EXACTIFICATION THEORY IN ANALYTIC NUMBER THEORY VIII: MOTIVIC IDENTITY SYSTEMS AND ARITHMETIC FIELD THEORIES: FROM TOWERS TO TYPES TO THEORIES – A UNIFIED FRAMEWORK FOR ARITHMETIC COHOMOTOPY

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Abstract. In this eighth paper of the Exactification Program, we introduce a motivic identity algebra and propose a framework for arithmetic field theories based on exactification towers. We treat each arithmetic function as a field, each exactification tower as its configuration space, and each cohomology group as its observable data.

We develop a motivic identity system encoding the higher paths and symmetries among arithmetic resolutions, and formulate a derived cohomotopy field theory (dCFT) over the moduli stack $\mathbb{EXACT}_\infty$. This provides a new interface between condensed motives, higher type theory, and the spectral Langlands landscape.

We conclude with the proposal of an Arithmetic Exactification Field Theory (AEFT) unifying all arithmetic flows into a cohomological, motivic, and homotopy-theoretic system.

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1. Exactification Towers as Fields: Toward a Derived Arithmetic Field Theory

1.1. **From Functions to Fields.** Let $f \in \mathcal{A}$ be an arithmetic function. Traditionally, f is a discrete-valued object.

In the exactification framework, we reinterpret:

- f as a *field configuration*;
- its exactification tower $\mathscr{E}^{[f],\bullet}$ as a *space of local states or resolutions*;
- its cohomology $H^i(\mathscr{E}^{[f]})$ as *physical observables*;
- its motivic lift $\mathbb{M}_f$ as the *global field class.*

Thus, every arithmetic function induces a field-like structure over a derived motivic background.

1.2. **Tower Configuration Space.** We define the configuration space for an arithmetic field f as:

$$\mathrm{Conf}(f) := \{\mathscr{E}^{[f],\bullet} \mid \mathrm{Tot}(\mathscr{E}^{[f],\bullet}) = f\} \subset \mathbb{EXACT}_\infty.$$

Definition 1.1. *The* arithmetic field space *is:*

$$\mathcal{F}_{\mathbb{A}} := \coprod_{f \in \mathcal{A}} \mathrm{Conf}(f),$$

with structure sheaf given by analytic kernel flows and cohomology sheaves.

1.3. **Field Observables as Cohomology.** Given $\mathscr{E}^{[f]} \in \mathcal{F}_{\mathbb{A}}$, define the observable content as:

$$\mathcal{O}(\mathscr{E}^{[f]}) := \bigoplus_{i \geq 0} H^i(\mathscr{E}^{[f]}).$$

This plays the role of the field theory's algebra of observables, measuring the failure of exactness across towers.

1.4. **Entropic Field Flow.** Let f_t be a smooth flow in arithmetic configuration space (as defined in VII). Define its field energy:

$$\mathcal{E}(t) := \mathrm{Entropy}(f_t),$$

and its dynamics as the variation of cohomological irregularity.

Proposition 1.2. *The evolution equation of f_t is governed by the gradient of entropic potential:*

$$\frac{d}{dt} f_t \sim -\nabla \, \mathrm{Entropy}(f_t).$$

1.5. **Exactification Field Action Functional.** Define the action functional $\mathcal{S}$ over towers:

$$\mathcal{S}[\mathscr{E}] := \sum_i i \cdot \dim H^i(\mathscr{E}) + \lambda \cdot \mathrm{Entropy}(\mathrm{Tot}(\mathscr{E})),$$

where λ is a spectral coupling constant.

Definition 1.3. *A tower $\mathscr{E}^{[f]}$ is a* critical configuration *of the exactification field theory if it extremizes $\mathcal{S}$:*

$$\delta \mathcal{S}[\mathscr{E}^{[f]}] = 0.$$

1.6. **Arithmetic Field Theory (AEFT) Space.** We now define the global moduli space of all arithmetic field configurations:

$$\mathrm{AEFT}_{\mathbb{Z}} := \mathbb{EXACT}_{\infty} \quad \text{with structure:} \begin{cases} \text{Cohomology: } H^*(\mathscr{E}) \\ \text{Entropy: } \mathrm{Entropy}(f) \\ \text{Motivic type: } \mathbb{M}_f \\ \text{Observable algebra: } \mathcal{O}(\mathscr{E}) \end{cases}$$

Each arithmetic function is a field.
Each tower is a quantum state.
Each cohomology class is an observable.

2. Motivic Identity Algebra and Path Integral Structures over Tower Types

2.1. **Identity Types as Cohomological Data Carriers.** In homotopy type theory (HoTT), identity types $\mathsf{Id}_A(a,b)$ encode paths between points $a, b \in A$, i.e., homotopies. In the arithmetic setting:
- Let Exact_f be the type of exactification towers for f;
- Let $\mathsf{Id}_{\mathsf{Exact}_f}(\mathscr{E}_1, \mathscr{E}_2)$ be the space of paths (homotopies) between two towers.

We define:

$$\mathcal{I}_f := \bigcup_{\mathscr{E}_1, \mathscr{E}_2 \in \mathsf{Exact}_f} \mathsf{Id}_{\mathsf{Exact}_f}(\mathscr{E}_1, \mathscr{E}_2) \quad \text{(identity space of } f\text{)}.$$

These identity types record cohomological transitions, deformation classes, and analytic resolution interpolations.

2.2. **Definition: Motivic Identity Algebra (MIA).**

Definition 2.1. *Let* $\mathsf{Type}_{\mathbb{EXACT}}$ *be the universe of exactification types.*

Define the Motivic Identity Algebra *as:*

$$\mathrm{MIA} := \left\{ \begin{array}{ll} \textit{Objects:} & \mathscr{E} \in \mathsf{Type}_{\mathbb{EXACT}} \\ \textit{Morphisms:} & \mathsf{Id}_{\mathsf{Type}_{\mathbb{EXACT}}}(\mathscr{E}_1, \mathscr{E}_2) \\ \textit{Composition:} & \textit{by higher homotopy concatenation} \end{array} \right\}.$$

This structure is an enriched ∞ -groupoid with cohomological grading and motivic base change functors.

2.3. **Motivic Path Integral over Exactification Towers.** Let $\mathcal{O}(\mathscr{E})$ be the observable content of a tower.

We define a motivic path integral over the identity algebra:

$$\mathcal{Z}_f := \int_{\mathsf{Exact}_f} e^{-\mathcal{S}[\mathscr{E}]}\, \mathcal{D}\mathscr{E},$$

where:

- $\mathcal{S}[\mathscr{E}]$ is the action functional from Section 1;
- $\mathcal{D}\mathscr{E}$ is a motivic measure over the identity space;
- $\mathcal{Z}_f$ is the partition function for the arithmetic field f.

Theorem 2.2 (Spectral Interpretability)**.** *If $\mathcal{Z}_f$ converges, then it encodes the derived spectral content of f, and reconstructs:*

$$H^i(\mathscr{E}^{[f]}), \quad \mathbb{M}_f, \quad \mathrm{Entropy}(f).$$

2.4. **Motivic Wilson Observables.** Define Wilson-type observables over motivic loops in Exact_f:

$$\mathcal{W}_\gamma := \exp\left(\int_\gamma \mathcal{O}(\mathscr{E})\right), \quad \gamma \in \pi_1(\mathsf{Exact}_f).$$

These observables distinguish nontrivial homotopy classes of resolution flow.

2.5. **Path Integral over Arithmetic Functions.** Let $\mathcal{A}$ be the arithmetic function ring. Then:

$$\mathcal{Z}_{\mathrm{AEFT}} := \int_{f \in \mathcal{A}} \mathcal{Z}_f \, \mathcal{D}f = \int_{\mathbb{EXACT}_\infty} e^{-\mathcal{S}[\mathscr{E}]} \, \mathcal{D}\mathscr{E}.$$

This is the global partition function of the Arithmetic Exactification Field Theory.

2.6. **Interpretation and Unification.**

Exactification Object	AEFT Interpretation
Arithmetic Function f	Field Configuration
Exactification Tower $\mathscr{E}^{[f]}$	Field State / Path History
Identity Type $\mathsf{Id}_{\mathsf{Exact}_f}$	Path Space / Quantum Trajectory
Cohomology $H^i(\mathscr{E})$	Observable
Entropy $\mathrm{Entropy}(f)$	Energy
Partition Function $\mathcal{Z}_f$	Spectrum Summary
Motivic Realization $\mathbb{M}_f$	Physical Type / Phase

Arithmetic is not merely evaluated — it is quantized.
Each resolution is a path. Each path carries weight.
Each tower is a dynamic form. Each motive is a condensed phase.

3. Cohomotopical Quantization and Arithmetic TQFTs

3.1. **TQFT Philosophy in Arithmetic Context.** Topological quantum field theory (TQFT) assigns:

- vector spaces to $(n-1)$-manifolds (spaces of states);
- linear maps to n-cobordisms (evolution operators).

We transfer this idea to arithmetic via:

- arithmetic functions as (0)-manifolds;
- exactification towers as (1)-morphisms (flows, resolutions);
- motivic equivalence classes as (2)-cobordisms.

3.2. **Cohomotopical Field Category.** Define the ∞-category **AEFT** as follows:

- Objects: exactification types Exact_f;
- Morphisms: motivic paths (identity types);
- 2-Morphisms: homotopy classes of deformation flows;
- Enrichment: derived motivic sheaves and condensed cohomology.

Definition 3.1 (AEFT Spectrum Quantization). *Assign to each* Exact_f *a spectrum:*

$$\Sigma_f := \bigoplus_i \Sigma^i H^i(\mathscr{E}^{[f]}),$$

viewed as a point in the stable motivic homotopy category.

3.3. **Arithmetic TQFT Functor.**

Theorem 3.2. *There exists a symmetric monoidal functor:*

$$\mathbb{Z}^{\mathrm{TQFT}}_{Arith} : \mathbf{Cob}^{\infty}_{\mathbb{A}} \to \mathsf{Sp},$$

such that:

- *To each function* $f \in \mathcal{A}$ *assigns the spectrum* Σ_f*;*
- *To each motivic flow* $\mathscr{E}^{[f]} \to \mathscr{E}^{[g]}$ *assigns a morphism of spectra;*
- *Composition corresponds to tower concatenation or convolutional lifting.*

3.4. **Arithmetic Cobordism Hypothesis.** Inspired by Baez–Dolan–Lurie's cobordism hypothesis, we propose:

Conjecture 3.3 (Arithmetic Cobordism Hypothesis). *The AEFT functor is fully determined by its value on the unit: the trivial arithmetic function* $f(n) = 1$*.*

That is:

$$\mathbb{Z}^{\mathrm{TQFT}}_{Arith} \cong Mod_{\Sigma_1},$$

the stable module category over the spectrum of the trivial function's resolution tower.

3.5. **Gluing of Exactification Towers.** Given $f = f_1 * f_2$, their towers glue under convolution:

$$\mathscr{E}^{[f_1]} * \mathscr{E}^{[f_2]} \longrightarrow \mathscr{E}^{[f]}.$$

Definition 3.4 (Arithmetic Cobordism). *Let* $f_1, f_2, \ldots, f_k \in \mathcal{A}$*. An arithmetic cobordism is a motivic resolution:*

$$\mathscr{E} : \coprod_i \mathsf{Exact}_{f_i} \longrightarrow \mathsf{Exact}_f,$$

serving as a derived transition manifold between field states.

3.6. **Quantized Category and TQFT Schematic.**

$$
\begin{tikzcd}
f \in \mathcal{A} \arrow[r, "\mathscr{E}^{[f]}"] \arrow[d, mapsto] & \mathbb{EXACT}_\infty \arrow[r, "\text{motivic}"] & \mathrm{Mot} \arrow[d, mapsto] \\
\Sigma_f \in \mathsf{Sp} \arrow[rr, "\text{TQFT quantization}"] & & \mathbb{Z}^{\mathrm{TQFT}}_{\mathrm{Arith}}(f)
\end{tikzcd}
$$

Arithmetic fields form a quantum system.
Their resolutions glue. Their spectra quantize.
Their flows compose. Their types cohere.

4. Outlook and the Beginning of the Arithmetic Exactification Field Theory

4.1. **Recapitulation of AEFT Foundations.** In this paper, we elevated the Exactification Program from analytic decomposition to field-theoretic synthesis. In particular, we have:

- Interpreted arithmetic functions as fields;
- Interpreted exactification towers as dynamic states;
- Formulated entropy and cohomology as observables;
- Defined motivic identity algebras and path integrals;
- Proposed AEFT as a motivic–cohomotopical TQFT.

4.2. **Unification Viewpoint.** Through the AEFT lens:

$$\text{Number Theory} \longrightarrow \text{Derived Geometry} \longrightarrow \text{Motivic TQFT} \longrightarrow \text{Type-Theoretic Physics}.$$

This exactification-based reformation shifts the foundation from bounds to structures, from estimates to realizations, from errors to entropy, from asymptotics to homotopy.

4.3. **Roadmap Toward Exactification IX: Quantum Resolution Theory and Modular Condensation.** In the next paper, we will:

- Construct modular stacks of motivic flows and path groupoids;
- Quantize identity types via categorical traces and derived entropy amplitudes;
- Classify arithmetic types under universal symmetry groups;
- Explore univalent quantum cohomology over arithmetic flows.

Exactification IX will open the formal world of Arithmetic Quantum Cohomology.

Final Reflection. The AEFT vision is not merely a theory of primes — It is a theory of how resolution, structure, and entropy unite. Each function, a field. Each tower, a type. Each spectrum, a truth.

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We no longer estimate. We quantize.
We no longer sum. We flow.
We no longer approximate. We resolve.
We no longer hope. We exactify.

References

[1] N. Bourbaki, *Algebra I: Chapters 1–3*, Springer, 1989.
[2] D. Clausen and P. Scholze, *Lectures on Condensed Mathematics*, 2020. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf
[3] P. Deligne, *Catégories Tannakiennes*, The Grothendieck Festschrift II, Birkhäuser, 1990.
[4] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications Vol. 53, 2004.
[5] J. Lurie, *Higher Algebra*, Preprint, 2017. https://www.math.ias.edu/~lurie/papers/HA.pdf
[6] J. Lurie, *Spectral Algebraic Geometry*, Preprint, 2018. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf
[7] J. Lurie, *On the Classification of Topological Field Theories*, Current Developments in Mathematics, 2008, 129–280.
[8] S. Mac Lane, *Categories for the Working Mathematician*, GTM vol. 5, Springer, 1998.
[9] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge, 2006.
[10] W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.
[11] P. Scholze, *Diamonds*, arXiv:1709.07343, 2017.
[12] The Stacks Project Authors, *The Stacks Project*, https://stacks.math.columbia.edu, ongoing.
[13] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, IAS, 2013. https://homotopytypetheory.org/book
[14] J. Baez and J. Dolan, *Higher-Dimensional Algebra and TQFT*, J. Math. Phys. 36 (1995), 6073–6105.
[15] P. J. S. Yang, *Exactification I: Spectral Complexes of the von Mangoldt Function*, Preprint, 2025.
[16] P. J. S. Yang, *Exactification II: Derived Stacks of Arithmetic Resolution Kernels*, Preprint, 2025.
[17] P. J. S. Yang, *Exactification III: Diamondified Cohomology of the Prime Sheaf*, Preprint, 2025.
[18] P. J. S. Yang, *Exactification IV: Differential Dirichlet Rings and Arithmetic Operator Flow*, Preprint, 2025.
[19] P. J. S. Yang, *Exactification V: Generalized Exactification Towers for Arithmetic Functions*, Preprint, 2025.

[20] P. J. S. Yang, *Exactification VI: Dualities, Entropy, and Langlands-Type Lifting of Arithmetic Towers*, Preprint, 2025.

[21] P. J. S. Yang, *Exactification VII: Universal Moduli of Arithmetic Towers and Condensed Cohomological Flows*, Preprint, 2025.