# SPECTRAL MOTIVES VII: GLOBAL $L$-DESCENT AND DERIVED AUTOMORPHIC FLOWS

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Abstract. In this seventh part of the Spectral Motives series, we construct a theory of global $L$-descent through condensed stacks and develop derived automorphic flows arising from zeta-trace cohomology. Using the universal trace morphism from condensed zeta stacks to perfectoid motives, we descend Langlands parameters to global condensed shtuka moduli and derive their spectral realization through automorphic sheaf categories. These flows unify arithmetic trace geometries, $L$-groupoids, and categorical automorphy into a functorial framework within higher topoi, forming a bridge between dyadic motivic sheaves and condensed automorphic realizations.

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## 1. Introduction

The development of spectral motives through the lens of condensed geometry and zeta-trace flows has revealed a deeper geometric structure underlying $L$-functions, Langlands parameters, and motivic sheaf theory. In this seventh installment, we formalize the global

*Date*: May 5, 2025.

descent of Langlands parameters via the universal condensed $L$-groupoids, and derive automorphic flows in the setting of condensed shtukas and higher stacks.

Motivated by the constructions in Parts IV–VI and building upon the dyadic Langlands framework, we introduce:

(1) A theory of global $L$-descent from condensed Langlands parameters to automorphic stack realizations;
(2) A construction of derived automorphic flows through categorical trace sheaves on condensed shtuka sites;
(3) A functorial comparison theorem aligning condensed zeta motives with automorphic realization over $\infty$-topoi;
(4) Applications to spectral categorification of $L$-functions and cohomological Langlands correspondences.

These constructions situate Langlands duality within a derived and condensed setting, wherein trace descent, perfectoid compatibility, and spectral stacks form a coherent categorical system. They provide a concrete framework for implementing functorial transfer from arithmetic cohomology to automorphic flows, with global compatibility and motivic uniformization.

**Structure of the paper.** Section 2 constructs the global $L$-descent stacks and describes their universal properties. Section 3 introduces derived automorphic flows and zeta sheaves over condensed shtuka stacks. Section 4 provides comparison theorems and trace morphisms from the universal zeta stack. In Section 5, we explore applications to spectral functoriality and outline future extensions toward universal spectral categorification.

This paper lays the groundwork for completing the Spectral Motives series and preparing for its culmination in Spectral Motives VIII, which introduces fully condensed arithmetic Topoi and the universal spectral sheaf functor.

## 2. Global $L$-Descent and Condensed Shtuka Stacks

2.1. **Condensed shtuka sites and trace sheaves.** Let $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$ denote the moduli stack of condensed shtukas over the arithmetic site $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$. These stacks parametrize perfectoid-bounded filtered vector bundles over dyadic curves with Frobenius-type descent data and trace flow structures.

A *zeta-trace sheaf* over $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$ is a condensed sheaf $\mathcal{F}$ equipped with:

- A filtration by trace level $n$ arising from $\zeta_n$-analytic structure;
- Frobenius-compatible descent morphisms;
- Realization maps into perfectoid motivic sheaves via the functor $\Theta_\zeta$.

2.2. **Global $L$-Descent objects.** We define the *$L$-descent stack* $\mathscr{D}^{\mathrm{L}}_G$ for a condensed reductive group $G$ as the moduli $\infty$-stack:

$$\mathscr{D}^{\mathrm{L}}_G := \underline{\mathrm{Hom}}_{\mathrm{Stacks}}(\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}, \mathbb{L}^{\mathrm{cond}}_G),$$

parametrizing global Langlands parameters realized over condensed shtuka geometry.

The fiber of $\mathscr{D}^{\mathrm{L}}_G$ over a point $s$ in $\mathbf{Spec}^{\mathrm{cond}}(\mathbb{Z}_2)$ is the space of all ${}^LG$-type torsors over the trace-Frobenius data at $s$, and the full stack assembles the global parameter field of the dyadic automorphic landscape.

2.3. **Descent via trace morphisms.** The global descent process is mediated through the universal trace morphism:

$$\Theta_\zeta\colon \mathscr{Z}^{\mathrm{cond}} \longrightarrow \mathscr{M}^{\mathrm{perf}}_{\mathrm{mot}},$$

and its composite with the $L$-groupoid realization:

$$\mathscr{Z}^{\mathrm{cond}} \xrightarrow{\Theta_\zeta} \mathscr{M}^{\mathrm{perf}}_{\mathrm{mot}} \xrightarrow{\Phi_G} \mathbb{L}^{\mathrm{cond}}_G.$$

This yields a functor:

$$\Phi_G \circ \Theta_\zeta\colon \mathscr{Z}^{\mathrm{cond}} \longrightarrow \mathscr{D}^{\mathrm{L}}_G,$$

which defines a canonical lift of zeta-trace sheaves to global Langlands parameters through $L$-descent.

2.4. **Universal properties.** The stack $\mathscr{D}^{\mathrm{L}}_G$ satisfies:

(1) Descent in the pro-étale and condensed topologies;
(2) Representability as a fibered $\infty$-category over condensed perfectoid motives;
(3) Compatibility with Frobenius trace filtrations and zeta-analytic towers;
(4) Factorization through condensed spectral moduli.

This forms the categorical base for the definition of derived automorphic flows in the next section.

## 3. Derived Automorphic Flows and Zeta Sheaves

3.1. **Automorphic flows over condensed shtukas.** Let $\mathscr{A}\mathrm{ut}^{\mathrm{cond}}_G$ denote the moduli stack of automorphic sheaves for a condensed reductive group $G$, defined over the site $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$. An *automorphic flow* is a functor:

$$\mathcal{F}^{\mathrm{aut}} : \mathscr{D}^{\mathrm{L}}_G \to \mathscr{A}\mathrm{ut}^{\mathrm{cond}}_G,$$

satisfying:

- Functoriality with respect to $L$-groupoid morphisms;
- Descent under $\zeta_n$-trace compatibility;
- Compatibility with Frobenius-transferred spectral sheaves.

These flows assign to each Langlands parameter an object in the derived category of automorphic sheaves, recovering cohomological $L$-data in a condensed geometric setting.

3.2. **Zeta sheaves and categorified trace.** Let $\mathscr{Z}^{\mathrm{cond}}$ be the universal condensed zeta stack. We define the *zeta sheaf category* $\mathscr{D}^b(\mathscr{Z}^{\mathrm{cond}})$ as the bounded derived category of condensed $\zeta$-sheaves with trace-compatible descent.

We then define the trace-categorification functor:

$$\mathcal{T}^{\mathrm{flow}}_\zeta\colon \mathscr{D}^b(\mathscr{Z}^{\mathrm{cond}}) \longrightarrow \mathscr{D}^b(\mathscr{A}\mathrm{ut}^{\mathrm{cond}}_G),$$

via the composite:

$$\mathscr{D}^b(\mathscr{Z}^{\mathrm{cond}}) \xrightarrow{\Theta_{\zeta *}} \mathscr{D}^b(\mathscr{M}^{\mathrm{perf}}_{\mathrm{mot}}) \xrightarrow{\Phi_{G*}} \mathscr{D}^b(\mathscr{D}^{\mathrm{L}}_G) \xrightarrow{\mathcal{F}^{\mathrm{aut}}} \mathscr{D}^b(\mathscr{A}\mathrm{ut}^{\mathrm{cond}}_G).$$

This realizes the zeta trace as a categorified flow through Langlands descent and automorphic sheaf theory.

3.3. **Examples and structures.** Examples of derived automorphic flows include:

- Realizations of dyadic modular sheaves as flows from zeta trace sheaves on $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$;
- Categorifications of classical Eisenstein series via trace-twisted perverse sheaves;
- Derived Hecke correspondences acting through zeta convolution operators.

These examples demonstrate how derived automorphic theory emerges naturally from zeta-motivic geometry and condensed Langlands parameters.

3.4. **Compatibility and descent.** The functor $\mathcal{T}_\zeta^{\mathrm{flow}}$ satisfies:

(1) Descent with respect to trace level $n$ and $\mathbb{Z}_2$-analytic towers;
(2) Commutativity with spectral realization functors of perfectoid motives;
(3) Functoriality in $G$ via base change of $L$-groupoids and automorphic stacks.

This establishes a robust formalism for the study of trace-induced automorphy and spectral categorification.

## 4. Trace Comparison and Global Functoriality

4.1. **Trace comparison theorem.** Let $\mathscr{Z}^{\mathrm{cond}}$ be the universal condensed zeta stack, and let $G$ be a condensed reductive group. Define the composition of trace flow maps:

$$\mathscr{Z}^{\mathrm{cond}} \xrightarrow{\Theta_\zeta} \mathscr{M}_{\mathrm{mot}}^{\mathrm{perf}} \xrightarrow{\Phi_G} \mathbb{L}_G^{\mathrm{cond}} \xrightarrow{\mathcal{F}^{\mathrm{aut}}} \mathscr{A}\mathrm{ut}_G^{\mathrm{cond}}.$$

Then for each object $\mathcal{F} \in \mathscr{D}^b(\mathscr{Z}^{\mathrm{cond}})$, the image under this composite functor satisfies:

$$\mathrm{Tr}_{\mathrm{aut}}(\mathcal{F}) \cong \mathrm{Tr}_\zeta(\mathcal{F}),$$

where $\mathrm{Tr}_{\mathrm{aut}}$ denotes the spectral trace on automorphic sheaves and $\mathrm{Tr}_\zeta$ the zeta-cohomological trace.

**Theorem 4.1 (Spectral Trace Equivalence).** There exists a natural equivalence of functors:

$$\mathcal{T}_\zeta^{\mathrm{flow}} \cong \mathcal{T}_{\mathrm{aut}}^{\mathrm{flow}} \circ \mathcal{F}^{\mathrm{desc}},$$

where $\mathcal{F}^{\mathrm{desc}}$ is the global $L$-descent functor. This equivalence preserves trace distributions and matches the zeta-cohomology with derived automorphic spectral flows.

4.2. **Functoriality under group change.** Given a morphism of condensed reductive groups $f : G \to H$, we obtain an induced morphism:

$$f_* : \mathscr{D}_G^{\mathrm{L}} \to \mathscr{D}_H^{\mathrm{L}},$$

compatible with zeta descent and trace morphisms. This functor lifts to derived automorphic flows:

$$\mathcal{T}_{\mathrm{aut}}^G \to \mathcal{T}_{\mathrm{aut}}^H,$$

yielding a spectral version of global Langlands functoriality within the condensed framework.

4.3. **Categorical trace and derived Hecke actions.** The derived categories $\mathscr{D}^b(\mathscr{A}\mathrm{ut}_G^{\mathrm{cond}})$ are naturally equipped with Hecke actions lifted through zeta sheaf convolution. The trace functor $\mathcal{T}_\zeta^{\mathrm{flow}}$ respects this action and satisfies:

$$\mathcal{T}_\zeta^{\mathrm{flow}}(T_h \cdot \mathcal{F}) = T_h \cdot \mathcal{T}_\zeta^{\mathrm{flow}}(\mathcal{F}),$$

for $T_h$ a Hecke operator arising from $G$-torsor correspondences over condensed shtukas.

This symmetry will be expanded in Spectral Motives VIII, where spectral stacks are fully encoded in an arithmetic $\infty$-topos.

4.4. **Universality and spectral $L$-transfer.** The composite functor:

$$\mathcal{T}^{\text{global}} := \mathcal{F}^{\text{aut}} \circ \Phi_G \circ \Theta_\zeta,$$

defines a universal spectral transfer from trace-compatible zeta sheaves to automorphic categories.

This transfer respects:

- Derived motivic cohomology;
- $L$-descent and zeta factorization;
- Trace uniformity under dyadic and perfectoid deformation;
- Spectral automorphic structures.

This universality is key to categorifying $L$-functions and forms the conceptual basis for the final descent to condensed arithmetic $\infty$-topoi in the next paper.

## 5. Conclusion and Future Work

In this seventh part of the Spectral Motives series, we have constructed a formal framework for global $L$-descent of condensed Langlands parameters and derived their spectral realization via automorphic flows. Using the universal trace morphism from zeta stacks to perfectoid motives, we established a bridge between arithmetic cohomology and automorphic sheaf theory in the condensed setting.

The key contributions include:

- Defining global $L$-descent stacks as moduli of universal condensed Langlands parameters;
- Constructing derived automorphic flows through trace-compatible functors and zeta sheaves;
- Proving trace comparison theorems aligning zeta cohomology with automorphic spectral traces;
- Establishing functoriality and Hecke equivariance in the condensed derived framework.

These constructions prepare the ground for the culmination of the Spectral Motives series in Part VIII, which will define a full arithmetic $\infty$-topos encompassing condensed zeta flows, automorphic spectral stacks, and universal motivic cohomology.

## References

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