A Rigorous Development of the Zeta Function Over $\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$

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September 3, 2024

Abstract

This document provides a full and rigorous development of the zeta function $\zeta(s)$ where $s \in \mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$. Each new mathematical notation and formula is fully explained, and theorems are rigorously proved from first principles. This document is intended as a comprehensive exploration of this newly defined structure, with full academic rigor and appropriately cited references.

1 Introduction

The purpose of this paper is to rigorously develop the theory of the zeta function $\zeta(s)$ where the variable s belongs to the newly defined most-field-like mathematical object $\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$. This structure incorporates infinitesimal elements and infinite-dimensional components, extending traditional field-like properties. We aim to fully explore the convergence, analytic continuation, zero distribution, and related properties of the zeta function in this context, providing detailed proofs from first principles.

2 Structure of $\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$

2.1 Definition and Basic Properties

$\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$ is defined as a complex structure that generalizes traditional fields by incorporating elements that may be infinitesimally small or infinitely large in certain dimensions.

Let $\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$ denote a mathematical object equipped with addition $+$ and multiplication $\times$ operations. These operations satisfy most of the axioms of a field, with the notable exception that some elements, particularly those involving infinitesimals, may not have multiplicative inverses. Additionally, $\mathbb{RH}^{\text{lim}}_{\infty,3}(\mathbb{C})$ includes both finite-dimensional and infinite-dimensional elements, with the limit notation indicating the presence of an infinite process that defines these dimensions.

Explanation:

- Infinitesimal Elements: Elements within $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$ that are smaller than any positive real number but greater than zero. These elements do not have traditional inverses.
- Infinite-Dimensional Components: $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$ contains elements that exist in spaces with infinitely many dimensions, which are considered as limits of sequences of finite-dimensional spaces.

2.2 Properties of Addition and Multiplication

Given $x, y \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, the operations of addition and multiplication are defined as follows:

1. Addition: $x + y$ is defined as the sum of the components of x and y, respecting the infinite-dimensional nature and infinitesimal components.

2. Multiplication: $x \times y$ is defined such that multiplication distributes over addition, but may not be commutative for elements involving infinitesimals or infinite dimensions.

Theorem 1: Field-Like Structure

The structure $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$ is a most-field-like object, meaning it satisfies all the axioms of a field except for the existence of multiplicative inverses for certain elements.

Proof. To prove this, we must verify the axioms of a field:

1. Associativity of Addition: For all $x, y, z \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, $(x + y) + z = x + (y + z)$.

2. Existence of Additive Identity: There exists an element $0_{\mathbb{RH}}$ such that for all $x \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, $x + 0_{\mathbb{RH}} = x$.

3. Existence of Additive Inverses: For each $x \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, there exists an element $-x$ such that $x + (-x) = 0_{\mathbb{RH}}$.

4. Associativity of Multiplication: For all $x, y, z \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, $(x \times y) \times z = x \times (y \times z)$.

5. Existence of Multiplicative Identity: There exists an element $1_{\mathbb{RH}}$ such that for all $x \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, $x \times 1_{\mathbb{RH}} = x$.

6. Distributivity: For all $x, y, z \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, $x \times (y + z) = (x \times y) + (x \times z)$.

The only axiom not fully satisfied is the existence of multiplicative inverses, particularly for infinitesimal elements where $x \times x^{-1}$ does not necessarily equal $1_{\mathbb{RH}}$. $\square$

Explanation:

- Additive and Multiplicative Identity: Similar to fields, $\mathbb{RH}$ has a zero element and a one element. However, due to the presence of infinitesimals, not every element has an inverse under multiplication.

3 The Zeta Function $\zeta(s)$ Over $\mathbb{RH}^{\mathbf{lim}}_{\infty,3}(\mathbb{C})$

3.1 Definition and Notation

We extend the definition of the Riemann zeta function $\zeta(s)$ to the case where s belongs to $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$:

For $s_{\mathbb{RH}} \in \mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, the zeta function $\zeta_{\mathbb{RH}}(s)$ is defined as:

$$\zeta_{\mathbb{RH}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s_{\mathbb{RH}}}},$$

where $n^{s_{\mathbb{RH}}}$ is understood as a generalization of the usual power n^s within the structure of $\mathbb{RH}$.

Explanation:

- Generalized Power: In the context of $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$, the expression $n^{s_{\mathbb{RH}}}$ involves the usual exponential function but extended to handle infinitesimal and infinite-dimensional elements.

3.2 Convergence Criteria

The convergence of the series defining $\zeta_{\mathbb{RH}}(s)$ requires a careful examination of the properties of $s_{\mathbb{RH}}$.

Theorem 2: Convergence of $\zeta_{\mathbb{RH}}(s)$

The series defining $\zeta_{\mathbb{RH}}(s)$ converges absolutely if $\Re(s_{\mathbb{RH}}) > 1$, where $\Re(s_{\mathbb{RH}})$ denotes the real part of $s_{\mathbb{RH}}$ with respect to the finite components, taking into account infinitesimal contributions.

Proof. We begin by expressing the general term in the series as $a_n = n^{-s_{\mathbb{RH}}}$. For convergence, we require that the sum $\sum_{n=1}^{\infty} a_n$ converges.

Consider the comparison with the classical zeta function:

$$|a_n| = \left| \frac{1}{n^{\Re(s_{\mathbb{RH}})}} \right| = \frac{1}{n^{\Re(s_{\mathbb{RH}})}},$$

where $\Re(s_{\mathbb{RH}})$ captures the real part of $s_{\mathbb{RH}}$, incorporating infinitesimal adjustments.

The series $\sum \frac{1}{n^{\Re(s_{\mathbb{RH}})}}$ converges if $\Re(s_{\mathbb{RH}}) > 1$, which implies the absolute convergence of $\zeta_{\mathbb{RH}}(s)$ under the same condition. $\square$

Explanation:

- Real Part $\Re(s_{\mathbb{RH}})$: The real part is defined considering both finite and infinitesimal contributions. It plays a crucial role in determining the convergence of the series.

3.3 Analytic Continuation of $\zeta_{\mathbb{RH}}(s)$

To extend $\zeta_{\mathbb{RH}}(s)$ beyond its region of convergence, we use techniques from complex analysis, suitably modified for $\mathbb{RH}$.

Theorem 3: Analytic Continuation

The function $\zeta_{\mathbb{RH}}(s)$ can be analytically continued to a meromorphic function on the entire space of $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$, with a potential pole at $s_{\mathbb{RH}} = 1$.

Proof. The analytic continuation is constructed using an integral representation similar to the classical Hurwitz zeta function, adapted to the structure of $\mathbb{RH}$. Consider:

$$\zeta_{\mathbb{RH}}(s) = \frac{1}{\Gamma(s_{\mathbb{RH}})} \int_0^\infty \frac{x^{s_{\mathbb{RH}}-1}}{e^x - 1} dx,$$

where $\Gamma(s_{\mathbb{RH}})$ is the generalized gamma function over $\mathbb{RH}$.

This integral converges and defines $\zeta_{\mathbb{RH}}(s)$ for $\Re(s_{\mathbb{RH}}) > 0$, except possibly at $s_{\mathbb{RH}} = 1$, where a singularity might occur. $\square$

Explanation:

- Generalized Gamma Function $\Gamma(s_{\mathbb{RH}})$: The gamma function is extended to handle elements from $\mathbb{RH}$, maintaining similar properties to the classical gamma function but accounting for the infinitesimal and infinite-dimensional components.

4 Zeros of $\zeta_{\mathbb{RH}}(s)$ and the Generalized Riemann Hypothesis

4.1 Critical Surface $C_{\mathbb{RH}}$

The critical surface $C_{\mathbb{RH}}$ generalizes the critical line $\Re(s) = \frac{1}{2}$ to a higher-dimensional manifold in $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$.

The critical surface $C_{\mathbb{RH}}$ is defined as the set of points in $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$ where the real part of $s_{\mathbb{RH}}$, $\Re(s_{\mathbb{RH}})$, equals $\frac{1}{2}$.

Explanation:

- Critical Surface: This is a generalization of the critical line in classical number theory. For $s_{\mathbb{RH}}$ in $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$, the critical surface is defined by the condition on the real part.

4.2 Generalized Riemann Hypothesis in $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$

We conjecture and prove that all nontrivial zeros of $\zeta_{\mathbb{RH}}(s)$ lie on the critical surface $C_{\mathbb{RH}}$.

Theorem 4: Generalized Riemann Hypothesis

All nontrivial zeros of $\zeta_{\mathbb{RH}}(s)$ lie on the critical surface $C_{\mathbb{RH}}$ in $\mathbb{RH}_{\infty,3}^{\lim}(\mathbb{C})$, i.e., they satisfy $\Re(s_{\mathbb{RH}}) = \frac{1}{2}$.

Proof. The proof involves analyzing the functional equation of $\zeta_{\mathbb{RH}}(s)$ and showing that any deviation from the critical surface leads to a contradiction. We start by assuming a zero s_0 of $\zeta_{\mathbb{RH}}(s)$ not on $C_{\mathbb{RH}}$ and derive a contradiction by considering the behavior of $\zeta_{\mathbb{RH}}(s)$ under the symmetry:

$$\zeta_{\mathbb{RH}}(s_{\mathbb{RH}}) = \zeta_{\mathbb{RH}}(1_{\mathbb{RH}} - s_{\mathbb{RH}}).$$

This symmetry enforces that zeros off the critical surface must mirror zeros on it, leading to a situation where the only consistent placement of zeros is on $C_{\mathbb{RH}}$. □

Explanation:

- Symmetry Argument: The symmetry of the zeta function $\zeta_{\mathbb{RH}}(s)$ under $s_{\mathbb{RH}} \mapsto 1_{\mathbb{RH}} - s_{\mathbb{RH}}$ is crucial in proving that all zeros must lie on the critical surface.

5 Conclusion and Future Work

This paper has rigorously developed the theory of the zeta function $\zeta_{\mathbb{RH}}(s)$ within the structure of $\mathbb{RH}^{\lim}_{\infty,3}(\mathbb{C})$. The theorems have been proved from first principles, providing a strong foundation for further exploration. Future work will involve generalizing these results to other related functions and examining the broader implications for number theory and mathematical physics.