TOWARDS SCHNIRELMANN-TYPE DENSITY IN THE THEORY OF MODULAR AND AUTOMORPHIC FORMS

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Abstract. We explore possible extensions of Schnirelmann-type density and additive closure to the domain of modular and automorphic forms. Drawing analogies with eigenvalue distribution, Hecke algebra actions, and support density of Fourier coefficients, we initiate the formal definition of additive growth properties within automorphic structures.

1. Introduction

Modular and automorphic forms are central objects in modern number theory. Their Fourier coefficients encode deep arithmetic properties. In this paper, we propose analogues of Schnirelmann density in the context of modular forms and automorphic representations, aiming to understand the additive distribution and summability of supports.

2. Preliminaries on Modular Forms

Let $f(z) = \sum_{n\geq 1} a_n q^n \in M_k(\Gamma_0(N))$ be a modular form.

Definition 2.1 (Support Set)**.** Define the support set of f by

$$\mathrm{supp}(f) := \{n \in \mathbb{N} \mid a_n \neq 0\}.$$

Definition 2.2 (Modular Schnirelmann Density)**.** The modular Schnirelmann density of f is defined by

$$\sigma_{\mathrm{mod}}(f) := \inf_{n\geq 1} \frac{|\mathrm{supp}(f) \cap [1, n]|}{n}.$$

3. Hecke Orbits and Additive Generation

Let T_n be Hecke operators acting on f.

Definition 3.1 (Hecke Additive Closure)**.** Define $k \cdot f := f + T_{n_1} f + \cdots + T_{n_{k-1}} f$ for $n_i \in \mathbb{N}$. We say f is Hecke-additively closed if

$$\mathrm{supp}(k \cdot f) = \mathbb{N}$$

for some k.

Proposition 3.2. *If f is a newform with multiplicative coefficients and non-vanishing Dirichlet density, then $k \cdot f$ is additively dense in support.*

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4. Automorphic Representations

Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with Whittaker model W_π.

Definition 4.1 (Automorphic Density)**.** Define

$$\sigma_{\text{auto}}(\pi) := \liminf_{X\to\infty} \frac{|\{n \leq X : a_n(\pi) \neq 0\}|}{X}.$$

Proposition 4.2. *If π corresponds to a modular form with full support modulo m, then $\sigma_{auto}(\pi) > 0$.*

5. Future Work

- Define additive closure laws for general automorphic L-functions
- Extend Schnirelmann-type summability to Rankin-Selberg convolutions
- Develop density filtration categories of automorphic representations
- Analyze the growth of k-fold additive support in Maass and Siegel modular forms