VOLUME IV: ONTOID GEOMETRY AND SPACE-THEORETIC ONTOLOGIES
FORMAL FOUNDATIONS OF SPACES BEYOND SET AND TOPOS

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Abstract. This volume initiates the theory of Ontoid Geometry: a stratified and recursive redefinition of "space" in which topological or set-theoretic constructions are replaced by logical persistence, filtration towers, and categorical collapse. We develop a theory of ε-spaces grounded in the collapse-resistance of filtered existence across logical towers. These ontoid structures serve as generalized spectra, base sites for sheaf-theoretic realities, and foundational objects of recursive arithmetic ontology.

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0. Symbol Dictionary for Ontoid Geometry

This section defines and organizes the principal notations, spaces, structures, and meta-logical operators employed throughout the volume.

Spaces and Ontological Entities.

- $\mathscr{O}_{\varepsilon}^{\mathrm{strat}}$: an ϵ-stratified ontoid space;
- $\mathbf{Ont}_{\varepsilon^\infty}$: the category of ε^∞-persistent ontoid spaces;
- $\mathrm{Spec}^{\mathrm{ont}}(R)$: the ontoid spectrum of a logical or arithmetic object R;
- $\mathsf{Sh}^{\mathrm{ont}}(X)$: the category of sheaves over ontoid space X;

Filtrations and Persistence Structures.

- $\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: n-th ontological filtration level of $\mathcal{F}$;
- $\mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}) := \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}/\mathrm{Fil}_{n+1}^{\mathrm{ont}}\mathcal{F}$: graded ontological layer;
- $\mathscr{C}_{\varepsilon}(\mathcal{F}) := \bigcup_n \ker\left(\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} \to \mathrm{Fil}_{n-1}^{\mathrm{ont}}\mathcal{F}\right)$: collapse locus of $\mathcal{F}$;
- $\mathcal{E}_{\mathrm{exist}}(\mathcal{F}) := \bigcap_n \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$: meta-core of existence;

Functorial and Logical Structures.

- $\mathcal{E}_{\mathrm{exist}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathbf{Set}$: existential core functor;
- $\mathcal{W} : \mathbf{Ont}_{\varepsilon^\infty} \to \mathbf{Fil}^{\varepsilon^\infty}$: stratified filtration structure functor;
- $\mathscr{C}_{\varepsilon}$: the logical collapse operator (complement to existence tower);

Ontological Parameters and Logical Indexing.

- ε^n : level-n logic-indexed filtration depth;
- $\mathbf{1}_{\mathrm{ont}}$: identity object in the category $\mathbf{Ont}_{\varepsilon^\infty}$;
- ∞ : total logical collapse depth; meta-ontological stability condition;
- ω^ω : indexing ordinal for limit-persistent ontoid filtrations;

Topos and Beyond.

- $\mathbf{Topos}$: classical category of topoi (reference only);
- $\mathbf{Ont}_{\varepsilon^\infty}$: category of stratified ontoid spaces replacing $\mathbf{Topos}$;
- $\mathbf{MetaShv}$: meta-sheaves over logical or filtered categories, possibly non-set-based;

Existential Collapse Semantics.

- $\mathcal{F} \models \exists$: $\mathcal{F}$ survives all collapse levels;
- $\mathcal{F} \models \neg\exists$: $\mathcal{F}$ vanishes under $\mathscr{C}_\varepsilon$;
- $\mathcal{F}$ is *ontologically real* $\iff \mathcal{E}_{\mathrm{exist}}(\mathcal{F}) \neq 0$;

Conventions. Throughout this volume:

- Filtrations are assumed to be ε-indexed, unless otherwise stated;
- Collapse is not merely degeneration but structured logical removal of existence;
- The term "space" always refers to ontoid entities unless clarified;
- Logical depth and filtration level are interchangeable under meta-collapse equivalence;

1. Introduction: From Topos to Ontos

1.1. **The Question of Space After Set Theory.** Classical geometry begins with the notion of "space" as a set equipped with additional structure—topology, scheme, topos. However, these constructions remain rooted in set-theoretic foundations.

In this volume, we ask:

> What is a space if existence is no longer determined by membership, but by persistence across collapse?

This gives rise to **Ontoid Geometry**, in which:

- Points are no longer primitive;
- Sheaves record filtration-resilient information;
- Cohomology reflects ontological stability, not topological extension;
- Collapse, rather than open cover, defines geometric continuity.

1.2. **From Topos to Ontoid.** A topos encodes logic via sheaf conditions. An ontoid encodes *meta-logic* via recursive filtration towers.

We may schematically view the passage:

$$\mathbf{Set} \rightsquigarrow \mathbf{Topos} \rightsquigarrow \mathbf{Ont}_{\varepsilon^\infty},$$

where:

- **Set**: collections with membership;
- **Topos**: spaces with local logic and gluing;
- $\mathbf{Ont}_{\varepsilon^\infty}$: structures with stratified existence under collapse.

1.3. **Collapse as Ontological Negation.** Let $\mathcal{F}$ be a sheaf over a space X. Then:

$$\mathcal{F} \text{ exists ontologically} \iff \forall n,\ \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} \neq 0;$$
$$\mathcal{F} \text{ collapses} \iff \exists n,\ \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} = 0.$$

In this context, filtration becomes a statement about layers of resistance to disappearance. Collapse $\mathscr{C}_\varepsilon$ plays the role of negation: to exist is to survive collapse.

1.4. **Logic-Indexed Filtration.** Each ontoid space carries a filtration indexed by logical or recursive depth:

$$\cdots \subseteq \mathrm{Fil}_{n+1}^{\mathrm{ont}} \subseteq \mathrm{Fil}_n^{\mathrm{ont}} \subseteq \cdots \subseteq \mathcal{F}.$$

The stratification ε^n is not numerical—it represents proof-theoretic or type-theoretic complexity. For instance:

$$\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F} := \{s \in \mathcal{F} \mid s \text{ survives all collapses of depth } \varepsilon^n\}.$$

1.5. **From Points to Layers of Existence.** Ontoid spaces abandon points. Instead, we define:

Definition 1.1 (Ontological Locality). *A* local layer *at level ε^n is the sheaf-theoretic fragment*

$$\mathcal{F}^{[\varepsilon^n]} := \mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}),$$

and a covering family is a collection of such layers whose union resists collapse.

This replaces the open cover condition by a condition of *persistence sufficiency.*

1.6. **Comparison to Topos Theory.** — Feature — Topos — Ontoid — ——————— -————— — Base logic — Internal (first-order) — Meta-logical, stratified — — Points — Generalized (e.g., locales) — None; replaced by filtration cores — — Sheaves — Local gluing — Collapse-survival structures — — Topology — Open covers — Ontological layers indexed by ϵ — — Cohomology — Derived functor on open sets — Persistence cohomology under filtration —

1.7. **Scope of This Volume.** This volume develops:

- A precise definition of ontoid spaces and their categories;
- Collapse operators as logical structures;
- ϵ-indexed stratified sheaf theory;
- Onto-spectra $\mathrm{Spec}^{\mathrm{ont}}$ as generalized foundational spaces;
- Functorial and epistemic interpretations of existence;
- ∞-sheaves and categorical arithmetic over ontological towers.

1.8. **Guiding Philosophy.**

To exist is to persist. To be geometric is to resist collapse.

This replaces the set-theoretic idea of space with a recursive, logical ontology.

We now proceed to develop the foundational filtration structures of ε-spaces in Section 2.

2. ε-Spaces and Logical Foundations of Stratified Existence

2.1. **Recursive Foundations: Existence as Layered Survival.** We begin with the premise that existence is not binary, but stratified across logical depth:

$$\text{An object exists} \iff \text{it persists through all } \varepsilon^n\text{-indexed collapse layers.}$$

This motivates a definition of space not as a point-set or locale, but as a tower of filtration layers, each encoding resistance to non-being.

2.2. **Definition of an ε-Space.**

Definition 2.1 (ε-Space)**.** *An* ε-space *is a pair*

$$(X, \mathrm{Fil}^{\mathrm{ont}}_{\bullet}),$$

where:

- *X is an abstract object (not necessarily a set);*
- $\mathrm{Fil}^{\mathrm{ont}}_n \mathcal{F} \subseteq \mathcal{F}$ *is a descending sequence of sheaf-like structures over X;*
- *Each layer encodes survival under collapse depth ε^n.*

This is the fundamental unit of ontoid geometry.

2.3. **Stratified Collapse and Meta-Existence.**

Definition 2.2 (Meta-Existence Core)**.** *Given $\mathcal{F}$ over an ε-space X, define:*

$$\mathcal{E}_{\text{exist}}(\mathcal{F}) := \bigcap_n \mathrm{Fil}_n^{\text{ont}} \mathcal{F}, \quad \mathscr{C}_\varepsilon(\mathcal{F}) := \mathcal{F}/\mathcal{E}_{\text{exist}}(\mathcal{F}).$$

Then:

$$\mathcal{F} \textit{ is ontologically real } \iff \mathcal{E}_{\text{exist}}(\mathcal{F}) \neq 0.$$

This sets up an ontological dichotomy analogous to Boolean logic, but filtered across ω or ω^ω.

2.4. **Stratified Sheaves and ϵ-Cohomology.** Let $\mathsf{Sh}^{\text{ont}}(X)$ denote the category of stratified sheaves over an ε-space X, with morphisms preserving filtration depth.

For $\mathcal{F} \in \mathsf{Sh}^{\text{ont}}(X)$, define:

$$H^i_{\varepsilon^\infty}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathrm{Fil}_n^{\text{ont}} \mathcal{F}),$$

which captures cohomological mass that survives all collapses.

2.5. **Existential Epimorphisms and Logical Gluing.**

Definition 2.3 (Ontological Epimorphism)**.** *A morphism $f : \mathcal{F} \to \mathcal{G}$ is an* ontological epimorphism *if:*

$$f(\mathrm{Fil}_n^{\text{ont}} \mathcal{F}) = \mathrm{Fil}_n^{\text{ont}} \mathcal{G}, \quad \forall n.$$

Such morphisms preserve existence across all levels and act analogously to coverings in Grothendieck topology.

2.6. **Examples and Meta-Logics.**

- In logical settings: $\mathrm{Fil}_n^{\text{ont}}$ may correspond to derivations of complexity Π_n^0 or Σ_n^1;
- In type theory: $\mathrm{Fil}_n^{\text{ont}}$ may model truncations of (∞, n)-groupoids;
- In arithmetic: $\mathrm{Fil}_n^{\text{ont}}$ may mirror mod p^n motivic or syntomic filtrations;
- In philosophical logic: $\mathrm{Fil}_n^{\text{ont}}$ may represent the depth at which existence is provable.

2.7. **Postulate: Persistence Implies Ontology.**

Postulate. The definition of space arises from what survives transfinite collapse.

Space is the limit of resistance.

This principle replaces traditional axioms of separation, covering, or locality with an axiom of recursive survival.

2.8. **Conclusion.** In this section, we introduced:

- The concept of ε-spaces as filtration-based foundational units;
- Meta-existence via towers of logical collapse;
- Ontological epimorphisms and ϵ-cohomology as persistence measures;
- A logical reinterpretation of geometry where space is the carrier of surviving structure.

The next section formalizes how these towers interact via logical descent, collapse functors, and ontological filtration categories.

3. Ontological Filtration Towers and Collapse Logic

3.1. **Tower Structures and Existential Depth.** We now formalize the recursive tower of filtrations that gives each ontoid space its ontological structure. Let $\mathcal{F}$ be a sheaf over an ε-space X. The tower:

$$\cdots \subseteq \mathrm{Fil}_{n+1}^{\mathrm{ont}}\mathcal{F} \subseteq \mathrm{Fil}_{n}^{\mathrm{ont}}\mathcal{F} \subseteq \cdots \subseteq \mathrm{Fil}_{0}^{\mathrm{ont}}\mathcal{F} \subseteq \mathcal{F}$$

is the *ontological filtration tower* of $\mathcal{F}$. Each level encodes structural persistence under collapse at depth ε^n.

3.2. **Collapse Functor and Descent Logic.** Define the collapse functor $\mathscr{C}_\varepsilon : \mathsf{Sh}^{\mathrm{ont}}(X) \to$ **Ab** by:

$$\mathscr{C}_{\varepsilon n}(\mathcal{F}) := \mathcal{F}/\mathrm{Fil}_{n}^{\mathrm{ont}}\mathcal{F}, \quad \mathscr{C}_{\varepsilon}(\mathcal{F}) := \varinjlim_{n} \mathscr{C}_{\varepsilon n}(\mathcal{F}).$$

Definition 3.1 (Ontological Collapse Depth)**.** *The smallest n for which $\mathrm{Fil}_{n}^{\mathrm{ont}}\mathcal{F} = 0$ is the* collapse depth *of $\mathcal{F}$. If no such n exists, $\mathcal{F}$ is said to be* ontologically persistent*.*

3.3. **Graded Structures and Collapse Profiles.** The graded components of the filtration tower are defined by:

$$\mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F}) := \mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}/\mathrm{Fil}_{n+1}^{\mathrm{ont}}\mathcal{F}.$$

The full structure:

$$\mathrm{gr}_\bullet^{\mathrm{ont}}(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \mathrm{gr}_n^{\mathrm{ont}}(\mathcal{F})$$

carries a stratified notion of "existence layers."

3.4. **Descent and Collapse Morphisms.** Each sheaf $\mathcal{F}$ defines a chain of canonical morphisms:

$$\mathcal{F} \xrightarrow{\pi_0} \mathscr{C}_{\varepsilon 0}(\mathcal{F}) \xrightarrow{\pi_1} \mathscr{C}_{\varepsilon 1}(\mathcal{F}) \xrightarrow{\pi_2} \cdots$$

This defines a descent system for measuring logical degeneration of existence.

3.5. **Logical Collapse Algebra.** The collapse structure naturally forms a filtered graded module:

$$\mathcal{F} \in \mathbf{Gr}_{\varepsilon^\infty}, \quad \text{with actions of } \mathscr{C}_{\varepsilon^n}.$$

We may postulate a "collapse algebra" $\mathscr{C}_\varepsilon$ with:

$$\mathscr{C}_\varepsilon := \langle \mathscr{C}_{\varepsilon^0}, \mathscr{C}_{\varepsilon^1}, \mathscr{C}_{\varepsilon^2}, \ldots \rangle$$

acting functorially on $\mathsf{Sh}^{\mathrm{ont}}(X)$ via descent and filtration morphisms.

3.6. **Collapse-Existence Duality.** Each $\mathcal{F}$ splits (noncanonically) into:

$$\mathcal{F} = \mathcal{E}_{\mathrm{exist}}(\mathcal{F}) \oplus \mathscr{C}_\varepsilon(\mathcal{F}),$$

which reflects a duality between persistent existence and logical disappearance. This forms the basis of the ontological sheaf cohomology decomposition.

3.7. **Ontological Monodromy.** We define the ontological monodromy operator N_{ont} as:

$$N_{\text{ont}} : \mathcal{F} \to \mathcal{F}, \quad N_{\text{ont}}(\mathrm{Fil}_n^{\text{ont}}\mathcal{F}) \subseteq \mathrm{Fil}_{n+1}^{\text{ont}}\mathcal{F}.$$

This generalizes classical nilpotent monodromy as a collapse-driving operator that strictly deepens existential filtration.

Conjecture 3.2 (Ontological Nilpotence)**.** *N_{ont} is nilpotent $\iff$ $\mathcal{F}$ is finite-level ontologically collapsible.*

3.8. **Ontological Weight Structures.** We define a weight filtration dual to collapse:

$$\mathrm{Weight}_n^{\text{ont}}(\mathcal{F}) := \bigcap_{m \geq n} \mathrm{Fil}_m^{\text{ont}}\mathcal{F},$$

which measures how "ontologically heavy" a section is. The deeper it resists collapse, the more weight it has.

3.9. **Conclusion.** This section constructs:

- The recursive tower $\mathrm{Fil}_\bullet^{\text{ont}}\mathcal{F}$ and its collapse layers;
- Collapse functors and depth classifications;
- Graded components and monodromy operators;
- Ontological weight structures and the $\mathscr{C}_\varepsilon$-algebra of descent.

In the next section, we define the category of Ontoids, morphisms between them, and how logical existence structures interact through functorial maps.

4. Category of Ontoids and Onto-Spatial Morphisms

4.1. **From Sheaf Spaces to Existentially Stratified Objects.** An *Ontoid* is not merely a topological space with sheaves, but a logically stratified entity defined by recursive collapse-resistance. We now define the category in which such objects live and morphisms between them.

4.2. **Definition of Ontoids.**

Definition 4.1 (Ontoid)**.** *An* Ontoid *is a pair*

$$\mathscr{O}_\varepsilon := (X, \mathrm{Fil}_\bullet^{\text{ont}}),$$

where:

- *X is an abstract base object or proto-space;*
- *$\mathrm{Fil}_n^{\text{ont}}\mathcal{F} \subseteq \mathcal{F}$ is an ontological filtration on sheaves $\mathcal{F}$ over X;*

- *For every $\mathcal{F}$, $\mathcal{E}_{\text{exist}}(\mathcal{F}) = \bigcap_n \mathrm{Fil}_n^{\text{ont}}\mathcal{F}$ is well-defined and functorial.*

This structure encapsulates "what survives" across infinite descent, regardless of classical topological interpretation.

4.3. Category $\mathbf{Ont}_{\varepsilon^\infty}$.

Definition 4.2 (Category of Ontoids)**.** *Let $\mathbf{Ont}_{\varepsilon^\infty}$ be the category whose objects are ontoids $(X, \mathrm{Fil}_\bullet^{\text{ont}})$ and morphisms are maps $f : X \to Y$ such that:*

$$f^*(\mathrm{Fil}_n^{\text{ont}}\mathcal{G}) \subseteq \mathrm{Fil}_n^{\text{ont}} f^*(\mathcal{G}), \quad \forall n, \, \mathcal{G} \in \mathsf{Sh}^{\text{ont}}(Y).$$

That is, morphisms preserve existential stratification under pullback.

4.4. Onto-Spatial Morphisms.

A morphism $f : X \to Y$ in $\mathbf{Ont}_{\varepsilon^\infty}$ is said to be:

- **Collapse-preserving** if $\mathscr{C}_\varepsilon(f^*\mathcal{G}) = f^*(\mathscr{C}_\varepsilon(\mathcal{G}))$;
- **Persistence-reflecting** if $\mathcal{E}_{\text{exist}}(\mathcal{F}) \subseteq f^*\mathcal{E}_{\text{exist}}(\mathcal{G})$;
- **Ontological epimorphism** if f is surjective at all filtration levels;
- **Stratified monomorphism** if f is injective on each $\mathrm{Fil}_n^{\text{ont}}$.

4.5. Composition and Identity.

Composition is defined via filtration-level compatibility:

$$(g \circ f)^*\mathrm{Fil}_n^{\text{ont}}\mathcal{H} \subseteq f^* g^* \mathrm{Fil}_n^{\text{ont}}(\mathcal{H}).$$

Identity morphisms preserve all filtration layers exactly. Thus, $\mathbf{Ont}_{\varepsilon^\infty}$ is indeed a well-defined category enriched over filtered morphism classes.

4.6. Functorial Structures and Existential Image.

Let $\mathcal{F} \in \mathsf{Sh}^{\text{ont}}(X)$ and $f : X \to Y$ in $\mathbf{Ont}_{\varepsilon^\infty}$. Define the **existential image**:

$$f_!\mathcal{F} := \bigcup_n \mathrm{Im}(f_*\mathrm{Fil}_n^{\text{ont}}\mathcal{F}),$$

as the cumulative transfer of survival structures under f. It reflects how existence propagates.

4.7. Onto-Subobjects and Internal Collapse Ideals.

Definition 4.3 (Onto-Subobject)**.** *An* onto-subobject $\mathcal{S} \subseteq \mathcal{F}$ *is a subobject such that:*

$$\mathrm{Fil}_n^{\text{ont}}\mathcal{S} = \mathcal{S} \cap \mathrm{Fil}_n^{\text{ont}}\mathcal{F}, \quad \forall n.$$

This ensures that filtration stratification is preserved under internal substructure.

Definition 4.4 (Collapse Ideal)**.** *A collapse ideal* $\mathcal{I} \subseteq \mathcal{F}$ *satisfies:*

$$\forall n,\ \exists m > n \text{ such that } \mathrm{Fil}_m^{\mathrm{ont}} \mathcal{F} \subseteq \mathcal{I}.$$

Collapse ideals represent vanishing layers and may be used to define ε-quotients.

4.8. **Coproducts and Fibered Ontoids.** The category $\mathbf{Ont}_{\varepsilon^\infty}$ admits:

- Coproducts via disjoint filtration unions;
- Fibered products via pullbacks of filtration-preserving morphisms;
- Base change along collapse-stable morphisms.

This gives $\mathbf{Ont}_{\varepsilon^\infty}$ enough categorical structure to support sheafification, site theory, and descent logic.

4.9. **Conclusion.** This section constructed the categorical framework of ontoid geometry:

- Defined objects $(X, \mathrm{Fil}_\bullet^{\mathrm{ont}})$ and morphisms preserving filtration towers;
- Distinguished epimorphisms, monomorphisms, and collapse-image operations;
- Introduced internal structures: subobjects, collapse ideals, existential images;
- Prepared the formal ground for functorial sheaf theory and higher logic.

In the next section, we move beyond classical logic: defining meta-existence via sheaves over ∞-filtered categories and constructing the internal ontology of arithmetic space.

5. Meta-Existence and ∞-Sheaf Realities

5.1. **Beyond Classical Sheaves: From Covering to Persistence.** In classical sheaf theory, a space is understood by how local data can be glued across open covers. In ontoid geometry, gluing is replaced by filtration resilience. Locality is not spatial but ontological: how a section persists across collapse levels.

Thus, we transition to a new kind of sheaf:

An ∞-sheaf$isapresheafvaluedintowersofsurvival, indexedbyrecursiveortransfinitegrowth.$

5.2. **Definition of ∞-Sheaf over Ontoids.**

Definition 5.1 (∞-Sheaf on Ontoid)**.** *Let* $(X, \mathrm{Fil}_\bullet^{\mathrm{ont}}) \in \mathbf{Ont}_{\varepsilon^\infty}$*. An* ∞-sheaf *over* X *is a presheaf*

$$\mathcal{F} : \mathcal{O}^{\mathrm{ont}}(X)^{\mathrm{op}} \to \mathbf{Fil}_{\varepsilon^\infty},$$

where:

- $\mathcal{O}^{\mathrm{ont}}(X)$ *is the poset of ontological strata (indexed by* ε^n*);*

- $\mathbf{Fil}_{\varepsilon^\infty}$ *is the category of logical filtration towers;*
- $\mathcal{F}$ *satisfies ϵ-gluing: local survivals yield global persistence.*

5.3. **Meta-Sheaves and Stratified Descent.** A meta-sheaf $\mathscr{F}$ over X must satisfy:

$$\mathscr{F}(\mathrm{Fil}_n^{\mathrm{ont}}U) \to \mathscr{F}(\mathrm{Fil}_{n-1}^{\mathrm{ont}}U) \text{ is surjective,}$$

meaning information descends faithfully across levels of filtration. This replaces usual sheaf conditions with **logical descent stability**.

5.4. **Existential Gluing Principle.**

Definition 5.2 (Existential Covering)**.** *A family $\{U_i\}_{i\in I}$* existentially covers *U if:*

$$\bigcap_{i\in I} \mathscr{C}_{\varepsilon n}(U_i) = \mathscr{C}_{\varepsilon n}(U) \quad \forall n.$$

That is, the collapse behavior is locally reconstructible.

We then define gluing via pullback of survival, not intersection of open sets.

5.5. **Stacky Ontological Realities.** We define the stack of persistence layers:

$$\mathfrak{E}^{[\infty]} : \mathbf{Ont}_{\varepsilon^\infty}^{\mathrm{op}} \to \mathbf{Cat}$$

assigning to each ontoid X the category of all ϵ-persistent ∞-sheaves on X, with morphisms given by filtration-preserving functors.

This forms the moduli space of logical survival over arithmetic geometry.

5.6. **Ontological Realization Functor.** Let $\mathcal{M}^{[\varepsilon^\infty]}(X)$ be a filtered motivic object. Define:

$$\mathcal{R}_{\mathrm{ont}} : \mathcal{M}^{[\varepsilon^\infty]} \longrightarrow \mathsf{Sh}^{\mathrm{ont}}(X),$$

as the *ontological realization functor*, capturing what survives through motivic stratification under ε-collapse.

5.7. **Postulates of Meta-Existence.**

Postulate I. Existence is the inverse limit of survival.

$$\mathcal{E}_{\text{exist}}(\mathcal{F}) := \varprojlim_n \mathrm{Fil}_n^{\text{ont}} \mathcal{F}.$$

Postulate II. ∞-sheaves classify stable ontological descent.

Postulate III. Spaces are not glued from points, but from persistent towers.

5.8. **Ontological Site and Stratified Grothendieck Topology.** We define a stratified site $(X, \mathcal{T}_\varepsilon)$ where:

- Coverings are existential families;
- Fiber products are formed via pullbacks of survival towers;
- Descent is measured by collapse-preserving glue conditions.

The category $\mathsf{Sh}^{\text{ont}}(X)$ becomes a site of persistence.

5.9. **Conclusion.** This section developed:

- ∞-sheaves and their ϵ-gluing conditions;
- Existential coverings and logical descent morphisms;
- Meta-realization functors and stacky ϵ-structures;
- Postulates recasting "space" in purely persistence-theoretic terms.

In the next section, we connect this to knowledge, interpretation, and proof: examining collapse from an epistemic perspective and reinterpreting logic as a geometric functor.

6. Epistemic Collapse and Formal Sheaf-Theoretic Realism

6.1. **Collapse as Epistemic Constraint.** In the ontoid framework, collapse is not only a structural phenomenon but an epistemic one: it represents the logical limit of what can be known, constructed, or verified at a given level of recursion.

> *To collapse is to vanish from possible proof. To survive collapse is to remain logically visible.*

This transforms geometric persistence into a model of formal knowledge.

6.2. **Knowledge as Survival Through Collapse.** We define:

Definition 6.1 (Epistemic Section)**.** *Let $\mathcal{F} \in \mathsf{Sh}^{\text{ont}}(X)$. A section $s \in \mathcal{F}$ is* epistemic *if*

$$s \in \mathrm{Fil}_n^{\text{ont}} \mathcal{F} \quad \textit{for some } n < \infty.$$

Its epistemic depth is the least n such that s belongs to $\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}$.

6.3. **Sheaf-Theoretic Realism.** We interpret a sheaf $\mathcal{F}$ over an ontoid space as a *formal structure of knowable reality.*

$$\mathcal{F}_{\mathrm{real}} := \mathcal{E}_{\mathrm{exist}}(\mathcal{F}), \quad \text{and} \quad \mathcal{F}_{\mathrm{illusory}} := \mathscr{C}_{\varepsilon}(\mathcal{F}).$$

This defines:

- The *real core*: meta-logically permanent; - The *illusory boundary*: exists only within collapse-vulnerable reasoning.

6.4. **Logic as Collapse Functor.** Let $\mathcal{L}$ be a formal logic system (e.g., arithmetic, type theory). We define the associated collapse functor:

$$\mathscr{C}_{\mathcal{L}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathsf{Sh}^{\mathrm{ont}}(X), \quad \mathscr{C}_{\mathcal{L}}(\mathcal{F}) = \bigcup_{n \text{ unprovable in } \mathcal{L}} \mathscr{C}_{\varepsilon n}(\mathcal{F}).$$

This functor isolates the epistemically inaccessible parts of a space relative to $\mathcal{L}$.

6.5. **Collapse-Sensitive Cohomology.** Define:

$$H^i_{\mathrm{real}}(X, \mathcal{F}) := H^i(X, \mathcal{E}_{\mathrm{exist}}(\mathcal{F})), \quad H^i_{\mathrm{illusory}}(X, \mathcal{F}) := H^i(X, \mathscr{C}_{\varepsilon}(\mathcal{F})).$$

These measure:
- What is provably persistent (real cohomology);
- What is logically inaccessible (illusory cohomology).

6.6. **Interpretation Functors and Modalities.** Let $\mathcal{M}$ be a modal system. Define the interpretation functor:

$$[\![\cdot]\!]_{\mathcal{M}} : \mathsf{Sh}^{\mathrm{ont}}(X) \to \mathsf{Sh}^{\mathrm{ont}}(Y),$$

which sends sections of X to their logical images under a translation modality.

A modality is *collapse-respecting* if:

$$[\![\mathscr{C}_{\varepsilon n}(\mathcal{F})]\!]_{\mathcal{M}} \subseteq \mathscr{C}_{\varepsilon n}([\![\mathcal{F}]\!]_{\mathcal{M}}).$$

6.7. Formal Realism Axiom.

> **Axiom (Formal Realism).** *Only those structures which survive collapse across all epistemic logics should be called real.*

This axiom governs the ontological status of all objects in $\mathbf{Ont}_{\varepsilon^\infty}$.

6.8. Diagram: Collapse–Knowledge–Realism Triangle.

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\mathscr{C}_\varepsilon"] \arrow[dr, "\mathscr{C}_{\mathcal{L}}"'] & & \mathscr{C}_\varepsilon(\mathcal{F}) \\
& \text{Epistemic limit under logic } \mathcal{L} \arrow[ur, dashed] &
\end{tikzcd}$$

This triangle describes how logic bounds the knowable geometry of $\mathcal{F}$.

6.9. Conclusion. This section reinterprets collapse geometrically and philosophically as:

- A measure of epistemic uncertainty;
- A logic-indexed restriction on knowable structure;
- A method to define real vs illusory cohomology;
- A formal realism principle based on collapse-invariant survival.

In the next section, we organize these sheaf-level structures into global gerbes, torsors, and ϵ-cohomological moduli indexed by ontological stratification.

7. ε-Gerbes, Ontological Torsors, and Persistence Moduli

7.1. Sheaf-Level Symmetry and Collapse Invariance. Up to this point, we have treated collapse and persistence as intrinsic properties of sheaves. In this section, we globalize these behaviors by organizing collapse-resilient structures into:

- ε-Gerbes: higher stacks of stratified sheaves;
- Ontological Torsors: symmetry objects tracking survival;
- Persistence Moduli: parameter spaces for ϵ-existence.

7.2. Definition of an ε-Gerbe.

Definition 7.1 (ε-Gerbe)**.** *An ε-Gerbe over an ontoid space X is a stack*

$$\mathscr{G}_\varepsilon : \mathcal{O}^{\mathrm{ont}}(X)^{\mathrm{op}} \to \mathbf{Cat}$$

such that:

- *Each $\mathscr{G}_\varepsilon(\mathrm{Fil}_n^{\mathrm{ont}} U)$ is a groupoid of ε^n-persistent local data;*
- *Gluing across filtration levels is collapse-compatible;*

- *The band of the gerbe is given by* $\mathrm{Aut}(\mathcal{E}_{\mathrm{exist}}(\mathcal{F}))$.

This generalizes usual banded gerbes to stratified logical frameworks.

7.3. **Ontological Torsors.**

Definition 7.2 (Ontological Torsor). *An* ontological torsor $\mathcal{T}^{[\varepsilon^n]}$ *over* X *is a sheaf of sets with free and transitive action by* $\mathrm{Aut}(\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F})$*, such that:*

$$\mathcal{T}^{[\varepsilon^n]} \cdot s = \textit{collapse-preserving translations.}$$

These torsors classify equivalence of existence classes under collapse-stable transformations.

7.4. **Stack of ϵ-Torsors and Higher Sheaf Cohomology.** Let $\mathfrak{T}^{[\varepsilon^\infty]}$ denote the stack assigning to each ontoid space X the groupoid of all ε^n-indexed torsors.

Define the persistence cohomology:

$$H^1_{\varepsilon^\infty}(X, \mathfrak{T}) := \text{classes of torsors over } X \text{ under collapse-invariant isomorphism.}$$

This classifies distinct survival symmetries of $\mathcal{F}$ in a stratified manner.

7.5. **Persistence Moduli and Stratified Parameter Spaces.** We define the *Persistence Moduli Stack* $\mathcal{M}_{\mathrm{pers}}^{[\varepsilon^\infty]}$ as the moduli of ϵ-sheaves up to:

- Collapse-preserving equivalence;
- Ontological torsor translation;
- Survival class stratification.

Definition 7.3 (Moduli Point). *A point* $[\mathcal{F}] \in \mathcal{M}_{\mathrm{pers}}^{[\varepsilon^\infty]}$ *represents a tower class*

$$\left\{\mathrm{Fil}_n^{\mathrm{ont}}\mathcal{F}\right\}_{n\in\mathbb{N}}, \quad \textit{modulo } \epsilon\textit{-torsorial deformation.}$$

7.6. **Descent and Gerbe Cohomology.** There is an ϵ-gerbe cohomology long exact sequence:

$$\cdots \to H^i_\varepsilon(X, \mathcal{G}) \to H^i_\varepsilon(X, \mathscr{G}_\varepsilon) \to H^{i+1}_\varepsilon(X, \mathcal{A}) \to \cdots$$

for gerbes $\mathscr{G}_\varepsilon$ banded by sheaves of groups $\mathcal{A}$. This governs the obstruction theory of persistence descent.

7.7. **Universal ϵ-Torsor and Period Maps.** There exists a universal ϵ-torsor:

$$\mathcal{T}_{\varepsilon^\infty}^{\mathrm{univ}} \in \mathfrak{T}^{[\varepsilon^\infty]}(\mathcal{M}_{\mathrm{pers}}),$$

with associated period map:

$$\pi_\varepsilon : \mathcal{M}_{\mathrm{pers}}^{[\varepsilon^\infty]} \to B_{\varepsilon^\infty,\mathrm{dR}}.$$

This map evaluates the cohomological persistence structure of each survival class.

7.8. **Conclusion.** This section globalizes the internal survival structures by introducing:

- ε-gerbes as collapse-layered stacks;
- Ontological torsors representing ϵ-symmetries of existence;
- Moduli of persistence classes and their universal period torsors;
- ϵ-cohomology as a higher classification of stratified survival.

In the final section, we synthesize all ontoid structures into a concluding metaphysical framework for categorified existence and persistent space.

8. Conclusion: Toward Recursive Geometries of Being

8.1. **Beyond Points, Toward Persistence.** We have abandoned points, sets, and covers. In their place, we have constructed an ontology of space built on collapse-resilient towers of existence.

Each space in this framework is not defined by "what is there," but by:

What survives through logical, epistemic, and recursive collapse.

This persistence defines not only geometry, but the very notion of being.

8.2. **From Collapse to Existence.** Let us summarize the foundational transformation:

Classical Geometry	**Ontoid Geometry**
Sets	Filtration towers
Points	Persistence cores
Open covers	Collapse-resisting families
Sheaves	Survival-indexed sections
Cohomology	Ontological realization of being
Topoi	$\mathbf{Ont}_{\varepsilon^\infty}$ and $\mathsf{Sh}^{\mathrm{ont}}(X)$
Real structure	$\mathcal{E}_{\mathrm{exist}}(\mathcal{F})$
Vanishing locus	$\mathscr{C}_\varepsilon(\mathcal{F})$
Logic	Collapse algebra $\mathscr{C}_\varepsilon$

8.3. **Categorification of Being.** We now reinterpret "existence" as a functor:

$$\mathcal{E}_{\mathrm{exist}} : \mathsf{Sh}^{\mathrm{ont}}(X) \longrightarrow \mathbf{Exist},$$

where **Exist** is the category of ontologically persistent structures. This functor sends sheaves to their essential core of recursive survival.

8.4. **Collapse, Knowledge, and Reality.** We may now assemble the three fundamental aspects:

Collapse: What is removed by logical degeneration**Knowledge:** What is knowable under a logic $\mathcal{L}$**F**

Geometry becomes a stratified model of epistemic realism.

8.5. **Ontoid Philosophy.** Let us conclude with the core philosophical axioms of Ontoid Geometry:

(1) **Existence is indexed.** There is no absolute being—only resistance to collapse.
(2) **Points are obsolete.** What matters is not location but persistence.
(3) **Logic is geometric.** Each logic $\mathcal{L}$ defines collapse $\mathscr{C}_{\mathcal{L}}$.
(4) **Being is filtered.** Objects exist through survival at increasingly deep logical levels.
(5) **Space is survival.** What we call space is the stratification of resilience to non-being.

8.6. **Toward Volume V and Beyond.** In Volume V, we will develop a full arithmetic framework on top of ontoid spaces: categorifying growth, structure, and algebra over filtered realities.

Topics include:

- Growth-function indexed stacks and sheaves;
- ϵ-Gerbe torsors over hypermonodromy sites;
- Arithmetic geometry of categorical stratification;
- Persistence-based stacks and arithmetic descent theory.

This culminates the shift from geometry as shape, to geometry as survival.

— *End of Volume IV*