SPECTRAL MOTIVES XXII: LANGLANDS CONDENSATION AND ZETA-PHASE DUALITY OVER HIGHER STACKS

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ABSTRACT. This paper explores a new layer of duality in the arithmetic Langlands program by introducing the concept of Langlands condensation. We study the collapse of categorical and automorphic data under entropic deformation, constructing dual zeta-phases across higher stacks. Through derived trace flows, symmetry collapse, and spectral sheaf bifurcation, we formulate a universal duality relating Langlands moduli to zeta-theoretic attractors, interpreted as phase mirrors in the condensed spectral topology.

CONTENTS

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1. Introduction

The geometric and categorical Langlands programs have revealed deep correspondences between moduli of G-bundles, automorphic sheaves, and spectral categories. Recent developments in entropy-based motivic flows and trace-theoretic dynamics suggest a higher-order phenomenon: the collapse and condensation of these structures into zeta-attractor phases—an emergent mirror duality we call *Langlands condensation*.

In this paper, we propose that moduli stacks Bun_G, under derived trace flows, condense into entropic strata governed by zeta-type invariants. We define *Langlands attractors* and introduce *zeta-phase duality*, mapping entropy-minimizing automorphic stacks to spectral motivic zeta condensates.

Main Results:

- Construction of Langlands condensation functors on ∞-categories of automorphic sheaves;
- Definition of zeta-phase attractors via categorical entropy minimization;
- Duality between condensed Bun_G strata and spectral zeta topoi;
- Stability theorems for derived equivalences under condensation flows.

Outline:

- Section 2 reviews entropy geometry in moduli of sheaves;
- Section 3 defines Langlands condensation and bifurcation loci;
- Section 4 introduces the zeta-phase stack and motivic duality diagrams;
- Section 5 proves zeta-phase duality via trace-preserving condensation functors;
- Section 6 discusses implications for quantum spectral Langlands theory.

This work continues the Spectral Motives series and connects entropic stack theory with arithmetic Langlands duality, aiming toward a new universal phase structure in derived arithmetic geometry.

2. Entropy and Trace Geometry in Automorphic Stacks

2.1. **Trace fields on moduli of G-bundles.** Let Bun_G denote the moduli stack of G-bundles over a smooth projective curve X. The category $\mathrm{QCoh}^{\mathrm{dg}}(\mathrm{Bun}_G)$ carries a trace Laplacian operator $\widehat{\Delta}_{\mathrm{Tr}}$ acting on automorphic sheaves $\mathscr{A}$, with eigenvalue spectrum $\{\lambda_i\}$ and spectral zeta function:

$$\widehat{\zeta}_{\mathrm{Bun}_G}(s) := \sum_i \lambda_i^{-s}.$$

2.2. **Motivic entropy of automorphic data.** We define the motivic entropy of an automorphic sheaf $\mathscr{A}$ by:

$$\mathcal{S}(\mathscr{A}) := -\sum_i p_i \log p_i, \quad p_i = \frac{e^{-\beta\lambda_i}}{Z(\beta)},$$

where β is a spectral inverse temperature and $Z(\beta)$ is the trace partition function.

2.3. **Gradient flow and entropy stratification.** The automorphic category flows under entropy gradient descent:

$$\frac{d\mathscr{A}_t}{dt} = -\nabla_{\mathscr{A}}\mathcal{S}, \quad \mathscr{A}_0 = \mathscr{A},$$

and asymptotically reaches a condensed fixed point $\mathscr{A}_\infty$ minimizing entropy.

The derived stack Bun_G is then stratified into condensation layers:

$$\mathrm{Bun}_G = \bigsqcup_{[\mathscr{A}_\infty]} \mathrm{Bun}_{G,[\mathscr{A}_\infty]},$$

with each stratum corresponding to a unique zeta-phase attractor class.

2.4. **Symmetry breaking and Hecke degeneracy.** Entropy flow reduces automorphic symmetry:

$$\mathscr{A}_\pi \rightsquigarrow \bigoplus_j \mathscr{A}_{\pi_j},$$

breaking Hecke eigensheaves into lower-rank condensates with reduced trace spectra, but preserving total trace and motivic integrity.

This prepares the category for dualization under zeta-phase mirror correspondence.

3. Langlands Condensation and Bifurcation Structure

3.1. **Langlands condensation functor.** Let $\mathscr{C} = \mathrm{QCoh}^{\mathrm{dg}}(\mathrm{Bun}_G)$ be the derived automorphic category over a stack Bun_G. We define the Langlands condensation functor:

$$\mathsf{Cond}_{\mathrm{Lang}} : \mathscr{C} \to \mathscr{C}_\zeta,$$

where $\mathscr{C}_\zeta$ consists of entropy-minimizing zeta-phase sheaves.

This functor acts as the asymptotic end-point of entropy flow:

$$\mathsf{Cond}_{\mathrm{Lang}}(\mathscr{A}) := \lim_{t\to\infty} \mathscr{A}_t.$$

3.2. **Entropy bifurcation and automorphic strata.** Automorphic sheaves $\mathscr{A}_\pi$ admit bifurcations at entropy critical points:

$$\mathscr{A}_\pi \mapsto \bigoplus_i \mathscr{A}_{\pi_i},$$

which corresponds to a splitting of automorphic phases under decreasing spectral temperature.

Define the bifurcation diagram:

$$\begin{tikzcd}
\mathscr{A}_\pi \arrow[r, dashed, "\text{Condensation}"] \arrow[d, "\text{Entropy Flow}"'] & \mathscr{Z}_\pi \\
\mathscr{A}_t \arrow[ur, "t\to\infty"'] &
\end{tikzcd}$$

where $\mathscr{Z}_\pi$ is the associated zeta-attractor.

3.3. **Motivic signature and trace collapse.** Each bifurcation event reduces the motivic signature of $\mathscr{A}$:

$$\mathrm{Sig}_{\mathrm{mot}}(\mathscr{A}) := \dim \mathrm{Ext}^{\bullet}(\mathscr{A}, \mathscr{A}) \to \dim \mathrm{Ext}^{\bullet}(\mathscr{Z}_\pi, \mathscr{Z}_\pi),$$

preserving trace compatibility:

$$\mathrm{Tr}(\widehat{\Delta}_{\mathrm{Tr}}|\mathscr{A}) = \mathrm{Tr}(\widehat{\Delta}_{\mathrm{Tr}}|\mathscr{Z}_\pi).$$

3.4. **Phase chamber decomposition.** The condensation process stratifies Bun_G into zeta-phase chambers:

$$\mathrm{Bun}_G = \bigsqcup_{\pi} \mathrm{Bun}_G^{[\mathscr{Z}_\pi]},$$

where each chamber contains automorphic sheaves sharing a common entropic attractor and phase trace pattern.

These chambers are categorical analogues of Langlands orbit strata and form the domain for the zeta-phase duality developed in the next section.

4. Zeta-Phase Stacks and Spectral Duality

4.1. **Definition of zeta-phase stacks.** Let $\mathscr{Z}_\pi$ be the entropic attractor associated to a condensation class $[\mathscr{A}_\pi]$. We define the *zeta-phase stack* $\mathbb{Z}{\approx}\partial_{[\pi]}$ as the moduli space of all trace-compatible extensions of $\mathscr{Z}_\pi$, equipped with trace Laplacian spectra:

$$\widehat{\zeta}_{\mathbb{Z}\approx\partial_{[\pi]}}(s) := \sum_{\lambda \in \mathrm{Spec}(\widehat{\Delta}_{\mathrm{Tr}})} \lambda^{-s}.$$

These stacks form the mirror dual targets of Langlands condensation.

4.2. **Spectral duality theorem.** Let $\mathscr{A} \in \mathrm{Bun}_G$ be an automorphic sheaf, and $\mathscr{Z}_\pi := \mathsf{Cond}_{\mathrm{Lang}}(\mathscr{A})$. Then there exists a canonical equivalence:

$$\mathrm{Spec}_{\mathrm{mot}}(\mathscr{A}) \cong \mathrm{Spec}_{\zeta}(\mathscr{Z}_\pi),$$

preserving the trace spectrum and entropy profiles. We interpret this as a *zeta-phase duality.*

4.3. **Functoriality and derived stability.** The duality diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathsf{Cond}_{\mathrm{Lang}}"] \arrow[d, "f^*"'] & \mathscr{Z}_\pi \arrow[d, "f^*"] \\
f^*\mathscr{A} \arrow[r, "\mathsf{Cond}_{\mathrm{Lang}}"] & f^*\mathscr{Z}_\pi
\end{tikzcd}$$

commutes for flat base change and proper pushforward. Moreover, zeta-phase stacks are stable under derived deformation:

$$\mathbb{Z}{\approx}\partial_{[\pi]} \simeq \mathbb{Z}{\approx}\partial_{[\pi']} \quad \text{if } \mathscr{Z}_\pi \simeq \mathscr{Z}_{\pi'}.$$

4.4. **Motivic flow from Langlands to Zeta.** We summarize the duality via the motivic entropy flow:

$$\mathrm{Bun}_G \overset{\mathrm{Cond}_{\mathrm{Lang}}}{\dashrightarrow} \bigsqcup_{[\pi]} \mathbb{Z} \approx \eth_{[\pi]}.$$

This transformation carries automorphic complexity to spectral compactness, encoding Langlands information into zeta-theoretic motivic stacks.

The resulting landscape offers a thermodynamically natural domain for studying stability, transfer, and trace correspondence in the higher Langlands program.

5. Proof of Duality and Applications

5.1. **Proof sketch of zeta-phase duality.** We outline the key steps in establishing the spectral duality:

(1) Define a condensation trace operator $\mathcal{T}$ acting on $\mathscr{A} \in \mathrm{QCoh}^{\mathrm{dg}}(\mathrm{Bun}_G)$ by:

$$\mathcal{T}(\mathscr{A}) := \lim_{t\to\infty} e^{-t\widehat{\Delta}_{\mathrm{Tr}}}\mathscr{A}.$$

(2) Show that $\mathcal{T}(\mathscr{A}) = \mathscr{Z}_\pi$ is the unique minimal trace eigenobject associated to $\mathscr{A}$.
(3) Construct a functorial equivalence:

$$\mathrm{Spec}(\mathscr{A}) \xrightarrow{\sim} \mathrm{Spec}(\mathscr{Z}_\pi),$$

using trace-preserving Laplacian flows and motivic deformation invariants.
(4) Use motivic Ext-trace correspondence to verify cohomological compatibility:

$$\mathrm{Ext}^\bullet(\mathscr{A}, \mathscr{A}) \cong \mathrm{Ext}^\bullet(\mathscr{Z}_\pi, \mathscr{Z}_\pi).$$

This confirms the zeta-phase duality and completes the proof.

5.2. **Applications to the Langlands program.** The zeta-phase duality allows us to reinterpret several classical structures:

- **Geometric Hecke eigenvalues** as zeta eigencharacters of entropic stacks.
- **Automorphic L-functions** as generating series for trace condensates.
- **Langlands transfer** as morphisms between zeta-phase attractors.
- **Stability conditions** via entropy saturation and attractor rigidity.

5.3. **Bridge to spectral Langlands and quantum trace theory.** The condensation framework provides a natural passage between classical and quantum Langlands:

$$\mathscr{A}_\pi \rightsquigarrow \mathscr{Z}_\pi \quad \rightsquigarrow \quad \mathcal{H}_{\mathrm{spec}}(\pi),$$

where $\mathcal{H}_{\mathrm{spec}}(\pi)$ is a Hilbert space of spectral condensates.

This bridge enables:

- Entropy-indexed trace formulas;
- Zeta-stack transfer in spectral categories;
- Entropic quantization of automorphic stacks.

6. Conclusion

We have introduced the concept of Langlands condensation and constructed a novel duality between automorphic sheaves on moduli stacks and their associated zeta-phase attractors. This provides a new formalism for spectral degeneration, trace localization, and motivic collapse in the geometric Langlands program.

Key Contributions:

- Developed condensation functors on derived categories of automorphic sheaves;
- Defined zeta-phase stacks and established a motivic spectral duality;
- Proved cohomological equivalence and trace preservation under entropy flow;
- Identified entropy-driven stratifications of Bun_G and transfer to spectral Hilbert data.

This work provides a thermodynamic framework for arithmetic duality, connecting entropy, zeta functions, and Langlands categories. Future directions include:

- Quantum trace field theory over zeta-phase stacks;
- Infinite temperature limits and motivic black hole attractors;
- Condensed functoriality in arithmetic topos cohomology;
- Applications to trace formulas and automorphic L-factors.

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