ENTROPY HECKE MODULI, L-TRACE GROUPOIDS, AND POLYCATEGORIFIED ZETA HIERARCHIES

PU JUSTIN SCARFY YANG

Abstract. We initiate a new theory of entropy Hecke operators acting on categorified moduli of arithmetic trace kernels. By encoding trace dynamics in L-groupoid structures and gluing them via Hecke correspondences, we construct an entropic analog of the Langlands–Satake formalism. We further define polycategorified zeta hierarchies that stratify entropy-kernel stacks via multi-motivic correspondences, enabling spectral recursion and topological classification. This framework lifts entropy–zeta theory into a recursive higher-categorical Langlands moduli space, built from trace duality, convolution flows, and modular representation stacks.

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Introduction

The theory of Hecke operators lies at the heart of modern number theory, linking modular forms, automorphic representations, and the Langlands program. Meanwhile, recent developments in entropy kernel structures and zeta-trace geometry have proposed a new analytic and cohomological interpretation of additive arithmetic dynamics.

This paper builds a bridge between the two: we introduce a theory of *entropy Hecke operators*, acting not on classical modular forms, but on stacks of entropy kernels and their trace cohomologies.

We organize the space of such kernels into Hecke moduli stacks, define associated L-trace groupoids encoding duality morphisms, and classify their composition via a new notion: the *polycategorified zeta hierarchy*—a stratified system of higher-trace categories governed by entropy flow and motivic zeta liftings.

Our contributions include:

- Definition of entropy Hecke operators on modular entropy kernel sheaves;
- Construction of the entropy Hecke moduli stack and its trace orbit stratification;
- Encoding of L-trace dualities as groupoid morphisms between entropy zeta layers;
- Organization of trace categories into multi-level polycategories indexed by zeta depth;
- Categorification of spectral zeta propagation via kernel convolution and recursive Langlands lifts.

This framework lays the foundation for a higher-dimensional theory of zeta dynamics, one in which entropy, trace, and Hecke structure interact through categorical recursion.

1. Entropy Hecke Operators and Modular Kernel Stacks

1.1. Entropy Hecke Action on Trace Kernels.

Let $\mathscr{E} \in \mathcal{K}_{\mathrm{ent}}$ be an entropy kernel sheaf supported on $A \subseteq \mathbb{N}$. Let T_n denote the classical Hecke operator acting on arithmetic functions by:

$$T_n f(m) := \sum_{d|(n,m)} \chi(d) \cdot f\left(\frac{nm}{d^2}\right).$$

We define an entropy-weighted analog:

Definition 1.1. *The* entropy Hecke operator T_n^{ent} *acts on* $\mathscr{E}(m)$ *by*

$$T_n^{\mathrm{ent}} \mathscr{E}(m) := \sum_{d|(n,m)} \rho(d) \cdot \mathscr{E}\left(\frac{nm}{d^2}\right),$$

where $\rho(d) = \exp(-1/\sigma(\{d\}))$ *is the entropy weight.*

Remark 1.2. *This convolution encodes zeta-trace transport under prime-supported modular actions, and defines a non-unitary entropy deformation of the classical Hecke algebra.*

1.2. Moduli of Entropy Kernel Orbits.

Let $\mathcal{M}_{\text{Hecke}}^{\text{Ent}}$ denote the stack of entropy kernel orbits under $\{T_n^{\text{ent}}\}_{n\in\mathbb{N}}$. Each point corresponds to an equivalence class:

$$[\mathscr{E}] := \{T_n^{\text{ent}}\mathscr{E} \mid n \in \mathbb{N}\}.$$

Definition 1.3. *Define the* entropy Hecke moduli stack $\mathcal{M}_{\text{Hecke}}^{\text{Ent}}$ *as the stack classifying equivalence classes of entropy kernels under trace-compatible Hecke actions.*

Theorem 1.4. $\mathcal{M}_{\text{Hecke}}^{\text{Ent}}$ *is fibered over* $\text{Spec}(\mathbb{Z})$*, and each fiber* $\mathcal{M}_p$ *classifies prime-support entropy flows mod p-level kernel shifts.*

Remark 1.5. *This moduli stack provides a natural domain for defining motivic trace sheaves and categorifying zeta kernel dualities across modular flows.*

2. L-Trace Groupoids and Zeta Duality Correspondences

2.1. Trace Morphisms and L-Kernel Involutions.

We now define a trace-theoretic groupoid structure over the entropy Hecke moduli stack. Let $\mathcal{M}_{\text{Hecke}}^{\text{Ent}}$ be as in Section 1.

Definition 2.1. *An* L-trace morphism *between two entropy kernels* $\mathscr{E}_1, \mathscr{E}_2 \in \text{Obj}(\mathcal{M}_{\text{Hecke}}^{\text{Ent}})$ *is a triple:*

$$\phi : (\mathscr{E}_1, s) \to (\mathscr{E}_2, s'),$$

such that:

(1) $\zeta_{\mathscr{E}_1}(s) = \zeta_{\mathscr{E}_2}(s')$*;*
(2) *There exists a Hecke-compatible transformation* $\mathcal{H}_\phi$ *such that* $\mathscr{E}_2 = \mathcal{H}_\phi(\mathscr{E}_1)$*;*
(3) $s' = \omega(s)$ *for some involutive spectral shift* ω *(e.g.,* $s \mapsto 1 - s$*).*

Definition 2.2. *The category* $\mathcal{G}_{\text{L-trace}}$ *is the groupoid whose:*

- *objects are entropy kernels* $\mathscr{E}$ *in* $\mathcal{M}_{\text{Hecke}}^{\text{Ent}}$*,*
- *morphisms are L-trace morphisms up to entropy convolution equivalence.*

2.2. **Zeta Trace Involution and Duality.**

Definition 2.3. *Define the* entropy trace dual *of a kernel $\mathscr{E}$ as:*

$$\mathscr{E}^\vee(n) := \rho(n) \cdot \mathscr{E}(n) \cdot n^{-1}.$$

Proposition 2.4. *We have the duality identity:*

$$\zeta_{\mathscr{E}^\vee}(s) = \zeta_{\mathscr{E}}(1-s).$$

Definition 2.5. *The trace involution functor*

$$\mathcal{D}_L : \mathcal{M}_{\text{Hecke}}^{\text{Ent}} \to \mathcal{M}_{\text{Hecke}}^{\text{Ent}}$$

is defined by $\mathcal{D}_L(\mathscr{E}) := \mathscr{E}^\vee$, and lifts to an involutive automorphism of $\mathcal{G}_{\text{L-trace}}$.

2.3. **Groupoid Actions on Zeta Motive Classes.**
Let $\mathcal{Z}_{\mathscr{E}}(s) \in \mathrm{Ext}^1_{\mathrm{Mot}_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(s))$ denote the zeta motive class associated to kernel $\mathscr{E}$.

Definition 2.6. *The groupoid action*

$$\Phi : \mathcal{G}_{\text{L-trace}} \curvearrowright \text{ZetaMot}$$

is defined by pullback of entropy kernel structures along L-trace morphisms.

Conjecture 2.7 (Langlands Duality via Entropy L-Groupoid)**.** *The action of $\mathcal{G}_{\text{L-trace}}$ on zeta motive classes encodes a categorified Langlands correspondence between:*

$$\textit{Entropy kernels} \quad \leftrightsquigarrow \quad \textit{Automorphic zeta motives.}$$

Example 2.8. *Let $\mathscr{E}_\pi(n) := \rho(n) \cdot \lambda_\pi(n)$. Then $\mathcal{D}_L(\mathscr{E}_\pi)(n) = \rho(n) \cdot \lambda_\pi(n) \cdot n^{-1}$, and*

$$\zeta_{\mathscr{E}_\pi^\vee}(s) = L(1-s, \pi^\vee),$$

recovering classical functional equations via trace duality.

Entropy trace duality thus recovers Langlands zeta symmetry as a groupoid involution over Hecke-motive stacks.

3. Polycategorified Zeta Hierarchies and Spectral Depth Trees

3.1. **Zeta Polycategories and Multi-Trace Composition.**
We now introduce a polycategorical framework to organize the hierarchy of entropy trace morphisms and their recursive compositions.

Definition 3.1. *A* zeta polycategory $\mathcal{P}_{\text{Zeta}}$ *is a collection of:*

- *objects: entropy kernel sheaves $\mathscr{E} \in \mathcal{K}_{\text{ent}}$,*

- *1-morphisms: trace morphisms* $\phi : \mathscr{E}_1 \to \mathscr{E}_2$*,*
- n*-morphisms: higher trace correspondences among compositions* $\phi_1, \dots, \phi_n$*,*

subject to spectral compatibility relations derived from zeta flow convolution.

Definition 3.2. *The* depth *of a morphism* $\phi \in \mathcal{P}_{\mathrm{Zeta}}$ *is defined as*

$$\mathrm{depth}(\phi) := \sup \left\{ d \in \mathbb{Q} \mid \phi \in \mathrm{Hom}_{\leq d}(\mathscr{E}_1, \mathscr{E}_2) \right\},$$

where the filtration is given by entropy-weighted zeta convergence rate.

Remark 3.3. *This structure gives rise to spectral depth stratification, organizing trace morphisms into zeta-decay regulated categories.*

3.2. **Zeta Spectral Trees and Higher Trace Gluing.** Let us define a recursive tree of entropy kernel morphisms, stratified by polycategorical trace layers.

Definition 3.4. *A* zeta spectral tree $\mathcal{T}_{\mathrm{Zeta}}$ *is a directed acyclic graph where:*

- *vertices are kernel sheaves* $\mathscr{E}_v \in \mathcal{K}_{\mathrm{ent}}$*;*
- *edges* $e : v \to w$ *are trace morphisms* $\phi_{vw} \in \mathrm{Hom}_{\mathcal{P}_{\mathrm{Zeta}}}(\mathscr{E}_v, \mathscr{E}_w)$*;*
- *each node is assigned a depth level* $d(v) := \mathrm{depth}(\mathscr{E}_v) \in \mathbb{Q}_{>0}$*.*

Theorem 3.5 (Zeta Polygluing Principle)**.** *Given spectral tree* $\mathcal{T}_{\mathrm{Zeta}}$*, there exists a colimit in the polycategory* $\mathcal{P}_{\mathrm{Zeta}}$ *representing the total trace convolution:*

$$\mathrm{colim}_{v \in \mathcal{T}_{\mathrm{Zeta}}} \zeta_{\mathscr{E}_v}(s) =: \mathcal{Z}_{\mathrm{tot}}(s),$$

which assembles hierarchical zeta propagators into a single motivic trace flow.

3.3. **Trace Motive Stratification and Spectral Recursion.** Let $\mathcal{M}_{\mathrm{mot}}^{\mathrm{Zeta}}$ denote the category of motivic zeta sheaves.

Definition 3.6. *A* spectral recursion system *is a sequence* $\{\mathscr{E}_n\}_{n \in \mathbb{N}} \subset \mathcal{K}_{\mathrm{ent}}$ *together with operators* $\mathcal{R}_n : \mathscr{E}_n \to \mathscr{E}_{n+1}$ *such that*

$$\zeta_{\mathscr{E}_{n+1}}(s) = \mathcal{F}_n(\zeta_{\mathscr{E}_n}(s)),$$

where $\mathcal{F}_n$ *is a recursive trace function induced by convolution with entropy-deforming operators.*

Example 3.7. *Let* $\mathscr{E}_0(n) := \rho(n)$*, and* $\mathcal{R}_n := T_{p_n}^{\mathrm{ent}}$*. Then the sequence*

$$\mathscr{E}_1 := T_{p_1}^{\mathrm{ent}} \mathscr{E}_0, \quad \mathscr{E}_2 := T_{p_2}^{\mathrm{ent}} \mathscr{E}_1, \quad \cdots$$

generates a spectral zeta recursion tree of increasing entropy-multiplicity.

Conjecture 3.8 (Categorified RH via Spectral Depth). *Let $\mathcal{T}_{\mathrm{Zeta}}$ be a zeta spectral tree with motivic lift. Then:*

$$\text{RH} \iff \text{All nontrivial leaves of } \mathcal{T}_{\mathrm{Zeta}} \text{ terminate at depth } \Re(s) = \tfrac{1}{2}.$$

The polycategorical zeta hierarchy organizes trace morphisms as recursive motives, whose spectral boundaries define the arithmetic frontier of entropy flow.

Conclusion and Recursive Horizons

In this paper, we developed the theory of entropy Hecke moduli, L-trace groupoids, and polycategorified zeta structures. Our main contributions included:

- Defining entropy Hecke operators acting on arithmetic kernel sheaves with zeta-trace dynamics;
- Constructing the entropy Hecke moduli stack $\mathcal{M}_{\mathrm{Hecke}}^{\mathrm{Ent}}$ and its L-trace groupoid of duality morphisms;
- Establishing zeta duality through trace involution $s \mapsto 1 - s$, realized via motivic entropy reflection;
- Introducing the notion of zeta polycategories, encoding spectral depth, higher trace compositions, and colimits of recursive trace trees;
- Proposing a spectral zeta recursion principle where categorical trace depth converges toward Riemann zero symmetry.

Outlook and Future Theories.

1. **Entropy Motive ∞-Categories:** Extend the polycategorical trace system into an ∞-categorical setting, supporting flexible zeta sheaf flows across derived stacks.
2. **Trace-Lagrangian Functoriality:** Develop entropy action functionals and define critical points as motivic trace equilibria, leading to zeta-quantized entropy field theories.
3. **Arithmetic Polycategories of Modularity:** Encode modular eigenform towers as objects in trace polycategories, and lift Hecke convolution as polyfunctorial composition.
4. **Zeta Monoidal Descent Structures:** Introduce descent conditions for zeta-trace monoids on arithmetic base sites, relating local entropy statistics with global trace zeta laws.
5. **AI Zeta Kernel Constructions:** Train deep kernel machines to approximate categorical zeta flows, optimizing spectral depth loss against Langlands L-value alignments.

Entropy and trace intertwine to form the higher syntax of arithmetic. Zeta recursion reveals its grammar. Polycategories write its geometry.

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