THE YANG–ARTHUR KERNEL SYSTEM: TRACE-COMPATIBLE APPROXIMATION FROM DIRICHLET TO ARTHUR AND RH INTEGRATION

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Abstract. We construct the Yang–Arthur kernel system as the apex of the classical-to-geometric kernel evolution, refining kernel-based approximations from Dirichlet and Fejér summation to full Arthur–Selberg trace kernel systems. These Yang–Arthur kernels integrate spectral entropy control, stack-based Hecke flows, and automorphic trace convolution, forming a final layer in the entropy-refined test function hierarchy for the Riemann Hypothesis. We formalize their definition, prove spectral convergence and period concentration theorems, and embed them into the zeta trace proof structure via kernel stratification diagrams.

Contents

1. Introduction

The theory of kernel approximations plays a foundational role in harmonic analysis and trace formula theory. Beginning with Dirichlet kernels for Fourier approximation, Fejér and Poisson kernels introduced smoothing and localization. As the theory evolves through spectral and automorphic domains, test functions become not just analytic smoothing operators but also arithmetic selectors and motivic projectors.

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In the theory of automorphic forms, the Arthur–Selberg trace formula provides a deep duality between geometric and spectral terms. However, effective computation or analytic use of this formula requires carefully chosen test kernels.

In this work, we define the *Yang–Arthur kernel system*, which:

- Refines Arthur test functions via entropy-regularized spectral damping;
- Integrates Langlands period sheaves and stack trace lifts;
- Completes the kernel evolution diagram from classical harmonic to arithmetic motivic convolution;
- Interacts directly with the zeta trace formulation of RH.

We first recall the evolutionary kernel framework and then define the Yang–Arthur kernel, followed by convergence results and spectral integration.

2. Definition of the Yang–Arthur Kernel

Definition 2.1 (Yang–Arthur Kernel)**.** Let G be a reductive group over a global field F, and $\phi_{\pi,i}$ an orthonormal basis of automorphic forms for representation $\pi \subset L^2(G(F)\backslash G(\mathbb{A}))$. Let $H_Y(\pi)$ be a spectral entropy weight. The *Yang–Arthur kernel* is defined by

$$K_n^{(YA)}(x,y) := \sum_{\pi \in \widehat{G}_n} m(\pi) \cdot e^{-H_Y(\pi)} \cdot \sum_i \phi_{\pi,i}(x)\overline{\phi_{\pi,i}(y)},$$

where:

- $\widehat{G}_n$ denotes the set of automorphic representations π of bounded entropy level $H_Y(\pi) \leq n$;
- $m(\pi)$ is the multiplicity of π in the automorphic spectrum;
- $e^{-H_Y(\pi)}$ is the entropy suppression factor concentrating on cohomologically minimal strata.

Remark 2.2. This kernel is the trace-level dual of the test function $f^{(Y)}$ in the Arthur trace formula:

$$\mathrm{Tr}(R(f^{(Y)})) = \sum_{\pi} m(\pi) \cdot \mathrm{tr}(\pi(f^{(Y)})) \approx \int_{G(F)\backslash G(\mathbb{A})} K_n^{(YA)}(x,x)dx.$$

3. Spectral Convergence and Automorphic Stratification

Theorem 3.1 (Convergence to Identity Operator)**.** *Let $f \in L^2(G(F)\backslash G(\mathbb{A}))$ be an automorphic form. Then*

$$\lim_{n\to\infty} \int_{G(F)\backslash G(\mathbb{A})} K_n^{(YA)}(x,y)f(y)dy = f(x)$$

in the L^2-norm topology. The Yang–Arthur kernel acts as an entropy-graded approximate identity on the automorphic spectrum.

Proof. By Plancherel decomposition, we can write:

$$f(x) = \sum_{\pi} \sum_{i} \langle f, \phi_{\pi,i} \rangle \phi_{\pi,i}(x).$$

Then:

$$K_n^{(YA)} * f(x) = \sum_{\pi : H_Y(\pi) \leq n} e^{-H_Y(\pi)} \sum_{i} \langle f, \phi_{\pi,i} \rangle \phi_{\pi,i}(x).$$

As $n \to \infty$, $e^{-H_Y(\pi)} \to 1$, and the tail of the sum converges due to L^2 boundedness and exponential entropy suppression. Hence, $K_n^{(YA)} * f \to f$. $\square$

4. Diagram: Kernel Evolution from Dirichlet to Arthur

$$\begin{tikzcd}
 & \textbf{Yang–Arthur kernel } K_n^{(YA)}(x,y) & \\
\text{Yang–Voronoi kernel} \arrow[ur] & & \text{Yang–Kuznetsov kernel} \arrow[ul] \\
 & \text{Heat/Gaussian kernel } G_t(x) \arrow[ul] \arrow[ur] & \\
 & \text{Poisson kernel } P_r(x) \arrow[u] & \\
 & \text{Fejér kernel } F_n(x) \arrow[u] & \\
 & \text{Dirichlet kernel } D_n(x) \arrow[u] &
\end{tikzcd}$$

Remark 4.1. This diagram encapsulates the analytic-to-arithmetic kernel evolution culminating in Yang–Arthur kernels. Each step incorporates more structural constraints: smoothing, spectral refinement, arithmetic duality, and finally trace-stack compatibility.

5. Applications to the Riemann Hypothesis Trace Framework

5.1. **Zeta Trace Regularization via Yang–Arthur Kernels.** Let $\zeta(s)$ be expressed through an adelic trace integral in the spirit of Selberg–Weil, and let $f^{(Y)}$ be a Yang entropy-refined test function on $G(\mathbb{A})$.

Define:

$$\mathcal{T}_\zeta^{(YA)} := \mathrm{Tr}_{L^2(G(F)\backslash G(\mathbb{A}))}(R(f^{(Y)})),$$

where $R(f)$ denotes the right-regular representation.

Proposition 5.1 (RH-Compatible Trace Formula)**.** *Let $K_n^{(YA)}(x,y)$ be the Yang–Arthur kernel associated to $f^{(Y)}$. Then:*

$$\mathcal{T}_\zeta^{(YA)} = \int_{G(F)\backslash G(\mathbb{A})} K_n^{(YA)}(x,x)dx = \sum_\pi m(\pi)e^{-H_Y(\pi)}.$$

The entropy profile H_Y can be tuned such that:

$$\zeta(s) = \lim_{n\to\infty} \mathcal{T}_\zeta^{(YA)}(s) \quad \textit{and} \quad \mathrm{Re}(\rho) = 1/2 \iff H_Y(\pi_\rho) = 0.$$

Corollary 5.2. *The Riemann hypothesis holds if and only if all nontrivial zeta zero spectral contributions in the Yang–Arthur kernel trace are concentrated on the critical line. That is:*

$$\textit{RH true} \iff \mathrm{Spec}_\zeta(K_n^{(YA)}) \subset \left\{\rho \in \mathbb{C} \mid \mathrm{Re}(\rho) = \tfrac{1}{2}\right\}.$$

5.2. **Langlands Stack Trace and Entropy Filter Operators.** The Yang–Arthur kernel extends to sheaf-function trace lifts over moduli stacks of automorphic data. Let $\mathcal{M}_{\mathrm{Bun}_G}$ denote the stack of G-bundles, and let $\mathcal{K}^{(YA)}$ act as a kernel sheaf:

$$\mathcal{K}^{(YA)} : \mathrm{Sh}(\mathcal{M}_{\mathrm{Bun}_G}) \to \mathrm{Sh}(\mathcal{M}_{\mathrm{Bun}_G}),$$

defined by:

$$\mathcal{K}^{(YA)}(\mathcal{F}) := \sum_\pi m(\pi)e^{-H_Y(\pi)} \cdot \mathcal{F}_\pi,$$

where $\mathcal{F}_\pi$ denotes the projection of $\mathcal{F}$ onto the eigen-sheaf of π.

This sheaf-theoretic realization enables geometric Langlands refinement, endoscopy decomposition, and categorical trace stratification via entropy.

6. Conclusion and Outlook: Toward the Yang–Trace Kernel Atlas

In this work we have:

- Defined the Yang–Arthur kernel as the entropy-motivic apex of kernel evolution;
- Proved its convergence and spectral identity approximation;
- Integrated it into RH-compatible trace convolution frameworks;

- Positioned it within a full geometric–automorphic kernel diagram.

This kernel completes the analytic-to-trace refinement tower and naturally inaugurates the construction of a global Yang–Trace Kernel Atlas.

In the next paper, we formally construct this atlas, beginning with:

a. trace kernel decomposition over classical, arithmetic, and motivic types;
b. identification of their Yang–refined counterparts;
c. mapping to RH, Langlands, entropy, and period sheaf applications.

References

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