VOLUME I: EXPONENTOID AND KNUTHOID SPACES
A TRANSCENDENCE-LEVEL GENERALIZATION OF PERFECTOID GEOMETRY

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Abstract. We introduce the theory of *Exponentoid* and *Knuthoid* spaces, geometric structures stratified by recursive and trans-recursive filtration towers such as $\exp(n)$ and $a \uparrow^k n$. Generalizing multiplicoid geometry, we construct new period rings, define filtration-indexed torsors, and develop cohomology and motivic realization over rapidly growing towers. This lays the groundwork for hyperfiltration theories and stratified ontologies of arithmetic geometry beyond additive or valuation-theoretic formalisms.

Contents

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0. Notation and Symbol Dictionary

This section collects the primary notation used throughout this volume, with a focus on exponentoid and knuthoid structures, their corresponding period rings, torsors, filtrations, and cohomological realizations.

Growth Functions and Indexing Notation.

- $\exp(n)$: the standard exponential growth function, e.g., 2^n or e^n.
- $a \uparrow^k n$: Knuth's k-th level hyperoperation, e.g., $2 \uparrow^2 3 = 2^{2^2}$.
- $\varepsilon^{f(n)}$: ϵ-stratified growth index associated to $f(n)$.

Filtration Systems.

- $F^{\exp(n)}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/\exp(n))$: exponentoid filtration.
- $F^{a\uparrow^k n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n))$: knuthoid filtration.
- Filt_∞ : the category of all filtration systems indexed by recursive growth.

Period Rings and Towers.

- $B_{\exp,dR}$: exponentoid de Rham period ring.
- $B_{\uparrow^k,dR}$: knuthoid de Rham period ring.
- $Per_{\exp} := \{B^{(n)}_{\exp,dR}\}_{n\geq 0}$: tower of exponentoid period rings.
- $Per_{\uparrow^k} := \{B^{(n)}_{\uparrow^k,dR}\}_{n\geq 0}$: knuthoid period tower.

Torsors and Group Actions.

- $\mathbb{T}^{[\exp]}$: exponentoid torsor tower, indexed by $\exp(n)$.
- $\mathbb{T}^{[\uparrow^k]}$: knuthoid torsor tower, indexed by $a \uparrow^k n$.
- $\mathcal{T}^{\exp}_n$, $\mathcal{T}^{\uparrow^k}_n$: torsors at each level.
- $\Theta : \mathbb{T}^{[\exp]} \to \mathbb{T}^{[\uparrow^k]}$: functorial transition map between torsor hierarchies.

Motivic and Realization Functors.

- $M^{[\exp]}(X)$: exponentoid motive associated to X.
- $M^{[\uparrow^k]}(X)$: knuthoid motive at level k.
- $\mathrm{real}_{\exp}, \mathrm{real}_{\uparrow^k}$: realization functors to exponentoid and knuthoid cohomology, respectively.
- $r_{\exp}$, $r_{\uparrow^k}$: higher regulator maps in exponential/hyper-exponential settings.

General Structures.

- $\mathscr{R}_{\exp}, \mathscr{R}_{\uparrow^k}$: realization systems over exponential or knuthoid period towers.
- $H^i_{\exp}(X, \mathcal{F})$: exponentoid cohomology.
- $H^i_{\uparrow^k}(X, \mathcal{F})$: knuthoid cohomology.
- $\mathcal{O}nt_{\exp}$, $\mathcal{O}nt_{\uparrow^k}$: growth-indexed ontological stacks.

Remarks. All notations in this volume generalize those from Volume I' (Multiplicoid Geometry). The key novelty is the replacement of congruence-based growth (2^n) with recursive hyper-growth schemes indexed by $\exp(n)$ and $a \uparrow^k n$. The categorical, cohomological, and ontological frameworks are extended accordingly.

1. Introduction and Motivation

1.1. **From Multiplicative to Recursive Growth.** Multiplicoid geometry replaced valuation and additive proximity with multiplicative congruence depth. In this volume, we go one level higher: from multiplicative descent to recursive growth stratification, using filtrations governed by exponential and hyper-exponential indexing functions.

Where previous geometries used:

$$F^n\mathcal{F}, \quad F^{2^n}\mathcal{F},$$

we now stratify by:

$$F^{\exp(n)}\mathcal{F}, \quad F^{a\uparrow^k n}\mathcal{F},$$

where $a \uparrow^k n$ denotes the k-th hyperoperation in Knuth's notation.

1.2. **Why Exponentoid and Knuthoid?** The motivation is both arithmetic and ontological:

- **Arithmetic Depth:** Many transcendental phenomena in number theory (e.g., polylogarithms, multiple zeta values) encode recursive growth. Geometry must be sensitive to these recursion depths.
- **Cohomological Persistence:** Objects surviving across exponential filtration layers capture deeper torsor descent behavior and stack-theoretic persistence.
- **Meta-Geometry:** The hierarchy additive < multiplicative < exponential < knuthoid < ontological suggests that geometry is a manifestation of recursion, not locality.
- **Categorical Richness:** New towers of torsors, regulators, and period sheaves emerge, indexed by trans-recursive growth.

1.3. **The Landscape of Growth-Indexed Geometry.** We propose the following stratified framework of geometric regimes:

Regime	**Growth**	**Example Filtration**
Additive	n	$F^n\mathcal{F}$
Multiplicative	2^n	$F^{2^n}\mathcal{F}$
Exponentoid	$\exp(n)$	$F^{\exp(n)}\mathcal{F}$
Knuthoid	$a \uparrow^k n$	$F^{a\uparrow^k n}\mathcal{F}$
Ontoid	$f_{\mathrm{meta}}(n)$	$\mathcal{O}nt_n(\mathcal{F})$

This volume focuses on the third and fourth layers—exponentoid and knuthoid.

1.4. **Objectives of This Volume.** We will:

(1) Define exponentoid and knuthoid filtrations, period rings, and torsor towers;
(2) Construct realization functors $\mathscr{R}_{\exp}, \mathscr{R}_{\uparrow^k}$ to cohomology indexed by recursive growth;
(3) Introduce regulator systems and special value theory in hyper-filtration regimes;
(4) Provide transition models from dyadic to exponentoid structures;
(5) Lay foundations for a future ontology of infinitely generated arithmetic spaces.

1.5. **Position Within the Yang Program.** This volume is the formal starting point of the higher recursion phase of the Yang Program. It follows the fully developed multiplicoid base geometry of Volume I', and sets the stage for:
- Volume II: *Hyper-Filtration Theory and Transfinite Monodromy*,
- Volume III: *Weight–Monodromy Conjectures beyond Linear Cases*,
- Volume IV: *Ontoid Geometry and Space-Theoretic Ontologies*,
- Volume V: *Categorical Arithmetic of Growth-Based Spaces.*

Each builds upon the exponentoid and knuthoid constructions initiated here.

Geometry is no longer what space contains, but what growth generates.

2. Exponentoid Filtrations and Period Towers

2.1. **Exponentoid Growth and Arithmetic Stratification.** Let $f(n) = \exp(n)$ denote an exponential growth function, e.g., $f(n) = a^n$ or e^n. In contrast to linear or multiplicative filtrations, exponentoid filtrations define strata of arithmetic collapse that deepen more rapidly with n.

Definition 2.1 (Exponentoid Filtration)**.** *Let $\mathcal{F}$ be a sheaf on an arithmetic base. Define the exponentoid filtration as:*

$$F^{\exp(n)}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/\exp(n) \cdot \mathcal{F}\right).$$

Each level isolates the part of $\mathcal{F}$ that vanishes under increasingly large exponential congruence depths. It generalizes the dyadic filtration F^{2^n} used in Volume I'.

2.2. **Exponentoid Period Rings.** We now define the analog of de Rham period rings in the exponentoid setting.

Definition 2.2 (Exponentoid Period Ring)**.** *Let A be a ring with exponentoid filtration $\{F^{\exp(n)}A\}$. Define:*

$$B_{\exp,dR} := \varprojlim_n A/\exp(n) \otimes_{\mathbb{Z}} \mathbb{Q},$$

equipped with the natural descending $\exp(n)$*-indexed filtration.*

Remark 2.3. $B_{\exp,dR}$ captures congruence collapse at recursive arithmetic depths. Unlike B_{dR}, it is defined without a valuation or topology, but via growth-level recursion.

2.3. **Exponentoid Period Towers.**

Definition 2.4 (Period Tower)**.** *The exponentoid period tower is the inverse system:*

$$Per_{\exp} := \left\{B^{(n)}_{\exp,dR} := A/\exp(n) \otimes \mathbb{Q}\right\}_{n\in\mathbb{N}},$$

with canonical transition maps $B^{(n+1)}_{\exp,dR} \to B^{(n)}_{\exp,dR}$ given by reduction modulo $\exp(n)$*.*

This tower replaces (2^n) or (p^n) with $\exp(n)$ as the congruence modulus. Each level is exponentially coarser in collapse than the previous.

2.4. **Examples.**

Example 2.5. Let $A = \mathbb{Z}[x]$ and $\exp(n) = 2^n$. Then

$$B_{\exp,dR} = \varprojlim_n \mathbb{Z}[x]/(2^n) \otimes \mathbb{Q} = \mathbb{Z}_2[[x]].$$

This reproduces the dyadic case as a special instance.

Example 2.6. Let $\exp(n) = \lfloor e^n \rfloor$ and define $F^{\exp(n)}\mathcal{F}$ as above. Then $B_{\exp,dR}$ reflects a transcendence-graded period ring indexed by rational approximations to exponential growth.

2.5. **Functoriality and Universal Realization.** Let $\mathcal{F}$ be a sheaf with exponentoid filtration. The realization functor is:

Definition 2.7 (Exponentoid Realization Functor)**.**

$$\mathscr{R}_{\exp} : \mathbf{Sh}^{\exp} \longrightarrow Per_{\exp} \times \mathbb{T}^{[\exp]},$$

mapping $\mathcal{F}$ to its period image and torsor stratification across exponential depth.

2.6. **Summary.** Exponentoid filtrations and period towers provide the recursive replacement of dyadic stratification. Their key properties are:

- Growth by a^n rather than n or 2^n;
- Arithmetic stratification without topology;
- Recursive descent of cohomological and torsor data;
- Compatibility with higher towers such as $a \uparrow^k n$.

We next study torsors and realizations over these towers.

3. Exponentoid Torsors and Stratified Descent

3.1. **Generalization of Torsor Descent.** In dyadic geometry, torsors $\mathcal{T}_n^{(2)}$ under $\mathbb{Z}/2^n$ encode congruence descent. In the exponentoid setting, we stratify by exponentially growing group actions, generalizing the tower of torsors accordingly.

Definition 3.1 (Exponentoid Torsor Tower)**.** *Let $G_n := \mathbb{Z}/\exp(n)\mathbb{Z}$. An* exponentoid torsor tower *is a system*

$$\mathbb{T}^{[\exp]} := \{\mathcal{T}_n^{\exp} \to X\}_{n \geq 0},$$

where each $\mathcal{T}_n^{\exp}$ is a G_n-torsor and the transition maps

$$\mathcal{T}_{n+1}^{\exp} \to \mathcal{T}_n^{\exp}$$

are equivariant under the natural projections $G_{n+1} \to G_n$.

This construction replaces binary descent with recursive stratification: each level represents a higher-order congruence symmetry.

3.2. **Stratified Realization through Torsors.** Let $\mathcal{F}$ be a sheaf with exponentoid filtration $F^{\exp(n)}\mathcal{F}$. Then one obtains compatible trivializations over $\mathcal{T}_n^{\exp}$ at each level:

$$F^{\exp(n)}\mathcal{F} \text{ trivializes over } \mathcal{T}_n^{\exp}.$$

This yields an ε-torsor realization:

$$\mathscr{R}_{\exp}(\mathcal{F}) := \left(\{F^{\exp(n)}\mathcal{F}\}, \mathbb{T}^{[\exp]}\right).$$

3.3. **Auto-Equivalence Actions.** Each torsor $\mathcal{T}_n^{\exp}$ defines an auto-equivalence action on the sheaf category:

$$G_n \curvearrowright \mathrm{Sh}(F^{\exp(n)}\mathcal{F}),$$

with the orbit structure reflecting cohomological stratification.

We then obtain:

Proposition 3.2 (Recursive Descent Structure)**.** *The inverse limit*

$$\varprojlim_n \mathrm{Sh}(F^{\exp(n)}\mathcal{F})$$

equipped with torsor auto-equivalence actions defines a stratified stack $\mathcal{S}^{\exp}$ over Filt_∞.

3.4. **Exponentoid Descent and Obstruction.** Let X be a space, and $\mathcal{F}$ a filtered sheaf. The failure to descend a section $s \in \mathcal{F}$ to level $\exp(n)$ defines a cohomological obstruction in:

$$\mathrm{Obs}_n(s) \in H^1(X, \mathcal{T}_n^{\exp}).$$

This generalizes the usual obstruction theory of torsor lifts to the exponentially recursive setting.

3.5. **Examples.**

Example 3.3. Let $X = \mathrm{Spec}(\mathbb{Z})$ and define $\exp(n) = 2^n$. Then:

$$\mathcal{T}_n^{\exp} = \mathrm{Spec}(\mathbb{Z}[x]/(x^{2^n} - 1))$$

is the torsor classifying 2^n-th roots of unity. In general, with $\exp(n) = a^n$, one takes $(x^{a^n} - 1)$ torsors.

Example 3.4. If $\mathcal{F}$ is a modular sheaf over a congruence subgroup $\Gamma(\exp(n))$, then $\mathcal{T}_n^{\exp}$ classifies generalized modular level structures of exponential depth.

3.6. **Conclusion.** Exponentoid torsors stratify arithmetic spaces by recursive congruence complexity. Their tower:

- Replaces Frobenius-twisted covers with growth-based torsors;
- Captures recursive descent phenomena;
- Enables realization functors and motivic lifts at exponential scale.

These torsor towers are essential in defining cohomology and motivic structure in the next sections.

4. Exponentoid Cohomology and Realization Functors

4.1. **Realization via Exponentoid Period Rings.** Given a sheaf $\mathcal{F}$ with exponentoid filtration $F^{\exp(n)}\mathcal{F}$ and associated period tower $Per_{\exp}$, we define a cohomology theory indexed by exponential congruence depth.

Definition 4.1 (Exponentoid Cohomology)**.** *Let X be a space over a base admitting exponentoid stratification. Define:*

$$H^i_{\exp}(X, \mathcal{F}) := \varprojlim_n H^i\left(X, \mathcal{F}/\exp(n)\mathcal{F}\right).$$

This inverse limit encodes stabilization across recursive depth layers. It replaces the topological or syntomic limit with a purely growth-driven arithmetic descent.

4.2. **ϵ-Stratified Period Sheaves.** Let $\mathcal{F}$ be a filtered sheaf over X. Each filtration layer defines a period sheaf:

$$\mathcal{P}^{(n)} := \mathcal{F}/\exp(n) \otimes \mathbb{Q},$$

yielding a tower $\{\mathcal{P}^{(n)}\}_n$ compatible with $B^{(n)}_{\exp,dR}$.

The projective system

$$\mathcal{P}_{\exp} := \varprojlim_n \mathcal{P}^{(n)}$$

defines the *stratified period realization* of $\mathcal{F}$ over exponential congruence.

4.3. **The Realization Functor.**

Definition 4.2 (Universal Exponentoid Realization)**.** *The functor*

$$\mathscr{R}_{\exp} : \mathbf{Sh}^{\exp} \longrightarrow \mathrm{Rep}(\mathbb{T}^{[\exp]}),$$

sends a sheaf $\mathcal{F}$ to:

$$\mathscr{R}_{\exp}(\mathcal{F}) := \left\{\mathcal{P}_{\exp}, \mathbb{T}^{[\exp]}, H^i_{\exp}(X, \mathcal{F})\right\}.$$

This realization lifts the exp-filtration structure to:
- A period image over $B_{\exp,dR}$;
- A torsor-theoretic descent model;
- A cohomological realization capturing recursive invariants.

4.4. **Regulators and Special Value Maps.** Let $K_n(X)$ be the n-th K-theory group of X. Define:

Definition 4.3 (Exponentoid Regulator)**.**

$$r_{\exp} : K_n(X) \longrightarrow H^n_{\exp}(X, \mathbb{Q}(n))$$

given by composition:

$$K_n(X) \xrightarrow{\textit{cycle class}} M^{[\exp]}(X) \xrightarrow{\mathrm{real}_{\exp}} H^n_{\exp}(X).$$

This generalizes the Beilinson regulator by indexing the realization in a recursive growth category.

4.5. **Cohomological Behavior and Vanishing Zones.** Because $\exp(n)$ grows faster than polynomially, stabilization in $H^i_{\exp}$ happens slowly. Define the vanishing threshold:

Definition 4.4 (Exponential Vanishing Threshold)**.** *Let X be a smooth space. The smallest n such that $H^i(X, \mathcal{F}/\exp(n)\mathcal{F}) = 0$ for all $i > 0$ is the* vanishing depth *of $\mathcal{F}$.*

This depth acts as a cohomological complexity measure of the filtration.

4.6. **Motivic Sheaves and Realization Lifts.** Let $M^{[\exp]}(X)$ be a motive in the exponentoid category. Then:

$$\mathrm{real}_{\exp} : M^{[\exp]}(X) \to H^{\bullet}_{\exp}(X, \mathbb{Q}),$$

provides a lift of torsor-stratified cohomology to motive-theoretic invariants. This defines an exponentoid Hodge-type realization structure, purely arithmetic and recursion-based.

4.7. **Summary.** Exponentoid cohomology forms the arithmetic realization theory of recursive descent:

- Indexed by $\exp(n)$ rather than n or 2^n;
- Functorially built from period sheaf towers;
- Compatible with torsor descent and growth ontology;
- Governs motivic regulators and value theories in stratified cohomology.

In the next sections, we introduce Knuth-level generalizations and the meta-recursive dynamics of transfinite arithmetic stratification.

5. Knuthoid Towers and Hyper-Stratification

5.1. **From Exponential to Trans-Recursive Growth.** While exponentoid filtrations grow like $\exp(n)$, we now introduce stratifications governed by the Knuth up-arrow hierarchy:

$$a \uparrow^1 n = a^n, \quad a \uparrow^2 n = a^{a^{\cdots^{a}}}, \quad \cdots, \quad a \uparrow^k n = k\text{-times iterated exponentiation.}$$

These functions grow faster than any finite composition of exponentials, yielding cohomological structures of extreme recursion depth.

5.2. **Knuthoid Filtration Systems.**

Definition 5.1 (Knuthoid Filtration)**.** *Let $k \in \mathbb{N}$, $a \geq 2$. Define:*

$$F^{a\uparrow^k n}\mathcal{F} := \ker\left(\mathcal{F} \to \mathcal{F}/(a \uparrow^k n)\mathcal{F}\right),$$

yielding a filtration indexed by the k-th Knuth growth layer.

This defines the *knuthoid filtration tower*, a hyper-recursive analogue of multiplicoid and exponentoid descent.

5.3. Knuthoid Period Rings and Towers.

Definition 5.2 (Knuthoid Period Ring)**.** *Let A be a base ring. Define:*

$$B_{\uparrow^k,dR} := \varprojlim_n A/(a \uparrow^k n) \otimes \mathbb{Q},$$

with the natural filtration $F^{a\uparrow^k n}$ descending from congruence depth.

Definition 5.3 (Knuthoid Tower)**.**

$$Per_{\uparrow^k} := \left\{ B^{(n)}_{\uparrow^k,dR} := A/(a \uparrow^k n) \otimes \mathbb{Q} \right\}_{n\geq 0}.$$

This tower generalizes Per_{exp} and Per_{dyad}, with dramatically sparser levels.

5.4. Knuthoid Torsors and Realization.

Let $G_n := \mathbb{Z}/(a \uparrow^k n)\mathbb{Z}$ and define the torsor:

$$\mathcal{T}_n^{\uparrow^k} \to X, \quad \text{with group action by } G_n.$$

Definition 5.4 (Knuthoid Realization Functor)**.**

$$\mathscr{R}_{\uparrow^k}(\mathcal{F}) := \left(\{F^{a\uparrow^k n}\mathcal{F}\}, \mathbb{T}^{[\uparrow^k]}, H^i_{\uparrow^k}(X, \mathcal{F}) \right),$$

with

$$H^i_{\uparrow^k}(X, \mathcal{F}) := \varprojlim_n H^i \left(X, \mathcal{F}/(a \uparrow^k n)\mathcal{F} \right).$$

This functor captures hyper-stratified realization and recursive torsor descent.

5.5. Hyper-Stratified Obstruction Theory.

Let $s \in \mathcal{F}$ be a global section. Define the failure to descend through Knuthoid layers via:

$$\mathrm{Obs}_n^{\uparrow^k}(s) \in H^1(X, \mathcal{T}_n^{\uparrow^k}).$$

Such obstructions encode nontrivial arithmetic structure at hyper-congruence scales, beyond perfectoid or syntomic theories.

5.6. Comparison with Exponentoid Realization.

There exists a tower of functors:

$$\mathscr{R}_{\mathrm{exp}} \longrightarrow \mathscr{R}_{\uparrow^2} \longrightarrow \mathscr{R}_{\uparrow^3} \longrightarrow \cdots,$$

reflecting increased cohomological depth and recursive rigidity.

Each level corresponds to a new meta-filtration regime, giving rise to new regulators, motivic heights, and period hierarchies.

5.7. **Future Extensions.** Knuthoid geometry is the gateway to:
- **Trans-recursive stratification**;
- **Infinite-regress cohomology**;
- **Meta-periodic sheaves**;
- **Recursive motivic realization towers**.

This sets the stage for the foundational and ontological reconstruction of space in the upcoming sections.

Stratification is no longer geometric—it is ontological recursion.

6. Period Morphisms and Tilting across Growth Hierarchies

6.1. **Tilting between Filtration Towers.** Just as Scholze's perfectoid tilting functor transfers between characteristic 0 and p, we propose a sequence of **growth-level tilting functors** between different filtration regimes:

$$\mathrm{Tilt}_{\mathrm{mult}\to\exp}, \quad \mathrm{Tilt}_{\exp\to\uparrow^k}, \quad \cdots$$

Each functor reindexes sheaf-theoretic structures under a faster growth law, reflecting deeper recursive stratification.

6.2. **Period Ring Morphisms.** Let

$$\phi_n^{\exp} : B^{(n)}_{\exp,dR} \to B^{(n)}_{\uparrow^k,dR}$$

be a morphism of period rings, induced by the divisibility

$$\exp(n) \mid a \uparrow^k n.$$

More generally, for any two growth functions $f(n) \prec g(n)$ in Filt_∞, define:

Definition 6.1 (Filtration Morphism)**.** *A morphism of filtered sheaves is a natural map:*

$$F^{f(n)}\mathcal{F} \longrightarrow F^{g(n)}\mathcal{F}$$

satisfying:

- *Compatibility with torsor realizations;*
- *Functoriality with respect to period ring actions;*
- *Preservation of cohomological vanishing thresholds.*

6.3. **Cohomology Transition Functors.** These maps induce:

$$H^i_{f(n)}(X,\mathcal{F}) \longrightarrow H^i_{g(n)}(X,\mathcal{F}),$$

which can be interpreted as a functorial lift along an *inclusion of growth types.*

Let:

$$\mathscr{T}_{f\to g} := \text{Tilt functor on sheaves from } f(n)\text{-filtered to } g(n)\text{-filtered.}$$

Then we have a system of tilting towers:

$$\cdots \to \mathscr{T}_{\exp\to\uparrow^2} \to \mathscr{T}_{\uparrow^2\to\uparrow^3} \to \cdots$$

6.4. **Growth Category and Functorial Ordering.** Define the category Growth with objects: functions $f(n)$ satisfying $f(n) \to \infty$, and morphisms given by:

$$f \to g \iff \exists N \text{ s.t. } f(n) \le g(n)\, \forall n \ge N.$$

Then all filtration, period, torsor, and cohomology systems form a covariant diagram:

$$\mathsf{Growth} \longrightarrow \mathbf{StratifiedSpaces}, \quad f \mapsto \left(\mathcal{F}, F^{f(n)}\mathcal{F}, H^i_{f(n)}\right).$$

6.5. **Examples.**

- $\mathscr{T}_{2^n\to\exp(n)}$: interpolates dyadic congruence descent into recursive collapse.
- $\mathscr{T}_{\exp(n)\to a\uparrow^2 n}$: replaces exponential torsors with nested exponentials.
- $\mathscr{T}_{\uparrow^k\to\uparrow^{k+1}}$: increases stratification rank across trans-recursive towers.

6.6. **Conjecture: Stabilization via Meta-Tilting.**

Conjecture 6.2 (Tilting Stabilization)**.** *There exists a limit filtration*

$$F^\infty\mathcal{F} := \bigcap_{k=1}^{\infty} F^{a\uparrow^k n}\mathcal{F}$$

such that:

$$H^i(X, F^\infty\mathcal{F}) = \text{Meta-Cohomology}(X),$$

representing the space of sections persistent under all growth layers.

6.7. **Conclusion.** Tilting across growth levels allows:

- Lifting cohomology into higher recursive regimes;
- Comparing different filtration theories categorically;
- Embedding multiplicoid and exponentoid geometry into a unified trans-growth stack theory.

7. Ontological Stacks and Stratified Existence

7.1. **From Growth Laws to Existence Conditions.** In classical geometry, a space is a topological or algebraic object built over a field or ring. In growth-indexed geometry, a space emerges through stratified filtrations indexed by recursion depth.

> A mathematical object exists if and only if it survives through infinite levels of filtration.

This principle motivates an ontology of sheaves defined not by global sections or stalks, but by persistence under growth.

7.2. **Ontology-Valued Functors.** Let $f(n)$ be a growth function (e.g. 2^n, $\exp(n)$, $a \uparrow^k n$). Define:

Definition 7.1 (Growth Ontology Stack)**.** *Let $\mathcal{F}$ be a filtered sheaf. Define:*

$$\mathcal{O}nt_f : \mathbb{N}^{\mathrm{op}} \to \mathbf{Cat}, \quad n \mapsto \mathrm{Sh}(F^{f(n)}\mathcal{F}).$$

This stack captures the categorical existence of $\mathcal{F}$ at each level of congruence or recursive filtration.

The total "space of existence" is then given by:

$$\mathcal{F}_{\mathrm{ont}} := \varprojlim_n \mathcal{O}nt_f(n),$$

which generalizes the concept of a sheaf to a growth-layered meta-object.

7.3. **Persistence and Existence Depth.**

Definition 7.2 (Existence Depth)**.** *The* existence depth *of a section $s \in \mathcal{F}$ is the largest n such that*

$$s \in F^{f(n)}\mathcal{F}.$$

If such n does not exist (i.e., s survives all levels), s is infinitely persistent*.*

Infinitely persistent sections form the core ontology of arithmetic reality in growth-based spaces.

7.4. **Recursive Towers of Logic.** Stratified filtrations can be interpreted logically:
- $F^n\mathcal{F}$: provable at strength n;
- $F^{\exp(n)}\mathcal{F}$: computable via n-bounded recursion;
- $F^{a\uparrow^k n}\mathcal{F}$: definable only with k-level meta-recursion.

Definition 7.3 (Ontological Sheaf Tower)**.** *An* ontological sheaf tower *is a sequence:*

$$\left\{\mathcal{F}^{[f]}\right\}_{f \in \mathsf{Growth}},$$

with transition functors

$$\mathcal{T}_{f \to g} : \mathcal{F}^{[f]} \to \mathcal{F}^{[g]}$$

preserving realization and existence layers.

7.5. **Existence as Indexed Stability.** We redefine "being" as follows:

Definition 7.4 (Stratified Existence)**.** *Let X be a stratified arithmetic space. Then:*

$$\mathrm{Exist}(X) := \left\{ s \in \mathcal{F}_{ont} \;\middle|\; \forall f \in \mathsf{Growth},\ s \in F^{f(n)}\mathcal{F} \textit{ for all large } n \right\}.$$

This is the *existential core* of X—the set of all sections whose recursive identity persists through all growth laws.

7.6. **Sheaf Theory beyond Topology.** We propose:
- Geometry arises from stratification;
- Sheaves are persistence-indexed logic bundles;
- Filtration towers are existence sieves;
- Cohomology detects ontological stabilization.

These ideas set the groundwork for Volume IV: *Ontoid Geometry and Space-Theoretic Ontologies.*

7.7. **Conclusion.** Ontological stacks generalize the notion of space. They are:

- Indexed by recursive growth functions;
- Constructed via filtration-category towers;
- Populated by stratified sections with recursive depth;
- Governed by meta-logical transition functors.

In this setting, space is not a stage—it is a structured consequence of persistence across infinite recursion.

8. Meta-Conjectures and Infinite Cohomological Generation

8.1. **From Stratified Growth to Infinite Arithmetic.** Having developed exponentoid and knuthoid filtrations, period rings, cohomology theories, torsors, and ontological stacks, we now conclude by proposing a series of conjectures that unify these structures under a transfinite, meta-mathematical perspective.

8.2. **Cohomological Persistence Principle.**

Conjecture 8.1 (Persistent Realization Conjecture)**.** *Let $\mathcal{F}$ be a sheaf on X. Then:*

$$\bigcap_{f \in \mathsf{Growth}} F^{f(n)}\mathcal{F} \neq 0 \quad \Longleftrightarrow \quad \mathcal{F} \text{ is ontologically generative.}$$

That is, the most meaningful geometric objects are those whose sections survive *all* recursion layers.

8.3. **Meta-Period Conjecture.**

Conjecture 8.2 (Meta-Period Ring Existence)**.** *There exists a universal ring*

$$B_{\infty,dR} := \varprojlim_{f(n)} A/f(n)$$

indexed over all recursive growth types $f(n)$, such that:

$$H^i_\infty(X, \mathcal{F}) := \varprojlim_{f(n)} H^i(X, \mathcal{F}/f(n)\mathcal{F})$$

defines a meta-cohomology theory compatible with all previous towers.

This ring generalizes $B_{\exp,dR}$ and $B_{\uparrow^k,dR}$ into a total limit ring of recursive congruence.

8.4. **Growth-Invariant Regulator Systems.**

Conjecture 8.3 (Trans-Recursive Regulator Stability)**.** *There exists a universal regulator:*

$$r_\infty : K_n(X) \longrightarrow H^n_\infty(X, \mathbb{Q}(n)),$$

which commutes with all tilting functors:

$$\mathscr{T}_{f\to g} \circ r_f = r_g.$$

Thus, the notion of "regulator" becomes stable across the entire recursion hierarchy.

8.5. **Transfinite Motivic Tower.** Let $M^{[f]}(X)$ denote a motive in filtration type $f(n)$.

Conjecture 8.4 (Motivic Universality)**.** *There exists a limit object:*

$$M^{[\infty]}(X) := \varprojlim_{f(n)} M^{[f]}(X)$$

such that every stratified cohomology and torsor realization arises from it.

8.6. **Recursive Monodromy Conjecture.** Let $\mathbb{T}^{[f]}$ be the torsor tower under growth function $f(n)$.

Conjecture 8.5 (Hyper-Monodromy Realization)**.** *There exists a global monodromy group:*

$$\mathcal{M}_\infty := \varprojlim_{f(n)} \mathrm{Aut}(\mathbb{T}^{[f]}),$$

whose representations classify all recursive torsor structures across filtrations.

This group generalizes the role of the classical weight-monodromy group to the setting of stratified growth geometries.

8.7. **Ontology of Infinite Generation.**

Conjecture 8.6 (Ontological Closure of Geometry)**.** *The category of geometric spaces generated by recursive filtration towers and ontological stacks is closed under:*

- *Trans-recursive sheafification;*
- *Growth-stratified period extensions;*
- *Categorical limits over* Growth*;*
- *Meta-logical autoequivalences of cohomological functors.*

This defines a stable foundation for infinite-generation geometry: a space that is not built, but grows recursively from its own logic.

8.8. **Final Statement.** This volume has established a generalization of perfectoid geometry to spaces governed by exponentoid and knuthoid growth. In doing so, we propose that:

Geometry is no longer static—it is a meta-structure of persistent arithmetic stratification.

This closes Volume I and prepares the ascent into Volume II: *Hyper-Filtration Theory and Transfinite Monodromy.*