K-Theory for Motives, Automorphic Forms, and L-Functions

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Contents

1 Introduction

In this document, we rigorously develop various forms of K-theory (algebraic, topological, motivic, equivariant, and twisted) for important mathematical objects: motives M, automorphic forms A, and L-functions L. We aim to systematically study the relationships between these K-theories and these objects, focusing on their algebraic, topological, and deeper structures.

2 Algebraic K-theory for Motives, Automorphic Forms, and L-Functions

2.1 Preliminaries on Algebraic K-Theory

We begin by recalling the basic setup of algebraic K-theory. Let X be a smooth projective variety over a field k. We define the K-groups $K_n(X)$ using the following construction:

$$K_0(X) = \text{Grothendieck group of vector bundles on } X.$$

$$K_n(X) = \pi_n(BGL(X)^+).$$

2.2 Algebraic K-theory of Motives

Let M be a motive over a field k. We can associate to M a variety X and define the algebraic K-groups $K_n(M)$ as:

$$K_n(M) = K_n(X),$$

where X is the variety associated to M. This allows us to study motivic K-theory as a natural extension of algebraic K-theory.

2.3 Algebraic K-theory of Automorphic Forms

Let A be an automorphic form associated with a group G. We can define the K-theory of A by considering the moduli space of automorphic forms, denoted by $\mathscr{M}_A$, and define:

$$K_n(A) = K_n(\mathscr{M}_A).$$

This gives us a way to explore the algebraic K-theory of automorphic forms.

2.4 Algebraic K-theory of L-functions

For an L-function L, we define its K-theory by associating L with a related variety or motive. Let $\mathscr{M}_L$ be the moduli space associated with the L-function. Then, we define:

$$K_n(L) = K_n(\mathscr{M}_L).$$

3 Topological K-theory for Motives, Automorphic Forms, and L-Functions

3.1 Preliminaries on Topological K-Theory

Topological K-theory is a cohomology theory based on vector bundles over topological spaces. For a topological space X, we define K-groups as:

$$K^0(X) = \text{Grothendieck group of vector bundles over } X.$$

$$K^1(X) = \pi_1(BGL(X)^+).$$

3.2 Topological K-theory of Motives

Let M be a motive. We can define a topological space X_M associated with M, and study its topological K-theory:

$$K^0(M) = K^0(X_M), \quad K^1(M) = K^1(X_M).$$

3.3 Topological K-theory of Automorphic Forms

For automorphic forms A, we can associate a topological space X_A, such as a moduli space of automorphic forms. We define the topological K-groups as:

$$K^0(A) = K^0(X_A), \quad K^1(A) = K^1(X_A).$$

3.4 Topological K-theory of L-functions

Let L be an L-function. We define a topological space X_L associated with L, and compute the topological K-groups:

$$K^0(L) = K^0(X_L), \quad K^1(L) = K^1(X_L).$$

4 Motivic K-theory for Motives, Automorphic Forms, and L-Functions

4.1 Preliminaries on Motivic K-theory

Motivic K-theory is defined as a theory that extends algebraic K-theory to motives and related objects. Let X be a smooth projective variety, and let

$\mathcal{M}(X)$ denote the category of motives. We define the motivic K-groups as:

$$K_{\mathrm{mot}}^n(X) = \text{higher } K\text{-theory of motives.}$$

4.2 Motivic K-theory of Motives

Let M be a motive. We define its motivic K-groups by considering the motives associated with M:

$$K_{\mathrm{mot}}^n(M) = K_{\mathrm{mot}}^n(X_M),$$

where X_M is the variety associated with the motive M.

4.3 Motivic K-theory of Automorphic Forms

Let A be an automorphic form. We define the motivic K-theory of A by associating A with a moduli space of automorphic forms:

$$K_{\mathrm{mot}}^n(A) = K_{\mathrm{mot}}^n(X_A).$$

4.4 Motivic K-theory of L-functions

For an L-function L, we define its motivic K-theory as:

$$K_{\mathrm{mot}}^n(L) = K_{\mathrm{mot}}^n(X_L).$$

5 Equivariant K-theory for Motives, Automorphic Forms, and L-Functions

5.1 Preliminaries on Equivariant K-theory

Equivariant K-theory is a variant of K-theory where one considers group actions. Let G be a group acting on a space X. The equivariant K-groups are defined as:

$$K_G^0(X) = \text{Grothendieck group of } G\text{-equivariant vector bundles.}$$

5.2 Equivariant K-theory of Motives

For a motive M with a group action G, we define the equivariant K-theory as:

$$K_G^0(M) = K_G^0(X_M).$$

5.3 Equivariant K-theory of Automorphic Forms

Let A be an automorphic form with a group action G. We define the equivariant K-theory as:

$$K_G^0(A) = K_G^0(X_A).$$

5.4 Equivariant K-theory of L-functions

For an L-function L with a group action G, we define the equivariant K-theory as:

$$K_G^0(L) = K_G^0(X_L).$$

6 Twisted K-theory for Motives, Automorphic Forms, and L-Functions

6.1 Preliminaries on Twisted K-theory

Twisted K-theory arises when the vector bundles on a space X are twisted by a class in cohomology. Let $H^3(X, \mathbb{Z})$ be a class, and the twisted K-groups are defined as:

$$K^0_{\text{tw}}(X, H) = \text{twisted Grothendieck group}.$$

6.2 Twisted K-theory of Motives

For a motive M, we define the twisted K-theory as:

$$K^0_{\text{tw}}(M, H) = K^0_{\text{tw}}(X_M, H).$$

6.3 Twisted K-theory of Automorphic Forms

Let A be an automorphic form twisted by a class H. We define the twisted K-theory as:

$$K^0_{\text{tw}}(A, H) = K^0_{\text{tw}}(X_A, H).$$

6.4 Twisted K-theory of L-functions

For an L-function L, we define its twisted K-theory by:

$$K^0_{\text{tw}}(L, H) = K^0_{\text{tw}}(X_L, H).$$