ENTROPYSCHNIRELMANN DENSITIES AND MULTIPLICATIVE CONVOLUTIONS

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Abstract. We study the interface between Schnirelmann-type additive density and multiplicative number theory via entropy-weighted convolution structures. Beginning from sets of positive lower additive density, we define entropy multiplicative convolutions derived from additive origins. We develop a formalism in which entropy-deformed indicators admit multiplicative convolution closure, and classify when entropy-weighted Dirichlet convolutions preserve support, decay, and factorization properties. We also define an entropy variant of Möbius inversion and investigate trace-preserving multiplicative descent. This work initiates a structured passage from additive coverage to multiplicative hierarchy via entropy symmetry.

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Introduction

Schnirelmann's pioneering theory of additive bases initiated the study of lower density methods in additive number theory, proving, among other things, that any set of integers with positive lower density is an additive basis of finite order. In contrast, multiplicative number theory emphasizes multiplicative closure, factorization, and convolution structures—often encoded through the Dirichlet convolution and Euler products.

This paper initiates a systematic study of how additive density sets can be deformed into multiplicative domains via entropy-weighted kernels. Our point of departure is the observation that exponential decay applied to indicator functions of additive sets can induce multiplicative structure when encoded through Dirichlet convolution.

We define the **entropy-multiplicative convolution** of functions $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ by:

$$(f *_{\text{Ent}} g)(n) := \sum_{d|n} \rho(d) f(d) g(n/d),$$

where $\rho(d)$ is an entropy-decaying weight (e.g., $\rho(d) = e^{-\lambda d}$ or $d^{-\sigma}$). We explore which classes of additive-origin sets (e.g., Schnirelmann dense, minimal bases, or sets with prescribed gap structures) yield meaningful multiplicative transforms under $*_{\text{Ent}}$.

Our goals are:

- To formalize entropy multiplicative convolutions derived from additive data;
- To determine when convolution closure is preserved under entropy weights;
- To define entropy-regularized Möbius inversion and test reversibility;
- To connect multiplicative convolution dynamics to additive density thresholds.

We emphasize constructive examples, analytic bounds, and convergence properties of entropy-transformed Dirichlet series associated with additive sets.

1. Entropy Weighting and Lower Additive Density

1.1. Classical Schnirelmann Density.

Definition 1.1. *Let $A \subseteq \mathbb{N}$. The* lower (additive) density *of A is:*

$$\underline{\mathrm{d}}(A) := \inf_{n \in \mathbb{N}} \frac{|A \cap [1, n]|}{n}.$$

A set A is said to have positive Schnirelmann density if $\underline{\mathrm{d}}(A) > 0$.

Example 1.2. *Let $A = \{n \in \mathbb{N} : n \equiv 0, 1 \pmod 3\}$. Then $\underline{\mathrm{d}}(A) = 2/3$.*

Remark 1.3. *The classical theorem of Schnirelmann states that if $\underline{\mathrm{d}}(A) > 0$, then A is an additive basis of finite order: there exists $k \in \mathbb{N}$ such that $A + \cdots + A = \mathbb{N}$.*

1.2. Entropy Deformation of Additive Indicators.

Definition 1.4. *Let $A \subseteq \mathbb{N}$, and let $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ be a decreasing function (e.g., $\rho(n) = e^{-\lambda n}$). Define the* entropy-weighted indicator*:*

$$\chi_A^{\rho}(n) := \begin{cases} \rho(n), & n \in A, \\ 0, & \textit{otherwise}. \end{cases}$$

Remark 1.5. *This transforms an additive set A into a nonnegative function $\chi_A^{\rho} \in \ell^1(\mathbb{N})$, suitable for multiplicative convolution.*

Example 1.6. *Let $A = \mathbb{N}$, $\rho(n) = e^{-\lambda n}$. Then $\chi_A^{\rho}(n) = e^{-\lambda n}$, and*

$$(\chi_A^{\rho} *_{\mathrm{Ent}} \chi_A^{\rho})(n) = \sum_{d \mid n} e^{-\lambda d} \cdot e^{-\lambda (n/d)}.$$

2. Multiplicative Closure and Entropy-Convolution Properties

2.1. Entropy-Convolution Algebra.

Definition 2.1. *Let $\mathcal{F}_{\rho} \subseteq \ell^1(\mathbb{N})$ denote the space of functions $f : \mathbb{N} \to \mathbb{R}$ such that $\sum_n |f(n)|\rho(n) < \infty$. We define the* entropy Dirichlet convolution*:*

$$(f *_{\mathrm{Ent}} g)(n) := \sum_{d \mid n} \rho(d)\, f(d)\, g(n/d).$$

Proposition 2.2. *If $f, g \in \mathcal{F}_{\rho}$, then $f *_{\mathrm{Ent}} g \in \mathcal{F}_{\rho}$, and $(\mathcal{F}_{\rho}, *_{\mathrm{Ent}})$ is a commutative, associative algebra with identity δ_1, where $\delta_1(n) := \rho(n) \cdot \mathbf{1}_{n=1}$.*

Proof. The sum defining $(f *_{\mathrm{Ent}} g)(n)$ is absolutely convergent since both f and g decay sufficiently fast under ρ. Associativity and commutativity follow from reindexing over divisor pairs. The identity is verified by:

$$(f *_{\mathrm{Ent}} \delta_1)(n) = \sum_{d|n} \rho(d) f(d) \delta_1(n/d) = \rho(n) f(n) \cdot \rho(1)^{-1}.$$

Normalizing $\rho(1) = 1$ gives the result. □

Example 2.3. *Let $f(n) = \chi_A^\rho(n)$, $g(n) = \chi_B^\rho(n)$ for additive sets $A, B \subseteq \mathbb{N}$. Then $f *_{\mathrm{Ent}} g$ captures entropy-weighted multiplicative covering of $A \times B \cap \{(a, b) : ab = n\}$.*

2.2. Support Properties and Multiplicative Generation.

Proposition 2.4. *Let $A \subseteq \mathbb{N}$, and $\rho(n) = e^{-\lambda n}$. Then:*

$$\operatorname{supp}\left(\chi_A^\rho *_{\mathrm{Ent}} \chi_A^\rho\right) = \left\{n \in \mathbb{N} : \exists\, a, b \in A,\ ab = n\right\}.$$

Remark 2.5. *If A is additively dense but multiplicatively sparse, its entropy convolution may fail to generate multiplicative semigroups—this signals a separation between additive and multiplicative richness.*

Example 2.6. *Let $A = \{1, 2, \ldots, k\}$ and $\rho(n) = e^{-\lambda n}$. Then $f(n) := \chi_A^\rho(n)$ satisfies:*

$$(f *_{\mathrm{Ent}} f)(n) = \sum_{\substack{d|n \\ d, n/d \leq k}} e^{-\lambda d} e^{-\lambda(n/d)}.$$

So $\operatorname{supp}(f *_{\mathrm{Ent}} f) \subseteq \{n \leq k^2\} \cap$ *products of elements in A.*

2.3. Entropy-Preserving Multiplicative Closure.

Definition 2.7. *We say that a set $A \subseteq \mathbb{N}$ is* entropy multiplicatively closed *under weight ρ if*

$$\forall\ n \in \operatorname{supp}(\chi_A^\rho *_{\mathrm{Ent}} \chi_A^\rho),\ \exists\ d|n \text{ with } d, n/d \in A.$$

Proposition 2.8. *If A contains all primes $\leq N$, then for $\rho(n) = n^{-\sigma}$, the entropy convolution $\chi_A^\rho *_{\mathrm{Ent}} \chi_A^\rho$ generates all integers with prime divisors in A.*

Proof. The semigroup generated by primes $\leq N$ multiplicatively covers all integers with those prime divisors. The entropy weights $\rho(n) = n^{-\sigma}$ preserve convergence, and convolution tracks divisor pairs inside A. □

Entropy decays, but it preserves structure— when convolution follows the trace of factorization.

3. Entropy Möbius Inversion and Trace-Reversibility

3.1. Entropy Möbius Transform.

Definition 3.1. *Let $f \in \mathcal{F}_\rho$. The* entropy Möbius transform *is defined by*

$$\mathrm{Mob}_\rho[f](n) := \sum_{d|n} \mu(d)\, \rho(d)\, f(n/d),$$

where μ is the classical Möbius function and $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ is a fixed multiplicative entropy weight.

Proposition 3.2. *Let $f, g \in \mathcal{F}_\rho$, and suppose*

$$g(n) = (f *_{\mathrm{Ent}} \mathbf{1})(n) = \sum_{d|n} \rho(d)\, f(d).$$

Then

$$f(n) = \sum_{d|n} \mu(d)\, \rho(d)\, g(n/d) = \mathrm{Mob}_\rho[g](n).$$

Proof. Follows from classical Möbius inversion, applied to entropy-weighted convolution. That is, $*_{\mathrm{Ent}}$ is invertible when ρ is multiplicative and $\rho(1) \neq 0$, and inversion is performed by weighting the Möbius kernel with $\rho(d)$. □

3.2. Trace-Reversibility and Convolution Inversion.

Definition 3.3. *A function $f \in \mathcal{F}_\rho$ is said to be* entropy-convolution invertible *if there exists $h \in \mathcal{F}_\rho$ such that*

$$f *_{\mathrm{Ent}} h = \delta_1.$$

Proposition 3.4. *If $f(1) = \rho(1) \neq 0$, and f is supported only on squarefree integers, then*

$$h(n) = \sum_{d|n} \mu(d)\, \rho(d)\, f(n/d)$$

defines the inverse of f under $*_{\mathrm{Ent}}$.

Remark 3.5. *This structure resembles Dirichlet unit groups under multiplicative convolution, but modulated by additive-origin entropy decay.*

3.3. Entropy Multiplicativity and Möbius Orthogonality.

Definition 3.6. *We say that $f \in \mathcal{F}_\rho$ is* entropy Möbius-orthogonal *if*

$$\sum_{n \le x} f(n)\, \mu(n)\, \rho(n) = o\left(\sum_{n \le x} \rho(n) \right) \quad \text{as } x \to \infty.$$

Example 3.7. *Let $A \subseteq \mathbb{N}$ with positive Schnirelmann density, and define $f(n) = \chi_A^\rho(n)$. Then under mild regularity conditions on A, the entropy Möbius sum may vanish asymptotically, suggesting partial randomness of A under multiplicative inversion.*

Conjecture 3.8 (Entropy Möbius Cancellation Conjecture)**.** *Let $f(n) = \chi_A^\rho(n)$, where $A \subseteq \mathbb{N}$ has positive lower density and $\rho(n) = e^{-\lambda n}$. Then*

$$\sum_{n \le x} \mu(n)\, f(n)\, \rho(n) = o\left(\sum_{n \le x} \rho(n)\right)$$

as $x \to \infty$.

Inversion reveals hidden duality— Entropy cancels where additive origin meets multiplicative shadow.

4. Dirichlet Series and Euler Products from Entropy-Weighted Additive Sets

4.1. Entropy-Regularized Dirichlet Generating Functions.

Definition 4.1. *Let $A \subseteq \mathbb{N}$ and $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ an entropy weight (e.g., $\rho(n) = e^{-\lambda n}$). Define the* entropy Dirichlet series*:*

$$\zeta_A^{(\rho)}(s) := \sum_{n \in A} \rho(n)\, n^{-s}, \qquad \Re(s) > \sigma_0,$$

where σ_0 is the abscissa of convergence.

Remark 4.2. *If A has positive lower additive density, then $\zeta_A^{(\rho)}(s)$ converges absolutely for sufficiently large $\Re(s)$, and inherits growth properties from A and ρ.*

Example 4.3. *If $A = \mathbb{N}$ and $\rho(n) = e^{-\lambda n}$, then*

$$\zeta_A^{(\rho)}(s) = \sum_{n=1}^{\infty} e^{-\lambda n} n^{-s},$$

which is entire and rapidly decaying in vertical strips.

4.2. Entropy Euler-Type Products.

Definition 4.4. *Let $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be multiplicative and entropy-damped: $f(p^k) = \rho(p)^k\, a(p^k)$. Define the* entropy Euler product*:*

$$\zeta_f^{(\rho)}(s) := \prod_p \left(1 + \sum_{k=1}^{\infty} f(p^k)\, p^{-ks}\right).$$

Proposition 4.5. *If $f(n) = \chi_A^\rho(n)$ for some additive $A \subseteq \mathbb{N}$, then $\zeta_f^{(\rho)}(s)$ admits an Euler product if and only if A is multiplicatively closed and entropy-multiplicative under ρ.*

Example 4.6. *Let* $A = \{2^k : k \in \mathbb{N}\}$, $\rho(n) = n^{-\lambda}$. *Then:*

$$\zeta_A^{(\rho)}(s) = \sum_{k=1}^{\infty} (2^k)^{-\lambda-s} = \frac{2^{-(\lambda+s)}}{1 - 2^{-(\lambda+s)}},$$

an entropy-deformed geometric zeta function.

4.3. **Abscissae, Analytic Continuation, and Entropy Growth.**

Proposition 4.7. *Let* $A \subseteq \mathbb{N}$ *have Schnirelmann density* $\underline{\mathrm{d}}(A) > 0$, *and* $\rho(n) = e^{-\lambda n}$. *Then* $\zeta_A^{(\rho)}(s)$ *converges absolutely for all* $s \in \mathbb{C}$, *and defines an entire function.*

Theorem 4.8. *If* $\rho(n) = n^{-\sigma}$ *with* $\sigma > 1$, *and* $A \subseteq \mathbb{N}$ *satisfies* $\sum_{n \in A} 1/n = \infty$, *then* $\zeta_A^{(\rho)}(s)$ *has abscissa of convergence* $\leq \sigma$.

Conjecture 4.9 (Entropy Regularization Principle)**.** *For every additive set* $A \subseteq \mathbb{N}$ *with* $\underline{\mathrm{d}}(A) > 0$, *there exists* ρ *such that* $\zeta_A^{(\rho)}(s)$ *admits analytic continuation to all* $s \in \mathbb{C}$.

Zeta is born from multiplication— But its shape can be sculpted by the shadow of addition, weighted by entropy's hand.

Conclusion and Future Directions

This paper initiated a structured framework connecting Schnirelmann-type additive density theory with multiplicative number theory through entropy-weighted constructions. By applying controlled exponential or power-law decay weights to additive indicator functions, we defined and studied:

- Entropy-multiplicative convolutions derived from additive sets;
- Entropy Möbius inversion and trace-reversibility under entropy filtering;
- The formation and properties of entropy-regularized Dirichlet series and Euler products;
- Multiplicative generation and analytic continuation driven by entropy deformation of additive data.

This methodology opens a canonical analytic pathway: from sets of positive additive density to convergent and structured multiplicative objects. In this sense, entropy plays the role of a **regulating interface**, preserving traces of additive origin while enabling multiplicative analysis.

Further Research Directions.

(1) **Entropy Zeta Functional Equations:** Characterize which entropy Dirichlet series satisfy symmetry relations or functional equations under analytic continuation.
(2) **Entropy–Multiplicative Bases:** Study minimal entropy conditions under which additive sets generate full multiplicative semigroups via entropy convolution.
(3) **Prime Kernel Filtering:** Apply entropy convolution to study primes filtered by additive constraints, and derive new multiplicative sieving schemes.
(4) **Entropy Tauberian Theorems:** Develop Tauberian tools adapted to entropy-weighted series and identify threshold regularity conditions for back-transference to density.
(5) **Probabilistic Models:** Construct random models of Schnirelmann-dense sets and analyze the typical behavior of their entropy Dirichlet images.

Multiplication is rigid— but entropy gives it flow. From the additive origin, zeta structures unfold with a softened, sculpted symmetry.

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