On a theory of prime producing sieves, II

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Basic Setup

Problem: Count primes in $\mathscr{A} \subset (x, 2x]$
Tools: Type I and Type II bounds

Three basic parameters: γ, θ, ν.

$$\sum_{m \leqslant x^{\gamma}} \tau(m)^B \left| \#\{a \in \mathscr{A} : m|a\} - \frac{|\mathscr{A}|}{m} \right| \ll_B \frac{|\mathscr{A}|}{(\log x)^B} \quad \text{(Type I bound)}.$$

For any divisor-bounded complex sequences $(\kappa_m), (\zeta_n)$,

$$\left| \sum_{\substack{x^{\theta} < m \leqslant x^{\theta+\nu} \\ x < mn \leqslant 2x}} \kappa_m \zeta_n \Big(1_{mn \in \mathscr{A}} - \frac{|\mathscr{A}|}{x} \Big) \right| \ll_B \frac{|\mathscr{A}|}{(\log x)^B} \quad \text{(Type II bound)}.$$

Our new approach: use all Type I/II information at once

Linnik's identity: $$\frac{\Lambda(n)}{\log n} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{\substack{n=d_1\cdots d_j \\ d_i \geqslant 2\ (1\leqslant i\leqslant j)}} 1.$$

Let $w_n = 1_{n\in\mathcal{A}} - \frac{|\mathscr{A}|}{x}$, average zero. Then

$$\sum_p w_p = \sum_n w_n \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{\substack{n=d_1\cdots d_j \\ d_i \geqslant 2\ (1\leqslant i\leqslant j)}} 1 \qquad \text{(Linnik; ignore prime powers)}$$

$$\approx \sum_n w_n \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{\substack{n=d_1\cdots d_j \\ 2\leqslant d_i \leqslant x^{1-\gamma}\ (1\leqslant i\leqslant j)}} 1 \qquad \text{(using Type-I for } d_i > x^{1-\gamma}\text{)}$$

$$\approx \sum_{n\in\mathcal{U}} w_n \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{\substack{n=d_1\cdots d_j \\ 2\leqslant d_i \leqslant x^{1-\gamma}\ (1\leqslant i\leqslant j)}} 1 \qquad \text{(using Type-II),}$$

where $\mathcal{U} = \{x < n \leqslant 2x : \underbrace{x^{1-\gamma} - \text{smooth}}_{\text{Type-I}}, \underbrace{\text{no divisor in } (x^{\theta}, x^{\theta+\nu}]}_{\text{Type-II}}\}$.

An asymptotic for the number of primes in $\mathscr{A}$

$$\sum_p w_p \approx \sum_{n\in\mathcal{U}} w_n \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{\substack{n=d_1\cdots d_j\\ 2\leqslant d_i\leqslant x^{1-\gamma}\ (1\leqslant i\leqslant j)}} 1.$$

where $\mathcal{U} = \{x < n \leqslant 2x : \underbrace{x^{1-\gamma} - \text{smooth}}_{\text{Type-I}}, \underbrace{\text{no divisor in } (x^\theta, x^{\theta+\nu}]}_{\text{Type-II}}\}$.

Corollary. If $\mathcal{U}$ is empty or tiny, then

$$0 \approx \sum_p w_p = \#\{p \in \mathscr{A}\} - \frac{|\mathscr{A}|}{x}\#\{x < p \leqslant 2x\},$$

and so

$$\#\{p \in \mathscr{A}\} \sim \frac{|\mathscr{A}|}{\log x}.$$

In fact, the asymptotic is guaranteed iff $\mathcal{U}$ is “empty or tiny”.

We need *constructions* of sets $\mathscr{A}$ satisfying Type I and Type II bounds with $\#\{p \in \mathscr{A}\}$ unusually large/small in order to show

(i) The asymptotic is not guaranteed when $\mathcal{U}$ is "substantial".
(ii) For some γ, θ, ν, it's possible that $\#\{p \in \mathscr{A}\} = 0$.
(iii) To show that our sieve bounds on $\#\{p \in \mathscr{A}\}$ are best possible.

Examples with no primes

Theorem [FM,2024]. For every $\gamma < 1$, there is a $\nu > 0$ so that for *any* θ, there are examples of $\mathscr{A}$ satisfying the Type I and Type II bounds but with no primes.

Selberg's example

$\mathscr{A} = \{x < n \leqslant 2x : n$ has an even number of prime factors$\}$ satisfies Type I for any $\gamma < 1$, but has no primes.

Constructions: from real sequences to sets

Consider a bounded, non-negative sequence $(v_n)_{x<n\leqslant 2x}$ satisfying

- $v_p = 0$ if p is prime;
- (Type I) $$\sum_{m\leqslant x^\gamma} \tau(m)^B \Big|\sum_{m|n}(v_n - 1)\Big| \ll_B \frac{x}{\log^B x}.$$
- (Type II) For any divisor-bounded complex sequences (κ_m), (ζ_n),

$$\sum_{\substack{x^\theta<m\leqslant x^{\theta+\nu} \\ x<mn\leqslant 2x}} \kappa_m \zeta_n (v_{mn} - 1) \ll_B \frac{x}{(\log x)^B}.$$

Let $V = \max_n v_n$, then choose $\mathscr{A}$ randomly from (v_n) by

$$\mathrm{Prob}(n \in \mathscr{A}) = \frac{v_n}{V} \qquad (x < n \leqslant 2x).$$

Then $\mathscr{A}$ contains no primes (since $v_p = 0$) and with high probability, $\mathscr{A}$ satisfies the Type I and Type II bounds.

Choosing v_n in terms of a vector function f

Take $v_n = 1 + f\left(\frac{\log p_1}{\log n}, \ldots, \frac{\log p_k}{\log n}\right)$ for $n = p_1 \cdots p_k$, where $f \in \mathscr{F}_\varepsilon(\gamma, \theta, \nu)$, the set of functions on variable-length vectors, supported on vectors with

- sum of components 1;
- no subset sum in $[\theta, \theta + \nu]$ (so $v_n = 1$ if $\exists d | n, d \in (x^\theta, x^{\theta+\nu}]$; Type II trivial),
- all components $\geqslant \varepsilon$ (so $v_n = 1$ if n has a prime factor $< n^\varepsilon$);

and additionally

- for each $k \geqslant 1$, $f(u_1, \ldots, u_k)$ is piecewise smooth and *symmetric in all variables*;
- f satisfies an analog of the Type I bound for (v_n):

$$\text{for } r \geqslant 0,\ \xi_1 + \cdots + \xi_r \leqslant \gamma, \qquad \sum_{k \geqslant r+1} \int \cdots \int_{\substack{\varepsilon \leqslant \xi_{r+1} \leqslant \cdots \leqslant \xi_k \\ \xi_1 + \cdots + \xi_k = 1}} \frac{f(\xi_1, \ldots, \xi_k)}{\xi_{r+1} \cdots \xi_k} = 0. \qquad (I_1)$$

$\mathscr{F}_\varepsilon(\gamma, \theta, \nu)$ forms a vector space!

Goal: Find $f \in \mathscr{F}_\varepsilon(\gamma, \theta, \nu)$ with

- $f(1) = -1$ (this makes $v_p = 0$ for primes p);
- $f(\mathbf{u}) \geqslant -1$ for all $\mathbf{u}$ (to ensure $v_n \geqslant 0$ for all n);

Tweaking the Type I integral equations

$$\forall \xi_1 + \cdots + \xi_r \leqslant \gamma, \quad \sum_{k \geqslant r+1} \underset{\substack{\varepsilon \leqslant \xi_{r+1} \leqslant \cdots \leqslant \xi_k \\ \xi_1 + \cdots + \xi_k = 1}}{\int \cdots \int} \frac{f(\xi_1, \ldots, \xi_k)}{\xi_{r+1} \cdots \xi_k} = 0 \qquad (I_1)$$

The $k = r+1$ term is $\alpha^{-1} f(\xi_1, \ldots, \xi_r, \alpha)$, where $\alpha = 1 - (\xi_1 + \cdots + \xi_r)$. Thus, (I_1) is equivalent to

$$f(\xi_1, \ldots, \xi_r, \alpha) = -\alpha \sum_{k \geqslant r+2} \underset{\substack{\varepsilon \leqslant \xi_{r+1} \leqslant \cdots \leqslant \xi_k \\ \xi_{r+1} + \cdots + \xi_k = \alpha}}{\int \cdots \int} \frac{f(\xi_1, \ldots, \xi_k)}{\xi_{r+1} \cdots \xi_k}. \qquad (I_1')$$

Here we have a “fragmentation” of α: $\quad \alpha \to (\xi_{r+1}, \ldots, \xi_k)$.

We have $\alpha \geqslant 1 - \gamma$. Now iterate (I_1'): fragment each component $\geqslant 1 - \gamma$ (the process is finite since all components are $\geqslant \varepsilon$).

The main fragmentation relation for f

After iterating (I_1'), (I_1) is equivalent to:

(I_2) For all $s \geqslant 0$, $\ell \geqslant 1$, all $\beta_1, \ldots, \beta_s \in [\varepsilon, 1-\gamma)$ and $\alpha_1, \ldots, \alpha_\ell \geqslant 1-\gamma$ and $\beta_1 + \cdots + \beta_s + \alpha_1 + \cdots + \alpha_\ell = 1$, we have

$$\frac{f(\boldsymbol{\beta}, \alpha_1, \ldots, \alpha_\ell)}{\alpha_1 \cdots \alpha_\ell} = \sum_{k_1, \ldots, k_\ell \geqslant 2} \quad \int \cdots \int_{\substack{\alpha_j = u_{j,1} + \cdots + u_{j,k_j} \\ \varepsilon \leqslant u_{j,1} \leqslant \cdots \leqslant u_{j,k_j} < 1-\gamma \\ 1 \leqslant j \leqslant \ell}} \frac{\text{Л}(\mathbf{u}_1) \cdots \text{Л}(\mathbf{u}_\ell) f(\boldsymbol{\beta}, \mathbf{u}_1, \ldots, \mathbf{u}_\ell)}{\prod_{j=1}^{\ell} \prod_{h=1}^{k_j} u_{j,h}},$$

where $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,k_j})$ for $1 \leqslant j \leqslant \ell$, and $\text{Л}(\mathbf{u}) \in \mathbb{Z}$ is a combinatorial factor, the vector analog of the truncated Linnik function.

- On the LHS, $f(\boldsymbol{\beta}, \alpha_1, \ldots, \alpha_\ell)$ has at least one component $\geqslant 1-\gamma$;
- On the RHS, $f(\boldsymbol{\beta}, \mathbf{u}_1, \ldots, \mathbf{u}_\ell)$ has *all* components $< 1-\gamma$.

- We may freely choose f on $\mathcal{R}_\varepsilon(\gamma, \theta, \nu)$, the set of vectors with all components in $[\varepsilon, 1-\gamma)$, sum 1, and no subset sum in $[\theta, \theta+\nu]$.
- Then f is uniquely determined on vectors with some component $\geqslant 1-\gamma$.

$\mathcal{R}_\varepsilon(\gamma, \theta, \nu)$ is the vector version of $\mathcal{U}$ (with the additional restriction of components $\geqslant \varepsilon$).

From $\nu = 0$ to $\nu > 0$

Theorem [FM,2024]. For every $\gamma < 1$, there is a $\nu > 0$ so that for *any* θ, there are examples of $\mathscr{A}$ satisfying the Type I and Type II bounds but with no primes.

A reduction to the case of no Type II information

Suppose that there is an $\varepsilon > 0$ and $\overbrace{f \in \mathscr{F}_\varepsilon(\gamma, 0, 0)}^{\text{no Type II}}$ such that

- $f(1) < -1$; and
- $f(\mathbf{u}) \geqslant -1$ for all $\mathbf{u}$ with at least 2 components.

Then the above theorem holds for this γ.

Idea: For small enough $\nu > 0$ and any θ, construct $\tilde{f} \in \mathscr{F}_\varepsilon(\gamma, \theta, \nu)$ from f:

(1) Let $\tilde{f} = f$ on $\mathcal{R}_\varepsilon(\gamma, \theta, \nu)$; a subset of $\mathcal{R}_\varepsilon(\gamma, 0, 0)$. For small ν, there is a "small" difference in the sets: $\mathcal{R}_\varepsilon(\gamma, \theta, \nu) = \{\mathbf{u} \in \mathcal{R}_\varepsilon(\gamma, 0, 0) : \mathbf{u} \text{ has no subsum in } [\theta, \theta + \nu]\}$.

(2) Define $\tilde{f}$ for other vectors by

$$\frac{\tilde{f}(\boldsymbol{\beta}, \alpha_1, \dots, \alpha_\ell)}{\alpha_1 \cdots \alpha_\ell} = \sum_{k_1, \dots, k_\ell \geqslant 2} \int \cdots \int_{\substack{\alpha_j = u_{j,1} + \cdots + u_{j,k_j} \\ \varepsilon \leqslant u_{j,1} \leqslant \cdots \leqslant u_{j,k_j} < 1 - \gamma \\ 1 \leqslant j \leqslant \ell}} \underbrace{\frac{\text{Л}(\mathbf{u}_1) \cdots \text{Л}(\mathbf{u}_\ell)}{\prod_{j=1}^{\ell} \prod_{h=1}^{k_j} u_{j,h}}}_{bounded} \tilde{f}(\boldsymbol{\beta}, \mathbf{u}_1, \dots, \mathbf{u}_\ell) \qquad (I_2)$$

Let $f(1) = -1 - \delta$. For small $\nu > 0$, $\tilde{f}(1) < -1 - \delta/2$, and for other $\mathbf{u}$, $\tilde{f}(\mathbf{u}) \geqslant -1 - \delta/3$.

(3) Rescale $\tilde{f}$ so that $\tilde{f}(1) = -1$.

Revisiting Selberg's example

A reduction to the case of no Type II information

Suppose that there is an $\varepsilon > 0$ and $f \in \mathscr{F}_\varepsilon(\gamma, 0, 0)$ such that

- $f(1) < -1$; and
- $f(\mathbf{u}) \geqslant -1$ for all $\mathbf{u}$ with at least 2 components.

Then the above theorem holds for this γ.

Selberg's example corresponds to $\varepsilon = 0$ and $f(u_1, \ldots, u_k) = (-1)^k$, the vector version of the Liouville function $\lambda(n)$.

This "just fails" because $f(1) = -1$ and $\varepsilon = 0$.

Can Selberg's example be tweaked to work?

A family of Liouville-type functions

For $0 < \varepsilon < c < 1/2$ define $\tilde{\lambda}^{(c,\varepsilon)}$ as follows:

- $\tilde{\lambda}^{(c,\varepsilon)}(u_1, \ldots, u_s) = (-1)^s$ if $\varepsilon \leqslant u_i < c$ for all i;
- $\tilde{\lambda}^{(c,\varepsilon)}(\mathbf{u}) = 0$ if any component is $< \varepsilon$;
- If $\beta_1, \ldots, \beta_s < c \leqslant \alpha_1, \ldots, \alpha_\ell$,

  $$\tilde{\lambda}^{(c,\varepsilon)}(\boldsymbol{\beta}, \boldsymbol{\alpha}) = (-1)^s M^{(c,\varepsilon)}(\alpha_1) \cdots M^{(c,\varepsilon)}(\alpha_\ell),$$

  where

  $$M^{(c,\varepsilon)}(\alpha) = \alpha \sum_{k \geqslant 1} \frac{(-1)^k}{k!} \int \cdots \int_{\substack{\alpha = u_1 + \cdots + u_k \\ \varepsilon < u_i < c \ (1 \leqslant i \leqslant k)}} \frac{\Pi_c(\mathbf{u})}{u_1 \cdots u_k}.$$

Lemma. If $\varepsilon < c \leqslant 1 - \gamma$, then $\tilde{\lambda}^{(c,\varepsilon)} \in \mathscr{F}_\varepsilon(\gamma, 0, 0)$;

If $\varepsilon < c \leqslant \frac{1-\gamma}{2}$ then $2^k \tilde{\lambda}^{(c,\varepsilon)}(u_1, \ldots, u_k) \in \mathscr{F}_\varepsilon(\gamma, 0, 0)$.

Endgame

Goal: there is an $\varepsilon > 0$ and $f \in \mathscr{F}_\varepsilon(\gamma, 0, 0)$ such that $f(1) < -1$ and $f(\mathbf{u}) \geqslant -1$ for all $\mathbf{u}$ with at least 2 components.

Lemma. $-1 \leqslant M^{(c,\varepsilon)}(\alpha) \leqslant -1 + (c\varepsilon)^{-1}\rho(c/\varepsilon - 1)$, ρ is Dickman's fcn.
Corollary. If $c \geqslant \frac{1-\gamma}{2}$ and $\varepsilon = \varepsilon(\gamma) > 0$ small enough then $-1 \leqslant M^{(c,\varepsilon)}(u) \leqslant -1 + 2^{-2/\varepsilon}$.
In particular, $|\tilde{\lambda}^{(c,\varepsilon)}(\mathbf{u})| \leqslant 1$ and $\operatorname{sgn} \tilde{\lambda}^{(c,\varepsilon)}(u_1, \ldots, u_k) = (-1)^k$ for all $\mathbf{u}$.

The proof: Let $g(u_1, \ldots, u_k) = (1 - 2^{k-3})\tilde{\lambda}^{((1-\gamma)/2,\varepsilon)}(\mathbf{u})$, $g_0 = \max |g(\mathbf{u})| \leqslant 2^{1/\varepsilon}$ and

$$f(\mathbf{u}) = \tilde{\lambda}^{(1-\gamma,\varepsilon)}(\mathbf{u}) + g_0^{-1} g(\mathbf{u}).$$

- If some $u_i < \frac{1-\gamma}{2}$ then $g(\mathbf{u}) = 0$ and $f(\mathbf{u}) \geqslant -1$;
- If $k \geqslant 3$ is odd and all $u_i \geqslant \frac{1-\gamma}{2}$, then $g(u_1, \ldots, u_k) \geqslant 0$ and $f(\mathbf{u}) \geqslant -1$;
- If $k \geqslant 2$ is even and all $u_i \geqslant \frac{1-\gamma}{2}$, then $\tilde{\lambda}^{(1-\gamma,\varepsilon)}(\mathbf{u}) \geqslant 0$ and $f(\mathbf{u}) \geqslant -1$;
- We have

$$f(1) = M^{(1-\gamma,\varepsilon)}(1) + g_0^{-1}(3/4)M^{((1-\gamma)/2,\varepsilon)}(1) \leqslant -1 + \frac{1}{10g_0} - \frac{1}{2g_0} < -1.$$