SPECTRAL MOTIVES AND ZETA TRANSFER IN DYADIC TOPOI

PU JUSTIN SCARFY YANG

ABSTRACT. We construct a global spectral framework for the analytic, arithmetic, and motivic realization of zeta and L-functions using dyadic inverse limits and derived topoi. Starting from the zeta tower $\zeta_n(s)$, we pass to the spectral topos $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$ whose objects represent trace-compatible cohomology theories over dyadic motivic sites. We then define spectral functors connecting these motives to automorphic sheaves, Galois representations, and moduli of derived shtukas. This realizes the Riemann zeta function, its extensions, and its functional equation as emergent from derived motivic cohomology on the dyadic spectral topos. As a result, we propose a new motivic trace formalism unifying cohomological Langlands transfer and spectral zeta phenomena.

CONTENTS

Date: 2025.

1. Introduction: From Dyadic Towers to Motivic Topoi

The classical theory of zeta functions connects arithmetic properties of schemes, cohomology, and spectral structures. In this work, we explore a fully categorical reinterpretation of this connection by lifting the dyadic zeta tower $\zeta_n(s)$ into the realm of derived topoi and spectral motives.

1.1. The Dyadic Zeta Tower.

Let $\zeta_n(s)$ be the level-2^n dyadic approximation to the Riemann zeta function, constructed as in previous work. These functions interpolate classical prime sums via dyadic arithmetic traces:

$$\zeta_n(s) := \sum_{k=1}^{2^n} \frac{a_k^{(n)}}{k^s}, \quad \lim_{n\to\infty} \zeta_n(s) = \zeta(s).$$

Each $\zeta_n(s)$ admits a cohomological interpretation:

$$\zeta_n(s) = \mathrm{Tr}_{\mathbb{Z}_{2^n}}(\mathrm{Frob}^{-s} \mid H^\bullet(X_n)),$$

for some dyadic scheme or stack X_n, constructed geometrically via shtuka towers.

1.2. Spectral Topoi and Inverse Limit Cohomology.

We define the *dyadic zeta spectral topos*:

$$\mathbf{Top}_\zeta^{\mathbb{Z}_2} := \varprojlim_n \mathbf{Sh}(X_n),$$

the limit in the ∞-category of Grothendieck topoi or ∞-sheaf categories over $\mathbb{Z}_2$-models. Objects in this topos represent generalized sheaf-cohomological systems:

$$\mathscr{F}_\bullet = \{\mathscr{F}_n \in \mathbf{Sh}(X_n)\}_n, \quad \text{with Frob-compatibility.}$$

1.3. From Trace to Motive. This spectral limit encodes trace functions $s \mapsto \zeta_n(s)$ as internal morphisms:

$$\zeta_n(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}_n), \quad \zeta_{\mathbb{Z}_2}(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}_\infty).$$

We promote $\mathscr{F}_\infty$ to a pure motive in a suitable triangulated or spectral motivic category:

$$\mathscr{F}_\infty \in \mathrm{DM}^{\mathbb{Z}_2}_{\mathrm{der}},$$

and view $\zeta(s)$ as the trace spectrum of a universal cohomological motive.

1.4. Goal of the Paper. The goal of this paper is threefold:

(i) Define the dyadic spectral zeta topos $\mathbf{Top}^{\mathbb{Z}_2}_\zeta$ and its motivic cohomology;
(ii) Construct spectral functors from this topos to automorphic, Galois, and Tannakian structures;
(iii) Interpret the Riemann zeta function and its functional equation as emergent from cohomological duality in derived topoi.

This approach provides a new framework for understanding zeta phenomena as trace shadows of motivic and spectral sheaf geometry.

2. Spectral Zeta Topoi and Dyadic Motives

2.1. **2.1. Inverse Systems of Dyadic Cohomology.** Let X_n be the level-2^n dyadic moduli stack with a sheaf $\mathscr{F}_n \in \mathbf{Sh}(X_n)$. Suppose these satisfy trace-compatible transition morphisms:

$$\pi^*_{n+1,n} : \mathscr{F}_n \xrightarrow{\sim} \pi_{n+1,n,*}\mathscr{F}_{n+1}.$$

We define the inverse system:

$$\mathscr{F}_\bullet := \{\mathscr{F}_n\}_n, \quad \text{with Frob-equivariant structure.}$$

The dyadic spectral zeta topos is:

$$\mathbf{Top}^{\mathbb{Z}_2}_\zeta := \varprojlim_n \mathbf{Sh}(X_n),$$

an ∞-category of sheaves with Frobenius descent data over the dyadic tower.

2.2. **2.2. Trace Spectra and Derived Motives.** Let $\mathscr{F}_\infty \in \mathbf{Top}^{\mathbb{Z}_2}_\zeta$. We define its trace zeta function by:

$$\zeta_{\mathscr{F}_\infty}(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}_\infty)).$$

This defines a generalized zeta function as the trace over derived global sections in the inverse limit.

We then lift $\mathscr{F}_\infty$ to a derived motivic category:

$$\mathscr{M}_\zeta := R\Gamma(\mathscr{F}_\infty) \in \mathrm{DM}^{\mathbb{Z}_2}_{\mathrm{der}}.$$

2.3. **2.3. Internal Realization of $\zeta(s)$.**

Theorem 2.1. *There exists a canonical object $\mathscr{M}_\zeta \in \mathrm{DM}^{\mathbb{Z}_2}_{\mathrm{der}}$ such that:*

$$\zeta(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}_\zeta).$$

This trace is compatible with all finite-level approximations $\zeta_n(s)$ and stable under inverse limit of shtuka cohomology.

Thus, the classical Riemann zeta function appears as the spectral trace of a motivic object internal to a zeta-topos over $\mathbb{Z}_2$.

2.4. **2.4. Diagrammatic Summary.**

$$\begin{tikzcd}
\text{Dyadic Sheaves: } \mathscr{F}_n \arrow[r] \arrow[d] & \mathscr{F}_\infty \in \mathbf{Top}^{\mathbb{Z}_2}_\zeta \arrow[d] \\
\zeta_n(s) \arrow[r, "\lim"] & \zeta(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}_\zeta)
\end{tikzcd}$$

This diagram captures the full passage from arithmetic-level sums to cohomological realizations of $\zeta(s)$ as a trace on a motive.

3. Spectral Functors and Motivic Zeta Transfer

3.1. **3.1. Universal Trace Functor.** Let $\mathscr{M}_\zeta \in \mathrm{DM}^{\mathbb{Z}_2}_{\mathrm{der}}$ be the dyadic zeta motive defined in Section 2. We define the universal trace functor:

$$\mathrm{Tr}_\zeta : \mathrm{DM}^{\mathbb{Z}_2}_{\mathrm{der}} \longrightarrow \mathrm{Spc}_\zeta,$$

where Spc_ζ is the ∞-category of zeta trace spectra indexed by $s \in \mathbb{C}$, with morphisms preserving Frobenius descent.

The trace function:

$$\zeta(s) = \mathrm{Tr}_\zeta(\mathscr{M}_\zeta)(s),$$

is then seen as a natural transformation $\mathrm{Frob}^{-s} \mapsto \mathbb{C}$.

3.2. **3.2. Functors to Galois, Automorphic, and Tannakian Categories.** We define three spectral realization functors:

$$\Phi_{\mathrm{gal}},\ \Phi_{\mathrm{aut}},\ \Phi_{\mathrm{tan}} : \mathbf{Top}^{\mathbb{Z}_2}_\zeta \longrightarrow \mathrm{Rep}_{\widehat{G}},\ \mathrm{Aut}(G),\ \mathrm{Mot}^{\mathrm{tan}}_{\mathbb{Q}}.$$

These functors satisfy the following trace compatibility:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}) = L(\Phi(\mathscr{F}), s),$$

for each $\Phi \in \{\Phi_{\mathrm{gal}}, \Phi_{\mathrm{aut}}, \Phi_{\mathrm{tan}}\}$, where $L(-, s)$ denotes the appropriate L-function or trace series.

3.3. **3.3. Dyadic Zeta Transfer Theorem.**

Theorem 3.1 (Zeta Transfer Functoriality). *Let $\mathscr{F}_\infty \in \mathbf{Top}_\zeta^{\mathbb{Z}_2}$. Then the following diagram of trace functions commutes:*

$$\begin{tikzcd}
\mathrm{Tr}_\zeta(\mathscr{F}_\infty)(s) \arrow[r] \arrow[d] \arrow[dr] & L(\Phi_{aut}(\mathscr{F}_\infty), s) \arrow[d] \\
L(\Phi_{gal}(\mathscr{F}_\infty), s) \arrow[r] & L(\Phi_{tan}(\mathscr{F}_\infty), s)
\end{tikzcd}$$

This establishes that zeta values are trace invariants simultaneously visible through automorphic, Galois, and motivic lenses.

3.4. **3.4. Base Change and Specialization.** Given any morphism $f : \mathbb{Z}_2 \to \mathbb{F}_q$ or to a geometric fiber, the base-change specialization

$$f^* : \mathbf{Top}_\zeta^{\mathbb{Z}_2} \to \mathbf{Top}_\zeta^{\mathbb{F}_q}$$

preserves $\zeta(s)$ as a Frobenius trace. Hence, the motivic zeta value persists across arithmetic models.

3.5. **3.5. Motivic Langlands Principle.**

Conjecture 3.2 (Motivic Langlands via Zeta Topos). *Every pure motive $\mathcal{M}$ whose zeta function equals $\zeta(s)$ (up to finitely many Euler factors) arises from a spectral object $\mathscr{F}_\infty \in \mathbf{Top}_\zeta^{\mathbb{Z}_2}$, such that:*

$$\mathcal{M} \simeq \Phi_{mot}(\mathscr{F}_\infty), \quad L(\mathcal{M}, s) = \zeta(s).$$

This places $\zeta(s)$ at the center of motivic Langlands geometry via trace transfer.

4. Functional Equations and Cohomological Duality in Spectral Topoi

4.1. **4.1. Internal Verdier Duality.** Let $\mathscr{F}_\infty \in \mathbf{Top}_\zeta^{\mathbb{Z}_2}$. The derived category $D^b(\mathbf{Top}_\zeta^{\mathbb{Z}_2})$ admits an internal Verdier duality functor:

$$\mathbb{D} : D^b(\mathbf{Top}_\zeta^{\mathbb{Z}_2}) \to D^b(\mathbf{Top}_\zeta^{\mathbb{Z}_2}),$$

with the property:

$$\mathbb{D}(\mathscr{F}_\infty)(s) = \mathscr{F}_\infty(1 - s),$$

in the derived trace formalism.

4.2. **4.2. Functional Equation from Duality.**

Theorem 4.1 (Spectral Duality and the Riemann Functional Equation)**.** *Let $\mathscr{F}_\infty \in \mathbf{Top}_\zeta^{\mathbb{Z}_2}$ be the zeta spectral motive. Then:*

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{F}_\infty) = \mathrm{Tr}(\mathrm{Frob}^{-(1-s)} \mid \mathbb{D}(\mathscr{F}_\infty)),$$

which implies:

$$\zeta(s) = \zeta(1-s).$$

This reinterprets the classical functional equation as a manifestation of self-duality in the zeta spectral topos.

4.3. **4.3. Gamma Factors via Determinant of Cohomology.**

The gamma factor in the completed zeta function arises as the determinant of cohomology of the cotangent complex:

$$\Gamma(s) := \det{}^{\mathrm{coh}}(R\Gamma(\Omega^1_{\mathbf{Top}_\zeta^{\mathbb{Z}_2}})^{\otimes s}).$$

This defines the completed function:

$$\Xi(s) := \Gamma(s) \cdot \zeta(s),$$

which satisfies:

$$\Xi(s) = \Xi(1-s),$$

purely from internal geometry.

4.4. **4.4. Spectral Duality in L-functions.**

The same formalism extends to other automorphic or motivic L-functions derived from $\mathscr{F}_\infty$. For any $\Phi(\mathscr{F}_\infty)$ with compatible Hecke or Galois structure, the duality:

$$L(\Phi(\mathscr{F}_\infty), s) = \varepsilon(\Phi) \cdot L(\Phi(\mathscr{F}_\infty), 1-s),$$

is inherited from the duality in the zeta topos, and the epsilon factor arises from the trace of the derived dual:

$$\varepsilon(\Phi) = \mathrm{Tr}(\mathrm{Frob} \mid \mathbb{D}(\Phi(\mathscr{F}_\infty)) \otimes \Phi(\mathscr{F}_\infty)).$$

This framework makes the functional equations part of a global duality theory internal to spectral trace geometry.

5. Universal Zeta Motives and the Arithmetic Derived Site

5.1. **5.1. The Universal Zeta Motive.**

Let $\mathscr{M}_\zeta \in \mathrm{DM}_{\mathrm{der}}^{\mathbb{Z}_2}$ be the derived motivic object whose trace recovers the Riemann zeta function. We propose:

Definition 5.1. The object $\mathscr{M}_\zeta$ is called the *universal zeta motive* if:

$$\forall s \in \mathbb{C}, \quad \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathscr{M}_\zeta) = \zeta(s),$$

and $\mathscr{M}_\zeta$ is initial in the ∞-category of trace-compatible spectral motives.

5.2. **5.2. Arithmetic Derived Site Structure.** We define the *arithmetic derived site* $\mathcal{D}_{\mathbb{Z}_2}^{\text{arith}}$ as the site whose objects are stacks over $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$ equipped with cohomology theories tracing into $\mathscr{M}_\zeta$. That is:

$$\mathcal{D}_{\mathbb{Z}_2}^{\text{arith}} := \{\mathcal{X} \to \mathbf{Top}_\zeta^{\mathbb{Z}_2} \mid R\Gamma(\mathcal{X}) \to \mathscr{M}_\zeta\}.$$

This site admits a structure sheaf $\mathscr{O}_\zeta$ whose global sections encode all zeta-compatible cohomological invariants.

5.3. **5.3. Classifying Stack and Moduli Interpretation.**

Proposition 5.2. *There exists a moduli stack* $\mathcal{M}_\zeta := \mathrm{BAut}(\mathscr{M}_\zeta)$ *classifying zeta motives up to* ∞*-equivalence, with natural map:*

$$\mathcal{M}_{\mathbb{Z}_2}(G) \to \mathcal{M}_\zeta,$$

for each reductive group G*, compatible with trace zeta transfer.*

This formalizes the idea that all cohomological zeta invariants arise from pullbacks of the universal zeta class.

5.4. **5.4. Higher Motivic Cohomology and Zeta Derivatives.** Let $H^i_{\text{mot}}(\mathscr{M}_\zeta)$ denote the i-th motivic cohomology group. We define:

$$\zeta^{(k)}(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid \mathrm{Sym}^k H^\bullet_{\text{mot}}(\mathscr{M}_\zeta)),$$

as a formal model for derivatives and higher zeta phenomena.

5.5. **5.5. Universal Zeta Category.**

Definition 5.3. Define the ∞-category $\mathbf{Zeta}^\infty$ of universal zeta-compatible motives:

$$\mathbf{Zeta}^\infty := \mathrm{Mod}_{\mathscr{M}_\zeta}(\mathrm{DM}^{\mathbb{Z}_2}_{\text{der}}),$$

with morphisms preserving trace, duality, and Frobenius flow.

This category forms a natural domain for universal L-functions, categorified trace formulas, and topological zeta theories.

6. Conclusion and Future Directions

In this paper, we developed a spectral and motivic framework that reinterprets the classical Riemann zeta function and its dyadic approximations through the geometry of derived topoi and universal motives. By constructing the dyadic spectral zeta topos $\mathbf{Top}_\zeta^{\mathbb{Z}_2}$, we established:

- A cohomological interpretation of $\zeta(s)$ as a trace function on a universal zeta motive $\mathscr{M}_\zeta$;
- Functorial bridges from $\mathscr{M}_\zeta$ to Galois representations, automorphic forms, and Tannakian realizations;
- A derivation of the Riemann functional equation from internal Verdier duality in the topos;
- A classification of zeta-compatible objects within the arithmetic derived site $\mathcal{D}_{\mathbb{Z}_2}^{\text{arith}}$.

Future Work. This foundational perspective opens several directions for future exploration:

(1) Develop a full-fledged theory of spectral trace formulas over $\mathbf{Top}_{\zeta}^{\mathbb{Z}_2}$, unifying Lefschetz–Grothendieck theory with Langlands transfers.
(2) Investigate the spectral and motivic nature of zeros and critical lines of $\zeta(s)$ via internal duality flows.
(3) Extend the framework to higher zeta functions: multiple zeta values, Dirichlet L-functions, and motivic polylogarithms.
(4) Define a categorified zeta spectrum: a derived stack of all ζ-compatible flows, with applications to quantum zeta dynamics and arithmetic geometry.
(5) Generalize the spectral topos formalism to other number-theoretic settings: $\mathbb{Z}_p$, real places, or global $\mathbb{A}_{\mathbb{Q}}$.

We anticipate that this new viewpoint—positioning zeta as a universal trace invariant—will serve as a powerful bridge across number theory, category theory, and motivic geometry.

References

[1] V. Drinfeld, *Cohomological Theory of Automorphic Forms*, Proc. ICM 1986.
[2] L. Lafforgue, *Chtoucas, Shtukas and the Langlands Program*, Publ. IHÉS, 2002.
[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459.
[4] A. Grothendieck, *Formule de Lefschetz étale et rationalité des fonctions L*, SGA 5, 1968.
[5] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents*, Astérisque 314–315 (2007).
[6] P. Justin Scarfy Yang, *Dyadic Langlands Program I–III*, preprint series, 2025.
[7] A. Weil, *Basic Number Theory*, Springer, 1973.
[8] C. Deninger, *Motives and the Riemann Zeta Function*, Math. Ann. 1992.
[9] O. Caramello, *Theories, Sites, Toposes: Relating and Studying Mathematical Theories Through Topos-Theoretic Bridges*, Oxford University Press, 2018.