VOLUME III: WEIGHT–MONODROMY CONJECTURES BEYOND LINEAR CASES
NEW CONJECTURES AND RESULTS IN MULTIPLICATIVE, EXPONENTIAL, AND HYPER-GROWTH GEOMETRIES

PU JUSTIN SCARFY YANG

ABSTRACT. This volume develops a trans-recursive generalization of the classical Weight–Monodromy Conjecture (WMC). By lifting the theory from linear and additive filtrations to multiplicative, exponential, and Knuth-level growth structures, we formulate and analyze a family of hyper-WMC conjectures governing the cohomological and motivic behavior of arithmetic spaces under non-linear and transfinite stratification. These new perspectives yield refined period morphisms, torsor regulators, and ε-geometric constraints, extending the Scholze–Deligne framework into infinite-depth growth regimes.

CONTENTS

Date: May 10, 2025.

0. Notation and Symbol Dictionary

This section catalogs the core symbols, towers, conjecture notation, and filtration structures used in this volume. We focus on stratified growth geometries beyond additive cases, including multiplicative, exponential, and higher hyperoperation settings.

Filtration Structures.

- $\mathrm{Fil}^g_n \mathcal{F}$: the n-th level filtration on $\mathcal{F}$ indexed by a growth function $g(n)$.
- $g(n)$: growth types such as n (additive), 2^n (multiplicative), e^n (exponential), $a \uparrow^k n$ (Knuth-type).
- $\mathrm{gr}^g_n \mathcal{F}$: graded piece corresponding to Fil^g_n.

Weight–Monodromy Conjecture Notation.

- $\mathrm{WMC}_{\mathrm{lin}}$: classical (additive/valuation-based) weight–monodromy conjecture.
- $\mathrm{WMC}_{\times}$: multiplicative WMC, defined over 2^n-indexed towers.
- $\mathrm{WMC}_{\exp}$: exponential WMC, over $\exp(n)$ stratifications.
- $\mathrm{WMC}_{\uparrow^k}$: Knuth-level WMC over $a \uparrow^k n$ indexing.

Weight and Monodromy Actions.

- $\mathrm{Weight}^{[g]}$: weight filtration associated to $g(n)$ growth.
- $\mathrm{Mono}^{[g]}$: monodromy operator indexed by growth function $g(n)$.
- Nil_g : the g-nilpotent monodromy space, i.e., where ${\mathrm{Mono}^{[g]}}^k = 0$ for finite k.

Cohomological and Motivic Towers.

- $H^i_{g(n)}(X, \mathcal{F})$: cohomology at depth $g(n)$.
- $M^{[g]}(X)$: motive indexed by $g(n)$-filtration layers.
- $\mathrm{real}_{\mathrm{wmc}}$: realization functor from $M^{[g]}(X)$ to cohomology with growth stratification.
- r_{wmc} : regulator associated with weight–monodromy structures in a non-linear filtration context.

Growth Categories.

- Growth : category of stratification growth types with $f \to g$ if $f(n) \leq g(n)$ eventually.
- WMC-Tors : category of torsors structured by monodromy descent under growth filtrations.

Torsors and Autoequivalences.

- $\mathcal{T}_n^{[g]}$: torsor at filtration level $g(n)$.
- $\mathrm{Aut}^{[g]}$: group of autoequivalences under monodromy indexed by $g(n)$.
- $\rho^{[g]}$: representation of $\mathcal{T}_n^{[g]}$ on cohomology groups with g-filtration.

General Conventions. Throughout this volume:

- All growth functions $g(n)$ are strictly increasing and recursive;
- Cohomology and motives are assumed to be defined over a suitable base (e.g., $\mathbb{Z}_p$, perfectoid towers, or motivic sites);
- ε continues to denote ε-stratified filtration layers defined in Volume II;
- Compatibility with $\mathcal{M}_{\mathrm{hyper}}$ and persistence stability under infinite descent is assumed unless stated otherwise.

1. Introduction: From Linear WMC to Recursive WMC

1.1. **From Classical to Non-Linear Stratification.** The classical *Weight–Monodromy Conjecture (WMC)* posits that for a proper smooth scheme over a p-adic field, the monodromy operator N acting on p-adic étale cohomology induces a filtration (the monodromy filtration) that corresponds to the weight filtration expected from Hodge theory or mixed motives. This setting is inherently linear:

$$\mathrm{Fil}^n H^i \sim \text{weight } \leq n, \quad N^k = 0, \quad \text{for some } k \in \mathbb{Z}.$$

In this volume, we ask:

> What becomes of the WMC in geometries where the filtration is multiplicative, exponential, or hyper-recursive in nature?

We aim to generalize the WMC beyond additive filtrations, into *recursive stratified geometries* with growth types like 2^n, $\exp(n)$, and $a \uparrow^k n$.

1.2. **Motivation from Hyperfiltration Theory.** Volume II developed ε-hyperfiltrations and the structure of transfinite towers, where sections and cohomology are not measured by length or valuation, but by their survival through recursive collapse layers. These structures naturally suggest:

- New torsors indexed by growth functions;
- Cohomology stabilized across exponential descent;
- Monodromy acting over non-additive towers;
- Realization functors compatible with recursive group actions.

This motivates a reformulation of WMC over these growth types.

1.3. **Core Concept: Growth-Typed Monodromy.** Let $g(n)$ be a growth function (e.g., 2^n, e^n, $a \uparrow^k n$). Then we define:

- A monodromy operator $\mathrm{Mono}^{[g]}$ acting at level $g(n)$;
- A corresponding weight filtration $\mathrm{Weight}^{[g]}$;
- A tower of sheaves or motives $M^{[g]}(X)$ stratified by Fil_n^g;
- A realization $\mathrm{real}_{\mathrm{wmc}}$ and regulator r_{wmc} compatible with this structure.

1.4. **Reformulated WMC over Growth Structures.** We propose a general **Growth-Based WMC**:

> Let X be an arithmetic space with filtration indexed by $g(n)$. Then the monodromy operator $\mathrm{Mono}^{[g]}$ defines a weight filtration $\mathrm{Weight}^{[g]}$ on H^i such that:
>
> $$\text{Graded pieces } \mathrm{gr}_n^g H^i \text{ are of weight } w = i - 2n.$$

This interpretation recovers the classical WMC when $g(n) = n$, and extends to all stratified geometries.

1.5. **Scope and Structure of This Volume.** This volume establishes:

- The *multiplicative WMC* ($\mathrm{WMC}_\times$), stratifying by 2^n towers;
- The *exponential WMC* ($\mathrm{WMC}_{\exp}$), based on e^n-growth filtrations;
- The *hyper-WMC* family ($\mathrm{WMC}_{\uparrow^k}$), over $a \uparrow^k n$ growth types;
- Meta-regulators, torsors, and monodromy compatibility across stratification types;
- Spectral consequences and examples across filtered cohomology classes.

1.6. **Outline of the Sections.**

Section 1: **Multiplicative Filtrations** and their induced torsor towers;
Section 2: **Exponential Period Structures** and weight-matching conjectures;
Section 3: **Knuth-Level Structures** with infinite-depth stratification;

Section 4: **Motivic Realizations** over stratified cohomology towers;
Section 5: **Hyper-Conjectures** generalizing WMC to ε-geometric and non-linear towers;
Section 6: **Persistence Collapse and ε-Geometric Criteria** for motivic weight filtration;
Section 7: **Unified Diagrammatic View** of monodromy, weight, growth, and realization.

The classical monodromy operator acts on valuation. The recursive monodromy acts on existence itself.

2. Multiplicative Filtrations and Non-Additive Weights

2.1. **From Additive to Multiplicative Filtration Structures.** Let us begin with the first level of generalization: replacing additive filtrations (e.g., Fil^n) with multiplicative stratifications indexed by 2^n.

Let $\mathcal{F}$ be a sheaf or a cohomological object. We define the multiplicative filtration by:

$$\mathrm{Fil}_n^{\times}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/2^n\mathcal{F}\right), \quad \text{with } \mathrm{Fil}_{n+1}^{\times} \subset \mathrm{Fil}_n^{\times}.$$

This filtration structure refines valuation-based descent by exponentially stratifying the congruence.

2.2. **Multiplicative Monodromy and Weight.** Let $N_\times := \mathrm{Mono}^{[\times]}$ denote the monodromy operator acting over this tower. We define:

- $\mathrm{Weight}_n^{[\times]}$:= the weight filtration level corresponding to 2^n-indexed stratification;
- $N_\times^k = 0$ for some k finite $\Rightarrow$ torsor trivialization over level 2^k;
- Graded pieces $\mathrm{gr}_\times^n := \mathrm{Fil}_n^\times/\mathrm{Fil}_{n+1}^\times$.

2.3. **Formulation of the Multiplicative WMC.**

Conjecture 2.1 (Multiplicative Weight–Monodromy Conjecture $\mathrm{WMC}_\times$)**.** *Let X be a proper smooth variety over a 2-adic or $\mathbb{Q}$-dyadic base, and let $\mathcal{F}$ be a sheaf with multiplicative filtration. Then:*

(1) There exists a monodromy operator $N_\times$ satisfying $N_\times^k = 0$;
(2) The associated graded $\mathrm{gr}_\times^n H^i(X, \mathcal{F})$ is pure of weight $w = i - 2n$;
(3) This filtration is realized through stratified cohomology descent over 2^n-torsors.

2.4. **Motivic Realization and Multiplicoid Period Rings.** Let $M^{[\times]}(X)$ be a motive in the multiplicative filtration category. Define:

$$\mathrm{real}_{\mathrm{wmc}}(M) := \varprojlim_n H^\bullet\left(X, M/2^nM\right), \quad r_{\mathrm{wmc}} : K_n(X) \to H^n_{2^n}(X, \mathbb{Q}(n)).$$

We introduce the period ring:

Definition 2.2 (Multiplicoid Period Ring)**.**

$$B_{\times,\mathrm{dR}} := \varprojlim_n A/2^n \otimes \mathbb{Q},$$

with a compatible filtration indexed by 2^n*.*

2.5. **Torsors and Monodromy Realization.** Each filtration level corresponds to a torsor $\mathcal{T}_n^{[\times]}$ under the group $\mathbb{Z}/2^n\mathbb{Z}$:

$$\mathcal{T}_n^{[\times]} \curvearrowright \mathrm{Fil}_n^{\times}\mathcal{F}.$$

The entire tower of monodromy action:

$$\mathrm{Torsor}_{\times} := \left\{\mathcal{T}_n^{[\times]}\right\}_{n\in\mathbb{N}}, \quad \mathcal{M}_{\mathrm{hyper}_{\times}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\times]})$$

acts as the symmetry group of multiplicative filtration geometry.

2.6. **Spectral Description and Comparison.** Let $\mathcal{F}$ be a cohomological object with $\mathrm{Fil}_n^{\times}$ filtration. Define the multiplicative spectral sequence:

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_{\times}^{p}\mathcal{F}) \quad \Rightarrow \quad H_{\times}^{p+q}(X, \mathcal{F}).$$

Its convergence reflects recursive stabilization under exponential torsor descent.

2.7. **Examples.**

- Perfectoid spaces naturally admit 2^n-indexed congruence towers, allowing reinterpretation under $\mathrm{WMC}_{\times}$.
- Mod-2^n syntomic cohomology fits within this stratification;
- Dyadic étale towers yield Galois torsors $\mathbb{Z}_2/2^n$ which correspond to $\mathcal{T}_n^{[\times]}$.

2.8. **Conclusion.** This section establishes:

- The foundation of multiplicative filtration theory;
- A conjectural weight–monodromy correspondence under 2^n-growth;
- Realization functors, period rings, and torsor symmetry under multiplicative depth;
- A stepping-stone toward exponential and hyper-growth versions.

In the next section, we generalize this structure to exponential filtration and define $\mathrm{WMC}_{\mathrm{exp}}$ with faster cohomological stratification.

3. Exponential Weight–Monodromy Towers

3.1. **Beyond Multiplicative Growth: Exponential Filtrations.** We now refine our geometric stratification further: from multiplicative layers (2^n) to exponential growth (e^n or similar). This allows for deeper torsor towers, faster convergence of period morphisms, and more subtle cohomological behavior.

Definition 3.1 (Exponential Filtration)**.** *Let $\mathcal{F}$ be a filtered sheaf or cohomological object. Define the exponential filtration:*

$$\mathrm{Fil}_n^{\exp}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\exp(n) \cdot \mathcal{F}\right),$$

with stratification indexed by the rapidly growing function $\exp(n)$*.*

This filtration refines both additive and multiplicative structures and fits naturally into ε-hyperfiltration theories.

3.2. **Exponential Monodromy and Weight Structure.** Let $N_{\exp} := \mathrm{Mono}^{[\exp]}$ act across $\mathrm{Fil}_n^{\exp}\mathcal{F}$. Define the weight filtration:

$$\mathrm{Weight}_n^{[\exp]} := \text{Image of } N_{\exp}^n \text{ on } \mathrm{Fil}_n^{\exp}\mathcal{F}.$$

We postulate:

Conjecture 3.2 (Exponential Weight–Monodromy Conjecture $\mathrm{WMC}_{\exp}$)**.** *Let X be an arithmetic space equipped with exponential stratification. Then:*

1. *There exists a nilpotent monodromy operator $N_{\exp}$ acting over* $\exp(n)$*-indexed torsors;*
2. *The graded pieces $\mathrm{gr}^n_{\exp}H^i(X,\mathcal{F})$ are pure of weight $w = i - 2n$;*
3. *All realizations stabilize under* $\exp(n)$*-growth descent.*

3.3. **Period Rings and Exponential Cohomology.** Define the exponential period ring:

$$B_{\exp,\mathrm{dR}} := \varprojlim_n A/\exp(n) \cdot A \otimes \mathbb{Q}.$$

It carries a natural $\exp(n)$-indexed filtration and admits comparison morphisms with both $B_{\times,\mathrm{dR}}$ and $B_{\varepsilon^\infty,\mathrm{dR}}$ from Volume II.

Let:

$$\mathrm{real}^{\exp}_{\mathrm{wmc}}(M) := \varprojlim_n H^i(X, M/\exp(n)M), \quad r^{\exp}_{\mathrm{wmc}} : K_n(X) \to H^n_{\exp}(X, \mathbb{Q}(n)).$$

3.4. **Torsor Actions and Monodromy Representation.** Let $\mathcal{T}_n^{[\exp]}$ denote torsors corresponding to congruences modulo $\exp(n)$:

$$\mathcal{T}_n^{[\exp]} \curvearrowright \mathrm{Fil}_n^{\exp}\mathcal{F}, \quad \mathcal{M}_{\mathrm{hyper}_{\exp}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\exp]}).$$

These torsors are less accessible via classical Galois descent, but may arise through syntomic–motivic–perfectoid hybrid categories or ε-stratified stacky realizations.

3.5. **Spectral Structures and Stabilization.**

Proposition 3.3 (Exponential Spectral Sequence). *Let $\mathcal{F}$ be exponentially filtered. Then:*

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}^p_{\exp}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\exp}(X, \mathcal{F}),$$

with differentials reflecting torsor-deviation between consecutive exponential levels.

3.6. **Motivic Realization in Exponential Tower.** Let $M^{[\exp]}(X)$ be the motive stratified by $\exp(n)$-filtration:

$$\mathrm{real}_{\mathrm{wmc}}(M^{[\exp]}) = \{\mathrm{Fil}_n^{\exp} H^i(M)\}_n.$$

We conjecture that:

- The exponential realization induces a fully faithful functor from K-theory;
- Regulator values in $B_{\exp,\mathrm{dR}}$ correspond to special exponential-type motivic periods;
- Stratification reflects not valuation but transcendental complexity class.

3.7. **Example: Iterated Polylogarithmic Sheaves.** The tower of polylogarithmic realizations $\mathcal{L}i_n$ naturally admits exponential decay in weight:

$$\mathrm{Fil}_n^{\exp}\mathcal{L}i := \mathrm{Span}\left\{\mathrm{Li}_k \mid k \geq \exp(n)\right\}.$$

This suggests that exponential stratification aligns with transcendence-weight filtration, motivating a reinterpretation of WMC in terms of period depth rather than cohomological dimension alone.

3.8. **Conclusion.** In this section, we developed:

- Exponential filtration towers $\mathrm{Fil}_n^{\exp}$;
- Torsor actions and period ring $B_{\exp,\mathrm{dR}}$;
- Exponential WMC $\mathrm{WMC}_{\exp}$ as a recursive-refined conjecture;
- Applications to transcendental realizations and meta-motivic regulators.

4. Higher Hyper-Operations and Knuth-Type Monodromy

4.1. **From Exponentiation to Hyperoperations.** After additive, multiplicative, and exponential filtrations, we now enter the domain of Knuth's *hyperoperation hierarchy*:

$$a \uparrow n, \quad a \uparrow\uparrow n, \quad a \uparrow\uparrow\uparrow n, \quad \dots$$

We use the notation $a \uparrow^k n$ for the k-th hyperoperation applied n times. These functions define ultra-rapidly growing filtration layers that model *trans-recursive descent* in cohomology and torsor stratification.

4.2. **Knuthoid Filtration Structures.**

Definition 4.1 (Knuth-Type Filtration)**.** *Let $\mathcal{F}$ be a sheaf. Fix $k \in \mathbb{N}$ and define:*

$$\mathrm{Fil}_n^{\uparrow^k} \mathcal{F} := \ker \left(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n) \cdot \mathcal{F} \right).$$

These layers generalize exponential filtrations and induce recursive collapse over deep logical strata.

4.3. **Monodromy Operators and Collapse Towers.** Define the Knuth-monodromy operator $N_{\uparrow^k} := \mathrm{Mono}^{[\uparrow^k]}$ such that:

$$N_{\uparrow^k}^m = 0 \quad \text{iff } \mathcal{F} \text{ becomes trivial at level } a \uparrow^k m.$$

The associated weight filtration is:

$$\mathrm{Weight}_n^{[\uparrow^k]} := \text{Image of } N_{\uparrow^k}^n.$$

4.4. **Hyper-WMC Conjecture.**

Conjecture 4.2 (Knuth-Type Weight–Monodromy Conjecture $\mathrm{WMC}_{\uparrow^k}$)**.** *Let X be an ε-geometric or recursively stratified space. Then:*

- *There exists a nilpotent operator $N_{\uparrow^k}$ compatible with the filtration $\mathrm{Fil}_n^{\uparrow^k}$;*
- *The graded pieces $\mathrm{gr}_{\uparrow^k}^n H^i(X, \mathcal{F})$ are pure of weight $w = i - 2n$;*
- *The associated torsor tower stabilizes under hyperoperation-indexed descent.*

4.5. **Torsor Towers and Meta-Galois Symmetry.** Define the tower of torsors:

$$\mathcal{T}_n^{[\uparrow^k]} := \text{Torsor at filtration level } a \uparrow^k n, \quad \mathcal{M}_{\mathrm{hyper}_{\uparrow^k}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}_n^{[\uparrow^k]}).$$

These torsors may arise from:

- Infinite congruence systems over generalized recursive Galois fields;
- ε-stratified period topologies with hyper-logarithmic moduli;
- Formal geometry over towers of logic-indexed stacks.

4.6. **Realizations and Period Structures.** Let $B_{\uparrow^k,\mathrm{dR}}$ denote the Knuthoid period ring:

$$B_{\uparrow^k,\mathrm{dR}} := \varprojlim_n A/(a \uparrow^k n) \cdot A \otimes \mathbb{Q}.$$

Then for a motive $M^{[\uparrow^k]}$, define:

$$\mathrm{real}^{\uparrow^k}_{\mathrm{wmc}}(M) := \varprojlim_n H^i(X, M/(a \uparrow^k n) \cdot M), \quad r^{\uparrow^k}_{\mathrm{wmc}} : K_n(X) \to H^n_{\uparrow^k}(X, \mathbb{Q}(n)).$$

4.7. **Spectral Stabilization and Recursive Spectra.** Each filtration layer gives rise to:

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}^p_{\uparrow^k}\mathcal{F}) \quad \Rightarrow \quad H^{p+q}_{\uparrow^k}(X, \mathcal{F}).$$

The complexity of differentials grows with k, corresponding to deeper logical relations among strata.

4.8. **Ontological Interpretation.** The $\mathrm{WMC}_{\uparrow^k}$ conjectures connect geometry to higher-order computability:

- For $k = 1$: exponential descent $\to$ analytic depth;
- For $k = 2$: hyper-exponential $\to$ provability towers;
- For $k \to \infty$: cohomological persistence aligns with meta-ontological logical foundations.

4.9. **Conclusion.** This section introduces:

- Knuth-type filtration systems $\mathrm{Fil}^{\uparrow^k}_n$;
- Generalized monodromy $N_{\uparrow^k}$, torsors $\mathcal{T}^{[\uparrow^k]}_n$, and symmetry groups $\mathcal{M}_{\mathrm{hyper}_{\uparrow^k}}$;
- Weight–Monodromy Conjectures over stratifications indexed by higher-order operations;
- Period realizations and speculative ontological parallels.

The next section focuses on how these towers interact with motivic structures and torsor regulators in the framework of recursive realization theories.

5. Stratified Motivic Realizations and Torsor Regulators

5.1. **Motivic Foundations for Growth-Based Filtrations.** In each of the monodromy contexts—additive, multiplicative, exponential, and Knuth-type—motives can be stratified according to their filtration depth. We consider towers of the form:

$$M^{[g]}(X) = \{\mathrm{Fil}^g_n M\}_n\,, \quad \text{with } g(n) \in \mathsf{Growth},$$

where $M \in \mathsf{DM}^{\varepsilon^\infty}$, the derived category of ε-stratified motives.

5.2. **Realization Functors and Stratified Cohomology.** The realization functor $\mathrm{real}_{\mathrm{wmc}}$ acts fiberwise across filtration strata:

$$\mathrm{real}^{[g]}_{\mathrm{wmc}} : M^{[g]}(X) \longrightarrow \left\{H^i_{g(n)}(X, \mathbb{Q})\right\}_n .$$

These realizations respect monodromy, torsor symmetry, and weight filtration conditions under each growth type $g(n)$.

5.3. **Stratified Torsors and Torsor Regulators.** Each filtration level defines a torsor $\mathcal{T}^{[g]}_n$ acting on $\mathrm{Fil}^g_n M$. Define:

$$\mathbb{T}^{[g]} := \left\{\mathcal{T}^{[g]}_n\right\}_n, \quad \mathcal{M}_{\mathrm{hyper}_{[g]}} := \varprojlim_n \mathrm{Aut}(\mathcal{T}^{[g]}_n).$$

Then, torsor regulators become natural maps:

Definition 5.1 (Torsor-Regulator Map)**.**

$$r^{[g]}_{wmc} : K_n(X) \longrightarrow H^n_{g(n)}(X, \mathbb{Q}(n)),$$

compatible with $\mathcal{M}_{hyper_{[g]}}$-actions and weight decompositions under $\mathrm{Weight}^{[g]}$.

5.4. **Cohomological Realization Towers.** Let $\mathcal{F} = \mathrm{real}^{[g]}_{\mathrm{wmc}}(M)$. Then:

$$H^i_{[g]}(X) := \varprojlim_n H^i(X, \mathrm{Fil}^g_n \mathcal{F})$$

is the cohomological realization tower indexed by $g(n)$. The filtration-stabilized cohomology measures the depth of motivic persistence.

5.5. **Growth-Filtrations on Period Rings.** Each motive $M^{[g]}$ admits a period realization into a filtered ring $B_{g,\mathrm{dR}}$, for example:

$$B_{\times,\mathrm{dR}}, \quad B_{\exp,\mathrm{dR}}, \quad B_{\uparrow^k,\mathrm{dR}}.$$

The sheaf of periods is a module over this filtered ring, and torsor symmetry transfers through:

$$\mathbb{T}^{[g]} \curvearrowright B_{g,\mathrm{dR}}.$$

5.6. **Duality and ε-Stratified Pairings.** We define pairings on realization layers:

$$\langle -, - \rangle_{\mathrm{mot}}^{[g]} : H^i_{g(n)}(X) \otimes H^{2d-i}_{g(n)}(X) \to B_{g,\mathrm{dR}},$$

compatible with monodromy weight shifts and motivic period interpretation.

5.7. **Motivic Period Torsors.** Period torsors encode how motives deform under regulator maps:

$$\mathcal{P}_n^{[g]} := \mathrm{Hom}_{\mathrm{mot}}\left(M^{[g]}(X), \mathbb{Q}(n)\right), \quad \mathcal{P}^{[g]} := \varprojlim_n \mathcal{P}_n^{[g]}.$$

These are naturally torsors under $\mathcal{M}_{\mathrm{hyper}_{[g]}}$, with fibers corresponding to regulator realizations.

5.8. **Example: Logarithmic Growth and Polylogarithmic Towers.** Let $\mathcal{L}i_n^{[\log]} := \mathrm{Span}(\mathrm{Li}_k \mid k \leq \log(n))$, then:

- $\mathcal{L}i_n^{[\log]}$ forms a partial realization of $\mathrm{gr}^n_{\uparrow^1}$;
- The monodromy action corresponds to iterations of differential operators (polylog recursion);
- The realization tower interpolates between weight stratification and transcendence degree.

5.9. **Conclusion.** This section develops the motivic side of growth-based monodromy theory:

- Motivic towers $M^{[g]}$, with growth-indexed filtration and realization;
- Stratified torsors $\mathbb{T}^{[g]}$, period rings $B_{g,\mathrm{dR}}$, and regulator maps $r^{[g]}_{\mathrm{wmc}}$;
- Duality, pairing, and spectral structures tied to growth levels;
- Meta-periods via torsor deformations and monodromy symmetry.

In the next section, we formulate generalized conjectures in each of these contexts—exponential, hyper-recursive, and ε-stratified—to construct a unified meta-WMC framework.

6. Hyper-Conjectures: WMC in Infinite Filtration Settings

6.1. **Beyond Finite Nilpotence: Toward Infinite Descent.** In classical and growth-indexed WMC, the monodromy operator N is nilpotent: $N^k = 0$ for finite k. In ε-stratified and transfinite settings, such an operator may instead converge only in a projective system or meta-limit sense.

We now propose **hyper-conjectures** extending WMC into infinite filtrations, ε-cohomology towers, and categorical arithmetic structures.

6.2. **Hyper-WMC Meta-Conjecture.**

Conjecture 6.1 (Meta Weight–Monodromy Conjecture). *Let X be an ε-stratified space with filtration tower $F^{\varepsilon^n}\mathcal{F}$. Then there exists a transfinite operator*

$$N_{\varepsilon^\infty} : \mathcal{F} \to \mathcal{F}$$

satisfying:

(1) For each n, $N_{\varepsilon^\infty}^{(n)} = Res_n(N_{\varepsilon^\infty})$ acts as monodromy over $F^{\varepsilon^n}\mathcal{F}$;
(2) The associated graded pieces $\mathrm{gr}_\varepsilon^n H^i(X, \mathcal{F})$ are pure of weight $i - 2n$;
(3) There exists a motivic realization $M^{[\varepsilon^\infty]}(X)$ whose image under real_{wmc} yields this tower.

6.3. **Regulator Stability and Meta-Monodromy.**

Conjecture 6.2 (Meta-Regulator Universality). *There exists a universal regulator:*

$$r_{wmc}^{[\infty]} : K_n(X) \longrightarrow H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n)),$$

compatible with all finite-level growth regulators, and functorial under stratified monodromy towers.

6.4. **Transfinite Duality and Stratified Pairing Stability.**

Conjecture 6.3 (ε-Stratified Meta-Duality). *For ε-persistent sheaves $\mathcal{F}, \mathcal{G}$, the pairing:*

$$\langle -, - \rangle_{\varepsilon^\infty} : H^\bullet_{\varepsilon^\infty}(X, \mathcal{F}) \otimes H^\bullet_{\varepsilon^\infty}(X, \mathcal{G}) \to B_{\varepsilon^\infty, \mathrm{dR}}$$

is perfect and respects meta-weight decompositions.

6.5. Collapse-Stability Conjecture.

Conjecture 6.4 (Spectral Collapse at Transfinite Depth)**.** *The spectral sequence induced by ε-stratified filtration:*

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_\varepsilon^p \mathcal{F}) \Rightarrow H_{\varepsilon^\infty}^{p+q}(X, \mathcal{F})$$

degenerates at a finite stage if and only if $\mathcal{F} \in \mathbf{Ont}_{\varepsilon^\infty}$, i.e., it belongs to the category of persistent ontological sheaves.

6.6. Diagrammatic Synthesis of WMC Structures.

$$\begin{tikzcd}
K_n(X) \arrow[r, "r_{\mathrm{wmc}}^{[\infty]}"] \arrow[d] & H^n_{\varepsilon^\infty}(X, \mathbb{Q}(n)) \arrow[d, "\mathrm{gr}_{\varepsilon^n}"] & \\
M^{[\varepsilon^\infty]}(X) \arrow[r, "\mathrm{real}_{\mathrm{wmc}}"] & \mathcal{F}^{[\varepsilon^\infty]} \arrow[r] & B_{\varepsilon^\infty,\mathrm{dR}}
\end{tikzcd}$$

6.7. Hyper-WMC Family as ε-Indexed Sheaf Theory.

Let:

$$\mathcal{WMC}_{[g]} := \left(\mathrm{Fil}_n^g, N_{[g]}, \mathrm{Weight}_n^{[g]}\right)$$

and define the family:

$$\mathcal{WMC}_{\varepsilon^\infty} := \left\{\mathcal{WMC}_{[g]} \mid g \in \mathsf{Growth}\right\}$$

Conjecture 6.5 (Stratified Sheaf–WMC Correspondence)**.** *There exists a functor:*

$$\mathsf{Sh}^{\varepsilon^\infty} \longrightarrow \mathsf{WMC}_{\varepsilon^\infty}$$

assigning to each persistent sheaf a complete transfinite weight–monodromy structure.

6.8. Conclusion.

In this section, we formulate:

- Universal versions of WMC over infinite ε-hyperfiltrations;
- Theories of transfinite regulator maps and stratified duality;
- Collapse stability principles and spectral conjectures;
- A categorical framework for WMC-family classification over Growth.

In the next section, we analyze cohomological collapse spectra and persistence profiles, tying ε-geometric survival directly to motivic monodromy patterns.

7. Cohomological Collapse, Persistence Spectra, and ε-Geometric WMC

7.1. Cohomological Collapse Across Filtration Towers. Let $\mathcal{F}$ be a sheaf equipped with a tower of filtrations $F^{\varepsilon^n}\mathcal{F}$. The sequence of cohomology groups:

$$H^i\left(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right) \longrightarrow H^i\left(X, \mathcal{F}/\varepsilon^{n-1} \cdot \mathcal{F}\right) \longrightarrow \cdots$$

forms a descending system. We say that **cohomological collapse** occurs when stabilization is observed at some finite stage.

Definition 7.1 (Collapse Depth)**.** *The minimal n such that*

$$H^i\left(X, \mathcal{F}/\varepsilon^k \cdot \mathcal{F}\right) \simeq H^i\left(X, \mathcal{F}/\varepsilon^n \cdot \mathcal{F}\right), \quad \forall k \geq n$$

is called the collapse depth *of $H^i(\mathcal{F})$.*

If no such n exists, the cohomology is said to be *transfinitely unstable.*

7.2. Persistence Spectrum and Collapse Profile. For each degree i, we define the **persistence spectrum** of $\mathcal{F}$:

$$\mathrm{PersSpec}_i(\mathcal{F}) := \left\{\alpha \in \mathbb{N} \cup \infty \mid F^{\varepsilon^\alpha} H^i(X, \mathcal{F}) \neq 0\right\}.$$

Definition 7.2 (Cohomological Persistence Profile)**.** *Let $\mathcal{F}$ be a stratified sheaf. Its persistence profile is the function:*

$$P_{\mathcal{F}}(i) := \sup\{\alpha \mid F^{\varepsilon^\alpha} H^i(X, \mathcal{F}) \neq 0\}.$$

This function measures how long cohomological components survive along the ε-stratification tower.

7.3. Persistence-Based Weight–Monodromy Classification. Given a sheaf $\mathcal{F}$ with persistence profile $P_{\mathcal{F}}(i)$, we define:

Definition 7.3 (Persistence-Classified WMC Type)**.** *A sheaf satisfies* $\mathrm{WMC}^{[\infty]}$ *of type (i, α) if*

$$P_{\mathcal{F}}(i) = \alpha, \quad \textit{and } \mathrm{gr}^{\alpha}_{\varepsilon} H^i(X, \mathcal{F}) \textit{ is pure of weight } i - 2\alpha.$$

This allows a classification of WMC-behavior indexed by persistence layers rather than growth type alone.

7.4. **Stratified Collapse Spectra.** Define the *collapse spectrum* $\mathcal{C}_{\mathcal{F}}$ as the collection:

$$\mathcal{C}_{\mathcal{F}} := \left\{(i, \alpha) \mid H^i(X, F^{\varepsilon^\alpha}\mathcal{F}) \neq 0\right\}.$$

We conjecture:

Conjecture 7.4 (ε-Geometric Collapse Rigidity)**.** *For $\mathcal{F} \in \mathbf{Ont}_{\varepsilon^\infty}$, the collapse spectrum $\mathcal{C}_{\mathcal{F}}$ is finite and algebraically stratified by $\mathcal{WMC}_{\varepsilon^\infty}$.*

7.5. **Diagrammatic Stratification of Collapse.**

$$\begin{tikzcd}
F^{\varepsilon^{n+1}}\mathcal{F} \arrow[r, hook] \arrow[d] & F^{\varepsilon^n}\mathcal{F} \arrow[r] \arrow[d] & \mathrm{gr}^n_\varepsilon \mathcal{F} \arrow[d, "\text{Weight } i-2n"] \\
H^i(X, F^{\varepsilon^{n+1}}\mathcal{F}) \arrow[r] & H^i(X, F^{\varepsilon^n}\mathcal{F}) \arrow[r] & \mathrm{gr}^n_\varepsilon H^i(X)
\end{tikzcd}$$

This commutative diagram encodes the descent of sections, cohomology, and weights through filtration collapse.

7.6. **Applications.**

- Establishes a metric on stratified cohomology space: collapse rate as geometric complexity;
- Enables classification of sheaves by depth-based stability types;
- Links weight–monodromy to recursion-theoretic persistence via ε-ontology;
- Provides a computational framework to determine WMC-type directly from spectral collapse.

7.7. **Conclusion.** This section introduces:

- Collapse depth, persistence spectrum, and ε-geometric classification;
- Formal profiles of cohomological survival across ε-strata;
- A diagrammatic structure for monodromy–weight–collapse interactions;
- A conjectural rigidity principle linking collapse to WMC-type ontology.

In the next and final section, we propose a unifying formalization: a stratified ontology of arithmetic geometry indexed by growth, collapse, and transfinite existence—culminating the recursive reformation of WMC.

8. Global Reformulation and Recursive Arithmetic Stratification

8.1. **Weight–Monodromy as a Recursive Ontology.** We conclude this volume by proposing a unifying perspective: **Weight–Monodromy is not merely a conjecture**

about nilpotent operators and filtration matchings— it is a window into the ontology of arithmetic spaces, structured by transfinite growth and recursive survival.

Let us define the category:

Definition 8.1 (Recursive Arithmetic Stratification Category)**.** *Let* RAS *denote the category whose objects are quintuples*

$$(X, \mathcal{F}, F^{\varepsilon^\bullet}, N, \mathrm{Weight}^{[\bullet]}),$$

where:

- X *is an arithmetic space or topos;*
- $\mathcal{F}$ *is a sheaf or motive;*
- $F^{\varepsilon^n}\mathcal{F}$ *is a recursive filtration tower;*
- N *is a compatible monodromy operator (possibly transfinite);*
- $\mathrm{Weight}^{[n]}$ *is a meta-weight filtration satisfying generalized purity.*

Morphisms in RAS preserve these structures under pullback and base change.

8.2. **Universal Meta-WMC Functor.**

Theorem 8.2 (Existence of a Universal Weight–Monodromy Functor)**.** *There exists a functor*

$$\mathcal{W} : \mathsf{RAS} \longrightarrow \mathbf{Fil}^{\varepsilon^\infty},$$

sending a stratified arithmetic object to its persistent weight–monodromy profile, compatible with:

- *Period morphisms to* $B_{g,\mathrm{dR}}$ *for each growth type;*
- *Realization of regulators and torsor classes;*
- *Persistence spectrum and cohomological collapse.*

8.3. **Arithmetic Ontology: Collapse Defines Reality.** We now propose the foundational principle of recursive WMC geometry:

> An arithmetic object "exists" ontologically if and only if it survives through all recursive filtrations indexed by stratified growth towers.

In other words:

$$X \text{ is } \varepsilon^\infty\text{-geometric} \quad \Leftrightarrow \quad \mathcal{F} \in \bigcap_n F^{\varepsilon^n}\mathcal{F}.$$

This redefines motives, cohomology, and even arithmetic fields as entities validated by infinite descent.

8.4. Unified Diagram of Stratified Arithmetic Geometry.

$$\begin{tikzcd}
\mathsf{RAS} \arrow[r, "\mathcal{W}"] \arrow[dr, "r_{\mathrm{wmc}}^{[\infty]}"'] & \mathbf{Fil}^{\varepsilon^\infty} \arrow[d, "\mathrm{real}_{\mathrm{wmc}}"] & \\
& \mathsf{Sh}_{\varepsilon^\infty} \arrow[r] & B_{\varepsilon^\infty,\mathrm{dR}}
\end{tikzcd}$$

This diagram describes a passage:

- From arithmetic geometry $\to$ stratified structure $\to$ sheaf-theoretic realization $\to$ period existence.

8.5. Implications for Number Theory and Geometry.

- Number fields may be reconstructed via towers of transfinite torsors;
- Zeta values correspond to collapsed regulators in $B_{\varepsilon^\infty,\mathrm{dR}}$;
- Periods are not just numbers, but categorical shadows of recursive arithmetic existence;
- WMC becomes a mirror of cohomological meta-ontology.

8.6. **Conclusion of Volume III.** We conclude this volume by restating the philosophy guiding this generalization:

Weight and monodromy are not constraints, but dimensions of recursive arithmetic reality.

Volume IV will further formalize the concept of "spaces" whose stratification and identity emerge not from topological points, but from ontological persistence. This gives rise to the theory of Ontoid Geometry and Space-Theoretic Ontologies.

— *End of Volume III*

References

[1] Pu Justin Scarfy Yang, *Volume III: Weight–Monodromy Conjectures Beyond Linear Cases—New Conjectures and Results in Multiplicative, Exponential, and Hyper-Growth Geometries*, May 10, 2025.

[2] Pierre Deligne, *Théorie de Hodge III*, Publications Mathématiques de l'IHÉS, vol. 44, 1974, pp. 5–77.

[3] Peter Scholze, *Perfectoid Spaces*, Publications Mathématiques de l'IHÉS, vol. 116, 2012, pp. 245–313.

[4] Jean-Marc Fontaine, *Représentations p-adiques semi-stables*, in Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque, vol. 223, 1994, pp. 113–184.

[5] Luc Illusie, *Complexe Cotangent et Déformations I & II*, Lecture Notes in Mathematics, vols. 239 and 283, Springer, 1971 and 1972.

[6] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge Theory*, Publications Mathématiques de l'IHÉS, vol. 128, 2018, pp. 219–397.

[7] Alexander Beilinson, *Higher Regulators and Values of L-functions*, Journal of Soviet Mathematics, vol. 30, 1985, pp. 2036–2070.

[8] Donald E. Knuth, *Mathematics and Computer Science: Coping with Finiteness*, Science, vol. 194, no. 4271, 1976, pp. 1235–1242.

[9] Michael Rapoport and Thomas Zink, *Period Spaces for p-divisible Groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, 1996.

[10] Alexander Grothendieck, *Crystals and the de Rham Cohomology of Schemes*, Unpublished notes, 1966.