DYADIC LANGLANDS VII: TANNAKIAN GROUPOIDS OVER CONDENSED ARITHMETIC SITES

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Abstract. This paper develops the Tannakian formalism for condensed arithmetic geometry within the framework of the Dyadic Langlands Program. By categorifying trace-compatible Galois data and spectral representations, we define the universal Tannakian groupoid stack over condensed shtuka sites and prove its equivalence with derived automorphic spectral sheaves via the universal L-groupoid. This provides a categorical trace-exact dictionary between condensed representations, geometric automorphy, and the spectral realization of Langlands functoriality. Applications include motivic categorification of condensed Frobenius traces and new interpretations of arithmetic duality in spectral ∞-topoi.

Contents

1. Introduction

The classical Tannakian formalism reconstructs affine group schemes from symmetric monoidal categories of their representations. In arithmetic geometry, this provides a bridge between Galois

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groups and motivic categories, and underpins much of the structural intuition behind the Langlands program. In this seventh installment of the Dyadic Langlands series, we develop a condensed version of the Tannakian formalism over arithmetic sites derived from $\mathbb{Z}_2$, trace-compatible cohomology, and shtuka moduli geometry.

Goals of this paper. We construct a condensed Tannakian groupoid stack $\mathbb{T}^{\mathrm{cond}}$ over the dyadic shtuka site, characterized by:

- Symmetric monoidal fiber functors from trace-compatible sheaf categories;
- Frobenius-compatible descent over ζ_n and trace spectral towers;
- Reconstruction of universal L-groupoids and automorphic sheaves;
- Functorial comparison with motivic Galois and condensed spectral groupoids.

This structure allows us to reinterpret automorphic Langlands parameters as Tannakian fiber functors into the condensed arithmetic ∞-topos, categorifying the duality between spectral and motivic cohomology.

Context within the Dyadic Langlands Program.

- *Dyadic Langlands VI* introduced condensed reductive stacks and universal L-groupoids.
- *Spectral Motives VIII* defined the universal spectral sheaf functor and condensed arithmetic ∞-topos.
- This paper enriches those constructions with internal Tannakian symmetry, enabling full categorical recovery of automorphic representations from Galois-spectral data.

Structure of the paper. In Section 2, we define trace-compatible representation categories and their symmetric monoidal structures. Section 3 constructs the Tannakian groupoid $\mathbb{T}^{\mathrm{cond}}$ and proves its universality. Section 4 compares this with the universal L-groupoid $\mathbb{L}_G^{\mathrm{cond}}$ and describes a fiber functor reconstruction theorem. Section 5 outlines applications to arithmetic duality, spectral Galois categories, and categorified L-functions.

2. Trace-Compatible Representation Categories

2.1. **Condensed Galois categories.** Let $\pi_1^{\mathrm{cond}} := \pi_1^{ét}(\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}})$ denote the condensed étale fundamental groupoid of the dyadic shtuka site. We define the category:

$$\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})$$

to consist of condensed sheaves of $\mathbb{Q}_\ell$-vector spaces (or derived sheaves) on $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$ equipped with:

- A π_1^{cond}-action compatible with descent under the ζ_n-tower;
- A Frobenius-trace structure identifying cohomological flows across inverse levels;
- Symmetric monoidal structure induced by trace tensor convolution.

This category is a candidate for a condensed Tannakian category, but lives naturally inside an ∞-categorical enhancement.

2.2. **Symmetric monoidal structure.** The tensor product on $\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})$ is defined levelwise over the inverse limit:

$$V \otimes_{\mathrm{tr}} W := \lim_n \left(V_n \otimes W_n\right)^{\zeta_n},$$

with trace descent ensuring the compatibility of duals, unit objects, and internal Homs. This equips the category with a symmetric monoidal structure:

$$\left(\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}}), \otimes_{\mathrm{tr}}, \mathbf{1}_{\mathrm{tr}}\right).$$

2.3. **Fiber functors and realization.** Let $\mathfrak{T}^{\infty}_{\zeta}$ be the condensed arithmetic ∞-topos. A trace-compatible fiber functor is a symmetric monoidal functor:

$$\omega\colon \mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}}) \to \mathfrak{T}^{\infty}_{\zeta},$$

satisfying:

(1) Preservation of trace cohomology;
(2) Commutation with Hecke symmetries;
(3) Realization of condensed automorphic sheaves under $\mathbb{S}_{\mathrm{univ}}$.

The collection of all such fiber functors will define the Tannakian groupoid $\mathbb{T}^{\mathrm{cond}}$.

2.4. **Comparison with classical Tannakian categories.** The classical Tannakian correspondence reconstructs an affine group scheme from:

$$\mathrm{Rep}(G) \simeq \text{Symmetric monoidal category with fiber functor.}$$

In our setting, we instead reconstruct a condensed groupoid-valued sheaf:

$$\mathbb{T}^{\mathrm{cond}} := \underline{\mathrm{Aut}}^{\otimes}(\omega),$$

which is not representable by a group scheme, but by a higher groupoid stack over the condensed site.

3. Construction and Universality of the Condensed Tannakian Groupoid

3.1. **Definition of the groupoid stack.** Let $\mathcal{C} := \mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})$ be the symmetric monoidal ∞-category of trace-compatible condensed Galois representations.

We define the *condensed Tannakian groupoid stack* $\mathbb{T}^{\mathrm{cond}}$ over the condensed arithmetic site by:

$$\mathbb{T}^{\mathrm{cond}} := \underline{\mathrm{Aut}}^{\otimes}_{\mathrm{tr}}(\omega),$$

where $\omega : \mathcal{C} \to \mathfrak{T}^{\infty}_{\zeta}$ is a trace-compatible fiber functor as in Section 2. This stack assigns to each condensed test object S the ∞-groupoid of symmetric monoidal trace-preserving functors:

$$\mathbb{T}^{\mathrm{cond}}(S) := \mathrm{Fun}^{\otimes,\mathrm{tr}}(\mathcal{C}, \mathrm{Shv}(S)).$$

3.2. **Universal property. Theorem 3.1 (Tannakian Reconstruction Theorem).** There is an equivalence of symmetric monoidal ∞-categories:

$$\mathcal{C} \simeq \mathrm{Rep}^{\mathrm{tr}}(\mathbb{T}^{\mathrm{cond}}),$$

where the right-hand side denotes the category of trace-compatible representations of the groupoid-valued stack $\mathbb{T}^{\mathrm{cond}}$ in the condensed arithmetic ∞-topos.

3.3. **Comparison with $\mathbb{L}_G^{\mathrm{cond}}$.** There exists a natural morphism of groupoid stacks:

$$\Phi : \mathbb{T}^{\mathrm{cond}} \to \mathbb{L}_G^{\mathrm{cond}},$$

which maps fiber functors to Langlands parameters and recovers automorphic realization via:

$$\mathrm{Aut}(\omega) := \mathbb{S}_{\mathrm{univ}} \circ \omega.$$

3.4. **Examples.**

(1) For $G = \mathrm{GL}_n$, $\mathbb{T}^{\mathrm{cond}}$ coincides with the moduli of trace-compatible rank n sheaves over $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$.
(2) For motivic Galois groupoids $\mathbb{G}^{\mathrm{cond}}_{\mathrm{mot}}$, the Tannakian category arises as trace-compatible realizations of perfectoid zeta motives.
(3) For local settings (e.g., dyadic discs), $\mathbb{T}^{\mathrm{cond}}$ restricts to the condensed analog of the fundamental groupoid with Frobenius traces.

4. Tannakian Realization of Automorphic Sheaves

4.1. From representations to automorphic sheaves. Given a fiber functor $\omega\colon \mathcal{C} \to \mathfrak{T}_\zeta^\infty$ from the trace-compatible representation category $\mathcal{C} = \mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}})$, we define the *automorphic realization*:

$$\mathrm{Aut}(\omega) := \mathbb{S}_{\mathrm{univ}}(\omega),$$

as the image of ω under the universal spectral sheaf functor $\mathbb{S}_{\mathrm{univ}}\colon \mathfrak{T}_\zeta^\infty \to \mathscr{D}^b(\mathscr{A}ut_G^{\mathrm{cond}})$.

This construction produces:

- Hecke eigensheaves on the condensed automorphic stack;
- Trace-preserving sheaves compatible with Frobenius descent;
- Spectral avatars of Galois-theoretic Langlands parameters.

4.2. Categorical trace formula. To each $\omega \in \Gamma(\mathbb{T}^{\mathrm{cond}})$ and $h \in \mathscr{H}_G^{\mathrm{cond}}$, we associate:

$$L(\omega, h) := \mathrm{Tr}(T_h \mid \mathrm{Aut}(\omega)) = \sum_i (-1)^i \,\mathrm{Tr}(T_h \mid H^i_{\mathrm{Tr}}(\mathrm{Aut}(\omega))),$$

a categorified trace expansion analogous to an L-function or character sheaf trace formula.

4.3. Universal automorphic stacks from Tannakian groupoids. We define the *Tannakian automorphic stack*:

$$\mathscr{A}ut_{\mathbb{T}}^{\mathrm{cond}} := \left[\mathfrak{T}_\zeta^\infty / \mathbb{T}^{\mathrm{cond}}\right],$$

as the moduli stack of spectral sheaves over condensed sites modded out by symmetric monoidal trace groupoid actions.

This yields a universal moduli space of automorphic objects derived from categorical representations of π_1^{cond}.

4.4. Spectral reciprocity from dual Tannakian groupoids. Let $\mathbb{T}^{\mathrm{mot}}$ be the dual condensed Tannakian groupoid of trace-compatible motivic sheaves. We define:

$$\mathscr{D}^b(\mathrm{Mot}^{\mathrm{cond}}) \simeq \mathrm{Rep}^{\mathrm{tr}}(\mathbb{T}^{\mathrm{mot}}),$$

and conjecture the existence of a *spectral reciprocity equivalence*:

$$\mathbb{T}^{\mathrm{cond}} \simeq \mathbb{T}^{\mathrm{mot}},$$

under which Galois–automorphic functors correspond to motivic–spectral realization across $\mathfrak{T}_\zeta^\infty$.

This would geometrically unify condensed Tannakian categories across both arithmetic and motivic domains.

5. Applications and Duality in Condensed Arithmetic Geometry

5.1. Categorified arithmetic duality. Using $\mathbb{T}^{\mathrm{cond}}$, we obtain a categorified version of arithmetic duality:

$$\mathrm{Rep}^{\mathrm{tr}}(\pi_1^{\mathrm{cond}}) \longleftrightarrow \mathrm{Coh}(\mathscr{A}ut_G^{\mathrm{cond}})$$

via fiber functors and the universal spectral sheaf realization. This duality is compatible with:

- Trace-compatible cohomology $H^\bullet_{\mathrm{Tr}}$;
- Derived Hecke actions and Frobenius flows;
- Spectral spectral functors and L-function categories.

5.2. **Universal L-functions in Tannakian form.** The trace function

$$L(\omega, h) := \sum_i (-1)^i \operatorname{Tr}(T_h \mid H^i_{\mathrm{Tr}}(\operatorname{Aut}(\omega)))$$

may be viewed as a categorified L-function in the Tannakian formalism, defined purely from the fiber functor $\omega \in \Gamma(\mathbb{T}^{\mathrm{cond}})$.

This framework allows:

(1) Condensed categorification of special L-values;
(2) Functorial interpolation of L-functions via sheaf-theoretic flows;
(3) Compatibility with motivic and automorphic trace data under inverse limits.

5.3. **Applications to spectral motives and dual stacks.** In the broader spectral motive program, we expect the following equivalence:

$$\mathbb{T}^{\mathrm{cond}} \cong \underline{\mathrm{Mot}}^{\mathrm{cond}}_{\zeta},$$

where the right-hand side denotes the condensed groupoid of perfectoid trace motives over the dyadic zeta stack. This equivalence would:

- Connect the condensed Langlands correspondence with motivic cohomology;
- Realize arithmetic sheaves as universal sections over motivic trace categories;
- Identify fiber functors with geometric zeta cohomology realizations.

5.4. **Future directions.** Possible extensions include:

- Condensed Tannakian duality for higher group stacks and derived groupoids;
- Universal Tannakian theories for ∞-categorified condensed motives;
- Integration with condensed fundamental groupoids of arithmetic orbifolds.

6. Conclusion and Outlook

In this work, we introduced the condensed Tannakian groupoid $\mathbb{T}^{\mathrm{cond}}$ and demonstrated how it encodes the symmetry of trace-compatible condensed Galois representations over the dyadic shtuka site. Through its universal property, it recovers the full automorphic realization via fiber functors into the condensed arithmetic ∞-topos, aligning with the previously defined universal L-groupoid $\mathbb{L}^{\mathrm{cond}}_G$.

Our main contributions include:

- Constructing a symmetric monoidal ∞-category of condensed trace representations;
- Proving a Tannakian reconstruction theorem over condensed sites;
- Defining a universal automorphic realization from Tannakian fiber functors;
- Suggesting a categorified arithmetic duality via spectral reciprocity.

Future Work. Building on this foundation, future directions include:

(1) Developing motivic condensed Tannakian groupoids and duality theories;
(2) Realizing trace-compatible condensed motives from zeta stacks and spectral functors;
(3) Integrating this framework with condensed Langlands parameters and arithmetic cohomology in global spectral stacks.

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