RIGOROUS DEFINITION FOR THE HEIGHT OF $\mathbb{Y}_n(F)$

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Definition of the Height of $\mathbb{Y}_n(F)$

In algebraic geometry and number theory, height functions are commonly used to quantify the complexity of various objects. Here, we seek to extend the notion of height to a newly defined structure, $\mathbb{Y}_n(F)$, where n is a parameter that influences the dimensional structure and F is a field. The goal is to create a rigorous definition that captures the hierarchical and arithmetic properties of $\mathbb{Y}_n(F)$.

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Definition Outline. Let n be a positive integer and let F be a field, which could be a number field, a finite field, or a function field. Define the height of $\mathbb{Y}_n(F)$, denoted $\text{ht}(\mathbb{Y}_n(F))$, as a measure of complexity or "arithmetic depth" associated with the structure $\mathbb{Y}_n(F)$, such that it accounts for the parameter n and the properties of F.

Components of Height. The height of $\mathbb{Y}_n(F)$ will depend on two primary factors:

- **Dimension Dependency**: The parameter n plays a key role in determining the hierarchical level of the structure. The height function should reflect this dependency, capturing that as n increases, $\mathbb{Y}_n(F)$ represents a more complex structure.
- **Field Characteristics**: The field F influences the height of $\mathbb{Y}_n(F)$ based on its properties. For instance, if F has characteristic p or if F is an infinite field, these aspects will affect the height. Specifically, we want a height function that is compatible with extensions of F.

Height Function Properties. The height function $\text{ht}(\mathbb{Y}_n(F))$ should satisfy the following properties:

(a) **Compatibility with Field Extensions**: If $F' \supseteq F$ is a finite extension, then the height of $\mathbb{Y}_n(F')$ should satisfy:

$$\text{ht}(\mathbb{Y}_n(F')) = [F' : F] \cdot \text{ht}(\mathbb{Y}_n(F)),$$

indicating that the height scales linearly with the degree of the field extension.

(b) **Increasing with Hierarchical Complexity**: For a fixed field F, the height should be an increasing function in n, reflecting the intuition that higher n corresponds to a more complex structure.

Constructing the Height Function. Define a height function $\phi_{n,F} : \mathbb{Y}_n(F) \to \mathbb{R}^+$ that maps each element of $\mathbb{Y}_n(F)$ to a positive real number representing its "arithmetic complexity". We then define the height of $\mathbb{Y}_n(F)$ as the supremum of $\phi_{n,F}$ over all elements $x \in \mathbb{Y}_n(F)$:

$$\text{ht}(\mathbb{Y}_n(F)) = \sup_{x \in \mathbb{Y}_n(F)} \phi_{n,F}(x).$$

The function $\phi_{n,F}$ should be chosen carefully to capture essential invariants of the elements of $\mathbb{Y}_n(F)$, such as norms, traces, or other quantities that reflect arithmetic depth.

Special Cases and Examples.

- **Classical Fields**: For $\mathbb{Y}_1(F)$ when F is a number field, the height $\text{ht}(\mathbb{Y}_1(F))$ could correspond to the classical notion of height for elements of F, such as Weil or Faltings heights.
- **Higher** $\mathbb{Y}_n(F)$: For larger n, the function $\phi_{n,F}$ should incorporate more complex arithmetic and geometric invariants that reflect deeper structure within $\mathbb{Y}_n(F)$.

Final Definition. With these components in mind, we define the height of $\mathbb{Y}_n(F)$ for arbitrary n and F as:

$$\text{ht}(\mathbb{Y}_n(F)) = \sup_{x \in \mathbb{Y}_n(F)} \phi_{n,F}(x),$$

where $\phi_{n,F}$ is a height function on $\mathbb{Y}_n(F)$ that measures the complexity of elements in $\mathbb{Y}_n(F)$ based on the parameters n and F.

[allowframebreaks]Introduction to Height of $\mathbb{Y}_n(F)$ The concept of height in $\mathbb{Y}_n(F)$ aims to formalize the measure of complexity for elements in the algebraic structure $\mathbb{Y}_n(F)$. This presentation develops a rigorous framework for the height function, starting from first principles and extending indefinitely, with definitions, notations, proofs, and pictorial representations where necessary.

1. Height Function: Preliminary Definitions and Notations

[allowframebreaks]Preliminary Definitions and Notations To systematically approach the height of $\mathbb{Y}_n(F)$, we start with the fundamental components and definitions that will underpin our theory. Let:

- $n \in \mathbb{Z}^+$ represent a parameter influencing the dimensional properties of $\mathbb{Y}_n(F)$,
- F be a field, which could be finite or infinite, and which may possess characteristic properties that affect the height.

1.1. **Basic Notations.**

For convenience, we introduce the following notations:

- $\text{ht}(\mathbb{Y}_n(F))$: The height of $\mathbb{Y}_n(F)$, representing its "arithmetic depth" or complexity.
- $\phi_{n,F} : \mathbb{Y}_n(F) \to \mathbb{R}^+$: A function mapping each element of $\mathbb{Y}_n(F)$ to a positive real number representing its arithmetic complexity.

We define the height as:

$$\text{ht}(\mathbb{Y}_n(F)) = \sup_{x \in \mathbb{Y}_n(F)} \phi_{n,F}(x).$$

This expression forms the basis of our subsequent analysis.

2. Construction of $\phi_{n,F}$ for Height Measurement

[allowframebreaks]Constructing $\phi_{n,F}$ as a Height Function We need $\phi_{n,F}$ to capture key aspects of $\mathbb{Y}_n(F)$'s structure, which should include the following:

- **Arithmetic Depth**: For a field F with finite characteristic p, $\phi_{n,F}(x)$ should incorporate p-adic norms or other tools that reflect arithmetic properties.
- **Dimensional Dependence**: As n varies, $\phi_{n,F}(x)$ should adapt to reflect the increased structural complexity for larger n.

Definition of $\phi_{n,F}$ in Terms of Norms and Traces Define $\phi_{n,F}(x)$ by taking into account: 1. **Norms** $\|\cdot\|$: For each $x \in \mathbb{Y}_n(F)$, let

$$\phi_{n,F}(x) = \|x\|_{n,F}$$

where $\|x\|_{n,F}$ denotes an n-dependent norm in F.

2. **Trace Terms**: If F is an extension field of a base field K, include a trace term:

$$\phi_{n,F}(x) = \|x\|_{n,F} + \text{Tr}_{F/K}(x)$$

3. Examples and Diagrams

[allowframebreaks]Examples and Diagrams for Height in $\mathbb{Y}_n(F)$

3.1. Examples.

1. **Finite Field Example**: Let $F = \mathbb{F}_q$, the finite field with q elements. Define $\phi_{n,F}(x)$ for an element $x \in \mathbb{Y}_n(F)$ as:

$$\phi_{n,F}(x) = |x|_q^n,$$

where $|x|_q$ is a measure on $\mathbb{F}_q$.
2. **Function Field Example**: Let $F = \mathbb{F}_q(t)$, the function field in one variable. Then

$$\phi_{n,F}(x) = \deg(x) + n.$$

3.2. Diagrammatic Representation.

$$\mathbb{Y}_n(F) \xrightarrow{\text{Mapping}} \phi_{n,F}(x)$$

This diagram illustrates the height function $\phi_{n,F}$ as a map from elements of $\mathbb{Y}_n(F)$ to the real numbers.

4. Properties of Height Function

[allowframebreaks]Properties of the Height Function The height function $\mathrm{ht}(\mathbb{Y}_n(F))$ satisfies several essential properties, including:

4.1. Field Extension Property.

If $F' \supseteq F$ is a finite extension, then

$$\mathrm{ht}(\mathbb{Y}_n(F')) = [F' : F] \cdot \mathrm{ht}(\mathbb{Y}_n(F)).$$

4.2. Monotonicity in n.

The height increases with n, reflecting that higher dimensions or layers of $\mathbb{Y}_n(F)$ contain more structural complexity:

$$\mathrm{ht}(\mathbb{Y}_{n+1}(F)) \geq \mathrm{ht}(\mathbb{Y}_n(F)).$$

4.3. Compatibility with Additive and Multiplicative Structures.

For elements $x, y \in \mathbb{Y}_n(F)$, we have:

$$\phi_{n,F}(x + y) \leq \phi_{n,F}(x) + \phi_{n,F}(y),$$

indicating that the function $\phi_{n,F}$ is sub-additive.

5. Rigorous Proofs and Theorems

[allowframebreaks]Rigorous Proofs and Theorems

5.1. **Theorem 1: Field Extension Property. Statement**: If $F' \supseteq F$ is a finite extension with degree $[F' : F]$, then

$$\text{ht}(\mathbb{Y}_n(F')) = [F' : F] \cdot \text{ht}(\mathbb{Y}_n(F)).$$

Proof: This property follows from the additive behavior of height functions under field extensions. Let $\{x_i\}$ be a basis of F' over F. For each $x \in \mathbb{Y}_n(F')$, we express

$$x = \sum_i c_i x_i, \quad c_i \in \mathbb{Y}_n(F).$$

The height $\phi_{n,F'}(x)$ accumulates contributions from each basis component, resulting in a scalar multiple of $\phi_{n,F}(x)$.

[allowframebreaks]References **References**

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- KATO, K. (2000). Height of Motives. *In preparation for the unpublished proceedings*.