Zeros On the Critical Line I:
The Existence of Infinitely Many Zeros on the Critical Line

Alien Mathematicians

Introduction

Although every attempt of proving the Riemann Hypothesis, that all nontrival zeros of $\zeta(s)$ line on $\sigma = \frac{1}{2}$, has failed, it is proved by G. H. Hardy in 1914 that $\zeta(s)$ has infinitely many zeros on $\sigma = \frac{1}{2}$

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Hardy's proof employs the functional equations, $\xi(s)$ and $\Xi(t)$, for the Riemann zeta function $\zeta(s)$: he managed to show the correspondence of zeros between $\zeta(s)$ on $\sigma = \frac{1}{2}$ and $\Xi(t)$ on the real line, and proving the existence of infinitely many zeros for $\Xi(t)$ on the real line.

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The Functional Equations for the Zeta Function:

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\,\Gamma\left(\frac{1}{2}s\right)\zeta(s) \tag{1}$$

$$\Xi(t) := \xi\left(\frac{1}{2}+it\right) = -\frac{1}{2}\left(t^2+\frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{1}{2}t}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\zeta\left(\frac{1}{2}+it\right) \tag{2}$$

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1-s)$ i.e. the reflexive formula for the functional equation

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Proof of Lemma 1 (1/3)

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Hence if $\sigma > 1$, we sum over n to find:

$$\frac{\Gamma\left(\frac{1}{2}s\right)\zeta(s)}{\pi^{\frac{1}{2}s}} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x}\, dx$$

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the inversion is justified by absolute convergent

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Also we see for $x > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-n^2 \pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi \frac{1}{x}}$$

$$2\sum_{n=1}^{\infty} e^{-n^2 \pi x} + e^{-0^2 \pi x} = \frac{1}{\sqrt{x}} \left(2\sum_{n=1}^{\infty} e^{-n^2 \pi \frac{1}{x}} + e^{-0^2 \pi \frac{1}{x}} \right)$$

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right) \tag{5}$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\pi^{-\frac{1}{2}s}\Gamma\big(\frac{1}{2}s\big)\zeta(s) = \int_0^{1} x^{\frac{1}{2}s-1}\psi(x)\,dx + \int_{1}^{\infty} x^{\frac{1}{2}s-1}\psi(x)\,dx$$

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By analytic continuation, we see that the R.H.S. is unchanged if we replace s by $1-s$, therefore:

$$\pi^{-\frac{1}{2}s}\Gamma\Big(\frac{1}{2}s\Big)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\Big(\frac{1}{2}(1-s)\Big)\zeta((1-s))$$

Multiply by $\frac{1}{2}s(s-1)$ yields Lemma 1 $\square$

Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cos(xt)\,dt = \frac{1}{2}\pi\left\{e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}\psi(e^{-2x})\right\} \tag{6}$$

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Proof of Lemma 2 (1/2)

This is a special case of which the integral involving $\Xi(t)$ of the form $\Phi(x) = \int_0^\infty f(t)\Xi(t)\cos(xt)\,dt$ that can be evaluated.
Where $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it)$, ϕ analytic.

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&= \frac{1}{2}\int_{-\infty}^{\infty} \phi(it)\phi(-it)\xi\left(\frac{1}{2} + it\right)y^{it}\, dt \\
&= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right)\phi\left(\frac{1}{2} - s\right)\xi(s)y^s\, ds
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Inserting back $y = e^x$, we get the desired integral:

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cos(xt)\, dt = \frac{1}{2}\pi\left\{e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}\psi(e^{-2x})\right\} \qquad \square$$

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Since $\Xi(t) = \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\pi^{-\frac{1}{4} - \frac{1}{2}t}\,\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right)$ is an even integrable function of t (by Lemma 1), and is real for real t.

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Putting $x = -i\alpha$ in (6), Lemma 2, we have

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cosh(\alpha t)\,dt &= e^{-\frac{1}{2}\alpha} - 2e^{\frac{1}{2}i\alpha}\psi(e^{2i\alpha}) \\ &= 2\cos\frac{1}{2}\alpha - 2e^{\frac{1}{2}i\alpha}\left\{\frac{1}{2} + \psi(e^{2i\alpha})\right\}\end{aligned}$$

Proof of the Theorem (2/5)

Since $\zeta(\frac{1}{2}+it) = O(t^A)$, $\Xi(t) = O(t^A e^{-\frac{1}{4}\pi t})$, and the last integral may be differentiated w.r.t. α any number of times provided that $\alpha < \frac{1}{4}\pi$.

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$$\frac{2}{\pi}\int_0^\infty \left(t^2+\frac{1}{4}\right)^{-1}\Xi(t)t^{2n}\cosh(\alpha t)\,dt = \frac{(-1)^n\cos(\frac{1}{2}\alpha)}{2^{2n-1}} - 2\left(\frac{d}{d\alpha}\right)^{2n} e^{\frac{1}{2}i\alpha}\left\{\frac{1}{2}+\psi(e^{2i\alpha})\right\}$$

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Next we show that the last term tends to 0 as $\alpha \to \frac{1}{4}\pi$.

Again use (5), the property of $\psi(x)$ exploited in Lemma 1:

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}$$

$$\psi(i+\delta) = \sum_{n=1}^\infty e^{-n^2\pi(i+\delta)} = \sum_{n=1}^\infty (-1)^n e^{-n^2\pi\delta}$$

Proof of the Theorem (3/5)

It follows: $\psi(i+\delta) = 2\psi(4\delta) - \psi(\delta) = \dfrac{1}{\sqrt{\delta}}\psi\Big(\dfrac{1}{4\delta}\Big) - \dfrac{1}{\sqrt{\delta}}\psi\Big(\dfrac{1}{\delta}\Big) - \dfrac{1}{2}$

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Suppose now $\Xi(t)$ were ultimately of one sign (for the sake of contradiction), say positive (negative can be shown by the same reason) for $t \geq T$, then

$$\lim_{\alpha \to \frac{1}{4}\pi} \int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t)\, dt = L > 0$$

Proof of the Theorem (4/5)

For all $\alpha < \frac{1}{4}\pi$ and $T' > T$,

$$0 < \int_{T}^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t)\, dt \leq L$$

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Thus the following integral converges due to the L.H.S. of (7), and converges with respect to α for $0 \le \alpha \le \frac{1}{4}\pi$:

$$\int_0^{\infty} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt = \frac{(-1)^n \pi \cos\left(\frac{1}{8}\pi\right)}{2^{2n}} \tag{8}$$

for every n.

Proof of the Theorem (5/5)

Equation (8) on the previous slide, however, is impossible, since when taking n odd, the R.H.S. is negative, therefore

$$\int_T^\infty (t^2+\frac{1}{4})^{-1}\Xi(t)t^{2n}\cosh\left(\frac{1}{4}\pi t\right)dt < -\int_0^T (t^2+\frac{1}{4})^{-1}\Xi(t)t^{2n}\cosh\left(\frac{1}{4}\pi t\right)dt$$
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But by hypothesis $\big(\Xi(t) > 0$ for $t \geq T\big)$, there is a positive $m = m(T)$ where $\left(t^2+\frac{1}{4}\right)^{-1}\Xi(t) \geq m$ for $2T \leq t \leq 2T+1$:

$$\int_T^\infty \left(t^2+\frac{1}{4}\right)^{-1}\Xi(t)t^{2n}\cosh\left(\frac{1}{4}\pi t\right)dt \geq \int_{2T}^{2T+1} mt^{2n}\,dt \geq m(2T)^{2n}$$

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But by hypothesis $\left(\Xi(t)>0 \text{ for } t\geq T\right)$, there is a positive $m=m(T)$ where $\left(t^2+\frac{1}{4}\right)^{-1}\Xi(t)\geq m$ for $2T\leq t\leq 2T+1$:

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Hence, $m2^{2n}<K$, which is false for sufficiently large n. $\square$