Exploring New Field Constructions from Generalized Motives and Applications

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Abstract

This paper explores the construction of novel fields derived from generalized motives, including mixed motives, complexified motives, and derived categories. We discuss their uniqueness, provide motivations for their study, and outline potential applications, particularly in the context of L-functions. Specific examples illustrate their use in algebraic geometry and number theory.

1 Introduction

In the realm of algebraic geometry and number theory, fields constructed from generalized motives offer unique perspectives and new structures beyond classical fields such as $\mathbb{R}$ and $\mathbb{Q}_p$. These constructions involve intricate algebraic and geometric methods, resulting in fields with distinctive properties.

2 Fields from Generalized Motives

2.1 Fields from Mixed Motives

2.1.1 Construction

Let $\mathbb{Q}_{\text{mot}}$ be a field constructed from the theory of mixed motives, incorporating both pure and mixed motives in algebraic geometry.

$$\text{Field: } \mathbb{Q}_{\text{mot}}$$

$$\text{Algebraic Closure: } \mathbb{Q}_{\text{mot, alg}}$$

$$\text{Completion (Archimedean): } \mathbb{Q}_{\text{mot, alg, arch}}$$

$$\text{Completion (Non-Archimedean): } \mathbb{Q}_{\text{mot, alg, non-arch}}$$

2.1.2 Motivation and Applications

Fields derived from mixed motives provide insights into the interplay between various types of motives in algebraic geometry. They are particularly useful in the study of L-functions, which are generating functions associated with algebraic varieties.

Example: L-functions Consider the L-function of a mixed motive, which encapsulates arithmetic information about the underlying variety. For a given variety X with a mixed motive, its L-function $L(s, X)$ can be expressed in terms of the completed field $\mathbb{Q}_{\text{mot, alg, arch}}$ or $\mathbb{Q}_{\text{mot, alg, non-arch}}$. The analytic properties of $L(s, X)$, such as its meromorphic continuation and functional equation, depend on the structure of these fields.

Concrete Example: Let X be a smooth projective variety over $\mathbb{Q}$ with an associated mixed motive. The L-function $L(s, X)$ can be expressed as:

$$L(s, X) = \prod_p L_p(s, X)$$

where $L_p(s, X)$ is the local L-function at a prime p, and its behavior at different completions $\mathbb{Q}_{\text{mot, alg, arch}}$ and $\mathbb{Q}_{\text{mot, alg, non-arch}}$ provides deep insights into the arithmetic of X.

2.2 Fields from Complexified Motives

2.2.1 Construction

$$\text{Field: } \mathbb{Q}_{\text{CM}}$$

$$\text{Algebraic Closure: } \mathbb{Q}_{\text{CM, alg}}$$

$$\text{Completion (Archimedean): } \mathbb{Q}_{\text{CM, alg, arch}}$$

$$\text{Completion (Non-Archimedean): } \mathbb{Q}_{\text{CM, alg, non-arch}}$$

2.2.2 Motivation and Applications

Fields from complexified motives often arise in the study of abelian varieties and complex multiplication. These fields are useful in understanding modular forms and their connection to Galois representations.

Example: Modular Forms For an abelian variety with complex multiplication, the associated modular form can be analyzed using $\mathbb{Q}_{\text{CM, alg, arch}}$. The completed field $\mathbb{Q}_{\text{CM, alg, arch}}$ can be used to understand the Fourier coefficients of modular forms and their connection to L-functions.

2.3 Fields from Derived Categories

2.3.1 Construction

$$\text{Field: } \mathbb{Q}_{\text{D}}$$

$$\text{Algebraic Closure: } \mathbb{Q}_{\text{D, alg}}$$

$$\text{Completion (Archimedean): } \mathbb{Q}_{\text{D, alg, arch}}$$

$$\text{Completion (Non-Archimedean): } \mathbb{Q}_{\text{D, alg, non-arch}}$$

2.3.2 Motivation and Applications

Fields from derived categories and perverse sheaves provide tools for studying the geometry of sheaves and their cohomology. These fields can be employed in the study of the geometric properties of algebraic varieties and their moduli.

Example: Perverse Sheaves The field $\mathbb{Q}_{\text{D, alg, arch}}$ can be used to analyze the cohomology of perverse sheaves on a variety. This analysis helps in understanding the structure of the derived category associated with the variety and can be applied to the study of derived categories in algebraic geometry.

3 References

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