OPERATOR-THEORETIC PERSPECTIVES ON THE ARITHMETIC DERIVATIVE UNDER DIRICHLET CONVOLUTION

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Abstract. We initiate an operator-theoretic study of the arithmetic derivative D defined over the convolution algebra of arithmetic functions. Viewing D as a linear operator acting on function spaces such as $\ell^2(\mathbb{N})$ or convolution modules, we explore its algebraic and spectral properties, commutator behavior, kernel structure, and potential extensions to dynamical or categorical settings. This study aims to establish an analytical backbone to the symbolic calculus of arithmetic functions.

Contents

1. Introduction

The arithmetic derivative D, when defined symbolically as a derivation over the Dirichlet convolution algebra of arithmetic functions, introduces a new analytic direction for multiplicative number theory. In this paper, we investigate D from the viewpoint of operator theory: as a (possibly unbounded) linear operator acting on infinite-dimensional function spaces over $\mathbb{N}$, including weighted ℓ^p-spaces, convolution modules, and symbolic algebras.

Our goal is to frame D within classical functional analysis:

- What is the domain and closure of D as an operator on $\ell^2(\mathbb{N})$?
- Can D be represented as a matrix or integral transform?
- Does D generate a semigroup or a flow?
- What is the structure of its spectrum, kernel, or commutators?

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This paper sets the groundwork for further spectral and dynamical analyses of arithmetic function flows.

2. Preliminaries and Function Spaces

We define the space of arithmetic functions:

$$\mathscr{A} := \{f : \mathbb{N} \to \mathbb{C}\},$$

with the Dirichlet convolution product:

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

We also consider:

- $\ell^2(\mathbb{N})$: Hilbert space with inner product $\langle f, g\rangle = \sum_{n=1}^{\infty} f(n)\overline{g(n)}$;
- $\mathscr{A}_*$: convolution algebra of arithmetic functions;
- $\mathscr{M} \subset \mathscr{A}$: multiplicative functions;
- $\mathscr{U} \subset \mathscr{A}_*$: convolution units f with $f(1) \neq 0$.

3. Definition and Basic Properties of the Operator D

We define $D\colon \mathscr{A} \to \mathscr{A}$ by:

$$D(f)(n) := \log(n)f(n),$$

or more generally via the Leibniz rule:

$$D(f * g) = D(f) * g + f * D(g).$$

Then D is a derivation over $(\mathscr{A}_*, *)$. Note:

$$D(\delta) = 0, \quad D(n^s) = s\log(n)n^s.$$

4. Matrix Representation and Domain Analysis

We analyze D as an infinite matrix on $\ell^2(\mathbb{N})$, indexed by natural numbers. Define basis vectors e_n with $e_n(k) = \delta_{nk}$, then:

$$D(e_n) = \log(n)e_n,$$

implying D is diagonal on the standard basis, unbounded, but densely defined.

4.1. Spectrum and Self-Adjointness.

The spectrum of D over ℓ^2 is $\sigma(D) = \overline{\{\log(n) : n \in \mathbb{N}\}}$. Since D is diagonal with real entries, it is essentially self-adjoint. However, D is not compact.

5. Commutator Structures

Define the commutator:

$$[D, T_f] := DT_f - T_fD,$$

where T_f is the operator of convolution with f. Study the algebra generated by D and such T_f, yielding an analogue of a Heisenberg-type Lie algebra for arithmetic function operators.

6. Advanced Operator-Theoretic Structures

We now develop four advanced extensions of the operator D in the context of arithmetic function spaces under Dirichlet convolution.

6.1. **Exponentials and Semigroup Generation.**

Definition 6.1. Let D be the diagonal operator on $\ell^2(\mathbb{N})$ defined by $D(f)(n) = \log(n)f(n)$. Define the exponential operator $e^{tD}\colon \ell^2 \to \ell^2$ for $t \in \mathbb{R}$ as:

$$(e^{tD}f)(n) := n^t f(n).$$

Proposition 6.2. *The family $\{e^{tD}\}_{t\in\mathbb{R}}$ forms a strongly continuous one-parameter semigroup of bounded operators on $\ell^2(\mathbb{N})$.*

Proof. For each $t \in \mathbb{R}$, e^{tD} acts as multiplication by n^t, which defines a bounded operator on ℓ^2 provided that $f(n)$ decays faster than any polynomial growth. The semigroup property follows from:

$$e^{sD}e^{tD}f(n) = n^s(n^t f(n)) = n^{s+t}f(n) = e^{(s+t)D}f(n).$$

The identity $e^{0D} = \mathrm{Id}$ is trivial. Strong continuity in t holds as $n^t \to 1$ pointwise when $t \to 0$. □

6.2. **Resolvents and Dirichlet-Type Green Operators.**

Definition 6.3. Let D be the diagonal operator as before. Define its resolvent at $z \in \mathbb{C} \setminus \overline{\log(\mathbb{N})}$ as:

$$R_z := (z - D)^{-1}, \quad \text{where } R_z f(n) := \frac{1}{z - \log(n)} f(n).$$

Theorem 6.4. *The resolvent operator R_z is a bounded linear operator on $\ell^2(\mathbb{N})$ for $z \notin \overline{\log(\mathbb{N})}$, and satisfies the identity:*

$$(z - D)R_z f = f.$$

Proof. Since D is diagonal and self-adjoint, the resolvent is well-defined for $z \notin \sigma(D)$. Then:

$$((z - D)R_z f)(n) = (z - \log(n)) \cdot \frac{1}{z - \log(n)} f(n) = f(n).$$

Boundedness follows from:

$$\|R_z f\|^2 = \sum_{n=1}^{\infty} \left| \frac{f(n)}{z - \log(n)} \right|^2 \le \frac{1}{\delta^2}\|f\|^2,$$

where $\delta := \inf_n |z - \log(n)| > 0$ since $z \notin \overline{\log(\mathbb{N})}$. □

6.3. **Extension to Convolution Modules.**

Definition 6.5. Let M be a commutative monoid. Define the space of functions $\mathscr{A}_M := \{f : M \to \mathbb{C}\}$. Define convolution $(f * g)(m) := \sum_{ab=m} f(a)g(b)$, and the logarithmic derivation:

$$D(f)(m) := \log(w(m))f(m),$$

where $w\colon M \to \mathbb{R}_{>0}$ is a weight function (e.g., $w(n) = n$ for $M = \mathbb{N}$).

Proposition 6.6. *D is a derivation on the convolution algebra $\mathscr{A}_M$ satisfying:*

$$D(f * g) = D(f) * g + f * D(g).$$

Proof.

$$D(f * g)(m) = \log(w(m)) \sum_{ab=m} f(a)g(b),$$

and using logarithmic multiplicativity:

$$= \sum_{ab=m} (\log(w(a)) + \log(w(b)))f(a)g(b) = \sum_{ab=m} \log(w(a))f(a)g(b) + \sum_{ab=m} f(a)\log(w(b))g(b),$$

which equals $(D(f) * g)(m) + (f * D(g))(m)$. □

6.4. **Categorical Embedding.**

Definition 6.7. Define the category $\mathscr{C}$ where:

- Objects are pairs (A, D), where A is a convolution algebra and $D\colon A \to A$ is a derivation;
- Morphisms $\phi\colon (A, D_A) \to (B, D_B)$ are algebra homomorphisms satisfying $\phi \circ D_A = D_B \circ \phi$.

Proposition 6.8. *The category $\mathscr{C}$ is a differential tensor category under convolution product $*$.*

Proof. The category $\mathscr{C}$ admits a symmetric monoidal structure via convolution tensoring: $(A, D_A) \otimes (B, D_B) := (A \otimes B, D_A \otimes \mathrm{Id} + \mathrm{Id} \otimes D_B)$, and morphisms preserve derivations by naturality. □

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