CATEGORIFIED SYNTOMIC ENTROPY MOTIVES AND ARITHMETIC THERMODYNAMICS

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Abstract. We construct a theory of categorified syntomic entropy motives: derived sheaves governed by Frobenius-periodic dynamics, Nygaard filtrations, and entropy flow functionals. These motives encode arithmetic thermodynamic data over p-adic and quantum bases. We develop a six-functor formalism for entropy-motivic stacks, construct entropy heat cohomology, and define arithmetic thermodynamic invariants via filtered trace descent. This framework provides a categorified model for zeta-flow quantization and entropy-periodic Langlands deformations.

Contents

1. Introduction

The unification of arithmetic geometry, cohomological thermodynamics, and zeta-periodic quantum structures gives rise to a new class of motivic objects: *categorified syntomic entropy motives*. These extend syntomic and prismatic cohomology to derived sheaf theories governed by entropy-weighted Frobenius dynamics, and encode quantum thermal behavior of arithmetic fields.

Key components of this theory include:

- Categorified filtered complexes with entropy-modulated trace flows;
- Derived stacks of entropy motives with six-functor operations;

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- Quantum thermal field structures over p-adic motives;
- Zeta-crystal realizations via entropy-regulated Nygaard descent.

These structures unify the ideas of entropy zeta flows, Frobenius dynamics, and thermodynamic descent within the framework of categorified arithmetic motives.

2. Entropy-Motivic Stacks and Frobenius-Filtered Categories

We now define the foundational category of entropy motives as filtered objects in the derived setting equipped with Frobenius-periodic flows and entropy functionals. These form the building blocks for categorified motivic sheaves over prismatic, syntomic, or zeta-theoretic moduli. We organize them into derived stacks admitting six-functor formalisms and thermodynamic trace flows.

2.1. **Categorified Frobenius-Filtered Motives.**

Let R be a derived p-adic base (e.g., a perfect formal scheme or condensed $\mathbb{Z}_p$-algebra). We define the category of entropy motives over R as follows:

Definition 2.1 (Category of Categorified Entropy Motives). *Let $\mathcal{M}^{\mathrm{ent}}(R)$ denote the stable ∞-category whose objects are triples $(C^\bullet, \varphi, \mathcal{S})$ where:*

- $C^\bullet \in \mathcal{D}_{\mathrm{perf}}(R)$ *is a perfect derived complex;*
- $\varphi : C^\bullet \to C^\bullet$ *is a Frobenius-periodic endomorphism;*
- $\mathcal{S} : C^\bullet \mapsto \mathbb{R}_{\geq 0}$ *is an entropy functional satisfying flow-invariance:*

$$\mathcal{S}(\varphi(C^\bullet)) = \mathcal{S}(C^\bullet).$$

Remark 2.2. The functional $\mathcal{S}$ induces a thermodynamic grading or norm on cohomological sheaves, and enables interpretation of syntomic filtrations in terms of entropy growth or heat decay.

2.2. **Entropy-Motivic Stacks.**

We globalize the category of entropy motives over a base stack X.

Definition 2.3 (Entropy-Motivic Stack). *Let X be a derived p-adic stack. The* entropy-motivic stack $\mathfrak{EM}_X$ *is the moduli stack:*

$$\mathfrak{EM}_X := \{(C^\bullet, \varphi, \mathcal{S})\}$$

where $(C^\bullet, \varphi, \mathcal{S})$ varies in $\mathcal{M}^{\mathrm{ent}}(\mathcal{O}_X)$.

Example 2.4. *Let $X = \mathrm{Spf}(\mathbb{Z}_p)$. Then $\mathfrak{EM}_X$ parametrizes entropy-weighted filtered Frobenius motives, whose periods are encoded in $\mathsf{Prism}_{/\mathbb{Z}_p}$ and whose dynamics follow entropy-deformed trace flow.*

2.3. **Frobenius–Entropy Flow and Sheaf Dynamics.**

Definition 2.5 (Frobenius–Entropy Operator)**.** *For any* $(C^\bullet, \varphi, \mathcal{S}) \in \mathfrak{EM}_X$*, define the* Frobenius–entropy operator*:*

$$\Delta_{\varphi,\mathcal{S}} := \log_\varphi - \nabla_{\mathcal{S}}$$

where $\log_\varphi := \sum_{n=1}^{\infty} \frac{1}{n}(\varphi^n - \mathrm{id})$ *and* $\nabla_{\mathcal{S}}$ *is the entropy gradient operator derived from* $\mathcal{S}$*.*

Remark 2.6. This operator governs the infinitesimal evolution of $C^\bullet$ under both Frobenius flow and entropy deformation, and defines a categorified analogue of the Hamiltonian in arithmetic thermodynamics.

2.4. **Functoriality and Six-Operation Framework.**

The stack $\mathfrak{EM}_X$ supports a six-functor formalism:

Theorem 2.7 (Entropy Motivic Six Functors)**.** *Let* $f : X \to Y$ *be a morphism of derived stacks. Then there exist adjoint functors:*

$$f^*, f_*, f^!, f_!, \otimes, \mathbb{R}\mathcal{H}om$$

on the category $\mathfrak{EM}_X$*, satisfying:*

- *Compatibility with entropy filtrations and Frobenius-periodic structure;*
- *Base change and projection formulas in the filtered derived category;*
- *Entropy-trace descent under syntomic and crystalline maps.*

Sketch. Extend the classical six-functor formalism on $\mathcal{D}_{\mathrm{qc}}(X)$ to filtered Frobenius sheaves with entropy deformation via derived descent and solid completion techniques. □