DYADIC LANGLANDS VI: CONDENSED REDUCTIVE STACKS AND UNIVERSAL L-GROUPOIDS

PU JUSTIN SCARFY YANG

Abstract. This sixth paper in the Dyadic Langlands series introduces a formalism for condensed reductive stacks and constructs universal L-groupoids over trace-compatible arithmetic sites. We show that condensed group actions on dyadic shtuka stacks can be extended to a geometric representation of global Langlands parameters as groupoid-valued sheaves. The theory unifies Frobenius trace descent, inverse zeta towers, and automorphic categories into a global condensed groupoid stack, providing the categorical infrastructure for the universal realization of automorphic-to-motivic reciprocity. Applications include spectral transfer functors, categorified L-data, and condensed Hecke orbit spaces.

Contents

1. Introduction

The Langlands program seeks to unify Galois representations and automorphic forms through a global correspondence mediated by L-groups and their parameters. In this sixth installment of the Dyadic Langlands series, we shift the geometric and categorical foundation of this correspondence into the setting of *condensed mathematics*, allowing the incorporation of inverse limit topologies and trace-descent geometries naturally emerging from dyadic cohomological flows.

Date: May 5, 2025.

Goals and Core Construction. This paper develops the categorical infrastructure needed to define *universal condensed L-groupoids* and establishes a moduli stack of *condensed reductive group actions* on dyadic shtuka stacks. Our key contributions include:

- Defining condensed reductive stacks $\mathscr{G}^{\mathrm{cond}}$ over trace sites derived from $\mathbb{Z}_2$-adic cohomology;
- Constructing ∞-groupoids $\mathbb{L}_G^{\mathrm{cond}}$ that encode Langlands parameters as sheaves with Frobenius-trace descent;
- Demonstrating how automorphic data arises functorially as sections over these universal L-groupoids;
- Describing the functorial properties of base change, Hecke symmetries, and spectral stack realization in the condensed setting.

Relation to Prior Work. Previous papers in this series constructed the dyadic shtuka sites, trace-compatible zeta flows, and derived automorphic stacks. The present paper unifies these developments under group-theoretic moduli geometry, forming the representation-theoretic core of the dyadic Langlands philosophy. In particular:

- *Dyadic Langlands III* introduced condensed shtuka stacks and trace Hecke cohomology;
- *Dyadic Langlands IV* studied derived automorphic stacks over trace flows;
- *Spectral Motives VIII* formalized the universal spectral sheaf functor into a condensed arithmetic ∞-topos.

Outline. In Section 2, we define condensed reductive stacks and trace actions. Section 3 constructs universal L-groupoids as moduli groupoids over shtuka descent stacks. Section 4 establishes functoriality, spectral descent, and compatibility with zeta flows. The final section connects these constructions to Langlands reciprocity, categorified trace functions, and future directions in dyadic representation theory.

2. Condensed Reductive Stacks and Trace Group Actions

2.1. **Reductive groups in condensed geometry.** Let G be a reductive group scheme defined over $\mathbb{Z}$. We define its condensed analogue as a sheaf of group objects:

$$G^{\mathrm{cond}} := \underline{G} \in \mathrm{Shv}_{\text{pro-ét}}(\mathrm{Cond}(\mathbb{Z}_2)),$$

where $\mathrm{Cond}(\mathbb{Z}_2)$ denotes the condensed site over the dyadic integers with pro-étale topology. This provides a geometric avatar of G compatible with Frobenius-trace structures.

2.2. **Moduli of condensed torsors.** We define the moduli stack of G^{cond}-torsors over the condensed dyadic shtuka site $\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}$:

$$\mathscr{G}^{\mathrm{cond}} := \mathrm{Bun}_{G^{\mathrm{cond}}}(\mathscr{S}_{\mathrm{sht}}^{\mathrm{cond}}),$$

which classifies condensed vector bundles with G-structure under ζ_n-trace-compatible descent.

This stack is equipped with:

- Frobenius-trace descent structure induced from inverse limits over n;
- Condensed automorphic flows via trace-shtuka morphisms;
- Sheaf-theoretic group actions from condensed Hecke correspondences.

2.3. **Trace group actions and dyadic symmetries.** A condensed trace group action on $\mathscr{G}^{\mathrm{cond}}$ is given by:

$$\mathcal{T}_h \colon \mathscr{G}^{\mathrm{cond}} \to \mathscr{G}^{\mathrm{cond}}, \quad h \in G(\mathbb{A}_f),$$

where h ranges over adelic points of G and acts via condensed Hecke operators defined by convolution of trace sheaves:

$$\mathcal{F} \mapsto \mathcal{F} \star \mathscr{H}_h^{\mathrm{cond}}.$$

These trace group actions preserve the descent structure and allow for defining spectral orbits and trace cohomology classes within $\mathscr{G}^{\mathrm{cond}}$.

2.4. Examples.

(1) For $G = \mathrm{GL}_2$, $\mathscr{G}^{\mathrm{cond}}$ classifies condensed shtuka-sheaves with rank 2 trace-compatible bundles;
(2) For $G = \mathrm{GSp}_{2g}$, the symplectic moduli problem includes additional trace-polarization conditions;
(3) For general G, representations are induced through categorical descent from the trace action on the inverse tower $\{\zeta_n\}$.

These examples are foundational for constructing Langlands parameters stacks and identifying trace-derived automorphic symmetries.

3. Universal L-Groupoids over Condensed Sites

3.1. Langlands parameters and groupoids. Given a condensed reductive stack $\mathscr{G}^{\mathrm{cond}}$ over the dyadic trace site $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$, we define the *universal L-groupoid* $\mathbb{L}^{\mathrm{cond}}_G$ as a functorial groupoid-valued sheaf encoding Langlands parameters:

$$\mathbb{L}^{\mathrm{cond}}_G \colon \mathfrak{T}^{\infty}_{\zeta} \to \infty\text{-Groupoids},$$

where $\mathfrak{T}^{\infty}_{\zeta}$ is the condensed arithmetic ∞-topos introduced in *Spectral Motives VIII*.

Each object in $\mathbb{L}^{\mathrm{cond}}_G$ corresponds to a system of:

- Trace-compatible representations of the dyadic Galois group;
- Descent data from ζ_n-cohomology to global spectral stacks;
- Automorphic realization via trace-preserving morphisms.

3.2. Construction via moduli of condensed shtukas. We define:

$$\mathbb{L}^{\mathrm{cond}}_G := \underline{\mathrm{Hom}}^{\otimes,\mathrm{tr}}(\pi_1^{\mathrm{cond}}, \widehat{G}^{\mathrm{cond}}),$$

where π_1^{cond} is the condensed étale fundamental groupoid of $\mathscr{S}^{\mathrm{cond}}_{\mathrm{sht}}$, and $\widehat{G}^{\mathrm{cond}}$ is the Langlands dual group stack with trace structure.

The superscript tr imposes trace-compatibility with Frobenius descent across all levels:

$$\mathrm{Tr}_{\zeta_n}(\rho_n) = \mathrm{Tr}_{\zeta_{n+1}}(\rho_{n+1}) \quad \forall\, n.$$

3.3. Functoriality and base change. Given a morphism of reductive groups $f \colon G \to H$, we obtain a canonical map of groupoids:

$$f_* \colon \mathbb{L}^{\mathrm{cond}}_G \to \mathbb{L}^{\mathrm{cond}}_H,$$

compatible with trace descent, Frobenius cohomology, and derived Hecke orbits.

This construction endows the category $\mathrm{Red}^{\mathrm{cond}}$ of condensed reductive stacks with a symmetric monoidal functor:

$$G \mapsto \mathbb{L}^{\mathrm{cond}}_G.$$

3.4. Properties of the universal L-groupoid.

- **Universality:** Every trace-compatible Langlands parameter factors uniquely through $\mathbb{L}^{\mathrm{cond}}_G$.
- **Stackiness:** $\mathbb{L}^{\mathrm{cond}}_G$ admits a derived stack structure over the condensed site.
- **Automorphic realization:** Sections of $\mathbb{L}^{\mathrm{cond}}_G$ correspond to automorphic trace sheaves via the universal spectral functor $\mathbb{S}_{\mathrm{univ}}$.

4. Applications to Automorphic Realization and Spectral Reciprocity

4.1. Automorphic sheaves over L-groupoids. Given a section $\rho \in \Gamma(\mathbb{L}_G^{\mathrm{cond}})$, we define its automorphic realization as:

$$\mathrm{Aut}(\rho) := \mathbb{S}_{\mathrm{univ}}(\rho),$$

where $\mathbb{S}_{\mathrm{univ}}$ is the universal spectral sheaf functor from $\mathscr{D}^b(\mathscr{X}^{\mathrm{cond}})$ to the condensed arithmetic ∞-topos $\mathfrak{T}_\zeta^\infty$.

The object $\mathrm{Aut}(\rho)$ lies in the derived automorphic category $\mathscr{D}^b(\mathscr{A}\mathrm{ut}_G^{\mathrm{cond}})$ and satisfies:

- Frobenius trace descent along ζ_n;
- Hecke eigenobject structure via $\mathscr{H}_h^{\mathrm{cond}}$;
- Motivic realization through $\mathscr{M}_{\mathrm{mot}}^{\mathrm{perf}}$.

4.2. Spectral reciprocity functors. To each morphism $\rho : \pi_1^{\mathrm{cond}} \to \widehat{G}^{\mathrm{cond}}$, we associate the *spectral reciprocity functor*:

$$\mathcal{R}_\rho \colon \mathrm{Rep}^{\mathrm{cond}}(\pi_1) \to \mathrm{Coh}^{\mathrm{tr}}(\mathscr{A}\mathrm{ut}_G^{\mathrm{cond}}),$$

sending condensed Galois representations to trace-compatible coherent automorphic sheaves. This is compatible with:

- Trace cohomology $H^\bullet_{\mathrm{Tr}}(-)$;
- Categorified L-functions via derived traces of Hecke flows;
- Functorial transfer under $G \to H$.

4.3. Langlands functoriality in condensed stacks. The following diagram illustrates functoriality of automorphic realization:

$$\begin{tikzcd}
\mathbb{L}_G^{\mathrm{cond}} \arrow[r, "f_*"] \arrow[d, "\mathbb{S}_{\mathrm{univ},G}"'] & \mathbb{L}_H^{\mathrm{cond}} \arrow[d, "\mathbb{S}_{\mathrm{univ},H}"] \\
\mathscr{A}\mathrm{ut}_G^{\mathrm{cond}} \arrow[r, "f_*"] & \mathscr{A}\mathrm{ut}_H^{\mathrm{cond}}
\end{tikzcd}$$

Theorem 4.1 (Trace-Compatible Langlands Functoriality). The above square commutes in the ∞-categorical sense, and preserves:

(1) Frobenius trace descent and spectral cohomology;
(2) Hecke symmetries and automorphic flows;
(3) Universal L-function categories over derived condensed sites.

4.4. Spectral condensation of automorphic L-data. The universal L-groupoid framework allows for defining *spectral L-data* as trace objects in $\mathfrak{T}_\zeta^\infty$:

$$\mathbb{L}(\rho) := \bigoplus_n \mathrm{Tr}(T_h \mid H^n_{\mathrm{Tr}}(\mathrm{Aut}(\rho))),$$

which generalizes classical L-functions to stable ∞-sheaf invariants over condensed arithmetic sites.

5. Conclusion and Outlook

We have introduced the formalism of condensed reductive stacks and universal L-groupoids over dyadic arithmetic sites, completing a critical layer in the infrastructure of the Dyadic Langlands Program. The groupoid-valued sheaves $\mathbb{L}_G^{\mathrm{cond}}$ integrate Frobenius-trace descent, shtuka symmetries, and automorphic realization into a unified geometric representation of Langlands parameters.

This formalism provides:

- A universal condensed moduli framework for Galois representations;
- Derived automorphic sheaf categories compatible with ζ_n-trace flows;

- Functorial Hecke actions and L-function structures encoded categorically.

Future Work. The next stage of development includes:

(1) Integration into condensed motivic cohomology via perfectoid zeta motives;
(2) Applications to categorified L-functions and trace formulas over condensed stacks;
(3) Defining arithmetic representations of condensed Tannakian groupoids;
(4) Formal synthesis with condensed automorphic ∞-topoi and global spectral functoriality.

These ingredients pave the way toward a fully spectral geometric version of the Langlands Program over $\mathbb{Z}_2$-condensed sites.

References

[1] D. Clausen and P. Scholze, *Condensed Mathematics*, 2020. `https://condensed-math.org`
[2] L. Fargues and P. Scholze, *Geometrization of the Local Langlands Correspondence*, Preprint, 2021.
[3] P. J. S. Yang, *The Dyadic Langlands Program I–V*, 2025.
[4] P. J. S. Yang, *Spectral Motives I–VIII*, 2025.
[5] J. Lurie, *Spectral Algebraic Geometry*, 2018.
[6] T. Kaletha, *Galois Categories and the Langlands Program*, in progress.
[7] T. Richarz and J. Scholbach, *The Geometric Satake Equivalence in Mixed Characteristic*, 2017.