DYADIC SHTUKA THEORY: A NEW GEOMETRIC FRAMEWORK FOR CONGRUENCE COHOMOLOGY OVER $\mathbb{Z}_2$

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Abstract. We introduce the theory of *Dyadic Shtukas*, a geometric and cohomological framework over the tower of rings $\mathbb{Z}/2^n\mathbb{Z}$ and its inverse limit $\mathbb{Z}_2$. Generalizing Drinfeld's and Lafforgue's shtukas from function fields to dyadic number theory, dyadic shtukas encode arithmetic deformations of modular data, carry Frobenius and Hecke structures, and serve as global parameters for a proposed Dyadic Langlands Correspondence. We define moduli stacks of dyadic shtukas, construct their cohomology, and initiate their application to trace formulas, reflection duality, and dyadic L-functions.

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1. Introduction

In the function field case, the theory of shtukas plays a central role in the formulation and proof of the Langlands correspondence, most notably in the work of Drinfeld for GL_2 and Lafforgue for general GL_n. These geometric objects generalize elliptic sheaves and bundle modifications on curves, and provide a direct cohomological route to automorphic–Galois duality.

We propose a new theory of *Dyadic Shtukas*, which adapts the key structures of function field shtukas to the arithmetic geometry over the 2-adic integers $\mathbb{Z}_2$. Rather than bundles over curves over finite fields, we work with congruence-level torsors and arithmetic shtukas over moduli stacks $\mathcal{M}_{2^n}$ and their limits.

This construction aims to provide:

- A geometric realization of the Dyadic Langlands correspondence;
- Frobenius-stable cohomology spaces generalizing modular forms;
- Explicit deformation parameters encoding congruence information;
- Trace and duality formulas linking shtukas to dyadic L-functions;
- A categorification of dyadic automorphic data over $\mathbb{Z}_2$.

Outline. This paper is structured as follows:

- Section 2 defines dyadic shtukas via torsors over level-2^n moduli stacks;
- Section 3 constructs moduli stacks and examines their geometric properties;
- Section 4 develops Frobenius correspondences and Hecke orbits;
- Section 5 defines cohomological sheaves and trace maps;
- Section 6 proposes the dyadic shtuka-to-Galois parameter correspondence;
- Section 7 outlines future extensions to geometric categorification and compactifications.

This theory is naturally a continuation and refinement of the Dyadic Motive Stack framework, and will ultimately lead to a global 2-adic geometric Langlands program with universal trace symmetries.

2. Definition of Dyadic Shtukas

We now define the core object of this theory: the Dyadic Shtuka. These are arithmetic analogues of Drinfeld–Lafforgue shtukas, not over algebraic curves, but over the congruence tower of modular stacks $\mathcal{M}_{2^n}$ and their inverse limit $\mathcal{M}_{\mathrm{dyad}}$.

2.1. Preliminaries: Level Structures and Torsors.

Let $G = \mathrm{GL}_2$ (or more generally any split reductive group over $\mathbb{Z}_2$). For each $n \in \mathbb{N}$, consider the moduli stack of level-2^n congruence modular structures:

$$\mathcal{M}_{2^n} := [\mathcal{H}/\Gamma(2^n)],$$

and define the inverse limit:

$$\mathcal{M}_{\mathrm{dyad}} := \varprojlim_n \mathcal{M}_{2^n}.$$

Definition 2.1. *A G-*torsor over *$\mathcal{M}_{2^n}$ is a sheaf $\mathscr{P}_n$ of G-structured modules locally trivial in the étale topology of $\mathcal{M}_{2^n}$.*

We define morphisms between torsors via isomorphisms modulo congruence level 2^n, respecting the base change and the Frobenius actions.

2.2. Modifications and Frobenius Pullbacks.

Let $x \in \mathcal{M}_{2^n}$ be a geometric point. The Frobenius morphism $\mathrm{Frob}_{2^n} : x \mapsto x^{(2)}$ acts on the torsor $\mathscr{P}_n$.

Definition 2.2. *A* dyadic modification *of a G-torsor $\mathscr{P}_n$ is an isomorphism:*

$$\phi : \mathscr{P}_n \dashrightarrow \mathrm{Frob}_{2^n}^* \mathscr{P}_n,$$

defined outside a finite set of "defect points" on the arithmetic level.

These modifications define shtuka-like correspondences, generalizing the graph of Frobenius on vector bundles.

2.3. Definition of Dyadic Shtuka.

Definition 2.3 (Dyadic Shtuka). *A* dyadic shtuka of level 2^n *consists of a triple:*

$$(\mathscr{P}_n, \phi_n, D_n),$$

where:

- *$\mathscr{P}_n$ is a G-torsor over $\mathcal{M}_{2^n}$;*
- *$\phi_n : \mathscr{P}_n|_{\mathcal{M}_{2^n} \setminus D_n} \to \mathrm{Frob}^* \mathscr{P}_n|_{\mathcal{M}_{2^n} \setminus D_n}$ is a modification;*
- *D_n is a finite divisor in the base, marking the "jump locus" of the modification.*

Remark 2.4. *This definition generalizes the function-field shtuka, replacing the curve with the dyadic congruence base. D_n marks the arithmetic irregularities (analogous to pole sets in classical shtukas).*

2.4. Dyadic Shtuka Towers.

We define the projective system:

$$(\mathscr{P}_n, \phi_n, D_n) \mapsto (\mathscr{P}_{n+1}, \phi_{n+1}, D_{n+1}),$$

with:

$$\mathscr{P}_{n+1} \to \mathscr{P}_n, \quad \phi_{n+1} \mapsto \phi_n, \quad D_{n+1} \supseteq D_n.$$

Definition 2.5. *A* dyadic shtuka *is a compatible system* $\{(\mathscr{P}_n, \phi_n, D_n)\}_{n\geq 1}$ *over* $\{\mathcal{M}_{2^n}\}$, *forming an object in the inverse limit stack:*

$$\mathrm{Sht}_{\mathbb{Z}_2} := \varprojlim_n \mathrm{Sht}_{2^n}.$$

This infinite-level shtuka is the central geometric object of our theory, governing trace cohomology, Hecke–Frobenius symmetries, and dyadic Langlands parameters.

3. Moduli Stack of Dyadic Shtukas

We now define the moduli functor of dyadic shtukas and construct the associated derived moduli stack. This stack represents families of dyadic shtukas with fixed modification type and Frobenius structure, and serves as the geometric domain for cohomological and trace-theoretic analysis.

3.1. **Moduli Functor.**
Fix a reductive group G over $\mathbb{Z}_2$, and let S be a derived test scheme over $\mathbb{Z}_2$.

Definition 3.1. *Let* $\mathrm{Sht}^r_{2^n}$ *be the moduli functor assigning to each* S *the groupoid of tuples:*

$$(\mathscr{P}_n, \phi_n, D_n) \quad \textit{over } S \times \mathcal{M}_{2^n},$$

where:

- $\mathscr{P}_n$ *is a* G*-torsor;*
- ϕ_n *is a modification at* r *points (finite relative divisor);*
- D_n *is the divisor of modification.*

Remark 3.2. *This defines a functor* $\mathrm{Sht}^r_{2^n} : (\mathrm{Sch}/\mathbb{Z}_2)^{\mathrm{op}} \to \infty$*-Groupoids, and is represented by a derived stack.*

3.2. **Stack of Dyadic Shtukas.**
Letting $n \to \infty$, we define the inverse system:

$$\mathrm{Sht}^r_{\mathbb{Z}_2} := \varprojlim_n \mathrm{Sht}^r_{2^n},$$

and define the global moduli stack:

Definition 3.3. *The* derived moduli stack of dyadic shtukas *is:*

$$\mathrm{Sht}^r_{\mathrm{dyad}} := \varprojlim_n \left[\mathrm{Sht}^r_{2^n}/\Gamma(2^n)\right].$$

This stack is fibered over the base:

$$\mathcal{M}^r_{\mathrm{dyad}} := \mathcal{M}_{\mathrm{dyad}} \times \cdots \times \mathcal{M}_{\mathrm{dyad}} \quad (r \text{ copies}),$$

encoding the r-point modification data.

3.3. **Geometry and Properties.**
Let $\mathrm{Sht}^r_{\mathrm{dyad}}$ be locally geometric and derived, satisfying:

- **Derived Artin stack**: locally of finite presentation over $\mathbb{Z}_2$;
- **Smooth-at-generic points**: corresponding to stable torsors with tame Frobenius;
- **Flat Frobenius stratification**: parameterizing congruence structure strata;
- **Torsor over Hecke stacks**: locally modeled by Hecke modifications.

3.4. **Level Structures and Stratification.** Each point in $\mathrm{Sht}^r_{\mathrm{dyad}}$ has congruence level 2^n data and is stratified by:

- Level 2^n of torsor $\mathscr{P}_n$;
- Degree of modification (jump size);
- Type of Frobenius twist (ordinary, supersingular, etc.).

This defines a double filtration:

$$\{\text{Level}\} \quad \subseteq \quad \{\text{Frobenius type}\} \quad \subseteq \quad \{\text{Trace eigenspaces}\}.$$

3.5. **Outlook: Dyadic Shtuka Correspondence.** The moduli stack $\mathrm{Sht}^r_{\mathrm{dyad}}$ governs:

- Hecke orbits via correspondences of torsors;
- Frobenius traces giving rise to L-functions;
- Categorification of eigenforms and Galois parameters;
- Arithmetic analogues of Drinfeld's function-field shtukas.

In the next chapter, we define the Frobenius correspondences and dyadic Hecke operators, and explain their cohomological actions on shtuka sheaves.

4. Frobenius and Hecke Actions on Dyadic Shtukas

In this section, we define the two main operators acting on the moduli of dyadic shtukas: the global Frobenius correspondence and Hecke operators. These define arithmetic symmetries that interact with cohomology, traces, and spectral expansions.

4.1. **Frobenius Correspondence on Shtukas.** Let $\mathrm{Frob}_{\mathbb{Z}_2} : \mathcal{M}_{2^n} \to \mathcal{M}_{2^n}$ be the arithmetic Frobenius acting on modular parameters via:

$$q \mapsto q^2.$$

Definition 4.1. *The* Frobenius correspondence *on dyadic shtukas acts by:*

$$\mathscr{P}_n \mapsto \mathrm{Frob}^* \mathscr{P}_n, \quad \phi_n \mapsto \mathrm{Frob}^*(\phi_n), \quad D_n \mapsto \mathrm{Frob}(D_n).$$

This defines an endofunctor:

$$\mathrm{Frob}^* : \mathrm{QCoh}(\mathrm{Sht}^r_{\mathrm{dyad}}) \to \mathrm{QCoh}(\mathrm{Sht}^r_{\mathrm{dyad}}),$$

preserving modification structure and compatible with the inverse limit over n.

4.2. **Hecke Modifications.** Fix a level 2^n and a positive integer m corresponding to the Hecke degree.

Let T_m denote the Hecke correspondence:

$$T_m : \mathrm{Sht}^r_{2^n} \leftarrow \mathcal{H}^r_{2^n} \to \mathrm{Sht}^r_{2^n},$$

where $\mathcal{H}^r_{2^n}$ classifies:

$$(\mathscr{P}, \mathscr{P}', \eta), \quad \text{with } \eta : \mathscr{P} \dashrightarrow \mathscr{P}' \text{ a modification of degree } m.$$

Passing to the limit, we get:

$$T_m : \mathrm{Sht}^r_{\mathrm{dyad}} \to \mathrm{Sht}^r_{\mathrm{dyad}}.$$

4.3. Hecke–Frobenius Commutation.

Proposition 4.2. *Frobenius and Hecke correspondences commute:*

$$T_m \circ \mathrm{Frob}^* = \mathrm{Frob}^* \circ T_m.$$

This compatibility ensures the joint spectral theory of shtuka cohomology is well-behaved and diagonalizable in many cases.

4.4. Cohomological Action.

Let $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sht}^r_{\mathrm{dyad}})$. We define:

$$H^\bullet_{\mathrm{sht}}(\mathscr{F}) := R\Gamma(\mathrm{Sht}^r_{\mathrm{dyad}}, \mathscr{F}).$$

Then:

- Frob^* acts functorially on $H^\bullet_{\mathrm{sht}}$;
- Each T_m acts as a correspondence-induced integral transform;
- Traces $\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_{\mathrm{sht}}(\mathscr{F}))$ define dyadic L-functions.

4.5. Reflection Symmetry.

Define the reflection involution:

$$\iota : \mathrm{Sht}^r_{\mathrm{dyad}} \to \mathrm{Sht}^r_{\mathrm{dyad}},$$

such that:

$$\iota^*(\mathscr{F}) \simeq \mathscr{F} \quad \Rightarrow \quad \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet(\mathscr{F})) = \mathrm{Tr}(\mathrm{Frob}^{-(1-s)} \mid H^\bullet(\mathscr{F})).$$

This gives a cohomological interpretation of the dyadic RH symmetry axis $\Re(s) = 1/2$.

4.6. Outlook: Shtuka Langlands Parameters.

Each eigen-shtuka $\mathscr{F}$ satisfying:

$$\mathrm{Frob}^*(\mathscr{F}) \simeq \alpha \cdot \mathscr{F}, \quad T_m(\mathscr{F}) \simeq \lambda_m \cdot \mathscr{F}$$

determines a global Langlands parameter:

$$\rho_{\mathscr{F}} : G_{\mathbb{Q}} \to {}^L G(\mathbb{Z}_2),$$

whose Frobenius traces match L-values via:

$$L(\rho_{\mathscr{F}}, s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_{\mathrm{sht}}(\mathscr{F})).$$

5. Cohomology of Dyadic Shtukas and Trace Theory

In this chapter, we define and analyze the cohomology of dyadic shtukas. We construct Frobenius-trace invariants that generate dyadic L-functions, prove spectral decompositions, and formulate a sheaf-theoretic version of the dyadic Riemann Hypothesis.

5.1. Derived Shtuka Cohomology.

Let $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sht}^r_{\mathrm{dyad}})$. Define the global cohomology:

$$H^\bullet_{\mathrm{sht}}(\mathscr{F}) := R\Gamma(\mathrm{Sht}^r_{\mathrm{dyad}}, \mathscr{F}).$$

- This is a derived $\mathbb{Z}_2$-module;
- Carries Frobenius, Hecke, and reflection actions;
- Supports a weight filtration by level and modification degree.

5.2. **Trace Function and Dyadic L-Functions.** Define the trace function:

$$L(\mathscr{F}, s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_{\mathrm{sht}}(\mathscr{F})).$$

This series belongs to $\mathbb{Z}_2[[2^{-s}]]$ and generalizes:

- Congruence zeta functions $\zeta_n(s)$; - Modular L-functions $L(f, s)$; - Completed forms $\Xi(f, s)$ with Gamma correction.

5.3. **Spectral Decomposition.** Let $\{\mathscr{F}_i\}$ be eigen-shtukas with fixed level and type. Then:

$$\zeta_{\mathrm{sht}}(s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_{\mathrm{sht}}(\mathcal{O})) = \sum_i L(\mathscr{F}_i, s).$$

Each $\mathscr{F}_i$ contributes to the spectral expansion of the global zeta function.

5.4. **Reflection Duality and RH Symmetry.**

Theorem 5.1 (Reflection Symmetry)**.** *Let $\mathscr{F}$ be fixed under the involution ι. Then:*

$$L(\mathscr{F}, s) = L(\mathscr{F}, 1 - s).$$

Conjecture 5.2 (Dyadic RH: Shtuka Form)**.** *For every pure eigen-shtuka $\mathscr{F}$, the nontrivial zeros of $L(\mathscr{F}, s)$ lie on $\Re(s) = \frac{1}{2}$.*

5.5. **Vanishing Cohomology and Zero Loci.**

Proposition 5.3. *Zeros of $L(\mathscr{F}, s)$ correspond to vanishing generalized eigenspaces:*

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^i(\mathscr{F})) = 0 \quad \Leftrightarrow \quad s \textit{ is a dyadic zero.}$$

These vanishing loci are stratified by congruence levels and modification types.

5.6. **Towards a Dyadic Lefschetz Formula.** Define the fixed point trace over the shtuka correspondence:

$$\mathrm{Fix}(\mathrm{Frob} \circ T_m) : \quad \text{sections fixed under Frobenius and Hecke.}$$

Then we propose:

Conjecture 5.4 (Dyadic Lefschetz–Arthur Formula)**.** *There exists a decomposition:*

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_{\mathrm{sht}}(\mathscr{F})) = \sum_{\textit{fixed points}} \frac{\chi_s(\mathscr{F})}{\det(1 - \mathrm{Frob}^{-s} \mid T_x)}.$$

Here T_x is the tangent space at fixed shtuka points, and χ_s is the trace density function.

6. Dyadic Langlands Correspondence via Shtukas

We now formulate a dyadic analogue of the global Langlands correspondence, based on the cohomological and geometric properties of eigen-shtukas. The correspondence matches automorphic data arising from Hecke–Frobenius stable shtukas with continuous Galois representations over $\mathbb{Z}_2$.

6.1. **Eigen-Shtukas as Automorphic Parameters.** Let $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sht}^r_{\mathrm{dyad}})$ be a Frobenius–Hecke eigen-sheaf:

$$\mathrm{Frob}^*(\mathscr{F}) \simeq \alpha \cdot \mathscr{F}, \quad T_m(\mathscr{F}) \simeq \lambda_m \cdot \mathscr{F}.$$

We call such a sheaf an *eigen-shtuka*, and interpret:

$$\lambda = \{\lambda_m\}_{m\geq 1} \quad \text{as Hecke eigenvalues,}$$

$$\alpha = \text{Frobenius eigenvalue.}$$

Definition 6.1. *Let* $\mathrm{EigSht}_{\mathrm{dyad}}$ *be the moduli stack of eigen-shtukas. It classifies isomorphism classes of shtukas with simultaneous Frobenius and Hecke diagonalizability.*

6.2. **Galois Parameters over $\mathbb{Z}_2$.** Let $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and fix the Langlands dual group ${}^LG$ over $\mathbb{Z}_2$.

Definition 6.2. *A* dyadic Langlands parameter *is a continuous group homomorphism:*

$$\rho : G_{\mathbb{Q}} \longrightarrow {}^LG(\mathbb{Z}_2),$$

unramified outside 2*, with compatible Frobenius and inertia actions.*

Such a representation gives rise to a compatible trace:

$$L(\rho, s) := \mathrm{Tr}(\rho(\mathrm{Frob}_\ell) \cdot \ell^{-s}),$$

and determines a local-global compatible L-function.

6.3. **Formulation of the Correspondence.** We postulate:

Conjecture 6.3 (Dyadic Langlands Correspondence)**.** *There is a natural bijection:*

$$\{\textit{Irreducible eigen-shtukas } \mathscr{F}\} \quad \longleftrightarrow \quad \left\{\textit{Continuous } \rho : G_{\mathbb{Q}} \to {}^LG(\mathbb{Z}_2)\right\},$$

satisfying:

$$L(\mathscr{F}, s) = L(\rho, s),$$

and preserving Frobenius traces, local types, and Hecke eigenvalues.

6.4. **Categorical Reformulation.** The correspondence is upgraded to an equivalence of Tannakian categories:

$$\mathrm{EigSht}^{\otimes}_{\mathrm{dyad}} \simeq \mathrm{Rep}_{\mathbb{Z}_2}(G_{\mathbb{Q}}).$$

That is, the symmetric monoidal category of Frobenius–Hecke eigen-shtukas is equivalent to the category of Galois representations over $\mathbb{Z}_2$.

6.5. **Trace Identity and Spectral Equivalence.** For each $\mathscr{F}$ with corresponding $\rho_{\mathscr{F}}$, we have:

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^{\bullet}_{\mathrm{sht}}(\mathscr{F})) = \mathrm{Tr}(\rho_{\mathscr{F}}(\mathrm{Frob}_\ell) \cdot \ell^{-s}).$$

6.6. **Outlook: Nonabelian Class Field Theory over $\mathbb{Z}_2$.** This correspondence is the first step in a broader program to formulate a full *nonabelian class field theory over* $\mathbb{Z}_2$, using dyadic shtukas as moduli spaces of nonabelian reciprocity data, cohomologically paired with Galois symmetry.

7. Compactification and Boundary Motives of Dyadic Shtukas

To fully describe the moduli geometry and cohomology of dyadic shtukas, we construct a compactification of the moduli stack $\mathrm{Sht}^r_{\mathrm{dyad}}$ and define boundary motives associated to its degenerations. These provide necessary ingredients for trace stability, reflection symmetry, and the dyadic RH at the infinite congruence boundary.

7.1. **Cuspidal Degenerations and Boundary Divisors.**

Let $r \geq 1$. The moduli stack $\mathrm{Sht}^r_{\mathrm{dyad}}$ admits stratification by modification type and degeneration locus. Define the cuspidal boundary divisor:

$$\partial\,\mathrm{Sht}^r_{\mathrm{dyad}} := \bigcup_i \mathrm{Sht}_{\mathrm{deg},i},$$

where each $\mathrm{Sht}_{\mathrm{deg},i}$ represents degenerations of torsors at points of maximal congruence colength (e.g., rank drops, non-smooth points, or infinite filtration jumps).

7.2. **Compactified Stack.**

Definition 7.1. *Let $\overline{\mathrm{Sht}}^r_{\mathrm{dyad}}$ be the derived compactification of $\mathrm{Sht}^r_{\mathrm{dyad}}$ along the congruence boundary:*

$$\overline{\mathrm{Sht}}^r_{\mathrm{dyad}} := \varprojlim_n \overline{\mathrm{Sht}}^r_{2^n},$$

where each $\overline{\mathrm{Sht}}^r_{2^n}$ includes singular and reducible torsors.

This compactification admits:

- A stratification by depth of degeneration;
- A boundary sheaf $\mathcal{O}_\partial$ supported on $\partial\,\mathrm{Sht}^r_{\mathrm{dyad}}$;
- Residual Frobenius structure and modified Hecke actions.

7.3. **Boundary Cohomology and Motives.**

Define the relative cohomology of shtuka sheaves:

$$H^\bullet_{\mathrm{bd}}(\mathscr{F}) := R\Gamma(\overline{\mathrm{Sht}}^r_{\mathrm{dyad}}, \mathscr{F})/R\Gamma(\mathrm{Sht}^r_{\mathrm{dyad}}, \mathscr{F}).$$

Definition 7.2. *The boundary motive $M_\partial(\mathscr{F})$ is the complex:*

$$M_\partial(\mathscr{F}) := H^\bullet_{\mathrm{bd}}(\mathscr{F}),$$

interpreted as the residual object in the category of dyadic motives.

7.4. **Reflection Symmetry at Infinity.**

The boundary motive satisfies a duality:

$$\mathbb{D}_\infty(M_\partial(\mathscr{F})) \simeq M_\partial(\iota^*\mathscr{F})^\vee,$$

and

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid M_\partial(\mathscr{F})) = \mathrm{Tr}(\mathrm{Frob}^{-(1-s)} \mid M_\partial(\mathscr{F})).$$

7.5. **Asymptotic Zeta Expansion and Final RH Form.** Combining interior and boundary, define the completed dyadic zeta function:

$$\widehat{\zeta}_{\mathrm{dyad}}(s) := \sum_{\mathscr{F} \in \mathrm{IrrEigSht}} \left[\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^{\bullet}_{\mathrm{sht}}(\mathscr{F})) + \mathrm{Tr}(\mathrm{Frob}^{-s} \mid M_{\partial}(\mathscr{F})) \right].$$

Then:

Theorem 7.3 (Functional Equation)**.** *The completed zeta satisfies:*

$$\widehat{\zeta}_{\mathrm{dyad}}(s) = \widehat{\zeta}_{\mathrm{dyad}}(1-s).$$

7.6. **Final Statement of Dyadic RH.**

Conjecture 7.4 (Dyadic RH — Geometric Form)**.** *All nontrivial zeros of $\widehat{\zeta}_{\mathrm{dyad}}(s)$ lie on the critical line:*

$$\Re(s) = \tfrac{1}{2}.$$

This form of the RH includes boundary contributions, derived duality, and compactified trace structures arising from the geometry of $\overline{\mathrm{Sht}}^{r}_{\mathrm{dyad}}$.

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