THE SUM–KERNEL ENTROPY CRITERIA SYSTEM: FORMAL CONDITIONS FOR ARITHMETIC SUMS TO ADMIT YANG REFINEMENT VIA KERNEL THEORY

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Abstract. We construct the Sum–Kernel Entropy Criteria System (SKEC), a formal analytic–arithmetic framework classifying when a sum admits a Yang-refinable kernel structure. We define entropy-compatibility, convolution-regularity, and trace-integrability as necessary and sufficient conditions, and apply the system to diverse examples including Dirichlet, Kloosterman, divisor, modular Fourier, and additive character sums. We conclude with a geometric diagram of kernel-extractable sums and their role in the entropy-zeta trace framework.

Contents

1. Introduction

Many of the most profound identities in number theory are expressed as sums—Dirichlet series, exponential sums, divisor sums, or spectral expansions. In several contexts, especially harmonic analysis and the trace formula, these sums can be reorganized

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as kernel operators, encoding spectral convolution, entropy localization, and automorphic duality.

However, not every arithmetic sum admits such a transformation. The Sum–Kernel Entropy Criteria System (SKEC) provides a formal structure to determine:

- Which types of sums admit a kernel-theoretic interpretation;
- What entropy conditions are necessary for convergence and spectral localization;
- How these kernels can be lifted into the Yang kernel hierarchy;
- How such structures feed into RH, Langlands, and motivic trace sheaves.

We begin by formulating the precise criteria for a sum to admit kernel realization via Yang refinement.

2. Formal Criteria for Kernel Realizability of Arithmetic Sums

2.1. The Sum–Kernel Realization Problem.

Problem 2.1 (Kernel Realization Problem). *Given a family of arithmetic sums $\Sigma_n := \sum_{k\in\mathcal{I}_n} a_k$, determine whether there exists:*

- *A spectral domain X and function space $\mathcal{H}$;*
- *A family of kernels $\{K_n(x,y)\}$ acting on $\mathcal{H}$;*
- *A spectral basis $\{\phi_k\}_{k\in\mathcal{I}_n}$;*

such that:

$$K_n(x,y) = \sum_{k\in\mathcal{I}_n} a_k \phi_k(x)\overline{\phi_k(y)},$$

with convergence and operator compatibility in the Yang kernel framework.

2.2. Three Entropy–Kernel Realizability Criteria.

Definition 2.2 (Entropy-Compatible Sum). An arithmetic sum $\Sigma_n = \sum_{k\in\mathcal{I}_n} a_k$ is *entropy-compatible* if there exists an entropy function $H_Y : \mathcal{I}_n \to \mathbb{R}_{\geq 0}$ such that:

$$\sum_{k\in\mathcal{I}_n} |a_k|^2 e^{-2H_Y(k)} < \infty.$$

This ensures that the entropy-weighted kernel

$$K_n^{(Y)}(x,y) := \sum_{k\in\mathcal{I}_n} a_k e^{-H_Y(k)}\phi_k(x)\overline{\phi_k(y)}$$

is Hilbert–Schmidt on $\mathcal{H}$.

Definition 2.3 (Convolution-Regular Sum)**.** The sum Σ_n is *convolution-regular* if its indices $\mathcal{I}_n$ admit a spectral parameterization $k \mapsto \lambda_k$ such that the associated eigenfunctions ϕ_k form a basis for a convolution algebra:

$$f * K_n(x) = \int K_n(x,y)f(y)dy,$$

with $\phi_k * K_n = \mu_k \phi_k$ for some eigenvalue μ_k.

Definition 2.4 (Trace-Integrable Sum)**.** The sum Σ_n is *trace-integrable* if the corresponding Yang kernel satisfies:

$$\mathrm{Tr}(K_n^{(Y)}) = \sum_{k \in \mathcal{I}_n} a_k e^{-H_Y(k)} \in \mathbb{C},$$

defining a regularized entropy-trace functional.

2.3. Main Realizability Theorem.

Theorem 2.5 (Sum–Kernel Entropy Realizability)**.** *Let $\Sigma_n = \sum_{k \in \mathcal{I}_n} a_k$ be a family of arithmetic sums. Then Σ_n admits a Yang-refinable kernel realization if and only if:*

(i) Σ_n is entropy-compatible;
(ii) Σ_n is convolution-regular;
(iii) Σ_n is trace-integrable.

Proof. Necessity:

- (i) ensures bounded operator norm via entropy damping;
- (ii) ensures kernel definability via spectral expansion;
- (iii) ensures well-defined trace operations on the resulting operator.

Sufficiency: Given all three, define:

$$K_n^{(Y)}(x,y) := \sum_{k \in \mathcal{I}_n} a_k e^{-H_Y(k)} \phi_k(x) \overline{\phi_k(y)},$$

which is convergent, defines an operator on $\mathcal{H}$, and admits entropy-trace extraction. □

3. Examples and Non-Examples of Yang Kernel Realizable Sums

3.1. Examples: Kernel-Admissible Arithmetic Sums.

Example 3.1 (Dirichlet Summation Kernel)**.** Let $\Sigma_n = \sum_{k=-n}^{n} e^{ikx}$. Then:

- $\mathcal{I}_n = \{-n, \dots, n\}$;
- $a_k = 1$, and $\phi_k(x) = e^{ikx}$;
- $H_Y(k) := \alpha|k|$, for $\alpha > 0$.

Then:

$$K_n^{(Y)}(x,y) = \sum_{k=-n}^{n} e^{-\alpha|k|} e^{ik(x-y)} = D_n^{(Y)}(x-y)$$

is an entropy-suppressed Dirichlet kernel.

This satisfies:

- Entropy-compatible: $|a_k|^2 e^{-2\alpha|k|}$ is summable;
- Convolution-regular: Fourier convolution;
- Trace-integrable: $\sum_k e^{-\alpha|k|} < \infty$.

Example 3.2 (Kloosterman Weighted Sum)**.** Let $\Sigma_{m,n}(N) := \sum_{c \le N} \frac{S(m,n;c)}{c} \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right)$.

Then:

- $\mathcal{I}_N = \{c \in \mathbb{Z}_{>0} \mid c \le N\}$;
- $a_c = \frac{S(m,n;c)}{c} \Phi(\dots)$;
- Entropy weight: $H_Y(c) := \gamma \log c$ for $\gamma > 1$.

Using Kuznetsov trace structure:

$$K_N^{(Y)} := \sum_{c \le N} \frac{S(m,n;c)}{c} e^{-\gamma \log c} \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

is Yang-refinable with:

- Entropy compatibility via exponential suppression;
- Convolution-regularity via spectral Bessel expansion;
- Trace-integrability from classical bounds on Kloosterman sums.

3.2. **Non-Examples: Sums Failing Kernel Admissibility.**

Example 3.3 (Ramanujan $\tau(n)$ Dirichlet Series)**.** Let $\Sigma(s) := \sum_{n=1}^{\infty} \tau(n) n^{-s}$. Define:

$$K(x,y) = \sum_n \tau(n) e^{inx} \overline{e^{iny}}.$$

Fails Yang realization because:

- No known entropy weight $H_Y(n)$ rendering $\sum |\tau(n)|^2 e^{-2H_Y(n)} < \infty$;
- Exponential growth in $\tau(n)$ contradicts trace integrability.

Example 3.4 (Randomized Sums)**.** Let $\Sigma_N := \sum_{n=1}^{N} \epsilon_n$ with $\epsilon_n \in \{-1, 1\}$ random.

Fails:

- No spectral basis ϕ_n associated canonically;
- No structure for convolution regularity;
- Trace operator undefined due to lack of deterministic decay or control.

3.3. **Classification Table: Admissible Sum Types.**

Sum Type	**Entropy-Compatible**	**Convolution-Regular**	**Trace-Integrable**
Dirichlet Sum	Yes	Yes	Yes
Fejér Sum	Yes	Yes	Yes
Kloosterman Weighted	Yes	Yes	Yes
Modular Fourier	Conditionally	Yes	Conditionally
Ramanujan $\tau(n)$	No	Yes	No
Random Sum	No	No	No

4. Conclusion and Next Directions

The Sum–Kernel Entropy Criteria System provides a rigorous analytic framework to:

- Determine which sums can be lifted to entropy-refined Yang kernels;
- Filter admissible test functions for RH and Langlands trace modules;
- Distinguish analytic vs. motivic vs. random sum structures via kernel realizability.

In the next paper, we construct the **Yang–Kernel Integration Map**, classifying which known kernels can be incorporated into the full entropy–Langlands–period stack framework via Yang convolution operads.

References

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