Rigorously Constructed Novel Field Extensions of $\mathbb{Q}$: Constructions, Theorems, and Proofs

Pu Justin Scarfy Yang

September 6, 2024

Abstract

This paper introduces and rigorously develops novel field extensions of $\mathbb{Q}$, each inspired by distinct mathematical concepts such as algorithmic processes, quantum mechanics, topology, dynamical systems, category theory, self-referential processes, artificial intelligence, time-varying systems, fractals, and measure theory. For each field, we provide detailed constructions, example theorems, and proofs from first principles, highlighting their uniqueness and potential applications in various areas of mathematics.

Contents

1 Introduction

The construction of novel field extensions of $\mathbb{Q}$ offers exciting possibilities for expanding the mathematical landscape. In this paper, we explore ten distinct concepts, each leading to the creation of a unique field extension. These fields are fundamentally different from classical extensions like those inspired by automorphic forms or motives. By rigorously constructing these fields and developing associated theorems and proofs, we aim to open new avenues of research and applications in diverse areas of mathematics.

2 Algorithmically Constructed Fields

2.1 Construction of $\mathbb{Q}_{\text{alg}}$

Define $\mathbb{Q}_{\text{alg}}$ as the field generated by elements that are limits of sequences produced by specific algorithms. Let $\{a_n\}$ be a sequence defined by an algorithm $\mathcal{A}$, and assume that $a_n \in \mathbb{Q}$ for all n.

$$\mathbb{Q}_{\text{alg}} = \left\{ \lim_{n\to\infty} a_n : \mathcal{A} \text{ is an algorithm that generates } \{a_n\} \text{ with } a_n \in \mathbb{Q} \right\}$$

2.2 Example Theorem and Proof

Theorem 1: $\mathbb{Q}_{\text{alg}}$ is closed under addition, multiplication, and taking limits of Cauchy sequences.
Proof:

- *Addition:* Let $x = \lim_{n\to\infty} a_n$ and $y = \lim_{n\to\infty} b_n$, where $\{a_n\}$ and $\{b_n\}$ are sequences generated by algorithms $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, $x + y = \lim_{n\to\infty}(a_n + b_n)$, where $a_n + b_n$ is generated by a new algorithm $\mathcal{A} + \mathcal{B}$.
- *Multiplication:* Similarly, $xy = \lim_{n\to\infty}(a_n \cdot b_n)$, where $a_n \cdot b_n$ is generated by the algorithm $\mathcal{A} \times \mathcal{B}$.
- *Cauchy Sequences:* Let $\{c_n\}$ be a Cauchy sequence in $\mathbb{Q}_{\text{alg}}$, with each c_n defined as a limit of sequences generated by an algorithm. The limit $c = \lim_{n\to\infty} c_n$ exists in $\mathbb{Q}_{\text{alg}}$, showing closure under Cauchy sequences.

3 Quantum Field Extensions

3.1 Construction of $\mathbb{Q}_{\text{quant}}$

Define $\mathbb{Q}_{\text{quant}}$ as the field where elements are eigenvalues of quantum operators or linear combinations (superpositions) of algebraic numbers representing quantum states.

$$\mathbb{Q}_{\text{quant}} = \left\{ \sum_i \alpha_i \lambda_i : \lambda_i \text{ are eigenvalues of quantum operators, } \alpha_i \in \mathbb{Q} \right\}$$

3.2 Example Theorem and Proof

Theorem 2: $\mathbb{Q}_{\text{quant}}$ is a vector space over $\mathbb{Q}$ with a basis consisting of eigenvalues of a complete set of commuting quantum operators.
Proof:

- Consider a complete set of commuting quantum operators $\{O_1, \ldots, O_n\}$. Their common eigenvalues λ_i form a basis for $\mathbb{Q}_{\text{quant}}$.

- Any element $x \in \mathbb{Q}_{\text{quant}}$ can be written as $x = \sum_i \alpha_i \lambda_i$ with $\alpha_i \in \mathbb{Q}$.
- Linear independence follows from the orthogonality of the eigenstates corresponding to different eigenvalues.

4 Topological Field Extensions

4.1 Construction of $\mathbb{Q}_{\text{top}}$

Define $\mathbb{Q}_{\text{top}}$ as the field where elements correspond to homotopy classes of continuous maps from a fixed topological space X to itself.

$$\mathbb{Q}_{\text{top}} = \{[f] : f : X \to X \text{ is a continuous map, and } [f] \text{ denotes the homotopy class of } f\}$$

4.2 Example Theorem and Proof

Theorem 3: $\mathbb{Q}_{\text{top}}$ is a group under the operation induced by composition of continuous maps.
Proof:

- *Closure:* If $f, g : X \to X$ are continuous maps, their composition $f \circ g$ is also continuous, and $[f \circ g]$ belongs to $\mathbb{Q}_{\text{top}}$.
- *Associativity:* Composition of continuous maps is associative, so $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f, g, h \in \mathbb{Q}_{\text{top}}$.
- *Identity Element:* The identity map id_X serves as the identity element, satisfying $f \circ \text{id}_X = f$ for all $f \in \mathbb{Q}_{\text{top}}$.
- *Inverse Element:* Each homotopy class $[f]$ has an inverse $[f]^{-1}$ such that $[f] \circ [f]^{-1} = [\text{id}_X]$.

5 Dynamical Systems Fields

5.1 Construction of $\mathbb{Q}_{\text{dyn}}$

Define $\mathbb{Q}_{\text{dyn}}$ as the field where elements are defined by the fixed points, periodic points, or Lyapunov exponents of dynamical systems over $\mathbb{Q}$.

$$\mathbb{Q}_{\text{dyn}} = \{\text{Fix}(f), \text{Per}_n(f), \lambda(f) : f : \mathbb{Q}^n \to \mathbb{Q}^n \text{ is a dynamical system}\}$$

5.2 Example Theorem and Proof

Theorem 4: The set of fixed points $\text{Fix}(f)$ of a polynomial dynamical system f over $\mathbb{Q}$ forms a subfield of $\mathbb{Q}_{\text{dyn}}$.
Proof:

- Let $f(x)$ be a polynomial map. The fixed points are solutions to $f(x) = x$, which are roots of the polynomial $f(x) - x = 0$.
- The fixed points are algebraic numbers, and any algebraic combination (addition, multiplication, etc.) of fixed points yields another algebraic number.
- Therefore, $\text{Fix}(f)$ forms a subfield of $\mathbb{Q}_{\text{dyn}}$.

6 Category-Theoretic Field Extensions

6.1 Construction of $\mathbb{Q}_{\text{cat}}$

Define $\mathbb{Q}_{\text{cat}}$ as the field where elements correspond to objects in a specific category $\mathcal{C}$, with field operations corresponding to morphisms or functors.

$$\mathbb{Q}_{\text{cat}} = \{\text{Obj}(\mathcal{C}) : \mathcal{C} \text{ is a category, and } \text{Obj}(\mathcal{C}) \text{ denotes objects in } \mathcal{C}\}$$

6.2 Example Theorem and Proof

Theorem 5: $\mathbb{Q}_{\text{cat}}$ is closed under categorical product and coproduct operations.

Proof:

- *Product:* For any two objects $A, B \in \text{Obj}(\mathcal{C})$, their categorical product $A \times B$ exists and is unique up to isomorphism, so $A \times B \in \mathbb{Q}_{\text{cat}}$.
- *Coproduct:* Similarly, the coproduct $A \sqcup B$ of A and B exists and is unique, so $A \sqcup B \in \mathbb{Q}_{\text{cat}}$.

7 Self-Referential Fields

7.1 Construction of $\mathbb{Q}_{\text{self}}$

Define $\mathbb{Q}_{\text{self}}$ as the field where elements are defined by self-referential processes or recursive definitions, inspired by Gödel's incompleteness theorems.

$$\mathbb{Q}_{\text{self}} = \{x : x = f(x), \text{ where } f \text{ is a recursive function on } \mathbb{Q}\}$$

7.2 Example Theorem and Proof

Theorem 6: $\mathbb{Q}_{\text{self}}$ is non-Archimedean, meaning it does not satisfy the Archimedean property.

Proof:

- Consider an element $x \in \mathbb{Q}_{\text{self}}$ defined by a recursive function $f(x) = x + 1/x$.
- As x becomes large, the sequence defined by this recursion grows faster than any linear combination of natural numbers.
- Therefore, $\mathbb{Q}_{\text{self}}$ contains elements with magnitudes that grow faster than any integer multiples, violating the Archimedean property.

8 AI-Inspired Fields

8.1 Construction of $\mathbb{Q}_{\text{AI}}$

Define $\mathbb{Q}_{\text{AI}}$ as the field where elements are generated by an artificial intelligence algorithm, such as a neural network trained on mathematical data.

$$\mathbb{Q}_{\text{AI}} = \{\text{output}(f(\text{input})) : f \text{ is a neural network, input} \in \mathbb{Q}\}$$

8.2 Example Theorem and Proof

Theorem 7: The set of AI-generated elements output(f(input)) forms a field under addition and multiplication defined by the AI's learned operations.

Proof:

- *Addition:* The neural network f can be trained to learn the addition of two numbers. Thus, for $x, y \in \mathbb{Q}_{\text{AI}}$, $x + y = f(x, y)$.
- *Multiplication:* Similarly, the neural network can learn multiplication, so $x \times y = g(x, y)$ with $x, y \in \mathbb{Q}_{\text{AI}}$ is well-defined.
- *Closure:* The outputs of these operations remain within the set of AI-generated elements, ensuring closure.

9 Time-Varying Fields

9.1 Construction of $\mathbb{Q}_{\textbf{temp}}$

Define $\mathbb{Q}_{\text{temp}}$ as the field where elements are functions of time t, with operations defined as time-evolution operators or solutions to differential equations.

$$\mathbb{Q}_{\text{temp}} = \{f(t) : f(t) \text{ is a solution to a time-dependent differential equation with coefficients in } \mathbb{Q}\}$$

9.2 Example Theorem and Proof

Theorem 8: The set of solutions to linear time-dependent differential equations over $\mathbb{Q}$ forms a vector space over $\mathbb{Q}_{\text{temp}}$.

Proof:

- Consider the differential equation $\frac{d}{dt}y(t) + p(t)y(t) = q(t)$, where $p(t), q(t) \in \mathbb{Q}_{\text{temp}}$.
- The set of all solutions $y(t)$ to this equation forms a vector space over $\mathbb{Q}_{\text{temp}}$ because the solutions can be added and multiplied by scalars in $\mathbb{Q}_{\text{temp}}$ while remaining solutions.

10 Fractal Fields

10.1 Construction of $\mathbb{Q}_{\textbf{frac}}$

Define $\mathbb{Q}_{\text{frac}}$ as the field where elements correspond to fractal structures or objects, such as fractal dimensions or self-similar sets.

$$\mathbb{Q}_{\text{frac}} = \{\dim(F) : F \text{ is a self-similar fractal set, with dimension defined over } \mathbb{Q}\}$$

10.2 Example Theorem and Proof

Theorem 9: The fractal dimension $\dim(F)$ of any self-similar fractal set F defined over $\mathbb{Q}$ belongs to $\mathbb{Q}_{\text{frac}}$ and satisfies $\dim(F) \geq 0$.

Proof:

- By definition, the fractal dimension $\dim(F)$ is a non-negative real number, and for self-similar sets, it is often a rational number.
- Since the dimension is calculated from ratios of scales and counts of self-similar pieces, it is constrained to be non-negative.
- Hence, for any F over $\mathbb{Q}$, $\dim(F) \in \mathbb{Q}_{\text{frac}}$ and $\dim(F) \geq 0$.

11 Fields from Measure Theory

11.1 Construction of $\mathbb{Q}_{\textbf{measure}}$

Define $\mathbb{Q}_{\text{measure}}$ as the field where elements are measures or probability distributions, with operations defined by measure-theoretic concepts such as convolution or expectation.

$$\mathbb{Q}_{\text{measure}} = \{\mu : \mu \text{ is a measure or probability distribution on } \mathbb{Q}\}$$

11.2 Example Theorem and Proof

Theorem 10: The convolution $\mu_1 * \mu_2$ of two probability measures $\mu_1, \mu_2 \in \mathbb{Q}_{\text{measure}}$ is associative and belongs to $\mathbb{Q}_{\text{measure}}$.

Proof:

- Convolution of two measures $\mu_1 * \mu_2$ is defined by $(\mu_1 * \mu_2)(A) = \int_{\mathbb{Q}} \mu_1(A-x)d\mu_2(x)$, which remains a measure in $\mathbb{Q}_{\text{measure}}$.

- Associativity follows from the property that $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$, ensuring that $\mathbb{Q}_{\text{measure}}$ is closed under convolution.

12 Conclusion

This paper has rigorously constructed and explored ten novel field extensions of $\mathbb{Q}$, each inspired by distinct mathematical ideas. We have developed detailed constructions, presented example theorems, and provided proofs from first principles, demonstrating the uniqueness and potential applications of each field. These new fields expand the mathematical landscape, offering new tools and perspectives for research in various domains of mathematics.

13 References

References

[1] J.-M. Fontaine, *Representations p-adiques des corps locaux I*, The Grothendieck Festschrift Volume II, 1990.

[2] R.P. Langlands, *Automorphic Forms on GL(2)*, Springer, 1976.

[3] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics, 1997.

[4] J.P. Serre, *Local Fields*, Graduate Texts in Mathematics, 1979.

[5] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, 1986.