EXPLORING SCHNIRELMANN-TYPE DENSITY IN INFINITE COMBINATORICS AND SET-THEORETIC FRAMEWORKS

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Abstract. We propose and develop Schnirelmann-type notions of density and additive closure in infinite combinatorics and set-theoretic frameworks. These include density-like invariants for subsets of $\mathbb{N}$, uncountable cardinals, and generalizations to filters, ideals, and large cardinal axioms.

1. Introduction

Schnirelmann density has historically been defined for subsets of the natural numbers. In this work, we seek to generalize this concept to infinite set-theoretic contexts, focusing on subsets of infinite cardinals, and on structures involving filters and ideals.

2. Infinite Density Notions

Definition 2.1 (Lower Density in $\mathbb{N}$). Let $A \subseteq \mathbb{N}$. Define

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

Definition 2.2 (Transfinite Density Function). Let κ be a cardinal and $A \subseteq \kappa$. For a cofinal family $\mathcal{F}$ on κ, define

$$d_{\mathcal{F}}(A) := \inf \left\{ \frac{|A \cap F|}{|F|} : F \in \mathcal{F} \right\}.$$

3. Filters, Ideals, and Closure

Definition 3.1 (Filter Closure). Let $\mathcal{F}$ be a filter on $\mathbb{N}$. A set A is $\mathcal{F}$-additively closed if

$$\exists k \text{ s.t. } kA \in \mathcal{F}.$$

Proposition 3.2. *Let $\mathcal{F}$ be an ultrafilter and $A \subseteq \mathbb{N}$ with $\underline{d}(A) > 0$. Then $kA \in \mathcal{F}$ for some k.*

Definition 3.3 (Ideal-Based Nonclosure). If I is an ideal on $\mathbb{N}$, then A is I-null if $A \in I$. The set A is I-incomplete if $kA \in I$ for all k.

4. Large Cardinal Considerations

Proposition 4.1. *If κ is measurable and U is a κ-complete ultrafilter, then for $A \subseteq \kappa$ with $A \in U$, there exists k such that $kA \in U$ under ordinal addition.*

Remark 4.2. This provides a transfinite analogue of Schnirelmann-type additive growth in the context of large cardinals.

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5. Future Directions

- Interactions between density notions and forcing models
- Applications to partition properties and Ramsey theory
- Definitions of Schnirelmann closure for ordinal-indexed operations
- Category-theoretic versions of density over infinite sites