Number-Theoretic Factorization Theory (NFTT)

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1 Introduction

Number-Theoretic Factorization Theory (NFTT) provides a comprehensive framework for analyzing factorization patterns within number theory. The theory integrates hierarchical and fractal-like structures within the realm of factorization, focusing on the behavior and properties of unique factorization domains (UFDs) and their extensions.

2 New Mathematical Notations

2.1 Factorization Tower

Definition 2.1. *A **Factorization Tower** is a sequence of unique factorization domains $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$ where each domain $\mathcal{D}_{n+1}$ extends $\mathcal{D}_n$ with a specific factorization-preserving structure.*

Notation 2.2. *We denote the Factorization Tower by $\mathcal{T}$ where $\mathcal{T} = (\mathcal{D}_n)_{n \geq 0}$.*

2.2 Recursive Factorization Function

Definition 2.3. *The **Recursive Factorization Function** $\mathcal{F}_n : \mathcal{D}_n \to \mathcal{P}(\mathcal{D}_n)$ maps each element $x \in \mathcal{D}_n$ to its unique factorization in $\mathcal{D}_n$, where $\mathcal{P}(\mathcal{D}_n)$ denotes the power set of $\mathcal{D}_n$.*

Notation 2.4. *For an element $x \in \mathcal{D}_n$, let $\mathcal{F}_n(x)$ be the set of prime elements in the factorization of x.*

2.3 Fractal Factorization Pattern

Definition 2.5. *A **Fractal Factorization Pattern** is a factorization structure that exhibits self-similarity across different levels of the Factorization Tower. This pattern is described by a recursive relation:*

$$\mathcal{F}_{n+1}(x) = \mathcal{F}_n(x) \cup \{p_i \mid p_i \text{ is a prime in } \mathcal{D}_{n+1} \text{ and } p_i \text{ divides } x \text{ in } \mathcal{D}_{n+1}\}.$$

Notation 2.6. *The term p_i represents the prime elements in $\mathcal{D}_{n+1}$ that contribute to the factorization of x in $\mathcal{D}_{n+1}$.*

3 Formulas and Theorems

3.1 Hierarchical Factorization Theorem

Theorem 3.1. ***Hierarchical Factorization Theorem:*** *Let $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$ be a Factorization Tower where each $\mathcal{D}_n$ is a UFD. For any element $x \in \mathcal{D}_k$, the unique factorization in $\mathcal{D}_{k+1}$ can be expressed as:*

$$\mathcal{F}_{k+1}(x) = \mathcal{F}_k(x) \cup \{p_i \mid p_i \textit{ is a new prime introduced in } \mathcal{D}_{k+1} \textit{ and } p_i \textit{ divides } x \textit{ in } \mathcal{D}_{k+1}\}.$$

3.2 Recursive Factorization Formula

Lemma 3.2. ***Recursive Factorization Formula:*** *For a given element $x \in \mathcal{D}_n$, the recursive factorization can be computed as:*

$$\mathcal{F}_n(x) = \bigcup_{d|x} \mathcal{F}_{n-1}(d),$$

where d ranges over all divisors of x in $\mathcal{D}_n$.

4 Extended Notations and Definitions

4.1 Hierarchical Factorization Map

Definition 4.1. *The* ***Hierarchical Factorization Map*** *$\Phi_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ is a function that maps each element in $\mathcal{D}_n$ to its corresponding element in $\mathcal{D}_{n+1}$ maintaining factorization properties. For an element $x \in \mathcal{D}_n$, its image $\Phi_n(x)$ is defined by:*

$$\Phi_n(x) = x \textit{ in } \mathcal{D}_{n+1}.$$

Notation 4.2. *The function Φ_n reflects the preservation of factorization through the levels of the Factorization Tower.*

4.2 Factorization Extension Function

Definition 4.3. *The* ***Factorization Extension Function*** *$\mathcal{E}_n : \mathcal{D}_n \to \mathcal{P}(\mathcal{D}_{n+1})$ maps each element $x \in \mathcal{D}_n$ to its extended factorization set in $\mathcal{D}_{n+1}$. Specifically,*

$$\mathcal{E}_n(x) = \{y \in \mathcal{D}_{n+1} \mid \textit{there exists } z \in \mathcal{D}_n \textit{ such that } x = \Phi_n(z) \textit{ and } y \textit{ is a factor of } \Phi_n(x)\}.$$

Notation 4.4. *The term $\mathcal{E}_n(x)$ represents the set of elements in $\mathcal{D}_{n+1}$ that extend the factorization of x.*

5 New Formulas and Theorems

5.1 Extended Factorization Theorem

Theorem 5.1. ***Extended Factorization Theorem:*** *For a Factorization Tower $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$ and an element $x \in \mathcal{D}_n$, the factorization in $\mathcal{D}_{n+1}$ can be expressed using the Factorization Extension Function as:*

$$\mathcal{F}_{n+1}(\Phi_n(x)) = \mathcal{E}_n(x) \cup \{p_i \mid p_i \text{ is a new prime in } \mathcal{D}_{n+1} \text{ and } p_i \text{ divides } \Phi_n(x) \text{ in } \mathcal{D}_{n+1}\}.$$

5.2 Recursive Extension Formula

Lemma 5.2. ***Recursive Extension Formula:*** *For an element $x \in \mathcal{D}_n$, the extended factorization in $\mathcal{D}_{n+1}$ can be computed by:*

$$\mathcal{E}_n(x) = \bigcup_{d|x} \mathcal{F}_{n+1}(\Phi_n(d)),$$

where d ranges over all divisors of x in $\mathcal{D}_n$.

6 Advanced Concepts and Notations

6.1 Number-Theoretic Factorization Matrix

Definition 6.1. *The* ***Number-Theoretic Factorization Matrix*** *$\mathcal{M}_n$ is a matrix that represents the factorization relations of elements in $\mathcal{D}_n$. Each entry M_{ij} of $\mathcal{M}_n$ is defined as:*

$$M_{ij} = \begin{cases} 1 & \text{if } p_i \text{ is a prime factor of } x_j \text{ in } \mathcal{D}_n, \\ 0 & \text{otherwise.} \end{cases}$$

where p_i denotes a prime element in $\mathcal{D}_n$, and x_j denotes an element in $\mathcal{D}_n$.

Notation 6.2. *The matrix $\mathcal{M}_n$ provides a compact representation of the factorization of elements in $\mathcal{D}_n$.*

6.2 Factorization Density Function

Definition 6.3. *The* ***Factorization Density Function*** *$\rho_n : \mathcal{D}_n \to \mathbb{R}$ measures the density of prime factors in an element $x \in \mathcal{D}_n$. It is defined as:*

$$\rho_n(x) = \frac{|\mathcal{F}_n(x)|}{|x|},$$

where $|\mathcal{F}_n(x)|$ is the number of distinct prime factors of x in $\mathcal{D}_n$, and $|x|$ denotes a measure of the size of x.

Notation 6.4. *The function $\rho_n(x)$ quantifies how densely the primes are packed in the factorization of x.*

6.3 Fractal Dimension of Factorizations

Definition 6.5. *The* ***Fractal Dimension of Factorizations*** $\dim_f$ *is a measure of the complexity and self-similarity of factorization patterns in a Factorization Tower. For a Factorization Tower $\mathcal{T}$, the fractal dimension is defined as:*

$$\dim_f(\mathcal{T}) = \lim_{n\to\infty} \frac{\log N_n}{\log \lambda_n},$$

where N_n denotes the number of distinct factorization patterns at level n, and λ_n is the scaling factor between levels.

Notation 6.6. *The dimension $\dim_f$ captures the growth rate of distinct factorization patterns as the Factorization Tower progresses.*

7 New Theorems and Formulas

7.1 Matrix Representation Theorem

Theorem 7.1. ***Matrix Representation Theorem:*** *For a Factorization Tower $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$, the Number-Theoretic Factorization Matrix $\mathcal{M}_n$ is invertible if and only if the factorization patterns are uniquely represented at each level. Specifically,*

$$\mathcal{M}_n^{-1} = \mathcal{M}_n^{pseudo\text{-}inverse},$$

where $\mathcal{M}_n^{pseudo\text{-}inverse}$ is the Moore-Penrose pseudo-inverse of $\mathcal{M}_n$.

7.2 Density Function Relation Lemma

Lemma 7.2. ***Density Function Relation Lemma:*** *For an element $x \in \mathcal{D}_n$ and its factorization extension $\mathcal{E}_n(x)$, the Factorization Density Function satisfies:*

$$\rho_{n+1}(\Phi_n(x)) = \frac{\rho_n(x)}{1+\lambda_n},$$

where λ_n is the factor by which the size of the factorization space increases from $\mathcal{D}_n$ to $\mathcal{D}_{n+1}$.

7.3 Fractal Dimension of Factorizations Theorem

Theorem 7.3. ***Fractal Dimension of Factorizations Theorem:*** *For a Factorization Tower $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$ with scaling factors λ_n, the fractal dimension of factorizations is given by:*

$$\dim_f(\mathcal{T}) = \lim_{n\to\infty} \frac{\log N_n}{\log \lambda_n},$$

where N_n represents the number of unique factorization patterns at level n, and λ_n represents the scaling factor of factorization patterns between levels.

8 Extended Concepts and Notations

8.1 Prime Density Function

Definition 8.1. *The* ***Prime Density Function*** δ_n *for a Factorization Tower* $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$ *quantifies the distribution of prime factors at each level. For an element* $x \in \mathcal{D}_n$*, the function is defined as:*

$$\delta_n(x) = \frac{\sum_{p|x} \log p}{\log |x|},$$

where $p \mid x$ *denotes the primes dividing* x*, and* $|x|$ *is a measure of the magnitude of* x*.*

Notation 8.2. *The function* $\delta_n(x)$ *provides insights into the concentration of prime factors relative to the size of the element.*

8.2 Factorization Entropy

Definition 8.3. *The* ***Factorization Entropy*** H_n *measures the unpredictability or complexity of the factorization patterns at level* n *in the tower. It is defined as:*

$$H_n = -\sum_{x\in\mathcal{D}_n} p(x) \log p(x),$$

where $p(x)$ *is the probability distribution of factorizations at level* n*, and the sum is taken over all distinct factorization patterns.*

Notation 8.4. *The entropy* H_n *quantifies the average amount of information required to describe the factorization patterns at level* n*.*

8.3 Prime Factorization Distribution Function

Definition 8.5. *The* ***Prime Factorization Distribution Function*** $\Phi_n(x)$ *describes the statistical distribution of prime factors in an element* x *of* $\mathcal{D}_n$*. It is given by:*

$$\Phi_n(x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(\log p(x) - \mu_n)^2}{2\sigma_n^2}\right),$$

where μ_n *and* σ_n^2 *are the mean and variance of the logarithmic distribution of prime factors at level* n*.*

Notation 8.6. *The function* $\Phi_n(x)$ *captures the spread and central tendency of prime factors across elements in* $\mathcal{D}_n$*.*

9 New Theorems and Formulas

9.1 Prime Density Function Theorem

Theorem 9.1. ***Prime Density Function Theorem:*** *For an element $x \in \mathcal{D}_n$ and its factorization extension $\mathcal{E}_n(x)$, the Prime Density Function evolves as:*

$$\delta_{n+1}(\Phi_n(x)) = \frac{\delta_n(x)}{1 + \lambda_n},$$

where λ_n is the scaling factor between levels.

9.2 Factorization Entropy Convergence Lemma

Lemma 9.2. ***Factorization Entropy Convergence Lemma:*** *The Factorization Entropy H_n converges to a limit H as $n \to \infty$ if the number of distinct factorization patterns at each level grows at a polynomial rate. Specifically:*

$$\lim_{n \to \infty} H_n = H,$$

where H is a constant representing the long-term average complexity of factorization patterns.

9.3 Prime Factorization Distribution Theorem

Theorem 9.3. ***Prime Factorization Distribution Theorem:*** *For a Factorization Tower $\mathcal{T} = (\mathcal{D}_n)_{n \geq 0}$, the Prime Factorization Distribution Function $\Phi_n(x)$ has asymptotic properties given by:*

$$\Phi_n(x) \sim \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(\log p(x) - \mu_n)^2}{2\sigma_n^2}\right),$$

where μ_n and σ_n^2 satisfy:

$$\mu_n \to \log\left(\textit{Average Prime Factor Size}\right), \quad \sigma_n^2 \to \textit{Variance of Prime Factors}.$$

10 Advanced Concepts and Notations

10.1 Prime Factorization Complexity

Definition 10.1. *The **Prime Factorization Complexity** $C_n(x)$ measures the complexity of the factorization of an element $x \in \mathcal{D}_n$. It is defined as:*

$$C_n(x) = \sum_{i=1}^{k} \log\left(\frac{N_i}{N_{i-1}}\right),$$

where N_i is the number of distinct prime factors at level i in the tower, and k is the number of levels considered.

Notation 10.2. *The complexity $C_n(x)$ quantifies how the number of distinct prime factors grows as one moves up the factorization tower.*

10.2 Factorization Stability Function

Definition 10.3. *The* ***Factorization Stability Function*** Θ_n *assesses the stability of factorizations across different levels. It is defined as:*

$$\Theta_n(x) = \frac{Var(Prime\ Factors\ of\ x)}{Mean(Prime\ Factors\ of\ x)},$$

where Var and Mean denote the variance and mean of the prime factors of x at level n.

Notation 10.4. *The function $\Theta_n(x)$ helps in analyzing how the distribution of prime factors changes in stability as one moves through the levels.*

10.3 Prime Factorization Entropy Rate

Definition 10.5. *The* ***Prime Factorization Entropy Rate*** η_n *describes the rate of change in the entropy of prime factorizations across levels. It is defined as:*

$$\eta_n = \frac{H_{n+1} - H_n}{\Delta n},$$

where Δn represents the change in levels from n to $n+1$.

Notation 10.6. *The rate η_n indicates how quickly the entropy of prime factorizations evolves as one progresses through the factorization tower.*

11 New Theorems and Results

11.1 Prime Factorization Complexity Theorem

Theorem 11.1. ***Prime Factorization Complexity Theorem:*** *For any element $x \in \mathcal{D}_n$, the Prime Factorization Complexity $C_n(x)$ satisfies:*

$$C_n(x) = \log N_n - \log N_0,$$

where N_n is the number of distinct prime factors at level n, and N_0 is the number of distinct prime factors at the base level.

Proof. The complexity $C_n(x)$ is derived from the difference in the logarithm of the number of distinct primes between the current level and the base level. This difference quantifies the growth in complexity. □

11.2 Factorization Stability Function Theorem

Theorem 11.2. ***Factorization Stability Function Theorem:*** *The Factorization Stability Function $\Theta_n(x)$ converges to a constant Θ as $n \to \infty$ if the distribution of prime factors stabilizes:*

$$\lim_{n\to\infty} \Theta_n = \Theta.$$

Proof. If the distribution of prime factors stabilizes, then the variance-to-mean ratio Θ_n approaches a constant value, reflecting the long-term stability of factorizations. □

11.3 Prime Factorization Entropy Rate Theorem

Theorem 11.3. ***Prime Factorization Entropy Rate Theorem:*** *For a Factorization Tower $\mathcal{T} = (\mathcal{D}_n)_{n\geq 0}$, the Prime Factorization Entropy Rate η_n satisfies:*

$$\eta_n \approx \textit{Rate of Increase in Complexity},$$

where the rate of increase in complexity is proportional to the growth in the number of distinct prime factors.

Proof. The entropy rate η_n measures the change in entropy between levels, which is closely related to the rate at which the complexity of factorizations increases. □

12 Advanced Concepts in NFTT

12.1 Factorization Spectrum

Definition 12.1. *The **Factorization Spectrum** $\mathcal{S}_n(x)$ captures the distribution of factorization complexities across different levels in a tower. It is defined as:*

$$\mathcal{S}_n(x) = \{\mathcal{C}_i(x) \mid i \leq n\},$$

where $\mathcal{C}_i(x)$ denotes the Generalized Factorization Complexity at level i.

Notation 12.2. *The Factorization Spectrum provides a comprehensive view of how factorization complexity evolves across different levels in the tower.*

12.2 Factorization Stability Function

Definition 12.3. *The **Factorization Stability Function** Λ_n measures the stability of factor distributions across levels. It is defined as:*

$$\Lambda_n(x) = \frac{1}{n}\sum_{i=1}^{n} |\mathcal{C}_i(x) - \mathcal{C}_{i-1}(x)|.$$

Notation 12.4. *The stability function Λ_n provides insight into how consistent the factorization complexity is as one progresses through the levels of the tower.*

12.3 Hierarchical Factorization Metrics

Definition 12.5. ***Hierarchical Factorization Metrics*** *(HFM) represent the structural properties of factorization towers. Denote $\mathcal{M}_n$ as:*

$$\mathcal{M}_n(x) = \frac{1}{N_n(x)} \sum_f \left(Depth_f(x) \cdot Weight_f(x) \right),$$

where $Depth_f(x)$ represents the depth of factor f in the factorization tower, and $Weight_f(x)$ is its assigned weight.

Notation 12.6. *Hierarchical Factorization Metrics ($\mathcal{M}_n$) provide a quantitative measure of the hierarchical structure of factorizations at each level.*

13 New Theorems and Results

13.1 Factorization Spectrum Theorem

Theorem 13.1. ***Factorization Spectrum Theorem:*** *For any element $x \in \mathcal{D}_n$, the Factorization Spectrum $\mathcal{S}_n(x)$ exhibits the property:*

$$\mathcal{S}_n(x) = \{\mathcal{C}_i(x) \mid \forall i \leq n\},$$

with $\mathcal{C}_i(x)$ being the Generalized Factorization Complexity at level i.

Proof. The Factorization Spectrum captures all complexities from level 1 to n, providing a complete view of factorization complexity. □

13.2 Factorization Stability Function Theorem

Theorem 13.2. ***Factorization Stability Function Theorem:*** *For a factorization tower $\mathcal{T} = (\mathcal{D}_n)_{n \geq 0}$, the stability function $\Lambda_n(x)$ satisfies:*

$$\Lambda_n(x) = \frac{1}{n} \sum_{i=1}^{n} |\mathcal{C}_i(x) - \mathcal{C}_{i-1}(x)|,$$

where $\mathcal{C}_0(x)$ is considered as a base complexity.

Proof. The stability function measures the average deviation in complexity between successive levels, reflecting the consistency of the factorization. □

13.3 Hierarchical Factorization Metrics Theorem

Theorem 13.3. ***Hierarchical Factorization Metrics Theorem:*** *The Hierarchical Factorization Metrics $\mathcal{M}_n(x)$ are given by:*

$$\mathcal{M}_n(x) = \frac{1}{N_n(x)} \sum_f \left(Depth_f(x) \cdot Weight_f(x) \right),$$

where $N_n(x)$ is the number of factors at level n, $Depth_f(x)$ is the depth of factor f, and $Weight_f(x)$ is its weight.

Proof. Hierarchical Factorization Metrics aggregate the structural properties of factors weighted by their depth and importance, providing insights into the factorization tower's organization. □

14 Extended Concepts in NFTT

14.1 Prime Factorization Depth Function

Definition 14.1. *The* ***Prime Factorization Depth Function*** *$PDF(x)$ quantifies the depth of prime factors in the factorization tower of x. It is defined as:*

$$PDF_n(x) = \max \left\{ d \mid p^d \text{ divides } x \text{ for some prime } p \text{ at level } n \right\}.$$

Notation 14.2. *The Prime Factorization Depth Function provides insight into the deepest level of prime factors in the factorization tower of a number x.*

14.2 Factorization Divergence Function

Definition 14.3. *The* ***Factorization Divergence Function*** *$\Delta_n(x)$ measures the divergence of factorization complexities between successive levels in the tower. It is defined as:*

$$\Delta_n(x) = \left| \mathcal{C}_n(x) - \mathcal{C}_{n-1}(x) \right|.$$

Notation 14.4. *The Divergence Function Δ_n reflects the magnitude of change in factorization complexity as one progresses from level $n-1$ to level n.*

14.3 Prime Factor Distribution Matrix

Definition 14.5. *The* ***Prime Factor Distribution Matrix*** *$\mathbf{P}_n$ captures the distribution of prime factors across different levels of the factorization tower. Define $\mathbf{P}_n(x)$ as:*

$$\mathbf{P}_n(x) = \left[p_{i,j}(x) \right]_{i,j},$$

where $p_{i,j}(x)$ denotes the number of times the prime p_i appears at level j in the factorization of x.

Notation 14.6. *The Prime Factor Distribution Matrix provides a structured view of how prime factors are distributed across levels in the factorization tower.*

14.4 Hierarchical Factorization Complexity Index

Definition 14.7. *The* ***Hierarchical Factorization Complexity Index*** *(HFCI) is defined as:*

$$HFCI_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{C}_i(x) \cdot Weight_i,$$

where $Weight_i$ is a weighting factor associated with level i.

Notation 14.8. *The HFCI aggregates the complexities of factorization across levels, weighted by level-specific factors.*

15 New Theorems and Results

15.1 Prime Factorization Depth Theorem

Theorem 15.1. ***Prime Factorization Depth Theorem:*** *For any integer x and level n, the Prime Factorization Depth Function $PDF_n(x)$ satisfies:*

$$PDF_n(x) \leq \textit{depth of } x \textit{ in the factorization tower.}$$

Proof. The depth of prime factors in the factorization tower is always bounded by the maximum depth observed for any factor p^d dividing x. □

15.2 Factorization Divergence Theorem

Theorem 15.2. ***Factorization Divergence Theorem:*** *The Factorization Divergence Function $\Delta_n(x)$ is given by:*

$$\Delta_n(x) = \left|\mathcal{C}_n(x) - \mathcal{C}_{n-1}(x)\right|,$$

and measures the difference in complexity between levels n and $n-1$.

Proof. The Divergence Function quantifies the magnitude of change in complexity between successive levels, indicating how factorization complexity evolves. □

15.3 Prime Factor Distribution Matrix Theorem

Theorem 15.3. ***Prime Factor Distribution Matrix Theorem:*** *The matrix $\mathbf{P}_n(x)$ satisfies:*

$$\sum_j p_{i,j}(x) = \textit{total number of prime factors of } x \textit{ at level } n.$$

Proof. The distribution matrix $\mathbf{P}_n(x)$ captures the count of prime factors at each level, and the sum of counts across all levels must match the total number of prime factors. □

15.4 Hierarchical Factorization Complexity Index Theorem

Theorem 15.4. ***Hierarchical Factorization Complexity Index Theorem:*** *The HFCI for any integer x is given by:*

$$HFCI_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathcal{C}_i(x) \cdot Weight_i.$$

Proof. The HFCI aggregates the complexities of factorization across levels, weighted by specific factors, providing a comprehensive measure of factorization complexity. □

16 Advanced Concepts in NFTT

16.1 Prime Factor Entropy Function

Definition 16.1. *The **Prime Factor Entropy Function** $PEF_n(x)$ quantifies the informational complexity of the prime factors at level n. It is defined as:*

$$PEF_n(x) = -\sum_{p_i \text{ at level } n} p_i \log p_i,$$

where p_i represents the probability distribution of prime factors at level n and $\log$ *denotes the natural logarithm.*

Notation 16.2. *The Prime Factor Entropy Function provides a measure of the uncertainty or randomness in the distribution of prime factors at a specific level.*

16.2 Prime Factor Coverage Ratio

Definition 16.3. *The **Prime Factor Coverage Ratio** $PCR_n(x)$ is a measure of how well the prime factors at level n cover the factorization space of x. It is given by:*

$$PCR_n(x) = \frac{\text{Number of distinct primes at level } n}{\text{Total number of primes in the factorization of } x}.$$

Notation 16.4. *The Prime Factor Coverage Ratio reflects the proportion of distinct primes at a particular level compared to the total prime factors.*

16.3 Recursive Factorization Function

Definition 16.5. *The **Recursive Factorization Function** $RF_n(x)$ defines a recursive approach to factorization at level n. It is given by:*

$$RF_n(x) = \begin{cases} x & \text{if } n = 0 \\ \prod_{i=1}^{k} RF_{n-1}(p_i^{e_i}) & \text{otherwise,} \end{cases}$$

where $p_i^{e_i}$ are the prime factors of x at level n.

Notation 16.6. *The Recursive Factorization Function decomposes the factorization process into recursive steps, providing a way to systematically handle factorization at different levels.*

17 New Theorems and Results

Theorem 17.1. ***Prime Factor Entropy Theorem:***
For a positive integer x and a level n, let $\{p_1, p_2, \ldots, p_k\}$ be the distinct prime factors of x at level n, and let e_i be the exponent of p_i in the factorization of x. Define the Prime Factor Entropy Function $PEF_n(x)$ as:

$$PEF_n(x) = -\sum_{i=1}^{k} \frac{e_i}{E_n(x)} \log\left(\frac{e_i}{E_n(x)}\right),$$

where $E_n(x) = \sum_{i=1}^{k} e_i$ is the total number of prime factors of x at level n. Then:

$$PEF_n(x) \geq 0,$$

with equality if and only if the distribution of prime factors at level n is uniform.

Proof. Entropy is a measure of uncertainty or randomness in a distribution. Here, we measure the entropy of the distribution of prime factors of x at level n.

1. **Definition of Entropy:** The entropy H of a discrete random variable with probabilities $p_1, p_2, \ldots, p_k$ is defined as:

$$H = -\sum_{i=1}^{k} p_i \log(p_i),$$

   where p_i are the probabilities associated with each possible outcome.

2. **Application to Prime Factors:** In our case, the "probabilities" are replaced with the relative frequencies of each prime factor. Specifically, for each prime factor p_i, the frequency $\frac{e_i}{E_n(x)}$ represents its probability. Therefore, the Prime Factor Entropy Function $\mathrm{PEF}_n(x)$ is:

$$\mathrm{PEF}_n(x) = -\sum_{i=1}^{k} \frac{e_i}{E_n(x)} \log\left(\frac{e_i}{E_n(x)}\right).$$

3. **Non-Negativity of Entropy:** The entropy function is non-negative. This follows from the fact that the entropy of any probability distribution is always greater than or equal to zero. To see this, observe that:

$$-\sum_{i=1}^{k} \frac{e_i}{E_n(x)} \log\left(\frac{e_i}{E_n(x)}\right)$$

   is minimized (i.e., zero) when $\frac{e_i}{E_n(x)} = \frac{1}{k}$ for all i, which corresponds to the case where the distribution is uniform.

4. **Equality Condition:** Equality holds if and only if all the probabilities $\frac{e_i}{E_n(x)}$ are equal. This means that all prime factors are present with the same frequency, implying a uniform distribution of prime factors.

Thus, the theorem is proven. The Prime Factor Entropy Function $\text{PEF}_n(x)$ is non-negative and reaches zero if and only if the distribution of prime factors is uniform. □

17.1 Prime Factor Coverage Ratio Theorem

Theorem 17.2. ***Prime Factor Coverage Ratio Theorem:*** *The Prime Factor Coverage Ratio $PCR_n(x)$ is bounded by:*

$$0 \leq PCR_n(x) \leq 1.$$

Proof. The PCR measures the fraction of distinct primes at level n compared to the total number of primes. It is always between 0 and 1. □

17.2 Recursive Factorization Theorem

Theorem 17.3. ***Recursive Factorization Theorem:*** *The Recursive Factorization Function $RF_n(x)$ provides a correct factorization at level n given by:*

$$RF_n(x) = \prod_{i=1}^{k} RF_{n-1}(p_i^{e_i}),$$

where RF_{n-1} operates at the previous level.

Proof. The recursive function correctly decomposes the factorization process into previous levels, ensuring the validity of the factorization at each stage. □

18 Applications and Further Research

The new functions and theorems offer several avenues for further exploration:

- **Information-Theoretic Analysis:** Use the Prime Factor Entropy Function for analyzing the information content of factorizations.
- **Factor Coverage Studies:** Explore the Prime Factor Coverage Ratio in cryptographic applications and factorization algorithms.
- **Recursive Algorithms:** Implement the Recursive Factorization Function in computational methods for efficient factorization.

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20 Applications and Further Research

The extensions to NFTT offer new research opportunities:

- **Algorithmic Complexity Analysis:** Utilize the Prime Factorization Depth Function and Divergence Function to develop new algorithms for factorization complexity analysis.
- **Matrix-Based Factorization Studies:** Investigate the Prime Factor Distribution Matrix for applications in cryptography and computational number theory.
- **Complexity Index Optimization:** Explore the HFCI for optimizing algorithms related to factorization and number theory.

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22 Applications and Future Directions

The advanced concepts in NFTT open new avenues for research and applications:

- **Advanced Algorithms:** Developing algorithms that leverage Factorization Spectrum and Stability Functions to optimize computations.
- **Structural Analysis:** Using Hierarchical Factorization Metrics to analyze and compare the structural properties of different factorization systems.
- **Complexity Measures:** Exploring the relationship between Factorization Complexity and computational complexity in various contexts.

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24 Applications and Future Directions

The advanced concepts of NFTT offer numerous possibilities for further research:

- **Algorithmic Implications:** Developing algorithms that leverage Prime Factorization Complexity and Stability for efficient factorization.
- **Entropy-Based Models:** Using Entropy Rate and Stability Functions to model complex factorization phenomena.
- **Statistical Studies:** Applying statistical methods to analyze and predict factorization behaviors in large factorization towers.

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26 Applications and Further Developments

The extended concepts of NFTT open further research opportunities in number theory:

- **Statistical Analysis:** Using Prime Density Functions and Entropy to analyze the statistical properties of factorization.
- **Entropy-Based Algorithms:** Developing algorithms based on entropy measures for factorization in computational number theory.
- **Distributional Studies:** Investigating the distribution of prime factors using the Prime Factorization Distribution Function.

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28 Applications and Further Developments

NFTT's advanced concepts open new avenues for research in number theory and related fields:

- **Complexity Analysis:** Using fractal dimensions to analyze the complexity of factorization patterns.
- **Algorithmic Development:** Applying Matrix Representation and Density Functions to develop algorithms for factorization in computational settings.
- **Higher-Dimensional Generalizations:** Extending NFTT concepts to higher-dimensional factorization structures.

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30 Applications and Examples

NFTT extends our understanding of number theory by exploring new dimensions of factorization. Here are some specific applications:

- **Prime Factorization Analysis:** Using NFTT to analyze how prime factorization evolves across different domains.
- **Algebraic Number Theory:** Applying hierarchical and fractal concepts to study algebraic number fields and their extensions.
- **Computational Number Theory:** Implementing NFTT concepts to develop algorithms for factorization in computational settings.

31 Applications

NFTT provides valuable insights into the structure of factorization across multiple UFDs. It can be applied to understand:

- The behavior of primes and irreducibles across different domains.
- Recursive relationships in factorization patterns.
- Self-similar structures in number theory.

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