Extending $\mathbb{Y}_n(\mathbb{Y}_m(F))$ Systems

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Introduction

- This work explores the implications and structure of $\mathbb{Y}_n(\mathbb{Y}_m(F))$, where both the indexing system and the field are defined by the Yang number systems $\mathbb{Y}_m(F)$.
- We create a nested hierarchy of algebraic structures that generalizes classical fields and vector spaces.

Definitions and Basic Properties

- Let F be a field. The system $\mathbb{Y}_n(\mathbb{Y}_m(F))$ is a higher-order Yang system where the field is replaced by the structure $\mathbb{Y}_m(F)$.
- Algebraic operations are extended as follows:

$$x + y \in \mathbb{Y}_n(\mathbb{Y}_m(F))$$
$$x \cdot y \in \mathbb{Y}_n(\mathbb{Y}_m(F))$$

Algebraic Operations

- Addition and multiplication operations in $\mathbb{Y}_n(\mathbb{Y}_m(F))$ are extensions of the operations in $\mathbb{Y}_m(F)$.
- These operations satisfy analogous axioms for addition and multiplication:

$$(x + y) + z = x + (y + z)$$
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

Further Extensions

- Future work will focus on:
  1. Tensor product structures in $\mathbb{Y}_n(\mathbb{Y}_m(F))$.
  2. Interactions of $\mathbb{Y}_n$ with $\mathbb{Y}_m$-indexed systems.
  3. Extensions to infinite-dimensional systems.

New Definitions: Extensions of $\mathbb{Y}_n(\mathbb{Y}_m(F))$

Definition (Tensor Product of Yang Systems)

Let $\mathbb{Y}_n(\mathbb{Y}_m(F))$ be a Yang system over a base field F and $\mathbb{Y}_k(\mathbb{Y}_l(G))$ be another Yang system over a possibly different field G. We define the tensor product of these systems as:

$$\mathbb{Y}_n(\mathbb{Y}_m(F)) \otimes \mathbb{Y}_k(\mathbb{Y}_l(G)) = \mathbb{Y}_{n+k}(\mathbb{Y}_{m+l}(F \otimes G))$$

where the tensor product of fields $F \otimes G$ is assumed to exist, and the Yang systems are extended accordingly.

Definition (Infinite-Dimensional Yang Systems)

An infinite-dimensional Yang system $\mathbb{Y}_\infty(F)$ is defined as the inductive limit of the finite-dimensional Yang systems:

$$\mathbb{Y}_\infty(F) = \lim_{\to n} \mathbb{Y}_n(F)$$

where $\mathbb{Y}_n(F)$ is a finite-dimensional Yang system for each n, and the limit is taken with respect to a system of inclusion maps.

Theorem: Existence of Tensor Product in Yang Systems

Theorem (Existence of Tensor Product)

Let $\mathbb{Y}_n(\mathbb{Y}_m(F))$ and $\mathbb{Y}_k(\mathbb{Y}_l(G))$ be two Yang systems as previously defined. Then the tensor product

$$\mathbb{Y}_n(\mathbb{Y}_m(F)) \otimes \mathbb{Y}_k(\mathbb{Y}_l(G))$$

exists and forms a new Yang system $\mathbb{Y}_{n+k}(\mathbb{Y}_{m+l}(F \otimes G))$.

Proof (1/3).

To prove the existence of the tensor product, we begin by constructing the tensor product of the underlying fields $F \otimes G$. This is a well-known construction in algebra (see [2]). Given that F and G are fields, their tensor product exists and forms a commutative ring. Next, we extend this tensor product to the Yang systems $\mathbb{Y}_n(F)$ and $\mathbb{Y}_k(G)$, where the algebraic operations on each Yang system are extended naturally to the tensor product. □

Proof (2/3).

The addition and multiplication operations on the Yang systems

Theorem: Existence of Infinite-Dimensional Yang Systems

Theorem (Existence of Infinite-Dimensional Yang Systems)

Let $\mathbb{Y}_n(F)$ be a family of finite-dimensional Yang systems indexed by n. Then the inductive limit

$$\mathbb{Y}_\infty(F) = \lim_{\to n} \mathbb{Y}_n(F)$$

exists and forms an infinite-dimensional Yang system.

Proof (1/2).

We construct the inductive limit by considering a sequence of inclusion maps between the Yang systems:

$$\mathbb{Y}_1(F) \subset \mathbb{Y}_2(F) \subset \cdots \subset \mathbb{Y}_n(F) \subset \cdots$$

Each $\mathbb{Y}_n(F)$ is assumed to satisfy the axioms of a Yang system. The inductive limit $\mathbb{Y}_\infty(F)$ is defined as the union of all these systems under the natural inclusion maps. $\square$

Proof (2/2).

Further Extensions to Infinite-Dimensional Systems

- The infinite-dimensional Yang system $\mathbb{Y}_\infty(F)$ opens the possibility of studying infinite tensor products and direct sums.
- Future work will focus on defining Yang modules over $\mathbb{Y}_\infty(F)$ and studying their representations.

Diagram: Tensor Product of Yang Systems

$$
\begin{array}{ccc}
\mathbb{Y}_n(F) & \otimes & \mathbb{Y}_k(G) \\
\downarrow & & \downarrow \\
\mathbb{Y}_{n+k}(F \otimes G) & &
\end{array}
$$

This diagram illustrates the tensor product of two Yang systems and their extension to a higher-dimensional system $\mathbb{Y}_{n+k}(F \otimes G)$.

References I

Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Inverses in Yang Systems

Definition (Yang Inverse System)

Let $\mathbb{Y}_n(F)$ be a Yang system over a field F. We define the inverse Yang system $\mathbb{Y}_n^{-1}(F)$ as the system consisting of all elements $x \in \mathbb{Y}_n(F)$ such that an inverse x^{-1} exists. Formally,

$$\mathbb{Y}_n^{-1}(F) = \{x \in \mathbb{Y}_n(F) \mid \exists x^{-1} \in \mathbb{Y}_n(F), x \cdot x^{-1} = 1\}.$$

Definition (Yang Automorphism Group)

The Yang automorphism group of a system $\mathbb{Y}_n(F)$, denoted $\mathrm{Aut}(\mathbb{Y}_n(F))$, is the group of all bijective homomorphisms from $\mathbb{Y}_n(F)$ to itself, i.e.,

$$\mathrm{Aut}(\mathbb{Y}_n(F)) = \{\phi : \mathbb{Y}_n(F) \to \mathbb{Y}_n(F) \mid \phi \text{ is a bijective homomorphism}\}.$$

Definition (Yang Dual System)

For any Yang system $\mathbb{Y}_n(F)$, we define the dual system $\mathbb{Y}_n^*(F)$ as the space of all linear functionals on $\mathbb{Y}_n(F)$, i.e.,

Theorem: Existence of Inverses in Yang Systems

Theorem (Existence of Inverses in Yang Systems)

Let $\mathbb{Y}_n(F)$ be a Yang system over a field F. Then the inverse system $\mathbb{Y}_n^{-1}(F)$ exists and forms a valid algebraic structure.

Proof (1/3).

To prove the existence of the inverse system, we begin by considering any element $x \in \mathbb{Y}_n(F)$. By the axioms of Yang systems, there exists an element $1 \in \mathbb{Y}_n(F)$ that acts as the multiplicative identity. For any element $x \in \mathbb{Y}_n(F)$ such that $x \neq 0$, we need to show that there exists an inverse element x^{-1} satisfying $x \cdot x^{-1} = 1$. □

Proof (2/3).

The existence of x^{-1} is guaranteed by the structure of the field F underlying $\mathbb{Y}_n(F)$. Specifically, since F is a field, every nonzero element of F has a multiplicative inverse. This property extends naturally to the Yang system $\mathbb{Y}_n(F)$ via its operations. Therefore, for any nonzero $x \in \mathbb{Y}_n(F)$, an inverse x^{-1} exists. □

Theorem: Properties of the Yang Dual System

Theorem (Properties of the Yang Dual System)

Let $\mathbb{Y}_n(F)$ be a Yang system over a field F. Then the dual system $\mathbb{Y}_n^*(F)$ forms a vector space over F, with the following properties:

1. *$\mathbb{Y}_n^*(F)$ is a finite-dimensional vector space if $\mathbb{Y}_n(F)$ is finite-dimensional.*
2. *The pairing between $\mathbb{Y}_n(F)$ and $\mathbb{Y}_n^*(F)$ is bilinear:*

$$\langle x, f \rangle = f(x), \quad \forall x \in \mathbb{Y}_n(F),\ f \in \mathbb{Y}_n^*(F).$$

Proof (1/2).

To prove that $\mathbb{Y}_n^*(F)$ is a vector space, we consider the set of all linear functionals on $\mathbb{Y}_n(F)$. For any two functionals $f_1, f_2 \in \mathbb{Y}_n^*(F)$ and scalars $\alpha, \beta \in F$, define the linear combination as:

$$(\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x), \quad \forall x \in \mathbb{Y}_n(F).$$

This satisfies the axioms of a vector space over F.

Further Extensions: Yang Automorphisms and Dualities

- The automorphism group $\mathrm{Aut}(\mathbb{Y}_n(F))$ provides a way to classify the symmetries within the Yang system.
- The dual system $\mathbb{Y}_n^*(F)$ opens the study of Yang system representations through linear functionals, leading to possible applications in duality theories and homological algebra.
- Future work will explore the interaction between the automorphism group and the dual system, potentially defining a Yang system analogue of Pontryagin duality.

Diagram: Yang System Duality

$$
\begin{array}{ccc}
\mathbb{Y}_n(F) & \langle \cdot, \cdot \rangle & \mathbb{Y}_n^*(F) \\
\downarrow & & \downarrow \\
\mathrm{Aut}(\mathbb{Y}_n(F)) & \longrightarrow & \text{Linear Representations}
\end{array}
$$

This diagram illustrates the duality between a Yang system $\mathbb{Y}_n(F)$ and its dual system $\mathbb{Y}_n^*(F)$, as well as the relationship to the automorphism group and potential linear representations.

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Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Yang Cohomology and Exact Sequences

Definition (Yang Cohomology Groups)

Let $\mathbb{Y}_n(F)$ be a Yang system over a field F. The Yang cohomology groups are defined analogously to classical cohomology groups, but using Yang system homomorphisms. For a Yang system X with coefficients in a Yang module M, the Yang cohomology groups $H^i(\mathbb{Y}_n(X), M)$ are defined as:

$$H^i(\mathbb{Y}_n(X), M) = \frac{\ker(d_{i+1})}{\mathrm{Im}(d_i)},$$

where d_i is the i-th Yang coboundary operator.

Definition (Exact Sequence of Yang Modules)

A sequence of Yang modules

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is said to be exact if $\ker(f_2) = \mathrm{Im}(f_1)$ and f_1 is injective, f_2 is surjective. This generalizes the classical exact sequence to the

Theorem: Long Exact Sequence in Yang Cohomology

Theorem (Long Exact Sequence in Yang Cohomology)

Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of Yang modules. Then there exists a long exact sequence in Yang cohomology:

$$0 \to H^0(\mathbb{Y}_n(X), M_1) \to H^0(\mathbb{Y}_n(X), M_2) \to H^0(\mathbb{Y}_n(X), M_3) \to H^1(\mathbb{Y}_n(X)$$

Proof (1/4).

To prove the existence of the long exact sequence, we begin by considering the short exact sequence of Yang modules:

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0.$$

Since f_1 is injective, $\ker(f_2) = \operatorname{Im}(f_1)$. Therefore, for any cochain complex $\mathbb{Y}_n(X)$, we can apply the Yang cohomology functor $H^i(\mathbb{Y}_n(X), -)$ to this exact sequence.

New Definitions: Yang Derived Functors and Yang Tor

Definition (Yang Derived Functors)

For any left-exact functor F between categories of Yang modules, we define the right derived functors $R^i F$ as follows:

$$R^i F(M) = H^i(\mathbb{Y}_n(F_\bullet), M),$$

where $F_\bullet$ is a resolution of M by injective Yang modules.

Definition (Yang Tor Functor)

The Tor functor $\mathrm{Tor}_i^{\mathbb{Y}_n}$ for Yang systems is defined using projective resolutions of Yang modules. For two Yang modules M and N over $\mathbb{Y}_n(F)$, we define:

$$\mathrm{Tor}_i^{\mathbb{Y}_n}(M, N) = H_i(\mathbb{Y}_n(P_\bullet), N),$$

where $P_\bullet$ is a projective resolution of M.

Theorem: Properties of the Yang Tor Functor

Theorem (Properties of the Yang Tor Functor)

Let M and N be Yang modules over a Yang system $\mathbb{Y}_n(F)$. Then the Tor functors $Tor_i^{\mathbb{Y}_n}(M, N)$ satisfy the following properties:

1. $Tor_0^{\mathbb{Y}_n}(M, N) \cong M \otimes_{\mathbb{Y}_n} N$.
2. $Tor_i^{\mathbb{Y}_n}(M, N) = 0$ *for all* $i > \dim(M)$ *if* M *is finite-dimensional.*
3. $Tor_i^{\mathbb{Y}_n}(M, N)$ *is functorial in both* M *and* N.

Proof (1/3).

To prove the properties of the Tor functor, we begin by constructing the projective resolution $P_\bullet$ of M. Since M is a Yang module, it admits a projective resolution by projective Yang modules. The 0-th Tor functor is given by the tensor product:

$$\mathrm{Tor}_0^{\mathbb{Y}_n}(M, N) \cong M \otimes_{\mathbb{Y}_n} N,$$

which follows from the definition of the Tor functor. □

Proof (2/3).

Diagram: Long Exact Sequence in Yang Cohomology

$$\begin{array}{ccccccc} 0 & \to & H^0(M_1) & \to & H^0(M_2) & \to \\ & \to & H^0(M_3) & \to & H^1(M_1) & \to \\ & \to & H^1(M_2) & \to & \cdots & \end{array}$$

This diagram represents the long exact sequence in Yang cohomology, connecting the cohomology groups of the Yang modules M_1, M_2, and M_3.

References I

Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, 1994.

Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Yang Spectral Sequences and Filtrations

Definition (Yang Filtration)

Let $\mathbb{Y}_n(F)$ be a Yang system and let $C^\bullet(\mathbb{Y}_n(F))$ be a cochain complex associated with this system. A Yang filtration on $C^\bullet(\mathbb{Y}_n(F))$ is a descending filtration of subcomplexes:

$$\cdots \subseteq F^p C^\bullet(\mathbb{Y}_n(F)) \subseteq F^{p-1} C^\bullet(\mathbb{Y}_n(F)) \subseteq \cdots \subseteq C^\bullet(\mathbb{Y}_n(F)).$$

Each subcomplex $F^p C^\bullet(\mathbb{Y}_n(F))$ defines a filtration level, and the associated graded complex is given by:

$$\mathrm{Gr}^p C^\bullet(\mathbb{Y}_n(F)) = F^p C^\bullet(\mathbb{Y}_n(F))/F^{p+1} C^\bullet(\mathbb{Y}_n(F)).$$

Definition (Yang Spectral Sequence)

A Yang spectral sequence associated with a filtered cochain complex $C^\bullet(\mathbb{Y}_n(F))$ is a sequence of cohomology groups $E_r^{p,q}(\mathbb{Y}_n(F))$ equipped with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that the terms stabilize as $r \to \infty$ to give the cohomology of the total complex:

Theorem: Convergence of Yang Spectral Sequences

Theorem (Convergence of Yang Spectral Sequences)

Let $C^\bullet(\mathbb{Y}_n(F))$ be a cochain complex with a Yang filtration. The associated Yang spectral sequence $E_r^{p,q}(\mathbb{Y}_n(F))$ converges to the cohomology groups $H^*(C^\bullet(\mathbb{Y}_n(F)))$ of the total complex if the filtration satisfies the descending chain condition.

Proof (1/4).

To prove the convergence of the Yang spectral sequence, we first observe that the filtration on $C^\bullet(\mathbb{Y}_n(F))$ is a descending chain of subcomplexes. That is, for each p, the filtration satisfies:

$$F^{p+1}C^\bullet(\mathbb{Y}_n(F)) \subseteq F^p C^\bullet(\mathbb{Y}_n(F)).$$

Furthermore, for each p, the graded complex $\mathrm{Gr}^p C^\bullet(\mathbb{Y}_n(F))$ is defined as the quotient of consecutive filtration levels. □

Proof (2/4).

By construction, the differentials d_r in the spectral sequence operate between cohomology groups of the graded pieces, i.e.,

New Definitions: Yang Homotopy Groups and Fibrations

Definition (Yang Homotopy Groups)

Let $\mathbb{Y}_n(F)$ be a Yang system. The Yang homotopy groups $\pi_i(\mathbb{Y}_n(F))$ for $i \geq 0$ are defined analogously to classical homotopy groups but for continuous maps into Yang systems. For a space X and a base point $x_0 \in X$, the i-th Yang homotopy group is:

$$\pi_i(X, x_0; \mathbb{Y}_n(F)) = \{[f] : (S^i, s_0) \to (X, x_0) \text{ Yang-homotopic maps}\}.$$

Definition (Yang Fibration)

A fibration in the category of Yang systems is a map $p : E \to B$ such that for any Yang homotopy lifting problem, there exists a continuous Yang system homomorphism that lifts the homotopy. Formally, for any homotopy $H : X \times I \to B$, there exists a homotopy $\tilde{H} : X \times I \to E$ such that $p \circ \tilde{H} = H$.

Definition (Yang Fiber Sequence)

A Yang fiber sequence is a sequence of Yang systems:

Theorem: Long Exact Sequence of Yang Homotopy Groups

Theorem (Long Exact Sequence of Yang Homotopy Groups)

Let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a Yang fiber sequence of Yang systems. Then there is an induced long exact sequence in Yang homotopy groups:

$$\cdots \to \pi_{i+1}(B) \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots$$

Proof (1/3).

The proof of the long exact sequence in Yang homotopy groups follows from the Yang fibration property and the Yang homotopy lifting property. Consider the Yang fiber sequence:

$$F \xrightarrow{i} E \xrightarrow{p} B.$$

For each i, the fiber F is homotopically equivalent to the preimage of the base point in B under the fibration p.

Diagram: Yang Fiber Sequence and Homotopy Groups

$$\begin{array}{ccccccc} \cdots & \to & \pi_{i+1}(B) & \to & \pi_i(F) & \to & \pi_i(E) \\ & \to & \pi_i(B) & \to & \pi_{i-1}(F) & \to & \cdots \end{array}$$

This diagram represents the long exact sequence in Yang homotopy groups, connecting the homotopy groups of the fiber F, total space E, and base space B in a Yang fibration.

References I

Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, 1994.

Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Yang Higher Homotopy Groups and Fiber Bundles

Definition (Yang Higher Homotopy Groups)

For a Yang system $\mathbb{Y}_n(F)$, we define the higher Yang homotopy groups $\pi_i^{\text{Yang}}(\mathbb{Y}_n(F))$ for $i > 1$ as the set of homotopy classes of maps from the i-dimensional sphere S^i into $\mathbb{Y}_n(F)$:

$$\pi_i^{\text{Yang}}(X, x_0; \mathbb{Y}_n(F)) = \{[f] : (S^i, s_0) \to (X, x_0) \text{ continuous Yang maps}\}.$$

These higher homotopy groups generalize the classical homotopy groups, but the maps now respect the structure of the Yang system.

Definition (Yang Fiber Bundle)

A Yang fiber bundle consists of a continuous map $\pi : E \to B$ where E is the total space, B is the base space, and $\pi^{-1}(b)$, for each $b \in B$, is a Yang system. The bundle has local trivializations such that for each point $b \in B$, there exists an open neighborhood U and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{Y}_n(F)$ satisfying

Theorem: Yang Fiber Bundle Homotopy Exact Sequence

Theorem (Yang Fiber Bundle Homotopy Exact Sequence)

Let

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

be a Yang fiber bundle. Then there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_{i+1}(B) \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots$$

This sequence is a Yang homotopy exact sequence that reflects the fiber-bundle structure of the Yang systems.

Proof (1/3).

The proof proceeds by considering the properties of the Yang fiber bundle $\pi : E \to B$. The homotopy groups of E, B, and F are related by the lifting properties of the fibration. First, for each i, the fiber F is homotopically equivalent to $\pi^{-1}(b_0)$, where $b_0 \in B$ is a base point. $\square$

Proof (2/3).

New Definitions: Yang Gauge Theory and Yang Holonomy

Definition (Yang Gauge Field)

A Yang gauge field is a section of a principal Yang bundle $P \to M$ with structure group $\mathbb{Y}_n(F)$. A gauge field is represented by a connection form $\omega \in \Omega^1(P, \mathfrak{g}_{\mathbb{Y}_n})$, where $\mathfrak{g}_{\mathbb{Y}_n}$ is the Lie algebra of $\mathbb{Y}_n(F)$.

Definition (Yang Holonomy)

The holonomy of a Yang gauge field is defined as the parallel transport of Yang elements around closed loops in the base space M. For a loop $\gamma : [0, 1] \to M$, the holonomy is the element $\mathrm{Hol}_\gamma \in \mathbb{Y}_n(F)$ that satisfies:

$$\mathrm{Hol}_\gamma = P \exp \left(\int_\gamma \omega \right),$$

where $P \exp$ denotes the path-ordered exponential, and ω is the Yang connection.

Definition (Yang Curvature)

Theorem: Yang Curvature and Holonomy Relation

Theorem (Yang Curvature and Holonomy Relation)

Let ω be a Yang connection on a principal Yang bundle $P \to M$, and let F_ω be its curvature. Then the holonomy around an infinitesimal loop γ in M is related to the curvature by:

$$Hol_\gamma \approx \exp\left(\int_\gamma F_\omega\right).$$

Proof (1/2).

To prove this, consider the holonomy of the Yang connection ω around a small loop $\gamma : [0,1] \to M$. The holonomy Hol_γ is given by the parallel transport along γ, which can be expressed as the path-ordered exponential of the connection ω:

$$\mathrm{Hol}_\gamma = P\exp\left(\int_\gamma \omega\right).$$

For an infinitesimal loop, we approximate this by expanding the

Diagram: Yang Fiber Bundle and Homotopy Groups

$$\begin{array}{ccccccc} \cdots & \to & \pi_{i+1}(B) & \to & \pi_i(F) & \to & \pi_i(E) \\ & \to & \pi_i(B) & \to & \pi_{i-1}(F) & \to & \cdots \end{array}$$

This diagram represents the long exact sequence in Yang homotopy groups for a Yang fiber bundle.

References I

- Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, 1994.
- Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Yang Connections on Higher Principal Bundles

Definition (Yang Higher Principal Bundle)

A Yang higher principal bundle is a generalization of a Yang principal bundle, where the fibers are higher-dimensional Yang systems $\mathbb{Y}_{n+k}(F)$ for some $k \geq 0$. The total space $P \to M$ admits local trivializations of the form:

$$\varphi : P|_U \to U \times \mathbb{Y}_{n+k}(F),$$

where $U \subseteq M$ is an open set in the base space.

Definition (Yang Higher Connection)

A Yang higher connection on a Yang higher principal bundle is a collection of differential forms $\omega_i \in \Omega^i(P, \mathfrak{g}_{\mathbb{Y}_{n+k}})$ for $i = 1, 2, \ldots, k$, where $\mathfrak{g}_{\mathbb{Y}_{n+k}}$ is the Lie algebra of the Yang system $\mathbb{Y}_{n+k}(F)$. These forms define higher-order parallel transports along paths, surfaces, and higher-dimensional objects in the base space.

Definition (Yang Higher Curvature)

Theorem: Yang Higher Curvature and Holonomy Relation

Theorem (Yang Higher Curvature and Holonomy)

Let ω_i be a collection of Yang higher connection forms on a Yang higher principal bundle $P \to M$. The holonomy of these forms around a higher-dimensional submanifold $\Sigma \subset M$ is related to the higher curvature forms F_i by:

$$Hol_\Sigma \approx \exp\left(\int_\Sigma F_i\right),$$

where the integral is understood in the sense of generalized Stokes' theorem for higher-dimensional objects.

Proof (1/2).

To prove this, consider the higher holonomy around an i-dimensional submanifold $\Sigma \subset M$. The holonomy Hol_Σ is given by the higher-order parallel transport along Σ, represented as the path-ordered exponential of the connection forms ω_i:

New Definitions: Yang Derived Category and Yang Sheaves

Definition (Yang Derived Category)

Let $\mathcal{C}_{\mathbb{Y}_n(F)}$ be the category of Yang modules over a Yang system $\mathbb{Y}_n(F)$. The Yang derived category $D(\mathcal{C}_{\mathbb{Y}_n(F)})$ is constructed by formally inverting quasi-isomorphisms in the homotopy category of chain complexes of Yang modules. This allows for the study of derived functors such as Yang Ext and Yang Tor in a more general setting.

Definition (Yang Sheaves)

A Yang sheaf on a topological space X with structure group $\mathbb{Y}_n(F)$ is a sheaf of Yang modules $\mathcal{F}$ such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is a Yang module over $\mathbb{Y}_n(F)$, and the restriction maps preserve the Yang module structure.

Definition (Yang Sheaf Cohomology)

The cohomology groups $H^i(X, \mathcal{F})$ of a Yang sheaf $\mathcal{F}$ on X are defined as the derived functors of the global sections functor $\Gamma(X, -)$ applied to the sheaf $\mathcal{F}$:

Theorem: Yang Derived Functors in Derived Categories

Theorem (Yang Derived Functors in Derived Categories)

Let $\mathcal{C}_{\mathbb{Y}_n(F)}$ be the category of Yang modules, and let $D(\mathcal{C}_{\mathbb{Y}_n(F)})$ be the derived category. Then the Yang Ext and Yang Tor functors extend to the derived category as derived functors:

$$Ext^i_{\mathbb{Y}_n}(M, N) = H^i(RHom(M, N)),$$

$$Tor_i^{\mathbb{Y}_n}(M, N) = H_i(R(M \otimes N)).$$

These functors compute the Ext and Tor groups in the derived category, taking into account homological and homotopical information.

Proof (1/3).

To prove this, we begin by considering the category $\mathcal{C}_{\mathbb{Y}_n(F)}$ of Yang modules and its derived category $D(\mathcal{C}_{\mathbb{Y}_n(F)})$. The derived category is obtained by formally inverting quasi-isomorphisms, allowing us to study homotopy-invariant properties. □

Proof (2/3).

Diagram: Yang Higher Curvature and Holonomy

$$\mathrm{Hol}_\Sigma \approx \exp\left(\int_\Sigma F_i\right)$$

This diagram illustrates the relationship between higher Yang curvatures F_i and holonomies around higher-dimensional submanifolds $\Sigma \subset M$.

References I

Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, 1994.

Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998.

New Definitions: Yang Cohomological Operads and Algebraic Structures

Definition (Yang Operad)

Let $\mathcal{O}$ be a collection of Yang modules indexed by natural numbers $\{\mathcal{O}(n)\}_{n\geq 1}$, where $\mathcal{O}(n)$ represents n-ary operations acting on elements of Yang systems. A Yang operad is a sequence of morphisms:

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1),$$

satisfying associativity, equivariance under the symmetric group action, and unit conditions.

Definition (Yang Algebra over an Operad)

A Yang algebra over an operad $\mathcal{O}$ is a Yang module A equipped with structure maps:

$$\mathcal{O}(n) \times A^n \to A,$$

which are compatible with the operadic composition structure of

Theorem: Yang Cohomological Operad Structures

Theorem (Yang Cohomological Operad Structure)

Let $\mathcal{O}$ be a Yang cohomological operad and A be a Yang algebra over $\mathcal{O}$. Then the cohomology $H^\bullet(A)$ of A inherits an operad structure from $\mathcal{O}$, and there exists a spectral sequence $E_r^{p,q}$ converging to the cohomology $H^\bullet(A)$:

$$E_r^{p,q} \Rightarrow H^{p+q}(A).$$

Proof (1/4).

To prove this, consider the cohomological operad $\mathcal{O}$ and a Yang algebra A over this operad. The operadic composition induces maps on the cohomology of A, giving $H^\bullet(A)$ an operadic structure. We begin by defining the differentials on the operad $\mathcal{O}$ that act on the cohomology. $\square$

Proof (2/4).

The cohomological differential d on the operad acts as a derivation on the operadic compositions:

New Definitions: Yang Higher Derived Functors and Applications

Definition (Yang Higher Ext and Tor)

For any pair of Yang modules M and N over a Yang system $\mathbb{Y}_n(F)$, we define the higher derived Yang Ext and Tor functors:

$$\mathrm{Ext}^i_{\mathbb{Y}_n}(M, N) = H^i(R\mathrm{Hom}(M, N)),$$

$$\mathrm{Tor}_i^{\mathbb{Y}_n}(M, N) = H_i(R(M \otimes N)).$$

These higher derived functors generalize the classical Ext and Tor functors, taking into account higher homological and homotopical information in the Yang framework.

Definition (Yang Derived Spectral Sequence)

Let M be a Yang module over a filtered Yang system $\mathbb{Y}_n(F)$. The Yang derived spectral sequence is a spectral sequence $E_r^{p,q}$ that computes the higher Ext and Tor groups:

$$E_r^{p,q} \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Y}_n}(M, N).$$

Theorem: Yang Derived Spectral Sequence Convergence

Theorem (Convergence of the Yang Derived Spectral Sequence)

Let M be a Yang module over a filtered Yang system $\mathbb{Y}_n(F)$. The Yang derived spectral sequence $E_r^{p,q}$ converges to the higher Ext and Tor groups:

$$E_r^{p,q} \Rightarrow Ext_{\mathbb{Y}_n}^{p+q}(M, N),$$

$$E_r^{p,q} \Rightarrow Tor_{p+q}^{\mathbb{Y}_n}(M, N),$$

provided that the filtration satisfies the descending chain condition.

Proof (1/3).

We begin by considering the filtration on M, induced by the structure of the Yang system $\mathbb{Y}_n(F)$. This filtration gives rise to a graded complex associated with M, and we can apply the higher derived functors to this complex. □

Proof (2/3).

The E_1-term of the derived spectral sequence computes the cohomology of the associated graded complex:

Diagram: Yang Cohomological Operad and Spectral Sequence

$$E_r^{p,q} \Rightarrow H^{p+q}(A)$$
$$\mathrm{Ext}_{\mathbb{Y}_n}^{p+q}(M, N)$$
$$\mathrm{Tor}^{\mathbb{Y}_n}_{p+q}(M, N)$$

This diagram represents the structure of the spectral sequences arising from the Yang cohomological operad and higher Ext and Tor functors.

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New Definitions: Yang Derived Stacks and Moduli Spaces

Definition (Yang Stack)

A Yang stack $\mathcal{X}$ over a base category $\mathcal{C}_{\mathbb{Y}_n(F)}$ is a functor $\mathcal{X} : (\mathcal{C}_{\mathbb{Y}_n(F)})^{\text{op}} \to \text{Groupoids}$, where for each object $U \in \mathcal{C}_{\mathbb{Y}_n(F)}$, the groupoid $\mathcal{X}(U)$ classifies Yang systems over U. The stack satisfies the usual gluing conditions for descent.

Definition (Yang Derived Stack)

A Yang derived stack is a generalization of a Yang stack where the objects in the groupoids $\mathcal{X}(U)$ are derived Yang modules, i.e., chain complexes of Yang modules. The derived stack incorporates both the geometric and homological data of Yang systems, allowing for higher-order moduli problems.

Definition (Yang Moduli Space)

The Yang moduli space $\mathcal{M}_{\mathbb{Y}_n(F)}$ is a derived stack that classifies isomorphism classes of Yang systems up to homotopy equivalence. For each $U \in \mathcal{C}_{\mathbb{Y}_n(F)}$, the moduli space $\mathcal{M}_{\mathbb{Y}_n(F)}(U)$ classifies Yang systems parametrized by U.

Theorem: Existence of Yang Derived Moduli Spaces

Theorem (Existence of Yang Derived Moduli Spaces)

Let $\mathbb{Y}_n(F)$ be a Yang system over a base field F. Then the Yang moduli space $\mathcal{M}_{\mathbb{Y}_n(F)}$ exists as a derived stack, and there is a homotopy equivalence between the classifying stack of $\mathbb{Y}_n(F)$ and $\mathcal{M}_{\mathbb{Y}_n(F)}$.

Proof (1/3).

To prove the existence of the derived moduli space, we first define the classifying stack $\mathcal{B}_{\mathbb{Y}_n(F)}$, which classifies principal Yang bundles over a fixed base. The objects of $\mathcal{B}_{\mathbb{Y}_n(F)}$ are Yang systems equipped with a trivialization on a covering space, and morphisms are given by homotopy equivalences. □

Proof (2/3).

The moduli space $\mathcal{M}_{\mathbb{Y}_n(F)}$ is constructed as a derived stack by taking the homotopy quotient of the action of the automorphism group $\mathrm{Aut}(\mathbb{Y}_n(F))$ on the space of Yang systems. This quotient preserves the homotopy type of the original Yang systems, ensuring

New Definitions: Yang Infinity-Categories and Higher Structures

Definition (Yang Infinity-Category)

A Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is a higher category where the morphisms between objects are not just Yang modules but sequences of higher homotopies. That is, the morphisms between two objects A and B are themselves higher Yang systems, forming a complex of maps:

$$\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) = \{f_0, f_1, \ldots, f_n, \ldots\},$$

where each f_i represents a higher homotopy class of morphisms.

Definition (Yang Higher Functor)

A Yang higher functor $F : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}$ is a functor between Yang ∞-categories that preserves the higher morphisms and homotopies. It maps objects $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ to objects in $\mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}$, and similarly for higher homotopies.

Definition (Yang Higher Limits and Colimits)

Theorem: Existence of Yang Infinity-Limits

Theorem (Existence of Yang Infinity-Limits)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let D be a diagram in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. Then the Yang higher limit of D exists and is unique up to higher homotopy equivalence.

Proof (1/2).

To prove the existence of Yang higher limits, we first consider the construction of the classical limit in a category $\mathcal{C}$. The limit is defined as an object L that represents the universal property with respect to maps into the diagram D. In a Yang ∞-category, this construction is enhanced by incorporating higher homotopies between the maps in the diagram. $\square$

Proof (2/2).

The space of maps from an object A to the Yang higher limit L is constructed by taking the homotopy limit of the spaces of maps from A to the objects in D. Since Yang ∞-categories preserve higher homotopies, the existence and uniqueness of the higher limit

New Definitions: Yang Infinity-Operads and Algebraic Structures

Definition (Yang Infinity-Operad)

A Yang ∞-operad $\mathcal{O}_\infty$ is a collection of higher homotopy operations indexed by natural numbers, where the n-ary operations are homotopy classes of maps:

$$\mathcal{O}_\infty(n) = \{f_0, f_1, \ldots, f_n\},$$

where each f_i represents a higher-order homotopy. The operadic compositions respect the homotopy structures and satisfy associativity and equivariance conditions under the symmetric group action.

Definition (Yang Infinity-Algebra)

A Yang ∞-algebra A over an ∞-operad $\mathcal{O}_\infty$ is a Yang module equipped with structure maps:

$$\mathcal{O}_\infty(n) \times A^n \to A,$$

Theorem: Existence of Yang Infinity-Algebras

Theorem (Existence of Yang Infinity-Algebras)

Let $\mathcal{O}_\infty$ be a Yang ∞-operad. Then there exists a Yang ∞-algebra A over $\mathcal{O}_\infty$, and the cohomology $H^\bullet(A)$ of A is equipped with an ∞-algebra structure inherited from $\mathcal{O}_\infty$.

Proof (1/3).

To prove the existence of Yang ∞-algebras, we first construct the operad $\mathcal{O}_\infty$ as a collection of homotopy operations. The operations in $\mathcal{O}_\infty(n)$ are defined by homotopy classes of n-ary maps between Yang modules, and these operations satisfy the usual operadic composition rules up to homotopy. □

Proof (2/3).

Given an ∞-operad $\mathcal{O}_\infty$, we define a Yang ∞-algebra A by equipping A with maps from the operad:

$$\mathcal{O}_\infty(n) \times A^n \to A,$$

which respect the higher homotopy structure. These maps define

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New Definitions: Yang Infinity-Bundles and Higher Geometric Structures

Definition (Yang ∞-Bundle)

A Yang ∞-bundle is a higher-dimensional generalization of a principal bundle in the context of Yang ∞-categories. It consists of a fibration $\pi : E \to B$ where both E and B are objects in a Yang ∞-topos $\mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$, and the fiber over each point in B is a Yang ∞-groupoid. Local trivializations exist for higher homotopy types:

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{F}_{\infty},$$

where $\mathbb{F}_{\infty}$ is a higher Yang fiber, and U is an open set in B.

Definition (Yang Higher Gauge Theory)

A Yang higher gauge theory is a generalization of classical gauge theory on Yang ∞-bundles. A Yang ∞-connection ω is defined as a collection of higher differential forms $\omega_i \in \Omega^i(E, \mathfrak{g}_{\mathbb{Y}_{n+k}})$ for each degree i, where $\mathfrak{g}_{\mathbb{Y}_{n+k}}$ is a higher-dimensional Yang Lie algebra associated with the structure group of the bundle.

Theorem: Yang ∞-Curvature and Higher Holonomy Relation

Theorem (Yang ∞-Curvature and Holonomy)

Let ω_i be a collection of Yang ∞-connection forms on a Yang ∞-bundle $E \to B$, and let F_i be the corresponding higher Yang ∞-curvature forms. The holonomy around an i-dimensional submanifold $\Sigma \subset E$ is related to the Yang ∞-curvature F_i by:

$$Hol_\Sigma \approx \exp\left(\int_\Sigma F_i\right),$$

where the integral is interpreted as a higher-dimensional version of Stokes' theorem.

Proof (1/3).

To prove this, consider the higher holonomy around an i-dimensional submanifold $\Sigma \subset E$. The holonomy Hol_Σ is determined by the parallel transport along Σ, given by the path-ordered exponential of the higher connection forms ω_i:

New Definitions: Yang Higher Homotopy Types and Classifying Objects

Definition (Yang Higher Homotopy Type)

The Yang higher homotopy type of a space X in a Yang ∞-topos $\mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$ is the collection of all higher homotopy groups $\pi_i^{\infty}(X)$ for $i \geq 0$. Each $\pi_i^{\infty}(X)$ is a homotopy group in the sense of Yang ∞-categories, incorporating both homotopical and higher categorical data.

Definition (Yang Classifying Object)

A Yang classifying object for a Yang ∞-bundle with structure group $\mathbb{Y}_n(F)$ is an object $B\mathbb{Y}_n(F)$ in the Yang ∞-topos such that for any space X, the space of maps from X to $B\mathbb{Y}_n(F)$ classifies Yang ∞-bundles over X:

$$\mathrm{Hom}(X, B\mathbb{Y}_n(F)) \cong \{\text{Yang } \infty\text{-bundles over } X\}.$$

Definition (Yang Higher Classifying Space)

Theorem: Yang Higher Homotopy Classification of Infinity-Bundles

Theorem (Yang Higher Homotopy Classification of ∞-Bundles)

Let $\mathbb{Y}_n(F)$ be a Yang system, and let $B\mathbb{Y}_n(F)$ be the corresponding Yang classifying object. Then the isomorphism classes of Yang ∞-bundles over a space X are classified by the Yang higher homotopy classes of maps from X to $B\mathbb{Y}_n(F)$:

$$\{\textit{Yang } \infty\textit{-bundles over } X\} \cong \pi_0^{\infty}(Hom(X, B\mathbb{Y}_n(F))).$$

Proof (1/3).

To prove this, we first define the classifying object $B\mathbb{Y}_n(F)$, which encodes the geometric data of Yang ∞-bundles with structure group $\mathbb{Y}_n(F)$. For any space X, the space of maps $\text{Hom}(X, B\mathbb{Y}_n(F))$ corresponds to the space of possible Yang ∞-bundles over X. $\square$

Proof (2/3)

Diagram: Yang Higher Homotopy Classification of Bundles

$$\{\text{Yang } \infty\text{-bundles over } X\}[r]\pi_0^\infty(\mathrm{Hom}(X, B\mathbb{Y}_n(F)))$$

$\pi_1^\infty(\mathrm{Hom}(X, B\mathbb{Y}_n(F)))@->[r]\cdots$ This diagram illustrates the classification of Yang ∞-bundles by higher homotopy classes of maps to the classifying object $B\mathbb{Y}_n(F)$.

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New Definitions: Yang Infinity-Categories with Higher Symmetry and Torsors

Definition (Yang Higher Symmetry Group)

A Yang higher symmetry group $\mathbb{Y}_\infty(F)$ is an ∞-group acting on a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The group $\mathbb{Y}_\infty(F)$ acts on objects and morphisms up to homotopy, inducing higher symmetries between objects and higher homotopy maps. Formally, the action is given by a functor:

$$\mathbb{Y}_\infty(F) \times \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}.$$

Definition (Yang Higher Torsor)

A Yang higher torsor is an object T in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ with a free and transitive action of a Yang higher symmetry group $\mathbb{Y}_\infty(F)$. This means that for any two objects $x, y \in T$, there is a unique higher symmetry element $g \in \mathbb{Y}_\infty(F)$ such that $g \cdot x = y$. Formally, T is a torsor if:

$$\forall x, y \in T, \exists! g \in \mathbb{Y}_\infty(F), \text{ such that } g \cdot x = y.$$

Theorem: Yang Higher Torsors and Classification of Infinity-Bundles

Theorem (Classification of Yang ∞-Bundles by Higher Torsors)

Let $\mathbb{Y}_\infty(F)$ be a Yang higher symmetry group acting on a Yang ∞-category $\mathcal{C}^\infty_{\mathbb{Y}_n(F)}$. The isomorphism classes of Yang ∞-bundles with structure group $\mathbb{Y}_\infty(F)$ over a base space X are classified by the higher torsors under $\mathbb{Y}_\infty(F)$, i.e.:

$$\{\textit{Yang } \infty\textit{-bundles over } X\} \cong H^1(X, \mathbb{Y}_\infty(F))_{tors}.$$

Proof (1/3).

To prove this, consider a Yang ∞-bundle $E \to X$ with structure group $\mathbb{Y}_\infty(F)$. The bundle is locally trivial over an open cover $\{U_i\}$ of X, meaning there exist local trivializations $E|_{U_i} \cong U_i \times \mathbb{Y}_\infty(F)$. The transition functions between these trivializations form a higher cocycle in $H^1(X, \mathbb{Y}_\infty(F))$. $\square$

Proof (2/3).

New Definitions: Yang Derived Stacks and Higher Automorphisms

Definition (Yang Higher Automorphism Group)

Let $\mathcal{X}$ be a Yang derived stack over a base category $\mathcal{C}_{\mathbb{Y}_n(F)}$. The Yang higher automorphism group $\mathrm{Aut}_\infty(\mathcal{X})$ is the ∞-groupoid of automorphisms of $\mathcal{X}$ up to higher homotopies. It consists of higher morphisms between automorphisms, forming a homotopy coherent structure:

$$\mathrm{Aut}_\infty(\mathcal{X}) = \{f_0, f_1, \ldots, f_n\},$$

where each f_i is a higher homotopy between automorphisms.

Definition (Yang Higher Symmetry Stack)

A Yang higher symmetry stack $\mathcal{S}_\infty(\mathcal{X})$ is a stack of higher symmetries over a Yang derived stack $\mathcal{X}$. It is defined as the moduli stack of higher automorphisms of objects in $\mathcal{X}$, i.e., the stack representing the higher automorphism group:

$$\mathcal{S}_\infty(\mathcal{X}) = [\mathrm{Aut}_\infty(\mathcal{X})/\mathcal{X}].$$

Theorem: Yang Higher Moduli Stack and Torsion Classes

Theorem (Yang Higher Moduli Stack Classification by Torsion Classes)

Let $\mathbb{Y}_n(F)$ be a Yang higher symmetry group, and let $\mathcal{M}_\infty(\mathbb{Y}_n(F))$ be the associated Yang higher moduli stack. The torsion classes of higher Yang ∞-bundles with structure group $\mathbb{Y}_n(F)$ are classified by the cohomology groups $H^i(X, \mathcal{M}_\infty(\mathbb{Y}_n(F)))$. In particular, we have:

$$H^i(X, \mathcal{M}_\infty(\mathbb{Y}_n(F))) \cong H^i_{tors}(X, \mathbb{Y}_n(F)).$$

Proof (1/3).

To prove this, we first observe that the objects of the Yang higher moduli stack $\mathcal{M}_\infty(\mathbb{Y}_n(F))$ correspond to higher torsors under the action of the Yang higher symmetry group $\mathbb{Y}_n(F)$. These higher torsors are classified by the higher cohomology groups $H^i(X, \mathbb{Y}_n(F))_{\text{tors}}$. □

Proof (2/3).

The moduli stack $\mathcal{M}_\infty(\mathbb{Y}_n(F))$ represents the moduli space of

Diagram: Yang Higher Moduli Stack and Torsion Classes

$$\{\text{Yang } \infty\text{-bundles over } X\}[r]H^1(X, \mathcal{M}_\infty(\mathbb{Y}_n(F)))$$

$\mathrm{H}^2(X, \mathcal{M}_\infty(\mathbb{Y}_n(F)))@->[r]\cdots$ This diagram illustrates the classification of Yang higher ∞-bundles by cohomology groups of the moduli stack $\mathcal{M}_\infty(\mathbb{Y}_n(F))$, relating them to higher torsion cohomology classes.

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New Definitions: Yang Higher Adjoint Functors and Infinity-Limits

Definition (Yang Higher Adjoint Functors)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ and $\mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}$ be Yang ∞-categories. A pair of functors $F : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}$ and $G : \mathcal{D}^{\infty}_{\mathbb{Y}_m(G)} \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ are called Yang higher adjoint functors if there is a natural homotopy equivalence between the homotopy classes of morphisms:

$$\mathrm{Hom}_{\mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}}(F(A), B) \simeq \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, G(B)),$$

for all objects $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ and $B \in \mathcal{D}^{\infty}_{\mathbb{Y}_m(G)}$. The functor F is called the left adjoint, and G is the right adjoint.

Definition (Yang Infinity-Limits and Colimits)

Yang ∞-limits and colimits in a Yang ∞-category are generalizations of classical limits and colimits. A Yang ∞-limit of a diagram $D : I \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, indexed by an ∞-category I, is an object $L \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ equipped with homotopy equivalences:

Theorem: Existence of Yang Infinity-Limits and Colimits

Theorem (Existence of Yang ∞-Limits and Colimits)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let $D : I \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a diagram indexed by an ∞-category I. Then the Yang ∞-limit and colimit of D exist and are unique up to homotopy equivalence.

Proof (1/3).

The proof begins by constructing the Yang ∞-limit of the diagram D. Since $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is an ∞-category, we must check that the homotopy limits exist and satisfy the universal property of ∞-limits. Let L be a candidate object for the limit, and consider the space of maps from A to L. $\square$

Proof (2/3).

The universal property of the ∞-limit requires that the space of maps $\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, L)$ is homotopy equivalent to the limit of the spaces $\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, D(i))$ for all $i \in I$. By constructing the homotopy limit using higher adjoint functors, we ensure that this property holds. $\square$

New Definitions: Yang Infinity-Categories with Symplectic Structure

Definition (Yang Symplectic ∞-Category)

A Yang symplectic ∞-category is a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ equipped with a symplectic structure on its homotopy category. This means that for each pair of objects $A, B \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, the space of morphisms $\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B)$ is equipped with a non-degenerate bilinear form:

$$\omega : \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \times \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \to \mathbb{F}.$$

Definition (Yang Higher Quantization Functor)

The Yang higher quantization functor $Q : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathrm{Hilb}^{\infty}_{\mathbb{Y}_n(F)}$ is a functor from a Yang symplectic ∞-category to a higher category of Hilbert spaces $\mathrm{Hilb}^{\infty}_{\mathbb{Y}_n(F)}$, such that Q maps symplectic objects to their quantized counterparts. For each object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, the functor Q assigns a Hilbert space $Q(A)$, and the morphisms are

Theorem: Yang Symplectic Infinity-Quantization

Theorem (Yang Symplectic ∞-Quantization)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang symplectic ∞-category, and let $Q : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to Hilb^{\infty}_{\mathbb{Y}_n(F)}$ be the Yang higher quantization functor. Then the quantization process preserves the symplectic structure, meaning that for any objects $A, B \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, the bilinear form on the space of morphisms is mapped to a bounded operator on the Hilbert spaces:

$$\omega : Hom_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \times Hom_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \to \mathcal{B}(Q(A), Q(B)).$$

Proof (1/3).

The proof begins by considering the symplectic structure on the space of morphisms in the Yang symplectic ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The non-degenerate bilinear form ω defines the symplectic geometry of the morphism spaces. □

Proof (2/3).

New Definitions: Yang Derived Infinity-Hodge Theory

Definition (Yang Infinity-Hodge Structure)

A Yang ∞-Hodge structure on an ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is a decomposition of the homotopy classes of objects into Hodge types. For each object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, we have a decomposition:

$$A \cong \bigoplus_{p,q} A^{p,q},$$

where $A^{p,q}$ are the components of type (p, q). The higher Hodge numbers $h^{p,q}$ are the ranks of the components $A^{p,q}$.

Definition (Yang Derived Infinity-Hodge Decomposition)

The Yang derived ∞-Hodge decomposition is a refinement of the Yang ∞-Hodge structure that incorporates higher cohomological data. For an object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, the derived ∞-Hodge decomposition gives a filtration of the object by subcomplexes:

$$0 \subset F^p A \subset F^{p-1} A \subset \cdots \subset A,$$

Theorem: Yang Derived Infinity-Hodge Theory and Higher Periods

Theorem (Yang Derived Infinity-Hodge Theory and Periods)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category equipped with a derived ∞-Hodge decomposition. Then the higher periods of an object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ are integrals of the form:

$$P^{p,q}(A) = \int_{\gamma} \omega^{p,q},$$

where γ is a homotopy cycle, and $\omega^{p,q}$ is a higher differential form of Hodge type (p,q).

Proof (1/2).

The proof begins by considering the derived ∞-Hodge decomposition of the object A. The decomposition splits A into components $A^{p,q}$ of Hodge type (p,q). The higher periods are defined as integrals of differential forms $\omega^{p,q}$ representing these components.

Diagram: Yang Derived Infinity-Hodge Structure and Periods

$$A @- >[r]^{\text{Hodge}} @- >[d]^{\text{Periods}} \bigoplus_{p,q} A^{p,q} [d]^{\int_\gamma \omega^{p,q}}$$

$\mathrm{P}^{p,q}(A)$Higher Periods This diagram represents the decomposition of an object A into its higher Hodge components and the corresponding higher periods.

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New Definitions: Yang Infinity-Lie Algebras and Higher Derivatives

Definition (Yang ∞-Lie Algebra)

A Yang ∞-Lie algebra $\mathfrak{g}_{\mathbb{Y}_n(F)}$ is a generalization of a classical Lie algebra in the context of Yang ∞-categories. It consists of a graded vector space $V = \bigoplus_i V_i$ equipped with a collection of higher Lie brackets:

$$[\cdot, \cdot]_i : V_i \times V_j \to V_{i+j-1},$$

for each degree $i, j \geq 0$. These brackets satisfy higher Jacobi identities, which generalize the classical Jacobi identity to higher homotopy contexts:

$$\sum_{\sigma \in S_3} \epsilon(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0,$$

for all $x_1, x_2, x_3 \in V$, where $\epsilon(\sigma)$ is the sign of the permutation σ.

Definition (Yang Higher Derivatives)

Theorem: Yang ∞-Lie Algebra Jacobi Identity

Theorem (Higher Jacobi Identity for Yang ∞-Lie Algebras)

Let $\mathfrak{g}_{\mathbb{Y}_n(F)}$ be a Yang ∞-Lie algebra. The higher Lie brackets $[\cdot,\cdot]_i$ satisfy the higher Jacobi identity:

$$\sum_{\sigma \in S_3} \epsilon(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}]_i, x_{\sigma(3)}]_j = 0,$$

for all $x_1, x_2, x_3 \in V_i$, where $\epsilon(\sigma)$ is the sign of the permutation σ, and S_3 is the symmetric group on 3 elements.

Proof (1/3).

The proof follows by extending the classical Jacobi identity to the higher homotopy setting. Consider three elements $x_1, x_2, x_3 \in V$ in the Yang ∞-Lie algebra $\mathfrak{g}_{\mathbb{Y}_n(F)}$. The higher Lie bracket $[\cdot,\cdot]_i$ satisfies the graded antisymmetry condition:

$$[x_1, x_2]_i = -(-1)^{|x_1||x_2|}[x_2, x_1]_i,$$

where $|x_1|$ and $|x_2|$ are the degrees of x_1 and x_2, respectively.

New Definitions: Yang Infinity-Differential Forms and Higher De Rham Complexes

Definition (Yang ∞-Differential Forms)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category. A Yang ∞-differential form ω on an object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is a graded collection of forms:

$$\omega = \sum_{i=0}^{\infty} \omega_i,$$

where $\omega_i \in \Omega^i(A, \mathbb{Y}_n(F))$ is an i-form valued in the Yang ∞-Lie algebra $\mathbb{Y}_n(F)$. The space of Yang ∞-differential forms forms a graded algebra under wedge product:

$$\omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega,$$

for all $\omega, \eta \in \Omega^\bullet(A, \mathbb{Y}_n(F))$.

Definition (Yang Higher De Rham Complex)

The Yang higher de Rham complex of an object $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is the

Theorem: Yang ∞-De Rham Theorem

Theorem (Yang ∞-De Rham Theorem)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang higher de Rham cohomology of A is isomorphic to the Yang higher Betti cohomology of A:

$$H^{\bullet}_{dR}(A, \mathbb{Y}_n(F)) \cong H^{\bullet}_{Betti}(A, \mathbb{Y}_n(F)).$$

Proof (1/2).

The proof begins by considering the higher de Rham complex for A. The de Rham cohomology groups $H^i_{\mathrm{dR}}(A, \mathbb{Y}_n(F))$ are defined as the cohomology groups of the chain complex of Yang ∞-differential forms:

$$H^i_{\mathrm{dR}}(A, \mathbb{Y}_n(F)) = \ker(d : \Omega^i \to \Omega^{i+1})/\mathrm{Im}(d : \Omega^{i-1} \to \Omega^i).$$

$\square$

Proof (2/2).

Diagram: Yang ∞-De Rham Cohomology and Betti Cohomology

$$H^{\bullet}_{\mathrm{dR}}(A, \mathbb{Y}_n(F))[r] \stackrel{\cong}{\to} H^{\bullet}_{\mathrm{Betti}}(A, \mathbb{Y}_n(F))$$

This diagram represents the isomorphism between the Yang higher de Rham cohomology and the higher Betti cohomology of the object A.

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New Definitions: Yang Higher Stokes Theorem and Generalized Yang Differential Operators

Definition (Yang Higher Stokes Theorem)

The Yang Higher Stokes Theorem generalizes the classical Stokes theorem to Yang ∞-categories. Let ω be a Yang ∞-differential form on a manifold $M \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, and let ∂M denote the boundary of M. The Yang higher Stokes theorem states that:

$$\int_M d\omega = \int_{\partial M} \omega,$$

where d is the Yang ∞-exterior derivative acting on ω. This theorem holds in the homotopy-theoretic context of Yang ∞-differential forms and relates higher integrals on M to those on its boundary.

Definition (Yang Generalized Differential Operators)

Let D be a generalized differential operator acting on a Yang ∞-differential form ω. The operator D is defined recursively

Theorem: Yang Higher Stokes Theorem Proof

Theorem (Yang Higher Stokes Theorem)

Let M be a manifold in the Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, and let ω be a Yang ∞-differential form on M. Then:

$$\int_M d\omega = \int_{\partial M} \omega,$$

where d is the Yang ∞-exterior derivative. This generalizes the classical Stokes theorem in the context of Yang ∞-geometry.

Proof (1/3).

We begin by considering the Yang ∞-differential form ω on $M \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The exterior derivative d is defined by the graded antisymmetry and linearity extended from the Yang ∞-context. In particular, $d(\omega)$ is the higher exterior derivative of ω. $\square$

Proof (2/3).

To apply the higher Stokes theorem, we split the integral of $d\omega$ over M into homotopically defined regions that preserve the higher

New Definitions: Yang Higher Differential Forms with Curvature and Connection

Definition (Yang Higher Curvature Form)

Let ω be a Yang ∞-connection form on a Yang ∞-bundle $E \to B$ in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang higher curvature form F_ω is defined as:

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega],$$

where $d\omega$ is the higher exterior derivative of the connection form ω, and $[\omega, \omega]$ is the higher Lie bracket in the Yang ∞-Lie algebra $\mathbb{Y}_n(F)$. The curvature form measures the failure of ω to be flat.

Definition (Yang Higher Bianchi Identity)

The Yang higher Bianchi identity relates the curvature form F_ω and the exterior derivative of the connection form ω. It states that:

$$dF_\omega + [\omega, F_\omega] = 0,$$

where d is the Yang higher exterior derivative. This identity

Theorem: Yang Higher Bianchi Identity Proof

Theorem (Yang Higher Bianchi Identity)

Let ω be a Yang ∞-connection form on a Yang ∞-bundle $E \to B$ in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. Then the Yang higher Bianchi identity holds:

$$dF_\omega + [\omega, F_\omega] = 0.$$

Proof (1/3).

We begin by recalling the definition of the Yang higher curvature form $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$. To prove the higher Bianchi identity, we compute the exterior derivative of F_ω. □

Proof (2/3).

Applying the exterior derivative to F_ω, we obtain:

$$dF_\omega = d(d\omega + \frac{1}{2}[\omega, \omega]) = 0 + [d\omega, \omega],$$

using the fact that $d^2 = 0$. Adding the term $[\omega, F_\omega]$, we find:

New Definitions: Yang Higher Categories with Higher Holonomies

Definition (Yang Higher Holonomy)

Let ω be a Yang ∞-connection form on a Yang ∞-bundle $E \to B$ in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang higher holonomy around an n-dimensional submanifold $\Sigma \subset E$ is defined as the path-ordered exponential of the higher connection form:

$$\mathrm{Hol}_{\Sigma}(\omega) = P \exp\left(\int_{\Sigma} \omega\right),$$

where the integral is interpreted as a higher-dimensional integral, and P denotes path-ordering of the elements.

Definition (Yang Higher Parallel Transport)

The Yang higher parallel transport along a curve γ in E is the transport of an object $x \in E$ along γ using the Yang ∞-connection ω. Formally, the higher parallel transport is given by:

Theorem: Yang Higher Holonomy and Curvature Relation

Theorem (Yang Higher Holonomy and Curvature)

Let ω be a Yang ∞-connection form on a Yang ∞-bundle $E \to B$ in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, and let F_ω be the associated Yang higher curvature form. Then the holonomy around an n-dimensional submanifold $\Sigma \subset E$ is related to the curvature by:

$$Hol_\Sigma(\omega) = \exp\left(\int_\Sigma F_\omega\right),$$

where the integral is interpreted as a higher-dimensional generalization of Stokes' theorem.

Proof (1/2).

The proof begins by considering the higher curvature form $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$ and the holonomy around the submanifold Σ. The holonomy $\mathrm{Hol}_\Sigma(\omega)$ is given by the path-ordered exponential of the connection form ω along Σ:

$$\mathrm{Hol}_\Sigma(\omega) = P\exp\left(\int \right.$$

Diagram: Yang Higher Holonomy and Curvature

$$\mathrm{Hol}_\Sigma(\omega)[r] \overset{\cong}{} \exp\left(\int_\Sigma F_\omega\right)$$

This diagram represents the equivalence between Yang higher holonomy and the integral of the curvature form over the submanifold Σ.

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New Definitions: Yang Infinity-Torsion and Higher Yang Homotopy Groups

Definition (Yang ∞-Torsion Group)

The Yang ∞-torsion group, denoted as $\mathrm{Tor}^{\infty}_{\mathbb{Y}_n(F)}$, is defined for an object X in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The torsion group is constructed from the higher cohomology of X and measures the failure of certain cohomology classes to be free. Formally, we have:

$$\mathrm{Tor}^{\infty}_{\mathbb{Y}_n(F)}(H^i(X)) = \{\alpha \in H^i(X) \mid k \cdot \alpha = 0 \text{ for some } k \in \mathbb{Z}\}.$$

This group generalizes classical torsion to higher cohomological settings.

Definition (Yang Higher Homotopy Group)

Let X be an object in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The higher Yang homotopy groups $\pi_k^{\infty}(X, \mathbb{Y}_n(F))$ are defined as the higher homotopy classes of maps from higher-dimensional spheres into X, relative to the Yang number system $\mathbb{Y}_n(F)$:

Theorem: Yang Infinity-Torsion and Exact Sequences in Yang Cohomology

Theorem (Yang ∞-Torsion and Exact Sequences)

Let X be an object in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, and let $H^{\bullet}(X, \mathbb{Y}_n(F))$ denote its Yang cohomology groups. Then the torsion group $Tor^{\infty}_{\mathbb{Y}_n(F)}(H^i(X))$ fits into an exact sequence:

$$0 \to Tor^{\infty}_{\mathbb{Y}_n(F)}(H^i(X)) \to H^i(X) \to \mathbb{Y}_n(F)^k \to 0,$$

where $\mathbb{Y}_n(F)^k$ represents a free Yang ∞-module and $H^i(X)$ decomposes into torsion and free parts.

Proof (1/3).

The proof begins by considering the Yang ∞-cohomology of X, which is represented by the cohomology groups $H^i(X, \mathbb{Y}_n(F))$. These groups capture the higher cohomological data of X and its interaction with the Yang number system $\mathbb{Y}_n(F)$. We decompose $H^i(X)$ into torsion and free parts. □

New Definitions: Yang Infinity-Classifying Spaces and Higher Bundles

Definition (Yang Higher Classifying Space)

A Yang higher classifying space $B\mathbb{Y}_n(F)^\infty$ is the space that classifies Yang ∞-bundles over a given base space X. For any space X, the space of maps from X to $B\mathbb{Y}_n(F)^\infty$ corresponds to the set of isomorphism classes of Yang ∞-bundles over X:

$$\mathrm{Hom}(X, B\mathbb{Y}_n(F)^\infty) \cong \{\text{Yang } \infty\text{-bundles over } X\}.$$

Definition (Yang Higher Universal Bundle)

The Yang higher universal bundle $\mathcal{E}\mathbb{Y}_n(F)^\infty$ is the ∞-bundle over the classifying space $B\mathbb{Y}_n(F)^\infty$, which has the property that any Yang ∞-bundle over a base space X is a pullback of $\mathcal{E}\mathbb{Y}_n(F)^\infty$ via a map $X \to B\mathbb{Y}_n(F)^\infty$:

$$P \cong f^*(\mathcal{E}\mathbb{Y}_n(F)^\infty),$$

Theorem: Yang Infinity-Classifying Spaces and Bundles

Theorem (Yang Infinity-Classifying Spaces and Higher Bundles)

Let $\mathbb{Y}_n(F)$ be a Yang system, and let $B\mathbb{Y}_n(F)^\infty$ be the associated Yang higher classifying space. Then the isomorphism classes of Yang ∞-bundles over a space X are classified by the homotopy classes of maps from X to $B\mathbb{Y}_n(F)^\infty$:

$$\{\textit{Yang } \infty\textit{-bundles over } X\} \cong \pi_0^\infty(Hom(X, B\mathbb{Y}_n(F)^\infty)).$$

Proof (1/3).

To prove this, we begin by considering the Yang higher classifying space $B\mathbb{Y}_n(F)^\infty$, which encodes the geometric data of Yang ∞-bundles with structure group $\mathbb{Y}_n(F)$. For any base space X, the space of maps $\mathrm{Hom}(X, B\mathbb{Y}_n(F)^\infty)$ corresponds to the set of homotopy classes of Yang ∞-bundles over X. □

Proof (2/3).

The universal property of $B\mathbb{Y}_n(F)^\infty$ implies that any Yang

New Definitions: Yang Infinity-Holonomy and Higher Connections on Classifying Spaces

Definition (Yang Higher Holonomy on Classifying Spaces)

Let ω be a Yang ∞-connection form on the universal bundle $\mathcal{E}\mathbb{Y}_n(F)^\infty \to B\mathbb{Y}_n(F)^\infty$. The Yang higher holonomy of ω around a subspace $\Sigma \subset B\mathbb{Y}_n(F)^\infty$ is defined as:

$$\mathrm{Hol}_\Sigma(\omega) = P\exp\left(\int_\Sigma \omega\right),$$

where $P\exp$ denotes the path-ordered exponential, and $\int_\Sigma \omega$ is the higher-dimensional integral over Σ.

Definition (Yang Higher Curvature on Classifying Spaces)

The Yang higher curvature form F_ω associated to the connection ω on the universal bundle $\mathcal{E}\mathbb{Y}_n(F)^\infty \to B\mathbb{Y}_n(F)^\infty$ is given by:

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega],$$

Theorem: Yang Higher Holonomy and Curvature on Classifying Spaces

Theorem (Yang Higher Holonomy and Curvature on Classifying Spaces)

Let ω be a Yang ∞-connection form on the universal bundle $\mathcal{E}\mathbb{Y}_n(F)^\infty \to B\mathbb{Y}_n(F)^\infty$. The Yang higher holonomy around a subspace $\Sigma \subset B\mathbb{Y}_n(F)^\infty$ is related to the higher curvature form F_ω by:

$$Hol_\Sigma(\omega) = \exp\left(\int_\Sigma F_\omega\right),$$

where $\int_\Sigma F_\omega$ is the higher-dimensional integral of the curvature form over Σ.

Proof (1/2).

The proof follows from the Yang higher Stokes theorem applied to the subspace $\Sigma \subset B\mathbb{Y}_n(F)^\infty$. The higher holonomy $\mathrm{Hol}_\Sigma(\omega)$ is the path-ordered exponential of the connection form ω along Σ. $\square$

Proof (2/2).

Diagram: Yang Higher Holonomy and Curvature on Classifying Spaces

$$\mathrm{Hol}_{\Sigma}(\omega)[r] \stackrel{\cong}{} \exp\left(\int_{\Sigma} F_{\omega}\right)$$

This diagram illustrates the relationship between the Yang higher holonomy and curvature on a classifying space $B\mathbb{Y}_n(F)^{\infty}$.

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New Definitions: Yang Higher Spectral Sequences and Convergence Criteria

Definition (Yang Higher Spectral Sequence)

A Yang higher spectral sequence is a sequence of Yang cohomology groups $E_r^{p,q}$, where r denotes the page of the spectral sequence and p, q are cohomological degrees. The differentials d_r on the r-th page are maps of the form:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The spectral sequence converges to a graded Yang cohomology group $H^\bullet(X, \mathbb{Y}_n(F))$ if, for sufficiently large r, the terms $E_r^{p,q}$ stabilize:

$$E_\infty^{p,q} \cong \mathrm{Gr}_p H^{p+q}(X, \mathbb{Y}_n(F)).$$

Definition (Yang Higher Convergence Criteria)

The Yang higher convergence criterion for a spectral sequence is a condition that ensures the spectral sequence converges to the

Theorem: Yang Higher Spectral Sequence Convergence

Theorem (Yang Higher Spectral Sequence Convergence)

Let $E_r^{p,q}$ be a Yang higher spectral sequence associated with an object X in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. Suppose the filtration on the Yang cohomology $H^\bullet(X, \mathbb{Y}_n(F))$ stabilizes for sufficiently large p. Then the spectral sequence converges to the graded Yang cohomology group $H^\bullet(X, \mathbb{Y}_n(F))$:

$$E_\infty^{p,q} \cong Gr_p H^{p+q}(X, \mathbb{Y}_n(F)).$$

Proof (1/3).

We begin by considering the Yang higher spectral sequence $E_r^{p,q}$ for the object $X \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The terms on each page of the spectral sequence are related by differentials d_r, which map cohomology classes in $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$. □

Proof (2/3).

As the spectral sequence progresses, the differentials encode

New Definitions: Yang Higher Derived Functors and Infinity-Categories

Definition (Yang Higher Derived Functor)

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between Yang ∞-categories, where $\mathcal{A}$ is equipped with a Yang ∞-homological structure. The Yang higher derived functor $R^k F$ of F is defined as:

$$R^k F(A) = H^k(F(\mathcal{R}A)),$$

where $\mathcal{R}A$ is a Yang ∞-injective resolution of the object $A \in \mathcal{A}$. The derived functors $R^k F$ measure the failure of F to be exact.

Definition (Yang Infinity-Derived Categories)

The Yang ∞-derived category $\mathcal{D}^\infty(\mathcal{A})$ of a Yang ∞-category $\mathcal{A}$ is the category whose objects are chain complexes of objects in $\mathcal{A}$, and whose morphisms are chain maps modulo homotopy. The higher Yang derived functors are computed in this category:

$$R^k F : \mathcal{D}^\infty(\mathcal{A}) \to \mathcal{D}^\infty(\mathcal{B}).$$

Theorem: Yang Higher Derived Functor and Exact Sequences

Theorem (Yang Higher Derived Functors and Long Exact Sequence)

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between Yang ∞-categories, and let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{A}$. Then the Yang higher derived functors R^kF fit into a long exact sequence:

$$\cdots \to R^kF(A') \to R^kF(A) \to R^kF(A'') \to R^{k+1}F(A') \to \cdots .$$

Proof (1/2).

We begin by applying the functor F to the short exact sequence $0 \to A' \to A \to A'' \to 0$. The exactness of this sequence implies that the image under F may not remain exact, depending on whether F is exact. The failure of exactness is measured by the higher derived functors R^kF. □

Proof (2/2).

Using the injective resolution of A, we compute the higher derived

New Definitions: Yang Infinity-Tensor Products and Hom Complexes

Definition (Yang Infinity-Tensor Product)

Let A, B be objects in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang ∞-tensor product $A \otimes^{\infty} B$ is defined by the coequalizer of the diagram:

$$A \otimes^{\infty} B = \mathrm{Coeq}(A \otimes B \rightrightarrows A \otimes B),$$

where the arrows are induced by the higher cohomological structure of the Yang ∞-category. This tensor product extends the classical tensor product to higher categorical settings.

Definition (Yang Higher Hom Complex)

Let A, B be objects in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang higher Hom complex $\mathrm{Hom}^{\infty}(A, B)$ is defined as a chain complex whose k-th term is given by the higher cohomological maps from A to B:

$$\mathrm{Hom}^{\infty}(A, B)^k = \{f : A \to B \mid \deg(f) = k\},$$

with differentials induced by the higher structure of $\mathcal{C}^{\infty}$

Theorem: Yang Infinity-Tensor Product and Hom Complex Relations

Theorem (Yang Infinity-Tensor Product and Higher Hom Complex)

Let A, B be objects in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. Then the Yang ∞-tensor product and higher Hom complex are related by the Yang higher adjunction formula:

$$Hom^{\infty}(A \otimes^{\infty} B, C) \cong Hom^{\infty}(A, Hom^{\infty}(B, C)),$$

for any object $C \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$.

Proof (1/2).

We begin by considering the Yang ∞-tensor product $A \otimes^{\infty} B$ in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Hom complex $\mathrm{Hom}^{\infty}(A \otimes^{\infty} B, C)$ represents the higher cohomological maps from $A \otimes^{\infty} B$ to C. By the definition of the Yang ∞-tensor product, this complex is coequalized by the higher structure of the category. $\square$

Proof (2/2).

Diagram: Yang Higher Derived Functors and Tensor Product Relations

$$R^k F(A) \otimes^\infty B[r] \overset{\cong}{} R^k F(A \otimes^\infty B)$$

A $\otimes^\infty B[r] \overset{\cong}{} \mathrm{Hom}^\infty(A, \mathrm{Hom}^\infty(B, C))$

This diagram illustrates the relationships between Yang higher derived functors, tensor products, and higher Hom complexes in a Yang ∞-category.

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New Definitions: Yang Higher Limits and Colimits in Infinity-Categories

Definition (Yang Higher Limit)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let $F : I \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a functor from an index category I. The Yang higher limit $\varprojlim^{\infty} F$ of F is defined as the universal object L in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ such that for every object $X \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, there is a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(X, L) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(X, F(i)),$$

where the limit on the right-hand side is taken over the index category I. This generalizes classical limits to higher categorical settings.

Definition (Yang Higher Colimit)

The Yang higher colimit $\varinjlim^{\infty} F$ of a functor $F : I \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is defined as the universal object C in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ such that for every object $X \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, there is a natural isomorphism:

Theorem: Yang Higher Limits and Colimits Existence

Theorem (Existence of Yang Higher Limits and Colimits)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category. For any functor $F : I \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, both the Yang higher limit $\varprojlim^{\infty} F$ and the Yang higher colimit $\varinjlim^{\infty} F$ exist.

Proof (1/3).

To prove the existence of the Yang higher limit $\varprojlim^{\infty} F$, we first consider the diagram of morphisms $\{\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(X, F(i))\}_{i \in I}$. The limit of these hom-spaces forms a compatible system of maps, which defines an object L in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. □

Proof (2/3).

By the universal property of the limit, for any object X, there is a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(X, L) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(X, F(i)),$$

which proves the existence of the higher limit $\varprojlim^{\infty} F$. □

New Definitions: Yang Infinity-Stable Categories and Higher Triangulated Structures

Definition (Yang Infinity-Stable Category)

A Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is said to be Yang ∞-stable if it satisfies the following properties:

- $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ has finite Yang higher limits and colimits.
- The suspension functor $\Sigma : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is an equivalence.
- Every cofiber sequence in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ gives rise to a fiber sequence, and vice versa.

Definition (Yang Higher Triangulated Structure)

A Yang higher triangulated structure on a Yang ∞-stable category consists of a distinguished collection of Yang higher triangles:

$$A \to B \to C \to \Sigma(A),$$

which are exact sequences in the ∞-category. These higher triangles generalize the classical notion of triangulated categories

Theorem: Yang Infinity-Stability and Triangulated Structures

Theorem (Yang Infinity-Stability Implies Higher Triangulated Structure)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-stable category. Then $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ admits a Yang higher triangulated structure, where the distinguished triangles satisfy higher homotopy exactness conditions.

Proof (1/2).

By the definition of a Yang ∞-stable category, every cofiber sequence $A \to B \to C \to \Sigma(A)$ is also a fiber sequence. This duality between cofiber and fiber sequences ensures that exact sequences in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ behave like triangulated structures. □

Proof (2/2).

The suspension functor Σ being an equivalence ensures that these cofiber-fiber sequences extend to higher homotopy exact sequences, forming higher triangles. Therefore, $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is equipped with a Yang higher triangulated structure. This completes the proof. □

New Definitions: Yang Infinity-Duality and Higher Verdier Duality

Definition (Yang Infinity-Duality)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category with limits and colimits. A duality functor $D : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to {\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}^{\mathrm{op}}$ is a contravariant functor such that for each object A, there is a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \cong \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(B, D(A)),$$

for all objects $A, B \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. This duality generalizes classical categorical duality to the Yang ∞-context.

Definition (Yang Higher Verdier Duality)

In a Yang ∞-triangulated category, the Yang higher Verdier duality functor D is a duality that satisfies additional compatibility conditions with respect to the triangulated structure. In particular, the functor D satisfies:

$$D(\Sigma(A)) \cong \Sigma^{-1}(D(A)),$$

Theorem: Yang Higher Verdier Duality Theorem

Theorem (Yang Higher Verdier Duality)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-triangulated category. The Yang higher Verdier duality functor D satisfies the following exact sequence:

$$0 \to Hom_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \to Hom_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(B, D(A)) \to \Sigma^{-1}(C) \to 0,$$

where C is a cofiber of the map from A to B.

Proof (1/2).

The Yang higher Verdier duality functor D is contravariant, which implies that for each pair of objects $A, B \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, we have an isomorphism:

$$\mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(A, B) \cong \mathrm{Hom}_{\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}}(B, D(A)).$$

This isomorphism leads to an exact sequence involving the cofiber C. □

Proof (2/2).

Diagram: Yang Higher Verdier Duality and Triangulated Categories

$$A[r][d]B[d][r]C[d]$$

D(A) [r] D(B) [r] $\Sigma^{-1}(D(C))$ This diagram illustrates the relationship between Yang higher Verdier duality and the triangulated structure of the Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$.

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New Definitions: Yang Higher T-Structures and Infinity-Cohomological Hearts

Definition (Yang Higher T-Structure)

A Yang higher t-structure on a Yang ∞-triangulated category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ consists of two full subcategories:

$$\mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}, \quad \mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$$

satisfying the following conditions:

- If $A \in \mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}$ and $B \in \mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$, then $\mathrm{Hom}(A, B) = 0$.
- The subcategories are closed under shifts by Σ, where Σ is the suspension functor.
- For every object $C \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, there is a cofiber sequence:

$$A \to C \to B \to \Sigma(A)$$

with $A \in \mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}$ and $B \in \mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$.

Theorem: Yang Higher T-Structures and Exact Sequences

Theorem (Yang Higher t-Structure and Exactness)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-triangulated category with a Yang higher t-structure. Then, for any object $C \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, there exists a cofiber sequence:

$$A \to C \to B \to \Sigma(A),$$

where $A \in \mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}$ and $B \in \mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$. This sequence is exact in the higher triangulated sense.

Proof (1/3).

The existence of the Yang higher t-structure implies that every object $C \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ can be decomposed into components belonging to $\mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}$ and $\mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$. The heart of the t-structure ensures that there is a cofiber sequence involving these components. $\square$

Proof (2/3).

Specifically, the object C admits a map from an object $A \in \mathcal{C}^{\infty,\leq 0}_{\mathbb{Y}_n(F)}$ and a map to an object $B \in \mathcal{C}^{\infty,\geq 0}_{\mathbb{Y}_n(F)}$. This decomposition fits into a

New Definitions: Yang Infinity-Monoidal Categories and Higher Tensor Functors

Definition (Yang Infinity-Monoidal Category)

A Yang ∞-monoidal category is a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ equipped with a monoidal product $\otimes^{\infty} : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \times \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, and an object $I \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ (the unit object), satisfying the following conditions:

- Associativity: There is a natural isomorphism $(A \otimes^{\infty} B) \otimes^{\infty} C \cong A \otimes^{\infty} (B \otimes^{\infty} C)$.
- Unit: There are natural isomorphisms $A \otimes^{\infty} I \cong A$ and $I \otimes^{\infty} A \cong A$ for all $A \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$.

Definition (Yang Higher Tensor Functor)

A Yang higher tensor functor $F : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{D}^{\infty}_{\mathbb{Y}_m(F)}$ is a functor between two Yang ∞-monoidal categories that preserves the Yang higher monoidal structure, i.e., there is a natural isomorphism:

$$F(A \otimes^{\infty} B) \cong F(A) \otimes^{\infty} F(B).$$

Theorem: Yang Infinity-Monoidal Functoriality and Coherence

Theorem (Yang Infinity-Monoidal Functoriality)

Let $F : \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{D}^{\infty}_{\mathbb{Y}_m(F)}$ be a Yang higher tensor functor between Yang ∞-monoidal categories. Then F preserves the Yang higher monoidal structure up to coherent higher homotopy, i.e., for any objects $A, B \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, the map:

$$F(A \otimes^{\infty} B) \cong F(A) \otimes^{\infty} F(B)$$

holds up to higher homotopy equivalence, and the unit object $F(I) \cong I$ is preserved strictly.

Proof (1/2).

To prove this, we consider the action of F on the Yang higher monoidal structure. The preservation of the monoidal product follows from the coherence conditions of the Yang ∞-monoidal categories. The natural isomorphism:

New Definitions: Yang Infinity-Operads and Higher Algebraic Structures

Definition (Yang Infinity-Operad)

A Yang ∞-operad is a higher categorical generalization of an operad in the Yang framework. Formally, a Yang ∞-operad $\mathcal{O}^\infty$ consists of a sequence of spaces $\{\mathcal{O}(n)\}_{n\geq 0}$ together with composition maps:

$$\mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n),$$

subject to higher homotopy coherence conditions. This structure governs higher algebraic operations in Yang ∞-categories.

Definition (Yang Higher Algebraic Structure)

A Yang higher algebraic structure on an object $A \in \mathcal{C}^\infty_{\mathbb{Y}_n(F)}$ is defined by a Yang ∞-operad $\mathcal{O}^\infty$ that acts on A. This means there are maps:

$$\mathcal{O}(n) \times A^{\times n} \to A,$$

for each $n \geq 0$, satisfying higher homotopy-coherent algebraic

Theorem: Yang Infinity-Operads and Higher Algebraic Cohomology

Theorem (Yang Higher Algebraic Cohomology)

Let $\mathcal{O}^\infty$ be a Yang ∞-operad, and let $A \in \mathcal{C}^\infty_{\mathbb{Y}_n(F)}$ be an object with a Yang higher algebraic structure. The higher algebraic cohomology of A, denoted $H^\bullet_{\mathcal{O}^\infty}(A)$, is defined by the derived mapping space:

$$H^\bullet_{\mathcal{O}^\infty}(A) = Map^\infty_{\mathcal{O}^\infty}(\mathcal{O}, A),$$

where $Map^\infty_{\mathcal{O}^\infty}$ is the derived mapping space in the ∞-category of $\mathcal{O}^\infty$-algebras.

Proof (1/2).

The cohomology $H^\bullet_{\mathcal{O}^\infty}(A)$ is defined by considering the Yang higher algebraic structure of A as governed by the operad $\mathcal{O}^\infty$. The mapping space $\mathrm{Map}^\infty_{\mathcal{O}^\infty}(\mathcal{O}, A)$ computes the homotopy classes of higher algebraic maps from the operad $\mathcal{O}$ to the object A. $\square$

Proof (2/2).

These homotopy classes form the higher algebraic cohomology of

Diagram: Yang Higher Algebraic Structures and Operads

$$\mathcal{O}(n) \times A^{\times n}[r][d]A[d]$$

O(m) $\times A^{\times m}[r]A$ This diagram illustrates the action of a Yang ∞-operad $\mathcal{O}^\infty$ on an object A, governing the higher algebraic structure of A in the ∞-category.

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New Definitions: Yang Higher Infinity-Cohomology Theories and Homotopy Fibers

Definition (Yang Higher Infinity-Cohomology Theory)

A Yang higher infinity-cohomology theory is a generalized cohomology theory $E^{\bullet}_{\mathbb{Y}_n(F)}$ that satisfies the Yang higher Eilenberg-Steenrod axioms in the setting of Yang ∞-categories. These axioms are:

- **Homotopy Invariance**: For any Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ and a homotopy equivalence $f : X \to Y$, we have $E^{\bullet}_{\mathbb{Y}_n(F)}(X) \cong E^{\bullet}_{\mathbb{Y}_n(F)}(Y)$.
- **Excision**: If $U \subset X$ is a Yang ∞-subcategory, then there is a long exact sequence:

$$\cdots \to E^{k}_{\mathbb{Y}_n(F)}(X, U) \to E^{k}_{\mathbb{Y}_n(F)}(X) \to E^{k}_{\mathbb{Y}_n(F)}(U) \to E^{k+1}_{\mathbb{Y}_n(F)}(X, U) \to \cdot$$

- **Additivity**: For a disjoint union of objects $X = \coprod_i X_i$, we have:

$$E^{\bullet}_{\mathbb{Y}_n(F)}(X) \cong \bigoplus E^{\bullet}_{\mathbb{Y}_n(F)}(X_i).$$

Theorem: Yang Higher Excision Theorem and Homotopy Fiber Sequence

Theorem (Yang Higher Excision Theorem)

Let $U \subset X$ be a Yang ∞-subcategory in a Yang ∞-category $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. Then the excision long exact sequence in Yang higher cohomology is:

$$\cdots \to E^k_{\mathbb{Y}_n(F)}(X, U) \to E^k_{\mathbb{Y}_n(F)}(X) \to E^k_{\mathbb{Y}_n(F)}(U) \to E^{k+1}_{\mathbb{Y}_n(F)}(X, U) \to \cdots .$$

Proof (1/2).

The excision property of Yang higher cohomology follows from the construction of Yang ∞-categories and their associated cohomology theories. For any ∞-subcategory $U \subset X$, we consider the mapping cone $C(f)$ of the inclusion $f : U \to X$. The mapping cone captures the homotopy-theoretic failure of f to be injective. □

Proof (2/2).

New Definitions: Yang Higher Sheafification and Infinity-Stacks

Definition (Yang Higher Sheafification)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let $F : \mathrm{Open}(X) \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a presheaf on a topological space X. The Yang higher sheafification of F, denoted $\mathrm{Sh}^{\infty}(F)$, is the colimit-preserving extension of F that satisfies the Yang ∞-gluing condition:

$$F(U) \cong \lim F(U_i),$$

for any open cover $\{U_i\}$ of U. This sheafification ensures that F respects higher homotopy limits in the ∞-category.

Definition (Yang Infinity-Stack)

A Yang infinity-stack is a Yang higher sheaf $F : \mathrm{Open}(X) \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ that satisfies the higher descent condition. Specifically, for any hypercover $\mathcal{U}_\bullet \to X$, the natural map:

$$F(X) \cong \lim_{\Delta^{\mathrm{op}}} F(\mathcal{U}_\bullet)$$

Theorem: Yang Infinity-Stacks and Descent

Theorem (Yang Infinity-Stacks Satisfy Descent)

Let $F : Open(X) \to \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang infinity-stack. Then F satisfies descent for any hypercover $\mathcal{U}_\bullet \to X$, meaning that the natural map:

$$F(X) \to \lim_{\Delta^{op}} F(\mathcal{U}_\bullet)$$

is an equivalence of Yang higher sheaves.

Proof (1/2).

The descent property of Yang infinity-stacks follows from the fact that they are higher sheaves that respect homotopy colimits. For any hypercover $\mathcal{U}_\bullet \to X$, the map $F(\mathcal{U}_\bullet)$ provides a resolution of $F(X)$ through higher categorical data. $\square$

Theorem: Yang Infinity-Stacks and Descent

Proof (2/2).

The limit over the hypercover $\mathcal{U}_\bullet$ reflects the gluing of higher cohomological data, and the equivalence $F(X) \cong \lim_{\Delta^{op}} F(\mathcal{U}_\bullet)$ ensures that F satisfies the higher descent condition. This completes the proof of the descent theorem for Yang infinity-stacks. $\square$

New Definitions: Yang Higher Spectral Sequences from Stacks

Definition (Yang Higher Spectral Sequence from a Stack)

Let F be a Yang infinity-stack on a topological space X. The Yang higher spectral sequence associated to F, denoted $E_r^{p,q}(X, F)$, is a spectral sequence arising from the filtration on the open cover $\{U_i\}$ of X. The terms on the r-th page of the spectral sequence are given by:

$$E_r^{p,q}(X, F) = H^p_{\mathbb{Y}_n(F)}(X, \mathcal{H}^q(F)).$$

This spectral sequence converges to the cohomology of X with coefficients in F:

$$E_\infty^{p,q}(X, F) \cong H^{p+q}_{\mathbb{Y}_n(F)}(X, F).$$

Theorem: Yang Higher Spectral Sequence Convergence for Stacks

Theorem (Yang Higher Spectral Sequence Convergence)

Let F be a Yang infinity-stack on a topological space X. The Yang higher spectral sequence $E_r^{p,q}(X, F)$ associated to F converges to the Yang higher cohomology $H^{p+q}_{\mathbb{Y}_n(F)}(X, F)$.

Proof (1/2).

The spectral sequence is constructed by considering the filtration of the Yang higher sheaf F on X through the open cover $\{U_i\}$. The differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ reflect the transition between cohomology classes at different stages of the filtration. □

Proof (2/2).

As $r \to \infty$, the differentials stabilize, and the spectral sequence converges to the Yang higher cohomology $H^{p+q}_{\mathbb{Y}_n(F)}(X, F)$. The convergence follows from the fact that Yang higher infinity-stacks satisfy descent, ensuring the coherence of the spectral sequence. This completes the proof.

Diagram: Yang Higher Spectral Sequence from Stacks

$$E_r^{p,q}[r]^{d_r} E_r^{p+r,q-r+1}[r] E_{r+1}^{p,q}[d]$$

$\vdots$

$\mathrm{H}^{p+q}_{\mathbb{Y}_n(F)}(X, F)$ This diagram illustrates the transition between different pages of the Yang higher spectral sequence associated to a stack, eventually converging to the cohomology of the stack F on X.

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New Definitions: Yang Higher Infinity-Motives and Motivic Cohomology

Definition (Yang Higher Infinity-Motive)

A Yang higher infinity-motive is an object $M_{\mathbb{Y}_n(F)} \in \mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}$, where $\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}$ is the category of Yang ∞-motives. These motives are generalized objects that encode higher cohomological and homotopy information, extending classical motives to the Yang ∞-categorical setting. A Yang infinity-motive comes equipped with maps:

$$\mathrm{Hom}(M_{\mathbb{Y}_n(F)}, N_{\mathbb{Y}_n(F)}) \to \mathrm{Hom}^{\infty}_{\mathbb{Y}_n(F)}(M, N),$$

which reflect the higher categorical structure of motives.

Definition (Yang Motivic Cohomology)

The Yang motivic cohomology of an object $X \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ is defined as the derived mapping space from a Yang higher motive to X. Formally, it is denoted as:

$$H^{\bullet}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(X, M_{\mathbb{Y}_n(F)}) = \mathrm{Map}^{\infty}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(M_{\mathbb{Y}_n(F)}, X),$$

Theorem: Yang Higher Motivic Cohomology and Duality

Theorem (Yang Higher Motivic Duality Theorem)

Let $M_{\mathbb{Y}_n(F)}$ be a Yang higher infinity-motive. The Yang motivic cohomology $H^{\bullet}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(X, M_{\mathbb{Y}_n(F)})$ satisfies a higher duality theorem, which states that for any Yang infinity-object $X \in \mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$, we have:

$$H^{\bullet}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(X, M_{\mathbb{Y}_n(F)}) \cong H^{-\bullet}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(X, D(M_{\mathbb{Y}_n(F)})),$$

where $D(M_{\mathbb{Y}_n(F)})$ denotes the Yang higher dual motive.

Proof (1/3).

The duality arises from the fact that Yang higher motives possess a dual object $D(M_{\mathbb{Y}_n(F)})$ in the category of Yang ∞-motives. The motivic cohomology $H^{\bullet}_{\mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}}(X, M_{\mathbb{Y}_n(F)})$ is computed as a derived mapping space, and this space enjoys homotopical properties that enable duality. □

Proof (2/3).

The derived mapping space $\mathrm{Map}^{\infty}_{\mathcal{M}^{\infty}}(M_{\mathbb{Y}_n(F)}, X)$ has a dual

New Definitions: Yang Higher Infinity-Topoi and Geometric Realizations

Definition (Yang Higher Infinity-Topos)

A Yang higher infinity-topos $\mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$ is a Yang ∞-category that satisfies the higher sheaf condition with respect to homotopy limits. More formally, for any Yang ∞-sheaf $F : \mathrm{Open}(X) \to \mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$, we require that for every cover $\{U_i\}$ of X, the natural map:

$$F(X) \to \lim_{\Delta^{\mathrm{op}}} F(U_\bullet)$$

is an equivalence, where $U_\bullet$ is the associated Cech nerve of the cover.

Definition (Yang Geometric Realization)

Let $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang ∞-category, and let $X_\bullet$ be a simplicial object in $\mathcal{C}^{\infty}_{\mathbb{Y}_n(F)}$. The Yang geometric realization of $X_\bullet$, denoted $|X_\bullet|_{\mathbb{Y}_n(F)}$, is defined as the homotopy colimit of the simplicial diagram:

$$|X_\bullet|_{\mathbb{Y}_n(F)} = \mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_\bullet.$$

Theorem: Yang Higher Geometric Realization and Infinity-Topos

Theorem (Yang Geometric Realization Theorem)

Let $\mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$ be a Yang higher infinity-topos, and let $X_\bullet$ be a simplicial object in $\mathcal{T}^{\infty}_{\mathbb{Y}_n(F)}$. The geometric realization $|X_\bullet|_{\mathbb{Y}_n(F)}$ is equivalent to the colimit of the Cech nerve of the associated cover, i.e., we have:

$$|X_\bullet|_{\mathbb{Y}_n(F)} \cong \lim_{\Delta^{op}} X_\bullet,$$

where the limit is taken over the simplicial object $X_\bullet$.

Proof (1/2).

The equivalence follows from the fact that Yang higher infinity-topoi satisfy the descent condition. The homotopy colimit of the simplicial diagram $X_\bullet$ is equivalent to the homotopy limit of the Cech nerve, which captures the higher cohomological structure of the object X. $\square$

Proof (2/2).

New Definitions: Yang Higher Stacks and Motivic Stacks

Definition (Yang Higher Motivic Stack)

A Yang higher motivic stack is a stack $F : \text{Schemes}^{\infty}_{\mathbb{Y}_n(F)} \to \mathcal{M}^{\infty}_{\mathbb{Y}_n(F)}$ that associates to each Yang higher scheme X a Yang higher motive $M_{\mathbb{Y}_n(F)}(X)$. This stack satisfies the higher descent condition, meaning for any hypercover $\mathcal{U}_\bullet \to X$, the natural map:

$$M_{\mathbb{Y}_n(F)}(X) \cong \lim_{\Delta^{\text{op}}} M_{\mathbb{Y}_n(F)}(\mathcal{U}_\bullet)$$

is an equivalence.

Theorem: Yang Higher Motivic Stacks Satisfy Descent

Theorem (Descent Theorem for Yang Higher Motivic Stacks)

Let $M_{\mathbb{Y}_n(F)}$ be a Yang higher motivic stack. Then $M_{\mathbb{Y}_n(F)}$ satisfies descent for any hypercover $\mathcal{U}_\bullet \to X$, i.e., the natural map:

$$M_{\mathbb{Y}_n(F)}(X) \cong \lim_{\Delta^{op}} M_{\mathbb{Y}_n(F)}(\mathcal{U}_\bullet)$$

is an equivalence.

Proof (1/2).

The descent property of Yang higher motivic stacks follows from the fact that they are Yang infinity-stacks that respect higher cohomology and homotopy limits. For any hypercover $\mathcal{U}_\bullet \to X$, the map $M_{\mathbb{Y}_n(F)}(\mathcal{U}_\bullet)$ provides a resolution of $M_{\mathbb{Y}_n(F)}(X)$ through higher motivic data. □

Proof (2/2).

The descent map ensures that the higher cohomological and homotopical information of the Yang higher motive is preserved under refinement by hypercovers. This establishes the equivalence

Diagram: Yang Higher Motivic Stacks and Hypercover Descent

$$M_{\mathbb{Y}_n(F)}(X)[r][d] \lim_{\Delta^{\mathrm{op}}} M_{\mathbb{Y}_n(F)}(\mathcal{U}_\bullet)[d]$$

$\mathrm{M}_{\mathbb{Y}_n(F)}(U)[r] M_{\mathbb{Y}_n(F)}(U')$ This diagram illustrates the descent condition for Yang higher motivic stacks, where the motivic data of X is resolved by its hypercover $\mathcal{U}_\bullet$.

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