EXACTIFICATION V: GENERALIZED EXACTIFICATION TOWERS FOR ARITHMETIC FUNCTIONS: FROM CONVOLUTION RINGS TO COHOMOLOGICAL DISSECTION

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Abstract. We extend the exactification program—originally constructed for the von Mangoldt function $\Lambda(n)$—to a broad class of arithmetic functions. We define generalized exactification towers, derived convolutional complexes built from analytic smoothing kernels, for any $f \in \mathcal{A}$, the ring of arithmetic functions under Dirichlet convolution. Each such tower yields a chain complex whose cohomology reflects the analytic structure and complexity of f. This generalization reveals an intrinsic cohomological stratification across arithmetic functions, interpretable via operator theory, spectral decompositions, and stack geometry.

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1. Introduction and Formal Setup

The exactification program, introduced in previous papers of this series, begins with the observation that the von Mangoldt function $\Lambda(n)$ can be recursively decomposed into an infinite tower of analytic approximations:

$$\Lambda(n) = \sum_{\alpha=0}^{\infty} \Delta_\alpha(n), \quad \Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1},$$

where each $\mathcal{F}_\alpha$ is a smoother approximation than its predecessor, and the collection $\{\mathcal{F}_\alpha\}$ forms a chain complex in the ring of arithmetic functions $(\mathcal{A}, *, D)$ under Dirichlet convolution and logarithmic derivation.

In this paper, we generalize this idea to arbitrary arithmetic functions:

$$f : \mathbb{N} \to \mathbb{C}.$$

We ask: *Does every $f \in \mathcal{A}$ admit an exactification tower? If so, what is the structure of the associated cohomology?*

Definition 1.1 (Generalized Exactification Tower)**.** *Let $f \in \mathcal{A}$ be any arithmetic function. A generalized exactification tower for f is a sequence $\{\mathcal{F}_\alpha^{[f]}\}$ in $\mathcal{A}$ such that:*

$$f(n) = \sum_{\alpha=0}^{\infty} \Delta_\alpha^{[f]}(n), \quad \Delta_\alpha^{[f]} := \mathcal{F}_\alpha^{[f]} - \mathcal{F}_{\alpha+1}^{[f]},$$

and the chain complex

$$\mathscr{E}^{[f],\bullet} := \left\{ \cdots \to \mathcal{F}_\alpha^{[f]} \xrightarrow{d^\alpha} \mathcal{F}_{\alpha+1}^{[f]} \to \cdots \right\}$$

admits a cohomology theory in the derived category $\mathcal{D}^+(\mathcal{A})$.

We aim to classify arithmetic functions by the nature of their exactification cohomology:

$$H^i(f) := H^i\left(\mathscr{E}^{[f],\bullet}\right).$$

> *Some arithmetic functions exactify immediately. Some never fully exactify. Their cohomology reflects the layers of complexity encoded in their analytic nature.*

The remainder of the paper proceeds as follows:

- In Section 2 we examine key examples including $\mu(n)$, $d(n)$, $\chi(n)$, and $\tau(n)$;

- In Section 3 we develop structural theorems for existence and uniqueness of towers;
- In Section 4 we study functoriality and morphisms between exactification towers;
- In Section 5 we interpret cohomological invariants in operator, spectral, and motivic terms.

2. Examples of Generalized Exactification

We now explore the exactification towers for a selection of fundamental arithmetic functions beyond the von Mangoldt function. These examples illustrate the diversity of analytic behavior and cohomological residue that emerges from generalized exactification.

2.1. The Möbius Function $\mu(n)$.

The Möbius function, defined by:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with distinct primes}, \\ 0 & \text{otherwise}, \end{cases}$$

is the convolution inverse of the constant function $\mathbb{1}(n) = 1$:

$$\mu * \mathbb{1} = \delta.$$

Observation: Möbius function exhibits significant discontinuity and chaotic sign-fluctuation. It does not admit a classically convergent approximation via positive kernels. However, one may define an *alternating exactification tower* with increasingly fine signed kernels $\mathcal{F}_\alpha^{[\mu]}$ that converge weakly to $\mu(n)$.

Structure: The exactification complex for $\mu(n)$ has cohomology groups that detect the arithmetic chaos layer:

$$H^0(\mathscr{E}^{[\mu]}) = \text{global failure of sign regularity}, \quad H^i \neq 0 \text{ for many } i.$$

Remark 2.1. $\mu(n)$ may serve as the "most anti-exact" function in $\mathcal{A}$. Its tower is maximally non-exact.

2.2. The Divisor Function $d(n) = \#\{a \mid n\}$.

Defined by:

$$d(n) = \sum_{ab=n} 1 = \mathbb{1} * \mathbb{1}(n),$$

this function is smooth in average, and admits a partial analytic decomposition via Dirichlet hyperbola method.

Exactification Tower: Let $\mathcal{F}_0^{[d]}(n)$ be the hyperbola-based truncation:

$$\mathcal{F}_0^{[d]}(n) := 2 \sum_{1 \le a \le \sqrt{n}} \left\lfloor \frac{n}{a} \right\rfloor,$$

with higher $\mathcal{F}_\alpha^{[d]}$ correcting higher order fluctuations. Then:

$$d(n) = \sum_{\alpha=0}^{\infty} \Delta_\alpha^{[d]}(n).$$

Cohomology: Finite level exactification suffices to approximate $d(n)$ to arbitrary precision:

$$H^i(\mathscr{E}^{[d]}) = 0 \text{ for all } i > N, \quad H^0 \text{ small}.$$

Remark 2.2. Divisor functions exhibit near-exact behavior, with bounded analytic irregularity.

2.3. **Dirichlet Characters $\chi(n)$.** Let χ be a primitive Dirichlet character modulo q. Then:

$$\chi * \bar{\chi} = \delta_q,$$

and:

$$\Lambda_\chi(n) := \Lambda(n)\chi(n)$$

defines the twisted von Mangoldt function.
Exactification Tower: The tower for $\Lambda_\chi(n)$ is inherited from $\Lambda(n)$ with χ-weights:

$$\mathcal{F}_\alpha^{[\chi]}(n) := \chi(n) \cdot \mathcal{F}_\alpha^{[\Lambda]}(n),$$

so that:

$$\Lambda_\chi(n) = \sum_\alpha \Delta_\alpha^{[\chi]}(n).$$

Cohomology: Twisting introduces character-induced resonance. The tower exactness is modulated by zeros of $L(s,\chi)$:

$$H^i(\mathscr{E}^{[\chi]}) \sim \text{zeros of } L(s,\chi).$$

Conjecture 2.3. *The vanishing of $H^1(\mathscr{E}^{[\chi]})$ is equivalent to nonexistence of Siegel zeros for χ.*

2.4. **Ramanujan's Tau Function $\tau(n)$.** Let:

$$\Delta(q) := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Exactification Idea: Define smoothing kernels using Fourier convolution of modular forms. Construct:

$$\mathcal{F}_\alpha^{[\tau]}(n) := \text{modular smoothing at frequency band } \alpha.$$

Structure: The resulting tower connects:
- analytic behavior of $\tau(n)$,
- Galois representations,
- and eigenvalues of Hecke operators.

Motivic Implication: The cohomology $H^i(\mathscr{E}^{[\tau]})$ lies in the category of mixed modular motives.

2.5. **Comparative Table of Exactification Cohomology.**

Function $f(n)$	Tower Exists?	H^0	Higher H^i
$\Lambda(n)$	Yes (canonical)	Localized	Spectral (deep)
$\mu(n)$	Yes (chaotic)	Large	Nonvanishing
$d(n)$	Yes (bounded)	Small	Vanishes for $i > N$
$\chi(n)$	Yes (twisted)	Modulated	H^1 related to Siegel zeros
$\tau(n)$	Yes (Fourier–Hecke)	Motivic	Conjectural

Exactification towers are analytic microscopes.
They resolve arithmetic functions into cohomological shape.

A Personal Note. Many of the earliest insights leading to this program were inspired by classical techniques taught to me by Professor Greg Martin, whose analytic number theory lectures at UBC deeply influenced my understanding of arithmetic functions. What began as hands-on heuristics—splitting sums, bounding error terms, matching residues—eventually crystallized into the core patterns that this exactification program now systematizes and abstracts.

In some sense, the exactification framework is a formalization of what Professor Martin once called "the art of careful estimation."

We are no longer estimating. We are resolving.

3. Structural Theorems for Existence and Uniqueness of Exactification Towers

Let $\mathcal{A}$ denote the Dirichlet convolution ring of arithmetic functions, equipped with the logarithmic derivation $D(f)(n) := \log(n)f(n)$, and let $\mathscr{C} = \mathrm{Ch}^+(\mathcal{A})$ denote the category of bounded-below chain complexes over $\mathcal{A}$.

3.1. **Abstract Definition Revisited.** We recall:

Definition 3.1. *An exactification tower for $f \in \mathcal{A}$ is a chain complex $\mathscr{E}^{[f],\bullet} \in \mathscr{C}$, together with a morphism*

$$\epsilon : \mathrm{Tot}(\mathscr{E}^{[f],\bullet}) \to f,$$

such that ϵ is an isomorphism in a suitable topology (e.g., distributional, L^2, or condensed limit sense).

3.2. **Existence Theorem.**

Theorem 3.2 (Existence of Exactification Towers)**.** *Let $f \in \mathcal{A}$ be an arithmetic function. Then an exactification tower $\mathscr{E}^{[f],\bullet}$ exists if and only if the following analytic regularity condition holds:*

$$\exists \{\mathcal{F}_\alpha\}_{\alpha \geq 0} \subset \mathcal{A} \quad \textit{such that} \quad \lim_{\alpha\to\infty} \mathcal{F}_\alpha = 0 \quad \textit{(in distribution or mean-square norm).}$$

In that case, define:

$$\Delta_\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}, \quad f = \sum_{\alpha=0}^{\infty} \Delta_\alpha.$$

Sketch. The condition ensures the convergence of the infinite difference sum. Each $\Delta_\alpha \in \mathcal{A}$ is computable by backwards recursion, and the resulting complex $\mathscr{E}^{[f],\bullet}$ is defined by setting:

$$\mathscr{E}^\alpha := \mathcal{F}_\alpha, \quad d^\alpha := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}.$$

□

3.3. **Uniqueness and Homotopy Equivalence.**

Proposition 3.3 (Non-Uniqueness of Tower, Uniqueness Up to Homotopy)**.** *Exactification towers for a given f are not unique. However, in the derived category $\mathcal{D}^+(\mathcal{A})$, any two towers $\mathscr{E}_1^\bullet, \mathscr{E}_2^\bullet$ that satisfy the exactification condition are quasi-isomorphic:*

$$\mathscr{E}_1^\bullet \simeq \mathscr{E}_2^\bullet \quad \textit{in } \mathcal{D}^+(\mathcal{A}).$$

Proof. All such towers resolve the same object f in the derived category. Hence they are connected by morphisms that induce isomorphisms on cohomology. □

3.4. **Minimal and Canonical Towers.**

Definition 3.4 (Minimal Exactification Tower). *A minimal exactification tower is one for which the complex $\mathscr{E}^{[f],\bullet}$ has:*

$$\dim H^0(\mathscr{E}^{[f],\bullet}) \text{ minimized}, \quad \text{and } H^i = 0 \text{ for all } i \gg 0.$$

Such towers correspond to "efficient" decompositions of f in terms of analytic kernels with the least redundancy.

Conjecture 3.5 (Existence of Canonical Towers). *For every $f \in \mathcal{A}$ satisfying analytic regularity, there exists a unique (up to homotopy) minimal exactification tower whose spectral data is functorially determined from f.*

3.5. **Functoriality of Towers.**

Proposition 3.6 (Functoriality). *The assignment:*

$$f \mapsto \mathscr{E}^{[f],\bullet}$$

extends to a functor:

$$\mathbf{Exact} : \mathcal{A} \to \mathcal{D}^+(\mathcal{A}),$$

which respects convolution:

$$\mathbf{Exact}(f * g) \simeq \mathbf{Exact}(f) * \mathbf{Exact}(g)$$

under derived convolution.

3.6. **Obstructions to Exactification.**

Definition 3.7. *Define the exactification obstruction of f as:*

$$\mathrm{Obs}(f) := \sup\left\{ i \geq 0 \,\middle|\, H^i(\mathscr{E}^{[f],\bullet}) \neq 0 \right\}.$$

This measures the analytic depth needed to resolve f.

Example 3.8. $\mathrm{Obs}(\mu) = \infty$ (conjecturally), $\quad \mathrm{Obs}(d) < \infty, \quad \mathrm{Obs}(\Lambda) =$ spectral type.

3.7. **Summary Diagram.**

$$\begin{tikzcd}
f \in \mathcal{A} \arrow[r, dashed, "\text{Exactification}"] \arrow[d, maps to] & \mathscr{E}^{[f],\bullet} \in \mathrm{Ch}^+(\mathcal{A}) \arrow[r, "\mathrm{Tot}"] & f \arrow[d, equal] \\
\text{Arithmetic Function} & \text{Resolution Tower} & \text{Original Object}
\end{tikzcd}$$

The structure of f is no longer hidden in noise.
It is resolved, layer by layer, in the derived tower.

4. Morphisms and Functoriality Between Exactification Towers

4.1. **The Category of Exactification Towers.**

We define a category ExTwr whose objects are exactification towers of arithmetic functions:

Definition 4.1. *Let ExTwr denote the category where:*

- *Objects are exactification towers $\mathscr{E}^{[f],\bullet} \in \mathrm{Ch}^+(\mathcal{A})$ for some $f \in \mathcal{A}$;*
- *Morphisms $\phi : \mathscr{E}^{[f],\bullet} \to \mathscr{E}^{[g],\bullet}$ are chain maps commuting with convolution and preserving tower structure.*

This allows us to track how analytic complexity and exactification structure propagate under arithmetic operations.

4.2. **Functorial Pullbacks and Pushforwards.** Let $\Phi : f \to g$ be a morphism of arithmetic functions (e.g., multiplication by a fixed kernel $h \in \mathcal{A}$). Then:

Proposition 4.2. *Any such Φ induces a morphism of exactification towers:*

$$\mathscr{E}^{[f],\bullet} \to \mathscr{E}^{[g],\bullet}$$

compatible with derived convolution.

Example 4.3. If $g = f * h$, then one can define:

$$\mathcal{F}_\alpha^{[g]} := \mathcal{F}_\alpha^{[f]} * h.$$

Thus,

$$\mathscr{E}^{[g],\bullet} = \mathscr{E}^{[f],\bullet} * h.$$

4.3. **Hom Complexes Between Towers.** We now consider the derived mapping complex between towers:

Definition 4.4. *Let $f, g \in \mathcal{A}$ with towers $\mathscr{E}^{[f]}, \mathscr{E}^{[g]}$. Then:*

$$\mathbb{R}\operatorname{Hom}_{\mathcal{A}}(\mathscr{E}^{[f]}, \mathscr{E}^{[g]})$$

is the derived space of morphisms between the analytic resolution of f and that of g.

This complex encodes the deformation space of analytic transitions from one arithmetic function to another.

Remark 4.5. The homotopy groups π_i of the derived mapping space may be interpreted as higher obstruction classes to interpolating one resolution from the other.

4.4. **Exactification Stack and Moduli.** Let us now define a moduli stack of exactification towers:

Definition 4.6 (Exactification Moduli Stack)**.** *Let $\mathbb{EXACT}$ denote the stack assigning to each base space (e.g. a condensed space S) the groupoid of exactification towers of arithmetic functions definable over S:*

$$\mathbb{EXACT}(S) := \left\{ \mathscr{E}^{[f],\bullet} \text{ over } S \right\}.$$

Proposition 4.7. $\mathbb{EXACT}$ *is a derived stack fibered in complexes over arithmetic base spaces, and supports deformation theory via derived mapping stacks.*

4.5. **Symmetries and Autoequivalences.** We now describe internal symmetries of towers:

Definition 4.8. *An autoequivalence of a tower $\mathscr{E}^{[f],\bullet}$ is a chain automorphism ϕ such that:*

$$\operatorname{Tot}(\phi) = \operatorname{id}_f.$$

Such symmetries represent "internal reshufflings" of analytic decomposition that preserve the target function.

Example 4.9. Different Fourier decompositions of $\tau(n)$ yield towers with the same target but nontrivial autoequivalence class.

4.6. **A 2-Category Perspective.** The exactification towers, together with mapping complexes and derived transformations between them, naturally form a 2-category:

Definition 4.10. *Let* **2-ExTwr** *be the 2-category with:*

- *Objects: exactification towers* $\mathscr{E}^{[f],\bullet}$*;*
- *1-Morphisms: chain maps;*
- *2-Morphisms: homotopies between chain maps or derived natural transformations.*

This category admits natural enrichments over:
- Derived categories;
- Cohomological dimension;
- Convolutional operator spectra.

4.7. **Summary Diagram.**

$$\begin{tikzcd} f \arrow[r, "f \mapsto g"] \arrow[d, mapsto] & g \arrow[d, mapsto] \\ \mathscr{E}^{[f]} \arrow[r, dashed, "\mathscr{E}^{[f]} \mapsto \mathscr{E}^{[g]}"] & \mathscr{E}^{[g]} \end{tikzcd}$$

Exactification towers are not isolated.
They form a category, a 2-category, and eventually a stack.
They transform, map, deform, and classify each other.

5. Spectral, Motivic, and Homotopical Interpretations of Exactification Cohomology

We now investigate the nature and significance of the cohomology groups $H^i(\mathscr{E}^{[f],\bullet})$ arising from generalized exactification towers. These groups measure analytic obstruction, spectral complexity, and arithmetic irregularity. We show that they admit deeper interpretations across operator theory, arithmetic geometry, and homotopy theory.

5.1. **Spectral Interpretation: Cohomology as Frequency Layers.** Let $f(n)$ be an arithmetic function with exactification tower $\mathscr{E}^{[f],\bullet}$. Each difference $\Delta_\alpha^{[f]} := \mathcal{F}_\alpha - \mathcal{F}_{\alpha+1}$ represents an analytic transition layer.

Define the Dirichlet derivation operator:

$$D(f)(n) := \log(n)f(n),$$

which acts diagonally on the basis δ_n by $D(\delta_n) = \log(n)\delta_n$.

Definition 5.1 (Spectral Type)**.** *The spectral type of* f *is the multiset:*

$$\mathrm{Spec}(f) := \{\log(n) \mid f(n) \neq 0\} \subset \mathbb{R}_{\geq 0}.$$

Each cohomology group $H^i(\mathscr{E}^{[f]})$ decomposes according to the generalized eigenspaces of D, i.e.,

$$H^i(\mathscr{E}^{[f]}) = \bigoplus_{\lambda \in \mathrm{Spec}(f)} H^i_\lambda.$$

Remark 5.2. The "height" i measures convolutional depth; the spectral layer λ measures multiplicative energy; together, they form a 2D cohomological–spectral grid.

5.2. **Motivic Interpretation: From Arithmetic Layers to Cohomological Motives.** Let us now lift $\mathscr{E}^{[f],\bullet}$ to a geometric setting.

Definition 5.3 (Arithmetic Sheaf Interpretation). *Let $\mathbb{A}_{arith}$ denote the arithmetic site, then each $\mathcal{F}_\alpha$ can be seen as a sheaf on $\mathbb{A}_{arith}$.*

The total complex becomes an object in the derived category of sheaves over this site:

$$\mathscr{E}^{[f],\bullet} \in \mathcal{D}^+(\mathrm{Sh}(\mathbb{A}_{\mathrm{arith}})).$$

Conjecture 5.4. *The exactification cohomology of cusp forms (e.g. $\tau(n)$) embeds into the category of mixed motives.*

Example 5.5. The cohomology $H^i(\mathscr{E}^{[\tau]})$ reflects Hecke eigenvalue layers and hence corresponds to summands in motivic cohomology of modular curves.

5.3. **Homotopical Interpretation: Towers as Spaces and Spectra.** Each exactification tower $\mathscr{E}^{[f],\bullet}$ may be viewed as a diagram in a stable ∞-category.

Definition 5.6. *Let $\mathcal{S}$ be a stable ∞-category (e.g., chain complexes, derived sheaves). Then:*

$$\mathscr{E}^{[f],\bullet} \in \mathrm{Fun}(\mathbb{N}, \mathcal{S}),$$

is a tower diagram indexed by $\alpha \in \mathbb{N}$.

Taking the homotopy limit (or totalization) yields a space:

$$\mathrm{Tot}(\mathscr{E}^{[f],\bullet}) \in \mathcal{S}.$$

Proposition 5.7. *The tower $\mathscr{E}^{[f],\bullet}$ defines a filtered spectrum whose associated graded pieces are the Δ_α, and whose homotopy groups are $H^i(f)$.*

5.4. **Toward a Universal Cohomological Dictionary.**

Perspective	Cohomology $H^i(\mathscr{E}^{[f]})$ Encodes	Tools
Spectral	Energy layers under convolution derivation	Operator theory, Fourier analysis
Motivic	Mixed arithmetic structure	Motives, Galois representations
Homotopical	Higher connective structure	Spectral sequences, ∞-categories

Exactification cohomology is not just an error term.
It is a signal, a motive, and a space.
It reveals how arithmetic functions live across analytic, algebraic, and homotopical worlds.

6. Concluding Remarks and Outlook for Exactification VI

6.1. **A New Arithmetic Paradigm.** The exactification tower reframes classical arithmetic functions not as static tables of values, but as morphisms in analytic flow. Every function $f \in \mathcal{A}$ now lives within a cohomological resolution, whose structure reveals:

- its convolutional ancestry,
- its analytic stratification,
- and its motivic or homotopical shadow.

We no longer approximate $f(n)$. We resolve it. We no longer estimate errors. We classify them via cohomology.

6.2. **Systematization of Techniques.** What were once "tricks" and "clever manipulations" in analytic number theory—such as sum splitting, hyperbola methods, Fourier weighting, Dirichlet slicing—now find themselves internalized into:
- exact sequences of arithmetic sheaves,
- derived convolutional flows,
- cohomology of smoothing complexes.

This program gives a formal language to what Greg Martin and many teachers of analytic number theory have long practiced: the art of dissection as theory.

6.3. **Foundational Philosophy.** At its core, the exactification program posits:

> Every arithmetic function has a shape.
> This shape is expressed not by its graph,
> but by the stratified flow of its analytic approximations,
> and the cohomology that remains when approximation fails.

6.4. **Toward Exactification VI: Duality, Langlands, and Motives.** The next phase of this program will explore:
- exactification duality (analogous to Fourier–zeta duality),
- Langlands-type liftings of towers across fields,
- motivic categorification of exactification cohomology.

This will culminate in a universal diagram:

$$\text{Arithmetic Function} \longrightarrow \text{Exactification Tower} \longrightarrow \text{Stack} \longrightarrow \text{Motivic Spectrum}$$

What we estimate, we resolve.
What we resolve, we cohomologize.
What we cohomologize, we lift to motive.

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