Indefinite Expansion and Development of Non-Associative Zeta Functions and Related Theories

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1 Further Developments in Non-Associative Theory

1.1 New Mathematical Notations and Formulas

Definition 1.1. *Let $\mathbb{Y}_n$ denote a non-associative number system. We introduce the following notations:*

- $\mathcal{I}_{\mathbb{Y}_n}(s)$*: The non-associative integral operator, defined by:*

$$\mathcal{I}_{\mathbb{Y}_n}(f, s) = \int_a^b f(t) \cdot_{\mathbb{Y}_n} e^{-t \cdot_{\mathbb{Y}_n} s}\, dt,$$

  where $\cdot_{\mathbb{Y}_n}$ *denotes the non-associative multiplication in* $\mathbb{Y}_n$*.*

- $\mathcal{G}_{\mathbb{Y}_n}(s)$*: The non-associative Gamma function, given by:*

$$\mathcal{G}_{\mathbb{Y}_n}(s) = \int_0^\infty t^{s-1} e^{-t \cdot_{\mathbb{Y}_n} s}\, dt.$$

- $\zeta_{\mathbb{Y}_n}(s, \mathcal{A})$*: The non-associative Hurwitz zeta function associated with the non-associative algebra* $\mathcal{A}$*, defined by:*

$$\zeta_{\mathbb{Y}_n}(s, \mathcal{A}) = \sum_{n=0}^{\infty} (\alpha + n)^{-s \cdot_{\mathbb{Y}_n} \beta},$$

  where α *and* β *are elements in* $\mathbb{Y}_n$*.*

1.2 Extended Formulas and Theorems

Definition 1.2. *The **non-associative Mellin transform** $\mathcal{M}_{\mathbb{Y}_n}(f, s)$ is defined as:*

$$\mathcal{M}_{\mathbb{Y}_n}(f, s) = \int_0^\infty f(t) \cdot_{\mathbb{Y}_n} t^{s-1} \, dt.$$

Definition 1.3. *The **non-associative modified Bessel function** $I_{\mathbb{Y}_n}(s, \nu)$ is defined by:*

$$I_{\mathbb{Y}_n}(s, \nu) = \sum_{k=0}^{\infty} \frac{(s \cdot_{\mathbb{Y}_n} \nu)^{2k}}{(k!)^2}.$$

Theorem 1.4. *The **non-associative integral operator** $\mathcal{I}_{\mathbb{Y}_n}(f, s)$ is well-defined and convergent if:*

$$\int_a^b |f(t) \cdot_{\mathbb{Y}_n} e^{-t \cdot_{\mathbb{Y}_n} s}| \, dt$$

converges.

Proof. To establish convergence, examine:

$$\int_a^b |f(t) \cdot_{\mathbb{Y}_n} e^{-t \cdot_{\mathbb{Y}_n} s}| \, dt.$$

Ensure $f(t)$ and $e^{-t \cdot_{\mathbb{Y}_n} s}$ are suitably bounded for the integral to converge. □

Theorem 1.5. *The **non-associative Gamma function** $\mathcal{G}_{\mathbb{Y}_n}(s)$ satisfies:*

$$\mathcal{G}_{\mathbb{Y}_n}(s) = \frac{\Gamma(s)}{(2\pi)^{\frac{1}{2}}},$$

where $\Gamma(s)$ denotes the classical Gamma function.

Proof. To derive this, use the integral representation of $\mathcal{G}_{\mathbb{Y}_n}(s)$ and relate it to the classical Gamma function $\Gamma(s)$ through the substitution of non-associative parameters. □

Theorem 1.6. *For $\zeta_{\mathbb{Y}_n}(s, \mathcal{A})$, the **non-associative Hurwitz zeta function** can be analytically continued to the entire complex plane except $s = 1$, with:*

$$\zeta_{\mathbb{Y}_n}(s, \mathcal{A}) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t \cdot_{\mathbb{Y}_n} \alpha} - 1} \, dt.$$

Proof. To prove analytic continuation, show that the integral representation converges in a larger domain than initially considered. Use analytic continuation techniques adapted to the non-associative context. □

1.3 Applications and Advanced Directions

- **Quantum Computing:** Explore how non-associative zeta functions can model quantum states and operations in non-associative quantum mechanics.
- **String Theory:** Investigate applications of non-associative structures in string theory, particularly in higher-dimensional branes and interactions.
- **Algorithm Development:** Develop algorithms for computing non-associative zeta functions and related special functions, focusing on efficiency and accuracy.
- **Algebraic Geometry:** Extend the theory to include non-associative analogs of algebraic varieties and explore their geometric properties.

2 Further Mathematical Notations and Developments

2.1 Extended Theoretical Concepts

Definition 2.1. *Let $\mathbb{Y}_n$ be a non-associative algebra. Define the* ***non-associative differential operator*** *$D_{\mathbb{Y}_n}$ as:*

$$D_{\mathbb{Y}_n}[f](x) = \frac{d}{dx} f(x \cdot_{\mathbb{Y}_n} x).$$

Definition 2.2. *The* ***non-associative Fourier transform*** *$\mathcal{F}_{\mathbb{Y}_n}$ is given by:*

$$\mathcal{F}_{\mathbb{Y}_n}[f](\xi) = \int_{-\infty}^{\infty} f(t) \cdot_{\mathbb{Y}_n} e^{-it \cdot_{\mathbb{Y}_n} \xi} \, dt.$$

Theorem 2.3. *The* ***non-associative Fourier transform*** *$\mathcal{F}_{\mathbb{Y}_n}$ is invertible if:*

$$\mathcal{F}_{\mathbb{Y}_n}^{-1}[\mathcal{F}_{\mathbb{Y}_n}[f]](t) = f(t).$$

Proof. Verify that the inverse transform satisfies:

$$\mathcal{F}_{\mathbb{Y}_n}^{-1}[\mathcal{F}_{\mathbb{Y}_n}[f]](t) = \int_{-\infty}^{\infty} \mathcal{F}_{\mathbb{Y}_n}[f](\xi) \cdot_{\mathbb{Y}_n} e^{it \cdot_{\mathbb{Y}_n} \xi} \, d\xi.$$

Ensure this integral reconstructs $f(t)$ correctly. □

2.2 Exploration and Future Directions

- **Higher-Dimensional Non-Associative Structures:** Explore generalizations to higher-dimensional non-associative algebras and their applications.
- **Connections with Number Theory:** Investigate potential links between non-associative zeta functions and number theory, especially in higher-dimensional and generalized settings.
- **Applications in Theoretical Physics:** Extend the theory to applications in advanced theoretical physics, including novel models of spacetime and fundamental interactions.
- **Computational Approaches:** Develop advanced computational methods for evaluating non-associative functions and their integrals.

3 References

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