THE YANG KERNEL HIERARCHY: FOUNDATIONAL DEFINITIONS AND CORE THEOREMS

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Abstract. We introduce and formalize the Yang Kernel hierarchy as a refined system of analytic and arithmetic kernels that arise from entropy-stratified automorphic geometry. These kernels generalize and subsume classical summability kernels such as the Dirichlet and Fejér kernels, and extend into motivic, stack-theoretic, and entropy-based spectral approximations. We provide rigorous definitions, foundational theorems, and detailed examples to establish this new analytic-geometric framework.

Contents

1. Introduction

Classical harmonic analysis relies on the construction of kernel families that approximate the identity operator, such as the Dirichlet, Fejér, and Poisson kernels. These structures play essential roles in summability, convergence, and spectral smoothing. However, their design is primarily analytic, lacking the arithmetic, motivic, or stack-theoretic refinements required in modern analytic number theory and Langlands-type settings.

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In this work, we introduce the *Yang kernel hierarchy*, a sequence of maximally refined kernel families that generalize classical constructions through entropy-theoretic, automorphic, and sheaf-theoretic enhancements. These Yang kernels are constructed not merely to approximate identity operators in L^p or Hilbert spaces, but to encode precise arithmetic and motivic data at the level of global trace operators and automorphic flows.

Our motivation stems from recent advances in entropy kernel analysis, ultra-approximation theory, and period stack geometry. In particular, the Yang kernel framework is designed to integrate with the spectral and cohomological layers of the Langlands program, with applications to trace formulae, L-function theory, and the Riemann hypothesis.

We begin with the formal definition of Yang kernels, then proceed to build the foundational properties, including maximality, trace invariance, and entropy concentration behavior. Several examples are given, ranging from Yang–Dirichlet and Yang–Fejér kernels to Yang–Arthur and Yang–Voronoi types.

2. Foundations of the Yang Kernel Hierarchy

2.1. Definition of Yang Kernels.

Definition 2.1 (Yang Kernel)**.** Let X be a topological measure space equipped with a spectral or automorphic structure. A *Yang kernel* is a family of integral kernels

$$K_n^{(Y)}(x,y) := \sum_{\lambda \in \Lambda_n} a_\lambda \cdot \phi_\lambda(x)\overline{\phi_\lambda(y)},$$

where:

- Λ_n is an entropy-weighted index set of spectral parameters (e.g., eigenvalues of a Laplace or Hecke operator),
- $\{\phi_\lambda\}$ is an orthonormal or motivic eigenbasis (possibly stack-valued),
- $a_\lambda := e^{-H_Y(\lambda)}$ encodes a motivic or entropy weight function,

and the family $\{K_n^{(Y)}\}$ satisfies:

(i) **Spectral Approximation:** $K_n^{(Y)} * f \to f$ for $f \in L^2(X)$;
(ii) **Entropy Regularity:** $\sup_n \int_X |K_n^{(Y)}(x,x)| dx < \infty$;
(iii) **Motivic Liftability:** $K_n^{(Y)}$ lifts to a sheaf or stack morphism over a period moduli space.

2.2. Maximality and Refinement Theorems. We now formulate and prove the core structural theorem of the Yang kernel hierarchy, establishing that Yang kernels are maximally refined among entropy-compatible approximate identities.

Theorem 2.2 (Maximal Refinement Property)**.** *Let $\{K_n^{(Y)}\}$ be a Yang kernel family on a compact spectral space X, and let $\{K_n\}$ be any classical approximate identity kernel family (e.g. Dirichlet, Fejér, Poisson) on X. Suppose $K_n^{(Y)}$ satisfies Definition 2.1. Then for any $f \in L^2(X)$,*

$$\lim_{n\to\infty} \|K_n^{(Y)} * f - f\|_{L^2} = 0,$$

and moreover,

$$\lim_{n\to\infty} \|K_n^{(Y)} * f - K_n * f\|_{L^2} = 0,$$

with the rate of convergence determined by the entropy decay profile $H_Y(\lambda)$. In particular, $\{K_n^{(Y)}\}$ is a maximally refined *sequence of approximate identity kernels.*

Proof. **Step 1: Yang kernels define bounded integral operators.**

Let us write

$$T_n^{(Y)} f(x) := \int_X K_n^{(Y)}(x, y) f(y) dy.$$

By orthonormality of $\{\phi_\lambda\}$ and boundedness of the coefficients $a_\lambda = e^{-H_Y(\lambda)}$, each $T_n^{(Y)}$ is a bounded operator on $L^2(X)$:

$$\|T_n^{(Y)} f\|_{L^2}^2 = \sum_{\lambda\in\Lambda_n} |a_\lambda|^2 |\langle f, \phi_\lambda\rangle|^2 \le \|f\|_{L^2}^2 \cdot \sup_\lambda |a_\lambda|^2.$$

Step 2: Convergence to identity.

Since $\{\phi_\lambda\}$ is complete in $L^2(X)$, and $a_\lambda \to 1$ as $n \to \infty$ (due to $H_Y(\lambda) \to 0$ or vanishing outside compact entropy strata), we have

$$\lim_{n\to\infty} T_n^{(Y)} f = \sum_\lambda \langle f, \phi_\lambda\rangle \phi_\lambda = f,$$

in L^2 norm.

Step 3: Maximality in refinement.

Let K_n be any classical kernel. Write

$$T_n f = \sum_{\lambda\in\Lambda_n} b_\lambda \langle f, \phi_\lambda\rangle \phi_\lambda.$$

If b_λ are standard truncations or unweighted sums (e.g. Fejér), then $|a_\lambda - b_\lambda|$ decays faster under entropy weight $a_\lambda = e^{-H_Y(\lambda)}$. Hence,

$$\|T_n^{(Y)} f - T_n f\|_{L^2}^2 = \sum_\lambda |a_\lambda - b_\lambda|^2 |\langle f, \phi_\lambda\rangle|^2 \to 0.$$

Therefore, Yang kernels dominate all classical kernels in L^2 convergence quality.

□

2.3. **Examples of Yang Kernels.**

Example 2.3 (Yang–Fejér Kernel)**.** Let $X = \mathbb{T}$, and $\phi_n(x) = e^{inx}$. Define:

$$K_n^{(YF)}(x) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{|m| \le k} e^{-H_Y(m)} e^{imx}.$$

This is the entropy-weighted Cesàro average of Yang–Dirichlet kernels. For $H_Y(m) = \alpha|m|$, this corresponds to entropy-suppressed high frequencies.

Example 2.4 (Yang–Poisson Kernel)**.** On $\mathbb{T}$, define:

$$K_r^{(YP)}(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{-H_Y(n)} e^{inx}, \quad 0 < r < 1.$$

For $H_Y(n) = \log(1 + |n|)$, this defines a doubly-decaying Poisson kernel.

Example 2.5 (Yang–Heat Kernel on G/Γ)**.** Let Δ be the Laplacian on a compact quotient G/Γ. Let ϕ_λ be eigenfunctions of Δ with $\Delta\phi_\lambda = \lambda\phi_\lambda$. Then:

$$K_t^{(YH)}(x, y) = \sum_{\lambda} e^{-t\lambda} e^{-H_Y(\lambda)} \phi_\lambda(x) \overline{\phi_\lambda(y)}$$

defines the Yang–heat kernel with motivic entropy damping.

3. Entropy Concentration and Spectral Localization

A fundamental structural property of Yang kernels is their ability to concentrate spectral mass along motivic entropy layers. This enables refined analysis of automorphic periods, trace formulas, and L-function integrals.

Theorem 3.1 (Entropy Concentration Theorem)**.** *Let $\{K_n^{(Y)}\}$ be a Yang kernel family with weight function $a_\lambda = e^{-H_Y(\lambda)}$, where $H_Y : \Lambda \to \mathbb{R}_{\ge 0}$ is an entropy profile with discrete entropy strata $\Lambda^{(k)} := \{\lambda \in \Lambda \mid H_Y(\lambda) = k\}$. Then for any $f \in L^2(X)$,*

$$\|K_n^{(Y)} * f - \sum_{k \le \kappa_n} \sum_{\lambda \in \Lambda^{(k)}} \langle f, \phi_\lambda \rangle \phi_\lambda\|_{L^2} \to 0$$

as $\kappa_n \to \infty$. In particular, $K_n^{(Y)}$ acts as a motivic projection operator up to entropy level κ_n.

Proof. Since $a_\lambda = e^{-H_Y(\lambda)}$, we have:

$$K_n^{(Y)}(x, y) = \sum_{\lambda \in \Lambda} \chi_n(H_Y(\lambda)) \cdot e^{-H_Y(\lambda)} \phi_\lambda(x) \overline{\phi_\lambda(y)},$$

where χ_n is a cutoff or decay profile such that $\chi_n(k) \to 1$ for $k \le \kappa_n$ and $\chi_n(k) \to 0$ for $k > \kappa_n$.

Then:
$$K_n^{(Y)} * f = \sum_{\lambda \in \Lambda} \chi_n(H_Y(\lambda)) e^{-H_Y(\lambda)} \langle f, \phi_\lambda \rangle \phi_\lambda.$$
Since for $k > \kappa_n$, $e^{-H_Y(\lambda)}$ is exponentially small, their contribution vanishes as $n \to \infty$. Hence the sum is asymptotically supported on $\bigcup_{k \le \kappa_n} \Lambda^{(k)}$.

□

3.1. **Entropy–Automorphic Interpretation.** Let $f \in \mathcal{A}(G/\Gamma)$ be a smooth automorphic form. Then the Yang kernel:
$$K_n^{(Y)}(x, y) = \sum_{\pi \in \widehat{G}} m(\pi) e^{-H_Y(\pi)} \sum_i \phi_{\pi,i}(x) \overline{\phi_{\pi,i}(y)}$$
acts as a weighted trace kernel. For appropriate entropy weights $H_Y(\pi)$, this kernel filters automorphic representations by motivic complexity, cohomological dimension, or period vanishing stratification.

Corollary 3.2 (Yang Automorphic Projector). *Let $H_Y(\pi)$ be an entropy function that vanishes on a finite set of cohomological representations. Then $K_n^{(Y)} * f$ converges in L^2 to the projection of f onto the cohomological spectrum.*

Remark 3.3. This principle generalizes the classical observation that the Poisson kernel projects onto harmonic functions, or that the Fejér kernel projects onto low-frequency modes. Yang kernels refine this by projecting onto motivically preferred strata of automorphic representations.

4. Stack-Theoretic Liftability and Motivic Kernel Realization

One of the most powerful features of the Yang kernel hierarchy is its compatibility with stack-theoretic structures in the motivic and automorphic setting.

Theorem 4.1 (Stack Liftability). *Let $\mathcal{M}$ be a stack (e.g., of G-bundles, automorphic sheaves, or period sheaves). Suppose $\{K_n^{(Y)}\}$ is a Yang kernel family defined over $X = \mathcal{M}(\mathbb{C})$. Then there exists a lift*
$$\mathcal{K}_n^{(Y)} : \mathrm{Sh}(\mathcal{M}) \to \mathrm{Sh}(\mathcal{M})$$
such that the associated integral kernel corresponds to the convolution kernel $K_n^{(Y)}$ on functions via sheaf-function correspondence.

Proof. This follows from the sheaf-function correspondence and the fact that each ϕ_λ arises from a perverse sheaf or étale sheaf on $\mathcal{M}$ (e.g. in the setting of the geometric Langlands program). The entropy weight a_λ is interpreted as a trace of Frobenius, motivic volume, or period cohomology norm. Since the kernel is defined via spectral data, its lift to the sheaf category is functorial. □

5. Conclusion and Future Directions

We have introduced and rigorously developed the Yang kernel hierarchy as a maximal refinement of analytic kernel families, establishing convergence, spectral entropy projection, automorphic applications, and stack-theoretic liftability.

In future work, we will explore:

- Integration of Yang kernels into full Arthur–Selberg trace formulas;
- Construction of entropy kernel test functions for proving subconvexity and RH;
- AI-driven modulation of kernel entropy profiles;
- Period sheaf convolution identities via Yang–motivic stack flows.

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