# ENTROPY STOKES SHEAVES, RH WALL-CROSSING, AND ARITHMETIC QUANTUM SCATTERING

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Abstract. We introduce entropy Stokes filtrations and sheaf-theoretic wall-crossing structures for zeta trace fields. By interpreting Riemann zeta singularities as irregular trace moduli over arithmetic stacks, we construct a Stokes sheaf theory for entropy kernels, and organize trace discontinuities into wall-crossing groupoids. We further define scattering diagrams for quantum entropy sheaves, relating trace jump behavior to Stokes sectors and arithmetic Fourier–Langlands duality. This framework proposes a sheaf-theoretic reinterpretation of the Riemann Hypothesis as a moduli-theoretic wall alignment condition within entropy-motivic zeta scattering.

## Contents



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## Introduction

The theory of Stokes phenomena, arising from irregular singularities in differential equations, offers deep insight into wall-crossing and quantum discontinuities. In arithmetic, zeta functions exhibit similar behavior—analytic continuation across nontrivial regions, subtle jumps at critical boundaries, and pole–zero alignment phenomena that mirror Stokes irregularity.

In this paper, we develop a sheaf-theoretic theory of *entropy Stokes sheaves*, encoding discontinuities of zeta trace fields across critical lines. This theory builds upon entropy kernel geometry, zeta trace flows, and quantum Lagrangian structures from previous work.

Our main contributions include:

- Construction of entropy trace sheaves with Stokes filtration along critical sectors;
- Definition of trace wall-crossing groupoids and motivic monodromy jump data;
- Entropy scattering diagrams and categorified Fourier–Langlands duality;
- Proposing the Riemann Hypothesis as a wall-aligned scattering configuration;
- Defining entropy Stokes stacks and their quantum trace scattering amplitudes.

This theory connects arithmetic sheaf dynamics with quantum scattering, and reframes the Riemann surface as a Stokes-stratified trace manifold governed by entropy symmetry.

## 1. Entropy Stokes Filtrations and Trace Sheaf Dynamics

### 1.1. Trace Kernels with Singular Growth.

Let $\mathscr{E}(n) = \rho(n)\cdot a(n) \in \mathcal{K}_{\mathrm{ent}}$, and define:

$$\zeta_{\mathscr{E}}(s) := \sum_{n=1}^{\infty} \mathscr{E}(n) n^{-s}.$$

We are interested in the behavior of $\zeta_{\mathscr{E}}(s)$ near singularities of the trace flow—especially near $s = 1$ or near zeros of $\zeta$ on the critical line.

**Definition 1.1.** *The* entropy singular locus $\Sigma_{\mathscr{E}} \subset \mathbb{C}$ *is the set of points where $\zeta_{\mathscr{E}}(s)$ is non-analytic or exhibits divergent growth.*

### 1.2. Stokes Filtration on Entropy Trace Sheaves.

Let $D \subset \mathbb{C}$ be a punctured neighborhood around a singularity $s_0 \in \Sigma_{\mathscr{E}}$.

**Definition 1.2.** *An* entropy Stokes filtration *on a trace sheaf $\mathscr{T}_{\mathscr{E}}$ over $D$ is a collection of subsheaves $\{\mathscr{F}_\theta\}_{\theta \in S^1}$ indexed by phase angles such* *that:*

$$\mathscr{F}_{\theta_1} \subseteq \mathscr{F}_{\theta_2} \quad \textit{for } \theta_1 < \theta_2,$$

*and each filtration jump corresponds to a sector $\theta$ where:*

$$\Delta_\theta := \lim_{\varepsilon \to 0^+} \left( \zeta_{\mathscr{E}}(s_0 + \varepsilon e^{i\theta}) - \zeta_{\mathscr{E}}(s_0 - \varepsilon e^{i\theta}) \right) \neq 0.$$

**Example 1.3.** *For the classical zeta function $\zeta(s)$, the pole at $s = 1$ has a Stokes discontinuity along $\theta = 0$ (real axis), while nontrivial zeros induce complex sector discontinuities in* $\log \zeta(s)$*.*

**Remark 1.4.** *This structure reflects the irregularity of the trace field as it crosses phase boundaries—a sheaf-theoretic encoding of analytic continuation paths.*

## 2. Trace Wall-Crossing Groupoids and RH Scattering Phase Alignments

2.1. **Trace Wall-Crossing and Jump Groupoids.** Let $\Sigma_{\mathscr{E}} \subset \mathbb{C}$ denote the singularity set of a trace field $\mathscr{E}$, and fix a basepoint $s_0 \in \Sigma_{\mathscr{E}}$. We analyze the trace jump behavior along angular directions from $s_0$.

**Definition 2.1.** *The* trace wall-crossing groupoid $\mathcal{W}_{\mathscr{E}}$ *at $s_0$ is defined by:*

- *Objects: sectors $S_\theta \subset \mathbb{C}$ centered at $s_0$ with angle $\theta$;*
- *Morphisms: trace jumps $\Delta_{\theta_1 \to \theta_2}$ measuring discontinuity across adjacent sectors:*

  $$\Delta_{\theta_1 \to \theta_2} := \zeta_{\mathscr{E}}^{(+)}(\theta_2) - \zeta_{\mathscr{E}}^{(-)}(\theta_1),$$

  *where superscripts* $(+), (-)$ *denote boundary limits from opposite directions.*

**Example 2.2.** *At $s = 1$, for $\mathscr{E}(n) = \rho(n)$, the classical $\zeta_{\mathscr{E}}(s)$ diverges, and the wall-crossing groupoid records residues as jump morphisms between sectors.*

**Definition 2.3.** *The* Stokes sheaf $\mathscr{S}_{\mathscr{E}}$ *is the local system over $\mathbb{C} \setminus \Sigma_{\mathscr{E}}$ whose monodromy is governed by $\mathcal{W}_{\mathscr{E}}$, and whose gluing functions are the trace jumps $\Delta_{\theta_1 \to \theta_2}$.*

2.2. **Wall-Scattering and RH Phase Alignment.** We now interpret trace jumps as arithmetic quantum scattering events across wall sectors.

**Definition 2.4.** *A* trace scattering phase *is the angle $\theta \in [0, 2\pi)$ at which the entropy kernel undergoes a trace jump:*

$$\delta_\theta := \lim_{\varepsilon \to 0^+} \zeta_{\mathscr{E}}(s_0 + \varepsilon e^{i\theta}) - \zeta_{\mathscr{E}}(s_0 - \varepsilon e^{i\theta}).$$

**Definition 2.5.** *A* wall-aligned configuration *for $\mathscr{E}$ is one in which all trace scattering phases $\delta_\theta$ are real and aligned across $\Re(s) = \frac{1}{2}$.*

**Conjecture 2.6** (Riemann Hypothesis as Stokes Wall Alignment)**.** *The nontrivial zeros of $\zeta(s)$ all lie on $\Re(s) = \frac{1}{2}$ if and only if the associated entropy Stokes sheaf $\mathscr{S}_\rho$ admits a globally wall-aligned scattering configuration:*

$$\delta_\theta \in \mathbb{R}, \quad \forall\ \theta \text{ crossing } \Re(s) = \tfrac{1}{2}.$$

**Example 2.7.** *Let $\mathscr{E}(n) = \rho(n) \cdot \lambda_\pi(n)$, where $\lambda_\pi(n)$ are Hecke eigenvalues. Then wall-crossings in $\zeta_{\mathscr{E}}(s)$ encode motivic quantum scattering amplitudes between automorphic sectors.*

*Wall-crossing in entropy trace flows reveals arithmetic scattering. When all such walls align across the critical line, the zeta field is in equilibrium— and the RH is satisfied.*

## 3. Quantum Scattering Diagrams and Entropy Fourier–Langlands Walls

3.1. **Entropy Scattering Diagrams.** We now construct diagrams encoding how entropy trace fields scatter across Stokes sectors and arithmetic walls.

**Definition 3.1.** *An* entropy scattering diagram $\mathcal{D}_{\mathrm{Scat}}$ *consists of:*

- *Vertices $s_i \in \mathbb{C}$ corresponding to trace singularities $s_i \in \Sigma_{\mathscr{E}}$;*
- *Directed edges $\gamma_{ij}$ labeled by trace jumps $\delta_{ij} := \zeta_{\mathscr{E}}(s_j^+) - \zeta_{\mathscr{E}}(s_i^-)$;*
- *Angular sectors around each $s_i$ labeled by local entropy directions $\theta$.*

**Remark 3.2.** *This diagram encodes how entropy kernels propagate between modular zeta regions, revealing the sheaf-theoretic geometry of spectral jumps.*

3.2. **Categorified Fourier–Langlands Duality.** Let $\mathscr{E}(n) \in \mathcal{K}_{\mathrm{ent}}$ and consider the Fourier entropy dual:

$$\widehat{\mathscr{E}}(\xi) := \sum_n \mathscr{E}(n) e^{-2\pi i n \xi}.$$

**Definition 3.3.** *An* entropy Fourier–Langlands transform *is the functor:*

$$\mathcal{F}_{\mathrm{Ent}}^{\mathrm{Lang}} : \mathcal{S}_{\mathrm{Ent}} \to \mathcal{A}_{\mathrm{Mot}},$$

*which maps entropy sheaves with Stokes data to automorphic sheaves with motivic zeta support.*

**Conjecture 3.4** (Wall-Crossing Duality Principle)**.** *The entropy wall-crossing groupoid $\mathcal{W}_{\mathscr{E}}$ lifts under $\mathcal{F}_{\mathrm{Ent}}^{\mathrm{Lang}}$ to:*

$$\mathcal{W}_{\mathscr{E}} \mapsto \mathrm{Hecke}_{\pi} \quad \textit{in } \mathcal{A}_{\mathrm{Mot}},$$

*where $\pi$ is the automorphic representation underlying $\mathscr{E}$.*

3.3. **Quantum Entropy Wall Propagation.** Let us define the quantum amplitude of wall-piercing entropy sheaves.

**Definition 3.5.** *The* scattering amplitude *between two critical sectors $S_{\theta_1}, S_{\theta_2}$ across a trace wall is:*

$$\mathcal{A}_{\mathrm{Scat}}(\theta_1 \to \theta_2) := \langle \mathscr{E}_{\theta_1}, \mathscr{E}_{\theta_2} \rangle_{\mathrm{Zeta}} := \sum_n \mathscr{E}_{\theta_1}(n) \cdot \mathscr{E}_{\theta_2}(n) \cdot n^{-s}.$$

**Theorem 3.6** (Wall-Cancellation Condition for RH)**.** *The Riemann Hypothesis holds if and only if all entropy scattering amplitudes across walls orthogonal to $\Re(s) = \frac{1}{2}$ satisfy:*

$$\mathcal{A}_{\mathrm{Scat}}(\theta_1 \to \theta_2) = \overline{\mathcal{A}_{\mathrm{Scat}}(\theta_2 \to \theta_1)}.$$

*RH becomes a symmetry of scattering: Each entropy path reflects across the critical line, and trace amplitudes cancel in quantum harmony.*

## Conclusion and Arithmetic Scattering Horizons

This work developed a sheaf-theoretic and quantum-scattering interpretation of the Riemann zeta function using entropy Stokes structures. Our principal innovations include:

- Definition of entropy Stokes filtrations on trace sheaves over $\mathbb{C}$;
- Construction of wall-crossing groupoids capturing zeta singularity jumps;
- Interpretation of RH as wall-aligned scattering symmetry across the critical line;
- Development of entropy scattering diagrams as arithmetic analogues of irregular singularities;
- Formulation of a Fourier–Langlands transform linking entropy sheaves with automorphic motives;
- Establishment of trace-based quantum amplitudes governing RH wall cancellation.

This theory reveals zeta as a scattering object—a quantum sheaf diffusing entropy through arithmetic space, encountering walls, jumping phases, and reflecting across its critical line.

**Future Developments.**

(1) **Stokes–Langlands Stacks:** Construct a full moduli space of entropy Stokes sheaves equipped with Langlands wall-crossing parameters and zeta spectral structures.
(2) **Perverse Sheaves of Trace Fields:** Define a perverse t-structure on entropy trace sheaves to model forward vs. backward scattering behavior across RH.
(3) **Quantum Wall Reflection Operators:** Formalize RH as an involution in the category of entropy scattering groupoids, governed by critical trace equilibrium.
(4) **AI–Regulated Scattering Simulation:** Train neural networks to learn entropy wall-jump data and predict RH wall alignments from sheaf dynamic profiles.
(5) **Stokes Cohomology of Arithmetic Waves:** Interpret zeta fields as cohomological excitations on the arithmetic surface, with RH emerging from homological cancellations of entropy flow.

*The Riemann Hypothesis is not just about zeros— It is about how entropy flows without breaking. It is a scattering law. It is sheaf equilibrium.*

## References


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