SPECTRAL MOTIVES XVIII: ARITHMETIC HOLOGRAPHY AND BOUNDARY TRACE GEOMETRY

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Abstract. We develop a motivic theory of arithmetic holography, where bulk spectral motives over derived stacks project onto boundary trace data encoded in categorical sheaves and L-trace flows. Using a formalism of trace-pairing duality and arithmetic boundary stacks, we define bulk-to-boundary correspondences in cohomology, Laplacians, and categorical entropy. This provides a blueprint for arithmetic analogues of holographic duality, spectral renormalization, and motivic boundary flows in derived geometric Langlands theory.

Contents

1. Introduction

The concept of holography — that bulk geometric or physical information is fully encoded in boundary data — has reshaped our understanding of space, information, and duality. In arithmetic geometry, a parallel phenomenon arises in the trace-theoretic formulation of motives: certain arithmetic invariants, defined globally over spectral stacks, can be recovered from the behavior of trace flows along categorical or derived boundaries.

Date: May 5, 2025.

In this eighteenth installment, we propose a motivic framework for **arithmetic holography**, where:

- Bulk spectral motives over higher stacks project to boundary trace categories;
- L-traces over boundary strata encode full arithmetic flow information;
- Entropic and Laplacian structures obey holographic renormalization dynamics;
- Bulk automorphic correspondences admit boundary factorization into trace sheaves.

This perspective unifies concepts from:

- Arithmetic AdS/CFT duality and moduli-theoretic trace structures;
- Langlands boundary data and cohomological Eisenstein series;
- Motivic trace sheaves and boundary ∞-categorical centers;
- Motivic renormalization and trace-theoretic entropy scaling.

Paper Overview:

- Section 2: Defines arithmetic boundary stacks and trace duality;
- Section 3: Constructs bulk-to-boundary Laplacian projections;
- Section 4: Derives entropy scaling and holographic trace laws;
- Section 5: Applies the theory to automorphic L-flows and categorical Langlands duality.

This provides a geometric realization of motivic holography, inviting deeper connections between arithmetic cohomology, trace dynamics, and higher categorical geometry.

2. Boundary Stacks and Trace-Pairing Duality

2.1. **Arithmetic boundary stacks.** Let $\mathscr{X}$ be a derived spectral arithmetic stack, such as a moduli stack of shtukas, perfectoid Shimura varieties, or L-parameter stacks. We define the *arithmetic boundary stack* $\partial\mathscr{X}$ via:

$$\partial\mathscr{X} := \operatorname{colim}_{Z\subset\mathscr{X},\ \mathrm{codim}(Z)=1} Z,$$

taken over boundary strata of codimension one or via log-geometry completions in derived settings.

This boundary captures the "infinity" behavior of motivic flows and provides a geometric substrate for holographic projection.

2.2. **Trace sheaves on boundary sites.** Let $\mathscr{F}$ be a dg-category-valued sheaf on $\mathscr{X}$. We define the boundary trace sheaf as the restriction:

$$\mathscr{F}_\partial := \mathscr{F}|_{\partial\mathscr{X}}.$$

In analogy with physical theories, this sheaf encodes the observable algebra or trace content seen from the "boundary perspective".

2.3. **Trace-pairing duality.** The core principle of motivic holography is the existence of a *trace-pairing*:

$$\langle -,-\rangle_{\mathrm{Tr}} : \mathscr{F}|_{\mathscr{X}} \otimes \mathscr{F}_\partial \longrightarrow \mathbf{1},$$

satisfying:

$$\mathrm{Tr}_{\mathscr{X}}(T) = \sum_i \langle \psi_i|_\partial, T\psi_i\rangle_{\mathrm{Tr}},$$

for trace eigenmodes ψ_i. This expresses the bulk trace as a pairing over the boundary.

2.4. **Examples: Drinfeld boundary and geometric Eisenstein data.** In the case of moduli of shtukas or stacks over function fields, the boundary stack often parametrizes degenerate level structures or parabolic reductions.

Trace-pairing duality manifests as the equality between constant term integrals of automorphic sheaves and traces over Eisenstein boundary data:

$$\mathrm{Tr}_{\mathscr{X}}(\mathscr{A}) = \mathrm{Tr}_{\partial\mathscr{X}}(\mathscr{E}),$$

for appropriate automorphic object $\mathscr{A}$ and Eisenstein sheaf $\mathscr{E}$.

This provides a natural geometric realization of motivic trace descent.

3. Laplacian Descent and Boundary Renormalization

3.1. **Bulk Laplacians and fluctuation spectra.** Let $\Delta_{\mathscr{X}}$ be the trace Laplacian acting on a motivic sheaf $\mathscr{F}$ over a derived stack $\mathscr{X}$. The bulk fluctuation spectrum $\{\lambda_i\}$ defines entropy and trace functionals central to arithmetic quantum geometry.

3.2. **Boundary restriction and spectral projection.** Under trace-pairing duality, eigenmodes ψ_i restricted to the boundary give:

$$\psi_i^\partial := \psi_i|_{\partial\mathscr{X}} \in \mathscr{F}_\partial,$$

and satisfy the holographic relation:

$$\langle \psi_i^\partial, \psi_j^\partial \rangle_{\mathrm{Tr}} = \delta_{ij}.$$

The spectrum of $\Delta_{\mathscr{X}}$ induces a projected Laplacian Δ_∂ via:

$$\Delta_\partial \psi_i^\partial := \lambda_i \psi_i^\partial,$$

i.e., the boundary encodes the entire spectral decomposition of the bulk.

3.3. **Motivic renormalization group flow.** Define a family of boundary effective Laplacians:

$$\Delta_\partial^{(\mu)} := \sum_{\lambda_i \le \mu} \lambda_i \cdot \psi_i^\partial \otimes \psi_i^\partial,$$

where μ is a spectral cutoff. This defines a motivic renormalization group flow:

$$\mu \mapsto \Delta_\partial^{(\mu)},$$

analogous to the Wilsonian flow in QFT, now applied to categorical trace data over arithmetic stacks.

3.4. **Entropy scaling and trace area laws.** Define the trace entropy function:

$$\mathcal{S}_\partial(\mu) := -\sum_{\lambda_i \le \mu} p_i \log p_i, \quad p_i := \frac{e^{-\lambda_i}}{\sum_{\lambda_j \le \mu} e^{-\lambda_j}}.$$

This satisfies scaling laws:

$$\mathcal{S}_\partial(\mu) \sim \alpha \cdot \mathrm{Vol}(\partial\mathscr{X}) + \mathcal{O}(\log \mu),$$

which mirrors holographic entropy-area relations and suggests a deep connection between motivic curvature, trace energy, and derived boundary volume.

4. Holographic Langlands and Motivic Boundary Correspondences

4.1. **Langlands stacks and spectral sheaves.** Let LocSys_G be the stack of G-local systems on a smooth projective curve X, and let Bun_G denote the moduli stack of G-bundles. The geometric Langlands correspondence relates sheaves on Bun_G with D-modules (or spectral sheaves) on LocSys_G.

From the spectral motive viewpoint, both stacks are equipped with derived trace Laplacians, and the trace content of the correspondence may be expressed via supertraces, entropy, and Laplacian eigenmodes.

4.2. **Boundary functors and Eisenstein holography.** Let $P \subset G$ be a parabolic subgroup, and consider:

$$\mathrm{Bun}_P \xrightarrow{\partial} \mathrm{Bun}_G,$$

as a boundary morphism. The Eisenstein functor Eis_P projects trace data from bulk automorphic sheaves to boundary parabolic strata.

The induced correspondence in trace cohomology is:

$$\mathrm{Tr}_{\mathrm{Bun}_G}(\mathscr{A}) = \mathrm{Tr}_{\mathrm{Bun}_P}(\mathrm{Eis}_P^* \mathscr{A}),$$

which constitutes a holographic identity for automorphic motives.

4.3. **Holographic duality for L-parameters.** Let $\phi : \pi_1(X) \to {}^L G$ be an L-parameter. The automorphic sheaf $\mathscr{F}_\phi$ admits restriction to the boundary stack of degenerate bundles or singular fibers.

We conjecture a motivic boundary duality:

$$\mathscr{F}_\phi \mapsto \mathscr{F}_\phi^\partial \in \mathscr{D}(\partial \mathrm{Bun}_G),$$

preserving the trace spectrum and satisfying:

$$\zeta_{\mathrm{mot}}(\mathscr{F}_\phi, s) = \zeta_\partial(\mathscr{F}_\phi^\partial, s),$$

where both sides are regularized trace zeta functions.

4.4. **Categorical Langlands-Ryu–Takayanagi identity.** Inspired by the Ryu–Takayanagi formula in holographic QFT, we propose a categorical identity:

$$\mathcal{S}_{\mathrm{bulk}}(\mathscr{F}) = \mathrm{Area}_{\partial \mathscr{X}}(\mathscr{F}^\partial) + \mathrm{Flux}_{\mathrm{Tr}}(\mathscr{F}),$$

where $\mathcal{S}_{\mathrm{bulk}}$ is motivic entropy, $\mathrm{Area}_{\partial \mathscr{X}}$ measures categorical trace volume, and $\mathrm{Flux}_{\mathrm{Tr}}$ captures trace nonconservation across the boundary.

This identity connects Langlands categories, trace entropy, and geometric flows across motivic interfaces.

5. Conclusion

We have developed a motivic formalism for arithmetic holography, wherein spectral motives over derived stacks project to boundary trace sheaves through trace-pairing duality and fluctuation descent.

Summary of Contributions:

- Defined arithmetic boundary stacks and their trace-theoretic structure;
- Constructed Laplacian descent and renormalization group flows over boundaries;
- Quantified entropy-area laws in motivic trace geometry;

- Proposed holographic Langlands dualities between bulk automorphic objects and boundary Eisenstein sheaves;
- Formulated a categorical Ryu–Takayanagi-type identity for trace entropy.

This work connects trace cohomology, geometric Langlands, and derived moduli in a boundary-aware framework, suggesting a broad generalization of arithmetic duality principles and new routes to understanding functoriality, spectral transfer, and entropy in motivic contexts.

Future directions include:

- Boundary trace descent in condensed spectral stacks;
- Derived holography over Shimura-Perfectoid towers;
- Motivic black hole entropy and trace curvature bounds;
- Categorical AdS/CFT analogues in global L-groupoids.

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