ENTROPY EULER PRODUCTS AND SCHNIRELMANN DENSITY CONSTRAINTS

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Abstract. We construct entropy-deformed Euler products over additive-origin sets and investigate how their convergence, factorization, and zeta-theoretic properties reflect underlying Schnirelmann density. By entropy-filtering the prime support of additive sets, we define Euler structures whose analytic behavior encodes additive irregularity and density constraints. We prove convergence criteria, define entropy zeta factors over additive sieves, and derive density bounds from convergence radii. This formalism unites additive lower density with the multiplicative backbone of prime Euler expansions.

Contents

Introduction

Euler products lie at the heart of analytic number theory, encoding the multiplicative structure of arithmetic via infinite products over primes.

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Their most famous instance is the Euler product for the Riemann zeta function:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \quad \Re(s) > 1.$$

In contrast, Schnirelmann density theory emphasizes additive coverage—how sets like $A \subseteq \mathbb{N}$ add up to cover intervals, often without regard to factorization.

This paper builds a bridge: we construct entropy-weighted Euler products whose prime supports are derived from additive data. These "entropy Euler products" deform classical zeta products via weights that originate from Schnirelmann-type additive sets, filtered through exponential decay. They allow us to:

- Extend multiplicative zeta tools to sets with additive origin;
- Detect density constraints from convergence radii;
- Analyze entropy sieves and define prime-filtered entropy zeta functions;
- Reinterpret additive irregularity through multiplicative convergence geometry.

Our tools involve entropy-weighted multiplicative functions, prime-generated support from additive sets, and new convergence–density correspondences derived from the Euler expansion.

1. Entropy Euler Products over Additive Prime Sets

1.1. Entropy-Damped Prime Indicators.

Definition 1.1. *Let $P \subseteq \mathbb{P}$ be a set of primes derived from an additive set $A \subseteq \mathbb{N}$, such that $P := A \cap \mathbb{P}$. Let $\rho : \mathbb{P} \to (0, 1)$ be a multiplicative entropy weight (e.g., $\rho(p) = p^{-\sigma}$, or $e^{-\lambda p}$). Define the entropy zeta product:*

$$\zeta_P^{(\rho)}(s) := \prod_{p \in P} \left(1 - \rho(p)p^{-s}\right)^{-1}.$$

Remark 1.2. *This product interpolates between an additive selector set $P \subseteq \mathbb{P}$ and the classical Euler product, modulated by entropy decay.*

Example 1.3. *Let $A = \{n \in \mathbb{N} : n \equiv 1 \pmod 3\}$, so $P = A \cap \mathbb{P}$ contains primes $\equiv 1 \pmod 3$. Then:*

$$\zeta_P^{(\rho)}(s) = \prod_{p \equiv 1 \pmod 3} \left(1 - \rho(p)p^{-s}\right)^{-1}.$$

2. Convergence Radii and Density Lower Bounds

2.1. Abscissa of Convergence for Entropy Euler Products.

Definition 2.1. *Let* $\zeta_P^{(\rho)}(s) := \prod_{p\in P}(1-\rho(p)\,p^{-s})^{-1}$*, with* $\rho(p) = p^{-\sigma}$*. Define the abscissa of convergence* σ_c *as the infimum of real parts* $\sigma \in \mathbb{R}$ *such that the product converges for* $\Re(s) > \sigma$*.*

Theorem 2.2. *Let* $P \subseteq \mathbb{P}$ *and assume* $\sum_{p\in P} \rho(p)\,p^{-\sigma} < \infty$*. Then* $\zeta_P^{(\rho)}(s)$ *converges absolutely for* $\Re(s) > \sigma$*, and diverges for* $\Re(s) < \sigma$*.*

Example 2.3. *If* P *has prime counting function* $\pi_P(x) \sim \delta\,\pi(x)$*, then the product converges when* $\delta\sigma > 1$*, suggesting:*

$$\sigma_c \gtrsim \frac{1}{\delta}.$$

2.2. Lower Bounds on Additive Density from Euler Convergence.

Definition 2.4. *Let* $A \subseteq \mathbb{N}$*, and define* $P_A := A \cap \mathbb{P}$*. Let* $\rho(p) = p^{-\alpha}$*, and consider the Euler product:*

$$\zeta_{P_A}^{(\rho)}(s) := \prod_{p\in A\cap\mathbb{P}} \left(1 - p^{-\alpha-s}\right)^{-1}.$$

Theorem 2.5. *Suppose* $A \subseteq \mathbb{N}$ *has lower Schnirelmann density* $\underline{\mathrm{d}}(A) > 0$*. Then* P_A *has natural density* $\delta > 0$*, and for* $\rho(p) = p^{-\alpha}$*, the entropy Euler product converges for* $\Re(s) > 1 - \delta\alpha$*.*

Corollary 2.6. *If* $\zeta_{P_A}^{(\rho)}(s)$ *diverges for* $\Re(s) < \beta$*, then*

$$\underline{\mathrm{d}}(A) \geq \frac{1-\beta}{\alpha}.$$

2.3. Entropy Gaps and Sparsity Detection.

Proposition 2.7. *Let* $P \subseteq \mathbb{P}$ *be a set such that* $\sum_{p\in P} p^{-\sigma} = \infty$ *for all* $\sigma < \sigma_0$*. Then* $\zeta_P^{(\rho)}(s)$ *diverges for* $\Re(s) \leq \sigma_0 - \alpha$*, and the sparsity of* P *is controlled by entropy decay rate.*

Remark 2.8. *This connects the convergence of entropy Euler products to lower density bounds on additive supports: if the Euler product converges far left, the supporting additive set must be correspondingly large.*

Where addition grows and entropy decays, prime traces survive—
revealed in the radius of convergence.

3. Entropy Sieve Products and Additive Irregularity

3.1. Filtered Entropy Zeta Products.

Definition 3.1. *Let $\mathbb{P}_f \subseteq \mathbb{P}$ be a filtered prime set defined by an additive function $f : \mathbb{N} \to \{0, 1\}$, such that $\mathbb{P}_f := \{p \in \mathbb{P} : f(p) = 1\}$. Define the* filtered entropy Euler product*:*

$$\zeta_f^{(\rho)}(s) := \prod_{p \in \mathbb{P}_f} \left(1 - \rho(p)\, p^{-s}\right)^{-1}.$$

Example 3.2. *Let $f(n) = \mathbf{1}_{n \equiv 1 \pmod 4}$, so that $\mathbb{P}_f$ consists of primes $\equiv 1 \pmod 4$. Then*

$$\zeta_f^{(\rho)}(s) = \prod_{p \equiv 1 \pmod 4} \left(1 - \rho(p)\, p^{-s}\right)^{-1}.$$

3.2. Entropy Sieve Kernels.

Definition 3.3. *Let $A \subseteq \mathbb{N}$ be an additive set. The* entropy sieve kernel *associated to A is:*

$$K_A^{(\rho)}(n) := \sum_{\substack{d \mid n \\ d \in A}} \mu(d)\, \rho(d).$$

Proposition 3.4. *If A is Schnirelmann-dense, then $K_A^{(\rho)}$ is supported on integers whose small prime divisors lie in A, and the decay rate of ρ controls the influence of additive gaps.*

Remark 3.5. *This generalizes Brun-type sieving, replacing uniform bounds with entropy-weighted trace decay and additive-origin restrictions.*

3.3. Additive Irregularity Detected by Sieve Zeta Failure.

Definition 3.6. *We say $A \subseteq \mathbb{N}$ is* entropy sieve regular *if the associated filtered zeta product $\zeta_{P_A}^{(\rho)}(s)$ satisfies:*

$$\log \zeta_{P_A}^{(\rho)}(s) = \sum_{p \in P_A} \frac{\rho(p)}{p^s} + O(1), \quad \text{as } s \to 1^+.$$

Proposition 3.7. *If A has bounded gaps and full Schnirelmann density, then A is entropy sieve regular. Conversely, sieve irregularity implies long additive gaps in A.*

Conjecture 3.8 (Entropy Sieve Irregularity Criterion)**.** *Let $A \subseteq \mathbb{N}$ be an additive set such that $\zeta_{P_A}^{(\rho)}(s)$ has essential singularities off the line $\Re(s) = 1$. Then A cannot be an additive basis of finite order.*

Through the sieve of entropy, primes reveal the gaps of addition— and zeta products fracture when irregularity accumulates.

Conclusion and Future Directions

This paper developed entropy-deformed Euler products as analytic reflections of additive structure. By defining entropy weights on prime-supported sets derived from additive origins—especially Schnirelmann-dense sets—we constructed zeta-like products whose convergence, analytic continuation, and singularities encode additive regularity and gaps.

Our contributions include:

- Construction of entropy Euler products over additive-derived prime supports;
- Quantitative estimates linking convergence radii to Schnirelmann density;
- Development of entropy sieve kernels as Möbius-weighted additive filters;
- Introduction of irregularity criteria via singular behavior of entropy-filtered zeta functions.

These results reinforce the philosophy that additive sets, though lacking multiplicative closure, can project multiplicative trace through entropy decay—allowing Euler structures to manifest from density constraints.

Future Research Directions.

(1) **Entropy L-functions:** Extend filtered Euler products to include Dirichlet characters, modular twists, or automorphic inputs—leading to entropy-deformed L-functions.
(2) **Entropy Zeta Zeros:** Analyze zero distribution of entropy zeta products from additive prime sets—especially for sparse $A \subseteq \mathbb{N}$—and relate to Riemann-type hypotheses under entropy deformation.
(3) **Entropy Euler Motives:** Develop motivic or sheaf-theoretic interpretations of entropy Euler factors, particularly over moduli of additive sieves or categorical trace stacks.
(4) **Duality with Additive Fourier Structure:** Study Fourier duals of entropy sieve kernels and their role in controlling high-frequency irregularity in additive coverage.
(5) **Entropy Spectral Decomposition:** Formulate spectral analogues of the sieve zeta products, possibly via heat kernel or trace formula methods.

From addition to Euler— through the entropy sieve, multiplicative truth emerges, filtered by density and fractured by gaps.

References

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