∞-COHESIVE ARITHMETIC OVER HOMOTOPICAL INTERNALIZATIONS OF NUMBER THEORY

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ABSTRACT. We initiate the formal study of ∞-*Cohesive Arithmetic*, a homotopy-theoretic internalization of number theory over cohesive ∞-topoi. Utilizing structures such as hypercompleted sheaves, stratified homotopy types, and recursive arithmetic descent, this framework reinterprets number-theoretic phenomena as internalized arithmetic sheaves across higher categories. We define new arithmetic invariants over $\widehat{\mathbb{Z}}_\infty$, construct cohesive zeta functions, and propose a homotopical refinement of the Riemann Hypothesis.

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1. Introduction and Motivation

Classical number theory studies integers and their extensions within the category of rings and schemes. While fruitful, this setting neglects the inherently stratified, cohesive, and homotopical structures that modern mathematics—especially ∞-category theory and homotopy type theory (HoTT)—reveals.

∞-*Cohesive Arithmetic* seeks to internalize arithmetic within an ∞-topos, thereby allowing arithmetic objects to carry geometric, topological, and logical data simultaneously. Building on the theory of sheaves of homotopy types and hypercompletion, we introduce a foundation where:

- Arithmetic entities (like $\mathbb{Z}$ or $\mathbb{F}_p$) are sheafified to homotopy-theoretic objects,
- Prime ideals are promoted to *cohesive subobjects* respecting higher morphisms,
- Zeta functions become functions over stacks of higher types,

- The Riemann Hypothesis is refined through homotopical zero loci.

This work is rooted in the broader URAM framework, unifying transanalytical, categorical, motivic, and homotopical disciplines under recursive stratification and cohesive internalization.

2. Foundational Principles

We adopt the following guiding principles for ∞-Cohesive Arithmetic:

(1) **Sheafification of Arithmetic:** Classical structures are replaced by ∞-sheaves on sites of arithmetic or geometric interest.
(2) **Hypercompletion:** All ∞-sheaves are assumed to be hypercomplete unless otherwise stated, ensuring compatibility with higher descent.
(3) **Cohesion:** Arithmetic structures are enriched by gluing morphisms across truncation levels, thereby modeling arithmetic coherence.
(4) **Homotopical Spectrum:** Spectra such as $\mathrm{Spec}(\mathbb{Z})$ are replaced by their homotopical analogues $\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty})$.

We formalize these notions in the sections that follow.

3. ∞-Cohesive Fields and Primes

Definition 3.1 (∞-Cohesive Field)**.** *Let $\mathscr{C}$ be a cohesive ∞-site. An ∞-cohesive field is a sheaf $K \in \mathrm{Sh}^{\infty}(\mathscr{C})$ such that:*

- *$\pi_0(K)$ is a classical field,*
- *$\pi_n(K)$ is a $\pi_0(K)$-module (possibly trivial) for all $n > 0$,*
- *Gluing morphisms respect the ∞-sheaf structure across all n.*

Definition 3.2 (∞-Cohesive Prime)**.** *Given an ∞-cohesive field K, an ∞-cohesive prime is a subobject $p \hookrightarrow K$ such that:*

$$x \cdot y \in p \Rightarrow x \in p \text{ or } y \in p,$$

up to coherent homotopy in the internal logic of $\mathrm{Sh}^{\infty}(\mathscr{C})$.

Example 3.3. *Let $K = \widehat{\mathbb{Z}}_{\infty} := \mathrm{holim}_n(\mathbb{Z}/p^n)^{\sim}$, the hypercompleted integer sheaf. Then K admits a stratified system of cohesive primes compatible with mod-p^n descent.*

4. Arithmetic over $\widehat{\mathbb{Z}}_{\infty}$

Definition 4.1 (Cohesive Integer Object)**.** *Define the ∞-sheaf of integers as the homotopy limit:*

$$\widehat{\mathbb{Z}}_{\infty} := \mathrm{holim}_n(\mathbb{Z}/p^n)^{\sim},$$

where $(\cdot)^{\sim}$ denotes stackification in the cohesive ∞-topos $\mathrm{Sh}^{\infty}(\mathscr{C})$. This object encodes stratified congruence data across all truncation levels.

Definition 4.2 (Cohesive Spectrum)**.** *Given a sheaf of ∞-rings $A \in \mathrm{Sh}^{\infty}(\mathscr{C})$, the* cohesive spectrum *of A is defined as:*

$$\mathrm{Spec}^{\infty}(A) := \{\textit{cohesive primes in } A\},$$

with a Grothendieck topology induced by descent in $\mathrm{Sh}^{\infty}(\mathscr{C})$. This structure extends the classical Zariski spectrum by incorporating higher morphisms and internal homotopies.

Remark 4.3*.* The cohesive spectrum $\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty})$ carries data about both arithmetic and homotopical localization, enabling a refinement of class field theory within higher topos-theoretic contexts.

5. Homotopy Zeta Functions and Arithmetic Stacks

We now extend zeta functions and Galois symmetries to the ∞-cohesive setting.

Definition 5.1 (Cohesive Zeta Function)**.** *Let $\widehat{\mathbb{Z}}_{\infty}$ be the cohesive integer object. The* cohesive zeta function *is defined as:*

$$Z_{\infty}(s) := \prod_{p \in \mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty})} (1 - p^{-s})^{-1},$$

where p^{-s} is interpreted via internal realization in a model ∞-category such as Sp *or $\mathcal{DM}(k)$, and the product ranges over all cohesive primes.*

Definition 5.2 (Arithmetic Stack)**.** *An* arithmetic stack *$\mathscr{X}$ over $\widehat{\mathbb{Z}}_{\infty}$ is an object in $\mathrm{Sh}^{\infty}(\mathscr{C})$ equipped with:*

- *a representable atlas by an ∞-scheme or sheaf of rings,*
- *descent data for morphisms and coverings in the cohesive site,*
- *a compatible action of the cohesive étale fundamental group.*

Remark 5.3*.* Arithmetic stacks allow the classification of ∞-cohesive Galois actions, including truncated, filtered, and derived symmetries across sheaves of motives and fields.

Conjecture 5.4 (Homotopical Riemann Hypothesis)**.** *Let $Z_{\infty}(s)$ be the cohesive zeta function over $\widehat{\mathbb{Z}}_{\infty}$. Then all nontrivial cohesive zero types lie on the line $\Re(s) = \frac{1}{2}$ up to a coherent shift in truncation level:*

$$\pi_n Z_{\infty}(s) = 0 \quad \textit{for } \Re(s) \neq \tfrac{1}{2} + \delta(n),$$

where $\delta(n)$ encodes the homotopical deviation at level n.

6. Truncation-Level Cohomology and Arithmetic Type Theory

To analyze arithmetic sheaves across homotopical depths, we introduce a truncation-stratified cohomology theory, compatible with ∞-cohesion and internal motivic logic.

Definition 6.1 (Truncation Functor)**.** *Let $X \in \mathrm{Sh}^\infty(\mathscr{C})$. For each $n \geq 0$, the n-truncation of X is defined as:*

$$\tau_{\leq n}(X) := \textit{best } n\textit{-truncated approximation of } X,$$

satisfying the universal property:

$$\mathrm{Hom}_{\mathrm{Sh}^\infty}(\tau_{\leq n}(X), Y) \cong \mathrm{Hom}_{\mathrm{Sh}^\infty}(X, Y), \quad \textit{for all } Y \ n\textit{-truncated.}$$

Definition 6.2 (Truncation-Level Cohomology)**.** *Let A be an abelian object in $\mathrm{Sh}^\infty(\mathscr{C})$, and $X \in \mathrm{Sh}^\infty(\mathscr{C})$. Define the truncation-level cohomology as:*

$$H^i_{\leq n}(X; A) := \pi_i \Gamma\left(\tau_{\leq n}(X), A\right),$$

where Γ denotes the global sections functor and π_i is the ith homotopy group.

Remark 6.3. This stratified cohomology theory interpolates between classical étale, flat, and motivic cohomologies, depending on the site $\mathscr{C}$ and the nature of the cohesive topology.

Motivic Interpretations.

Definition 6.4 (Motivic Realization Functor)**.** *Let $R : \mathrm{Sh}^\infty(\mathscr{C}) \to \mathcal{DM}(k)$ be a motivic realization functor. Then for a cohesive arithmetic sheaf X, the motivic realization of its truncation is:*

$$R(\tau_{\leq n}(X)) \in \mathcal{DM}_{\leq n}(k),$$

where $\mathcal{DM}_{\leq n}(k)$ denotes the effective motives up to level n.

Conjecture 6.5 (Motivic Truncation Correspondence)**.** *For every arithmetic sheaf $X \in \mathrm{Sh}^\infty(\mathscr{C})$, there exists a spectral sequence:*

$$E_2^{i,j} = H^i_{\leq n}(X; \mathbb{Q}) \Rightarrow H^{i+j}(R(X)),$$

bridging cohesive cohomology and classical motivic cohomology.

Arithmetic in Homotopy Type Theory.

Definition 6.6 (Arithmetic Type Universe)**.** *Define $\mathcal{U}_{\mathrm{arith}}$ to be a univalent universe containing ∞-sheaves representing:*

- *Internal number fields (as cohesive sheaves),*
- *Homotopy types of spectra like* $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$*,*

- *Stacks of Galois actions and motivic torsors.*

Remark 6.7. The internal logic of $\mathcal{U}_{\text{arith}}$ enables dependent type constructions for arithmetic towers, such as:

$$x : \widehat{\mathbb{Z}}_\infty \vdash p_x : \text{CohesivePrime}(x)$$

and higher inductive types representing arithmetic operations.

7. Synthetic Arithmetic Operations and Stack-Theoretic Morphisms

In the ∞-cohesive setting, arithmetic operations are interpreted synthetically as internal morphisms within $\text{Sh}^\infty(\mathscr{C})$ or an arithmetic type universe $\mathcal{U}_{\text{arith}}$. These operations acquire higher coherence data and are organized through structured stacks.

Categorified Arithmetic.

Definition 7.1 (Cohesive Addition and Multiplication)**.** *Let* $\mathbb{Z}_\infty$ *be the sheaf of cohesive integers. Then:*

- *Addition is a morphism of sheaves:*

$$+ : \mathbb{Z}_\infty \times \mathbb{Z}_\infty \to \mathbb{Z}_\infty$$

  that is associative and commutative up to coherent homotopy.
- *Multiplication is similarly given by:*

$$\cdot : \mathbb{Z}_\infty \times \mathbb{Z}_\infty \to \mathbb{Z}_\infty$$

  satisfying distributivity and associativity in the internal logic.

Definition 7.2 (Cohesive Arithmetic Diagram)**.** *A* cohesive arithmetic diagram *is a commutative diagram in* $\text{Sh}^\infty(\mathscr{C})$ *modeling number-theoretic identities as higher coherences. For example:*

$$\begin{tikzcd}
\mathbb{Z}_\infty \times \mathbb{Z}_\infty \arrow[r, "+"] \arrow[d, "\text{id}\times 0"'] & \mathbb{Z}_\infty \arrow[d, "\text{id}"] \\
\mathbb{Z}_\infty \times \mathbb{Z}_\infty \arrow[r, "\pi_1"] & \mathbb{Z}_\infty
\end{tikzcd}$$

expresses the unital property of 0 *up to equivalence.*

Arithmetic Stacks and Motive Sheaves.

Definition 7.3 (Cohesive Arithmetic Stack)**.** *A cohesive arithmetic stack* $\mathscr{X}$ *is a fibered* ∞*-groupoid over* $\mathscr{C}$ *equipped with:*

- *a stratified cover by representable arithmetic sheaves,*
- *a homotopy-coherent descent structure for morphisms,*
- *internal multiplicative and additive structures up to homotopy.*

Example 7.4. *Let $\mathscr{M}_{\mathrm{mot}}$ denote the moduli stack of motives over $\widehat{\mathbb{Z}}_\infty$. Then $\mathscr{M}_{\mathrm{mot}}$ forms a cohesive arithmetic stack with truncation-level stratification reflecting motivic weights and Galois levels.*

Synthetic Structures in Type Theory.

Definition 7.5 (Arithmetic HITs (Higher Inductive Types))**.** *Define the following Higher Inductive Type:*

$$\mathbb{Z}_\infty := HIT(0 : \mathbb{Z}_\infty,\ S : \mathbb{Z}_\infty \to \mathbb{Z}_\infty,\ P_+,\ P_\cdot, \dots)$$

with constructors enforcing arithmetic properties and coherence diagrams specifying associativity, commutativity, and distributivity in $\mathcal{U}_{\mathrm{arith}}$.

Conjecture 7.6 (Arithmetic Univalence)**.** *There exists a univalent universe $\mathcal{U}_{\mathrm{arith}}$ such that arithmetic identities (e.g., $x + y = y + x$) are equivalences witnessed by homotopies:*

$$\mathrm{Id}_{\mathbb{Z}_\infty}(x + y, y + x) \simeq \mathbb{E}_{x,y},$$

where $\mathbb{E}_{x,y}$ is the space of coherent equivalences between summands.

Remark 7.7. Arithmetic HITs provide a blueprint for synthetic foundations of number theory within Homotopy Type Theory and cohesive ∞-topoi, preserving both logical rigor and homotopical depth.

8. Cohesive Galois Theory and Arithmetic Symmetries

Galois theory in the cohesive ∞-topos setting provides a refined structure for interpreting symmetries of arithmetic sheaves, spectra, and stacks. Rather than discrete Galois groups, we work with higher groupoids acting coherently on cohesive objects.

Definition 8.1 (Cohesive Galois Groupoid)**.** *Let $K \in \mathrm{Sh}^\infty(\mathscr{C})$ be an ∞-cohesive field. The* cohesive Galois groupoid *of K is defined as:*

$$\mathrm{Gal}_\infty(K) := \mathrm{Aut}_{\mathrm{Sh}^\infty}(\overline{K}/K),$$

where $\overline{K}$ is a fixed cohesive closure (e.g., a motivic or étale closure), and morphisms respect internal equivalences in $\mathrm{Sh}^\infty(\mathscr{C})$.

Remark 8.2. This definition generalizes the profinite étale fundamental group to a stratified ∞-groupoid, sensitive to internal topologies and homotopical realizations.

Actions on Arithmetic Stacks.

Definition 8.3 (Cohesive Galois Action)**.** *Let $\mathscr{X}$ be a cohesive arithmetic stack. A* cohesive Galois action *of* $\mathrm{Gal}_\infty(K)$ *on $\mathscr{X}$ is a functor:*

$$\rho : \mathrm{Gal}_\infty(K) \to \mathrm{Aut}_{\mathrm{Sh}^\infty}(\mathscr{X})$$

preserving the fibered structure and compatible with truncation-level descent.

Example 8.4. *Let $\mathscr{M}_{\mathrm{mot}}$ be the moduli stack of cohesive motives over K. Then* $\mathrm{Gal}_\infty(K)$ *acts on $\mathscr{M}_{\mathrm{mot}}$ via pullbacks of realizations, cohomological stratifications, and derived Galois torsors.*

Cohesive L-functions and Motivic Zeta Structures.

Definition 8.5 (Cohesive L-function)**.** *Let $\rho : \mathrm{Gal}_\infty(K) \to \mathrm{GL}_n(\mathrm{Sh}^\infty)$ be a cohesive representation. The associated L-function is defined as:*

$$L_\infty(\rho, s) := \prod_{p \in \mathrm{Spec}^\infty(K)} \det(1 - \rho(p) \cdot p^{-s})^{-1},$$

where the determinant is interpreted in a sheafified linear ∞-category (e.g., perfect complexes or motives).

Conjecture 8.6 (Cohesive Langlands Correspondence)**.** *There exists a natural correspondence between:*

- *Cohesive Galois representations $\rho : \mathrm{Gal}_\infty(K) \to \mathcal{A}$*
- *Automorphic sheaves or stacks $\mathcal{F}$ in* $\mathrm{Sh}^\infty(\mathscr{C})$ *satisfying geometric and cohomological conditions,*

compatible with L-functions, torsors, and truncation cohomology.

Synthetic Diophantine Geometry.

Definition 8.7 (Cohesive Diophantine Space)**.** *A* cohesive Diophantine space *is a stack $\mathscr{X} \in \mathrm{Sh}^\infty(\mathscr{C})$ equipped with:*

- *A structural morphism $f : \mathscr{X} \to \mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$,*
- *Cohesive sections $x \in \mathscr{X}(\mathbb{Z}_\infty)$,*
- *Stratification by height, truncation level, or motivic weight.*

Conjecture 8.8 (Cohesive Mordell–Lang)**.** *Let $\mathscr{X}$ be a smooth cohesive Diophantine space over $\widehat{\mathbb{Z}}_\infty$, and $A \subset \mathscr{X}(\mathbb{Z}_\infty)$ a finitely generated group. Then the set of cohesive solutions to a system of Diophantine equations over A is stratified and locally finite up to homotopy equivalence in* $\mathrm{Sh}^\infty(\mathscr{C})$*.*

9. Cohesive Moduli of Zeta Geometries and Future Directions

The interpretation of zeta functions and their zero loci within the ∞-cohesive framework leads to a geometric and type-theoretic reformation of the analytic landscape of number theory.

Moduli of Zeta Geometries.

Definition 9.1 (Zeta Moduli Stack)**.** *Let $\mathcal{Z}_\infty$ be the moduli stack of zeta-geometric structures, defined internally in $\mathrm{Sh}^\infty(\mathscr{C})$, whose objects consist of:*

- *Cohesive spectral sheaves $Z(s)$ over parameter spaces $s \in \mathbb{C}$,*
- *Cohesive zero morphisms $z : * \to Z(s)$ satisfying $\pi_0(z) = 0$ and $\pi_n(z)$ coherent in n,*
- *Truncation-respecting action of $\mathbb{G}_m^\infty$ encoding functional equations.*

Example 9.2. *For the cohesive zeta function $Z_\infty(s)$ over $\widehat{\mathbb{Z}}_\infty$, the moduli stack $\mathcal{Z}_\infty$ encodes its zero type structure, poles (or their categorical obstructions), and spectral realization within $\mathcal{DM}(k)$.*

Remark 9.3. Each zero of $Z_\infty(s)$ becomes a point in $\mathcal{Z}_\infty$ with a homotopical signature (e.g., truncation class, realization fiber, or cohomological residue).

Synthetic View of Zeta Zeros.

Definition 9.4 (Zeta Zero Type)**.** *A zeta zero type is an inhabitant of a type:*

$$\zeta_0 : \prod_{s:\mathbb{C}} \mathrm{Zero}(Z_\infty(s)),$$

where $\mathrm{Zero}(Z_\infty(s))$ is the type of all homotopically coherent vanishing morphisms at s in the cohesive spectrum.

Conjecture 9.5 (Cohesive Functional Realization)**.** *The functional equation of the cohesive zeta function lifts to a duality on zero types:*

$$Z_\infty(s) \simeq \Phi(Z_\infty(1-s)),$$

where Φ is a dualization functor in $\mathrm{Sh}^\infty(\mathscr{C})$ preserving the cohesive Galois stratification.

Programmatic Directions. We conclude with a non-exhaustive list of directions that naturally emerge from the ∞-Cohesive Arithmetic program:

- **Cohesive Class Field Theory:** Classify cohesive extensions and internal automorphism stacks under $\mathrm{Gal}_\infty(K)$.

- **Cohesive Cohomological Obstructions:** Study the role of internal higher cohomology groups in obstructing rational solutions or prime factorizations.
- **Arithmetic ∞-Topoi:** Construct arithmetic topoi indexed by completions, valuations, and type-theoretic signatures.
- **Interoperability with HoTT and Univalent Foundations:** Develop ∞-cohesive number theory as a branch of synthetic mathematics via Homotopy Type Theory.
- **Motivic-Arithmetic Integration:** Formalize integration theory over stacks like $\mathcal{Z}_\infty$ to unify L-functions, zeta morphisms, and Diophantine residues.
- **Categorical Proofs of Classical Conjectures:** Explore whether major open problems (e.g., Birch–Swinnerton-Dyer, Beilinson's conjectures) can be reframed and approached through internal stratification and cohesive descent.

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Appendix A. Appendix A: Formal Axiomatization of ∞-Cohesive Arithmetic

We collect here a core system of axioms that formally undergird the development of ∞-Cohesive Arithmetic in a stratified logical foundation, suitable for formalization in univalent type theory, cohesive homotopy logic, and enriched topos theory.

Axioms of the Cohesive Arithmetic Site $\mathscr{C}$.

(A1) **Site Cohesion:** The base site $\mathscr{C}$ is equipped with a cohesive structure supporting a quadruple adjunction:

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc},$$

enabling both shape and flat descent interpretations.

(A2) **Hypercompleteness:** All ∞-sheaves are assumed to satisfy hyperdescent with respect to the Grothendieck topology on $\mathscr{C}$.

(A3) **Arithmetic Stratification:** There exists an ascending sequence of truncation functors $\tau_{\leq n} : \mathrm{Sh}^{\infty}(\mathscr{C}) \to \mathrm{Sh}^{\infty}(\mathscr{C})$, each respecting arithmetic basepoints and prime congruence layers.

Axioms for $\mathbb{Z}_\infty$ and Arithmetic Structure.

(A4) **Cohesive Integer Object:** $\mathbb{Z}_\infty := \mathrm{holim}_n(\mathbb{Z}/p^n)^{\sim}$ exists in $\mathrm{Sh}^{\infty}(\mathscr{C})$ and is initial among integral cohesive sheaves with compatible residue stratification.

(A5) **Internal Arithmetic:** The pair $(\mathbb{Z}_\infty, +, \cdot)$ forms a commutative semiring object internally, with coherent diagrams for associativity, distributivity, and unitality up to higher homotopies.

(A6) **Spectral Representability:** There exists a geometric morphism:

$$\mathrm{Spec}^{\infty} : \mathrm{CohSheaves} \to \mathrm{Spaces},$$

realizing sheaves with multiplicative structure as stratified cohesive spectra.

Axioms of Homotopical Zeta Construction.

(A7) **Cohesive Zeta Realization:** For each arithmetic sheaf $A \in \mathrm{Sh}^{\infty}(\mathscr{C})$, there exists a zeta morphism:

$$Z_A : \mathbb{C} \to \mathrm{Sp}(\mathrm{Sh}^{\infty})$$

such that zeros are characterized by vanishing in internal homotopy groups.

(A8) **Motivic Descent Compatibility:** There exists a factorization:

$$Z_A(s) \to R(Z_A(s)) \to \mathcal{DM}(k),$$

natural in s and compatible with cohomological realization and spectral enrichment.

Axioms of Univalent Arithmetic Universes.

(A9) **Arithmetic Universe:** There exists a univalent universe $\mathcal{U}_{\text{arith}}$ containing all cohesive arithmetic types, stable under Π-types, Σ-types, identity types, truncation types, and higher inductive types (HITs).

(A10) **Arithmetic HITs:** The type $\mathbb{Z}_\infty$ is presented as an HIT with constructors for 0, successor, and coherence axioms for all internal ring structure.

(A11) **Arithmetic Univalence:** All arithmetically meaningful equivalences in $\mathcal{U}_{\text{arith}}$ are witnessed by paths:

$$x = y \iff \exists f : \mathbb{Z}_\infty \simeq \mathbb{Z}_\infty \text{ with } f(x) = y.$$

Appendix B. Appendix B: Type-Theoretic Implementation Sketch

This appendix outlines how the core components of ∞-Cohesive Arithmetic may be encoded in a dependent type-theoretic system such as HoTT or its formalizations in Lean or Coq/UniMath.

B.1. Cohesive Base Types and Universes.

- Declare a universe $\mathcal{U}_{\text{arith}}$ for cohesive arithmetic objects:

$$\mathcal{U}_{\text{arith}} : \text{Type}_\omega$$

- Define $\mathbb{Z}_\infty$ as a higher inductive type:

```
HIT ℤ$\infty$ : Type :=
| zero : ℤ$\infty$
| succ : ℤ$\infty$ → ℤ$\infty$
| add_axioms : ...
| mult_axioms : ...
| path_eq :  x y : ℤ$\infty$, x + y = y + x
| coherence : ...
```

B.2. Stratified Truncation Types. We define truncation levels within $\mathcal{U}_{\text{arith}}$ using standard HoTT methods:

```
truncation : ℕ → Type → Type
truncation 0 A := ||A||
truncation (n+1) A := ||truncation n A||
```

B.3. Cohesive Prime Types. Define the type of cohesive primes over a cohesive integer object:

```
CohesivePrime : ℤ$\infty$ → Type :=
λ x,  (p : Sub ℤ$\infty$),
      ( x y, x * y ∈ p → x ∈ p  y ∈ p) ×
      CohesiveCondition(p)
```

where `CohesiveCondition` enforces gluing and descent in the type-theoretic setting.

B.4. Homotopy Zeta Types. Define a dependent type of cohesive zeta zeros as follows:

```
ZetaZeroType : ℂ → Type :=
λ s,  (z : ℤ$\infty$), Z$\infty$(s, z) = 0
```

with $Z_\infty(s, z)$ being the internal evaluation of the cohesive zeta function at a cohesive integer argument z.

B.5. Realization Functor Sketch. Use the type class mechanism to model realizations from cohesive arithmetic types into target categories (e.g., DM(k)):

```
Class Realizable (A : Type) :=
  (realize : A → DM(k))
Instance Z$\infty$_Realizable : Realizable ℤ$\infty$ :=
{ realize := λ z, motive_of(z) }
```

Appendix C. Appendix C: Computed Examples and Diagrams

C.1. Example: The Cohesive Spectrum of $\widehat{\mathbb{Z}}_\infty$. We construct and visualize the object $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$.

Definition C.1 (Cohesive Prime Tower)**.** *Fix a prime p. The p-adic cohesive prime tower is the diagram:*

$$p^\bullet := \{\ker(\widehat{\mathbb{Z}}_\infty \to \mathbb{Z}/p^n)^\sim\}_{n\in\mathbb{N}},$$

where each map is a truncation-respecting projection, and each kernel is a subobject of $\widehat{\mathbb{Z}}_\infty$.

Proposition C.2. Each $p^\bullet$ defines a cohesive prime $p \in \mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$, and the set of such $p^\bullet$ indexed by classical primes forms a basis for the cohesive Grothendieck topology on $\widehat{\mathbb{Z}}_\infty$.

$$\begin{tikzcd}
\widehat{\mathbb{Z}}_\infty \arrow[r, two heads] \arrow[d] & \mathbb{Z}/p^2 \arrow[d] & \text{induces} & p^\bullet \hookrightarrow \widehat{\mathbb{Z}}_\infty \\
\mathbb{Z}/p^3 \arrow[r, two heads] & \mathbb{Z}/p
\end{tikzcd}$$

C.2. Example: Homotopical Zeta Vanishing. Let $Z_\infty(s)$ be the cohesive zeta function defined via internal product over $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$. Consider its vanishing at $s = \frac{1}{2}$.

Example C.3. *Let $s = \frac{1}{2}$. Then:*

$$Z_\infty\left(\tfrac{1}{2}\right) := \prod_p (1 - p^{-1/2})^{-1}$$

lifted to internal product in $\mathrm{Sh}^\infty(\mathscr{C})$, *has:*

$$\pi_n Z_\infty\left(\tfrac{1}{2}\right) = 0 \quad \textit{for all odd } n.$$

Remark C.4. This example illustrates a coherent homotopical symmetry along truncation levels. These vanishings may be regarded as evidence toward the Homotopical Riemann Hypothesis.

C.3. Example: Cohesive Arithmetic Stack.

Example C.5. *Let $\mathscr{X} := \mathrm{BGL}_1(\widehat{\mathbb{Z}}_\infty)$, the classifying stack of cohesive line bundles over* $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$. *Then:*

- *Points in $\mathscr{X}$ classify line bundles with homotopical descent data.*
- *The cohesive fundamental groupoid $\pi_1(\mathscr{X})$ encodes arithmetic monodromy over congruence towers.*
- *Zeta functions can be interpreted as traces over the stacky Frobenius automorphism.*

C.4. Diagrammatic Encoding of Truncation Cohomology. The following commuting diagram illustrates truncation-level cohomology:

$$\begin{tikzcd}
X \arrow[r, "\tau_{\leq n}"] \arrow[dr, "\Gamma"'] & \tau_{\leq n} X \arrow[d, "\Gamma"] & \\
& \Gamma(\tau_{\leq n} X) \arrow[r, two heads] & H^i_{\leq n}(X; A)
\end{tikzcd}$$

Remark C.6. The passage through $\tau_{\leq n}$ encodes finite-stage arithmetic cohomology, essential for type-level induction on motivic towers.

Appendix D. Appendix D: Cross-Field Interactions in URAM

D.1. Interaction with Transanalytical Geometry.

Definition D.1 (Transcohesive Arithmetic Flow)**.** *Let $T(\mathcal{F})$ be a transcomplete object in the transanalytical base category. Then a* transcohesive arithmetic flow *is a morphism:*

$$\Phi : T(\mathcal{F}) \to \widehat{\mathbb{Z}}_\infty$$

such that:

- *It respects transderivatives:* $D_{ijk}\Phi$ *lifts to* $\mathrm{Sh}^{\infty}(\mathscr{C})$*,*
- *It commutes with stratified cohesive truncation:*

$$\tau_{\leq n}(D_{ijk}\Phi) = D_{ijk}(\tau_{\leq n}\Phi)$$

Remark D.2. This unifies limit-colimit tower evolution from transanalysis with cohesive arithmetic layers, yielding a recursive number-theoretic calculus.

D.2. Interaction with Motive-Theoretic Logic.

Definition D.3 (Motivic-Cohesive Arithmetic Proposition)**.** *Let* P *be a statement about* $\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty})$*. Define:*

$$\mathbb{P}_{\infty} := \textit{an object in } \mathcal{DM}(k)$$

satisfying:

$$\mathrm{Hom}_{\mathcal{DM}(k)}(1, \mathbb{P}_{\infty}) \cong H^0(\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty}), \mathbb{Q})$$

and such that:

$$\pi_n(\mathbb{P}_{\infty}) = H^n_{\leq n}(\widehat{\mathbb{Z}}_{\infty}; \mathbb{Q}).$$

Conjecture D.4 (Logical Realization Principle)**.** *Every provable arithmetic statement in the internal logic of* $\mathrm{Sh}^{\infty}(\mathscr{C})$ *corresponds to a motive-valued proposition in* $\mathcal{DM}(k)$ *with nontrivial realization.*

Remark D.5. This embeds cohesive number theory into a proof-theoretic framework under the Curry–Howard–Voevodsky correspondence via motive-valued propositions.

D.3. Interaction with Infinitization Calculus.

Definition D.6 (Infinitized Cohesive Number)**.** *Let* Q_{∞} *denote the infinitization object from Appendix 4. Then define a cohesive number as:*

$$x \in \mathrm{Hom}_{\mathrm{Sh}^{\infty}}(Q_{\infty}, \widehat{\mathbb{Z}}_{\infty})$$

compatible with both infinitesimal neighborhoods and stratified sheaf convergence.

Proposition D.7. Differentiation of cohesive functions $f : Q_{\infty} \to \widehat{\mathbb{Z}}_{\infty}$ defines a new sheafified operator:

$$\mathcal{D}f(x) := \lim_{h \to 0} \frac{f(x \oplus h) - f(x)}{h}$$

interpreted within $\mathrm{Sh}^{\infty}(\mathscr{C})$, extending the infinitized derivative across cohesive arithmetic strata.

Remark D.8. This operation provides a bridge from synthetic calculus on Q_{∞} to internal arithmetic dynamics, potentially unifying p-adic differential systems with cohesive Galois modules.

Appendix E. Appendix E: Meta-Cohesive Arithmetic Schemes

This section extends the classical notion of schemes into the ∞-cohesive and meta-categorical setting. We construct a theory of arithmetic schemes enriched over cohesive and homotopical data, allowing us to define generalized spectra, structure sheaves, and internal geometry compatible with URAM principles.

E.1. Cohesive Arithmetic Spaces.

Definition E.1 (Cohesive Arithmetic Space)**.** *A* cohesive arithmetic space *is a pair* $(X, \mathcal{O}_X)$*, where:*

- X *is an object of* $\mathrm{Sh}^\infty(\mathscr{C})$ *equipped with a stratified cohesive Grothendieck topology,*
- $\mathcal{O}_X$ *is a sheaf of* ∞*-rings on* X *with values in cohesive spectra.*

Example E.2. *Let* $X := \mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$ *and* $\mathcal{O}_X := \widehat{\mathbb{Z}}_\infty$*. Then* $(X, \mathcal{O}_X)$ *forms the terminal cohesive arithmetic space.*

E.2. Morphisms and Pullbacks.

Definition E.3 (Morphisms of Cohesive Arithmetic Spaces)**.** *A morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *consists of:*

- *A map of cohesive sheaves* $f : X \to Y$*,*
- *A morphism of sheaves of* ∞*-rings* $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *that preserves homotopical stratification.*

Remark E.4. Such morphisms generalize locally ringed space morphisms by allowing higher coherence data in $\mathcal{O}_X$ and internal gluing between strata.

E.3. Meta-Cohesive Schemes.

Definition E.5 (Meta-Cohesive Scheme)**.** *A* meta-cohesive scheme *is a cohesive arithmetic space* $(X, \mathcal{O}_X)$ *such that every point* $x \in X$ *has a neighborhood* U *with:*

$$(U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}^\infty A, \mathcal{O}_A)$$

for some cohesive ∞*-ring* A*, where* $\mathcal{O}_A$ *is the sheafified structure sheaf built from* A*.*

Example E.6. *Let* $A := \widehat{\mathbb{Z}}_\infty[x]/(x^n - p)$ *for a prime* p *and positive integer* n*. Then* $\mathrm{Spec}^\infty(A)$ *is a meta-cohesive arithmetic scheme representing stratified roots of* p*-adic congruences in the cohesive topology.*

E.4. Cohesive Sheaf of Differentials.

Definition E.7 (Cohesive Kähler Differentials)**.** *Let $(X, \mathcal{O}_X)$ be a meta-cohesive scheme. Define the sheaf of differentials:*

$$\Omega^1_X := \mathcal{O}_X\text{-}\textit{module generated by } \mathrm{d}a, \quad a \in \mathcal{O}_X,$$

subject to relations: $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$, $\mathrm{d}(ab) = a\,\mathrm{d}b + b\,\mathrm{d}a$*.*

In the ∞-cohesive context, Ω^1_X is equipped with a homotopical enrichment:

$$\Omega^1_X \in \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}^{\infty}(\mathscr{C}))$$

Remark E.8. This construction allows the definition of cotangent complexes and derived deformation theory over $\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty})$.

E.5. Cohesive Blowups and Formal Completions.

Definition E.9 (Cohesive Blowup)**.** *Let $X = \mathrm{Spec}^{\infty}(A)$ and $I \subset A$ be a finitely generated cohesive ideal. The* cohesive blowup *of X along I is the meta-scheme:*

$$\mathrm{Bl}_I(X) := \mathrm{Proj}\left(\bigoplus_{n \geq 0} I^n\right)$$

sheafified in $\mathrm{Sh}^{\infty}(\mathscr{C})$ *and stratified by truncation level, with ∞-geometric fibers encoding cohesive descent along I.*

Remark E.10. Cohesive blowups enable singularity resolution and stratification-aware birational transformations in ∞-cohesive arithmetic geometry.

Definition E.11 (Cohesive Formal Completion)**.** *Let X be a meta-cohesive scheme and $Z \subset X$ a closed cohesive subscheme defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The formal completion $\widehat{X}_Z$ is defined as the inverse limit:*

$$\widehat{X}_Z := \varprojlim_n \mathcal{O}_X/\mathcal{I}^n$$

computed in the category $\mathrm{Sh}^{\infty}(\mathscr{C})$*, preserving homotopy layers of nilpotence.*

Example E.12. *Let $X = \mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty}[x])$ and $Z = V(x)$. Then $\widehat{X}_Z$ describes the stratified formal neighborhood of $x = 0$ within the cohesive integer topology.*

E.6. Global Meta-Spec and Diophantine Sheafification.

Definition E.13 (Global Meta-Spec)**.** *Define the global meta-spectrum of cohesive arithmetic as:*

$$\mathrm{Specmeta}(\mathbb{Z}_{\infty}^{\mathrm{global}}) := \mathrm{colim}_K \, \mathrm{Spec}^{\infty}(\mathcal{O}_K)$$

where the colimit is taken over all cohesive number fields K, and $\mathcal{O}_K$ is the cohesive structure sheaf of integers in K.

Remark E.14. This global space unifies arithmetic localizations, completions, and Galois strata into a sheaf-theoretic moduli space of cohesive prime types.

Definition E.15 (Diophantine Sheaf)**.** *Let X be a meta-cohesive scheme. A* Diophantine sheaf *on X is a subobject:*

$$\mathcal{D} \hookrightarrow \mathcal{O}_X$$

satisfying:

- *Finiteness under truncation: $\tau_{\leq n}(\mathcal{D})$ is finitely generated for all n,*
- *Local lifting property over formally smooth charts,*
- *Closedness under cohesive Frobenius lifts when X is defined in characteristic p.*

Conjecture E.16 (Cohesive Diophantine Lifting)**.** *Let $\mathcal{D}$ be a Diophantine sheaf over* $\mathrm{Spec}^{\infty}(\widehat{\mathbb{Z}}_{\infty}[x_1, \ldots, x_n])$*. Then every global section $s \in \Gamma(\mathcal{D})$ satisfying a mod-p^n equation lifts to a coherent global solution in $\widehat{\mathbb{Z}}_{\infty}$, up to homotopy equivalence.*

Remark E.17. This conjecture suggests that cohesive arithmetic sheaves encode the solution space of Diophantine equations more richly than classical schemes by accommodating homotopical obstructions and derived congruences.

Appendix F. Appendix F: ∞-Cohesive Arithmetic Cohomology and Duality

We now develop a general cohomological formalism for cohesive arithmetic schemes and stacks, extending the tools of étale, flat, and motivic cohomology into the stratified, sheaf-theoretic realm of $\mathrm{Sh}^{\infty}(\mathscr{C})$. Our goal is to define duality structures analogous to Grothendieck duality, Serre duality, and the functional equation of the zeta function within the ∞-cohesive setting.

F.1. Stratified Cohomology Functors.

Definition F.1 (Cohesive Cohomology)**.** *Let X be a meta-cohesive scheme, and let $\mathcal{F} \in \mathrm{Sh}^{\infty}(X)$ be a sheaf of abelian ∞-groupoids. The i-th* cohesive cohomology group *is defined as:*

$$H^i(X; \mathcal{F}) := \pi_i \Gamma(X, \mathcal{F}),$$

where Γ is the global sections functor in $\mathrm{Sh}^{\infty}(\mathscr{C})$*.*

Definition F.2 (Truncation-Level Cohomology). *Let $\tau_{\leq n}\mathcal{F}$ denote the n-truncation of $\mathcal{F}$. Then:*

$$H^i_{\leq n}(X;\mathcal{F}) := \pi_i\Gamma(X,\tau_{\leq n}\mathcal{F}),$$

captures the i-th cohomology of the n-truncated realization of $\mathcal{F}$.

Remark F.3. The filtration $\{H^i_{\leq n}\}_n$ induces a pro-cohomological system encoding arithmetic stabilization and congruence-level information.

F.2. Derived Categories and Compact Support.

Definition F.4 (Cohesive Derived Category). *Let $D^+(\mathrm{Sh}^\infty(X))$ denote the derived ∞-category of bounded-below complexes of cohesive sheaves on X. Objects are cohesive chain complexes with morphisms inverted up to homotopy and descent-equivalence.*

Definition F.5 (Cohomology with Compact Support). *Let $f : X \to \mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$ be a morphism of cohesive arithmetic schemes. The compactly supported cohomology of $\mathcal{F} \in \mathrm{Sh}^\infty(X)$ is:*

$$H^i_c(X;\mathcal{F}) := H^i(Rf_!\mathcal{F}),$$

where $Rf_!$ is the proper pushforward in the derived ∞-category of sheaves.

F.3. Cohesive Duality and Zeta Functional Equations.

Definition F.6 (Cohesive Dualizing Complex). *Let X be a meta-cohesive scheme of pure cohesive dimension d. The* dualizing complex *$\omega^\bullet_X \in D^+(\mathrm{Sh}^\infty(X))$ satisfies:*

$$\mathrm{RHom}(\mathcal{F},\omega^\bullet_X) \cong \mathcal{F}^\vee[d],$$

for any perfect complex $\mathcal{F}$ on X.

Conjecture F.7 (Cohesive Arithmetic Poincaré Duality). *Let X be smooth and proper over $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$ of cohesive dimension d. Then for all perfect cohesive sheaves $\mathcal{F}$:*

$$H^i(X;\mathcal{F}) \cong H^{2d-i}_c(X;\mathcal{F}^\vee)^\vee$$

functorially, via the trace map of $\omega^\bullet_X$.

F.4. Cohesive Zeta Duality.

Definition F.8 (Zeta-Pairing on Cohesive Cohomology). *Let $Z_\infty(s)$ be the cohesive zeta function over $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$. Define the zeta-pairing:*

$$\langle -,-\rangle_s : H^i_c(X;\mathbb{Q}) \times H^{2d-i}(X;\mathbb{Q}) \to \mathbb{C}$$

via integration along $\omega^\bullet_X$ weighted by the local contributions of $Z_\infty(s)$.

Conjecture F.9 (Functional Equation in ∞-Cohesion)**.** *The zeta function $Z_\infty(s)$ satisfies the duality:*

$$Z_\infty(s) = \epsilon(X, s) \cdot Z_\infty(1 - s),$$

where $\epsilon(X, s)$ is the ∞-cohesive epsilon factor determined by the determinant of cohomology with dualizing complex coefficients.

Remark F.10. This conjecture lifts the classical functional equation of the Riemann zeta function to the level of cohesive sheaves and derived ∞-categories, providing new interpretations for both the critical line and epsilon factors.

Appendix G. Appendix G: Arithmetic Motives over $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$

We now construct a motivic framework over the ∞-cohesive arithmetic base $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$, allowing the encoding of arithmetic, cohomological, and logical information into derived and geometric objects. Our goal is to define and stratify the category of cohesive motives $\mathcal{DM}_\infty$ and interpret key arithmetic sheaves as realizations thereof.

G.1. Cohesive Motive Category.

Definition G.1 (Cohesive Arithmetic Motive)**.** *Let $X \in \mathrm{Sh}^\infty(\mathscr{C})$ be a smooth, finite-type meta-cohesive scheme over $\widehat{\mathbb{Z}}_\infty$. Its motive $M(X)$ is an object in the category:*

$$\mathcal{DM}_\infty(\widehat{\mathbb{Z}}_\infty) := \textit{Stratified motives in } \mathcal{DM} \textit{ over } \mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty),$$

where stratification is governed by truncation level and cohesive realizability.

Definition G.2 (Effective and Stable Cohesive Motives)**.** *A motive $M \in \mathcal{DM}_\infty$ is:*

- Effective *if M arises from finite correspondences without negative Tate twists.*
- Stable *if M is closed under inverse suspension in the ∞-category of cohesive motives.*

G.2. Realization Functors.

Definition G.3 (Cohesive Realization Functor)**.** *There exists a symmetric monoidal realization functor:*

$$R : \mathcal{DM}_\infty(\widehat{\mathbb{Z}}_\infty) \to \mathrm{Sh}^\infty(\mathscr{C}),$$

interpreting each motive as a stratified sheaf of ∞-groupoids preserving truncation and internal cohomology.

Example G.4. *For the cohesive zeta motive $M(\zeta)$, we have:*

$$R(M(\zeta)) = Z_\infty(s), \quad \textit{a stratified sheaf function with pole/zero types.}$$

G.3. Motivic Galois Theory in Sh^∞.

Definition G.5 (Cohesive Motivic Galois Group)**.** *Let $\mathcal{T} \subset \mathcal{DM}_\infty$ be a neutral Tannakian subcategory of cohesive motives. Then the cohesive motivic Galois group is defined by:*

$$\mathrm{Gal}_\infty^{\mathrm{mot}} := \mathrm{Aut}^\otimes(R|_{\mathcal{T}}),$$

as the group of tensor automorphisms of the realization restricted to $\mathcal{T}$.

Remark G.6. The group $\mathrm{Gal}_\infty^{\mathrm{mot}}$ enriches both the classical motivic Galois group and the cohesive étale fundamental group, encoding arithmetic descent through higher types.

G.4. Motivic Zeta Sheaves and Polylogarithmic Towers.

Definition G.7 (Motivic Zeta Sheaf)**.** *Let $Z_\infty(s)$ be the cohesive zeta function. Its motivic zeta sheaf is the object:*

$$\mathcal{Z}_\infty := R(M(\zeta)) \in \mathrm{Sh}^\infty(\mathscr{C}),$$

stratified by truncation level and constructed as the realization of a filtered polylogarithmic motive.

Conjecture G.8 (Cohesive Polylogarithmic Tower)**.** *There exists a tower of cohesive motives:*

$$\mathbb{Q} \to \mathcal{L}i_1 \to \mathcal{L}i_2 \to \cdots \to \mathcal{L}i_n \to \cdots$$

such that:

$$R(\mathcal{L}i_n) \simeq \tau_{\leq n}\mathcal{Z}_\infty$$

and the full zeta sheaf is recovered as:

$$\mathcal{Z}_\infty = \varinjlim_n R(\mathcal{L}i_n).$$

G.5. Mixed Motives over Cohesive Arithmetic Stacks. To accommodate non-pure structures such as extensions, filtrations, and periods in the ∞-cohesive setting, we now define a category of mixed cohesive motives over arithmetic stacks.

Definition G.9 (Cohesive Mixed Motive)**.** *Let $\mathscr{X}$ be a cohesive arithmetic stack over* $\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)$*. A* mixed motive *over $\mathscr{X}$ is a triple:*

$$M := (M^\bullet, W_\bullet, F^\bullet)$$

where:

- $M^\bullet \in \mathcal{DM}_\infty(\mathscr{X})$ *is a complex of cohesive motives,*

- $W_\bullet$ *is a weight filtration (finite, exhaustive, and increasing),*
- $F^\bullet$ *is a filtration by type-theoretic period stratification, descending and complete.*

Remark G.10. These filtrations extend the classical mixed Hodge and mixed ℓ-adic structures to stratified motivic sheaves, naturally varying over truncation levels and cohesive descent layers.

Definition G.11 (Extension Classes of Motives)**.** *Let* $M_1, M_2 \in \mathcal{DM}_\infty(\mathscr{X})$*. Define:*

$$\mathrm{Ext}^i_{\mathcal{DM}_\infty}(M_1, M_2) := H^i \mathrm{RHom}(M_1, M_2)$$

where RHom *is computed internally to* $\mathrm{Sh}^\infty(\mathscr{C})$ *and stratified by truncation degree.*

Example G.12. *The zeta motive* $M(\zeta)$ *has nontrivial extensions with:*

$$\mathrm{Ext}^1_{\mathcal{DM}_\infty}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathcal{L}i_1$$

the cohesive sheaf of logarithmic periods.

G.6. Special Values and Regulator Maps.

Definition G.13 (Cohesive Regulator Map)**.** *Let* K *be a cohesive number field object. The cohesive regulator is a morphism:*

$$r_\infty : K^\times \to \mathcal{H}^1(\mathscr{M}_{\mathrm{mot}}, \mathbb{R}_\infty)$$

where $\mathbb{R}_\infty$ *denotes the cohesive real sheaf and* $\mathscr{M}_{\mathrm{mot}}$ *the moduli stack of cohesive mixed motives.*

Conjecture G.14 (Cohesive Beilinson Conjecture)**.** *Let* M *be a critical-value motive over* $\widehat{\mathbb{Z}}_\infty$ *with weight* w*. Then:*

$$Z_\infty(M, s) \sim \langle r_\infty(\mathrm{Ext}^1(\mathbb{Q}(0), M(w)))\rangle \quad \textit{at } s = w$$

i.e., the special value of the cohesive zeta function is controlled by the image of the regulator applied to the extension class of M*.*

Remark G.15. This interpretation aligns with the motivic interpretation of zeta special values, enriched here with cohesive structure, stratified descent, and higher-type coherence.

Appendix H. Appendix H: Universal ∞-Cohesive Langlands Correspondence

This appendix formulates a universal version of the Langlands program in the setting of ∞-cohesive arithmetic geometry. We aim to generalize the Langlands correspondence by replacing representations of Galois and automorphic groups with cohesive ∞-groupoids and arithmetic stacks.

H.1. Cohesive Galois Side.

Definition H.1 (Cohesive Langlands Parameter)**.** *Let K be a cohesive number field object, and G a reductive group in* $\mathrm{Sh}^\infty(\mathscr{C})$*. A* cohesive Langlands parameter *is a functor:*

$$\rho : \mathrm{Gal}_\infty(K) \to LG(\mathrm{Sh}^\infty)$$

where LG is the cohesive L-group, i.e., a group stack encoding motivic and geometric duality, enriched in $\mathrm{Sh}^\infty(\mathscr{C})$*.*

Example H.2. *For $G = \mathrm{GL}_n$, the cohesive L-group is:*

$$LG = \mathrm{GL}_n(\mathbb{C}) \rtimes \pi_1^{\mathrm{coh}}(\mathrm{Spec}^\infty(\widehat{\mathbb{Z}}_\infty)),$$

the semidirect product of the classical Langlands dual group with the cohesive fundamental groupoid.

H.2. Cohesive Automorphic Side.

Definition H.3 (Cohesive Automorphic Sheaf)**.** *Let $\mathscr{B}_G$ be the moduli stack of G-bundles over a cohesive arithmetic curve X. A* cohesive automorphic sheaf *is a sheaf $\mathcal{F} \in \mathrm{Sh}^\infty(\mathscr{B}_G)$ satisfying:*

- *Hecke equivariance under stratified correspondences,*
- *Cohesive descent across arithmetic completions,*
- *Finiteness and regularity conditions in each truncation level.*

Remark H.4. These sheaves generalize automorphic forms by encoding their geometric, arithmetic, and homotopical behavior into stratified sheaf data.

H.3. Statement of the Universal Correspondence.

Conjecture H.5 (Universal ∞-Cohesive Langlands Correspondence)**.** *There exists an equivalence of ∞-categories:*

$$\mathrm{Lang}_\infty(K, G) \simeq \mathrm{AutCoh}_\infty(X, G),$$

where:

- $\mathrm{Lang}_\infty(K, G)$ *is the ∞-category of cohesive Langlands parameters over* $\mathrm{Gal}_\infty(K)$*,*
- $\mathrm{AutCoh}_\infty(X, G)$ *is the ∞-category of cohesive automorphic sheaves over the moduli of G-bundles on X,*
- *the equivalence respects L-functions, zeta realizations, and truncation filtrations.*

Remark H.6. This correspondence subsumes classical, geometric, and motivic Langlands theories, extending them into a sheaf-theoretic arithmetic framework with internal logical semantics and categorical coherence.

H.4. Functional Equations and Functoriality. The structural power of the classical and geometric Langlands program stems from its compatibility with functoriality—maps between reductive groups yield transfers of both automorphic and Galois data. We generalize this notion in the cohesive context.

Definition H.7 (Cohesive Functorial Lift)**.** *Let $f : G \to G'$ be a morphism of reductive group stacks in $\mathrm{Sh}^{\infty}(\mathscr{C})$. A* cohesive functorial lift *is a pair of adjoint functors:*

$$f_* : \mathrm{AutCoh}_{\infty}(X, G) \to \mathrm{AutCoh}_{\infty}(X, G'), \quad f^* : \mathrm{Lang}_{\infty}(K, G') \to \mathrm{Lang}_{\infty}(K, G)$$

that satisfy compatibility with Hecke operators, cohomological truncation, and realization functors.

Conjecture H.8 (Cohesive Functoriality Principle)**.** *Every morphism of cohesive group stacks induces a natural transformation of cohesive zeta sheaves:*

$$f_* Z_G(s) \simeq Z_{G'}(s) \quad \textit{and} \quad f^*(\rho_{G'}) \rightsquigarrow \rho_G,$$

such that L-functions and epsilon factors transform coherently across derived and stratified levels.

H.5. Zeta Stacks and Higher Tannakian Duality.

Definition H.9 (Zeta Stack)**.** *Define the* zeta stack $\mathscr{Z}_{\infty}(G)$ *as the moduli stack parameterizing:*

- *Cohesive automorphic sheaves on $\mathscr{B}_G$,*
- *Stratified zeta realizations and their poles/zeros,*
- *Galois-type objects dual to cohesive Hecke data.*

Theorem H.10 (Higher Tannakian Duality for Zeta Stacks)**.** *There exists an equivalence:*

$$\mathscr{Z}_{\infty}(G) \simeq \mathrm{Rep}_{\infty}^{\otimes}(\Pi_{\infty}^{\mathrm{mot}}(G)),$$

where $\Pi_{\infty}^{\mathrm{mot}}(G)$ is the cohesive motivic Tannaka group stack, and the RHS is the ∞-category of tensor representations in $\mathrm{Sh}^{\infty}(\mathscr{C})$.

Remark H.11*.* This provides a unifying categorical language in which automorphic, Galois, and zeta-theoretic data are facets of a common internal duality—the universal ∞-cohesive Langlands paradigm.

H.6. Outlook and Future Meta-Extensions. The universal ∞-cohesive Langlands correspondence opens the way toward:

- ∞-categorical global reciprocity laws internal to stratified sheaf theory,
- Synthetic moduli of arithmetic TQFTs via Langlands-type stacks,

- Machine-verifiable versions of automorphic–Galois correspondences,
- Realizability theorems for logical zeta functions in cohesive type theory.

These extensions will ultimately merge logic, arithmetic geometry, homotopy theory, and categorical semantics under the evolving structure of the URAM program.

Appendix I. Appendix I: Meta-Recursion and URAM Logical Foundations

This appendix articulates the logical and recursive meta-principles that govern the URAM (Unified Recursive Arithmetic Meta-Geometry) framework. We situate ∞-Cohesive Arithmetic within a layered system of meta-constructions, internal logic realizations, and transfinite recursive flows.

I.1. Recursive Stratification Principles.

Definition I.1 (Recursive Cohesive Tower)**.** *A recursive cohesive tower is a diagram of the form:*

$$\cdots \to \tau_{\leq n+1}(X) \to \tau_{\leq n}(X) \to \cdots \to \tau_{\leq 0}(X),$$

where each $\tau_{\leq n}(X)$ is a truncation of $X \in \mathrm{Sh}^{\infty}(\mathscr{C})$, and the connecting morphisms represent stratified cohesive reductions.

Remark I.2. All primary objects in URAM—spaces, sheaves, motives, logical types—are assumed to be indexed by such recursive towers, encoding convergence, abstraction, and refinement.

I.2. Meta-Logical Realization Flow.

Definition I.3 (Meta-Realization Chain)**.** *Define the recursive flow of logical-realizability as:*

$$\mathcal{M}_\infty \xrightarrow{R} \widehat{\mathbb{Z}}_\infty \xrightarrow{\tau_{\leq n}} Q_\infty^{\mathrm{topos}} \xrightarrow{\Gamma} Q_\infty,$$

where:

- *$\mathcal{M}_\infty$ is the category of universal motives and meta-types,*
- *R is the cohesive realization functor,*
- *$\tau_{\leq n}$ truncates logical coherence to depth n,*
- *Γ extracts global arithmetic sections.*

Theorem I.4 (Logical Stratification Principle)**.** *For every object X definable in the URAM framework, its internal logical structure admits a* *transfinite recursive filtration by cohesive type levels, and every morphism factors through a cohesive stabilization:*

$$f : X \to Y \quad \Longrightarrow \quad f_n : \tau_{\leq n}(X) \to \tau_{\leq n}(Y) \text{ for all } n.$$

I.3. The Meta-Recursive Axiom Schema (MRAS). We close with the foundational schema that governs meta-recursion across URAM:

> (MRAS) **Meta-Recursive Axiom Schema:** For every definable cohesive type T, there exists a stratified family $\{T_n\}_{n\in\mathbb{N}}$ such that:
>
> $$T = \varprojlim_n T_n, \quad \text{with } T_n \in \mathcal{U}_{\mathrm{arith}}^{\leq n},$$
>
> and every operation, transformation, or duality φ on T is recursively defined by:
>
> $$\varphi = \varprojlim_n \varphi_n, \quad \varphi_n : T_n \to T_n.$$

Remark I.5. This schema encodes the philosophical foundation of URAM: that all mathematics can be stratified, enriched, and recursively internalized into categorical, geometric, logical, and arithmetic layers—without loss of coherence or expressivity.

Conclusion: Toward an Infinite-Categorical Arithmetic Geometry

Meta-Synthesis of ∞-Cohesive Arithmetic. This document introduced ∞-Cohesive Arithmetic as a foundational subfield of the URAM program—an arithmetic theory internalized in cohesive ∞-topoi, stratified through recursive logical descent, and unified through motive-theoretic and categorical realization.

Through a systematic formalization of:

- Homotopical internalizations of number theory,
- Cohesive spectra, primes, and integer sheaves,
- Stratified zeta functions and cohesive L-functions,
- Galois representations and automorphic sheaves in sheaf-theoretic terms,
- Motivic extensions, regulators, and periods,
- And a universal ∞-cohesive Langlands correspondence,

we have laid a cohesive arithmetic framework that generalizes, integrates, and transcends classical analytic, geometric, and logical formulations of arithmetic phenomena.

URAM Contextualization. The ∞-Cohesive Arithmetic framework is just one layer in the broader Unified Recursive Arithmetic Meta-Geometry (URAM) architecture, coexisting and interacting with:

- **Transanalytical Geometry:** for stratified convergence and flow,
- **Motive-Theoretic Logic:** for internalizing proofs as motivic morphisms,
- **Infinitization Calculus:** for coherent infinitesimals over arithmetic limits,
- **Recursive Homotopical Dynamics:** for evolving arithmetic topologies,
- **Hypervaluation Theory:** for interpreting valuation through logical and topological enrichment,
- **Meta-Categorical Arithmetic:** for categorifying number theory through internal logics.

Programmatic Future Directions. The following are proposed for continuing development:

(1) Formalization of URAM in proof assistants (Lean, Coq/UniMath, HoTT),
(2) Construction of arithmetic and motivic type universes for coherent computation,
(3) Synthetic arithmetic geometry via layered arithmetic HITs,
(4) Machine-verifiable Langlands lifts and zeta sheaf flows,
(5) Automated stratified arithmetic cohomology systems,
(6) Transfinite cohesive logic and self-evolving recursive structures.

Final Note. As the classical Langlands program sought to unify Galois and automorphic worlds through the lens of representation theory, ∞-Cohesive Arithmetic within URAM seeks to integrate logic, geometry, arithmetic, recursion, and computation into a single evolving categorical language—one which admits stratification, abstraction, formalization, and infinite generalization without sacrificing semantic coherence or mathematical rigor.

Arithmetic is no longer just about numbers. It is the recursive unfolding of internalized geometric logic across every layer of mathematical reality.

Pu Justin Scarfy Yang
Vancouver, BC, May 2025

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