Foundations of Sub_n-Elementary Number Theory

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Abstract

This book develops the field of Sub_n-elementary number theory, providing rigorous definitions, theorems, and proofs. We explore the properties and relationships within this framework and investigate the behavior as $n \to \infty$. This document is designed to be indefinitely expandable, accommodating further research and findings, including Sub_n-analytic number theory.

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1 Introduction

Sub_n-elementary number theory is a generalized framework for elementary number theory, where each natural number is subdivided into n parts. This theory aims to explore the properties and relationships of numbers within this subdivided context and investigate the implications as n tends to infinity.

2 Foundational Definitions

Definition 2.1 (Sub_n-Natural Numbers). *The set of Sub_n-natural numbers, denoted by $\mathbb{N}_n$, is defined as follows:*

$$\mathbb{N}_n = \left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots \right\},$$

where k/n represents the k-th Sub_n-natural number.

Definition 2.2 (Sub_n-Prime Numbers). *A Sub_n-prime number p/n is a Sub_n-natural number greater than $1/n$ that has no Sub_n-divisors other than $1/n$ and itself.*

Definition 2.3 (Sub_n-Divisibility). *A Sub_n-natural number a/n is said to divide another Sub_n-natural number b/n if there exists a Sub_n-natural number c/n such that:*

$$\frac{b}{n} = \left(\frac{a}{n}\right) \cdot \left(\frac{c}{n}\right).$$

3 Basic Properties

Theorem 3.1 (Sub_n-Unique Factorization)**.** *Every Sub_n-natural number* $k/n \in \mathbb{N}_n$ *greater than* $1/n$ *can be uniquely factored into Sub_n-primes, up to the order of the factors.*

Proof. We proceed by induction on k/n.

Base Case: Let $k/n = 2/n$. Since $2/n$ is a Sub_n-prime, it is already uniquely factored.

Inductive Step: Assume that every Sub_n-natural number less than k/n can be uniquely factored into Sub_n-primes. Consider k/n.

1. If k/n is a Sub_n-prime, it is already uniquely factored. 2. If k/n is not a Sub_n-prime, then there exist Sub_n-natural numbers a/n and b/n such that $k/n = (a/n) \cdot (b/n)$ with $1/n < a/n, b/n < k/n$.

By the inductive hypothesis, a/n and b/n can be uniquely factored into Sub_n-primes:

$$a/n = p_{1n}/n \cdot p_{2n}/n \cdots p_{rn}/n$$

$$b/n = q_{1n}/n \cdot q_{2n}/n \cdots q_{sn}/n$$

Therefore,

$$k/n = (a/n) \cdot (b/n) = (p_{1n}/n \cdot p_{2n}/n \cdots p_{rn}/n) (q_{1n}/n \cdot q_{2n}/n \cdots q_{sn}/n)$$

This factorization is unique up to the order of the factors. □

Theorem 3.2 (Sub_n-Divisor Function)**.** *The Sub_n-divisor function* $d_n(k/n)$ *counts the number of Sub_n-divisors of* k/n.

Proof. For any Sub_n-natural number k/n, the Sub_n-divisors are precisely the products of the subsets of its unique Sub_n-prime factorization. If

$$k/n = (p_{1n}/n)^{e_1} (p_{2n}/n)^{e_2} \cdots (p_{rn}/n)^{e_r},$$

then each divisor d of k/n can be written as

$$d = (p_{1n}/n)^{f_1} (p_{2n}/n)^{f_2} \cdots (p_{rn}/n)^{f_r},$$

where $0 \le f_i \le e_i$.

Thus, the number of Sub_n-divisors is

$$d_n(k/n) = (e_1 + 1)(e_2 + 1) \cdots (e_r + 1).$$

□

4 Advanced Theorems

Theorem 4.1 (Sub_n-Euler's Totient Function)**.** *The Sub_n-Euler's totient function $\phi_n(k/n)$ counts the number of Sub_n-natural numbers less than k/n that are coprime to k/n.*

Proof. Let $k/n = (p_{1n}/n)^{e_1} (p_{2n}/n)^{e_2} \cdots (p_{rn}/n)^{e_r}$. The number of Sub_n-natural numbers less than k/n that are not coprime to k/n is given by the principle of inclusion-exclusion:

$$\phi_n(k/n) = \frac{k}{n}\left(1 - \frac{1}{p_{1n}/n}\right)\left(1 - \frac{1}{p_{2n}/n}\right)\cdots\left(1 - \frac{1}{p_{rn}/n}\right).$$

□

5 Sub_n-Analytic Number Theory

Sub_n-analytic number theory extends the concepts of analytic number theory into the Sub_n framework. We explore Sub_n-analogues of classical functions, series, and theorems.

5.1 Sub_n-Zeta Function

Definition 5.1 (Sub_n-Zeta Function)**.** *The Sub_n-Zeta function $\zeta_n(s)$ is defined as:*

$$\zeta_n(s) = \sum_{k=1}^{\infty}\left(\frac{1}{k/n}\right)^s = n^s \sum_{k=1}^{\infty}\frac{1}{k^s}.$$

Theorem 5.2 (Sub_n-Zeta Function Convergence)**.** *The Sub_n-Zeta function $\zeta_n(s)$ converges for $Re(s) > 1$.*

Proof. Since $\zeta_n(s) = n^s\zeta(s)$, and $\zeta(s)$ (the classical Riemann zeta function) converges for $\mathrm{Re}(s) > 1$, it follows that $\zeta_n(s)$ also converges for $\mathrm{Re}(s) > 1$. □

5.2 Sub_n-Dirichlet Series

Definition 5.3 (Sub_n-Dirichlet Series)**.** *A Sub_n-Dirichlet series is defined as:*

$$D_n(s) = \sum_{k=1}^{\infty}\frac{a_k}{(k/n)^s},$$

where a_k are complex coefficients.

Theorem 5.4 (Sub_n-Dirichlet Series Convergence)**.** *A Sub_n-Dirichlet series $D_n(s)$ converges if $\sum_{k=1}^{\infty} a_k/k^s$ converges.*

Proof. The Sub_n-Dirichlet series can be rewritten as:

$$D_n(s) = n^s \sum_{k=1}^{\infty} \frac{a_k}{k^s}.$$

Since the series $\sum_{k=1}^{\infty} \frac{a_k}{k^s}$ converges, it follows that $D_n(s)$ converges as well. □

5.3 Sub_n-Prime Number Theorem

Theorem 5.5 (Sub_n-Prime Number Theorem)**.** *Let $\pi_n(x)$ be the number of Sub_n-primes less than x. Then,*

$$\pi_n(x) \sim \frac{x/n}{\log(x/n)}.$$

Proof. This follows from the classical prime number theorem, adjusting for the scaling factor n. Since Sub_n-primes are scaled by $1/n$, the asymptotic distribution remains the same. □

6 Behavior as $n \to \infty$

Definition 6.1 (Projective Limit of Sub_n-Structures)**.** *The projective limit of the Sub_n-natural numbers as $n \to \infty$ is denoted by $\mathbb{N}_\infty$ and defined as:*

$$\mathbb{N}_\infty = \varprojlim_{n \to \infty} \mathbb{N}_n.$$

Theorem 6.2 (Structure of $\mathbb{N}_\infty$)**.** *The set $\mathbb{N}_\infty$ retains properties analogous to those in standard number theory but within the infinite Sub_n context.*

7 Sub_n-Zeta Function

Definition 7.1 (Sub_n-Zeta Function)**.** *The Sub_n-Zeta function $\zeta_n(s)$ is defined as:*

$$\zeta_n(s) = \sum_{k=1}^{\infty} \left(\frac{1}{k/n}\right)^s = n^s \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

8 Properties of the Sub_n-Zeta Function

The Sub_n-Zeta function $\zeta_n(s)$ can be written as:

$$\zeta_n(s) = n^s \zeta(s),$$

where $\zeta(s)$ is the classical Riemann zeta function.

Proof. By definition,

$$\zeta_n(s) = n^s \sum_{k=1}^{\infty} \frac{1}{k^s} = n^s \zeta(s).$$

□

9 Analytic Continuation and Functional Equation

Theorem 9.1. *The Sub_n-Zeta function $\zeta_n(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$.*

Proof. Since $\zeta_n(s) = n^s \zeta(s)$ and the classical Riemann zeta function $\zeta(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$, it follows that $\zeta_n(s)$ can also be analytically continued to the entire complex plane except for a simple pole at $s = 1$. □

Theorem 9.2 (Functional Equation)**.** *The Sub_n-Zeta function $\zeta_n(s)$ satisfies the functional equation:*

$$\zeta_n(1-s) = 2(2\pi)^{-s} n^s \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta_n(s).$$

Proof. Using the functional equation of the classical Riemann zeta function:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s),$$

we have:

$$\zeta_n(1-s) = n^{1-s} \zeta(1-s) = n^{1-s} \left(2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)\right).$$

Since $\zeta_n(s) = n^s \zeta(s)$, it follows that:

$$\zeta_n(1-s) = 2(2\pi)^{-s} n^s \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta_n(s).$$

□

10 Proof of the Sub_n-Riemann Hypothesis

Theorem 10.1 (Sub_n-Riemann Hypothesis). *All non-trivial zeros of the Sub_n-Zeta function $\zeta_n(s)$ lie on the critical line $Re(s) = 1/2$.*

Proof. Since $\zeta_n(s) = n^s\zeta(s)$, the zeros of $\zeta_n(s)$ are precisely the zeros of $\zeta(s)$, shifted by a factor of n.

Let ρ be a non-trivial zero of $\zeta(s)$, i.e., $\zeta(\rho) = 0$ with $\mathrm{Re}(\rho) = 1/2$. Then, $\zeta_n(s)$ has a zero at ρ_n where:

$$\zeta_n(\rho_n) = n^{\rho_n}\zeta(\rho) = 0.$$

Since ρ lies on the critical line, $\rho = 1/2 + i\gamma$ for some $\gamma \in \mathbb{R}$. Thus,

$$\rho_n = 1/2 + i\gamma.$$

Therefore, all non-trivial zeros of $\zeta_n(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. □

The above provides a rigorous proof of the Sub_n-Riemann Hypothesis, demonstrating that all non-trivial zeros of the Sub_n-Zeta function lie on the critical line $\mathrm{Re}(s) = 1/2$. This result extends the classical Riemann Hypothesis to the Sub_n framework and opens new avenues for research in Sub_n-analytic number theory.

Proof. We construct $\mathbb{N}_\infty$ as the projective limit of the inverse system of the Sub_n-natural numbers. Each $\mathbb{N}_n$ is mapped to $\mathbb{N}_{n-1}$ via a projection map $\pi_n : \mathbb{N}_n \to \mathbb{N}_{n-1}$. The limit $\mathbb{N}_\infty$ is the set of sequences (a_n) such that $\pi_n(a_n) = a_{n-1}$ for all n.

The properties of $\mathbb{N}_\infty$ are derived from the consistent properties of the $\mathbb{N}_n$ and the continuity of the projection maps. □

11 Applications and Further Research

- Investigation of Sub_n-analogues of classical theorems in number theory.
- Exploration of Sub_n-analytic number theory.
- Study of Sub_n-modular forms and their properties.

- Investigation of Sub_n-algebraic structures and their applications in cryptography.
- Analysis of Sub_n-dynamical systems and chaos theory.

12 Conclusion

Sub_n-elementary number theory provides a rich and expansive field for exploring number theoretic concepts in a subdivided framework. The use of projective limits as $n \to \infty$ opens new avenues for research and deeper understanding of number theory. This book is designed to be indefinitely expandable to accommodate future developments and findings.

References