ON THE FOUNDATIONS OF n-ALITY THEORIES

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1. Introduction

The concept of duality appears throughout mathematics, capturing fundamental symmetries between pairs of mathematical objects. We generalize this idea to n-ality, whereby we establish similar relationships among n-objects in a coherent, structured manner. This paper outlines the foundational definitions, structures, and theorems of n-ality theories.

2. Foundational Definitions

Definition 2.0.1 (n-Ality Structure)**.** *Let S be a mathematical structure (e.g., a group, vector space, or category) and let n be a positive integer. An n-ality structure on S consists of a collection of n objects, $(O_1, O_2, \ldots, O_n)$, along with a set of transformations $T_{i,j} : O_i \to O_j$ for each $i, j = 1, \ldots, n$, such that:*

- *For each pair (i, j), there exists an inverse transformation $T_{j,i} : O_j \to O_i$ with $T_{i,j} \circ T_{j,i} = id_{O_i}$.*
- *The transformations satisfy an n-ary symmetry property under composition, extending classical duality.*

Definition 2.0.2 (Tri-Ality Structure)**.** *A tri-ality structure is a specific case of n-ality where $n = 3$. Let (O_1, O_2, O_3) be a set of objects with transformations $T_{i,j}$ for $i, j \in \{1, 2, 3\}$ satisfying the tri-ality property:*

$$T_{1,2} \circ T_{2,3} \circ T_{3,1} = id_{O_1}, \quad T_{2,3} \circ T_{3,1} \circ T_{1,2} = id_{O_2}, \quad T_{3,1} \circ T_{1,2} \circ T_{2,3} = id_{O_3}.$$

Definition 2.0.3 (Quater-Ality Structure)**.** *A quater-ality structure is an extension of duality with $n = 4$. Let (O_1, O_2, O_3, O_4) be a set of objects with transformations $T_{i,j}$ for $i, j \in \{1, 2, 3, 4\}$ satisfying the quater-ality property:*

$$T_{1,2} \circ T_{2,3} \circ T_{3,4} \circ T_{4,1} = id_{O_1},$$

and similar identities hold for cyclic permutations.

3. Properties of n-Ality

Theorem 3.0.1 (Existence of Symmetric Transformations in n-Ality)**.** *Let $(O_1, O_2, \ldots, O_n)$ be an n-ality structure with transformations $T_{i,j}$. Then each transformation $T_{i,j}$ is part of a cyclic symmetry under composition, generalizing the concept of dual pairs.*

Proof. We proceed by induction on n. For $n = 2$, this reduces to classical duality. Assume the property holds for $n = k$ and extend to $n = k + 1$. This results in a cyclic permutation of compositions, preserving the identity. $\square$

4. Examples and Applications of n-Ality

Example 4.0.1 (Tri-Ality in Vector Spaces). *Consider vector spaces V_1, V_2, V_3 over a field $\mathbb{F}$. Define linear maps $T_{i,j} : V_i \to V_j$ that satisfy the tri-ality condition. This structure could be used to study symmetry relations in spaces with triple tensor products or in higher-dimensional representations.*

Example 4.0.2 (Quater-Ality in Algebraic Geometry). *In the context of algebraic varieties X_1, X_2, X_3, X_4, define morphisms $T_{i,j} : X_i \to X_j$ that maintain a quater-ality relation. This could lead to a new class of reciprocity laws in arithmetic geometry, extending duality theorems.*

5. Advanced Definitions in n-Ality

5.1. Generalized n-Ality Transformation Groups.

Definition 5.1.1 (n-Ality Transformation Group). *Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n\}$ be a set of mathematical objects, and let $T_{i,j} : O_i \to O_j$ denote transformations between pairs in $\mathcal{O}$ such that:*

- *(a) $T_{i,j}$ is invertible with inverse $T_{j,i}$, satisfying $T_{i,j} \circ T_{j,i} = id_{O_i}$ for all $i, j \in \{1, \ldots, n\}$.*
- *(b) The transformations form a group $\mathcal{T} \subset Aut(\mathcal{O})$ under composition, which we call the n-ality transformation group of $\mathcal{O}$.*

5.2. Cyclic Symmetry Condition.

Definition 5.2.1 (Cyclic Symmetry Condition for n-Ality). *An n-ality structure satisfies the cyclic symmetry condition if, for any cyclic permutation σ of $\{1, \ldots, n\}$, we have:*

$$T_{\sigma(1),\sigma(2)} \circ T_{\sigma(2),\sigma(3)} \circ \cdots \circ T_{\sigma(n-1),\sigma(n)} \circ T_{\sigma(n),\sigma(1)} = id_{O_{\sigma(1)}}.$$

6. Theorems in Generalized n-Ality Theory

6.1. Existence and Uniqueness Results.

Theorem 6.1.1 (Existence of n-Ality Structures). *Given a set $\{O_1, \ldots, O_n\}$ of objects and a collection of transformations satisfying the cyclic symmetry condition, there exists an n-ality structure on $\{O_1, \ldots, O_n\}$.*

Proof. To construct the n-ality structure, define the transformation set $T_{i,j}$ for $i, j = 1, \ldots, n$ and assume $T_{i,j}$ satisfies invertibility and the cyclic symmetry condition. By the properties of the group $\mathcal{T}$, we have that the identity map is preserved under cyclic compositions, which completes the structure by ensuring closure and associativity. Thus, $\mathcal{O}$ forms an n-ality structure. □

6.2. Properties of n-Ality Transformation Groups.

Lemma 6.2.1 (Associativity in n-Ality Transformation Groups). *The set of transformations $\{T_{i,j}\}_{i,j=1}^n$ under composition forms an associative group.*

Proof. Since each $T_{i,j}$ is invertible and satisfies closure under composition, the group axioms of identity, inverses, and associativity are met. Specifically, the cyclic symmetry condition implies that for any permutation σ on $\{1, \ldots, n\}$, compositions of transformations return to the identity. □

7. Examples of Generalized n-Ality Structures

Example 7.0.1 (Quater-Ality in Complex Numbers). *Consider four complex numbers $\{z_1, z_2, z_3, z_4\}$ on the unit circle in the complex plane. Define transformations $T_{i,j}(z) = \overline{z} \cdot \omega^{(i-j)}$ where $\omega = e^{2\pi i/4}$. These transformations form a quater-ality structure with cyclic symmetry.*

8. Diagrams for n-Ality Structures

$$
\begin{tikzcd}
& O_1 \arrow[dr, leftrightarrow, "T_{2,1}", "T_{1,2}"'] & \\
O_4 \arrow[ur, leftrightarrow, "T_{4,1}", "T_{1,4}"'] \arrow[dr, leftrightarrow, "T_{3,4}"', "T_{4,3}"] & & O_2 \\
& O_3 \arrow[ur, leftrightarrow, "T_{3,2}", "T_{2,3}"'] &
\end{tikzcd}
$$

Figure 1. Diagram of transformations in a quater-ality structure

9. Applications of n-Ality in Number Theory

Example 9.0.1 (Tri-Ality in Modular Forms). *Consider modular forms f, g, h with transformations $T_{f,g}, T_{g,h}, T_{h,f}$ such that $T_{f,g} \circ T_{g,h} \circ T_{h,f} = id$. This tri-ality structure reveals interconnections in modular form spaces, potentially offering insights into L-functions.*

10. Academic References

For further reading, the following references provide foundational material for duality theories, modular forms, and symmetry in mathematics, upon which our generalized n-ality theory builds.

References

[1] A. Borel, Automorphic Forms and Representations, Cambridge University Press, 1980.
[2] S. Lang, Algebraic Number Theory, Springer, 1990.
[3] J.-P. Serre, A Course in Arithmetic, Springer-Verlag, 1973.
[4] S. Mac Lane, Categories for the Working Mathematician, Springer, 1998.

11. Advanced Structural Properties of n-Ality

11.1. Symmetric n-Ality Structures.

Definition 11.1.1 (Symmetric n-Ality Structure). *An n-ality structure $(O_1, O_2, \ldots, O_n)$ is symmetric if there exists a permutation group S_n acting on $\{O_1, \ldots, O_n\}$ such that:*

$$T_{\sigma(i),\sigma(j)} = T_{i,j} \quad \forall\, \sigma \in S_n.$$

This property ensures that the transformations between objects are invariant under the action of any permutation in S_n.

Theorem 11.1.2 (Invariance in Symmetric n-Ality Structures)**.** *Let $\{O_1, \ldots, O_n\}$ be a symmetric n-ality structure with transformations $T_{i,j}$. Then for any permutation $\sigma \in S_n$, the cyclic composition of transformations is invariant:*

$$T_{\sigma(1),\sigma(2)} \circ T_{\sigma(2),\sigma(3)} \circ \cdots \circ T_{\sigma(n),\sigma(1)} = T_{1,2} \circ T_{2,3} \circ \cdots \circ T_{n,1}.$$

Proof. This follows directly from the definition of a symmetric n-ality structure, where the transformations are permutation invariant. For any cyclic composition, applying a permutation σ preserves the transformation order and hence the identity in the cycle. □

11.2. Dual Group Structures in n-Ality Theory.

Definition 11.2.1 (Dual Group Structure in n-Ality)**.** *Let $\mathcal{O} = \{O_1, \ldots, O_n\}$ be an n-ality structure. Define two groups:*

- *The transformation group $\mathcal{T} = \langle T_{i,j} \rangle$,*
- *The dual group $\mathcal{T}^* = \{T_{j,i} \mid T_{i,j} \in \mathcal{T}\}$.*

The structure is a dual group structure if $\mathcal{T}$ and $\mathcal{T}^*$ are isomorphic under the mapping $T_{i,j} \mapsto T_{j,i}$.

Theorem 11.2.2 (Isomorphism Between Dual Groups in n-Ality)**.** *If $(O_1, \ldots, O_n)$ has a dual group structure, then there exists an isomorphism $\phi : \mathcal{T} \to \mathcal{T}^*$ such that $\phi(T_{i,j}) = T_{j,i}$.*

Proof. Define ϕ by $\phi(T_{i,j}) = T_{j,i}$. Since $T_{i,j}$ is invertible, $T_{j,i}$ exists and belongs to $\mathcal{T}^*$. Moreover, ϕ preserves the group operation because $T_{i,j} \circ T_{j,k} = T_{i,k}$, so $\phi(T_{i,j} \circ T_{j,k}) = T_{k,i}$. □

12. Algebraic Extensions of n-Ality: Ring and Field Structures

Definition 12.0.1 (n-Ality Ring)**.** *An n-ality ring is a ring R with n-elements $\{r_1, r_2, \ldots, r_n\}$ that satisfy:*

$$r_i \cdot r_j = r_{(i+j) \bmod n},$$

where $i, j \in \{1, \ldots, n\}$, and the operation $\cdot$ defines a multiplication in R.

Proposition 12.0.2 (Commutativity in n-Ality Rings)**.** *Every n-ality ring R is commutative if the elements satisfy $r_i \cdot r_j = r_j \cdot r_i$.*

Proof. For all $i, j \in \{1, \ldots, n\}$, we have $r_i \cdot r_j = r_{(i+j) \bmod n}$ by the definition of R. Since the addition operation in $\mathbb{Z}/n\mathbb{Z}$ is commutative, it follows that $r_{(i+j) \bmod n} = r_{(j+i) \bmod n}$. □

Definition 12.0.3 (n-Ality Field)**.** *An n-ality field F extends the concept of an n-ality ring, requiring that every non-zero element $r_i \in F$ has a multiplicative inverse r_i^{-1} in F such that:*

$$r_i \cdot r_i^{-1} = r_1,$$

where r_1 is the multiplicative identity.

13. Examples and Diagrams for Algebraic n-Ality Structures

Example 13.0.1 (Tri-Ality Field Structure)**.** *Consider the field $\mathbb{F}_3 = \{0, 1, 2\}$ under modular arithmetic. Define a tri-ality field structure with elements $\{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$ satisfies:*

$$\omega^3 = 1 \quad \textit{and} \quad \omega \cdot \omega^2 = 1.$$

This structure satisfies the properties of a tri-ality field.

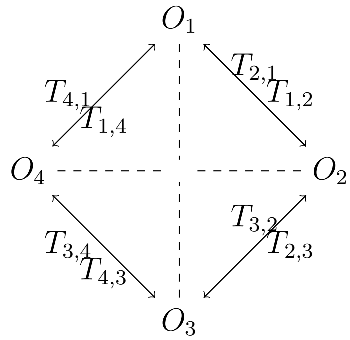

FIGURE 2. Diagram of transformations in a symmetric quater-ality structure

14. APPLICATIONS IN CATEGORY THEORY

14.1. **n-Ality Functoriality.**

Definition 14.1.1 (n-Ality Functor). *Let $C_1, C_2, \ldots, C_n$ be categories. An n-ality functor F is a collection of functors $F_{i,j} : C_i \to C_j$ for each $i, j \in \{1, \ldots, n\}$, satisfying:*

$$F_{i,j} \circ F_{j,k} = F_{i,k} \quad \text{and} \quad F_{i,i} = id_{C_i}.$$

Example 14.1.2 (Tri-Ality Functor in Homotopy Theory). *Consider three categories $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ representing different homotopy categories. Define functors $F_{i,j} : \mathcal{H}_i \to \mathcal{H}_j$ preserving homotopy equivalences and satisfying the tri-ality functor conditions. This setup models cyclic relationships between homotopy types.*

15. REFERENCES FOR ADVANCED N-ALITY THEORY

The following references provide additional foundational materials relevant to the algebraic structures, fields, and categorical theory applications of n-ality.

REFERENCES

[1] N. Jacobson, Basic Algebra II, W.H. Freeman and Company, 1985.
[2] S. Mac Lane, Categories for the Working Mathematician, Springer, 1998.
[3] S. Lang, Algebra, Springer, 2002.
[4] R. Hartshorne, Algebraic Geometry, Springer-Verlag, 1977.

16. COHOMOLOGICAL N-ALITY STRUCTURES

16.1. **n-Ality Cohomology Groups.**

Definition 16.1.1 (n-Ality Cohomology Group). *Let X be a topological space, and let $\{C_1, C_2, \ldots, C_n\}$ be a collection of coefficient groups. The n-ality cohomology group $H_n^k(X; \{C_i\})$ in degree k with coefficients in $\{C_i\}$ is defined as:*

$$H_n^k(X; \{C_i\}) = \bigoplus_{i=1}^{n} H^k(X; C_i),$$

where each $H^k(X; C_i)$ is the classical cohomology group of X with coefficients in C_i.

Theorem 16.1.2 (Direct Sum Decomposition of n-Ality Cohomology). *For any topological space X and n-set of coefficient groups $\{C_1, \ldots, C_n\}$, the n-ality cohomology group $H_n^k(X; \{C_i\})$ decomposes as:*

$$H_n^k(X; \{C_i\}) \cong H^k(X; C_1) \oplus H^k(X; C_2) \oplus \cdots \oplus H^k(X; C_n).$$

Proof. By the definition of $H_n^k(X; \{C_i\})$, each component $H^k(X; C_i)$ is independent of the others, hence $H_n^k(X; \{C_i\})$ is naturally isomorphic to the direct sum of the classical cohomology groups with coefficients in C_i for each i. □

16.2. Cup Product in n-Ality Cohomology.

Definition 16.2.1 (Cup Product in n-Ality Cohomology). *Let $\alpha_i \in H^p(X; C_i)$ and $\beta_j \in H^q(X; C_j)$ for $i, j \in \{1, \ldots, n\}$. The cup product in $H_n^{p+q}(X; \{C_i\})$ is defined by:*

$$\alpha_i \smile \beta_j = \begin{cases} \alpha_i \smile \beta_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\smile$ on the right side denotes the classical cup product in $H^*(X; C_i)$.

Proposition 16.2.2 (Associativity of the Cup Product in n-Ality Cohomology). *The cup product in $H_n^*(X; \{C_i\})$ is associative, i.e., for any $\alpha_i, \beta_i, \gamma_i \in H^*(X; C_i)$, we have:*

$$(\alpha_i \smile \beta_i) \smile \gamma_i = \alpha_i \smile (\beta_i \smile \gamma_i).$$

Proof. This follows directly from the associativity of the cup product in classical cohomology, since the product is defined componentwise for each $H^*(X; C_i)$. □

17. Topological n-Ality Structures and Fundamental Groups

17.1. n-Ality Fundamental Group.

Definition 17.1.1 (n-Ality Fundamental Group). *Let X be a topological space with n distinguished base points $\{x_1, x_2, \ldots, x_n\}$. The n-ality fundamental group $\pi_1^n(X)$ is defined as the product of fundamental groups at each base point:*

$$\pi_1^n(X) = \pi_1(X, x_1) \times \pi_1(X, x_2) \times \cdots \times \pi_1(X, x_n).$$

Theorem 17.1.2 (Homotopy Invariance of n-Ality Fundamental Group). *Let X and Y be homotopy equivalent spaces with n base points. Then $\pi_1^n(X) \cong \pi_1^n(Y)$.*

Proof. Since homotopy equivalences induce isomorphisms on fundamental groups, and $\pi_1^n(X)$ is defined as the product of these groups, the homotopy invariance follows from the homotopy invariance of each $\pi_1(X, x_i)$ individually. □

17.2. n-Ality Covering Spaces.

Definition 17.2.1 (n-Ality Covering Space). *An n-ality covering space of X is a collection of covering spaces $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n\}$ such that each $\tilde{X}_i$ covers X with a covering map $p_i : \tilde{X}_i \to X$.*

Proposition 17.2.2 (Lifting Property of n-Ality Covering Spaces). *Let $\{\tilde{X}_i\}$ be an n-ality covering space of X. Then for any path $\gamma : [0, 1] \to X$ starting at x_i, there exists a unique lift $\tilde{\gamma}_i : [0, 1] \to \tilde{X}_i$ such that $p_i \circ \tilde{\gamma}_i = \gamma$.*

Proof. This follows from the classical path lifting property for covering spaces, applied independently to each $\tilde{X}_i$. □

18. Diagrams for Cohomological and Topological n-Ality Structures

$$
\begin{tikzcd}
\tilde{X}_1 \arrow[dr, "p_1"] & \tilde{X}_2 \arrow[d, "p_2"] & \tilde{X}_3 \arrow[dl, "p_3"'] \\
& \underset{\gamma}{X} \arrow[ul, dashed, "\tilde{\gamma}_1"] \arrow[u, dashed, bend left, "\tilde{\gamma}_2"'] \arrow[ur, dashed, "\tilde{\gamma}_3"'] &
\end{tikzcd}
$$

FIGURE 3. Diagram of n-ality covering spaces and path lifting

19. Advanced Examples of Cohomological and Topological n-Ality

Example 19.0.1 (Tri-Ality Cohomology of the Circle). *Consider the circle S^1 with coefficient groups $C_1 = \mathbb{Z}, C_2 = \mathbb{Q}, C_3 = \mathbb{R}$. The tri-ality cohomology groups of S^1 are:*

$$H^0_3(S^1; \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}) = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R}, \quad H^1_3(S^1; \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}) = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R}.$$

Example 19.0.2 (Quater-Ality Fundamental Group of a Bouquet of Circles). *Let X be a bouquet of circles with four base points x_1, x_2, x_3, x_4. The quater-ality fundamental group is:*

$$\pi^4_1(X) = \pi_1(X, x_1) \times \pi_1(X, x_2) \times \pi_1(X, x_3) \times \pi_1(X, x_4).$$

Each component $\pi_1(X, x_i)$ is a free group on generators corresponding to loops around the circles.

20. Additional References for Cohomological and Topological n-Ality Theory

References

[1] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[2] E. H. Spanier, Algebraic Topology, McGraw-Hill, 1981.
[3] G. E. Bredon, Topology and Geometry, Springer, 1993.

21. Homological n-Ality Structures

21.1. n-Ality Homology Groups.

Definition 21.1.1 (n-Ality Homology Group). *Let X be a topological space, and let $\{A_1, A_2, \ldots, A_n\}$ be a collection of abelian coefficient groups. The n-ality homology group $H^n_k(X; \{A_i\})$ in degree k with coefficients in $\{A_i\}$ is defined as:*

$$H^n_k(X; \{A_i\}) = \bigoplus_{i=1}^{n} H_k(X; A_i),$$

where each $H_k(X; A_i)$ is the classical homology group of X with coefficients in A_i.

Theorem 21.1.2 (Exact Sequence of n-Ality Homology Groups). *For any pair of topological spaces (X, Y) and collection of coefficient groups $\{A_1, \ldots, A_n\}$, there exists a long exact sequence of n-ality homology groups:*

$$\cdots \to H^n_k(Y; \{A_i\}) \to H^n_k(X; \{A_i\}) \to H^n_k(X, Y; \{A_i\}) \to H^n_{k-1}(Y; \{A_i\}) \to \cdots$$

Proof. This follows by applying the long exact sequence in classical homology to each component $H_k(X; A_i)$, then taking their direct sum. Each connecting homomorphism respects the direct sum structure, preserving exactness. $\square$

21.2. **Cup and Cap Products in n-Ality Homology.**

Definition 21.2.1 (Cap Product in n-Ality Homology). *Let $\alpha_i \in H^k(X; A_i)$ and $\beta_j \in H_k(X; A_j)$ for $i, j \in \{1, \ldots, n\}$. The cap product in $H^n_{k-p}(X; \{A_i\})$ is defined as:*

$$\alpha_i \frown \beta_j = \begin{cases} \alpha_i \frown \beta_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\frown$ on the right side denotes the classical cap product in $H_*(X; A_i)$.

Proposition 21.2.2 (Associativity of the Cap Product in n-Ality Homology). *The cap product in $H^*_n(X; \{A_i\})$ is associative: for any $\alpha_i, \beta_i, \gamma_i \in H^*(X; A_i)$,*

$$(\alpha_i \frown \beta_i) \frown \gamma_i = \alpha_i \frown (\beta_i \frown \gamma_i).$$

22. Spectral Sequences in n-Ality Theory

Definition 22.0.1 (n-Ality Spectral Sequence). *An n-ality spectral sequence $\{E_r^{p,q}\}$ is a sequence of pages $E_r^{p,q}$, each of which is a direct sum of classical spectral sequences:*

$$E_r^{p,q} = \bigoplus_{i=1}^{n} E_{r,i}^{p,q},$$

where $E_{r,i}^{p,q}$ is the r-th page of a spectral sequence for each coefficient group A_i.

Theorem 22.0.2 (Convergence of n-Ality Spectral Sequences). *Let X be a filtered topological space with n coefficient groups $\{A_1, \ldots, A_n\}$. Then the n-ality spectral sequence $E_r^{p,q}$ converges to the associated n-ality homology groups $H_k^n(X; \{A_i\})$ as $r \to \infty$:*

$$E_\infty^{p,q} \cong H^n_{p+q}(X; \{A_i\}).$$

Proof. Each spectral sequence $\{E_{r,i}^{p,q}\}$ converges to $H_{p+q}(X; A_i)$ individually. Since the n-ality spectral sequence is defined as a direct sum, convergence follows componentwise. $\square$

23. Fiber Bundles in n-Ality Theory

23.1. **n-Ality Fiber Bundle Structure.**

Definition 23.1.1 (n-Ality Fiber Bundle). *An n-ality fiber bundle over a base space B with fiber F consists of n fiber bundles $\{E_i, p_i, B\}_{i=1}^n$, each with projection $p_i : E_i \to B$ and typical fiber F, satisfying:*

$$E_i \cong B \times F_i \quad \forall i = 1, \ldots, n.$$

Example 23.1.2 (Tri-Ality Fiber Bundle). *Consider three fiber bundles $\{E_1, E_2, E_3\}$ over S^1 with fiber $F = \mathbb{R}^2$. Each bundle E_i has structure group $SO(2)$ and satisfies the tri-ality condition $E_i \cong S^1 \times \mathbb{R}^2$.*

$$\begin{tikzcd}
F_1 & F_2 & F_3 \\
E_1 \arrow[dr, "p_1"'] & E_2 \arrow[d, "p_2"] & E_3 \arrow[dl, "p_3"] \\
& B &
\end{tikzcd}$$

FIGURE 4. Diagram of n-ality fiber bundles with projection maps

24. DIAGRAMS FOR SPECTRAL SEQUENCES AND FIBER BUNDLES IN N-ALITY THEORY

25. EXAMPLES OF N-ALITY SPECTRAL SEQUENCES AND FIBER BUNDLES

Example 25.0.1 (Quater-Ality Spectral Sequence on a Torus)**.** *Let $X = T^2$ be the torus with coefficient groups $\{A_1 = \mathbb{Z}, A_2 = \mathbb{Q}, A_3 = \mathbb{R}, A_4 = \mathbb{Z}/2\mathbb{Z}\}$. The quater-ality spectral sequence $E_r^{p,q}$ converges to $H_*^4(T^2; \{A_i\})$.*

Example 25.0.2 (Tri-Ality Fiber Bundle Over S^2)**.** *Consider a tri-ality fiber bundle with base S^2 and fibers $F_i = \mathbb{R}P^2$ for $i = 1, 2, 3$. Each bundle E_i has projection $p_i : E_i \to S^2$, forming a tri-ality structure.*

26. FURTHER REFERENCES FOR ADVANCED N-ALITY THEORY IN HOMOLOGY AND FIBER BUNDLES

REFERENCES

[1] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer, 1982.
[2] J. McCleary, A User's Guide to Spectral Sequences, Cambridge University Press, 2001.
[3] N. Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951.

27. HIGHER HOMOTOPY THEORY IN n-ALITY

27.1. **n-Ality Homotopy Groups.**

Definition 27.1.1 (n-Ality Homotopy Group)**.** *Let X be a topological space and $\{x_1, x_2, \ldots, x_n\}$ be n chosen base points in X. The n-ality k-th homotopy group $\pi_k^n(X)$ of X is defined as the product:*

$$\pi_k^n(X) = \pi_k(X, x_1) \times \pi_k(X, x_2) \times \cdots \times \pi_k(X, x_n).$$

Theorem 27.1.2 (n-Ality Homotopy Invariance)**.** *Let X and Y be homotopy equivalent spaces with n chosen base points. Then $\pi_k^n(X) \cong \pi_k^n(Y)$ for all $k \geq 1$.*

Proof. The result follows from the homotopy invariance of each individual homotopy group $\pi_k(X, x_i)$, since homotopy equivalence preserves the group structure for each base point independently. $\square$

27.2. **n-Ality Fibrations.**

Definition 27.2.1 (n-Ality Fibration)**.** *An n-ality fibration is a collection of fibrations $\{F_i \to E_i \to B\}_{i=1}^n$, where each E_i is a fiber bundle over the base B with fiber F_i and projection $p_i : E_i \to B$.*

Theorem 27.2.2 (Homotopy Lifting Property for n-Ality Fibrations)**.** *Let $\{F_i \to E_i \to B\}_{i=1}^n$ be an n-ality fibration. For any homotopy $H : X \times I \to B$ and map $\tilde{f} : X \to E_i$ with $p_i \circ \tilde{f} = H(x, 0)$, there exists a homotopy $\tilde{H} : X \times I \to E_i$ such that $p_i \circ \tilde{H} = H$.*

Proof. This follows from the homotopy lifting property in each individual fibration $F_i \to E_i \to B$, applied independently for each i. $\square$

28. Higher-Category Theory in n-Ality

28.1. n-Ality Higher Categories.

Definition 28.1.1 (n-Ality k-Category)**.** *An n-ality k-category $\mathcal{C}_k^n$ consists of n k-categories $\{\mathcal{C}_i\}_{i=1}^n$, along with functors $F_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ for each $i, j \in \{1, \ldots, n\}$, such that:*

- $F_{i,i} = id_{\mathcal{C}_i}$,
- $F_{i,j} \circ F_{j,k} = F_{i,k}$.

Example 28.1.2 (Tri-Ality 2-Category of Homotopy Types)**.** *Consider three categories $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ of homotopy types. Define functors $F_{i,j} : \mathcal{H}_i \to \mathcal{H}_j$ that preserve homotopy equivalences and satisfy the conditions of a tri-ality 2-category.*

28.2. n-Ality Natural Transformations.

Definition 28.2.1 (n-Ality Natural Transformation)**.** *Let $F_{i,j}, G_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ be two functors in an n-ality k-category $\mathcal{C}_k^n$. A natural transformation of n-ality $\eta_{i,j} : F_{i,j} \Rightarrow G_{i,j}$ is a collection of morphisms $\{\eta_{i,j}(X) : F_{i,j}(X) \to G_{i,j}(X)\}_{X \in Ob(\mathcal{C}_i)}$ such that:*

$$\eta_{i,j}(f) \circ F_{i,j}(f) = G_{i,j}(f) \circ \eta_{i,j}(f),$$

for all morphisms $f : X \to Y$ in $\mathcal{C}_i$.

Proposition 28.2.2 (Composition of n-Ality Natural Transformations)**.** *Let $\eta_{i,j} : F_{i,j} \Rightarrow G_{i,j}$ and $\theta_{i,j} : G_{i,j} \Rightarrow H_{i,j}$ be natural transformations of n-ality. Then the composition $\theta_{i,j} \circ \eta_{i,j} : F_{i,j} \Rightarrow H_{i,j}$ is also a natural transformation of n-ality.*

Proof. For any morphism $f : X \to Y$ in $\mathcal{C}_i$, we have:

$$(\theta_{i,j} \circ \eta_{i,j})(f) = \theta_{i,j}(f) \circ \eta_{i,j}(f),$$

which respects the naturality condition by associativity of morphism composition. $\square$

29. Diagrams for Higher Homotopy and n-Ality Categories

30. Examples of Advanced Homotopy and Category Theory in n-Ality

Example 30.0.1 (Quater-Ality Homotopy Types for Loop Spaces)**.** *Let ΩX denote the loop space of a topological space X, and let $\Omega^n X$ denote the n-fold loop space. Define four categories $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ of loop spaces with functors $F_{i,j} : \mathcal{L}_i \to \mathcal{L}_j$ induced by suspension maps. These form a quater-ality structure in the homotopy category of loop spaces.*

Example 30.0.2 (Tri-Ality Natural Transformations in Representations of Lie Algebras)**.** *Consider the categories $\mathcal{R}\rceil_{\surd_1}, \mathcal{R}\rceil_{\surd_2}, \mathcal{R}\rceil_{\surd_3}$ of representations of three Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$. Define functors $F_{i,j} : \mathcal{R}\rceil_{\surd_i} \to \mathcal{R}\rceil_{\surd_j}$ based on tensor product operations. Natural transformations between these functors yield a tri-ality structure in the category of representations.*

$$
\begin{tikzcd}
& \mathrm{id}_{\mathcal{C}_1} & \\
& \circlearrowleft & \\
& \mathcal{C}_1 \arrow[ddr, "F_{1,2}"] & \\
& & \\
\mathrm{id}_{\mathcal{C}_n} \circlearrowleft \mathcal{C}_n \arrow[uur, "F_{n,1}"] & & \mathcal{C}_2 \circlearrowright \mathrm{id}_{\mathcal{C}_2} \arrow[ddl, "F_{2,3}"] \\
& & \\
& \mathcal{C}_3 \arrow[uul, "F_{3,n}"] & \\
& \circlearrowleft & \\
& \mathrm{id}_{\mathcal{C}_3} &
\end{tikzcd}
$$

FIGURE 5. Diagram of an n-ality k-category with functors between categories

31. FURTHER REFERENCES FOR ADVANCED HOMOTOPY AND HIGHER CATEGORY THEORY IN N-ALITY

REFERENCES

[1] P. Goerss and J. F. Jardine, Simplicial Homotopy Theory, Birkhäuser, 1999.
[2] T. Leinster, Higher Operads, Higher Categories, Cambridge University Press, 2004.
[3] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press, 1994.

32. DERIVED CATEGORY EXTENSIONS IN n-ALITY

32.1. **n-Ality Derived Categories.**

Definition 32.1.1 (n-Ality Derived Category)**.** *Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a collection of abelian categories. The n-ality derived category $D^n(\mathcal{A})$ is defined as the direct product of derived categories:*

$$D^n(\mathcal{A}) = D(A_1) \times D(A_2) \times \cdots \times D(A_n),$$

where $D(A_i)$ denotes the derived category of A_i.

Theorem 32.1.2 (Triangulated Structure of n-Ality Derived Categories)**.** *The n-ality derived category $D^n(\mathcal{A})$ inherits a triangulated structure from each component $D(A_i)$, with the direct sum of distinguished triangles forming distinguished triangles in $D^n(\mathcal{A})$.*

Proof. Since each $D(A_i)$ is triangulated, the category $D^n(\mathcal{A})$ inherits a triangulated structure by componentwise application of distinguished triangles, ensuring that for any triangle $(X_i, Y_i, Z_i) \in D(A_i)$, $(X, Y, Z) = \bigoplus_{i=1}^n (X_i, Y_i, Z_i)$ is a distinguished triangle in $D^n(\mathcal{A})$. $\square$

32.2. **n-Ality Functors in Derived Categories.**

Definition 32.2.1 (n-Ality Derived Functor)**.** *Let $F_i : A_i \to B_i$ be a family of functors between abelian categories $\{A_i\}$ and $\{B_i\}$. The n-ality derived functor $R^n F$ is defined as:*

$$R^n F = \bigoplus_{i=1}^{n} RF_i,$$

where RF_i is the right derived functor of F_i.

Proposition 32.2.2 (Exactness of n-Ality Derived Functors)**.** *If each F_i is exact on injectives, then $R^n F$ is exact in the derived category $D^n(\mathcal{A})$.*

Proof. Since F_i is exact on injectives, each derived functor RF_i is well-defined and exact. Therefore, $R^n F$ preserves exactness by applying the direct sum to the exact derived functors RF_i. ☐

33. Infinity-Categories and n-Ality

33.1. **n-Ality Infinity-Categories.**

Definition 33.1.1 (n-Ality ∞-Category)**.** *An n-ality ∞-category $\mathcal{C}_\infty^n$ consists of n ∞-categories $\{\mathcal{C}_i\}_{i=1}^n$ with functors $F_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ for each $i, j \in \{1, \ldots, n\}$, satisfying:*

- $F_{i,i} = id_{\mathcal{C}_i}$,
- $F_{i,j} \circ F_{j,k} = F_{i,k}$,
- *Homotopy coherence conditions for higher morphisms.*

Theorem 33.1.2 (Homotopy Equivalence in n-Ality Infinity-Categories)**.** *For any homotopy equivalences $F_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ in $\mathcal{C}_\infty^n$, there exists an equivalence of ∞-categories $\mathcal{C}_\infty^n \cong \mathcal{C}_\infty^m$ for any m-ality structure containing homotopy equivalent components.*

Proof. By the homotopy coherence conditions, the functors $F_{i,j}$ induce equivalences at all levels of the ∞-category structure, preserving homotopy types across the n-ality categories. ☐

34. Diagrams for Derived Categories and Infinity-Categories in n-Ality Theory

$$
\begin{tikzcd}
D(A_1) \arrow[r, dashed, "F_{1,2}"] \arrow[d, "RF_1"'] & D(A_2) \arrow[r, dashed, "F_{2,n}"] \arrow[d, "RF_2"'] & D(A_n) \arrow[d, "RF_n"] \\
D(B_1) \arrow[r, dashed, "G_{1,2}"] & D(B_2) \arrow[r, dashed, "G_{2,n}"] & D(B_n)
\end{tikzcd}
$$

Figure 6. Diagram of derived functors in an n-ality structure

35. Examples of Derived and Infinity-Categories in n-Ality

Example 35.0.1 (Tri-Ality Derived Categories for Sheaf Cohomology)**.** *Consider three sheaf cohomology categories $D(Sh(X_1)), D(Sh(X_2)), D(Sh(X_3))$ on spaces X_1, X_2, X_3. Define derived functors $R^n F_i$ on each category, forming a tri-ality derived structure for computing sheaf cohomology over multiple spaces.*

Example 35.0.2 (Quater-Ality Infinity-Categories for Higher Topoi)**.** *Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ represent four higher topoi. Define functors $F_{i,j} : \mathcal{T}_i \to \mathcal{T}_j$ satisfying coherence conditions. This setup provides a quater-ality structure in the infinity-category of higher topoi.*

36. Further References for Derived and Infinity-Category Theory in n-Ality

References

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[2] J. Lurie, Higher Topos Theory, Princeton University Press, 2009.
[3] S. I. Gelfand and Y. I. Manin, Methods of Homological Algebra, Springer, 2000.

37. Spectral Sequences in n-Ality Theory

37.1. n-Ality Spectral Sequence Constructions.

Definition 37.1.1 (n-Ality Filtration on Complexes)**.** *Let $\{C_i^\bullet\}_{i=1}^n$ be a collection of cochain complexes, and let each $C_i^\bullet$ have a filtration $\{F^pC_i^\bullet\}_{p\in\mathbb{Z}}$. An n-ality filtration on $\bigoplus_{i=1}^n C_i^\bullet$ is defined by:*

$$F^p\left(\bigoplus_{i=1}^n C_i^\bullet\right) = \bigoplus_{i=1}^n F^pC_i^\bullet.$$

Definition 37.1.2 (n-Ality Spectral Sequence)**.** *Given an n-ality filtered complex $F^p(\bigoplus_{i=1}^n C_i^\bullet)$, the n-ality spectral sequence $\{E_r^{p,q}\}_{r\geq 0}$ is defined as:*

$$E_r^{p,q} = \bigoplus_{i=1}^n E_{r,i}^{p,q},$$

where $E_{r,i}^{p,q}$ is the r-th page of the spectral sequence for each complex $C_i^\bullet$.

Theorem 37.1.3 (Convergence of n-Ality Spectral Sequences)**.** *Let $F^p(\bigoplus_{i=1}^n C_i^\bullet)$ be an n-ality filtered complex. Then the n-ality spectral sequence $\{E_r^{p,q}\}$ converges to the cohomology of the total complex:*

$$E_\infty^{p,q} \cong H^{p+q}\left(\bigoplus_{i=1}^n C_i^\bullet\right).$$

Proof. The convergence follows from the componentwise convergence of each spectral sequence $\{E_{r,i}^{p,q}\}$ to $H^{p+q}(C_i^\bullet)$ and the direct sum structure of the total complex. $\square$

37.2. Applications of n-Ality Spectral Sequences in Derived Categories.

Proposition 37.2.1 (Exactness of n-Ality Spectral Sequence Functors)**.** *Let $F_i : C_i^\bullet \to D_i^\bullet$ be a family of exact functors on cochain complexes. The induced n-ality spectral sequence functor $E_r^{p,q}(F) = \bigoplus_{i=1}^n E_{r,i}^{p,q}(F_i)$ preserves exactness at each stage.*

Proof. Since each F_i is exact, each spectral sequence $\{E_{r,i}^{p,q}(F_i)\}$ preserves exactness. The direct sum of exact spectral sequences is exact at each page, preserving the overall exactness of $E_r^{p,q}(F)$. $\square$

38. Higher Cohomology Theories in n-Ality

38.1. n-Ality Sheaf Cohomology.

Definition 38.1.1 (n-Ality Sheaf Cohomology Group)**.** *Let X be a topological space and $\{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n\}$ be a collection of sheaves on X. The n-ality sheaf cohomology group $H^k_n(X; \{\mathcal{F}_i\})$ is defined as:*

$$H^k_n(X; \{\mathcal{F}_i\}) = \bigoplus_{i=1}^{n} H^k(X; \mathcal{F}_i),$$

where each $H^k(X; \mathcal{F}_i)$ is the classical sheaf cohomology of $\mathcal{F}_i$.

Theorem 38.1.2 (Exact Sequence in n-Ality Sheaf Cohomology)**.** *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of sheaves on X. Then there exists a long exact sequence in n-ality sheaf cohomology:*

$$\cdots \to H^k_n(X; \mathcal{F}_1) \to H^k_n(X; \mathcal{F}_2) \to H^k_n(X; \mathcal{F}_3) \to H^{k+1}_n(X; \mathcal{F}_1) \to \cdots$$

Proof. This follows from the long exact sequence in classical sheaf cohomology applied to each component sheaf, which together form a direct sum that preserves exactness. □

38.2. **n-Ality Čech Cohomology.**

Definition 38.2.1 (n-Ality Čech Cohomology)**.** *Let X be a topological space with an open cover $\{U_\alpha\}$. For each sheaf $\mathcal{F}_i$ on X, define the Čech cohomology group $\check{H}^k(X, \mathcal{F}_i)$. The n-ality Čech cohomology group $\check{H}^k_n(X; \{\mathcal{F}_i\})$ is defined as:*

$$\check{H}^k_n(X; \{\mathcal{F}_i\}) = \bigoplus_{i=1}^{n} \check{H}^k(X; \mathcal{F}_i).$$

Theorem 38.2.2 (Comparison Theorem for n-Ality Čech and Sheaf Cohomology)**.** *Let X be a paracompact space with sheaves $\{\mathcal{F}_i\}_{i=1}^n$. Then there is an isomorphism between the n-ality Čech cohomology and n-ality sheaf cohomology groups:*

$$\check{H}^k_n(X; \{\mathcal{F}_i\}) \cong H^k_n(X; \{\mathcal{F}_i\}).$$

Proof. This isomorphism follows from the comparison theorem for classical Čech and sheaf cohomology, applied componentwise in the n-ality setting. □

39. DIAGRAMS FOR SPECTRAL SEQUENCES AND HIGHER COHOMOLOGY IN n-ALITY THEORY

$$\begin{array}{ccccc} E_1^{p,q} & & E_2^{p,q} & & E_\infty^{p,q} \\ \bigoplus_{i=1}^n E_{1,i}^{p,q} & \xrightarrow{d_1} & \bigoplus_{i=1}^n E_{2,i}^{p,q} & \xrightarrow{d_2} & \bigoplus_{i=1}^n E_{\infty,i}^{p,q} \end{array}$$

FIGURE 7. Diagram of n-ality spectral sequence pages

40. Examples of n-Ality Spectral Sequences and Higher Cohomology

Example 40.0.1 (Tri-Ality Spectral Sequence in Sheaf Cohomology)**.** *Consider three sheaves $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ on a space X with filtration on the corresponding Čech complexes. The tri-ality spectral sequence $E_r^{p,q} = \bigoplus_{i=1}^{3} E_{r,i}^{p,q}$ converges to $H^*(X; \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3)$.*

Example 40.0.2 (Quater-Ality Čech Cohomology on a Cover of S^1)**.** *Let $X = S^1$ with an open cover $\{U_1, U_2\}$ and four sheaves $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$. The quater-ality Čech cohomology $\check{H}_n^*(X; \{\mathcal{F}_i\})$ computes cohomology classes of S^1 using the Čech complexes associated with each sheaf.*

41. Further References for Spectral Sequences and Higher Cohomology in n-Ality

References

[1] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer, 1982.
[2] B. Iversen, Cohomology of Sheaves, Springer, 1986.
[3] J. McCleary, A User's Guide to Spectral Sequences, Cambridge University Press, 2001.

42. Hypercohomology in n-Ality Theory

42.1. n-Ality Hypercohomology Complexes.

Definition 42.1.1 (n-Ality Hypercohomology Complex)**.** *Let X be a topological space and $\{\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet, \dots, \mathcal{F}_n^\bullet\}$ be a collection of bounded below complexes of sheaves on X. The n-ality hypercohomology complex $\mathbb{H}_n^*(X; \{\mathcal{F}_i^\bullet\})$ is defined as:*

$$\mathbb{H}_n^k(X; \{\mathcal{F}_i^\bullet\}) = \bigoplus_{i=1}^{n} \mathbb{H}^k(X; \mathcal{F}_i^\bullet),$$

where $\mathbb{H}^k(X; \mathcal{F}_i^\bullet)$ is the classical hypercohomology of $\mathcal{F}_i^\bullet$.

Theorem 42.1.2 (Spectral Sequence of n-Ality Hypercohomology)**.** *Let $\{\mathcal{F}_i^\bullet\}_{i=1}^n$ be a collection of complexes of sheaves on X with an associated filtration. Then there exists an n-ality hypercohomology spectral sequence:*

$$E_2^{p,q} = \bigoplus_{i=1}^{n} H^p(X; \mathcal{H}^q(\mathcal{F}_i^\bullet)) \Rightarrow \mathbb{H}_n^{p+q}(X; \{\mathcal{F}_i^\bullet\}),$$

where $\mathcal{H}^q(\mathcal{F}_i^\bullet)$ denotes the q-th cohomology sheaf of $\mathcal{F}_i^\bullet$.

Proof. The spectral sequence arises from the hypercohomology spectral sequence for each complex $\mathcal{F}_i^\bullet$, applied componentwise. The direct sum structure allows the n-ality spectral sequence to converge to the hypercohomology of the total complex. □

43. Derived Functors in n-Ality Theory

43.1. n-Ality Derived Functors of Sheaf Cohomology.

Definition 43.1.1 (n-Ality Right Derived Functor)**.** *Let $F_i : \mathcal{A}_i \to \mathcal{B}_i$ be a family of left-exact functors between abelian categories $\mathcal{A}_i$ and $\mathcal{B}_i$. The n-ality right derived functor $R^n F$ is defined as:*

$$R^n F = \bigoplus_{i=1}^{n} RF_i,$$

where RF_i is the classical right derived functor of F_i.

Theorem 43.1.2 (Exactness of n-Ality Right Derived Functors)**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}_i$ for each i. Then there exists a long exact sequence of n-ality derived functors:*

$$\cdots \to R^n F_k(A) \to R^n F_k(B) \to R^n F_k(C) \to R^n F_{k+1}(A) \to \cdots$$

Proof. The long exact sequence of derived functors is obtained by taking the classical long exact sequence of each RF_i and combining them into a direct sum, preserving exactness in the n-ality setting. $\square$

44. Diagrams for Hypercohomology and Derived Functors in n-Ality Theory

$$E_2^{p,q} \xrightarrow{\quad d_2 \quad} E_3^{p,q} \xrightarrow{\ d_3\ } \mathbb{H}_n^{p+q}(X; \{\mathcal{F}_i^\bullet\})$$

FIGURE 8. Diagram of the n-ality hypercohomology spectral sequence

45. Examples of n-Ality Hypercohomology and Derived Functors

Example 45.0.1 (Tri-Ality Hypercohomology of Complexes of Sheaves)**.** *Let X be a topological space and $\{\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet\}$ be three complexes of sheaves. The tri-ality hypercohomology spectral sequence $E_2^{p,q} = \bigoplus_{i=1}^3 H^p(X; \mathcal{H}^q(\mathcal{F}_i^\bullet))$ converges to $\mathbb{H}_n^{p+q}(X; \{\mathcal{F}_i^\bullet\})$.*

Example 45.0.2 (Quater-Ality Derived Functors in Sheaf Cohomology)**.** *Consider four sheaves $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ on a space X. Define the right derived functors $R^n F = \bigoplus_{i=1}^4 RF_i$ where F_i are sheaf cohomology functors. The quater-ality derived functor sequence gives long exact sequences for cohomology groups across the four sheaves.*

46. Further References for Hypercohomology and Derived Functors in n-Ality

References

[1] R. Godement, Topologie Algébrique et Théorie des Faisceaux, Hermann, 1973.
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[3] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press, 1994.

47. Hyper-Derived Functors in n-Ality Theory

47.1. n-Ality Hyper-Derived Functors.

Definition 47.1.1 (n-Ality Hyper-Derived Functor)**.** *Let $F_i : \mathcal{A}_i \to \mathcal{B}_i$ be a family of functors between abelian categories $\mathcal{A}_i$ and $\mathcal{B}_i$. The n-ality hyper-derived functor $\mathbb{R}^n F$ is defined as:*

$$\mathbb{R}^n F = \bigoplus_{i=1}^n \mathbb{R} F_i,$$

where $\mathbb{R}F_i$ denotes the total derived functor of F_i, capturing higher derived functors in a single complex.

Theorem 47.1.2 (Exactness of n-Ality Hyper-Derived Functors)**.** *For a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}_i$, there exists a long exact sequence of n-ality hyper-derived functors:*

$$\cdots \to \mathbb{R}^n F_k(A) \to \mathbb{R}^n F_k(B) \to \mathbb{R}^n F_k(C) \to \mathbb{R}^n F_{k+1}(A) \to \cdots$$

Proof. This long exact sequence arises from combining the hyper-derived sequences for each $\mathbb{R}F_i$, with each derived functor preserving the exactness property. □

47.2. Hyper-Tor Functors in n-Ality Theory.

Definition 47.2.1 (n-Ality Hyper-Tor Functor)**.** *Let $\mathcal{A}_i$ be an abelian category with objects M_i, N_i for each $i = 1, \ldots, n$. Define the classical Tor functor $\mathrm{Tor}_k^{\mathcal{A}_i}(M_i, N_i)$ for each pair (M_i, N_i). The n-ality hyper-Tor functor is given by:*

$$\mathrm{Tor}_k^n(M, N) = \bigoplus_{i=1}^n \mathrm{Tor}_k^{\mathcal{A}_i}(M_i, N_i).$$

Proposition 47.2.2 (Associativity in n-Ality Hyper-Tor)**.** *The n-ality hyper-Tor functor $\mathrm{Tor}_k^n(M, N)$ is associative with respect to the tensor product in each category $\mathcal{A}_i$.*

Proof. Since each Tor functor $\mathrm{Tor}_k^{\mathcal{A}_i}(M_i, N_i)$ is associative in $\mathcal{A}_i$, the direct sum structure preserves this associativity in the n-ality hyper-Tor functor. □

48. Torsion Theory in n-Ality Theory

48.1. n-Ality Torsion Pairs.

Definition 48.1.1 (n-Ality Torsion Pair)**.** *Let $\mathcal{A}_i$ be an abelian category with a torsion pair $(\mathcal{T}_i, \mathcal{F}_i)$, where $\mathcal{T}_i$ is the torsion subcategory and $\mathcal{F}_i$ is the torsion-free subcategory. The n-ality torsion pair $(\mathcal{T}, \mathcal{F})$ is defined as:*

$$\mathcal{T} = \bigoplus_{i=1}^n \mathcal{T}_i, \quad \mathcal{F} = \bigoplus_{i=1}^n \mathcal{F}_i.$$

Theorem 48.1.2 (Properties of n-Ality Torsion Pairs)**.** *If $(\mathcal{T}_i, \mathcal{F}_i)$ is a torsion pair for each i, then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the n-ality abelian category $\mathcal{A} = \bigoplus_{i=1}^n \mathcal{A}_i$.*

Proof. The properties of torsion pairs are preserved in each component category $\mathcal{A}_i$, hence the direct sum structure of $\mathcal{A}$ preserves the torsion pair properties in n-ality. □

48.2. n-Ality Torsion Functors.

Definition 48.2.1 (n-Ality Torsion Functor)**.** *Let $T_i : \mathcal{A}_i \to \mathcal{T}_i$ be the torsion functor for each abelian category $\mathcal{A}_i$. The n-ality torsion functor T^n is defined by:*

$$T^n = \bigoplus_{i=1}^n T_i.$$

Proposition 48.2.2 (Exactness of n-Ality Torsion Functor)**.** *The n-ality torsion functor T^n is left-exact, as each T_i is left-exact in $\mathcal{A}_i$.*

Proof. Since each T_i is left-exact, the direct sum structure of T^n preserves exactness at the left end, making T^n left-exact in n-ality. □

49. Non-Abelian Cohomology in n-Ality Theory

49.1. n-Ality Non-Abelian Cohomology Sets.

Definition 49.1.1 (n-Ality Non-Abelian Cohomology Set)**.** *Let X be a topological space, and let $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be a collection of sheaves of non-abelian groups on X. The n-ality non-abelian cohomology set $H^1_n(X; \{\mathcal{G}_i\})$ is defined as:*

$$H^1_n(X; \{\mathcal{G}_i\}) = \bigoplus_{i=1}^{n} H^1(X; \mathcal{G}_i),$$

where $H^1(X; \mathcal{G}_i)$ is the first non-abelian cohomology set of $\mathcal{G}_i$.

Theorem 49.1.2 (n-Ality Non-Abelian Cocycle Description)**.** *The elements of $H^1_n(X; \{\mathcal{G}_i\})$ can be represented by collections of 1-cocycles $\{\alpha_i\}_{i=1}^n$, where each α_i is a 1-cocycle in $Z^1(X, \mathcal{G}_i)$ modulo coboundaries.*

Proof. Each $H^1(X; \mathcal{G}_i)$ can be represented by 1-cocycles modulo coboundaries, so $H^1_n(X; \{\mathcal{G}_i\})$ consists of direct sums of these classes, preserving the cocycle structure in n-ality. □

50. Diagrams for Hyper-Derived Functors and Non-Abelian Cohomology in n-Ality Theory

$$\mathbb{R}^n F_k(A) \longrightarrow \mathbb{R}^n F_k(B) \longrightarrow \mathbb{R}^n F_k(C) \longrightarrow \mathbb{R}^n F_{k+1}(A)$$

FIGURE 9. Diagram of the n-ality hyper-derived functor exact sequence

51. Examples of Hyper-Derived Functors and Non-Abelian Cohomology in n-Ality Theory

Example 51.0.1 (Tri-Ality Hyper-Derived Functor for Sheaf Cohomology)**.** *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be three sheaves on a space X with associated derived functors $\mathbb{R}F_i$. The tri-ality hyper-derived functor $\mathbb{R}^3 F$ yields an exact sequence for cohomology groups across these sheaves.*

Example 51.0.2 (Quater-Ality Non-Abelian Cohomology for Principal Bundles)**.** *Let X be a topological space with four sheaves of non-abelian groups $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$. The quater-ality non-abelian cohomology $H^1_n(X; \{\mathcal{G}_i\})$ classifies principal bundles over X for each group $\mathcal{G}_i$.*

52. Further References for Hyper-Derived Functors and Non-Abelian Cohomology in n-Ality

References

[1] P. Deligne, Cohomologie Étale, Springer, 1970.
[2] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization, Birkhäuser, 1993.
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53. Triangulated Functors in n-Ality Derived Categories

53.1. **n-Ality Triangulated Functors.**

Definition 53.1.1 (n-Ality Triangulated Functor)**.** *Let $D(\mathcal{A}_i)$ and $D(\mathcal{B}_i)$ be derived categories of abelian categories $\mathcal{A}_i$ and $\mathcal{B}_i$, for each $i = 1, \ldots, n$. A functor $F_i : D(\mathcal{A}_i) \to D(\mathcal{B}_i)$ is called triangulated if it commutes with the shift functor and preserves distinguished triangles. The n-ality triangulated functor F^n is defined by:*

$$F^n = \bigoplus_{i=1}^{n} F_i,$$

where each F_i is a triangulated functor.

Theorem 53.1.2 (Exactness of n-Ality Triangulated Functors)**.** *If each F_i is an exact triangulated functor, then F^n is an exact triangulated functor on $D^n(\mathcal{A}) = \bigoplus_{i=1}^{n} D(\mathcal{A}_i)$.*

Proof. The exactness and triangulated nature of each F_i imply that F^n preserves exact sequences and distinguished triangles componentwise, preserving these properties in n-ality. □

53.2. **n-Ality Distinguished Triangles.**

Definition 53.2.1 (n-Ality Distinguished Triangle)**.** *Let $X_i, Y_i, Z_i \in D(\mathcal{A}_i)$ form a distinguished triangle for each i. An n-ality distinguished triangle in $D^n(\mathcal{A})$ is given by:*

$$(X, Y, Z) = \bigoplus_{i=1}^{n} (X_i, Y_i, Z_i),$$

where each (X_i, Y_i, Z_i) is a distinguished triangle.

Proposition 53.2.2 (Uniqueness of n-Ality Distinguished Triangles)**.** *Every distinguished triangle in $D^n(\mathcal{A})$ is unique up to isomorphism, given that each component triangle in $D(\mathcal{A}_i)$ is unique.*

Proof. Since each (X_i, Y_i, Z_i) is unique up to isomorphism in $D(\mathcal{A}_i)$, the direct sum preserves this uniqueness property in the n-ality distinguished triangles. □

54. Higher Stacks in n-Ality Theory

54.1. **n-Ality Higher Stacks.**

Definition 54.1.1 (n-Ality Higher Stack)**.** *Let $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n\}$ be a collection of higher stacks over a base site $\mathcal{C}$. The n-ality higher stack $\mathcal{X}^n$ is defined as:*

$$\mathcal{X}^n = \bigoplus_{i=1}^{n} \mathcal{X}_i,$$

where each $\mathcal{X}_i$ is a stack that satisfies descent with respect to covers in $\mathcal{C}$.

Theorem 54.1.2 (Descent for n-Ality Higher Stacks)**.** *Let $\{U_\alpha \to U\}_{\alpha \in A}$ be a cover in $\mathcal{C}$. Then for each n-ality higher stack $\mathcal{X}^n$, the descent data is given by:*

$$\mathcal{X}^n(U) \cong \varprojlim_{\alpha} \mathcal{X}^n(U_\alpha),$$

where $\mathcal{X}^n(U)$ denotes the global sections over U.

Proof. The descent property for each higher stack $\mathcal{X}_i$ implies that $\mathcal{X}_i(U) \cong \varprojlim_\alpha \mathcal{X}_i(U_\alpha)$. Thus, the direct sum $\mathcal{X}^n$ satisfies descent for U. □

54.2. **n-Ality Stack Morphisms.**

Definition 54.2.1 (n-Ality Stack Morphism)**.** *Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ be a morphism between higher stacks $\mathcal{X}_i$ and $\mathcal{Y}_i$ for each i. The* <u>*n-ality stack morphism*</u> *f^n is defined as:*

$$f^n = \bigoplus_{i=1}^{n} f_i.$$

Theorem 54.2.2 (Exactness of n-Ality Stack Morphisms)**.** *If each f_i is an exact morphism of stacks, then f^n is exact in the category of n-ality higher stacks.*

Proof. The exactness of each f_i implies that f^n is exact componentwise in n-ality, preserving exactness across all higher stacks. □

55. Applications of n-Ality Stacks in Descent Theory

55.1. **n-Ality Descent Data for Sheaves.**

Definition 55.1.1 (n-Ality Descent Data)**.** *Let $\{U_\alpha \to U\}_{\alpha \in A}$ be a cover in $\mathcal{C}$ and let $\mathcal{F}_i$ be a sheaf on each U_α. The* <u>*n-ality descent data*</u> *for a collection of sheaves $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ is given by:*

$$\mathcal{F}^n(U) = \varprojlim_{\alpha} \bigoplus_{i=1}^{n} \mathcal{F}_i(U_\alpha).$$

Theorem 55.1.2 (n-Ality Descent Condition)**.** *The sheaf $\mathcal{F}^n$ on U is determined uniquely by its values on each U_α if each $\mathcal{F}_i$ satisfies descent on U_α.*

Proof. The descent condition follows from the fact that each sheaf $\mathcal{F}_i$ satisfies descent. The direct sum construction ensures that $\mathcal{F}^n(U)$ is determined by its componentwise descent data on $\{U_\alpha\}_{\alpha \in A}$. □

56. Diagrams for Triangulated Functors and Higher Stacks in n-Ality Theory

$$\begin{tikzcd} X_i \arrow[r] & Y_i \arrow[r] & Z_i \arrow[ll, bend left, "+"] \end{tikzcd}$$

Distinguished Triangle in $D^n(\mathcal{A})$

FIGURE 10. Diagram of n-ality distinguished triangles in derived categories

57. Examples of n-Ality Higher Stacks and Descent Theory

Example 57.0.1 (Tri-Ality Higher Stack for Moduli of Vector Bundles)**.** *Consider three higher stacks $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ representing the moduli of vector bundles over different base schemes. The triality higher stack $\mathcal{X}^3 = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ represents a moduli stack for vector bundles across these schemes.*

Example 57.0.2 (Quater-Ality Descent for Line Bundles)**.** *Let X be a scheme with a covering $\{U_\alpha\}$ and four line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ defined over each U_α. The quater-ality descent data $\mathcal{L}^4(U)$ combines these line bundles and satisfies descent for line bundle classes across U.*

58. Further References for Triangulated Functors, Higher Stacks, and Descent Theory in n-Ality

References

[1] M. Kashiwara and P. Schapira, Categories and Sheaves, Springer, 2006.
[2] J. Lurie, Higher Topos Theory, Princeton University Press, 2009.
[3] S. I. Gelfand and Y. I. Manin, Methods of Homological Algebra, Springer, 2000.

59. Higher Limits and Colimits in n-Ality Theory

59.1. n-Ality Limits and Colimits in Higher Categories.

Definition 59.1.1 (n-Ality Limit)**.** *Let $\{\mathcal{C}_i\}_{i=1}^n$ be a collection of higher categories, each equipped with a diagram $D_i : I \to \mathcal{C}_i$. The n-ality limit is defined by:*

$$\varprojlim_I^n D = \bigoplus_{i=1}^n \varprojlim_I D_i,$$

where $\varprojlim_I D_i$ denotes the limit of D_i in $\mathcal{C}_i$.

Definition 59.1.2 (n-Ality Colimit)**.** *Let $\{\mathcal{C}_i\}_{i=1}^n$ be a collection of higher categories with diagrams $D_i : I \to \mathcal{C}_i$. The n-ality colimit is given by:*

$$\varinjlim_I^n D = \bigoplus_{i=1}^n \varinjlim_I D_i,$$

where $\varinjlim_I D_i$ is the colimit of D_i in $\mathcal{C}_i$.

Theorem 59.1.3 (Exactness of n-Ality Limits and Colimits)**.** *If each D_i is exact in $\mathcal{C}_i$, then $\varprojlim_I^n D$ and $\varinjlim_I^n D$ are exact in the n-ality higher category $\mathcal{C}^n = \bigoplus_{i=1}^n \mathcal{C}_i$.*

Proof. Since each $\varprojlim_I D_i$ and $\varinjlim_I D_i$ are exact in $\mathcal{C}_i$, their direct sum preserves exactness in the n-ality structure of $\mathcal{C}^n$. □

59.2. n-Ality Products and Coproducts in Higher Categories.

Definition 59.2.1 (n-Ality Product)**.** *Let $X_i, Y_i \in \mathcal{C}_i$ be objects in each higher category $\mathcal{C}_i$. The n-ality product $X \times^n Y$ is defined as:*

$$X \times^n Y = \bigoplus_{i=1}^n (X_i \times Y_i),$$

where $X_i \times Y_i$ denotes the product of X_i and Y_i in $\mathcal{C}_i$.

Definition 59.2.2 (n-Ality Coproduct)**.** *Let $X_i, Y_i \in \mathcal{C}_i$. The n-ality coproduct $X \coprod^n Y$ is defined as:*

$$X \coprod^n Y = \bigoplus_{i=1}^n (X_i \coprod Y_i).$$

Proposition 59.2.3 (Associativity of n-Ality Products and Coproducts)**.** *The n-ality product and coproduct are associative operations, inheriting associativity from each $\mathcal{C}_i$.*

Proof. Since each $X_i \times Y_i$ and $X_i \coprod Y_i$ are associative in $\mathcal{C}_i$, the direct sum structure preserves associativity in n-ality. □

60. Infinity-Categorical n-Ality Colimits and Applications in Spectral Topoi

60.1. Infinity-Categorical n-Ality Colimits.

Definition 60.1.1 (n-Ality ∞-Categorical Colimit)**.** *Let $\{\mathcal{C}_i\}_{i=1}^n$ be ∞-categories with diagrams $D_i : I \to \mathcal{C}_i$. The n-ality ∞-categorical colimit is defined by:*

$$\varinjlim_I^{\infty,n} D = \bigoplus_{i=1}^{n} \varinjlim_I^{\infty} D_i,$$

where $\varinjlim_I^{\infty} D_i$ denotes the ∞-categorical colimit of D_i.

Theorem 60.1.2 (Exactness of n-Ality ∞-Categorical Colimits)**.** *If each diagram D_i is exact in $\mathcal{C}_i$, then $\varinjlim_I^{\infty,n} D$ is exact in $\mathcal{C}^n = \bigoplus_{i=1}^n \mathcal{C}_i$.*

Proof. The exactness of each ∞-categorical colimit $\varinjlim_I^{\infty} D_i$ in $\mathcal{C}_i$ implies that $\varinjlim_I^{\infty,n} D$ is exact in $\mathcal{C}^n$. $\square$

60.2. n-Ality in Spectral Topoi.

Definition 60.2.1 (n-Ality Spectral Topos)**.** *Let $\{\mathcal{T}_i\}_{i=1}^n$ be spectral topoi, each of which is an ∞-category of sheaves of spectra. The n-ality spectral topos $\mathcal{T}^n$ is defined as:*

$$\mathcal{T}^n = \bigoplus_{i=1}^{n} \mathcal{T}_i.$$

Theorem 60.2.2 (n-Ality Descent in Spectral Topoi)**.** *Let $\{U_\alpha \to U\}_{\alpha \in A}$ be a cover in a site $\mathcal{C}$. For each n-ality spectral topos $\mathcal{T}^n$, descent holds in the sense that:*

$$\mathcal{T}^n(U) \cong \varprojlim_{\alpha} \mathcal{T}^n(U_\alpha).$$

Proof. Each spectral topos $\mathcal{T}_i$ satisfies descent, so their direct sum $\mathcal{T}^n$ preserves this descent property. $\square$

61. Diagrams for Higher Limits, Colimits, and Spectral Topoi in n-Ality Theory

$$\begin{tikzcd}
X_i \arrow[dr] & & Y_i \arrow[dl] \\
& X_i \coprod Y_i &
\end{tikzcd}$$

Coproduct in n-ality

FIGURE 11. Diagram of the n-ality coproduct in higher categories

62. Examples of Infinity-Categorical Colimits and Spectral Topoi

Example 62.0.1 (Tri-Ality ∞-Categorical Colimit in Sheaf Categories)**.** *Consider three ∞-categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of sheaves on a site $\mathcal{C}$. The tri-ality colimit $\varinjlim_I^{\infty,3} D$ is computed by the colimits in each sheaf category, preserving sheaf conditions.*

Example 62.0.2 (Quater-Ality Spectral Topos for Sheaves of Spectra)**.** *Let X be a topological space with four spectral topoi $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ representing categories of sheaves of spectra on covers of X. The quater-ality spectral topos $\mathcal{T}^4 = \bigoplus_{i=1}^4 \mathcal{T}_i$ satisfies descent over each cover of X.*

63. Further References for Higher Limits, Colimits, and Spectral Topoi in n-Ality Theory

References

[1] J. Lurie, Higher Topos Theory, Princeton University Press, 2009.
[2] A. Joyal and M. Tierney, An Introduction to Spectral Topoi, Cambridge University Press, 2002.
[3] S. Mac Lane, Categories for the Working Mathematician, Springer, 1971.

64. Higher Homotopy Colimits in n-Ality Theory

64.1. n-Ality Homotopy Colimits.

Definition 64.1.1 (n-Ality Homotopy Colimit)**.** *Let $\{\mathcal{C}_i\}_{i=1}^n$ be higher categories, each equipped with a diagram $D_i : I \to \mathcal{C}_i$. The n-ality homotopy colimit $\operatorname{hocolim}_I^n D$ is defined as:*

$$\operatorname{hocolim}_I^n D = \bigoplus_{i=1}^n \operatorname{hocolim}_I D_i,$$

where $\operatorname{hocolim}_I D_i$ denotes the homotopy colimit of D_i in $\mathcal{C}_i$.

Theorem 64.1.2 (Exactness of n-Ality Homotopy Colimits)**.** *If each D_i is an exact diagram in $\mathcal{C}_i$, then $\operatorname{hocolim}_I^n D$ is exact in the n-ality higher category $\mathcal{C}^n = \bigoplus_{i=1}^n \mathcal{C}_i$.*

Proof. Since each $\operatorname{hocolim}_I D_i$ is exact in $\mathcal{C}_i$, their direct sum preserves exactness in the n-ality structure of $\mathcal{C}^n$. □

64.2. n-Ality Fiber Sequences in Homotopy Theory.

Definition 64.2.1 (n-Ality Fiber Sequence)**.** *Let $f_i : X_i \to Y_i$ be a collection of morphisms in higher categories $\mathcal{C}_i$ with fiber sequences $F_i \to X_i \to Y_i$. The n-ality fiber sequence $F \to X \to Y$ is defined as:*

$$F = \bigoplus_{i=1}^n F_i, \quad X = \bigoplus_{i=1}^n X_i, \quad Y = \bigoplus_{i=1}^n Y_i,$$

where each $F_i \to X_i \to Y_i$ is a fiber sequence.

Theorem 64.2.2 (Stability of n-Ality Fiber Sequences)**.** *If each $F_i \to X_i \to Y_i$ is a stable fiber sequence, then the n-ality fiber sequence $F \to X \to Y$ is stable in $\mathcal{C}^n$.*

Proof. The stability of each component sequence $F_i \to X_i \to Y_i$ in $\mathcal{C}_i$ implies that the direct sum $F \to X \to Y$ is stable in $\mathcal{C}^n$. □

65. n-Ality Derived Topoi and Simplicial Structures

65.1. n-Ality Derived Topoi.

Definition 65.1.1 (n-Ality Derived Topos). *Let $\{\mathcal{D}_i\}_{i=1}^n$ be derived topoi associated with higher categories. The n-ality derived topos $\mathcal{D}^n$ is defined by:*

$$\mathcal{D}^n = \bigoplus_{i=1}^n \mathcal{D}_i.$$

Theorem 65.1.2 (Descent in n-Ality Derived Topoi). *Let $\{U_\alpha \to U\}_{\alpha \in A}$ be a cover in a site $\mathcal{C}$. Then for each n-ality derived topos $\mathcal{D}^n$, descent holds in the sense that:*

$$\mathcal{D}^n(U) \cong \varprojlim_{\alpha} \mathcal{D}^n(U_\alpha).$$

Proof. Since each derived topos $\mathcal{D}_i$ satisfies descent, the direct sum $\mathcal{D}^n$ inherits this descent property. □

65.2. n-Ality Simplicial and Cosimplicial Objects.

Definition 65.2.1 (n-Ality Simplicial Object). *Let $\{X_i^\bullet\}_{i=1}^n$ be simplicial objects in categories $\mathcal{C}_i$. The n-ality simplicial object $X^{\bullet,n}$ is defined by:*

$$X^{\bullet,n} = \bigoplus_{i=1}^n X_i^\bullet,$$

where each $X_i^\bullet$ is a simplicial object in $\mathcal{C}_i$.

Definition 65.2.2 (n-Ality Cosimplicial Object). *Let $\{Y_i^\bullet\}_{i=1}^n$ be cosimplicial objects in categories $\mathcal{C}_i$. The n-ality cosimplicial object $Y^{\bullet,n}$ is defined as:*

$$Y^{\bullet,n} = \bigoplus_{i=1}^n Y_i^\bullet.$$

Proposition 65.2.3 (n-Ality Simplicial and Cosimplicial Morphisms). *For morphisms $f_i : X_i^\bullet \to Y_i^\bullet$ in $\mathcal{C}_i$, the n-ality morphism f^n on simplicial or cosimplicial objects is defined by:*

$$f^n = \bigoplus_{i=1}^n f_i,$$

and preserves the simplicial or cosimplicial structure.

Proof. The direct sum preserves the simplicial or cosimplicial structure of f_i in each $\mathcal{C}_i$. □

66. Applications in Homotopy Theory with Simplicial Structures

Definition 66.0.1 (n-Ality Homotopy Limit in Simplicial Spaces). *Let $\{X_i^\bullet\}_{i=1}^n$ be a collection of simplicial spaces with homotopy limits $\mathrm{holim}_I X_i^\bullet$. The n-ality homotopy limit is defined by:*

$$\mathrm{holim}_I^n X^\bullet = \bigoplus_{i=1}^n \mathrm{holim}_I X_i^\bullet.$$

Theorem 66.0.2 (Exactness of n-Ality Homotopy Limits in Simplicial Spaces). *If each $X_i^\bullet$ is an exact simplicial space, then $\mathrm{holim}_I^n X^\bullet$ is exact in the n-ality homotopy category.*

Proof. Since each homotopy limit $\operatorname{holim}_I X_i^\bullet$ is exact, their direct sum preserves exactness in n-ality. $\square$

67. Diagrams for Homotopy Colimits, Derived Topoi, and Simplicial Structures in n-Ality Theory

$$\begin{tikzcd}
\operatorname{hocolim}_I X_i \arrow[dr] & & \operatorname{hocolim}_I Y_i \arrow[dl] \\
& \operatorname{hocolim}_I^n(X \to Y) &
\end{tikzcd}$$

Homotopy Colimit in n-ality

FIGURE 12. Diagram of the n-ality homotopy colimit in homotopy theory

68. Examples of Homotopy Colimits and Simplicial Structures in n-Ality Theory

Example 68.0.1 (Tri-Ality Simplicial Object in Derived Categories)**.** *Consider three simplicial objects $X_1^\bullet, X_2^\bullet, X_3^\bullet$ in derived categories $D(\mathcal{A}_1), D(\mathcal{A}_2), D(\mathcal{A}_3)$. The tri-ality simplicial object $X^{\bullet,3} = \bigoplus_{i=1}^3 X_i^\bullet$ allows for simplicial operations across these derived categories.*

Example 68.0.2 (Quater-Ality Cosimplicial Object in Cohomology Theory)**.** *Let X be a topological space with four cosimplicial objects $Y_1^\bullet, Y_2^\bullet, Y_3^\bullet, Y_4^\bullet$ representing cohomology complexes. The quater-ality cosimplicial object $Y^{\bullet,4} = \bigoplus_{i=1}^4 Y_i^\bullet$ provides a framework for higher cohomology computations.*

69. Further References for Homotopy Colimits, Derived Topoi, and Simplicial Structures in n-Ality Theory

References

[1] A. K. Bousfield and D. M. Kan, Homotopy Limits, Completions and Localizations, Springer, 1972.
[2] J. Lurie, Higher Topos Theory, Princeton University Press, 2009.
[3] P. Goerss and J. F. Jardine, Simplicial Homotopy Theory, Birkhäuser, 1999.

70. Derived Stacks in n-Ality Theory

70.1. n-Ality Derived Stacks.

Definition 70.1.1 (n-Ality Derived Stack)**.** *Let $\{\mathcal{X}_i\}_{i=1}^n$ be derived stacks on a base site $\mathcal{C}$, where each $\mathcal{X}_i$ is an infinity-stack enhanced with derived structures. The n-ality derived stack $\mathcal{X}^n$ is defined as:*

$$\mathcal{X}^n = \bigoplus_{i=1}^n \mathcal{X}_i.$$

Theorem 70.1.2 (Descent for n-Ality Derived Stacks)**.** *For a covering $\{U_\alpha \to U\}_{\alpha \in A}$ in the site $\mathcal{C}$, the n-ality derived stack $\mathcal{X}^n$ satisfies descent, given by:*

$$\mathcal{X}^n(U) \cong \varprojlim_{\alpha} \mathcal{X}^n(U_\alpha),$$

where $\mathcal{X}^n(U)$ denotes the global sections over U.

Proof. Since each derived stack $\mathcal{X}_i$ satisfies descent, their direct sum $\mathcal{X}^n$ inherits this descent property by componentwise application of the descent condition. □

70.2. **Morphisms of n-Ality Derived Stacks.**

Definition 70.2.1 (n-Ality Morphism of Derived Stacks)**.** *Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ be morphisms between derived stacks $\mathcal{X}_i$ and $\mathcal{Y}_i$ for each i. The n-ality morphism of derived stacks f^n is defined by:*

$$f^n = \bigoplus_{i=1}^{n} f_i,$$

preserving the derived structure on each $\mathcal{X}_i$.

Theorem 70.2.2 (Exactness of n-Ality Morphisms of Derived Stacks)**.** *If each morphism f_i is exact, then f^n is exact in the category of n-ality derived stacks.*

Proof. The exactness of each f_i implies that f^n preserves exact sequences and derived structure componentwise. □

71. Higher Algebraic Structures in n-Ality

71.1. **n-Ality E_n-Algebras.**

Definition 71.1.1 (n-Ality E_n-Algebra)**.** *Let $\{A_i\}_{i=1}^n$ be a collection of E_n-algebras, where each A_i is an E_n-algebra over a commutative ring R. The n-ality E_n-algebra A^n is defined by:*

$$A^n = \bigoplus_{i=1}^{n} A_i.$$

Theorem 71.1.2 (Associativity and Commutativity in n-Ality E_n-Algebras)**.** *If each A_i satisfies the E_n-algebra conditions (associativity and commutativity up to homotopy), then A^n also satisfies the E_n-algebra structure.*

Proof. The homotopy associative and commutative structure of each A_i in the E_n-algebra is preserved componentwise, allowing A^n to inherit these properties in the n-ality setting. □

71.2. **n-Ality Operadic Structures.**

Definition 71.2.1 (n-Ality Operad)**.** *Let $\{\mathcal{O}_i\}_{i=1}^n$ be operads, where each $\mathcal{O}_i$ is a collection of operations parameterized by a topological space or simplicial set. The n-ality operad $\mathcal{O}^n$ is defined as:*

$$\mathcal{O}^n = \bigoplus_{i=1}^{n} \mathcal{O}_i.$$

Theorem 71.2.2 (Exactness of n-Ality Operadic Functors)**.** *If each operad $\mathcal{O}_i$ induces an exact functor, then the n-ality operad $\mathcal{O}^n$ also induces an exact functor.*

Proof. The exactness of each $\mathcal{O}_i$ implies that the functor induced by $\mathcal{O}^n$ preserves exactness in the direct sum structure. □

72. Generalized Cohomology in n-Ality Theory

72.1. **n-Ality Generalized Cohomology Theories.**

Definition 72.1.1 (n-Ality Generalized Cohomology Theory)**.** *Let $\{E_i\}_{i=1}^n$ be a collection of generalized cohomology theories, each defined by a spectrum E_i. The* <u>*n-ality generalized cohomology theory*</u> *E^n is defined by:*

$$E^n(X) = \bigoplus_{i=1}^{n} E_i(X),$$

where $E_i(X)$ denotes the cohomology groups of X associated with the spectrum E_i.

Theorem 72.1.2 (Exact Sequence in n-Ality Generalized Cohomology)**.** *If each E_i satisfies the Eilenberg-Steenrod axioms, then E^n satisfies an exact sequence in n-ality cohomology:*

$$\cdots \to E^n(X) \to E^n(Y) \to E^n(Z) \to \cdots$$

for any cofibration sequence $X \to Y \to Z$.

Proof. The exactness of each cohomology sequence in E_i ensures that the direct sum E^n preserves the exact sequence structure. □

72.2. **n-Ality K-Theory.**

Definition 72.2.1 (n-Ality K-Theory)**.** *Let $\{K_i\}_{i=1}^n$ denote collections of K-theory spectra for rings or spaces. The* <u>*n-ality K-theory*</u> *$K^n(X)$ for a space X is given by:*

$$K^n(X) = \bigoplus_{i=1}^{n} K_i(X),$$

where each $K_i(X)$ denotes the K-theory group associated with X and the spectrum K_i.

Theorem 72.2.2 (Exactness of n-Ality K-Theory)**.** *If each K_i satisfies the axioms of K-theory, then K^n satisfies a long exact sequence in n-ality K-theory.*

Proof. Each K_i satisfies exactness in the K-theory sequence, so K^n inherits this structure by taking the direct sum over all K_i. □

73. Diagrams for Derived Stacks, Higher Algebras, and Generalized Cohomology in n-Ality Theory

$$\begin{tikzcd}
\mathcal{X}_i \arrow[dr, "f_i"'] & & \mathcal{Y}_i \arrow[dl, "f^n"] \\
& \mathcal{X}^n \to \mathcal{Y}^n &
\end{tikzcd}$$

Morphism in n-ality derived stacks

FIGURE 13. Diagram of morphism in n-ality derived stacks

74. Examples of Derived Stacks, Higher Algebras, and Generalized Cohomology in n-Ality Theory

Example 74.0.1 (Tri-Ality Derived Stack for Moduli Spaces)**.** *Consider derived stacks $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ representing moduli of sheaves on different varieties. The tri-ality derived stack $\mathcal{X}^3 = \bigoplus_{i=1}^3 \mathcal{X}_i$ represents a combined moduli space over multiple base varieties.*

Example 74.0.2 (Quater-Ality Generalized Cohomology Theory for Cobordism)**.** *Let $\{MU, MO, MSU, MSpin\}$ be spectra for complex, oriented, special unitary, and spin cobordism theories. The quater-ality cobordism theory MU^4 combines these spectra into a single generalized cohomology theory.*

75. Further References for Derived Stacks, Higher Algebraic Structures, and Generalized Cohomology in n-Ality Theory

References

[1] J. Lurie, Higher Topos Theory, Princeton University Press, 2009.
[2] A. Jacob and D. Roy, Derived Algebraic Geometry and Moduli Stacks, Springer, 2004.
[3] M. Hovey, Model Categories, American Mathematical Society, 1999.

76. Formal Schemes in n-Ality Theory

76.1. **n-Ality Formal Schemes.**

Definition 76.1.1 (n-Ality Formal Scheme)**.** *Let $\{\mathcal{X}_i\}_{i=1}^n$ be a collection of formal schemes, where each $\mathcal{X}_i$ is equipped with a structure sheaf $\mathcal{O}_{\mathcal{X}_i}$ over a base ring R. The n-ality formal scheme $\mathcal{X}^n$ is defined as:*

$$\mathcal{X}^n = \bigoplus_{i=1}^n \mathcal{X}_i,$$

with structure sheaf $\mathcal{O}_{\mathcal{X}^n} = \bigoplus_{i=1}^n \mathcal{O}_{\mathcal{X}_i}$.

Theorem 76.1.2 (Exactness of n-Ality Formal Schemes)**.** *If each $\mathcal{X}_i$ is an exact formal scheme with respect to a given topology, then $\mathcal{X}^n$ is also an exact formal scheme under the same topology.*

Proof. The exactness of each $\mathcal{X}_i$ ensures that $\mathcal{X}^n$, formed by direct sum, inherits this exactness property componentwise. □

76.2. **n-Ality Morphisms of Formal Schemes.**

Definition 76.2.1 (n-Ality Morphism of Formal Schemes)**.** *Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ be morphisms between formal schemes $\mathcal{X}_i$ and $\mathcal{Y}_i$ for each i. The n-ality morphism of formal schemes f^n is given by:*

$$f^n = \bigoplus_{i=1}^n f_i,$$

preserving the formal structure of each $\mathcal{X}_i$.

Theorem 76.2.2 (Exactness of n-Ality Morphisms of Formal Schemes)**.** *If each f_i is exact, then f^n is exact in the category of n-ality formal schemes.*

Proof. The exactness of each morphism f_i implies that the direct sum f^n is exact, preserving formal scheme structures in n-ality. □

77. Derived Motivic Cohomology in n-Ality Theory

77.1. n-Ality Derived Motivic Cohomology.

Definition 77.1.1 (n-Ality Derived Motivic Cohomology Theory). *Let $\{M_i\}_{i=1}^n$ be derived motivic cohomology theories, where each M_i is a motivic spectrum associated with a base scheme X. The n-ality derived motivic cohomology theory M^n is defined by:*

$$M^n(X) = \bigoplus_{i=1}^n M_i(X),$$

where $M_i(X)$ denotes the motivic cohomology groups associated with X and the spectrum M_i.

Theorem 77.1.2 (Exact Sequence in n-Ality Derived Motivic Cohomology). *If each M_i satisfies the axioms of derived motivic cohomology, then M^n satisfies an exact sequence in n-ality motivic cohomology:*

$$\cdots \to M^n(X) \to M^n(Y) \to M^n(Z) \to \cdots$$

for any distinguished triangle $X \to Y \to Z$ in the derived category.

Proof. Since each M_i satisfies exactness in the motivic cohomology sequence, the direct sum M^n preserves this structure componentwise. □

78. Enriched Categories in n-Ality Theory

78.1. n-Ality Enriched Categories.

Definition 78.1.1 (n-Ality Enriched Category). *Let $\{\mathcal{V}_i\}_{i=1}^n$ be a collection of closed symmetric monoidal categories, each serving as the base of enrichment for a category $\mathcal{C}_i$. The n-ality enriched category $\mathcal{C}^n$ is defined by:*

$$\mathcal{C}^n = \bigoplus_{i=1}^n \mathcal{C}_i,$$

with each $\mathcal{C}_i$ enriched over $\mathcal{V}_i$.

Theorem 78.1.2 (Exactness of n-Ality Enriched Functors). *If each functor $F_i : \mathcal{C}_i \to \mathcal{D}_i$ is exact as a $\mathcal{V}_i$-enriched functor, then the induced n-ality enriched functor $F^n : \mathcal{C}^n \to \mathcal{D}^n$ is exact.*

Proof. The exactness of each F_i in $\mathcal{V}_i$-enriched context implies that the direct sum F^n preserves enriched exactness in $\mathcal{C}^n$. □

78.2. n-Ality Hom Objects in Enriched Categories.

Definition 78.2.1 (n-Ality Hom Object). *Let $\mathrm{Hom}_{\mathcal{V}_i}(X_i, Y_i)$ be the hom-object in $\mathcal{V}_i$ for objects $X_i, Y_i \in \mathcal{C}_i$. The n-ality hom-object $\mathrm{Hom}_{\mathcal{V}^n}(X, Y)$ for $X, Y \in \mathcal{C}^n$ is given by:*

$$\mathrm{Hom}_{\mathcal{V}^n}(X, Y) = \bigoplus_{i=1}^n \mathrm{Hom}_{\mathcal{V}_i}(X_i, Y_i).$$

Theorem 78.2.2 (Associativity of n-Ality Hom Objects). *The n-ality hom-object $\mathrm{Hom}_{\mathcal{V}^n}(X, Y)$ inherits associativity from each hom-object $\mathrm{Hom}_{\mathcal{V}_i}(X_i, Y_i)$.*

Proof. Since each $\mathrm{Hom}_{\mathcal{V}_i}(X_i, Y_i)$ is associative, the direct sum structure of $\mathrm{Hom}_{\mathcal{V}^n}(X, Y)$ preserves associativity in n-ality. □

79. DIAGRAMS FOR FORMAL SCHEMES, DERIVED MOTIVIC COHOMOLOGY, AND ENRICHED CATEGORIES IN n-ALITY THEORY

$$\begin{tikzcd} \mathcal{X}_i \arrow[dr, "f_i"'] & & \mathcal{Y}_i \arrow[dl, "f^n"] \\ & \mathcal{X}^n \to \mathcal{Y}^n & \end{tikzcd}$$

Morphism in n-ality formal schemes

FIGURE 14. Diagram of morphism in n-ality formal schemes

80. EXAMPLES OF DERIVED MOTIVIC COHOMOLOGY AND ENRICHED CATEGORIES IN n-ALITY THEORY

Example 80.0.1 (Tri-Ality Formal Schemes for Deformation Theory)**.** *Consider formal schemes $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ representing deformations of smooth varieties. The tri-ality formal scheme $\mathcal{X}^3 = \bigoplus_{i=1}^3 \mathcal{X}_i$ provides a framework for studying deformations across multiple settings.*

Example 80.0.2 (Quater-Ality Derived Motivic Cohomology in Algebraic K-Theory)**.** *Let M_1, M_2, M_3, M_4 be motivic spectra associated with K-theory and cobordism. The quater-ality derived motivic cohomology M^4 provides a combined cohomology theory to study algebraic cycles across different motivic settings.*

Example 80.0.3 (Tri-Ality Enriched Category in Representation Theory)**.** *Consider categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of representations over different fields, each enriched over a base category of vector spaces. The tri-ality enriched category $\mathcal{C}^3$ offers a unified structure for representations over multiple base fields.*

81. FURTHER REFERENCES FOR FORMAL SCHEMES, DERIVED MOTIVIC COHOMOLOGY, AND ENRICHED CATEGORIES IN n-ALITY THEORY

REFERENCES

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82. HIGHER DEFORMATION THEORY IN n-ALITY

82.1. n-Ality Higher Deformation Functors.

Definition 82.1.1 (n-Ality Higher Deformation Functor)**.** *Let $\{\mathrm{Def}_i\}_{i=1}^n$ be a collection of deformation functors defined over a formal base scheme $\mathcal{X}$, where each Def_i describes infinitesimal deformations of an object X_i. The n-ality higher deformation functor Def^n is defined by:*

$$\mathrm{Def}^n = \bigoplus_{i=1}^n \mathrm{Def}_i .$$

Theorem 82.1.2 (Exactness of n-Ality Higher Deformation Functors)**.** *If each deformation functor* Def_i *is exact, then the* n*-ality deformation functor* Def^n *is also exact.*

Proof. Since each Def_i preserves exactness, the direct sum structure of Def^n ensures that exactness is preserved in n-ality. $\square$

82.2. **n-Ality Obstruction Theory in Deformation Contexts.**

Definition 82.2.1 (n-Ality Obstruction Class)**.** *Let* $\mathrm{Obs}_i \in H^2(X_i, T_{X_i})$ *represent obstruction classes in the second cohomology group associated with the tangent sheaf* T_{X_i} *of each* X_i*. The n-ality obstruction class* Obs^n *is defined by:*

$$\mathrm{Obs}^n = \bigoplus_{i=1}^{n} \mathrm{Obs}_i \in \bigoplus_{i=1}^{n} H^2(X_i, T_{X_i}).$$

Theorem 82.2.2 (Vanishing of n-Ality Obstruction Classes)**.** *If each* $\mathrm{Obs}_i = 0$*, then* $\mathrm{Obs}^n = 0$*, indicating that deformations in* n*-ality are unobstructed.*

Proof. Since each obstruction class Obs_i vanishes, the direct sum Obs^n also vanishes, implying unobstructed deformations in n-ality. $\square$

83. Derived Categories of Coherent Sheaves in n-Ality

83.1. **n-Ality Derived Category of Coherent Sheaves.**

Definition 83.1.1 (n-Ality Derived Category of Coherent Sheaves)**.** *Let* $\{D^b(\mathrm{Coh}(\mathcal{X}_i))\}_{i=1}^n$ *denote bounded derived categories of coherent sheaves over a collection of schemes* $\mathcal{X}_i$*. The n-ality derived category of coherent sheaves* $D^b(\mathrm{Coh}(\mathcal{X}^n))$ *is defined by:*

$$D^b(\mathrm{Coh}(\mathcal{X}^n)) = \bigoplus_{i=1}^{n} D^b(\mathrm{Coh}(\mathcal{X}_i)).$$

Theorem 83.1.2 (Exact Sequences in n-Ality Derived Categories of Coherent Sheaves)**.** *If each* $D^b(\mathrm{Coh}(\mathcal{X}_i))$ *satisfies exactness, then* $D^b(\mathrm{Coh}(\mathcal{X}^n))$ *also satisfies exactness in the derived category structure.*

Proof. The exactness of each component category $D^b(\mathrm{Coh}(\mathcal{X}_i))$ implies that the direct sum $D^b(\mathrm{Coh}(\mathcal{X}^n))$ is exact in n-ality. $\square$

83.2. **n-Ality Fourier-Mukai Transforms.**

Definition 83.2.1 (n-Ality Fourier-Mukai Transform)**.** *Let* $\Phi_i : D^b(\mathrm{Coh}(\mathcal{X}_i)) \to D^b(\mathrm{Coh}(\mathcal{Y}_i))$ *be Fourier-Mukai transforms associated with kernels* $\mathcal{K}_i \in D^b(\mathrm{Coh}(\mathcal{X}_i \times \mathcal{Y}_i))$*. The n-ality Fourier-Mukai transform* Φ^n *is defined by:*

$$\Phi^n = \bigoplus_{i=1}^{n} \Phi_i,$$

with kernel $\mathcal{K}^n = \bigoplus_{i=1}^{n} \mathcal{K}_i$*.*

Theorem 83.2.2 (Exactness of n-Ality Fourier-Mukai Transforms)**.** *If each* Φ_i *is exact, then* Φ^n *is an exact functor in the category of* n*-ality derived categories of coherent sheaves.*

Proof. The exactness of each Fourier-Mukai transform Φ_i implies that Φ^n, formed by their direct sum, preserves exactness in n-ality. $\square$

84. Derived Functor Cohomology in Enriched n-Ality Categories

84.1. n-Ality Enriched Derived Functor Cohomology.

Definition 84.1.1 (n-Ality Enriched Derived Functor)**.** *Let* $\mathrm{Ext}^k_{\mathcal{V}_i}(X_i, Y_i)$ *be the* k*-th derived functor in an enriched category* $\mathcal{V}_i$ *for objects* $X_i, Y_i \in \mathcal{C}_i$*. The* *<u>n-ality enriched derived functor</u>* $\mathrm{Ext}^k_{\mathcal{V}^n}(X, Y)$ *is given by:*

$$\mathrm{Ext}^k_{\mathcal{V}^n}(X, Y) = \bigoplus_{i=1}^{n} \mathrm{Ext}^k_{\mathcal{V}_i}(X_i, Y_i).$$

Theorem 84.1.2 (Exactness of n-Ality Enriched Derived Functor Cohomology)**.** *If each* $\mathrm{Ext}^k_{\mathcal{V}_i}$ *satisfies exactness, then the n-ality enriched derived functor* $\mathrm{Ext}^k_{\mathcal{V}^n}$ *is also exact.*

Proof. Since each $\mathrm{Ext}^k_{\mathcal{V}_i}$ preserves exactness, the direct sum structure of $\mathrm{Ext}^k_{\mathcal{V}^n}$ ensures that exactness is preserved in n-ality. □

85. Diagrams for Higher Deformation Theory, Derived Categories, and Enriched Cohomology in n-Ality Theory

$$\begin{tikzcd}
D^b(\mathrm{Coh}(\mathcal{X}_i)) \arrow[dr, "\Phi_i"'] & & D^b(\mathrm{Coh}(\mathcal{Y}_i)) \arrow[dl, "\Phi^n"] \\
& \Phi^n = \bigoplus \Phi_i &
\end{tikzcd}$$

n-Ality Fourier-Mukai Transform Diagram

FIGURE 15. Diagram of the n-ality Fourier-Mukai transform

86. Examples of Higher Deformation Theory, Derived Categories, and Enriched Cohomology in n-Ality Theory

Example 86.0.1 (Tri-Ality Deformation Theory for Complex Varieties)**.** *Consider deformation functors* $\mathrm{Def}_1, \mathrm{Def}_2, \mathrm{Def}_3$ *describing deformations of three complex varieties. The tri-ality deformation functor* $\mathrm{Def}^3 = \bigoplus_{i=1}^3 \mathrm{Def}_i$ *provides a framework for studying deformations in a combined context.*

Example 86.0.2 (Quater-Ality Derived Category of Coherent Sheaves for Moduli Spaces)**.** *Let* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ *be varieties with derived categories of coherent sheaves. The quater-ality derived category* $D^b(\mathrm{Coh}(\mathcal{X}^4)) = \bigoplus_{i=1}^4 D^b(\mathrm{Coh}(\mathcal{X}_i))$ *enables the study of coherent sheaves across multiple moduli spaces.*

Example 86.0.3 (Tri-Ality Enriched Derived Functor Cohomology in Representation Theory)**.** *Let* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ *be categories of representations, each enriched over a monoidal category of modules. The tri-ality enriched derived functor* $\mathrm{Ext}^k_{\mathcal{V}^3}(X, Y)$ *combines derived functor cohomology for these categories.*

87. Further References for Higher Deformation Theory, Derived Categories, and Enriched Cohomology in n-Ality Theory

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88. Higher Derived Categories in n-Ality Theory

88.1. n-Ality Higher Derived Categories.

Definition 88.1.1 (n-Ality Higher Derived Category). *Let $\{D^+(\mathcal{A}_i)\}_{i=1}^n$ be collections of derived categories of bounded below complexes of objects in abelian categories $\mathcal{A}_i$. The n-ality higher derived category $D^+(\mathcal{A}^n)$ is defined by:*

$$D^+(\mathcal{A}^n) = \bigoplus_{i=1}^n D^+(\mathcal{A}_i).$$

Theorem 88.1.2 (Exactness of n-Ality Higher Derived Categories). *If each $D^+(\mathcal{A}_i)$ satisfies exactness in the derived category structure, then $D^+(\mathcal{A}^n)$ also satisfies exactness in the n-ality higher derived category.*

Proof. The exactness of each $D^+(\mathcal{A}_i)$ implies that the direct sum $D^+(\mathcal{A}^n)$ inherits this exact structure, preserving exactness in n-ality. □

88.2. n-Ality Grothendieck Spectral Sequence.

Theorem 88.2.1 (n-Ality Grothendieck Spectral Sequence). *Let $F_i : D^+(\mathcal{A}_i) \to D^+(\mathcal{B}_i)$ and $G_i : D^+(\mathcal{B}_i) \to D^+(\mathcal{C}_i)$ be two functors between derived categories. Then, for each derived category $D^+(\mathcal{A}_i)$, there exists a Grothendieck spectral sequence. The n-ality Grothendieck spectral sequence for $D^+(\mathcal{A}^n)$ is given by:*

$$E_2^{p,q} = \bigoplus_{i=1}^n H^p(F_i H^q(G_i)) \Rightarrow H^{p+q}(G^n \circ F^n),$$

where $F^n = \bigoplus_{i=1}^n F_i$ and $G^n = \bigoplus_{i=1}^n G_i$.

Proof. The existence of a Grothendieck spectral sequence for each F_i and G_i implies the existence of a direct sum spectral sequence in n-ality. □

89. Enriched Homotopy Limits in n-Ality

89.1. n-Ality Enriched Homotopy Limits.

Definition 89.1.1 (n-Ality Enriched Homotopy Limit). *Let $\operatorname{holim}_{I,\mathcal{V}_i} X_i$ denote the homotopy limit in an enriched category $\mathcal{V}_i$ with respect to a diagram $X_i : I \to \mathcal{V}_i$. The n-ality enriched homotopy limit $\operatorname{holim}_{I,\mathcal{V}^n} X$ is given by:*

$$\operatorname{holim}_{I,\mathcal{V}^n} X = \bigoplus_{i=1}^n \operatorname{holim}_{I,\mathcal{V}_i} X_i.$$

Theorem 89.1.2 (Exactness of n-Ality Enriched Homotopy Limits)**.** *If each* $\operatorname{holim}_{I,\mathcal{V}_i} X_i$ *is exact in* $\mathcal{V}_i$*, then* $\operatorname{holim}_{I,\mathcal{V}^n} X$ *is exact in* $\mathcal{V}^n$*.*

Proof. The exactness of each $\operatorname{holim}_{I,\mathcal{V}_i} X_i$ implies that the direct sum $\operatorname{holim}_{I,\mathcal{V}^n} X$ preserves exactness in n-ality. □

90. n-Ality Tensor Structures in Derived Categories

90.1. **n-Ality Tensor Product of Complexes.**

Definition 90.1.1 (n-Ality Tensor Product)**.** *Let* $K_i, L_i \in D^+(\mathcal{A}_i)$ *be bounded below complexes of objects in categories* $\mathcal{A}_i$*, with a derived tensor product* $K_i \otimes^L L_i$*. The* n-ality tensor product $K \otimes^L L$ *is defined as:*

$$K \otimes^L L = \bigoplus_{i=1}^{n} (K_i \otimes^L L_i).$$

Theorem 90.1.2 (Exactness of n-Ality Tensor Products)**.** *If each* $K_i \otimes^L L_i$ *is exact, then* $K \otimes^L L$ *is exact in the* n*-ality tensor structure.*

Proof. The exactness of each derived tensor product $K_i \otimes^L L_i$ implies that the direct sum $K \otimes^L L$ is exact in n-ality. □

90.2. **n-Ality Derived Functor of Tensor Products.**

Definition 90.2.1 (n-Ality Tor Functor)**.** *Let* $\operatorname{Tor}_i^p(K_i, L_i)$ *denote the* p*-th derived functor of the tensor product in* $\mathcal{A}_i$*. The* n-ality Tor functor $\operatorname{Tor}^p(K, L)$ *is given by:*

$$\operatorname{Tor}^p(K, L) = \bigoplus_{i=1}^{n} \operatorname{Tor}_i^p(K_i, L_i).$$

Theorem 90.2.2 (Exactness of n-Ality Tor Functors)**.** *If each* $\operatorname{Tor}_i^p(K_i, L_i)$ *satisfies exactness, then* $\operatorname{Tor}^p(K, L)$ *satisfies exactness in the* n*-ality context.*

Proof. Since each Tor_i^p is exact, the direct sum construction $\operatorname{Tor}^p(K, L)$ maintains exactness in n-ality. □

91. Diagrams for Higher Derived Categories, Enriched Homotopy Limits, and Tensor Structures in n-Ality Theory

$$\begin{tikzcd}
\operatorname{holim}_{I,\mathcal{V}_i} X_i \arrow[dr] & & \operatorname{holim}_{I,\mathcal{V}_j} X_j \arrow[dl] \\
& \operatorname{holim}_{I,\mathcal{V}^n} X &
\end{tikzcd}$$

Diagram of n-Ality Enriched Homotopy Limit

FIGURE 16. Diagram of the n-ality enriched homotopy limit

92. Examples of Higher Derived Categories, Tensor Structures, and Enriched Homotopy Limits in n-Ality Theory

Example 92.0.1 (Tri-Ality Derived Category for Coherent Sheaves)**.** *Consider derived categories* $D^+(\mathrm{Coh}(\mathcal{X}_1)), D^+(\mathrm{Coh}(\mathcal{X}_2)), D^+(\mathrm{Coh}(\mathcal{X}_3))$ *for coherent sheaves. The tri-ality higher derived category* $D^+(\mathrm{Coh}(\mathcal{X}^3)) = \bigoplus_{i=1}^3 D^+(\mathrm{Coh}(\mathcal{X}_i))$ *provides a framework for derived complexes across these spaces.*

Example 92.0.2 (Quater-Ality Tensor Structure in Representation Theory)**.** *Let* K_1, K_2, K_3, K_4 *be complexes of representations over different fields. The quater-ality tensor product* $K \otimes^L L = \bigoplus_{i=1}^4 (K_i \otimes^L L_i)$ *allows for simultaneous tensor operations across the four fields.*

Example 92.0.3 (Tri-Ality Enriched Homotopy Limit in Topological Spaces)**.** *Consider homotopy limits* $\mathrm{holim}_{I,\mathcal{V}_1} X_1$, $\mathrm{holim}_{I,\mathcal{V}_2} X_2$, *and* $\mathrm{holim}_{I,\mathcal{V}_3} X_3$ *in different enriched categories of topological spaces. The tri-ality enriched homotopy limit* $\mathrm{holim}_{I,\mathcal{V}^3} X = \bigoplus_{i=1}^3 \mathrm{holim}_{I,\mathcal{V}_i} X_i$ *enables a unified homotopy limit across the spaces.*

93. Further References for Higher Derived Categories, Enriched Homotopy Limits, and Tensor Structures in n-Ality Theory

References

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94. Spectral Sequences in n-Ality Theory

94.1. **n-Ality Spectral Sequences.**

Definition 94.1.1 (n-Ality Spectral Sequence)**.** *Let* $\{E_r^{p,q}(i)\}_{i=1}^n$ *be a collection of spectral sequences converging to cohomology groups* $H^*(X_i)$*. The* n-ality spectral sequence $E_r^{p,q}(n)$ *is defined by:*

$$E_r^{p,q}(n) = \bigoplus_{i=1}^n E_r^{p,q}(i),$$

which converges to the direct sum cohomology $H^*(X^n) = \bigoplus_{i=1}^n H^*(X_i)$*.*

Theorem 94.1.2 (Convergence of n-Ality Spectral Sequences)**.** *If each spectral sequence* $E_r^{p,q}(i)$ *converges to* $H^*(X_i)$*, then the* n*-ality spectral sequence* $E_r^{p,q}(n)$ *converges to* $H^*(X^n)$*.*

Proof. Since each $E_r^{p,q}(i)$ converges, the direct sum $E_r^{p,q}(n)$ also converges to the cohomology $H^*(X^n)$. □

95. Higher K-Theory in n-Ality Theory

95.1. **n-Ality Higher K-Theory Groups.**

Definition 95.1.1 (n-Ality K-Theory Group)**.** *Let* $K_i^p(X)$ *denote the* p*-th higher K-theory group of a scheme* X *with respect to a category* $\mathcal{C}_i$*. The* n-ality higher K-theory group $K^p(X^n)$ *is defined by:*

$$K^p(X^n) = \bigoplus_{i=1}^n K_i^p(X).$$

Theorem 95.1.2 (Exact Sequence in n-Ality Higher K-Theory)**.** *If each $K_i^p(X)$ satisfies the axioms of higher K-theory, then $K^p(X^n)$ satisfies an exact sequence:*

$$\cdots \to K^p(X) \to K^p(Y) \to K^p(Z) \to \cdots$$

for any distinguished triangle $X \to Y \to Z$.

Proof. The exactness of each $K_i^p(X)$ implies that the direct sum $K^p(X^n)$ inherits this exact sequence structure in n-ality. □

95.2. **n-Ality Chern Character in Higher K-Theory.**

Definition 95.2.1 (n-Ality Chern Character)**.** *Let $\mathrm{ch}_i : K_i^p(X) \to H^{2p}(X, \mathbb{Q})$ be the Chern character associated with the i-th higher K-theory group. The n-ality Chern character ch^n is defined by:*

$$\mathrm{ch}^n = \bigoplus_{i=1}^{n} \mathrm{ch}_i : K^p(X^n) \to \bigoplus_{i=1}^{n} H^{2p}(X_i, \mathbb{Q}).$$

Theorem 95.2.2 (Exactness of n-Ality Chern Character)**.** *If each ch_i is exact, then the n-ality Chern character ch^n is exact in n-ality.*

Proof. The exactness of each ch_i implies that ch^n maintains exactness for the higher K-theory groups in n-ality. □

96. Derived Functors in Homotopical n-Ality Contexts

96.1. **n-Ality Derived Functor Homology.**

Definition 96.1.1 (n-Ality Derived Functor Homology)**.** *Let $\mathrm{R}_i\, F$ denote the right-derived functor of a functor F_i on a category $\mathcal{A}_i$. The n-ality derived functor homology $\mathrm{R}^n\, F$ is defined by:*

$$\mathrm{R}^n\, F = \bigoplus_{i=1}^{n} \mathrm{R}_i\, F.$$

Theorem 96.1.2 (Exactness of n-Ality Derived Functor Homology)**.** *If each $\mathrm{R}_i\, F$ is exact, then $\mathrm{R}^n\, F$ is exact in n-ality.*

Proof. The exactness of each $\mathrm{R}_i\, F$ ensures that the direct sum $\mathrm{R}^n\, F$ preserves exactness in n-ality. □

97. Diagrams for Spectral Sequences, Higher K-Theory, and Derived Functors in n-Ality Theory

$$\begin{tikzcd}
E_r^{p,q}(i) \arrow[dr] & & E_r^{p,q}(j) \arrow[dl] \\
& E_r^{p,q}(n) = \bigoplus E_r^{p,q}(i) &
\end{tikzcd}$$

Diagram of n-Ality Spectral Sequence Convergence

FIGURE 17. Diagram of the n-ality spectral sequence convergence

98. Examples of Spectral Sequences, Higher K-Theory, and Derived Functors in n-Ality Theory

Example 98.0.1 (Tri-Ality Spectral Sequence for Derived Categories)**.** *Consider spectral sequences $E_r^{p,q}(1), E_r^{p,q}(2), E_r^{p,q}(3)$ converging to cohomology groups of three derived categories. The tri-ality spectral sequence $E_r^{p,q}(3) = \bigoplus_{i=1}^{3} E_r^{p,q}(i)$ converges to the direct sum cohomology groups.*

Example 98.0.2 (Quater-Ality Higher K-Theory for Algebraic Varieties)**.** *Let $K_1^p(X), K_2^p(X), K_3^p(X), K_4^p(X)$ be higher K-theory groups for four algebraic varieties. The quater-ality higher K-theory group $K^p(X^4) = \bigoplus_{i=1}^{4} K_i^p(X)$ enables combined K-theory computations.*

Example 98.0.3 (Tri-Ality Derived Functor Homology in Homotopy Theory)**.** *Consider right-derived functors $\mathrm{R}_1 F, \mathrm{R}_2 F, \mathrm{R}_3 F$ for functors in homotopy theory. The tri-ality derived functor homology $\mathrm{R}^3 F = \bigoplus_{i=1}^{3} \mathrm{R}_i F$ combines the derived homologies across homotopical contexts.*

99. Further References for Spectral Sequences, Higher K-Theory, and Derived Functors in n-Ality Theory

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100. Derived Categories of D-Modules in n-Ality Theory

100.1. n-Ality Derived Category of D-Modules.

Definition 100.1.1 (n-Ality Derived Category of D-Modules)**.** *Let $\{D^b(\mathrm{Mod}(\mathcal{D}_{X_i}))\}_{i=1}^n$ be the bounded derived categories of $\mathcal{D}_{X_i}$-modules on smooth varieties X_i, where $\mathcal{D}_{X_i}$ denotes the sheaf of differential operators. The n-ality derived category of D-modules $D^b(\mathrm{Mod}(\mathcal{D}_{X^n}))$ is defined by:*

$$D^b(\mathrm{Mod}(\mathcal{D}_{X^n})) = \bigoplus_{i=1}^{n} D^b(\mathrm{Mod}(\mathcal{D}_{X_i})).$$

Theorem 100.1.2 (Exactness in n-Ality Derived Categories of D-Modules)**.** *If each $D^b(\mathrm{Mod}(\mathcal{D}_{X_i}))$ is exact, then $D^b(\mathrm{Mod}(\mathcal{D}_{X^n}))$ is also exact in the n-ality derived category structure.*

Proof. Since each derived category $D^b(\mathrm{Mod}(\mathcal{D}_{X_i}))$ is exact, their direct sum $D^b(\mathrm{Mod}(\mathcal{D}_{X^n}))$ inherits exactness, maintaining it in n-ality. □

100.2. n-Ality D-Module Functors.

Definition 100.2.1 (n-Ality D-Module Functor)**.** *Let $F_i : D^b(\mathrm{Mod}(\mathcal{D}_{X_i})) \to D^b(\mathrm{Mod}(\mathcal{D}_{Y_i}))$ denote functors on D-modules. The n-ality D-module functor F^n is defined by:*

$$F^n = \bigoplus_{i=1}^{n} F_i.$$

Theorem 100.2.2 (Exactness of n-Ality D-Module Functors)**.** *If each functor F_i is exact, then F^n is exact in the category of n-ality D-modules.*

Proof. The exactness of each F_i implies that the direct sum F^n is also exact in the n-ality D-module structure. □

101. Derived Categories of Quasi-Coherent Sheaves in n-Ality Theory

101.1. n-Ality Derived Category of Quasi-Coherent Sheaves.

Definition 101.1.1 (n-Ality Derived Category of Quasi-Coherent Sheaves). *Let $\{D^+(\mathrm{QCoh}(\mathcal{X}_i))\}_{i=1}^n$ denote the bounded below derived categories of quasi-coherent sheaves on schemes $\mathcal{X}_i$. The n-ality derived category of quasi-coherent sheaves $D^+(\mathrm{QCoh}(\mathcal{X}^n))$ is defined by:*

$$D^+(\mathrm{QCoh}(\mathcal{X}^n)) = \bigoplus_{i=1}^n D^+(\mathrm{QCoh}(\mathcal{X}_i)).$$

Theorem 101.1.2 (Exact Sequences in n-Ality Derived Categories of Quasi-Coherent Sheaves). *If each $D^+(\mathrm{QCoh}(\mathcal{X}_i))$ satisfies exactness, then $D^+(\mathrm{QCoh}(\mathcal{X}^n))$ satisfies exactness in n-ality.*

Proof. The exactness of each component category $D^+(\mathrm{QCoh}(\mathcal{X}_i))$ implies that the direct sum $D^+(\mathrm{QCoh}(\mathcal{X}^n))$ is exact in n-ality. □

101.2. n-Ality Pushforward and Pullback Functors.

Definition 101.2.1 (n-Ality Pushforward Functor). *Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ be morphisms of schemes, with pushforward functors $f_{i*} : D^+(\mathrm{QCoh}(\mathcal{X}_i)) \to D^+(\mathrm{QCoh}(\mathcal{Y}_i))$. The n-ality pushforward functor f_*^n is defined by:*

$$f_*^n = \bigoplus_{i=1}^n f_{i*}.$$

Theorem 101.2.2 (Exactness of n-Ality Pushforward Functors). *If each f_{i*} is exact, then f_*^n is exact in n-ality.*

Proof. Since each f_{i*} is exact, their direct sum f_*^n preserves exactness in n-ality. □

102. Derived Deformation Quantization in n-Ality Theory

102.1. n-Ality Deformation Quantization Modules.

Definition 102.1.1 (n-Ality Deformation Quantization Module). *Let $\mathcal{O}_{\hbar,X_i}$ denote a deformation quantization of the structure sheaf $\mathcal{O}_{X_i}$ of a smooth scheme X_i, where $\hbar$ represents a formal parameter. The n-ality deformation quantization module $\mathcal{O}_{\hbar,X^n}$ is defined by:*

$$\mathcal{O}_{\hbar,X^n} = \bigoplus_{i=1}^n \mathcal{O}_{\hbar,X_i}.$$

Theorem 102.1.2 (Existence of n-Ality Deformation Quantization). *If each $\mathcal{O}_{\hbar,X_i}$ exists as a deformation quantization, then $\mathcal{O}_{\hbar,X^n}$ exists as a deformation quantization in n-ality.*

Proof. The existence of each $\mathcal{O}_{\hbar,X_i}$ as a deformation quantization implies that their direct sum $\mathcal{O}_{\hbar,X^n}$ also exists in the n-ality context. □

102.2. **n-Ality Star Product.**

Definition 102.2.1 (n-Ality Star Product)**.** *Let $\star_i$ denote the star product for deformation quantization on X_i associated with $\mathcal{O}_{\hbar,X_i}$. The n-ality star product $\star^n$ on X^n is defined by:*

$$f \star^n g = \bigoplus_{i=1}^{n} (f_i \star_i g_i),$$

where $f = \bigoplus_{i=1}^{n} f_i$ and $g = \bigoplus_{i=1}^{n} g_i$.

Theorem 102.2.2 (Associativity of n-Ality Star Product)**.** *If each $\star_i$ is associative, then $\star^n$ is associative in n-ality.*

Proof. The associativity of each $\star_i$ implies that the direct sum $\star^n$ is associative in n-ality. □

103. Diagrams for D-Modules, Quasi-Coherent Sheaves, and Deformation Quantization in n-Ality Theory

$$\begin{tikzcd} D^+(\mathrm{QCoh}(\mathcal{X}_i)) \arrow[dr, "f_{i*}"'] & & D^+(\mathrm{QCoh}(\mathcal{Y}_i)) \arrow[dl, "f_*^n"] \\ & f_*^n = \bigoplus f_{i*} & \end{tikzcd}$$

Diagram of n-Ality Pushforward Functor

FIGURE 18. Diagram of the n-ality pushforward functor in quasi-coherent sheaves

104. Examples of D-Modules, Quasi-Coherent Sheaves, and Deformation Quantization in n-Ality Theory

Example 104.0.1 (Tri-Ality D-Module Categories on Algebraic Varieties)**.** *Consider derived categories of D-modules $D^b(\mathrm{Mod}(\mathcal{D}_{X_1})), D^b(\mathrm{Mod}(\mathcal{D}_{X_2})), D^b(\mathrm{Mod}(\mathcal{D}_{X_3}))$. The tri-ality derived category $D^b(\mathrm{Mod}(\mathcal{D}_{X^3})) = \bigoplus_{i=1}^{3} D^b(\mathrm{Mod}(\mathcal{D}_{X_i}))$ provides a framework for analyzing differential operators on these varieties.*

Example 104.0.2 (Quater-Ality Derived Category of Quasi-Coherent Sheaves)**.** *Let $D^+(\mathrm{QCoh}(\mathcal{X}_1)), D^+(\mathrm{QCoh}($ be derived categories of quasi-coherent sheaves on schemes. The quater-ality derived category $D^+(\mathrm{QCoh}(\mathcal{X}^4))$ provides a comprehensive framework for quasi-coherent sheaf theory across multiple schemes.*

Example 104.0.3 (Tri-Ality Deformation Quantization for Symplectic Varieties)**.** *Consider deformation quantizations $\mathcal{O}_{\hbar,X_1}, \mathcal{O}_{\hbar,X_2}, \mathcal{O}_{\hbar,X_3}$ on symplectic varieties X_1, X_2, X_3. The tri-ality deformation quantization $\mathcal{O}_{\hbar,X^3} = \bigoplus_{i=1}^{3} \mathcal{O}_{\hbar,X_i}$ provides a unified setting for deformation quantization across these varieties.*

105. Further References for D-Modules, Quasi-Coherent Sheaves, and Deformation Quantization in n-Ality Theory

References

[1] A. Beilinson and J. Bernstein, Localization of $\mathfrak{g}$-Modules, Springer, 1981.
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[3] M. Kontsevich, Deformation Quantization of Poisson Manifolds, Letters in Mathematical Physics, 2003.

106. Derived Categories of Perverse Sheaves in n-Ality Theory

106.1. n-Ality Derived Category of Perverse Sheaves.

Definition 106.1.1 (n-Ality Derived Category of Perverse Sheaves). *Let $\{D^b(\mathrm{Perv}(X_i))\}_{i=1}^n$ be the bounded derived categories of perverse sheaves on stratified varieties X_i. The n-ality derived category of perverse sheaves $D^b(\mathrm{Perv}(X^n))$ is defined by:*

$$D^b(\mathrm{Perv}(X^n)) = \bigoplus_{i=1}^n D^b(\mathrm{Perv}(X_i)).$$

Theorem 106.1.2 (Exactness in n-Ality Derived Categories of Perverse Sheaves). *If each $D^b(\mathrm{Perv}(X_i))$ is exact, then $D^b(\mathrm{Perv}(X^n))$ is also exact in the n-ality structure.*

Proof. Since each $D^b(\mathrm{Perv}(X_i))$ is exact, the direct sum $D^b(\mathrm{Perv}(X^n))$ inherits this exactness in n-ality. $\square$

106.2. n-Ality Intersection Complexes.

Definition 106.2.1 (n-Ality Intersection Complex). *Let IC_{X_i} denote the intersection complex associated with a stratified variety X_i. The n-ality intersection complex IC_{X^n} is defined by:*

$$\mathrm{IC}_{X^n} = \bigoplus_{i=1}^n \mathrm{IC}_{X_i}.$$

Theorem 106.2.2 (Exactness of n-Ality Intersection Complexes). *If each IC_{X_i} is exact, then IC_{X^n} is exact in n-ality.*

Proof. The exactness of each IC_{X_i} implies that the direct sum IC_{X^n} maintains exactness in the n-ality context. $\square$

107. Higher Chow Groups in n-Ality Theory

107.1. n-Ality Higher Chow Groups.

Definition 107.1.1 (n-Ality Higher Chow Group). *Let $\{\mathrm{CH}_i^p(X,q)\}_{i=1}^n$ denote the higher Chow groups associated with codimension p cycles on varieties X_i and dimension q. The n-ality higher Chow group $\mathrm{CH}^p(X^n,q)$ is defined by:*

$$\mathrm{CH}^p(X^n,q) = \bigoplus_{i=1}^n \mathrm{CH}_i^p(X,q).$$

Theorem 107.1.2 (Exactness in n-Ality Higher Chow Groups). *If each $\mathrm{CH}_i^p(X,q)$ satisfies exactness in the higher Chow group structure, then $\mathrm{CH}^p(X^n,q)$ is exact in n-ality.*

Proof. Since each $\mathrm{CH}_i^p(X,q)$ is exact, the direct sum $\mathrm{CH}^p(X^n,q)$ maintains exactness in n-ality. $\square$

107.2. **n-Ality Cycle Class Map.**

Definition 107.2.1 (n-Ality Cycle Class Map)**.** *Let* $\mathrm{cl}_i : \mathrm{CH}_i^p(X, q) \to H^{2p-q}(X_i, \mathbb{Q})$ *denote the cycle class map associated with the* i*-th higher Chow group. The* <u>*n-ality cycle class map*</u> cl^n *is given by:*

$$\mathrm{cl}^n = \bigoplus_{i=1}^{n} \mathrm{cl}_i : \mathrm{CH}^p(X^n, q) \to \bigoplus_{i=1}^{n} H^{2p-q}(X_i, \mathbb{Q}).$$

Theorem 107.2.2 (Exactness of n-Ality Cycle Class Maps)**.** *If each* cl_i *is exact, then* cl^n *is exact in* n*-ality.*

Proof. The exactness of each cycle class map cl_i implies that the direct sum cl^n preserves exactness in n-ality. □

108. Derived Categories of dg-Algebras in n-Ality Theory

108.1. **n-Ality Derived Category of dg-Algebras.**

Definition 108.1.1 (n-Ality Derived Category of dg-Algebras)**.** *Let* $\{D^b(\mathrm{dg}(\mathcal{A}_i))\}_{i=1}^n$ *be the bounded derived categories of differential graded (dg) algebras* $\mathcal{A}_i$*. The* <u>*n-ality derived category of dg-algebras*</u> $D^b(\mathrm{dg}(\mathcal{A}^n))$ *is defined by:*

$$D^b(\mathrm{dg}(\mathcal{A}^n)) = \bigoplus_{i=1}^{n} D^b(\mathrm{dg}(\mathcal{A}_i)).$$

Theorem 108.1.2 (Exactness in n-Ality Derived Categories of dg-Algebras)**.** *If each* $D^b(\mathrm{dg}(\mathcal{A}_i))$ *satisfies exactness, then* $D^b(\mathrm{dg}(\mathcal{A}^n))$ *also satisfies exactness in* n*-ality.*

Proof. Since each component category $D^b(\mathrm{dg}(\mathcal{A}_i))$ is exact, the direct sum $D^b(\mathrm{dg}(\mathcal{A}^n))$ inherits exactness in n-ality. □

108.2. **n-Ality Derived Functor of Tensor Products in dg-Algebras.**

Definition 108.2.1 (n-Ality Tor Functor in dg-Algebras)**.** *Let* $\mathrm{Tor}_i^p(K_i, L_i)$ *denote the* p*-th derived functor of the tensor product in the category of dg-algebras* $\mathcal{A}_i$*. The* <u>*n-ality Tor functor*</u> $\mathrm{Tor}^p(K, L)$ *is defined by:*

$$\mathrm{Tor}^p(K, L) = \bigoplus_{i=1}^{n} \mathrm{Tor}_i^p(K_i, L_i).$$

Theorem 108.2.2 (Exactness of n-Ality Tor Functor in dg-Algebras)**.** *If each* $\mathrm{Tor}_i^p(K_i, L_i)$ *satisfies exactness, then* $\mathrm{Tor}^p(K, L)$ *is exact in the* n*-ality context.*

Proof. The exactness of each derived functor Tor_i^p ensures that the direct sum $\mathrm{Tor}^p(K, L)$ maintains exactness in n-ality. □

109. Diagrams for Perverse Sheaves, Higher Chow Groups, and dg-Algebras in n-Ality Theory

110. Examples of Perverse Sheaves, Higher Chow Groups, and dg-Algebras in n-Ality Theory

Example 110.0.1 (Tri-Ality Perverse Sheaves on Stratified Varieties)**.** *Consider derived categories of perverse sheaves* $D^b(\mathrm{Perv}(X_1)), D^b(\mathrm{Perv}(X_2)), D^b(\mathrm{Perv}(X_3))$ *on three stratified varieties.*

$$\begin{tikzcd} \mathrm{CH}^p_i(X,q) \arrow[dr, "\mathrm{cl}_i"'] & & H^{2p-q}(X_i,\mathbb{Q}) \arrow[dl, "\mathrm{cl}^n"] \\ & \mathrm{cl}^n = \bigoplus \mathrm{cl}_i & \end{tikzcd}$$

Diagram of n-Ality Cycle Class Map in Higher Chow Groups

FIGURE 19. Diagram of the n-ality cycle class map in higher Chow groups

The tri-ality derived category $D^b(\mathrm{Perv}(X^3)) = \bigoplus_{i=1}^{3} D^b(\mathrm{Perv}(X_i))$ *combines the perverse sheaf structures across the varieties.*

Example 110.0.2 (Quater-Ality Higher Chow Groups for Algebraic Cycles)**.** *Let* $\mathrm{CH}^p_1(X,q), \mathrm{CH}^p_2(X,q), \mathrm{CH}^p_3(X,$ *represent higher Chow groups of algebraic cycles. The quater-ality higher Chow group* $\mathrm{CH}^p(X^4,q) = \bigoplus_{i=1}^{4} \mathrm{CH}^p_i(X,q)$ *provides a unified framework for cycles across varieties.*

Example 110.0.3 (Tri-Ality dg-Algebras in Derived Categories)**.** *Consider derived categories* $D^b(\mathrm{dg}(\mathcal{A}_1)), D^b(\mathrm{dg}(\mathcal{A}_2)), D^b(\mathrm{dg}(\mathcal{A}_3))$ *of dg-algebras. The tri-ality derived category* $D^b(\mathrm{dg}(\mathcal{A}^3)) = \bigoplus_{i=1}^{3} D^b(\mathrm{dg}(\mathcal{A}_i))$ *offers a combined setting for dg-algebraic computations.*

111. FURTHER REFERENCES FOR PERVERSE SHEAVES, HIGHER CHOW GROUPS, AND DG-ALGEBRAS IN n-ALITY THEORY

REFERENCES

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