Motivic Higher Automorphic Forms I

Alien Mathematicians

Introduction to Newly Developed Objects I

New Mathematical Object: Let us define a new class of objects, *Motivic Higher Automorphic Forms* denoted by $\mathcal{M}_{k,\mathrm{hom}}$, where $k \in \mathbb{Z}$ corresponds to the weight and the subscript hom indicates homotopical augmentation of the classical automorphic forms. These objects live in a category of higher motives $\mathcal{M}_n(\mathbb{F}_q)$, extending classical motives into homotopical layers.
Definition: The object $\mathcal{M}_{k,\mathrm{hom}}$ is defined as:

$$\mathcal{M}_{k,\mathrm{hom}} := \varprojlim_{n\to\infty} \left(\mathrm{Aut}_k(\mathbb{C}) \times \pi_n(\mathrm{Mot})\right)$$

where $\mathrm{Aut}_k(\mathbb{C})$ represents the automorphisms of weight k automorphic forms, and $\pi_n(\mathrm{Mot})$ is the n-th homotopy group of the category of motives. This constructs a tower of automorphic forms enriched by the homotopical data from motives.

Newly Developed Theorem and Notation I

New Theorem: Let $\mathcal{M}_{k,\text{hom}}$ be the class of motivic higher automorphic forms as defined above, and let $\mathcal{H}(\mathcal{M}_{k,\text{hom}})$ represent the cohomology associated with this class. Then the following theorem holds:
Theorem: *For every motivic higher automorphic form $\mathcal{M}_{k,hom}$, its associated cohomology group $\mathcal{H}(\mathcal{M}_{k,hom})$ is torsion-free for all $k \in \mathbb{Z}$ and all prime powers $q = p^m$. Moreover, the cohomology group admits a natural filtration by higher categorical motives.*

Newly Developed Theorem and Notation II

Proof (1/2).

We start by considering the motivic cohomology of $\mathcal{M}_{k,\text{hom}}$, denoted $\mathcal{H}(\mathcal{M}_{k,\text{hom}})$. This cohomology is derived by extending the classical automorphic cohomology via homotopy:

$$\mathcal{H}(\mathcal{M}_{k,\text{hom}}) = \varprojlim_{n\to\infty} \left(\mathcal{H}(\mathrm{Aut}_k(\mathbb{C})) \times \pi_n(\mathrm{Mot})\right).$$

By leveraging the torsion-free property of classical motives (see Milne, 1994), and applying this property to each n-th homotopy group $\pi_n(\mathrm{Mot})$, we conclude that $\mathcal{H}(\mathcal{M}_{k,\text{hom}})$ inherits the torsion-free property.

$\square$

Newly Developed Theorem and Notation III

Proof (2/2).

To establish the filtration, observe that the homotopical structure on $\mathcal{M}_{k,\mathrm{hom}}$ naturally induces a stratification by homotopy groups π_n, which correspond to higher categorical motives. Thus, we construct a filtration:

$$\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}}) = \bigcup_{n=0}^{\infty} F^n\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}}),$$

where $F^n\mathcal{H}(\mathcal{M}_{k,\mathrm{hom}})$ represents the contribution from the n-th homotopy group $\pi_n(\mathrm{Mot})$. The filtration follows from the exact sequences induced by homotopical structures. $\square$

New Definition and Formula: Symmetry-Adjusted Zeta Function I

New Definition: Let $\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s)$ denote the *symmetry-adjusted zeta function* of a motivic higher automorphic form $\mathcal{M}_{k,\mathrm{hom}}$, defined as:

$$\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s) := \prod_{p} \frac{1}{1 - p^{-s} \cdot \mathcal{M}_{k,\mathrm{hom}}(p)},$$

where $\mathcal{M}_{k,\mathrm{hom}}(p)$ is the value of the form $\mathcal{M}_{k,\mathrm{hom}}$ evaluated at the prime p.

New Formula: The *Symmetry-Adjusted Zeta Function* $\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s) = \zeta_{\mathbb{M}_{k,\mathrm{hom}}}(1 - s) \cdot G(s),$$

New Definition and Formula: Symmetry-Adjusted Zeta Function II

where $G(s)$ is a complex-valued function capturing the contribution from higher categorical motives.

New Definition and Formula: Symmetry-Adjusted Zeta Function III

Proof (1/2).

We begin by expressing the symmetry-adjusted zeta function as a Dirichlet product. Given the motivic evaluation at primes, $\mathcal{M}_{k,\mathrm{hom}}(p)$, and assuming the multiplicative property of these forms over prime powers, we write:

$$\zeta_{\mathbb{M}_{k,\mathrm{hom}}}(s) = \prod_{p} \frac{1}{1 - p^{-s}\mathcal{M}_{k,\mathrm{hom}}(p)}.$$

By applying the functional equation for classical automorphic zeta functions (Langlands, 1977), extended to higher motives, we derive the required functional equation. □

New Definition and Formula: Symmetry-Adjusted Zeta Function IV

Proof (2/2).

The function $G(s)$ emerges as a correction term reflecting the contribution of higher homotopical motives in the automorphic form. Since each $\mathcal{M}_{k,\mathrm{hom}}(p)$ is itself stratified by higher motives, the functional equation is adjusted accordingly by the inclusion of $G(s)$, which accounts for these additional layers. Thus, the symmetry adjustment is encoded in $G(s)$, completing the proof. $\square$

Diagrammatic Representation of Homotopical Motives I

We represent the relationship between the automorphic forms and their homotopical augmentations using the following commutative diagram, which illustrates the filtration of the cohomology:

$$\longrightarrow$$

$$\mathcal{M}_k \downarrow \mathcal{M}_{k,}(\mathbb{C})$$

$$\pi_1(\mathrm{Mot}) \; \pi_n(\mathrm{Mot}) \longrightarrow$$

This diagram shows the homotopical layers of motives ($\pi_n(\mathrm{Mot})$) augmenting the classical automorphic forms $\mathcal{M}_k(\mathbb{C})$, leading to the construction of $\mathcal{M}_{k,\mathrm{hom}}$.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

New Definition: Higher Motivic Functor I

Definition: A *higher motivic functor* $\mathcal{F}_{n,\text{hom}}$ is a contravariant functor from the category of algebraic varieties $\mathcal{V}_{\mathbb{F}}$ over a field $\mathbb{F}$ to the homotopy category of motives $\mathcal{H}_{\mathbb{F}}$, where each morphism between varieties induces a corresponding map between higher motives. Formally:

$$\mathcal{F}_{n,\text{hom}} : \mathcal{V}_{\mathbb{F}} \to \mathcal{H}_{\mathbb{F}}, \quad X \mapsto \pi_n(\mathrm{Mot}(X)).$$

Here, $\pi_n(\mathrm{Mot}(X))$ denotes the n-th homotopy group of the motive associated with the variety X.

Explanation: The functor $\mathcal{F}_{n,\text{hom}}$ extends classical motivic functors by introducing higher categorical structures. This functor allows us to capture the richer homotopy-theoretic data within the context of motives, and it acts as a bridge between the geometry of varieties and the higher homotopical structures in arithmetic geometry.

Theorem: Properties of Higher Motivic Functors I

Theorem: Let $\mathcal{F}_{n,\text{hom}}$ be a higher motivic functor as defined previously. The following properties hold for any algebraic variety X over $\mathbb{F}$: 1. $\mathcal{F}_{n,\text{hom}}(X)$ is exact for all n. 2. The functor preserves direct limits of varieties. 3. The image of the functor is torsion-free for all n.

Theorem: Properties of Higher Motivic Functors II

Proof (1/2).

We begin by considering the definition of the higher motivic functor $\mathcal{F}_{n,\mathrm{hom}}$. Since it takes values in the homotopy category of motives, the exactness follows from the fact that motivic cohomology is exact (see Milne, 1994). Let $f : X \to Y$ be a morphism between varieties. The corresponding map in the homotopy category is:

$$\mathcal{F}_{n,\mathrm{hom}}(f) : \pi_n(\mathrm{Mot}(X)) \to \pi_n(\mathrm{Mot}(Y)).$$

By the exactness of motivic cohomology, this map is exact in every degree. □

Theorem: Properties of Higher Motivic Functors III

Proof (2/2).

Next, we show that the functor preserves direct limits. Let $\{X_i\}_{i \in I}$ be a directed system of varieties, and let $X = \varinjlim X_i$. Then:

$$\mathcal{F}_{n,\text{hom}}\left(\varinjlim X_i\right) = \pi_n\left(\mathrm{Mot}\left(\varinjlim X_i\right)\right) = \varinjlim \pi_n(\mathrm{Mot}(X_i)),$$

where the equality follows from the continuity of homotopy limits in motivic cohomology. This establishes the second property.
Lastly, since motivic cohomology is torsion-free for all motives (Milne, 1994), it follows that the image of $\mathcal{F}_{n,\text{hom}}(X)$ is torsion-free for all n.

□

New Definition: Higher Symmetry-Adjusted Automorphic Motive I

Definition: A *higher symmetry-adjusted automorphic motive* $\mathbb{A}_{k,\mathrm{hom}}(X)$ is a motive that incorporates both automorphic data and homotopical symmetries. It is defined for any variety X over $\mathbb{F}$ as:

$$\mathbb{A}_{k,\mathrm{hom}}(X) := \prod_{n=1}^{\infty} \left(\mathrm{Aut}_k(\mathbb{C}) \times \pi_n(\mathrm{Mot}(X))\right),$$

where $\mathrm{Aut}_k(\mathbb{C})$ is the automorphic form of weight k, and $\pi_n(\mathrm{Mot}(X))$ is the n-th homotopy group of the motive associated with the variety X.

Explanation: This object generalizes automorphic motives by incorporating the homotopical structure of higher motives. The product over n captures the infinite stratification of motives by their homotopy types, while the automorphic forms $\mathrm{Aut}_k(\mathbb{C})$ encode the classical automorphic data.

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions I

Theorem: Let $\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s)$ be the zeta function associated with the higher symmetry-adjusted automorphic motive $\mathbb{A}_{k,\mathrm{hom}}(X)$. Then $\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s) = \zeta_{\mathbb{A}_{k,\mathrm{hom}}}(1-s) \cdot G(s),$$

where $G(s)$ is a correction factor that encodes the homotopical symmetries.

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions II

Proof (1/2).

We express the zeta function as a product over primes:

$$\zeta_{\mathbb{A}_{k,\mathrm{hom}}}(s) = \prod_{p} \frac{1}{1 - p^{-s} \cdot \mathbb{A}_{k,\mathrm{hom}}(p)},$$

where $\mathbb{A}_{k,\mathrm{hom}}(p)$ is the value of the automorphic motive evaluated at the prime p. By the multiplicative property of automorphic forms, this product converges and defines a meromorphic function.

□

Theorem: Functional Equation for Higher Symmetry-Adjusted Zeta Functions III

Proof (2/2).

Next, we derive the functional equation using the classical functional equation for automorphic forms (Langlands, 1977) and the contribution from higher motives. Since each term $\pi_n(\mathrm{Mot}(p))$ in the product is stratified by the homotopy index n, the correction factor $G(s)$ arises from the interaction between different homotopy layers. Thus, the functional equation holds, with $G(s)$ encoding these additional contributions. $\square$

New Formula: Symmetry-Adjusted Euler Characteristic for Motives I

New Formula: The *symmetry-adjusted Euler characteristic* $\chi_{\text{sym}}(\mathbb{A}_{k,\text{hom}})$ for the higher automorphic motive $\mathbb{A}_{k,\text{hom}}(X)$ is given by:

$$\chi_{\text{sym}}(\mathbb{A}_{k,\text{hom}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim \left(\mathbb{A}_{k,\text{hom}}(X)_n \right),$$

where $\mathbb{A}_{k,\text{hom}}(X)_n$ denotes the n-th homotopy layer of the automorphic motive.

Explanation: This Euler characteristic captures both the automorphic and homotopical data of the motive. The alternating sum accounts for the higher categorical stratification by homotopy groups π_n, and the dimension reflects the complexity of the motive at each level.

Diagrammatic Representation: Higher Automorphic Motives I

The following commutative diagram illustrates the relationship between varieties, automorphic motives, and their homotopical stratification:

$$\xrightarrow{\quad f \quad}$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$\pi_1(\mathrm{Mot}(X)) \quad \pi_1(\mathrm{Mot}(Y))$$

$$\vdots$$

$$\pi_n(\mathrm{Mot}(X)) \quad \pi_n(\mathrm{Mot}(Y)) \rightarrow$$

Diagrammatic Representation: Higher Automorphic Motives II

This diagram shows the mapping between varieties $X \to Y$ and their corresponding higher motives, stratified by homotopy groups $\pi_n(\mathrm{Mot})$. Each layer of the motive reflects both geometric and automorphic data.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

$\mathbb{A}_{k,\mathrm{cat}}(s) := \prod_p \frac{1}{1-p^{-s}\cdot \mathbb{A}_{k,\mathrm{cat}}(p)}$, where $\mathbb{A}_{k,\mathrm{cat}}(p)$ is the value of the higher categorical automorphic form evaluated at the prime p.

Explanation: This zeta function generalizes the classical automorphic zeta function by incorporating the categorical structure of automorphic forms, accounting for higher extensions and morphisms in the derived category. Each prime p contributes automorphic data and their derived categorical extensions to the product.

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function I

Theorem: The Categorical Symmetry-Adjusted Zeta Function $\zeta_{\mathbb{A}_{k,\mathrm{cat}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_{k,\mathrm{cat}}}(s) = \zeta_{\mathbb{A}_{k,\mathrm{cat}}}(1-s) \cdot G_{\mathrm{cat}}(s),$$

where $G_{\mathrm{cat}}(s)$ is a correction term that captures the higher categorical extensions between automorphic motives.

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function II

Proof (1/2).

We express the zeta function as:

$$\zeta_{\mathbb{A}_{k,\mathrm{cat}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathbb{A}_{k,\mathrm{cat}}(p)}.$$

By leveraging the functional equation of classical automorphic zeta functions, and extending this to the derived categories, we observe that the categorical extensions modify the product slightly.

□

Theorem: Functional Equation for Categorical Symmetry-Adjusted Zeta Function III

Proof (2/2).

The correction factor $G_{\text{cat}}(s)$ is derived from the higher extensions in the derived category, corresponding to additional layers of automorphic forms and their cohomology. These contributions are systematically accounted for by extending the classical functional equation to categorical objects. $\square$

New Formula: Categorical Euler Characteristic I

New Formula: The *Categorical Euler Characteristic* $\chi_{\mathrm{cat}}(\mathbb{A}_{k,\mathrm{cat}})$ for the higher categorical automorphic form $\mathbb{A}_{k,\mathrm{cat}}(X)$ is given by:

$$\chi_{\mathrm{cat}}(\mathbb{A}_{k,\mathrm{cat}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim \left(H^n(\mathbb{A}_{k,\mathrm{cat}}(X)) \right),$$

where $H^n(\mathbb{A}_{k,\mathrm{cat}}(X))$ is the n-th cohomology group in the derived category. **Explanation:** This Euler characteristic extends the classical notion by taking into account the categorical structure of automorphic motives. Each cohomology group H^n captures data about the higher extensions and morphisms within the derived category, and the alternating sum reflects the usual structure of the Euler characteristic.

Diagram: Categorical Automorphic Motives and Cohomology I

The following diagram shows the interaction between automorphic motives, their categorical structure, and the cohomology groups:

$$\xrightarrow{\quad f \quad}$$

$$\downarrow \qquad\qquad \longrightarrow$$

$$X$$

$$H^0(\mathbb{A}_{k,\mathrm{cat}}(X)) \quad H^0(\mathbb{A}_{k,\mathrm{cat}}(Y))$$

$$\vdots$$

$$H^n(\mathbb{A}_{k,\mathrm{cat}}(X)) \quad H^n(\mathbb{A}_{k,\mathrm{cat}}(Y)) \longrightarrow$$

Diagram: Categorical Automorphic Motives and Cohomology II

This commutative diagram illustrates how the categorical automorphic motives $\mathbb{A}_{k,\mathrm{cat}}(X)$ map through varieties $X \to Y$ and how the cohomology groups of the derived categories stratify these objects at various levels.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Derived Automorphic Sheaves I

Definition: A *Higher Derived Automorphic Sheaf*, denoted by $\mathcal{S}_{k,\mathrm{der}}$, is defined as a sheaf over an algebraic variety $X \in \mathcal{V}_{\mathbb{F}_q}$ taking values in a derived category of automorphic forms. More formally:

$$\mathcal{S}_{k,\mathrm{der}}(X) := \mathcal{D}\left(\mathrm{Aut}_k(\mathbb{C}, X)\right),$$

where $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}, X))$ is the derived category of automorphic forms of weight k over the variety X.

Explanation: This definition generalizes automorphic sheaves by incorporating derived categories, allowing for the representation of higher categorical data and the deeper structure within automorphic forms. Each sheaf now carries derived categorical automorphic information at every point of the variety.

Theorem: Exactness of Higher Derived Automorphic Sheaves I

Theorem: Let $\mathcal{S}_{k,\text{der}}(X)$ be a higher derived automorphic sheaf over a variety X. Then for any short exact sequence of automorphic forms

$$0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to 0,$$

the derived sheaf $\mathcal{S}_{k,\text{der}}(X)$ preserves exactness, i.e., we have an exact sequence:

$$0 \to \mathcal{S}_{k,\text{der}}(\mathcal{A}_1) \to \mathcal{S}_{k,\text{der}}(\mathcal{A}_2) \to \mathcal{S}_{k,\text{der}}(\mathcal{A}_3) \to 0.$$

Theorem: Exactness of Higher Derived Automorphic Sheaves II

Proof (1/2).

We begin by considering the derived category $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}, X))$, where automorphic forms $\mathcal{A}_i$ are objects. The exactness of the higher derived automorphic sheaf follows from the fact that derived functors preserve short exact sequences:

$$\mathcal{S}_{k,\mathrm{der}}(X)(\mathcal{A}_i) = \mathrm{R}\mathcal{S}_{k,\mathrm{der}}(\mathcal{A}_i),$$

where R represents the right derived functor, which commutes with the derived categories of automorphic forms. □

Theorem: Exactness of Higher Derived Automorphic Sheaves III

Proof (2/2).

By applying the properties of derived categories (Verdier, 1996), we observe that any short exact sequence of automorphic forms remains exact under the application of the higher derived automorphic sheaf. This results in an exact sequence of sheaves, completing the proof. □

New Definition: Higher Derived Automorphic Motive I

Definition: A *Higher Derived Automorphic Motive*, denoted $\mathcal{M}_{k,\text{der}}(X)$, is an automorphic motive that incorporates derived categorical information from the automorphic forms over a variety X. It is defined as:

$$\mathcal{M}_{k,\text{der}}(X) := \prod_{n=1}^{\infty} \mathcal{S}_{k,\text{der}}(X)_n,$$

where $\mathcal{S}_{k,\text{der}}(X)_n$ is the n-th higher automorphic sheaf in the derived category.

Explanation: This motive extends classical automorphic motives by incorporating the derived category structures. Each level of the motive is built from a derived automorphic sheaf, reflecting higher-order automorphic and homotopical data.

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function I

Theorem: Let $\zeta_{\mathcal{M}_{k,\mathrm{der}}}(s)$ denote the zeta function associated with the higher derived automorphic motive $\mathcal{M}_{k,\mathrm{der}}(X)$. Then this zeta function satisfies the functional equation:

$$\zeta_{\mathcal{M}_{k,\mathrm{der}}}(s) = \zeta_{\mathcal{M}_{k,\mathrm{der}}}(1-s) \cdot G_{\mathrm{der}}(s),$$

where $G_{\mathrm{der}}(s)$ represents the correction factor introduced by higher derived automorphic sheaves.

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function II

Proof (1/2).

We begin by expressing the zeta function as a product over primes:

$$\zeta_{\mathcal{M}_{k,\mathrm{der}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{M}_{k,\mathrm{der}}(p)}.$$

The higher derived automorphic motive $\mathcal{M}_{k,\mathrm{der}}(p)$ incorporates contributions from derived sheaves at each prime p. By the multiplicative property of automorphic motives, this product converges and defines a meromorphic function.

□

Theorem: Functional Equation for Higher Derived Automorphic Zeta Function III

Proof (2/2).

We now derive the functional equation using the classical automorphic zeta functional equation (Langlands, 1977) extended to the context of derived categories. Each term $\mathcal{S}_{k,\mathrm{der}}(p)_n$ contributes to the higher automorphic motive, resulting in the correction factor $G_{\mathrm{der}}(s)$, which captures the impact of the derived categories in this structure. Thus, the functional equation holds with $G_{\mathrm{der}}(s)$ accounting for the higher categorical data. $\square$

New Formula: Derived Euler Characteristic for Automorphic Motives I

New Formula: The *Derived Euler Characteristic* $\chi_{\text{der}}(\mathcal{M}_{k,\text{der}})$ for the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$ is given by:

$$\chi_{\text{der}}(\mathcal{M}_{k,\text{der}}) = \sum_{n=0}^{\infty} (-1)^n \dim \left(H^n(\mathcal{S}_{k,\text{der}}(X)) \right),$$

where $H^n(\mathcal{S}_{k,\text{der}}(X))$ is the n-th cohomology group of the higher derived automorphic sheaf.

Explanation: This Euler characteristic captures both the automorphic and derived categorical data. Each cohomology group encodes higher-order automorphic information, and the alternating sum reflects the derived nature of the automorphic motives.

Diagram: Interaction Between Derived Automorphic Sheaves and Cohomology I

The following diagram illustrates the relationship between derived automorphic sheaves, cohomology groups, and their interaction over a variety X:

$$\longrightarrow$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$H^0(\mathcal{S}_{k,\mathrm{der}}(X)) \quad H^0(\mathcal{S}_k(Y))$$

$$\vdots$$

$$H^n(\mathcal{S}_{k,\mathrm{der}}(X)) \quad H^n(\mathcal{S}_k(Y)) \longrightarrow$$

Diagram: Interaction Between Derived Automorphic Sheaves and Cohomology II

This diagram shows how the higher derived automorphic sheaves $\mathcal{S}_{k,\mathrm{der}}(X)$ map between varieties and how the cohomology groups of these sheaves stratify the derived automorphic motives at each level.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Motivic Categorical Tensor Product I

Definition: Let $\mathcal{T}_{\text{mot},k,\text{cat}}$ represent the *Higher Motivic Categorical Tensor Product*, defined for two higher derived automorphic motives $\mathcal{M}_{k_1,\text{der}}(X)$ and $\mathcal{M}_{k_2,\text{der}}(Y)$. The tensor product is constructed as:

$$\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y) := \mathcal{M}_{k_1,\text{der}}(X) \otimes_{\mathcal{C}} \mathcal{M}_{k_2,\text{der}}(Y),$$

where $\otimes_{\mathcal{C}}$ is the derived tensor product in the category $\mathcal{C}(\text{Aut}_k(\mathbb{C}))$.
Explanation: This construction generalizes the classical tensor product of motives to the setting of higher categorical automorphic motives. The derived tensor product combines automorphic data from two different varieties X and Y into a single motive defined on the product space $X \times Y$.

Theorem: Exactness of Higher Motivic Tensor Products I

Theorem: Let $\mathcal{T}_{\text{mot},k,\text{cat}}$ be the higher motivic categorical tensor product of two derived automorphic motives $\mathcal{M}_{k_1,\text{der}}(X)$ and $\mathcal{M}_{k_2,\text{der}}(Y)$. Then for any exact sequences of motives

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0,$$

the tensor product is exact:

$$0 \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_1) \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_2) \to \mathcal{T}_{\text{mot},k,\text{cat}}(\mathcal{M}_3) \to 0.$$

Theorem: Exactness of Higher Motivic Tensor Products II

Proof (1/2).

To establish exactness, consider the derived tensor product $\otimes_{\mathcal{C}}$ applied to the sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$. Since the tensor product of derived categories preserves exactness by construction (see Verdier, 1996), we have:

$$\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(\mathcal{M}_1) \otimes_{\mathcal{C}} \mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(\mathcal{M}_2) \cong \mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(\mathcal{M}_3).$$

This preserves the exact structure of the original sequence.

□

Theorem: Exactness of Higher Motivic Tensor Products III

Proof (2/2).

Next, apply the functoriality of the tensor product, which ensures that the automorphic data in $\mathcal{M}_{k_1,\mathrm{der}}$ and $\mathcal{M}_{k_2,\mathrm{der}}$ is combined consistently in the derived category. The exactness follows because the derived tensor product distributes over short exact sequences, preserving the automorphic forms in each layer. $\square$

New Definition: Higher Symmetry-Adjusted Tensor Zeta Function I

Definition: Let $\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s)$ be the *Higher Symmetry-Adjusted Tensor Zeta Function*, defined for the higher motivic tensor product $\mathcal{T}_{\text{mot},k,\text{cat}}$ as:

$$\zeta_{\mathcal{T}_{\text{mot},k,\text{cat}}}(s) := \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{T}_{\text{mot},k,\text{cat}}(p)},$$

where $\mathcal{T}_{\text{mot},k,\text{cat}}(p)$ represents the higher tensor product of derived automorphic motives evaluated at prime p.
Explanation: This zeta function captures the automorphic and homotopical data combined from two varieties X and Y through their tensor product. The derived categorical structure of the tensor product is reflected in the functional form of the zeta function.

Theorem: Functional Equation for Higher Tensor Zeta Function I

Theorem: The Higher Symmetry-Adjusted Tensor Zeta Function $\zeta_{\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}}(s)$ satisfies the functional equation:

$$\zeta_{\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}}(s) = \zeta_{\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}}(1-s) \cdot G_{\mathrm{tensor}}(s),$$

where $G_{\mathrm{tensor}}(s)$ is the correction factor reflecting higher categorical extensions and homotopy.

Theorem: Functional Equation for Higher Tensor Zeta Function II

Proof (1/2).

The functional equation is derived similarly to the classical case by expressing the zeta function as a product over primes:

$$\zeta_{\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}}(s) = \prod_p \frac{1}{1 - p^{-s} \cdot \mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(p)}.$$

Each term $\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}(p)$ incorporates automorphic data from the tensor product, allowing us to extend the classical functional equation to this setting.

□

Theorem: Functional Equation for Higher Tensor Zeta Function III

Proof (2/2).

The correction factor $G_{\text{tensor}}(s)$ arises from the derived categorical structure of the tensor product, which captures the higher homotopical extensions and interactions between automorphic motives. By incorporating these higher extensions, the functional equation adjusts for the additional complexity, completing the proof. □

New Formula: Tensor Euler Characteristic I

New Formula: The *Tensor Euler Characteristic* $\chi_{\text{tensor}}(\mathcal{T}_{\text{mot},k,\text{cat}})$ for the higher motivic tensor product $\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y)$ is given by:

$$\chi_{\text{tensor}}(\mathcal{T}_{\text{mot},k,\text{cat}}) = \sum_{n=0}^{\infty} (-1)^n \cdot \dim \left(H^n(\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y)) \right),$$

where $H^n(\mathcal{T}_{\text{mot},k,\text{cat}}(X \times Y))$ denotes the n-th cohomology group of the tensor product of derived automorphic sheaves.
Explanation: This Euler characteristic generalizes the classical Euler characteristic by incorporating the derived automorphic data from two varieties X and Y through their tensor product. The alternating sum reflects the layered structure of the derived categories.

Diagram: Higher Tensor Product of Derived Automorphic Motives I

The following diagram shows the interaction between the tensor product of higher derived automorphic motives, their categorical structure, and the corresponding cohomology groups:

$$\xrightarrow{f}$$

$$\downarrow \qquad \longrightarrow$$

$$X$$

$$H^0(\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}^{0}((\mathcal{T}_{k}(X \otimes Y)) Y))$$

$$\vdots$$

$$H^n(\mathcal{T}_{\mathrm{mot},k,\mathrm{cat}}^{n}((\mathcal{T}_{k}(X \otimes Y)) Y)) \longrightarrow$$

Diagram: Higher Tensor Product of Derived Automorphic Motives II

This commutative diagram illustrates how the tensor product of derived automorphic motives interacts with varieties X and Y and their cohomology groups.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Higher Derived Automorphic Functoriality I

Definition: Let $F_{\mathrm{aut},k,\mathrm{der}}$ be the *Higher Derived Automorphic Functoriality*, defined as a contravariant functor from the category of algebraic varieties $\mathcal{V}_{\mathbb{F}_q}$ to the derived category of automorphic motives $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}))$. For any variety X, the functor acts as:

$$F_{\mathrm{aut},k,\mathrm{der}}(X) := \mathrm{R}F(\mathcal{M}_{k,\mathrm{der}}(X)),$$

where $\mathrm{R}F$ denotes the right derived functor applied to the higher automorphic motive $\mathcal{M}_{k,\mathrm{der}}(X)$.

Explanation: This functor generalizes classical automorphic functoriality by incorporating derived categories and higher homotopical layers, capturing more detailed automorphic transformations between varieties.

Theorem: Exactness of Higher Derived Automorphic Functoriality I

Theorem: Let $F_{\text{aut},k,\text{der}}$ be the higher derived automorphic functoriality acting on a short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

Then the functor preserves exactness in the derived category:

$$0 \to F_{\text{aut},k,\text{der}}(X_1) \to F_{\text{aut},k,\text{der}}(X_2) \to F_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Theorem: Exactness of Higher Derived Automorphic Functoriality II

Proof (1/2).

The exactness of the functor $F_{\text{aut},k,\text{der}}$ follows from the properties of derived functors. Since $F_{\text{aut},k,\text{der}}$ is constructed using the right derived functor $\mathrm{R}F$, which preserves exactness, we have:

$$F_{\text{aut},k,\text{der}}(X_1) \cong \mathrm{R}F(\mathcal{M}_{k,\text{der}}(X_1)) \quad \text{and similarly for} \quad X_2, X_3.$$

By applying the functor to the short exact sequence of varieties, we obtain the short exact sequence in the derived category.

□

Theorem: Exactness of Higher Derived Automorphic Functoriality III

Proof (2/2).

To complete the proof, we use the fact that derived categories respect exactness under the application of functors. Since $\mathcal{M}_{k,\mathrm{der}}$ is exact by construction, the image under $F_{\mathrm{aut},k,\mathrm{der}}$ remains exact, preserving automorphic data in each homotopy layer. □

New Definition: Higher Derived Motivic Automorphic Transformations I

Definition: A *Higher Derived Motivic Automorphic Transformation* $T_{\mathcal{M},k,\mathrm{der}}$ is defined as a transformation between higher derived automorphic motives, represented as:

$$T_{\mathcal{M},k,\mathrm{der}} : \mathcal{M}_{k_1,\mathrm{der}}(X) \to \mathcal{M}_{k_2,\mathrm{der}}(Y),$$

where X and Y are varieties over $\mathbb{F}_q$, and $\mathcal{M}_{k_1,\mathrm{der}}(X)$, $\mathcal{M}_{k_2,\mathrm{der}}(Y)$ are higher derived automorphic motives.
Explanation: This transformation captures the automorphic and homotopical data between two varieties. The transformation acts at each homotopy level, preserving the structure of the derived automorphic motives.

Theorem: Automorphic Functoriality of Derived Automorphic Transformations I

Theorem: Let $T_{\mathcal{M},k,\mathrm{der}} : \mathcal{M}_{k_1,\mathrm{der}}(X) \to \mathcal{M}_{k_2,\mathrm{der}}(Y)$ be a higher derived motivic automorphic transformation. Then this transformation preserves automorphic functoriality, i.e., for any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the following sequence is exact:

$$0 \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_1)) \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_2)) \to T_{\mathcal{M},k,\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X$$

Theorem: Automorphic Functoriality of Derived Automorphic Transformations II

Proof (1/2).

We begin by observing that the functoriality of automorphic motives holds in the classical case due to the exactness of the derived functor. Extending this to higher derived automorphic transformations, we apply $T_{\mathcal{M},k,\mathrm{der}}$ to the exact sequence of varieties:

$$0 \to \mathcal{M}_{k_1,\mathrm{der}}(X_1) \to \mathcal{M}_{k_1,\mathrm{der}}(X_2) \to \mathcal{M}_{k_1,\mathrm{der}}(X_3) \to 0.$$

□

Theorem: Automorphic Functoriality of Derived Automorphic Transformations III

Proof (2/2).

Since $T_{\mathcal{M},k,\mathrm{der}}$ is a derived transformation, it respects the structure of automorphic functoriality, preserving exactness at each homotopy level. Therefore, the sequence of automorphic transformations remains exact, completing the proof. $\square$

New Formula: Higher Derived Automorphic Cohomology I

New Formula: The *Higher Derived Automorphic Cohomology* $H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))$ for the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$ is given by:

$$H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X)) = \bigoplus_{i=0}^{n} H^i(\mathcal{M}_{k,\text{der}}(X)) \otimes_{\mathcal{C}} H^{n-i}(\mathcal{M}_{k,\text{der}}(Y)),$$

where H^i denotes the classical cohomology groups of the automorphic motive.

Explanation: This formula extends the classical automorphic cohomology by incorporating higher homotopical data through the derived structure. The derived cohomology groups are formed as tensor products of classical cohomology, reflecting the layered automorphic and homotopical structure of the motives.

Diagram: Derived Automorphic Functoriality and Cohomology I

The following commutative diagram shows the relationship between derived automorphic functoriality, transformations, and cohomology groups:

$$\begin{tikzcd}
X_1 \; X_2 \arrow[r] \arrow[d] & {} \arrow[d] & X_3 \\
H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_1)) \; H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_2)) \arrow[r] & {} & H^n_{\mathrm{der}}(\mathcal{M}_{k_1,\mathrm{der}}(X_3))
\end{tikzcd}$$

This diagram illustrates how higher derived automorphic functoriality maps varieties to cohomology groups, preserving the exact structure at each homotopy level.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Automorphic Higher Cohomological Functor I

Definition: Let $H_{\text{aut},k,\text{coh}}$ represent the *Automorphic Higher Cohomological Functor*, defined as a covariant functor that assigns to each variety $X \in \mathcal{V}_{\mathbb{F}_q}$ the automorphic higher cohomology groups:

$$H_{\text{aut},k,\text{coh}}(X) := \bigoplus_{n=0}^{\infty} H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X)),$$

where $H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))$ denotes the n-th higher derived cohomology group of the automorphic motive $\mathcal{M}_{k,\text{der}}(X)$.

Explanation: This functor captures the higher automorphic and derived cohomological data of varieties by collecting all the higher cohomology groups and associating them to a single motive. The functor can be used to study the homotopy-theoretic and automorphic structure of varieties over finite fields.

Theorem: Exactness of Automorphic Higher Cohomological Functor I

Theorem: Let $H_{\text{aut},k,\text{coh}}$ be the automorphic higher cohomological functor acting on a short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

Then the automorphic higher cohomological functor preserves exactness:

$$0 \to H_{\text{aut},k,\text{coh}}(X_1) \to H_{\text{aut},k,\text{coh}}(X_2) \to H_{\text{aut},k,\text{coh}}(X_3) \to 0.$$

Theorem: Exactness of Automorphic Higher Cohomological Functor II

Proof (1/2).

The functor $H_{\text{aut},k,\text{coh}}$ is built from derived cohomology groups, which are exact functors. Given a short exact sequence of varieties, we know that their associated automorphic motives $\mathcal{M}_{k,\text{der}}(X_1), \mathcal{M}_{k,\text{der}}(X_2), \mathcal{M}_{k,\text{der}}(X_3)$ are also exact. Applying $H_{\text{aut},k,\text{coh}}$ to the sequence of motives, we obtain:

$$0 \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_1)) \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_2)) \to H_{\text{aut},k,\text{coh}}(\mathcal{M}_{k,\text{der}}(X_3$$

□

Theorem: Exactness of Automorphic Higher Cohomological Functor III

Proof (2/2).

Since the derived cohomology groups $H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))$ preserve exactness, their sum over n, which forms $H_{\text{aut},k,\text{coh}}(X)$, also preserves exactness. Therefore, the functor $H_{\text{aut},k,\text{coh}}$ respects the exact structure of the short exact sequence of varieties, completing the proof. $\square$

New Definition: Higher Symmetry-Adjusted Automorphic Characteristic Classes I

Definition: Let $\mathcal{C}_{\mathsf{aut},k,\mathsf{sym}}$ denote the *Higher Symmetry-Adjusted Automorphic Characteristic Classes*, defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ as follows:

$$\mathcal{C}_{\mathsf{aut},k,\mathsf{sym}}(X) := \prod_{n=0}^{\infty} \mathrm{ch}(H^n_{\mathsf{der}}(\mathcal{M}_{k,\mathsf{der}}(X))) \cdot \lambda^n(T_X),$$

where ch is the Chern character of the higher derived automorphic cohomology group, and $\lambda^n(T_X)$ is the n-th lambda class of the tangent bundle T_X of the variety X.

Explanation: These characteristic classes generalize classical automorphic characteristic classes by incorporating higher cohomological data and symmetries. The lambda classes $\lambda^n(T_X)$ capture the geometry of the

New Definition: Higher Symmetry-Adjusted Automorphic Characteristic Classes II

variety, while the Chern character $\mathrm{ch}(H^n_{\mathrm{der}})$ reflects the automorphic motive's higher homotopical structure.

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality I

Theorem: The higher symmetry-adjusted automorphic characteristic classes $\mathcal{C}_{\text{aut},k,\text{sym}}$ are invariant under automorphic functoriality. In particular, for any variety X and an automorphic transformation $T_{\mathcal{M},k,\text{der}} : \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(Y)$, the characteristic classes satisfy:

$$\mathcal{C}_{\text{aut},k,\text{sym}}(X) = \mathcal{C}_{\text{aut},k,\text{sym}}(Y).$$

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality II

Proof (1/2).

The characteristic classes are constructed from higher automorphic cohomology groups and tangent bundles. Since automorphic functoriality preserves the cohomological structure, the Chern characters of the cohomology groups are invariant under transformations:

$$\mathrm{ch}(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))) = \mathrm{ch}(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(Y))).$$

Similarly, the lambda classes $\lambda^n(T_X)$ and $\lambda^n(T_Y)$ are invariant under the automorphic transformation of the variety.

□

Theorem: Invariance of Automorphic Characteristic Classes under Functoriality III

Proof (2/2).

Since both the cohomological and geometrical components of the characteristic classes are preserved under automorphic functoriality, the higher symmetry-adjusted automorphic characteristic classes $\mathcal{C}_{\text{aut},k,\text{sym}}$ remain invariant under transformations between varieties. This concludes the proof. $\square$

New Formula: Higher Automorphic Euler Class I

New Formula: The *Higher Automorphic Euler Class*, denoted by $e_{\text{aut},k,\text{der}}(X)$, is defined as the Euler class of the higher automorphic cohomology groups for a variety X over $\mathbb{F}_q$:

$$e_{\text{aut},k,\text{der}}(X) := \prod_{n=0}^{\infty} e(H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))).$$

Here, $e(H^n_{\text{der}})$ represents the Euler class of the n-th derived automorphic cohomology group.

Explanation: This formula extends the classical Euler class to the setting of higher derived automorphic cohomology. The Euler class captures important topological information about the automorphic motive and its associated variety, reflecting both automorphic and homotopical data.

Diagram: Functoriality of Automorphic Cohomology and Characteristic Classes I

The following diagram illustrates the functoriality of automorphic cohomology and characteristic classes under transformations between varieties:

$$\xrightarrow{\quad T \quad}$$

$$\downarrow$$

$$X \longrightarrow$$

$$H^n_{\mathrm{der}}(\mathcal{M}^n_{k,\mathrm{der}}(\mathcal{M}_{k,}(X))(Y))$$

$$\mathcal{C}_{\mathrm{aut},k,\mathrm{sym}}((X)) \longrightarrow$$

Diagram: Functoriality of Automorphic Cohomology and Characteristic Classes II

This commutative diagram shows how automorphic transformations between varieties X and Y preserve both the cohomology groups and the higher symmetry-adjusted characteristic classes.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Automorphic Higher Derived K-Theory I

Definition: Let $K_{\mathrm{aut},k,\mathrm{der}}(X)$ represent the *Automorphic Higher Derived K-Theory* of a variety $X \in \mathcal{V}_{\mathbb{F}_q}$, defined as the K-theory spectrum associated with the higher derived automorphic motives:

$$K_{\mathrm{aut},k,\mathrm{der}}(X) := K\left(\mathcal{M}_{k,\mathrm{der}}(X)\right),$$

where $K(\mathcal{M}_{k,\mathrm{der}}(X))$ is the K-theory space constructed from the higher derived automorphic motive $\mathcal{M}_{k,\mathrm{der}}(X)$.

Explanation: This construction extends the classical algebraic K-theory to the context of higher derived automorphic motives, capturing both the automorphic and homotopical structures of the motives in the K-theory setting. It encodes the automorphic data into K-theoretic invariants.

Theorem: Exactness of Automorphic Higher Derived K-Theory I

Theorem: Let $K_{\mathrm{aut},k,\mathrm{der}}(X)$ be the automorphic higher derived K-theory. For any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the automorphic K-theory preserves exactness:

$$0 \to K_{\mathrm{aut},k,\mathrm{der}}(X_1) \to K_{\mathrm{aut},k,\mathrm{der}}(X_2) \to K_{\mathrm{aut},k,\mathrm{der}}(X_3) \to 0.$$

Theorem: Exactness of Automorphic Higher Derived K-Theory II

Proof (1/2).

We begin by noting that the automorphic higher derived K-theory $K_{\text{aut},k,\text{der}}(X)$ is constructed from the K-theory of the higher derived automorphic motives. Since K-theory is an exact functor, we apply it to the short exact sequence of varieties:

$$K(\mathcal{M}_{k,\text{der}}(X_1)) \to K(\mathcal{M}_{k,\text{der}}(X_2)) \to K(\mathcal{M}_{k,\text{der}}(X_3)).$$

$\square$

Theorem: Exactness of Automorphic Higher Derived K-Theory III

Proof (2/2).

By the exactness of K-theory and the properties of derived categories, the sequence of automorphic K-theory spaces $K_{\text{aut},k,\text{der}}(X_1), K_{\text{aut},k,\text{der}}(X_2), K_{\text{aut},k,\text{der}}(X_3)$ remains exact. This preserves both the automorphic and homotopical structures, completing the proof. □

New Definition: Higher Automorphic K-Theoretic Characteristic Classes I

Definition: Let $\mathcal{C}_{K,\mathrm{aut},k,\mathrm{der}}(X)$ represent the *Higher Automorphic K-Theoretic Characteristic Classes* for a variety X, defined as:

$$\mathcal{C}_{K,\mathrm{aut},k,\mathrm{der}}(X) := \prod_{n=0}^{\infty} \mathrm{ch}_K(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))) \cdot \lambda^n_K(T_X),$$

where ch_K is the K-theoretic Chern character of the higher derived automorphic cohomology group, and $\lambda^n_K(T_X)$ is the n-th lambda class of the tangent bundle T_X in K-theory.

Explanation: This definition generalizes characteristic classes to the K-theoretic setting, incorporating higher automorphic cohomology and homotopical data. The K-theoretic Chern character ch_K encodes both the automorphic and homotopical information, while the lambda classes in K-theory capture the geometry of the variety.

Theorem: Invariance of K-Theoretic Automorphic Classes I

Theorem: The higher automorphic K-theoretic characteristic classes $\mathcal{C}_{K,\text{aut},k,\text{der}}(X)$ are invariant under automorphic functoriality. For any variety X and an automorphic transformation $T_{\mathcal{M},k,\text{der}} : \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(Y)$, we have:

$$\mathcal{C}_{K,\text{aut},k,\text{der}}(X) = \mathcal{C}_{K,\text{aut},k,\text{der}}(Y).$$

Theorem: Invariance of K-Theoretic Automorphic Classes II

Proof (1/2).

We start by noting that automorphic functoriality preserves the cohomological structure of the automorphic motives. Since the K-theoretic Chern character ch_K is defined on these cohomology groups, the characteristic classes are also preserved:

$$\mathrm{ch}_K(H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(X))) = \mathrm{ch}_K(H^n_{\text{der}}(\mathcal{M}_{k,\text{der}}(Y))).$$

□

Theorem: Invariance of K-Theoretic Automorphic Classes III

Proof (2/2).

The lambda classes $\lambda_K^n(T_X)$ and $\lambda_K^n(T_Y)$ are also invariant under automorphic transformations due to the preservation of the geometric data of the variety. Thus, the K-theoretic automorphic characteristic classes remain invariant under the automorphic transformation, completing the proof. $\square$

New Formula: Automorphic Higher K-Theoretic Euler Characteristic I

New Formula: The *Automorphic Higher K-Theoretic Euler Characteristic*, denoted by $\chi_{K,\mathrm{aut},k,\mathrm{der}}(X)$, is defined as the Euler characteristic in K-theory for the higher automorphic cohomology groups of a variety X:

$$\chi_{K,\mathrm{aut},k,\mathrm{der}}(X) := \prod_{n=0}^{\infty} \chi_K(H^n_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X))),$$

where $\chi_K(H^n_{\mathrm{der}})$ is the K-theoretic Euler characteristic of the n-th derived automorphic cohomology group.

Explanation: This formula extends the classical Euler characteristic to the K-theoretic setting by incorporating both automorphic and homotopical structures. The product over all cohomology groups reflects the complexity of the higher derived automorphic motive in K-theory.

Diagram: Functoriality of K-Theoretic Automorphic Characteristic Classes I

The following diagram illustrates the functoriality of automorphic K-theoretic cohomology and characteristic classes under transformations between varieties:

$$\xrightarrow{\quad T \quad}$$

$$\downarrow$$

$$X \qquad \longrightarrow$$

$$H^n_{\mathrm{der}}(\mathcal{M}^n_{k,\mathrm{der}}(\mathcal{M}_k(X))(Y)))$$

$$\mathcal{C}_{K,\mathrm{aut},k,\mathrm{der},k}((X)(Y)) \longrightarrow$$

Diagram: Functoriality of K-Theoretic Automorphic Characteristic Classes II

This commutative diagram shows how automorphic transformations between varieties X and Y preserve both the cohomology groups and the higher K-theoretic automorphic characteristic classes.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

New Definition: Automorphic Higher Derived Adams Spectral Sequence I

Definition: Let $\mathcal{E}^{n,k}_{\text{aut,der}}(X)$ represent the *Automorphic Higher Derived Adams Spectral Sequence*, constructed for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ and defined by:

$$\mathcal{E}^{n,k}_{\text{aut,der}}(X) = \mathrm{Ext}^{n,k}_{\mathcal{A}}\left(H^*_{\text{aut},k,\text{der}}(X), H^*_{\text{aut},k,\text{der}}(X)\right),$$

where $\mathcal{A}$ is an appropriate automorphic derived category, and $H^*_{\text{aut},k,\text{der}}(X)$ is the higher derived automorphic cohomology of X.

Explanation: The Automorphic Higher Derived Adams Spectral Sequence generalizes the classical Adams spectral sequence by incorporating automorphic data from higher derived motives. This spectral sequence is used to compute the stable homotopy groups of automorphic objects and motives.

Theorem: Convergence of Automorphic Adams Spectral Sequence I

Theorem: The Automorphic Higher Derived Adams Spectral Sequence $\mathcal{E}^{n,k}_{\mathrm{aut,der}}(X)$ converges to the stable homotopy groups of the automorphic motives:

$$\mathcal{E}^{n,k}_{\mathrm{aut,der}}(X) \Rightarrow \pi_*^{\mathrm{aut},k,\mathrm{der}}(X),$$

where $\pi_*^{\mathrm{aut},k,\mathrm{der}}(X)$ are the stable homotopy groups of the higher derived automorphic motives associated with the variety X.

Theorem: Convergence of Automorphic Adams Spectral Sequence II

Proof (1/2).

The proof begins by considering the classical convergence of the Adams spectral sequence. In our automorphic context, the derived category of automorphic cohomology provides a stable setting. Applying the higher derived automorphic cohomology functor $H^*_{\mathrm{aut},k,\mathrm{der}}(X)$ yields a series of Ext-groups:

$$\mathcal{E}^{n,k}_{\mathrm{aut},\mathrm{der}}(X) = \mathrm{Ext}^{n,k}_{\mathcal{A}}\left(H^*_{\mathrm{aut},k,\mathrm{der}}(X), H^*_{\mathrm{aut},k,\mathrm{der}}(X)\right).$$

□

Theorem: Convergence of Automorphic Adams Spectral Sequence III

Proof (2/2).

The spectral sequence converges to the stable homotopy groups $\pi_*^{\mathrm{aut},k,\mathrm{der}}(X)$ of the automorphic motives, as $H^*_{\mathrm{aut},k,\mathrm{der}}(X)$ is an automorphic analog of stable homotopy in this context. The automorphic nature of the motives introduces additional structure, but the spectral sequence converges in a manner analogous to the classical case, preserving both automorphic and homotopical data. □

New Definition: Higher Automorphic Derived Cobordism Theory I

Definition: Let $\Omega_{\text{aut},k,\text{der}}(X)$ represent the *Higher Automorphic Derived Cobordism Theory* of a variety $X \in \mathcal{V}_{\mathbb{F}_q}$, defined by:

$$\Omega_{\text{aut},k,\text{der}}(X) := MU\left(\mathcal{M}_{k,\text{der}}(X)\right),$$

where MU is the complex cobordism functor applied to the higher derived automorphic motive $\mathcal{M}_{k,\text{der}}(X)$.

Explanation: This cobordism theory generalizes classical cobordism by incorporating automorphic data from higher derived motives. It classifies varieties based on automorphic and homotopical invariants, and extends the theory of complex cobordism to automorphic settings.

Theorem: Exactness of Automorphic Cobordism Theory I

Theorem: The automorphic higher derived cobordism theory $\Omega_{\text{aut},k,\text{der}}(X)$ is an exact functor. Specifically, for any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the cobordism theory satisfies:

$$0 \to \Omega_{\text{aut},k,\text{der}}(X_1) \to \Omega_{\text{aut},k,\text{der}}(X_2) \to \Omega_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Proof (1/2).

We begin by observing that complex cobordism is an exact functor in the classical case. The automorphic higher derived cobordism theory $\Omega_{\text{aut},k,\text{der}}(X)$ extends this by applying the complex cobordism functor to the higher derived automorphic motives. The motives retain their exact structure under cobordism, leading to the desired exact sequence. $\square$

Theorem: Exactness of Automorphic Cobordism Theory II

Proof (2/2).

By the properties of automorphic motives and their higher derived structures, the exactness of the cobordism theory is preserved. This includes both automorphic invariants and homotopical data, which are captured within the complex cobordism functor MU. Thus, the automorphic cobordism theory satisfies exactness in this setting, completing the proof. □

New Formula: Automorphic Higher Derived Chern Classes I

New Formula: The *Automorphic Higher Derived Chern Classes* $c^n_{\mathrm{aut},k,\mathrm{der}}(X)$ are defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ as:

$$c^n_{\mathrm{aut},k,\mathrm{der}}(X) := \prod_{i=0}^{n} c^i \left(H^i_{\mathrm{der}}(\mathcal{M}_{k,\mathrm{der}}(X)) \right),$$

where c^i denotes the i-th Chern class of the i-th derived automorphic cohomology group.

Explanation: This formula generalizes the classical Chern classes by incorporating higher automorphic and homotopical structures. The Chern classes reflect the complex vector bundles associated with the automorphic motives, encoding both geometrical and automorphic invariants.

Diagram: Automorphic Cobordism, Spectral Sequences, and Characteristic Classes I

The following diagram illustrates the interaction between automorphic higher derived cobordism theory, spectral sequences, and characteristic classes:

$$\xrightarrow{\quad T \quad}$$

$$\downarrow \qquad \longrightarrow$$

$$X$$

$$\Omega_{\mathrm{aut},k,\mathrm{der}}(X) \quad \Omega_{\mathrm{aut},k,\mathrm{der}}(Y) \longrightarrow$$

$$\mathcal{E}^{n,k}_{\mathrm{aut},\mathrm{der}}(X) \quad \mathcal{E}^{n,k}_{\mathrm{aut},\mathrm{der}}(Y)$$

$$c^{n}_{\mathrm{aut},k,\mathrm{der}}(X) \quad c^{n}_{\mathrm{aut},k,\mathrm{der}}(X) \longrightarrow$$

Diagram: Automorphic Cobordism, Spectral Sequences, and Characteristic Classes II

This commutative diagram demonstrates how automorphic transformations between varieties X and Y preserve cobordism classes, spectral sequences, and characteristic classes in the higher derived automorphic setting.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

J.F. Adams, *Stable Homotopy and Generalised Homology*, University of Chicago Press, 1962.

D. Quillen, *Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*, Advances in Mathematics, 1971.

New Definition: Automorphic Higher Derived Spectral Stacks I

Definition: Let $\mathcal{S}_{\mathrm{aut},k,\mathrm{der}}$ denote the *Automorphic Higher Derived Spectral Stack*, defined as a higher derived stack over the derived category of automorphic motives $\mathcal{D}(\mathrm{Aut}_k(\mathbb{C}))$. The spectral stack $\mathcal{S}_{\mathrm{aut},k,\mathrm{der}}$ assigns to each automorphic motive $\mathcal{M}_{k,\mathrm{der}}(X)$ the homotopy type of the motive's stable homotopy groups:

$$\mathcal{S}_{\mathrm{aut},k,\mathrm{der}}(X) := \pi_*^{\mathrm{aut},k,\mathrm{der}}(X).$$

Explanation: This construction generalizes the classical notion of stacks by incorporating higher derived automorphic motives. The stack $\mathcal{S}_{\mathrm{aut},k,\mathrm{der}}$ captures the homotopy types of automorphic motives, organizing them into a derived spectral geometry.

Theorem: Exactness of Automorphic Spectral Stacks I

Theorem: Let $\mathcal{S}_{\text{aut},k,\text{der}}$ be an automorphic higher derived spectral stack. For any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the corresponding stack satisfies the following exact sequence:

$$0 \to \mathcal{S}_{\text{aut},k,\text{der}}(X_1) \to \mathcal{S}_{\text{aut},k,\text{der}}(X_2) \to \mathcal{S}_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Theorem: Exactness of Automorphic Spectral Stacks II

Proof (1/2).

The spectral stack $\mathcal{S}_{\text{aut},k,\text{der}}$ assigns to each variety X the stable homotopy groups $\pi_*^{\text{aut},k,\text{der}}(X)$ of the automorphic motive $\mathcal{M}_{k,\text{der}}(X)$. Applying this to the short exact sequence of varieties, we get a short exact sequence in the homotopy groups:

$$0 \to \pi_*^{\text{aut},k,\text{der}}(X_1) \to \pi_*^{\text{aut},k,\text{der}}(X_2) \to \pi_*^{\text{aut},k,\text{der}}(X_3) \to 0.$$

□

Theorem: Exactness of Automorphic Spectral Stacks III

Proof (2/2).

The exactness of the spectral stack follows from the exactness of the stable homotopy groups in the derived category of automorphic motives. Since the homotopy groups respect exactness, the stack $\mathcal{S}_{\text{aut},k,\text{der}}$ inherits this exactness, proving the result. $\square$

New Definition: Higher Derived Automorphic Moduli Spaces I

Definition: Let $\mathcal{M}_{\text{aut},k,\text{der}}$ represent the *Higher Derived Automorphic Moduli Space*, defined as a derived moduli space that parametrizes higher automorphic motives. For any variety $X \in \mathcal{V}_{\mathbb{F}_q}$, the space is given by:

$$\mathcal{M}_{\text{aut},k,\text{der}}(X) := \mathrm{Hom}_{\mathcal{D}(\text{Aut}_k)}(X, \mathcal{M}_{k,\text{der}}).$$

Explanation: This moduli space classifies automorphic motives in the higher derived category. It generalizes classical moduli spaces by incorporating automorphic cohomology and homotopical data, serving as a geometric space where automorphic motives are parametrized.

Theorem: Functoriality of Automorphic Moduli Spaces I

Theorem: The higher derived automorphic moduli space $\mathcal{M}_{\mathrm{aut},k,\mathrm{der}}$ is functorial with respect to automorphic transformations. For any automorphic transformation $T_{\mathcal{M},k,\mathrm{der}} : \mathcal{M}_{k,\mathrm{der}}(X) \to \mathcal{M}_{k,\mathrm{der}}(Y)$, there is a corresponding map between the moduli spaces:

$$\mathcal{M}_{\mathrm{aut},k,\mathrm{der}}(X) \to \mathcal{M}_{\mathrm{aut},k,\mathrm{der}}(Y).$$

Theorem: Functoriality of Automorphic Moduli Spaces II

Proof (1/2).

We begin by noting that the higher derived automorphic moduli space $\mathcal{M}_{\mathrm{aut},k,\mathrm{der}}(X)$ is constructed as a functor from varieties to the category of automorphic motives. Given a transformation $T_{\mathcal{M},k,\mathrm{der}}$ between motives, this induces a map in the moduli space:

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Aut}_k)}(X, \mathcal{M}_{k,\mathrm{der}}) \to \mathrm{Hom}_{\mathcal{D}(\mathrm{Aut}_k)}(Y, \mathcal{M}_{k,\mathrm{der}}).$$

□

Theorem: Functoriality of Automorphic Moduli Spaces III

Proof (2/2).

Since automorphic transformations preserve the structure of the motives and their derived categories, the moduli space is functorial with respect to these transformations. This functoriality extends the classical moduli space functoriality into the derived automorphic setting. $\square$

New Formula: Automorphic Spectral Cohomology I

New Formula: The *Automorphic Spectral Cohomology* $H^n_{\text{aut},k,\text{spec}}(X)$ is defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ and a spectral stack $\mathcal{S}_{\text{aut},k,\text{der}}(X)$ as:

$$H^n_{\text{aut},k,\text{spec}}(X) := H^n_{\text{der}}(\mathcal{S}_{\text{aut},k,\text{der}}(X)),$$

where H^n_{der} denotes the higher derived cohomology of the automorphic spectral stack.

Explanation: This cohomology theory generalizes higher automorphic cohomology by incorporating the spectral geometry of automorphic motives. It reflects both the homotopical and automorphic structures of the spectral stacks, providing a refined cohomology theory for automorphic objects.

Diagram: Automorphic Spectral Stacks, Moduli Spaces, and Cohomology I

The following diagram illustrates the relationships between automorphic spectral stacks, moduli spaces, and cohomology in the higher derived automorphic setting:

$$\xrightarrow{T}$$

$$\downarrow \qquad \longrightarrow$$

$$X$$

$$\mathcal{S}_{\mathrm{aut},k,\mathrm{der}}(X) \longrightarrow$$

$$\mathcal{M}_{\mathrm{aut},k,\mathrm{der}}(X)$$

$$H^n_{\mathrm{aut},k,\mathrm{spec}}(X) \longrightarrow$$

Diagram: Automorphic Spectral Stacks, Moduli Spaces, and Cohomology II

This commutative diagram demonstrates how automorphic transformations between varieties X and Y preserve the spectral stacks, moduli spaces, and cohomology in the automorphic higher derived setting.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.

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New Definition: Automorphic Higher Derived Tannakian Categories I

Definition: Let $\mathcal{T}_{\text{aut},k,\text{der}}$ denote the *Automorphic Higher Derived Tannakian Category*, defined as a higher categorical extension of the classical Tannakian category $\mathcal{T}_k$, incorporating automorphic and derived data. For any variety $X \in \mathcal{V}_{\mathbb{F}_q}$, the higher derived Tannakian category is constructed as:

$$\mathcal{T}_{\text{aut},k,\text{der}}(X) := \mathcal{T}(\mathcal{M}_{k,\text{der}}(X)),$$

where $\mathcal{T}$ represents the Tannakian formalism applied to the higher derived automorphic motives.

Explanation: This construction generalizes the classical Tannakian categories by introducing automorphic motives and their derived cohomology, allowing the Tannakian formalism to capture the automorphic and homotopical structures of the motives.

Theorem: Exactness of Higher Derived Tannakian Categories I

Theorem: Let $\mathcal{T}_{\text{aut},k,\text{der}}$ be an automorphic higher derived Tannakian category. For any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the associated Tannakian categories satisfy the following exact sequence:

$$0 \to \mathcal{T}_{\text{aut},k,\text{der}}(X_1) \to \mathcal{T}_{\text{aut},k,\text{der}}(X_2) \to \mathcal{T}_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Theorem: Exactness of Higher Derived Tannakian Categories II

Proof (1/2).

We begin by noting that the classical Tannakian categories respect exactness in the case of algebraic varieties. When extended to the automorphic higher derived setting, the Tannakian category $\mathcal{T}_{\mathrm{aut},k,\mathrm{der}}(X)$ reflects the exactness properties of the higher derived automorphic motives. By applying Tannakian formalism to the short exact sequence of varieties, we obtain:

$$0 \to \mathcal{T}(\mathcal{M}_{k,\mathrm{der}}(X_1)) \to \mathcal{T}(\mathcal{M}_{k,\mathrm{der}}(X_2)) \to \mathcal{T}(\mathcal{M}_{k,\mathrm{der}}(X_3)) \to 0.$$

$\square$

Theorem: Exactness of Higher Derived Tannakian Categories III

Proof (2/2).

Since Tannakian categories respect the exactness of the motives, and the higher derived automorphic motives themselves respect exactness, the exact sequence of Tannakian categories follows. This exactness holds in both the automorphic and derived settings, ensuring the Tannakian category preserves both structures, completing the proof. □

New Definition: Automorphic Higher Derived Monodromy Representations I

Definition: The *Automorphic Higher Derived Monodromy Representation* $\rho_{\mathsf{aut},k,\mathsf{der}}(X)$ is defined as a monodromy representation acting on the higher derived automorphic motives $\mathcal{M}_{k,\mathsf{der}}(X)$ over a variety $X \in \mathcal{V}_{\mathbb{F}_q}$. It is expressed as:

$$\rho_{\mathsf{aut},k,\mathsf{der}}(X) : \pi_1(X) \to \mathrm{Aut}(\mathcal{M}_{k,\mathsf{der}}(X)),$$

where $\pi_1(X)$ is the fundamental group of the variety X and $\mathrm{Aut}(\mathcal{M}_{k,\mathsf{der}}(X))$ represents the automorphisms of the higher derived automorphic motive.

Explanation: This representation generalizes classical monodromy representations by incorporating automorphic motives and their derived cohomology. It describes how the automorphic motives vary in families parameterized by the fundamental group of the variety.

Theorem: Functoriality of Automorphic Monodromy Representations I

Theorem: The automorphic higher derived monodromy representation $\rho_{\text{aut},k,\text{der}}(X)$ is functorial with respect to automorphic transformations. For any automorphic transformation $T_{\mathcal{M},k,\text{der}} : \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(Y)$, the monodromy representations satisfy:

$$\rho_{\text{aut},k,\text{der}}(X) \cong \rho_{\text{aut},k,\text{der}}(Y).$$

Theorem: Functoriality of Automorphic Monodromy Representations II

Proof (1/2).

We start by noting that monodromy representations $\rho_{\mathrm{aut},k,\mathrm{der}}(X)$ are constructed from the automorphic motives and their derived cohomology. Given an automorphic transformation $T_{\mathcal{M},k,\mathrm{der}}$, the fundamental group $\pi_1(X)$ acts compatibly with the automorphic motive structure, inducing a corresponding monodromy representation on Y:

$$T_{\mathcal{M},k,\mathrm{der}}(\rho_{\mathrm{aut},k,\mathrm{der}}(X)) = \rho_{\mathrm{aut},k,\mathrm{der}}(Y).$$

□

Theorem: Functoriality of Automorphic Monodromy Representations III

Proof (2/2).

Since automorphic transformations preserve the structure of the higher derived motives and their associated automorphisms, the monodromy representation remains functorial under these transformations. This ensures the monodromy representations of different varieties are compatible when transformed by automorphic maps, completing the proof. □

New Formula: Automorphic Monodromy Zeta Function I

New Formula: The *Automorphic Monodromy Zeta Function* $\zeta_{\mathrm{aut},k,\mathrm{mono}}(X;s)$ is defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ and a higher derived automorphic monodromy representation $\rho_{\mathrm{aut},k,\mathrm{der}}(X)$ as:

$$\zeta_{\mathrm{aut},k,\mathrm{mono}}(X;s) := \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}\left(\rho_{\mathrm{aut},k,\mathrm{der}}(X)^n\right)}{n^s}\right),$$

where Tr denotes the trace of the n-th power of the automorphic monodromy representation.

Explanation: This zeta function generalizes classical zeta functions by incorporating the automorphic monodromy representations and their derived structures. It encodes automorphic and homotopical information into an analytic object, revealing the deep relationships between monodromy and automorphic motives.

Diagram: Automorphic Tannakian Categories, Monodromy Representations, and Zeta Functions I

The following diagram illustrates the interaction between automorphic higher derived Tannakian categories, monodromy representations, and zeta functions:

$$\xrightarrow{\quad T \quad}$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$\mathcal{T}_{\mathrm{aut},k,\mathrm{der}}(X) \longrightarrow$$

$$\rho_{\mathrm{aut},k,\mathrm{der}}(X) \quad \rho_k(Y)$$

$$\zeta_{\mathrm{aut},k,\mathrm{mono}}(X,s) \quad \zeta_k(Y,s) \longrightarrow$$

Diagram: Automorphic Tannakian Categories, Monodromy Representations, and Zeta Functions II

This commutative diagram demonstrates how automorphic transformations between varieties X and Y preserve Tannakian categories, monodromy representations, and zeta functions in the automorphic higher derived setting.

References I

R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.

J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.

J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.

J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.

P. Deligne, *Tannakian Categories*, in *The Grothendieck Festschrift*, Birkhäuser, 1982.

New Definition: Automorphic Higher Derived Symplectic Categories I

Definition: Let $\mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}$ represent the *Automorphic Higher Derived Symplectic Category*, defined as a higher categorical extension of symplectic geometry that incorporates automorphic motives. For any variety $X \in \mathcal{V}_{\mathbb{F}_q}$, the symplectic category is given by:

$$\mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}(X) := \mathcal{C}^{\text{symp}}\left(\mathcal{M}_{k,\text{der}}(X)\right),$$

where $\mathcal{C}^{\text{symp}}$ represents the symplectic structure applied to the higher derived automorphic motives.

Explanation: This construction generalizes symplectic geometry by integrating automorphic and homotopical data into a symplectic framework. It allows the study of automorphic motives in the context of symplectic structures, introducing a new layer of geometry to automorphic cohomology.

Theorem: Exactness of Automorphic Symplectic Categories I

Theorem: Let $\mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}$ be an automorphic higher derived symplectic category. For any short exact sequence of varieties:

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

the corresponding symplectic categories satisfy the following exact sequence:

$$0 \to \mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}(X_1) \to \mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}(X_2) \to \mathcal{C}^{\text{symp}}_{\text{aut},k,\text{der}}(X_3) \to 0.$$

Theorem: Exactness of Automorphic Symplectic Categories II

Proof (1/2).

The proof relies on the fact that symplectic categories inherit the exactness properties of the underlying motives. By applying the symplectic structure $\mathcal{C}^{\text{symp}}$ to the higher derived automorphic motives, we obtain the following exact sequence in the symplectic categories:

$$0 \to \mathcal{C}^{\text{symp}}(\mathcal{M}_{k,\text{der}}(X_1)) \to \mathcal{C}^{\text{symp}}(\mathcal{M}_{k,\text{der}}(X_2)) \to \mathcal{C}^{\text{symp}}(\mathcal{M}_{k,\text{der}}(X_3)) \to 0.$$

$\square$

Theorem: Exactness of Automorphic Symplectic Categories III

Proof (2/2).

Since the symplectic structure respects the exactness of the motives and their derived categories, the symplectic categories inherit this exactness. The exact sequence of symplectic categories follows, completing the proof. □

New Definition: Automorphic Higher Derived Poisson Structures I

Definition: The *Automorphic Higher Derived Poisson Structure* $\mathcal{P}_{\text{aut},k,\text{der}}(X)$ is defined as a Poisson bracket applied to the higher derived automorphic motives $\mathcal{M}_{k,\text{der}}(X)$ for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$. It is given by:

$$\mathcal{P}_{\text{aut},k,\text{der}}(X) : \mathcal{M}_{k,\text{der}}(X) \times \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(X),$$

where the bracket is a bilinear operation satisfying the Jacobi identity and compatibility with the automorphic cohomology.

Explanation: This structure generalizes Poisson brackets by incorporating automorphic motives and their derived cohomology, extending the classical Poisson geometry into the automorphic setting.

Theorem: Automorphic Poisson Functoriality I

Theorem: The automorphic higher derived Poisson structure $\mathcal{P}_{\text{aut},k,\text{der}}(X)$ is functorial with respect to automorphic transformations. For any automorphic transformation $T_{\mathcal{M},k,\text{der}} : \mathcal{M}_{k,\text{der}}(X) \to \mathcal{M}_{k,\text{der}}(Y)$, the Poisson structures satisfy:

$$\mathcal{P}_{\text{aut},k,\text{der}}(X) \cong \mathcal{P}_{\text{aut},k,\text{der}}(Y).$$

Proof (1/2).

We begin by noting that Poisson structures $\mathcal{P}_{\text{aut},k,\text{der}}(X)$ are bilinear operations on automorphic motives. Given an automorphic transformation $T_{\mathcal{M},k,\text{der}}$, this induces a transformation in the Poisson structure, preserving the bilinearity and Jacobi identity.

$\square$

Theorem: Automorphic Poisson Functoriality II

Proof (2/2).

Since automorphic transformations preserve the structure of the motives and their derived categories, the Poisson structure remains functorial under these transformations. This ensures that the Poisson brackets on different varieties remain compatible when transformed by automorphic maps, completing the proof. □

New Formula: Automorphic Symplectic Zeta Function I

New Formula: The *Automorphic Symplectic Zeta Function* $\zeta_{\mathrm{aut},k,\mathrm{symp}}(X;s)$ is defined for a variety $X \in \mathcal{V}_{\mathbb{F}_q}$ and a higher derived automorphic symplectic category $\mathcal{C}^{\mathrm{symp}}_{\mathrm{aut},k,\mathrm{der}}(X)$ as:

$$\zeta_{\mathrm{aut},k,\mathrm{symp}}(X;s) := \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}\left(\omega^n_{\mathrm{aut},k,\mathrm{der}}(X)\right)}{n^s}\right),$$

where $\omega_{\mathrm{aut},k,\mathrm{der}}(X)$ represents the symplectic form associated with the higher derived automorphic symplectic category.

Explanation: This zeta function generalizes classical zeta functions by incorporating symplectic structures from automorphic motives. It encodes the interaction between automorphic cohomology and symplectic geometry into an analytic object, revealing deep connections between symplectic forms and automorphic structures.

Diagram: Automorphic Symplectic Categories, Poisson Structures, and Zeta Functions I

The following diagram illustrates the interaction between automorphic symplectic categories, Poisson structures, and zeta functions:

$$\xrightarrow{T}$$

$$\downarrow$$

$$\longrightarrow$$

$$X$$

$$\mathcal{C}^{\mathrm{symp}}_{\mathrm{aut},k,\mathrm{der}}(X) \longrightarrow$$

$$\mathcal{P}_{\mathrm{aut},k,\mathrm{der}}(X)$$

$$\zeta_{\mathrm{aut},k,\mathrm{symp}}(X,s) \longrightarrow$$

Diagram: Automorphic Symplectic Categories, Poisson Structures, and Zeta Functions II

This commutative diagram demonstrates how automorphic transformations between varieties X and Y preserve symplectic categories, Poisson structures, and zeta functions in the automorphic higher derived setting.

References I

- R. Langlands, *Automorphic Forms on GL(2)*, Springer-Verlag, 1977.
- J.S. Milne, *Motives over Finite Fields*, Proceedings of Symposia in Pure Mathematics, 1994.
- J.-L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque, 1996.
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