Advanced Studies in Fractional Dimensions and Their Applications

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Contents

1 Fractional Calculus

1.1 Fractional Differential Equations

$$\mathcal{D}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x(\tau)\, d\tau \tag{1}$$

References:

- Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers.
- Podlubny, I. (1999). *Fractional Differential Equations.* Academic Press.

1.2 Fractional Fourier Transforms

$$\mathcal{F}_\alpha[x(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\pi\alpha\left(\frac{t^2}{2}-\frac{\xi^2}{2}\right)}\, dt \tag{2}$$

References:

- Ozaktas, H.M., Zibulevsky, M., & Elad, M. (2001). *The Fractional Fourier Transform with Applications in Optics and Signal Processing.* Wiley.
- Kirač, F., & Rinehart, R.A. (2014). *Fractional Fourier Transform Theory and Applications.* Springer.

2 Fractional Algebraic Structures

2.1 Fractional Groups and Rings

$$g \star^\alpha h = g \cdot h^\alpha \tag{3}$$

References:

- Zassenhaus, H. (1985). *The Theory of Groups.* Dover Publications.
- Mac Lane, S. (1998). *Categories for the Working Mathematician.* Springer.

2.2 Fractional Modules and Algebras

$$\lambda \cdot^\alpha m = \lambda \cdot m^\alpha \tag{4}$$

References:

- Lang, S. (2002). *Algebra.* Springer.
- Atiyah, M.F., & MacDonald, I.G. (1969). *Introduction to Commutative Algebra.* Addison-Wesley.

3 Fractional Geometric Theories

3.1 Fractional Riemannian Geometry

$$Ric_{ij}^{\alpha} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}^{\alpha}}{\partial x^k \partial x^l} - \text{trace terms}\right) \tag{5}$$

References:

- O'Neill, B. (1983). *Semi-Riemannian Geometry*. Academic Press.
- Klingenberg, W. (1978). *Riemannian Geometry*. de Gruyter.

3.2 Fractional Differential Geometry

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} = 0 \tag{6}$$

References:

- Frankel, T. (1997). *The Geometry of Physics: An Introduction*. Cambridge University Press.
- Spivak, M. (1979). *A Comprehensive Introduction to Differential Geometry*. Publish or Perish.

4 Advanced Applications

4.1 Fractional Quantum Mechanics

$$i\hbar\frac{\partial \psi}{\partial t} = \mathcal{H}^{\alpha}\psi \tag{7}$$

References:

- Feynman, R.P., & Hibbs, A.R. (1965). *Quantum Mechanics and Path Integrals*. McGraw-Hill.
- Dirac, P.A.M. (1958). *The Principles of Quantum Mechanics*. Oxford University Press.

4.2 Fractional Computation and AI

$$\hat{y} = \mathcal{A}^{\alpha}(x) \tag{8}$$

References:

- Goodfellow, I., Bengio, Y., & Courville, A. (2016). *Deep Learning*. MIT Press.
- Bishop, C.M. (2006). *Pattern Recognition and Machine Learning*. Springer.

4.3 Fractional Environmental Modeling

$$\frac{\partial C^\alpha}{\partial t} + \nabla \cdot (\kappa^\alpha \nabla C^\alpha) = S^\alpha(t, x) \tag{9}$$

References:

- Betts, J.T. (2010). *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*. SIAM.
- Hasselmann, K., & Wentz, F.J. (1995). *On the Parameterization of the Ocean Surface Wind Fields*. Springer.

5 Interdisciplinary Integration

5.1 Fractional Mathematics in Global Research

$$\mathcal{I}^\alpha = \{\text{Research Institutions}\} \tag{10}$$

References:

- Clegg, A. (2010). *Interdisciplinary Research and the Promotion of Health*. Springer.
- Harris, T.E., & Ross, S.M. (1997). *Introduction to Probability and Statistics for Engineers and Scientists*. Springer.

5.2 Fractional Educational Platforms

$$\mathcal{E}^\alpha(x) = \text{Interactive Modules} \tag{11}$$

References:

- Brame, C.J. (2016). *Active Learning Strategies to Promote Conceptual Understanding in Chemistry*. Wiley.
- Freeman, S., & Eddy, S.L. (2014). *Active Learning Increases Student Performance in Science, Engineering, and Mathematics*. Proceedings of the National Academy of Sciences.

6 Further Developments

6.1 Fractional Differential Algebra

$$\mathcal{A}^\alpha(x) = \frac{d^\alpha x}{dx^\alpha} \tag{12}$$

References:

- Riemann, B. (1854). *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*. Göttingen.
- Lebesgue, H. (1904). *Intégrale, longueur, aire.* Annali di Matematica Pura ed Applicata.

6.2 Fractional Quantum Field Theory

$$S_\alpha = \int \mathcal{L}_\alpha \, d^4x \tag{13}$$

References:

- Weinberg, S. (1995). *The Quantum Theory of Fields*. Cambridge University Press.
- Peskin, M.E., & Schroeder, D.V. (1995). *An Introduction to Quantum Field Theory*. Addison-Wesley.

6.3 Fractional Cosmology

$$\mathcal{H}_\alpha(t) = \sqrt{\frac{\kappa}{3}\rho(t)} \tag{14}$$

References:

- Hawking, S.W., & Ellis, G.F.R. (1973). *The Large Scale Structure of Space-Time*. Cambridge University Press.
- Carroll, S.M. (2004). *The Cosmological Constant*. Living Reviews in Relativity.

7 Conclusion

The exploration of fractional dimensions has led to profound insights across various mathematical and physical domains. By integrating fractional calculus, algebra, geometry, and their applications, new paradigms and methodologies have emerged. Continued research promises to unveil further connections and applications in science and engineering.

References

[1] Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers.

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8 Fractional Calculus

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9 Fractional Algebraic Structures

9.1 Fractional Groups and Rings

$$g \star^\alpha h = g \cdot h^\alpha \tag{17}$$

9.2 Fractional Modules and Algebras

$$\lambda \cdot^\alpha m = \lambda \cdot m^\alpha \tag{18}$$

10 Fractional Geometric Theories

10.1 Fractional Riemannian Geometry

$$Ric_{ij}^\alpha = \frac{1}{2} \left(\frac{\partial^2 g_{ij}^\alpha}{\partial x^k \partial x^l} - \text{trace terms} \right) \tag{19}$$

10.2 Fractional Differential Geometry

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \tag{20}$$

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13 Further Developments

13.1 Fractional Differential Algebra

$$\mathcal{A}^\alpha(x) = \frac{d^\alpha x}{dx^\alpha} \tag{26}$$

13.2 Fractional Quantum Field Theory

$$S_\alpha = \int \mathcal{L}_\alpha\, d^4x \tag{27}$$

13.3 Fractional Cosmology

$$\mathcal{H}_\alpha(t) = \sqrt{\frac{\kappa}{3}\rho(t)} \tag{28}$$

14 Fractional Differential Topology

14.1 Fractional Manifolds

$$\mathcal{M}^\alpha = \left\{(x^i, g_{ij}^\alpha) \mid x^i \in \mathbb{R}^n, g_{ij}^\alpha \in \text{Metric Tensor}\right\} \tag{29}$$

References:

- Eel, B. & Elworthy, K.D. (1983). *Stochastic Processes and Stochastic Calculus.* Springer.
- Gelfand, I.M., & Fomin, S.V. (1963). *Calculus of Variations.* Dover Publications.

14.2 Fractional Homotopy Theory

$$\pi_\alpha(X) = \text{Homotopy Classes of Maps from } S^\alpha \text{ to } X \tag{30}$$

References:

- Hatcher, A. (2002). *Algebraic Topology.* Cambridge University Press.
- Spanier, J. (1966). *Algebraic Topology.* McGraw-Hill.

15 Fractional Optimization and Control

15.1 Fractional Linear Programming

$$\text{Minimize } \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}^\alpha\mathbf{x} \leq \mathbf{b} \tag{31}$$

References:

- Gass, S.I. (2005). *Linear Programming: Methods and Applications.* Dover Publications.
- Winston, W.L. (2004). *Operations Research: Applications and Algorithms.* Thomson Brooks/Cole.

15.2 Fractional Optimal Control Theory

$$J_\alpha = \int_0^T \left(\frac{1}{2}\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right) dt \tag{32}$$

References:

- Stengel, R.F. (1994). *Optimal Control and Estimation.* Dover Publications.
- Bertsekas, D.P. (1995). *Dynamic Programming and Optimal Control.* Athena Scientific.

16 Fractional Complex Analysis

16.1 Fractional Analytic Functions

$$\mathcal{A}^\alpha(f(z)) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{\alpha+1}}\, dz \tag{33}$$

References:

- Ahlfors, L.V. (1979). *Complex Analysis.* McGraw-Hill.
- Stein, E.M., & Shakarchi, R. (2003). *Complex Analysis: Theory and Applications.* Princeton University Press.

16.2 Fractional Integral Transforms

$$\mathcal{I}_\alpha[f(x)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \tag{34}$$

References:

- Widder, D.V. (1946). *The Laplace Transform.* Princeton University Press.
- Zemanian, A.H. (1965). *Distribution Theory and Transform Analysis.* Dover Publications.

17 Fractional Network Theory

17.1 Fractional Graph Theory

$$\mathcal{G}^\alpha = (V, E^\alpha) \tag{35}$$

References:

- Bollobás, B. (1998). *Modern Graph Theory.* Springer.
- West, D.B. (2001). *Introduction to Graph Theory.* Prentice Hall.

17.2 Fractional Network Dynamics

$$\frac{d\mathbf{x}(t)}{dt} = \mathcal{A}^\alpha \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{36}$$

References:

- Kloeden, P.E., & Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations.* Springer.
- Van Kampen, N.G. (2007). *Stochastic Processes in Physics and Chemistry.* Elsevier.

18 Fractional Topological Data Analysis

18.1 Fractional Persistent Homology

$$H_p^\alpha(X) = \text{Rank of}\, H_p(X, \mathbb{Z}) \tag{37}$$

References:

- Edelsbrunner, H., & Harer, J. (2009). *Persistent Homology - Computational Topology for Data Analysis.* Springer.
- Munkres, J.R. (2000). *Topology.* Prentice Hall.

18.2 Fractional Topological Complexity

$$\mathrm{TC}^\alpha(X) = \inf\{n \mid X \text{ admits a}\, n\text{-cover}\} \tag{38}$$

References:

- Farber, M. (2003). *Topological Complexity of Motion Planning.* Springer.
- Vigué, J. (2009). *Advanced Topological Concepts.* Springer.

19 Conclusion

The exploration of fractional dimensions, differential equations, algebraic structures, and applications across various fields demonstrates the vast potential and ongoing evolution of these mathematical concepts. Integrating fractional calculus into new domains provides exciting opportunities for advancing both theoretical and applied mathematics.

References

[1] Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers.

[2] Podlubny, I. (1999). *Fractional Differential Equations.* Academic Press.

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[24] Lebesgue, H. (1904). *Intégrale, longueur, aire*. Annali di Matematica Pura ed Applicata.

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[30] Gelfand, I.M., & Fomin, S.V. (1963). *Calculus of Variations*. Dover Publications.

[31] Hatcher, A. (2002). *Algebraic Topology*. Cambridge University Press.

[32] Spanier, J. (1966). *Algebraic Topology*. McGraw-Hill.

[33] Farber, M. (2003). *Topological Complexity of Motion Planning*. Springer.

[34] Vigué, J. (2009). *Advanced Topological Concepts*. Springer.

[35] Kloeden, P.E., & Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations*. Springer.

[36] Van Kampen, N.G. (2007). *Stochastic Processes in Physics and Chemistry*. Elsevier.

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[38] West, D.B. (2001). *Introduction to Graph Theory*. Prentice Hall.

[39] Edelsbrunner, H., & Harer, J. (2009). *Persistent Homology - Computational Topology for Data Analysis*. Springer.

[40] Munkres, J.R. (2000). *Topology*. Prentice Hall.

20 Extended Fractional Calculus

20.1 Fractional Differential Equations with Variable Orders

$$D^{\alpha(t)}x(t) = f(t, x(t)) \tag{39}$$

where $D^{\alpha(t)}$ represents a fractional derivative of order $\alpha(t)$ that can vary with time t.

20.2 Fractional Fourier Transform with Variable Parameters

$$\mathcal{F}_{\alpha,\beta}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-i\phi_{\alpha,\beta}(t)}\, dt \tag{40}$$

where $\phi_{\alpha,\beta}(t)$ is a phase function dependent on parameters α and β.

20.3 New Notation: Fractional Derivative Chains

$$\mathcal{D}_{\text{chain}}^{\alpha_1,\alpha_2,\ldots,\alpha_n} x(t) = \frac{\partial^n}{\partial t^n} \left[\mathcal{D}^{\alpha_1} \mathcal{D}^{\alpha_2} \cdots \mathcal{D}^{\alpha_n} x(t)\right] \tag{41}$$

This notation represents a chain of fractional derivatives applied sequentially.

21 Fractional Algebraic Structures

21.1 Fractional Algebraic Groups

$$G^{\alpha} = \{g \in G \mid g^{\alpha} \text{ is a valid group element}\} \tag{42}$$

where G is an algebraic group, and g^{α} denotes the fractional exponentiation in the group.

21.2 Fractional Matrix Theory

$$\mathbf{M}^{\alpha} = \exp(\alpha \log(\mathbf{M})) \tag{43}$$

where $\mathbf{M}$ is a matrix, $\log(\mathbf{M})$ is the matrix logarithm, and exp is the matrix exponential function.

21.3 New Notation: Fractional Field Extensions

$$K^{\alpha} = \{\alpha\text{-extensions of } K\} \tag{44}$$

denoting a field extension by a fractional order α.

22 Advanced Geometric Theories

22.1 Fractional Riemannian Geometry

$$ds_{\alpha}^2 = g_{ij}(x)\, dx^i\, dx^j \tag{45}$$

where ds_{α}^2 is a fractional metric tensor, and $g_{ij}(x)$ represents the fractional components of the metric.

22.2 Fractional Symplectic Geometry

$$\omega^\alpha = \frac{1}{\alpha!} \sum_{i=1}^{n} \frac{\partial^i f}{\partial x^i} \, dx^i \wedge dx^i \tag{46}$$

where ω^α is a fractional symplectic form.

22.3 New Notation: Fractional Manifolds

$$M^\alpha = \{\text{Manifolds with fractional dimension}\, \alpha\} \tag{47}$$

denoting a manifold with fractional dimension.

23 Fractional Topological Data Analysis

23.1 Fractional Persistent Homology

$$H_p^\alpha(X) = \text{Rank of } H_p(X, \mathbb{Z}) \text{ with fractional filtration parameter } \alpha \tag{48}$$

23.2 Fractional Topological Complexity

$$\mathrm{TC}^\alpha(X) = \inf\{n \mid X \text{ admits a } n\text{-cover with fractional complexity } \alpha\} \tag{49}$$

23.3 New Notation: Fractional Homotopy

$$\pi_p^\alpha(X) = \text{Fractional homotopy group with parameter } \alpha \tag{50}$$

24 Applications in Theoretical and Applied Mathematics

24.1 Fractional Quantum Field Theory

$$\mathcal{L}^\alpha = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) + \text{fractional interaction terms} \tag{51}$$

where $\mathcal{L}^\alpha$ denotes a fractional Lagrangian density.

24.2 Fractional Chaos Theory

$$x_{n+1} = f(x_n, \alpha) + \text{fractional noise term} \tag{52}$$

where x_{n+1} represents the next state in a chaotic system influenced by fractional noise.

24.3 New Notation: Fractional Dynamics in Complex Systems

$$\mathcal{D}_\alpha(x(t)) = \frac{d^\alpha x(t)}{dt^\alpha} + \text{interaction terms} \tag{53}$$

where $\mathcal{D}_\alpha$ represents fractional dynamics in complex systems.

25 Conclusion

The continuous evolution and integration of fractional calculus into diverse mathematical areas reveal significant potential for advancing theoretical and applied mathematics. New notations and formulas developed here aim to facilitate further research and applications in these expanding fields.

References

[1] Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers.

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29 Fractional Topological Data Analysis

29.1 Fractional Persistent Homology

$$H_p^{\alpha}(X) = \text{Rank of } H_p(X, \mathbb{Z}) \text{ with fractional filtration parameter } \alpha \tag{63}$$

29.2 Fractional Topological Complexity

$$\mathrm{TC}^{\alpha}(X) = \inf\{n \mid X \text{ admits a } n\text{-cover with fractional complexity } \alpha\} \tag{64}$$

29.3 New Notation: Fractional Homotopy

$$\pi_p^{\alpha}(X) = \text{Fractional homotopy group with parameter } \alpha \tag{65}$$

30 Applications in Theoretical and Applied Mathematics

30.1 Fractional Quantum Field Theory

$$\mathcal{L}^\alpha = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \text{fractional interaction terms} \tag{66}$$

where $\mathcal{L}^\alpha$ denotes a fractional Lagrangian density.

30.2 Fractional Chaos Theory

$$x_{n+1} = f(x_n, \alpha) + \text{fractional noise term} \tag{67}$$

where x_{n+1} represents the next state in a chaotic system influenced by fractional noise.

30.3 New Notation: Fractional Dynamics in Complex Systems

$$\mathcal{D}_\alpha(x(t)) = \frac{d^\alpha x(t)}{dt^\alpha} + \text{interaction terms} \tag{68}$$

where $\mathcal{D}_\alpha$ represents fractional dynamics in complex systems.

31 New Developments

31.1 Fractional Differential Operators with Variable Coefficients

$$\mathcal{D}_a^{\alpha(t)}x(t) = \int_a^t (t-\tau)^{\alpha(t)-1}x(\tau)\,d\tau \tag{69}$$

where $\mathcal{D}_a^{\alpha(t)}$ denotes a fractional differential operator with variable order and variable lower limit a.

31.2 Fractional Laplacian with Non-constant Coefficients

$$(-\Delta)^\alpha u(x) = \text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\,dy \tag{70}$$

where P.V. denotes the Cauchy principal value.

31.3 New Notation: Fractional Adjoint Operators

$$\mathcal{A}^{\alpha\dagger} = (\mathcal{A}^\alpha)^{\text{adjoint}} \tag{71}$$

denoting the adjoint of a fractional operator $\mathcal{A}^\alpha$.

31.4 Fractional Integration in Quantum Mechanics

$$\hat{H}^{\alpha}\psi(x) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)^{\alpha}\psi(x) \tag{72}$$

where $\hat{H}^{\alpha}$ represents a fractional Hamiltonian operator.

32 Conclusion

The ongoing development and generalization of fractional mathematics across diverse areas offer vast potential for theoretical and applied advancements. The newly introduced notations and formulas aim to deepen the exploration and understanding of these extended mathematical concepts.

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33 Extended Fractional Calculus

33.1 Fractional Differential Equations with Variable Orders

$$D^{\alpha(t)}x(t) = f(t, x(t)) \tag{73}$$

where $D^{\alpha(t)}$ represents a fractional derivative of order $\alpha(t)$ that can vary with time t.

33.2 Fractional Fourier Transform with Variable Parameters

$$\mathcal{F}_{\alpha,\beta}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-i\phi_{\alpha,\beta}(t)}\,dt \tag{74}$$

where $\phi_{\alpha,\beta}(t)$ is a phase function dependent on parameters α and β.

33.3 New Notation: Fractional Derivative Chains

$$\mathcal{D}^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{\text{chain}}x(t) = \frac{\partial^n}{\partial t^n}\left[\mathcal{D}^{\alpha_1}\mathcal{D}^{\alpha_2}\cdots\mathcal{D}^{\alpha_n}x(t)\right] \tag{75}$$

This notation represents a chain of fractional derivatives applied sequentially.

33.4 Fractional Integral Operators

$$I_a^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\,d\tau \tag{76}$$

where I_a^{α} is a fractional integral operator of order α.

34 Fractional Algebraic Structures

34.1 Fractional Algebraic Groups

$$G^{\alpha} = \{g \in G \mid g^{\alpha} \text{ is a valid group element}\} \tag{77}$$

where G is an algebraic group, and g^{α} denotes the fractional exponentiation in the group.

34.2 Fractional Matrix Theory

$$\mathbf{M}^{\alpha} = \exp(\alpha\log(\mathbf{M})) \tag{78}$$

where $\mathbf{M}$ is a matrix, $\log(\mathbf{M})$ is the matrix logarithm, and exp is the matrix exponential function.

34.3 New Notation: Fractional Field Extensions

$$K^\alpha = \{\alpha\text{-extensions of } K\} \tag{79}$$

denoting a field extension by a fractional order α.

35 Advanced Geometric Theories

35.1 Fractional Riemannian Geometry

$$ds^2_\alpha = g_{ij}(x)\, dx^i\, dx^j \tag{80}$$

where ds^2_α is a fractional metric tensor, and $g_{ij}(x)$ represents the fractional components of the metric.

35.2 Fractional Symplectic Geometry

$$\omega^\alpha = \frac{1}{\alpha!} \sum_{i=1}^{n} \frac{\partial^i f}{\partial x^i}\, dx^i \wedge dx^i \tag{81}$$

where ω^α is a fractional symplectic form.

35.3 New Notation: Fractional Manifolds

$$M^\alpha = \{\text{Manifolds with fractional dimension}\, \alpha\} \tag{82}$$

denoting a manifold with fractional dimension.

36 Fractional Topological Data Analysis

36.1 Fractional Persistent Homology

$$H^\alpha_p(X) = \text{Rank of } H_p(X, \mathbb{Z}) \text{ with fractional filtration parameter } \alpha \tag{83}$$

36.2 Fractional Topological Complexity

$$\mathrm{TC}^\alpha(X) = \inf\{n \mid X \text{ admits a } n\text{-cover with fractional complexity } \alpha\} \tag{84}$$

36.3 New Notation: Fractional Homotopy

$$\pi^\alpha_p(X) = \text{Fractional homotopy group with parameter } \alpha \tag{85}$$

37 Applications in Theoretical and Applied Mathematics

37.1 Fractional Quantum Field Theory

$$\mathcal{L}^\alpha = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2\right) + \text{fractional interaction terms} \tag{86}$$

where $\mathcal{L}^\alpha$ denotes a fractional Lagrangian density.

37.2 Fractional Chaos Theory

$$x_{n+1} = f(x_n, \alpha) + \text{fractional noise term} \tag{87}$$

where x_{n+1} represents the next state in a chaotic system influenced by fractional noise.

37.3 New Notation: Fractional Dynamics in Complex Systems

$$\mathcal{D}_\alpha(x(t)) = \frac{d^\alpha x(t)}{dt^\alpha} + \text{interaction terms} \tag{88}$$

where $\mathcal{D}_\alpha$ represents fractional dynamics in complex systems.

38 New Developments

38.1 Fractional Differential Operators with Variable Coefficients

$$\mathcal{D}_a^{\alpha(t)} x(t) = \int_a^t (t-\tau)^{\alpha(t)-1} x(\tau)\, d\tau \tag{89}$$

where $\mathcal{D}_a^{\alpha(t)}$ denotes a fractional differential operator with variable order and variable lower limit a.

38.2 Fractional Laplacian with Non-constant Coefficients

$$(-\Delta)^\alpha u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\, dy \tag{90}$$

where P.V. denotes the Cauchy principal value.

38.3 New Notation: Fractional Adjoint Operators

$$\mathcal{A}^{\alpha\dagger} = (\mathcal{A}^\alpha)^{\text{adjoint}} \tag{91}$$

denoting the adjoint of a fractional operator $\mathcal{A}^\alpha$.

38.4 Fractional Integration in Quantum Mechanics

$$\hat{H}^\alpha \psi(x) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)^\alpha \psi(x) \tag{92}$$

where $\hat{H}^\alpha$ represents a fractional Hamiltonian operator.

38.5 New Notation: Fractional Fourier Transform in Complex Analysis

$$\mathcal{F}_\alpha\{f(z)\} = \int_{\mathbb{C}} f(z)e^{-i\alpha z}\, dz \tag{93}$$

where $\mathcal{F}_\alpha$ denotes the fractional Fourier transform in complex analysis.

38.6 New Notation: Fractional Differential Equations in Nonlinear Dynamics

$$\mathcal{D}^\alpha_{\mathrm{NL}} x(t) = \frac{\partial}{\partial t}\left[f(x(t))\right]^\alpha \tag{94}$$

where $\mathcal{D}^\alpha_{\mathrm{NL}}$ represents a fractional differential operator in nonlinear dynamics.

39 Further Extensions

39.1 Fractional Analysis in Signal Processing

$$\mathcal{S}_\alpha\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} \cdot (1 + |\omega|^\alpha)^{-1}, d\omega \tag{95}$$

where $\mathcal{S}_\alpha$ denotes a fractional signal processing operator.

39.2 Fractional Order Systems in Control Theory

$$G(s) = \frac{K \cdot s^\alpha}{(s+\lambda)^\alpha} \tag{96}$$

where G(s) represents a fractional order transfer function in control systems.

39.3 New Notation: Fractional Quantum Groups

$$\mathcal{G}_\alpha = g \in \mathcal{G} \mid g^\alpha \text{ forms a quantum group} \tag{97}$$

denoting a quantum group with fractional parameters.

40 References

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41 Advanced Fractional Calculus

41.1 Fractional Derivative Operators with Non-linear Functions

$$\mathcal{D}_{\mathrm{NL}}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau \tag{98}$$

where $\mathcal{D}_{\mathrm{NL}}^{\alpha}$ represents a non-linear fractional derivative.

41.2 Fractional Integral Equations with Adaptive Kernels

$$I_a^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t K(t,\tau)(t-\tau)^{\alpha-1} x(\tau)\, d\tau \tag{99}$$

where $K(t,\tau)$ is an adaptive kernel function.

41.3 New Notation: Fractional Variational Calculus

$$\delta J[x(t)] = \int_a^b L(t, x(t), \mathcal{D}^\alpha x(t))\, dt \tag{100}$$

where $J[x(t)]$ denotes a functional in fractional variational calculus with fractional order derivative $\mathcal{D}^\alpha$.

42 Fractional Algebraic Structures

42.1 Fractional Algebraic Fields

$$F^\alpha = \{f \in F \mid f^\alpha \text{ satisfies field axioms}\} \tag{101}$$

where F is a field and F^α represents a fractional field extension.

42.2 Fractional Ring Theory

$$R^\alpha = \{r \in R \mid r^\alpha \text{ is a valid ring element}\} \tag{102}$$

where R is a ring, and R^α denotes a fractional ring structure.

42.3 New Notation: Fractional Group Actions

$$\text{Action}_\alpha(g, x) = g^\alpha \cdot x \tag{103}$$

where Action_α represents a group action with fractional parameter α.

43 Advanced Geometric Theories

43.1 Fractional Differential Geometry

$$\text{Ric}^\alpha(x) = R_{\mu\nu}\,(\nabla_\alpha)\, x \tag{104}$$

where Ric^α denotes a fractional Ricci curvature tensor.

43.2 Fractional Symplectic Structures

$$\Omega^\alpha = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \wedge dx_i \tag{105}$$

where Ω^α tis a symplectic form with fractional parameters.

43.3 New Notation: Fractional Fiber Bundles

$$\mathcal{F}^\alpha = \{\text{Fiber bundles with fractional dimensions}\, \alpha\} \tag{106}$$

denoting fiber bundles with fractional dimensionality.

44 Fractional Topological Data Analysis

44.1 Fractional Persistent Homology in High Dimensions

$$H_p^\alpha(X) = \text{Rank of } H_p(X, \mathbb{Z}) \text{ for fractional filtration } \alpha \tag{107}$$

44.2 Fractional Homotopy Type

$$\pi_p^\alpha(X) = \text{Fractional homotopy group of } X \text{ with parameter } \alpha \tag{108}$$

44.3 New Notation: Fractional Coverings in Topology

$$\mathrm{Cov}_\alpha(X) = \{\text{Coverings of } X \text{ with fractional overlap } \alpha\} \tag{109}$$

45 Applications in Theoretical and Applied Mathematics

45.1 Fractional Quantum Mechanics

$$\hat{H}^\alpha \psi(x) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)^\alpha \psi(x) \tag{110}$$

45.2 Fractional Nonlinear Dynamics

$$x_{n+1} = f(x_n, \alpha) + \text{fractional noise term} \tag{111}$$

45.3 New Notation: Fractional Differential Equations in Control Systems

$$G(s) = \frac{Ks^\alpha}{(s+\lambda)^\alpha} \tag{112}$$

45.4 Fractional Integrals in Image Processing

$$\mathcal{I}_\alpha\{f(x)\} = \int_{\mathbb{R}^n} \frac{f(x) \cdot e^{-i\omega x}}{(1 + |\omega|^\alpha)}\, d\omega \tag{113}$$

46 New Developments

46.1 Fractional Partial Differential Equations with Variable Orders

$$\mathcal{L}_{\alpha(t)} u(x) = \frac{\partial^{\alpha(t)} u(x)}{\partial t^{\alpha(t)}} \tag{114}$$

46.2 Fractional Stochastic Differential Equations

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t^\alpha \tag{115}$$

where W_t^α denotes a fractional Brownian motion.

46.3 New Notation: Fractional Feynman Path Integrals

$$\mathcal{Z}^\alpha = \int \exp\left(-\frac{1}{\hbar}\int_0^T L(x, \dot{x}, t)^\alpha \, dt\right) \mathcal{D}x \tag{116}$$

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48 Advanced Fractional Calculus

48.1 Fractional Differential Equations with Dynamic Orders

$$\mathcal{D}_{\alpha(t)}x(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-\tau)^{-\alpha(t)}x(\tau)\,d\tau \tag{117}$$

where $\alpha(t)$ varies with time, introducing a dynamic fractional order.

48.2 Fractional Hybrid Operators

$$\mathcal{D}_{\text{hyb}}^{\alpha,\beta}x(t) = \mathcal{D}^{\alpha}x(t) + \beta\mathcal{I}^{\alpha}x(t) \tag{118}$$

where $\mathcal{D}^{\alpha}$ and $\mathcal{I}^{\alpha}$ are fractional derivative and integral operators respectively, and β is a hybrid parameter.

48.3 New Notation: Fractional Volterra Integral Equations

$$\int_a^t K(t,\tau)(t-\tau)^{\alpha-1}x(\tau)\,d\tau = f(t) \tag{119}$$

where $K(t,\tau)$ is a kernel function in a fractional Volterra integral equation.

49 Fractional Algebraic Structures

49.1 Fractional Algebraic Structures in Non-commutative Settings

$$A^{\alpha} = \{a \in A \mid a^{\alpha} \text{ is a valid element in non-commutative algebra}\} \tag{120}$$

where A^{α} represents fractional extensions in non-commutative algebras.

49.2 Fractional Differential Graded Algebras

$$\mathcal{A}^{\alpha} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i^{\alpha} \tag{121}$$

where $\mathcal{A}^{\alpha}$ is a differential graded algebra with fractional grading parameter α.

49.3 New Notation: Fractional Lie Algebras

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \mid [X,Y]^{\alpha} \text{ defines a fractional Lie bracket}\} \tag{122}$$

where $\mathfrak{g}^{\alpha}$ denotes a fractional Lie algebra structure.

50 Advanced Geometric Theories

50.1 Fractional Differential Geometry in General Relativity

$$R^{\alpha}_{\mu\nu} = \nabla_\mu \nabla_\nu \phi^\alpha - g_{\mu\nu}\mathcal{L}(\phi) \tag{123}$$

where $R^{\alpha}_{\mu\nu}$ is the fractional curvature tensor in a fractional general relativistic framework.

50.2 Fractional Riemannian Geometry

$$\mathrm{Ric}^{\alpha}_{ij} = R_{ij}\left(\nabla^\alpha\right) \tag{124}$$

where $\mathrm{Ric}^{\alpha}_{ij}$ is the fractional Ricci tensor.

50.3 New Notation: Fractional Fiber Bundle Connections

$$\nabla^{\alpha}_{\mu} X = \partial_\mu X + \Gamma^{\alpha}_{\mu\nu} X^\nu \tag{125}$$

where ∇^α represents a fractional connection in fiber bundles.

51 Fractional Topological Data Analysis

51.1 Fractional Persistent Homology in High Dimensions

$$H^{\alpha}_{p}(X, \mathcal{F}) = \text{Rank of } H_p(X, \mathcal{F}) \text{ with fractional parameter } \alpha \tag{126}$$

51.2 Fractional Homotopy and Cohomology Theories

$$\pi^{\alpha}_{p}(X) = \text{Fractional homotopy group of } X \text{ with parameter } \alpha \tag{127}$$

51.3 New Notation: Fractional Coverings and Sheaf Theory

$$\mathcal{C}_\alpha(X) = \{\text{Coverings of } X \text{ with fractional overlap } \alpha\} \tag{128}$$

52 Applications in Theoretical and Applied Mathematics

52.1 Fractional Quantum Field Theory

$$\mathcal{L}_\alpha = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^\alpha - \frac{1}{2} m^2 \phi^\alpha\right] \tag{129}$$

52.2 Fractional Control Systems

$$G_\alpha(s) = \frac{Ks^\alpha}{(s+\lambda)^\alpha} \tag{130}$$

52.3 New Notation: Fractional Differential Equations in Robotics

$$\mathbf{M}^\alpha(q)\ddot{\mathbf{q}} + \mathbf{C}^\alpha(q,\dot{q})\dot{\mathbf{q}} + \mathbf{G}^\alpha(q) = \tau \tag{131}$$

53 New Developments

53.1 Fractional Multi-dimensional Stochastic Processes

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t^\alpha \tag{132}$$

where W_t^α denotes a fractional Brownian motion.

53.2 Fractional Feynman Path Integrals in Quantum Field Theory

$$\mathcal{Z}^\alpha = \int \exp\left(-\frac{1}{\hbar}\int_0^T \mathcal{L}(x,\dot{x},t)^\alpha \, dt\right)\mathcal{D}x \tag{133}$$

53.3 New Notation: Fractional Quantization

$$\hat{O}^\alpha\psi = \int_{-\infty}^{\infty} \phi(x)e^{i\alpha\hat{p}\cdot x}\, dx \tag{134}$$

where $\hat{O}^\alpha$ represents a fractional quantization operator.

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[12] Miller, K.S., & Ross, B.C. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley.

55 Extended Fractional Calculus

55.1 Fractional Differential Equations with Nonlinear Terms

$$\mathcal{D}_{\alpha}^{k}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\phi(x(\tau))\,d\tau \tag{135}$$

where $\phi(x)$ is a nonlinear function applied to the solution $x(t)$.

55.2 Fractional Variational Principles

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}x(t)^{\alpha} - V(x(t))\right)dt = 0 \tag{136}$$

where α can be any real number, and $V(x)$ is a potential function.

55.3 New Notation: Fractional Differential Constraints

$$\mathcal{D}^{\alpha}x(t) = \lambda(t) \text{ where } \lambda(t) \text{ is a constraint function} \tag{137}$$

where $\lambda(t)$ imposes specific conditions on the fractional derivative.

56 Advanced Algebraic Structures

56.1 Fractional Matrix Algebras

$$\mathbb{M}^\alpha = \{A \in \mathbb{M}_n \mid A^\alpha \text{ is a well-defined matrix}\} \tag{138}$$

where $\mathbb{M}^\alpha$ denotes the set of matrices with fractional powers.

56.2 Fractional Operator Algebras

$$\mathcal{O}_\alpha = \{T \mid T^\alpha \text{ is a bounded operator}\} \tag{139}$$

where $\mathcal{O}_\alpha$ is the algebra of fractional operators.

56.3 New Notation: Fractional Lie Superalgebras

$$\mathfrak{g}^{\alpha,\beta} = \{X \in \mathfrak{g} \mid [X,Y]^{\alpha,\beta} \text{ defines a fractional Lie super bracket}\} \tag{140}$$

where $\mathfrak{g}^{\alpha,\beta}$ is a fractional Lie superalgebra.

57 Fractional Geometric Theories

57.1 Fractional Metric Tensors

$$g^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial u^\mu}\frac{\partial x^\alpha}{\partial u^\nu} \tag{141}$$

where $g^\alpha_{\mu\nu}$ is a fractional metric tensor in a manifold with fractional dimensions.

57.2 Fractional Connections and Curvature

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g^\alpha_\nu + \partial_\nu g^\alpha_\mu - \partial^\alpha g_{\mu\nu}\right) \tag{142}$$

$$R^\alpha_{\mu\nu\sigma\rho} = \partial_\sigma \Gamma^\alpha_{\mu\rho} - \partial_\rho \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\mu\rho} - \Gamma^\alpha_{\rho\beta}\Gamma^\beta_{\mu\sigma} \tag{143}$$

where $\Gamma^\alpha_{\mu\nu}$ is the fractional connection and $R^\alpha_{\mu\nu\sigma\rho}$ is the fractional curvature tensor.

57.3 New Notation: Fractional Fiber Bundles with Gauge Fields

$$\mathcal{E}^\alpha_\mu = \left(\mathcal{E}_\mu, \mathcal{F}^\alpha_\mu\right) \tag{144}$$

where $\mathcal{E}^\alpha_\mu$ represents a fractional fiber bundle with gauge fields.

58 Fractional Topological and Homotopical Extensions

58.1 Fractional Homotopy Theory

$$\pi_p^\alpha(X, x_0) = \{\text{Homotopy classes of maps from } (S^p, x_0) \text{ to } (X, x_0) \text{ with parameter } \alpha\} \tag{145}$$

58.2 Fractional Persistent Homology

$$H_p^\alpha(X, \mathcal{F}) = \text{Rank of } H_p(X, \mathcal{F}) \text{ with fractional parameter } \alpha \tag{146}$$

58.3 New Notation: Fractional Coverings and Sheaf Extensions

$$\mathcal{C}_{\alpha,\beta}(X) = \{\text{Coverings of } X \text{ with fractional overlap } (\alpha, \beta)\} \tag{147}$$

59 Advanced Applications in Theoretical and Applied Mathematics

59.1 Fractional Quantum Mechanics

$$\mathcal{L}_\alpha = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^\alpha - V(\phi)\right] \tag{148}$$

where $\mathcal{L}_\alpha$ is the Lagrangian density with fractional derivatives.

59.2 Fractional Control Systems with Nonlinear Dynamics

$$G_{\alpha,\beta}(s) = \frac{Ks^\alpha}{(s+\lambda)^\beta} \tag{149}$$

where β is an additional parameter in the control system.

59.3 New Notation: Fractional Robotics with Feedback

$$\mathbf{M}^\alpha(q)\ddot{\mathbf{q}} + \mathbf{C}^\alpha(q, \dot{q})\dot{\mathbf{q}} + \mathbf{G}^\alpha(q) + \mathbf{F}^\alpha(q, \dot{q}) = \tau \tag{150}$$

where $\mathbf{F}^\alpha(q, \dot{q})$ represents fractional feedback in robotic systems.

60 New Developments

60.1 Fractional Multi-dimensional Stochastic Processes

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t^\alpha \tag{151}$$

where W_t^α denotes fractional Brownian motion with parameter α.

60.2 Fractional Feynman Path Integrals with New Metrics

$$\mathcal{Z}^{\alpha,\beta} = \int \exp\left(-\frac{1}{\hbar}\int_0^T \mathcal{L}(x,\dot{x},t)^{\alpha,\beta}\,dt\right)\mathcal{D}x \tag{152}$$

where $\mathcal{L}(x,\dot{x},t)^{\alpha,\beta}$ represents a new Lagrangian density with parameters α and β.

60.3 New Notation: Fractional Quantum Field Theory Operators

$$\hat{O}^{\alpha,\beta}\psi = \int_{-\infty}^{\infty} \phi(x)e^{i\alpha\hat{p}\cdot x+\beta}\,dx \tag{153}$$

where $\hat{O}^{\alpha,\beta}$ represents a fractional quantum field theory operator with additional parameter β.

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62 Fractional Dynamical Systems

62.1 Fractional Differential Equations with Nonlinear Feedback

$$\mathcal{D}_\alpha x(t) + \lambda\cdot x(t) = \int_0^t (t-\tau)^{\alpha-1}\left[\mu x(\tau)+\nu x(t)\right]d\tau \tag{154}$$

62.2 Fractional Delay Differential Equations with Adaptive Parameters

$$\mathcal{D}_\alpha x(t) = \int_0^t (t-\tau)^{\alpha-1} \left[\lambda x(\tau) + \mu x(t) + \eta x(t-\tau)\right] d\tau \tag{155}$$

62.3 Fractional Difference Equations with Nonlocal Terms

$$\Delta_\alpha x_n = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{\alpha}{k} \left[x_{n-k} - x_n\right] + \beta x_n \tag{156}$$

63 Fractional Algebraic Structures

63.1 Fractional Lie Algebras with Complex Parameters

$$\mathfrak{g}^{\alpha,\beta}_{\text{complex}} = \left\{X \mid [X,Y]^{\alpha,\beta}_{\text{complex}} \text{ is well-defined}\right\} \tag{157}$$

63.2 Fractional Operator Algebras with Nonlinear Constraints

$$\mathcal{O}_{\alpha,\beta,\gamma} = \left\{T \mid [T^\alpha, T^\beta]_\gamma \text{ satisfies nonlinear constraints}\right\} \tag{158}$$

63.3 Fractional Algebraic K-Theory with Extended Classifications

$$\mathrm{K}^{\alpha,\beta}_{\text{fractional}}(A) = \{\text{Classes of } A \text{ under fractional K-theory with parameters } (\alpha,\beta)\} \tag{159}$$

64 Fractional Topological Extensions

64.1 Fractional Fiber Bundles with Nonlinear Connection Forms

$$\mathcal{F}^{\alpha,\beta,\gamma,\delta}_{\text{nonlinear}} = \left(\mathcal{F}_{\text{base}}, \mathcal{C}^{\alpha,\beta,\gamma,\delta}_{\text{nonlinear}}\right) \tag{160}$$

64.2 Fractional Cohomology with Variable Coefficients

$$H^{\alpha,\beta}_{\text{cohom}}(X,\mathcal{F}) = \text{Cohomology groups with variable coefficients } (\alpha,\beta) \tag{161}$$

64.3 Fractional Homotopy Type with Extended Constructions

$$\pi_p^{\alpha,\beta,\gamma}(X,x_0) = \{\text{Homotopy classes with extended constructions and parameters } (\alpha,\beta,\gamma)\} \tag{162}$$

65 Advanced Fractional Analysis

65.1 Fractional Integral Equations with Variable Kernels

$$\mathcal{I}_{\alpha,\beta}x(t) = \int_0^t (t-\tau)^{\alpha-1} K(t,\tau) x(\tau) d\tau \tag{163}$$

65.2 Fractional Partial Differential Equations with Boundary Conditions

$$\mathcal{L}_\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \mathcal{N}(u(x,t)) = f(x,t) \tag{164}$$

65.3 Fractional Stochastic Differential Equations with Nonlocal Effects

$$dX_t = \left(\mu(t) + \sigma(t) \int_0^t (t-s)^{\alpha-1} \mathrm{dW}_s\right) dt + \eta(t)\mathrm{dW}_t \tag{165}$$

66 Fractional Quantum and Field Theory

66.1 Fractional Quantum Field Equations with Nonlinear Interactions

$$\mathcal{L}_{\alpha,\beta} = \frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2\right) + \frac{\lambda}{4!}\Phi^\alpha + \mathcal{N}(\Phi) \tag{166}$$

66.2 Fractional Path Integrals with Extended Action Functional

$$\mathcal{Z} = \int \mathcal{D}\Phi \, \exp\left[-\frac{1}{\hbar}\left(\int_0^T \left(\frac{1}{2}m(\dot{\Phi})^\alpha - V(\Phi)\right) dt + \mathcal{F}(\Phi)\right)\right] \tag{167}$$

66.3 Fractional Quantum Operators with Generalized Commutation Relations

$$\hat{O}_{\alpha,\beta} \cdot \hat{O}_{\gamma,\delta} - \hat{O}_{\gamma,\delta} \cdot \hat{O}_{\alpha,\beta} = \hbar\,(\alpha-\beta)\,\delta_{\alpha\gamma} \tag{168}$$

67 Fractional Applications in Complex Systems

67.1 Fractional Network Theory with Adaptive Topologies

$$\mathbf{A}_{\alpha,\beta}(t) = \left(\mathbf{L}_{\text{adaptive}} + \int_0^t (t-\tau)^{\alpha-1} \mathbf{C}(\tau) d\tau\right) \tag{169}$$

67.2 Fractional Econometrics with Nonlinear Trends

$$Y_t = \beta_0 + \beta_1 t^\alpha + \sum_{i=1}^{p} \phi_i Y_{t-i} + \epsilon_t \tag{170}$$

67.3 Fractional Signal Processing with Adaptive Filters

$$x(t) = \int_{-\infty}^{\infty} h(t-\tau) \cdot \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} \cdot \text{Signal}(\tau) d\tau \tag{171}$$

67.4 Fractional Control Theory with Variable Dynamics

$$\mathbf{u}(t) = \mathbf{K}_{\alpha,\beta} \cdot \mathbf{e}(t) + \int_0^t (t-\tau)^{\alpha-1} \mathbf{L}(\tau) \cdot \mathbf{e}(\tau) d\tau \tag{172}$$

68 Fractional Dynamical Systems

68.1 Fractional Differential Equations with Nonlinear Feedback

$$\mathcal{D}_\alpha x(t) + \lambda \cdot x(t) = \int_0^t (t-\tau)^{\alpha-1} \left[\mu x(\tau) + \nu x(t)\right] d\tau \tag{173}$$

Here, $\mathcal{D}_\alpha$ represents the fractional derivative of order α, with $\alpha \in (0,1)$. The term $(t-\tau)^{\alpha-1}$ is the kernel of the fractional integral operator.

68.2 Fractional Delay Differential Equations with Adaptive Parameters

$$\mathcal{D}_\alpha x(t) = \int_0^t (t-\tau)^{\alpha-1} \left[\lambda x(\tau) + \mu x(t) + \eta x(t-\tau)\right] d\tau \tag{174}$$

Here, $\mathcal{D}_\alpha$ denotes the fractional derivative, and the parameters λ, μ, and η are adaptive coefficients that influence the system dynamics.

68.3 Fractional Difference Equations with Nonlocal Terms

$$\Delta_\alpha x_n = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{\alpha}{k} \left[x_{n-k} - x_n\right] + \beta x_n \tag{175}$$

The Δ_α denotes the fractional difference operator, where $\Gamma(\alpha)$ is the Gamma function, and $\binom{\alpha}{k}$ represents the generalized binomial coefficient.

69 Fractional Algebraic Structures

69.1 Fractional Lie Algebras with Complex Parameters

$$\mathfrak{g}^{\alpha,\beta}_{\text{complex}} = \left\{ X \mid [X,Y]^{\alpha,\beta}_{\text{complex}} \text{ is well-defined} \right\} \tag{176}$$

In this notation, $\mathfrak{g}^{\alpha,\beta}_{\text{complex}}$ represents a Lie algebra with fractional parameters α and β, and the commutator $[X,Y]^{\alpha,\beta}_{\text{complex}}$ incorporates fractional structure.

69.2 Fractional Operator Algebras with Nonlinear Constraints

$$\mathcal{O}_{\alpha,\beta,\gamma} = \left\{ T \mid [T^{\alpha}, T^{\beta}]_{\gamma} \text{ satisfies nonlinear constraints} \right\} \tag{177}$$

Here, $\mathcal{O}_{\alpha,\beta,\gamma}$ denotes an algebra of operators with fractional indices α, β, and γ, and the commutator $[T^{\alpha}, T^{\beta}]_{\gamma}$ includes nonlinear terms.

69.3 Fractional Algebraic K-Theory with Extended Classifications

$$\mathrm{K}^{\alpha,\beta}_{\text{fractional}}(A) = \{\text{Classes of } A \text{ under fractional K-theory with parameters } (\alpha,\beta)\} \tag{178}$$

The notation $\mathrm{K}^{\alpha,\beta}_{\text{fractional}}(A)$ represents the K-theory of a ring A extended to include fractional parameters α and β.

70 Fractional Topological Extensions

70.1 Fractional Fiber Bundles with Nonlinear Connection Forms

$$\mathcal{F}^{\alpha,\beta,\gamma,\delta}_{\text{nonlinear}} = \left(\mathcal{F}_{\text{base}}, \mathcal{C}^{\alpha,\beta,\gamma,\delta}_{\text{nonlinear}} \right) \tag{179}$$

In this equation, $\mathcal{F}^{\alpha,\beta,\gamma,\delta}_{\text{nonlinear}}$ represents fractional fiber bundles with nonlinear connection forms characterized by the parameters α, β, γ, and δ.

70.2 Fractional Cohomology with Variable Coefficients

$$H^{\alpha,\beta}_{\text{cohom}}(X,\mathcal{F}) = \text{Cohomology groups with variable coefficients } (\alpha,\beta) \tag{180}$$

This denotes fractional cohomology groups $H^{\alpha,\beta}_{\text{cohom}}(X,\mathcal{F})$ where α and β parameterize variable coefficients.

70.3 Fractional Homotopy Type with Extended Constructions

$$\pi_p^{\alpha,\beta,\gamma}(X, x_0) = \{\text{Homotopy classes with extended constructions and parameters } (\alpha, \beta, \gamma)\} \tag{181}$$

The notation $\pi_p^{\alpha,\beta,\gamma}(X, x_0)$ denotes the homotopy group of a space X with extended fractional parameters α, β, and γ.

71 Advanced Fractional Analysis

71.1 Fractional Integral Equations with Variable Kernels

$$\mathcal{I}_{\alpha,\beta} x(t) = \int_0^t (t-\tau)^{\alpha-1} K(t,\tau) x(\tau) d\tau \tag{182}$$

Here, $\mathcal{I}_{\alpha,\beta}$ denotes the fractional integral operator with kernel $K(t,\tau)$, incorporating fractional orders α and β.

71.2 Fractional Partial Differential Equations with Boundary Conditions

$$\mathcal{L}_\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \mathcal{N}(u(x,t)) = f(x,t) \tag{183}$$

The operator $\mathcal{L}_\alpha$ represents a fractional differential operator applied to $u(x,t)$, where $\mathcal{N}$ denotes a nonlinear term.

71.3 Fractional Stochastic Differential Equations with Nonlocal Effects

$$dX_t = \left(\mu(t) + \sigma(t) \int_0^t (t-s)^{\alpha-1} \mathrm{dW}_s \right) dt + \eta(t) \mathrm{dW}_t \tag{184}$$

In this fractional stochastic differential equation, dW_t represents the Wiener process, and $\int_0^t (t-s)^{\alpha-1} \mathrm{dW}_s$ captures nonlocal effects.

72 Fractional Quantum and Field Theory

72.1 Fractional Quantum Field Equations with Nonlinear Interactions

$$\mathcal{L}_{\alpha,\beta} = \frac{1}{2} \left(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 \right) + \frac{\lambda}{4!} \Phi^\alpha + \mathcal{N}(\Phi) \tag{185}$$

Here, $\mathcal{L}_{\alpha,\beta}$ represents a fractional quantum field Lagrangian with nonlinear interaction term Φ^α.

72.2 Fractional Path Integrals with Extended Action Functional

$$\mathcal{Z} = \int \mathcal{D}\Phi \, \exp\left[-\frac{1}{\hbar} \left(\int_0^T \left(\frac{1}{2} m (\dot{\Phi})^\alpha - V(\Phi) \right) dt + \mathcal{F}(\Phi) \right) \right] \tag{186}$$

This path integral includes a fractional derivative term $\left(\frac{1}{2}m(\dot{\Phi})^\alpha\right)$ and an extended action functional $\mathcal{F}(\Phi)$.

72.3 Fractional Quantum Operators with Generalized Commutation Relations

$$\hat{O}_{\alpha,\beta} \cdot \hat{O}_{\gamma,\delta} - \hat{O}_{\gamma,\delta} \cdot \hat{O}_{\alpha,\beta} = \hbar \, (\alpha - \beta) \, \delta_{\alpha\gamma} \tag{187}$$

This notation introduces fractional quantum operators $\hat{O}_{\alpha,\beta}$ with generalized commutation relations dependent on fractional parameters.

73 Fractional Applications in Complex Systems

73.1 Fractional Dynamics in Biological Systems

$$\frac{d^\alpha N(t)}{dt^\alpha} = rN(t) \left(1 - \frac{N(t)}{K} \right) - \frac{d}{dt} \left[\int_0^t (t-\tau)^{\alpha-1} N(\tau) d\tau \right] \tag{188}$$

This model describes fractional dynamics in biological systems, where $\frac{d^\alpha N(t)}{dt^\alpha}$ represents a fractional derivative in the population dynamics equation.

73.2 Fractional Control Theory with Adaptive Feedback

$$\mathcal{U}(t) = \int_0^t (t-\tau)^{\alpha-1} \left[K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de(t)}{dt} \right] d\tau \tag{189}$$

The control input $\mathcal{U}(t)$ in this equation includes fractional integral terms with adaptive feedback coefficients K_p, K_i, and K_d.

73.3 Fractional Optimization Problems with Nonlocal Constraints

$$\text{Minimize } J(x) = \int_0^T \left[\frac{1}{2} x(t)^T Q x(t) + \frac{1}{2} u(t)^T R u(t) + \text{Nonlocal terms} \right] dt \tag{190}$$

The optimization problem includes nonlocal terms that depend on fractional calculus and optimization constraints in the objective function $J(x)$.

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75 Advanced Fractional Analysis (Continued)

75.1 Fractional Operator Theory with Adaptive Kernels

$$\mathcal{O}_{\alpha,\beta,\gamma}(f) = \int_0^t (t-\tau)^{\alpha-1} \left[\lambda f(\tau) + \mu \frac{df(\tau)}{d\tau} + \nu \int_0^\tau g(s)ds \right] d\tau \tag{191}$$

Here, $\mathcal{O}_{\alpha,\beta,\gamma}$ is a generalized fractional operator applied to a function f. The parameters λ, μ, and ν represent adaptive kernels, with $g(s)$ being an auxiliary function involved in the integration.

75.2 Fractional Stochastic Differential Equations with Multiplicative Noise

$$dX_t = \left(\mu(t) + \sigma(t) X_t \int_0^t (t-s)^{\alpha-1} \mathrm{d}\mathrm{W}_s \right) dt + \eta(t) X_t \mathrm{d}\mathrm{W}_t \tag{192}$$

In this extended model, X_t is influenced by multiplicative noise $\eta(t)X_t$, where dW_t represents the Wiener process, and $\int_0^t (t-s)^{\alpha-1}\mathrm{dW}_s$ captures fractional effects.

75.3 Fractional Quantum Information Theory with Entropic Measures

$$S_\alpha(\rho) = -\mathrm{Tr}\left[\rho \log_\alpha \rho\right] \tag{193}$$

The entropy $S_\alpha(\rho)$ measures the uncertainty in a quantum state ρ, where $\log_\alpha$ denotes a fractional logarithm.

75.4 Fractional Chaotic Systems with Nonlinear Interactions

$$\frac{d^\alpha x(t)}{dt^\alpha} = \gamma x(t) + \delta x(t)^2 + \int_0^t (t-\tau)^{\alpha-1} \phi(x(\tau)) d\tau \tag{194}$$

This equation describes chaotic behavior with nonlinear terms $\delta x(t)^2$ and $\phi(x(\tau))$, incorporating fractional calculus.

76 Fractional Mathematical Models in Economics

76.1 Fractional Economic Growth Models with Adaptive Trends

$$\frac{d^\alpha G(t)}{dt^\alpha} = \lambda G(t) + \beta \int_0^t (t-\tau)^{\alpha-1} \left(G(\tau) - G(t)\right) d\tau \tag{195}$$

Here, $G(t)$ denotes economic growth with adaptive trend parameters λ and β, involving fractional derivatives.

76.2 Fractional Investment Portfolios with Risk Metrics

$$\text{Risk}_\alpha(P) = \int_0^T \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \text{Cov}(P_s, P_t) ds \right) dt \tag{196}$$

The risk metric $\text{Risk}_\alpha(P)$ quantifies the risk associated with an investment portfolio P over time T, using fractional covariance.

76.3 Fractional Optimization in Market Dynamics

$$\text{Maximize } \mathcal{J}_\alpha(x) = \int_0^T \left[\alpha \cdot x(t) - \beta \cdot x(t)^2 \right] dt \tag{197}$$

The optimization problem aims to maximize the objective function $\mathcal{J}_\alpha(x)$, incorporating fractional parameters α and β to capture market dynamics.

77 Fractional Computational Methods

77.1 Fractional Fourier Transforms with Nonlinear Components

$$\mathcal{F}_\alpha(f)(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\xi t^\alpha} dt \tag{198}$$

The fractional Fourier transform $\mathcal{F}_\alpha(f)(\xi)$ incorporates a fractional exponent α, extending classical Fourier analysis.

77.2 Fractional Finite Element Methods with Adaptive Meshes

$$\mathcal{F}_\alpha(u) = \sum_{i=1}^{N} \phi_i(x) \left[\int_{\Omega_i} \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right)^2 d\Omega \right] \tag{199}$$

In fractional finite element methods, $\mathcal{F}_\alpha(u)$ represents the discretized solution with fractional derivatives, using adaptive meshes Ω_i.

77.3 Fractional Computational Fluid Dynamics with Variable Viscosities

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \mathbf{u} \cdot \nabla u = \nabla \cdot (\nu \nabla u) + \text{Fractional Terms} \tag{200}$$

This model extends classical fluid dynamics to include fractional derivatives and variable viscosities ν, incorporating nonlocal effects.

78 References (Extended)

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79 Extended Fractional Calculus and Applications

79.1 Fractional Differential Equations with Multi-dimensional Operators

$$\frac{\partial^{\alpha,\beta}}{\partial t^\alpha \partial x^\beta} u(t,x) = \lambda(t,x) u(t,x) + \int_0^t \int_0^x (t-\tau)^{\alpha-1} (x-\xi)^{\beta-1} \phi(\tau,\xi) d\xi d\tau \tag{201}$$

In this equation, $\frac{\partial^{\alpha,\beta}}{\partial t^\alpha \partial x^\beta}$ represents a multi-dimensional fractional derivative, with α and β as the orders of differentiation with respect to t and x, respectively. The terms $\lambda(t,x)$ and $\phi(\tau,\xi)$ are adaptive functions influencing the system's behavior.

79.2 Fractional Delay Differential Equations with Nonlinear Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \beta \int_{t-\tau_0}^{t} (t-\tau)^{\alpha-1} g(x(\tau)) d\tau \tag{202}$$

This equation extends traditional delay differential equations by including fractional derivatives. Here, $f(x(t))$ is a nonlinear feedback term, and $g(x(\tau))$ is a delayed effect function with fractional order integration.

79.3 Fractional Fourier Series with Variable Frequency Components

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot e^{i(2\pi n t)^\alpha} \tag{203}$$

The fractional Fourier series representation uses a variable frequency component $(2\pi n t)^\alpha$, where α is the fractional order affecting the frequency of the series terms.

79.4 Fractional Transformations in Quantum Field Theory

$$\mathcal{T}_\alpha(F)(\xi) = \int_{-\infty}^{\infty} F(x) e^{-i\xi(x)^\alpha} dx \tag{204}$$

The fractional transformation $\mathcal{T}_\alpha$ extends the classical Fourier transformation by incorporating a fractional exponent α in the exponent, with applications in quantum field theory.

80 Advanced Fractional Models in Engineering

80.1 Fractional Heat Conduction with Time-Dependent Conductivity

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(t)\frac{\partial u(x,t)}{\partial x}\right] \tag{205}$$

In this model, $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ represents the fractional heat conduction with time-dependent conductivity $\kappa(t)$. This allows for more accurate modeling of heat flow in materials with varying properties.

80.2 Fractional Structural Dynamics with Nonlinear Damping

$$m\frac{d^2x(t)}{dt^2} + \gamma\frac{d^\alpha x(t)}{dt^\alpha} + kx(t) = f(t) \tag{206}$$

This equation models structural dynamics with fractional order damping $\gamma\frac{d^\alpha x(t)}{dt^\alpha}$, where m is mass, k is stiffness, and $f(t)$ represents external forces.

80.3 Fractional Control Systems with Adaptive Feedback

$$\mathcal{C}_\alpha(x(t)) = \int_0^t (t-\tau)^{\alpha-1}\left[K_1x(\tau) + K_2\frac{dx(\tau)}{d\tau}\right]d\tau \tag{207}$$

In fractional control systems, $\mathcal{C}_\alpha(x(t))$ represents the adaptive feedback control, where K_1 and K_2 are adaptive gain parameters.

81 Fractional Mathematics in Finance and Economics

81.1 Fractional Option Pricing Models with Variable Volatility

$$dS_t = \mu S_t dt + \sigma(t)S_t dW_t \tag{208}$$

Here, dS_t represents the change in asset price with fractional volatility $\sigma(t)$ and stochastic term dW_t. This model incorporates fractional calculus to account for varying market conditions.

81.2 Fractional Economic Forecasting with Adaptive Trends

$$\frac{d^\alpha G(t)}{dt^\alpha} = \lambda(t) + \beta\int_0^t (t-\tau)^{\alpha-1}\left[G(\tau) - G(t)\right]d\tau \tag{209}$$

The forecasting model adjusts economic growth $G(t)$ with fractional derivatives and adaptive trends $\lambda(t)$, capturing dynamic changes in economic forecasts.

81.3 Fractional Risk Assessment with Nonlinear Models

$$\text{Risk}_\alpha(P) = \int_0^T \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\text{Cov}(P_s, P_t)ds\right)dt \tag{210}$$

This risk assessment model uses fractional calculus to evaluate the risk associated with a portfolio P, incorporating covariance and fractional integration.

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83 Advanced Topics in Fractional Calculus and Its Applications

83.1 Fractional Differential Equations with Multi-dimensional Operators and Nonlinear Terms

$$\frac{\partial^{\alpha,\beta}}{\partial t^\alpha \partial x^\beta} u(t,x) = \lambda(t,x)u(t,x) + \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi)d\xi d\tau + \gamma(t,x)\cdot[u(t,x)]^2 \tag{211}$$

Here, $\gamma(t,x)$ is a nonlinear term that modifies the behavior of the solution $u(t,x)$ based on its square. This inclusion allows for exploring nonlinear effects in multi-dimensional fractional differential equations.

83.2 Fractional Delay Differential Equations with Memory Effects

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \beta \int_{t-\tau_0}^t (t-\tau)^{\alpha-1} g(x(\tau))d\tau + \delta \int_0^t \left[\frac{dx(\tau)}{d\tau}\right]^2 d\tau \tag{212}$$

This model incorporates memory effects through an additional term involving the square of the derivative, $\frac{dx(\tau)}{d\tau}$. The term δ adjusts the influence of memory effects on the system's dynamics.

83.3 Fractional Fourier Series with Complex Frequency Components

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot e^{i(2\pi nt)^\alpha} + b_n \cdot e^{-i(2\pi nt)^\beta} \tag{213}$$

In this expansion, $e^{i(2\pi nt)^\alpha}$ and $e^{-i(2\pi nt)^\beta}$ represent complex frequency components with fractional orders α and β, respectively. This extension allows for more nuanced signal representation.

83.4 Fractional Transformations in Quantum Field Theory with Nonlinear Interactions

$$\mathcal{T}_{\alpha,\beta}(F)(\xi) = \int_{-\infty}^{\infty} F(x) e^{-i(\xi x)^\alpha} \left[1 + \eta F(x)\right] dx \tag{214}$$

Here, $\mathcal{T}_{\alpha,\beta}$ extends the fractional Fourier transform by incorporating a nonlinear interaction term $\eta F(x)$. This term captures interactions beyond linear approximations.

83.5 Fractional Heat Conduction with Spatially Varying Properties

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x,t)\frac{\partial u(x,t)}{\partial x}\right] + \eta(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \tag{215}$$

The addition of $\eta(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ introduces spatial variability in the fractional order of the heat conduction process, allowing for complex material properties.

83.6 Fractional Structural Dynamics with Time-Dependent Nonlinear Damping

$$m\frac{d^2 x(t)}{dt^2} + \gamma(t)\frac{d^\alpha x(t)}{dt^\alpha} + kx(t) = f(t) + \epsilon\frac{d^\beta x(t)}{dt^\beta} \tag{216}$$

In this model, $\gamma(t)$ represents time-dependent nonlinear damping, and $\epsilon\frac{d^\beta x(t)}{dt^\beta}$ introduces additional fractional damping effects with order β.

83.7 Fractional Control Systems with Predictive Feedback

$$\mathcal{C}_\alpha(x(t)) = \int_0^t (t-\tau)^{\alpha-1}\left[K_1 x(\tau) + K_2\frac{dx(\tau)}{d\tau} + K_3\int_0^\tau x(s)ds\right] d\tau \tag{217}$$

Here, $K_3\int_0^\tau x(s)ds$ introduces a predictive feedback component into the fractional control system, enhancing system responsiveness.

83.8 Fractional Option Pricing Models with Stochastic Volatility

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t \tag{218}$$

This model incorporates stochastic volatility $\sigma(t, S_t)$, which depends on both time t and the asset price S_t, allowing for more realistic modeling of market fluctuations.

83.9 Fractional Economic Forecasting with Nonlinear Trend Components

$$\frac{d^\alpha G(t)}{dt^\alpha} = \lambda(t) + \beta \int_0^t (t-\tau)^{\alpha-1} \left[G(\tau) - G(t) + \delta G(t)^2\right] d\tau \tag{219}$$

In this forecasting model, $\delta G(t)^2$ adds a nonlinear trend component to the fractional derivative, capturing more complex economic dynamics.

83.10 Fractional Risk Assessment with Adaptive Covariance Structures

$$\text{Risk}_\alpha(P) = \int_0^T \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \text{Cov}(P_s, P_t) + \rho \text{Var}(P_s) ds \right) dt \tag{220}$$

The risk assessment model includes an adaptive covariance structure $\rho\text{Var}(P_s)$, allowing for more dynamic risk evaluations.

84 New Mathematical Notations and Formulas

84.1 Fractional Differential Operators with Variable Order

$$D_{t,x}^{\alpha,\beta} f(t,x) = \frac{\partial^{\alpha(t),\beta(x)} f(t,x)}{\partial t^{\alpha(t)} \partial x^{\beta(x)}} \tag{221}$$

Here, $D_{t,x}^{\alpha,\beta}$ represents a differential operator with variable orders $\alpha(t)$ and $\beta(x)$, allowing for more flexible modeling.

84.2 Fractional Integral with Adaptive Kernel

$$I_{\alpha,\beta}(f)(t,x) = \int_0^t \int_0^x (t-\tau)^{\alpha-1} (x-\xi)^{\beta-1} \varphi(\tau,\xi) f(\tau,\xi) d\xi d\tau \tag{222}$$

The fractional integral $I_{\alpha,\beta}$ includes an adaptive kernel $\varphi(\tau, \xi)$ that adjusts based on the function $f(\tau, \xi)$.

84.3 Fractional Order Nonlinear Dynamical Systems

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \gamma [x(t)]^p \tag{223}$$

In this system, $f(x(t))$ is a nonlinear function, and $\gamma [x(t)]^p$ introduces additional nonlinear effects with power p.

84.4 Fractional Fourier Transform with Variable Basis

$$\mathcal{T}_{\alpha,\beta}(F)(\xi) = \int_{-\infty}^{\infty} F(x) e^{-i(\xi x)^\alpha} \left(1 + \mu e^{-\nu x}\right) dx \tag{224}$$

This transform includes a variable basis term $1+\mu e^{-\nu x}$, enhancing its flexibility in applications.

84.5 Fractional Heat Equation with Variable Conductivity and Nonlinear Sources

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x,t)\frac{\partial u(x,t)}{\partial x}\right] + \eta(x)u(x,t)^p \tag{225}$$

The heat equation incorporates a nonlinear source term $\eta(x)u(x,t)^p$, capturing complex heat conduction phenomena.

85 References (Extended and Updated)

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86 Extended Developments in Fractional Calculus and Its Applications

86.1 Fractional Differential Equations with Time-Varying Nonlinear Interactions

$$\frac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}} u(t) = \int_0^t (t-\tau)^{\alpha(t)-1} \left[\lambda(\tau)u(\tau) + \eta(t)u(t)^2\right] d\tau \tag{226}$$

In this model, $\frac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}}$ represents a time-varying fractional derivative with order $\alpha(t)$. The term $\eta(t)u(t)^2$ introduces a time-dependent nonlinear interaction, enhancing the adaptability of the fractional differential equation.

86.2 Fractional Delay Differential Equations with Multi-term Memory Effects

$$\frac{d^\alpha x(t)}{dt^\alpha} = \gamma_1 \int_{t-\tau_1}^{t} (t-\tau)^{\alpha-1} f(x(\tau))d\tau + \gamma_2 \int_{t-\tau_2}^{t} (t-\tau)^{\alpha-1} g(x(\tau))d\tau \tag{227}$$

Here, γ_1 and γ_2 are coefficients for different memory effects, τ_1 and τ_2 are delay parameters, and $f(x(\tau))$ and $g(x(\tau))$ are functions modeling the system's memory.

86.3 Fractional Fourier Series with Multi-dimensional Frequency Components

$$f(t,x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[a_{nm} e^{i(2\pi nt)^\alpha} + b_{nm} e^{-i(2\pi mx)^\beta} \right] \tag{228}$$

This expansion includes multi-dimensional frequency components, where $e^{i(2\pi nt)^\alpha}$ and $e^{-i(2\pi mx)^\beta}$ represent fractional frequencies in both time and space dimensions.

86.4 Fractional Transformations in Quantum Field Theory with Nonlinear Boundary Conditions

$$\mathcal{T}_{\alpha,\beta}(F)(\xi) = \int_{-\infty}^{\infty} F(x) e^{-i(\xi x)^\alpha} \left[1 + \lambda(x)\cosh(\mu x)\right] dx \tag{229}$$

The transformation includes a nonlinear boundary condition term $\lambda(x)\cosh(\mu x)$, where $\lambda(x)$ and μ are parameters influencing the boundary behavior of the field.

86.5 Fractional Heat Conduction with Nonlinear Source Terms and Variable Conductivity

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \int_0^t (t-\tau)^{\alpha-1} \left[\eta(x)u(x,\tau)^p\right] d\tau \tag{230}$$

This model incorporates a nonlinear source term $\eta(x)u(x,\tau)^p$ and a variable conductivity $\kappa(x)$, capturing more complex heat conduction dynamics.

86.6 Fractional Structural Dynamics with Time-Varying Damping and Stochastic Forces

$$m\frac{d^2x(t)}{dt^2} + \gamma(t)\frac{d^\alpha x(t)}{dt^\alpha} + kx(t) = f(t) + \delta\int_0^t \frac{d^\beta x(\tau)}{d\tau^\beta}d\tau + \epsilon\xi(t) \tag{231}$$

The model includes time-varying damping $\gamma(t)$, an additional fractional damping term, and a stochastic force term $\epsilon\xi(t)$, where $\xi(t)$ represents a stochastic process.

86.7 Fractional Control Systems with Adaptive Nonlinear Feedback

$$\mathcal{C}_{\alpha,\beta}(x(t)) = \int_0^t (t-\tau)^{\alpha-1}\left[K_1x(\tau) + K_2\frac{dx(\tau)}{d\tau} + K_3\int_0^\tau x(s)ds\right]d\tau + \varphi(x(t)) \tag{232}$$

Here, $\varphi(x(t))$ represents an adaptive nonlinear feedback term, modifying the system's response based on the current state $x(t)$.

86.8 Fractional Option Pricing Models with Stochastic Volatility and Nonlinear Trends

$$dS_t = \mu S_t dt + \sigma(t,S_t)S_t dW_t + \lambda S_t^2 dt \tag{233}$$

In this model, $\lambda S_t^2 dt$ introduces a nonlinear trend component into the fractional option pricing model, alongside stochastic volatility $\sigma(t, S_t)$.

86.9 Fractional Economic Forecasting with Adaptive Trend and Seasonality

$$\frac{d^\alpha G(t)}{dt^\alpha} = \lambda(t)+\beta\int_0^t (t-\tau)^{\alpha-1}\left[G(\tau) - G(t) + \delta G(t)^2 + \varphi(t)\cos(\psi t)\right]d\tau \tag{234}$$

The inclusion of $\varphi(t)\cos(\psi t)$ adds adaptive trend and seasonality effects to the forecasting model, where $\varphi(t)$ and ψ are parameters controlling these effects.

86.10 Fractional Risk Assessment with Dynamic Covariance and Correlation

$$\text{Risk}_\alpha(P) = \int_0^T \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\text{Cov}(P_s,P_t) + \rho(t)\text{Cor}(P_s,P_t)\right)dt \tag{235}$$

The model incorporates a dynamic correlation term $\rho(t)\text{Cor}(P_s, P_t)$, reflecting changing correlations over time.

86.11 Fractional Differential Operators with Variable Orders and Nonlinear Terms

$$D_{t,x}^{\alpha(t),\beta(x)} f(t,x) = \frac{\partial^{\alpha(t),\beta(x)} f(t,x)}{\partial t^{\alpha(t)} \partial x^{\beta(x)}} + \varphi(t,x) f(t,x) \tag{236}$$

Here, $D_{t,x}^{\alpha(t),\beta(x)}$ is a fractional differential operator with variable orders, and $\varphi(t,x)$ represents a nonlinear modification.

86.12 Fractional Integral with Dynamic Kernel and Nonlinear Feedback

$$I_{\alpha,\beta}(f)(t,x) = \int_0^t \int_0^x (t-\tau)^{\alpha-1} (x-\xi)^{\beta-1} \varphi(\tau,\xi) f(\tau,\xi) d\xi d\tau + \psi(t,x) f(t,x) \tag{237}$$

The fractional integral includes a dynamic kernel $\varphi(\tau,\xi)$ and an additional nonlinear feedback term $\psi(t,x)f(t,x)$.

86.13 Fractional Order Nonlinear Dynamical Systems with Adaptive Controls

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \gamma(t) x(t)^p \tag{238}$$

The system incorporates an adaptive control term $\gamma(t)x(t)^p$, where $\gamma(t)$ is a time-dependent coefficient influencing the nonlinearity.

86.14 Fractional Fourier Transform with Adjustable Phase Shifts

$$\mathcal{T}_{\alpha,\beta}(F)(\xi) = \int_{-\infty}^{\infty} F(x) e^{-i(\xi x)^\alpha} \left[1 + \lambda e^{i\phi(x)}\right] dx \tag{239}$$

This transform includes an adjustable phase shift $\lambda e^{i\phi(x)}$, where $\phi(x)$ represents a phase function.

86.15 Fractional Heat Equation with Complex Boundary Conditions and Nonlinear Source Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x} \left[\kappa(x) \frac{\partial u(x,t)}{\partial x}\right] + \int_0^t (t-\tau)^{\alpha-1} \left[\eta(x) u(x,\tau)^p + \zeta(x,\tau)\right] d\tau \tag{240}$$

The model includes complex boundary conditions and nonlinear source terms, enhancing the description of heat dynamics.

86.16 Fractional Differential Equations with Multi-scale Analysis and Adaptive Nonlinearity

$$\frac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}}u(t) = \int_0^t (t-\tau)^{\alpha(t)-1} \left[\lambda(\tau)u(\tau) + \eta(t)u(t)^2 + \varphi(t,\tau)\right] d\tau \tag{241}$$

In this model, $\varphi(t,\tau)$ introduces adaptive nonlinearity across different scales, capturing more complex behaviors.

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87 Advanced Developments in Fractional Calculus and Nonlinear Dynamics

87.1 Fractional Differential Operators with Nonlinear Boundary Conditions

$$\mathcal{D}_{t,x}^{\alpha,\beta} f(t,x) = \frac{\partial^\alpha f(t,x)}{\partial t^\alpha} + \frac{\partial^\beta f(t,x)}{\partial x^\beta} + \psi(t,x)f(t,x) \tag{242}$$

In this formulation, $\mathcal{D}_{t,x}^{\alpha,\beta}$ is a fractional differential operator with orders α and β for t and x, respectively. The term $\psi(t,x)$ represents a nonlinear boundary condition function, introducing additional complexity to the differential operator.

87.2 Fractional Order Nonlinear Partial Differential Equations with Adaptive Dynamics

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \nabla \cdot (\kappa(x) \nabla u(t,x)) + \lambda(t) u(t,x) + \gamma(t) u(t,x)^2 + \delta(t) \sin(\theta x) \quad (243)$$

This equation integrates fractional order temporal differentiation with adaptive dynamics terms. The term $\lambda(t)u(t,x)$ represents a linear adaptive component, $\gamma(t)u(t,x)^2$ is a nonlinear term, and $\delta(t)\sin(\theta x)$ introduces an adaptive sinusoidal perturbation.

87.3 Fractional Stochastic Differential Equations with Nonlinear Feedback

$$dX_t = [\mu(t)X_t + \sigma(t, X_t)X_t]\, dt + \eta(t) X_t^\alpha dW_t \quad (244)$$

Here, $\eta(t)X_t^\alpha dW_t$ introduces fractional stochastic effects with feedback depending on X_t and α. $\mu(t)$ and $\sigma(t, X_t)$ represent drift and diffusion components, respectively.

87.4 Fractional Integral Equations with Dynamic Nonlinear Kernels

$$I_{\alpha,\beta}\left[f(t,x)\right] = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi) f(\tau,\xi) d\xi d\tau + \psi(t,x) \quad (245)$$

In this integral equation, $\phi(\tau, \xi)$ represents a dynamic nonlinear kernel, and $\psi(t,x)$ is an additional term capturing more complex interactions.

87.5 Fractional Heat Equation with Variable Conductivity and Nonlinear Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x) \frac{\partial u(x,t)}{\partial x}\right] + \beta(x) u(x,t)^p + \gamma(t) \quad (246)$$

The model includes variable conductivity $\kappa(x)$, a nonlinear term $\beta(x)u(x,t)^p$, and an additional time-dependent term $\gamma(t)$.

87.6 Fractional Control Systems with Dynamic Nonlinear Feedback

$$\mathcal{C}_{\alpha,\beta}(x(t)) = \int_0^t (t-\tau)^{\alpha-1}\left[K_1 x(\tau) + K_2 \frac{dx(\tau)}{d\tau} + K_3 \int_0^\tau x(s) ds\right] d\tau + \varphi(t, x(t)) \quad (247)$$

This control system model incorporates dynamic nonlinear feedback $\varphi(t, x(t))$ and fractional order integration terms.

87.7 Fractional Dynamical Systems with Nonlinear Adaptive Controls

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \gamma(t)x(t)^\beta + \eta(t)\cos(\phi t) \tag{248}$$

In this model, $\gamma(t)x(t)^\beta$ represents nonlinear adaptive control, and $\eta(t)\cos(\phi t)$ introduces additional periodic effects.

87.8 Fractional Order Nonlinear Optics with Complex Boundary Effects

$$\frac{\partial^\alpha E(t,x)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial E(t,x)}{\partial x}\right] + \lambda(t)E(t,x) + \mu E(t,x)^2 + \nu\sin(\omega x) \tag{249}$$

This equation models fractional order nonlinear optics, incorporating boundary effects with parameters $\lambda(t)$, μ, and ν.

87.9 Fractional Statistical Mechanics with Variable Interaction Terms

$$Z_\alpha(T,V) = \int_0^V \int_0^T \exp\left[-\frac{\beta(x)\phi(t)}{x^\alpha}\right][1 + \lambda(t)\cos(\delta x)]\,dxdt \tag{250}$$

The partition function $Z_\alpha(T,V)$ includes variable interaction terms $\beta(x)$ and $\lambda(t)$, and introduces fractional dynamics into statistical mechanics.

87.10 Fractional Order Nonlinear Systems with Adaptive Noise

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu x(t) + \sigma(t)x(t)^\gamma + \eta(t)\xi(t) \tag{251}$$

This system incorporates adaptive noise $\eta(t)\xi(t)$, with $\sigma(t)$ representing time-dependent variability and γ denoting the nonlinearity.

87.11 Fractional Quantum Mechanics with Time-Dependent Nonlinear Potentials

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^2\psi(t,x) + V(t,x)\psi(t,x) \tag{252}$$

where $V(t,x)$ is a time-dependent nonlinear potential, introducing additional complexity into fractional quantum mechanics.

87.12 Fractional Financial Models with Adaptive Risk Factors

$$dS_t = [\mu S_t + \sigma(t)S_t]\,dt + \eta(t)S_t^\alpha dW_t \tag{253}$$

In the financial model, $\sigma(t)S_t$ introduces adaptive risk factors, while $\eta(t)S_t^\alpha dW_t$ captures stochastic behavior.

87.13 Fractional Control Theory with Multi-Scale Dynamics

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + B\int_0^t (t-\tau)^{\alpha-1}\left[Cx(\tau) + D\frac{dx(\tau)}{d\tau}\right]d\tau \tag{254}$$

Here, A, B, C, and D are parameters controlling the multi-scale dynamics in the fractional control theory.

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88 Advanced Developments in Fractional Calculus and Nonlinear Dynamics (Continued)

88.1 Fractional Order Nonlinear Volterra Integral Equations

$$I_{\alpha,\beta}\left[f(t)\right] = \int_0^t (t-\tau)^{\alpha-1}\left[\int_0^\tau (t-\xi)^{\beta-1}\phi(\tau,\xi)f(\xi)d\xi\right]d\tau + \psi(t) \tag{255}$$

Here, $I_{\alpha,\beta}$ denotes a fractional integral operator, and $\phi(\tau,\xi)$ represents a nonlinear kernel. The function $\psi(t)$ adds additional complexity.

88.2 Fractional Nonlinear Partial Differential Equations with Adaptive Temporal Kernels

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \nabla \cdot [\kappa(x)\nabla u(t,x)] + \lambda(t)u(t,x) + \gamma(t)u(t,x)^2 + \delta(t)\cos(\theta x) \quad (256)$$

In this model, $\kappa(x)$ represents a spatially varying diffusion coefficient, and $\lambda(t)$, $\gamma(t)$, and $\delta(t)$ are time-dependent coefficients introducing nonlinear and adaptive components.

88.3 Fractional Stochastic Processes with Nonlinear Drift and Diffusion

$$dX_t = [\mu(t)X_t + \sigma(t, X_t)X_t]\,dt + \eta(t)X_t^\alpha dW_t \quad (257)$$

Here, $\mu(t)$ represents a time-dependent drift term, $\sigma(t, X_t)$ is a nonlinear diffusion coefficient, and $\eta(t)X_t^\alpha dW_t$ introduces stochastic noise with fractional order.

88.4 Fractional Integral Equations with Dynamic Nonlinear Kernels and Feedback

$$I_{\alpha,\beta}\left[f(t,x)\right] = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi)f(\tau,\xi)d\xi d\tau + \psi(t,x) + \rho(t)\cos(\theta x) \quad (258)$$

This equation introduces a dynamic nonlinear kernel $\phi(\tau, \xi)$ and a feedback term $\rho(t)\cos(\theta x)$, extending the standard fractional integral equation.

88.5 Fractional Heat Equations with Variable Conductivity and Nonlinear Boundary Conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \beta(x)u(x,t)^p + \gamma(t) + \delta(t)\sin(\lambda x) \quad (259)$$

Here, $\kappa(x)$ represents spatially varying conductivity, $\beta(x)u(x,t)^p$ introduces nonlinear effects, $\gamma(t)$ and $\delta(t)\sin(\lambda x)$ represent additional boundary and time-dependent terms.

88.6 Fractional Control Systems with Multi-Scale Dynamics and Adaptive Feedback

$$\mathcal{C}_{\alpha,\beta}(x(t)) = \int_0^t (t-\tau)^{\alpha-1}\left[K_1 x(\tau) + K_2\frac{dx(\tau)}{d\tau} + K_3\int_0^\tau x(s)ds + \psi(\tau, x(\tau))\right]d\tau \quad (260)$$

This control system incorporates multi-scale dynamics and an adaptive feedback term $\psi(\tau, x(\tau))$.

88.7 Fractional Dynamical Systems with Nonlinear Adaptive Controls and Stochastic Perturbations

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)) + \gamma(t)x(t)^\beta + \eta(t)\cos(\phi t) + \zeta(t)\frac{dx(t)}{dt} \tag{261}$$

In this model, $\gamma(t)x(t)^\beta$ introduces adaptive nonlinear controls, $\eta(t)\cos(\phi t)$ adds periodic effects, and $\zeta(t)\frac{dx(t)}{dt}$ represents stochastic perturbations.

88.8 Fractional Order Nonlinear Optics with Complex Boundary and Adaptive Effects

$$\frac{\partial^\alpha E(t,x)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial E(t,x)}{\partial x}\right] + \lambda(t)E(t,x) + \mu E(t,x)^2 + \nu\sin(\omega x) + \xi(t)E(t,x) \tag{262}$$

This equation models fractional order nonlinear optics with additional boundary effects and adaptive terms.

88.9 Fractional Statistical Mechanics with Adaptive Interaction Terms and Time-Dependent Potential

$$Z_\alpha(T,V) = \int_0^V \int_0^T \exp\left[-\frac{\beta(x)\phi(t)}{x^\alpha}\right][1 + \lambda(t)\cos(\delta x)]\, dxdt \tag{263}$$

The partition function $Z_\alpha(T,V)$ includes adaptive interaction terms and a time-dependent potential $\phi(t)$.

88.10 Fractional Order Nonlinear Systems with Adaptive Noise and Nonlinear Drift

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu x(t) + \sigma(t)x(t)^\gamma + \eta(t)\xi(t) + \theta(t)x(t) \tag{264}$$

This system features adaptive noise $\eta(t)\xi(t)$, nonlinear drift $\sigma(t)x(t)^\gamma$, and an additional term $\theta(t)x(t)$.

88.11 Fractional Quantum Mechanics with Time-Dependent Nonlinear Boundary Conditions

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^2\psi(t,x) + V(t,x)\psi(t,x) + \lambda(t)\psi(t,x)^2 \tag{265}$$

In this fractional quantum mechanics model, $V(t,x)$ represents a time-dependent nonlinear potential and $\lambda(t)\psi(t,x)^2$ introduces nonlinear boundary effects.

88.12 Fractional Financial Models with Adaptive Risk Factors and Nonlinear Pricing

$$dS_t = [\mu S_t + \sigma(t)S_t]\,dt + \eta(t)S_t^{\alpha}dW_t + \phi(t)S_t \tag{266}$$

In this model, $\sigma(t)S_t$ and $\phi(t)S_t$ account for adaptive risk factors and nonlinear pricing effects.

88.13 Fractional Control Theory with Nonlinear Feedback and Multi-Scale Dynamics

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + B\int_0^t (t-\tau)^{\alpha-1}\left[Cx(\tau) + D\frac{dx(\tau)}{d\tau} + \psi(\tau, x(\tau))\right]d\tau \tag{267}$$

This model incorporates nonlinear feedback $\psi(\tau, x(\tau))$ and multi-scale dynamics with parameters A, B, C, and D.

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89 Expanded Developments in Fractional Calculus and Nonlinear Dynamics

89.1 Fractional Nonlinear Diffusion Equation with Adaptive Feedback

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \lambda(x)u(x,t) + \gamma(t)u(x,t)^2 + \delta(t)\sin(\beta x) \tag{268}$$

Notation: Here, $\kappa(x)$ is the spatially dependent diffusion coefficient, $\lambda(x)$ is a nonlinear feedback term, $\gamma(t)$ and $\delta(t)$ are time-dependent coefficients, and β represents a spatial frequency component. This equation models diffusion with spatially varying properties and nonlinear effects.

89.2 Fractional Nonlinear Delay Differential Equation with Dynamic Parameters

$$\frac{d^\alpha x(t)}{dt^\alpha} = a(t)x(t) + b(t)x(t-\tau) + c(t)x(t)^2 + d(t)\sin(\epsilon t) \tag{269}$$

Notation: $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are time-dependent parameters, with $x(t-\tau)$ introducing delay effects and $\sin(\epsilon t)$ a time-varying periodic function. This model explores dynamics with delays and adaptive parameters.

89.3 Fractional Stochastic Differential Equation with Nonlinear Drift and Fractional Noise

$$dX_t = \left[\mu(t)X_t + \sigma(t)X_t^\beta\right] dt + \eta(t)X_t^\alpha dW_t \tag{270}$$

Notation: $\mu(t)$ and $\sigma(t)$ are drift and diffusion coefficients, α and β are fractional exponents, and dW_t represents a Wiener process. This equation incorporates nonlinear drift and fractional noise into a stochastic framework.

89.4 Fractional Integral Equations with Multi-Scale Kernels

$$I_{\alpha,\beta}\left[f(t,x)\right] = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi)f(\tau,\xi)d\xi d\tau + \psi(t,x) \tag{271}$$

Notation: $I_{\alpha,\beta}$ denotes a fractional integral operator with a multi-scale kernel $\phi(\tau,\xi)$. The term $\psi(t,x)$ introduces additional complexity.

89.5 Fractional Control Systems with Adaptive Feedback and Nonlinear Dynamics

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + B(t)\int_0^t (t-\tau)^{\alpha-1}\left[Cx(\tau) + D\frac{dx(\tau)}{d\tau} + \psi(\tau, x(\tau))\right] d\tau \tag{272}$$

Notation: $A(t)$ and $B(t)$ are time-dependent control coefficients, with C, D, and $\psi(\tau, x(\tau))$ introducing feedback effects. This model addresses fractional control with adaptive and nonlinear components.

89.6 Fractional Quantum Mechanics with Nonlinear Perturbations

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^2\psi(t,x) + V(t,x)\psi(t,x) + \lambda(t)\psi(t,x)^2 + \xi(t)\psi(t,x) \tag{273}$$

Notation: $V(t,x)$ represents a time-dependent potential, $\lambda(t)$ adds nonlinear perturbations, and $\xi(t)$ introduces additional time-dependent effects.

89.7 Fractional Statistical Mechanics with Adaptive Interactions

$$Z_\alpha(T,V) = \int_0^V \int_0^T \exp\left[-\frac{\beta(x)\phi(t)}{x^\alpha}\right][1+\lambda(t)\sin(\delta x)]\,dxdt \tag{274}$$

Notation: $Z_\alpha(T,V)$ is the partition function with $\beta(x)$ and $\phi(t)$ representing interaction terms, and $\lambda(t)\sin(\delta x)$ introduces time-dependent effects.

89.8 Fractional Order Nonlinear Dynamics with Periodic and Adaptive Terms

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu(t)x(t) + \sigma(t)x(t)^\gamma + \eta(t)\cos(\phi t) + \theta(t)x(t) \tag{275}$$

Notation: $\mu(t)$, $\sigma(t)$, and $\eta(t)$ are coefficients, with $x(t)^\gamma$ introducing nonlinear effects and $\cos(\phi t)$ a periodic term. $\theta(t)x(t)$ represents additional adaptive dynamics.

89.9 Fractional Heat Equation with Adaptive Source Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \lambda(t)u(x,t) + \gamma(x)u(x,t)^2 + \delta(t)\cos(\eta x) \tag{276}$$

Notation: $\kappa(x)$ is a spatially varying conductivity term, $\lambda(t)$ is a time-dependent source term, $\gamma(x)$ introduces nonlinearity, and $\delta(t)\cos(\eta x)$ accounts for additional periodic effects.

89.10 Fractional Dynamical Systems with Multi-Scale Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = \int_0^t (t-\tau)^{\alpha-1}\left[Ax(\tau) + B\frac{dx(\tau)}{d\tau} + C\int_0^\tau x(s)ds + \psi(\tau, x(\tau))\right]d\tau \tag{277}$$

Notation: A, B, and C are coefficients with $\psi(\tau, x(\tau))$ representing adaptive feedback. This system models multi-scale dynamics with various feedback components.

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90 Continued Expansion in Advanced Fractional Calculus and Nonlinear Dynamics

90.1 Fractional Heat Conduction with Nonlinear Source Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \lambda(t)u(x,t)^2 + \gamma(x)\frac{\partial u(x,t)}{\partial x} + \delta(t)\cos(\eta x) \tag{278}$$

Notation:

- $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo fractional time derivative of order α.
- $\kappa(x)$ is the spatially dependent thermal conductivity.
- $\lambda(t)$ is a time-dependent coefficient for the nonlinear source term.
- $\gamma(x)$ represents a spatially varying gradient effect.
- $\delta(t)$ and η introduce time-dependent and spatially varying periodic effects.

90.2 Fractional Nonlinear Delay Differential Equation with Nonlocal Interaction

$$\frac{d^\alpha x(t)}{dt^\alpha} = a(t)x(t) + b(t)x(t-\tau) + \int_{t-\tau}^{t} k(t,s)x(s)ds + \lambda(t)x(t)^2 \tag{279}$$

Notation:

- $a(t)$ and $b(t)$ are time-dependent coefficients.
- $x(t-\tau)$ introduces delay effects.
- $k(t,s)$ is a kernel function describing nonlocal interactions.
- $\lambda(t)$ is a time-dependent coefficient for the nonlinear term.

90.3 Fractional Stochastic Dynamics with Adaptive Drift and Diffusion

$$dX_t = \left[\mu(t)X_t + \sigma(t)X_t^\beta\right]dt + \eta(t)X_t^\alpha dW_t \tag{280}$$

Notation:

- $\mu(t)$ and $\sigma(t)$ are time-dependent drift and diffusion coefficients.
- X_t^β and X_t^α introduce nonlinear effects in drift and diffusion.
- dW_t represents the increment of a Wiener process.

90.4 Fractional Integral Equations with Time-Dependent Kernels

$$I_{\alpha,\beta}[f(t,x)] = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi)f(\tau,\xi)d\xi d\tau + \psi(t,x) \quad (281)$$

Notation:

- $I_{\alpha,\beta}$ is the fractional integral operator with order (α, β).
- $\phi(\tau, \xi)$ is a time-dependent kernel function.
- $\psi(t, x)$ represents additional terms or perturbations.

90.5 Fractional Control Systems with Dynamic Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + B(t)\int_0^t (t-\tau)^{\alpha-1}\left[Cx(\tau) + D\frac{dx(\tau)}{d\tau} + \psi(\tau, x(\tau))\right] d\tau \quad (282)$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent control coefficients.
- C and D are constants representing feedback effects.
- $\psi(\tau, x(\tau))$ is an additional feedback term dependent on both time and state.

90.6 Fractional Quantum Mechanics with Adaptive Potentials

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^2\psi(t,x) + V(t,x)\psi(t,x) + \lambda(t)\psi(t,x)^2 + \xi(t)\psi(t,x) \quad (283)$$

Notation:

- ∇^2 represents the Laplacian operator.
- $V(t, x)$ is a time-dependent potential function.
- $\lambda(t)$ introduces nonlinear perturbations.
- $\xi(t)$ adds additional time-dependent effects.

90.7 Fractional Statistical Mechanics with Multi-Scale Interaction Terms

$$Z_\alpha(T,V) = \int_0^V \int_0^T \exp\left[-\frac{\beta(x)\phi(t)}{x^\alpha}\right] [1 + \lambda(t)\sin(\delta x)]\, dxdt \tag{284}$$

Notation:

- $Z_\alpha(T,V)$ is the partition function.
- $\beta(x)$ and $\phi(t)$ represent interaction terms.
- $\lambda(t)\sin(\delta x)$ introduces additional time-dependent and spatially varying effects.

90.8 Fractional Order Nonlinear Dynamics with Time-Dependent Nonlinear Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu(t)x(t) + \sigma(t)x(t)^\gamma + \eta(t)\cos(\phi t) + \theta(t)x(t) + \rho(t)\exp(\zeta x) \tag{285}$$

Notation:

- $\mu(t)$, $\sigma(t)$, and $\eta(t)$ are coefficients.
- $x(t)^\gamma$ introduces nonlinear feedback.
- $\cos(\phi t)$ represents periodic effects.
- $\theta(t)x(t)$ is additional time-dependent dynamics.
- $\rho(t)$ and $\exp(\zeta x)$ introduce exponential growth effects.

90.9 Fractional Heat Equation with Complex Source Terms and Boundary Conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \lambda(t)u(x,t) + \gamma(x)u(x,t)^2 + \delta(t)\sin(\eta x) + \alpha(t)u(x,t)^\beta \tag{286}$$

Notation:

- $\kappa(x)$ is a spatially varying thermal conductivity.
- $\lambda(t)$, $\gamma(x)$, and $\delta(t)$ are coefficients.
- $\sin(\eta x)$ introduces periodic spatial effects.
- $\alpha(t)$ and $u(x,t)^\beta$ represent additional nonlinear source terms.

90.10 Fractional Dynamical Systems with Complex Feedback and Delay Effects

$$\frac{d^\alpha x(t)}{dt^\alpha} = \int_0^t (t-\tau)^{\alpha-1} \left[Ax(\tau) + B\frac{dx(\tau)}{d\tau} + C\int_0^\tau x(s)ds + \psi(\tau, x(\tau)) + \lambda(\tau)x(\tau - \tau_0) \right] d\tau \tag{287}$$

Notation:

- A, B, and C are coefficients.
- $\psi(\tau, x(\tau))$ is an additional feedback term.
- $\lambda(\tau)$ introduces delay effects with τ_0 as the delay parameter.

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91 Further Developments in Fractional Calculus and Nonlinear Dynamics

91.1 Fractional Schrödinger Equation with Adaptive Potentials and Nonlinear Feedback

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^\alpha\psi(t,x) + V(t,x)\psi(t,x) + \lambda(t)\psi(t,x)^2 + \xi(t)\psi(t,x) + \zeta(t)\frac{\partial\psi(t,x)}{\partial x} \tag{288}$$

Notation:

- ∇^α represents the fractional Laplacian operator of order α.
- $V(t,x)$ denotes a time-dependent potential function.
- $\lambda(t)$ represents a time-dependent nonlinear coefficient.
- $\xi(t)$ introduces additional time-dependent perturbations.
- $\zeta(t)$ accounts for a time-dependent gradient effect.

91.2 Fractional Delay Differential Equation with Multiple Time Scales

$$\frac{d^\alpha x(t)}{dt^\alpha} = a_1(t)x(t)+a_2(t)x(t-\tau_1)+a_3(t)x(t-\tau_2)+\int_{t-\tau_2}^{t} k(t,s)x(s)ds+\lambda(t)x(t)^2 \tag{289}$$

Notation:

- $a_1(t)$, $a_2(t)$, and $a_3(t)$ are time-dependent coefficients.
- $x(t-\tau_1)$ and $x(t-\tau_2)$ introduce multiple delay effects.
- $k(t,s)$ is a kernel function describing nonlocal interactions.
- $\lambda(t)$ is a time-dependent coefficient for the nonlinear term.

91.3 Fractional Order Stochastic Differential Equation with Adaptive Drift

$$dX_t = \left[\mu(t)X_t + \sigma(t)X_t^\beta\right] dt + \eta(t)X_t^\alpha dW_t + \xi(t)\sin(\phi t)dt \tag{290}$$

Notation:

- $\mu(t)$ and $\sigma(t)$ are time-dependent drift and diffusion coefficients.
- X_t^β and X_t^α introduce nonlinear drift and diffusion effects.
- dW_t represents the Wiener process increment.
- $\xi(t)$ and $\sin(\phi t)$ represent additional periodic effects.

91.4 Fractional Integral Equations with Variable Order Kernels

$$I_{\alpha,\beta}[f(t,x)] = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(x-\xi)^{\beta-1}\phi(\tau,\xi)f(\tau,\xi)d\xi d\tau + \psi(t,x) \tag{291}$$

Notation:

- $I_{\alpha,\beta}$ is the fractional integral operator with orders (α,β).
- $\phi(\tau,\xi)$ is a kernel function describing interactions.
- $\psi(t,x)$ is an additional term or perturbation.

91.5 Fractional Control Systems with Time-Dependent Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t)+B(t)\int_0^t (t-\tau)^{\alpha-1}\left[Cx(\tau)+D\frac{dx(\tau)}{d\tau}+\psi(\tau,x(\tau))\right]d\tau+\lambda(t)x(t) \tag{292}$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent coefficients.
- C and D represent feedback coefficients.
- $\psi(\tau, x(\tau))$ is an additional feedback term.
- $\lambda(t)$ introduces a time-dependent control effect.

91.6 Fractional Quantum Mechanics with Multi-Scale Interaction

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\nabla^\alpha\psi(t,x)+V(t,x)\psi(t,x)+\lambda(t)\psi(t,x)^2+\xi(t)\psi(t,x)+\zeta(t)\frac{\partial\psi(t,x)}{\partial x}+\eta(t)\cos(\phi x) \tag{293}$$

Notation:

- ∇^α denotes the fractional Laplacian operator.
- $V(t,x)$ is a time-dependent potential function.
- $\lambda(t)$ and $\xi(t)$ represent nonlinear and additional time-dependent effects.
- $\zeta(t)$ introduces gradient effects.
- $\eta(t)\cos(\phi x)$ accounts for periodic spatial variations.

91.7 Fractional Statistical Mechanics with Adaptive Interactions

$$Z_\alpha(T,V) = \int_0^V \int_0^T \exp\left[-\frac{\beta(x)\phi(t)}{x^\alpha}\right][1+\lambda(t)\sin(\delta x)]\,dxdt+\mu(T)\exp(-\gamma V) \tag{294}$$

Notation:

- $Z_\alpha(T,V)$ is the partition function with fractional order α.
- $\beta(x)$ and $\phi(t)$ describe interaction terms.
- $\lambda(t)$ introduces additional time-dependent effects.
- $\mu(T)$ and $\exp(-\gamma V)$ are additional terms representing exponential decay effects.

91.8 Fractional Nonlinear Dynamics with Complex Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu(t)x(t) + \sigma(t)x(t)^\gamma + \eta(t)\cos(\phi t) + \theta(t)x(t) + \rho(t)\exp(\zeta x) + \lambda(t)\frac{dx(t)}{dt} \tag{295}$$

Notation:

- $\mu(t)$, $\sigma(t)$, and $\eta(t)$ are coefficients.
- $x(t)^\gamma$ introduces nonlinear effects.
- $\cos(\phi t)$ and $\exp(\zeta x)$ represent periodic and exponential terms.
- $\theta(t)$ and $\lambda(t)$ account for additional time-dependent dynamics.

91.9 Fractional Heat Equation with Complex Boundary Conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x,t)}{\partial x}\right] + \lambda(t)u(x,t) + \gamma(x)u(x,t)^2 + \delta(t)\sin(\eta x) + \alpha(t)\exp(\beta x) \tag{296}$$

Notation:

- $\kappa(x)$ is the spatially varying thermal conductivity.
- $\lambda(t)$ and $\gamma(x)$ are coefficients.
- $\sin(\eta x)$ introduces periodic spatial effects.
- $\alpha(t)$ and $\exp(\beta x)$ represent additional boundary terms.

91.10 Fractional Dynamical Systems with Complex Feedback and Delays

$$\frac{d^\alpha x(t)}{dt^\alpha} = \int_0^t (t-\tau)^{\alpha-1}\left[Ax(\tau) + B\frac{dx(\tau)}{d\tau} + C\int_0^\tau x(s)ds + \psi(\tau, x(\tau)) + \lambda(\tau)x(\tau-\tau_0) + \mu(\tau)\cos(\phi\tau)\right]d\tau \tag{297}$$

Notation:

- A, B, and C are coefficients.
- $\psi(\tau, x(\tau))$ is an additional feedback term.
- $\lambda(\tau)$ introduces delays.
- $\mu(\tau)$ and $\cos(\phi\tau)$ represent periodic effects.

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92 Advanced Topics in Fractional Dynamics and Complex Systems

92.1 Fractional Order Reaction-Diffusion Systems with Nonlinear Feedback

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\nabla^\beta u(x,t)+\alpha u(x,t)+\beta u(x,t)^2+\gamma\int_0^x \exp(-\delta(x-\xi))u(\xi,t)d\xi+\lambda(t)\sin(\mu x) \tag{298}$$

Notation:

- D is the diffusion coefficient.
- ∇^β is the fractional Laplacian of order β.
- α and β represent linear and nonlinear feedback coefficients.
- $\exp(-\delta(x-\xi))$ is a decaying kernel function for spatial interaction.
- $\lambda(t)$ and $\sin(\mu x)$ add periodic spatial effects.

92.2 Fractional Stochastic Partial Differential Equations with Adaptive Kernels

$$dU(x,t) = \left[\mu(t)U(x,t) + \sigma(t)\nabla^\gamma U(x,t) + \int_0^x k(t,\xi)U(\xi,t)d\xi\right] dt + \eta(t)U(x,t)dW_t \tag{299}$$

Notation:

- $\mu(t)$ and $\sigma(t)$ are time-dependent coefficients.
- ∇^γ denotes the fractional Laplacian of order γ.
- $k(t,\xi)$ is a time-dependent kernel function.
- $\eta(t)$ is a time-dependent diffusion coefficient.
- dW_t represents the increment of a Wiener process.

92.3 Fractional Quantum Field Theory with Nonlinear Interactions

$$i\frac{\partial\phi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha\phi(x,t) + V(x,t)\phi(x,t) + \lambda(t)\phi(x,t)^3 + \xi(x,t)\phi(x,t) \tag{300}$$

Notation:

- ∇^α is the fractional Laplacian of order α.
- $V(x,t)$ represents a time-dependent potential.
- $\lambda(t)$ introduces a nonlinear interaction term.
- $\xi(x,t)$ accounts for additional perturbations.

92.4 Fractional Order Optimal Control with Nonlinear Dynamics

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + B(t)\left[\int_0^t \exp(-\lambda(t-\tau))x(\tau)d\tau\right] + \eta(t)x(t)^2 \tag{301}$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent coefficients.
- $\exp(-\lambda(t-\tau))$ describes a decaying memory effect.
- $\eta(t)$ introduces a time-dependent nonlinear control term.

92.5 Fractional Chaos Theory with Adaptive Interactions

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu(t)x(t) + \sigma(t)\int_0^t \phi(t-\tau)x(\tau)d\tau + \lambda(t)\sin(\eta t) + \gamma(t)x(t)^3 \quad (302)$$

Notation:

- $\mu(t)$ and $\sigma(t)$ are time-dependent coefficients.
- $\phi(t-\tau)$ is a memory kernel describing temporal interactions.
- $\lambda(t)$ introduces periodic forcing.
- $\gamma(t)$ represents a nonlinear feedback term.

92.6 Fractional Thermodynamics with Variable Interaction Coefficients

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \kappa(x)\nabla^\beta T(x,t) + \lambda(x,t)\frac{\partial T(x,t)}{\partial x} + \xi(x)T(x,t) + \mu(t)\cos(\nu x) \quad (303)$$

Notation:

- $\kappa(x)$ is the spatially varying thermal conductivity.
- ∇^β denotes the fractional Laplacian of order β.
- $\lambda(x,t)$ is a time-dependent gradient coefficient.
- $\xi(x)$ introduces a spatially varying heat source term.
- $\mu(t)$ and $\cos(\nu x)$ account for periodic temperature variations.

92.7 Fractional Electro-Magnetic Dynamics with Adaptive Potentials

$$\frac{\partial^\alpha \mathbf{E}(x,t)}{\partial t^\alpha} = \nabla\cdot[\sigma(x,t)\nabla\mathbf{E}(x,t)] + \phi(x,t)\mathbf{E}(x,t) + \lambda(t)\exp(-\mu x) + \xi(t)\cos(\phi t) \quad (304)$$

Notation:

- ∇ denotes the gradient operator.
- $\sigma(x,t)$ is a time-dependent conductivity function.
- $\phi(x,t)$ is a time-dependent potential function.
- $\lambda(t)$ introduces an exponential term for spatial decay.
- $\xi(t)$ and $\cos(\phi t)$ represent additional periodic effects.

92.8 Fractional Quantum Optics with Complex Field Interactions

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha\psi(x,t)+V(x)\psi(x,t)+\lambda(t)\psi(x,t)^2+\eta(x)\frac{\partial\psi(x,t)}{\partial x}+\gamma(t)\exp(-\delta x) \tag{305}$$

Notation:

- ∇^α is the fractional Laplacian operator of order α.
- $V(x)$ represents the potential function.
- $\lambda(t)$ introduces a nonlinear interaction term.
- $\eta(x)$ describes a spatial gradient effect.
- $\gamma(t)$ and $\exp(-\delta x)$ represent additional spatial decay effects.

92.9 Fractional Geometric Dynamics with Adaptive Curvature

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t) + \mu(x)\mathbf{R}(x,t) + \sigma(t)\int_0^x \exp(-\lambda(x-\xi))\mathbf{R}(\xi,t)d\xi \tag{306}$$

Notation:

- ∇^β is the fractional Laplacian of order β.
- $\mathbf{R}(x,t)$ represents a geometric field.
- $\mu(x)$ and $\sigma(t)$ are coefficients for curvature and interaction.
- $\exp(-\lambda(x-\xi))$ describes a decaying kernel function for spatial interactions.

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93 Further Developments in Fractional Dynamics and Complex Systems

93.1 Fractional Order Hyperbolic Dynamics with Variable Damping

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) - \lambda(t)\frac{\partial^2 u(x,t)}{\partial x^2} + \sigma(t)\frac{\partial u(x,t)}{\partial t} + \gamma(t)\int_0^x \phi(t-\xi)u(\xi,t)d\xi \tag{307}$$

Notation:

- ∇^β denotes the fractional Laplacian of order β.
- $\lambda(t)$ is a time-dependent damping coefficient.
- $\sigma(t)$ represents a time-dependent dissipation term.
- $\gamma(t)$ is a time-dependent interaction coefficient.
- $\phi(t-\xi)$ is a kernel function describing temporal interaction effects.

93.2 Fractional Order Nonlinear Schrödinger Equation with Adaptive Potentials

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha \psi(x,t) + V(x,t)\psi(x,t) + \lambda(t)\psi(x,t)^2 + \mu(t)\exp(-\nu x) \tag{308}$$

Notation:

- ∇^α is the fractional Laplacian of order α.
- $V(x,t)$ represents a time-dependent potential.
- $\lambda(t)$ introduces a nonlinear interaction term.
- $\mu(t)$ and $\exp(-\nu x)$ represent additional spatial effects.

93.3 Fractional Optimal Control Systems with Nonlinear Dynamics

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + B(t)\left[\int_0^t \phi(t-\tau)x(\tau)d\tau\right] + \gamma(t)x(t)^2 + \delta(t)\sin(\eta t) \tag{309}$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent coefficients.
- $\phi(t-\tau)$ describes a memory kernel for interaction.
- $\gamma(t)$ introduces a nonlinear feedback term.
- $\delta(t)$ represents a periodic forcing function.
- $\sin(\eta t)$ accounts for periodic effects.

93.4 Fractional Chaotic Systems with Adaptive Couplings

$$\frac{d^\alpha x(t)}{dt^\alpha} = \mu(t)x(t) + \sigma(t)\int_0^t \exp(-\lambda(t-\tau))x(\tau)d\tau + \theta(t)\cos(\phi t) + \xi(t)x(t)^3 \tag{310}$$

Notation:

- $\mu(t)$ and $\sigma(t)$ are time-dependent coefficients.
- $\exp(-\lambda(t-\tau))$ is a kernel function for memory effects.
- $\theta(t)$ introduces periodic components.
- $\xi(t)$ represents a nonlinear feedback term.

93.5 Fractional Thermodynamics with Anisotropic Diffusion

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \kappa(x,t)\nabla^\beta T(x,t)+\lambda(x,t)\frac{\partial T(x,t)}{\partial x}+\xi(x)T(x,t)+\mu(t)\cos(\nu x)+\delta(x)\int_0^x \exp(-\eta(x-\xi))T(\xi,t)d\xi \tag{311}$$

Notation:

- $\kappa(x,t)$ is an anisotropic diffusion coefficient.
- ∇^β denotes the fractional Laplacian of order β.
- $\lambda(x,t)$ is a gradient coefficient.
- $\xi(x)$ represents a spatially varying heat source.
- $\mu(t)$ and $\cos(\nu x)$ account for periodic temperature variations.
- $\delta(x)$ introduces an additional spatial decay effect.
- $\exp(-\eta(x-\xi))$ describes a kernel function for spatial interaction.

93.6 Fractional Quantum Optics with Nonlinear Couplings

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha\psi(x,t)+V(x,t)\psi(x,t)+\lambda(t)\psi(x,t)^2+\mu(x,t)\exp(-\nu x)+\xi(t)\frac{\partial\psi(x,t)}{\partial x} \tag{312}$$

Notation:

- ∇^α is the fractional Laplacian operator of order α.
- $V(x,t)$ represents a time-dependent potential.
- $\lambda(t)$ introduces a nonlinear interaction term.
- $\mu(x,t)$ and $\exp(-\nu x)$ account for spatial effects.
- $\xi(t)$ is a term describing spatial gradients.

93.7 Fractional Geometric Dynamics with Complex Curvatures

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t) + \mu(x,t)\mathbf{R}(x,t) + \sigma(t)\int_0^x \exp(-\lambda(x-\xi))\mathbf{R}(\xi,t)d\xi + \theta(x,t)\cos(\phi t) \tag{313}$$

Notation:

- ∇^β is the fractional Laplacian of order β.
- $\mathbf{R}(x,t)$ represents a geometric field.
- $\mu(x,t)$ is a time-dependent curvature coefficient.
- $\sigma(t)$ describes a temporal interaction term.
- $\theta(x,t)$ introduces additional periodic effects.
- $\exp(-\lambda(x-\xi))$ is a spatial kernel function.

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94 Indefinite Expansion of Fractional Dynamics and Complex Systems

94.1 Extended Fractional Nonlinear Schrödinger Equation with Hybrid Potentials

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha \psi(x,t) + (V_1(x,t) + V_2(x,t))\,\psi(x,t) + \lambda(t)\psi(x,t)^2 + \mu(t)\psi(x,t)^3 + \xi(t)\sin(\eta x) \tag{314}$$

Notation:

- ∇^α denotes the fractional Laplacian of order α.

- $V_1(x,t)$ and $V_2(x,t)$ are two different time-dependent potentials.
- $\lambda(t)$ introduces a quadratic nonlinearity.
- $\mu(t)$ introduces a cubic nonlinearity.
- $\xi(t)$ represents a periodic spatial term.
- $\sin(\eta x)$ accounts for additional spatial variation.

94.2 Fractional Stochastic Differential Equations with Adaptive Parameters

$$d^\alpha X(t) = \mu(t)X(t)\,dt + \sigma(t)\nabla^\beta X(t)\,dW(t) + \lambda(t)\int_0^t \exp(-\gamma(t-\tau))X(\tau)d\tau \tag{315}$$

Notation:

- $d^\alpha X(t)$ denotes the fractional differential operator of order α.
- $\mu(t)$ and $\sigma(t)$ are time-dependent drift and diffusion coefficients.
- ∇^β denotes the fractional Laplacian of order β.
- $dW(t)$ represents a differential Wiener process.
- γ is a decay parameter in the kernel function.

94.3 Fractional Optimal Control with Time-Varying Constraints

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t)+B(t)\left[\int_0^t \phi(t-\tau)x(\tau)d\tau\right]+C(t)\frac{dx(t)}{dt}+\lambda(t)x(t)^2+\eta(t)\exp(-\xi t) \tag{316}$$

Notation:

- $A(t)$, $B(t)$, and $C(t)$ are time-dependent matrices.
- $\phi(t-\tau)$ is a kernel function describing memory effects.
- $\lambda(t)$ introduces nonlinear control terms.
- $\eta(t)$ represents an exponential decay term.
- $\exp(-\xi t)$ describes additional time-dependent effects.

94.4 Fractional Multi-Scale Systems with Cross-Interactions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \int_0^x \Phi(x-\xi)u(\xi,t)d\xi + \lambda(x)u(x,t)^2 + \sigma(t)\frac{\partial u(x,t)}{\partial x} + \theta(x,t)\cos(\phi t) \tag{317}$$

Notation:

- ∇^β denotes the fractional Laplacian.
- $\Phi(x-\xi)$ is a cross-interaction kernel function.
- $\lambda(x)$ introduces a spatially varying nonlinear term.
- $\sigma(t)$ is a time-dependent gradient term.
- $\theta(x,t)$ represents additional periodic effects.
- $\cos(\phi t)$ accounts for time-dependent periodic variations.

94.5 Fractional Wave Equation with Anisotropic Nonlinearities

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \lambda(x)\frac{\partial^2 u(x,t)}{\partial x^2} + \mu(t)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 + \sigma(x)\exp(-\gamma t) \tag{318}$$

Notation:

- ∇^β is the fractional Laplacian operator.
- $\lambda(x)$ represents an anisotropic diffusion term.
- $\mu(t)$ introduces a time-dependent nonlinear term.
- $\sigma(x)$ is a spatially varying coefficient.
- $\exp(-\gamma t)$ represents exponential decay effects.

94.6 Fractional Thermodynamic Systems with Nonlinear Boundary Conditions

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \kappa(x)\nabla^\beta T(x,t) + \lambda(x)\frac{\partial T(x,t)}{\partial x} + \mu(t)T(x,t)^2 + \xi(x)\cos(\eta t) + \delta(x)\int_0^x \exp(-\theta(x-\xi))T(\xi,t)d\xi \tag{319}$$

Notation:

- $\kappa(x)$ is a spatially varying diffusion coefficient.
- ∇^β is the fractional Laplacian of order β.
- $\lambda(x)$ denotes a boundary condition coefficient.
- $\mu(t)$ introduces a quadratic temperature term.

- $\xi(x)$ and $\cos(\eta t)$ account for periodic effects.
- $\delta(x)$ represents additional spatial interaction.
- $\exp(-\theta(x-\xi))$ describes a spatial kernel function.

94.7 Fractional Quantum Systems with Hybrid Nonlinearities

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha\psi(x,t)+(V_1(x)+V_2(x,t))\,\psi(x,t)+\lambda(t)\psi(x,t)^2+\mu(t)\psi(x,t)^3+\xi(x)\exp(-\gamma t) \tag{320}$$

Notation:

- ∇^α is the fractional Laplacian operator.
- $V_1(x)$ and $V_2(x,t)$ are time-dependent and spatially varying potentials.
- $\lambda(t)$ introduces a quadratic nonlinearity.
- $\mu(t)$ introduces a cubic nonlinearity.
- $\xi(x)$ accounts for spatial variations.
- $\exp(-\gamma t)$ represents temporal decay effects.

94.8 Fractional Geometric Dynamics with Complex Interactions

$$\frac{\partial^\alpha\mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta\mathbf{R}(x,t)+\mu(x,t)\mathbf{R}(x,t)+\sigma(t)\int_0^x \exp(-\lambda(x-\xi))\mathbf{R}(\xi,t)d\xi+\theta(x)\sin(\phi t) \tag{321}$$

Notation:

- ∇^β denotes the fractional Laplacian operator.
- $\mathbf{R}(x,t)$ represents a geometric field with complex interactions.
- $\mu(x,t)$ is a time-dependent curvature term.
- $\sigma(t)$ introduces additional interaction effects.
- $\theta(x)$ and $\sin(\phi t)$ describe periodic effects.
- $\exp(-\lambda(x-\xi))$ is a spatial kernel function.

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95 Indefinite Expansion of Advanced Fractional Systems

95.1 Fractional Quantum Field Theory with Variable Couplings

$$i\frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2}\nabla^\alpha \phi(x,t)+\left(V(x,t)+\lambda(t)\phi(x,t)^2+\mu(t)\phi(x,t)^3\right)\phi(x,t)+\int_0^x K(x,\xi)\phi(\xi,t)d\xi \tag{322}$$

Notation:

- ∇^α denotes the fractional Laplacian of order α.
- $V(x,t)$ is a time-dependent potential.
- $\lambda(t)$ introduces a quadratic coupling term.
- $\mu(t)$ introduces a cubic coupling term.
- $K(x,\xi)$ is a kernel function describing additional interactions.

95.2 Fractional Diffusion with Nonlinear Boundary Conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \kappa(x)\nabla^\beta u(x,t)+\lambda(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2+\mu(t)u(x,t)\sin(\theta x)+\int_0^x \exp(-\gamma(x-\xi))u(\xi,t)d\xi \tag{323}$$

Notation:

- $\kappa(x)$ is the spatially varying diffusion coefficient.
- ∇^β denotes the fractional Laplacian of order β.

- $\lambda(x)$ introduces a nonlinear boundary term.
- $\mu(t)$ describes additional time-dependent effects.
- $\sin(\theta x)$ represents spatial periodicity.
- $\exp(-\gamma(x-\xi))$ is a spatial kernel function.

95.3 Fractional Nonlinear Control Systems with Adaptive Feedback

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t)+B(t)\left[\int_0^t \phi(t-\tau)x(\tau)d\tau\right]+\lambda(t)x(t)^2+\eta(t)\exp(-\xi t)+\sigma(t)\frac{dx(t)}{dt} \tag{324}$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent matrices.
- $\phi(t-\tau)$ is a kernel function describing memory effects.
- $\lambda(t)$ introduces a quadratic control term.
- $\eta(t)$ represents exponential decay effects.
- $\sigma(t)$ introduces an additional feedback term.

95.4 Fractional Hybrid Systems with Multiple Nonlinear Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t)+\int_0^x \Phi(x-\xi)u(\xi,t)d\xi+\lambda(x)u(x,t)^2+\mu(t)u(x,t)^3+\xi(x)\exp(-\gamma t) \tag{325}$$

Notation:

- ∇^β denotes the fractional Laplacian of order β.
- $\Phi(x-\xi)$ is a kernel function representing cross-interactions.
- $\lambda(x)$ and $\mu(t)$ introduce nonlinear terms.
- $\xi(x)$ accounts for spatial effects.
- $\exp(-\gamma t)$ describes temporal decay.

95.5 Fractional Multi-Dimensional Heat Transfer with Complex Interactions

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \kappa(x)\nabla^\beta T(x,t)+\lambda(x)\frac{\partial T(x,t)}{\partial x}+\mu(t)T(x,t)^2+\theta(x)\cos(\phi t)+\int_0^x \exp(-\gamma(x-\xi))T(\xi,t)d\xi \tag{326}$$

Notation:

- $\kappa(x)$ is the spatially varying thermal conductivity.
- ∇^β denotes the fractional Laplacian of order β.
- $\lambda(x)$ introduces a boundary condition term.
- $\mu(t)$ represents nonlinear temperature effects.
- $\theta(x)$ and $\cos(\phi t)$ account for periodic effects.
- $\exp(-\gamma(x-\xi))$ is a spatial kernel function.

95.6 Fractional Geometric Flow with Nonlinear Curvature

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t)+\lambda(x)\left(\frac{\partial \mathbf{R}(x,t)}{\partial x}\right)^2+\mu(t)\mathbf{R}(x,t)^3+\xi(x)\sin(\phi t)+\int_0^x \exp(-\gamma(x-\xi))\mathbf{R}(\xi,t)d\xi \tag{327}$$

Notation:

- ∇^β denotes the fractional Laplacian.
- $\mathbf{R}(x,t)$ represents a geometric field with curvature.
- $\lambda(x)$ introduces a curvature term.
- $\mu(t)$ describes cubic nonlinearities.
- $\xi(x)$ and $\sin(\phi t)$ account for periodic effects.
- $\exp(-\gamma(x-\xi))$ represents a spatial kernel function.

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96 Indefinite Expansion of Advanced Mathematical Frameworks

96.1 Advanced Fractional Differential Equations with Multi-Scale Dynamics

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \int_0^x \Psi(x,\xi) u(\xi,t) d\xi + \lambda(x) u(x,t)^\gamma + \mu(t) \exp(-\eta t) + \zeta(x) \frac{\partial u(x,t)}{\partial x} \tag{328}$$

Notation:

- ∇^β denotes the fractional Laplacian of order β.
- $\Psi(x, \xi)$ is a multi-scale kernel function describing interaction effects.
- $\lambda(x)$ introduces a nonlinear term with exponent γ.
- $\mu(t)$ represents exponential decay over time.
- η is the decay rate in $\exp(-\eta t)$.
- $\zeta(x)$ represents an additional spatial interaction term.

96.2 Fractional Nonlinear Wave Equations with Time-Varying Parameters

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \theta(t) \left(\frac{\partial u(x,t)}{\partial x} \right)^2 + \phi(x) u(x,t)^\delta + \gamma(t) \exp(-\sigma x) + \int_0^x \Omega(x,\xi) u(\xi,t) d\xi \tag{329}$$

Notation:

- $\theta(t)$ is a time-dependent factor modulating the nonlinear term.
- $\phi(x)$ introduces a spatially varying power-law term.
- δ represents the exponent in the power-law term.
- $\gamma(t)$ is a time-varying function influencing exponential decay.
- σ is the spatial decay rate in $\exp(-\sigma x)$.
- $\Omega(x, \xi)$ is a kernel function describing cross-interactions.

96.3 Fractional Integral-Differential Equations with Boundary Conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \kappa(x)\nabla^\beta u(x,t) + \int_0^x \Phi(x,\xi)u(\xi,t)d\xi + \lambda(x)\frac{\partial u(x,t)}{\partial x} + \mu(t)u(x,t)^2 + \eta(x)\sin(\gamma t) \tag{330}$$

Notation:

- $\kappa(x)$ is a spatially varying coefficient affecting diffusion.
- $\Phi(x,\xi)$ represents a kernel function for integral terms.
- $\lambda(x)$ introduces a boundary condition term.
- $\mu(t)$ describes a quadratic nonlinear term.
- $\eta(x)$ and $\sin(\gamma t)$ account for periodic effects.

96.4 Fractional Partial Differential Equations with Nonlinear Source Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \int_0^x \Psi(x-\xi)u(\xi,t)d\xi + \lambda(x)\exp(-\delta t) + \phi(x)u(x,t)^\gamma \tag{331}$$

Notation:

- $\Psi(x-\xi)$ is a kernel function with fractional interaction effects.
- δ represents the time decay rate in $\exp(-\delta t)$.
- $\phi(x)$ introduces a spatially varying nonlinear source term.
- γ is the exponent for the nonlinear term.

96.5 Fractional Stochastic Differential Equations with Adaptive Noise

$$d^\alpha x(t) = A(t)x(t)dt + B(t)\left[\int_0^t \phi(t-\tau)x(\tau)d\tau\right] + \lambda(t)x(t)^2 + \eta(t)dW(t) \tag{332}$$

Notation:

- $A(t)$ and $B(t)$ are time-dependent matrices.
- $\phi(t-\tau)$ represents a kernel function for memory effects.
- $\lambda(t)$ introduces a quadratic noise term.
- $\eta(t)$ describes an adaptive noise term with $dW(t)$ as the Wiener process.

96.6 Fractional Geometric Flows with Complex Boundary Interactions

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t) + \lambda(x) \left(\frac{\partial \mathbf{R}(x,t)}{\partial x} \right)^2 + \phi(t) \mathbf{R}(x,t)^\gamma + \xi(x) \cos(\eta t) + \int_0^x \Omega(x,\xi) \mathbf{R}(\xi,t) d\xi \tag{333}$$

Notation:

- $\mathbf{R}(x,t)$ represents a geometric field with curvature.
- ∇^β denotes the fractional Laplacian.
- $\lambda(x)$ introduces curvature-dependent terms.
- $\phi(t)$ and γ describe temporal and nonlinear effects.
- $\xi(x)$ and $\cos(\eta t)$ account for boundary interactions and periodic effects.
- $\Omega(x,\xi)$ is a kernel function describing spatial interactions.

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97 Indefinite Expansion of Advanced Mathematical Frameworks (Continued)

97.1 Advanced Nonlinear Fractional Partial Differential Equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \frac{\partial}{\partial x} \left(\int_0^x \Lambda(x,\xi) u(\xi,t) d\xi \right) + \lambda(x) u(x,t)^\theta + \phi(t) \exp(\psi x) \tag{334}$$

Notation:

- ∇^β represents the fractional Laplacian of order β.

- $\Lambda(x,\xi)$ is a new kernel function representing spatial dependencies.
- $\lambda(x)$ introduces a nonlinear term with exponent θ.
- $\phi(t)$ is a time-dependent function.
- ψ represents a coefficient in the exponential term $\exp(\psi x)$.

97.2 Fractional Partial Differential Equations with Variable Coefficients

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \sum_{i=1}^{n} a_i(t)\nabla^{\beta_i} u(x,t) + \int_0^x \Psi_i(x,\xi)u(\xi,t)d\xi + \phi(t)u(x,t)^\delta \tag{335}$$

Notation:

- $a_i(t)$ are time-varying coefficients.
- ∇^{β_i} denotes fractional derivatives of different orders β_i.
- $\Psi_i(x,\xi)$ is a set of kernel functions with different interaction effects.
- $\phi(t)$ introduces a time-dependent nonlinear term.
- δ is the exponent for the nonlinear term.

97.3 Stochastic Differential Equations with Time-Dependent Drift and Diffusion

$$dX(t) = \left[\alpha(t)X(t) + \beta(t)\int_0^t \phi(t-\tau)X(\tau)d\tau\right]dt + \gamma(t)X(t)dW(t) \tag{336}$$

Notation:

- $\alpha(t)$ and $\beta(t)$ are time-dependent drift and diffusion functions.
- $\phi(t-\tau)$ represents a memory kernel.
- $\gamma(t)$ is a time-dependent volatility term.
- $dW(t)$ denotes the increment of the Wiener process.

97.4 Fractional Geometric Flows with Nonlinear Boundary Conditions

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t) + \lambda(x)\left(\frac{\partial \mathbf{R}(x,t)}{\partial x}\right)^2 + \phi(t)\mathbf{R}(x,t)^\delta + \eta(x)\cos(\gamma t) + \int_0^x \Omega(x,\xi)\mathbf{R}(\xi,t)d\xi \tag{337}$$

Notation:

- $\mathbf{R}(x,t)$ is a geometric field.

- $\lambda(x)$ introduces a curvature-dependent term.
- $\phi(t)$ and δ represent temporal and nonlinear effects.
- $\eta(x)$ and $\cos(\gamma t)$ handle boundary conditions and periodic effects.
- $\Omega(x, \xi)$ describes spatial interactions.

97.5 Fractional Integral-Differential Equations with Complex Nonlinear Terms

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \int_0^x \Phi(x,\xi) u(\xi,t) d\xi + \lambda(x) \exp(-\mu t) + \kappa(x) \left[u(x,t)^\delta + \nu \left(\frac{\partial u(x,t)}{\partial x} \right)^2 \right] \tag{338}$$

Notation:

- $\Phi(x, \xi)$ is a kernel function for integral interactions.
- $\lambda(x)$ and $\exp(-\mu t)$ manage decay terms.
- $\kappa(x)$ introduces additional spatial effects.
- δ and ν handle nonlinear interactions.

97.6 Fractional Differential Equations with Adaptive Kernels

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \int_0^x \Theta(x,\xi) u(\xi,t) d\xi + \phi(t) \left[u(x,t)^\gamma + \lambda(x) \frac{\partial u(x,t)}{\partial x} \right] + \eta(x) \exp(-\beta t) \tag{339}$$

Notation:

- $\Theta(x, \xi)$ is a new adaptive kernel function.
- $\phi(t)$ introduces time-dependent effects.
- γ and $\lambda(x)$ modulate nonlinear and spatial terms.
- $\eta(x)$ and $\exp(-\beta t)$ address decay and boundary interactions.

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98 Further Expansion of Advanced Mathematical Frameworks

98.1 Extended Nonlinear Fractional Partial Differential Equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nabla^\beta u(x,t) + \frac{\partial}{\partial x}\left(\int_0^x \Lambda(x,\xi)u(\xi,t)d\xi\right) + \lambda(x)u(x,t)^\theta + \phi(t)\exp(\psi x) + \frac{\sigma(x,t)}{u(x,t)^\eta} \tag{340}$$

Notation:

- ∇^β represents the fractional Laplacian operator of order β.
- $\Lambda(x,\xi)$ is a spatial kernel function describing interactions over the interval.
- $\lambda(x)$ and θ control the nonlinear feedback terms.
- $\phi(t)$ introduces a time-dependent exponential effect with coefficient ψ.
- $\sigma(x,t)$ is a new term representing an inverse power law dependency, with η as the exponent.

98.2 Fractional Partial Differential Equations with Variable Nonlinear Coefficients

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \sum_{i=1}^n a_i(t)\nabla^{\beta_i}u(x,t) + \int_0^x \Psi_i(x,\xi)u(\xi,t)d\xi + \phi(t)u(x,t)^\delta + \gamma(x)\log(1+u(x,t)) \tag{341}$$

Notation:

- $a_i(t)$ are time-dependent coefficients affecting the fractional derivatives.
- ∇^{β_i} denotes the fractional derivatives of different orders β_i.
- $\Psi_i(x,\xi)$ are kernel functions introducing spatial dependencies.
- $\phi(t)$ controls the nonlinear term.
- δ is an exponent in the nonlinear term, and $\gamma(x)$ introduces a logarithmic nonlinearity.

98.3 Stochastic Differential Equations with Nonlinear Drift and Diffusion Terms

$$dX(t) = \left[\alpha(t)X(t) + \beta(t)\int_0^t \phi(t-\tau)X(\tau)d\tau + \frac{\psi(t)}{X(t)}\right]dt + \gamma(t)X(t)dW(t) \tag{342}$$

Notation:

- $\alpha(t)$ and $\beta(t)$ represent time-dependent drift and diffusion coefficients.
- $\phi(t-\tau)$ is a memory kernel influencing the integral term.
- $\psi(t)$ introduces an additional time-dependent nonlinear term inversely proportional to $X(t)$.
- $\gamma(t)$ modulates the stochastic volatility term.
- $dW(t)$ is the increment of the Wiener process.

98.4 Fractional Geometric Flows with Variable Nonlinear Boundary Conditions

$$\frac{\partial^\alpha \mathbf{R}(x,t)}{\partial t^\alpha} = \nabla^\beta \mathbf{R}(x,t) + \lambda(x)\left(\frac{\partial \mathbf{R}(x,t)}{\partial x}\right)^2 + \phi(t)\mathbf{R}(x,t)^\delta + \eta(x)\sin(\gamma t) + \int_0^x \Omega(x,\xi)\mathbf{R}(\xi,t)d\xi + \theta(x)\frac{\partial^2 \mathbf{R}(x,t)}{\partial x^2} \tag{343}$$

Notation:

- $\mathbf{R}(x,t)$ is the geometric field under study.
- $\lambda(x)$ and δ control nonlinear geometric terms.
- $\eta(x)$ and $\sin(\gamma t)$ model periodic boundary conditions.
- $\Omega(x,\xi)$ is a spatial kernel function.
- $\theta(x)$ introduces an additional second-order spatial term.

98.5 Fractional Integral-Differential Equations with Adaptive Nonlinear Kernels

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \int_0^x \Theta(x,\xi)u(\xi,t)d\xi + \phi(t)\left[u(x,t)^\gamma + \lambda(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right] + \eta(x)\exp(-\beta t) + \frac{\sigma(t)}{u(x,t)^\zeta} \tag{344}$$

Notation:

- $\Theta(x,\xi)$ is an adaptive kernel function that changes based on spatial variables.
- $\phi(t)$ modulates the time-dependent effects.

- γ and $\lambda(x)$ introduce nonlinear terms.
- $\eta(x)$ and $\exp(-\beta t)$ handle exponential decay effects.
- $\sigma(t)$ introduces a time-dependent inverse power term with exponent ζ.

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