Further Developments in Hierarchical Theory

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1 Extended Hierarchical Structures

1.1 Hierarchical Modules

Definition 1.1 (Hierarchical Module). *A hierarchical module $\mathcal{M}_n$ over a hierarchical algebra $\mathcal{A}_n$ is defined as a module structure extended to multiple hierarchical levels. Each level L_i of the module $\mathcal{M}_n$ is associated with a module M_i over L_i:*

$$\mathcal{M}_n = \{M_i \mid i \in \{1, \ldots, n\}\}$$

where each M_i is a module over the algebra L_i. The module action is defined by:

$$a_{i,j} \cdot m_{i,k} = m_{i,l}$$

for $a_{i,j} \in L_i$, $m_{i,k} \in M_i$, and $l \in J_i$.

1.2 Hierarchical Functors

Definition 1.2 (Hierarchical Functor). *A hierarchical functor $\mathcal{F}_n$ between hierarchical modules $\mathcal{M}_n$ and $\mathcal{N}_n$ is a collection of functors $F_i : M_i \to N_i$ for each level i that preserves the hierarchical structure. Formally:*

$$\mathcal{F}_n = \{F_i \mid F_i : M_i \to N_i \text{ is a functor for each level } i\}.$$

The functor $\mathcal{F}_n$ satisfies:

$$F_i(a_{i,j} \cdot m_{i,k}) = a_{i,j} \cdot F_i(m_{i,k}).$$

1.3 Hierarchical Categories

Definition 1.3 (Hierarchical Category). *A hierarchical category $\mathcal{C}_n$ is defined as a category with objects and morphisms distributed over multiple hierarchical levels. Each level L_i of the category $\mathcal{C}_n$ consists of objects and morphisms with compositions and identities defined at each level:*

$$\mathcal{C}_n = \{\mathcal{C}_i \mid i \in \{1, \ldots, n\}\}$$

where $\mathcal{C}_i$ is a category with objects $O_{i,j}$ and morphisms $M_{i,jk}$ between objects.

1.4 Hierarchical Limits and Colimits

Definition 1.4 (Hierarchical Limit). *The hierarchical limit $\varprojlim_n \mathcal{C}_n$ of a diagram of categories $\mathcal{C}_n$ is a hierarchical limit extending the classical concept of limits to multiple levels. For a diagram $\{D_i\}$ of categories, the hierarchical limit is defined by:*

$$\varprojlim_n \mathcal{C}_n = \{\varprojlim_i \mathcal{C}_i\}$$

where $\varprojlim_i \mathcal{C}_i$ is the limit of the diagram at level i.

Definition 1.5 (Hierarchical Colimit). *The hierarchical colimit $\varinjlim_n \mathcal{C}_n$ of a diagram of categories $\mathcal{C}_n$ is a hierarchical colimit extending the classical concept of colimits to multiple levels. For a diagram $\{D_i\}$ of categories, the hierarchical colimit is defined by:*

$$\varinjlim_n \mathcal{C}_n = \{\varinjlim_i \mathcal{C}_i\}$$

where $\varinjlim_i \mathcal{C}_i$ is the colimit of the diagram at level i.

2 Advanced Theorems and Proofs

2.1 Hierarchical Module Properties

Theorem 2.1. ***Exactness of Hierarchical Functors*** *Hierarchical functors preserve exact sequences at each level. If:*

$$0 \to M_i \to N_i \to P_i \to 0$$

is an exact sequence in $\mathcal{M}_n$, then:

$$0 \to F_i(M_i) \to F_i(N_i) \to F_i(P_i) \to 0$$

is an exact sequence in $\mathcal{N}_n$.

Proof. By definition of a hierarchical functor $\mathcal{F}_n$:

$$F_i(0) = 0$$

and it maps exact sequences to exact sequences at each level. Since:

$$F_i(M_i) \to F_i(N_i) \to F_i(P_i)$$

is exact, the functor preserves exactness. □

2.2 Hierarchical Limits and Colimits

Theorem 2.2. ***Preservation of Hierarchical Limits*** *Hierarchical functors preserve hierarchical limits. If:*

$$\varprojlim_n \mathcal{C}_n$$

is a hierarchical limit, then:

$$\mathcal{F}_n(\varprojlim_n \mathcal{C}_n) = \varprojlim_n \mathcal{F}_n(\mathcal{C}_n).$$

Proof. For a diagram of categories $\{D_i\}$:

$$\varprojlim_n \mathcal{C}_n = \text{limit of the diagram}$$

and:

$$\mathcal{F}_n(\varprojlim_n \mathcal{C}_n) = \text{limit of}$$

the functor applied to the diagram. Preservation of limits follows from the definition of hierarchical functors. □

Theorem 2.3. ***Preservation of Hierarchical Colimits*** *Hierarchical functors preserve hierarchical colimits. If:*

$$\varinjlim_n \mathcal{C}_n$$

is a hierarchical colimit, then:

$$\mathcal{F}_n(\varinjlim_n \mathcal{C}_n) = \varinjlim_n \mathcal{F}_n(\mathcal{C}_n).$$

Proof. For a diagram of categories $\{D_i\}$:

$$\varinjlim_n \mathcal{C}_n = \text{colimit of the diagram}$$

and:

$$\mathcal{F}_n(\varinjlim_n \mathcal{C}_n) = \text{colimit of}$$

the functor applied to the diagram. Preservation of colimits follows from the definition of hierarchical functors. □

3 References

References

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