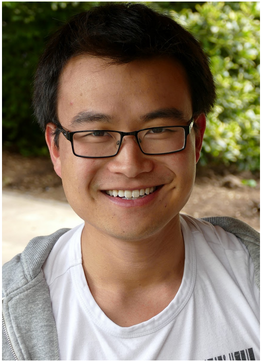
Smooth Discrepancy and Littlewood's Conjecture

by Niclas Technau

October 24, 2024

UNIVERSITÄT BONN

Collaborator



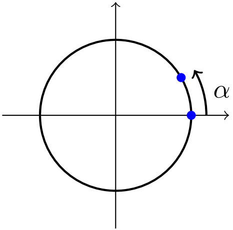
Sam Chow (Warwick)

In a nutshell

 $=$  $*$  $*$  $*$ 

Kronecker sequences

- Let
$$\mathbb{T} := [0,1) \cong \mathbb{R}/\mathbb{Z}.$$
- Let $R_\alpha(x+\mathbb{Z}) := \alpha + x + \mathbb{Z}$.



- $R_\alpha(0+\mathbb{Z}) = \alpha + \mathbb{Z}$, its iterate
$$R_\alpha^{\circ 2}(0+\mathbb{Z}) = \alpha + (\alpha + \mathbb{Z}) + \mathbb{Z} = 2\alpha + \mathbb{Z}.$$
- Generally,
$$R_\alpha^{\circ n}(0+\mathbb{Z}) = n\alpha + \mathbb{Z} \cong n\alpha \bmod 1.$$

Equidistribution

Definition 1 (Equidistribution).

A sequence $\mathbf{x} = (x_n)_n \subseteq \mathbb{T}$ is equidistributed if, for any (closed) interval $\mathfrak{B} \subseteq \mathbb{T}$, we have

$$\sum_{n \leq N} \mathbf{1}_{\mathfrak{B}}(x_n) \sim \mathrm{vol}(\mathfrak{B})N \qquad (N \to \infty).$$

Remark 1.

$\mathbf{x}$ is equidistributed iff the local discrepancy

$$D_N(\mathbf{x}, \mathfrak{B}) := \sum_{n \in \mathbb{Z}} \mathbf{1}_{[1,N]}(n)\mathbf{1}_{\mathfrak{B}}(x_n) - \mathrm{vol}(\mathfrak{B})N = o_{\mathfrak{B}}(N).$$

Equidistribution

Remark 2.

The Kronecker sequence $(n\alpha \bmod 1)_n$ is equidistributed iff $\alpha \notin \mathbb{Q}$.

L. Kronecker

Follows by unique ergodicity of irrational rotations, as

$$\frac{1}{N}\sum_{n\leq N} 1_{\mathfrak{B}}(R_\alpha^{\circ n}(0+\mathbb{Z})) \to \int_{\mathbb{T}} 1_{\mathfrak{B}}(x)\,\mathrm{d}x = \mathrm{vol}(\mathfrak{B}).$$

Weyl sums: need to check

$$\sum_{n\leq N} e(\ell\alpha n) = o_\ell(N) \qquad \forall_{\ell\in\mathbb{Z}\setminus\{0\}}.$$

Discrepancy

Definition 2 (Discrepancy).

Recalling

$$D_N(\mathbf{x}, \mathfrak{B}) = \sum_{n \in \mathbb{Z}} \mathbf{1}_{[1,N]}(n) \mathbf{1}_{\mathfrak{B}}(x_n) - \mathrm{vol}(\mathfrak{B}) N,$$

define

$$D_N(\mathbf{x}) := \sup_{\substack{\mathfrak{B} \subseteq \mathbb{T} \\ \mathfrak{B} \text{ is an interval}}} |D_N(\mathbf{x}, \mathfrak{B})|.$$

A conjecture of van der Corput

Conjecture 1 (van der Corput [10]; 1935).

Is $D_N(\mathbf{x})$ unbounded.

van Aardenne–Ehrenfest

Theorem 1 (van Aardenne–Ehrenfest [8]; 1945).

Conjecture 1 is true: $D_N(\mathbf{x})$ is unbounded.

T. van Aardenne–Ehrenfest

Refinements

Theorem 2 (van Aardenne–Ehrenfest [8]; 1949).

In fact

$$D_N(\mathbf{x}) = \Omega\Big(\frac{\log\log N}{\log\log\log N}\Big)$$

meaning there exists $C > 0$ so that

$$D_N(\mathbf{x}) \geq C\frac{\log\log N}{\log\log\log N} \qquad \text{for infinitely many } N \geq 1.$$

Theorem 3 (K. F. Roth [7]; 1954).

$D_N(\mathbf{x}) = \Omega(\sqrt{\log N})$.

Two basic mechanisms

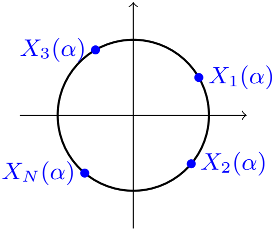
Spot irregularities of $D_N(\mathbf{x}, \mathfrak{B}) = \sum_{n \in \mathbb{Z}} \mathbf{1}_{[1,N]}(n) \mathbf{1}_{\mathfrak{B}}(x_n) - \mathrm{vol}(\mathfrak{B})N$ from

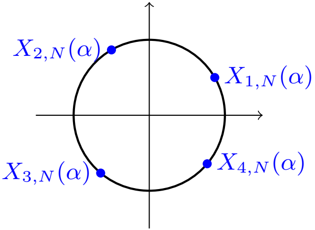
1. (under-count): a $\mathfrak{B}$ with large volume and $\mathfrak{B} \cap \{x_n : n \leq N\} = \emptyset$. Then
$$D_N(\mathbf{x}, \mathfrak{B}) = -\mathrm{vol}(\mathfrak{B})N.$$
2. (over-count): a $\mathfrak{B}$ with small volume, say, $O(1)$ and $\#(\mathfrak{B} \cap \{x_n : n \leq N\})$ large. Then
$$D_N(\mathbf{x}, \mathfrak{B}) \gg \#(\mathfrak{B} \cap \{x_n : n \leq N\}).$$

Random Model

- Let $X_n : \mathbb{T} \to \mathbb{T}$ be independent, $[0,1]$-uniformly distributed RV's.
- Unfold:

$$\{X_1(\alpha), \ldots, X_N(\alpha)\} = \{X_{1,N}(\alpha) \leq \ldots \leq X_{N,N}(\alpha)\}.$$





- Average gap is

$$\frac{1}{N}\sum_{n\leq N}(X_{n+1,N}(\alpha) - X_{n,N}(\alpha)) = \frac{1}{N}$$

where $X_{N+1,N}(\alpha) := 1 + X_{1,N}(\alpha)$ for consistency.

Random Model

- Consider re-normalised gaps $g_{n,N}(\alpha) := N(X_{n+1,N}(\alpha) - X_{n,N}(\alpha))$. Almost surely,

$$\frac{1}{N}\#\{n \leq N : g_{n,N}(\alpha) \in I\} \to \int_I e^{-s}\mathrm{d}s$$

  for any fixed interval $I \subseteq [0, \infty)$.

- So,

$$\#\{n \leq N : g_{n,N}(\alpha) \in [\log N, \infty)\} \approx N \int_{\log N}^{\infty} e^{-s}\mathrm{d}s = 1.$$

The truth in $\mathbb{T}$

Theorem 4 (W. Schmidt [7]; 1972).

The random model is true: $D_N(\mathbf{x}) = \Omega(\log N)$.

Remark 3.

Well-known that

$$D_N((n\alpha \bmod 1)_n) \ll \log N$$

when, e.g., α is a quadratic irrational.

Higher-dimensional discrepancy

Definition 3.

Let

$$\mathfrak{B} := \prod_{i \leq d} [\gamma_i - \rho_i, \gamma_i + \rho_i] \subseteq \mathbb{T}^d. \tag{1}$$

Given $\mathbf{x} \subseteq \mathbb{T}^d$, put

$$D_N(\mathbf{x}, \mathfrak{B}) := \sum_{n \in \mathbb{Z}} \mathbf{1}_{[1,N]}(n) \mathbf{1}_{\mathfrak{B}}(x_n) - \mathrm{vol}(\mathfrak{B}) N$$

and

$$D_N(\mathbf{x}) := \sup_{\mathfrak{B} \text{ as in } (1)} |D_N(\mathbf{x}, \mathfrak{B})|.$$

Roth's Method

Theorem 5 (K. F. Roth [6]).

$D_N(\mathbf{x}) = \Omega_d((\log N)^{\frac{d}{2}})$ *for* $\mathbf{x} \subseteq \mathbb{T}^d$ *and* $d \geq 1$.

Theorem 6 (J. Beck [1]).

$D_N(\mathbf{x}) = \Omega_\varepsilon(\log N(\log\log N)^{\frac{1}{8}-\varepsilon})$ *for* $\mathbf{x} \subseteq \mathbb{T}^2$ *and* $\varepsilon > 0$.

Theorem 7 (Bilyk, Lacey, Vagharshakyan [3]).

$D_N(\mathbf{x}) = \Omega_d((\log N)^{\frac{d}{2}+\eta_d})$ *for some* $\eta_d > 0$, *any* $\mathbf{x} \subseteq \mathbb{T}^d$ *and* $d \geq 1$.

Main Conjecture

Conjecture 2.

Is $D_N(\mathbf{x}) = \Omega_d((\log N)^d)$*?*

Beck's Result

Theorem 8 (J. Beck; 1994).

Let $g : [1, \infty) \to [1, \infty)$ be increasing. Then,

$$D_N((n\boldsymbol{\alpha} \bmod 1)_n) \ll (\log N)^d g(\log \log N) \quad \text{for a.e. } \boldsymbol{\alpha} \in \mathbb{T}^d$$

if and only if

$$\sum_{n \geq 1} \frac{1}{g(n)} < \infty.$$

Corollary 1.

Taking $g_\pm(x) = x(\log x)^{1\pm\varepsilon}$ implies: a.e. $\boldsymbol{\alpha} \in \mathbb{T}^d$ satisfies

$$D_N((n\boldsymbol{\alpha} \bmod 1)_n) \ll (\log N)^d \log \log N \, (\log \log \log N)^{1+\varepsilon}$$

and

$$D_N((n\boldsymbol{\alpha} \bmod 1)_n) = \Omega((\log N)^d \log \log N \, (\log \log \log N)^{1-\varepsilon}).$$

Broad outline of Beck's argument

- Write as lattice point counting problem for $\Lambda_{\boldsymbol{\alpha}} = \begin{pmatrix} I_{d\times d} & \boldsymbol{\alpha} \\ 0 & 1 \end{pmatrix} \mathbb{Z}^{d+1}$ in boxes $\mathfrak{B}$.
- Almost surely smoothing the counting function.
- Local-to-global principle: "the "global irregularities" come from the "local irregularities.""
- E.g. if

$$n_0\alpha \bmod 1 \le \frac{\varepsilon}{n_0}$$

then for $N_0 := \varepsilon^{-1/2} n_0$ the count $\#([0, \frac{\varepsilon^{1/2}}{n_0}] \cap \{n\alpha \bmod 1 : n \le N_0\}) > \varepsilon^{-1/2}$ is large since

$$\{mn_0\alpha \bmod 1 : m \le \varepsilon^{-1/2}\} \subseteq [0, \frac{\varepsilon^{1/2}}{n_0}].$$

Dirichlet's Theorem

Theorem 9 (Dirichlet).

Let $N \geq 1$ and $\boldsymbol{\alpha} \in \mathbb{T}^d$. Then there exists $n \leq N$ with

$$\max_{i \leq d} \|n\alpha_i\| \leq \frac{1}{N^{1/d}} \leq \frac{1}{n^{1/d}} \tag{2}$$

where $\|\cdot\| = \mathrm{dist}(\cdot, \mathbb{Z})$.

Proof.

By Minkowski's 1st Theorem, $\Lambda_{\boldsymbol{\alpha}} = \begin{pmatrix} I_{d\times d} & \boldsymbol{\alpha} \\ 0 & 1 \end{pmatrix} \mathbb{Z}^{d+1}$ has a non-zero lattice point $\begin{pmatrix} I_{d\times d} & \boldsymbol{\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ n \end{pmatrix} = \begin{pmatrix} a_1 + n\alpha_1, & \ldots & , a_d + n\alpha_d, & n \end{pmatrix}^T$ in $B = \left[-\frac{1}{N^{1/d}}, \frac{1}{N^{1/d}}\right] \times \ldots \times \left[-\frac{1}{N^{1/d}}, \frac{1}{N^{1/d}}\right] \times \left[-N, N\right]$. □

Badly approximables

Remark 4.

Dirichlet's theorem is optimal (up to a constant), since

$$\mathrm{Bad}_d := \left\{ \boldsymbol{\alpha} \in \mathbb{T}^d : \liminf_{n\to\infty} n^{1/d} \cdot (\|n\alpha_1\| + \ldots + \|n\alpha_d\|) > 0 \right\}$$

is non-empty; its elements are called badly approximable vectors. Consider

$$\mathrm{Bad}_d^{\times} := \left\{ \boldsymbol{\alpha} \in \mathbb{T}^d : \liminf_{n\to\infty} n \cdot (\|n\alpha_1\| \cdot \ldots \cdot \|n\alpha_d\|) > 0 \right\}.$$

Clearly,

$$\mathrm{Bad}_d^{\times} \subseteq \mathrm{Bad}_1 \times \ldots \times \mathrm{Bad}_1.$$

Littlewood's Conjecture

Conjecture 3 (Littlewood, ca. 1930).

$\mathrm{Bad}_2^\times$ *is empty.*

E. Littlewood

What's known?

Remark 5.

(a) Einsiedler, Katok, and E. Lindenstrauss [5]: $\mathrm{Bad}_2^\times$ has zero Hausdorff dimension.

(b) Pollington and S. Velani [11]: For any $\alpha_1 \in \mathrm{Bad}_1$ there exists a set of $\alpha_2 \in \mathrm{Bad}_1$, with full Hausdorff dimension, so that

$$n \cdot \|n\alpha_1\| \cdot \|n\alpha_2\| < \frac{1}{\log n} \quad \text{infinitely often.}$$

Smooth weights

Definition 4.

We smooth in all

$$k := d + 1$$

variables. Let $\mathcal{G}_k$ be the set of $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_k) : \mathbb{R}^k \to [0, \infty)^k$ with

$$\omega_i : \mathbb{R} \to [0, \infty), \qquad \omega_i \in C^\infty, \qquad \mathrm{supp}(\omega_i) \subseteq [-2, 2],$$

and

$$\widehat{\omega}_i(x) := \int_{\mathbb{R}} \omega_i(y) e(-xy) \mathrm{d}y > 0$$

for all $i \leq k$.

Smooth Discrepancy

Definition 5.

Let $\boldsymbol{\omega} \in \mathcal{G}_k$. Given

$$\mathfrak{B} = \prod_{i \leq d} [\gamma_i - \rho_i, \gamma_i + \rho_i], \qquad \boldsymbol{\rho} \in [0, 1/2)^d, \quad \text{and} \quad \boldsymbol{\gamma} \in \mathbb{R}^d \qquad (3)$$

let

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}, \mathfrak{B}) := \sum_{(n,\mathbf{a}) \in \mathbb{Z}^k} \omega_k\left(\frac{n}{N}\right) \prod_{i \leq d} \omega_i\left(\frac{n\alpha_i + a_i - \gamma_i}{\rho_i}\right) - N\mathrm{vol}(\mathfrak{B}) \prod_{i \leq d} \widehat{\omega}_i(0)$$

and

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}) := \sup_{\mathfrak{B} \text{ as in } (3)} |D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}, \mathfrak{B})|.$$

Further,

$$H(\mathbf{y}) := \prod_{i \leq d} \max(1, |y_i|).$$

Upper-bound

Theorem 10 (Sam Chow, N.T.).

Let $\boldsymbol{\omega} \in \mathcal{G}_k$. *Suppose* $\phi : [1, \infty) \to [1, \infty)$ *is increasing so that*

$$\|\boldsymbol{\alpha} \cdot \boldsymbol{n}\| \geq \frac{1}{H(\boldsymbol{n})\phi(H(\boldsymbol{n}))} \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}.$$

Define $L : [\phi(1), \infty) \to [1, \infty)$ *via* $L(x)\phi(L(x)) = x$. *Then*

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}) \ll_{\boldsymbol{\omega}} \phi(L(N)),$$

provided $\phi(2x) \ll \phi(x)$.

Lower bound

Theorem 11 (Sam Chow, N.T.).

Let $\boldsymbol{\omega} \in \mathcal{G}_k$. *If*

$$\|\boldsymbol{\alpha} \cdot \boldsymbol{n}\| < \frac{1}{H(\boldsymbol{n})\phi(H(\boldsymbol{n}))} \quad \text{for } \infty - \text{many } \boldsymbol{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$$

for some increasing $\phi : [1, \infty) \to [1, \infty)$ *and let* $L(x)\phi(L(x)) = x$, *then*

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}) \gg_{\boldsymbol{\omega}} \phi(L(N)).$$

Consequence

Corollary 2 (Sam Chow, N.T.).

Littlewood's conjecture is true if and only if the C^3-discrepancy of any Kronecker sequence in $\mathbb{T}^2$ is unbounded.

Inflation and Geometry of Numbers

- Suppose $\mathrm{vol}(\mathfrak{B}) \leq \phi(L(N))$. Take $\mathfrak{B}'$ with $\mathfrak{B} \subseteq \mathfrak{B}'$ and $\mathrm{vol}(\mathfrak{B}') = \phi(L(N))$.
- Put
$$w_{\mathfrak{B}}(\mathbf{y}) := \omega_k\left(\frac{y_k}{N}\right) \prod_{i \leq d} \omega_i\left(\frac{y_i - \gamma_i}{\rho_i}\right) \geq 0.$$
- Notice
$$-N\mathrm{vol}\mathfrak{B} \prod_{i \leq d} \widehat{\omega}_i(0) \leq D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}, \mathfrak{B}) = \sum_{\boldsymbol{\lambda} \in \Lambda_{\boldsymbol{\alpha}}} w_{\mathfrak{B}}(\boldsymbol{\lambda}) - N\mathrm{vol}\mathfrak{B} \prod_{i \leq d} \widehat{\omega}_i(0)$$
and
$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha}, \mathfrak{B}) \leq \sum_{\boldsymbol{\lambda} \in \Lambda_{\boldsymbol{\alpha}}} w_{\mathfrak{B}}(\boldsymbol{\lambda}).$$

Inflation and Geometry of Numbers

- Upshot

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha},\mathfrak{B}) \leq D_{4\cdot N}(\boldsymbol{\alpha},\mathfrak{B}') + \phi(L(N)).$$

- Use geometry of numbers to show the Bohr set

$$\#(\mathfrak{B}' \cap \Lambda_{\boldsymbol{\alpha}}) = \#\{n \leq 4N : \ \|n\alpha_i - \gamma_i\| \leq \rho_i \quad (i \leq d)\}$$

has at most $O(\mathrm{vol}(\mathfrak{B}'))$ elements.

Fourier analysis and Gap Argument

- Suppose $\mathrm{vol}(\mathfrak{B}) > \phi(L(N))$. By Poisson summation,

$$\sum_{\boldsymbol{\lambda}\in\Lambda_{\boldsymbol{\alpha}}} w_{\mathfrak{B}}(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\lambda}^*\in\Lambda_{\boldsymbol{\alpha}}^*} \widehat{w_{\mathfrak{B}}}(\boldsymbol{\lambda}^*)$$

where $\Lambda_{\boldsymbol{\alpha}}^* = \begin{pmatrix} I_{d\times d} & 0 \\ -\boldsymbol{\alpha} & 1 \end{pmatrix} \mathbb{Z}^k$ is the dual $\Lambda_{\boldsymbol{\alpha}}$.

- Thus,

$$D_{N,\boldsymbol{\omega}}(\boldsymbol{\alpha},\mathfrak{B}) = \sum_{\boldsymbol{\lambda}^*\in\Lambda_{\boldsymbol{\alpha}}^*\setminus\{\mathbf{0}\}} \widehat{w_{\mathfrak{B}}}(\boldsymbol{\lambda}^*).$$

Analyze the right hand side by a gap principle.

The End

Thank you very much for your attention!

References

J. Beck: *A two-dimensional van Aardenne-Ehrenfest theorem in irregularities of distribution*, Compositio Math. 72 (1989), 269–339

J. Beck: *Probabilistic diophantine approximation, I. Kronecker sequences,* Annals of Mathematics 140.2 (1994): 449-502.

D. Bilyk, M. T. Lacey, and A. Vagharshakyan: *On the small ball inequality in all dimensions,* Journal of Functional Analysis 254.9 (2008): 2470-2502.

S. Chow, and N. Technau: *Smooth discrepancy and Littlewood's conjecture*, arXiv preprint arXiv:2409.17006 (2024).

M. Einsiedler, A. Katok, and E. Lindenstrauss: *Invariant measures and the set of exceptions to Littlewood's conjecture*, Annals of Mathematics (2006): 513-560.

K. F. Roth:*On irregularities of distribution."* Mathematika 1.2 (1954): 73-79.

W. Schmidt: *Irregularities of distribution VII,* Acta Arithmetica 21.1 (1972): 45-50.

T. van Aardenne-Ehrenfest: *Proof of the impossibility of a just distribution of an infinite sequence of points over an interval*, Proc. Kon. Ned. Akad. v. Wetensch 48 (1945): 266-271.

T. van Aardenne-Ehrenfest: *On the impossibility of a just distribution*, Proc. Kon. Ned. Akad. v. Wetensch 52 (1949): 734-739.

J. G. van der Corput: *Verteilungsfunktionen i.* Proc. Kon. Ned. Akad. v. Wetensch. 38 (1935): 813-821.

A. D. Pollington, and S. L. Velani: *On a problem in simultaneous diophantine approximation: Littlewood's conjecture,* Acta Mathematica 185.2 (2000): 287-306.