On the isogeny invariance of the Bloch-Kato's Tamagawa Numbers conjecture:

a K-theoretic point of view

by

Justin Scarfy

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Abstract

A major task in mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both. The arithmetic properties of various interesting objects (e.g. number fields, varieties, or even a motive, i.e. discrete objects) are all encoded in their respective L-functions, i.e. continuous objects, of which the Riemann zeta function is the simplest example. The understanding of these L-functions undergoes three phases [Kat1993a]:

1. analytic properties and the rationality of values,
2. algebraic properties illuminated by the p-adic properties of values,
3. arithmetic-geometric point of view of interpreting the values.

To date much of these have been achieved for the Riemann zeta function [CRSS2015] (except the locations of the zeros, or the zero free region). The deep conjectures of Deligne [De1979] and Beilinson [Be1985] were among the first to place these problems in a broad framework. Ambiguity in the work of Beilinson for interpreting these values, as they are interpreted only up to nonzero rational number multiples. In their article published in the Grothendieck Festschrift [BK1990], Spencer Bloch and Kazuya Kato removed this ambiguity in their formulation of what is now called the Tamagawa number conjecture (or the Bloch-Kato conjecture for motives). They also proved that the conjecture is invariant under isogeny, and the book [CRSS2015] exploits this isogeny invariance condition in the optic of K-theory for the Riemann zeta function.

This thesis investigates the isogeny invariance for specific motives and elaborates on accompanying results in cohomology and K-theory.

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To my parents

parents.jpg

Introduction

0.1 The leading role played by arithmetic within mathematics and recent breakthroughs

Arithmetic enjoys a privileged position within mathematics as a fertile source of fundamental questions. Among the seven Millennium problems listed by the Clay Institute [Clay], not fewer than three: the Birch and Swinnerton-Dyer conjecture, the Hodge conjecture, and the Riemann hypothesis, were handed down by the Queen of Mathematics. Even by the standards of a subject which has remained vibrant since the days of Fermat and Gauß, the last two decades have witnessed a real golden age, with landmarks too numerous to list completely: such as the striking progress on the Birch and Swinnerton-Dyer conjecture arising from the work of Gross-Zagier [GZ1986], Kolyvagin [Kol1989], and Kato [Kat2004]; the proofs of the Shimura-Taniyama-Weil conjecture [BCDT2001], Serre's conjectures [KW2009], the Fontaine-Mazur conjecture for two-dimensional Galois representations [Kis2009], and the Sato-Tate conjectures [CHR2008] which grew out of Wiles' epoch-making proof of Fermat's Last Theorem [Wil1995], [TW1995]; the revolutionary ideas of Bourgain [Bo2008] and Gowers [Go2007] blending techniques in harmonic analysis and additive combinatorics, the Fields-medal winning breakthrough of Green and Tao on primes in arithmetic progressions [GT2008], and the work of Goldston, Pintz, and Yıldırım [GPY2009], [GPY2010], and its spectacular recent strengthenings by Zhang [Zha2014], and Maynard [May2015] and Tao [Poly2014], on bounded gaps between primes. Recent innovations in arithmetic geometry by the innovation of Perfectoid spaces [Scho2012], and subsequent topological realization of the absolute Galois group [Scho2016] by Peter Scholze also shed new lights on the Langlands Programme, a web of conjectures that connects number theory, harmonic analysis, and geometry.

0.2 On the Riemann Hypothesis

Little essential progress has been made to the Riemann Hypothesis in the past two decades for the Riemann zeta function (or the Generalized Riemann Hypothesis for automorphic L-functions), which predicts that all the zeros of such L-functions are critical: They lie on the real line $\mathrm{Re}\, s = \frac{1}{2}$ with multiplicity one. The sharpest result for the Riemann zeta function is due to Feng [Fe2012], proving that at least 41.28% of the zeros of the Riemann zeta function,

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \qquad (s := \sigma + it),$$

lie on the critical line, by introducing a new mollifier and applying the original method of Levinson [Lev1974] and its subsequent strengthening by Conrey [Con1989], which gave at least 34.20% and 40.88% of the zeros of $\zeta(s)$ are critical, respectively. The best zero-free region to date for the Riemann zeta function is obtained by Ford [For2002] with

$$\sigma \geq 1 - \frac{1}{57.54(\log|t|)^{2/3}(\log\log|t|)^{1/3}}, \qquad |t| \geq 3.$$

A better zero-free region would imply a stronger result in the error term of the prime number theorem,

Other industries including computing the higher moments of these automorphic L-functions,

the attempt of using random matrices to explain the spacing between the zeros

Montgomery pair-correlation conjecture

the Siegel-Walfisz theorem and the large sieve developed by Bombieri are often used in place of the Generalized Riemann Hypothesis for Dirichlet L-functions to prove theorems and do estimates.

0.3 On the Hodge conjecture

Algebraic cycles

Chow groups

Kähler manifold

Known cases:

Generalizations:

Main difficulties/ best result in hoped cases.

0.4 On the Birch and Swinnerton-Dyer conjecture

Recall the Dirichlet class number formula for a number field K [Neu1999],

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w|d_K|^{1/2}} hR,$$

where the left hand side is the Dedekind zeta function,

$$\zeta_K(s) := \sum_{\mathfrak{a}\subset\mathcal{O}_K} \frac{1}{\mathfrak{N}(\mathfrak{a})},$$

with $\mathfrak{a}$ ideals in $\mathcal{O}_K$, the ring of integers of K, and $\mathfrak{N}$ its norm. On the right hand side, r_1 and r_2 the numbers of real and complex embeddings of K, respectively. h denotes the class number of $\mathcal{O}_K$, which measures the failure of $\mathcal{O}_K$ being a unique factorization domain, R the regulator, the determinant w the root number, and d_K the discriminate of the number field K

The Birch and Swinnerton-Dyer conjecture (BSD), formulated by B. Birch and Swinnerton-Dyer (1965) when studying the asymptotics of

$$\prod_{p\le x} \frac{\#E(\mathcal{F}_p)}{p},$$

has seen tremendous progress since the first breakthrough of Coates and Wiles.

The BSD conjecture

(a) (Weak BSD)

$$\mathrm{ord}_{s=1}\, L(E/K,s) = r_K,$$

(b) (Strong BSD)

$$\lim_{s\to 1}\frac{L(E/K,s)}{(s-1)^{r_p}} = \Omega_{E/K}\times \mathrm{Reg}_{\infty,K}(E)\times \frac{|\text{Ш}_K(E)|\prod_{p\le\infty}[E(K_p):E_0(K_p)]}{\sqrt{\Delta_K}\times|E(K)|^2_{\mathrm{tors}}},$$

can be seen as a generalization of Dirichlet class number formula, in the sense Ш(E) measures the failure of

Note Iwasawa main conjecture, proved in cases

1. $\mathbb{Q}$ [MW1984]

2. totally real number fields [Wil1990]

3. imaginary quadratic fields [Rub1988] [Rub1991]

4. Dirichlet characters [HK2003]

5. CM elliptic curves at supersingular primes [PR2004]

6. for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions [BD2005].

7. non-commutative main conjecture for totally real p-adic Lie extension of a number field [Kak2013] [CSSV2013].

8. (automorphic) GL_2 [SU2014]

One promising view
algebraic K-theory,
Suslin and Voevodsky,
Bloch
Motivic cohomology

Euler systems [Rub2000]

Classical Euler systems:

1. Siegels' cyclotomic units gives Kubota-Leopoldt p-adic L function, but no BSD application [CS2006].

2. Elliptic units Coates and Wiles' homomorphism

3. Heegner points gives anticyclotomic p-adic L-function of

Kato's Euler systems:

1. Beilinson-Kato elements

2. Beilinson-Flach elements

3. Gross-Kudla-Schoen cycles

Note that the monograph [Del2008] is devoted to the study of the BSD conjecture over the universal deformation rings of an elliptic curve.

Known cases of BSD

1. Coates and Wiles (1977) proved that if E is a curve over a number field F with complex multiplication by an imaginary quadratic field K of class number 1, $F = K$ or $\mathbb{Q}$, and $L(E, 1)$ is not 0 then $E(F)$ is a finite group. This was extended to the case where F is any finite abelian extension of K by Arthaud (1978).

2. Gross and Zagier (1986) showed that if a modular elliptic curve has a $L'(E, 1) = 0$ then it has a rational point of infinite order; see Gross-Zagier theorem.

3. Kolyvagin (1989) showed that a modular elliptic curve E for which $L(E, 1)$ is not zero has rank 0, and a modular elliptic curve E for which $L(E, 1)$ has a first-order zero at $s = 1$ has rank 1.

4. Rubin (1991) showed that for elliptic curves defined over an imaginary quadratic field K with complex multiplication by K, if the L-series of the elliptic curve was not zero at $s = 1$, then the p-part of the Tate-Shafarevich group had the order predicted by the Birch and Swinnerton-Dyer conjecture, for all primes $p > 7$.

5. Breuil et al. (2001), extending work of Wiles (1995), proved that all elliptic curves defined over the rational numbers are modular, which extends results 2 and 3 to all elliptic curves over the rationals, and shows that the L-functions of all elliptic curves over $\mathbb{Q}$ are defined at $s = 1$.

6. Bhargava and Shankar (2015) proved that the average rank of the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$ is bounded above by 7/6. Combining this with the p-parity theorem of Nekovár (2009) and Dokchitser & Dokchitser (2010) and with the proof of the main conjecture of Iwasawa theory for GL(2) by Skinner & Urban (2014), they conclude that a positive proportion of elliptic curves over $\mathbb{Q}$ have analytic rank zero, and hence, by Kolyvagin (1989), satisfy the Birch and Swinnerton-Dyer conjecture.

Chapter 1

Bloch-Kato's Tamagawa Numbers conjecture and the isogeny invariance

1.1 From the Birch and Swinnerton-Dyer conjecture to the Bloch-Kato conjecture

The Bloch-Kato's Tamagawa Numbers conjecture [BK1990], can be seen as a generalization of the Birch and Swinnerton-Dyer conjecture for motives.

Need: Fontaine's topological period rings B_{dR} and B_{crys}, where the former is a complete valued field with residue field $\mathbb{C}_p$

For a motivic pair (V, D) with weights $\leq w$ and a finite set of places Ω of $\mathbb{Q}$ containing ∞, the L-function $L_\Omega(V, s)$ is defined to be as the Euler product

$$L_\Omega(V, s) := \prod_{p \notin \Omega} P_p(V, p^{-s})^{-1},$$

it is absolute convergent for $\mathrm{Re}(s) > \frac{w}{2} + 1$.

Fixing a $\mathbb{Z}$-lattice $M \subset V$

1.2 Motives, Fontaine's p-adic period rings, and other essential gadgets

To define Tamagawa measures one needs groups $A(\mathbb{Q}_P), p \leq \infty$, and $A(\mathbb{Q})$. Bloch and Kato define such groups for a motivic pair (V, D) as follows: Assuming the motivic pair (V, D) has weight ≤ -1

Fontaine's p-adic period rings [Fon1982], [FM1987],

Tamagawa number [Wei1982]

1.3 The isogeny invariance explained

1.4 K-theoretic background

Reference [HB-K1], [HB-K2], [Sri1993], [TT1990], [Wei2013], [Wal1987a] , [Wal1987b]

We will need to use higher K-theory, where K_0 was invented by Grothendieck in proving the Riemann-Roch Theorem,

Quillen higher K-theory

the + construction

the Q construction for schemes

Waldhausen K-theory

K-theory of Thomason & Trobaugh.

Chapter 2

The isogeny invariance for elliptic curves with complex multiplication

2.1 CM elliptic curves

Reference: [Hid2013], [Sil2009], [Sil1994], [Kob1993], [Kna1992], [Ca1991], [Mil2006], [CVG1999], [dSh1987]

2.2 The isogeny invariance for CM elliptic curves

2.3 The K-theoretic viewpoint

Chapter 3

The isogeny invariance for elliptic curves without complex multiplication

3.1 non-CM elliptic curves

3.2 The isogeny invariance for non-CM elliptic curves

3.3 The K-theoretic viewpoint

Chapter 4

The isogeny invariance for modular forms

4.1 Modular forms

Reference [Miy1989], [DS2005], [AF1995]

4.2 The isogeny invariance for modular forms

4.3 The K-theoretic viewpoint

Chapter 5

The isogeny invariance for abelian varieties with complex multiplication

5.1 CM abelian varieties

References: [Shi1998], [Mum1974], [BL2004]

5.2 The isogeny invariance for CM abelian varieties

5.3 The K-theoretic viewpoint

Chapter 6

Proposed isogeny invariance for tori and motif

All of the above geometric objects are in the form of a pure motive: which are smooth projective varieties.

To date many problems occur in mixed motives due to standard conjectures.

The best understanding to date is due to Voevodsky's motivic cohomology [MWV2006]

Conclusion

Categorification?

Derived version?

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