SCHNIRELMANN-TYPE DENSITY IN STRUCTURED ALGEBRAIC SYSTEMS: FINITE FIELDS AND MODULAR GROUPS

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ABSTRACT. We introduce generalizations of Schnirelmann density to structured algebraic systems, including finite fields $\mathbb{F}_q$ and modular groups $\mathbb{Z}/n\mathbb{Z}$. We define algebraic analogues of additive closure, explore their properties, and compare with classical density theory on $\mathbb{N}$.

1. INTRODUCTION

Schnirelmann density has been foundational in additive number theory over $\mathbb{N}$. To extend this theory to algebraic systems, we must adapt its definition to account for modular structure, finite cardinality, and the behavior of additive subsets in non-linear contexts. In this paper, we explore:

- Finite field analogues
- Density on $\mathbb{Z}/n\mathbb{Z}$
- Behavior under group action and coset partitioning

2. SCHNIRELMANN DENSITY ON FINITE GROUPS

Definition 2.1 (Group Density)**.** Let G be a finite abelian group, and $A \subseteq G$. Define

$$\sigma_G(A) := \frac{|A|}{|G|}.$$

Definition 2.2 (Additive Closure in Groups)**.** Let $k \in \mathbb{N}$ and define

$$kA := \{a_1 + \cdots + a_k : a_i \in A\} \subseteq G.$$

We say A is *k-additively closed* if $kA = G$.

Proposition 2.3. *If $A \subseteq \mathbb{Z}/n\mathbb{Z}$ with $\sigma(A) > \frac{1}{2}$, then $2A = \mathbb{Z}/n\mathbb{Z}$.*

Proof. Standard result in additive combinatorics (Cauchy–Davenport-type theorem) implies this for cyclic groups. □

3. DENSITY IN FINITE FIELDS

Let $\mathbb{F}_q$ be a finite field of order q.

Definition 3.1 (Vector Space Density)**.** Let $V = \mathbb{F}_q^d$ be a vector space. For $A \subseteq V$, define

$$\sigma_{q,d}(A) := \frac{|A|}{q^d}.$$

Date: May 5, 2025.

Theorem 3.2 (Finite Field Additive Closure). *Let $A \subseteq \mathbb{F}_q^d$ with $\sigma_{q,d}(A) > 1 - \epsilon$. Then for sufficiently large k, the sumset $kA = \mathbb{F}_q^d$.*

Remark 3.3. This is a finite-field analogue of Schnirelmann's theorem, with tight bounds depending on the additive energy and structure of A.

4. Modular Structures and Generalizations

Definition 4.1 (Modular Schnirelmann Density). Let $A \subseteq \mathbb{Z}/n\mathbb{Z}$, define

$$\sigma_{\text{mod}}(A) := \frac{|A|}{n}, \quad \text{and define } kA \text{ modulo } n.$$

Proposition 4.2. *If $\sigma_{mod}(A) > \frac{1}{\sqrt{k}}$, then kA covers a positive proportion of $\mathbb{Z}/n\mathbb{Z}$.*

Example 4.3. Let $A = \{1, 2, 3, \ldots, m\} \subset \mathbb{Z}/p\mathbb{Z}$. As $m \to p$, $kA = \mathbb{Z}/p\mathbb{Z}$ for smaller k.

5. Future Directions

- Nonabelian group extensions: define density under conjugation classes
- Multiplicative analogues of Schnirelmann density
- Connections with sum-product theorems
- Generalization to algebraic structures: rings, modules, coset spaces