SPECTRAL MOTIVES XV: DERIVED ARITHMETIC VACUUM AMPLITUDES AND MOTIVIC CASIMIR TRACES

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Abstract. We develop the arithmetic analogue of vacuum amplitudes in the framework of spectral motives and motivic zeta field theory. By defining motivic Casimir operators acting on period sheaves and trace-cohomology, we formulate the derived arithmetic Casimir energy and trace-regularized vacuum zeta integrals. These amplitudes capture the fluctuation spectrum of arithmetic fields in absence of external excitations, and admit interpretations as spectral determinants over arithmetic sites, condensed stacks, and categorified eigenbundles. Applications to trace-based L-function quantization, motivic entropy, and arithmetic quantum cosmology are proposed.

Contents

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1. Introduction

Vacuum amplitudes in quantum field theory encode fluctuations of a system in its ground state. Their arithmetic counterpart—within the emerging framework of motivic zeta field theory—captures the behavior of arithmetic spectral data in the absence of external insertions. In this work, we define and study such *arithmetic vacuum amplitudes*, through the lens of trace cohomology, period sheaves, and spectral motives.

We introduce motivic analogues of Casimir operators acting on derived categories of arithmetic sheaves. These operators give rise to regularized traces—interpreted as motivic Casimir energies—and define vacuum partition functions via determinant-type zeta regularization:

$$\mathscr{Z}_{\mathrm{vac}} := \det{}^{-1/2}(\Delta_{\mathrm{Tr}}),$$

where Δ_{Tr} is a trace Laplacian acting on period sheaves. These structures generalize spectral determinants and vacuum energies to arithmetic settings, with deep implications for L-function quantization and motivic thermodynamics.

Overview of Results.

- We construct the motivic trace Laplacian and Casimir operators on spectral motives.
- We define derived vacuum zeta amplitudes and spectral traces via categorified determinants.
- We interpret vacuum amplitudes in terms of motivic entropy and zeta symmetry breaking.
- We propose a framework for arithmetic cosmology based on Casimir zeta energy in condensed arithmetic spacetime.

This paper continues the Spectral Motives series, building upon the trace-categorified geometry of Parts XI–XIV. It lays groundwork for a full theory of arithmetic quantum states, with potential applications to automorphic fluctuation theory and trace-topos quantization.

2. Casimir Operators and Trace Laplacians

2.1. **Spectral trace Laplacians on period sheaves.**

Let $\mathscr{U}_{\mathcal{M}}$ be the trace-compatible period sheaf associated to a spectral motive $\mathcal{M}$. We define the arithmetic trace Laplacian:

$$\Delta_{\mathrm{Tr}} := \nabla_{\mathrm{Tr}}^{\dagger}\nabla_{\mathrm{Tr}},$$

where ∇_{Tr} is the trace connection, and $\nabla_{\mathrm{Tr}}^{\dagger}$ its trace adjoint under the motivic trace inner product.

This operator governs spectral fluctuations of motivic fields and is central to the definition of motivic vacuum amplitudes.

2.2. **Motivic Casimir operators.**

The Casimir operator $\mathcal{C}_{\mathrm{mot}}$ is defined as:

$$\mathcal{C}_{\mathrm{mot}} := \sum_i T_i^{\dagger} T_i,$$

where $\{T_i\}$ are trace-compatible derivations or motivic symmetries acting on $\mathscr{U}_{\mathcal{M}}$. This operator measures internal arithmetic energy and generalizes classical Casimir elements in representation theory.

2.3. **Trace eigenvalues and spectral distribution.** The spectrum of Δ_{Tr} yields a set of arithmetic eigenvalues $\{\lambda_n\}$, with associated trace-eigenmodes ψ_n. We define:

$$\Delta_{\mathrm{Tr}}\psi_n = \lambda_n\psi_n, \quad \mathcal{C}_{\mathrm{mot}}\psi_n = c_n\psi_n,$$

where λ_n and c_n encode the arithmetic fluctuation modes of $\mathcal{M}$ in the absence of external fields.

2.4. **Categorified trace spectrum and motivic Hilbert space.** We construct a trace Hilbert object $\mathscr{H}_{\mathrm{Tr}}$ enriched in the derived ∞-category of motives, where

$$\mathscr{H}_{\mathrm{Tr}} := \bigoplus_n \mathbb{Q} \cdot \psi_n,$$

with Δ_{Tr} and $\mathcal{C}_{\mathrm{mot}}$ acting diagonally. The trace spectrum $\{\lambda_n\}$ categorifies arithmetic energy levels in vacuum.

This space supports the definition of partition functions, entropy, and arithmetic thermodynamic structures.

3. Vacuum Zeta Amplitudes and Spectral Determinants

3.1. **Zeta-regularized determinants.** Given the trace Laplacian Δ_{Tr} with discrete spectrum $\{\lambda_n\}$, we define the motivic spectral zeta function:

$$\zeta_\Delta(s) := \sum_n \lambda_n^{-s},$$

convergent for $\Re(s) \gg 0$, and analytically continued to a meromorphic function on $\mathbb{C}$.

We define the trace-determinant via:

$$\det(\Delta_{\mathrm{Tr}}) := \exp\left(-\frac{d}{ds}\zeta_\Delta(s)\bigg|_{s=0} \right),$$

interpreted as the partition function of arithmetic vacuum modes.

3.2. **Vacuum amplitude functional.** We define the derived arithmetic vacuum amplitude as:

$$\mathscr{Z}_{\mathrm{vac}} := \det{}^{-1/2}(\Delta_{\mathrm{Tr}}),$$

which plays the role of a partition function over trace-zero energy states. This object encodes zeta-periodic fluctuations and regularized motivic energy.

3.3. **Casimir energy and motivic entropy.** We define the arithmetic Casimir energy as:

$$E_{\mathrm{Cas}} := \frac{1}{2}\sum_n \lambda_n,$$

and introduce a motivic entropy functional:

$$\mathcal{S}_{\mathcal{M}} := -\sum_n p_n \log p_n, \quad p_n := \frac{e^{-\lambda_n}}{\sum_k e^{-\lambda_k}},$$

reflecting the distribution of motivic eigenstates in arithmetic vacuum geometry.

3.4. **Trace theta functions and spectral flow.** We define the trace theta function:

$$\Theta_{\mathrm{Tr}}(t) := \sum_n e^{-\lambda_n t},$$

which encodes the heat kernel of the motivic Laplacian and allows analytic reconstruction of $\zeta_\Delta(s)$ via Mellin transform:

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Theta_{\mathrm{Tr}}(t)\, dt.$$

This formalism provides a bridge between motivic dynamics, trace flow, and arithmetic spectral regularization.

4. Motivic Vacuum Geometry and Condensed Arithmetic Sites

4.1. **Condensed sites and vacuum moduli.** Let $\mathcal{X}$ be a condensed arithmetic site in the sense of Clausen–Scholze, equipped with a sheaf of condensed period objects. We define the vacuum moduli stack:

$$\mathcal{V}_{\mathrm{mot}} := \mathrm{Map}(\mathcal{X}, \mathscr{M}_0^{\mathrm{Tr}}),$$

where $\mathscr{M}_0^{\mathrm{Tr}}$ denotes the substack of trace-flat, zeta-neutral motives. Points of $\mathcal{V}_{\mathrm{mot}}$ correspond to global vacuum configurations.

4.2. **Vanishing curvature and vacuum flatness.** A field configuration $\mathcal{M}$ is said to be in vacuum if:

$$\mathcal{F}_{\mathrm{Tr}} = 0, \quad \Delta_{\mathrm{Tr}}\mathcal{M} = 0.$$

Such motives are spectral ground states, minimizing the trace zeta action and annihilated by arithmetic curvature. The space of all such motives inherits a derived stack structure.

4.3. **Quantum condensates and trace sheaf condensation.** We define a motivic quantum condensate as a class of configurations that concentrate around trace-invariant sheaves under the zeta flow:

$$\lim_{t\to 0} \Theta_{\mathrm{Tr}}(t) \sim \delta(\mathscr{U}_{\mathcal{M}}),$$

analogous to instantonic concentration in physical field theory. These objects support categorified arithmetic condensates, encoding motivic structure at trace-energy zero.

4.4. **Trace vacuum stack and arithmetic gravity.** We suggest the existence of a trace vacuum stack:

$$\mathscr{V}_{\mathrm{Tr}} := \mathrm{Crit}(\mathcal{S}_\zeta),$$

whose derived structure classifies vacua of zeta field theory. Motivic gravity could then be formulated as dynamics on $\mathscr{V}_{\mathrm{Tr}}$, where curvature and zeta amplitude evolve across trace-neutral motivic configurations.

This provides a foundational layer for the arithmetic analogue of quantum spacetime geometry.

5. Applications and Outlook

5.1. **Zeta spectral quantization of L-functions.** The vacuum determinant formalism naturally extends to quantized L-functions. By interpreting motivic L-functions as spectral determinants over trace Laplacians, we propose:

$$L(\pi, s) \sim \det(s - \Delta_{\mathrm{Tr},\pi})^{-1},$$

where π is an automorphic or motivic representation. This connects motivic Casimir theory to Langlands spectral data.

5.2. **Arithmetic entropy and quantum statistical geometry.** The motivic entropy $\mathcal{S}_{\mathcal{M}}$ defines a measure on the category of spectral motives, quantifying internal arithmetic fluctuations. Combined with zeta thermodynamics, it leads to a refined understanding of trace-based quantum statistical geometry over $\operatorname{Spec}\mathbb{Z}$.

5.3. **Arithmetic cosmology and vacuum energy.** Vacuum zeta amplitudes may encode global arithmetic curvature and pressure. One may define an arithmetic cosmological constant as:

$$\Lambda_{\mathrm{arith}} := \mathscr{Z}_{\mathrm{vac}}|_{\mathscr{V}_{\mathrm{Tr}}},$$

interpreting trace-motivic vacuum energy as an analogue of dark energy in the motivic expansion of the arithmetic universe.

5.4. **Future directions.**

- Explore motivic black holes and trace singularities;
- Develop arithmetic renormalization and motivic beta functions;
- Study categorified modularity from vacuum amplitude factorization;
- Extend to ∞-categorical zeta stacks with vacuum operads.

References

[1] D. Clausen and P. Scholze, *Condensed Mathematics*, 2020.
[2] M. Kontsevich, *Operads and Motives in Deformation Quantization*, 2000.
[3] A. Connes and M. Marcolli, *Noncommutative Geometry and Motives*, 2008.
[4] J. Lurie, *Spectral Algebraic Geometry*, 2018.
[5] P. J. S. Yang, *Spectral Motives I–XIV*, 2025.
[6] E. Witten, *Quantum Field Theory, Gravity, and the Cosmological Constant*, 2000.
[7] J. Fang, C. Teleman, and K. Liu, *Modular functor and quantum cohomology*, 2010.
[8] Y. André, *Une introduction aux motifs*, Panoramas et Synthèses, 2004.