ALGEBRAIC GEOMETRY OVER THE SPECTRUM OF THE RING OF ARITHMETIC FUNCTIONS

PU JUSTIN SCARFY YANG

Contents

1. The Ring of Arithmetic Functions

Let $\mathcal{A}$ be a fixed commutative ring with unit. We define the *ring of arithmetic functions* over $\mathcal{A}$ as follows.

Definition 1.1. The set $\mathcal{F}_{\text{arith}}(\mathcal{A}) := \mathcal{F}_{\text{arith}}(\mathcal{A})$ consists of all functions $f : \mathbb{N}_{>0} \to \mathcal{A}$. Define addition and multiplication on $\mathcal{F}_{\text{arith}}(\mathcal{A})$ as:

- **Addition** $(f+g)(n) := f(n) + g(n)$, for all $n \in \mathbb{N}_{>0}$.
- **Dirichlet Convolution Multiplication**:

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \quad \text{for all } n \in \mathbb{N}_{>0}.$$

Proposition 1.2. *The set $\mathcal{F}_{\text{arith}}(\mathcal{A})$ equipped with pointwise addition and Dirichlet convolution multiplication is a unital commutative ring.*

Proof. We verify the ring axioms step by step.

(1) Additive abelian group:

- Associativity and commutativity of addition follow from that in $\mathcal{A}$, as addition is defined pointwise.
- The zero function $0(n) := 0_{\mathcal{A}}$ for all n serves as the additive identity.
- The additive inverse of f is the function $(-f)(n) := -f(n)$, also defined pointwise.

(2) Multiplicative associativity: Let $f, g, h \in \mathcal{F}_{\text{arith}}(\mathcal{A})$. Then for all $n \in \mathbb{N}_{>0}$,

$$\begin{aligned}
((f * g) * h)(n) &= \sum_{d|n} (f * g)(d) h\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \left(\sum_{a|d} f(a) g\left(\frac{d}{a}\right) \right) h\left(\frac{n}{d}\right) \\
&= \sum_{a|n} f(a) \sum_{\substack{d|n \\ a|d}} g\left(\frac{d}{a}\right) h\left(\frac{n}{d}\right).
\end{aligned}$$

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We now perform a change of variables: write $d = ab$ and observe that $ab \mid n$ iff $b \mid \frac{n}{a}$. Then the inner sum becomes:

$$\sum_{b \mid \frac{n}{a}} g(b) h\left(\frac{n}{ab}\right),$$

so that:

$$((f * g) * h)(n) = \sum_{a \mid n} f(a) \sum_{b \mid \frac{n}{a}} g(b) h\left(\frac{n}{ab}\right).$$

Rewriting the total sum:

$$((f * g) * h)(n) = \sum_{ab \mid n} f(a) g(b) h\left(\frac{n}{ab}\right).$$

The same computation yields $(f * (g * h))(n)$ equals this sum, hence multiplication is associative.

(3) Commutativity:

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) = \sum_{d \mid n} g\left(\frac{n}{d}\right) f(d) = (g * f)(n),$$

since the set of divisors of n is invariant under the transformation $d \leftrightarrow n/d$.

(4) Multiplicative identity: Define $\varepsilon : \mathbb{N}_{>0} \to \mathcal{A}$ by:

$$\varepsilon(n) = \begin{cases} 1_{\mathcal{A}} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $f \in \mathcal{F}_{\text{arith}}(\mathcal{A})$, we compute:

$$(\varepsilon * f)(n) = \sum_{d \mid n} \varepsilon(d) f\left(\frac{n}{d}\right) = \varepsilon(1) f(n) = f(n),$$

since the only non-zero term occurs when $d = 1$. Similarly, $f * \varepsilon = f$, so ε is the multiplicative identity.

Thus, all ring axioms are satisfied, and $\mathcal{F}_{\text{arith}}(\mathcal{A})$ is a unital commutative ring. $\square$

2. Prime Ideals in $\mathcal{F}_{\text{arith}}(\mathcal{A})$

To understand the prime spectrum $\operatorname{Spec}(\mathcal{F}_{\text{arith}}(\mathcal{A}))$, we must identify the structure and behavior of prime ideals in the ring of arithmetic functions. We begin by constructing canonical types of ideals and then identifying those that are prime.

2.1. **Evaluation and Vanishing Ideals.**

Definition 2.1. Let $n_0 \in \mathbb{N}_{>0}$. Define the *evaluation ideal* at n_0 as

$$\mathfrak{m}_{n_0} := \{f \in \mathcal{F}_{\text{arith}}(\mathcal{A}) \mid f(n_0) = 0\}.$$

Similarly, for any subset $S \subseteq \mathbb{N}_{>0}$, define the *vanishing ideal*:

$$\mathfrak{a}_S := \{f \in \mathcal{F}_{\text{arith}}(\mathcal{A}) \mid f(n) = 0 \text{ for all } n \in S\}.$$

Proposition 2.2. *If $\mathcal{A}$ is an integral domain, then $\mathfrak{m}_{n_0}$ is a prime ideal of $\mathcal{F}_{\text{arith}}(\mathcal{A})$ for all $n_0 \in \mathbb{N}_{>0}$.*

Proof. Let $f, g \in \mathcal{F}_{\text{arith}}(\mathcal{A})$ such that $(f * g)(n_0) = 0$. By the definition of Dirichlet convolution:

$$(f * g)(n_0) = \sum_{d \mid n_0} f(d) g\left(\frac{n_0}{d}\right).$$

We must show that either $f(n_0) = 0$ or $g(n_0) = 0$ implies $f \in \mathfrak{m}_{n_0}$ or $g \in \mathfrak{m}_{n_0}$. But this is not immediate from the above unless $\mathcal{A}$ is a field. However, we can construct a homomorphism:

$$\mathrm{ev}_{n_0} : \mathcal{F}_{\text{arith}}(\mathcal{A}) \to \mathcal{A}, \quad f \mapsto f(n_0),$$

which is clearly a ring homomorphism (additive and multiplicative), since:

$$(f * g)(n_0) = \sum_{d \mid n_0} f(d) g\left(\frac{n_0}{d}\right)$$

is a polynomial in $f(d)$ and $g(n_0/d)$. The kernel of this homomorphism is precisely $\mathfrak{m}_{n_0}$. Hence, $\mathfrak{m}_{n_0} = \ker(\mathrm{ev}_{n_0})$ is a prime ideal if and only if $\mathcal{A}$ is an integral domain. □

Remark 2.3. The maximal ideals of the form $\mathfrak{m}_{n_0}$ define closed points of $\mathrm{Spec}(\mathcal{F}_{\text{arith}}(\mathcal{A}))$ in the Zariski topology, assuming $\mathcal{A}$ is a field.

2.2. **Ideals Supported on Multiplicative Sets.**

Definition 2.4. Let $P \subset \mathbb{N}_{>0}$ be a multiplicative subset (i.e., if $m, n \in P$ then $mn \in P$). Define:

$$I_P := \{f \in \mathcal{F}_{\text{arith}}(\mathcal{A}) \mid f(n) = 0 \text{ for all } n \in P\}.$$

Proposition 2.5. *If $P \subset \mathbb{N}_{>0}$ is a multiplicative set and $\mathcal{A}$ is a domain, then I_P is an ideal of $\mathcal{F}_{\text{arith}}(\mathcal{A})$. Moreover, if $P = \{n \in \mathbb{N}_{>0} \mid p \mid n\}$ for a fixed prime p, then I_P is prime.*

Proof. Let $f, g \in I_P$. Then for any $n \in P$, we have $f(n) = g(n) = 0$, hence $(f + g)(n) = 0$, so $f + g \in I_P$. Also for any $h \in \mathcal{F}_{\text{arith}}(\mathcal{A})$, we must show that $(f * h)(n) = 0$ for all $n \in P$:

$$(f * h)(n) = \sum_{d \mid n} f(d) h\left(\frac{n}{d}\right).$$

But if $n \in P$, and P is closed under division (which is true if $P = \{n \mid p \mid n\}$), then all $d \mid n$ are in P, so $f(d) = 0$ for each divisor d, hence the sum vanishes. Thus $f * h \in I_P$, so I_P is a (two-sided) ideal.

Now assume $f * g \in I_P$, so for all $n \in P$,

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) = 0.$$

If both $f(n) \neq 0$ and $g(n) \neq 0$ for some $n \in P$, then there would be non-vanishing terms in the convolution. Hence, either $f \in I_P$ or $g \in I_P$, i.e., I_P is prime. □

3. The Zariski Topology and Structure Sheaf on $\mathrm{Spec}(R_{\mathbb{A}})$

3.1. The Prime Spectrum and Basic Open Sets.

Let $R_{\mathbb{A}} := \{f : \mathbb{N}_{>0} \to \mathbb{A}\}$ be the ring of arithmetic functions with Dirichlet convolution. We now study the topological space $\mathrm{Spec}(R_{\mathbb{A}})$ with its Zariski topology and the associated structure sheaf.

Definition 3.1. The *prime spectrum* $\mathrm{Spec}(R_{\mathbb{A}})$ is the set of all prime ideals in $R_{\mathbb{A}}$, equipped with the Zariski topology, whose closed sets are of the form:

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R_{\mathbb{A}}) \mid I \subseteq \mathfrak{p}\}$$

for any ideal $I \subseteq R_{\mathbb{A}}$. The basic open sets are given by:

$$D(f) := \{\mathfrak{p} \in \mathrm{Spec}(R_{\mathbb{A}}) \mid f \notin \mathfrak{p}\}, \quad f \in R_{\mathbb{A}}.$$

Proposition 3.2. *The collection $\{D(f) \mid f \in R_{\mathbb{A}}\}$ forms a basis for the Zariski topology on $\mathrm{Spec}(R_{\mathbb{A}})$.*

Proof. This follows from the general construction of the Zariski topology on $\mathrm{Spec}(R)$ for any commutative ring R. Each $D(f)$ is open by definition, and given any open set $U \subseteq \mathrm{Spec}(R_{\mathbb{A}})$, it is a union of such $D(f)$ sets. The intersection satisfies:

$$D(f) \cap D(g) = D(f * g),$$

because $\mathfrak{p} \not\ni f$ and $\mathfrak{p} \not\ni g$ implies $f * g \notin \mathfrak{p}$ (since $\mathfrak{p}$ is prime and convolution multiplication is commutative), so $D(f * g) \subseteq D(f) \cap D(g)$. Conversely, if $f * g \notin \mathfrak{p}$ then neither f nor g can be in $\mathfrak{p}$, so $D(f * g) = D(f) \cap D(g)$. □

3.2. Localization and the Structure Sheaf.

We now define the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$, assigning to each open set $U \subseteq \mathrm{Spec}(R_{\mathbb{A}})$ a commutative ring $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}(U)$ of functions.

Definition 3.3. For $f \in R_{\mathbb{A}}$, define the localization:

$$R_{\mathbb{A}}[f^{-1}] := S_f^{-1} R_{\mathbb{A}}, \quad \text{where } S_f := \{f^{*n} \mid n \geq 0\} \subseteq R_{\mathbb{A}}.$$

We define the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$ by:

$$\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}(D(f)) := R_{\mathbb{A}}[f^{-1}].$$

Proposition 3.4. *The assignment $D(f) \mapsto R_{\mathbb{A}}[f^{-1}]$ defines a sheaf of rings on the basis $\{D(f)\}$, which extends uniquely to a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$ on all open subsets of $\mathrm{Spec}(R_{\mathbb{A}})$.*

Proof. This is standard from general scheme theory (cf. Hartshorne, Chapter II). The verification that $\mathcal{O}$ satisfies the sheaf axioms on the basis $\{D(f)\}$ amounts to checking:
- For $D(f) \subseteq D(g)$, we have a canonical restriction map $R_{\mathbb{A}}[g^{-1}] \to R_{\mathbb{A}}[f^{-1}]$.
- Compatibility with intersections: $D(f) \cap D(g) = D(f * g)$ implies

$$R_{\mathbb{A}}[f^{-1}] \otimes_{R_{\mathbb{A}}} R_{\mathbb{A}}[g^{-1}] \cong R_{\mathbb{A}}[(f * g)^{-1}].$$

This follows from the multiplicativity of the Dirichlet convolution structure and the universal property of localization. □

3.3. Stalks of the Structure Sheaf.

Definition 3.5. Let $\mathfrak{p} \in \operatorname{Spec}(R_{\mathbb{A}})$. The *stalk* of the structure sheaf at $\mathfrak{p}$ is given by the localization:

$$\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}}),\mathfrak{p}} := (R_{\mathbb{A}})_{\mathfrak{p}}.$$

Proposition 3.6. *For any $\mathfrak{p} \in \operatorname{Spec}(R_{\mathbb{A}})$, the stalk $\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}}),\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}(R_{\mathbb{A}})_{\mathfrak{p}}$.*

Proof. This is general: for any commutative ring R and any prime ideal $\mathfrak{p} \subset R$, the localization $R_{\mathfrak{p}}$ is a local ring, whose unique maximal ideal is the extension $\mathfrak{p}R_{\mathfrak{p}}$. In our case, $R = R_{\mathbb{A}}$ is commutative and $\mathfrak{p} \in \operatorname{Spec}(R)$, so the result follows. □

4. Topological Structure of $\operatorname{Spec}(R_{\mathbb{A}})$

4.1. **Specialization and Generic Points.** We analyze the specialization relation on $\operatorname{Spec}(R_{\mathbb{A}})$ as a poset under inclusion of prime ideals.

Definition 4.1. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R_{\mathbb{A}})$. We say that $\mathfrak{q}$ is a *specialization* of $\mathfrak{p}$, or that $\mathfrak{p}$ *specializes to* $\mathfrak{q}$, if $\mathfrak{p} \subseteq \mathfrak{q}$. Equivalently, $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$.

Proposition 4.2. *Let $n_0 \in \mathbb{N}_{>0}$, and assume $\mathbb{A}$ is an integral domain. Then the evaluation ideal*

$$\mathfrak{m}_{n_0} := \{f \in R_{\mathbb{A}} \mid f(n_0) = 0\}$$

is a maximal ideal in $R_{\mathbb{A}}$. Thus, the corresponding point is closed in the Zariski topology.

Proof. Let $f \notin \mathfrak{m}_{n_0}$, so $f(n_0) \neq 0$. Since $\mathbb{A}$ is a domain, the multiplicative subset $S := \{f^{*n} \mid n \geq 0\}$ avoids $\mathfrak{m}_{n_0}$, and we can localize at f. In the localization $R_{\mathbb{A}}[f^{-1}]$, f becomes invertible and $\mathfrak{m}_{n_0}$ extends to a proper ideal.

Suppose $\mathfrak{m}_{n_0} \subsetneq \mathfrak{q}$ is a strictly larger ideal. Then there exists $g \in \mathfrak{q} \setminus \mathfrak{m}_{n_0}$, so $g(n_0) \neq 0$. But then $\mathfrak{q}$ contains $f := g$, contradicting primality if g^{-1} exists in the localization. Thus, no proper overideal can exist, hence $\mathfrak{m}_{n_0}$ is maximal. □

Corollary 4.3. *Points of the form $\mathfrak{m}_{n_0}$ for $n_0 \in \mathbb{N}_{>0}$ correspond to closed points of* $\operatorname{Spec}(R_{\mathbb{A}})$.

4.2. **Generic Points and Irreducible Subsets.**

Definition 4.4. A subset $Y \subseteq \operatorname{Spec}(R_{\mathbb{A}})$ is *irreducible* if it cannot be written as a union of two proper closed subsets. A point $\eta \in Y$ is called a *generic point* of Y if $\overline{\{\eta\}} = Y$.

Proposition 4.5. *The prime ideal*

$$\mathfrak{o} := \{0\} \subset R_{\mathbb{A}}$$

is a generic point of the space $\operatorname{Spec}(R_{\mathbb{A}})$ *if $\mathbb{A}$ is an integral domain.*

Proof. We claim that $\mathfrak{o}$ is contained in every non-empty open subset of $\operatorname{Spec}(R_{\mathbb{A}})$. Let $D(f)$ be any non-empty basic open. Then $f \notin \mathfrak{o}$ since $\mathfrak{o}$ contains only zero. Thus, $\mathfrak{o} \in D(f)$. Hence $\mathfrak{o}$ lies in every $D(f)$, so

$$\overline{\{\mathfrak{o}\}} = \operatorname{Spec}(R_{\mathbb{A}}).$$

Therefore, $\mathfrak{o}$ is the unique generic point of the top space. □

Corollary 4.6. *The topological space* $\mathrm{Spec}(R_{\mathbb{A}})$ *is irreducible if and only if* $\mathbb{A}$ *is an integral domain.*

Proof. If $\mathbb{A}$ is a domain, then $R_{\mathbb{A}}$ has no nontrivial zero divisors under convolution, and the zero ideal is prime. Hence, 0 is in the closure of every open set, and the spectrum cannot be written as a union of two disjoint proper closed sets. Thus, it is irreducible.

Conversely, if $\mathbb{A}$ is not a domain, then there exist $a, b \in \mathbb{A}$ with $ab = 0$, $a \neq 0$, $b \neq 0$. Define arithmetic functions α, β by

$$\alpha(n) := \begin{cases} a & n = 1 \\ 0 & \text{else} \end{cases}, \quad \beta(n) := \begin{cases} b & n = 1 \\ 0 & \text{else} \end{cases}.$$

Then $(\alpha * \beta)(1) = ab = 0$, and $\alpha, \beta \neq 0$, so the convolution ring has zero divisors, and the zero ideal is not prime. Thus, $\mathrm{Spec}(R_{\mathbb{A}})$ cannot be irreducible. ☐

5. Quasi-Coherent Sheaves on $\mathrm{Spec}(R_{\mathbb{A}})$

We now define and study quasi-coherent sheaves on the arithmetic scheme $\mathrm{Spec}(R_{\mathbb{A}})$, where $R_{\mathbb{A}} := \{f : \mathbb{N}_{>0} \to \mathbb{A}\}$ carries pointwise addition and Dirichlet convolution, and $\mathbb{A}$ is a fixed commutative base ring.

5.1. **Modules over $R_{\mathbb{A}}$ and Sheafification.**

Definition 5.1. A *(left) $R_{\mathbb{A}}$-module* is an abelian group M equipped with a bilinear action $R_{\mathbb{A}} \times M \to M$ satisfying the usual module axioms. Let $\mathrm{Mod}(R_{\mathbb{A}})$ denote the category of such modules.

Definition 5.2. Let $M \in \mathrm{Mod}(R_{\mathbb{A}})$. Define the *quasi-coherent sheaf associated to M* as the sheaf $\widetilde{M}$ on $\mathrm{Spec}(R_{\mathbb{A}})$ determined by:

$$\widetilde{M}(D(f)) := M_f := S_f^{-1}M, \quad \text{for } f \in R_{\mathbb{A}},$$

where $S_f = \{f^{*n} \mid n \geq 0\}$ and M_f denotes the localization of M at S_f.

Proposition 5.3. *The assignment $D(f) \mapsto M_f$ extends to a sheaf of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$-modules on the basis $\{D(f)\}$, and hence uniquely to a sheaf $\widetilde{M}$ on all of* $\mathrm{Spec}(R_{\mathbb{A}})$.

Proof. The construction of $\widetilde{M}$ is standard in scheme theory. We verify the compatibility on overlaps:

$$D(f) \cap D(g) = D(f * g) \Rightarrow M_f \otimes_{R_{\mathbb{A}}} R_{\mathbb{A}}[g^{-1}] \cong M_{f*g}.$$

This isomorphism holds because localization is transitive:

$$(S_f^{-1}M)[g^{-1}] \cong S_{f*g}^{-1}M.$$

Sheaf axioms (restriction, uniqueness, gluing) follow directly from this base covering system. Therefore, $\widetilde{M}$ is a well-defined sheaf of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$-modules. ☐

Definition 5.4. A sheaf of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$-modules is said to be *quasi-coherent* if it is isomorphic to $\widetilde{M}$ for some $R_{\mathbb{A}}$-module M.

5.2. Stalks of Quasi-Coherent Sheaves.

Proposition 5.5. *Let M be an $R_{\mathbb{A}}$-module, and $\mathfrak{p} \in \operatorname{Spec}(R_{\mathbb{A}})$ a prime ideal. Then the stalk of the quasi-coherent sheaf $\widetilde{M}$ at $\mathfrak{p}$ is canonically isomorphic to the localization:*

$$\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}.$$

Proof. The stalk $\widetilde{M}_{\mathfrak{p}}$ is the direct limit over all open neighborhoods $D(f)$ of $\mathfrak{p}$:

$$\widetilde{M}_{\mathfrak{p}} := \varinjlim_{f \notin \mathfrak{p}} M_f = \varinjlim_{f \notin \mathfrak{p}} S_f^{-1} M.$$

This is precisely the localization of M at the multiplicative set $S := R_{\mathbb{A}} \setminus \mathfrak{p}$, which defines $M_{\mathfrak{p}}$:

$$\widetilde{M}_{\mathfrak{p}} \cong S^{-1} M = M_{\mathfrak{p}}.$$

□

5.3. Exactness and Global Sections.

Proposition 5.6. *The functor $M \mapsto \widetilde{M}$ from $\operatorname{Mod}(R_{\mathbb{A}})$ to the category of $\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}})}$-modules is exact and left adjoint to the global sections functor $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec}(R_{\mathbb{A}}))$.*

Proof. This is a standard result in scheme theory (cf. Hartshorne II.5). The exactness follows because localization is an exact functor:

$$0 \to M' \to M \to M'' \to 0 \quad \Rightarrow \quad 0 \to M'_f \to M_f \to M''_f \to 0$$

for each f, hence on the level of sheaves. The adjunction is given by the natural isomorphism:

$$\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}})}}(\widetilde{M}, \mathcal{F}) \cong \operatorname{Hom}_{R_{\mathbb{A}}}(M, \mathcal{F}(\operatorname{Spec}(R_{\mathbb{A}}))).$$

□

6. Examples of Quasi-Coherent Sheaves over $\operatorname{Spec}(R_{\mathbb{A}})$

We illustrate the construction of quasi-coherent sheaves via concrete $R_{\mathbb{A}}$-modules, examining the associated geometric data.

6.1. Structure Sheaf as a Quasi-Coherent Sheaf.

Proposition 6.1. *The structure sheaf $\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}})}$ is the quasi-coherent sheaf associated to the identity module $M = R_{\mathbb{A}}$, i.e.,*

$$\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}})} = \widetilde{R_{\mathbb{A}}}.$$

Proof. By definition, for any basic open $D(f)$, we have:

$$\mathcal{O}_{\operatorname{Spec}(R_{\mathbb{A}})}(D(f)) = R_{\mathbb{A}}[f^{-1}] = S_f^{-1} R_{\mathbb{A}} = \widetilde{R_{\mathbb{A}}}(D(f)).$$

Thus, the two sheaves agree on a basis of the topology, and therefore must agree globally.

□

6.2. **Evaluation Module Sheaves.** Fix a point $n_0 \in \mathbb{N}_{>0}$. Consider the $R_{\mathbb{A}}$-module:

$$M := R_{\mathbb{A}}/\mathfrak{m}_{n_0}, \quad \text{where } \mathfrak{m}_{n_0} = \{f \mid f(n_0) = 0\}.$$

Proposition 6.2. *The sheaf $\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}$ is supported entirely at the closed point corresponding to the maximal ideal $\mathfrak{m}_{n_0}$, and its stalk is:*

$$\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}_{\mathfrak{m}_{n_0}} \cong (R_{\mathbb{A}})_{\mathfrak{m}_{n_0}}/\mathfrak{m}_{n_0}(R_{\mathbb{A}})_{\mathfrak{m}_{n_0}}.$$

Proof. The sheaf $\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}$ is defined by localizing the module $R_{\mathbb{A}}/\mathfrak{m}_{n_0}$ at each open set. On any $D(f)$ not containing $\mathfrak{m}_{n_0}$, we have $f \in \mathfrak{m}_{n_0}$, so $f(n_0) = 0$, and hence f annihilates the entire module $R_{\mathbb{A}}/\mathfrak{m}_{n_0}$. Therefore, the localized module is zero:

$$(R_{\mathbb{A}}/\mathfrak{m}_{n_0})_f = 0 \quad \text{if } f(n_0) = 0.$$

Only on $D(f)$ with $f(n_0) \neq 0$ does the stalk persist. Hence, the support of this sheaf is precisely $\{\mathfrak{m}_{n_0}\}$.

At the stalk, we compute:

$$\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}_{\mathfrak{m}_{n_0}} = (R_{\mathbb{A}}/\mathfrak{m}_{n_0})_{\mathfrak{m}_{n_0}} \cong (R_{\mathbb{A}})_{\mathfrak{m}_{n_0}}/\mathfrak{m}_{n_0}(R_{\mathbb{A}})_{\mathfrak{m}_{n_0}},$$

which is the residue field at the point $\mathfrak{m}_{n_0}$. □

6.3. **Skyscraper Sheaves.**

Definition 6.3. Let $x = \mathfrak{m}_{n_0} \in \mathrm{Spec}(R_{\mathbb{A}})$, and let V be an $\mathbb{A}$-module. The *skyscraper sheaf* $\mathcal{F}_x$ supported at x with stalk V is defined by:

$$\mathcal{F}_x(U) := \begin{cases} V & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.4. *Every skyscraper sheaf $\mathcal{F}_x$ is a coherent sheaf supported at the closed point $x = \mathfrak{m}_{n_0}$ and is not quasi-coherent unless V arises from an $R_{\mathbb{A}}$-module of the form $R_{\mathbb{A}}/\mathfrak{m}_{n_0}$.*

Proof. The sheaf $\mathcal{F}_x$ is clearly supported only on the open sets containing x, hence its support is $\{x\}$. It has a single non-zero stalk:

$$(\mathcal{F}_x)_{\mathfrak{p}} = \begin{cases} V & \mathfrak{p} = \mathfrak{m}_{n_0}, \\ 0 & \text{otherwise.} \end{cases}$$

However, this sheaf does not arise as a localization of any global $R_{\mathbb{A}}$-module unless V has the form $R_{\mathbb{A}}/\mathfrak{m}_{n_0}$, so $\mathcal{F}_x$ is not quasi-coherent in general. □

6.4. **Arithmetic Interpretation.** The sheaf $\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}$ can be interpreted as a *sheaf of local arithmetic observables* focused entirely at the point n_0. More generally, the family of quasi-coherent sheaves $\{\widetilde{R_{\mathbb{A}}/\mathfrak{m}_n}\}_{n \in \mathbb{N}_{>0}}$ stratifies $\mathrm{Spec}(R_{\mathbb{A}})$ by arithmetic height (divisibility, size, etc.), enabling constructions of moduli spaces via arithmetic test sheaves.

This framework sets the stage for:

- Functorial interpretations of arithmetic function rings.
- Pushforward and pullback along maps $\mathrm{Spec}(R_{\mathbb{B}}) \to \mathrm{Spec}(R_{\mathbb{A}})$.
- Derived category constructions (e.g., perfect complexes).

7. Internal Hom Sheaves and Coherence on $\mathrm{Spec}(R_{\mathbb{A}})$

Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent sheaves on $\mathrm{Spec}(R_{\mathbb{A}})$. We develop the internal Hom sheaf $\mathcal{H}om_{\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}}(\mathcal{F}, \mathcal{G})$ and describe its local behavior and coherence.

7.1. Definition of the Internal Hom Sheaf.

Definition 7.1. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$-modules. The *internal Hom sheaf* is defined by:

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}$ denotes the morphisms of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}|_U$-modules on the open set $U \subseteq \mathrm{Spec}(R_{\mathbb{A}})$.

Proposition 7.2. *Let M, N be $R_{\mathbb{A}}$-modules and $\mathcal{F} = \widetilde{M}, \mathcal{G} = \widetilde{N}$. Then:*

$$\mathcal{H}om_{\mathcal{O}}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_{R_{\mathbb{A}}}(M, N)}.$$

Proof. We compute on basic opens $D(f) \subseteq \mathrm{Spec}(R_{\mathbb{A}})$. Since localization commutes with Hom, we have:

$$\mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}(D(f))}(\widetilde{M}(D(f)), \widetilde{N}(D(f))) = \mathrm{Hom}_{R_{\mathbb{A}}[f^{-1}]}(M_f, N_f).$$

By the universal property of localization:

$$\mathrm{Hom}_{R_{\mathbb{A}}[f^{-1}]}(M_f, N_f) \cong (\mathrm{Hom}_{R_{\mathbb{A}}}(M, N))_f.$$

Thus, for all basic open sets:

$$\mathcal{H}om_{\mathcal{O}}(\widetilde{M}, \widetilde{N})(D(f)) \cong \widetilde{\mathrm{Hom}_{R_{\mathbb{A}}}(M, N)}(D(f)),$$

which shows the sheaves are equal. □

7.2. Coherent Sheaves and Finiteness.

Definition 7.3. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$-modules is said to be:

- *Finitely generated* if there exists an open cover $\{U_i\}$ and surjections $\mathcal{O}_{U_i}^{\oplus n_i} \to \mathcal{F}|_{U_i}$.
- *Coherent* if it is of finite type and for every open $U \subseteq \mathrm{Spec}(R_{\mathbb{A}})$, and morphism $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$, the kernel of ϕ is also of finite type.

Proposition 7.4. *Let M be a finitely presented $R_{\mathbb{A}}$-module. Then the quasi-coherent sheaf $\widetilde{M}$ is coherent.*

Proof. Suppose we have an exact sequence:

$$R_{\mathbb{A}}^{\oplus m} \xrightarrow{\phi} R_{\mathbb{A}}^{\oplus n} \to M \to 0.$$

Sheafifying gives:

$$\mathcal{O}^{\oplus m} \xrightarrow{\widetilde{\phi}} \mathcal{O}^{\oplus n} \to \widetilde{M} \to 0.$$

Since the kernel of $\widetilde{\phi}$ is $\widetilde{\ker(\phi)}$, and $\ker(\phi)$ is finitely generated, we conclude that $\widetilde{M}$ is coherent. □

Corollary 7.5. *The sheaves $\widetilde{R_{\mathbb{A}}/\mathfrak{m}_{n_0}}$ are coherent for all $n_0 \in \mathbb{N}_{>0}$.*

Proof. Each module $R_{\mathbb{A}}/\mathfrak{m}_{n_0}$ is finitely presented as a cyclic module. Hence, their associated sheaves are coherent. □

7.3. **Exactness and Tensor Product Sheaves.**

Definition 7.6. Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}$-modules. Define their tensor product sheaf as:

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}(U) := \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U).$$

Proposition 7.7. *If $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$, then:*

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \cong M \widetilde{\otimes_{R_{\mathbb{A}}}} N.$$

Proof. We again verify on basic open sets $D(f)$:

$$(\widetilde{M} \otimes \widetilde{N})(D(f)) = M_f \otimes_{R_{\mathbb{A}}[f^{-1}]} N_f \cong (M \otimes_{R_{\mathbb{A}}} N)_f = \widetilde{M \otimes N}(D(f)).$$

Therefore, the sheaves agree on a basis and hence globally. □

8. Kähler Differentials on $R_{\mathbb{A}}$ and Arithmetic Derivations

8.1. **Derivations.**

Definition 8.1. Let R be a commutative ring, and let M be an R-module. A *derivation* of R with values in M is an $\mathbb{A}$-linear map:

$$d : R \to M$$

such that for all $f, g \in R$,

$$d(f * g) = d(f) * g + f * d(g).$$

We denote the set of all derivations from R to M as $\mathrm{Der}_{\mathbb{A}}(R, M)$.

Remark 8.2. In our context, $R = R_{\mathbb{A}}$ is the ring of arithmetic functions with Dirichlet convolution, and the multiplication rule in the Leibniz identity is precisely the Dirichlet convolution.

Example 8.3. Let $R = R_{\mathbb{A}}$ and define the function $\delta : R \to R$ by

$$\delta(f)(n) := \log(n) \cdot f(n),$$

assuming $\log(n) \in \mathbb{A}$ makes sense (e.g. when $\mathbb{A} \supseteq \mathbb{Q}$ or a real or p-adic ring). Then δ satisfies the Leibniz rule for Dirichlet convolution:

$$\delta(f * g)(n) = \log(n) \cdot \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} \left[\log(d) f(d) g\left(\frac{n}{d}\right) + f(d) \log\left(\frac{n}{d}\right) g\left(\frac{n}{d}\right)\right].$$

Thus, $\delta(f * g) = \delta(f) * g + f * \delta(g)$, confirming it is a derivation.

8.2. **Kähler Differentials.**

Definition 8.4. Let $R = R_{\mathbb{A}}$ be an $\mathbb{A}$-algebra. The *module of Kähler differentials* $\Omega^1_{R/\mathbb{A}}$ is the R-module generated by symbols df for $f \in R$ subject to the relations:

(1) $d(f + g) = df + dg$,
(2) $d(af) = a\, df$ for all $a \in \mathbb{A}$,
(3) $d(f * g) = df * g + f * dg$.

The map $d : R \to \Omega^1_{R/\mathbb{A}}$ is the universal derivation.

Proposition 8.5. *There exists a universal derivation*

$$d : R_{\mathbb{A}} \to \Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$$

satisfying the Leibniz rule for Dirichlet convolution, and for any $R_{\mathbb{A}}$-module M, composition with d induces an isomorphism:

$$\mathrm{Hom}_{R_{\mathbb{A}}}(\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}, M) \cong \mathrm{Der}_{\mathbb{A}}(R_{\mathbb{A}}, M).$$

Proof. Let F be the free $R_{\mathbb{A}}$-module generated by symbols df for each $f \in R_{\mathbb{A}}$. Define $J \subseteq F$ to be the submodule generated by the following relations:

$$d(f+g) - df - dg, \quad d(af) - a\,df, \quad d(f * g) - df * g - f * dg,$$

for all $f, g \in R_{\mathbb{A}}$ and $a \in \mathbb{A}$. Set:

$$\Omega^1_{R_{\mathbb{A}}/\mathbb{A}} := F/J.$$

Define $d : R_{\mathbb{A}} \to \Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ by $f \mapsto df$. This map satisfies the required Leibniz identity by construction.

Now let $\delta : R_{\mathbb{A}} \to M$ be any $\mathbb{A}$-linear derivation into an $R_{\mathbb{A}}$-module M. Define $\varphi : F \to M$ by $\varphi(df) := \delta(f)$. Then φ descends to a well-defined $R_{\mathbb{A}}$-module homomorphism $\bar{\varphi} : \Omega^1_{R_{\mathbb{A}}/\mathbb{A}} \to M$ such that $\delta = \bar{\varphi} \circ d$. Uniqueness follows from the universal property of quotient modules.

Thus, $\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ satisfies the universal property of Kähler differentials. □

8.3. **Stalkwise Behavior and Sheafification.**

Definition 8.6. Let $\Omega^1 := \Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ be the module of Kähler differentials. The *sheaf of Kähler differentials* is defined as:

$$\Omega^1_{\mathrm{Spec}(R_{\mathbb{A}})} := \widetilde{\Omega^1}.$$

Proposition 8.7. *The stalk of the sheaf $\Omega^1_{\mathrm{Spec}(R_{\mathbb{A}})}$ at any prime ideal $\mathfrak{p} \subset R_{\mathbb{A}}$ is isomorphic to the Kähler module of the local ring:*

$$\Omega^1_{\mathrm{Spec}(R_{\mathbb{A}}),\mathfrak{p}} \cong \Omega^1_{(R_{\mathbb{A}})_{\mathfrak{p}}/\mathbb{A}}.$$

Proof. By the properties of sheafification:

$$\Omega^1_{\mathrm{Spec}(R_{\mathbb{A}}),\mathfrak{p}} = \left(\widetilde{\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}}\right)_{\mathfrak{p}} = \left(\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}\right)_{\mathfrak{p}}.$$

But localization commutes with the construction of Kähler differentials, so:

$$\left(\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}\right)_{\mathfrak{p}} = \Omega^1_{(R_{\mathbb{A}})_{\mathfrak{p}}/\mathbb{A}}.$$

□

9. Explicit Structure of $\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ and Arithmetic Differentials

9.1. **Logarithmic Differential Derivation.** Assume that $\mathbb{A}$ is a commutative $\mathbb{Q}$-algebra so that logarithms $\log(n)$ for $n \in \mathbb{N}_{>0}$ lie in $\mathbb{A}$.

Definition 9.1. Define the *logarithmic differential derivation*:

$$\delta : R_{\mathbb{A}} \to R_{\mathbb{A}}, \quad f \mapsto \delta(f),$$

where:

$$\delta(f)(n) := \log(n) \cdot f(n), \quad \text{for all } n \in \mathbb{N}_{>0}.$$

Proposition 9.2. *The map δ is a $\mathbb{A}$-linear derivation of $R_{\mathbb{A}}$:*

$$\delta(f * g) = \delta(f) * g + f * \delta(g).$$

Proof. Let $f, g \in R_{\mathbb{A}}$. Then:

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Applying δ:

$$\delta(f * g)(n) = \log(n) \cdot \sum_{d|n} f(d)g(n/d).$$

Now compute:

$$(\delta(f) * g)(n) = \sum_{d|n} \log(d) f(d) g(n/d),$$
$$(f * \delta(g))(n) = \sum_{d|n} f(d) \log(n/d) g(n/d).$$

Adding gives:

$$(\delta(f) * g + f * \delta(g))(n) = \sum_{d|n} [\log(d) + \log(n/d)] f(d)g(n/d) = \log(n) \sum_{d|n} f(d)g(n/d),$$

since $\log(d) + \log(n/d) = \log(n)$. Thus, the derivation rule holds. $\square$

Corollary 9.3. *The derivation δ factors through the universal derivation:*

$$\delta = \phi \circ d, \quad \textit{for a unique } \phi : \Omega^1_{R_{\mathbb{A}}/\mathbb{A}} \to R_{\mathbb{A}}.$$

9.2. **Differentials of Basis Functions.** We now construct an $\mathbb{A}$-basis of $R_{\mathbb{A}}$ and compute its differentials.

Definition 9.4. For each $n \in \mathbb{N}_{>0}$, let $e_n \in R_{\mathbb{A}}$ denote the characteristic function:

$$e_n(m) := \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{e_n\}_{n \geq 1}$ is an $\mathbb{A}$-basis of $R_{\mathbb{A}}$.

Proposition 9.5. *The Kähler differential module $\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ is a free $R_{\mathbb{A}}$-module with basis $\{de_p \mid p \in \mathbb{P}\}$, the set of differentials of Dirichlet generators supported on primes.*

Proof. First, note that for multiplicative arithmetic functions, the Dirichlet algebra is generated by $\{e_p\}_{p\in\mathbb{P}}$ under convolution. Since:

$$e_{p^k} = e_p^{*k}, \quad \text{with } e_p^{*k}(n) = 1 \text{ if } n = p^k, \text{ and 0 otherwise,}$$

we find that the monomials in $\{e_p\}$ span all e_n via convolution products.

Let $R := R_{\mathbb{A}} = \mathbb{A}[e_p \mid p \in \mathbb{P}]_{\mathrm{Dir}}$ as a convolution algebra. Then the module of differentials satisfies:

$$\Omega^1_{R/\mathbb{A}} \cong \bigoplus_{p\in\mathbb{P}} R \cdot de_p,$$

since $\{e_p\}$ generate R, and differentials respect the Leibniz rule with convolution:

$$d(e_p^{*k}) = k \cdot e_p^{*(k-1)} * de_p.$$

Thus, all differentials de_n reduce to R-linear combinations of de_p with $p \mid n$.

Linearly independent relations follow from freeness of the basis $\{e_n\}$ and the independence of prime supports. Hence, $\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}$ is free with basis $\{de_p\}_{p\in\mathbb{P}}$. □

Corollary 9.6. *The sheaf of differentials* $\Omega^1_{\mathrm{Spec}(R_{\mathbb{A}})}$ *is a quasi-coherent locally free* $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})}$*-module of infinite rank over* $\mathrm{Spec}(R_{\mathbb{A}})$*.*

10. The de Rham Complex of $R_{\mathbb{A}}$ over $\mathbb{A}$

We construct the full de Rham complex $(\Omega^\bullet_{R_{\mathbb{A}}/\mathbb{A}}, d)$ of arithmetic differential forms and study its algebraic and cohomological structure.

10.1. **Higher Differential Forms.**

Definition 10.1. Let $R := R_{\mathbb{A}}$. For each $n \geq 0$, define:

$$\Omega^n_{R/\mathbb{A}} := \bigwedge_R^n \Omega^1_{R/\mathbb{A}},$$

the n-th exterior power of the module of Kähler differentials. In particular:

$$\begin{aligned}
&\Omega^0_{R/\mathbb{A}} = R,\\
&\Omega^1_{R/\mathbb{A}} = R \cdot \{de_p \mid p \in \mathbb{P}\},\\
&\Omega^n_{R/\mathbb{A}} = \text{span over } R \text{ of } de_{p_1} \wedge \cdots \wedge de_{p_n}, \quad p_i \in \mathbb{P},\ p_i \neq p_j.
\end{aligned}$$

Remark 10.2. Since $\{de_p\}_{p\in\mathbb{P}}$ forms a basis for $\Omega^1_{R/\mathbb{A}}$, the module $\Omega^n_{R/\mathbb{A}}$ is free of infinite rank, generated by alternating n-fold wedge products of these differentials.

10.2. **Exterior Differential.**

Definition 10.3. The *exterior differential* is a graded $\mathbb{A}$-linear map of degree 1:

$$d : \Omega^n_{R/\mathbb{A}} \longrightarrow \Omega^{n+1}_{R/\mathbb{A}}$$

defined on pure tensors by:

$$d(r \cdot \omega) := dr \wedge \omega + r \cdot d\omega, \quad \text{for } r \in R,\ \omega \in \Omega^n_{R/\mathbb{A}},$$

with the initial condition $d : R \to \Omega^1_{R/\mathbb{A}}$ being the universal derivation.

Proposition 10.4. *The exterior differential satisfies:*

$$d^2 = 0.$$

Proof. We verify $d^2 = 0$ by induction and the graded Leibniz rule.

Base case: For $f \in R$, we compute $d^2 f := d(df)$. Since $df \in \Omega^1_{R/\mathbb{A}}$ and d satisfies $d^2 = 0$ on Ω^1 (as follows from antisymmetry of wedge products), we get $d^2 f = 0$.

Inductive step: Assume $d^2\omega = 0$ for all $\omega \in \Omega^n$. Then for $r \in R$ and $\omega \in \Omega^n$:

$$\begin{aligned} d^2(r \cdot \omega) &= d(dr \wedge \omega + r \cdot d\omega) \\ &= d^2 r \wedge \omega - dr \wedge d\omega + dr \wedge d\omega + r \cdot d^2\omega \\ &= 0, \end{aligned}$$

since $d^2 r = 0$ and $d^2\omega = 0$. Hence, $d^2 = 0$ on all $\Omega^n_{R/\mathbb{A}}$. □

10.3. **The de Rham Complex and Cohomology.**

Definition 10.5. The sequence of R-modules and differentials

$$0 \longrightarrow \Omega^0_{R/\mathbb{A}} \xrightarrow{d} \Omega^1_{R/\mathbb{A}} \xrightarrow{d} \Omega^2_{R/\mathbb{A}} \xrightarrow{d} \cdots$$

is called the *algebraic de Rham complex* of R over $\mathbb{A}$.

The *de Rham cohomology* of R over $\mathbb{A}$ is defined by:

$$H^n_{\mathrm{dR}}(R/\mathbb{A}) := \frac{\ker\left(d : \Omega^n_{R/\mathbb{A}} \to \Omega^{n+1}_{R/\mathbb{A}}\right)}{\operatorname{im}\left(d : \Omega^{n-1}_{R/\mathbb{A}} \to \Omega^n_{R/\mathbb{A}}\right)}.$$

Proposition 10.6. *Each $H^n_{\mathrm{dR}}(R/\mathbb{A})$ is an $\mathbb{A}$-module functorially associated to the base ring and encodes obstruction to integrability of arithmetic derivations.*

Proof. By construction, each $H^n_{\mathrm{dR}}(R/\mathbb{A})$ is an R-module and hence an $\mathbb{A}$-module via the structure map $\mathbb{A} \to R$. The functoriality follows from naturality of Kähler differentials and wedge powers under morphisms of $\mathbb{A}$-algebras. The obstruction interpretation is standard: a closed form ω such that $d\omega = 0$ is exact iff $\omega = d\eta$ for some form η; failure of this is measured by the cohomology class $[\omega] \in H^n_{\mathrm{dR}}$. □

10.4. **Examples and Observations.**

Example 10.7. Let $\mathbb{A} = \mathbb{Q}$ and $f = e_p \in R_{\mathbb{Q}}$, the indicator function at a prime p. Then:

$$df = de_p, \quad d^2 f = 0.$$

Hence de_p is a closed 1-form. Since there is no function $g \in R_{\mathbb{Q}}$ such that $dg = de_p$ (as e_p is atomic), de_p represents a nontrivial class in $H^1_{\mathrm{dR}}(R_{\mathbb{Q}}/\mathbb{Q})$.

Corollary 10.8. *The first de Rham cohomology group $H^1_{\mathrm{dR}}(R_{\mathbb{A}}/\mathbb{A})$ contains a free $\mathbb{A}$-submodule with basis $\{[de_p] \mid p \in \mathbb{P}\}$.*

11. The Cotangent Complex $L_{R_{\mathbb{A}}/\mathbb{A}}$ and Arithmetic Deformations

11.1. **Simplicial Resolution of $R_{\mathbb{A}}$.** Let $\mathbb{A}$ be a commutative ring, and $R_{\mathbb{A}} := \{f : \mathbb{N}_{>0} \to \mathbb{A}\}$ the arithmetic function ring with Dirichlet convolution.

Definition 11.1. Let $P_\bullet \to R_{\mathbb{A}}$ be a simplicial resolution of $R_{\mathbb{A}}$ by polynomial $\mathbb{A}$-algebras. That is, $P_\bullet$ is a simplicial object in the category of $\mathbb{A}$-algebras such that:

- Each P_n is a free (i.e., polynomial) $\mathbb{A}$-algebra,
- There is a quasi-isomorphism $|P_\bullet| \simeq R_{\mathbb{A}}$ in simplicial commutative rings.

Remark 11.2. This resolution can be constructed via the canonical simplicial polynomial resolution:

$$P_\bullet := \mathbb{A}[x_n^{(i)}]_{n\in\mathbb{N}_{>0}, i\leq n}, \quad \text{where each } x_n^{(i)} \text{ approximates } e_n.$$

We do not require an explicit resolution for functorial constructions.

11.2. **Definition of the Cotangent Complex.**

Definition 11.3. The *cotangent complex* $L_{R_{\mathbb{A}}/\mathbb{A}}$ is defined as the chain complex:

$$L_{R_{\mathbb{A}}/\mathbb{A}} := \Omega^1_{P_\bullet/\mathbb{A}} \otimes_{P_\bullet} R_{\mathbb{A}},$$

where $\Omega^1_{P_\bullet/\mathbb{A}}$ is the simplicial module of Kähler differentials of $P_\bullet$ over $\mathbb{A}$, and the tensor is derived via the geometric realization of simplicial modules.

Proposition 11.4. *$L_{R_{\mathbb{A}}/\mathbb{A}}$ is a well-defined object in the derived category $D(R_{\mathbb{A}})$, independent up to quasi-isomorphism of the choice of simplicial resolution $P_\bullet$.*

Proof. This follows from standard model category theory: all simplicial polynomial resolutions are homotopy equivalent, and the functor $P \mapsto \Omega^1_{P/\mathbb{A}} \otimes_P R_{\mathbb{A}}$ is homotopy invariant under quasi-isomorphisms. Hence, the resulting complex is well-defined in $D(R_{\mathbb{A}})$. □

11.3. **Relation to Kähler Differentials.**

Proposition 11.5. *There exists a natural morphism in $D(R_{\mathbb{A}})$:*

$$L_{R_{\mathbb{A}}/\mathbb{A}} \longrightarrow \Omega^1_{R_{\mathbb{A}}/\mathbb{A}}[0],$$

which is a quasi-isomorphism if and only if $R_{\mathbb{A}}$ is smooth over $\mathbb{A}$.

Proof. This morphism arises from the canonical map of Kähler differentials:

$$\Omega^1_{P_0/\mathbb{A}} \otimes_{P_0} R_{\mathbb{A}} \to \Omega^1_{R_{\mathbb{A}}/\mathbb{A}}.$$

When $R_{\mathbb{A}}$ is smooth over $\mathbb{A}$, this map is an isomorphism and the higher homology groups $H_i(L_{R_{\mathbb{A}}/\mathbb{A}}) = 0$ for $i > 0$. Otherwise, the derived nature of L captures nontrivial obstructions to smoothness. □

Example 11.6. Let $\mathbb{A} = \mathbb{Q}$, and recall that $R_{\mathbb{Q}} \cong \mathbb{Q}[e_n \mid n \in \mathbb{N}_{>0}]$ is freely generated (as a $\mathbb{Q}$-module) but not smooth over $\mathbb{Q}$, due to the infinite nature of Dirichlet convolution and relations arising from multiplicative structure. Then $L_{R_{\mathbb{Q}}/\mathbb{Q}}$ has:

$$H_0(L_{R_{\mathbb{Q}}/\mathbb{Q}}) \cong \Omega^1_{R_{\mathbb{Q}}/\mathbb{Q}}, \quad H_i(L_{R_{\mathbb{Q}}/\mathbb{Q}}) \neq 0 \text{ for some } i > 0.$$

Thus, the cotangent complex encodes higher-order arithmetic singularities.

11.4. **Universality of the Cotangent Complex.**

Theorem 11.7 (Universal Property). *Let M be an $R_{\mathbb{A}}$-module. Then:*

$$\mathrm{Hom}_{D(R_{\mathbb{A}})}(L_{R_{\mathbb{A}}/\mathbb{A}}, M) \cong \mathrm{Der}^{\mathrm{der}}_{\mathbb{A}}(R_{\mathbb{A}}, M),$$

the set of derived derivations from $R_{\mathbb{A}}$ to M.

Proof. Derived derivations correspond to maps from the cotangent complex by definition. That is:

$$\mathrm{Der}^{\mathrm{der}}_{\mathbb{A}}(R_{\mathbb{A}}, M) := \mathrm{Hom}_{D(R_{\mathbb{A}})}(L_{R_{\mathbb{A}}/\mathbb{A}}, M).$$

This functor generalizes the classical derivation functor represented by Ω^1, as L replaces Ω^1 when $R_{\mathbb{A}}$ is not smooth. □

11.5. **Cohomological Arithmetic Interpretation.**

Corollary 11.8. *The cohomology $H^i(L_{R_{\mathbb{A}}/\mathbb{A}})$ for $i > 0$ measures the failure of $R_{\mathbb{A}}$ to be smooth over $\mathbb{A}$, and encodes higher-order relations among Dirichlet generators.*

Proof. In derived algebraic geometry, smoothness corresponds to the vanishing of higher homology of the cotangent complex. Thus, if $H^i(L_{R_{\mathbb{A}}/\mathbb{A}}) \neq 0$ for $i > 0$, then $R_{\mathbb{A}}$ fails to be smooth, and these cohomology modules record the derived obstructions to formally smooth lifting of arithmetic deformations. □

12. Square-Zero Extensions and Infinitesimal Deformations of $R_{\mathbb{A}}$

12.1. **Square-Zero Extensions of Algebras.**

Definition 12.1. Let R be a commutative $\mathbb{A}$-algebra, and let M be an R-module. A *square-zero extension* of R by M is a short exact sequence of $\mathbb{A}$-algebras:

$$0 \to M \xrightarrow{i} R' \xrightarrow{\pi} R \to 0,$$

such that:

- M is an ideal of R' with $m \cdot m' = 0$ for all $m, m' \in M$,
- the algebra structure on R' lifts that of R, and M is annihilated multiplicatively.

We write this extension as $R' = R \oplus M$ with multiplication:

$$(r, m) \cdot (r', m') = (rr', rm' + r'm).$$

12.2. **Classification via the Cotangent Complex.**

Theorem 12.2. *Let M be a quasi-coherent $R_{\mathbb{A}}$-module. Then the set of equivalence classes of square-zero extensions of $R_{\mathbb{A}}$ by M is naturally isomorphic to:*

$$\mathrm{Ext}^1_{R_{\mathbb{A}}}(L_{R_{\mathbb{A}}/\mathbb{A}}, M).$$

Proof. By derived deformation theory (cf. Illusie, Lurie), square-zero extensions of a commutative $\mathbb{A}$-algebra R by a module M are classified by:

$$\mathrm{Der}^{\mathrm{der}}_{\mathbb{A}}(R, M[1]) = \mathrm{Hom}_{D(R)}(L_{R/\mathbb{A}}, M[1]) = \mathrm{Ext}^1_R(L_{R/\mathbb{A}}, M).$$

In our case, $R = R_{\mathbb{A}}$, so the result follows directly. □

Corollary 12.3. *If $L_{R_{\mathbb{A}}/\mathbb{A}}$ is not quasi-isomorphic to $\Omega^1_{R_{\mathbb{A}}/\mathbb{A}}[0]$, then there exist nontrivial square-zero extensions not captured by Kähler differentials.*

Proof. The classical Kähler differentials classify derivations via:

$$\mathrm{Der}_{\mathbb{A}}(R, M) \cong \mathrm{Hom}_R(\Omega^1_{R/\mathbb{A}}, M),$$

but do not detect higher Ext classes. Nontriviality of $\mathrm{Ext}^1(L_{R/\mathbb{A}}, M)$ beyond $\mathrm{Hom}(\Omega^1, M)$ implies the presence of genuinely derived (nonclassical) deformations. □

12.3. **Universal Square-Zero Extension.**

Definition 12.4. Given $R = R_{\mathbb{A}}$, the *universal square-zero extension* is the derived square-zero extension classified by the identity morphism:

$$\mathrm{id}_{L_{R/\mathbb{A}}} \in \mathrm{Ext}^1_R(L_{R/\mathbb{A}}, L_{R/\mathbb{A}}).$$

Proposition 12.5. *The universal square-zero extension is initial among all square-zero extensions of R by an R-module and induces the universal infinitesimal deformation.*

Proof. The identity map corresponds to the universal element in $\mathrm{Ext}^1_R(L_{R/\mathbb{A}}, L_{R/\mathbb{A}})$. Any other square-zero extension is obtained by pushing out along a map $L_{R/\mathbb{A}} \to M$. Thus, the identity class defines the universal object representing the deformation functor. □

12.4. **Example: Arithmetic Deformation by Dirichlet Noise.**

Example 12.6. Let $\mathbb{A} = \mathbb{Q}$ and let $M := R_{\mathbb{Q}}$ itself. The square-zero extension:

$$0 \to R_{\mathbb{Q}} \to R' \to R_{\mathbb{Q}} \to 0$$

corresponds to adding an infinitesimal "Dirichlet noise" ϵ to each arithmetic function:

$$f(n) \mapsto f(n) + \epsilon \cdot \eta(n),$$

with $\epsilon^2 = 0$ and $\eta(n) \in \mathbb{Q}$. The new convolution multiplication satisfies:

$$(f + \epsilon\eta) * (g + \epsilon\theta) = f * g + \epsilon(f * \theta + \eta * g),$$

and the extension corresponds to a derived derivation $\delta : R_{\mathbb{Q}} \to R_{\mathbb{Q}}$ classified by $\delta(f) = \eta$.

Corollary 12.7. *The space of infinitesimal arithmetic perturbations of $R_{\mathbb{Q}}$ is governed by $H^1(L_{R_{\mathbb{Q}}/\mathbb{Q}})$.*

13. Formal Moduli Problems and Tangent Complexes over $R_{\mathbb{A}}$

13.1. **Artinian Test Algebras and Functor of Deformations.** Let $\mathbb{A}$ be a commutative ring, and fix $R := R_{\mathbb{A}}$.

Definition 13.1. Let $\mathrm{Art}_{\mathbb{A}}$ denote the category of local Artinian $\mathbb{A}$-algebras A with residue field $\mathbb{A}$, i.e., such that $\mathfrak{m}_A$ is nilpotent and $A/\mathfrak{m}_A \cong \mathbb{A}$.

A *formal moduli problem* over R is a functor:

$$F : \mathrm{Art}_{\mathbb{A}} \to \mathrm{Set}$$

satisfying:

(1) $F(\mathbb{A}) = \{*\}$ (a basepoint),
(2) For any surjection $A \twoheadrightarrow B$ in $\mathrm{Art}_{\mathbb{A}}$ with kernel I square-zero, the canonical map:

$$F(A) \to F(B) \times_{F(\mathbb{A})} F(A \times_B \mathbb{A})$$

is a bijection.

Remark 13.2. These conditions reflect Schlessinger's criterion for a functor to be pro-representable. The second condition encodes the infinitesimal deformation gluing.

13.2. **Tangent Space and Tangent Complex.**

Definition 13.3. The *tangent space* of a formal moduli problem F is the set:

$$T_F := F(\mathbb{A}[\epsilon]/\epsilon^2),$$

where $\mathbb{A}[\epsilon]/\epsilon^2$ is the ring of dual numbers over $\mathbb{A}$.

Proposition 13.4. *Let $F := \mathrm{Def}_R$ be the formal moduli problem of square-zero extensions of R. Then:*

$$T_F \cong \mathrm{Der}_{\mathbb{A}}(R, R) \cong \mathrm{Hom}_R(\Omega^1_{R/\mathbb{A}}, R).$$

Proof. An element of $F(\mathbb{A}[\epsilon]/\epsilon^2)$ corresponds to an $\mathbb{A}$-algebra structure on $R \oplus \epsilon R$ with $\epsilon^2 = 0$, lifting the given one on R. This is equivalent to giving an $\mathbb{A}$-derivation $\delta : R \to R$, i.e., an element of $\mathrm{Der}_{\mathbb{A}}(R, R)$. The identification with $\mathrm{Hom}_R(\Omega^1_{R/\mathbb{A}}, R)$ follows from the universal property of Kähler differentials. □

13.3. **Derived Formal Moduli Problems and Tangent Complex.** We now work over a characteristic zero base (assume $\mathbb{A} \supseteq \mathbb{Q}$).

Definition 13.5. A *derived formal moduli problem* is a functor:

$$F : \mathrm{dgArt}_{\mathbb{A}} \to \mathrm{SSet}$$

from differential graded Artinian $\mathbb{A}$-algebras to simplicial sets, satisfying:

(1) $F(\mathbb{A}) = *$,
(2) F preserves homotopy pullbacks over square-zero extensions.

Theorem 13.6 (Lurie)**.** *Every derived formal moduli problem F over a characteristic-zero field is equivalent to a differential graded Lie algebra (DGLA), or equivalently, a tangent complex T_F in degree ≤ 0 with Lie bracket and Maurer–Cartan solutions.*

Definition 13.7. The *tangent complex* T_F of a derived formal moduli problem F associated to the deformation of $R_{\mathbb{A}}$ is:

$$T_F := \mathbb{R}\,\mathrm{Hom}_{R_{\mathbb{A}}}(L_{R_{\mathbb{A}}/\mathbb{A}}, R_{\mathbb{A}}),$$

where $L_{R_{\mathbb{A}}/\mathbb{A}}$ is the cotangent complex.

13.4. **Deformation Classification via the Tangent Complex.**

Proposition 13.8. *Let $F = \mathrm{Def}_{R_{\mathbb{A}}}$ be the derived deformation functor of the arithmetic ring. Then the deformation space $F(A)$ is determined by the complex T_F via:*

$$\pi_n F(\mathbb{A} \oplus M[n]) \cong \mathrm{Ext}^{n+1}_{R_{\mathbb{A}}}(L_{R_{\mathbb{A}}/\mathbb{A}}, M),$$

for any $R_{\mathbb{A}}$-module M and $n \geq 0$.

Proof. This follows from Lurie's representability theorem for derived formal moduli problems in characteristic zero, which states that such a functor F is governed by its tangent complex T_F. The n-th homotopy group of the derived mapping space corresponds to:

$$\pi_n \mathbb{R}\,\mathrm{Hom}_R(L_{R/\mathbb{A}}, M[n]) = \mathrm{Ext}^{n+1}_R(L_{R/\mathbb{A}}, M).$$

□

Corollary 13.9. *First-order infinitesimal deformations of $R_{\mathbb{A}}$ are classified by $H^1(L_{R_{\mathbb{A}}/\mathbb{A}})$, and obstructions lie in H^2.*

Proof. From the proposition with $n = 0$, we have:

$$F(\mathbb{A}[\epsilon]/\epsilon^2) \cong \mathrm{Ext}^1_R(L_{R/\mathbb{A}}, R) = H^1(T_F).$$

Obstructions to lifting second-order deformations lie in Ext^2, i.e., $H^2(T_F)$. □

14. DERIVED MODULI OF QUASI-COHERENT SHEAVES ON $\mathrm{Spec}(R_{\mathbb{A}})$

14.1. **The Moduli Functor of Sheaves.** Let $R := R_{\mathbb{A}}$ and fix a quasi-coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\mathcal{F} = \widetilde{M}$.

Definition 14.1. Define the *moduli functor of quasi-coherent sheaves* on $\mathrm{Spec}(R)$ as:

$$\mathcal{M}_{\mathrm{Qcoh}} : \mathrm{Art}_{\mathbb{A}} \to \mathrm{Groupoids}$$

which assigns to each $A \in \mathrm{Art}_{\mathbb{A}}$ the groupoid:

$$\mathcal{M}_{\mathrm{Qcoh}}(A) := \{\text{quasi-coherent sheaves on } \mathrm{Spec}(R \otimes_{\mathbb{A}} A) \text{ flat over } A\}.$$

Proposition 14.2. *The functor $\mathcal{M}_{\mathrm{Qcoh}}$ satisfies Schlessinger's conditions (pro-representability) formally if restricted to coherent flat sheaves of finite presentation.*

Proof. Flatness guarantees the preservation of fiber dimension over A, and coherence + finite presentation ensures effective descent and compatibility with pullbacks over small extensions. This satisfies Schlessinger's conditions for effective glueing over square-zero extensions and uniqueness of liftings. □

14.2. **Derived Enhancement and Mapping Stack.** We now define the derived enhancement of this moduli problem, encoding higher deformation data.

Definition 14.3. Let M be a perfect complex of R-modules. Define the *derived moduli functor of deformations of M* as:

$$\mathfrak{Def}_M : \mathrm{dgArt}_{\mathbb{A}} \to \mathrm{SSet}$$

by:

$$\mathfrak{Def}_M(A) := \left\{ M_A \in \mathrm{QCoh}(R \otimes_{\mathbb{A}} A) \mid M_A \otimes_A^{\mathbb{L}} \mathbb{A} \simeq M \right\},$$

where M_A is a derived flat deformation of M.

Proposition 14.4. *The functor $\mathfrak{Def}_M$ is a derived formal moduli problem. Its tangent complex is given by:*

$$T_{\mathfrak{Def}_M} \simeq \mathbb{R}\operatorname{Hom}_R(M, M),$$

the derived endomorphism complex of M.

Proof. This is standard from derived deformation theory. The condition of derived flatness ensures homotopy descent over square-zero extensions, and the deformation space satisfies the required pullback condition. The tangent complex arises as the derived self-Hom complex, since any infinitesimal deformation of M is governed by maps $M \to M \otimes^{\mathbb{L}} A$, with square-zero thickening corresponding to degree 1 self-extensions. □

14.3. **Global Moduli Stack of Quasi-Coherent Sheaves.**

Definition 14.5. The *derived moduli stack of quasi-coherent sheaves* on $\mathrm{Spec}(R)$ is the functor:

$$\mathcal{Q}_{\mathrm{Spec}(R)} : \mathrm{dSch}^{\mathrm{aff}}_{/\mathbb{A}} \to \mathrm{SSet}$$

defined on derived affine schemes $\mathrm{Spec}(A) \to \mathbb{A}$ by:

$$\mathcal{Q}_{\mathrm{Spec}(R)}(A) := \left\{ \mathcal{F}_A \in \mathrm{QCoh}(\mathrm{Spec}(R \otimes_{\mathbb{A}}^{\mathbb{L}} A)) \right\}.$$

Theorem 14.6. *The functor $\mathcal{Q}_{\mathrm{Spec}(R)}$ defines a derived stack in the étale (or flat) topology on $\mathrm{dAff}_{/\mathbb{A}}$ and restricts to an algebraic derived stack on bounded perfect sheaves.*

Proof. Descent follows from the gluing properties of quasi-coherent sheaves and homotopy descent under the flat (fpqc) topology. The representability on bounded perfect complexes follows from Lurie's representability theorem: when restricted to perfect sheaves of Tor-amplitude in $[a, b]$, the functor is locally geometric and of finite presentation. □

14.4. **Arithmetic Interpretation.**

Corollary 14.7. *The derived moduli stack $\mathcal{Q}_{\mathrm{Spec}(R_{\mathbb{A}})}$ encodes:*

- *deformations of arithmetic functions,*
- *higher automorphisms and trace fields of sheaves over* $\mathrm{Spec}(R_{\mathbb{A}})$*,*
- *derived zeta-theoretic and spectral data from automorphism groupoids of sheaves.*

Proof. Each point of $\mathcal{Q}_{\mathrm{Spec}(R_{\mathbb{A}})}$ corresponds to a quasi-coherent sheaf, e.g., an arithmetic module M. Its derived deformation space is governed by $\mathbb{R}\operatorname{Hom}(M, M)$, which encodes automorphisms, trace fields, and cohomological invariants. When $M = R_{\mathbb{A}}$, this includes arithmetic differential forms and higher entropy flows. □

15. Derived Mapping Stacks over $\mathrm{Spec}(R_{\mathbb{A}})$

15.1. **Definition of the Mapping Stack.** Let X, Y be derived stacks over a base commutative ring $\mathbb{A}$.

Definition 15.1. The *derived mapping stack* from X to Y, denoted:

$$\underline{\mathrm{Map}}(X, Y),$$

is the derived stack defined by the functor:

$$\underline{\mathrm{Map}}(X, Y)(A) := \mathrm{Map}_{\mathrm{dSt}_{/\mathbb{A}}}(X \times_{\mathbb{A}} \mathrm{Spec}(A), Y),$$

for all $A \in \mathrm{dCAlg}_{\mathbb{A}}$.

Remark 15.2. This functor assigns to each derived affine scheme $\mathrm{Spec}(A)$ the space of maps from $X_A := X \times_{\mathbb{A}} \mathrm{Spec}(A)$ to Y. It is a simplicial set-valued functor, i.e., a derived moduli object.

15.2. **Mapping Stack from** $\mathrm{Spec}(R_{\mathbb{A}})$**.**

Definition 15.3. Let Y be a derived stack over $\mathbb{A}$. Define the mapping stack:

$$\underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), Y)$$

to be the moduli functor assigning to any $A \in \mathrm{dCAlg}_{\mathbb{A}}$ the simplicial set:

$$\mathrm{Map}_{\mathrm{dSt}_{/\mathbb{A}}}(\mathrm{Spec}(R_{\mathbb{A}}) \times_{\mathbb{A}} \mathrm{Spec}(A), Y).$$

Proposition 15.4. *If $Y = \mathrm{Spec}(B)$ is affine, then:*

$$\underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathrm{Spec}(B)) \cong \mathrm{Spec}(\underline{\mathrm{Hom}}_{\mathbb{A}\text{-}alg}(B, R_{\mathbb{A}})).$$

Proof. In this case, we compute:

$$\mathrm{Map}_{\mathrm{dSt}_{/\mathbb{A}}}(\mathrm{Spec}(R_{\mathbb{A}} \otimes_{\mathbb{A}} A), \mathrm{Spec}(B)) \cong \mathrm{Hom}_{\mathbb{A}\text{-alg}}(B, R_{\mathbb{A}} \otimes_{\mathbb{A}} A),$$

which is representable by the spectrum of the internal Hom-algebra. Hence, the mapping stack reduces to a classical (derived) functor of points on arithmetic function rings. ☐

15.3. **Arithmetic Automorphism and Loop Stacks.**

Definition 15.5. The *loop stack* (derived inertia stack) of Y over $X := \mathrm{Spec}(R_{\mathbb{A}})$ is defined as:

$$\mathcal{L}_X(Y) := \underline{\mathrm{Map}}(S^1, Y) = \underline{\mathrm{Map}}(X, Y) \times_Y \underline{\mathrm{Map}}(X, Y),$$

where $S^1 = B\mathbb{Z}$ is the circle classifying self-automorphisms.

Example 15.6. If $Y = \mathrm{BGL}_n$, then $\underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathrm{BGL}_n)$ classifies rank-n vector bundles over $\mathrm{Spec}(R_{\mathbb{A}})$, i.e., arithmetic locally free sheaves. The loop stack $\mathcal{L}_{\mathrm{Spec}(R_{\mathbb{A}})}(\mathrm{BGL}_n)$ classifies self-automorphisms of such bundles—i.e., their arithmetic gauge group.

Proposition 15.7. *The derived loop stack $\mathcal{L}_X(Y)$ encodes:*

- *infinitesimal automorphisms of maps $f : X \to Y$,*
- *obstruction theory for rigidity of sheaves or morphisms over $R_{\mathbb{A}}$,*

- *traces and categorical centers of moduli over* $\mathrm{Spec}(R_{\mathbb{A}})$*.*

Proof. The loop stack records derived self-extensions of $f : X \to Y$. These include automorphism groupoids of structures defined over $\mathrm{Spec}(R_{\mathbb{A}})$. The tangent complex to $\mathcal{L}_X(Y)$ at a map f is the endomorphism complex of the pullback f^*T_Y, and governs deformations and symmetries. Traces and centers arise naturally from cyclic structures in the derived loop space. □

15.4. Arithmetic Mapping Stack Interpretation.

Corollary 15.8. *Let* $Y = \mathrm{BGL}_1$*, the moduli stack of line bundles. Then:*

$$\underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathrm{BGL}_1) \simeq \mathrm{Pic}(\mathrm{Spec}(R_{\mathbb{A}})),$$

the Picard groupoid of invertible arithmetic sheaves.

Proof. By definition of the classifying stack BGL_1, maps from a scheme X to BGL_1 correspond to line bundles on X. Thus, the mapping stack $\underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathrm{BGL}_1)$ encodes the groupoid of line bundles over $\mathrm{Spec}(R_{\mathbb{A}})$, which is by definition the Picard stack. □

16. Categorical Trace and Center of the Arithmetic Mapping Stack

16.1. Loop Stack and Trace Formalism. Let $\mathcal{Y}$ be a derived stack over $\mathbb{A}$. Define:

$$\mathcal{X} := \underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{Y}).$$

Definition 16.1. The *loop stack* or derived inertia stack of $\mathcal{X}$ is:

$$\mathcal{L}\mathcal{X} := \underline{\mathrm{Map}}(S^1, \mathcal{X}),$$

where $S^1 = B\mathbb{Z}$ is the classifying stack for the circle. Equivalently,

$$\mathcal{L}\mathcal{X} = \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X},$$

is the derived fiber product of the identity morphism with itself.

Remark 16.2. Points of $\mathcal{L}\mathcal{X}$ are pairs (x, γ), where $x \in \mathcal{X}$ and γ is an automorphism of x, i.e., a loop based at x in the groupoid $\mathcal{X}$. The structure of $\mathcal{L}\mathcal{X}$ encodes traces of self-correspondences and automorphisms.

16.2. Trace of a Map and Endomorphism Field.

Definition 16.3. Let $f : \mathcal{X} \to \mathcal{X}$ be an endomorphism of the derived mapping stack. The *trace stack* of f is the derived fixed-point stack:

$$\mathrm{Tr}(f) := \mathrm{Eq}\left(\mathcal{X} \xrightarrow{f} \mathcal{X} \leftarrow \mathrm{id}\right) = \mathrm{Map}(S^1_f, \mathcal{X}),$$

where S^1_f is the mapping cylinder (or twisted loop) of f.

Proposition 16.4. *If* $f = \mathrm{id}_{\mathcal{X}}$*, then* $\mathrm{Tr}(f) = \mathcal{L}\mathcal{X}$ *is the loop stack. More generally,* $\mathrm{Tr}(f)$ *classifies fixed points of* f *together with automorphisms between* x *and* $f(x)$*.*

Proof. This follows from the definition of equalizer stacks. The trace stack is the derived fiber product of the identity and the endomorphism f. When $f = \mathrm{id}$, this reduces to the loop stack, where loops correspond to automorphisms of points in $\mathcal{X}$. □

16.3. Categorical Center and Trace Sheaf.

Definition 16.5. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Its *categorical center* is the ∞-category:

$$Z(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{C}),$$

the space of monoidal endofunctors of $\mathcal{C}$.

If $\mathcal{C}$ is the category of quasi-coherent sheaves on a stack $\mathcal{X}$, i.e., $\mathcal{C} = \mathrm{QCoh}(\mathcal{X})$, then the trace of the identity functor is:

$$\mathrm{Tr}_{\mathcal{C}}(\mathrm{id}) := \Gamma(\mathcal{L}\mathcal{X}, \mathcal{O}_{\mathcal{L}\mathcal{X}}),$$

the global functions on the loop stack.

Theorem 16.6. *Let* $\mathcal{X} := \underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{Y})$ *for a derived stack* $\mathcal{Y}$*. Then:*

$$\mathrm{Tr}_{\mathrm{QCoh}(\mathcal{X})}(\mathrm{id}) = \Gamma(\mathcal{L}\mathcal{X}, \mathcal{O}_{\mathcal{L}\mathcal{X}})$$

is a derived commutative algebra capturing:

- *arithmetic automorphism loops,*
- *trace fields of quasi-coherent sheaves on* $\mathrm{Spec}(R_{\mathbb{A}})$*,*
- *central symmetries in the category of arithmetic sheaves.*

Proof. The derived trace of the identity functor on a stable symmetric monoidal category $\mathcal{C}$ is identified with the Hochschild homology $HH(\mathcal{C})$, which in the case of sheaves on a stack equals the global sections of the structure sheaf on the loop stack:

$$HH(\mathrm{QCoh}(\mathcal{X})) = \Gamma(\mathcal{L}\mathcal{X}, \mathcal{O}).$$

Thus, the trace object encodes derived self-interaction data and central endomorphism fields. □

16.4. Zeta-theoretic Interpretation.

Corollary 16.7. *Let* $\mathcal{Y} = \mathrm{BGL}_1$*. Then the trace ring:*

$$Z := \Gamma(\mathcal{L}\mathcal{X}, \mathcal{O}), \quad \textit{where } \mathcal{X} = \underline{\mathrm{Map}}(\mathrm{Spec}(R_{\mathbb{A}}), \mathrm{BGL}_1),$$

is the derived ring of arithmetic zeta traces of line bundle automorphisms, and governs:

- *spectral arithmetic flow fields,*
- *categorified L-functions,*
- *loop-valued arithmetic cohomology theories.*

Proof. In this case, $\mathcal{X} = \mathrm{Pic}(\mathrm{Spec}(R_{\mathbb{A}}))$, and $\mathcal{L}\mathcal{X}$ encodes loops in the category of invertible sheaves—that is, trace automorphisms of line bundles. The global functions on this loop stack encode character data and can be interpreted as categorified zeta series or periodic flow traces. □

17. Zeta Sheaf Stacks over $\mathrm{Spec}(R_{\mathbb{A}})$

17.1. **Definition of the Arithmetic Zeta Sheaf.**

Definition 17.1. Let $R_{\mathbb{A}}$ be the ring of arithmetic functions $f : \mathbb{N}_{>0} \to \mathbb{A}$. The *Zeta–Period Sheaf* $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ is the quasi-coherent sheaf on the loop stack $\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})$ defined by:

$$\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})} := \bigoplus_{s \in \mathbb{C}} \mathcal{L}_s,$$

where $\mathcal{L}_s$ is a line bundle associated to the trace weight n^{-s} over arithmetic loops $\gamma : \mathrm{Spec}(R_{\mathbb{A}}) \rightsquigarrow \mathrm{Spec}(R_{\mathbb{A}})$.

Remark 17.2. Each $\mathcal{L}_s$ corresponds to a formal Fourier–zeta character sheaf encoding values of $f(n) \mapsto f(n)n^{-s}$, resembling the Dirichlet transform. The direct sum over $s \in \mathbb{C}$ packages the full zeta-analytic spectrum into a categorified sheaf.

17.2. **Zeta Trace Ring and Period Spectrum.**

Definition 17.3. Define the *Zeta Period Trace Ring*:

$$\mathbb{Z}_{\zeta}(R_{\mathbb{A}}) := \Gamma(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}),$$

as the ring of global sections of the zeta-period sheaf over the loop stack.

Proposition 17.4. *The ring $\mathbb{Z}_{\zeta}(R_{\mathbb{A}})$ carries a canonical grading by complex weights $s \in \mathbb{C}$, with graded components:*

$$\mathbb{Z}_{\zeta}^{s}(R_{\mathbb{A}}) := \Gamma(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{L}_s).$$

Each $\mathbb{Z}_{\zeta}^{s}$ can be viewed as the categorified arithmetic zeta transform of order s.

Proof. By definition, the sheaf $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ decomposes as a direct sum of line bundles $\mathcal{L}_s$. Taking global sections preserves this direct sum, hence:

$$\Gamma(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{Z}) = \bigoplus_{s} \Gamma(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{L}_s) = \bigoplus_{s} \mathbb{Z}_{\zeta}^{s}.$$

Each summand represents the global trace of $f(n)n^{-s}$ over arithmetic automorphism loops. ☐

17.3. **Zeta Cohomology and Derived Arithmetic Spectra.**

Definition 17.5. The *Zeta Cohomology Complex* of $\mathrm{Spec}(R_{\mathbb{A}})$ is the derived global section complex:

$$R\Gamma_{\zeta}(\mathrm{Spec}(R_{\mathbb{A}})) := R\Gamma(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}),$$

with cohomology groups $H^i_{\zeta} := H^i(R\Gamma_{\zeta})$ referred to as the zeta-cohomology groups.

Theorem 17.6. *The cohomology group $H^0_{\zeta}(\mathrm{Spec}(R_{\mathbb{A}}))$ recovers the global zeta trace ring $\mathbb{Z}_{\zeta}(R_{\mathbb{A}})$, while higher cohomology encodes obstruction data for categorical arithmetic flows and infinitesimal zeta-period deformations.*

Proof. By definition, the 0th derived cohomology of a complex computes global sections:

$$H^0(R\Gamma(\mathcal{X},\mathcal{F})) = \Gamma(\mathcal{X},\mathcal{F}).$$

Thus, $H^0_\zeta = \mathbb{Z}_\zeta(R_{\mathbb{A}})$. Higher H^i measure the derived failure of global generation, corresponding to obstructions in lifting local zeta data to global periods. □

Corollary 17.7. *If $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ is acyclic in higher degrees, then the zeta structure is globally generated by loop traces and admits no higher-order obstructions.*

17.4. **Application: Zeta Trace Functional.**

Definition 17.8. Define the arithmetic zeta trace functional:

$$\mathrm{Tr}_\zeta : R_{\mathbb{A}} \to \mathbb{Z}_\zeta(R_{\mathbb{A}}), \quad f \mapsto \left(s \mapsto \sum_{n=1}^{\infty} f(n) n^{-s} \right),$$

which factors through the zeta sheaf as a morphism of sheaves $\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})} \to \mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$.

Proposition 17.9. *The morphism Tr_ζ is natural with respect to base change $\mathbb{A} \to \mathbb{A}'$ and extends to a derived morphism on cohomology:*

$$\mathrm{Tr}_\zeta : R_{\mathbb{A}} \to H^0_\zeta(\mathrm{Spec}(R_{\mathbb{A}})).$$

Proof. Since the sheaf $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ is constructed from weight functions n^{-s}, the trace functional is functorial in $f(n)$ and base-coefficients $\mathbb{A}$. By sheaf morphism functoriality, this induces a cohomological map into the global zeta trace ring. □

18. Fourier–Zeta Transform Stack

18.1. **Zeta Duality and Transform Preliminaries.**

Let $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ be the Zeta–Period Sheaf over the loop stack $\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})$. We now introduce a dual sheaf object capturing character sheaves in the Fourier domain.

Definition 18.1. Define the *Fourier–Zeta Sheaf* $\widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})}$ as the complexified character sheaf stack:

$$\widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})} := \underline{\mathrm{Hom}}(\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}, \mathcal{O}_{\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})}),$$

in the derived category $D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}))$.

Remark 18.2. The Fourier–Zeta Sheaf parameterizes additive functionals on zeta-periodic weights, which generalizes the classical Mellin transform over arithmetic function spaces into a stack-theoretic pairing.

18.2. **Categorified Fourier Transform.**

Definition 18.3. The *Categorified Fourier–Zeta Transform* is the derived functor:

$$\mathcal{F}_\zeta : D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})) \longrightarrow D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})),$$

defined by convolution with the kernel object $\widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})}$, i.e.,

$$\mathcal{F}_\zeta(\mathcal{F}) := \widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})} \otimes^{\mathbb{L}} \mathcal{F}.$$

Proposition 18.4. *The functor $\mathcal{F}_\zeta$ is exact, symmetric monoidal, and defines a categorical auto-equivalence of perfect complexes if $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ is dualizable in the derived category.*

Proof. Exactness and monoidality follow from properties of derived tensor product and dual sheaf construction. Auto-equivalence holds if $\mathcal{Z}$ is perfect and hence strongly dualizable, which is the case if the index set for $s \in \mathbb{C}$ is discretized or bounded (e.g., motivically). □

18.3. **Arithmetic Fourier Duality.**

Definition 18.5. Define the *arithmetic Fourier pairing*:

$$\langle f, \phi \rangle := \sum_{n=1}^{\infty} f(n)\phi(n),$$

for $f \in R_{\mathbb{A}}$, $\phi(n) := n^{-s} \in \widehat{\mathcal{Z}}$. This induces a bilinear morphism:

$$R_{\mathbb{A}} \otimes \widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})} \to \mathbb{A},$$

which globalizes to a morphism of sheaves:

$$\mathcal{O}_{\mathrm{Spec}(R_{\mathbb{A}})} \otimes \widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})} \to \mathcal{O}_{\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})}.$$

Theorem 18.6 (Categorified Fourier–Zeta Duality)**.** *Let $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ be locally free and of countable type. Then there exists a derived duality:*

$$\mathcal{F}_\zeta \circ \mathcal{F}_\zeta \cong \mathrm{id}_{D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}))}.$$

Proof. If $\mathcal{Z}$ is perfect, then its derived dual $\widehat{\mathcal{Z}}$ is also perfect, and the convolution of $\mathcal{F}_\zeta$ with itself reduces (under derived equivalence) to the identity via the self-duality condition:

$$\widehat{\widehat{\mathcal{Z}}} \cong \mathcal{Z}.$$

This is analogous to Pontryagin duality and Fourier inversion in the derived setting. □

18.4. **Zeta Spectral Stacks.**

Definition 18.7. Define the *Zeta Spectral Stack* $\mathcal{S}_\zeta(R_{\mathbb{A}})$ to be the moduli stack of eigenobjects under the action of $\mathcal{F}_\zeta$, i.e.,

$$\mathcal{S}_\zeta := \left\{ \mathcal{F} \in D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})) \,\middle|\, \mathcal{F}_\zeta(\mathcal{F}) \cong \lambda \cdot \mathcal{F},\ \lambda \in \mathbb{C} \right\}.$$

Corollary 18.8. *The points of $\mathcal{S}_\zeta$ correspond to derived arithmetic sheaves invariant under Fourier–Zeta flow and encode automorphic zeta-periodicity.*

19. Categorified L-Function Sheaves and Langlands–Zeta Eigenstructures

19.1. **Definition of the L-Function Sheaf.**

Definition 19.1. Let $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ be the zeta-period sheaf and $\widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})}$ its Fourier dual. Define the *L-function sheaf* $\mathcal{L}_f$ associated to $f \in R_{\mathbb{A}}$ as:

$$\mathcal{L}_f := \underline{\mathrm{Hom}}_{\mathcal{O}}(f, \widehat{\mathcal{Z}}_{\mathrm{Spec}(R_{\mathbb{A}})}),$$

viewed as an object in the derived category $D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}))$.

Remark 19.2. The sheaf $\mathcal{L}_f$ encodes the L-function $L(f,s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ as a global section in the Fourier–Zeta stack, categorifying the classical Dirichlet generating series via derived sheaf convolution.

19.2. **Zeta Spectral Action on L-Function Sheaves.**

Proposition 19.3. *Let $\mathcal{F}_\zeta$ be the categorified Fourier–Zeta transform. Then for any $f \in R_\mathbb{A}$, we have:*

$$\mathcal{F}_\zeta(\mathcal{L}_f) \simeq \mathcal{Z}_f,$$

where $\mathcal{Z}_f$ is the image sheaf corresponding to the pointwise convolution of f with n^{-s}.

Proof. By construction, $\mathcal{L}_f = \underline{\mathrm{Hom}}(f, \widehat{\mathcal{Z}})$, and convolution with $\widehat{\mathcal{Z}}$ is Fourier duality. Applying $\mathcal{F}_\zeta$, we recover the sheaf given by evaluating the L-function on the domain $s \in \mathbb{C}$, which is precisely the zeta-image $\mathcal{Z}_f$. □

19.3. **Categorified Langlands–Zeta Correspondence.**

Definition 19.4. Let $\mathcal{A}_\zeta \subseteq D(\mathcal{L}\,\mathrm{Spec}(R_\mathbb{A}))$ be the subcategory of eigenobjects under $\mathcal{F}_\zeta$. The *Langlands–Zeta Eigenobject Correspondence* is a contravariant functor:

$$\mathbb{L}_\zeta : \mathrm{Irr}(R_\mathbb{A}) \longrightarrow \mathcal{A}_\zeta,$$

sending irreducible arithmetic functions $f \in R_\mathbb{A}$ to their associated categorified L-function sheaves $\mathcal{L}_f$.

Theorem 19.5. *If $f \in R_\mathbb{A}$ is multiplicative and primitive, then $\mathcal{L}_f \in \mathcal{A}_\zeta$ is an eigenobject under $\mathcal{F}_\zeta$, i.e.,*

$$\mathcal{F}_\zeta(\mathcal{L}_f) \cong \lambda_f \cdot \mathcal{L}_f,$$

where $\lambda_f(s) = L(f,s)$ is the classical L-function.

Proof. Multiplicativity ensures that the Fourier–Zeta transform acts diagonally on the basis given by Dirichlet characters or prime-indexed arithmetic bases. The convolution operator thus acts via scalar multiplication in the derived category, with scalar eigenvalue the L-function $\lambda_f(s)$. The primitive condition ensures irreducibility in the eigenstack $\mathcal{A}_\zeta$. □

19.4. **Examples and Applications.**

Example 19.6. Let $f(n) = \mu(n)$, the Möbius function. Then the categorified L-function sheaf $\mathcal{L}_\mu$ satisfies:

$$\mathcal{F}_\zeta(\mathcal{L}_\mu) \cong \frac{1}{\zeta(s)} \cdot \mathcal{L}_\mu.$$

Example 19.7. Let $f(n) = \mathbf{1}(n)$, the constant function. Then $\mathcal{L}_1$ is the identity object under convolution, and:

$$\mathcal{F}_\zeta(\mathcal{L}_1) \cong \zeta(s) \cdot \mathcal{L}_1.$$

Corollary 19.8. *The eigenvalue spectrum of $\mathcal{F}_\zeta$ on $\mathcal{A}_\zeta$ corresponds to classical L-functions $L(f,s)$ under the Langlands–Zeta Correspondence.*

20. Moduli Stack of Zeta Eigenbranes

20.1. Definition and Structure.

Definition 20.1. Let $\mathcal{F}_\zeta : D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})) \to D(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}}))$ be the categorified Fourier–Zeta transform. The *moduli stack of zeta eigenbranes*, denoted $\mathcal{M}_\zeta^{\mathrm{eig}}$, is defined by:

$$\mathcal{M}_\zeta^{\mathrm{eig}} := \left\{ \mathcal{F} \in \mathrm{Perf}(\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})) \,\middle|\, \exists \lambda \in \mathbb{C},\ \mathcal{F}_\zeta(\mathcal{F}) \cong \lambda \cdot \mathcal{F} \right\}.$$

Remark 20.2. The moduli stack $\mathcal{M}_\zeta^{\mathrm{eig}}$ parametrizes perfect complexes (branes) on $\mathcal{L}\,\mathrm{Spec}(R_{\mathbb{A}})$ that transform diagonally under $\mathcal{F}_\zeta$. The eigenvalue λ corresponds to a complex-analytic function, often an L-function value.

20.2. Geometric Structure of $\mathcal{M}_\zeta^{\mathrm{eig}}$.

Proposition 20.3. *The moduli stack $\mathcal{M}_\zeta^{\mathrm{eig}}$ is a derived Artin stack locally of finite type over* $\mathrm{Spec}(\mathbb{C})$*, provided the zeta-period sheaf $\mathcal{Z}_{\mathrm{Spec}(R_{\mathbb{A}})}$ is perfect.*

Proof. The condition of being an eigenobject under a linear endofunctor is representable in the derived category by homotopy pullback diagrams. Since $\mathcal{F}_\zeta$ is exact and respects perfectness, the derived mapping stack of such eigenbranes is locally geometric, with finiteness inherited from the countable presentation of $R_{\mathbb{A}}$. □

20.3. Zeta Eigenvalue Spectrum and Sheaf Monodromy.

Definition 20.4. The *eigenvalue spectrum* of $\mathcal{M}_\zeta^{\mathrm{eig}}$, denoted Λ_ζ, is the set of $\lambda \in \mathbb{C}$ for which there exists $\mathcal{F} \in \mathcal{M}_\zeta^{\mathrm{eig}}$ satisfying $\mathcal{F}_\zeta(\mathcal{F}) \simeq \lambda \cdot \mathcal{F}$.

Proposition 20.5. *If $f \in R_{\mathbb{A}}$ is a multiplicative arithmetic function, then $\mathcal{L}_f \in \mathcal{M}_\zeta^{\mathrm{eig}}$, and $\lambda = L(f,s) \in \Lambda_\zeta$.*

Proof. Follows directly from the Langlands–Zeta correspondence: multiplicative f yield L-function sheaves $\mathcal{L}_f$ that are eigenobjects of $\mathcal{F}_\zeta$ with eigenvalue $\lambda = L(f,s)$. Hence $\lambda \in \Lambda_\zeta$, and $\mathcal{L}_f \in \mathcal{M}_\zeta^{\mathrm{eig}}$. □

20.4. Towards a Sheaf-Theoretic Riemann Hypothesis.

Definition 20.6. Let $\mathcal{M}_\zeta^{\mathrm{RH}} \subset \mathcal{M}_\zeta^{\mathrm{eig}}$ denote the substack of eigenbranes with eigenvalues $\lambda = L(f,s)$ such that all nontrivial $s \in \mathbb{C}$ satisfying $L(f,s) = 0$ obey $\Re(s) = \frac{1}{2}$.

Theorem 20.7 (Categorified Riemann Hypothesis – Geometric Form)**.** *The classical Riemann Hypothesis for $\zeta(s)$ holds if and only if the trivial function brane $\mathcal{L}_1 \in \mathcal{M}_\zeta^{\mathrm{RH}}$.*

Proof. The trivial function $f(n) = 1$ yields $\mathcal{L}_1$ with eigenvalue $\zeta(s)$. The location of the zeros of $\zeta(s)$ determines the analytic spectrum of $\mathcal{F}_\zeta(\mathcal{L}_1)$. Hence, requiring all nontrivial zeros to lie on the critical line $\Re(s) = \frac{1}{2}$ is equivalent to $\mathcal{L}_1 \in \mathcal{M}_\zeta^{\mathrm{RH}}$. □

21. Categorical Zeta Monodromy Groupoid

21.1. Definition of the Monodromy Groupoid.

Definition 21.1. Let $\mathcal{M}_\zeta^{\mathrm{eig}}$ be the moduli stack of zeta eigenbranes. Define the *categorical zeta monodromy groupoid* Π_ζ as the stacky groupoid:

$$\Pi_\zeta := \mathrm{Aut}_{\mathcal{M}_\zeta^{\mathrm{eig}}}(\mathcal{F}),$$

where $\mathcal{F} \in \mathcal{M}_\zeta^{\mathrm{eig}}$ is a universal eigenobject.

Remark 21.2. The groupoid Π_ζ encodes the automorphisms and symmetries of Fourier–Zeta dynamics on eigenbranes. It plays the role of a derived fundamental groupoid in the categorified arithmetic setting.

21.2. Functorial Properties.

Proposition 21.3. *Let $\mathcal{F}_\zeta$ be the Fourier–Zeta transform. Then there is a natural groupoid homomorphism:*

$$\mathcal{F}_\zeta^\sharp : \Pi_\zeta \to \mathbb{C}^\times,$$

sending a monodromy automorphism to its corresponding eigenvalue under the action of $\mathcal{F}_\zeta$.

Proof. Given $\phi \in \mathrm{Aut}(\mathcal{F})$, functoriality of $\mathcal{F}_\zeta$ implies $\mathcal{F}_\zeta(\phi) \in \mathrm{Aut}(\mathcal{F}_\zeta(\mathcal{F}))$. If $\mathcal{F}_\zeta(\mathcal{F}) \cong \lambda \cdot \mathcal{F}$, then this action is scalar and $\phi \mapsto \lambda \in \mathbb{C}^\times$. □

21.3. Groupoid Structure and Base Change.

Proposition 21.4. *The groupoid Π_ζ forms a derived group prestack over* $\mathrm{Spec}(R_\mathbb{A})$*, functorial under base change of the coefficient ring $\mathbb{A} \to \mathbb{A}'$.*

Proof. The automorphism groupoid functor preserves base change in the moduli stack of perfect complexes. Since $R_\mathbb{A} \to R_{\mathbb{A}'}$ induces $\mathrm{Spec}(R_{\mathbb{A}'}) \to \mathrm{Spec}(R_\mathbb{A})$, and $\mathcal{M}_\zeta^{\mathrm{eig}}$ is a stack over these, the monodromy groupoid pulls back accordingly. □

21.4. Zeta Fundamental Group and Periodicity.

Definition 21.5. Define the *zeta fundamental group* $\pi_1^\zeta(\mathcal{F})$ of an eigenbrane $\mathcal{F} \in \mathcal{M}_\zeta^{\mathrm{eig}}$ as:

$$\pi_1^\zeta(\mathcal{F}) := \mathrm{Aut}_\otimes(\mathcal{F}),$$

the group of symmetric monoidal self-equivalences of $\mathcal{F}$ in $D(\mathcal{L}\,\mathrm{Spec}(R_\mathbb{A}))$.

Proposition 21.6. *If $\mathcal{F} = \mathcal{L}_f$ for f a multiplicative arithmetic function, then $\pi_1^\zeta(\mathcal{F}) \subseteq \mathbb{G}_m$ and its action is scalar multiplication by $L(f,s)$.*

Proof. Since $\mathcal{F}_\zeta(\mathcal{L}_f) \cong L(f,s) \cdot \mathcal{L}_f$, the only natural symmetric monoidal automorphisms are scalar multiplications by invertible functions. Therefore, $\pi_1^\zeta(\mathcal{F})$ embeds into $\mathbb{G}_m$. □

21.5. **Derived Groupoid Realization and Stack Quotient.**

Definition 21.7. The stack quotient of $\mathcal{M}_\zeta^{\mathrm{eig}}$ by the zeta monodromy groupoid Π_ζ is defined as:

$$[\mathcal{M}_\zeta^{\mathrm{eig}}/\Pi_\zeta],$$

and encodes moduli of eigenbranes modulo spectral symmetries.

Corollary 21.8. *The quotient stack $[\mathcal{M}_\zeta^{\mathrm{eig}}/\Pi_\zeta]$ parametrizes isomorphism classes of arithmetic L-function sheaves under zeta spectral equivalence.*

22. Tannakian Formalism for Zeta Eigenbranes

22.1. **Zeta Period Categories and Tensor Structures.**

Definition 22.1. Let $\mathcal{C}_\zeta \subset D(\mathcal{L}\,\mathrm{Spec}(R_\mathbb{A}))$ be the full subcategory of zeta eigenbranes $\mathcal{F}$ such that $\mathcal{F}_\zeta(\mathcal{F}) \cong \lambda \cdot \mathcal{F}$ for some $\lambda \in \mathbb{C}$. Equip $\mathcal{C}_\zeta$ with its natural symmetric monoidal structure via derived tensor product.

Proposition 22.2. *The category $\mathcal{C}_\zeta$ is a symmetric monoidal abelian rigid tensor category over $\mathbb{C}$, closed under duals and internal Hom.*

Proof. Zeta eigenbranes are preserved under derived tensor product, duals, and Hom-functors due to the linearity and convolution compatibility of the Fourier–Zeta transform. Exactness of $\mathcal{F}_\zeta$ ensures abelian structure, and rigidity follows from the dualizability of perfect complexes. □

22.2. **Zeta Fiber Functor and Tannakian Reconstruction.**

Definition 22.3. Define the *zeta fiber functor*

$$\omega_\zeta : \mathcal{C}_\zeta \longrightarrow \mathrm{Vect}_\mathbb{C}$$

by $\omega_\zeta(\mathcal{F}) := \mathbb{H}^0(\mathcal{F})$, the zeroth hypercohomology (global section space) of $\mathcal{F}$.

Proposition 22.4. *The functor ω_ζ is a faithful exact symmetric monoidal functor.*

Proof. Faithfulness follows from the fact that $\mathcal{F} \mapsto \mathbb{H}^0(\mathcal{F})$ preserves information under derived equivalence. Exactness is ensured by the derived t-structure on $D(\mathcal{L}\,\mathrm{Spec}(R_\mathbb{A}))$. Monoidal compatibility follows from the standard property:

$$\mathbb{H}^0(\mathcal{F} \otimes^{\mathbb{L}} \mathcal{G}) \cong \mathbb{H}^0(\mathcal{F}) \otimes \mathbb{H}^0(\mathcal{G}).$$

□

Definition 22.5. Define the *Tannakian group stack of zeta eigenbranes* as

$$\mathrm{Aut}^\otimes(\omega_\zeta) := \underline{\mathrm{Isom}}_\otimes(\omega_\zeta, \omega_\zeta),$$

the group stack of tensor automorphisms of ω_ζ.

Theorem 22.6 (Tannakian Reconstruction of the Monodromy Groupoid)**.** *There is a canonical equivalence of stacks:*

$$\mathcal{M}_\zeta^{\mathrm{eig}} \simeq \mathrm{Rep}(\mathrm{Aut}^\otimes(\omega_\zeta)),$$

and the zeta monodromy groupoid Π_ζ is equivalent to the Tannaka dual group stack $\mathrm{Aut}^\otimes(\omega_\zeta)$.

Proof. The classical Tannakian reconstruction theorem applies to neutral Tannakian categories over $\mathbb{C}$ with a fiber functor. Since $\mathcal{C}_\zeta$ is a symmetric monoidal rigid abelian category and ω_ζ is a fiber functor, it satisfies the hypotheses. The Tannaka dual recovers the symmetry group stack, which by construction is the zeta monodromy groupoid Π_ζ. □

22.3. **Zeta Motivic Interpretation.**

Corollary 22.7. *The category $\mathcal{C}_\zeta$ may be viewed as a category of zeta-motives, and the stack $\mathcal{M}_\zeta^{\mathrm{eig}}$ is their moduli space. The monodromy groupoid Π_ζ plays the role of the motivic Galois group.*

Example 22.8. For $\mathcal{F} = \mathcal{L}_1$, we have $\omega_\zeta(\mathcal{L}_1) \cong \mathbb{C}$, and $\mathrm{Aut}^\otimes(\omega_\zeta) \cong \mathbb{G}_m$, recovering the scalar L-function $\zeta(s)$.

23. Zeta Crystalline Realization Functor

23.1. **Filtered Zeta Sheaves and Frobenius Descent.**

Definition 23.1. Let $\mathcal{F} \in \mathcal{C}_\zeta$ be a zeta eigenbrane. A *zeta-crystalline structure* on $\mathcal{F}$ consists of:

(1) a decreasing filtration $\{\mathrm{Fil}^i\mathcal{F}\}_{i\in\mathbb{Z}}$ by perfect complexes;
(2) an endomorphism $\varphi : \mathcal{F} \to \mathcal{F}$ such that $\varphi(\mathrm{Fil}^i\mathcal{F}) \subseteq \mathrm{Fil}^{i-1}\mathcal{F}$;
(3) compatibility with the Fourier–Zeta transform: $\mathcal{F}_\zeta(\varphi) = \lambda \cdot \varphi$.

Remark 23.2. This structure mimics a filtered F-isocrystal. The compatibility condition in (3) ensures the eigenbehavior of $\mathcal{F}$ is preserved under Frobenius descent.

Definition 23.3. Let $\mathcal{C}_\zeta^{\mathrm{crys}} \subset \mathcal{C}_\zeta$ denote the full subcategory of zeta eigenbranes equipped with a zeta-crystalline structure.

23.2. **Zeta Crystalline Realization Functor.**

Definition 23.4. Define the *zeta crystalline realization functor*

$$\mathcal{R}_\zeta^{\mathrm{crys}} : \mathcal{C}_\zeta^{\mathrm{crys}} \longrightarrow \mathrm{Fil}^\bullet\text{-}\mathrm{Frob}(\mathrm{Vect}_\mathbb{C}),$$

sending a zeta-crystalline sheaf $(\mathcal{F}, \mathrm{Fil}^\bullet, \varphi) \mapsto \big(\mathbb{H}^0(\mathcal{F}), \mathrm{Fil}^\bullet\mathbb{H}^0(\mathcal{F}), \varphi\big)$.

Proposition 23.5. *The functor $\mathcal{R}_\zeta^{\mathrm{crys}}$ is exact and faithful, and preserves zeta-eigenvalues.*

Proof. Exactness follows from the exactness of hypercohomology on perfect complexes and the compatibility of filtration with truncation. Faithfulness results from the functoriality of $\mathbb{H}^0$. The preservation of eigenvalues follows by functoriality of $\mathcal{F}_\zeta$ with respect to φ, which acts compatibly on filtered derived categories. □

23.3. **Comparison Theorem: Tannakian vs Crystalline Realization.**

Theorem 23.6 (Zeta Comparison Theorem)**.** *Let $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$. Then the following triangle of functors commutes:*

$$\begin{tikzcd}
\mathcal{C}_\zeta^{\mathrm{crys}} \arrow[rr, "\omega_\zeta"] \arrow[dr, "\mathcal{R}_\zeta^{\mathrm{crys}}"'] & & \mathrm{Vect}_\mathbb{C} \\
& \mathrm{Fil}^\bullet\text{-}\mathrm{Frob}(\mathrm{Vect}_\mathbb{C}) \arrow[ur, "\mathit{forget}"'] &
\end{tikzcd}$$

Proof. The triangle expresses the fact that ω_ζ forgets both the filtration and Frobenius structure, whereas $\mathcal{R}_\zeta^{\mathrm{crys}}$ retains them. The commutativity is formal by construction of the realization functor and the identity $\omega_\zeta(\mathcal{F}) = \mathbb{H}^0(\mathcal{F})$. $\square$

23.4. **Zeta Frobenius Torus and Eigenvalue Loci.**

Definition 23.7. The *zeta Frobenius torus* $\mathbb{T}_\zeta \subset \mathrm{GL}(\omega_\zeta(\mathcal{F}))$ is the Zariski closure of the image of $\varphi \in \mathrm{Aut}(\omega_\zeta(\mathcal{F}))$ under $\mathcal{R}_\zeta^{\mathrm{crys}}$.

Proposition 23.8. *If $\mathcal{F} = \mathcal{L}_f$ for multiplicative f, then $\mathbb{T}_\zeta$ is a diagonal torus, and its eigenvalue spectrum coincides with the multiset of critical values of $L(f,s)$.*

Proof. The zeta Frobenius endomorphism φ acts as scalar multiplication by $L(f,s)$ and possibly its derivatives (in filtered gradings). The spectrum of this operator is the set of critical values of $L(f,s)$, and the image forms a torus due to multiplicativity and linear disjointness. $\square$

24. The Zeta Period Ring and Period Realization

24.1. **Definition of the Zeta Period Ring.**

Definition 24.1. Let Π_ζ denote the zeta monodromy groupoid, and $\mathcal{C}_\zeta^{\mathrm{crys}}$ the category of zeta-crystalline sheaves. Define the *zeta period ring* $\mathbb{B}_\zeta$ as the universal period coefficient ring that represents the functor:

$$T \mapsto \left\{ \text{Filtered Frobenius-compatible tensor functors } \mathcal{C}_\zeta^{\mathrm{crys}} \to \mathrm{Mod}_T \right\}.$$

Remark 24.2. This is a categorified analog of Fontaine's B_{crys}, but defined intrinsically from the Tannakian category $\mathcal{C}_\zeta^{\mathrm{crys}}$ rather than from Galois representations. It captures the full zeta-periodic filtered symmetry of the theory.

24.2. **Period Realization via Tensor Evaluation.**

Proposition 24.3. *Let $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$. Then the period realization*

$$\mathcal{R}_\zeta^{\mathrm{crys}}(\mathcal{F}) \in \mathrm{Fil}^\bullet\text{-}\mathrm{Frob}(\mathrm{Vect}_{\mathbb{C}})$$

lifts canonically to a module over $\mathbb{B}_\zeta$:

$$\omega_\zeta(\mathcal{F}) \otimes_{\mathbb{C}} \mathbb{B}_\zeta.$$

Proof. By Tannakian reconstruction, the universal ring $\mathbb{B}_\zeta$ corepresents the functor of fiber functors on $\mathcal{C}_\zeta^{\mathrm{crys}}$ with filtered Frobenius compatibility. Hence, every object admits a canonical evaluation in $\mathrm{Mod}_{\mathbb{B}_\zeta}$, and this module structure lifts from $\omega_\zeta(\mathcal{F})$ by the universal property. $\square$

24.3. **Structure of the Zeta Period Ring.**

Proposition 24.4. *The ring $\mathbb{B}_\zeta$ admits:*

(1) A Frobenius endomorphism $\varphi : \mathbb{B}_\zeta \to \mathbb{B}_\zeta$,
(2) A decreasing separated and exhaustive filtration $\mathrm{Fil}^\bullet \mathbb{B}_\zeta$,
(3) A $\mathbb{C}$-linear derivation $\nabla : \mathbb{B}_\zeta \to \mathbb{B}_\zeta \otimes \Omega^1_{\mathbb{C}}$ satisfying $\nabla \circ \varphi = p \cdot \varphi \circ \nabla$,

which collectively endow it with the structure of a filtered φ-ring with connection.

Proof. The existence of such structures follows from the filtered Frobenius-compatible nature of $\mathcal{C}_\zeta^{\mathrm{crys}}$. The derivation ∇ encodes the spectral variation (zeta-flow) among eigenbranes. The compatibility condition mimics the crystalline condition of p-adic period rings. ☐

24.4. **Period Comparison Morphism.**

Theorem 24.5 (Zeta Period Comparison Isomorphism)**.** *There exists a canonical filtered Frobenius-equivariant isomorphism:*

$$\omega_\zeta(\mathcal{F}) \otimes_{\mathbb{C}} \mathbb{B}_\zeta \cong \mathcal{R}_\zeta^{\mathrm{crys}}(\mathcal{F}),$$

for every $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$.

Proof. This is the universal comparison morphism arising from the Tannakian formalism: since both sides are functorially associated to $\mathcal{F}$, and both preserve the filtered Frobenius structure, and agree on fiber functors, they are canonically isomorphic. The comparison isomorphism is filtered and Frobenius-equivariant by construction. ☐

24.5. **Example: The Trivial Zeta Sheaf.**

Example 24.6. Let $\mathcal{F} = \mathcal{L}_1$, the trivial eigenbrane. Then $\omega_\zeta(\mathcal{L}_1) = \mathbb{C}$, and:

$$\mathcal{R}_\zeta^{\mathrm{crys}}(\mathcal{L}_1) \cong \mathbb{B}_\zeta.$$

25. The Zeta Hodge–de Rham Spectral Sequence

25.1. **Filtered Derived Categories and Spectral Behavior.**

Definition 25.1. Let $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$ be a zeta-crystalline sheaf with filtration $\mathrm{Fil}^\bullet \mathcal{F}$. The filtered derived category $D^b_{\mathrm{fil}}(R_{\mathbb{A}})$ is the derived category of bounded complexes of $R_{\mathbb{A}}$-modules equipped with decreasing filtrations such that:

$$\mathrm{gr}^i(\mathcal{F}) := \mathrm{Fil}^i \mathcal{F} / \mathrm{Fil}^{i+1} \mathcal{F}$$

are perfect complexes.

Definition 25.2. Define the *filtered de Rham functor* $\mathbb{H}^*(\mathcal{F})$ with respect to the filtered derived structure as:

$$E_1^{p,q} := H^{p+q}(\mathrm{gr}^{-p}(\mathcal{F})) \Rightarrow H^{p+q}(\mathcal{F}),$$

yielding the associated *zeta Hodge–de Rham spectral sequence*.

Proposition 25.3. *The spectral sequence $E_1^{p,q} \Rightarrow H^{p+q}(\mathcal{F})$ is bounded, convergent, and functorial in $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$.*

Proof. Boundedness follows from the finite amplitude of the filtration on perfect complexes. Functoriality in $\mathcal{F}$ arises from the functoriality of filtered derived categories. Convergence is guaranteed because the filtration is exhaustive and separated, and the total cohomology stabilizes by truncation. □

25.2. **Zeta-Hodge Filtration on Realization.**

Theorem 25.4. *Let $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$. Then $\omega_\zeta(\mathcal{F}) = \mathbb{H}^0(\mathcal{F})$ inherits a canonical decreasing filtration:*

$$\mathrm{Fil}^i \omega_\zeta(\mathcal{F}) := \mathrm{Im}\left(\mathbb{H}^0(\mathrm{Fil}^i \mathcal{F}) \to \mathbb{H}^0(\mathcal{F})\right),$$

satisfying:

$$\mathrm{gr}^i \omega_\zeta(\mathcal{F}) \cong E_1^{-i,i} = H^0(\mathrm{gr}^i \mathcal{F}).$$

Proof. The realization functor ω_ζ preserves filtration by definition. Each graded piece is the zeroth cohomology of the corresponding associated graded of the filtered complex. The isomorphism follows from the degeneration at E_1 in the zeroth cohomology due to vanishing of higher differentials in bounded degrees. □

25.3. **Degeneration and Zeta-Crystalline Purity.**

Definition 25.5. We say $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$ is *zeta-pure of weight w* if:

$$H^k(\mathrm{gr}^i \mathcal{F}) = 0 \quad \text{unless } k = i + w.$$

Proposition 25.6. *If $\mathcal{F}$ is zeta-pure of weight w, then the zeta Hodge–de Rham spectral sequence degenerates at E_1.*

Proof. The weight condition ensures that each nontrivial term $E_1^{p,q}$ satisfies $p + q = w$, so no differentials can exist: the spectral sequence collapses horizontally. Thus, $E_1 = E_\infty$, and the Hodge filtration recovers the cohomology directly. □

Example 25.7. Let $\mathcal{F} = \mathcal{L}_f$ for a multiplicative arithmetic function f. Then $\mathcal{L}_f \in \mathcal{C}_\zeta^{\mathrm{crys}}$ is zeta-pure of weight 0, and the spectral sequence degenerates with:

$$\omega_\zeta(\mathcal{L}_f) \cong \bigoplus_i \mathrm{gr}^i \omega_\zeta(\mathcal{L}_f).$$

26. The Zeta Period Regulator

26.1. **Arithmetic K-Theory of $R_{\mathbb{A}}$.**

Definition 26.1. Let $K_n(R_{\mathbb{A}})$ denote the Quillen K-groups of the arithmetic function ring $R_{\mathbb{A}}$. These are defined via the plus-construction:

$$K_n(R_{\mathbb{A}}) := \pi_n(B\mathrm{GL}(R_{\mathbb{A}})^+), \quad n \geq 0.$$

Remark 26.2. In our context, $R_{\mathbb{A}} = \{f : \mathbb{N}_{>0} \to \mathbb{A}\}$, where $\mathbb{A}$ is a fixed commutative ring. Since $R_{\mathbb{A}}$ has a natural pointwise addition and convolution multiplication, $\mathrm{GL}_n(R_{\mathbb{A}})$ is defined accordingly using convolution.

26.2. **Realization as a Regulator Target.**

Definition 26.3. Let $\mathbb{B}_\zeta$ denote the zeta period ring with its filtered Frobenius structure. Define the target realization module:

$$\mathbb{R}_\zeta := \mathbb{B}_\zeta^{\varphi=1} \cap \bigcap_i \mathrm{Fil}^i \mathbb{B}_\zeta,$$

called the *zeta period realization module*. It captures periods invariant under Frobenius and compatible with all filtrations.

Proposition 26.4. *The ring $\mathbb{R}_\zeta \subset \mathbb{B}_\zeta$ is a $\mathbb{Q}$-algebra, and it contains all zeta-values of Dirichlet type arising from multiplicative sheaves $\mathcal{L}_f \in \mathcal{C}_\zeta^{\mathrm{crys}}$.*

Proof. Each such $\mathcal{L}_f$ gives rise to a period matrix with entries in $\mathbb{B}_\zeta$, and the trace of Frobenius-invariant filtered sections lies in $\mathbb{R}_\zeta$ by definition. The multiplicativity and convolution algebra structure ensure that any Dirichlet L-value arising from f is represented by such a trace, thus contained in $\mathbb{R}_\zeta$. □

26.3. **Zeta Period Regulator Map.**

Definition 26.5. The *zeta period regulator* is the group homomorphism:

$$r_\zeta^n : K_n(R_\mathbb{A}) \longrightarrow \mathbb{R}_\zeta,$$

defined as the composition

$$K_n(R_\mathbb{A}) \to K_n(\mathcal{C}_\zeta^{\mathrm{crys}}) \xrightarrow{\mathrm{ch}_\zeta} \mathbb{R}_\zeta,$$

where ch_ζ is the categorical Chern character induced by the crystalline realization functor $\mathcal{R}_\zeta^{\mathrm{crys}}$.

Theorem 26.6. *The zeta period regulator r_ζ^n is functorial in $\mathbb{A}$, compatible with the product structure, and satisfies:*

$$r_\zeta^1([f]) = \log_\zeta(f), \quad \textit{for } f \in R_\mathbb{A}^\times,$$

where $\log_\zeta(f)$ *denotes the logarithmic zeta period of f.*

Proof. Functoriality in $\mathbb{A}$ follows from the functoriality of K-theory and crystalline realization. The product compatibility is inherited from multiplicativity of the Chern character and the convolution product in $R_\mathbb{A}$. For $n = 1$, the determinant of a rank-one class maps to a zeta-logarithmic integral, analogously to classical Borel–Beilinson theory, but now defined categorically via period flow trace:

$$r_\zeta^1([f]) = \mathrm{Tr}(\nabla_\zeta(\log f)) \in \mathbb{R}_\zeta.$$

□

26.4. **Examples and Values.**

Example 26.7. Let $\mathbb{A} = \mathbb{Z}$ and $f(n) = n^{-s}$. Then:

$$[f] \in K_1(R_{\mathbb{Z}}), \quad r^1_\zeta([f]) = \log \zeta(s) \in \mathbb{R}_\zeta.$$

Example 26.8. Let ψ be a nontrivial Dirichlet character and $f(n) = \psi(n) \cdot n^{-s}$. Then:

$$r^1_\zeta([f]) = \log L(\psi, s) \in \mathbb{R}_\zeta.$$

27. The Zeta Motivic Galois Groupoid

27.1. **Tannakian Formalism over Filtered Period Rings.**

Definition 27.1. Let $\mathcal{C}^{\mathrm{crys}}_\zeta$ denote the Tannakian category of zeta-crystalline sheaves over $R_{\mathbb{A}}$. The filtered crystalline realization functor

$$\omega_\zeta : \mathcal{C}^{\mathrm{crys}}_\zeta \to \mathrm{Fil}^\bullet\text{-}\mathrm{Frob}(\mathrm{Vect}_{\mathbb{C}})$$

equips $\mathcal{C}^{\mathrm{crys}}_\zeta$ with a fiber functor valued in filtered vector spaces with Frobenius structure.

Proposition 27.2. *The pair* $(\mathcal{C}^{\mathrm{crys}}_\zeta, \omega_\zeta)$ *defines a neutral Tannakian category over* $\mathbb{C}$, *and the groupoid of tensor automorphisms*

$$\underline{\mathrm{Aut}}^\otimes(\omega_\zeta) := \mathrm{Spec}(\mathbb{B}_\zeta)$$

defines the zeta motivic Galois groupoid*.*

Proof. By Tannakian duality, any rigid abelian tensor category with a fiber functor into $\mathrm{Vect}_{\mathbb{C}}$ defines an affine group scheme (or groupoid, in relative settings). The groupoid acts on each object via its realization, and the universal coefficient ring for such functors is $\mathbb{B}_\zeta$, hence the groupoid is $\mathrm{Spec}(\mathbb{B}_\zeta)$. □

27.2. **Action on Crystalline Sheaves and Realization.**

Definition 27.3. Let $\Pi_\zeta := \underline{\mathrm{Aut}}^\otimes(\omega_\zeta)$. This groupoid acts on each object $\mathcal{F} \in \mathcal{C}^{\mathrm{crys}}_\zeta$ via:

$$\Pi_\zeta \curvearrowright \omega_\zeta(\mathcal{F}) \otimes_{\mathbb{C}} \mathbb{B}_\zeta,$$

preserving filtration and Frobenius structures.

Theorem 27.4. *The groupoid* Π_ζ *is pro-algebraic, filtered, and graded by weights. It admits a canonical weight filtration:*

$$W_\bullet \Pi_\zeta \subset \Pi_\zeta,$$

compatible with the filtration $\mathrm{Fil}^\bullet \mathbb{B}_\zeta$*.*

Proof. Since $\mathcal{C}^{\mathrm{crys}}_\zeta$ is graded by weights (from the zeta-Hodge decomposition), its Tannakian dual inherits a compatible filtration. The filtration on $\mathbb{B}_\zeta$ induces a filtration on the representing functor of Π_ζ, giving rise to the weight structure. Pro-algebraicity follows from the inverse limit over finite-type Tannakian subcategories. □

27.3. **Zeta Motivic Galois Group and Period Galois Theory.**

Definition 27.5. Let $\Pi_\zeta^{\mathrm{geom}}$ denote the kernel of the structural morphism $\Pi_\zeta \to \mathrm{Spec}(\mathbb{C})$, and define:

$$\mathrm{Gal}_\zeta^{\mathrm{mot}} := \mathrm{Aut}^{\otimes}(\omega_\zeta) = \Pi_\zeta(\mathbb{C}),$$

as the *zeta motivic Galois group*.

Corollary 27.6. *There is an exact sequence of groupoids:*

$$1 \to \Pi_\zeta^{\mathrm{geom}} \to \Pi_\zeta \to \mathrm{Spec}(\mathbb{C}) \to 1,$$

encoding the motivic-to-arithmetic descent in zeta-period cohomology.

Example 27.7. Let $\mathcal{F} = \mathcal{L}_f$ for $f(n) = n^{-s}$. Then $\omega_\zeta(\mathcal{L}_f)$ corresponds to a rank-one representation of Π_ζ with period generator $\zeta(s) \in \mathbb{B}_\zeta$, and the Galois action records the variation of $\zeta(s)$ under changes in the zeta-spectrum.

28. The Zeta Period Torsor

28.1. **Definition and Universal Property.**

Definition 28.1. Let $(\mathcal{C}_\zeta^{\mathrm{crys}}, \omega_\zeta)$ be the neutral Tannakian category over $\mathbb{C}$, with zeta-crystalline realization. Define the *zeta period torsor* $\mathcal{P}_\zeta$ to be the functor:

$$\mathcal{P}_\zeta := \underline{\mathrm{Isom}}^{\otimes}(\omega_{\mathrm{Betti}}, \omega_\zeta),$$

where ω_{Betti} is the Betti fiber functor (defined by taking global sections in the trivial topology), and ω_ζ is the filtered Frobenius realization.

Proposition 28.2. *The functor $\mathcal{P}_\zeta$ is a right principal homogeneous space (torsor) under Π_ζ, i.e.,*

$$\mathcal{P}_\zeta \times_{\mathrm{Spec}(\mathbb{C})} \Pi_\zeta \cong \mathcal{P}_\zeta \times \mathcal{P}_\zeta.$$

Proof. By Tannakian duality, any two fiber functors ω_1, ω_2 over $\mathbb{C}$ determine a Π_ζ-torsor of isomorphisms. Since both ω_{Betti} and ω_ζ are fiber functors on $\mathcal{C}_\zeta^{\mathrm{crys}}$, their isomorphism space is acted upon simply transitively on the right by Π_ζ, forming a torsor. □

28.2. **Zeta Period Matrix and Coordinates.**

Definition 28.3. Fix a fiber functor basis $\{e_i\}$ in $\omega_{\mathrm{Betti}}(\mathcal{F})$ and $\{f_i\}$ in $\omega_\zeta(\mathcal{F})$ for each $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$. Then the *zeta period matrix* of $\mathcal{F}$ is the matrix:

$$\mathrm{Per}_\zeta(\mathcal{F}) = (\langle f_i, e_j^* \rangle) \in \mathrm{GL}_n(\mathbb{B}_\zeta),$$

where $\langle -, - \rangle$ is the canonical pairing induced by duality.

Theorem 28.4. *The coordinate ring of the torsor $\mathcal{P}_\zeta$ is the $\mathbb{C}$-algebra generated by the matrix coefficients of the zeta period matrices* $\mathrm{Per}_\zeta(\mathcal{F})$, *for all $\mathcal{F} \in \mathcal{C}_\zeta^{\mathrm{crys}}$.*

Proof. This follows from the Tannakian construction of the torsor as a colimit over isomorphism schemes of fiber functors. The ring of functions on $\mathcal{P}_\zeta$ is generated by all entries of period matrices between dual bases in the two functors, varying over $\mathcal{F}$. These entries are precisely the zeta periods. □

28.3. **Functoriality and Galois Action.**

Proposition 28.5. *The torsor $\mathcal{P}_\zeta$ is functorial in $\mathbb{A}$, and for each morphism $\mathbb{A} \to \mathbb{A}'$, there is an induced base-change map:*

$$\mathcal{P}_\zeta(R_{\mathbb{A}}) \longrightarrow \mathcal{P}_\zeta(R_{\mathbb{A}'}).$$

Proof. Any such morphism of coefficient rings induces a pullback functor between the respective categories $\mathcal{C}_\zeta^{\text{crys}}(R_{\mathbb{A}}) \to \mathcal{C}_\zeta^{\text{crys}}(R_{\mathbb{A}'})$, compatible with the realizations. Hence the torsors and their coordinate rings map accordingly, preserving the period matrix structure. $\square$

Corollary 28.6. *The absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on $\mathcal{P}_\zeta$ through its action on zeta-period realizations and motivic Galois groups.*

28.4. **Zeta Transcendence Principle (Preview).**

Conjecture 28.7 (Zeta Period Transcendence)**.** *If $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are linearly independent in $\mathcal{C}_\zeta^{\text{crys}}$, then the corresponding zeta periods $\zeta(s_1), \ldots, \zeta(s_n)$ are algebraically independent over $\mathbb{Q}$ inside $\mathbb{B}_\zeta$, unless forced by functional identities.*

29. Zeta Period Stratification on $\mathrm{Spec}(R_{\mathbb{A}})$

29.1. **Motivic Period Sheaf and Stratification Functor.**

Definition 29.1. Let $\mathscr{P}_\zeta$ denote the period sheaf on the Zariski site $\mathrm{Spec}(R_{\mathbb{A}})$, defined by:

$$\mathscr{P}_\zeta(U) := \left\{ \mathrm{Per}_\zeta(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}_\zeta^{\text{crys}}(U) \right\} \subset \mathbb{B}_\zeta^\times,$$

where $\mathrm{Per}_\zeta(\mathcal{F})$ denotes the zeta-period matrix (or scalar) associated to the sheaf $\mathcal{F}$ over the open affine $U \subseteq \mathrm{Spec}(R_{\mathbb{A}})$.

Definition 29.2. Define the *Zeta Period Stratification Functor* as:

$$\mathfrak{Strat}_\zeta : \mathrm{Spec}(R_{\mathbb{A}}) \longrightarrow \mathcal{S}_\zeta,$$

where $\mathcal{S}_\zeta$ is the category of period types, with objects given by isomorphism classes of fiber functor data $(\omega_{\text{Betti}}, \omega_\zeta, \mathrm{Per}_\zeta)$, and morphisms given by change-of-period equivalences.

Remark 29.3. Two points $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R_{\mathbb{A}})$ lie in the same stratum if and only if the associated zeta period data of all sheaves over $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic under Π_ζ-torsor transport.

29.2. **Construction of the Stratification.**

Definition 29.4. Let $\mathcal{S}_\zeta^{(r)} \subset \mathrm{Spec}(R_{\mathbb{A}})$ be the locally closed subset defined by:

$$\mathcal{S}_\zeta^{(r)} := \left\{ \mathfrak{p} \in \mathrm{Spec}(R_{\mathbb{A}}) \mid \mathrm{rk}\left(\mathscr{P}_\zeta(\mathfrak{p})\right) = r \right\}.$$

We call $\mathcal{S}_\zeta^{(r)}$ the *r-th zeta-period stratum*.

Theorem 29.5. *Each stratum $\mathcal{S}_\zeta^{(r)} \subset \operatorname{Spec}(R_{\mathbb{A}})$ is a constructible subset in the Zariski topology, and the collection $\{\mathcal{S}_\zeta^{(r)}\}_{r\in\mathbb{N}}$ forms a finite stratification of $\operatorname{Spec}(R_{\mathbb{A}})$ by zeta-period type.*

Proof. For each r, the condition that the fiber rank of $\mathscr{P}_\zeta$ equals r is equivalent to the vanishing of all $(r+1)\times(r+1)$ minors of the period matrix and non-vanishing of at least one $r\times r$ minor. These are polynomial conditions on coefficients $f(n) \in R_{\mathbb{A}}$, hence define constructible sets. Finiteness follows from boundedness of motivic ranks. □

29.3. **Stratification and Motivic Galois Orbits.**

Definition 29.6. Define the *Galois period orbit* of a point $\mathfrak{p} \in \operatorname{Spec}(R_{\mathbb{A}})$ as the Π_ζ-orbit of the torsor section $\mathscr{P}_\zeta(\mathfrak{p})$. Denote:

$$\mathcal{O}_\zeta(\mathfrak{p}) := \Pi_\zeta \cdot \mathscr{P}_\zeta(\mathfrak{p}) \subset \mathbb{B}_\zeta.$$

Proposition 29.7. *The Galois period orbits $\mathcal{O}_\zeta(\mathfrak{p})$ determine an invariant decomposition of the stratification, with each $\mathcal{S}_\zeta^{(r)}$ decomposing as a union of Galois orbit closures:*

$$\mathcal{S}_\zeta^{(r)} = \bigsqcup_{[\mathcal{O}_\zeta]\in\mathcal{S}_\zeta^{(r)}/\Pi_\zeta} \overline{\mathcal{O}_\zeta}.$$

Proof. This follows from the torsor action structure: each point in $\mathcal{S}_\zeta^{(r)}$ corresponds to a fiber of rank r, and the period torsor over that point admits a transitive Π_ζ-action. Hence the orbits partition the stratum, and their Zariski closures define the decomposition. □

29.4. **Examples of Zeta Stratification.**

Example 29.8. Let $\mathbb{A} = \mathbb{Q}$, and $f(n) = n^{-s}$ with $s \in \mathbb{C}$. Then the closed point $\mathfrak{p}_s$ corresponding to f lies in the rank-one stratum $\mathcal{S}_\zeta^{(1)}$, and

$$\mathscr{P}_\zeta(\mathfrak{p}_s) = \zeta(s)\cdot\mathbb{C}^\times \subset \mathbb{B}_\zeta.$$

Example 29.9. Let $f(n) = \psi(n)n^{-s}$, where ψ is a Dirichlet character of conductor N. Then the associated stratum depends on the order of ψ, and ζ-period torsor takes values in $\mathbb{Q}(\zeta_N)[\zeta(s)]$, hence defines a Galois orbit under $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

30. THE ZETA PERIOD STACK

30.1. **Stack of Zeta Period Data.**

Definition 30.1. Define the fibered category $\mathcal{P}_\zeta^{\text{stack}} \to \operatorname{Spec}(R_{\mathbb{A}})$ by assigning to each open $U \subset \operatorname{Spec}(R_{\mathbb{A}})$ the groupoid:

$$\mathcal{P}_\zeta^{\text{stack}}(U) := \left\{(\mathcal{F}, \operatorname{Per}_\zeta(\mathcal{F})) \mid \mathcal{F} \in \mathcal{C}_\zeta^{\text{crys}}(U)\right\}.$$

Morphisms are isomorphisms of objects in $\mathcal{C}_\zeta^{\text{crys}}(U)$ compatible with period data.

Proposition 30.2. *The fibered category $\mathcal{P}_\zeta^{\text{stack}}$ is a prestack on $\operatorname{Spec}(R_{\mathbb{A}})$ with respect to the Zariski topology.*

Proof. Descent for morphisms follows from the sheaf property of hom-sets in $\mathcal{C}_\zeta^{\mathrm{crys}}$, and descent of objects holds because period matrices glue locally on overlaps in the open cover. Hence, this is a prestack. □

30.2. **Stackification and Geometric Structure.**

Definition 30.3. The stackification $\mathbf{P}_\zeta := \mathscr{S}(\mathcal{P}_\zeta^{\mathrm{stack}})$ is called the *Zeta Period Stack*, and it forms an fpqc stack over $\mathrm{Spec}(R_{\mathbb{A}})$, classifying zeta-crystalline objects and their periods up to equivalence.

Theorem 30.4. *The stack $\mathbf{P}_\zeta$ is an Artin stack locally of finite presentation over $\mathrm{Spec}(R_{\mathbb{A}})$, equipped with a smooth atlas from the moduli of realizations in $\mathcal{C}_\zeta^{\mathrm{crys}}$.*

Proof. Let $\mathcal{M}_\zeta \to \mathbf{P}_\zeta$ be the moduli space of objects in $\mathcal{C}_\zeta^{\mathrm{crys}}$ together with bases of their Betti and zeta-realizations. Since $\mathcal{M}_\zeta$ is representable by an affine scheme locally of finite type (as realizations vary in vector bundles), and the isomorphism condition of fiber functors is a smooth equivalence relation, the diagonal is smooth and the stack is Artin. □

30.3. **Period Morphism and Stacky Galois Action.**

Definition 30.5. Define the universal period morphism:

$$\mathrm{Per}_\zeta^{\mathrm{univ}} : \mathbf{P}_\zeta \longrightarrow [\mathrm{GL}_n/\Pi_\zeta],$$

sending a sheaf with zeta-period data to its equivalence class of period matrices modulo Π_ζ-action.

Proposition 30.6. *The morphism $\mathrm{Per}_\zeta^{\mathrm{univ}}$ is representable and faithfully flat over its image, defining a quotient moduli problem for period torsors.*

Proof. By construction, the isomorphism classes of period data in the stack correspond to orbits under the motivic Galois group Π_ζ. Since torsor data descends, the morphism defines a representable fpqc cover. □

30.4. **Zeta Period Stack Stratification.**

Definition 30.7. Define the rank filtration stratification:

$$\mathbf{P}_\zeta = \bigsqcup_{r\in\mathbb{N}} \mathbf{P}_\zeta^{(r)},$$

where $\mathbf{P}_\zeta^{(r)}$ is the substack classifying objects whose zeta-period matrix has rank r.

Theorem 30.8. *Each substack $\mathbf{P}_\zeta^{(r)}$ is locally closed in the fppf topology and admits a coarse moduli space given by the orbit closure:*

$$|\mathbf{P}_\zeta^{(r)}| = \overline{\mathcal{O}}_r \subset \mathrm{Spec}(R_{\mathbb{A}}) \otimes \mathbb{B}_\zeta.$$

Proof. As in the earlier theorem on period stratification, rank conditions are expressed via determinantal equations, which are algebraic. The orbit closures define coarse moduli for stack points up to equivalence, and the fppf topology refines the Zariski one to capture faithfully flat families. □

31. The Zeta Period Groupoid and Descent Structure

31.1. Definition of the Zeta Period Groupoid.

Definition 31.1. Let $\mathcal{G}_\zeta \rightrightarrows \mathbf{P}_\zeta$ be the groupoid in stacks defined as follows:

- Objects are pairs $(\mathcal{F}, \mathrm{Per}_\zeta(\mathcal{F}))$ in $\mathbf{P}_\zeta(U)$, for $U \subset \mathrm{Spec}(R_\mathbb{A})$,
- Morphisms between $(\mathcal{F}, \mathrm{Per}_\zeta(\mathcal{F})) \to (\mathcal{F}', \mathrm{Per}_\zeta(\mathcal{F}'))$ are isomorphisms $\phi : \mathcal{F} \to \mathcal{F}'$ in $\mathcal{C}_\zeta^{\mathrm{crys}}(U)$ such that:
$$\mathrm{Per}_\zeta(\mathcal{F}') = \phi \cdot \mathrm{Per}_\zeta(\mathcal{F}).$$

Remark 31.2. This groupoid encodes symmetries of the zeta-period data, and acts fiber-wise on the stack $\mathbf{P}_\zeta$. The automorphism group of an object in $\mathcal{G}_\zeta$ is a subgroup of the motivic Galois group Π_ζ.

31.2. Descent Groupoid and Stackification.

Proposition 31.3. *The groupoid $\mathcal{G}_\zeta$ defines a groupoid object in the 2-category of fibered categories over* $\mathrm{Spec}(R_\mathbb{A})$*, and admits a classifying stack:*
$$[\mathbf{P}_\zeta / \mathcal{G}_\zeta] \simeq \mathbf{P}_\zeta.$$

Proof. The groupoid axioms (source, target, identity, composition) follow from functoriality of the category $\mathcal{C}_\zeta^{\mathrm{crys}}$ and compatibility of isomorphisms with period matrices. Since $\mathbf{P}_\zeta$ is already a stack under the fpqc topology, the quotient by this internal groupoid structure reproduces the same stack. □

31.3. Moduli Interpretation via Groupoid Atlases.

Definition 31.4. Let $\mathcal{U}_\zeta \to \mathbf{P}_\zeta$ be a smooth presentation by a scheme (or higher Artin space) representing trivialized zeta-period data. The groupoid atlas is:
$$\mathcal{G}_\zeta := \mathcal{U}_\zeta \times_{\mathbf{P}_\zeta} \mathcal{U}_\zeta \rightrightarrows \mathcal{U}_\zeta.$$

Proposition 31.5. *The stack $\mathbf{P}_\zeta$ is equivalent to the quotient stack of the groupoid atlas:*
$$\mathbf{P}_\zeta \simeq [\mathcal{U}_\zeta / \mathcal{G}_\zeta],$$
with descent data along the groupoid encoding equivalence of zeta-period configurations.

Proof. By Artin's theorem, every algebraic stack with smooth cover can be written as the quotient of a smooth groupoid in schemes. Since $\mathcal{U}_\zeta \to \mathbf{P}_\zeta$ is smooth and faithfully flat, the groupoid structure encodes equivalences of objects with trivializations, giving the quotient structure. □

31.4. Groupoid Cohomology and Period Invariants.

Definition 31.6. Let $\mathcal{F}_\zeta$ be a quasi-coherent sheaf on $\mathbf{P}_\zeta$. The groupoid cohomology is defined as:
$$\mathrm{H}^i(\mathcal{G}_\zeta, \mathcal{F}_\zeta) := \mathrm{Ext}^i_{\mathbf{P}_\zeta}(\mathcal{O}, \mathcal{F}_\zeta),$$
computed via the simplicial nerve of the groupoid.

Proposition 31.7. *For a rank-one zeta-period sheaf $\mathcal{F}_\zeta$, we have:*

$$\mathrm{H}^0(\mathcal{G}_\zeta, \mathcal{F}_\zeta) = \Gamma(\mathrm{Spec}(R_{\mathbb{A}}), \mathcal{F}_\zeta)^{\Pi_\zeta}, \quad \mathrm{H}^1(\mathcal{G}_\zeta, \mathcal{F}_\zeta) = \text{Obstruction class.}$$

Proof. The zeroth cohomology is the invariants under the groupoid action, i.e., sections fixed by the motivic Galois symmetry. The first cohomology classifies obstructions to descent or extensions in the torsor structure, as computed by Čech cohomology over the groupoid cover. □