VOLUME II: HYPER-FILTRATION THEORY AND TRANSFINITE MONODROMY
FROM ADDITIVE NILPOTENTS TO KNUTH-LEVEL DYNAMICS

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ABSTRACT. This volume introduces the theory of ε-hyperfiltrations and transfinite arithmetic cohomology. Extending the exponentoid and knuthoid filtrations of Volume I, we define hyper-monodromy groups, meta-period rings, and ontologically accelerated torsor structures. These constructions form the foundation for arithmetic cohomology and motivic realization beyond exponential and recursive layers, initiating the study of transfinite descent geometry and infinite cohomological generation.

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0. Notation and Symbol Dictionary

This section compiles the primary notation used throughout Volume II, with emphasis on ε-hyperfiltrations, transfinite stratification, meta-periodic towers, and cohomological constructions beyond recursion.

Growth Indices and Hyper-Operators.

- ε^n : denotes n-fold stratified filtration levels under recursive or transfinite indexing.

- $a \uparrow^k n$: Knuth's k-level hyper-operator, e.g., $\uparrow^2 = \exp^{\circ n}$.
- $f \prec g$: $f(n)$ grows asymptotically slower than $g(n)$.

Hyper-Filtration Systems.

- $F^{\varepsilon^n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/\varepsilon^n \cdot \mathcal{F})$: ε-hyperfiltration.
- $F^{\varepsilon^\infty} := \bigcap_n F^{\varepsilon^n}$: limit filtration across all stratified levels.
- $\mathsf{Filt}_{\varepsilon^\infty}$: category of sheaves filtered over ε-indexed transfinite towers.

Period Rings and Cohomology.

- $B_{\infty,dR} := \varprojlim_f A/f(n) \otimes \mathbb{Q}$: meta-period ring over all recursive growth functions.
- $H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_n H^\bullet(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F})$: ε-hypercohomology.
- $H^\bullet_\infty(X)$: transfinite cohomology across all filtration types.

Torsor and Monodromy Structures.

- $\mathbb{T}^{[\varepsilon^\infty]}$: transfinite ε-torsor tower over a stratified space.
- $\mathcal{M}_{\text{hyper}}$: hyper-monodromy group of stratified recursion actions.
- $\mathbb{T}^{[\varepsilon^n]}$: torsor layer at level ε^n.

Motivic Realizations.

- $M^{[\varepsilon^\infty]}(X)$: ε-hypermotivic object of a space X.
- $\text{real}_{\text{hyper}}$: realization functor from motives to ε-cohomology.
- $r_{\varepsilon^\infty} : K_n(X) \to H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n))$: trans-recursive regulator map.

Ontological Structures.

- $\mathcal{O}nt_{\varepsilon^\infty}(n) := \text{Sh}(F^{\varepsilon^n}\mathcal{F})$: categorical layer of sheaf ontology at recursion depth ε^n.
- $\mathcal{O}nt_{\varepsilon^\infty} := \varprojlim_n \mathcal{O}nt_{\varepsilon^\infty}(n)$: ontological limit of stratified categorical space.

Meta-Theoretic Notation.

- Growth : category of growth functions ordered by asymptotic domination.
- $\mathcal{O}nt^{\text{Meta}}$: meta-stack of logic-indexed cohomological objects.
- $\lim_{\uparrow^k \to \infty} \mathcal{T}_n^{\uparrow^k}$: transfinite torsor collapse.

Conventions. Throughout, unless specified otherwise:
- All sheaves are assumed to be $\mathbb{Q}$-linear;
- All towers are assumed to be cofiltered and compatible with filtered colimits;
- All filtrations are indexed by growth functions in Growth or transfinite ε-layers.

1. Introduction and Foundational Setup

1.1. **From Recursive Geometry to Hyper-Stratification.** Volume I introduced exponentoid and knuthoid geometries, where stratification was indexed by recursive growth functions like $\exp(n)$ and $a \uparrow^k n$. These generalized multiplicoid filtrations and suggested a new framework: *geometry as stratified recursion.*

In this volume, we take a further step: we introduce ε-hyperfiltrations, transfinite torsor towers, and cohomology defined not by convergence, but by infinite persistence under layered recursion.

> This is not merely "higher geometry"—this is geometry redefined as a structure of trans-recursive depth.

1.2. **The Need for ε-Hyperfiltration.** Classical filtrations (e.g., Hodge, valuation, syntomic) stratify spaces through additive or multiplicative approximations. However, they are bounded by linear or exponential growth.

But what if:

- The meaningful structures persist only at transfinite depth?
- The true invariants arise not at finite filtration levels, but as *limits of growth types*?
- Geometry emerges only when recursion becomes unbounded?

This leads to the core objects of this volume:

- **ε-Hyperfiltrations**: infinite towers of sheaves filtered by iterated logic.
- **Hyper-Monodromy**: generalization of classical monodromy to stratified automorphism towers.
- **Transfinite Period Rings**: period structures stabilized under all recursive depth levels.
- **Ontological Stratification**: existence defined by recursive stability, not spatial position.

1.3. **Position Within the Yang Program.** This volume is the second in a series of recursive geometric foundations:

1. Volume I: *Exponentoid and Knuthoid Spaces*, defined filtered arithmetic geometry over recursion.
2. **Volume II:** defines geometry over transfinite recursion.
3. Volume III: will study weight–monodromy and motivic consequences in these settings.
4. Volume IV: will elevate the recursion stack into ontological logic categories.
5. Volume V: will integrate all layers into a categorical arithmetic theory.

Each level deepens the structure: from topological, to recursive, to meta-logical.

1.4. **Objectives of This Volume.** In this volume, we aim to:

- Define the theory of ε-stratified filtrations;
- Introduce and classify ε^∞-torsors and their cohomology;
- Construct $B_{\infty,dR}$ as the transfinite period ring;
- Build the functor $\mathrm{real}_{\mathrm{hyper}}$ from ε-hypermotives to ε-cohomology;
- Formulate conjectures on infinite persistence, regulator collapse, and meta-motivic realization.

These constructions pave the way for a trans-recursive arithmetic geometry beyond perfectoid, syntomic, and motivic theories.

1.5. **Overview.** The structure of this volume is as follows:

- Section 2: introduces ε-hyperfiltrations and recursive descent models;
- Section 3: defines transfinite filtration towers and persistence depth;
- Section 4: constructs hyper-monodromy groups and automorphism stacks;
- Section 5: builds realization theory for ε-motives;
- Section 6: develops hyper-cohomology and regulator systems;
- Section 7: formulates transfinite period morphisms and cohomological stabilization;
- Section 8: concludes with meta-conjectures about stratified ontological foundations.

Geometry is no longer where we are—it is what survives forever.

2. ε-Hyperfiltrations and Recursive Depth Structures

2.1. **Beyond Polynomial and Exponential Filtrations.** Previous filtration regimes (additive, multiplicoid, exponentoid, knuthoid) stratified sheaves and cohomology by functions such as:

$$n, \quad 2^n, \quad e^n, \quad a \uparrow^k n.$$

These correspond to recursive hierarchies of finite depth. We now pass to a higher framework:
- ε**-filtrations** index layers by meta-operators;
- ε**-depth** replaces modulus with recursion level;
- **Hyperfiltration towers** capture infinite progression of cohomological collapse.

2.2. **Definition of ε-Hyperfiltration.** Let ε^n denote the n-th level of trans-recursive stratification. We define:

Definition 2.1 (ε-Hyperfiltration)**.** *Let $\mathcal{F}$ be a sheaf. Define the ε-hyperfiltration:*

$$F^{\varepsilon^n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right).$$

Here, ε^n acts as a formal growth index, not a number. Its meaning is induced by a logic-encoded filtration structure, such as an ordinal tower, proof-theoretic strength, or recursive complexity class.

2.3. **Transfinite Growth Category.** We define a new indexing system for hyperfiltrations:

Definition 2.2 (ε-Growth Category)**.** *Let* $\mathsf{Growth}_{\varepsilon^\infty}$ *be the category whose objects are functions* $f : \mathbb{N} \to \mathbb{N}$ *such that:*

$$\forall m,\ f(n) \succ a \uparrow^m n \quad \textit{for } n \gg 0.$$

Morphisms are asymptotic domination: $f \to g$ if $f(n) \leq g(n)$ for sufficiently large n.

This category replaces Growth from Volume I, extending beyond primitive recursive operators.

2.4. **Tower of ε-Layers and Stability.** We define the ε-hyperfiltration tower as:

$$\mathcal{F} \supset F^{\varepsilon^1}\mathcal{F} \supset F^{\varepsilon^2}\mathcal{F} \supset \cdots \supset F^{\varepsilon^n}\mathcal{F} \supset \cdots$$

Each level corresponds to:
- A deeper layer of torsor symmetry collapse;
- A longer persistence under recursion;
- A smaller cohomological shadow.

2.5. **Limit Filtration and Persistence Core.**

Definition 2.3 (Limit ε-Filtration)**.** *Define:*

$$F^{\varepsilon^\infty}\mathcal{F} := \bigcap_n F^{\varepsilon^n}\mathcal{F}.$$

Definition 2.4 (ε-Persistent Section)**.** *A section $s \in \mathcal{F}$ is said to be* ε-persistent *if $s \in F^{\varepsilon^\infty}\mathcal{F}$.*

This defines the ontological core of a sheaf—the set of sections surviving infinite hyperfiltration.

2.6. **Examples.**

Example 2.5 (Arithmetic ε-Stratification). Let $A = \mathbb{Z}[x]$, and define:

$$\varepsilon^n := \text{least common multiple of } \{1, 2, \ldots, a \uparrow^n 1\}.$$

Then $F^{\varepsilon^n}A$ corresponds to functions vanishing modulo ever-expanding arithmetic depth.

Example 2.6 (Proof-Theoretic Interpretation). If $\mathcal{F}$ is a logic-sheaf on a type-theoretic universe, $F^{\varepsilon^n}\mathcal{F}$ may encode provability under meta-logical strength Π_n or consistency rank ε_n.

2.7. **Summary.** The ε-hyperfiltration defines:

- Transfinite stratification indexed by meta-growth;
- Infinite descent towers into the persistence core;
- A universal framework for recursive cohomology and logic-encoded geometry;
- The entry point into ε-torsors and transfinite motivic realizations.

In the next section, we define filtration towers and depth categories based on these ε-structures.

3. Transfinite Filtration Towers and Ontological Acceleration

3.1. **From Finite Depth to Meta-Stratified Geometry.** In classical geometry, filtrations like Hodge or Newton decay polynomially. In multiplicoid and exponentoid geometry, filtration depth increases exponentially.

In ε-hyperfiltration theory, growth is no longer numeric—it is logical. Each filtration level represents a *meta-recursive collapse*, such as:

$$F^{\varepsilon^n}\mathcal{F} \;\sim\; \text{Descent through all torsors with growth below } a \uparrow^n 1.$$

This demands a new tower model of existence.

3.2. **The ε-Stratified Filtration Tower.** Let $\mathcal{F}$ be a sheaf over a space X. Define the tower:

$$\mathcal{F} \to \cdots \to \mathcal{F}/\varepsilon^3 \to \mathcal{F}/\varepsilon^2 \to \mathcal{F}/\varepsilon^1$$

Each quotient records the failure of a section to persist under increasingly deep recursive congruences.

We define:

Definition 3.1 (Transfinite Filtration Tower)**.**

$$F^{\varepsilon^\bullet}\mathcal{F} := \left\{F^{\varepsilon^n}\mathcal{F}\right\}_{n\in\mathbb{N}}.$$

This is a cofiltered inverse system whose limit defines the persistent ε-sheaf.

3.3. **Existence Acceleration.** Let $s \in \mathcal{F}$. Its **existence decay speed** is the smallest n such that:

$$s \notin F^{\varepsilon^n}\mathcal{F}.$$

We define a filtration-indexed weight function:

Definition 3.2 (Existence Acceleration Rank)**.**

$$\alpha(s) := \min\left\{n \mid s \notin F^{\varepsilon^n}\mathcal{F}\right\}.$$

The faster s disappears under filtration, the "shallower" its ontological content.

The ε^∞-core of $\mathcal{F}$ is:

$$F^{\varepsilon^\infty}\mathcal{F} := \{s \in \mathcal{F} \mid \alpha(s) = \infty\}.$$

3.4. **Sheaf Ontology via Stratified Collapse.** We define the ε-indexed ontology stack:

$$\mathcal{O}nt_{\varepsilon^\infty}(n) := \mathrm{Sh}(F^{\varepsilon^n}\mathcal{F}), \quad \mathcal{O}nt_{\varepsilon^\infty} := \varprojlim_n \mathcal{O}nt_{\varepsilon^\infty}(n).$$

Definition 3.3 (Ontology Layer)**.** *$\mathcal{O}nt_{\varepsilon^\infty}(n)$ represents the logic-indexed categorical reality of $\mathcal{F}$ at collapse level ε^n.*

This reframes geometry as **recursive ontology**: to be a geometric object is to survive through meta-stratified existence sieves.

3.5. **Stratified Limits and Categorical Stability.** The tower $F^{\varepsilon^\bullet}\mathcal{F}$ forms an inverse system with transition morphisms:

$$F^{\varepsilon^{n+1}} \hookrightarrow F^{\varepsilon^n}, \quad \mathcal{F}/\varepsilon^{n+1} \twoheadrightarrow \mathcal{F}/\varepsilon^n.$$

We define:

$$\mathcal{F}^{[\infty]} := \varprojlim_n \mathcal{F}/\varepsilon^n$$

as the "ε-sheaf of infinite persistence."

Proposition 3.4 (Existence as Categorical Limit)**.** *The persistent object $\mathcal{F}^{[\infty]}$ exists if and only if all transition maps stabilize in the limit category $\mathbf{Sh}_{\varepsilon^\infty}$.*

3.6. **Applications and Interpretations.**
- In arithmetic: defines deep regulators and zeta-congruence stratification;
- In logic: classifies definability across meta-theoretic layers;
- In geometry: encodes collapse to transcendence-core layers of moduli;
- In computation: filters spaces of proof and verification depth.

3.7. **Conclusion.** This section defines the full tower of ε-stratified filtrations as:
- A transfinite collapse structure;
- A functorial ontology stack;
- A generator of meta-categorical arithmetic spaces.

In the next section, we use this to define ε-hypermonodromy groups and the corresponding realization actions.

4. Hyper-Monodromy Groups and Autoequivalence Stacks

4.1. **From Classical to Recursive Monodromy.** In classical geometry, monodromy measures the action of fundamental groups on fibers of local systems. In perfectoid or p-adic geometry, it arises from Frobenius liftings or Galois actions on period sheaves.

In ε-hyperfiltration theory, monodromy becomes:
- A meta-action of growth-based automorphisms;
- A recursive tower of symmetries acting across ε^n-strata;
- An *auto-equivalence structure* on towers of torsors and realizations.

4.2. **Torsor Actions and Persistence Symmetry.** Let $\mathcal{F}$ be a filtered sheaf, and define $\mathcal{T}_n^\varepsilon$ as the torsor at filtration level ε^n:

$$\mathcal{T}_n^\varepsilon \curvearrowright F^{\varepsilon^n}\mathcal{F}.$$

Each such action represents a symmetry collapse at depth ε^n.

We define:

Definition 4.1 (ε-Torsor Tower)**.**

$$\mathbb{T}^{[\varepsilon^\infty]} := \{\mathcal{T}_n^\varepsilon \to X\}_{n\in\mathbb{N}}, \quad \textit{with } \mathcal{T}_n^\varepsilon \textit{ acting on } F^{\varepsilon^n}\mathcal{F}.$$

4.3. **The Hyper-Monodromy Group.** We now define the group classifying all recursive torsor actions across ε-depth:

Definition 4.2 (Hyper-Monodromy Group)**.**

$$\mathcal{M}_{hyper} := \varprojlim_n \operatorname{Aut}(\mathcal{T}_n^\varepsilon),$$

the inverse limit of automorphism groups of each torsor layer.

This group acts simultaneously on all filtration levels, and defines an automorphism of the entire ε-filtration tower.

4.4. **Autoequivalence of Sheaf Categories.** Each $\mathcal{T}_n^\varepsilon$ induces an autoequivalence:

$$\theta_n : \operatorname{Sh}(F^{\varepsilon^n}\mathcal{F}) \longrightarrow \operatorname{Sh}(F^{\varepsilon^n}\mathcal{F}).$$

The full stack of such actions defines:

Definition 4.3 (Autoequivalence Stack)**.**

$$\operatorname{Aut}^{\varepsilon^\infty} := \left\{\theta_n \in \operatorname{Eq}(\operatorname{Sh}(F^{\varepsilon^n}\mathcal{F}))\right\}_{n\in\mathbb{N}}.$$

The limit of this stack under stabilization defines:

$$\operatorname{Eq}_\infty := \varprojlim_n \operatorname{Eq}(\operatorname{Sh}(F^{\varepsilon^n}\mathcal{F})).$$

4.5. **Realization Actions via Hyper-Monodromy.** Let $\mathrm{real}_{\mathrm{hyper}}$ denote the realization functor from motives or K-theory objects to cohomology:

$$\mathrm{real}_{\mathrm{hyper}} : M^{[\varepsilon^\infty]}(X) \to H^\bullet_{\varepsilon^\infty}(X).$$

Then $\mathcal{M}_{\mathrm{hyper}}$ acts compatibly on both the source and target via:

$$\theta \cdot \mathrm{real}_{\mathrm{hyper}}(M) = \mathrm{real}_{\mathrm{hyper}}(\theta \cdot M).$$

This symmetry extends to all derived categories of ε-stratified sheaves.

4.6. **Stratified Representation Theory.** Let $\mathrm{Rep}(\mathcal{M}_{\mathrm{hyper}})$ denote the category of hypermonodromy representations.

Definition 4.4 (Stratified Realization Representation)**.** *For each sheaf $\mathcal{F}$, its realization functor defines a representation:*

$$\rho_{\mathcal{F}} : \mathcal{M}_{hyper} \to \mathrm{Aut}(\mathrm{real}_{hyper}(\mathcal{F})).$$

4.7. **Cohomological Consequences.**
- Each $H^i_{\varepsilon^n}$ is naturally an $\mathcal{M}_{\mathrm{hyper}}$-module;
- Stabilization implies derived invariance under $\mathcal{M}_{\mathrm{hyper}}$-actions;
- Duality and spectral sequences must be reinterpreted via group-theoretic stratification.

4.8. **Conclusion.** The hyper-monodromy group $\mathcal{M}_{\mathrm{hyper}}$:

- Generalizes Galois/Frobenius symmetry to transfinite torsor action;
- Classifies all autoequivalences of ε-stratified sheaf categories;
- Controls cohomological descent, realization dynamics, and motivic persistence;
- Is the central symmetry group of the entire hyperfiltration framework.

In the next section, we define ε-stratified motives and their realization theories over these towers.

5. ε-Stratified Motives and Realization Theory

5.1. **Motives across Recursive Collapse.** Let X be a space admitting an ε-hyperfiltration tower. We wish to define motives not over schemes or topological spaces per se, but over stratified towers indexed by trans-recursive growth.

Definition 5.1 (ε-Hypermotive)**.** *An* ε-hypermotive $M^{[\varepsilon^\infty]}(X)$ *is an object in a triangulated category* $\mathsf{DM}_{\varepsilon^\infty}$ *satisfying:*

- *A tower of ε-filtration functors $F^{\varepsilon^n}M$;*
- *Realization to ε-cohomology:*

$$\mathrm{real}_{hyper}(M) := \varprojlim_n H^\bullet(X, F^{\varepsilon^n}M);$$

- *Compatibility with hypermonodromy actions.*

These motives reflect "infinite cohomological generation," with structures defined through recursive collapse instead of schemes.

5.2. **ε-Stratified Realization Functor.** We extend the realization functor to an ε-parameterized target:

$$\mathrm{real}_{\mathrm{hyper}} : \mathsf{DM}_{\varepsilon^\infty} \longrightarrow \mathbf{Sh}_{\varepsilon^\infty}, \quad M \mapsto \{F^{\varepsilon^n}\mathrm{real}_{\mathrm{hyper}}(M)\}.$$

This structure reflects not just cohomological grading, but stratified survival under filtration towers.

5.3. **Regulator Maps and ε-Cohomology.** Let $K_n(X)$ be the n-th K-group. Define:

Definition 5.2 (ε-Stratified Regulator)**.**

$$r_{\varepsilon^\infty} : K_n(X) \longrightarrow H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n))$$

given by:

$$K_n(X) \xrightarrow{class} M^{[\varepsilon^\infty]}(X) \xrightarrow{\mathrm{real}_{hyper}} H^n_{\varepsilon^\infty}.$$

This generalizes Beilinson, syntomic, and p-adic regulators to infinite-stratification depth.

5.4. **ε-Motivic Periods and Special Values.** Let $\mathcal{P}^{(n)} := F^{\varepsilon^n} M^{[\varepsilon^\infty]}(X)$. Then define the motivic period tower:

$$\mathcal{P}_{\varepsilon^\infty} := \varprojlim_n \mathcal{P}^{(n)}.$$

Definition 5.3 (ε-Motivic Period Ring)**.**

$$B_{\varepsilon^\infty, dR} := \varprojlim_n \mathrm{End}(\mathcal{P}^{(n)}),$$

equipped with ε-stratified filtration and $\mathcal{M}_{hyper}$-action.

Special values of L-functions and regulators are conjectured to live in $B_{\varepsilon^\infty, dR}$.

5.5. **ε-Stratified Spectral Filtration.** Let $\mathcal{F}$ be a stratified sheaf. The filtration layers define a spectral sequence:

$$E_1^{p,q} = H^{p+q}(F^{\varepsilon^p}\mathcal{F}/F^{\varepsilon^{p+1}}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\varepsilon^\infty}(X, \mathcal{F}).$$

5.6. **Example: Infinite Depth Polylogarithms.** Consider the object $M_{\mathrm{Li}}^{[\varepsilon^\infty]}$ encoding polylogarithmic motives. The realization tower reflects:
- Finite-depth: $\zeta(n)$, Li_n;
- Recursive-depth: multiple zeta values;
- Hyper-depth: Euler sums, associators, and motivic correlators.

Each filtration level F^{ε^n} corresponds to truncation in transcendental weight complexity.

5.7. **Summary.** In this section we have constructed:

- The triangulated category $\mathsf{DM}_{\varepsilon^\infty}$ of ε-stratified motives;
- Realization functors $\mathrm{real}_{\mathrm{hyper}}$ from motives to sheaves;
- Hyper-stratified regulators r_{ε^∞};
- Motivic period rings $B_{\varepsilon^\infty,dR}$;
- Spectral structures across transfinite filtration layers.

In the next section, we develop hyper-cohomology and examine stabilization, torsor descent, and growth-based dualities.

6. Cohomological Towers and Periodic Stability

6.1. **Cohomology as Stratified Descent.** In the context of ε-hyperfiltration, cohomology no longer simply measures global-to-local discrepancies. It becomes a record of *recursive persistence*—tracking how cohomological information survives across transfinite filtration layers.

We recall:

$$H^i_{\varepsilon^\infty}(X,\mathcal{F}) := \varprojlim_n H^i(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F}).$$

This ε-hypercohomology captures stable invariants that are invisible under any finite filtration.

6.2. **The Cohomological Tower.** Let $\mathcal{F}$ be an ε-stratified sheaf. The associated tower is:

$$\cdots \to H^i(X,\mathcal{F}/\varepsilon^{n+1}\mathcal{F}) \to H^i(X,\mathcal{F}/\varepsilon^{n}\mathcal{F}) \to \cdots \to H^i(X,\mathcal{F}/\varepsilon^{1}\mathcal{F}).$$

We define:

Definition 6.1 (Cohomological Stability Depth)**.** *The smallest n such that:*

$$H^i(X,\mathcal{F}/\varepsilon^{n+k}\mathcal{F}) \xrightarrow{\sim} H^i(X,\mathcal{F}/\varepsilon^{n}\mathcal{F})$$

for all $k \geq 0$, is the stability depth *of $\mathcal{F}$ in degree i.*

If such an n exists, the tower stabilizes, and we say that $\mathcal{F}$ is **cohomologically recursive-finite** in degree i.

6.3. **Persistent Torsor Classes.** Each class in $H^1(X, \mathcal{T}_n^\varepsilon)$ defines a torsor that trivializes $\mathcal{F}/\varepsilon^n$. These classes form a compatible system:

$$\cdots \to H^1(X, \mathcal{T}_{n+1}^\varepsilon) \to H^1(X, \mathcal{T}_n^\varepsilon) \to \cdots$$

Definition 6.2 (Persistent Torsor Descent). *An ε-torsor class $\tau \in \varprojlim_n H^1(X, \mathcal{T}_n^\varepsilon)$ is* persistent *if it stabilizes the tower of realizations:*

$$\forall n, \quad \mathcal{F}^\tau/\varepsilon^n \simeq \textit{constant}.$$

This captures "cohomologically invisible" torsors that act uniformly across all recursive depths.

6.4. **Dualities and Stratified Ext-Groups.** Let $\mathcal{F}$, $\mathcal{G}$ be two ε-stratified sheaves. Then:

$$\mathrm{Ext}^i_{\varepsilon^\infty}(\mathcal{F}, \mathcal{G}) := \varprojlim_n \mathrm{Ext}^i(\mathcal{F}/\varepsilon^n, \mathcal{G}/\varepsilon^n)$$

defines an ε-hyper Ext-group, tracking recursive extensions. Under certain compactness assumptions, these satisfy:

$$\mathrm{Ext}^i_{\varepsilon^\infty}(\mathcal{F}, \mathcal{G}) \simeq H^i_{\varepsilon^\infty}(X, \mathcal{F}^\vee \otimes \mathcal{G}).$$

6.5. **Hyperfiltration Spectral Sequences.** We may define a spectral sequence arising from the filtration tower:

$$E_1^{p,q} = H^{p+q}\left(X, \mathrm{gr}_{\varepsilon^p} \mathcal{F}\right) \quad \Rightarrow \quad H^{p+q}_{\varepsilon^\infty}(X, \mathcal{F}).$$

The differentials capture how cohomological mass migrates across ε-layers. Stabilization implies degeneration at finite E_r.

6.6. **Growth-Invariance and Universal Collapse.**

Conjecture 6.3 (Growth-Invariant Collapse). *If $f(n), g(n) \in \mathsf{Growth}_{\varepsilon^\infty}$ satisfy $f(n) \sim g(n)$ asymptotically, then:*

$$H^i_{\varepsilon^{f(n)}}(X, \mathcal{F}) \simeq H^i_{\varepsilon^{g(n)}}(X, \mathcal{F}).$$

This suggests cohomological towers depend not on fine growth rate, but on transfinite stratification class.

6.7. **Summary.** We have now defined:

- ε-hypercohomology towers and their stabilization;
- Persistent torsor descent;
- Stratified Ext-groups across recursive collapse;
- Spectral sequences for sheaves over filtration depth;
- Growth-invariance conjectures of hyper-descent structures.

In the next section, we formulate transfinite period morphisms and meta-cohomological structures governing infinite stabilization.

7. Transfinite Period Morphisms and Meta-Cohomology

7.1. **From Growth Towers to Period Morphisms.** In traditional cohomological settings, comparison morphisms (e.g., Betti–de Rham, étale–de Rham) relate cohomologies indexed by topological or valuation-theoretic structures. In ε-hyperfiltration theory, we define comparison morphisms across entire towers of recursive collapse.

Definition 7.1 (Transfinite Period Morphism)**.** *Let $\mathcal{F}$ be an ε-stratified sheaf. A* transfinite period morphism *is a natural system:*

$$\Phi_n^{f\to g} : H^i_{f(n)}(X,\mathcal{F}) \longrightarrow H^i_{g(n)}(X,\mathcal{F}),$$

indexed by morphisms $f \to g$ in the category $\mathsf{Growth}_{\varepsilon^\infty}$.

7.2. **The Universal Period System.** Let $B_{\infty,dR} := \varprojlim_{f(n)} A/f(n) \otimes \mathbb{Q}$ denote the meta-period ring across all ε-growth types. We define:

$$\mathcal{P}^{[\infty]} := \varprojlim_{f(n)} \mathrm{real}_{\mathrm{hyper}}(\mathcal{F}/f(n)), \quad H^\bullet_\infty := \varprojlim_{f(n)} H^\bullet_{f(n)}(X,\mathcal{F}).$$

Definition 7.2 (Meta-Cohomology)**.** *The* meta-cohomology *of $\mathcal{F}$ is the universal cohomological object*

$$H^\bullet_{\varepsilon^\infty}(X,\mathcal{F}) := H^\bullet_\infty = \varprojlim_{f(n)} H^\bullet(X,\mathcal{F}/f(n)\mathcal{F}),$$

with transition maps defined by period morphisms $\Phi_n^{f\to g}$.

7.3. **Tilting across Growth Classes.** Let $\mathscr{T}_{f\to g} : \mathcal{F}^{[f]} \to \mathcal{F}^{[g]}$ be a tilting functor between filtration types. Then:

$$\mathrm{real}_{\mathrm{hyper}} \circ \mathscr{T}_{f\to g} \simeq \Phi_*^{f\to g} \circ \mathrm{real}_{\mathrm{hyper}},$$

meaning all realization and cohomological behavior is preserved under stratified transition.

7.4. Meta-Stability and Convergence.

Definition 7.3 (Meta-Stability)**.** *A sheaf $\mathcal{F}$ is* meta-stable *if its cohomology tower*

$$H^i_{f(n)}(X, \mathcal{F}) \xrightarrow{\sim} H^i_{g(n)}(X, \mathcal{F})$$

stabilizes for all $f(n) \sim g(n)$ in the limit category.

Such sheaves admit well-defined meta-realization and ε-independent special value theory.

7.5. Meta-Period Sheaves and Realizations.

Define the period sheaf tower:

$$\mathcal{P}_f := \mathcal{F}/f(n)\mathcal{F}, \quad \mathcal{P}_\infty := \varprojlim_f \mathcal{P}_f.$$

Then the transfinite realization is:

$$\mathrm{Real}_\infty : \mathsf{Sh}^{\varepsilon^\infty} \to \mathbf{Mod}_{B_{\infty,dR}}, \quad \mathcal{F} \mapsto \mathcal{P}_\infty.$$

7.6. Meta-Cohomological Duality.

Let $\mathcal{F}, \mathcal{G}$ be two meta-stable sheaves. Then there exists a canonical pairing:

$$\langle -, - \rangle_\infty : H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) \otimes H^\bullet_{\varepsilon^\infty}(X, \mathcal{G}) \longrightarrow B_{\infty,dR},$$

compatible with all finite-level regulators and filtrations.

7.7. Applications and Generalizations.

- Transfinite motivic regulators;
- ε^∞-indexed special value conjectures;
- Collapse of infinite filtrations to arithmetic data in $B_{\infty,dR}$;
- Meta-theoretic comparison theorems over ontological categories.

7.8. Conclusion.

We have constructed the theory of meta-cohomology:

- $H^\bullet_{\varepsilon^\infty}$ unifies all recursive filtration cohomology theories;
- Period morphisms $\Phi^{f \to g}$ define comparison and stability;
- The meta-period ring $B_{\infty,dR}$ contains all ε-period realizations;
- Stratified duality and meta-regulators extend beyond motivic sheaf theory.

In the final section, we synthesize the ontology of persistence, cohomology, and propose a hierarchy of infinite generation conjectures.

8. Stratified Meta-Conjectures and Infinite Descent Structures

8.1. **Recursive Collapse as Ontological Criterion.** In traditional settings, a space is geometric if it carries topological or algebraic structure. In ε-hyperfiltration theory, we propose a stronger criterion:

> A mathematical object is *ontologically geometric* if it persists under all transfinite filtration collapses.

This gives rise to a hierarchy of infinite descent conjectures that unify persistence, meta-cohomology, torsor symmetry, and special values.

8.2. **Stratified Cohomology Conjecture.**

Conjecture 8.1 (Stratified Limit Realization). *Let $\mathcal{F}$ be a sheaf over X. Then the meta-cohomology*

$$H^{\bullet}_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_{f(n)} H^{\bullet}(X, \mathcal{F}/f(n)\mathcal{F})$$

admits:

- *A motivic origin from an object $M^{[\varepsilon^\infty]}(X)$;*
- *A regulator map r_{ε^∞} compatible with all lower-depth systems;*
- *A unique ε^∞-period realization in $B_{\infty,dR}$.*

8.3. **Persistence Principle.**

Conjecture 8.2 (Infinite Persistence Principle). *A section $s \in \mathcal{F}$ is ontologically stable if and only if*

$$s \in \bigcap_n F^{\varepsilon^n}\mathcal{F},$$

i.e., s survives all levels of recursive stratification.

These sections define the intrinsic "reality layer" of a sheaf in ε^∞-geometry.

8.4. **Universal Torsor Collapse.**

Conjecture 8.3 (Recursive Torsor Limit). *Let $\mathcal{T}_n^{\varepsilon}$ be the torsors acting on $F^{\varepsilon^n}\mathcal{F}$. Then the limit*

$$\mathbb{T}^{[\varepsilon^\infty]} := \varprojlim_n \mathcal{T}_n^{\varepsilon}$$

admits a canonical trivialization if and only if $\mathcal{F}$ is ε-flat (i.e., splits across all recursive torsors).

8.5. Meta-Period Special Value Conjecture.

Conjecture 8.4 (Transfinite Special Values). *For every ε-stratified motive M, the special value of its L-function satisfies:*

$$L^{\star}(M,n) \in \operatorname{Im}\left(r_{\varepsilon^\infty} : K_n(M) \to B_{\infty,dR}\right).$$

This generalizes Deligne–Beilinson–Bloch–Kato conjectures into a trans-recursive setting.

8.6. Ontological Closure Conjecture.

Conjecture 8.5 (Closure of ε-Geometric Objects). *The full subcategory* $\mathbf{Ont}_{\varepsilon^\infty}$ *of persistent geometric objects is closed under:*

- *Filtered colimits and inverse limits;*
- *Tensor products and duals;*
- *Homological functors and sheafification;*
- *Transfinite realization functors.*

Thus, recursive stability is not accidental—it generates a universe of geometry.

8.7. Diagram of Infinite-Generation Structures.

$$\begin{tikzcd}
K_n(X) \arrow[r, "r_{\varepsilon^\infty}"] \arrow[d] & H^n_{\varepsilon^\infty}(X,\mathbb{Q}(n)) \arrow[d] & \\
M^{[\varepsilon^\infty]}(X) \arrow[r, "\mathrm{real}_{\mathrm{hyper}}"] & \mathcal{P}^{[\varepsilon^\infty]} \arrow[r] & B_{\infty,dR}
\end{tikzcd}$$

This illustrates how motives, regulators, cohomology, and periods integrate across infinite stratification.

8.8. Final Statement. Volume II has established the foundations of ε-hyperfiltration theory. We have:

- Defined stratified towers indexed by trans-recursive growth;
- Constructed hypermonodromy and torsor symmetry;
- Built ε-motives and meta-cohomology;
- Formulated conjectures on persistent realization and transfinite regulators;
- Recast geometry as a logic-indexed ontology of infinite descent.

The geometry of the future is not shaped by coordinates—but by survival through collapse.

In Volume III, we turn to the ε-weighted analogues of the Weight–Monodromy Conjecture, constructing new towers of arithmetic realization and torsor representation for non-linear and meta-recursive cases.

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