# A STRUCTURAL AND HOMOTOPICAL CLASSIFICATION OF $p$-ADIC RINGS BEYOND THE SCHOLZE FRAMEWORK

PU JUSTIN SCARFY YANG

ABSTRACT. We present a systematic and categorical classification of $p$-adic rings, including both the families studied in depth by Scholze and the categories of rings yet untouched by his foundational frameworks. These include non-Noetherian, non-commutative, non-complete, Banach, and model-theoretic $p$-adic rings. We propose extensions of prismatic and perfectoid frameworks and analyze potential homotopy-theoretic invariants associated with exotic $p$-adic algebraic structures.

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## 1. Overview of Scholze's Framework on $p$-adic Rings

The modern theory of $p$-adic geometry and cohomology has undergone profound transformations through the contributions of Peter Scholze. At the heart of his framework lie several interwoven classes of $p$-adic rings, each serving as a foundational base for geometric, cohomological, or homotopical constructions. This section provides a systematic overview of the types of $p$-adic rings explicitly or implicitly treated in Scholze's works.

### 1.1. Perfectoid Rings and Periodic Geometry.

**Definition 1.1** (Perfectoid Ring)**.** Let $R$ be a $p$-adically complete Tate ring of characteristic 0 with an open and bounded subring $R_0 \subset R$. Then $R$ is said to be a *perfectoid ring* if:

(1) the subring $R_0$ is $p$-adically complete and integrally closed in $R$;
(2) there exists a pseudo-uniformizer $\varpi \in R_0$ such that $\varpi^p \mid p$ in $R_0$;
(3) the Frobenius map $\phi : R_0/p \to R_0/p$ is surjective.

Perfectoid rings enable tilting methods, Fontaine's period rings, and prisms. Scholze's foundational paper *"Perfectoid Spaces"* introduces this class as the keystone for new developments in $p$-adic Hodge theory.

*Example* 1.2. The completed perfect closure of $\mathbb{Z}_p[[T]]$ in the $p$-adic topology:

$$R := \mathbb{Z}_p\widehat{[[T^{1/p^\infty}]]}$$

is a classical example of a perfectoid ring in characteristic 0.

### 1.2. $p$-adically Complete Rings and Prismatic Cohomology.

Scholze and Bhatt generalize the base rings of crystalline and de Rham cohomology through the prism construction.

**Definition 1.3** (Prism)**.** A *bounded prism* is a pair $(A, I)$ consisting of a $\delta$-ring $A$ and an ideal $I \subset A$ such that:

(1) $A$ is $p$-adically complete and $I$ is principal;
(2) $A/I$ has bounded $p^\infty$-torsion;
(3) $A$ is derived $(p, I)$-complete.

The prism $(A_{\text{inf}}, \ker\theta)$ associated to a perfectoid ring serves as the base for prismatic cohomology. These rings are typically non-Noetherian but highly structured.

### 1.3. Topological Cyclic Homology and Trace Methods.

In [2], Bhatt–Morrow–Scholze demonstrate that the $K$-theory of smooth $p$-adic formal schemes over $\mathbb{Z}_p$ can be computed via topological cyclic homology (TC), using base rings like:

$$R = \mathbb{Z}_p[[T_1, \ldots, T_d]]$$

which are $p$-adically complete and formally smooth.

**Theorem 1.4** (BMS Comparison Theorem)**.** Let $R$ be a $p$-adic formally smooth $\mathbb{Z}_p$-algebra. Then there exists a natural isomorphism:

$$\pi_*\mathrm{TC}^-(R; \mathbb{Z}_p) \cong \widehat{\mathrm{dR}}(R)$$

where the left-hand side is topological cyclic homology and the right-hand side is the $p$-adic completion of the de Rham complex.

1.4. **Limitations of the Scholze Framework.** Despite the immense generality of perfectoid and prismatic methods, they share several common assumptions:

- Base rings are $p$-adically complete and often torsion-free;
- The rings admit a well-behaved Frobenius or tilt;
- The underlying topology is derived from the $p$-adic valuation.

Thus, the framework leaves out a variety of exotic or logically constructed $p$-adic rings. In the following sections, we will systematize and explore these underdeveloped directions.

Outline of This Paper

- **Section 2**: Non-complete and non-Henselian $p$-adic rings
- **Section 3**: Mixed-characteristic Dedekind rings beyond $\mathbb{Z}_p$
- **Section 4**: Non-Noetherian and infinite Krull-dimension examples
- **Section 5**: Non-commutative $p$-adic Banach algebras
- **Section 6**: $p$-adic analytic and functional-analytic generalizations
- **Section 7**: Model-theoretic, ultraproduct, and logical constructions
- **Section 8**: Homotopical and derived extensions of these classes

## 2. Non-complete and Non-Henselian $p$-adic Rings

While Scholze's foundational work relies critically on $p$-adically complete rings, especially for prismatic and perfectoid structures, there exists a broad class of $p$-adic rings that are either not complete or not Henselian. These rings arise naturally in arithmetic geometry, algebraic number theory, and deformation theory but remain largely outside the current reach of trace and prismatic methods.

### 2.1. Definition and Examples.

**Definition 2.1.** Let $A$ be a commutative ring and $p$ a fixed prime. Then $A$ is said to be a *$p$-adic ring* if it admits a topology in which the ideal $(p)$ is topologically nilpotent. If $A = \varprojlim A/p^n A$, then $A$ is *$p$-adically complete.*

Rings which are $p$-adic but not complete may appear as:

- localizations of $\mathbb{Z}_p$-algebras at non-$p$-adic primes;
- polynomial or power series rings over $\mathbb{Z}_p$ without $p$-adic completion;
- subrings of $\mathbb{Q}_p$ such as $\mathbb{Z}[1/p]$ or truncated valuation rings.

*Example* 2.2. The ring $A = \mathbb{Z}_p[T]$ with the $(p)$-adic topology is not complete. Its completion is $\widehat{A} = \mathbb{Z}_p[[T]]$.

*Example* 2.3. Let $K/\mathbb{Q}_p$ be a finite extension. The ring $\mathcal{O}_K$ of integers in $K$ is Henselian but a subring like $\mathcal{O}_K[T]/(T^2 - p)$ is not, unless completed.

2.2. **Failure of Scholze's Techniques.** Scholze's prismatic and perfectoid frameworks fundamentally depend on the derived completeness of base rings. Many foundational constructions break down without it:

- The $\delta$-structure of prismatic cohomology is undefined without completeness;
- The tilting equivalence between characteristic 0 and $p$ fails in non-complete settings;
- Almost mathematics and Fontaine's period rings require derived $(p)$-complete bases.

Thus, alternative methods must be developed to handle these rings.

2.3. **Henselization vs Completion.** Given a local ring $A$ with maximal ideal $\mathfrak{m}$, one may distinguish between the *Henselization* $A^h$ and the *completion* $\widehat{A}$. The two constructions serve distinct purposes:

- $A^h$ retains the algebraic structure while enforcing Hensel's lemma;
- $\widehat{A}$ provides the topological structure essential for analytic methods;
- In general, $A^h \hookrightarrow \widehat{A}$ is faithfully flat but not an isomorphism.

*Example* 2.4. For $A = \mathbb{Z}_p[T]_{(p,T)}$, the Henselization $A^h$ is the smallest Henselian local ring containing $A$, while $\widehat{A} \cong \mathbb{Z}_p[[T]]$ is its $p$-adic completion.

2.4. **Potential Applications and Research Directions.** The study of non-complete and non-Henselian $p$-adic rings could open up new directions in:

- **Arithmetic jet spaces**: differential-geometric structures over $p$-adic jets often begin with non-complete power series rings;
- **Moduli stacks**: formal neighborhoods in algebraic stacks often correspond to non-complete $p$-adic base rings;
- **Deformation theory**: first-order deformations naturally live over Artinian non-complete $p$-adic rings;
- **Functorial prismatic extensions**: developing prismatic analogues for $p$-adically separated, non-complete rings.

**Conjecture 2.5.** There exists a generalized prismatic cohomology theory defined on the category of separated $p$-adic rings, functorial in henselization and satisfying syntomic descent after derived completion.

We view the extension of prismatic and derived methods to these non-complete and non-Henselian structures as an essential step in broadening the landscape of $p$-adic geometry.

## 3. Mixed-characteristic Dedekind Rings Beyond $\mathbb{Z}_p$

While $\mathbb{Z}_p$ and its perfectoid extensions form the canonical base for $p$-adic Hodge theory, many naturally occurring Dedekind domains of mixed characteristic lie beyond the scope of Scholze's primary framework. These include integer rings of number fields, ramified extensions, and generalized one-dimensional Noetherian domains, many of which do not admit direct prismatic or perfectoid interpretations.

### 3.1. **Definition and General Framework.**

**Definition 3.1.** A *Dedekind domain* is a Noetherian, integrally closed, one-dimensional commutative ring in which every nonzero prime ideal is maximal.

In mixed characteristic $(0, p)$, such a Dedekind domain $A$ often arises as the ring of integers $\mathcal{O}_K$ of a finite extension $K/\mathbb{Q}$, or as a localization thereof at a non-archimedean place above $p$.

*Example* 3.2. Let $K = \mathbb{Q}(\sqrt{p})$. Then $\mathcal{O}_K = \mathbb{Z}[\sqrt{p}]$ is a Dedekind domain of mixed characteristic with multiple primes above $p$.

*Example* 3.3. Let $L/\mathbb{Q}_p$ be a wildly ramified extension. Then $\mathcal{O}_L$ is a local Dedekind domain whose residue field has positive characteristic, but its ramification index may obstruct tilting.

3.2. **Obstructions to Perfectoid and Prismatic Techniques.** Scholze's framework relies on base rings admitting:

- $p$-adic completeness;
- $p$-torsion freeness;
- existence of Frobenius-lifts (e.g., $\delta$-ring structures);
- perfectoid uniformizers (in the perfectoid case).

For general Dedekind domains, these assumptions fail in several ways:

(1) $\mathcal{O}_K$ is not $p$-adically complete unless explicitly completed;
(2) $\mathcal{O}_K$ may contain $p$-torsion in its étale extensions or integral closures;
(3) Wild ramification introduces non-liftable Frobenius actions;

(4) There is no canonical perfectoid structure on $\mathcal{O}_K$ without deep modification.

### 3.3. Examples of Interest.

- **Ramified Rings**: For $K = \mathbb{Q}_p(\zeta_{p^n})$, the integer ring $\mathcal{O}_K$ has high ramification. It may admit period maps, but only after highly nontrivial descent.
- **Localizations of Number Rings**: The localized ring $\mathbb{Z}[1/N]_{(p)}$ appears frequently in modular and automorphic deformation theory, yet lacks prismatic lifting unless completed.
- **Towers of Dedekind Extensions**: The infinite direct limit of $\mathcal{O}_{K_n}$ in the cyclotomic tower lacks Noetherianity and has non-finite residue degree, yet is geometrically meaningful.

### 3.4. Open Directions and Future Research.

**Problem 3.4.** Develop a prismatic site or period sheaf theory over Dedekind domains of mixed characteristic that are not $p$-complete, potentially through relative or stacky prismatics.

**Problem 3.5.** Construct a version of syntomic or Nygaard filtration for $\mathcal{O}_K$ with wild ramification, possibly using logarithmic prismatic theory.

**Conjecture 3.6.** There exists a prismatic-style cohomology theory over arbitrary Dedekind domains in mixed characteristic $(0, p)$ that reduces to classical crystalline cohomology on their formally smooth fibers after henselization and completion.

*Remark* 3.7. This theory could build upon log-geometry, Breuil–Kisin modules, and the integral $p$-adic Hodge theory developed by Caruso, Liu, and Bhatt–Scholze, with generalizations to the stack of torsors over $\mathrm{Spec}\mathcal{O}_K$.

These extensions are necessary for bridging arithmetic geometry, Iwasawa theory, and the cohomology of Shimura varieties in their integral models.

## 4. Non-Noetherian and Infinite Krull-Dimension $p$-adic Rings

While many constructions in algebraic geometry and number theory traditionally rely on Noetherian hypotheses, a significant range of meaningful $p$-adic rings are non-Noetherian. These include infinite power series rings, derived inverse limits, and valuation rings of infinite Krull dimension. Scholze's work does involve some non-Noetherian rings (e.g., $A_{\text{inf}}$), but the general theory of such rings within the $p$-adic framework remains underdeveloped.

### 4.1. Non-Noetherianity in $p$-adic Geometry.

**Definition 4.1.** A ring $A$ is *non-Noetherian* if it does not satisfy the ascending chain condition on ideals. A ring is said to have *infinite Krull dimension* if there exists an infinite strictly increasing chain of prime ideals:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq \cdots$$

In the $p$-adic setting, non-Noetherian rings often arise via:

- Inverse limits of Noetherian rings (e.g., $\varprojlim_n \mathbb{Z}_p[T]/(T^{p^n})$);
- Infinite power series rings, such as $\mathbb{Z}_p[[T_1, T_2, \dots]]$;
- Perfectoid completions involving towers of ramified extensions;
- Rings of analytic functions over open $p$-adic disks with overconvergent radii.

*Example* 4.2. The perfectoid ring $R = \widehat{\mathbb{Z}_p[T^{1/p^\infty}]}$ is non-Noetherian, as it is not finitely generated over $\mathbb{Z}_p$.

*Example* 4.3. Let $A = \mathbb{Z}_p[[T_1, T_2, \dots]]$. Then $A$ is non-Noetherian and has infinite Krull dimension, as the chain of prime ideals $(T_1) \subset (T_1, T_2) \subset \cdots$ never stabilizes.

### 4.2. Limitations of Derived and Prismatic Techniques.

Scholze's approach to non-Noetherian rings has so far been selective:

- $A_{\text{inf}} = W(R^\flat)$, while non-Noetherian, retains strong completeness and perfectoid control;
- The prism $(A_{\text{inf}}, \ker\theta)$ is derived $(p, \ker\theta)$-complete;
- Infinite-dimensional base rings such as $\mathbb{Z}_p[[T_1, T_2, \dots]]$ generally fall outside current prismatic formalisms due to the lack of finite presentation.

Moreover, the cotangent complex and Nygaard filtrations can behave pathologically in infinite-dimensional settings, impeding the application of THH and TC.

### 4.3. Examples of Non-Noetherian $p$-adic Contexts.

- **Towers of Perfectoid Algebras**: For instance, $\varinjlim_n \mathbb{Z}_p[[T^{1/p^n}]]$ has no finite presentation and fails Noetherianity.
- **Overconvergent Function Rings**: Rings of analytic functions on annuli or open discs with radii in $(0, 1)$ often fall outside the realm of Noetherian rigid spaces.

- **Infinite Jet Spaces**: In arithmetic differential geometry, the ring of infinite $p$-adic jets may be modeled on non-Noetherian inverse limits.
- **Non-standard Ultraproducts**: Rings like $\prod_{n\in\mathbb{N}} \mathbb{Z}_p/\mathcal{U}$ (with $\mathcal{U}$ a non-principal ultrafilter) are typically non-Noetherian and can have uncountable Krull dimension.

4.4. **New Directions and Homotopical Challenges.**

**Problem 4.4.** Develop a prismatic site over non-Noetherian $p$-adic rings that supports derived completions and filtered Frobenius lifts.

**Problem 4.5.** Construct a variant of the Nygaard filtration adapted to infinite Krull dimension, possibly through pro-categories or spectral descent.

**Conjecture 4.6.** The homotopy fixed points of THH over non-Noetherian $p$-adic rings still detect integral structures in a filtered colimit sense, and TC admits a pro-descent interpretation.

*Remark* 4.7. This area may benefit from the theory of condensed mathematics, where completeness and presentation can be encoded in terms of morphism sheaves rather than finite generation. It also aligns with the use of ind-objects and $\infty$-categorical limits.

A deeper understanding of non-Noetherian $p$-adic rings is essential for modeling universal deformations, arithmetic stacks, and infinite-level Shimura varieties beyond finite-type structures.

## 5. Non-commutative $p$-adic Banach Algebras

Thus far, the theory of $p$-adic cohomology and arithmetic geometry has been overwhelmingly commutative. However, numerous natural and arithmetic contexts give rise to non-commutative $p$-adic rings, particularly in representation theory, $p$-adic quantum groups, and completed enveloping algebras. Scholze's theory has only minimally addressed these cases, leaving open a rich landscape for non-commutative $p$-adic geometry.

5.1. **Definitions and Foundational Examples.**

**Definition 5.1.** A *$p$-adic Banach algebra* is a topological ring $A$ over $\mathbb{Q}_p$ equipped with a non-archimedean norm $\|\cdot\|$ such that:

1. $A$ is complete with respect to the norm topology;
2. $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$;
3. There exists a submultiplicative open unit ball making $A$ a topological $\mathbb{Q}_p$-algebra.

If $A$ is not commutative, we call it a *non-commutative $p$-adic Banach algebra.*

*Example* 5.2. The Iwasawa algebra $\Lambda(G) := \mathbb{Z}_p[[G]]$ for a profinite group $G$ is a topological non-commutative ring. For example, $G = \mathbb{Z}_p$ yields $\Lambda(\mathbb{Z}_p) \cong \mathbb{Z}_p[[T]]$, which is commutative, but more general $G$ yield genuinely non-commutative algebras.

*Example* 5.3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}_p$. The $p$-adic completion of the universal enveloping algebra $U(\mathfrak{g})$ is a non-commutative $p$-adic Banach algebra, denoted $\widehat{U(\mathfrak{g})}_p$.

### 5.2. Limitations of Scholze's Framework.

In the perfectoid and prismatic literature, the foundational constructions assume commutativity at each stage:

- Prisms are built on commutative $\delta$-rings;
- Frobenius lifts do not extend canonically to non-commutative settings;
- Tilt and untilt operations rely on multiplicative identities.

Moreover, trace methods such as THH and TC are generally constructed using symmetric monoidal categories of commutative ring spectra. Extending these to non-commutative base rings requires different homotopical machinery, such as non-commutative motives or $E_n$-algebras.

### 5.3. Examples in Arithmetic and Representation Theory.

- **Iwasawa Algebras**: For a non-abelian $p$-adic Lie group $G$, the ring $\Lambda(G)$ is non-commutative and plays a central role in non-commutative Iwasawa theory.
- **Distribution Algebras**: The $\mathbb{Q}_p$-algebra of locally analytic distributions $\mathcal{D}(G)$ for $G$ a $p$-adic analytic group is non-commutative and important in $p$-adic representation theory.
- **Quantum Groups**: $p$-adic analogues of quantum enveloping algebras (e.g., $U_q(\mathfrak{g})$ for $q$ a $p$-adic unit) are naturally non-commutative and could support motivic deformation theory.
- **$p$-adic Operator Algebras**: Certain $p$-adic completions of operator algebras over $\mathbb{Q}_p$ arise in non-archimedean functional analysis, yet have no analogues in Scholze's commutative period theory.

### 5.4. Future Extensions and Non-commutative Period Theories.

**Problem 5.4.** Develop a homotopical framework for non-commutative prisms or "Frobenius-periodic" $E_1$-rings in mixed characteristic.

**Problem 5.5.** Construct a variant of topological cyclic homology $\mathrm{TC}^{\mathrm{nc}}$ compatible with non-commutative $p$-adic Banach algebras and distribution algebras.

**Conjecture 5.6.** There exists a prismatic trace theory for $p$-adic completed $E_1$-algebras, whose syntomic invariants classify extensions of $p$-adic representations in the derived non-commutative setting.

*Remark* 5.7. Techniques from non-commutative motives, cyclic homology, and condensed algebra could be brought together to define a "prismatic center" for a non-commutative $p$-adic algebra $A$, generalizing crystalline and syntomic centers in the commutative case.

The development of non-commutative $p$-adic period theories could connect derived arithmetic representation theory with quantum geometry, operator algebras, and $p$-adic analytic dualities.

## 6. $p$-adic Analytic and Functional-Analytic Generalizations

The realm of $p$-adic geometry extends far beyond algebraic or formal structures. A vast and powerful world of nonarchimedean analysis exists, developed originally by Tate, Amice, and others, and further advanced through rigid analytic geometry and Berkovich spaces. These structures involve $p$-adic Banach algebras and functional-analytic categories not fully subsumed by the prismatic or perfectoid formalisms. Scholze's framework intersects only partially with this analytic universe.

### 6.1. Foundations of $p$-adic Functional Analysis.

**Definition 6.1.** A *$p$-adic Banach space* is a $\mathbb{Q}_p$-vector space $V$ equipped with a norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying:

(1) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ (ultrametric inequality),
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{Q}_p$,
(3) $V$ is complete with respect to this norm.

Such spaces generalize the analytic behavior of $\mathbb{Q}_p$, allowing the development of series expansions, operator theory, and spectral methods in the nonarchimedean setting.

### 6.2. Analytic Structures Not Covered by Scholze's Theory.

Despite the powerful machinery of perfectoid spaces, many important analytic constructions are not encompassed by the prismatic or perfectoid framework:

- **Amice Spaces**: Spaces of $p$-adic continuous or locally analytic functions over $\mathbb{Z}_p$ are Fréchet spaces not arising as perfectoid or formal schemes.
- **Tate Algebras**: Algebras like $\mathbb{Q}_p\langle T_1, \ldots, T_n\rangle$—convergent power series on the closed $p$-adic disk—are Banach but not perfectoid unless deeply extended.
- **Nuclear Fréchet and LB-spaces**: These general topological vector spaces arise in duality theory and distribution analysis but are not currently accessible to prismatic methods.
- **Spaces of Overconvergent Modular Forms**: These require strict control over radius of convergence and interpolation, not naturally expressed in Scholze's geometry.

6.3. **Examples and Applications.**

*Example* 6.2. The Tate algebra $\mathbb{Q}_p\langle T\rangle$ consists of power series $\sum_{n=0}^{\infty} a_n T^n$ with $a_n \in \mathbb{Q}_p$ and $|a_n| \to 0$. It is a commutative $p$-adic Banach algebra, but not a perfectoid algebra.

*Example* 6.3. The space $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbb{Q}_p)$ of locally analytic functions on $\mathbb{Z}_p$ is a nuclear Fréchet space over $\mathbb{Q}_p$, relevant for $p$-adic representation theory and interpolation of $L$-functions.

*Example* 6.4. Let $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$ denote the distribution algebra of a $p$-adic analytic group $G$. This noncommutative Banach algebra governs the locally analytic representations of $G$ on nuclear $p$-adic spaces.

6.4. **Incompatibility with Frobenius and $\delta$-structures.**

- Frobenius lifts $\phi$ are generally not defined on analytic functions, since $\phi(f(x)) = f(x^p)$ may leave the domain of convergence;
- No canonical $\delta$-ring structure exists on Tate algebras unless they are extended or modified;
- Derived completions in this category often require nuclearity or Montel-type properties, which are absent in algebraic constructions.

6.5. **Directions for Generalized Period and Analytic Cohomology.**

**Problem 6.5.** Develop a nonarchimedean analytic version of prismatic cohomology, adapted to Banach or Fréchet base algebras, potentially using condensed or analytic motives.

**Problem 6.6.** Construct a $p$-adic Hodge filtration on spaces of overconvergent modular forms via homotopical methods, possibly generalizing the Nygaard filtration.

**Conjecture 6.7.** There exists a functorial extension of the Hodge–Tate and de Rham period sheaves to categories of Banach–analytic or nuclear Fréchet–analytic $\mathbb{Q}_p$-algebras, satisfying continuity under completed tensor products.

*Remark* 6.8. The theory of condensed sets, as developed by Clausen–Scholze, may offer the right formalism for constructing coherent period sheaves in the analytic $p$-adic category, integrating homological algebra with functional analysis.

A complete understanding of $p$-adic geometry must integrate these analytic generalizations, which support the representation theory of $p$-adic groups, $p$-adic modular forms, and noncommutative functional analysis in arithmetic contexts.

## 7. Model-theoretic, Ultraproduct, and Logical Constructions of $p$-adic Rings

Beyond traditional algebraic and analytic frameworks, modern logic and model theory offer alternative constructions of $p$-adic rings through ultraproducts, nonstandard models, and valuation-theoretic languages. These rings often defy Noetherianity, completeness, or finite type conditions, yet they provide profound insights into uniformity, definability, and arithmetic behaviors over $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Scholze's theory has not yet addressed these logical extensions.

### 7.1. Ultraproducts of $p$-adic Rings.

**Definition 7.1.** Let $(A_i)_{i\in I}$ be a family of rings, and let $\mathcal{U}$ be a non-principal ultrafilter on $I$. The *ultraproduct* $\prod_{\mathcal{U}} A_i$ is the quotient:

$$\left(\prod_{i\in I} A_i\right) \Big/ \mathfrak{U}$$

where $\mathfrak{U} := \{(a_i) \in \prod A_i \mid \{i \in I \mid a_i = 0\} \in \mathcal{U}\}$.

When each $A_i$ is a $p$-adic ring (e.g., $\mathbb{Z}_{p_i}$), the resulting ultraproduct is a large, typically non-Noetherian ring of infinite Krull dimension.

*Example* 7.2. Let $A_i = \mathbb{Z}_p$ for each $i \in \mathbb{N}$ and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then $A = \prod_{\mathcal{U}} \mathbb{Z}_p$ is a nonstandard $p$-adic ring with uncountable dimension and no natural Frobenius lift.

*Remark* 7.3. Such constructions are central to Hrushovski–Kazhdan motivic integration and asymptotic $p$-adic geometry.

7.2. **Model Theory of Valued Fields.** The theory of algebraically closed valued fields (ACVF) and $p$-adically closed fields (PCF) provides logical descriptions of $p$-adic structures.

- The language $\mathcal{L}_{\text{val}} = \{+, -, \cdot, 0, 1, \text{val}\}$ encodes valuation;
- Quantifier elimination and cell decomposition theorems enable tame behavior;
- Nonstandard $p$-adic fields and rings appear naturally as elementary extensions.

*Example* 7.4. The field $\mathbb{Q}_p^{\text{alg}}$ admits many nonstandard elementary extensions within the model-theoretic framework of valued fields. Their valuation rings yield exotic $p$-adic structures not arising from algebraic geometry.

7.3. **Definable Sets and Logical Cohomology.** Recent work in model-theoretic cohomology and motivic integration defines "definable cohomology theories" for sets and structures not expressible in usual schemes or formal spaces.

- Hrushovski–Kazhdan: motivic integration on definable sets in ACVF;
- Cluckers–Loeser: motivic functions on $p$-adic definable subanalytic sets;
- These techniques allow the study of $p$-adic zeta functions, moduli spaces, and cohomology in logical terms.

7.4. **Limitations of Current Cohomological Frameworks.** Existing cohomological tools like crystalline, prismatic, or de Rham theories assume algebraic or formal underpinnings. They generally fail to apply to:

- Nonstandard ultraproducts;
- Elementary extensions of $\mathbb{Q}_p$;
- Definable $p$-adic sets with no scheme-theoretic realization.

*Remark* 7.5. These limitations suggest a need for model-theoretic cohomological frameworks compatible with ultraproducts and definable valuation geometry.

7.5. **Open Problems and Future Directions.**

**Problem 7.6.** Construct a derived motivic cohomology theory for ultraproducts of $p$-adic rings that detects uniform asymptotic properties of $\mathbb{Z}_p$-algebras.

**Problem 7.7.** Formulate a prismatic-style cohomology on the category of definable $p$-adic sets in the language of valued fields.

**Conjecture 7.8.** There exists a motivic period sheaf functor $\mathscr{P}$ from the category of definable $p$-adic rings (in $\mathcal{L}_{\mathrm{val}}$) to derived condensed sheaves of $\mathbb{Z}_p$-modules, compatible with definable integration and asymptotic THH-like traces.

These logical constructions provide a foundation for bridging nonstandard arithmetic, model theory, and cohomological geometry. They represent a frontier in the generalization of $p$-adic geometry beyond traditional algebraic boundaries.

## 8. Homotopical and Derived Extensions of Exotic $p$-adic Rings

In recent years, homotopy-theoretic and derived-algebraic techniques have revolutionized algebraic geometry, particularly through the advent of derived stacks, spectral algebraic geometry, and condensed mathematics. These tools offer a unifying framework for integrating the exotic classes of $p$-adic rings previously described—especially those that defy classical algebraic geometry—into a coherent and cohomologically computable context. Scholze's work in condensed mathematics and the prismatic theory provides the scaffolding for these extensions, but many directions remain open.

### 8.1. **Spectral and Derived $p$-adic Rings.**

**Definition 8.1.** A *derived $p$-adic ring* is an object $A \in \mathrm{CAlg}(\mathcal{D}(\mathbb{Z}_p))$, the category of commutative algebra objects in the derived $\infty$-category of $\mathbb{Z}_p$-modules, equipped with a $p$-adic filtration:

$$A \simeq \varprojlim_n A/p^n A$$

as a derived inverse limit.

These objects generalize classical $p$-adic rings, incorporating higher homotopy and cohomological data.

*Example* 8.2. The derived de Rham complex $\widehat{L\Omega^\bullet_{R/\mathbb{Z}_p}}$ of a simplicial $\mathbb{Z}_p$-algebra $R$ is a derived $p$-adic ring.

8.2. **Condensed and Solid $p$-adic Rings.** Condensed mathematics provides a topological-enriched categorical framework for analyzing topological $\mathbb{Z}_p$-modules and algebras.

**Definition 8.3.** A *condensed $p$-adic ring* is a sheaf of commutative rings on the pro-étale site of $\mathrm{Comp}_{\mathrm{Haus}}$, the category of compact Hausdorff spaces, satisfying $p$-adic completeness in the condensed topology.

*Example* 8.4. The sheaf $\underline{\mathbb{Z}}_p$ assigning $\mathrm{Map}_{\mathrm{cont}}(S, \mathbb{Z}_p)$ to each compact Hausdorff $S$ is a condensed $\mathbb{Z}_p$-algebra.

Condensed rings naturally encode infinite sums, derived limits, and topological cohomology. They extend the usability of period sheaves to Banach, Fréchet, and model-theoretic settings.

8.3. **Extension to Exotic Classes.** Using the language of $\infty$-categories and spectral algebra, one may extend cohomological structures over the previously identified exotic $p$-adic rings:

- **Ultraproducts**: Define THH or TC on pro-categories or condensed limits of $\mathbb{Z}_p$-algebras using ultraproduct descent.
- **Non-Noetherian Rings**: Use ind-coherent and formal completions in derived categories to compute local cohomology and prismatic analogues.
- **Non-commutative $E_1$-rings**: Extend syntomic filtrations and Hochschild-type traces to derived non-commutative settings.
- **Analytic Banach Rings**: Use solid modules and derived condensed tensor products to reconstruct period sheaves over functional-analytic base rings.

8.4. **Derived Prismatic and Nygaard Formalisms.**

**Problem 8.5.** Construct a universal $\infty$-topos of prismatic sheaves over the derived site of $p$-adically complete commutative ring spectra.

**Conjecture 8.6.** There exists a derived prismatic cohomology functor:

$$\mathrm{Prism}_\infty(-) : \mathrm{CAlg}^{\mathrm{der}}_{\mathbb{Z}_p} \to \mathcal{D}(\mathbb{Z}_p)$$

extending the Scholze–Bhatt construction to all derived $p$-adic rings, including those which are non-Noetherian, non-Henselian, or non-discrete.

*Remark* 8.7. This conjecture is inspired by the ideas of spectral algebraic geometry (Lurie), condensed rings (Clausen–Scholze), and derived prismatic crystals (Bhatt–Lurie).

8.5. **Future Outlook.** Derived and homotopical methods promise to unify all categories of $p$-adic rings—including those not directly accessible through classical means—under a common cohomological umbrella. This allows for:

- Fully functorial trace theories ($K$, THH, TC, TP) on generalized spectra;
- Period sheaves over infinite-dimensional or logical $p$-adic sites;
- Prismatic deformation theories over noncommutative, analytic, and ultraproduct bases;

- Higher-categorical generalizations of $p$-adic Hodge and syntomic filtrations.

The future of $p$-adic geometry thus lies not only in deeper refinement of classical structures, but in a wholesale derived expansion into categories long regarded as inaccessible.

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