ENTROPY LAGRANGIANS, ZETA FIELD ACTIONS, AND QUANTUM TRACE FUNCTIONALS

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Abstract. We initiate the theory of entropy Lagrangians for arithmetic trace kernels and construct a variational framework for zeta dynamics. By modeling entropy kernels as fields on arithmetic sites and assigning Lagrangian densities determined by trace flow energy, we define an action principle whose critical paths encode prime-regularized zeta equations. We introduce quantum zeta field functionals via entropy path integrals and propose a geometric reinterpretation of the Riemann Hypothesis in terms of stationary entropy-trace critical configurations. This theory connects entropy flow, functional analysis, and arithmetic physics through the lens of variational trace geometry.

Contents

Introduction

The structure of the Riemann zeta function suggests a deep dynamical architecture within arithmetic. Entropy kernels—trace-weighted

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arithmetic functions—have recently emerged as natural vehicles for transporting zeta information across modular, spectral, and motivic domains.

In this paper, we propose a new direction: viewing entropy kernels as *fields*, their zeta-trace behavior as *Lagrangian energy densities*, and their compositions as *quantum flows* over arithmetic configuration spaces.

We aim to define:

- Lagrangian functionals $\mathcal{L}[\mathscr{E}]$ whose variation yields zeta equations;
- Action functionals $\mathcal{A}[\mathscr{E}] = \int \mathcal{L}[\mathscr{E}]\, ds$ interpreted over entropy spectra;
- Critical entropy kernels satisfying variational zeta conditions;
- Path integral representations of zeta values via trace kernel fluctuations;
- Entropy–quantum analogs of classical field theory in arithmetic geometry.

This variational and quantum approach reframes zeta theory as a physics of arithmetic flow, with entropy kernels tracing the stationary energy paths of modular information.

1. Zeta Trace Fields and Entropy Lagrangians

1.1. **Entropy Kernel Fields.** Let $\mathscr{E} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an entropy kernel. We interpret $\mathscr{E}$ as a scalar field over the arithmetic site $\mathrm{Spec}(\mathbb{Z})$, with discrete space-time given by natural numbers.

Definition 1.1. *A* zeta trace field *is a function $\mathscr{E}(n) = \rho(n) \cdot a(n)$, where $\rho(n)$ is an entropy weight and $a(n) \in \mathbb{R}$ is an arithmetic potential.*

Remark 1.2. *The function $\zeta_{\mathscr{E}}(s) = \sum_{n=1}^{\infty} \mathscr{E}(n) n^{-s}$ acts as a spectral observable of the field $\mathscr{E}$.*

1.2. **Entropy Lagrangian Density.** We now define the Lagrangian density of an entropy field as a function measuring local trace tension.

Definition 1.3. *The* entropy Lagrangian density *is defined by:*

$$\mathcal{L}(\mathscr{E}; s) := \sum_n \left| \frac{d}{ds} \left(\mathscr{E}(n) n^{-s} \right) \right|^2 .$$

Example 1.4. *Let $\mathscr{E}(n) = \rho(n)$, then:*

$$\mathcal{L}(\rho; s) = \sum_{n=1}^{\infty} \rho(n)^2 \cdot \log^2(n) \cdot n^{-2s},$$

which captures exponential decay of entropy under logarithmic variation.

Remark 1.5. *The integrand encodes entropy field resistance to spectral scaling. High $\mathcal{L}$ indicates irregular spectral contribution; minimizers encode smooth zeta propagation.*

1.3. **Action Functional and Euler–Zeta Equations.**

Definition 1.6. *The* entropy action functional *is*

$$\mathcal{A}[\mathscr{E}] := \int_{\mathbb{R}_+} \mathcal{L}(\mathscr{E}; s)\, ds,$$

defined over a suitable vertical strip $s \in [\sigma_0, \sigma_1] \subset \mathbb{R}$.

Theorem 1.7 (Euler–Zeta Field Equation)**.** *The critical points of $\mathcal{A}$ satisfy the equation:*

$$\sum_{n=1}^{\infty} \rho(n)^2 \log^2(n) n^{-2s} = \lambda,$$

for some constant $\lambda \in \mathbb{R}$, i.e., the field maintains uniform spectral trace curvature.

The zeta action measures how trace fields vary through entropy—a calculus of arithmetic propagation.

2. Quantum Path Integrals and Zeta Trace Quantization

2.1. **Entropy Kernel Path Space.** To quantize the entropy kernel field theory, we define a configuration space over which quantum fluctuations of arithmetic fields are evaluated.

Definition 2.1. *Let $\mathcal{F}_{\text{ent}}$ denote the space of admissible entropy kernels:*

$$\mathcal{F}_{\text{ent}} := \left\{ \mathscr{E} : \mathbb{N} \to \mathbb{R}_{\geq 0} \,\middle|\, \mathscr{E}(n) = \rho(n) \cdot a(n),\ a(n) \textit{ arithmetic}, \zeta_{\mathscr{E}}(s) \textit{ convergent on } D \subset \mathbb{C} \right\}.$$

Remark 2.2. *The space $\mathcal{F}_{\text{ent}}$ serves as the arithmetic analogue of field configuration space in quantum theory.*

2.2. **Zeta Trace Path Integral.**

Definition 2.3. *The* zeta trace path integral *over $\mathcal{F}_{\text{ent}}$ is defined by:*

$$Z_{\text{tr}}(s) := \int_{\mathcal{F}_{\text{ent}}} \exp\left(-\mathcal{A}[\mathscr{E}]\right) \cdot \zeta_{\mathscr{E}}(s)\, \mathcal{D}\mathscr{E},$$

where $\mathcal{D}\mathscr{E}$ is a formal entropy-weighted measure on kernel fields.

Example 2.4. *Let $\mathscr{E}_\lambda(n) := \rho(n) \cdot e^{-\lambda n}$. Then*

$$Z_{\text{tr}}(s) = \int_0^{\infty} \zeta_{\mathscr{E}_\lambda}(s) \cdot e^{-\mathcal{A}[\mathscr{E}_\lambda]}\, d\lambda.$$

This regularizes the zeta divergence via exponential kernel damping.

2.3. **Quantized Trace Observables.**

Definition 2.5. *Given a functional* $\mathcal{O} : \mathcal{F}_{\text{ent}} \to \mathbb{R}$*, its quantum expectation is*

$$\langle \mathcal{O} \rangle_s := \frac{1}{Z_{\text{tr}}(s)} \int_{\mathcal{F}_{\text{ent}}} \mathcal{O}[\mathscr{E}] \cdot e^{-\mathcal{A}[\mathscr{E}]} \zeta_{\mathscr{E}}(s) \, \mathcal{D}\mathscr{E}.$$

Proposition 2.6. *Let* $\mathcal{O}[\mathscr{E}] := \mathrm{Tr}(\mathscr{E}) = \sum_n \mathscr{E}(n)$*. Then*

$$\langle \mathrm{Tr} \rangle_s \sim \frac{\partial}{\partial s} \log Z_{\text{tr}}(s).$$

2.4. **Critical Entropy Fluctuations and Quantum Zeros.**

Definition 2.7. *Define the* quantum zeta fluctuation field *by:*

$$\delta_{\mathrm{q}}[\mathscr{E}](s) := \zeta_{\mathscr{E}}(s) - \langle \zeta \rangle_s.$$

Conjecture 2.8 (Quantum Zeta Vanishing Principle)**.** *Let* $\mathscr{E} \in \mathcal{F}_{\text{ent}}$*. Then the expected variance*

$$\mathrm{Var}_s[\zeta] := \langle \zeta^2 \rangle_s - \langle \zeta \rangle_s^2$$

is minimized at $\Re(s) = \frac{1}{2}$*, i.e., the critical line corresponds to stationary quantum entropy.*

The zeta zeros arise not from singularities—but from cancellations across the path integral of entropy arithmetic fields.

Conclusion and Entropic Field Perspectives

This paper has initiated a variational and quantum formalism for entropy kernel structures and zeta trace dynamics. By treating entropy-weighted arithmetic functions as scalar fields over $\mathbb{N}$, we introduced:

- Entropy Lagrangian densities encoding trace fluctuation energy;
- Action functionals whose stationary points represent spectral balance;
- A quantum path integral formulation over entropy kernel configuration space;
- Fluctuation fields and critical trace expectations;
- A conjectural reformulation of the Riemann Hypothesis as a vanishing principle for zeta entropy variance at $\Re(s) = \frac{1}{2}$.

This framework brings together arithmetic analysis, functional field theory, and statistical zeta behavior into a cohesive dynamical structure.

Future Pathways.

(1) **Quantum Entropy TQFTs:** Elevate the entropy action formalism into a topological quantum field theory with arithmetic boundary conditions.
(2) **Trace Spectrum Flow Equations:** Derive differential equations for $Z_{\mathrm{tr}}(s)$ analogous to renormalization group flows in entropy-modified zeta settings.
(3) **Quantum Cohomology of Kernel Fields:** Construct derived categories of trace sheaves with quantum fluctuation functors and entropy curvature.
(4) **Numerical Zeta Path Simulation:** Implement Python-based integrators for path-sampled entropy kernel flows over $s \in \mathbb{C}$, and empirically test trace fluctuation conjectures.
(5) **Entropy–Riemann Duality Field Theory:** Unify all entropy kernel dynamics, zeta functionals, and quantum actions into a geometric quantization of the Riemann surface itself.

The zeta function is not static. It flows— through entropy, through action, through arithmetic space. To understand it is to trace the paths it draws across number theory.

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