Constructing Fields Larger than $\mathbb{C}$ Using Automorphic Forms and Motives

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Abstract

This paper explores various modifications of automorphic forms and motives to construct fields larger than $\mathbb{C}$. We extend classical constructions through the introduction of infinitesimals, p-adic numbers, non-commutative geometries, and other advanced mathematical frameworks.

1 Introduction

Fields constructed from $\mathbb{Q}$ using automorphic forms and motives are typically subfields of $\mathbb{C}$. In this work, we seek to extend these constructions to produce fields that are larger than $\mathbb{C}$, by introducing various advanced mathematical frameworks such as infinitesimals, p-adic numbers, non-commutative geometries, and more.

2 Infinitesimal and Hyperreal Extensions

2.1 Infinitesimal Extensions

Let $\mathbb{R}^*$ be the hyperreal field, which includes infinitesimal elements ϵ such that $\epsilon^2 = 0$. Consider an automorphic form f defined over $\mathbb{Q}$, and extend its range to $\mathbb{R}^*$ by allowing $f(\tau)$ to take values in the hyperreals. The resulting field $K_f = \mathbb{Q}(f(\tau) \mid \tau \in \mathbb{H}^*)$, where $\mathbb{H}^*$ is the hyperreal upper half-plane, is a hyperreal extension of $\mathbb{Q}$.

2.2 Hyperreal Automorphic Forms

Similarly, we can consider hyperreal analogues of motives by extending their coefficients to $\mathbb{R}^*$. This results in a field extension K_M that is larger than $\mathbb{C}$ and contains both real and infinitesimal elements.

3 P-adic and Adelic Numbers

3.1 P-adic Automorphic Forms

Consider an automorphic form f_p defined over a p-adic field $\mathbb{Q}_p$. By constructing the field $K_{f_p} = \mathbb{Q}(f_p(\tau_p) \mid \tau_p \in \mathbb{H}_p)$, where $\mathbb{H}_p$ is the p-adic upper half-plane, we obtain a p-adic field extension of $\mathbb{Q}$.

3.2 Adelic Motives

We define motives over the adèles $\mathbb{A}$ by considering motives M with coefficients in $\mathbb{A}$. The field $K_M = \mathbb{Q}(M \mid M \in \mathbb{A})$ is a global extension encompassing all completions of $\mathbb{Q}$, thus extending beyond $\mathbb{C}$.

4 Category-Theoretic and Topos-Theoretic Generalizations

4.1 Topos-Theoretic Motives

Topos theory allows for a generalized approach to motives, where we consider motives defined in a topos $\mathcal{T}$. The corresponding field $K_{\mathcal{T}}$ is constructed by extending $\mathbb{Q}$ with objects in the topos, potentially yielding a field larger than $\mathbb{C}$.

4.2 Category-Theoretic Automorphic Forms

We define automorphic forms in a higher categorical context, where the values are objects in a derived category $\mathcal{D}$. The field $K_f = \mathbb{Q}(\mathrm{Hom}(\mathcal{D}))$ extends $\mathbb{C}$ to include these higher categorical elements.

5 Non-commutative Geometry

5.1 Non-commutative Automorphic Forms

By defining automorphic forms over non-commutative algebras, we obtain a non-commutative field $K_{\mathrm{nc}} = \mathbb{Q}(\mathcal{A})$, where $\mathcal{A}$ is a non-commutative algebra. This field is inherently larger and structurally different from $\mathbb{C}$.

5.2 Non-commutative Motives

Motives defined in a non-commutative geometry context yield fields $K_{\mathrm{ncM}} = \mathbb{Q}(\text{Motives over } \mathcal{A})$ that extend beyond the classical field of complex numbers.

6 Quantum Field Theory and String Theory Generalizations

6.1 Quantum Automorphic Forms

We extend automorphic forms into quantum field theory by considering automorphic forms that are compatible with quantum symmetries. The resulting field K_q is constructed by including quantum operators, potentially leading to a field larger than $\mathbb{C}$.

6.2 String-Theoretic Motives

Incorporating motives into string theory frameworks, we construct fields $K_s = \mathbb{Q}(\text{Motives over String Compactifications})$, which may include additional structures or dimensions not present in $\mathbb{C}$.

7 Infinite-Dimensional Constructs

7.1 Infinite-Dimensional Automorphic Forms

We define automorphic forms in the context of infinite-dimensional algebras, such as loop groups. The field K_∞ constructed in this way is an infinite-dimensional extension of $\mathbb{Q}$.

7.2 Infinite-Dimensional Motives

Motives over infinite-dimensional spaces yield fields $K_{\infty M}$ that transcend the traditional confines of $\mathbb{C}$.

8 Non-Arithmetic Extensions

8.1 Transcendental Extensions

By considering automorphic forms and motives that yield transcendental numbers, we construct fields $K_{\mathrm{trans}} = \mathbb{Q}(\text{Transcendental Values})$ that are algebraically larger than $\mathbb{C}$.

8.2 Non-Arithmetic Motives

Extending motives to include non-algebraic elements yields fields $K_{\mathrm{NA}} = \mathbb{Q}(\text{Non-Arithmetic Motives})$, which cannot be contained within $\mathbb{C}$.

9 Conclusion

This paper has presented several methods to construct fields larger than $\mathbb{C}$ using automorphic forms and motives, each leveraging advanced mathematical frameworks. Future work could explore the implications of these constructions and their applications in various mathematical and physical theories.