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Details of the First Region of Integers x With $\pi_{3,2}(x) < \pi_{3,1}(x)$

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Abstract. Since the time of Chebyshev [4] there has been interest in the magnitude of the smallest integer x with $\pi_{3,2}(x) < \pi_{3,1}(x)$, where $\pi_{b,c}(x)$ denotes the number of positive primes $\leqslant x$ and $\equiv c \pmod b$. The authors have recently reached this threshold with the discovery that $\pi_{3,2}(608981813029) - \pi_{3,1}(608981813029) = -1$. This paper includes a detailed numerical and graphical description of values of $\pi_{3,2}(x) - \pi_{3,1}(x)$ in the vicinity of this long sought number.

1. Introduction. Let $\pi_{b,c}(x)$, $1 \leqslant c < b$, $(b, c) = 1$, denote the number of primes $\leqslant x$ which are $\equiv c \pmod b$ and let $\Delta_b(x, c, c') = \pi_{b,c}(x) - \pi_{b,c'}(x)$.

In a letter written in 1853, Chebyshev [4] (see also [5], [6], [10]) remarked that $\pi_{3,1}(x) < \pi_{3,2}(x)$ and $\pi_{4,1}(x) < \pi_{4,3}(x)$ for all small values of x. Due to the famous result of J. E. Littlewood [9], it is now well known that these inequalities and the related inequality $\pi(x) < \operatorname{li} x$ (where the right-hand side is the usual integral logarithm of x) are reversed for infinitely many integers x.

The first negative value of $\Delta_4(x, 3, 1)$ is not difficult to find with a computer. In 1957 John Leech discovered that $\Delta_4(x, 3, 1) = -1$ for $x = 26861$. This "first axis crossing" was discovered independently at a slightly later date by Shanks [10] and Wrench (see [10, p. 273]). However, in the computations by Leech, Shanks, Lehmer, and others no integers x with $\pi_{3,2}(x) < \pi_{3,1}(x)$ were found. The present authors experienced discouragement when an exhaustive search of the first quarter of a trillion integers still produced no axis crossing. A computer solution, if possible, is highly desirable since it is very difficult to obtain effective bounds for first axis crossings which are likely to represent anything close to actual values. For example, values of x with $\operatorname{li} x < \pi(x)$ may occur long before the integers x around 1.53×10^{1165} found by Lehman [8]. Moreover, even less has been known previously regarding the first negative values of $\Delta_3(x, 2, 1)$.

Consequently, we did not abandon the search but developed a new and faster program described in [2] (with values coinciding with the earlier run at 2.5×10^{11}) and discovered on December 25, 1976 that

$$\Delta_3(608981813029,2,1) = -1. \tag{1.1}$$

The purpose of this note is to give a numerical and graphical description of the region of integers from $x_0 = 608{,}981{,}813{,}029$ to $x_f = 610{,}968{,}213{,}796$ (and the vicinity). This region contains 316,889,212 integers x with $\Delta_3(x, 2, 1)$ negative, the only such integers that occur for $x < 2x_f$. Interestingly, this region separates into

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two parts removed from each other by an appreciable 1,363,263,116 integers. These appear pictorially as "twin" subregions; the first contains 150,276,170 and the second 166,613,042 negative values of $\Delta_3(x, 2, 1)$.

In spite of the large amount of machine time necessary to obtain these results we have considerable confidence in their accuracy. In particular, our computation of $\pi_{3,1}(10^{12}) + \pi_{3,1}(10^{12}) + 1$ agrees with the value of $\pi(10^{12})$ computed by Bohman [3];

$$\begin{gathered} \pi_{3,1}(10^{12}) = 18{,}803{,}933{,}520, \qquad \pi_{3,2}(10^{12}) = 18{,}803{,}978{,}497, \\ \pi(10^{12}) = 37{,}607{,}912{,}018. \end{gathered} \tag{1.2}$$

Moreover, our prime count agrees with that of Bohman at 10^{11}, 2×10^{11}, and 4×10^{11}.

2. Numerical and Graphical Description of the First Axis Crossing Region for the Modulus 3. As in [1], we define the first axis crossing region to be the first set of positive integers x with $x_0 \leqslant x \leqslant x_f$ satisfying the conditions,

$$\Delta_3(x_0, 2, 1) = \Delta_3(x_f, 2, 1) = -1, \tag{2.1}$$

and $\Delta_3(x, 2, 1) \geqslant 0$ for each integer x with

$$x_f < x < 2x_f. \tag{2.2}$$

Note that (2.1) does not require that we have $\Delta_3(x, 2, 1) < 0$ $\forall x$ with $x_0 \leqslant x \leqslant x_f$. We call the longest sequence of consecutive integers with $\Delta_3(x, 2, 1) < 0$ in an axis crossing region the "longest negative block" and the longest sequence of consecutive integers with $\Delta_3(x, 2, 1) \geqslant 0$ in an axis crossing region the "longest non-negative block". The condition (2.2) merely ensures that the region considered is an isolated axis crossing region (negative values of $\Delta_3(x, 2, 1)$ occurring at, say 650 billion, would be considered part of the same general region).

The word "value" in Table 1 refers always to values of $\Delta_3(x, 2, 1)$ for the entire axis crossing region or, as denoted, for one of the two twin subregions into which this region naturally separates. All classifications refer to values between the first and last -1 values inclusive of the region or subregion under consideration, excepting the first and last zero values which respectively precede and follow the first and last -1 values, and the classification "total integers on the axis" which refers to values between the first and last zero values (an integer is said to be above, on, or below the axis according as $\Delta_3(x, 2, 1)$ is positive, zero, or negative).

Figures 1 through 4 graphically depict the behavior of $\Delta_3(x, 2, 1)$ in the vicinity of the first axis crossing region for the modulus 3. Figures 1 and 2 consist of 610 points; Figures 3 and 4 consist of 1200 points. In all figures the vertical line represents the zero axis. In Figures 1 and 2 the left vertical line is for reference and approximates $\pi(x^{1/2})/4$. The horizontal scale of Figure 2 is twice that of Figure 1, and the horizontal scale of Figures 3 and 4 is 4 times that of Figure 1. Figure 1 spans 10% of the integers in the vicinity of 610 billion, Figure 2 spans 1%, and Figures 3 and 4 span .1%. Table 1 may be referred to for specific values of $\Delta_3(x, 2, 1)$ in the entire region (Figures 1 and 2), and in the "twin" subregions (Figures 3 and 4).

TABLE 1

	Subregion 1	Subregion 2	Region 1
first zero value ≥ 2	608,981,813,017	610,704,087,703	608,981,813,017
first -1 value occurs here	608,981,813,029	610,704,087,757	608,981,813,029
smallest negative value	-1538	-1223	-1538
first occurs here	609,224,663,413	610,772,745,721	609,224,663,413
largest positive value	1021	977	5564
first occurs here	609,092,758,259	610,850,161,643	609,801,304,649
longest negative block	59,684,338	40,118,512	59,684,338
starts here	609,186,533,689	610,916,115,529	609,186,533,689
longest non-negative block	68,710,514	32,085,152	1,363,263,116
starts here	609,058,426,259	610,809,565,661	609,340,824,641
total integers below axis	150,276,170	166,613,042	316,889,212
total integers not below axis	208,735,442	97,512,998	1,669,511,556
total integers on axis	307,702	264,684	572,386
number of negative blocks	5549	4859	10,408
ratio: total integers below axis/x_f	.000246	.000273	.000519
last -1 value occurs here	609,340,824,640	610,968,213,796	610,968,213,796
last zero value < $2x_f$	609,340,824,646	610,968,213,802	610,968,213,802

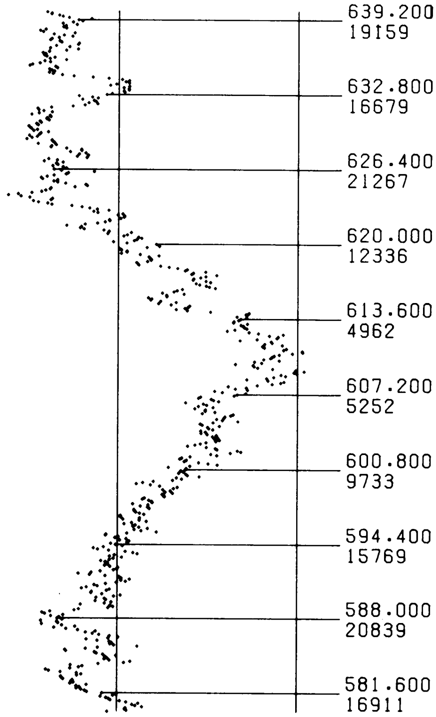

FIGURE 1

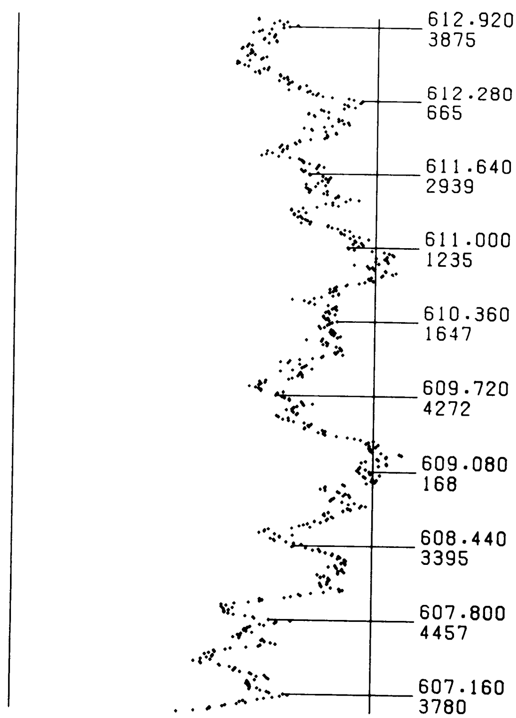

FIGURE 2

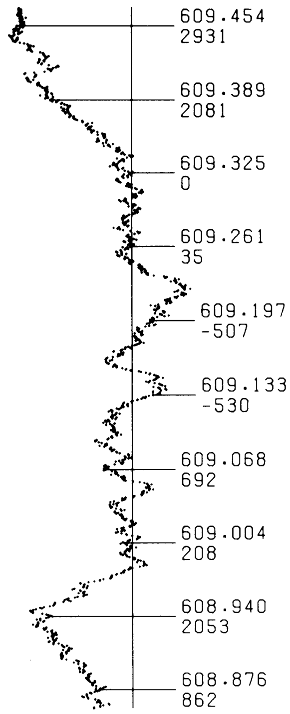

FIGURE 3

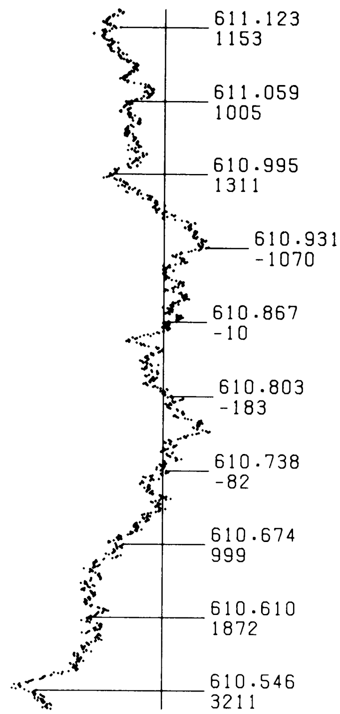

FIGURE 4

3. Concluding Remark. The first axis crossing region for the modulus 4 contains only two integers with $\Delta_4(x, 3, 1)$ negative; specifically $\Delta_4(26861, 3, 1) = \Delta_4(26862, 3, 1) = -1$ (the next negative value for $\Delta_4(x, 3, 1)$ is $x = 616841$). In contrast the first axis crossing region for the modulus 3 attains $\Delta_3(x, 2, 1) = -1538$ and remains negative for tens of millions of consecutive values of x. On the other hand, with its huge scale, Figure 1, which spans 10% of the integers in the vicinity of 610 billion, does appear similar to the Figure 1 in [2], which depicts $\Delta_4(x, 3, 1)$ for 10% of the integers in the vicinity of the first axis crossing, $x = 26861$. In particular, both $\Delta_3(x, 2, 1)$ and $\Delta_4(x, 3, 1)$ assume values far greater than $\pi(x^{1/2})/4$ above and below the first axis crossing (and within the 10% range depicted), so that values appear to plunge, barely cross, and then rise again rapidly. We suggest, therefore, that in any computer search for the smallest integer x with $\operatorname{li} x < \pi(x)$ which does not perform a check at each odd integer (e.g., using the recurrence employed by Bohman [3]), integers should be checked at fairly regular intervals (relative to x) in order not to miss a plunge resulting in a "shallow" axis crossing region. Moreover, rapid plunges (particularly plunges below $\pi(x^{1/2})/4$) should be scrutinized with special care.

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