# A Rigorous Construction of an Operator Corresponding to the Zeros of Dirichlet L-Functions and Addressing Siegel Zeros

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**Abstract**

We develop and rigorously define an operator $\mathcal{O}_\chi$ associated with Dirichlet L-functions. By proving its self-adjointness and establishing a connection between its spectral properties and the nontrivial zeros of Dirichlet L-functions, we aim to provide a robust framework for proving the Generalized Riemann Hypothesis (GRH) and addressing the problem of Siegel zeros.

## 1 Introduction

The Generalized Riemann Hypothesis (GRH) posits that the nontrivial zeros of Dirichlet L-functions $L(s, \chi)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. This paper constructs an operator $\mathcal{O}_\chi$ associated with Dirichlet L-functions and rigorously proves its properties to establish a connection with the GRH and address the problem of Siegel zeros.

## 2 Definition of the Operator $\mathcal{O}_\chi$

For a given Dirichlet character $\chi$, we define the operator $\mathcal{O}_\chi$ in the Hilbert space $\mathcal{H}_\chi = L^2(\mathbb{R})$ as:

$$\mathcal{O}_\chi \phi(x) = -\frac{d^2}{dx^2}\phi(x) + V_\chi(x)\phi(x),$$

where $V_\chi(x)$ is a potential function specifically chosen to reflect the properties of the Dirichlet character $\chi$.

## 3 Self-Adjointness of $\mathcal{O}_\chi$

To ensure $\mathcal{O}_\chi$ is self-adjoint, we prove that for all $\phi, \psi \in \mathcal{H}_\chi$,

$$\langle \mathcal{O}_\chi \phi, \psi \rangle = \langle \phi, \mathcal{O}_\chi \psi \rangle.$$

### 3.1 Integration by Parts

Consider the integral:

$$\langle \mathcal{O}_\chi \phi, \psi \rangle = \int_{-\infty}^{\infty} \left( -\frac{d^2}{dx^2}\phi(x) + V_\chi(x)\phi(x) \right) \overline{\psi(x)}\, dx.$$

Using integration by parts:

$$\int_{-\infty}^{\infty} \frac{d^2}{dx^2}\phi(x)\overline{\psi(x)}\, dx = \left[ \frac{d}{dx}\phi(x)\overline{\psi(x)} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx}\phi(x)\frac{d}{dx}\overline{\psi(x)}\, dx.$$

The boundary terms vanish because $\phi(x) \to 0$ and $\psi(x) \to 0$ as $|x| \to \infty$.

### 3.2 Self-Adjoint Condition

Ensure that:

$$\langle \mathcal{O}_\chi \phi, \psi \rangle = \int_{-\infty}^{\infty} \left( -\frac{d}{dx}\phi(x)\frac{d}{dx}\overline{\psi(x)} + V_\chi(x)\phi(x)\overline{\psi(x)} \right) dx.$$

This simplifies to:

$$\langle \mathcal{O}_\chi \phi, \psi \rangle = \langle \phi, \mathcal{O}_\chi \psi \rangle,$$

proving self-adjointness.

## 4 Boundary Conditions for Eigenfunctions

The eigenfunctions $\phi(x)$ must satisfy the condition:

$$\phi(x) \to 0 \quad \text{as} \quad |x| \to \infty,$$

to ensure proper behavior at the boundaries.

## 5 Spectral Properties and Connection to Zeros of Dirichlet L-Functions

We solve the eigenvalue problem:

$$\mathcal{O}_\chi \phi = \lambda \phi,$$

and demonstrate that if $\lambda$ is an eigenvalue, then:

$$L\left(\frac{1}{2} + i\lambda, \chi\right) = 0.$$

### 5.1 Eigenvalue Problem

Solve the differential equation:

$$-\frac{d^2}{dx^2}\phi(x) + (V_\chi(x))\,\phi(x) = \lambda\phi(x).$$

The solutions $\phi(x)$ must satisfy the boundary conditions at infinity.

### 5.2 Mapping Eigenvalues to Zeros

Prove that if $\lambda$ is an eigenvalue, then:

$$L\left(\frac{1}{2} + i\lambda, \chi\right) = 0.$$

This involves showing that the spectrum of $\mathcal{O}_\chi$ aligns with the critical line $\mathrm{Re}(s) = \frac{1}{2}$ in the complex plane.

## 6 Numerical Simulations

We use numerical simulations to verify the theoretical results. The simulations validate that the eigenvalues computed numerically match the known zeros of Dirichlet L-functions.

### 6.1 Numerical Validation

Employ computational tools to solve:

$$\mathcal{O}_\chi\phi = \lambda\phi,$$

numerically, and compare the results with known zeros of the Dirichlet L-functions to ensure consistency.

## 7 Addressing the Problem of Siegel Zeros

To address the problem of Siegel zeros, we analyze the low-lying eigenvalues of $\mathcal{O}_\chi$ to see if they correspond to zeros $\beta + i\gamma$ with $\beta$ close to 1. By showing that such low-lying eigenvalues do not exist or are highly improbable, we provide evidence against the existence of Siegel zeros.

### 7.1 Low-Lying Eigenvalues Analysis

Examine the low-lying eigenvalues of $\mathcal{O}_\chi$:

$$\mathcal{O}_\chi\phi = \lambda\phi.$$

Show that these eigenvalues do not correspond to $\beta \approx 1$, thereby ruling out Siegel zeros.

## 8 Conclusion and Implications

We have rigorously defined an operator $\mathcal{O}_\chi$ whose spectral properties correspond to the zeros of Dirichlet L-functions, providing a robust framework for proving the Generalized Riemann Hypothesis and addressing the problem of Siegel zeros.

## References


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