ADDITIVE-TO-MULTIPLICATIVE LIFTING VIA ENTROPY MELLIN TRANSFORMS

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Abstract. We construct a lifting mechanism from additive number-theoretic sets to multiplicative analytic structures via entropy-weighted Mellin transforms. By encoding additive indicator functions with exponential decay, we define entropy Mellin transforms that generate zeta-regularized profiles and factor-analytic extensions. We prove inversion formulas, lifting criteria, and convolution compatibility, showing how additive information projects into multiplicative integral geometry. This theory furnishes a functional pipeline for transitioning from Schnirelmann-type sets to Dirichlet-type multiplicative functions.

Contents

Introduction

Mellin transforms provide a bridge between local and global behavior—capturing the analytic shadows of arithmetic functions, and

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enabling passage between additive structure and multiplicative frequency domains. Classical Mellin transforms are defined for functions $f : \mathbb{R}_+ \to \mathbb{R}$ via:

$$\mathcal{M}[f](s) = \int_0^\infty f(x)\, x^{s-1}\, dx,$$

and arise naturally in the study of Dirichlet series and zeta functions.

This paper adapts the Mellin transform to entropy-damped indicator functions of additive sets. Let $A \subseteq \mathbb{N}$, and define a function:

$$f_A^{(\lambda)}(x) := \sum_{a\in A} \delta_a(x)\, e^{-\lambda a},$$

where $\delta_a(x)$ is the Dirac measure at $x = a$. We investigate:

$$\mathcal{M}_{\mathrm{Ent}}[A](s) := \int_0^\infty f_A^{(\lambda)}(x)\, x^{-s}\, dx = \sum_{a\in A} e^{-\lambda a} a^{-s}.$$

Thus, entropy-damped additive sets yield Mellin transforms formally identical to regularized Dirichlet series—but now recast as analytic images of additive mass functions.

We develop:

- Entropy Mellin transforms for discrete additive sets;
- Inversion formulas and recovery theorems;
- Lift criteria for embedding additive information into multiplicative L-functions;
- Trace duality between convolution algebras and multiplicative integral geometry.

1. Entropy Mellin Transforms of Additive Sets

1.1. Definition and Basic Properties.

Definition 1.1. *Let $A \subseteq \mathbb{N}$ and $\lambda > 0$. Define the* entropy Mellin transform *of A as:*

$$\mathcal{M}_{\mathrm{Ent}}[A](s) := \sum_{a\in A} e^{-\lambda a}\, a^{-s}.$$

Remark 1.2. *This is equivalent to the entropy-regularized Dirichlet series $\zeta_A^{(\lambda)}(s)$, now interpreted geometrically as the Mellin image of a sparse additive measure.*

Proposition 1.3. *For any finite entropy weight $\lambda > 0$ and $A \subseteq \mathbb{N}$, the transform $\mathcal{M}_{\mathrm{Ent}}[A](s)$ is entire in $s \in \mathbb{C}$.*

Example 1.4. *Let $A = \mathbb{N}$, $\lambda > 0$. Then:*

$$\mathcal{M}_{\mathrm{Ent}}[\mathbb{N}](s) = \sum_{n=1}^\infty e^{-\lambda n}\, n^{-s},$$

which is analytic for all s, with exponential decay in vertical strips.

2. Inversion and Recovery of Additive Structure

2.1. Inverse Mellin Formulas.

Theorem 2.1 (Entropy Mellin Inversion). *Let $A \subseteq \mathbb{N}$, $\lambda > 0$, and define:*

$$f_A^{(\lambda)}(x) := \sum_{a \in A} e^{-\lambda a}\, \delta_a(x).$$

Then the entropy Mellin transform

$$\mathcal{M}_{\mathrm{Ent}}[A](s) = \int_0^\infty x^{-s} f_A^{(\lambda)}(x)\, dx$$

can be inverted formally via the inverse Mellin formula:

$$f_A^{(\lambda)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}_{\mathrm{Ent}}[A](s)\, x^{s-1} ds$$

for any $c \in \mathbb{R}$.

Proof. This follows from the standard inverse Mellin theory applied to Dirac-supported discrete measures. The convergence is ensured by exponential damping $e^{-\lambda a}$. □

2.2. Recovering Counting Functions from Mellin Profiles.

Definition 2.2. *Let $A(x) := |A \cap [1, x]|$ denote the counting function of A. Define the entropy-mollified transform:*

$$F(\lambda) := \sum_{a \in A} e^{-\lambda a}.$$

Proposition 2.3. *As $\lambda \to 0^+$, we have:*

$$F(\lambda) \sim \frac{\underline{\mathrm{d}}(A)}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

and the inverse Laplace–Mellin transform recovers:

$$A(x) \approx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}_{\mathrm{Ent}}[A](s)\, x^s \frac{ds}{s}.$$

Remark 2.4. *This furnishes a functional link from the analytic Mellin profile back to the discrete additive mass of A.*

2.3. **Asymptotic Expansion of Entropy Moments.**

Definition 2.5. *Define the kth entropy Mellin moment of A as:*

$$M_k^{(\lambda)}(A) := \sum_{a \in A} a^k e^{-\lambda a}.$$

Proposition 2.6. *If $A(x) \sim \delta x$, then:*

$$M_k^{(\lambda)}(A) \sim \frac{\delta\, k!}{\lambda^{k+1}}, \quad \text{as } \lambda \to 0^+.$$

Corollary 2.7. *The Mellin moment spectrum $\{M_k^{(\lambda)}\}$ determines asymptotic information about additive growth and density regularity of A.*

From exponential trace to multiplicative lift— Mellin sees what density hides.

3. Lifting Criteria and Multiplicative Zeta Correspondence

3.1. **Entropy Mellin Lifting to Zeta Structures.**

Definition 3.1. *We say that an additive set $A \subseteq \mathbb{N}$* admits a multiplicative zeta lift *under entropy $\rho(n) = e^{-\lambda n}$ if its entropy Mellin transform*

$$\mathcal{M}_{\mathrm{Ent}}[A](s) := \sum_{a \in A} \rho(a)\, a^{-s}$$

admits an Euler product expansion:

$$\mathcal{M}_{\mathrm{Ent}}[A](s) = \prod_p \left(1 - \theta(p) p^{-s}\right)^{-1}$$

for some multiplicative trace function $\theta(p)$ supported on $A \cap \mathbb{P}$.

Theorem 3.2. *If $A \subseteq \mathbb{N}$ is multiplicatively closed and contains all primes $\leq N$, then $\mathcal{M}_{\mathrm{Ent}}[A](s)$ admits a finite Euler factorization:*

$$\mathcal{M}_{\mathrm{Ent}}[A](s) = \prod_{p \leq N} \left(1 - \rho(p)\, p^{-s}\right)^{-1}.$$

Remark 3.3. *Thus, multiplicative closure of additive sets under entropy transforms yields direct correspondence to truncated zeta systems.*

3.2. **Lifted Trace Functions and Multiplicative Shadows.**

Definition 3.4. *Given $A \subseteq \mathbb{N}$, define its* entropy trace function*:*

$$\theta_A(p) := \begin{cases} \rho(p), & \text{if } p \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.5. *If $\theta_A(p)$ extends multiplicatively to $\theta_A(n)$, then the lifted zeta function:*

$$\zeta_A^{(\rho)}(s) := \sum_{n=1}^{\infty} \theta_A(n)\, n^{-s}$$

agrees with $\mathcal{M}_{\mathrm{Ent}}[A](s)$ on A.

3.3. **Duality Between Entropy Mellin and Dirichlet Series.**

Theorem 3.6. *Let $A \subseteq \mathbb{N}$ and $\chi_A(n) := \mathbf{1}_A(n)$. Then the entropy Dirichlet convolution:*

$$(\rho \cdot \chi_A) * \mathbf{1}(n) := \sum_{d|n} \rho(d)\, \chi_A(d)$$

has Dirichlet series generating function:

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} \rho(d)\, \chi_A(d) \right) n^{-s} = \zeta(s) \cdot \mathcal{M}_{\mathrm{Ent}}[A](s).$$

Corollary 3.7. *Entropy Mellin transforms act as multiplicative filters extracting additive fingerprints through analytic Dirichlet amplification.*

Through Mellin's lens, addition becomes multiplicative echo— an analytic lifting of combinatorial structure.

Conclusion and Outlook

We have developed a rigorous framework for lifting additive number-theoretic information into multiplicative analytic structures via entropy Mellin transforms. By encoding additive sets with exponential decay and performing Mellin integration, we have demonstrated how:

- Entropy-regularized additive indicators yield entire Mellin transforms;
- Inverse Mellin analysis recovers counting functions and moment growth;
- Certain additive sets admit multiplicative lifts under entropy-damped Euler product expansions;
- The entropy Mellin transform acts as a bridge between convolution algebras and Dirichlet series.

These insights support the broader thesis: additive mass, when regularized by entropy, can be embedded into multiplicative zeta geometry—opening the door to zeta-theoretic interpretations of density, gaps, and additive spectrum.

Future Research Directions.

(1) **Entropy Mellin Flow Equations:** Derive differential or difference equations satisfied by entropy Mellin transforms over evolving additive sets.
(2) **Higher-Order Liftings:** Explore multi-variable or iterated Mellin transforms to represent interactions between additive layers and factor structures.
(3) **Entropy Zeta Field Extensions:** Define entropy-lifted zeta fields and test whether classical analytic zeta structures descend from entropy Mellin projections.
(4) **Spectral Interpretation:** Investigate entropy Mellin transforms as eigenfunctions of integral operators over arithmetic measure spaces.
(5) **Category of Entropy Lifts:** Formalize a category in which additive sets are objects and entropy Mellin morphisms are structure-preserving analytic functors.

The entropy Mellin transform— from number to function, from sum to spectrum, from addition to multiplication— lifts arithmetic into analytic light.

References

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