conjecture, to Professor Ákos Mayar on his lectures on the Green-Tao Theorem, to Professor (MORE TO BE ADDED) , and to every faculty member here at UBC whom I have either taken or sat in on a course with, and for PIMS giving me the use of an office as a luxury to do uninterrupted work.

Studying at the Department of Mathematics at UBC has been wonderful with the extremely helpful and friendly department staff, particularly Roseann Kinsey and Marlowe Dickson. Involvement in the graduate student and number theory community has been a great personal and educational experience, punctuated with many deep (non-

)mathematical discussions, for which I would like to thank (COLLEAGUES) amongst many others.

Simply put, I am only here because of my parents, who have been selflessly supportive throughout my life, let alone my education. I dedicate this thesis to them, as their hardships and struggles have always been for my future well-being.

To my parents

Introduction

0.1 The leading role played by arithmetic within mathematics and recent breakthroughs

Arithmetic enjoys a privileged position within mathematics as a fertile source of fundamental questions. Among the seven Millennium problems listed by the Clay Institute [Clay], not fewer than three: the Birch and Swinnerton-Dyer conjecture, the Hodge conjecture, and the Riemann hypothesis, were handed down by the Queen of Mathematics. Even by the standards of a subject which has remained vibrant since the days of Fermat and Gauß, the last two decades have witnessed a real golden age, with landmarks too numerous to list completely: such as the striking progress on the Birch and Swinnerton-Dyer conjecture arising from the work of Gross-Zagier [GZ1986], Kolyvagin [Kol1989], and Kato [Kat2004]; the proofs of the Shimura-Taniyama-Weil conjecture [BCDT2001], Serre's conjectures [KW2009], the Fontaine-Mazur conjecture for two-dimensional Galois representations [Kis2009], and the Sato-Tate conjectures [CHR2008] which grew out of Wiles' epoch-making proof of Fermat's Last Theorem [Wil1995], [TW1995]; the revolutionary ideas of Bourgain [Bo2008] and Gowers [Go2007] blending techniques in harmonic analysis and additive combinatorics, the Fields-medal winning breakthrough of Green and Tao on primes in arithmetic progressions [GT2008], and the work of Goldston, Pintz, and Yıldırım [GPY2009], [GPY2010], and its spectacular recent strengthenings by Zhang [Zha2014], and Maynard [May2015] and Tao [Poly2014], on bounded gaps between primes.

0.2 On the Riemann Hypothesis

Little essential progress has been made to the Riemann Hypothesis in the past two decades for the Riemann zeta function (or the Generalized Riemann Hypothesis for automorphic L-functions), which predicts that all the zeros of such L-functions are critical: They lie on the real line $\mathrm{Re}\, s = \frac{1}{2}$ with multiplicity one. The sharpest result for the Riemann zeta function is due to Feng [Fe2012], proving that at least 41.28% of the zeros of the Riemann zeta function,

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \qquad (s := \sigma + it),$$

lie on the critical line, by introducing a new mollifier and applying the original method of Levinson [Lev1974] and its subsequent strengthening by Conrey [Con1989], which gave at least 34.20% and 40.88% of the zeros of $\zeta(s)$ are critical, respectively. The best zero-free region to date for the Riemann zeta function is obtained by Ford [For2002] with

$$\sigma \geq 1 - \frac{1}{57.54(\log|t|)^{2/3}(\log\log|t|)^{1/3}}, \qquad |t| \geq 3.$$

A better zero-free region would imply a stronger result in the error term of the prime number theorem,

Other industries including computing the higher moments of these automorphic L-functions,

the attempt of using random matrices to explain the spacing between the zeros

Montgomery pair-correlation conjecture

the Siegel-Walfisz theorem and the large sieve developed by Bombieri are often used in place of the Generalized Riemann Hypothesis for Dirichlet L-functions to prove theorems and do estimates.

0.3 On the Hodge conjecture

Algebraic cycles

Chow groups

Kähler manifold

Known cases:

Generalizations:

Main difficulties/ best result in hoped cases.

0.4 On the Birch and Swinnerton-Dyer conjecture

The Birch and Swinnerton-Dyer conjecture (BSD), formed by B. Birch and Swinnerton-Dyer when studying the asymptotics of

$$\prod_{p\leq x} \frac{\#E(\mathcal{F}_p)}{p},$$

has seen tremendous progress since the first breakthrough of Coates and Wiles.

Recall the Dirichlet class number formula for a number field K [Neu1999],

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w|d_K|^{1/2}} hR.$$

The BSD conjecture

(a) (Weak BSD)

$$\mathrm{ord}_{s=1}\, L(E/K,s) = r_K,$$

(b) (Strong BSD)

$$\lim_{s\to 1} \frac{L(E/K,s)}{(s-1)^{r_p}} = \Omega_{E/K} \times \mathrm{Reg}_{\infty,K}(E) \times \frac{|\text{Ш}_K(E)| \prod_{p\leq\infty}[E(K_p):E_0(K_p)]}{\sqrt{\Delta_K}\times |E(K)|^2_{\mathrm{tors}}}$$

Note Iwasawa main conjecture, proved in cases

1. $\mathbb{Q}$ [MW1984]
2. totally real number fields [Wil1990]
3. imaginary quadratic fields [Rub1988] [Rub1991]
4. Dirichlet characters [HK2003]
5. CM elliptic curves at supersingular primes [PR2004]
6. for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions [BD2005].

7. non-commutative main conjecture for totally real p-adic Lie extension of a number field [Kak2013] [CSSV2013].

8. (automorphic) GL_2 [SU2014]

One promising view
algebraic K-theory,
Suslin and Voevodsky,
Bloch
Motivic cohomology

Euler systems [Rub2000]

Classical Euler systems:

1. Siegels' cyclotomic units gives Kubota-Leopoldt p-adic L function, but no BSD application [CS2006].

2. Elliptic units Coates and Wiles' homomorphism

3. Heegner points gives anticyclotomic p-adic L-function of

Kato's Euler systems:

1. Beilinson-Kato elements

2. Beilinson-Flach elements

3. Gross-Kudla-Schoen cycles

Note that the monograph [Del2008] is devoted to the study of the BSD conjecture over the universal deformation rings of an elliptic curve.

Chapter 1

Bloch-Kato's Tamagawa Numbers conjecture and the isogeny invariance

1.1 From the Birch and Swinnerton-Dyer conjecture to the Bloch-Kato conjecture

The Bloch-Kato's Tamagawa Numbers conjecture [BK1990], can be seen as a generalization of the Birch and Swinnerton-Dyer conjecture for motives.

Need: Fontaine's topological period rings B_{dR} and B_{crys}, where the former is a complete valued field with residue field $\mathbb{C}_p$

For a motivic pair (V, D) with weights $\leq w$ and a finite set of places Ω of $\mathbb{Q}$ containing ∞, the L-function $L_\Omega(V, s)$ is defined to be as the Euler product

$$L_\Omega(V, s) := \prod_{p \notin \Omega} P_p(V, p^{-s})^{-1},$$

it is absolute convergent for $\mathrm{Re}(s) > \frac{w}{2} + 1$.

Fixing a $\mathbb{Z}$-lattice $M \subset V$

1.2 Motives, Fontaine's p-adic period rings, and other essential gadgets

To define Tamagawa measures one needs groups $A(\mathbb{Q}_P), p \leq \infty$, and $A(\mathbb{Q})$. Bloch and Kato define such groups for a motivic pair (V, D) as follows: Assuming the motivic pair (V, D) has weight ≤ -1

Fontaine's p-adic period rings [Fon1982], [FM1987],

Tamagawa number [Wei1982]

1.3 The isogeny invariance explained

1.4 K-theoretic background

Reference [HB-K1] [HB-K2] [Sri1993], [Wei2013], [Wal1987a] [Wal1987b]

We will need to use higher K-theory, where K_0 was invented by Grothendieck in proving the Riemann-Roch Theorem

Quillen higher K-theory

the + construction

the Q construction for schemes

Waldhausen K-theory

Chapter 2

The isogeny invariance for elliptic curves with complex multiplication

2.1 CM elliptic curves

Reference: [Hid2013], [Sil2009], [Sil1994], [Kob1993], [Kna1992], [Ca1991], [Mil2006], [CVG1999], [dSh1987]

2.2 The isogeny invariance for CM elliptic curves

2.3 The K-theoretic viewpoint

Chapter 3

The isogeny invariance for elliptic curves without complex multiplication

3.1 non-CM elliptic curves

3.2 The isogeny invariance for non-CM elliptic curves

3.3 The K-theoretic viewpoint

Chapter 4

The isogeny invariance for modular forms

4.1 Modular forms

Reference [Miy1989], [DS2005], [AF1995]

4.2 The isogeny invariance for modular forms

4.3 The K-theoretic viewpoint

Chapter 5

The isogeny invariance for abelian varieties with complex multiplication

5.1 CM abelian varieties

References: [Shi1998], [Mum1974]

5.2 The isogeny invariance for CM abelian varieties

5.3 The K-theoretic viewpoint

Chapter 6

Proposed isogeny invariance for tori and motif

All of the above geometric objects are in the form of a pure motive: which are smooth projective varieties.

To date many problems occur in mixed motives due to standard conjectures.

The best understanding is due to Voevodsky's motivic cohomology [MWV2006]

Conclusion

Categorification?

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