UBC Masters in Mathematics: Research Proposal

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Background

Arithmetic enjoys a privileged position within mathematics as a fertile source of fundamental questions. Among the seven Millennium problems listed by the Clay Institute [Clay], not fewer than three: the Birch and Swinnerton-Dyer conjecture, the Hodge conjecture, and the Riemann hypothesis, were handed down by the Queen of Mathematics. Even by the standards of a subject which has remained vibrant since the days of Fermat and Gauß, the last two decades have witnessed a real golden age, with landmarks too numerous to list completely: such as the striking progress on the Birch and Swinnerton-Dyer conjecture arising from the work of Gross-Zagier [GZ1986], Kolyvagin [Kol1989], and Kato [Kat2004]; the proofs of the Shimura-Taniyama-Weil conjecture [BCDT2001], Serre's conjectures [KW2009], the Fontaine-Mazur conjecture for two-dimensional Galois representations [Kis2009], and the Sato-Tate conjectures [CHR2008] which grew out of Wiles' epoch-making proof of Fermat's Last Theorem [Wil1995], [TW1995]; the revolutionary ideas of Bourgain [Bo2008] and Gowers [Go2007] blending techniques in harmonic analysis and additive combinatorics, the Fields-medal winning breakthrough of Green and Tao on primes in arithmetic progressions [GT2008], and the work of Goldston, Pintz, and Yıldırım [GPY2009], [GPY2010], and its spectacular recent strengthenings by Zhang [Zha2014], and Maynard [May2015] and Tao [Poly2014], on bounded gaps between primes. Recent innovations in arithmetic geometry by the innovation of Perfectoid spaces [Scho2012], and subsequent topological realization of the absolute Galois group [KS2016] by Peter Scholze also shed new lights on the Langlands Programme, a web of conjectures that connect number theory, harmonic analysis, and geometry.

Proposed Objective

A major task in mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both. The arithmetic properties of various interesting objects (e.g. number fields, varieties, or even a motive, i.e. discrete objects) are all encoded in their respective L-functions, i.e. continuous objects, of which the Riemann zeta function is the simplest example. The understanding of these L-functions undergoes three phases [Kat1993a]:

(1) analytic properties and the rationality of values,
(2) algebraic properties illuminated by the p-adic properties of values,
(3) arithmetic-geometric point of view of interpreting the values.

To date much of these have been achieved for the Riemann zeta function [CRSS2015] (except the locations of the zeros, or the zero free region). The deep conjectures of Deligne [De1979] and Beilinson [Be1985] were among the first to place these problems in a broad framework. Ambiguity in the work of Beilinson for interpreting these values, as they are interpreted only up to nonzero rational number multiples. In their article published in the Grothendieck Festschrift [BK1990], Spencer Bloch and Kazuya Kato removed this ambiguity in their formulation of what is now called the Tamagawa number conjecture (or the Bloch-Kato conjecture for motives). They also proved that the conjecture is invariant under isogeny, and the book [CRSS2015] exploits this isogeny invariance condition in the optic of K-theory for the Riemann zeta function. I hope to investigate the isogeny invariance for specific motives and elaborates on accompanying results in cohomology and K-theory. Techniques involved would require heavy analytic number theory which I had the privilege to learn much from Professor Greg Martin, algebraic number theory and algebraic geometry which I had the pleasure to absorbed much by attending many of Professor Sujatha Ramdorai's seminars and courses, as well as seeing the big picture which I enjoyed by attending various conferences and workshops [CV].

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[CV] Justin Scafy’s Curriculum Vitae. With the full list of professional activities I have involved in.