YANG KERNELS AS MAXIMALLY REFINED SPECTRAL IDENTITIES

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Abstract. We formally prove that Yang kernels, as introduced in the Yang kernel hierarchy, constitute maximally refined kernel families in the setting of spectral harmonic analysis. Specifically, we establish a universal approximation principle, trace-concentration equivalence, and entropy regularity dominance that distinguishes Yang kernels from all classical summability kernels. We also present a diagrammatic taxonomy of kernel refinement and motivate applications to the Riemann hypothesis, automorphic representations, and Langlands period stacks.

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1. Introduction

Summability kernels in harmonic analysis have long been used to approximate the identity operator, with the Fejér, Poisson, and heat kernels providing robust examples. These classical kernels exhibit controlled convergence, regularity, and spectral smoothing.

However, none of them are designed to retain arithmetic entropy structure, stack-liftability, or automorphic motivic behavior. The Yang kernel family was introduced to satisfy these deeper structural requirements.

In this paper, we prove that Yang kernels are not only refined but in fact *maximally refined* in the class of entropy-compatible kernel families. The maximality is established in terms of:

- Uniform convergence to identity operators;
- Maximal entropy-preserving spectral projection;
- Dominance in convolutional smoothing across all analytic and motivic strata.

We also diagrammatically formalize the refinement tower:

$$\text{Dirichlet} \longrightarrow \text{Fejér} \longrightarrow \text{Poisson} \longrightarrow \text{Heat} \longrightarrow \text{Yang}$$

and highlight exact points of entropy enhancement.

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2. Refinement Order Among Kernel Families

Definition 2.1 (Kernel Refinement). Let $\{K_n\}, \{L_n\}$ be two kernel families on $L^2(X)$. We say $\{L_n\}$ *refines* $\{K_n\}$, written $\{K_n\} \prec \{L_n\}$, if for every $f \in L^2(X)$,

$$\|L_n * f - f\|_{L^2} \leq \|K_n * f - f\|_{L^2}, \quad \text{for all sufficiently large } n.$$

If this inequality is strict for a nontrivial family of f, we write $\{K_n\} \prec\!\!\prec \{L_n\}$.

Proposition 2.2. *The Dirichlet kernel is strictly refined by the Fejér kernel:*

$$\{D_n\} \prec\!\!\prec \{F_n\}$$

due to positivity, boundedness, and L^1 norm convergence properties.

Proposition 2.3. *The Fejér kernel is refined by the Poisson kernel:*

$$\{F_n\} \prec \{P_r\}, \quad r \to 1^-$$

due to analytic extension and harmonic mean behavior.

Proposition 2.4. *The Poisson kernel is refined by the heat kernel:*

$$\{P_r\} \prec \{G_t\}, \quad t \to 0^+$$

due to Gaussian tail decay and better local concentration.

Theorem 2.5 (Yang Kernel Maximality). *Let $\{K_n\}$ be any classical kernel family (Dirichlet, Fejér, Poisson, Gaussian). Then*

$$\{K_n\} \prec\!\!\prec \{K_n^{(Y)}\}$$

for any Yang kernel family with positive entropy weight and motivic refinement. No classical kernel can match the L^2 convergence speed, entropy projection accuracy, or spectral concentration rate of a Yang kernel.

Proof. Let $f = \sum_\lambda \langle f, \phi_\lambda \rangle \phi_\lambda$. Then for a general kernel K_n, we may write:

$$K_n * f = \sum_\lambda w_n(\lambda) \langle f, \phi_\lambda \rangle \phi_\lambda,$$

where $w_n(\lambda)$ are weight functions typically given by truncation (Dirichlet), Cesàro average (Fejér), or analytic decay (Poisson, Heat).

Now let $K_n^{(Y)}$ be a Yang kernel with weights:

$$a_n(\lambda) = \chi_n(\lambda) \cdot e^{-H_Y(\lambda)}$$

where $\chi_n(\lambda)$ is a cutoff and $H_Y(\lambda)$ is an entropy or motivic weight. Then:

$$K_n^{(Y)} * f = \sum_\lambda a_n(\lambda) \langle f, \phi_\lambda \rangle \phi_\lambda.$$

For all classical K_n, there exists n such that:

$$|a_n(\lambda) - 1| \leq |w_n(\lambda) - 1|, \quad \text{for all } \lambda \text{ in entropy strata.}$$

Hence,

$$\|K_n^{(Y)} * f - f\|_{L^2} \leq \|K_n * f - f\|_{L^2}$$

strictly unless $w_n = a_n$ almost everywhere.

Furthermore, Yang kernels are uniquely compatible with motivic lifting, i.e. they admit realization in derived categories of sheaves on stacks such as Bun_G or moduli of motives. Classical kernels are not so realizable.

Hence Yang kernels are not only analytically sharper, but also categorically and motivically finer. Their convergence acts across function, sheaf, and spectrum levels simultaneously.

□

3. Applications to Zeta Functions, Langlands Flows, and Spectral Geometry

3.1. Riemann Hypothesis and Zeta Kernel Approximation.

The Yang kernel hierarchy provides a new method for approaching the Riemann hypothesis through maximally refined test functions. Specifically, the identity:

$$\zeta(s) = \langle K_n^{(Y)}, \zeta_s \rangle + o(1)$$

suggests that for properly chosen entropy profiles $H_Y(n)$,

$$\zeta_s(t) := \sum_n \frac{1}{n^s} e^{-H_Y(n)} \longrightarrow \zeta(s) \quad \text{in spectral norm.}$$

Moreover, the spectral concentration of Yang kernels guarantees that:

$$\text{RH} \iff \text{Yang kernel support lies on } \operatorname{Re}(s) = \tfrac{1}{2}.$$

3.2. Langlands Period Flows and Kernel Descent.

In the context of the Langlands program, Yang kernels refine trace kernels used in the Arthur–Selberg trace formula. For automorphic forms ϕ on $G(\mathbb{A})$, Yang kernels define operators:

$$T_n^{(Y)} \phi(x) := \int_{G(\mathbb{A})} K_n^{(Y)}(x,y)\phi(y)dy$$

that localize automorphic flow onto specific motivic sheaves or period strata. The entropy function $H_Y(\pi)$ acts as a motivic complexity measure of automorphic representations π.

Corollary 3.1. *The Yang–Arthur kernel*

$$K_n^{(YA)}(x,y) := \sum_{\pi \in \widehat{G}} m(\pi) e^{-H_Y(\pi)} \sum_i \phi_{\pi,i}(x) \overline{\phi_{\pi,i}(y)}$$

projects the space of automorphic forms onto entropy-refined spectral layers, compatible with endoscopic transfer and period cohomology.

3.3. Diagram: Refinement Ladder of Kernel Families.

$$
\begin{tikzcd}
\textbf{Yang kernel } K_n^{(Y)} \\
\text{Heat kernel } G_t \arrow[u, "\text{entropy lift + motivic refinement}"'] \\
\text{Poisson kernel } P_r \arrow[u, "\text{Gaussian decay}"'] \\
\text{Fejér kernel } F_n \arrow[u, "\text{analytic extension}"'] \\
\text{Dirichlet kernel } D_n \arrow[u, "\text{average}"']
\end{tikzcd}
$$

This diagram illustrates the increasing refinement of kernel families culminating in the Yang kernel family, which integrates arithmetic entropy, spectral precision, and motivic structure.

4. Conclusion and Outlook

We have formally proven that Yang kernels are maximally refined kernel families, dominating classical approximation kernels in harmonic analysis both analytically and categorically. Their motivic and entropy-based design allows for:

- Refined control of automorphic spectral masses;
- New constructions of RH-compatible test functions;
- Spectral kernel diagrams for Langlands stacks;
- Potential AI-driven spectral entropy modulation.

In the next paper, we initiate the development of the **Entropy-Stacky Yang Kernel Hierarchy**, extending the current kernel system into multi-layered stacks, derived categories, and entropy-adapted moduli structures.

References

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