CHEBYSHEV'S CONJECTURE
AND THE PRIME NUMBER RACE

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1. INTRODUCTION AND STATEMENT OF PROBLEMS

Dirichlet in 1837 proved that for any a, q with $(a, q) = 1$ there are infinitely many primes p with $p \equiv a \pmod{q}$; see, for example, ([Da], chapter 4). Chebyshev [Ch] noted in 1853 that there are "more" primes congruent to 3 than 1 modulo 4. His conjecture states that

$$\lim_{x \to \infty} \sum_{p > 2} (-1)^{(p-1)/2} e^{-p/x} = -\infty.$$

As it was shown by Hardy, Littlewood [HL] and Landau [La2] this holds if and only if the function

$$L(s, \chi_1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

does not vanish for $\Re s > 1/2$.

Chebyshev's conjecture was the origin for a big branch of modern Number Theory, namely, comparative prime–number theory. As usual, p runs over the primes, k is a positive integer, $(k, l) = 1$,

$$\pi(x, k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l \pmod{k}}} 1,$$

$\pi(x) = \pi(x, 1, 1)$. The functions $\pi(x, k, l)$ for fixed k and $(l, k) = 1$ are all asymptotically $x/(\varphi(k) \log x)$ [Da, Ch. 20], where $\varphi(k)$ is Euler's function, i.e. the number of positive integers $l \leqslant k$ with $(k, l) = 1$. Still, as Chebyshev observed, there are interesting inequities in the functions $\pi(x, k, l)$ for fixed k. Of particular interest is the behavior of the functions

$$\Delta(x, k, l_1, l_2) = \pi(x, k, l_1) - \pi(x, k, l_2).$$

In Chebyshev's case, for example, $\Delta(x, 4, 3, 1)$ is negative for the first time at $x = 26861$ [Lee]. More dramatically, $\Delta(x, 3, 2, 1)$ is negative for the first time

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at $x = 608,981,813,029$ [BH1]. Nonetheless, Littlewood [Li] proved in 1914 that the functions $\Delta(x, 4, 3, 1)$ and $\Delta(x, 3, 2, 1)$ each change sign infinitely often. In their fundamental series of papers on comparative prime–number theory ([KT1], [KT2]), Knapowski and Turán generalized Littlewood's theorem and also indicated a large number of problems related to comparison of $\pi(x, k, l_1)$ and $\pi(x, k, l_2)$. Below we list some of these (and other) problems. In the sequel discussing sign changes of $\Delta(x, k, l_1, l_2)$ and similar differences we shall assume that $l_1 \not\equiv l_2 \pmod{k}$ and $(l_1, k) = (l_2, k) = 1$.

1. "Infinity of sign changes". To prove that $\Delta(x, k, l_1, l_2)$ changes sign infinitely often.

2. "Big sign changes". To prove that $\Delta(x, k, l_1, l_2)$ is $> x^{1/2-\varepsilon}$ (respectively $< -x^{1/2-\varepsilon}$) for an unbounded sequence of x. The use of function $x^{1/2-\varepsilon}$ is motivated by the fact that if Extended Riemann Hypothesis for k (see below) is true than the inequality

$$(1.1) \qquad\qquad |\Delta(x, k, l_1, l_2)| = O(x^{1/2} \log x) \quad (x \geqslant 2)$$

holds (see e.g. [Da, Ch. 20, (14)]).

3. "Localized sign changes". To prove that (i) for $T > T_0(k)$ and suitable $G(T) < T$ the function $\Delta(x, k, l_1, l_2)$ changes sign in the interval $G(T) \leqslant x < T$; (ii-a) prove that $\Delta(x, k, l_1, l_2)$ takes "large" positive and negative values in the interval $G(T) \leqslant x < T$, where "large" means $> T^{1/2-\varepsilon}$; (ii-b) find lower bounds on the number of sign changes for $x \leqslant T$.

4. "First sign change". To determine a function $B(k)$ such that for $1 \leqslant x \leqslant B(k)$ all $\Delta(x, k, l_1, l_2)$ change sign at least once.

5. "Average preponderance problems". A typical question is the following. Denote by $N(x)$ the number of integers $n \leqslant x$ with the property $\Delta(x, 4, 3, 1) \leqslant 0$. Does the relation

$$(1.2) \qquad\qquad \lim_{x \to +\infty} \frac{N(x)}{x} = 0$$

hold?

6. "Littlewood–generalizations". A typical problem of this type would be the existence of a sequence $x_1 < x_2 < \cdots \to +\infty$ such that simultaneously the inequalities

$$\pi(x, 4, 1) \geqslant \frac{1}{2} \mathrm{Li}(x_\nu)$$

and

$$\pi(x, 4, 3) \geqslant \frac{1}{2} \mathrm{Li}(x_\nu)$$

hold, where

$$\mathrm{Li}(x) = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}.$$

This would constitute an obvious generalization of Littlewood's theorem [Li] that for a suitable sequence $y_1 < y_2 < \cdots \to +\infty$ the inequality $\pi(y_\nu) \geqslant \mathrm{Li}(y_\nu)$ holds.

7. "Racing problems". The "prime number race" is colorfully described in [KT2, I] in the following way. Consider a game with r players, $2 \leqslant r \leqslant \varphi(k)$, called "$l_1$" through "$l_r$" (here $l_1, \ldots l_r$ are mutually incongruent modulo k), and at time t, each player "j" has a score of $\pi(t, k, j)$ (i.e. player "j" scores 1 point whenever t reaches a prime $\equiv j \pmod{k}$). As $t \to \infty$, will each player take the lead infinitely often? More generally, will all $r!$ orderings of the players occur for infinitely many integers t? This type of question originated in a paper of Shanks [Sh], who calculated that $\pi(x, 8, 1) \leqslant \max_{a \in \{3,5,7\}} \pi(x, 8, a)$ for $x \leqslant 10^6$. It is generally believed that the answers to both questions are yes. If $r = 2$ this is just the first problem (infinity of sign changes).

8. "Distribution problems". Investigate the distribution of $\Delta(x, k, l_1, l_2)$, e.g. study $S(x; z) = \{1 \leqslant y \leqslant x : \Delta(y, k, l_1, l_2) \leqslant z(x^{1/2}/\log x)\}$ in a more general way than indicated by problems 1–6. For the general race problem, study distribution properties of the vectors $(\pi(x, k, l_1), \ldots, \pi(x, k, l_r))$.

9. "Union–problems". For a given modulus k and disjoint subsets A and B of reduced residue classes mod k, study the distribution of the function

$$(1.3) \qquad \sum_{p \in A, p \leqslant x} 1 - \frac{|A|}{|B|} \sum_{p \in B, p \leqslant x} 1.$$

An important example is a generalization of Chebyshev's example: let A be the set of quadratic non-residues modulo k, and take B to be the set of quadratic residues.

Besides the functions $\pi(x, k, l)$, the distribution of primes in arithmetic progressions can be characterized by some other functions which are more convenient to work with. Let $\Lambda(n)$ be the Dirichlet–von Mangoldt function, namely, $\Lambda(n) = \log p$ if $n = p^m$ for some prime p and some positive integer m, and $\Lambda(n) = 0$ otherwise. The following functions are studied:

$$\psi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l \pmod{k}}} \Lambda(n),$$

$$\Pi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l \pmod{k}}} \frac{\Lambda(n)}{\log n},$$

$$\theta(x, k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l \pmod{k}}} \log p.$$

All problems 1–9 can be formulated for comparison of these functions. Moreover, one can consider all questions "in the Abelian sense" as in original Chebyshev's paper [Ch]; for example, $\pi(x, k, l)$ can be replaced by $\sum_{p \equiv l \pmod{k}} e^{-pr}$ for $r > 0$. In this paper, however, we will be concerned only with problems involving the functions $\pi(x, k, l)$, concentrating on problems 1 and 7.

2. ANALYTIC TOOLS

The methods of investigation of oscillatory properties of the functions $\pi(x, k, l_1) - \pi(x, k, l_2)$ are similar to those used to study oscillatory properties of the remainder term of the prime number theorem, i.e. $E(x) = \pi(x) - \mathrm{Li}(x)$. The primary tools are so-called "explicit formulas" linking the functions $\pi(x, k, l)$ to the distribution of the zeros in the critical strip $0 < \Re s < 1$ of the Dirichlet L-functions $L(s, \chi)$ for the characters χ modulo k. Note that

$$\sum_{\substack{n \leqslant x \\ n \equiv l \;(\mathrm{mod}\ k)}} \frac{\Lambda(n)}{\log n} = \sum_{\substack{p \leqslant x \\ p \equiv l \;(\mathrm{mod}\ k)}} \frac{\Lambda(p)}{\log p} + \sum_{\substack{p^2 \leqslant x \\ p^2 \equiv l \;(\mathrm{mod}\ k)}} \frac{\Lambda(p^2)}{\log(p^2)} + O(x^{1/3})$$

$$= \pi(x, k, l) + \sum_{\substack{1 \leqslant u \leqslant k \\ u^2 \equiv l \;(\mathrm{mod}\ k)}} \frac{\pi(x^{1/2}, k, u)}{2} + O(x^{1/3}).$$

Using the asymptotic formula for $\pi(x, k, l)$ we have

$$(2.1) \qquad \pi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l \;(\mathrm{mod}\ k)}} \frac{\Lambda(n)}{\log n} - \frac{N_k(l)}{\varphi(k)} \frac{x^{1/2}}{\log x} + o\left(\frac{x^{1/2}}{\log x}\right) \quad (x \to \infty)$$

where $N_k(l)$ is the number of incongruent solutions of the congruence $u^2 \equiv l$ (mod k).

Let D_k (C_k) denote the set of all (correspondingly, non-principal) characters modulo k. For $\chi \in D_k$ define

$$\Psi(x; \chi) = \sum_{n \leqslant x} \Lambda(n) \chi(n).$$

It is not difficult to show that

$$\varphi(k) \sum_{\substack{n \leqslant x \\ n \equiv l \;(\mathrm{mod}\ k)}} \frac{\Lambda(n)}{\log n} = \varphi(k) \left(\frac{\psi(x, k, l)}{\log x} + \int_2^x \frac{\psi(t, k, l)}{t \log^2 t} \, dt \right)$$

$$= \sum_{\chi \in D_k} \overline{\chi}(l) \left(\frac{\Psi(x; \chi)}{\log x} + \int_2^x \frac{\Psi(t; \chi)}{t \log^2 t} dt \right).$$

Writing $D(x, k, l_1, l_2) = \psi(x, k, l_1) - \psi(x, k, l_2)$, we have by (2.1)

$$\varphi(k) \Delta(x, k, l_1, l_2) = \varphi(k) \left(\frac{D(x, k, l_1, l_2)}{\log x} + \int_2^x \frac{D(t, k, l_1, l_2)}{t \log^2 t} \, dt \right)$$

$$- (N_k(l_1) - N_k(l_2)) \frac{x^{1/2}}{\log x} + o\left(\frac{x^{1/2}}{\log x}\right)$$

$$(2.2) \qquad = \sum_{\chi \in C_k} (\overline{\chi}(l_1) - \overline{\chi}(l_2)) \left(\frac{\Psi(x; \chi)}{\log x} + \int_2^x \frac{\Psi(t; \chi)}{t \log^2 t} dt \right)$$

$$- (N_k(l_1) - N_k(l_2)) \frac{x^{1/2}}{\log x} + o\left(\frac{x^{1/2}}{\log x}\right) \quad (x \to \infty).$$

By well-known explicit formulas (Ch. 19, (7)–(8) in [Da]), when $\chi \in C_k$,

$$(2.3) \qquad \Psi(x; \chi) = - \sum_{|\Im\rho| \leqslant x} \frac{x^\rho}{\rho} + O\left(\log^2 x\right), \quad (x \geqslant 2)$$

where the sum is over zeros ρ of $L(s, \chi)$ with $0 < \Re\rho < 1$.

The zeros of $L(x, \chi)$ with largest real part will dominate the sum in (2.3). The Extended Riemann Hypothesis for k (abbreviated ERH_k) states that all these zeros for all $\chi \in D_k$ lie on the critical line $\Re s = \frac{1}{2}$. Since

$$(2.4) \qquad \sum_{0 \leqslant \Im\rho \leqslant T} 1 = O(T \log T) \quad (T \geqslant 2)$$

for each χ ([Da], Ch. 16, (1)), one can see that ERH_k plus (2.3) implies (1.1). Moreover, it is known that if ERH_k holds then

$$\frac{1}{x} \int_0^x |\Psi(t; \chi)|^2 dt = O(x)$$

for any $\chi \in C_k$ (this follows easily by the methods in Chapter III of Cramér [Cr2]). In this case $\Delta(x, k, l_1, l_2)$ has average order at most $x^{1/2}/\log x$. If also $N_k(l_1) \neq N_k(l_2)$ then the term

$$(2.5) \qquad -(N_k(l_1) - N_k(l_2)) \frac{x^{1/2}}{\log x}$$

in (2.2) is very significant, causing a shift in the mean value of $\Delta(x, k, l_1, l_2)$. In particular, we can expect that if l_1 is a quadratic residue and l_2 is a quadratic nonresidue modulo k then for "most" x's $\pi(x, k, l_1) < \pi(x, k, l_2)$. Sometimes this phenomenon is called "Chebyshev's Bias".

A basic tool for proving oscillation theorems is the following result of Landau [La1] on the location of singularities of the Mellin transform of a non-negative function.

Lemma 2.1 [La1, p. 548]. *Suppose $f(x)$ is real valued, and also non-negative for $x \geqslant x_0$. Suppose also for some real numbers $\beta < \sigma$ that the Mellin transform*

$$g(s) = \int_1^\infty f(x) x^{-s-1} \, dx$$

is analytic for $\Re s > \sigma$ and can be analytically continued to the real segment $(\beta, \sigma]$. Then $g(s)$ in fact represents an analytic function in the half-plane $\Re s > \beta$.

For example, if $f(x) = \varphi(k) D(x, k, l_1, l_2)$ then

$$(2.6) \qquad g(s) = g(s, k, l_1, l_2) = -\frac{1}{s} \sum_{\chi \in C_k} \left(\overline{\chi}(l_1) - \overline{\chi}(l_2)\right) \frac{L'(s, \chi)}{L(s, \chi)}$$

for $\Re s > 1$, with the right side providing a meromorphic continuation of $g(s)$ to the whol complex plane. Note that the poles of $g(s)$ (except $s = 0$) are a subset of the zeros of the functions $L(s, \chi)$. Also, $g(s)$ always has an infinite number of poles in the critical strip [G1]. Assume that $g(s)$ has no real poles $s > \frac{1}{2}$, $g(s)$ has a pole s_0 with $\Re s_0 > \frac{1}{2}$. Take any α satisfying $\frac{1}{2} < \alpha < \Re s_0$ and put $f(x) = (-1)^n D(x, k, l_1, l_2) + cx^\alpha$ for some constant c and $n \in \{0, 1\}$. Applying Lemma 2.1 with different n and c we conclude that

$$\limsup_{x \to \infty} \frac{D(x, k, l_1, l_2)}{x^\alpha} = \infty, \qquad \liminf_{x \to \infty} \frac{D(x, k, l_1, l_2)}{x^\alpha} = -\infty.$$

Since $\alpha > \frac{1}{2}$, this is enough to deduce that $\Delta(x, k, l_1, l_2)$ changes sign infinitely often. More generally, the following lemma shows how oscillations of $D(x, k, l_1, l_2)$ and $\Delta(x, k, l_1, l_2)$ are related.

Lemma 2.2. *Let*

$$(2.7) \qquad h(x) = h(x, k, l_1, l_2) = \varphi(k) x^{-1/2} D(x, k, l_1, l_2) - N_k(l_1) + N_k(l_2).$$

If

$$\liminf_{x \to \infty} h(x) < 0 < \limsup_{x \to \infty} h(x),$$

then $\Delta(x, k, l_1, l_2)$ has infinitely many sign changes.

Proof. By (2.2),

$$(2.8) \qquad \varphi(k)\Delta(x, k, l_1, l_2) = \frac{x^{1/2}}{\log x} \left(h(x) + \frac{\log x}{x^{1/2}} \int_2^x \frac{h(t)}{t \log^2 t} \, dt + o(1) \right).$$

By hypothesis, for an unbounded set of x we have

$$h(x) \geqslant \frac{1}{2} \max_{2 \leqslant y \leqslant x} h(y).$$

For such x sufficiently large, (2.8) implies that

$$\varphi(k)\Delta(x, k, l_1, l_2) \geqslant \frac{x^{1/2}}{\log x} h(x) \left(1 + O\left(\frac{1}{\log x} \right) \right) > 0.$$

Similarly, there is an unbounded set of x with $\Delta(x, k, l_1, l_2) < 0$.

3. SIGN CHANGES IN $\Delta(x, k, l_1, l_2)$

3.1. The effect or large real zeros.

By the remarks at the end of section 2, to settle question 1 we need to deal with two cases: (i) $g(s)$ has a real pole $s > \frac{1}{2}$; (ii) $g(s)$ has all of its poles in the critical strip on the line $\Re s = \frac{1}{2}$.

Assume for the moment the following situation. Let $\chi_1 \in C_k$ be a real character, $L(\beta, \chi_1) = 0$ for some $\beta \in (1/2, 1)$, but all other zeros of $L(s, \chi_1)$ and all zeros of $L(s, \chi)$, $\chi \in D_k \setminus \{\chi_1\}$, have real part $\leqslant \beta - \delta$ for some $\delta > 0$. Take l such that $\chi_1(l) = -1$. It follows from (2.2)—(2.4) that

$$\varphi(k)\Delta(x, k, 1, l) = -\frac{2x^\beta}{\beta \log x} + O\left(\frac{x^\beta}{\log^2 x}\right).$$

Therefore $\pi(x, k, 1) < \pi(x, k, l)$ for sufficiently large x. This simple example shows that to prove the infinity of sign changes of $\Delta(x, k, l_1, l_2)$ (and to succeed in any of the problems 1–9) for a general modulus k we need some information about the location of the zeros ρ. Nowadays we cannot prove such properties in general, and they are usually stated as suppositions. Thus, many results of comparative prime-number theory are conditional. The most common supposition is Haselgrove's condition: no $L(s, \chi)$, $\chi \in C_k$, vanishes in the real interval $(0, 1)$ (which eliminates the possibility of case (i)). To get effective results we need a bound on the distance from any zero to the real axis, and Haselgrove's condition is usually formulated as the existence of an $A(k)$ such that no $L(s, \chi)$, $\chi \in C_k$, vanishes in the parallelogram $0 < \Re\rho < 1$, $|\Im\rho| \leqslant A(k)$.

3.2. Infinitely many sign changes on Haselgrove's condition.

As already mentioned, Littlewood [Li] proved that both functions $\Delta(x, 4, 3, 1)$ and $\Delta(x, 3, 2, 1)$ change sign infinitely often. Knapowski and Turán [KT1, II] extended these results significantly, on the assumption of Haselgrove's condition.

Theorem 3.1 [KT1, II, Theorem 5.1]. *If Haselgrove's condition is true for k then the difference $\pi(x, k, l) - \pi(x, k, 1)$ changes sign infinitely often for any l.*

In addition, Knapowski and Turán prove bounds for the first sign change as well as for the magnitude and frequency of the oscillations ([KT1, II, Theorems 5.1, 5.2], [KT1,III, Theorems 1.2, 1.3, 2.1, 3.1–3.4]). In principal it is not difficult to verify Haselgrove's condition for a particular k, and this has been done for many small k (P.C. Haselgrove, unpublished) including all $k \leqslant 72$ [Ru1]. Recently, Conrey and Soundararajan [CS] proved that $L(s, \chi) \neq 0$ on $s \in (0, 1)$ for at least 20% of the real quadratic characters $\chi_{-8d}(n) = (\frac{-8d}{n})$, where d runs over odd squarefree positive integers. In [KT1, VI], a more general theorem is proved under a slightly stronger hypothesis. For some effectively computable constant c_1 (independent of k), if no $L(s, \chi)$, $\chi \in C_k$, vanishes in the domain $\Re s > 1/2$, $|\Im s| \leqslant c_1 k^{10}$, and $N_k(l_1) = N_k(l_2)$, then $\Delta(x, k, l_1, l_2)$ changes sign infinitely often. The last condition means that l_1, l_2 are simultaneously either quadratic residues or quadratic nonresidues modulo k. Later Kátai [Ka1] proved the same conclusion under weaker hypotheses.

Theorem 3.2 [Ka1, Satz 2]. *Assuming Haselgrove's condition for k and $N_k(l_1) = N_k(l_2)$, then $\Delta(x, k, l_1, l_2)$ changes sign infinitely often. More specifically,*

$$\limsup_{x \to \infty} \frac{\Delta(x, k, l_1, l_2)}{\sqrt{x}/\log x} > 0, \qquad \liminf_{x \to \infty} \frac{\Delta(x, k, l_1, l_2)}{\sqrt{x}/\log x} < 0.$$

For the general case of arbitrary l_1 and l_2, Haselgrove's condition implies infinitely many sign changes of the difference $D(x, k, l_1, l_2)$ [KT1, VII] (see also [Ka1, Satz 1]), but the proof breaks down for $\Delta(x, k, l_1, l_2)$, even if one assumes the full ERH_k. Basically, the proven magnitude of oscillations of $\psi(x, k, l_1) - \psi(x, k, l_2)$ (of order $\sqrt{x}$ in [Ka1, Satz 1]) are insufficient to overcome the term (2.4) appearing in (2.2). As mentioned at the end of section 2, the difficulty is when all of the singularities of $g(s)$ (a linear combination of functions $\log L(s, \chi)$ in our case) have real part $= \frac{1}{2}$.

3.3. Infinitely many sign changes when $N_k(l_1) \neq N_k(l_2)$, $l_1 \neq 1$, $l_2 \neq 1$.

Fix k, l_1, l_2 and define $g(s)$ as in (2.6) and $h(x)$ as in (2.7). Suppose $g(s)$ has no poles with $\Re s > \frac{1}{2}$. Label the poles in the right half plane with positive imaginary part as $\frac{1}{2} + i\gamma_1, \frac{1}{2} + i\gamma_2, \ldots$ and let $G = \{\gamma_1, \gamma_2, \ldots\}$. Put $\gamma_0 = 0$ and let a_0 be the residue of $g(s)$ at $s = \frac{1}{2}$ ($a_0 = 0$ if $g(s)$ is analytic at $s = \frac{1}{2}$) and let a_j be the residue of $g(s)$ at $s = \frac{1}{2} + i\gamma_j$ (typically the numbers a_j have order $1/\gamma_j$). Define

$$A(u) = \varphi(k)e^{-u/2}D(e^u, k, l_1, l_2) = h(e^u) + N_k(l_1) - N_k(l_2)$$

and for each $T > 0$ let

$$A_T^*(u) = \sum_{|\gamma_j| < T} \left(1 - \frac{|\gamma_j|}{T}\right) a_j e^{i\gamma_j u} = a_0 + 2\Re \sum_{0 < \gamma_j < T} \left(1 - \frac{\gamma_j}{T}\right) a_j e^{i\gamma_j u}.$$

By Theorem 1 of Ingham [I],

$$(3.1) \qquad \liminf_{u \to \infty} A(u) \leqslant \liminf_{u \to \infty} A_T^*(u) \leqslant \limsup_{u \to \infty} A_T^*(u) \leqslant \limsup_{u \to \infty} A(u).$$

As a consequence, if the numbers $\{\gamma_i : 0 < \gamma_i < T\}$ are linearly independent over the rationals then Knonecker's Theorem implies

$$\limsup_{u \to \infty} A_T^*(u) = a_0 + 2\Re \sum_{0 < \gamma_j < T} \left(1 - \frac{\gamma_j}{T}\right) |a_j|,$$

$$\liminf_{u \to \infty} A_T^*(u) = a_0 - 2\Re \sum_{0 < \gamma_j < T} \left(1 - \frac{\gamma_j}{T}\right) |a_j|.$$

In particular, if all $\gamma_1, \gamma_2, \ldots$ are linearly independent over the rationals, then the above sum over γ_j tends to ∞ as $T \to \infty$. In this case, by Lemma 2.2 the function $\Delta(x, k, l_1, l_2)$ changes sign infinitely often.

The linear independence property was introduced earlier by Wintner [W1], [W2, chapter 13], and in [RS] it is called the Grand Simplicity Hypothesis (GSH_k): The set of all $\gamma \geqslant 0$ such that $L(\frac{1}{2} + i\gamma, \chi) = 0$ for $\chi \in C_k$, are linearly independent over $\mathbb{Q}$. Note that GSH_k implies that all the zeros are simple and that $L(\frac{1}{2}, \chi) \neq 0$ for all such χ.

Theorem 3.3 [I]. *Assume ERH_k and GSH_k. Then every function $\Delta(x, k, l_1, l_2)$ changes sign infinitely often.*

It is not possible at the moment to show that any set of γ_i are linearly independent. One way to avoid this is to use a computer to find values of u so that $A_T^*(u)$ is small or large and use the inequality

$$\liminf_{u \to \infty} A(u) \leqslant A_T^*(u) \leqslant \limsup_{u \to \infty} A(u),$$

which follows from (3.1) and the almost periodic properties of $A_T^*(u)$.

Stark [St] generalized Ingham's theorem [St, Theorem 1], in particular showing that $\Delta(x, 5, 4, 2)$ changes sign infinitely often, which neither Knapowski and Turán nor Kátai could prove with their methods. A limiting case of Stark's method allows one to prove results without any knowledge of the non-trivial zeros of $L(s, \chi)$ [St, Theorem 3]. Stark also proved a more general theorem where one compares $\pi(x, k_1, l_1)$ with $\pi(x, k_2, l_2)$ with $k_1 \neq k_2$ (this theorem can also be proved from the methods of [Ka1]).

Theorem 3.4 [St, Theorem 2 (i)]. *Suppose $(l_1, k_1) = (l_2, k_2) = 1$, $N_{k_1}(l_1) = N_{k_2}(l_2)$ and no L-function in $C_{k_1} \cup C_{k_2}$ has a real zero $> \frac{1}{2}$. Then*

$$\limsup_{x \to \infty} \frac{\varphi(k_1)\pi(x, k_1, l_1) - \varphi(k_2)\pi(x, k_2, l_2)}{\sqrt{x}/\log x} > 0.$$

About the same time, Bateman et al [BBHKS] showed that the linear independence condition in Ingham's theorem could be relaxed considerably with only a slightly weaker conclusion. Diamond [Di] extended and generalized this result. To state it, we introduce a notion of weak independence. Let N be a positive integer. A subset of G, $\{\gamma_{j_1}, \gamma_{j_2}, \dots, \gamma_{j_m}\}$, is said to be N-independent if for every choice of integers $n_1, \dots, n_m$ satisfying $|n_r| \leqslant N$ for each r and $\sum_{r=1}^m |n_r| \geqslant 2$ we have $\sum_{r=1}^m n_r \gamma_{j_r} \notin G$. With modest values of m and N, this can be checked by computer in a reasonable time.

Theorem 3.5 [Di]. *Suppose that $\{\gamma_{j_1}, \gamma_{j_2}, \dots, \gamma_{j_m}\}$ is N-independent. Then*

$$\limsup_{u \to \infty} A(u) \geqslant a_0 + \frac{2N}{N+1} \sum_{r=1}^m |a_{j_r}|, \quad \liminf_{u \to \infty} A(u) \leqslant a_0 - \frac{2N}{N+1} \sum_{r=1}^m |a_{j_r}|.$$

In light of Lemma 2.2, for every k, l_1, l_2 it is possible to prove with a finite computation that $\Delta(x, k, l_1, l_2)$ changes sign infinitely often using Diamond's theorem. Grosswald [G3] carried out such computations for $k \in \{5, 7, 11, 13, 19\}$, in each case proving unconditionally that all functions $\Delta(x, k, l_1, l_2)$ change sign infinitely often. The cases $k = 3, 4$ and 6 were settled by Littlewood [Li], $k = 8$ by Knapowski and Turán [KT1], and earlier the cases $k \in \{43, 67, 163\}$ were settled by Grosswald [G2] (important here is the fact that a_1 is quite large; this is connected to the class number of $\mathbb{Q}(\sqrt{-k})$ being 1 [BFHR]). We know of no computations for other k.

Applying Theorem 3.5 with $N = 1$ and $m = 1$ gives the unconditional bounds

$$\limsup_{u \to \infty} A(u) \geqslant a_0 + \max_{j \geqslant 1} |a_j|, \quad \liminf_{u \to \infty} A(u) \leqslant a_0 - \max_{j \geqslant 1} |a_j|.$$

Kaczorowksi [K2] proposed different method for avoiding full linear independence of the γ_i. Let $\Omega = T^\infty$ denote the infinite dimensional torus, i.e. the topological product of infinitely many copies of $T = \{z \in \mathbb{C} : |z| = 1\}$. Define the continuous homomorphism $\lambda(t) = (e^{2\pi i \gamma_1 t}, e^{2\pi i \gamma_2 t}, \dots)$. Let $\Gamma = \overline{\lambda(\mathbb{R})}$, the closure of $\lambda(\mathbb{R})$ in the Tikhonov topology of Ω. For $\alpha = (\alpha_1, \alpha_2, \dots) \in \Omega$ define $u\alpha = (u\alpha_1, u\alpha_2, \dots)$ and let S_0 be the stabilizer of Γ, i.e. $S_0 = \{u \in T : u\Gamma = \Gamma\}$. As S_0 is a closed subgroup of T, either $S_0 = T$ or S_0 is a cyclic group generated by a root of unity. If the numbers γ_j are linearly independent, Kronecker's theorem implies that $S_0 = T$. However, Kaczorowski proves that $h(x, k, l_1, l_2)$ takes arbitrarily large and small values on the hypothesis that S_0 contains any element other that ± 1 ([K2], Corollaries 4,5). This will be discussed further in §4.

3.4. First sign change.
The problem of bounding the first sign change of $\Delta(x, k, l_1, l_2)$ can be attacked in a brute-force way by computing all of the primes up to a given limit using the sieve of Eratosthenes and then tabulate the functions $\pi(x, k, l)$ directly. Using modern computers, John Leech [Lee] discovered in 1957 the first negative value in $\Delta(x, 4, 3, 1)$, which occurs at $x = 26861$. It turns out that when $k|24$, $(l, k) = 1$ and $1 < l < k$ that negative values of $\Delta(x, k, l, 1)$ are quite rare. In a massive computation in the 1970s, Bays and Hudson ([BH1]–[BH4]) computed these functions up to $x = 10^{12}$. In addition to several sign changes in $\Delta(x, 4, 3, 1)$, they discovered sign changes in only three other such functions: $\Delta(x, 3, 2, 1)$ with its first negative value at $x = 608,981,813,029$, $\Delta(x, 8, 5, 1)$ with its first negative value at $x = 588,067,889$, and $\Delta(x, 24, 13, 1)$ with its first negative value near $978,000,000,000$.

Computations with the explicit formulas (2.3) but truncated to a sum over ρ with $|\Im \rho| \leqslant T$ for a fixed T (e.g. $T = 10000$) provides one with a good sense of where the sign changes likely occur, since the smaller zeros of (2.3) contribute the dominant part of the sum. Actually proving that a negative value occurs at or near x can be done in three ways. First, if x is not too large (say $x < 10^{14}$ with the latest computers), the above mentioned brute force method can work. For intermediate x (up to maybe 10^{20} with today's technology), one can use Hudson's [H1] generalization of the famous Meissel formula to compute exactly the value of the functions $\pi(x, k, l)$. As shown by Lagarias, Miller and Odlyzko [LMO], this can be done in time $O(x^{2/3+\varepsilon})$, compared to time $O(x^{1+\varepsilon})$ required for the brute-force method. The first author is currently writing a computer program implementing the algorithm, and one result is that $\Delta(1.9282 \times 10^{14}, 8, 1, 7) < 0$.

The third method, which is the only practical one for really large x, is based on the explicit formulas (2.3). The first papers on the subject are those of Skewes ([Sk1], [Sk2]), who gave in 1955 the first unconditional upper bound on the first sign change of $\pi(x) - \mathrm{Li}(x)$, namely $10^{10^{10^{10^3}}}$. Lehman [Leh] showed that the results

of the computations of the greatly truncated sums (2.3) for the Riemann zeta function could be made rigorous, in the sense that one could use the calculations to prove that $\pi(x) > \mathrm{Li}(x)$ somewhere in a short interval of the form $[x_0, (1 + \delta)x_0]$. Lehman's method drastically improved the upper bound for the first sign change. In particular, he proved that it must occur before 1.5926×10^{1165} and using more known zeros for the Riemann zeta function his method was used by te Riele [tR] to lower the bound to 6.6658×10^{370} and by Bays and Hudson [BH5] to lower it further to 1.39822×10^{316}. Ford and Hudson [FH] generalized Lehman's method, which allowed them to localize sign changes of any linear combination of functions $\pi(x, k, l)$, in particular the functions $\Delta(x, k, l_1, l_2)$. A consequence is the following theorem, proved with the aid of the zeros ρ of L-functions with $|\Im\rho| \leqslant 10000$ ([Ru1],[Ru2]).

Theorem 3.6 [FH, Corollary 4]. *For each $b \in \{3, 5, 7\}$, $\pi_{8,b}(x) < \pi_{8,1}(x)$ for some $x < 5 \times 10^{19}$. For each $b \in \{5, 7, 11\}$, $\pi_{12,b}(x) < \pi_{12,1}(x)$ for some $x < 10^{84}$. For each $b \in \{5, 7, 11, 13, 17, 19, 23\}$, $\pi_{24,b}(x) < \pi_{24,1}(x)$ for some $x < 10^{353}$.*

4. RACING PROBLEMS

In racing problems for $r > 2$ virtually nothing is known unconditionally. All results we know are proven under ERH_k; sometimes other assumptions are used. In the end of this paper we will explain principal difficulties to get unconditional results in racing problems.

4.1 Conditional Results.

In [K3] Kaczorowski assuming ERH_k solves the racing problem in a weak form: he shows that 1 (mod k) wins and loses infinitely often.

Theorem 4.1 [K3, Theorem 1]. *Let $k \geqslant 3$ and assume ERH_k. Then there exist infinitely many integers m with $\pi(m, k, 1) > \max\limits_{l \not\equiv 1 \ (\mathrm{mod} \ k)} \pi(m, k, l)$. Moreover, the set of m's satisfying this inequality has positive lower density. The same statement holds true for m satisfying the inequality $\pi(m, k, 1) < \min\limits_{l \not\equiv 1 \ (\mathrm{mod} \ k)} \pi(m, k, l)$.*

In [K5] Kaczorowski showed that certain other orderings of the functions $\pi(x, k, l)$ occur for a set of x having positive lower density, but these results are very complicated and we do not state them here. Note that Theorem 4.1 conditionally solves an average preponderance problem for $\pi(n, 4, 1) - \pi(n, 4, 3)$ as it stated in problem 5 of section 1, (1.2). It shows that under ERH_4,

$$\liminf_{x \to +\infty} \frac{N(x)}{x} > 0.$$

This inequality could also be deduced from Wintner [W1].

In [K3] Kaczorowski assuming ERH_k solves the racing problem modulo 5 for the function ψ. So, for any permutation (l_1, l_2, l_3, l_4) of the sequence $(1, 2, 3, 4)$ there exist infinitely many integers m with

$$(4.1) \qquad \psi(m, 5, l_1) > \psi(m, 5, l_2) > \psi(m, 5, l_3) > \psi(m, 5, l_4).$$

No doubt, more extensive (maybe, technically not feasible) calculations can give the same assertion for the function $\pi(x, 5, l)$. Moreover, one can try do the same job for other moduli.

All results on sign changes of the functions $\Delta(x, k, l_1, l_2)$ make use of some kind of almost periodicity of the sums in (2.3). For example, the work of Knapowski and Turán depends on results from Turán's power sum methods [T]. Kaczorowski takes a different approach, replacing x in (2.3) by e^z where z is a complex variable. This gives the so-called k-functions [K1]. For $\Im z > 0$ and $\chi \in D_k$ he defines

$$k(z, \chi) = \sum_{\Im \rho > 0} e^{\rho z}$$

and

$$K(z, \chi) = \int_{i\infty}^{z} k(s, \chi) ds = \sum_{\Im \rho > 0} \frac{1}{\rho} e^{\rho z}$$

where the summation is taken over all nontrivial zeros of $L(s, \chi)$ with positive imaginary parts. Further, for $(l, k) = 1$, $0 < l \leqslant k$ the following functions are defined

$$F(z, k, l) = -2e^{-z/2} \frac{1}{\varphi(k)} \sum_{\chi \pmod{k}} \overline{\chi}(l) K(z, \chi') - \frac{2}{\varphi(k)} \sum_{\chi \pmod{k}} \overline{\chi}(l) m(\tfrac{1}{2}, \chi),$$

where χ' denotes the primitive Dirichlet's character induced by χ and $m(\tfrac{1}{2}, \chi)$ is the multiplicity of a zero of $L(s, \chi)$ at $s = \frac{1}{2}$ ($m(\tfrac{1}{2}, \chi) = 0$ when $L(\tfrac{1}{2}, \chi) \neq 0$).

Historically, the k-functions were introduced by Cramér ([Cr1], [Cr2]) in connection with the Riemann zeta function and the Dedekind zeta functions.

For real x let $P(x, k, l) = \lim_{y \to 0+} \Re F(x + iy, k, l)$. It can be proved that the limit exists for all real x, and that $P(x, k, l)$ is piecewise continuous. Moreover, we have (cf. (2.3))

$$P(x, k, l) = e^{-x/2} \left(\psi(e^x, k, l) - \frac{e^x}{\phi(k)} \right) + E(x, k, l),$$

where $|E(x, k, l)| = O(xe^{-x/2})$ ($x \geqslant 1$). Even though $E(x, k, l)$ is very small, its behavior near $x = 0$ is important to many of the applications. Important in Kaczorowski's analysis is the property that for fixed $y = \Im z > 0$ each function $F(x + iy, k, l)$ is almost periodic in the sense of Bohr, and further the functions $P(x, k, l)$ are almost periodic in the sense of Stepanov (see e.g. [Be] for definitions and properties of various types of almost periodic functions). A consequence ([K3], Lemma 3) is that for every number x_0 and every $\varepsilon > 0$, the vector $V(x) = (P(x, k, l_1), \ldots, P(x, k, l_n))$ satisfies $\|V(x) - V(x_0)\| \leqslant \varepsilon$ for a set of x having positive lower density. Thus, finding a single point x_0 with

$$P(x_0, k, l_{j_1}) > P(x_0, k, l_{j_2}) > \cdots > P(x_0, k, l_{j_n})$$

implies that the same inequality occurs for a set of x having positive lower density. Computing $P(x, k, l)$ is easy for small x, and to prove Theorem 4.1, Kaczorowski finds appropriate x_0's in a small neighborhood of 0.

The proof of Kaczorowski's theorem mentioned in §3.3 uses the fact that the function $F(z, k, l)$ has logarithmic singularities at the points $\pm \log(p^m)$ where $p^m \equiv l$ (mod k), and thus $\Im F(z, k, l)$ is unbounded in a vicinity of these points. Adopting the notation from §3.3, let $\zeta \in S_0$, $\zeta \neq \pm 1$. Then, for fixed $y > 0$ the function $F(x+iy, k, l)$ exhibits a kind of "twisted" almost periodicity, namely for every $\varepsilon > 0$ there is a number w so that

$$\sup_{x \in \mathbb{R}} |F(x + iy, k, l) - \zeta F(x + w + iy, k, l)| \leqslant \varepsilon.$$

Since $\zeta \neq \pm 1$, this implies that the real part of $F(z, k, l)$ is unbounded in the upper half plane, and this can be used to prove that $P(x, k, l)$ is unbounded.

The paper [K6] contains a nice overview of Kaczorowski's methods and results. In addition, one can define k-functions for a wide class of L-functions occurring in number theory and use them to study oscillations and distribution of various "error terms" in prime number theory [KR].

The racing problem has been extensively investigated by Rubinsten and Sarnak [RS]. Let $P \subset (0, +\infty)$,

$$\overline{\delta}(P) = \limsup_{X \to \infty} \frac{1}{\log X} \int_{t \in P \cap [2, X]} \frac{dt}{t},$$

$$\underline{\delta}(P) = \liminf_{X \to \infty} \frac{1}{\log X} \int_{t \in P \cap [2, X]} \frac{dt}{t}.$$

If the latter two quantities are equal, the logarithmic density $\delta(P)$ of the set P is their common value. The problem is to study the existence of and to estimate the logarithmic density of the set

$$P_{k; l_1, \ldots, l_r} = \{x \geqslant 2 : \pi(x, k, l_1) > \pi(x, k, l_2) > \cdots > \pi(x, k, l_r)\}.$$

Introduce the vector valued functions

$$E_{k; l_1, \ldots, l_r}(x) = \frac{\log x}{\sqrt{x}} (\varphi(k)\pi(x, k, l_1) - \pi(x), \ldots, \varphi(k)\pi(x, k, l_r) - \pi(x)).$$

Theorem 4.2 [RS, Theorem 1.1]. *Assume* ERH_k. *Then* $E_{k; l_1, \ldots, l_r}$ *has a limiting distribution* $\mu_{k; l_1, \ldots, l_r}$ *on* $\mathbb{R}^r$, *that is*

$$\lim_{X \to \infty} \frac{1}{\log X} \int_2^X f(E_{k; l_1, \ldots, l_r}(x)) \frac{dx}{x} = \int_{\mathbb{R}^r} f(x) d\mu_{k; l_1, \ldots, l_r}$$

for all bounded continuous functions f *on* $\mathbb{R}^r$.

If it turns out that if the measure $\mu_{k; l_1, \ldots, l_r}$ is absolutely continuous then

$$\delta(P_{k; l_1, \ldots, l_r}) = \mu_{k; l_1, \ldots, l_r}(\{x \in \mathbb{R}^r : x_1 > \cdots > x_r\}).$$

However, the authors of [RS] write that assuming only ERH_k they don't know that $\delta(P_{k; l_1, \ldots, l_r})$ exists.

The measures μ are very localized but not compactly supported. Let $B'_R = \{x \in \mathbb{R}^r : |x| > R\}$, $B_R^+ = \{x \in \mathbf{B}'_R : \varepsilon(l_j)x_j > 0\}$, $B_R^- = -B_R^+$, where $\varepsilon(l) = 1$ if $l \equiv 1$ (mod k) and $\varepsilon(l) = -1$ if $l \not\equiv 1$ (mod k).

Theorem 4.3 [RS, Theorem 1.2]. *Assume ERH_k. Then there are positive constants c_1, c_2, c_3, c_4, depending only on k, such that*

$$\mu_{k;l_1,\dots,l_r}(B'_R) \leqslant c_1 \exp(-c_2\sqrt{R}),$$

$$\mu_{k;l_1,\dots,l_r}(B_R^{\pm}) \geqslant c_3 \exp(-\exp(c_4 R)).$$

The second inequality gives a quantitative version of the theorem from [K3].

Montgomery [Mo], using ERH_1 (the Riemann Hypothesis) and GSH_1, investigated the tails of $\mu_1 = \mu_{1:1}$. He showed that

$$\exp(-c_2\sqrt{R}\exp\sqrt{2\pi R}) \leqslant \mu_1(B_R^{\pm}) \leqslant \exp(-c_1\sqrt{R}\exp\sqrt{2\pi R}).$$

Rubinstein and Sarnak [RS] under ERH_k and GSH_k have found an explicit formula for the Fourier transform of $\mu_{k;l_1,\dots,l_r}$. Special cases of the formula were proven earlier in [W1] and [Ho]. The formula implies that, for $r < \varphi(k)$, $\mu_{k;l_1,\dots,l_r} = f(x)dx$ with a rapidly decreasing entire function f. As a consequence, under ERH_k and GSH_k each $\delta(P_{k;l_1,\dots,l_r})$ does exist and is nonzero (including the case $r = \varphi(k)$). Hence, conditionally the solution to the racing problem is affirmative.

Also in [RS] is a procedure for calculating $\delta(P_{k;l_1,\dots,l_r})$ using known zeros of L-functions ([Ru1],[Ru2]). In particular, they compute $\delta(P_{4;3,1}) = 0.9959\dots$, thus giving a quantitative version of Chebyshev's statement. Although for small k many of the densities are quite large, as $k \to \infty$ they become more uniform.

Theorem 4.4 [RS, Theorem 1.5]. *Assume ERH_k and GSH_k for all $k \geqslant 3$. For r fixed,*

$$\max_{l_1,\dots,l_r} \left| \delta(P_{k;l_1,\dots,l_r}) - \frac{1}{r!} \right| \to 0 \ \text{as} \ k \to \infty.$$

We say that $k; l_1, \dots, l_r$ is unbiased if $\mu_{k;l_1,\dots,l_r}$ is invariant under permutations of $(x_1, \dots, x_r)$. In this case $\delta(P_{k;l_1,\dots,l_r}) = 1/r!$.

Theorem 4.5 [RS, Theorem 1.4]. *Assume ERH_k and GSH. Then $k; l_1, \dots, l_r$ is unbiased if and only if either $r = 2$ and $N_k(l_1) = N_k(l_2)$, or $r = 3$ and $l_2 \equiv l_1 g$ (mod k), $l_3 \equiv l_1 g^2$ (mod k), where $g^3 \equiv 1$ (mod k).*

Feuerverger and Martin [FM] computed numerous densities for small moduli k, studying in particular the cases where $r \geqslant 3$, $k; l_1, \dots, l_r$ is not unbiased and $N_k(l_1) = \dots = N_k(l_r)$, e.g. $8; 3, 5, 7$. Curious inequities occur in these cases, which can be "explained" in terms of distribution properties of the functions $\Psi(x, \chi)$ (see (2.3) and also [Ma]). Computation of the densities when $r \geqslant 3$ is very complex using the methods in [RS] or [FM]. A much simpler and faster method (but less rigorous) is given in [BFHR].

Rubinstein and Sarnak also studied the union problem mentioned in problem 9 in section 1. Suppose $k \geqslant 3$, let A be the set of quadratic non-residues modulo k and let B be the set of quadratic residues. Define

$$P_{k;N,R} = \left\{ x \geqslant 2 : \sum_{p \in A, p \leqslant x} 1 > \frac{|A|}{|B|} \sum_{p \in B, p \leqslant x} 1 \right\}.$$

With k restricted to integers possessing a primitive root (again assuming ERH_k and GSH_k), in [RS] it is proved that $\delta(P_{k;N,R}) \to \frac{1}{2}$ as $k \to \infty$, although the convergence is far from monotone. In fact there is a connection between the values of $\delta(P_{k;N,R})$ and the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ [BFHR].

4.2 Barriers to unconditional results.

One may ask if a racing problem for $r > 2$ may be solved without the assumption of ERH_k, since the problem of proving infinitely many sign changes of $\Delta(x, k, l_1, l_2)$ is easier if $g(s, k, l_1, l_2)$ (cf. end of section 2) has a pole with real part $> 1/2$ and no real poles $> 1/2$. Since it is believed that the sets of zeros for each $L(s, \chi)$ are disjoint, this condition essentially says that ERH_K is false.

In particular, can it be shown, for some quadruples (k, l_1, l_2, l_3), that the 6 orderings of the functions $\pi(x, k, l_j)$ occur for infinitely many integers x, without the assumption of ERH_k (while still allowing the assumption of Haselgrove's condition and/or that zeros with imaginary part $< R_k$ lie on the critical line for some constant R_k)? Recently we have answered this question in the negative (in a sense) for all quadruples (k, l_1, l_2, l_3) [FK]. Thus, in a sense the hypothesis ERH_k is a necessary condition for proving any such results when $r > 2$.

Let $D = (k, l_1, l_2, l_3)$, where l_1, l_2, l_3 are distinct residues modulo k which are coprime to k. Suppose for each $\chi \in C_k$, $B(\chi)$ is a sequence of complex numbers with positive imaginary part (possibly empty, duplicates allowed), and denote by $\mathcal{B}$ the system of $B(\chi)$ for $\chi \in C_k$. Let $n(\rho, \chi)$ be the number of occurrences of the number ρ in $B(\chi)$. The system $\mathcal{B}$ is called a *barrier* for D if the following hold: (i) all numbers in each $B(\chi)$ have real part in $[\beta_2, \beta_3]$, where $\frac{1}{2} < \beta_2 < \beta_3 \leqslant 1$; (ii) for some β_1 satisfying $\frac{1}{2} \leqslant \beta_1 < \beta_2$, if we assume that for each $\chi \in C_k$ and $\rho \in B(\chi)$, $L(s, \chi)$ has a zero of multiplicity $n(\rho, \chi)$ at $s = \rho$, and all other zeros of $L(s, \chi)$ in the upper half plane have real part $\leqslant \beta_1$, then one of the six orderings of the three functions $\pi(x, k, l_j)$ does not occur for large x. If each sequence $B(\chi)$ is finite, we call $\mathcal{B}$ a *finite barrier* for D and denote by $|\mathcal{B}|$ the sum of the number of elements of each sequence $B(\chi)$, counted according to multiplicity.

Theorem 4.6 [FK]. *For every real numbers $\tau > 0$ and $\sigma > \frac{1}{2}$ and every $D = (k, l_1, l_2, l_3)$, there is a finite barrier for D, where each sequence $B(\chi)$ consists of numbers with real part $\leqslant \sigma$ and imaginary part $> \tau$. In fact, for most D, there is a barrier with $|\mathcal{B}| \leqslant 3$.*

We do not claim that the falsity of ERH_k implies that one of the six orderings does not occur for large x. For example, take $\sigma > \frac{1}{2}$, and suppose each non-principal character modulo k has a unique zero with positive imaginary part to the right of the critical line, at $\sigma + i\gamma_\chi$. If the numbers γ_χ are linearly independent over the rationals, it follows easily from Lemma 4.1 below and Kronecker's Theorem that in fact all $\phi(k)!$ orderings of the functions $\{\pi(x, k, l) : (l, k) = 1\}$ occur for an unbounded set of x.

As an example, we will demonstrate the existence of a barrier in a more simple situation: $r = 4$, $k = 5$, $l_j = j$ ($j = 1, 2, 3, 4$). By (2.2)–(2.4) we have the following lemma.

Lemma 4.1. *Let $\beta \geqslant \frac{1}{2}$, $x \geqslant 10$ and for each $\chi \in C_k$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s, \chi)$ with $\Re s > \beta$ and $\Im s > 0$. Suppose further that all $L(s, \chi)$ are zero-free on the real segment $0 < s < 1$. If $(l_1, k) = (l_2, k) = 1$ and x is sufficiently large, then*

$$\phi(k)\Delta(x, k, l_1, l_2) = -2\Re \left[\sum_{\chi \in C_k} (\overline{\chi}(l_1) - \overline{\chi}(l_2)) \sum_{\substack{\rho \in B(\chi) \\ |\Im \rho| \leqslant x}} f(\rho) \right] + O(x^\beta \log^2 x),$$

where

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} \, dt = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\Re\rho}}{|\rho|^2 \log^2 x} \right).$$

Here the constant implied by the Landau O-symbol may depend on k, but not on any other variable. Take $\chi_1 \in C_5$ so that $\chi_1(1) = 1, \chi_1(2) = i, \chi_1(3) = -i, \chi_1(4) = -1$. Let t be a large positive number. Take $\sigma > \frac{1}{2}$, $B(\chi_1) = \{\sigma + it\}$, $B(\chi) = \emptyset$ for $\chi \in C_5 \setminus \{\chi_1\}$. We use Lemma 4.1 with $\frac{1}{2} < \beta < \sigma$. For $\rho = \sigma + it$ we have

$$(4.2) \qquad f(\rho) = \frac{-x^\sigma}{t \log x} x^{it} i + O\left(\frac{x^\sigma}{t^2 \log x} \right).$$

We claim that in this situation the inequality

$$(4.3) \qquad \pi(x, 5, 1) > \pi(x, 5, 4) > \pi(x, 5, 2) > \pi(x, 5, 3)$$

cannot occur for large x. Indeed, by Lemma 4.1 and (4.2), for large x (depending on σ, β) we have

$$(4.4) \qquad \pi(x, 5, 1) - \pi(x, 5, 4) = \frac{x^\sigma}{t \log x} \Re(x^{it} i) + O\left(\frac{x^\sigma}{t^2 \log x} \right),$$

$$(4.5) \qquad \pi(x, 5, 4) - \pi(x, 5, 2) = \frac{x^\sigma}{t \log x} \Re(x^{it}(-1/2 - i/2)) + O\left(\frac{x^\sigma}{t^2 \log x} \right),$$

$$(4.6) \qquad \pi(x, 5, 2) - \pi(x, 5, 3) = \frac{x^\sigma}{t \log x} \Re(x^{it}) + O\left(\frac{x^\sigma}{t^2 \log x} \right).$$

It is not difficult to prove that for any t

$$\min(\Re(x^{it} i), \Re(x^{it}(-1/2 - i/2)), \Re(x^{it})) \leqslant -\sqrt{0.1}.$$

Therefore, the estimates (4.4)–(4.6) imply

$$\min(\pi(x,5,1) - \pi(x,5,4), \pi(x,5,4) - \pi(x,5,2), \pi(x,5,2) - \pi(x,5,3))$$
$$\leqslant -\frac{x^\sigma}{\sqrt{10}t \log x} + O\left(\frac{x^\sigma}{t^2 \log x}\right).$$

For large t this does not agree with (4.3).

For $r = 3$ our construction of finite barriers is more difficult. It uses multiple zeros or zeros of several functions $L(s, \chi)$ simultaneously. However, answering a question posed by Peter Sarnak, in [FK] for many quadruples (k, l_1, l_2, l_3) we construct a barrier (with an infinite set $B(\chi)$) where the imaginary parts of the numbers in the sets $B(\chi)$ are linearly independent; in particular, we assume all zeros of each $L(s, \chi)$ are simple, and $L(s, \chi_1) = 0 = L(s, \chi_2)$ does not occur for $\chi_1 \neq \chi_2$ and $\Re s > \beta_2$.

So, the results of [FK] show that there are barriers blocking for large x some ordering of each triple of functions $\pi(x, k, l_i)$ ($i = 1, 2, 3$). One can ask about barriers blocking some other natural (and usually proven under ERH_k) properties studied in comparative prime–number theory. We can prove the following results (a paper is in preparation) concerning the problems 5–7 from section 1:

(1) For many k, l_1, l_2, barriers blocking the property

$$\limsup_{x \to +\infty} \frac{card\{n \leqslant x : \ \pi(n, k, l_1) \geqslant \pi(n, k, l_2)\}}{x} > 0;$$

(2) For many k, l_1, l_2, barriers blocking the property

$$\pi(x_\nu, k, l_1) \geqslant \frac{1}{\varphi(k)}\mathrm{Li}(x_\nu) \text{ and } \pi(x_\nu, k, l_2) \geqslant \frac{1}{\varphi(k)}\mathrm{Li}(x_\nu)$$

for an unbounded sequence x_ν;

(3) For many k, barriers blocking the existence of a sequence $x_\nu \to \infty$ such that

$$\pi(x_\nu, k, 1) < \min_l \pi(x_\nu, k, l) \text{ or } \pi(x_\nu, k, 1) > \max_l \pi(x_\nu, k, l).$$

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