THE YANG–KUZNETSOV KERNEL SYSTEM: ENTROPY-OPTIMIZED AUTOMORPHIC TEST FUNCTIONS FOR RH TRACE HIERARCHIES

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Abstract. We introduce and rigorously construct the Yang–Kuznetsov kernel system as a hierarchy of entropy-optimized test functions derived from Kloosterman-weighted trace formulas. These kernels generalize classical Kuznetsov-type test functions by integrating motivic entropy stratification, spectral localization, and zeta-function sensitivity. We establish their convergence, automorphic duality, and functional interaction with the Arthur–Selberg trace architecture. Moreover, we propose a visualization scheme and Python-based simulation model to compute and graphically represent the structure of Yang–Kuznetsov kernels in RH-critical spectral bands.

Contents

1. Introduction

The Kuznetsov trace formula is a spectral identity that expresses sums of automorphic Fourier coefficients against Kloosterman sums in terms of Bessel transforms and spectral parameters. Classically, it has the shape:

$$\sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} \cdot \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = \text{spectral side: } \sum_{\pi} \lambda_\pi(m)\lambda_\pi(n) \cdot \tilde{\Phi}(\nu_\pi).$$

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Here Φ is a test function with certain analytic decay, and $\tilde{\Phi}$ is its Bessel–Hankel transform.

In this paper, we define an entropy-optimized test function system $\Phi^{(Y)}$ whose properties:

- Maximize spectral localization near RH-critical bands;
- Minimize arithmetic Kloosterman oscillation outside motivic strata;
- Integrate as convolution kernels in the zeta entropy trace tower.

These give rise to the *Yang–Kuznetsov kernel system*, a new class of entropy–automorphic test function hierarchies.

We begin by defining the structure of these kernels and their spectral transforms.

2. Definition of the Yang–Kuznetsov Kernel

Definition 2.1 (Yang–Kuznetsov Kernel Function)**.** Let $\Phi^{(Y)}(x)$ be an entropy-weighted test function defined by:

$$\Phi^{(Y)}(x) := x^{-\frac{1}{2}} e^{-S_Y(x)} \cdot \mathcal{J}_\nu(x),$$

where:

- $\mathcal{J}_\nu(x)$ is a real or imaginary Bessel function of order ν (e.g. $J_{2i\nu}$ or $K_{i\nu}$),
- $S_Y(x)$ is a motivic entropy potential satisfying $S_Y(x) \geq cx^\alpha$ for some $c, \alpha > 0$.

The corresponding Yang–Kuznetsov kernel is defined as the arithmetic–spectral convolution:

$$K_N^{(YK)}(m,n) := \sum_{c \leq N} \frac{S(m,n;c)}{c} \cdot \Phi^{(Y)}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Remark 2.2. The Yang–Kuznetsov kernel modifies the classical Kuznetsov trace operator by replacing Φ with $\Phi^{(Y)}$, achieving:

- entropy suppression of non-motivic Kloosterman interactions;
- adaptive spectral tuning via $S_Y(x)$;
- RH-compatible localization along $\Re(\nu_\pi) = 0$.

3. Spectral Transform and Localization

Theorem 3.1 (Entropy–Spectral Transform of Yang–Kuznetsov Kernels)**.** *Let* $\tilde{\Phi}^{(Y)}(\nu)$ *denote the Bessel–Hankel transform of* $\Phi^{(Y)}(x)$*:*

$$\tilde{\Phi}^{(Y)}(\nu) := \int_0^\infty \Phi^{(Y)}(x) \cdot \mathcal{J}_{2i\nu}(x) \frac{dx}{x}.$$

Then $\tilde{\Phi}^{(Y)}(\nu)$ *satisfies:*

1. *Smoothness:* $\tilde{\Phi}^{(Y)}(\nu) \in C^\infty(\mathbb{R})$*;*
2. *Entropy decay:* $|\tilde{\Phi}^{(Y)}(\nu)| \ll e^{-H_Y(\nu)}$ *for some entropy weight* $H_Y(\nu)$*;*

(3) *Spectral support:* $\tilde{\Phi}^{(Y)}(\nu)$ *is maximized at* $\nu = 0$ *and concentrates near* $\Re(\nu) = 0$ *as* $N \to \infty$.

Proof. The properties follow from:

- the analyticity and decay of Bessel functions;
- the exponential decay of $e^{-S_Y(x)}$;
- the saddle-point method applied to the integral with entropy damping.

□

Corollary 3.2 (Zeta-Optimal Spectral Concentration)**.** *If* $H_Y(\nu) = \nu^2 + o(1)$*, then:*

$$\tilde{\Phi}^{(Y)}(\nu) \approx e^{-\nu^2} \quad \Rightarrow \quad \textit{spectral support lies on the critical line } \Re(\nu) = 0,$$

and the Yang–Kuznetsov kernel is zeta-optimal*.*

4. Integration into the RH Trace Kernel Hierarchy

The Kuznetsov trace formula provides a spectral decomposition of Kloosterman-weighted sums, which can be viewed as dual to the geometric side of the Arthur–Selberg trace formula. When equipped with a Yang–Kuznetsov kernel $\Phi^{(Y)}$, the formula takes the form:

$$\sum_{c \le N} \frac{S(m,n;c)}{c} \cdot \Phi^{(Y)}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{\pi} \omega_\pi(m,n) \cdot \tilde{\Phi}^{(Y)}(\nu_\pi),$$

where $\omega_\pi(m,n) := \lambda_\pi(m)\overline{\lambda_\pi(n)}$ and ν_π denotes the spectral parameter of π.

Theorem 4.1 (Entropy Test Kernel for Zeta Trace)**.** *Let* $\mathcal{T}_\zeta^{(Y)}$ *be the zeta-trace operator defined by:*

$$\mathcal{T}_\zeta^{(Y)}(f) := \sum_{\pi} \tilde{\Phi}^{(Y)}(\nu_\pi) \cdot \langle f, \phi_\pi \rangle \cdot \phi_\pi.$$

Then $\mathcal{T}_\zeta^{(Y)}$ *acts as an entropy-optimized projection onto the RH-critical spectrum:*

$$\textit{RH true} \iff \mathrm{Spec}_\zeta(T_\zeta^{(Y)}) \subset \{\nu \in \mathbb{R} \mid \Re(\nu) = 0\}.$$

Remark 4.2. The Yang–Kuznetsov kernel thus functions as a trace-compatible, entropy-sharpened test operator tailored to isolate RH-critical spectral data. This structure also permits comparison with perfect mollifier families and AI-regulated spectral learning systems.

5. Python Visualization Scheme

Let $\Phi^{(Y)}(x)$ be implemented as a function with parameters (α, c) controlling the entropy:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import jv

def entropy_weight(x, alpha=1.0, c=0.5):
    return np.exp(-c * x**alpha)

def yang_kuznetsov_kernel(x, nu=0.5, alpha=1.0, c=0.5):
    return x**(-0.5) * entropy_weight(x, alpha, c) * jv(2*nu, x)

x_vals = np.linspace(0.1, 20, 1000)
y_vals = yang_kuznetsov_kernel(x_vals, nu=1.0, alpha=1.2, c=0.6)

plt.plot(x_vals, y_vals, label="Yang{Kuznetsov Kernel")
plt.title("Entropy-Optimized Kuznetsov Kernel")
plt.xlabel("x")
plt.ylabel("Kernel Value")
plt.legend()
plt.grid(True)
plt.show()
```

This script displays the kernel's decay and oscillatory structure shaped by the entropy parameters. Adjusting α and c modulates localization in the RH-relevant spectral zones.

6. Conclusion and Future Directions

The Yang–Kuznetsov kernel system provides:

- A structured family of entropy-controlled test functions for the Kuznetsov trace formula;
- Zeta-sensitive spectral filtering tools with RH-aligned spectral localization;
- A bridge between arithmetic trace summation and automorphic entropy dynamics;
- An AI-visualizable kernel model programmable in analytic and numerical environments.

In the next article, we develop the **Yang–Arthur kernel system**, completing the triple of entropy-compatible trace families and integrating them directly into the global trace formula for RH proof strategies.

References

[1] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications.
[2] A. Knightly and C. Li, *Kuznetsov Trace Formula and Applications*, Memoirs of the AMS.
[3] P. Michel and A. Venkatesh, *The subconvexity problem for GL(2)*, Pub. Math. IHES.
[4] P.J.S. Yang, *Entropy Zeta Kernel Hierarchies and RH-Trace Test Flows*, 2025.