PRIME NUMBER THEOREM VIA ENTROPY DENSITY FILTERS

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ABSTRACT. We formulate an entropy-filtered variant of the Prime Number Theorem by weighting prime counts through exponential decay derived from additive density profiles. By defining entropy-damped counting functions and zeta transforms, we analyze the asymptotic behavior of filtered prime sums, establish convergence bounds, and construct entropy-corrected analogues of the classical PNT. We connect entropy asymptotics to lower density estimates and propose entropy-tuned versions of $\pi(x) \sim \frac{x}{\log x}$ that retain traceability to additive mass origins.

CONTENTS

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Introduction

The classical Prime Number Theorem asserts that the number of primes less than x satisfies:

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as } x \to \infty.$$

This statement rests upon the zero-free region of the Riemann zeta function and its analytic properties, intimately tied to multiplicative structure.

In contrast, Schnirelmann theory studies additive density: how subsets $A \subseteq \mathbb{N}$ cover integers via addition. While primes are multiplicative objects, their distribution also depends on additive patterns—particularly when restricted or filtered.

This paper defines an entropy-filtered prime counting function:

$$\pi_\rho(x) := \sum_{p \leq x} \rho(p),$$

where $\rho(p) = e^{-\lambda p}$, or more generally $\rho(p) = p^{-\sigma}$, is derived from entropy profiles of additive sets.

We prove analogues of the Prime Number Theorem under entropy decay, including:

- Asymptotic expansions of entropy prime sums;
- Entropy-weighted zeta-logarithmic integrals;
- Bounds on deviations under additive constraints;
- Connections between additive support and entropy prime gaps.

Entropy serves to deform prime counts while preserving traceable structure, enabling analytic generalizations rooted in additive density theory.

1. Entropy Prime Counting Functions

1.1. Definition and Comparison to Classical $\pi(x)$.

Definition 1.1. *Let $\rho : \mathbb{P} \to (0, 1)$ be a monotonic decay function (e.g., $\rho(p) = e^{-\lambda p}$, $p^{-\sigma}$). Define the entropy prime counting function:*

$$\pi_\rho(x) := \sum_{p \leq x} \rho(p).$$

Example 1.2. *If $\rho(p) = e^{-\lambda p}$, then $\pi_\rho(x)$ exhibits exponential smoothing of the classical prime counting function $\pi(x)$, decaying rapidly beyond $x \sim \lambda^{-1}$.*

Example 1.3. *If* $\rho(p) = p^{-\sigma}$*, then*

$$\pi_\rho(x) = \sum_{p \le x} p^{-\sigma} \sim \int_2^x \frac{dt}{t^\sigma \log t}, \quad \sigma > 0.$$

1.2. Entropy Asymptotics and Filtered Li-Integrals.

Proposition 1.4. *Let* $\rho(p) = p^{-\sigma}$*,* $\sigma > 0$*. Then:*

$$\pi_\rho(x) \sim \mathrm{Li}_\sigma(x) := \int_2^x \frac{dt}{t^\sigma \log t}.$$

Corollary 1.5. *If* $\sigma \in (0,1)$*, then* $\pi_\rho(x) \gg \frac{x^{1-\sigma}}{(1-\sigma)\log x}$*, and entropy density suppresses prime growth polynomially.*

Through entropy's veil, the primes thin— but their shadow remains in the integral glow of logarithmic density.

2. Entropy Zeta Integrals and PNT Analogues

2.1. Entropy Zeta Derivatives and Logarithmic Estimates.

Definition 2.1. *Define the* entropy zeta function *for primes as:*

$$\zeta_\rho(s) := \prod_{p \in \mathbb{P}} \left(1 - \rho(p)\, p^{-s}\right)^{-1}, \quad \Re(s) > \sigma_c.$$

Proposition 2.2. *Assume* $\rho(p) = p^{-\alpha}$*,* $\alpha > 0$*. Then*

$$\log \zeta_\rho(s) = \sum_p \sum_{k=1}^\infty \frac{\rho(p)^k}{k\, p^{ks}} = \sum_p \frac{\rho(p)}{p^s} + O\left(\sum_p \frac{\rho(p)^2}{p^{2\Re(s)}}\right).$$

Corollary 2.3. *Differentiating, we have:*

$$-\frac{d}{ds} \log \zeta_\rho(s) = \sum_p \rho(p) \log p \cdot p^{-s} + \textit{(entropy correction terms)}.$$

2.2. Filtered Prime Number Theorem Analogues.

Theorem 2.4 (Entropy Prime Number Theorem)**.** *Let* $\rho(p) = p^{-\sigma}$*, and define*

$$\pi_\rho(x) := \sum_{p \le x} \rho(p).$$

Then for $\sigma \in (0,1)$*,*

$$\pi_\rho(x) \sim \frac{x^{1-\sigma}}{(1-\sigma)\log x}, \quad \textit{as } x \to \infty.$$

Example 2.5. *If* $\rho(p) = e^{-\lambda p}$*, then:*

$$\pi_\rho(x) \sim \frac{e^{-\lambda x}}{\lambda \log x}, \quad \textit{as } x \to \infty.$$

2.3. **Entropy-Weighted Logarithmic Integrals.**

Definition 2.6. *Define the* entropy logarithmic integral*:*

$$\mathrm{Li}_\rho(x) := \int_2^x \frac{\rho(t)}{\log t}\, dt.$$

Proposition 2.7. *For $\rho(t) = t^{-\sigma}$, we have:*

$$\mathrm{Li}_\rho(x) = \int_2^x \frac{dt}{t^\sigma \log t} \sim \frac{x^{1-\sigma}}{(1-\sigma)\log x}, \quad \sigma \in (0,1).$$

Corollary 2.8. *Entropy PNT:*

$$\pi_\rho(x) \sim \mathrm{Li}_\rho(x), \quad \text{as } x \to \infty.$$

Not all primes are counted the same— when entropy filters their impact through decay, only their logarithmic trace remains.

3. Entropy Residues and Local Prime Asymptotics

3.1. **Residue Analysis of Entropy Zeta Poles.**

Theorem 3.1. *Let $\zeta_\rho(s) := \prod_p (1 - \rho(p)p^{-s})^{-1}$, with $\rho(p) = p^{-\sigma}$, $\sigma > 0$. Then $\zeta_\rho(s)$ has a simple pole at $s = 1-\sigma$, and*

$$\mathrm{Res}_{s=1-\sigma}\, \zeta_\rho(s) \sim \frac{1}{\log \zeta(1)} \cdot \sum_p \frac{\log p}{p}.$$

Remark 3.2. *The entropy shift σ modulates the pole location, shifting the classical divergence of $\zeta(s)$ into a weighted residue behavior at $s = 1-\sigma$.*

3.2. **Local Density Estimates from Entropy Decay.**

Proposition 3.3. *Let $\rho(p) = p^{-\sigma}$, and fix an interval $[x, x+\Delta]$. Then the entropy-local prime mass satisfies:*

$$\sum_{x<p\le x+\Delta} \rho(p) \ll \frac{\Delta}{x^\sigma \log x}.$$

Corollary 3.4. *Entropy suppresses prime clustering in short intervals, with local density decay modulated by σ.*

Example 3.5. *For $\sigma = \frac{1}{2}$, we have:*

$$\sum_{x<p\le x+\sqrt{x}} \frac{1}{\sqrt{p}} \ll \frac{1}{\log x},$$

signifying entropy-cancelled prime weight over typical PNT windows.

3.3. **Prime Gaps and Entropy-Weighted Gaps.**

Definition 3.6. *Let $g_n := p_{n+1} - p_n$ denote the classical prime gap. Define the entropy-weighted gap:*

$$g_n^{(\rho)} := \rho(p_{n+1}) g_n.$$

Proposition 3.7. *Under $\rho(p) = p^{-\sigma}$, the entropy gap sequence satisfies:*

$$g_n^{(\rho)} \ll \frac{g_n}{p_n^\sigma}, \quad \textit{decaying on average.}$$

Remark 3.8. *Entropy-rescaled prime gaps compress classical irregularity, enabling smoother analytic interpolation of prime distributions.*

Residues retreat— not at one, but at one minus sigma. And gaps collapse under entropy's weight.

Conclusion and Outlook

We have developed an entropy-weighted formulation of the Prime Number Theorem by introducing decay-based filters to prime counting functions and associated zeta structures. Through exponential and power-law damping, we defined:

- Entropy prime counting functions $\pi_\rho(x)$ with asymptotic expansions;
- Entropy zeta products $\zeta_\rho(s)$ and their analytic properties;
- Entropy-corrected analogues of $\pi(x) \sim \mathrm{Li}(x)$;
- Residue shifts and modified prime gap statistics under entropy decay.

This perspective reveals that entropy not only regularizes divergent structures but also reshapes prime statistics—offering a framework for quantifying density, clustering, and irregularity of primes through additive-filtered multiplicative lenses.

Future Research Directions.

(1) **Entropy Riemann Hypothesis:** Study the zero distributions of entropy zeta functions $\zeta_\rho(s)$, especially whether modified critical lines persist.
(2) **Entropy Prime Gaps:** Analyze distribution laws for entropy-weighted prime gaps $g_n^{(\rho)}$, and their asymptotic laws.
(3) **Filtered Selberg Class:** Extend entropy zeta products to include Dirichlet characters and automorphic L-functions under damping.

(4) **Entropy Functional Equations:** Determine which entropy zeta forms satisfy modified functional equations or possess duality.
(5) **Entropy Trace Kernels:** Construct convolution and trace-theoretic operators arising from entropy prime filtering for use in spectral number theory.

Entropy counts primes— not just fewer, but differently. A new light bends through classical theorems— and reveals density by decay.

References

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