# SPECTRAL MOTIVES XIII: TRACE-CATEGORIFIED QUANTUM COHOMOLOGY AND PERIODIC MOTIVE OPERATORS

PU JUSTIN SCARFY YANG

Abstract. We construct a trace-categorified theory of quantum cohomology for spectral motives, where period sheaves and trace flows admit deformation quantizations governed by derived arithmetic path integrals. Periodic operators on motivic stacks are introduced to generalize classical quantum differential equations. These operators encode derived zeta flows, Frobenius periodicities, and categorified enumerative structures. Applications include trace-deformed Gromov–Witten invariants, motivic quantum connections, and trace-phase dualities between arithmetic zeta domains.

## Contents



*Date*: May 5, 2025.

## 1. Introduction

Quantum cohomology enriches classical cohomological structures by introducing deformation parameters and enumerative invariants tied to Gromov–Witten theory. In the motivic and arithmetic realm, such deformations acquire new layers through spectral trace flows and categorified arithmetic structures.

In this paper, we propose a theory of *trace-categorified quantum cohomology*, where:

- Spectral motives evolve under trace-compatible quantum operators;
- Period sheaves are deformed via arithmetic path integrals;
- Zeta flows, Frobenius periodicities, and quantum groupoids interact categorically;
- Enumerative invariants are interpreted through derived motivic moduli.

This framework builds upon the thermodynamic and period-dynamic structures introduced in Spectral Motives XI–XII and extends them by formally quantizing motivic flows with operator-theoretic techniques.

**Outline.** Section 2 develops the deformation-quantization framework and motivic quantum spaces. Section 3 introduces trace-periodic operators acting on period sheaves and flows. Section 4 analyzes quantum cohomological equations and categorified zeta integrability. Section 5 presents applications to arithmetic Gromov–Witten theory and spectral dualities.

## 2. Trace Quantization and Motivic Quantum Geometry

2.1. **Deformation quantization over trace stacks.** Let $\mathscr{M}^{\mathrm{Tr}}$ be the stack of trace-compatible spectral motives. We define a deformation quantization functor:

$$\mathcal{Q}_\hbar : \mathscr{M}^{\mathrm{Tr}} \rightsquigarrow \mathscr{M}_\hbar^{\mathrm{Tr}},$$

parameterized by a formal variable $\hbar$ governing trace-phase interactions. The functor $\mathcal{Q}_\hbar$ acts on:

- Period sheaves: $\mathscr{U}_{\mathcal{M}} \mapsto \mathscr{U}_{\mathcal{M},\hbar}$;
- Trace flows: $\Theta_{\mathrm{Tr}} \mapsto \widehat{\Theta}_{\mathrm{Tr},\hbar}$;
- Derived moduli: $\mathscr{P}^\infty \mapsto \mathscr{P}_\hbar^\infty$.

2.2. **Quantum period sheaves and Heisenberg–Frobenius structure.** The quantized period sheaf $\mathscr{U}_{\mathcal{M},\hbar}$ satisfies:

$$[\widehat{x}, \widehat{y}] = \hbar \cdot \omega_{\mathrm{Tr}}(x, y),$$

for a derived trace symplectic form $\omega_{\mathrm{Tr}}$. Operators $\widehat{x}, \widehat{y}$ encode deformed period integrations and Frobenius-twisted trace differentials.

This induces a Heisenberg-like structure over the motivic moduli governed by arithmetic flow invariants.

2.3. **Trace-phase space and quantum groupoids.** We define the *trace-phase stack* $\mathscr{T}_\hbar^\infty$ as the moduli of quantized trace vector fields and arithmetic periods:

$$\mathscr{T}_\hbar^\infty := \mathbf{B}\left(\widehat{\Theta}_{\mathrm{Tr},\hbar}, \mathscr{U}_{\mathcal{M},\hbar}\right).$$

Objects of $\mathscr{T}_\hbar^\infty$ correspond to quantum groupoids governing the interaction of motivic dynamics with derived Frobenius symmetries.

2.4. **Path integrals and motivic quantization functors.** We construct a path integral representation for the quantum deformation:

$$\mathcal{Q}_\hbar(\mathcal{M}) := \int_{\mathscr{P}^\infty} e^{\frac{i}{\hbar}\mathcal{A}_{\mathrm{Tr}}(\mathcal{M})} \mathscr{D}\mu,$$

where $\mathcal{A}_{\mathrm{Tr}}$ is the arithmetic trace action functional and $\mathscr{D}\mu$ is a categorified measure over the spectral period space.

This path integral interprets motivic quantization as a trace-averaged cohomological deformation.

## 3. Periodic Operators and Quantum Trace Dynamics

3.1. **Trace-periodic operators on motivic sheaves.** We define a trace-periodic operator $\widehat{Z}_{\mathcal{M}}$ acting on a quantized period sheaf $\mathscr{U}_{\mathcal{M},\hbar}$ as:

$$\widehat{Z}_{\mathcal{M}} := \sum_{n=1}^{\infty} e^{2\pi i n t/\hbar} \cdot T_{\mathrm{Fr}}^n,$$

where $T_{\mathrm{Fr}}$ is the Frobenius trace evolution operator. These operators generate periodic flows reminiscent of arithmetic Fourier modes over motivic cohomology.

3.2. **Spectral commutation and arithmetic Heisenberg algebras.** Let $\widehat{\mathscr{H}_{\mathrm{Tr}}}$ denote the algebra generated by $\widehat{Z}_{\mathcal{M}}$, $\widehat{\Theta}_{\mathrm{Tr},\hbar}$, and period multiplications. The relations

$$[\widehat{\Theta}_{\mathrm{Tr},\hbar}, \widehat{Z}_{\mathcal{M}}] = \lambda \cdot \widehat{Z}_{\mathcal{M}},$$

define a trace-categorified Heisenberg structure encoding arithmetic spectral flow in a deformation-theoretic context.

3.3. **Quantum trace evolution equation.** The evolution of quantized periods follows the motivic Schrödinger-type equation:

$$i\hbar \cdot \frac{d}{dt}\mathscr{U}_{\mathcal{M},\hbar}(t) = \widehat{Z}_{\mathcal{M}} \cdot \mathscr{U}_{\mathcal{M},\hbar}(t),$$

where the operator $\widehat{Z}_{\mathcal{M}}$ governs arithmetic oscillations and zeta-phase evolution over the motivic tower.

3.4. **Eigenmodes and categorified trace harmonics.** Solutions to the quantum trace evolution decompose into spectral eigenmodes:

$$\mathscr{U}_{\mathcal{M},\hbar}(t) = \sum_{\rho} c_\rho \cdot e^{2\pi i \rho t/\hbar} \cdot \mathscr{E}_\rho,$$

where $\rho$ are trace eigenvalues and $\mathscr{E}_\rho$ are trace-categorified harmonic sheaves.

These harmonic decompositions generalize motivic Fourier analysis and serve as building blocks for zeta-modulated arithmetic flows.

## 4. Quantum Cohomological Equations and Zeta Integrability

**4.1. Quantum period differential equations.** Quantized period flows satisfy the differential relation:

$$\hbar^2 \cdot \frac{d^2}{dt^2} \mathscr{U}_{\mathcal{M},\hbar} = \mathcal{P}_{\mathcal{M}}(t) \cdot \mathscr{U}_{\mathcal{M},\hbar},$$

where $\mathcal{P}_{\mathcal{M}}(t)$ is a potential derived from motivic trace curvature. This mirrors quantum differential equations in mirror symmetry, now expressed over trace-periodic stacks.

**4.2. Categorified WDVV equations.** We formulate a motivic analogue of the Witten–Dijkgraaf–Verlind (WDVV) equation:

$$\frac{\partial^3 \mathcal{F}_{\mathcal{M}}}{\partial \tau_i \partial \tau_j \partial \tau_k} = \sum_{\ell} \eta^{\ell m} \cdot \frac{\partial^3 \mathcal{F}_{\mathcal{M}}}{\partial \tau_i \partial \tau_j \partial \tau_\ell} \cdot \frac{\partial^3 \mathcal{F}_{\mathcal{M}}}{\partial \tau_k \partial \tau_m \partial \tau_n},$$

where $\mathcal{F}_{\mathcal{M}}$ is the motivic generating function for trace Gromov–Witten invariants and $\tau_i$ parametrize motivic deformations.

**4.3. Zeta-integrable quantum flows.** We define a trace-categorified zeta integral for quantum evolution:

$$Z_{\mathcal{M}}(\hbar) := \int_{\mathscr{P}^\infty} \exp\left(-\frac{1}{\hbar} \cdot \zeta_{\mathrm{Tr}}(\mathcal{M})\right) \mathscr{D}\mu,$$

which serves as the partition function encoding arithmetic periods, trace weights, and derived zeta structures.

This function interpolates between quantum arithmetic invariants and the classical values of $\zeta(s)$ in the trace limit.

**4.4. Motivic quantization and modularity.** We conjecture that $\mathscr{U}_{\mathcal{M},\hbar}$ admits a modular-type transformation law:

$$\mathscr{U}_{\mathcal{M},-1/\hbar} \cong \mathcal{S} \cdot \mathscr{U}_{\mathcal{M},\hbar},$$

for a trace-S-duality operator $\mathcal{S}$, categorifying duality between arithmetic flow and quantum periodicity.

Such duality offers a categorified extension of modular forms over motivic stacks.

## 5. Applications to Motivic Enumerative Geometry and Trace Dualities

**5.1. Trace-deformed Gromov–Witten invariants.** Let $\mathcal{M}$ be a spectral motive with associated quantized period sheaf $\mathscr{U}_{\mathcal{M},\hbar}$. We define the trace-deformed Gromov–Witten invariants via:

$$\langle \tau_{d_1}(\gamma_1) \cdots \tau_{d_n}(\gamma_n) \rangle_{\mathcal{M},\hbar} := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \cdot \mathrm{Tr}_\hbar(\gamma_i),$$

where $\psi_i$ are trace-categorified cotangent classes and $\mathrm{Tr}_\hbar$ lifts motivic classes into the quantum trace phase space.

**5.2. Motivic quantum connections.** The quantum connection $\nabla^\hbar_{\mathcal{M}}$ is defined as:

$$\nabla^\hbar_{\mathcal{M}} := \hbar \cdot d + \sum_i \Phi_i(\tau) \cdot d\tau_i,$$

acting on $\mathscr{U}_{\mathcal{M},\hbar}$, where $\Phi_i(\tau)$ encodes trace-fluctuation in motivic moduli. Flatness conditions $[\nabla^\hbar_i, \nabla^\hbar_j] = 0$ correspond to integrability of quantum zeta flows.

5.3. **Arithmetic mirror symmetry and trace-phase duality.** The trace-categorified mirror map relates:

$$\mathscr{U}_{\mathcal{M},\hbar} \longleftrightarrow \mathscr{F}_{\check{\mathcal{M}}}(\hbar^{-1}),$$

where $\check{\mathcal{M}}$ is the mirror dual motive in the arithmetic moduli and $\mathscr{F}_{\check{\mathcal{M}}}$ is its Fourier–Laplace transform over zeta-geometric periods.

This realizes a duality between arithmetic expansions and quantum trace condensation.

5.4. **Categorified string-theoretic implications.** The developed structures suggest:

- A motivic string field theory governed by trace-evolving stacks;
- Arithmetic modularity from quantum motivic propagators;
- A bridge between enumerative geometry, motivic cohomology, and trace thermodynamics.

These ideas offer a unifying categorical language to study trace-formal zeta symmetries, quantum motivic fluctuations, and arithmetic flows.

## 6. Conclusion

We have developed a trace-categorified framework for quantum cohomology in the arithmetic and motivic setting. By quantizing period sheaves, constructing trace-periodic operators, and encoding zeta dynamics in differential flow equations, we have extended the theory of spectral motives into the domain of arithmetic quantum deformation.

**Summary of Contributions:**

- Defined quantum deformation functors acting on trace-compatible motives;
- Introduced arithmetic analogues of Heisenberg algebras and quantum flow equations;
- Formulated categorified Gromov–Witten invariants and motivic quantum connections;
- Proposed dualities between quantum trace evolution and zeta-functional geometry.

This paper forms the foundation for future study of categorified quantum zeta theories, motivic path integral formulations, and arithmetic modularity in derived motivic stacks. Subsequent works may explore quantized L-functions, arithmetic brane categories, and higher trace quantization over condensed $\infty$-topoi.

## References

[1] M. Kontsevich and Y. Manin, *Gromov–Witten classes, quantum cohomology, and enumerative geometry*, 1994.
[2] K. Behrend, *Gromov–Witten invariants in algebraic geometry*, 1997.
[3] E. Frenkel and E. Witten, *Geometric Endoscopy and Mirror Symmetry*, 2007.
[4] D. Clausen and P. Scholze, *Condensed Mathematics*, 2020. `https://condensed-math.org`
[5] P. J. S. Yang, *Spectral Motives I–XII*, 2025.
[6] J. Lurie, *Spectral Algebraic Geometry*, 2018.
[7] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math., 2012.
[8] Y. André, *Une Introduction aux Motifs*, Panoramas et Synthèses, 2004.