UNIFIED FRAMEWORK FOR ALGEBRAIC AND DIFFERENTIAL STRUCTURES IN MULTIPLICATIVE NUMBER THEORY

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ABSTRACT. We propose a unified theory integrating two complementary perspectives on arithmetic functions: the algebraic structure of multiplicative functions under Dirichlet convolution, and a formal differential calculus defined within the same algebraic framework. This work synthesizes structure-theoretic classifications with a symbolic calculus involving derivative-like operations and functional expansions. The resulting framework yields dynamic insight into the behavior of arithmetic functions, illuminating connections to zeta identities, operator theory, and dynamical number theory.

CONTENTS

1. INTRODUCTION AND MOTIVATION

Classical number theory distinguishes between two complementary approaches to the study of arithmetic functions. On one hand, the *structure-theoretic* perspective categorizes functions based on multiplicativity, periodicity, and convolution identities. On the other hand, the *analytic* perspective interprets these functions as coefficients of Dirichlet series, subject to convergence, transformation, and mean value theorems.

This paper aims to synthesize these two viewpoints into a common language. Building upon two companion works—*Structure-Theoretic Multiplicative Number Theory* and *Arithmetic Function Calculus under Dirichlet Convolution*—we propose a formal system that:

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(1) categorizes arithmetic functions under algebraic structures;
(2) defines differential operators analogous to classical calculus, within the Dirichlet convolution algebra;
(3) identifies compatibility laws and dynamic behaviors linking structure and calculus;
(4) and introduces a higher symbolic framework that may serve as a basis for future developments in arithmetic dynamics and representation theory.

Throughout, we let $*$ denote the Dirichlet convolution, and work over arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. The identity function is $\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$.

2. The Algebraic Universe of Multiplicative Functions

2.1. **Basic Definitions.** A function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{whenever } \gcd(m, n) = 1.$$

It is *completely multiplicative* if this holds for all m, n. Under Dirichlet convolution, the space of arithmetic functions forms a unital commutative ring.

We define key classes:

- $\mathcal{M}$: multiplicative functions;
- $\mathcal{CM} \subset \mathcal{M}$: completely multiplicative;
- $\mathcal{AF}$: arbitrary arithmetic functions;
- $\mathcal{U}$: unit group under $*$, i.e., functions f with $f(1) \neq 0$.

2.2. **Lattice Structures and Poset Relations.** We propose a partial order $f \preceq g$ if $f * h = g$ for some h, analogous to additive subgroups under convolution. This induces a lattice structure on subsets of $\mathcal{AF}$, where convolution acts as a join.

3. Differential Calculus under Dirichlet Convolution

3.1. **Arithmetic Derivative.** Define the arithmetic derivative D via

$$D(f)(n) := (\log n)f(n),$$

or more generally, via convolutional Leibniz rule:

$$D(f * g) = D(f) * g + f * D(g).$$

This operator extends linearly and defines a derivation on the convolution algebra.

3.2. **Formal Calculus Rules.** We formalize:

$$\begin{aligned} D(\delta) &= 0, \\ D(\mu) &= -\mu * \Lambda, \\ D(\log) &= \text{arithmetic derivative of } \log(n), \\ D^k(f) &= \text{iterated convolutional derivative.} \end{aligned}$$

4. Compatibility and Unified Dynamics

We explore when structure-theoretic classes (e.g., $\mathcal{M}, \mathcal{CM}$) are preserved under D, and when arithmetic analogues of exponential or trigonometric identities exist.

5. Further Directions

Potential extensions include:

- Operator-theoretic interpretations of D;
- Spectral theory of zeta-operators $\zeta(D)$;
- Symbolic dynamics on multiplicative function spaces;
- Type-theoretic or category-theoretic formalizations.

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