SPECTRAL MOTIVES XXI: NONABELIAN CONDENSATION AND MOTIVIC ENTROPIC STACKS

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Abstract. We introduce a theory of nonabelian motivic condensation as a mechanism for entropy minimization across derived arithmetic stacks. By deforming derived categories through entropic flows, we construct entropic stacks and study their condensation transitions, trace localization, and cohomological collapse. These structures lead to new invariants for moduli of sheaves, spectral data on higher stacks, and symmetry-breaking phenomena in derived Langlands systems.

Contents

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1. Introduction

Motivic structures over arithmetic stacks exhibit intricate trace dynamics, particularly under spectral entropy and derived quantum deformation. In this paper, we study how derived sheaves condense into entropic attractors, giving rise to *nonabelian motivic condensation*—a phenomenon where entropy-minimizing flows collapse moduli categories into localized, symmetry-reduced entropic stacks.

This framework generalizes classical Higgsing, symmetry breaking, and renormalization flows to a categorical motivic setting, governed by:

- Entropy gradient descent on derived trace functionals;
- Symmetry-breaking bifurcations in ∞-categorical sheaves;
- Collapse of higher Ext structures into condensed strata;
- Localization of motivic Laplacians around entropy minima.

Main Contributions:

- Defined motivic entropic stacks as attractors of trace curvature flows;
- Constructed condensation diagrams for nonabelian symmetry quotients;
- Proved entropy collapsing theorems and trace spectrum reconfiguration;
- Introduced cohomological invariants classifying condensed strata.

Structure of the Paper:

- Section 2 formalizes entropic stacks and trace curvature descent;
- Section 3 introduces the nonabelian condensation functor and stack stratification;
- Section 4 describes symmetry reduction and entropy bifurcations;
- Section 5 defines condensation invariants and motivic attractor structures.

These tools lay the foundation for entropy-theoretic moduli theory, offering a new lens on degeneracy phenomena in motivic and Langlands-theoretic settings.

2. Entropic Stacks and Trace Curvature Flows

2.1. **Definition of entropic stacks.** Let $\mathscr{X}$ be a derived moduli stack with a Laplacian operator $\widehat{\Delta}_{\mathrm{Tr}}$ acting on a sheaf $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{dg}}(\mathscr{X})$. The trace entropy functional is defined by:

$$\mathcal{S}_{\mathscr{X}}(\mathscr{F}) := -\sum_i p_i \log p_i, \quad p_i = \frac{e^{-\beta\lambda_i}}{Z(\beta)}.$$

We call $\mathscr{X}_{\min} \subset \mathscr{X}$ an *entropic stack* if $\mathscr{F}_{\min}$ minimizes $\mathcal{S}_{\mathscr{X}}(\mathscr{F})$ in its deformation class.

2.2. **Trace curvature gradient flow.** We define a gradient flow on the derived stack:

$$\frac{d\mathscr{F}_t}{dt} = -\nabla_{\mathscr{F}} \mathcal{S}_{\mathscr{X}}(\mathscr{F}_t),$$

which evolves the sheaf $\mathscr{F}_t$ toward lower entropy.

Fixed points of this flow correspond to entropic attractors $\mathscr{F}_\infty$, and the associated substacks $\mathscr{X}_\infty$ become canonical condensed strata.

2.3. **Examples: Rank stratification and degeneration loci.** Let $\mathscr{M}_n$ be the moduli stack of rank n vector bundles over a curve. The entropy functional $\mathcal{S}_n$ induces:

$$\mathscr{M}_n \rightsquigarrow \mathscr{M}_r \quad \text{for } r < n,$$

as entropy-minimizing degenerations via subbundle condensation.

Similarly, in derived categories of shtukas, trace localization along Frobenius-fixed strata yields:

$$\mathrm{Sht}_{G,n} \to \mathrm{Sht}_{G,r}^{\mathrm{ent}},$$

where entropy localizes in non-reduced fixed-point loci of Langlands trace flows.

2.4. **Cohomological collapse and Ext vanishing.** Along condensation flows, higher Ext-structures may vanish:

$$\mathrm{Ext}^k(\mathscr{F}_t, \mathscr{F}_t) \to 0, \quad \text{as } t \to \infty, \text{ for } k > 0,$$

indicating cohomological flattening. The limiting condensed sheaf $\mathscr{F}_\infty$ is then a perfect complex supported on a minimal entropic component.

This collapse signals symmetry breaking and trace factorization across the stack.

3. Nonabelian Condensation Functors and Symmetry Stratification

3.1. **Condensation functors on derived categories.** Let $\mathscr{C} := \mathrm{Perf}(\mathscr{X})$ be a stable ∞-category of perfect complexes on a derived stack $\mathscr{X}$, equipped with a trace Laplacian $\widehat{\Delta}_{\mathrm{Tr}}$. We define the *condensation functor*:

$$\mathsf{Cond} : \mathscr{C} \to \mathscr{C}_{\min},$$

as the asymptotic flow functor induced by the entropy gradient:

$$\mathsf{Cond}(\mathscr{F}) := \lim_{t\to\infty} \mathscr{F}_t, \quad \text{with } \frac{d\mathscr{F}_t}{dt} = -\nabla_{\mathscr{F}} \mathcal{S}_{\mathscr{X}}.$$

3.2. **Symmetry-breaking stratification of stacks.** Let a derived stack $\mathscr{X}$ admit an action by a nonabelian group G. The entropy-minimizing loci may reduce symmetry:

$$\mathscr{X} \to \mathscr{X}^{G'} \subset \mathscr{X}, \quad G' \subsetneq G,$$

yielding strata:

$$\mathscr{X} = \bigsqcup_i \mathscr{X}_{(G_i)},$$

where each component represents a condensation class with stabilizer G_i.

These strata form the condensation pattern, analogous to Higgs vacua in gauge theory.

3.3. **Diagram of nonabelian condensation.** We summarize the process as:

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\mathsf{Cond}"] \arrow[dr, "\text{trace flow}"'] & & \mathscr{X}_{\min} \\
& \mathscr{X}_t \arrow[ur, "t\to\infty"'] &
\end{tikzcd}$$

This diagram captures entropy collapse and flow to motivic attractors.

3.4. **Functorial compatibility and motivic stability.** The condensation functor is compatible with base change and proper pushforward:

$$f^* \circ \mathsf{Cond}_{\mathscr{Y}} = \mathsf{Cond}_{\mathscr{X}} \circ f^*, \quad \text{for } f : \mathscr{X} \to \mathscr{Y}.$$

Moreover, condensed sheaves $\mathscr{F}_\infty$ are canonically stabilized under motivic trace evolution:

$$\widehat{\Delta}_{\mathrm{Tr}} \mathscr{F}_\infty = \lambda_{\min} \cdot \mathscr{F}_\infty,$$

making them natural eigenmotives for entropy-degenerate stacks.

4. Entropy Bifurcations and Motivic Phase Transitions

4.1. **Entropy potential and criticality.** Let the entropy functional $\mathcal{S}_{\mathscr{X}}(\mathscr{F})$ be treated as a potential function on a derived moduli space. A point $\mathscr{F}_c$ is called a critical point if:

$$\nabla_{\mathscr{F}} \mathcal{S}_{\mathscr{X}}(\mathscr{F}_c) = 0.$$

We define a *bifurcation point* if the Hessian of $\mathcal{S}$ has a kernel, i.e.,

$$\mathrm{Hess}_{\mathscr{F}_c}(\mathcal{S}) \text{ is degenerate.}$$

At these points, the moduli space locally splits into entropy valleys, corresponding to distinct motivic phases.

4.2. **Phase diagrams of motivic stacks.** Consider a family of stacks $\{\mathscr{X}_\alpha\}$ parameterized by a spectral temperature β^{-1}. The motivic phase diagram is the stratification:

$$\mathrm{Phase}(\mathscr{X}) = \bigsqcup_i \mathscr{X}_{\beta_i},$$

where β_i are critical values separating stable condensed regions.

4.3. **Langlands bifurcation and automorphic splitting.** Let Bun_G be the moduli of G-bundles and let $\mathcal{A}ut_\pi$ be the automorphic sheaf attached to a representation π. Entropic deformation induces:

$$\mathcal{A}ut_\pi \rightsquigarrow \bigoplus_j \mathcal{A}ut_{\pi_j},$$

interpreted as a motivic phase transition in the automorphic category, preserving trace under:

$$\sum_j \mathrm{Tr}_{\pi_j} = \mathrm{Tr}_\pi .$$

4.4. **Condensation transitions and motivic thermodynamics.** As entropy evolves, derived categories undergo phase shifts:

$$\mathrm{Perf}(\mathscr{X}_{\mathrm{hot}}) \to \mathrm{Perf}(\mathscr{X}_{\mathrm{cold}}),$$

analogous to symmetry-breaking in thermal systems. Each phase has distinct cohomological behavior and trace eigenbases.

We interpret motivic condensation as a *thermodynamic renormalization group flow*, with condensation criticality marking transition thresholds in derived moduli spaces.

5. Condensation Invariants and Motivic Attractors

5.1. **Condensation index and entropic dimension.** Let $\mathscr{F} \in \mathrm{Perf}(\mathscr{X})$, and let $\mathscr{F}_\infty := \mathsf{Cond}(\mathscr{F})$ be its condensed limit. We define the *condensation index*:

$$\mathrm{CI}(\mathscr{F}) := \dim \mathrm{Ext}^*(\mathscr{F}, \mathscr{F}) - \dim \mathrm{Ext}^*(\mathscr{F}_\infty, \mathscr{F}_\infty).$$

This measures cohomological collapse during the condensation process.

The *entropic dimension* of a moduli stack $\mathscr{X}$ is then:

$$\dim_{\mathrm{ent}}(\mathscr{X}) := \inf_{\mathscr{F}} \mathrm{CI}(\mathscr{F}).$$

5.2. **Trace attractors and entropy saturation.** We define a *trace attractor* to be a sheaf $\mathscr{F}_\infty$ such that:

$$\widehat{\Delta}_{\mathrm{Tr}} \mathscr{F}_\infty = \lambda_{\min} \cdot \mathscr{F}_\infty,$$

and $\mathcal{S}_{\mathscr{X}}(\mathscr{F}_\infty)$ is minimal in its deformation orbit.

Trace attractors are entropy-saturating objects, acting as fixed points of entropy gradient descent. The collection of all such attractors forms the *motivic entropy skeleton* of $\mathscr{X}$.

5.3. **Condensation classes and equivalence relations.** Define an equivalence relation:

$$\mathscr{F} \sim \mathscr{F}' \iff \mathsf{Cond}(\mathscr{F}) \simeq \mathsf{Cond}(\mathscr{F}'),$$

partitioning the category $\mathrm{Perf}(\mathscr{X})$ into condensation classes.

These equivalence classes organize the motivic landscape by shared asymptotic entropy and symmetry-breaking limits.

5.4. **Entropy cohomology and attractor stratification.** We define *entropy cohomology*:

$$H^\bullet_{\mathrm{ent}}(\mathscr{X}) := \bigoplus_i \mathrm{Ext}^i(\mathscr{F}_\infty, \mathscr{F}_\infty),$$

where $\mathscr{F}_\infty$ ranges over all condensation attractors. This cohomology detects the residual structure preserved under maximal entropy flow.

The decomposition

$$\mathscr{X} = \bigsqcup_{\mathscr{F}_\infty} \mathscr{X}_{[\mathscr{F}_\infty]},$$

stratifies $\mathscr{X}$ into entropic phases, each governed by a distinguished trace eigenobject.

6. Conclusion

We introduced the concept of nonabelian motivic condensation as a universal mechanism driving entropy reduction, trace spectrum localization, and cohomological collapse across derived moduli stacks. This framework generalizes symmetry breaking, Higgs condensation, and quantum attractor dynamics into the motivic and arithmetic setting, leveraging the geometry of ∞-categories and derived sheaves.

Summary of Contributions:

- Formulated entropy gradient flows and constructed entropic stacks;
- Defined condensation functors compatible with derived morphisms;
- Analyzed phase transitions via motivic entropy bifurcations;
- Introduced condensation invariants and entropy skeletons;
- Stratified moduli stacks into trace-attractor classes.

Future directions include exploring motivic condensation in:

- Langlands dual moduli under trace deformation;
- p-adic and condensed arithmetic geometries;
- spectral-to-motivic stack correspondences;
- derived thermodynamic categories and categorical black holes.

This opens a path toward entropy-driven cohomology theory, motivic statistical geometry, and stack-theoretic renormalization flows in arithmetic quantum field theory.

References

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