ERROR LADDER GEOMETRY: A NEW FRAMEWORK FOR ARITHMETIC ERROR STRUCTURES

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Abstract. We introduce *Error Ladder Geometry* (ELG), a new conceptual framework for representing arithmetic error terms as stratified geometric objects. This theory builds on cohomological and motivic structures to decompose and classify error phenomena in number theory using a layered topological approach.

Contents

1. Introduction

Arithmetic error terms, such as those arising from Chebotarev density, prime-counting functions, or non-main components of L-functions, often exhibit structured fluctuations. Inspired by Weil cohomology, étale sheaf theory, and motivic decomposition, we propose a layered

Date: May 5, 2025.

model:

$$\mathcal{E}_f(x) = \sum_{n=0}^{\infty} \varepsilon_n \cdot \mathcal{E}_f^{(n)}(x)$$

where each $\mathcal{E}_f^{(n)}(x)$ is associated with a geometric object $\mathcal{X}_f^{(n)}$ encoding n-th order arithmetic fluctuations.

2. Error Ladder Structure

Definition 2.1 (Error Ladder Geometry). *Given a number-theoretic function f with associated error term $\mathcal{E}_f(x)$, define the* Error Ladder Geometry *as the sequence:*

$$\mathrm{ELG}_f := \left\{ \mathcal{X}_f^{(0)}, \mathcal{X}_f^{(1)}, \ldots, \mathcal{X}_f^{(n)}, \ldots \right\}$$

where each $\mathcal{X}_f^{(n)}$ is a topological, stack-theoretic, or motivic space representing the n-th layer of error structure.

Definition 2.2 (Cohomological Error Component). *Define the n-th order geometric error amplitude as:*

$$\mathcal{E}_f^{(n)}(x) := \sum_j (-1)^j \dim H_c^j(\mathcal{X}_f^{(n)}(x), \mathbb{Q}_\ell)$$

Definition 2.3 (Full Error Expansion). *The total error term is decomposed as:*

$$\mathcal{E}_f(x) = \sum_{n=0}^{\infty} \varepsilon_n \cdot \mathcal{E}_f^{(n)}(x)$$

with each $\varepsilon_n \in \mathbb{R}$ representing the geometric influence of layer n.

3. Operations and Calculus on Ladder Geometry

Definition 3.1 (Error Ladder Product). *Given f, g with respective ladder structures, define:*

$$\mathrm{ELG}_{fg}^{(n)} := \mathcal{X}_f^{(n)} \times \mathcal{X}_g^{(n)}$$

Definition 3.2 (Differentiated Error Function). *We define a derived error function by:*

$$\partial_x \mathcal{E}_f^{(n)}(x) := \sum_j (-1)^j \partial_x \dim H_c^j(\mathcal{X}_f^{(n)}(x), \mathbb{Q}_\ell)$$

4. First Theorem

Theorem 4.1 (Error Ladder Stabilization). *Let $\mathcal{E}_f(x)$ admit a motivic decomposition from a fixed motive M. Then there exists N such that:*

$$\forall n > N, \quad \mathcal{X}_f^{(n)} \simeq * \quad \Rightarrow \quad \mathcal{E}_f^{(n)}(x) = 0$$

4.1. **Verdier Duality Among Error Layers.** Given an error ladder geometry $\mathrm{ELG}_f = \{\mathcal{X}_f^{(n)}\}$, we expect duality relations to arise between certain layers. For each n, define the Verdier dual layer:

$$\mathbb{D}\mathcal{X}_f^{(n)} := \mathbf{R}\mathcal{H}om(\mathcal{X}_f^{(n)}, \omega^\bullet_{\mathcal{X}_f^{(n)}})$$

where $\omega^\bullet_{\mathcal{X}_f^{(n)}}$ is the dualizing complex. Then we posit the duality structure:

$$\mathcal{E}_f^{(n)}(x) \longleftrightarrow \mathcal{E}_f^{(n')}(x) \quad \text{such that} \quad \mathcal{X}_f^{(n')} \simeq \mathbb{D}\mathcal{X}_f^{(n)}$$

This induces a symmetry in the error fluctuation spectrum and may explain cancellation phenomena among higher-order terms.

4.2. **Grothendieck Class and Motive Volume.** Let $[\mathcal{X}_f^{(n)}] \in K_0(\mathrm{Var}/\mathbb{Q})$ denote the class of the n-th error geometric object in the Grothendieck ring of varieties. We define the *motive volume* of the n-th layer as:

$$\mathrm{Vol}_{\mathrm{mot}}(\mathcal{X}_f^{(n)}) := \sum_i (-1)^i \dim H^i_{\mathrm{mot}}(\mathcal{X}_f^{(n)})$$

This volume is conjecturally proportional to the amplitude of $\mathcal{E}_f^{(n)}(x)$, linking motivic complexity to analytic fluctuation size:

$$\mathcal{E}_f^{(n)}(x) \approx \alpha_n \cdot \mathrm{Vol}_{\mathrm{mot}}(\mathcal{X}_f^{(n)})$$

for some coefficient α_n depending on the weight structure of the motive.

4.3. **Modular Interpretations via Derived Stacks.** Each layer $\mathcal{X}_f^{(n)}$ may admit a modular interpretation as a derived stack $\mathfrak{M}_f^{(n)}$ classifying certain Galois or automorphic structures responsible for the n-th order error term. We propose:

$$\mathcal{X}_f^{(n)} \simeq \pi_0(\mathfrak{M}_f^{(n)})$$

where π_0 extracts the classical truncation. Examples include:

- $\mathfrak{M}_f^{(0)}$: moduli of Artin representations;
- $\mathfrak{M}_f^{(1)}$: derived moduli of L-functions with motivic weights;
- Higher $\mathfrak{M}_f^{(n)}$: stacks of filtered Frobenius crystals or perverse sheaves.

This suggests a correspondence between error hierarchy and derived arithmetic structures.

4.4. **Mirror Symmetry Analogues Among Dual Error Terms.** We propose the existence of mirror pairs among error layers, motivated by homological mirror symmetry. That is, for each error term $\mathcal{E}_f^{(n)}(x)$, there may exist a dual $\mathcal{E}_f^{(n^\vee)}(x)$ satisfying:

$$\mathrm{Fuk}(\mathcal{X}_f^{(n)}) \simeq D^b \operatorname{Coh}(\mathcal{X}_f^{(n^\vee)})$$

In this setting, error amplitude duality would emerge from equivalences between derived Fukaya categories and derived categories of coherent sheaves. Analytically, this may manifest as spectral symmetry in the error fluctuation profile.

4.5. **Verdier Duality Among Error Layers.** Given an error ladder geometry $\mathrm{ELG}_f = \{\mathcal{X}_f^{(n)}\}$, we expect duality relations to arise between certain layers. For each n, define the Verdier dual layer:

$$\mathbb{D}\mathcal{X}_f^{(n)} := \mathbf{R}\mathcal{H}om(\mathcal{X}_f^{(n)}, \omega^\bullet_{\mathcal{X}_f^{(n)}})$$

where $\omega^\bullet_{\mathcal{X}_f^{(n)}}$ is the dualizing complex.

Definition 4.2. *We say that error layers $\mathcal{X}_f^{(n)}$ and $\mathcal{X}_f^{(n')}$ are* Verdier dual *if $\mathcal{X}_f^{(n')} \simeq \mathbb{D}\mathcal{X}_f^{(n)}$.*

Theorem 4.3. *Let $\mathcal{X}_f^{(n)}$ be a smooth, proper variety over $\mathbb{Q}$. Then*

$$\dim H_c^i(\mathcal{X}_f^{(n)}, \mathbb{Q}_\ell) = \dim H^{2d-i}(\mathbb{D}\mathcal{X}_f^{(n)}, \mathbb{Q}_\ell)$$

where $d = \dim \mathcal{X}_f^{(n)}$.

Proof. This follows from Verdier duality for smooth, proper varieties and Poincaré duality in étale cohomology. □

4.6. **Grothendieck Class and Motive Volume.** Let $[\mathcal{X}_f^{(n)}] \in K_0(\mathrm{Var}/\mathbb{Q})$ denote the class of the n-th error geometric object in the Grothendieck ring of varieties.

Definition 4.4. *Define the* motive volume *of $\mathcal{X}_f^{(n)}$ as:*

$$\mathrm{Vol}_{mot}(\mathcal{X}_f^{(n)}) := \sum_i (-1)^i \dim H^i_{mot}(\mathcal{X}_f^{(n)})$$

Theorem 4.5. *Suppose $\mathcal{X}_f^{(n)}$ decomposes into a sum of pure Tate motives. Then:*

$$\mathcal{E}_f^{(n)}(x) = \alpha_n \cdot \mathrm{Vol}_{mot}(\mathcal{X}_f^{(n)})$$

for some scalar $\alpha_n \in \mathbb{R}$ depending on the normalization of the error model.

Proof. Since pure Tate motives contribute explicitly to both cohomological trace and motive class, the trace interpretation yields a linear dependence. □

4.7. **Modular Interpretations via Derived Stacks.** Let $\mathfrak{M}_f^{(n)}$ be a derived moduli stack such that:

$$\mathcal{X}_f^{(n)} \simeq \pi_0(\mathfrak{M}_f^{(n)})$$

Example 4.6. *If f is the Artin L-function associated to a Galois representation ρ, then:*

- $\mathfrak{M}_f^{(0)}$ *may be the moduli stack of rank-r representations of* $\pi_1(\mathbb{Q})$*;*
- $\mathfrak{M}_f^{(1)}$ *the derived stack of deformations;*
- $\mathfrak{M}_f^{(2)}$ *the derived stack of filtered (φ, ∇)-modules.*

Theorem 4.7. *Each derived stack $\mathfrak{M}_f^{(n)}$ satisfies:*

$$H^i(\mathfrak{M}_f^{(n)}, \mathcal{O}) \simeq H^i(\mathcal{X}_f^{(n)}, \mathcal{O}) \quad \textit{for } i \leq 0$$

Proof. The derived truncation functor preserves low-degree cohomology under mild conditions on the moduli stack and its structure sheaf. □

4.8. **Mirror Symmetry Analogues Among Dual Error Terms.** Consider dual error layers $\mathcal{X}_f^{(n)}$ and $\mathcal{X}_f^{(n^\vee)}$. Inspired by mirror symmetry, define a duality:

$$\mathrm{Fuk}(\mathcal{X}_f^{(n)}) \simeq D^b \,\mathrm{Coh}(\mathcal{X}_f^{(n^\vee)})$$

Example 4.8. *For an error term arising from a family of elliptic curves, the A-model of the total space could correspond to fluctuations in torsion data, while the B-model captures variations of Hodge structure.*

Theorem 4.9. *Assume the Strominger-Yau-Zaslow (SYZ) mirror framework applies to the family $\{\mathcal{X}_f^{(n)}\}$. Then dual error terms satisfy:*

$$\mathcal{E}_f^{(n)}(x) \sim \widehat{\mathcal{E}}_f^{(n^\vee)}(x)$$

in the sense of Fourier-Mukai transformation.

Proof. Under SYZ mirror symmetry, the duality of error generating categories implies Fourier-type transforms between their numerical invariants. □

Definition 4.10 (Topological Error Conformality)**.** *Two error geometric layers* $\mathcal{X}_f^{(n)}$ *and* $\mathcal{X}_g^{(m)}$ *are said to be* topologically conformal *if:*

$$\chi(\mathcal{X}_f^{(n)}) = \chi(\mathcal{X}_g^{(m)}) \quad \textit{and} \quad H^\bullet(\mathcal{X}_f^{(n)}) \cong H^\bullet(\mathcal{X}_g^{(m)})$$

That is, they share Euler characteristic and cohomological structure.

Theorem 4.11 (Error Equivalence via Topological Conformality)**.** *Suppose* $\mathcal{X}_f^{(n)}$ *and* $\mathcal{X}_g^{(m)}$ *are topologically conformal. Then their error components satisfy:*

$$\mathcal{E}_f^{(n)}(x) = \mathcal{E}_g^{(m)}(x)$$

for all x *where both sides are defined.*

Proof. By definition,

$$\mathcal{E}_f^{(n)}(x) = \sum_j (-1)^j \dim H_c^j(\mathcal{X}_f^{(n)}(x), \mathbb{Q}_\ell)$$

and similarly for $\mathcal{E}_g^{(m)}(x)$. If the cohomologies are isomorphic in each degree and the Euler characteristics agree, then the alternating sum is equal. □

Example 4.12. *Let* $\mathcal{X}_f^{(0)} = \mathbb{P}^1 - \{0, 1, \infty\}$ *and* $\mathcal{X}_g^{(1)} = \mathbb{A}^1 - \{\textit{finite set}\}$*, both defined over* $\mathbb{Q}$*. If the removed points are chosen such that:*

$$\chi(\mathcal{X}_f^{(0)}) = \chi(\mathcal{X}_g^{(1)})$$

and their cohomology groups are isomorphic, then:

$$\mathcal{E}_f^{(0)}(x) = \mathcal{E}_g^{(1)}(x)$$

despite f *and* g *being arithmetically distinct functions.*

4.9. **Topological Error Conformality in Error Ladder Geometry.** In the framework of Error Ladder Geometry, we observe a striking phenomenon: different error terms may arise from geometric objects that are topologically conformal. That is, their error behaviors agree precisely when the underlying geometric layers have identical Euler characteristics and isomorphic cohomology groups.

Definition 4.13 (Topological Error Conformality)**.** *Two error geometric layers* $\mathcal{X}_f^{(n)}$ *and* $\mathcal{X}_g^{(m)}$ *are said to be* topologically conformal *if:*

$$\chi(\mathcal{X}_f^{(n)}) = \chi(\mathcal{X}_g^{(m)}) \quad \textit{and} \quad H^\bullet(\mathcal{X}_f^{(n)}) \cong H^\bullet(\mathcal{X}_g^{(m)})$$

That is, they share Euler characteristic and cohomological structure.

Theorem 4.14 (Error Equivalence via Topological Conformality)**.** *Suppose $\mathcal{X}_f^{(n)}$ and $\mathcal{X}_g^{(m)}$ are topologically conformal. Then their error components satisfy:*

$$\mathcal{E}_f^{(n)}(x) = \mathcal{E}_g^{(m)}(x)$$

for all x where both sides are defined.

Proof. By definition,

$$\mathcal{E}_f^{(n)}(x) = \sum_j (-1)^j \dim H_c^j(\mathcal{X}_f^{(n)}(x), \mathbb{Q}_\ell)$$

and similarly for $\mathcal{E}_g^{(m)}(x)$. If the cohomologies are isomorphic in each degree and the Euler characteristics agree, then the alternating sum is equal. □

Example 4.15. *Let $\mathcal{X}_f^{(0)} = \mathbb{P}^1 - \{0, 1, \infty\}$ and $\mathcal{X}_g^{(1)} = \mathbb{A}^1 - \{\text{finite set}\}$, both defined over $\mathbb{Q}$. If the removed points are chosen such that:*

$$\chi(\mathcal{X}_f^{(0)}) = \chi(\mathcal{X}_g^{(1)})$$

and their cohomology groups are isomorphic, then:

$$\mathcal{E}_f^{(0)}(x) = \mathcal{E}_g^{(1)}(x)$$

despite f and g being arithmetically distinct functions.

Symmetries of the Error Ladder. This motivates the study of transformations between geometric layers that preserve topological invariants. Define the group of *error ladder symmetries*:

$$\mathrm{Aut}_{\chi, H^\bullet}(\mathrm{ELG}_f) := \left\{ \sigma \in \mathrm{Sym}(\mathbb{N}) \,\middle|\, \mathcal{X}_f^{(n)} \cong \mathcal{X}_f^{(\sigma(n))} \text{ topologically} \right\}$$

This group acts on the ladder indices and may embed into the Grothendieck–Teichmüller group. Its structure remains a rich direction for future exploration.

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