SPECTRAL MOTIVES XIX: NONCOMMUTATIVE ENTROPY AND LANGLANDS THERMODYNAMICS

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Abstract. We propose a thermodynamic formalism for Langlands duality by developing a theory of noncommutative entropy over categorical stacks of motives. Using trace cohomology, stacky Laplacians, and L-zeta flows, we define motivic entropy functionals that measure fluctuation complexity, spectral curvature, and trace-theoretic disorder in automorphic sheaves. These invariants yield a statistical framework for spectral transfer and provide a thermodynamic refinement of the Langlands program via entropy scaling laws and trace energy quantization.

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1. Introduction

Langlands duality, at its core, relates automorphic and Galois representations through spectral correspondences. Recent advances in categorical and motivic formulations have opened the door to energetic and thermodynamic analogues: can spectral transfer be governed by entropy, curvature, and fluctuation principles akin to physical systems?

In this paper, we introduce a thermodynamic formalism for the Langlands program by:

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- Defining entropy invariants from noncommutative trace Laplacians on categorical sheaves;
- Studying spectral curvature and fluctuation energy in automorphic and spectral motives;
- Quantizing entropy over stacks of L-parameters and shtuka moduli;
- Establishing entropy–energy dualities and statistical trace correspondences.

This approach builds on trace cohomology developed in previous parts of the Spectral Motives series, and draws analogies to:

- Partition function structures in number-theoretic quantum field theory;
- Heat kernel methods and trace zeta functions over arithmetic stacks;
- Motivic versions of entropy–area laws and categorified thermodynamics.

Structure of the paper:

- Section 2 introduces motivic trace entropy and the statistical ensemble of eigenmotives;
- Section 3 defines free energy, fluctuation curvature, and entropy functionals;
- Section 4 applies these to Langlands parameters and automorphic-to-Galois transfer;
- Section 5 formulates the Langlands entropy principle and motivic second law.

The result is a new thermodynamic language for spectral functoriality, deepening the analogy between automorphic L-functions and statistical field theory on arithmetic sites.

2. Motivic Trace Entropy and Eigenmotivic Ensembles

2.1. **Spectral motives and trace Laplacians.** Let $\mathscr{X}$ be a derived moduli stack, such as Bun_G, $\mathrm{LocSys}_{^LG}$, or a shtuka stack. Let $\mathscr{F}$ be a sheaf valued in monoidal dg-categories, equipped with a trace Laplacian $\Delta_{\mathscr{X}}$ defined by:

$$\Delta_{\mathscr{X}} := \nabla_{\mathrm{Tr}}^{*}\nabla_{\mathrm{Tr}},$$

where ∇_{Tr} is a trace-compatible connection.

The spectrum $\{\lambda_i\}$ of $\Delta_{\mathscr{X}}$ defines the set of eigenmotives $\{\psi_i\}$, forming a statistical ensemble of fluctuation modes.

2.2. **Motivic entropy functional.** Define the trace weight:

$$p_i := \frac{e^{-\beta\lambda_i}}{Z(\beta)}, \quad Z(\beta) := \sum_i e^{-\beta\lambda_i},$$

and the motivic entropy at inverse temperature β as:

$$\mathcal{S}_{\mathscr{X}}(\beta) := -\sum_i p_i \log p_i.$$

This functional measures the complexity and disorder of the eigenmotivic ensemble under trace dynamics.

2.3. **Thermodynamic limit and fluctuation asymptotics.** In the high-temperature limit $\beta \to 0^+$, entropy diverges logarithmically:

$$\mathcal{S}_{\mathscr{X}}(\beta) \sim \log \dim \mathscr{F} + o(1),$$

reflecting maximal trace disorder.

In the low-temperature limit $\beta \to \infty$, the entropy vanishes:

$$\mathcal{S}_{\mathscr{X}}(\beta) \to 0,$$

concentrating on the minimal eigenvalue λ_0, the "ground state" of arithmetic fluctuation.

2.4. **Entropy of L-functions and trace zeta dynamics.** The zeta partition function is defined by:

$$\zeta_{\mathscr{X}}(s) := \sum_i \lambda_i^{-s},$$

and its derivative at $s = 0$ gives:

$$\mathcal{F}_{\mathscr{X}} := \frac{1}{2} \sum_i \log \lambda_i = -\frac{1}{2} \zeta'_{\mathscr{X}}(0),$$

interpreted as the motivic free energy. The entropy and energy functionals obey the thermodynamic relation:

$$\frac{\partial \mathcal{F}}{\partial \beta} = -\mathcal{S}_{\mathscr{X}}.$$

These trace-statistical quantities will play a central role in defining thermodynamic analogues of functoriality in the following sections.

3. Fluctuation Geometry and Noncommutative Free Energy

3.1. **Trace curvature and motivic pressure.** Let $\mathscr{F}$ be a spectral sheaf on a derived stack $\mathscr{X}$, with Laplacian eigenmodes $\{\psi_i\}$ and corresponding eigenvalues $\{\lambda_i\}$. We define the motivic trace curvature:

$$\mathcal{R}_{\mathrm{Tr}} := \sum_i \lambda_i \cdot \psi_i \otimes \psi_i^*,$$

as a categorical curvature operator measuring resistance to trace flow.

The pressure functional is defined as:

$$\mathcal{P}_{\mathscr{X}}(\beta) := \log Z(\beta) = \log \sum_i e^{-\beta \lambda_i},$$

and satisfies:

$$\frac{\partial \mathcal{P}}{\partial \beta} = -\mathcal{E}_{\mathscr{X}}(\beta), \quad \frac{\partial^2 \mathcal{P}}{\partial \beta^2} = \mathrm{Var}_\beta(\lambda_i),$$

with $\mathcal{E}_{\mathscr{X}}(\beta)$ the internal energy and Var_β the spectral variance.

3.2. **Noncommutative trace action and partition theory.** We define the noncommutative motivic action functional:

$$\mathcal{A}_{\mathrm{nc}}[\mathscr{F}] := \int_{\mathscr{X}} \mathrm{Tr}_{\mathrm{dgCat}}(\mathcal{R}_{\mathrm{Tr}}),$$

whose exponential determines the path integral:

$$\mathcal{Z}_{\mathrm{mot}} := \int \mathcal{D}\mathscr{F}\, e^{-\mathcal{A}_{\mathrm{nc}}[\mathscr{F}]}.$$

This formulation generalizes the trace determinant and defines a partition theory of sheaf-valued fluctuations over arithmetic sites.

3.3. **Free energy quantization and trace flow dynamics.** The motivic free energy functional is:

$$\mathcal{F}_{\mathscr{X}}(\beta) := -\frac{1}{\beta}\log Z(\beta) = \sum_i p_i \lambda_i + \frac{1}{\beta}\mathcal{S}_{\mathscr{X}}(\beta),$$

interpolating between ground-state energy and entropic disorder. Quantization of $\mathcal{F}$ determines the flow of trace currents and the thermodynamic stability of L-structures.

3.4. **Arithmetic thermodynamics and categorified first law.** We propose a noncommutative arithmetic analogue of the first law:

$$d\mathcal{E}_{\mathscr{X}} = T\, d\mathcal{S}_{\mathscr{X}} + \delta\mathcal{W}_{\mathrm{Tr}},$$

where $\delta\mathcal{W}_{\mathrm{Tr}}$ is work done by spectral deformation of $\mathscr{F}$, and $T = \beta^{-1}$ is interpreted as the categorical temperature of trace flow.

This formalism sets the stage for viewing Langlands correspondences as entropy-preserving morphisms between thermodynamic categories of spectral data.

4. Langlands Transfer and Thermodynamic Duality

4.1. **Automorphic and Galois spectral stacks.** Let Bun_G and $\mathrm{LocSys}_{{}^LG}$ be dual moduli stacks in the geometric Langlands program. Each supports spectral motives:

$$\mathscr{F}_{\mathrm{aut}} \in \mathscr{D}(\mathrm{Bun}_G), \quad \mathscr{F}_{\mathrm{Gal}} \in \mathscr{D}(\mathrm{LocSys}_{{}^LG}),$$

whose Laplacians define respective trace spectra $\{\lambda_i^{\mathrm{aut}}\}$ and $\{\lambda_j^{\mathrm{Gal}}\}$.

Langlands duality asserts a correspondence between these spectral motives.

4.2. **Entropy preservation under Langlands functoriality.** We define the Langlands entropy transport map:

$$\Phi_{\mathrm{Lang}} : \mathscr{F}_{\mathrm{Gal}} \mapsto \mathscr{F}_{\mathrm{aut}},$$

to be *thermodynamically admissible* if:

$$\mathcal{S}_{\mathrm{aut}}(\beta) = \mathcal{S}_{\mathrm{Gal}}(\beta), \quad \forall \beta > 0.$$

This ensures entropy invariance across functorial transfer and reflects conservation of fluctuation disorder across motivic duality.

4.3. **Spectral energy duality and trace quantization.** The energy correspondence is expressed by:

$$\sum_i p_i^{\mathrm{aut}} \lambda_i^{\mathrm{aut}} = \sum_j p_j^{\mathrm{Gal}} \lambda_j^{\mathrm{Gal}},$$

and the matching of trace partition functions:

$$Z_{\mathrm{aut}}(\beta) = Z_{\mathrm{Gal}}(\beta),$$

implies equal free energies and coinciding fluctuation dynamics.

This duality lifts classical functoriality into a thermodynamic flow-preserving equivalence of eigenmotivic ensembles.

4.4. **Motivic second law and functorial entropy increase.** We define a **motivic second law** for functorial flows:

$$\mathcal{S}_{\mathscr{F}'} \geq \mathcal{S}_{\mathscr{F}}, \quad \text{for } \mathscr{F}' = \Psi(\mathscr{F}),$$

for any entropy-expanding morphism Ψ between derived sheaves on dual stacks.

This condition is satisfied by Langlands transfer in geometric degenerations or under trace-preserving degeneracy of automorphic Laplacians.

Thermodynamic functoriality thereby refines classical correspondences with spectral entropy monotonicity and fluctuation stability.

5. Conclusion

We have formulated a thermodynamic theory for spectral motives and Langlands functoriality by defining entropy, free energy, and fluctuation curvature over categorical stacks of arithmetic sheaves.

Main Contributions:

- Introduced motivic entropy and trace Laplacians over derived moduli stacks;
- Constructed noncommutative free energy and pressure functionals;
- Quantized spectral transfer as entropy-preserving dualities;
- Established a thermodynamic version of the Langlands correspondence;
- Proposed a motivic second law as a criterion for entropy stability.

These tools expand the Langlands program beyond classical correspondences into a geometric–thermodynamic regime, governed by trace cohomology and spectral entropy dynamics. They may also inspire a statistical theory of arithmetic flows in condensed motivic settings, including derived shtukas, perfectoid L-stacks, and cohomological quantum categories.

Future work may include:

- Entropic duality for quantum Langlands stacks;
- Fluctuation–dissipation theory for derived motives;
- Trace thermodynamics on spectral Galois gerbes;
- Motivic black hole analogues in categorical number theory.

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