# SPECTRAL THEORY OF ZETA OPERATORS $\zeta(D)$: OPERATOR-VALUED DIRICHLET FUNCTIONS AND THEIR SPECTRA

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ABSTRACT. We initiate a spectral analysis of zeta-type operators defined as operator-valued Dirichlet functions evaluated at the arithmetic derivative operator $D$. We introduce and investigate $\zeta(D)$, $L_f(D)$, and more general spectral transforms. Using functional calculus, we analyze convergence, domain extensions, spectra, and functional identities for such operators on arithmetic function spaces. This approach unifies Dirichlet series theory with operator theory and may offer new insights into the analytical structure of zeta functions.

## CONTENTS



## 1. INTRODUCTION

Zeta functions lie at the heart of analytic number theory, while operator theory provides the analytic backbone of infinite-dimensional dynamics. This paper bridges the two by developing a spectral theory of *operator-valued* zeta functions evaluated at the arithmetic derivative operator $D$, previously introduced in symbolic calculus over Dirichlet convolution algebras.

We define:

$$\zeta(D) := \sum_{n=1}^{\infty} \frac{1}{n^D}, \quad \text{interpreted formally as an operator series,}$$

and study its convergence, spectral decomposition, and algebraic properties when acting on function spaces such as $\ell^2(\mathbb{N})$, or more generally, on convolution modules.

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## 2. Preliminaries and Setup

Let $D\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the diagonal operator defined by:

$$(Df)(n) = \log(n) f(n).$$

Let $f \in \ell^2(\mathbb{N})$, and define operator exponentiation:

$$n^{-D} f(m) := e^{-\log(n)\cdot\log(m)} f(m) = m^{-\log(n)} f(m).$$

This defines an operator $T_n := n^{-D}$ acting diagonally.

## 3. Definition and Convergence of $\zeta(D)$

**Definition 3.1.** We define the zeta operator:

$$\zeta(D) := \sum_{n=1}^{\infty} n^{-D},$$

with the sum interpreted in the strong operator topology on $\ell^2(\mathbb{N})$, provided convergence.

**Proposition 3.2.** *The series defining $\zeta(D)$ converges strongly on the dense subspace $\mathscr{S} := \{f \in \ell^2(\mathbb{N}) : |f(n)| \le Cn^{-\alpha}$ for some $\alpha > 0\}$.*

*Proof.* We observe that:

$$\left|\left(n^{-D} f\right)(m)\right| = m^{-\log(n)} |f(m)| = e^{-\log(n)\log(m)} |f(m)| = n^{-\log(m)} |f(m)|.$$

Then:

$$\sum_{n=1}^{\infty} \left|\left(n^{-D} f\right)(m)\right| \le |f(m)| \sum_{n=1}^{\infty} n^{-\log(m)},$$

which converges for all $m \ge 2$. So pointwise convergence follows, and boundedness on $\mathscr{S}$ implies strong convergence. □

## 4. Spectral Identity and Eigenanalysis

**Theorem 4.1.** *For each standard basis vector $e_m$, we have:*

$$\zeta(D) e_m = \sum_{n=1}^{\infty} m^{-\log(n)} e_m =: \zeta_m \cdot e_m.$$

*Thus, each $e_m$ is an eigenvector with eigenvalue $\zeta_m$.*

**Corollary 4.2.** *The spectrum $\sigma(\zeta(D))$ is the closure of $\{\sum_{n=1}^{\infty} m^{-\log(n)} : m \in \mathbb{N}\}$, which accumulates at infinity.*

*Proof.* The action of $\zeta(D)$ on each $e_m$ yields a scalar, so the operator is diagonal with eigenvalues $\zeta_m$. The series grows with $m$, so the spectrum is unbounded. □

## 5. Operator Zeta Functions $L_f(D)$

**Definition 5.1.** For $f \in \mathscr{A}$, define the operator Dirichlet transform:

$$L_f(D) := \sum_{n=1}^{\infty} \frac{f(n)}{n^D}.$$

**Proposition 5.2.** *If $f(n) = \mu(n)$, then $L_f(D)$ acts as an inverse to $\zeta(D)$ on basis vectors:*

$$L_\mu(D)\zeta(D)e_m = e_m.$$

*Proof.* Note:

$$L_\mu(D)e_m = \sum_{n=1}^{\infty} \mu(n)m^{-\log(n)}e_m = \sum_{n=1}^{\infty} \mu(n)n^{-\log(m)} = \frac{1}{\zeta(\log(m))},$$

and hence:

$$L_\mu(D)\zeta(D)e_m = \frac{\zeta(\log(m))}{\zeta(\log(m))}e_m = e_m.$$

□

## 6. Further Operator-Theoretic Investigations of $\zeta(D)$

### 6.1. Towards a Functional Equation for $\zeta(D)$.

**Definition 6.1.** Let $J \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the reflection operator defined by:

$$(Jf)(n) := f\left(\left\lfloor \frac{1}{n} \right\rfloor\right) \quad \text{(formal)},$$

or, more concretely, define an involutive map $\mathcal{I} \colon \mathbb{N} \to \mathbb{R}_{>0}$ by $\mathcal{I}(n) := e^{\frac{1}{\log(n)}}$, and consider the induced operator:

$$(\mathscr{F}f)(n) := f\left(\mathcal{I}(n)\right).$$

**Proposition 6.2** (Formal Analogue)**.** *If one could define $\chi(D) := \pi^{-D/2}\Gamma(D/2)$, then a formal analogue of the Riemann zeta functional equation is:*

$$\chi(D)\zeta(D) = \zeta(1 - D).$$

*Heuristic Justification.* Recall the classical identity:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1 - s)/2)\zeta(1 - s),$$

holds for scalar $s$. If one defines *operator functional calculus* for analytic functions $f \mapsto f(D)$, then:

$$\pi^{-D/2}\Gamma(D/2)\zeta(D) = \zeta(1 - D)$$

may formally hold. However, this requires an extension of the domain of definition of $\Gamma(D)$ and $\zeta(D)$, likely through zeta-regularization or pseudodifferential calculus. □

### 6.2. $\zeta(D)$ and Multiplicative Random Walks.

**Definition 6.3.** Define a multiplicative random walk on $\mathbb{N}$ as a Markov chain $\{X_n\}$ where $X_{n+1} = X_n \cdot Y_n$, with $Y_n \sim \mu$, a distribution over $\mathbb{N}_{>1}$. Let $P_n(k) := \mathbb{P}(X_n = k)$.

**Proposition 6.4.** *Let $f_k(m) := \mathbb{P}(X_n = m \mid X_0 = k)$, and define operator*

$$\mathcal{Z}_n := \sum_{k=1}^{\infty} f_k \cdot k^{-D}.$$

*Then $\mathcal{Z}_n$ encodes the expected growth of $X_n$ via multiplicative harmonic weightings.*

**Remark 6.5.** This suggests $\zeta(D)$ acts analogously to a moment-generating operator for multiplicative stochastic systems, with a weighting corresponding to the logarithmic structure of $D$.

### 6.3. Spectral Properties and Classical Zeta Theory.

**Theorem 6.6** (Speculative Bridge)**.** *Let $\sigma(\zeta(D))$ denote the spectrum of $\zeta(D)$. Then:*

$$\sigma(\zeta(D)) = \overline{\left\{ \sum_{n=1}^{\infty} m^{-\log(n)} \mid m \in \mathbb{N} \right\}},$$

*accumulates towards infinity and reflects a "dual" to $\zeta(s)$'s behavior at $s \to 1^+$.*

*Interpretive Argument.* We know:

$$\zeta(s) \sim \frac{1}{s-1}, \quad \text{as } s \to 1^+,$$

while:

$$\zeta(D)e_m = \sum_{n=1}^{\infty} m^{-\log(n)} e_m = \sum_{n=1}^{\infty} n^{-\log(m)} = \zeta(\log m).$$

Thus, the spectrum of $\zeta(D)$ contains values of $\zeta(s)$ where $s = \log m$. As $m \to \infty$, $\log m \to \infty$, and $\zeta(\log m) \to 1$, reproducing analytic behavior in transformed form. □

### 6.4. Operator-Valued Euler Products.

**Definition 6.7.** Let $\zeta(D) = \sum_{n=1}^{\infty} n^{-D}$. Define a formal Euler product over primes $p$:

$$\zeta(D) = \prod_{p \in \mathbb{P}} (1 - p^{-D})^{-1}.$$

**Theorem 6.8** (Operator Euler Identity)**.** *The identity*

$$\zeta(D) = \prod_{p \in \mathbb{P}} (1 - p^{-D})^{-1}$$

*holds in the strong operator topology on a domain of functions with finite Dirichlet energy.*

*Proof.* Each operator $(1 - p^{-D})^{-1}$ acts as diagonal multiplication by $(1 - p^{-\log n})^{-1}$. Their product converges if:

$$\prod_{p} \left(1 - p^{-\log(n)}\right)^{-1} = \zeta(\log n),$$

which is consistent with earlier diagonal results:

$$\zeta(D)e_n = \zeta(\log n)e_n.$$

Thus, the product converges pointwise on basis vectors, hence extends to bounded operator on dense domains. □

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