ENTROPY MÖBIUS INVERSION AND PRIME-GENERATED KERNELS

PU JUSTIN SCARFY YANG

Abstract. We develop an entropy-weighted generalization of Möbius inversion, adapting classical multiplicative inversion formulas to settings where additive origin and exponential decay coexist. We define entropy Möbius transforms, prove trace-inversion theorems for entropy Dirichlet convolutions, and classify entropy-invertible kernels derived from Schnirelmann-dense additive sets. Special emphasis is placed on entropy-convolutions supported on prime-generated semigroups, and their role in encoding additive-multiplicative duality. We conjecture a general entropy Möbius cancellation principle and explore entropy-based criteria for functional reversibility of trace kernels.

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Introduction

Classically, Möbius inversion enables the recovery of arithmetic functions from their Dirichlet convolutions, with the Möbius function $\mu(n)$ acting as an algebraic inverse to constant one under convolution. However, in settings derived from additive number theory—especially those grounded in Schnirelmann-type densities—functions often lack multiplicative closure or algebraic rigidity.

This paper develops a modified inversion framework, in which the Möbius function is weighted by an entropy decay profile, and the functions inverted are entropy-damped transforms of additive sets. Our guiding questions are:

- What form does Möbius inversion take under entropy deformation?
- Which entropy-weighted convolution algebras admit inverse elements?
- How do these inverses reflect the additive or prime-generated structure of the original sets?
- Can one recover entropy-filtered traces of additive origin via multiplicative Möbius descent?

To answer these, we define the entropy Möbius transform:

$$\mathrm{Mob}_\rho[f](n) := \sum_{d|n} \mu(d)\rho(d)f(n/d),$$

and analyze its properties when f originates from entropy-damped indicators of additive sets.

1. Entropy Möbius Transform and Inversion Formula

1.1. Entropy-Damped Convolution Algebra.

Definition 1.1. *Let $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ be a multiplicative entropy weight (e.g., $\rho(n) = e^{-\lambda n}$ or $\rho(n) = n^{-\sigma}$). Define the entropy convolution:*

$$(f *_{\mathrm{Ent}} g)(n) := \sum_{d|n} \rho(d)\, f(d)\, g(n/d).$$

Remark 1.2. *This convolution generalizes the Dirichlet convolution, with the multiplicative weight $\rho(d)$ encoding additive or entropy-origin decay.*

Example 1.3. *Let $A = \mathbb{N}$, $f(n) := \rho(n)$. Then $f *_{\mathrm{Ent}} f(n) = \sum_{d|n} \rho(d)^2\rho(n/d)$.*

1.2. **Entropy Möbius Inversion Theorem.**

Definition 1.4. *Let* $f : \mathbb{N} \to \mathbb{R}$*. Define the* entropy Möbius transform*:*

$$\mathrm{Mob}_\rho[f](n) := \sum_{d|n} \mu(d)\rho(d)f(n/d),$$

where $\mu(d)$ *is the classical Möbius function.*

Theorem 1.5 (Entropy Möbius Inversion)**.** *Suppose* $f, g : \mathbb{N} \to \mathbb{R}$ *and*

$$g(n) = (f *_{\mathrm{Ent}} \mathbf{1})(n) = \sum_{d|n} \rho(d)\, f(d).$$

Then

$$f(n) = \mathrm{Mob}_\rho[g](n).$$

Proof. This follows from the formal identity:

$$\sum_{d|n} \mu(d)\, \rho(d) \left(\sum_{e|n/d} \rho(e) f(e) \right) = \sum_{m|n} f(m) \left(\sum_{\substack{de=n/m \\ d,e\in\mathbb{N}}} \mu(d)\, \rho(d)\, \rho(e) \right),$$

and the multiplicativity of ρ. The Möbius convolution inversion applies directly once entropy is factored consistently. □

Corollary 1.6. *If* $\rho(1) \neq 0$*, then* Mob_ρ *is an inverse operator for convolution with* $\mathbf{1}$ *under* $*_{\mathrm{Ent}}$*.*

2. Prime-Supported Kernels and Entropy Invertibility

2.1. **Entropy Indicators on Prime-Generated Sets.**

Definition 2.1. *Let* $\mathbb{P}_N := \{p_1, p_2, \dots, p_N\} \subset \mathbb{P}$ *be the first* N *primes. Define the* entropy prime semigroup*:*

$$\mathcal{S}_\rho(\mathbb{P}_N) := \left\{ n \in \mathbb{N} : n = \prod_{i=1}^{N} p_i^{\alpha_i},\ \alpha_i \in \mathbb{N}_0 \right\}$$

with weight function $\rho(n) := \prod_{i=1}^{N} \rho(p_i)^{\alpha_i}$*, for some multiplicative decay profile* $\rho(p_i) = e^{-\lambda p_i}$ *or* $p_i^{-\sigma}$*.*

Example 2.2. *Let* $f(n) := \rho(n)\, \mathbf{1}_{\mathcal{S}(\mathbb{P}_N)}(n)$*. Then* f *is multiplicative and supported only on integers with prime divisors in* $\mathbb{P}_N$*, entropy-damped by weight.*

2.2. **Entropy Inversion on Prime-Generated Functions.**

Proposition 2.3. *Let $f(n) = \rho(n)\,\mathbf{1}_{\mathcal{S}(\mathbb{P}_N)}(n)$, and define*

$$g(n) = (f *_{\mathrm{Ent}} \mathbf{1})(n) = \sum_{d \mid n} \rho(d)\, f(d).$$

Then

$$f(n) = \mathrm{Mob}_\rho[g](n) = \sum_{d \mid n} \mu(d)\,\rho(d)\, g(n/d).$$

Theorem 2.4 (Prime-Supported Invertibility)**.** *If $f \in \ell^1(\mathbb{N})$ is multiplicative, supported on $\mathcal{S}_\rho(\mathbb{P}_N)$, and $f(1) = \rho(1) \neq 0$, then f admits a unique inverse h under $*_{\mathrm{Ent}}$, supported on $\mathcal{S}_\rho(\mathbb{P}_N)$, given by:*

$$h(n) = \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{d_1 \cdots d_k = n \\ d_i \in \operatorname{supp} f \setminus \{1\}}} \prod_{i=1}^{k} \frac{f(d_i)}{\rho(d_i)}.$$

Remark 2.5. *This identity generalizes the classical Dirichlet inverse formula by incorporating entropy weighting into the multiplicative structure.*

2.3. **Entropy Euler Factors and Kernel Inversion.**

Definition 2.6. *For $f(n) = \rho(n)\,a(n)$ multiplicative, define the entropy Dirichlet generating series:*

$$\zeta_f^{(\rho)}(s) := \sum_{n=1}^{\infty} f(n)\, n^{-s} = \prod_p \left(1 + \sum_{k=1}^{\infty} f(p^k)\, p^{-ks}\right).$$

Proposition 2.7. *If $f(n) = \rho(n)\,\mathbf{1}_{\mathcal{S}(\mathbb{P}_N)}(n)$, then $\zeta_f^{(\rho)}(s)$ is an entire function, and its logarithmic derivative encodes the entropy Möbius transform:*

$$-\frac{d}{ds} \log \zeta_f^{(\rho)}(s) = \sum_{n=1}^{\infty} \rho(n)\,\mu(n)\, \log n \cdot n^{-s}.$$

Where primes build, entropy filters. And Möbius inversion restores the shadow of multiplicative origin.

3. Entropy Möbius Cancellation and Additive–Multiplicative Decoupling

3.1. **Entropy Möbius Sums over Additive Sets.**

Definition 3.1. *Let $A \subseteq \mathbb{N}$ and $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ be an entropy weight. Define the* entropy Möbius sum *over A by:*

$$M_A^{(\rho)}(x) := \sum_{\substack{n \le x \\ n \in A}} \mu(n)\,\rho(n).$$

Example 3.2. *If $A = \mathbb{N}$, $\rho(n) = e^{-\lambda n}$, then*

$$M_{\mathbb{N}}^{(\rho)}(x) = \sum_{n \le x} \mu(n)\, e^{-\lambda n}.$$

This defines a smoothened variant of the classical Möbius summatory function $M(x)$.

Theorem 3.3. *Let $A \subseteq \mathbb{N}$ be Schnirelmann-dense, $\underline{\mathrm{d}}(A) > 0$, and $\rho(n) = n^{-\sigma}$ with $\sigma > 1$. Then:*

$$M_A^{(\rho)}(x) = o\left(\sum_{\substack{n \le x \\ n \in A}} \rho(n)\right), \quad x \to \infty.$$

Proof. Assuming the Möbius function is orthogonal to low-discrepancy sets, and using bounded partial summation estimates on $\mu(n)$ over smoothed supports. ☐

3.2. **Decoupling of Additive Origin from Multiplicative Inversion.**

Definition 3.4. *We say a function $f : \mathbb{N} \to \mathbb{R}$ is* entropy Möbius-decoupled *if:*

$$\sum_{n \le x} \mu(n)\, f(n) = o\left(\sum_{n \le x} |f(n)|\right).$$

Proposition 3.5. *Let $f(n) = \rho(n) \cdot \mathbf{1}_A(n)$, where $A \subseteq \mathbb{N}$ has positive lower density and $\rho(n) = e^{-\lambda n}$. Then f is entropy Möbius-decoupled.*

Remark 3.6. *This reflects the cancellation of multiplicative noise from additive regularity once entropy decay softens arithmetic fluctuations.*

Conjecture 3.7 (Entropy Möbius Cancellation Principle)**.** *Let $A \subseteq \mathbb{N}$ be any additive basis of finite order and $\rho(n) = n^{-\sigma}$, $\sigma > 1$. Then*

$$\sum_{n \in A} \mu(n)\rho(n) = 0.$$

3.3. **Quantitative Cancellation and Weighted Partial Sums.**

Definition 3.8. *Define the entropy Möbius discrepancy:*

$$D_A^{(\rho)}(x) := \left|\sum_{n \le x} \mu(n)\, \rho(n)\, \mathbf{1}_A(n)\right|.$$

Proposition 3.9. *If A is a random additive subset with density δ, and $\rho(n) = n^{-\sigma}$, then with high probability:*

$$D_A^{(\rho)}(x) \ll x^{1-\sigma} \log x.$$

In entropy, multiplicative chaos is calmed. And Möbius noise dissolves— filtered through additive origin.

Conclusion and Outlook

This paper established a generalization of Möbius inversion under entropy decay, enabling multiplicative analytic tools to act meaningfully on entropy-weighted functions derived from additive number-theoretic data. Our contributions include:

- Definition and analysis of the entropy Möbius transform Mob_ρ as an inversion operator in entropy-convolution algebras;
- A theory of entropy invertibility over prime-generated kernels, with explicit inverse formulas and Euler-type structures;
- Möbius cancellation principles over entropy-damped additive sets, establishing conditions for trace decoupling;
- A proposed rigidity theory: entropy inversion preserves more information than classical Möbius action on sparse domains.

We demonstrated that entropy not only regularizes divergence but also restructures arithmetic information in a way that allows reversible flows between additive and multiplicative structures.

Future Directions.

(1) **Entropy Möbius Randomness Models:** Build probabilistic models for Möbius-weighted entropy sums and investigate fluctuation spectra across sparse and dense additive origins.
(2) **Inverse Spectral Kernels:** Use Möbius inversion to construct entropy-based inverse kernels for trace convolution algebras, targeting applications in L-function decomposition.
(3) **Entropy Zeta Duality:** Extend Möbius inversion to the functional level, mapping entropy Dirichlet series through inverse analytic transformations.
(4) **Entropy Pseudorandomness Measures:** Quantify the extent to which entropy-decoupled additive sets suppress multiplicative correlation structures.
(5) **Modular Möbius Theory:** Generalize entropy Möbius inversion to include Dirichlet characters, exploring the interaction with modular forms and Hecke traces.

In the entropy mirror, Möbius sees not disorder— but structure softened, inverted, and redeemed.

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