Field Extensions, Completions, and Galois Groups in Yang Number Systems

Pu Justin Scarfy Yang

August 5, 2024

To leverage the Yang number systems $\mathbb{Y}_n$ with the base field being the rationals $\mathbb{Q}$ and to contribute to the study of algebraic number theory, we consider the following approach:

1. Yang Number Systems: Define $\mathbb{Y}_n$ as a field extension of $\mathbb{Q}$ obtained by adjoining n algebraically independent elements over $\mathbb{Q}$. Specifically, $\mathbb{Y}_n = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are algebraically independent over $\mathbb{Q}$.

2. Processes of Completions and Closures: Consider two different alternating processes of closures and completions applied to $\mathbb{Y}_n$:

- Closure First, Then Completion: Start with $\mathbb{Y}_n$ and first take its algebraic closure to obtain $\overline{\mathbb{Y}}_n$. Then, complete $\overline{\mathbb{Y}}_n$ with respect to a given valuation or metric to get a completed field $\mathbb{Y}_n^{\text{complete}}$.

- Completion First, Then Closure: Start with $\mathbb{Y}_n$ and first complete it with respect to a given valuation or metric to obtain $\mathbb{Y}_n^{\text{complete}}$. Then, take the algebraic closure of this completed field to obtain $\overline{\mathbb{Y}}_n^{\text{complete}}$.

3. Arbitrary Series of Processes: Explore arbitrary (including infinite) series of combinations of the above processes:

- Series of Completions and Closures: Consider an infinite sequence where each step involves alternately taking completions followed by closures or vice versa. For example:

$$\mathbb{Y}_n^{(0)} = \mathbb{Y}_n$$
$$\mathbb{Y}_n^{(1)} = \text{Closure}(\mathbb{Y}_n^{(0)})$$
$$\mathbb{Y}_n^{(2)} = \text{Completion}(\mathbb{Y}_n^{(1)})$$
$$\mathbb{Y}_n^{(3)} = \text{Closure}(\mathbb{Y}_n^{(2)})$$
$$\mathbb{Y}_n^{(4)} = \text{Completion}(\mathbb{Y}_n^{(3)})$$

and so on. Similarly,

$$\mathbb{Y}_n^{(0)} = \mathbb{Y}_n$$
$$\mathbb{Y}_n^{(1)} = \text{Completion}(\mathbb{Y}_n^{(0)})$$
$$\mathbb{Y}_n^{(2)} = \text{Closure}(\mathbb{Y}_n^{(1)})$$
$$\mathbb{Y}_n^{(3)} = \text{Completion}(\mathbb{Y}_n^{(2)})$$

$$\mathbb{Y}_n^{(4)} = \text{Closure}(\mathbb{Y}_n^{(3)})$$

and so on.

4. Galois Groups and Field Extensions: Study the Galois groups associated with these fields and their extensions:

- Galois Group of Intermediate Fields: Analyze the Galois group $\text{Gal}(\mathbb{Y}_n/\mathbb{Q})$ for different n. This group can provide insights into the structure of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by understanding how $\mathbb{Y}_n$ behaves relative to $\mathbb{Q}$.

- Galois Group of Closures and Completions: Examine the Galois groups $\text{Gal}(\mathbb{Y}_n^{\text{complete}}/\mathbb{Q})$ and $\text{Gal}(\overline{\mathbb{Y}}_n^{\text{complete}}/\mathbb{Q})$ to understand the impact of completions and closures on field structures.

5. Specific Galois Groups: Consider the specific Galois groups $\text{Gal}(\mathbb{Y}_n(F)/\mathbb{Y}_0(\mathbb{Q}))$, where F is a field of interest:

- Group $\text{Gal}(\mathbb{Y}_n(F)/\mathbb{Y}_0(\mathbb{Q}))$: This group captures the automorphisms of the field $\mathbb{Y}_n(F)$ over the base field $\mathbb{Y}_0(\mathbb{Q})$. By studying these groups, we can gain insights into the behavior of Galois groups over different layers and extensions.

6. Contribution to Algebraic Number Theory: By studying these Galois groups, we contribute to algebraic number theory in the following ways:

- Local and Global Perspectives: Fields obtained through different completions and closures provide local perspectives that can be related to the global Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Local Galois groups can reveal global structures and vice versa.

- Infinite Variations: The processes of closures and completions can be applied in various sequences and combinations, leading to an infinite number of ways to construct fields. Each approach may offer new insights into the absolute Galois group.

7. Infinite Series of Processes: The variety of processes and field constructions allows for potentially infinite sequences of fields, each contributing uniquely to the understanding of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By analyzing these fields and their corresponding Galois groups, one can approach the study of the absolute Galois group from different angles and obtain a richer understanding of its structure and properties.

8. Taking Limits as $n \to \infty$

Taking the limit as $n \to \infty$ in the context of field extensions and completions can be approached in several distinct ways, each offering different perspectives on the resulting field. Note that there are potentially an infinite number of ways to take $n \to \infty$ in this context. Here's a detailed exploration of different methods to consider:

1. Direct Limit of Field Extensions

- Sequential Extensions: Start with $\mathbb{Q}$ and define $\mathbb{Y}_n$ as $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are algebraically independent over $\mathbb{Q}$.

- Limit Field: The direct limit in this case is:

$$\mathbb{Y}_\infty = \bigcup_{n=1}^{\infty} \mathbb{Y}_n$$

This field contains all elements from the fields $\mathbb{Y}_n$ for finite n and captures the union of all such extensions.

2. Completion Followed by Algebraic Closure

- Completion First: Start with $\mathbb{Q}$ and iteratively complete it to get fields like $\mathbb{R}$ and $\mathbb{Q}_p$. Then take algebraic closures.
- For example: $\mathbb{R}$ completed to $\mathbb{C}$, $\mathbb{Q}_p$ completed to $\bar{\mathbb{Q}}_p$.
- Limit Field: The limit here involves considering fields that are both completed and closed algebraically:

$$\mathbb{Y}_{\text{comp-cl}} = \bigcup_{n=1}^{\infty} \text{completion of (closure of } \mathbb{Q}(x_1, \ldots, x_n))$$

3. Algebraic Closure Followed by Completion

- Closure First: Start with $\mathbb{Q}$, take algebraic closure to get $\bar{\mathbb{Q}}$. Then iteratively apply completions.
- For example: Taking $\bar{\mathbb{Q}}$, completing it to various completions like $\bar{\mathbb{Q}}_p$ or others.
- Limit Field: The limit here involves completing the algebraic closure:

$$\mathbb{Y}_{\text{cl-comp}} = \bigcup_{n=1}^{\infty} \text{completion of (closure of } \mathbb{Q}(x_1, \ldots, x_n))$$

4. Nested Completions and Closures

- Nested Process: Alternately apply completions and closures in various orders. For instance, complete the field, then take its algebraic closure, and repeat.
- Limit Field: This limit can be expressed as:

$$\mathbb{Y}_{\text{nested}} = \bigcup_{n=1}^{\infty} \text{nested processes of completion and closure}$$

5. Iterative Application of Different Valuations

- Different Valuations: Apply completions with respect to different valuations (like p-adic valuations) and then consider the algebraic closures.
- Limit Field: The resulting field can be:

$$\mathbb{Y}_{\text{val}} = \bigcup_{n=1}^{\infty} \text{completions with different valuations and their closures}$$

Summary

There are several ways to take the limit of $n \to \infty$ in field theory, each yielding potentially different results:

1. Direct Limit of Extensions: $\mathbb{Y}_\infty = \bigcup_{n=1}^{\infty} \mathbb{Y}_n$
2. Completion Followed by Closure: $\mathbb{Y}_{\text{comp-cl}}$
3. Closure Followed by Completion: $\mathbb{Y}_{\text{cl-comp}}$
4. Nested Completions and Closures: $\mathbb{Y}_{\text{nested}}$
5. Iterative Valuation Applications: $\mathbb{Y}_{\text{val}}$

Each approach offers a different perspective on the fields you construct and can reveal various aspects of their structure and properties. The variety of methods highlights that there are potentially an infinite number of ways to approach the limit as $n \to \infty$, each providing unique insights into the behavior of the resulting fields.