WEIGHTED, LOCAL, AND SEGMENTED VARIANTS OF SCHNIRELMANN DENSITY

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Abstract. This paper introduces and explores generalizations of Schnirelmann density, including weighted density, local density, and segmented density. These variants aim to provide more precise control in sparse and structured subsets of the natural numbers, enabling finer analysis in additive number theory.

1. Introduction

The classical Schnirelmann density is defined globally and uniformly across all natural numbers. However, many important sets in number theory are either sparse, irregularly distributed, or have local structural properties. In this paper, we define three generalizations:

- **Weighted Schnirelmann density**
- **Local Schnirelmann density**
- **Segmented Schnirelmann density**

Each concept offers distinct advantages when dealing with additive bases, pseudorandom sets, or computational estimations.

2. Weighted Schnirelmann Density

Definition 2.1 (Weighted Schnirelmann Density)**.** Let $w : \mathbb{N} \to \mathbb{R}_{>0}$ be a weight function. Define the weighted Schnirelmann density of $A \subseteq \mathbb{N}$ by

$$\sigma_w(A) := \inf_{n \geq 1} \frac{\sum_{k \in A \cap [1,n]} w(k)}{\sum_{k=1}^{n} w(k)}.$$

Example 2.2. If $w(k) = 1/k$, the density emphasizes lower integers. If $w(k) = k$, the density highlights higher terms.

Proposition 2.3. *If w is non-increasing, then $\sigma_w(A)$ is dominated by the behavior of A in small intervals.*

3. Local Schnirelmann Density

Definition 3.1 (Local Schnirelmann Density)**.** Fix $h \in \mathbb{N}$ and define, for each $n \in \mathbb{N}$:

$$\sigma_{\mathrm{local}}(A; n, h) := \frac{|A \cap [n, n+h-1]|}{h}.$$

Then define the lower local density as

$$\underline{\sigma}_{\mathrm{local}}(A; h) := \inf_{n \geq 1} \sigma_{\mathrm{local}}(A; n, h).$$

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Remark 3.2. Local Schnirelmann density captures whether A has large gaps over short intervals.

4. Segmented Schnirelmann Density

Definition 4.1 (Segmented Schnirelmann Density)**.** Partition $\mathbb{N}$ into disjoint intervals $I_k = [n_k, n_{k+1})$. Define

$$\sigma_{\text{seg}}(A; \{I_k\}) := \inf_k \frac{|A \cap I_k|}{|I_k|}.$$

Example 4.2. Using dyadic intervals $I_k = [2^k, 2^{k+1})$ reveals how A behaves exponentially across scales.

Proposition 4.3. *If A has full segmented density for dyadic intervals, then A intersects every scale densely.*

5. Comparison and Unification

Proposition 5.1. *Each of the new densities satisfies:*

$$0 \leq \sigma_*(A) \leq 1,$$

where $*$ denotes w, local, or segmented.

Remark 5.2. In sparse or irregular settings, $\sigma(A) = 0$ but $\sigma_{\text{seg}}(A) > 0$, motivating the use of these refined densities.

6. Future Work

We propose further developments:

- Generalizing to measure-theoretic settings over $\mathbb{Z}^d$.
- Combining segmented density with additive combinatorics.
- Establishing additive closure theorems under new density hypotheses.