ENTROPYPRISMATIC TRACE STACKS AND PERIODIC QUANTUM COHOMOLOGY

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Abstract. We introduce entropy–prismatic trace stacks, combining prismatic cohomology, filtered Frobenius flows, and quantum entropy theory. This structure governs the moduli of periodic filtered Frobenius motives and defines a universal framework for entropy-regulated quantum cohomology. We construct trace filtrations arising from entropy potentials and define a categorified correspondence between entropy zeta spectra and prismatic period sheaves over arithmetic and quantum stacks. Our theory links thermodynamic flow equations, filtered p-adic Hodge theory, and derived quantum Langlands motives.

Contents

1. Introduction

The integration of entropy into arithmetic and cohomological frameworks has given rise to powerful new directions in motivic geometry,

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trace theory, and quantum number theory. Building upon the prismatic foundations developed by Bhatt–Scholze and the noncommutative trace structures introduced in prior work, we now incorporate entropy quantization into the geometry of Frobenius flows and filtered trace stacks.

We develop the notion of **entropy–prismatic trace stacks**, which classify filtered prismatic sheaves and cyclotomic complexes regulated by entropy potential functionals. These stacks serve as moduli of periodic cohomological dynamics and define derived categories of entropy–zeta sheaves.

Key constructions include:

- Entropy trace filtrations on Nygaard-complete prismatic cohomology;
- Categorified entropy stacks over p-adic or quantum base spectra;
- Frobenius-entropy flow fields and cohomological thermodynamics;
- Derived zeta–prismatic correspondences for entropy-periodic sheaf theories.

This framework synthesizes ideas from arithmetic dynamics, quantum trace theory, filtered derived geometry, and thermodynamic motivic flows.

2. Entropy-Filtered Prisms and Thermodynamic Frobenius Dynamics

To incorporate entropy into the prismatic and Frobenius framework, we introduce a new class of filtered prismatic objects: *entropy-filtered prisms*. These are equipped not only with a Frobenius lift and Nygaard filtration, but also with an entropy potential that regulates the flow of period sheaves and syntomic complexes. This allows us to define a thermodynamic trace formalism in prismatic cohomology.

2.1. Entropy-Filtered Prisms.

Let (A, I) be a classical prism, where A is a p-adically complete δ-ring and $I \subset A$ is a distinguished ideal. We enhance this structure by adding an entropy functional.

Definition 2.1 (Entropy-Filtered Prism)**.** *An* entropy-filtered prism *is a triple* $(A, I, \mathcal{S})$ *where:*

- (A, I) *is a bounded prism;*
- $\mathcal{N}^\bullet$ *is a Nygaard-type filtration on* $\mathsf{Prism}_{R/A}$*;*
- $\mathcal{S} : \mathcal{N}^\bullet \to \mathbb{R}_{\geq 0}$ *is an* entropy potential functional*, satisfying:*
  (1) Monotonicity: $\mathcal{S}(\mathcal{N}^i) \leq \mathcal{S}(\mathcal{N}^{i+1})$*;*
  (2) Additivity: $\mathcal{S}(\mathcal{N}^{i+j}) \leq \mathcal{S}(\mathcal{N}^i) + \mathcal{S}(\mathcal{N}^j)$*;*
  (3) Frobenius invariance: $\mathcal{S}(\varphi(\mathcal{N}^i)) = \mathcal{S}(\mathcal{N}^{pi})$.

Example 2.2. *Let $A = A_{\mathrm{inf}}$ and R be a perfectoid ring. Then $\mathsf{Prism}_{R/A}$ carries a natural Nygaard filtration, and we define:*

$$\mathcal{S}(\mathcal{N}^i) := \log_p \dim_{\mathbb{F}_p} H^0(\mathcal{N}^i/p).$$

This entropy counts the logarithmic growth of torsion dimensions in prismatic cohomology.

2.2. **Thermodynamic Frobenius Flow Fields.** We interpret the entropy functional as inducing a flow along the Frobenius filtration, generating a thermodynamic cohomology theory.

Definition 2.3 (Frobenius–Entropy Flow Field)**.** *Given a prism $(A, I, \mathcal{S})$ and a module M over $\mathsf{Prism}_{R/A}$, define the* Frobenius–entropy flow *by:*

$$\partial_\varphi^{\mathcal{S}}(M) := \lim_{n\to\infty} \frac{1}{p^n} \left(\varphi^n(M) - M\right),$$

measured with respect to the entropy-weighted norm induced by $\mathcal{S}$.

Remark 2.4. This operator measures the "thermodynamic velocity" of M under Frobenius iteration, incorporating the effect of entropy shearing in filtrations.

2.3. **Entropy-Stable Prismatic Complexes.** We define a class of complexes that are invariant under the entropy flow, providing fixed points of thermodynamic trace evolution.

Definition 2.5. *A prismatic complex $C^\bullet$ is* entropy-stable *if*

$$\partial_\varphi^{\mathcal{S}}(C^\bullet) = 0.$$

Such complexes model equilibrium states in cohomological thermodynamics.

Example 2.6. *The syntomic cohomology complex* $\mathrm{Syn}(R)$ *is entropy-stable when R is semistable and the filtration $\mathcal{N}^\bullet$ admits logarithmic Frobenius invariance.*

2.4. **Spectral Decomposition via Entropy Operators.** We interpret the Frobenius–entropy flow as a spectral operator on the derived category.

Theorem 2.7 (Entropy Spectral Decomposition)**.** *Let $C^\bullet$ be a perfect complex over $\mathsf{Prism}_{R/A}$ with entropy functional $\mathcal{S}$. Then:*

$$C^\bullet \simeq \bigoplus_{\lambda \in \mathrm{Spec}(\partial_\varphi^{\mathcal{S}})} C_\lambda^\bullet$$

where each summand corresponds to an eigenspace of the entropy–Frobenius flow.

Sketch. Use the solid derived category structure of $\mathrm{Cond}(\mathbb{Z}_p)$ to define the flow operator $\partial_\varphi^{\mathcal{S}}$ and apply the spectral decomposition theorem for compact self-adjoint operators in Banach-derived settings. $\square$

This spectral theory forms the basis for defining entropy-zeta correspondences and periodic cohomological dynamics, which we develop in the next section.

3. Entropy–Zeta Correspondence and Periodic Quantum Cohomology

Building upon the entropy-filtered prismatic formalism and Frobenius flow structures, we now define a correspondence between entropy-regulated spectral data and zeta-function-type invariants. This structure enables the construction of quantum-periodic cohomology theories, whose dynamics are governed by entropy operators and Frobenius-periodic trace flows.

3.1. **Entropy Zeta Functions from Prismatic Cohomology.**

We introduce a zeta function encoding entropy-weighted Frobenius eigenvalues across prismatic cohomological degrees.

Definition 3.1 (Entropy Zeta Function)**.** *Let $C^\bullet$ be a perfect prismatic complex over an entropy-filtered prism $(A, I, \mathcal{S})$. The associated* entropy zeta function *is:*

$$\zeta_\varphi^{\mathcal{S}}(C^\bullet, s) := \prod_i \det\left(1 - p^{-s}\varphi \middle| H^i(C^\bullet)\right)^{-(-1)^i \cdot \mathcal{S}(H^i(C^\bullet))}.$$

Remark 3.2. The weighting by $\mathcal{S}$ interprets the zeta function as an entropy-regulated trace determinant, capturing refined thermodynamic fluctuations in cohomological growth.

3.2. **Entropy–Zeta Trace Equivalence.**

The Frobenius trace over $\mathsf{Prism}_{R/A}$ can be expressed in terms of an entropy–zeta integral.

Theorem 3.3 (Entropy–Zeta Trace Identity)**.** *Let $C^\bullet$ be an entropy-stable complex. Then:*

$$\mathrm{Tr}_\varphi(C^\bullet) = -\frac{d}{ds} \log \zeta_\varphi^{\mathcal{S}}(C^\bullet, s)\bigg|_{s=0}.$$

Sketch. This follows from the relation between determinant traces and logarithmic derivatives, applied to the entropy-weighted spectrum of φ acting on filtered cohomology. $\square$

3.3. **Quantum Periodic Cohomology Theories.** Using entropy–zeta flows, we define cohomology theories with built-in periodicity and thermodynamic regularization.

Definition 3.4 (Quantum Periodic Prismatic Cohomology)**.** *Let X be a derived p-adic stack and $(A, I, \mathcal{S})$ an entropy-filtered prism. The* quantum periodic cohomology *of X is:*

$$\mathrm{QPC}^\bullet(X) := \bigoplus_{n\in\mathbb{Z}} H^\bullet_{\mathsf{prism}}(X) \otimes e^{2\pi i n \partial^{\mathcal{S}}_{\varphi}},$$

where the exponential defines a periodic flow by entropy–Frobenius eigenvalues.

Conjecture 3.5 (Prismatic–Quantum Correspondence)**.** *There exists an equivalence of filtered derived categories:*

$$\mathrm{QPC}^\bullet(X) \simeq \mathrm{D}^{\mathrm{per}}(\mathscr{Z}_{\mathcal{S}}),$$

where $\mathscr{Z}_{\mathcal{S}}$ is the category of entropy–zeta sheaves over X, and $\mathrm{D}^{\mathrm{per}}$ *denotes the derived category of periodic Frobenius sheaves.*

3.4. **Entropy Motives and Zeta-Flow Stacks.** We define entropy motive stacks whose cohomology realizes zeta flows:

Definition 3.6. *The* entropy–prismatic motive stack $\mathfrak{Mot}_{\mathcal{S}}$ *parametrizes objects $(C^\bullet, \varphi, \mathcal{S})$ with:*

- *Prismatic complex $C^\bullet$ with filtered Frobenius lift;*
- *Entropy potential $\mathcal{S}$ compatible with Nygaard-type growth;*
- *Periodic trace dynamics under φ-iteration.*

Example 3.7. *The stack $\mathfrak{Mot}_{\mathcal{S}}$ over* $\mathrm{Spf}(\mathbb{Z}_p)$ *classifies entropy-regulated zeta sheaves with period sheaf realization in $\mathbb{B}_{\mathrm{HT}}$ and syntomic deformation in $\mathbb{B}_{\mathrm{syn}}$.*

3.5. **Outlook: Zeta Crystals and Quantum Trace Fields.** This framework leads toward a broader program:

- Define zeta-crystalline sheaves via entropy-modulated prismatic periods;
- Quantize the space of zeta functions as spectral stacks with filtered trace flows;
- Classify periodic entropy motives under automorphic and Galois representations;
- Construct quantum Langlands zeta categories with Frobenius thermodynamics.

Conjecture 3.8 (Entropy–Langlands–Zeta Synthesis)**.** *There exists a fully faithful functor from periodic p-adic Langlands motives to entropy zeta crystals over $\mathfrak{Mot}_{\mathcal{S}}$, intertwining Frobenius dynamics, entropy flow, and automorphic spectral decomposition.*