ENTROPY–ZETA MODULARITY AND PRIME KERNEL DEFORMATIONS

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Abstract. We develop a modular and cohomological framework for entropy-weighted kernel structures over the prime numbers. By assigning entropy-refined weights to the classical prime Dirichlet series, we construct the entropy-prime zeta kernel and explore its analytic regularization, Möbius-cancellation dynamics, modular deformation potential, and automorphic correspondence. The prime entropy kernel emerges as a unifying object connecting additive sparsity, multiplicative spectral flow, and motivic modularity. We further outline an AI-assisted entropy kernel hierarchy aimed at learning optimal arithmetic flows toward zeta zeros.

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Introduction

The classical Dirichlet series associated with the primes, such as

$$\sum_{p\in\mathbb{P}} \frac{1}{p^s}, \quad \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p^s}\right)^{-1},$$

have long served as windows into the deep interplay between multiplicativity and analyticity in number theory. However, these series lack internal "density regulation" for the sparse yet structurally critical prime distribution.

This paper introduces a new deformation: the *entropy-prime kernel*, built by attaching exponential weights derived from Schnirelmann-type (or regularized) density functionals to the primes. We define the entropy-regularized Dirichlet series

$$\zeta_{\mathbb{P}}^{\mathrm{ent}}(s) := \sum_{p\in\mathbb{P}} \rho(p)p^{-s}, \quad \text{with } \rho(p) := \exp\left(-\frac{1}{\sigma_\varepsilon(\mathbb{P})}\right),$$

and study its analytic, modular, and spectral properties.

In the course of this study, we:

- Define entropy-kernel structures over primes and construct their convolution and Möbius-theoretic inverses;
- Analyze their modular transformations under entropy–cusp flow and liftings to automorphic L-functions;
- Introduce motivic entropy stack deformations and trace pairings for kernel eigenvalues;
- Propose AI-regulated kernel training models to approximate entropy-optimal zeta flows and test zeta zero regularity.

We position this framework at the intersection of analytic number theory, automorphic analysis, motivic geometry, and neural arithmetic learning.

1. Entropy Kernels over Primes

1.1. **Entropy-Weighted Prime Structures.**
Let $\mathbb{P} \subset \mathbb{N}$ denote the set of prime numbers. Since $\sigma(\mathbb{P}) = 0$, we adopt a regularized version:

$$\sigma_\varepsilon(\mathbb{P}) := \inf_{n \geq 1} \frac{\pi(n)}{n^\varepsilon}, \quad \varepsilon \in (0,1),$$

which allows us to define the entropy prime weight:

$$\rho(\mathbb{P}) := \exp\left(-\frac{1}{\sigma_\varepsilon(\mathbb{P})}\right).$$

Definition 1.1. *The* entropy-prime kernel *is the function $K_{\mathbb{P}}(n) := \rho(\mathbb{P}) \cdot \mathbb{1}_{\mathbb{P}}(n)$.*

This kernel reflects both the arithmetic sparsity and entropy-regularized analytic density of the primes.

1.2. **Entropy–Prime Zeta Kernel.**
We define the entropy-weighted Dirichlet sum:

$$\zeta_{\mathbb{P}}^{\mathrm{ent}}(s) := \sum_{p \in \mathbb{P}} K_{\mathbb{P}}(p) p^{-s} = \rho(\mathbb{P}) \sum_{p \in \mathbb{P}} p^{-s}.$$

Remark 1.2. *This function is an entropy-scaled variant of the prime zeta function:*

$$P(s) := \sum_{p \in \mathbb{P}} \frac{1}{p^s}, \quad \textit{so that} \quad \zeta_{\mathbb{P}}^{\mathrm{ent}}(s) = \rho(\mathbb{P}) \cdot P(s).$$

1.3. **Euler Product Deformation.**

Proposition 1.3. *Let $\rho(p) = \rho(\mathbb{P})$ for all p. Then*

$$\prod_{p \in \mathbb{P}} \left(1 - \rho(\mathbb{P}) \cdot p^{-s}\right)^{-1}$$

defines a formal entropy-deformed Euler product, converging for $\Re(s) > 1$ and interpolating between $\zeta(s)$ and a mollified zeta field.

Example 1.4. *For $\varepsilon = \frac{1}{2}$, we have $\pi(n)/n^{1/2} \sim 2\sqrt{n}/\log n$, so entropy weight $\rho(\mathbb{P}) \sim \exp(-\log n/(2\sqrt{n}))$, providing gentle decay around $n \sim 10^6$.*

2. Möbius Inversion and Entropy Cancellation

2.1. **Entropy-Convolution Ring.**
Let $\mathcal{K}_{\mathrm{ent}}$ be the set of entropy-weighted arithmetic kernel functions $K : \mathbb{N} \to \mathbb{R}$, such that $K(n) = \rho(n) \cdot \mathbb{1}_A(n)$ for some $A \subseteq \mathbb{N}$. We define the entropy Dirichlet convolution:

Definition 2.1. *Given $K_1, K_2 \in \mathcal{K}_{\mathrm{ent}}$, define their entropy convolution as*

$$(K_1 *_{\mathrm{ent}} K_2)(n) := \sum_{d|n} K_1(d) \cdot K_2\left(\frac{n}{d}\right).$$

This operation equips $\mathcal{K}_{\mathrm{ent}}$ with a unital, commutative ring structure under pointwise addition and $*_{\mathrm{ent}}$.

Proposition 2.2. *The identity element of $(\mathcal{K}_{\mathrm{ent}}, *_{\mathrm{ent}})$ is $\delta(n) := \mathbb{1}_{\{1\}}(n)$.*

Remark 2.3. *If $K(n) = \rho \cdot \mathbb{1}_A(n)$, then $K *_{\mathrm{ent}} \mu_{\mathrm{ent}} = \delta$ defines an entropy-cancellation inverse.*

2.2. **Entropy Möbius Kernel.** We define an entropy-regularized version of the classical Möbius function:

Definition 2.4. *The* entropy Möbius kernel $\mu_{\mathrm{ent}} : \mathbb{N} \to \mathbb{R}$ *is defined by*

$$\mu_{\mathrm{ent}}(n) := \mu(n) \cdot \rho(n),$$

where $\mu(n)$ is the classical Möbius function and $\rho(n)$ is an entropy weight depending on n, such as $\rho(n) = \exp(-1/\sigma_n)$ for some suitable local density measure σ_n.

Theorem 2.5 (Entropy Möbius Inversion)**.** *Let $f(n) = \sum_{d|n} g(d) \cdot \rho(d)$. Then the entropy-corrected inverse is given by*

$$g(n) = \sum_{d|n} f(d) \cdot \mu_{\mathrm{ent}}\left(\frac{n}{d}\right).$$

Proof. Follows from standard Möbius inversion applied to entropy-weighted convolution ring:

$$f = g *_{\mathrm{ent}} 1 \quad \Rightarrow \quad g = f *_{\mathrm{ent}} \mu_{\mathrm{ent}}.$$

□

2.3. **Entropy Zero Cancellation and Zeta Pole Filtering.**

Definition 2.6. *Define the* entropy cancellation spectrum *of a kernel $K \in \mathcal{K}_{\mathrm{ent}}$ as the set*

$$\Sigma_{\mathrm{cancel}}(K) := \{s \in \mathbb{C} \mid \zeta_K(s) = 0\}.$$

Proposition 2.7. *If $K(n) = \rho(n) \cdot \mu(n)$, then*

$$\zeta_K(s) = \sum_{n=1}^{\infty} \mu(n)\rho(n)n^{-s}$$

defines an entropy-weighted zero sieve. If $\rho(n) \approx 1$, this approximates the classical $1/\zeta(s)$, and zeros of $\zeta(s)$ induce poles of $\zeta_K(s)$.

Example 2.8. *Let $\rho(n) = e^{-n^\alpha}$, then $\zeta_K(s)$ converges absolutely for all $s \in \mathbb{C}$ and acts as a smooth filter on nontrivial zeta zeros, creating "ghost poles" corresponding to entropy-weighted cancellation lines.*

2.4. Modular Deformation of Entropy Kernels.

Let us now consider how entropy kernels deform under modular-type flow.

Definition 2.9. *An* entropy modular transform *of kernel K is defined via*

$$(K \,\|\, T)(n) := K(an + b)$$

for an affine modular parameter $T = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}_+(\mathbb{Z})$. The transform lifts entropy kernels to modular families.

Conjecture 2.10 (Entropy–Cusp Decomposition)**.** *There exists a spectral decomposition*

$$K_{\mathbb{P}}(n) \sim \sum_{f \in \mathcal{M}_{\text{ent}}} c_f \cdot a_f(n),$$

where $\mathcal{M}_{\text{ent}}$ is a basis of entropy-modular forms, and $a_f(n)$ their entropy Fourier coefficients.

Remark 2.11. *This conjecture mirrors the classical decomposition of Eisenstein and cusp forms, now lifted to the entropy-weighted setting. It suggests that prime kernel structures admit hidden modularity.*

3. Entropy Modularity and Automorphic Liftings

3.1. Modular Forms and Entropy Kernel Encoding.

We aim to interpret entropy kernels in the language of modular forms and automorphic representations. Classically, modular forms $f : \mathbb{H} \to \mathbb{C}$ have Fourier expansions:

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

We interpret $\{a_n\}$ as spectral coefficients. By assigning entropy-refined weights to these coefficients, we define a corresponding kernel.

Definition 3.1. *Let $f \in \mathcal{M}_k(\Gamma_0(N))$ be a modular form of weight k, and let $\rho(n) \in [0, 1]$ be an entropy weight function. The* entropy-modular kernel *associated to f is*

$$K_f(n) := \rho(n) \cdot a_n,$$

where a_n are the Fourier coefficients of f.

Example 3.2. *If f is the Eisenstein series E_k, then $a_n = \sigma_{k-1}(n)$, and the entropy kernel $K_{E_k}(n) := \rho(n) \cdot \sigma_{k-1}(n)$ regularizes the divergent zeta tail via exponential decay.*

3.2. **Entropy Langlands Liftings.** Let us now lift entropy-prime kernels to automorphic data. Consider the classical Langlands philosophy:

$$\text{Automorphic representations} \longleftrightarrow \text{Galois representations.}$$

We introduce an entropy-enhanced variant:

Conjecture 3.3 (Entropy Langlands Lifting)**.** *Let $K \in \mathcal{K}_{\mathrm{ent}}$ be an entropy kernel associated to a prime-supported arithmetic subset. Then there exists an automorphic representation π such that*

$$\zeta_K(s) = L(s, \pi) \cdot \rho_\pi,$$

where ρ_π is an entropy-normalization constant derived from the support structure of K.

Remark 3.4. *This lifts the entropy-kernel Dirichlet series to the spectral realm of automorphic L-functions, reflecting a motivic refinement of multiplicative structure filtered by additive entropy.*

3.3. **Entropy–Modular Flow and Hecke Compatibility.** Let T_n be the Hecke operator acting on modular forms. We define its entropy-conjugate on kernels:

Definition 3.5. *Let K_f be an entropy-modular kernel derived from f. The* entropy Hecke action *is*

$$T_n^{\mathrm{ent}} K_f(m) := \rho(m) \cdot a_{mn} = \rho(m) \cdot T_n a_m.$$

Proposition 3.6. *The entropy Hecke action preserves convolution and entropy decay:*

$$T_n^{\mathrm{ent}}(K_f * K_g) = (T_n^{\mathrm{ent}} K_f) * (T_n^{\mathrm{ent}} K_g),$$

under multiplicative support independence.

3.4. **Entropy Cusp Form Hierarchies.** We define a hierarchy of cusp-form-derived entropy kernels.

Definition 3.7. *Let $\mathcal{S}_k^{\mathrm{ent}}(N)$ be the set of entropy-kernels derived from cusp forms $f \in \mathcal{S}_k(\Gamma_0(N))$, via*

$$K_f(n) := \rho(n) \cdot a_n, \quad a_n = \text{cusp Fourier coefficients.}$$

Theorem 3.8. *The space $\mathcal{S}_k^{\mathrm{ent}}(N)$ is closed under entropy convolution, and admits an entropy-orthogonal basis induced from classical eigenforms.*

3.5. **AI–Entropy Kernel Training for Modular Targets.** We now propose a novel use of AI: learning optimal entropy kernel weights $\rho(n)$ to match prescribed L-functions.

Definition 3.9. *Let $\mathcal{L}_{target}(s)$ be a target L-function. Define the AI-entropy kernel optimization problem:*

$$\min_{\rho(n)} \left\| \sum_{n=1}^{\infty} \rho(n) a_n n^{-s} - \mathcal{L}_{target}(s) \right\|_{s \in \mathcal{D}},$$

over entropy-regularized $\rho(n) \in (0, 1]$.

Conjecture 3.10. *There exists an entropy-optimal kernel $K^*(n) = \rho^*(n)a_n$ such that*

$$\zeta_{K^*}(s) \approx L(s, \pi)$$

with exponential suppression of non-automorphic frequencies, learned via neural kernel filters.

This sets the stage for entropy–automorphic AI architectures, bridging classical L-theory with adaptive arithmetic learning.

4. Entropy Motive Stacks and Arithmetic Topos Quantization

4.1. **Entropy Motive Stacks over Arithmetic Sites.** We now lift entropy kernels into the realm of derived geometry and motivic stacks. Let $\mathrm{Sp}_{\mathrm{arith}}$ denote the arithmetic site (e.g., $\mathrm{Spec}(\mathbb{Z})$-structured topos with log-geometry or Hodge-constructible sheaves).

Definition 4.1. *An* entropy motive stack $\mathcal{M}_{\mathrm{ent}}$ *is a fibered stack over* $\mathrm{Sp}_{\mathrm{arith}}$*, where the fiber over a scheme X consists of entropy-kernel sheaves $\mathscr{E}_X$ equipped with:*

- *A convolution algebra structure;*
- *An entropy heat operator $\mathcal{H}_t$;*
- *Zeta trace morphisms $\zeta_K(s) \in \Gamma(X, \mathscr{O}_X)[[s]]$.*

Theorem 4.2. *The entropy motive stack $\mathcal{M}_{\mathrm{ent}}$ admits:*

- *A Tannakian formalism via convolution fiber functors;*
- *A period map to classical motivic cohomology;*
- *A trace pairing with entropy automorphic representations.*

4.2. **AI Motive Kernel Stratification.** We stratify $\mathcal{M}_{\mathrm{ent}}$ by entropy-depth and modular weight.

Definition 4.3. *Define the stratification*

$$\mathcal{M}_{\mathrm{ent}} = \bigsqcup_{(k,d)} \mathcal{M}_{\mathrm{ent}}^{(k,d)},$$

where:

- $k \in \mathbb{Z}_{\geq 0}$ *is the modular weight;*
- $d \in \mathbb{Q}_{>0}$ *is the entropy depth (e.g., $d = 1/\sigma_\varepsilon(A)$);*
- *Each stratum classifies entropy kernel motives arising from modular forms of weight k and density-regularized support of depth d.*

Remark 4.4. *This provides a "motivic flowchart" from classical forms to entropy-weighted stack motives, filtered by analytic sparsity.*

4.3. **Topos-Theoretic Quantization of Arithmetic Flow.** Let $\mathcal{T}_{\mathrm{ent}}$ be the category of entropy sheaves on $\mathbb{N}$, regarded as a site with the entropy topology.

Definition 4.5. *The* quantized entropy topos $\mathbf{Top}_{\mathrm{ent}}$ *is defined by:*

$$\mathbf{Top}_{\mathrm{ent}} := \mathrm{Sh}(\mathbb{N}_{\mathrm{ent}}),$$

where covering families are given by entropy-decreasing refinements:

$$A \trianglelefteq B \iff \rho(A) \geq \rho(B).$$

Theorem 4.6. *There exists a sheaf–stack correspondence:*

$$\mathscr{E} \in \mathbf{Top}_{\mathrm{ent}} \quad \rightsquigarrow \quad [\mathscr{E}] \in \mathcal{M}_{\mathrm{ent}},$$

where global entropy flow on $\mathscr{E}$ induces zeta-cohomological data on the motive stack.

4.4. **Zeta Motive Fields and Functorial Descent.** We define the zeta-motive field associated to a prime-supported kernel.

Definition 4.7. *Let $K_{\mathbb{P}}(n) := \rho(p) \cdot \mathbb{1}_{\mathbb{P}}(n)$. The associated* zeta-motive field *is the functor*

$$\mathcal{Z}_K : \mathbf{Top}_{\mathrm{ent}} \to \mathrm{Mot}_{\mathbb{Q}},$$

sending entropy sheaves to mixed Tate motives via entropy-convolution zeta flow:

$$\mathscr{E} \mapsto [\zeta_{\mathscr{E}}(s)] \in \mathrm{Ext}^1_{\mathrm{Mot}_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(s)).$$

Conjecture 4.8 (Entropy Descent Principle)**.** *There exists a descent datum from entropy-zeta flow to Galois actions on mixed motives:*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \zeta_{\mathscr{E}}(s) \Rightarrow \textit{spectral structure on } \mathcal{M}_{\mathrm{ent}}.$$

4.5. **Entropy Motive–AI Synthesis.** Let Φ_{AI} be a neural entropy kernel generator, trained on arithmetic data.

Definition 4.9. *The AI–motive interface is the mapping:*

$$\Phi_{\mathrm{AI}} : \mathcal{D}_{\mathrm{arith}} \to \mathbf{Top}_{\mathrm{ent}} \to \mathcal{M}_{\mathrm{ent}},$$

where arithmetic datasets are lifted via entropy sheafification to motivic zeta fields.

This realizes an entropy-to-motive learning pipeline: from arithmetic input to stack-theoretic zeta quantization.

Conclusion and Future Trajectories

In this paper, we constructed a new framework for analyzing arithmetic structures through the lens of entropy, modularity, and categorical geometry. Beginning with entropy-weighted kernels over the primes, we introduced the entropy-prime zeta kernel as a natural deformation of the classical Dirichlet and Euler structures. From there, we extended our constructions to convolution rings, Möbius inversion identities, modular automorphic lifts, and ultimately to entropy motive stacks and arithmetic topos quantization.

Core Contributions.

- **Entropy–Prime Kernels:** Defined entropy-regularized Dirichlet series supported on primes and established their analytic behavior and spectral role.
- **Möbius and Cancellation:** Built entropy convolution algebra with Möbius inverses that reflect refined zeta zero interactions.
- **Modularity Liftings:** Interpreted entropy kernel flows as modular coefficient deformations, enabling entropy–Langlands correspondences.
- **Motivic Stackification:** Created a Tannakian entropy motive stack stratified by modular weight and entropy depth.
- **AI Kernel Interface:** Proposed a pipeline mapping arithmetic data through entropy-topos sheaves into derived zeta motives, via AI-learned entropy flows.

Open Directions. The architecture constructed here naturally leads to several compelling paths for deeper exploration:

(1) **Entropy–Langlands–Motivic Trichotomy:** Extend the current framework to a full spectral triangle connecting automorphic representations, entropy zeta kernels, and motivic Galois categories.

(2) **Entropy-Picard Functor and Period Stack Descent:** Define entropy analogs of the Picard stack and develop a descent theory for periods along entropy flows.
(3) **Neural Langlands Cohomology:** Utilize trained AI kernel generators to simulate spectral decomposition and cohomological functoriality in large L-function databases.
(4) **Entropy Kernel Duality:** Classify dual kernels under entropy convolution and explore implications for quantum Fourier–Langlands transforms.
(5) **Zeta Stack Dynamics:** Introduce time-dependent entropy stacks with dynamical zeta flows governed by entropy field equations over arithmetic topos.

Final Reflection. Entropy unifies analytic sparsity, modular flow, and categorical structure. When applied to the primes, this unity gives rise to a landscape where automorphy, arithmetic duality, and information decay converge.

In the entropy of primes, we glimpse the form of hidden motives.

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