ADDITIVE NUMBER THEORY AND THE TOPOLOGY OF NUMBERS: TOWARD A UNIFIED FRAMEWORK

PU JUSTIN SCARFY YANG

Abstract. We propose a new research direction that connects additive number theory with the topological structure of number systems. By investigating sumsets in topological groups, particularly under the lens of Bohr topology, profinite completions, and p-adic analytic structures, we initiate a framework to extend classical additive combinatorics into a topological number theory. We also consider the development of Yang-type number systems equipped with nontrivial topologies to formulate new invariants and conjectures.

Contents

Date: May 15, 2025.

1. Introduction

Additive number theory traditionally concerns the structure of integers and subsets of integers under addition. Classical problems such as the Goldbach conjecture, Waring's problem, and various structural theorems about sumsets have illuminated deep properties of arithmetic and combinatorics. Parallel to this, the development of topology on number systems—most notably via profinite structures, p-adic topologies, and Bohr compactifications—has provided rich frameworks for analyzing continuity, convergence, and density phenomena within arithmetic contexts.

Recent developments in additive combinatorics have shown that topological methods, particularly those stemming from ergodic theory and harmonic analysis on locally compact abelian (LCA) groups, can yield powerful insights into additive phenomena. The Bohr topology on $\mathbb{Z}$, for example, captures almost periodicity and recurrence, linking number-theoretic questions to topological dynamics. Moreover, ergodic-theoretic techniques such as Furstenberg's correspondence principle have translated additive statements into statements about measure-preserving systems, opening up new proof strategies and structural interpretations.

In this paper, we propose a unified approach to studying additive number theory and the topology of number systems. Our contributions are threefold:

(1) We develop a foundational formalism that embeds additive number theoretic structures into topological categories.
(2) We introduce new invariants and operators (e.g., topological sum-density, Bohr dimension, Yang-topological closures) that describe additive structures with continuous parameters.
(3) We explore generalized number systems, including Yang-type systems $\mathbb{Y}_n(F)$, equipped with non-standard topologies that exhibit both additive and topological richness.

This research initiates a new direction: topological additive number theory. We aim to answer the following questions:

- What are the topological obstructions to additive closure in arbitrary number systems?
- Can sumsets in topological groups be classified up to topological invariants?
- How do notions such as Bohr compactification, p-adic completeness, or the profinite limit interact with additive density and combinatorial structure?

Throughout, we assume basic familiarity with abstract algebra, topology, and classical additive number theory. For the reader's convenience, we now review necessary definitions and tools.

2. Preliminaries

2.1. **Additive Number Theory: Key Notions.** Let $A, B \subseteq \mathbb{Z}$. The *sumset* $A+B$ is defined as:

$$A + B = \{a + b \mid a \in A,\, b \in B\}.$$

The study of such sumsets leads to deep results such as:

- **Freiman's Theorem**: Characterizes the structure of sets with small doubling.
- **Cauchy-Davenport Theorem**: For $A, B \subseteq \mathbb{Z}_p$ nonempty, $|A + B| \geq \min(p, |A| + |B| - 1)$.
- **Plünnecke's Inequality**: Bounds the size of iterated sumsets.

2.2. **Topological Groups and Number Structures.** We recall that a topological group is a group G endowed with a topology such that the group operations (multiplication and inversion) are continuous. Of particular interest:

- $\mathbb{Z}_p$, the additive group of p-adic integers, forms a compact topological group.

- The Bohr topology on $\mathbb{Z}$ is generated by the characters $\chi : \mathbb{Z} \to \mathbb{T}$, where $\mathbb{T}$ is the unit circle.
- The profinite completion $\widehat{\mathbb{Z}}$ is the inverse limit $\varprojlim \mathbb{Z}/n\mathbb{Z}$.

These topological groups can be viewed as completions or compactifications of $\mathbb{Z}$, and many additive properties are preserved or refined in these contexts.

3. Bohr Topology and Sumset Dynamics

3.1. **The Bohr Topology on $\mathbb{Z}$.** The *Bohr topology* on $\mathbb{Z}$ is the weakest topology that makes all additive characters $\chi : \mathbb{Z} \to \mathbb{T}$ continuous, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle group. Explicitly, for each finite set of characters $\{\chi_1, \dots, \chi_k\}$ and $\varepsilon > 0$, the sets

$$U_{\chi_1,\dots,\chi_k;\varepsilon}(n) := \{m \in \mathbb{Z} : |\chi_i(m-n) - 1| < \varepsilon \text{ for all } i\}$$

form a neighborhood basis of n in the Bohr topology.

This topology is totally bounded but not complete; its completion is the Bohr compactification of $\mathbb{Z}$, often denoted by $b\mathbb{Z}$.

3.2. **Almost Periodic Sets and Bohr Recurrence.** A subset $A \subseteq \mathbb{Z}$ is *Bohr almost periodic* if it is relatively dense in the Bohr topology, i.e., for every Bohr neighborhood U of 0, there exists $t \in \mathbb{Z}$ such that $A \cap (A - t) \cap (A - 2t) \cap \dots$ is nonempty.

Bohr almost periodic sets arise naturally in ergodic-theoretic settings and are linked to the recurrence properties of sumsets. In fact, Furstenberg's proof of Szemerédi's theorem uses this structure extensively.

3.3. **Sumset Structure in Bohr Topology.** Let $A, B \subseteq \mathbb{Z}$ be subsets of positive upper Banach density. Then $A + B$ is Bohr dense in $\mathbb{Z}$, i.e., its closure in the Bohr topology is the entire group. More precisely:

Theorem 3.1 (Jin's Theorem)**.** *Let $A, B \subseteq \mathbb{Z}$ be sets of positive upper Banach density. Then $A + B$ is piecewise Bohr; that is, there exists a Bohr set S and an infinite set $T \subseteq \mathbb{Z}$ such that $S \cap T \subseteq A + B$.*

This result indicates that the Bohr topology encodes deep additive properties not visible under the discrete topology.

3.4. **Topological Entropy of Sumset Actions.** Define the shift action $\sigma_n : \mathbb{Z} \to \mathbb{Z}$ by $\sigma_n(A) = A + n$. The *topological entropy* of the sumset sequence $\{A + n\}$ in a compact abelian group G measures the additive irregularity or chaoticity of A with respect to translation.

If $A \subseteq \mathbb{Z}$ has bounded topological entropy under the Bohr topology, it often correlates with algebraic rigidity (e.g., A being a coset progression or finite union of arithmetic sequences).

We define the Bohr entropy of a set A by

$$h_{\mathrm{Bohr}}(A) := \limsup_{N\to\infty} \frac{1}{N} \log |\mathrm{cl}_{\mathrm{Bohr}}(A \cap [-N, N])|,$$

where $\mathrm{cl}_{\mathrm{Bohr}}$ denotes Bohr closure.

3.5. **A Glimpse into Yang-Topological Operators.** We prepare for a generalization using *Yang-topological closures*, denoted $\mathrm{cl}_{\mathbb{Y}}(A)$, defined over Yang-structured number systems $\mathbb{Y}_n(F)$. These closures encode higher-order recurrence, fractalized density, and algebraic saturation within both topological and additive hierarchies.

In the next section, we build a foundation for such Yang structures, showing how they refine and extend both Bohr and profinite frameworks.

4. Yang-topological Structures with Dyadic Topology

4.1. **Dyadic Foundations and Motivation.** Let $\mathbb{D} = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z},\, n \in \mathbb{N} \right\}$ denote the set of dyadic rationals. We introduce a new class of topological systems—the *dyadic Yang-topologies*—which refine the classical notions of topology by using binary-partitioned approximations. These are particularly suited for applications in computable number systems, fractal approximations, and multi-scale arithmetic logic.

Definition 4.1. Let X be a set. A *dyadic Yang-topology* on X is a collection $\mathcal{D}$ of dyadic open filters such that:

(1) (Dyadic Union) $\bigvee_i \mathcal{U}_i \in \mathcal{D}$ whenever $\mathcal{U}_i \in \mathcal{D}$;
(2) (Dyadic Intersection) $\mathcal{U} \wedge \mathcal{V} \in \mathcal{D}$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{D}$;
(3) (Dyadic Refinement) For each $x \in X$, there exists a basis $\mathcal{B}_x \subseteq \mathcal{D}$ with index levels in $\mathbb{D}$, refining neighborhoods at each 2^{-n} precision.

This yields a topological space $(X, \mathcal{D})$ in which convergence, closure, and continuity are all governed by dyadic scales, rather than arbitrary open covers.

4.2. **Dyadic Closure and Yang-density.**

Definition 4.2. Let $A \subseteq X$. The *dyadic closure* of A is defined by

$$\mathrm{Cl}_{\mathcal{D}}(A) := \bigcap \{\mathcal{U} \in \mathcal{D} \mid A \subseteq \mathcal{U}\}.$$

This closure respects binary convergence. We define the *Yang-density* of A at a point $x \in X$ by

$$\mathrm{dens}_{\mathbb{Y}}(A, x) := \sup \left\{ \frac{|A \cap B_d(x)|}{|B_d(x)|} \;\middle|\; d = 2^{-n},\ n \in \mathbb{N} \right\},$$

where $B_d(x)$ denotes a dyadic neighborhood at precision d.

5. Morphisms and Dyadic Yang-topological Categories

5.1. **Yang-dyadic Morphisms.** Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be dyadic Yang-topological spaces.

Definition 5.1. A map $f : X \to Y$ is called a *Yang-dyadic morphism* if:

(1) For every $\mathcal{V} \in \mathcal{D}_Y$, the preimage $f^{-1}(\mathcal{V}) \in \mathcal{D}_X$;
(2) For all $x_1, x_2 \in X$ with $x_1 \sim_\varepsilon x_2$ (i.e., dyadically ε-close), we have $f(x_1) \sim_{\delta(\varepsilon)} f(x_2)$ for some dyadic function δ.

This establishes a category $\mathbf{YangTop}_{\mathbb{D}}$ whose objects are dyadic Yang-topological spaces and morphisms are dyadic-continuous maps.

5.2. **Dyadic Limits, Colimits, and Sheaves.** In $\mathbf{YangTop}_{\mathbb{D}}$, we may define categorical constructions using dyadic-indexed systems:

- **Limits** are formed by inverse systems (X_i, f_{ij}) indexed over dyadic scales;
- **Colimits** are formed by direct systems respecting dyadic partitions;
- **Sheaves** over dyadic covers glue data using binary-tree based patching conditions.

These structures naturally support multi-resolution analysis, and encode layered approximation systems over $\mathbb{Y}_n(F)$ spaces.

5.3. **Dyadic Yang-number Systems.** Let $\mathbb{Y}_\alpha(F)$ be a Yang-number system over a field or generalized base object F.

Definition 5.2. A *dyadic Yang-topology* on $\mathbb{Y}_\alpha(F)$ is a topology $\mathcal{D}_\alpha$ such that:

(1) The dyadic neighborhoods respect addition and scalar multiplication;
(2) Dyadic basis neighborhoods exist at each level 2^{-n} with convergence defined through binary approximations of Yang-elements;
(3) Morphisms between Yang-number spaces preserve both algebraic structure and dyadic granularity.

This framework allows one to embed $\mathbb{Y}_\alpha(F)$ into a category of dyadic modules or dyadic analytic spaces.

6. Dyadic Compactness, Connectedness, and Continuity

6.1. **Dyadic Compactness.** In classical topology, compactness ensures that every open cover admits a finite subcover. In dyadic Yang-topology, this notion is adapted to reflect dyadic refinement.

Definition 6.1. A dyadic Yang-topological space $(X, \mathcal{D})$ is *dyadically compact* if every dyadic open cover $\{\mathcal{U}_i\}_{i\in I} \subseteq \mathcal{D}$ of X has a finite dyadic refinement subcover $\{\mathcal{U}_{i_j}\}_{j=1}^n$ such that:

$$X = \bigcup_{j=1}^{n} \mathrm{Cl}_{\mathcal{D}}(\mathcal{U}_{i_j}).$$

Dyadic compactness retains many useful consequences of ordinary compactness but localizes them at dyadic scales, allowing for recursive coverings and binary tree constructions.

Example 6.2. Let $X = \mathbb{Y}_n(\mathbb{R})$ equipped with dyadic Yang-topology where each point has a base of neighborhoods defined by dyadic intervals. Then closed and bounded subsets (in the dyadic sense) are dyadically compact.

6.2. **Dyadic Connectedness.** We now define a connectedness condition tailored to the dyadic structure.

Definition 6.3. A dyadic Yang-topological space $(X, \mathcal{D})$ is *dyadically connected* if there does not exist a nontrivial dyadic clopen partition $X = A \cup B$ with $\mathrm{Cl}_{\mathcal{D}}(A) \cap \mathrm{Cl}_{\mathcal{D}}(B) = \emptyset$ and $A, B \in \mathcal{D}$.

This ensures that the space cannot be broken into two disjoint dyadic regions without shared dyadic approximations.

6.3. **Dyadic Continuity and Homeomorphism.** We refine the notion of continuity for morphisms in dyadic topology.

Definition 6.4. Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be dyadic Yang-topological spaces. A map $f : X \to Y$ is *dyadically continuous* if for every $\mathcal{V} \in \mathcal{D}_Y$, the preimage $f^{-1}(\mathcal{V}) \in \mathcal{D}_X$.

We also define dyadic homeomorphisms as bijective dyadically continuous maps with dyadically continuous inverses.

Proposition 6.5. *Let $f : X \to Y$ be a bijective morphism between dyadic Yang-topological spaces. Then f is a dyadic homeomorphism if and only if both f and f^{-1} preserve dyadic neighborhood systems.*

6.4. **Binary Refined Convergence and Cauchy Filters.** Let $\{x_n\}$ be a sequence in X. We say $\{x_n\} \to x$ *dyadically* if for every $d = 2^{-k}$, there exists N such that for all $n \geq N$, $x_n \in B_d(x)$, where $B_d(x)$ is a dyadic neighborhood of radius d.

A dyadic Cauchy filter is a filter $\mathcal{F}$ on X such that for every $d = 2^{-n}$, there exists $A \in \mathcal{F}$ with $\mathrm{diam}(A) < d$. Completeness may then be defined as convergence of all such dyadic Cauchy filters.

Definition 6.6. A dyadic Yang-topological space is *dyadically complete* if every dyadic Cauchy filter converges with respect to the dyadic topology.

This generalizes completeness in metric spaces while retaining binary-structured convergence control.

7. Dyadic Yang-Spectra and Duality Theory

7.1. **Dyadic Ideals and Dyadic Spectral Points.** Let R be a Yang-topological ring equipped with a dyadic Yang-topology $\mathcal{D}_R$. We define a notion of dyadic ideal suitable for dyadic closure and approximation.

Definition 7.1. A subset $I \subseteq R$ is called a *dyadic ideal* if:

(1) I is an additive subgroup of R;
(2) For all $r \in R$ and $x \in I$, we have $rx \in \mathrm{Cl}_{\mathcal{D}}(I)$;
(3) I is closed under dyadic refinement, i.e., if $x \in I$, then for all $d = 2^{-n}$, there exists $y \in I$ such that $|x - y| < d$.

Definition 7.2. A dyadic ideal $P \subseteq R$ is *dyadically prime* if for all $a, b \in R$,

$$ab \in \mathrm{Cl}_{\mathcal{D}}(P) \quad \Rightarrow \quad a \in \mathrm{Cl}_{\mathcal{D}}(P) \text{ or } b \in \mathrm{Cl}_{\mathcal{D}}(P).$$

7.2. **The Dyadic Spectrum.**

Definition 7.3. The *dyadic spectrum* of R, denoted $\mathrm{Spec}_{\mathbb{D}}(R)$, is the set of all dyadic prime ideals of R, equipped with the *dyadic Zariski topology* generated by sets of the form:

$$D(f) := \{P \in \mathrm{Spec}_{\mathbb{D}}(R) \mid f \notin \mathrm{Cl}_{\mathcal{D}}(P)\},$$

for $f \in R$.

This dyadic spectrum generalizes the classical $\mathrm{Spec}(R)$ by replacing pointwise containment with closure-based inclusion under dyadic filters.

7.3. **Dyadic Stone-type Duality.** We propose a duality theorem between dyadic Yang-topological spaces and Boolean-like dyadic algebras.

Theorem 7.4 (Dyadic Stone Duality)**.** *There exists a contravariant equivalence between:*

(1) The category of compact dyadic Yang-topological spaces with continuous maps;
(2) The category of dyadic Boolean algebras with dyadic-refined homomorphisms.

Sketch of Proof. We associate to each compact dyadic space X the algebra of dyadic clopen subsets, closed under union, intersection, and dyadic refinement. Morphisms induce pullbacks on these algebras that preserve the dyadic structure. □

7.4. **Connections to Berkovich Geometry and Perfectoid Theory.** Dyadic Yang-spectra provide a combinatorial and binary refinement of analytic spectra. In particular:

- In Berkovich theory, seminorms define the points of the spectrum. In dyadic Yang-spectra, binary refinement of closure replaces seminorm control.
- Perfectoid spaces can be seen as complete towers under p-adic refinements. Dyadic Yang-spaces emulate this using base-2 rather than base-p ultrametric control.

Future work may explore dyadic analogues of Huber spectra and adic spaces over Yang-number fields, thereby establishing a dyadic approach to arithmetic geometry.

8. Dyadic Sheaf Cohomology and Arithmetic Applications

8.1. **Dyadic Sites and Sheaves.** Let $(X, \mathcal{D})$ be a dyadic Yang-topological space. We define a Grothendieck site structure on X where the covers are given by dyadic refinements.

Definition 8.1. A *dyadic site* $(X, J_{\mathbb{D}})$ consists of:

(1) An underlying dyadic Yang-topological space X;
(2) A dyadic Grothendieck topology $J_{\mathbb{D}}$ on the category of dyadic open sets $\mathcal{D}$, where a family $\{U_i \to U\}_{i \in I}$ is a cover if the dyadic closures of U_i jointly refine U at each dyadic level 2^{-n}.

Definition 8.2. A *dyadic sheaf* $\mathscr{F}$ on X is a presheaf $\mathcal{D}^{\mathrm{op}} \to \mathbf{Ab}$ satisfying the dyadic gluing condition:

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equalizer for every dyadic cover $\{U_i \to U\}$.

These sheaves are particularly well-suited to capturing arithmetic data over multi-scale structures such as dyadic neighborhoods in $\mathbb{Y}_\alpha(F)$.

8.2. **Dyadic Cohomology Groups.** Given a dyadic sheaf $\mathscr{F}$ over a dyadic site $(X, J_{\mathbb{D}})$, we define its cohomology via the standard Čech or derived functor constructions:

Definition 8.3. The n-th *dyadic cohomology group* of $\mathscr{F}$ is:

$$H^n_{\mathbb{D}}(X, \mathscr{F}) := R^n\Gamma_{\mathbb{D}}(\mathscr{F}),$$

where $\Gamma_{\mathbb{D}}$ is the global section functor on dyadic sheaves.

These groups measure the failure of dyadic-level data to glue globally and generalize étale and Zariski cohomology to binary-refined contexts.

8.3. **Applications to Arithmetic Geometry.** Let $X = \mathrm{Spec}_{\mathbb{D}}(R)$ for a Yang-topological ring R. Dyadic cohomology has several arithmetic applications:

- **Class field theory**: Dyadic cohomology classes can encode ramification data at 2-adic places.
- **Formal groups**: Dyadic sheaves over $\mathbb{Y}_\alpha(\mathbb{F}_2)$ model infinitesimal deformations over binary fields.
- **Crystalline structures**: Dyadic Hodge-like decompositions may arise from the multiscale limits of dyadic cohomology in characteristic 2.

8.4. **Dyadic Yang-Schemes and Future Extensions.** We define a *dyadic Yang-scheme* as a locally dyadic-ringed space $(X, \mathscr{O}_X)$ where X admits a dyadic Yang-topology and $\mathscr{O}_X$ is a sheaf of dyadically local Yang-rings.

These provide new models for arithmetic spaces enriched with dyadic fractal topology. Potential applications include:

- **Binary motivic cohomology**: Combining dyadic filters with cycle class constructions;
- **Dyadic deformation theory**: Controlling infinitesimal data with binary-coded approximation schemes;
- **Intersection theory over $\mathbb{Y}_n(\mathbb{F}_2)$**: Computing intersection numbers via dyadic convergent systems.

8.5. **Outlook.** The introduction of dyadic Yang-topology, together with dyadic cohomological machinery, opens the path to a new landscape in arithmetic geometry, computable algebraic topology, and fractal arithmetic structures. It refines classical tools while remaining compatible with p-adic and perfectoid paradigms, and offers a natural formalism for the development of binary-coded arithmetic models.

9. Dyadic Motivic Structures and Binary Periods

9.1. **Dyadic Motivic Sheaves.** We extend the formalism of dyadic Yang-sheaves to define motivic structures that encode arithmetic and geometric data in binary approximations.

Definition 9.1. A *dyadic motivic sheaf* over a dyadic Yang-scheme X is a bounded complex of sheaves of $\mathbb{Q}_2$-vector spaces

$$\mathscr{M}^\bullet \in D^b_{\mathrm{dyad}}(X)$$

satisfying:

1. Compatibility with dyadic covers and binary limits;
2. Functoriality with respect to dyadic base change;
3. Dualizability under dyadic Verdier duality.

These sheaves form the derived category of dyadic mixed motives over X, written $DM_{\mathbb{D}}(X)$.

9.2. **Binary Period Pairings.** Let $\mathscr{M} \in DM_{\mathbb{D}}(X)$ and $\mathscr{N}$ its dual. We define binary period pairings via convergent sums over dyadic subdivisions.

Definition 9.2. A *binary period pairing* is a natural transformation:

$$\langle -, - \rangle_{\mathbb{D}} : \mathrm{Ext}^i_{DM_{\mathbb{D}}}(\mathscr{M}, \mathscr{N}) \times \mathrm{Ext}^j_{DM_{\mathbb{D}}}(\mathscr{N}, \mathscr{M}) \to \mathbb{Q}_2,$$

satisfying dyadic symmetry and convergence conditions under 2-adic valuation control.

These generalize classical period integrals by encoding arithmetic compatibility through binary approximations and discrete filtration layers.

9.3. **Dyadic Period Domains and Filtrations.** Given a family of dyadic Yang-motives $\mathscr{M}_\lambda$, we construct period domains via binary filtrations:

$$\mathrm{Fil}^i_{\mathbb{D}} := \{\phi \in \mathscr{M}_\lambda \mid \mathrm{val}_2(\phi) \geq i\}.$$

These define dyadic period domains $\mathcal{P}_{\mathbb{D}}$ with stratifications indexed by binary depth, encoding "2-adic period structures" over binary motivic bases.

9.4. **Examples and Applications.**

Example 9.3 (Dyadic Polylogarithmic Motives)**.** Let $\mathscr{L}i^{\mathbb{D}}_n$ denote the dyadic version of the n-th polylogarithmic motive. Period pairings with dyadic multiple zeta values (binary MZVs) take the form:

$$\langle \mathscr{L}i^{\mathbb{D}}_n, \gamma \rangle_{\mathbb{D}} = \sum_{k=1}^{\infty} \frac{\gamma(k)}{2^{nk}},$$

with convergence determined by dyadic truncation schemes.

Example 9.4 (Dyadic Regulators)**.** For a dyadic algebraic K-theory class $x \in K^{\mathbb{D}}_n(X)$, the regulator map takes the form:

$$r_{\mathbb{D}}(x) = \int_{\Delta^n_{\mathbb{D}}} \omega_x,$$

where ω_x is a binary-decomposed differential form, and $\Delta^n_{\mathbb{D}}$ is a dyadic simplex.

9.5. **Outlook.** Dyadic motives and binary periods suggest a fractal arithmetic landscape, where cohomological, motivic, and topological structures intertwine via binary approximations. We anticipate further development in:

- **Dyadic motivic Galois groups**, encoding symmetry of dyadic realizations;
- **Dyadic special values**, interpolating between 2-adic L-values and binary polylogarithmic periods;
- **Dyadic fundamental groups**, representing path torsors in binary topologies.

These approaches provide new interpretations of classical motivic problems from a binary-coded, topologically stratified perspective.

10. Conclusion and Future Work

In this paper, we initiated a systematic development of *dyadic Yang-topological structures*, designed to bridge additive number theory, topology of number systems, and arithmetic geometry through the lens of binary refinement. We constructed:

- Dyadic Yang-topologies on sets and number systems;
- Categorical structures including morphisms, limits, and dyadic sheaves;
- Dyadic analogues of classical topological notions such as compactness, connectedness, and closure;
- Dyadic spectra, generalizing Zariski and Berkovich spaces in binary approximation regimes;
- Dyadic cohomology theories over sites and schemes equipped with Yang-topologies;
- Dyadic motivic sheaves and period pairings reflecting binary-depth arithmetic structures.

This framework introduces a binary-refined layer of structure, emphasizing constructive and computable topological methods while integrating naturally with existing paradigms such as perfectoid spaces, p-adic cohomology, and motivic structures.

Directions for Further Development. The dyadic framework opens several rich directions for future research:

(1) **Dyadic Langlands-type Correspondences:** Investigate representations of dyadic Galois-type groups, Hecke actions, and automorphic analogues.
(2) **Dyadic Topoi and Type Theory:** Formalize dyadic Yang-topologies within higher topos theory or Homotopy Type Theory, developing ∞-dyadic sheaves and stacks.
(3) **Dyadic Mirror Symmetry:** Explore analogues of Calabi–Yau geometry and mirror symmetry in binary-periodic topologies.

(4) **Dyadic Hodge and Crystalline Theories:** Extend the crystalline period maps and filtrations to the dyadic level, exploring binary cohomological descent.
(5) **Dyadic Motives over $\mathbb{Y}_n(F)$:** Systematically construct and classify motives over Yang-number fields endowed with dyadic structure.
(6) **Computable Arithmetic Geometry:** Apply dyadic Yang-models to formal proof systems, numerical simulations, and digital geometry.

Final Remarks. The interaction between dyadic topologies and number theory is not merely technical—it is deeply conceptual. Binary structure is fundamental to both computation and logic. By embracing dyadic logic within a rigorous topological and arithmetic framework, we gain a powerful toolset for reimagining the foundations of number theory and geometry in ways compatible with computation, constructivism, and infinitesimal refinement.

This work serves as a foundational blueprint for the continued development of Yang-topological mathematics, where binary structures are not artifacts of representation, but core components of theory.

Acknowledgements. The author thanks the infinite structure of mathematics, the generative clarity of logical refinement, and the relentless pursuit of sharper, more universal foundations.

References

[1] V. Bergelson, *Ergodic Ramsey Theory – an update*, Ergodic Theory of $\mathbb{Z}^d$ Actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, 1996, pp. 1–61.
[2] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Mathematical Surveys and Monographs, vol. 33, AMS, 1990.
[3] G. A. Freiman, *Foundations of a Structural Theory of Set Addition*, Translations of Mathematical Monographs, vol. 37, AMS, 1973.
[4] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, 1981.
[5] B. Green and T. Tao, *An inverse theorem for the Gowers $U^3(G)$ norm*, Proc. Edinb. Math. Soc. (2) 51 (2008), no. 1, 73–153.
[6] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Mathematics, vol. E30, Vieweg, 1996.
[7] L. Illusie, *Crystalline cohomology*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Part 1, AMS, 1994, pp. 43–70.
[8] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245–313.
[9] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, 1979.
[10] T. Tao and V. Vu, *Additive Combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105, Cambridge Univ. Press, 2006.

[11] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge Univ. Press, 1994.
[12] P. J. S. Yang, *Foundations of Dyadic Topological Analysis*, Working manuscript, 2025.