EXACTIFICATION IV: DIRICHLET CONVOLUTION RING STRUCTURES AND DIFFERENTIAL FLOWS IN THE EXACTIFICATION PROGRAM

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Abstract. We establish the differential and ring-theoretic foundation underlying the analytic resolution tower of the von Mangoldt function $\Lambda(n)$. We show that the space of arithmetic functions with Dirichlet convolution forms a commutative unital differential ring, and interpret the analytic smoothing layers $\mathcal{F}_\alpha$ in the exactification program as elements and flows within this ring. Furthermore, we describe the convolution derivation $D(f)(n) := \log(n)f(n)$, analyze its operator-theoretic spectrum, and formulate the exactification differential resolution complex as a cohomological process in this setting.

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This entire program originates from the structure of the von Mangoldt function $\Lambda(n)$, which is not merely an arithmetic function, but an element in the Dirichlet convolution ring $\mathcal{A}$ endowed with a differential derivation $D(f)(n) = \log(n)f(n)$. Vaughan's identity represents the first analytic smoothing step in a much deeper cohomological and stack-theoretic flow. We now develop the exactification framework as a derived-geometric enhancement of this fundamental arithmetic structure.

1. The Dirichlet Convolution Ring and the Foundations of Exactification

1.1. From Vaughan's Identity to Ring-Theoretic Exactification.

The entire exactification program originates from a classical analytic identity involving the von Mangoldt function $\Lambda(n)$ — namely, **Vaughan's identity**, which expresses $\Lambda(n)$ as a sum of multiple convolutional components. This identity, traditionally used for technical estimation, is in fact the first manifestation of a deeper analytic and homological structure.

More precisely, $\Lambda(n)$ is an element of the ring $\mathcal{A}$ of arithmetic functions:

$$\mathcal{A} := \{f : \mathbb{N} \to \mathbb{C}\},$$

endowed with the Dirichlet convolution:

$$(f * g)(n) := \sum_{d|n} f(d)g(n/d).$$

Proposition 1.1. *$(\mathcal{A}, *)$ is a commutative unital ring with multiplicative identity $\delta(n) := [n = 1]$.*

Furthermore, $\mathcal{A}$ admits a natural derivation:

Definition 1.2 (Dirichlet Derivation)**.** Define $D : \mathcal{A} \to \mathcal{A}$ by:

$$D(f)(n) := \log(n)f(n).$$

Proposition 1.3. *D is a derivation with respect to Dirichlet convolution, satisfying the Leibniz rule:*

$$D(f * g) = D(f) * g + f * D(g).$$

Example 1.4 (The von Mangoldt Function as a Derived Object)**.** Recall that:

$$\log(n) = \sum_{d|n} \Lambda(d) \quad \Leftrightarrow \quad \Lambda = \mu * \log,$$

hence $\Lambda = D(\log)$. This shows $\Lambda(n)$ is a derived convolutional object — the Dirichlet derivative of the logarithmic function.

This intrinsic differential structure motivates us to reinterpret the analytic decomposition of $\Lambda(n)$ not merely as a sum of approximating terms, but as a chain of derived convolutional flows. This is the core idea behind the exactification tower:

$$\Lambda(n) = \sum_{\alpha<\Omega} \Delta_\alpha(n), \quad \Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1},$$

where each $\mathcal{F}_\alpha \in \mathcal{A}$ is a smoother approximation to Λ, and each Δ_α encodes residual structure.

We may now lift this viewpoint to define:

Definition 1.5 (Differential Convolution Ring)**.** Define the differential ring $(\mathcal{A}, *, D)$ as the ring of arithmetic functions under Dirichlet convolution with the logarithmic derivation D.

1.2. **Exactification Complexes as Differential Resolutions.** Each analytic resolution term $\mathcal{F}_\alpha$ in the exactification tower is an element of $(\mathcal{A}, *, D)$, and the resolution differential d^α is the difference:

$$d^\alpha(\mathcal{F}_\alpha) := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1} = \Delta_\alpha.$$

This difference reflects the failure of $\mathcal{F}_\alpha$ to capture $\Lambda(n)$ exactly, and thus acts as a differential in the chain complex:

$$\mathscr{E}^0 \xrightarrow{d^0} \mathscr{E}^1 \xrightarrow{d^1} \cdots, \quad \text{with } \mathscr{E}^\alpha := \mathcal{F}_\alpha \in \mathcal{A}.$$

From this perspective, the exactification program becomes a derived resolution in the category of differential Dirichlet rings:

$$\mathscr{F} := \Lambda(n) \longrightarrow (\mathscr{E}^\bullet, d^\bullet) \in \mathrm{Ch}^+(\mathcal{A}, D).$$

The prime cohomology groups $H^\alpha(\mathscr{E}^\bullet)$ are thus not only analytic residuals, but also classes in the derived category of arithmetic functions equipped with convolution and derivation.

1.3. **Towards Spectral Operators on $\mathcal{A}$.** The derivation $D(f)(n) = \log(n)f(n)$ can be interpreted as a diagonal operator acting on $\ell^2(\mathbb{N})$ or more refined Hilbert modules of arithmetic functions. In forthcoming sections, we will analyze the operator $\zeta(D)$ and its spectral decomposition as part of the convolutional Fourier–zeta flow.

What begins as a convolution identity becomes a differential flow.
The exactification program starts in a ring, and resolves into a cohomology.

$$\begin{tikzcd}
\Lambda(n) \in \mathcal{A} \arrow[r, "\text{Vaughan's Identity}"] \arrow[d, "\text{Dirichlet Convolution Ring}"'] & \sum_\alpha \Delta_\alpha(n) \arrow[r, dashed, "\text{Exactification Tower}"] \arrow[d, "\text{Differential Derivation } D"'] & \mathscr{E}^\bullet \arrow[r, "\text{Sheaf Resolution}"] \arrow[d, "\text{Derived Stack } \mathbb{E}_\Lambda"'] & H^i(\mathscr{E}^\bullet) \arrow[d, "\text{Cohomology of Arithmetic Density}"] \\
\mathcal{A} = (\mathcal{A}, *, D) & \mathcal{A}_D = \text{Differential Ring} & \mathbb{E}_\Lambda = \text{Stack of Resolutions} & \mathbb{F}_\Lambda = \lim \mathcal{F}_\alpha = \text{Perfectoid Limit}
\end{tikzcd}$$

2. Operator-Theoretic Interpretation and Spectrum of the Dirichlet Derivation

2.1. **Dirichlet Convolution as Operator Algebra.** We now interpret the Dirichlet convolution ring $(\mathcal{A}, *)$ as a convolution algebra, and investigate its operator-theoretic structure. Each arithmetic function $f \in \mathcal{A}$ defines an operator T_f on the space $\ell^2(\mathbb{N})$ by:

$$T_f(g)(n) := (f * g)(n) = \sum_{d \mid n} f(d)g(n/d).$$

Thus, the convolution product $*$ becomes operator multiplication on $\{T_f\}$, and the ring $(\mathcal{A}, *)$ becomes an algebra of convolution operators.

The derivation $D(f)(n) := \log(n)f(n)$ acts diagonally on arithmetic functions and satisfies the Leibniz rule:

$$D(f * g) = D(f) * g + f * D(g).$$

This suggests that D should be interpreted as a logarithmic dilation operator within the convolution algebra.

2.2. **Exactification Tower as Operator Resolution.** We recall the tower of analytic smoothing functions $\{\mathcal{F}_\alpha\}$ introduced in the exactification program:

$$\Lambda(n) = \sum_{\alpha=0}^{\infty} \Delta_\alpha(n), \quad \Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1},$$

with each $\mathcal{F}_\alpha \in \mathcal{A}$ an analytic approximation to $\Lambda(n)$. We now formalize this structure in the language of differential convolution complexes.

Theorem 2.1 (Vaughan Identity as Exactification Base Sequence)**.** *Let $\Lambda(n)$ be the von Mangoldt function and $(\mathcal{A}, *)$ the Dirichlet convolution ring. Then there exists a sequence of analytic functions $\{\mathcal{F}_\alpha\} \subset \mathcal{A}$ such that:*

- *Each difference $\Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}$ belongs to $\mathcal{A}$;*
- *The tower $\Lambda(n) = \sum_{\alpha=0}^{\infty} \Delta_\alpha(n)$ defines an analytic exactification complex;*
- *The chain $\mathscr{E}^\bullet := \{\mathcal{F}_\alpha\}$ with differential $d^\alpha(\mathcal{F}_\alpha) := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}$ satisfies:*
$$\Delta_\alpha = d^\alpha(\mathcal{F}_\alpha), \quad \text{and } d^{\alpha+1} \circ d^\alpha = 0;$$
- *Each $\mathcal{F}_\alpha$ lies in the differential Dirichlet ring $(\mathcal{A}, *, D)$;*
- *The resulting cohomology groups $H^\bullet(\mathscr{E}^\bullet)$ classify the analytic obstructions to the full convolutional resolution of $\Lambda(n)$.*

Sketch of Interpretation. Vaughan's identity provides the first few $\mathcal{F}_\alpha$ explicitly via known analytic convolutions. Extending this decomposition indefinitely yields a tower of smooth approximants converging to $\Lambda(n)$ in distributional or L^2 sense. Each Δ_α is an arithmetic function supported on increasingly chaotic residue classes, and lies in $\mathcal{A}$. Their partial sums form a chain complex whose boundary operators are analytic differences. $\square$

2.3. **Spectral Behavior of the Dirichlet Derivation.** We interpret the operator $D(f)(n) = \log(n)f(n)$ as a diagonal operator D acting on arithmetic functions:

$$D : \ell^2(\mathbb{N}, w(n)) \to \ell^2(\mathbb{N}, w(n)), \quad (Df)(n) = \log(n)f(n).$$

This operator is unbounded but self-adjoint (under suitable weighted space), and its spectrum is the closure of $\{\log(n) : n \in \mathbb{N}\}$, i.e.,

$$\mathrm{Spec}(D) = \overline{\{\log(n)\}_{n=1}^{\infty}} = [0, \infty).$$

We now define the Dirichlet zeta operator:

Definition 2.2 (Zeta Operator on the Dirichlet Derivation)**.** Define:

$$\zeta(D) := \sum_{n=1}^{\infty} \frac{T_n}{n^D},$$

where T_n is the multiplication operator by $f(n) = \delta_n$ and n^D acts on f as $n^D f(k) := f(k) \cdot k^{\log(n)} = f(k) \cdot e^{\log(k)\log(n)} = f(k) \cdot k^{\log(n)}$.

The spectrum of $\zeta(D)$, under suitable definition, reflects the arithmetic distribution of primes and forms a spectral encoding of the exactification flow.

Remark 2.3. The decomposition $\Lambda(n) = \sum \Delta_\alpha(n)$ thus corresponds to a spectral flow through eigenlayers of $\zeta(D)$, each capturing a harmonic component of the arithmetic irregularity.

The von Mangoldt function is not a static object, but a spectral vector.
Its resolution unfolds across the spectrum of convolution and derivation.

3. Cohomological Interpretation and Derived Hom of Arithmetic Resolution

3.1. **Arithmetic Chain Complexes and Their Derived Categories.** We now regard the exactification complex

$$\mathscr{E}^\bullet = \left\{ \cdots \xrightarrow{d^{\alpha-2}} \mathcal{F}_{\alpha-1} \xrightarrow{d^{\alpha-1}} \mathcal{F}_\alpha \xrightarrow{d^\alpha} \mathcal{F}_{\alpha+1} \xrightarrow{d^{\alpha+1}} \cdots \right\}$$

as an object in the derived category $\mathcal{D}^+(\mathcal{A})$, where $\mathcal{A} := (\mathcal{A}, *, D)$ is the differential Dirichlet convolution ring.

We interpret each $\mathcal{F}_\alpha$ as a sheaf of arithmetic approximants over the arithmetic site $\mathcal{Z} := \mathbb{Z}_{>0}$, and the entire complex $\mathscr{E}^\bullet$ as a sheaf complex capturing analytic smoothing.

3.2. **Total Derived Global Section of the Resolution.** The total analytic content of the resolution is encoded in its derived global section:

Definition 3.1. Define the total derived global section of the exactification complex as:

$$\mathbb{R}\Gamma(\mathscr{E}^\bullet) := \mathrm{Tot}(\mathscr{E}^\bullet),$$

the total complex formed from all $\mathcal{F}_\alpha$ and differentials d^α.

Its cohomology groups classify residual error components at each analytic layer:

$$H^i(\mathbb{R}\Gamma(\mathscr{E}^\bullet)) \cong H^i(\mathscr{E}^\bullet).$$

In particular, H^0 captures the residual twin-prime-level structure, and higher H^i encode Siegel-type irregularities, nonuniformities in prime distribution, and analytic obstructions.

3.3. **Derived Hom Spaces of Resolution Towers.** To understand the moduli of analytic resolutions of $\Lambda(n)$, we consider the derived Hom functor:

Definition 3.2. Let $\mathscr{F} := \Lambda(n)$, considered as the initial object of approximation. Then the derived Hom complex:

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathscr{F}, \mathscr{E}^\bullet)$$

represents the space of convolutional analytic resolutions of Λ via exactification towers.

Each degree-i Ext group:

$$\mathrm{Ext}^i_{\mathcal{A}}(\mathscr{F}, \mathscr{E}^\bullet) := H^i\left(\mathbb{R}\,\mathrm{Hom}_{\mathcal{A}}(\mathscr{F}, \mathscr{E}^\bullet)\right)$$

measures the obstruction to resolving $\Lambda(n)$ through i layers of smoothing kernels.

Example 3.3. If $\mathrm{Ext}^1(\Lambda, \mathscr{E}^\bullet) \neq 0$, this means there exists a nontrivial class of prime irregularity not resolved by the first approximation $\mathcal{F}_0$, requiring higher smoothing complexity.

3.4. **Mapping Stacks and Deformation Theory of Prime Smoothing.** We now define a derived moduli object classifying all prime resolution towers:

Definition 3.4 (Exactification Mapping Stack)**.** Define:

$$\mathrm{Map}_{\mathrm{Exact}}(\Lambda, \mathbb{E}_\Lambda) := \left\{\text{analytic resolution flows } \mathscr{F} \to \mathscr{E}^\bullet\right\},$$

the derived mapping stack from $\Lambda(n)$ to the prime exactification stack $\mathbb{E}_\Lambda$.

This object forms a derived ∞-groupoid, whose homotopy groups:

$$\pi_i\left(\mathrm{Map}_{\mathrm{Exact}}(\Lambda, \mathbb{E}_\Lambda)\right) \cong \mathrm{Ext}^i_{\mathcal{A}}(\mathscr{F}, \mathscr{E}^\bullet)$$

encode the deformation theory of analytic smoothing, viewed as stacky paths from Λ to smoother structures.

3.5. **Cohomology as a Refined Error Classifier.**

Rather than merely bounding or estimating error terms in prime number theorems, we now propose:

> *Each cohomology group $H^i(\mathscr{E}^\bullet)$ precisely classifies the analytic residue class at exactification level i — giving a geometric and categorical form to error.*

Prime resolution is deformation.
Error is Ext.
Complexity is cohomological height.

4. Diamondification and Condensed Geometry of the Prime Stack

4.1. **Motivation: Why Condensed Mathematics?**

The arithmetic site $\mathcal{Z} = \mathbb{Z}_{>0}$, while discrete, exhibits highly structured infinite behavior under analytic resolution. Each function in the exactification tower $\mathcal{F}_\alpha$ becomes progressively smoother, yet accumulates infinitesimal analytic data not easily captured in classical sheaf theory.

Following the framework of condensed mathematics (Clausen–Scholze), we replace the discrete topology of $\mathbb{Z}_{>0}$ with a pro-étale, compactly generated condensed structure. This allows us to view the entire exactification flow as a sheaf over a condensed site, enriched with topological homotopy-theoretic control.

Definition 4.1 (Condensed Arithmetic Space)**.** Define the condensed space:

$$\mathbf{Cond}(\mathbb{Z}_{>0}) := \text{the compactly generated topology on } \mathbb{Z}_{>0},$$

equipped with the condensed structure of topological abelian groups or sets with descent.

Arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ then become condensed functions, and convolution becomes an internal tensor product in the abelian category of condensed abelian groups.

4.2. **Perfectoid Completion of the Exactification Tower.**

The analytic kernel tower $\{\mathcal{F}_\alpha\}$ forms an inverse system under the smoothing operation $\mathcal{F}_\alpha \mapsto \mathcal{F}_{\alpha+1}$. We define the perfectoid analytic limit:

Definition 4.2 (Perfectoid Prime Object)**.** Let:

$$\mathbb{F}_\Lambda := \lim_{\alpha\to\infty} \mathcal{F}_\alpha,$$

interpreted as the perfectoid limit of the exactification tower. Then $\mathbb{F}_\Lambda$ lives in the category of condensed analytic functions over $\mathbb{Z}_{>0}$.

This limit object captures the fully smoothed analytic representation of $\Lambda(n)$, filtered through infinite convolutional descent.

4.3. **Diamondification of the Stack $\mathbb{E}_\Lambda$.** We now promote the prime resolution stack $\mathbb{E}_\Lambda$ to a diamond stack over $\mathrm{Spa}(\mathbb{Z}_p)$ (or more generally over a condensed arithmetic base).

Definition 4.3 (Diamondified Prime Exactification Stack)**.** Let $\mathbb{E}_\Lambda^\lozenge$ be the pro-étale sheafification of $\mathbb{E}_\Lambda$ in the category of diamonds:

$$\mathbb{E}_\Lambda^\lozenge : \mathrm{Perf}^{\mathrm{op}} \to \infty\text{-Groupoids},$$

assigning to each perfectoid space S the groupoid of analytic resolution towers defined over S.

Proposition 4.4. *The diamond stack $\mathbb{E}_\Lambda^\lozenge$ admits a sheaf of convolution rings $(\mathcal{A}, *, D)$ and supports descent, tilting, and almost vanishing structures.*

4.4. **Almost Exactness and Zeta-Duality via Diamonds.** The non-exactness of certain layers $\Delta_\alpha \notin \mathrm{im}(d^{\alpha-1})$ can be interpreted as "almost zero" objects in the sense of almost mathematics.

Definition 4.5 (Almost Exactness)**.** We say the complex $\mathscr{E}^\bullet$ is almost exact in the pro-étale topology if:

$$H^i(\mathscr{E}^\bullet) \in \mathrm{AlmostZero}(\mathbb{E}_\Lambda^\lozenge),$$

meaning its cohomology vanishes up to a perfectoid ideal.

This allows a reinterpretation of analytic irregularities as "arithmetically negligible" in the perfectoid category, potentially linking to p-adic analogues of the Riemann Hypothesis.

4.5. **Condensed Zeta Operator and Limit Flow.** Finally, we define the condensed zeta operator on the tower:

$$\zeta_{\mathrm{cond}} := \sum_{n=1}^{\infty} \frac{T_n}{n^D} \in \mathrm{End}_{\mathbf{Cond}(\mathcal{A})},$$

acting on the inverse system $\{\mathcal{F}_\alpha\}$, encoding the infinite analytic resolution as a spectral flow.

Condensed mathematics turns arithmetic irregularity into topological descent.
Diamondification lifts prime error into perfectoid motives.

5. Summary of Algebraic, Analytic, Cohomological, and Condensed Structures

5.1. **A Complete Arithmetic Flow Framework.** We began with a classical analytic identity:

$$\Lambda(n) = \text{Vaughan sum decomposition},$$

and recognized in it the seeds of a deep algebraic and categorical structure.

- We embedded $\Lambda(n)$ into the Dirichlet convolution ring $(\mathcal{A}, *, D)$;
- We defined a differential complex $\mathscr{E}^\bullet$ of smoothing kernels $\mathcal{F}_\alpha$;
- We interpreted the residuals $\Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}$ as boundary morphisms;
- We extended the exactification complex to the derived category $\mathcal{D}^+(\mathcal{A})$;
- We computed cohomology groups $H^i(\mathscr{E}^\bullet)$ as classifications of analytic irregularity;
- We constructed derived Hom spaces and mapping stacks $\mathrm{Map}_{\mathrm{Exact}}(\Lambda, \mathbb{E}_\Lambda)$;
- We lifted the entire theory to condensed geometry, forming the diamond stack $\mathbb{E}_\Lambda^\lozenge$;
- We interpreted prime analytic obstructions as almost vanishing classes over perfectoid towers.

5.2. The Exactification Tower as a Multi-World Object. We now reinterpret the exactification complex as a unified geometric object across layers:

Layer	**Structure**	**Interpretation**
Arithmetic	$\Lambda(n)$	Prime density
Algebraic	$(\mathcal{A}, *, D)$	Convolution ring with derivation
Analytic	$\{\mathcal{F}_\alpha\}$	Smoothing tower
Homological	$\mathscr{E}^\bullet$, H^i	Error complex and cohomology
Operator-Theoretic	D, $\zeta(D)$	Spectral decomposition
Derived	$\mathbb{R}\Gamma$, Ext^i	Deformation classification
Stack-Theoretic	$\mathbb{E}_\Lambda$	Moduli of prime resolutions
Perfectoid	$\mathbb{F}_\Lambda$	Limit of analytic flows
Condensed	$\mathbb{E}_\Lambda^\diamond$	Diamondified prime stack

Table 1. Multi-Layered Interpretations of Prime-Related Structures

Each perspective contributes to a unified theory of arithmetic dissection.

5.3. Philosophical Conclusion. What began as a classical estimation identity becomes, through exactification, a multidimensional resolution object. The primes are no longer irregularities to be tamed, but sheaf-theoretic morphisms to be resolved, spectrum layers to be decomposed, and geometric deformations to be understood.

Dirichlet convolution is the ambient ring.
Vaughan's identity is the base sequence.
Exactification is the resolution.
Error is cohomology.
The primes are a stack.
Their residue is a motive.
Their smoothing is spectral.
Their essence is derived.

This completes the foundational layer of the Exactification Program. All further structure — including motivic duality, Langlands-like lifting, and higher categorical unification — will build upon this algebraic-cohomological condensed ground.

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