Zeros On the Critical Line II: A Finite Proportion of the Zeros lie on the Critical Line

Alien Mathematicians

Introduction

Let $N_0(T)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. $\quad (T > 0)$
Last time, we saw that $N_0(T) \to \infty$ as $T \to \infty$.

Introduction

Let $N_0(T)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. $\quad (T > 0)$
Last time, we saw that $N_0(T) \to \infty$ as $T \to \infty$.

Hardy and Littlewood (1921) proved $N_0(T) \gg T$: Their essential idea is to divide the interval $(T, 2T)$ in pairs of abutting intervals, and prove that each abutting interval contains a zero

Introduction

Let $N_0(T)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. $\quad (T > 0)$
Last time, we saw that $N_0(T) \to \infty$ as $T \to \infty$.

Hardy and Littlewood (1921) proved $N_0(T) \gg T$: Their essential idea is to divide the interval $(T, 2T)$ in pairs of abutting intervals, and prove that each abutting interval contains a zero

Selberg later improved this, first (1942a) to $N_0(T) \gg T \log\log T$ and then (1942b) $N_0(T) \gg T \log T$, so that a positive proportion of the zeros are on the critical line.

Introduction

Let $N_0(T)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. $\quad (T > 0)$
Last time, we saw that $N_0(T) \to \infty$ as $T \to \infty$.

Hardy and Littlewood (1921) proved $N_0(T) \gg T$: Their essential idea is to divide the interval $(T, 2T)$ in pairs of abutting intervals, and prove that each abutting interval contains a zero

Selberg later improved this, first (1942a) to $N_0(T) \gg T \log\log T$ and then (1942b) $N_0(T) \gg T \log T$, so that a positive proportion of the zeros are on the critical line.

On the Riemann Hypothesis:

$$N_0(T) = N(T) \sim \frac{1}{2\pi} T \log T$$

Introduction

Let $N_0(T)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. $\quad (T > 0)$
Last time, we saw that $N_0(T) \to \infty$ as $T \to \infty$.

Hardy and Littlewood (1921) proved $N_0(T) \gg T$: Their essential idea is to divide the interval $(T, 2T)$ in pairs of abutting intervals, and prove that each abutting interval contains a zero

Selberg later improved this, first (1942a) to $N_0(T) \gg T \log\log T$ and then (1942b) $N_0(T) \gg T \log T$, so that a positive proportion of the zeros are on the critical line.

On the Riemann Hypothesis:

$$N_0(T) = N(T) \sim \frac{1}{2\pi} T \log T$$

Selberg's (1942b) proof modifies Hardy and Littlewood's (1921) proof by employing a general case of the Fourier transformations used in their (1921) paper. He then maneuvered these transforms to prove this Theorem

Fourier Transformations

Let $F(u), f(y)$ be functions related by the Fourier formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) e^{iyu}\, dy \qquad f(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u) e^{-iyu}\, du$$

Fourier Transformations

Let $F(u), f(y)$ be functions related by the Fourier formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) e^{iyu}\, dy \qquad f(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u) e^{-iyu}\, du$$

Integrating over $(t, t+H)$, we obtain

$$\int_t^{t+H} F(u)\, du = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) \frac{e^{iyH} - 1}{iy} e^{iyt}\, dy,$$

so that $\displaystyle\int_t^{t+H} F(u)\, du$ and $\displaystyle f(y)\frac{e^{iyH} - 1}{iy}$ are Fourier transforms.

Fourier Transformations

Let $F(u), f(y)$ be functions related by the Fourier formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) e^{iyu}\, dy \qquad f(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u) e^{-iyu}\, du$$

Integrating over $(t, t+H)$, we obtain

$$\int_t^{t+H} F(u)\, du = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) \frac{e^{iyH} - 1}{iy} e^{iyt}\, dy,$$

so that $\displaystyle\int_t^{t+H} F(u)\, du$ and $\displaystyle f(y)\frac{e^{iyH} - 1}{iy}$ are Fourier transforms.

Parseval's Theorem of the normal form gives (for $F(u)$ real, $f(y)$ even):

$$\int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)\, du \right|^2 dt = \int_{-\infty}^{\infty} \left| f(y) \frac{e^{iyH} - 1}{iy} \right|^2 dy \tag{1}$$

Fourier Transformations

Let $F(u), f(y)$ be functions related by the Fourier formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) e^{iyu}\, dy \qquad f(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u) e^{-iyu}\, du$$

Integrating over $(t, t+H)$, we obtain

$$\int_t^{t+H} F(u)\, du = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) \frac{e^{iyH} - 1}{iy} e^{iyt}\, dy,$$

so that $\int_t^{t+H} F(u)\, du$ and $f(y) \frac{e^{iyH} - 1}{iy}$ are Fourier transforms.

Parseval's Theorem of the normal form gives (for $F(u)$ real, $f(y)$ even):

$$\int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)\, du \right|^2 dt = \int_{-\infty}^{\infty} \left| f(y) \frac{e^{iyH} - 1}{iy} \right|^2 dy = \int_{-\infty}^{\infty} |f(y)|^2 \frac{4 \sin^2(\frac{1}{2} Hy)}{y^2}\, dy$$

$$\leq 2H^2 \int_0^{1/H} |f(y)|^2\, dy + 8 \int_{1/H}^{\infty} \frac{|f(y)|^2}{y^2}\, dy \tag{1}$$

Some Definitions Selberg used to Improve the Result

Define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \quad \alpha_1 = 1$$

Then we see from the Euler product that $\alpha_\mu \, \alpha_\nu = \alpha_{\mu\nu}$ if $(\mu, \, \nu) = 1$.
(i.e. α is multiplicative!)

Some Definitions Selberg used to Improve the Result

Define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \quad \alpha_1 = 1$$

Then we see from the Euler product that $\alpha_\mu \, \alpha_\nu = \alpha_{\mu\nu}$ if $(\mu, \, \nu) = 1$.
(i.e. α is multiplicative!)

Since the series for $(1-z)^{-\frac{1}{2}}$ dominates that for $(1-z)^{\frac{1}{2}}$:

If $\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha'_\nu}{\nu^s}, \quad \alpha'_1 = 1$, then $|\alpha_\nu| \leq \alpha'_\nu \leq 1$

Some Definitions Selberg used to Improve the Result

Define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \quad \alpha_1 = 1$$

Then we see from the Euler product that $\alpha_\mu \, \alpha_\nu = \alpha_{\mu\nu}$ if $(\mu, \, \nu) = 1$.
(i.e. α is multiplicative!)

Since the series for $(1-z)^{-\frac{1}{2}}$ dominates that for $(1-z)^{\frac{1}{2}}$:

If $\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha'_\nu}{\nu^s}, \quad \alpha'_1 = 1$, then $|\alpha_\nu| \le \alpha'_\nu \le 1$

$$\beta_\nu := \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right) \qquad (1 \le \nu < X)$$

Hence, all sums involving β_ν run over $[1, \, X]$ become:
(as we may assume $\beta_\nu = 0$ for $\nu \ge X$)

$$\phi(s) := \sum \frac{\beta_\nu}{\nu^s}$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \qquad \text{where } c>1$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \qquad \text{where } c > 1$$

Move the line of integration to $\sigma = \frac{1}{2}$ and evaluate the residue at $s = 1$:

$$\Phi(z) = \frac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \qquad \text{where } c > 1$$

Move the line of integration to $\sigma = \frac{1}{2}$ and evaluate the residue at $s = 1$:

$$\begin{aligned}\Phi(z) &= \frac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \\ &= \frac{1}{2}z\phi(1)\phi(0) + \frac{z^{\frac{1}{2}}}{2\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}\left|\phi\left(\frac{1}{2}+it\right)\right|^2 z^{it}\,dt\end{aligned}$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \qquad \text{where } c > 1$$

Move the line of integration to $\sigma = \frac{1}{2}$ and evaluate the residue at $s = 1$:

$$\begin{aligned}\Phi(z) &= \frac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \\ &= \frac{1}{2}z\phi(1)\phi(0) + \frac{z^{\frac{1}{2}}}{2\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}\left|\phi\left(\frac{1}{2}+it\right)\right|^2 z^{it}\,dt\end{aligned}$$

On the other hand, by definition of ϕ:

$$\Phi(z) = \frac{1}{4\pi i}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\beta_\mu\beta_\nu\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\frac{z^s}{n^s\mu^s\nu^{1-s}}z^s\,ds \tag{2}$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \qquad \text{where } c > 1$$

Move the line of integration to $\sigma = \frac{1}{2}$ and evaluate the residue at $s = 1$:

$$\begin{aligned}\Phi(z) &= \frac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \\ &= \frac{1}{2}z\phi(1)\phi(0) + \frac{z^{\frac{1}{2}}}{2\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}\left|\phi\left(\frac{1}{2}+it\right)\right|^2 z^{it}\,dt\end{aligned}$$

On the other hand, by definition of ϕ:

$$\begin{aligned}\Phi(z) &= \frac{1}{4\pi i}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\beta_\mu\beta_\nu\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\frac{z^s}{n^s\mu^s\nu^{1-s}}z^s\,ds \\ &= \sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right) \qquad (2)\end{aligned}$$

Preliminaries 2/3

Putting $z = e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta) - y}$, it follows that the functions:

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \frac{\Xi(t)}{t^2 + \frac{1}{4}} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t} \tag{3}$$

$$f(y) = \frac{1}{2} z^{\frac{1}{2}} \phi(1)\phi(0) - z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} \right) \tag{4}$$

are Fourier transforms.

Preliminaries 2/3

Putting $z = e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta) - y}$, it follows that the functions:

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \frac{\Xi(t)}{t^2 + \frac{1}{4}} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t} \tag{3}$$

$$f(y) = \frac{1}{2} z^{\frac{1}{2}} \phi(1)\phi(0) - z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} \right) \tag{4}$$

are Fourier transforms.

Inserting $y = \log x$, $G = e^{1/H}$ in (1) with $F(t)$ and $f(y)$ defined above for $H \leq 1$, the first integral on the right equals to

$$\int_1^G \left| \frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x} \phi(1)\phi(0) - \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \right) \right|^2 dx$$

Preliminaries 2/3

Putting $z = e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta) - y}$, it follows that the functions:

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \frac{\Xi(t)}{t^2 + \frac{1}{4}} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t} \tag{3}$$

$$f(y) = \frac{1}{2} z^{\frac{1}{2}} \phi(1)\phi(0) - z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} \right) \tag{4}$$

are Fourier transforms.

Inserting $y = \log x$, $G = e^{1/H}$ in (1) with $F(t)$ and $f(y)$ defined above for $H \leq 1$, the first integral on the right equals to

$$\int_1^G \left| \frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x} \phi(1)\phi(0) - \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \right) \right|^2 dx$$

Calling the triple sum $g(x)$, then the above equation is not greater than:

$$2 \int_1^G \frac{|\phi(1)\phi(0)|^2}{4x^2} \, dx + 2 \int_1^G |g(x)|^2 \, dx < \frac{1}{2} |\phi(1)\phi(0)|^2 + 2 \int_1^G |g(x)|^2 \, dx$$

Preliminaries 3/3

$$g(x) := \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\Big(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \Big)$$

Preliminaries 3/3

$$g(x) := \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\Big(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \Big)$$

Similarly, we can show that the second integral in (1) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G \log^2 G} + 2 \int_G^\infty \frac{|g(x)|^2}{\log^2 x} \, dx$$

Preliminaries 3/3

$$g(x) := \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\Big(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \Big)$$

Similarly, we can show that the second integral in (1) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G \log^2 G} + 2 \int_G^{\infty} \frac{|g(x)|^2}{\log^2 x} \, dx$$

We thus obtained upper bounds for integrals (1) as $\delta \to 0$,

Preliminaries 3/3

$$g(x) := \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(- \frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \right)$$

Similarly, we can show that the second integral in (1) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G \log^2 G} + 2 \int_G^\infty \frac{|g(x)|^2}{\log^2 x} \, dx$$

We thus obtained upper bounds for integrals (1) as $\delta \to 0$, but if we consider:

$$J(x,\, \theta) := \int_x^\infty |g(u)|^2 u^{-\theta} \, du \qquad (0 < \theta \le \frac{1}{2},\ 1 \le x)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda \nu} \int_x^\infty \exp\left\{ - \pi \Big(\frac{m^2 \kappa^2}{\lambda^2} + \frac{n^2 \mu^2}{\nu^2} \Big) u^2 \sin\delta + i\pi \Big(\frac{m^2 \kappa^2}{\lambda^2} - \frac{n^2 \mu^2}{\nu^2} \Big) u^2 \cos\delta \right\} \frac{du}{u^\theta}$$

and let $\sum_1$ denote the sum of those terms in which $m\kappa/\lambda = n\mu/\nu$, and $\sum_2$ the remainder.

Estimates of $\sum_1$ and $\sum_2$ for $J(x,\,\theta)$ converges uniformly

$$\sum\nolimits_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^{\infty} \exp\left\{ -2\pi \left(\frac{m^2 \kappa^2}{\lambda^2} \right) u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

$$\sum\nolimits_2 := J(x,\theta) - \sum\nolimits_1$$

Estimates of $\sum_1$ and $\sum_2$ for $J(x,\,\theta)$ converges uniformly

$$\sum_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^\infty \exp\left\{ -2\pi\left(\frac{m^2\kappa^2}{\lambda^2}\right) u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

$$\sum_2 := J(x,\theta) - \sum_1$$

It is shown that

$$\sum_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right) + \mathcal{O}\left\{\frac{(\delta^{\frac{1}{2}} x X^2)^\theta}{\delta^{\frac{1}{2}} \theta x^\theta \log X}\right\} + \mathcal{O}\left\{\frac{x^{1-\theta}\log(X/\theta)}{\theta} X^2 \log^2 X\right\}$$

Estimates of $\sum_1$ and $\sum_2$ for $J(x,\ \theta)$ converges uniformly

$$\sum\nolimits_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu} \int_x^\infty \exp\left\{ -2\pi\left(\frac{m^2\kappa^2}{\lambda^2}\right)u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

$$\sum\nolimits_2 := J(x,\theta) - \sum\nolimits_1$$

It is shown that

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right) + \mathcal{O}\left\{\frac{(\delta^{\frac{1}{2}} x X^2)^\theta}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right\} + \mathcal{O}\left\{\frac{x^{1-\theta}\log(X/\theta)}{\theta} X^2 \log^2 X\right\}$$

We may ultimately take $X = \delta^{-c}$, and $H = (a \log X)^{-1}$, where a and c are suitable positive constant. Then $G = X^a = \delta^{-ac}$.

Estimates of $\sum_1$ and $\sum_2$ for $J(x, \theta)$ converges uniformly

$$\sum\nolimits_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^{\infty} \exp\left\{ -2\pi\left(\frac{m^2\kappa^2}{\lambda^2}\right) u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

$$\sum\nolimits_2 := J(x,\theta) - \sum\nolimits_1$$

It is shown that

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right) + \mathcal{O}\left\{\frac{(\delta^{\frac{1}{2}} x X^2)^\theta}{\delta^{\frac{1}{2}} \theta x^\theta \log X}\right\} + \mathcal{O}\left\{\frac{x^{1-\theta}\log(X/\theta)}{\theta} X^2 \log^2 X\right\}$$

We may ultimately take $X = \delta^{-c}$, and $H = (a \log X)^{-1}$, where a and c are suitable positive constant. Then $G = X^a = \delta^{-ac}$. If $x \leq G$, the last two terms can be omitted with the first if $GX^2 = \mathcal{O}(\delta^{-\frac{1}{2}})$ i.e. if $(a+2)c \leq \frac{1}{4}$, we have:

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right)$$

Estimates of $\sum_1$ and $\sum_2$ for $J(x, \theta)$ converges uniformly

$$\sum\nolimits_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^\infty \exp\left\{ -2\pi\left(\frac{m^2\kappa^2}{\lambda^2}\right) u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

$$\sum\nolimits_2 := J(x, \theta) - \sum\nolimits_1$$

It is shown that

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right) + \mathcal{O}\left\{\frac{(\delta^{\frac{1}{2}} x X^2)^\theta}{\delta^{\frac{1}{2}} \theta x^\theta \log X}\right\} + \mathcal{O}\left\{\frac{x^{1-\theta}\log(X/\theta)}{\theta} X^2 \log^2 X\right\}$$

We may ultimately take $X = \delta^{-c}$, and $H = (a \log X)^{-1}$, where a and c are suitable positive constant. Then $G = X^a = \delta^{-ac}$. If $x \leq G$, the last two terms can be omitted with the first if $GX^2 = \mathcal{O}(\delta^{-\frac{1}{2}})$ i.e. if $(a+2)c \leq \frac{1}{4}$, we have:

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right)$$

and for $X = \delta^{-c}$, with $0 \leq c \leq \frac{1}{8}$

$$\sum\nolimits_2 = \mathcal{O}\left\{\frac{1}{x^\theta} \sum_{\kappa\lambda\mu\nu} \left(\frac{\lambda}{\kappa} \log\frac{1}{\delta} + \frac{1}{\kappa\mu} \log^2\frac{1}{\delta}\right)\right\} = \mathcal{O}\left(\frac{X^4}{x^\theta} \log^2\frac{1}{\delta}\right)$$

Estimates for integrals over the Fourier formulae

Lemma 3

Under the assumption of the estimate of $\sum_1$ and $\sum_2$ from the previous slide, we can deduce that

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+H} F(u)\, du \right|^2 dt = \mathcal{O}\Big(\frac{h}{\delta^{\frac{1}{2}} \log X}\Big) \tag{5}$$

Estimates for integrals over the Fourier formulae

Lemma 3

Under the assumption of the estimate of $\sum_1$ and $\sum_2$ from the previous slide, we can deduce that

$$\int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)\, du \right|^2 dt = \mathcal{O}\Big(\frac{h}{\delta^{\frac{1}{2}} \log X} \Big) \tag{5}$$

Lemma 4

Similarly, one can show that:

$$\int_{-\infty}^{\infty} \left\{ \int_t^{t+H} |F(u)|\, du \right\}^2 dt = \mathcal{O}\Big(\frac{h^2 \log(1/\delta)}{\delta^{\frac{1}{2}} \log X} \Big) \tag{6}$$

Estimates for integrals over the Fourier formulae

Lemma 3

Under the assumption of the estimate of $\sum_1$ and $\sum_2$ from the previous slide, we can deduce that

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+H} F(u)\, du \right|^2 dt = \mathcal{O}\Big(\frac{h}{\delta^{\frac{1}{2}} \log X} \Big) \tag{5}$$

Lemma 4

Similarly, one can show that:

$$\int_{-\infty}^{\infty} \left\{ \int_{t}^{t+H} |F(u)|\, du \right\}^2 dt = \mathcal{O}\Big(\frac{h^2 \log(1/\delta)}{\delta^{\frac{1}{2}} \log X} \Big) \tag{6}$$

The proofs of these Lemmas use the property that $J(x, \theta)$ is uniformly convergent with respect to θ, and then taking special values of the x's and the θ's.

Theorem (Selberg 1942b)

$$N_0(T) \gg T \log T$$

Theorem (Selberg 1942b)

$$N_0(T) \gg T \log T$$

Proof of the Theorem (1/3)

Let E be the subset of $(0, T)$, where

$$\int_t^{t+h} |F(u)|\, du > \left| \int_t^{t+h} F(u)\, du \right|$$

For such values t, $F(u)$ must change sign in $(t, t+h)$, then so does $\Xi(t)$.

Theorem (Selberg 1942b)

$$N_0(T) \gg T \log T$$

Proof of the Theorem (1/3)

Let E be the subset of $(0, T)$, where

$$\int_t^{t+h} |F(u)|\, du > \left| \int_t^{t+h} F(u)\, du \right|$$

For such values t, $F(u)$ must change sign in $(t, t+h)$, then so does $\Xi(t)$.
Since the two sides of the following are equal expect in E:

$$\int_E dt \int_t^{t+h} |F(u)|\, du \geq \int_E \left\{ \int_t^{t+h} |F(u)|\, du - \left| \int_t^{t+h} F(u)\, du \right| \right\} dt \tag{7}$$

Theorem (Selberg 1942b)

$$N_0(T) \gg T \log T$$

Proof of the Theorem (1/3)

Let E be the subset of $(0, T)$, where

$$\int_t^{t+h} |F(u)|\, du > \left| \int_t^{t+h} F(u)\, du \right|$$

For such values t, $F(u)$ must change sign in $(t, t+h)$, then so does $\Xi(t)$. Since the two sides of the following are equal expect in E:

$$\begin{aligned} \int_E dt \int_t^{t+h} |F(u)|\, du &\geq \int_E \left\{ \int_t^{t+h} |F(u)|\, du - \left| \int_t^{t+h} F(u)\, du \right| \right\} dt \\ &= \int_0^T \left\{ \int_t^{t+h} |F(u)|\, du - \left| \int_t^{t+h} F(u)\, du \right| \right\} dt \\ &> AhT^{\frac{3}{4}} - \int_0^T \left| \int_t^{t+h} F(u)\, du \right| dt \end{aligned} \tag{7}$$

Proof of the Theorem (2/3)

By Lemma 4 with $\delta = 1/T$, we obtain an upper bound:

$$\int_E dt \int_t^{t+h} |F(u)|\, du \leq \left\{ \int_E dt \int_E \left(\int_t^{t+h} |F(u)|\, du \right)^2 dt \right\}^{\frac{1}{2}} \quad \mathrm{Cauchy-Schwarz}$$

(8)

Proof of the Theorem (2/3)

By Lemma 4 with $\delta = 1/T$, we obtain an upper bound:

$$
\begin{aligned}
\int_E dt \int_t^{t+h} |F(u)|\,du &\leq \left\{ \int_E dt \int_E \left(\int_t^{t+h} |F(u)|\,du \right)^2 dt \right\}^{\frac{1}{2}} \quad \text{Cauchy} - \text{Schwarz} \\
&\leq \left\{ m(E) \int_{-\infty}^{\infty} \left(\int_t^{t+h} |F(u)|\,du \right)^2 dt \right\}^{\frac{1}{2}} \\
&< A\{m(E)\}^{\frac{1}{2}} h T^{\frac{1}{4}} \left(\frac{\log T}{\log X} \right)^{\frac{1}{2}} \qquad (8)
\end{aligned}
$$

Proof of the Theorem (2/3)

By Lemma 4 with $\delta = 1/T$, we obtain an upper bound:

$$\begin{aligned}
\int_E dt \int_t^{t+h} |F(u)|\, du &\leq \left\{ \int_E dt \int_E \left(\int_t^{t+h} |F(u)|\, du \right)^2 dt \right\}^{\frac{1}{2}} \quad \text{Cauchy} - \text{Schwarz} \\
&\leq \left\{ m(E) \int_{-\infty}^{\infty} \left(\int_t^{t+h} |F(u)|\, du \right)^2 dt \right\}^{\frac{1}{2}} \\
&< A\{m(E)\}^{\frac{1}{2}} h T^{\frac{1}{4}} \left(\frac{\log T}{\log X} \right)^{\frac{1}{2}} \qquad (8)
\end{aligned}$$

By Lemma 3, we see that

$$\begin{aligned}
\int_0^T \left\{ \int_t^{t+h} |F(u)|\, du - \left| \int_t^{t+h} F(u)\, du \right| \right\} dt &\leq \left\{ \int_0^T dt \int_0^T \left| \int_t^{t+h} F(u)\, du \right|^2 dt \right\}^{\frac{1}{2}} \\
&< \frac{A h^{\frac{1}{2}} T^{\frac{3}{4}}}{\log^{\frac{1}{2}} X} \qquad (9)
\end{aligned}$$

Proof of the Theorem (3/3)

Therefore, by (7), (8), (9)

$$\{m(E)\}^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T}\right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

where A_1 and A_2 denote the particular constants which occur.

Proof of the Theorem (3/3)

Therefore, by (7), (8), (9)

$$\{m(E)\}^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T}\right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

where A_1 and A_2 denote the particular constants which occur.

Since $X = T^c$ and $h = (a \log X)^{-1}$:

$$\{m(E)\}^{\frac{1}{2}} > A_1^{\frac{1}{2}} T^{\frac{1}{2}} - A_2 (ac)^{\frac{1}{2}} T^{\frac{1}{2}}$$

Taking a small enough, it follows: $m(E) > A_3 T$.

Proof of the Theorem (3/3)

Therefore, by (7), (8), (9)

$$\{m(E)\}^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T}\right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

where A_1 and A_2 denote the particular constants which occur.

Since $X = T^c$ and $h = (a \log X)^{-1}$:

$$\{m(E)\}^{\frac{1}{2}} > A_1^{\frac{1}{2}} T^{\frac{1}{2}} - A_2 (ac)^{\frac{1}{2}} T^{\frac{1}{2}}$$

Taking a small enough, it follows: $m(E) > A_3 T$.

Hence, of the partitions $(0, h)$, $(h, 2h)$, ... contained in $(0, T)$, at least $[A_3 T/h]$ must contain points of E. If $(nh, (n+1)h)$ contains a point t of E, a zero of $\zeta(\frac{1}{2} + iu)$ must be in $(t, t+h)$, and so does $(nh, (n+2)h)$

Proof of the Theorem (3/3)

Therefore, by (7), (8), (9)

$$\{m(E)\}^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T}\right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

where A_1 and A_2 denote the particular constants which occur.

Since $X = T^c$ and $h = (a \log X)^{-1}$:

$$\{m(E)\}^{\frac{1}{2}} > A_1^{\frac{1}{2}} T^{\frac{1}{2}} - A_2 (ac)^{\frac{1}{2}} T^{\frac{1}{2}}$$

Taking a small enough, it follows: $m(E) > A_3 T$.

Hence, of the partitions $(0, h)$, $(h, 2h)$, ... contained in $(0, T)$, at least $[A_3 T/h]$ must contain points of E. If $(nh, (n+1)h)$ contains a point t of E, a zero of $\zeta(\frac{1}{2} + iu)$ must be in $(t, t+h)$, and so does $(nh, (n+2)h)$

Recall the fact that each zero might be counted twice this way, there must be at least $\frac{1}{2}[A_3 T/h] > AT \log T$ zeros in $(0, T)$ □