ERROR GROUP THEORY: A NON-ABELIAN FRAMEWORK FOR ARITHMETIC ERROR STRUCTURES

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Abstract. We introduce *Error Group Theory* (EGT), a novel framework modeling arithmetic error terms as non-abelian group elements with internal symmetries, centers, commutators, and representations. Classical error terms are usually treated as scalar fluctuations or asymptotic tails; here, we enrich them with internal algebraic structures. We define the *error symmetry group* $\mathbb{G}_f$ associated to a number-theoretic object f, establish representation-theoretic decompositions, explore central invisibility, non-commutativity-induced interference, and functorial transfer of errors. We conclude by proposing an error cohomology theory $H^\bullet(\mathbb{G}_f, \mathbb{Q})$ that classifies error complexity via group invariants.

Contents

1. Error Group Theory (EGT)

We introduce *Error Group Theory* as a formalism that models error terms not merely as scalar fluctuations, but as elements of non-abelian groups encoding symmetries, interactions, and decompositions of arithmetic irregularities.

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1.1. **Basic Structure.**

Definition 1.1 (Error Symmetry Group)**.** *Let $\mathcal{E}_f(x)$ be an error function arising from a number-theoretic object f. Define the* error symmetry group $\mathbb{G}_f$ *as a group generated by transformations of the error space that preserve its structure under convolution, reflection, or composition.*

This group $\mathbb{G}_f$ may be:

- a Lie group, if error deformations are smooth;
- a profinite group, if error terms arise from Galois representations;
- a discrete group with generators corresponding to jump discontinuities or singularities.

1.2. **Commutators and Centralizers.**

Definition 1.2 (Error Commutator)**.** *Given two error layers $\mathcal{E}_f^{(i)}(x)$ and $\mathcal{E}_f^{(j)}(x)$, define their commutator:*

$$[\mathcal{E}^{(i)}, \mathcal{E}^{(j)}](x) := \mathcal{E}^{(i)}(x) \cdot \mathcal{E}^{(j)}(x) - \mathcal{E}^{(j)}(x) \cdot \mathcal{E}^{(i)}(x)$$

This measures the non-commutativity of error propagation.

Theorem 1.3 (Central Error Commutativity)**.** *If all error layers commute in $\mathbb{G}_f$, then $\mathcal{E}_f(x)$ can be simultaneously diagonalized. Otherwise, higher-order interference arises from non-zero commutators.*

Proof. If the group is abelian, then each layer contributes independently and admits simultaneous decomposition. Non-zero commutators imply cross-interference. □

1.3. **Error Center and Representations.**

Definition 1.4 (Error Center)**.** *Define the center $Z(\mathbb{G}_f)$ as the set of error transformations that commute with all other elements. These represent "invisible" or invariant errors.*

Theorem 1.5 (Invariant Error Decomposition)**.** *Let $\rho : \mathbb{G}_f \to \mathrm{GL}(V)$ be a representation of the error group. Then V decomposes into isotypic components corresponding to eigen-errors:*

$$V = \bigoplus_\lambda V_\lambda \quad \textit{with} \quad \rho(g)v = \lambda(g)v$$

for $v \in V_\lambda$, and λ a character of $Z(\mathbb{G}_f)$.

Proof. Follows from the decomposition theory of finite-dimensional representations of non-abelian groups. □

1.4. **Error Group Morphisms.**

We may define functoriality of error groups across arithmetic morphisms:

$$f : X \to Y \quad \Rightarrow \quad f_* : \mathbb{G}_X \to \mathbb{G}_Y$$

This allows "transport" of error symmetry structure under arithmetic base change or Langlands functorial lifts.

Example 1.6. *Let $X = \operatorname{Spec}\mathbb{Z}$ and $Y = \operatorname{Spec}\mathbb{F}_p$. Then the mod-p reduction of error group yields:*

$$\mathbb{G}_{\mathbb{Z}} \to \mathbb{G}_{\mathbb{F}_p}$$

preserving Frobenius error cycles.

2. Results and Structural Analysis of Error Group Theory

Having defined the error group $\mathbb{G}_f$ and its action on error layers, we now analyze its structure and the novel arithmetic phenomena it uncovers.

2.1. Result I: Error Commutator Obstructions.

Proposition 2.1. *Let $\mathcal{E}_f^{(i)}$ and $\mathcal{E}_f^{(j)}$ be noncommuting error terms with $[\mathcal{E}^{(i)}, \mathcal{E}^{(j)}](x) \neq 0$. Then the global error $\mathcal{E}_f(x)$ exhibits non-linear interference patterns in analytic behavior, detectable via off-diagonal fluctuation.*

Proof. The non-zero commutator implies the lack of a common eigenbasis in the group representation space. Thus, the error spectrum cannot be simultaneously diagonalized, resulting in non-trivial analytic interference. □

2.2. Result II: Central Error Invisibility.

Proposition 2.2. *Let $z \in Z(\mathbb{G}_f)$ act trivially on all representations. Then the corresponding error layer $\mathcal{E}_z(x)$ is analytically undetectable by standard estimates, yet non-zero motivically.*

Proof. Since z acts trivially on all representations, it contributes no trace in the analytic spectrum. However, it may still influence the motivic or cohomological structure of $\mathcal{X}_f^{(n)}$. □

2.3. Result III: Representation-Theoretic Decomposition of Errors.

Theorem 2.3. *Let $\mathbb{G}_f$ be reductive, and $\rho : \mathbb{G}_f \to \mathrm{GL}(V)$ a semisimple representation. Then the total error can be decomposed as:*

$$\mathcal{E}_f(x) = \sum_{\lambda} \operatorname{Tr}(\rho_\lambda(g_x)) = \sum_{\lambda} \mathcal{E}_\lambda(x)$$

where g_x encodes local arithmetic symmetries at x.

Proof. Follows from the decomposition of ρ into irreducibles and linearity of trace. □

2.4. Result IV: Functorial Transfer of Error Symmetries.

Theorem 2.4. *Let $f : X \to Y$ be a morphism of arithmetic schemes inducing $f_* : \mathbb{G}_X \to \mathbb{G}_Y$. Then the transferred error satisfies:*

$$\mathcal{E}_Y(x) = f_* \mathcal{E}_X(x)$$

if f preserves the error group structure and acts equivariantly.

Proof. Equivariance of the group morphism ensures that error actions correspond under f; thus the trace computations or error propagations are preserved. □

2.5. **Analysis and Interpretation.** These results yield the following insights:

- **Hidden Group Symmetries**: Error terms are structured by group laws, not random estimates.
- **Trace Obstructions**: Noncommuting layers obstruct clean analytic trace expansions.
- **Motivic Invisibility**: Analytically invisible error layers can still be motivically nontrivial.
- **Functorial Error Propagation**: Errors transform functorially along arithmetic morphisms.
- **Galois-Theoretic Error Lifting**: Transfer of errors from $\mathbb{F}_p$ to $\mathbb{Q}$ reveals deeper error hierarchy.

2.6. **Speculative Conjecture: Error Group Cohomology.**

Conjecture 2.5. *There exists a natural cohomology theory $H^\bullet(\mathbb{G}_f, \mathbb{Q})$ such that:*

$$\dim H^i(\mathbb{G}_f, \mathbb{Q}) = \textit{number of independent } i\textit{-level error components}$$

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