NONCOMMUTATIVE PRISMATIC MOTIVES OVER OPERATOR ALGEBRAS AND FROBENIUS TRACE HOMOTOPY

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Abstract. We propose a theory of noncommutative prismatic motives defined over p-adic Banach and operator algebras. Extending prismatic and syntomic cohomology to E_1-algebras, we construct trace-compatible period sheaves, filtrations, and Frobenius actions in derived categories of noncommutative condensed motives. This allows us to define a syntomic–prismatic correspondence for noncommutative geometry and quantum period sheaves. Applications include topological cyclic homology over distribution algebras, categorical trace flows on noncommutative stacks, and new links to p-adic quantum Langlands theory.

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1. Introduction

Recent progress in p-adic geometry and homotopy theory—most notably through prismatic cohomology and condensed mathematics—has created a powerful language for period sheaves, trace maps, and integral p-adic Hodge theory. However, these constructions rely crucially on commutativity and p-adic completeness.

In this paper, we initiate a noncommutative prismatic theory. We study p-adic E_1-algebras, operator completions, and distribution algebras as generalized base rings for cohomology and define *noncommutative prismatic motives*. These extend syntomic–crystalline–de Rham structures to noncommutative settings by lifting Frobenius-periodic geometry to stable ∞-categories of A_∞-modules, operator-derived stacks, and cyclotomic traces.

We formulate:

- a Frobenius-compatible filtration on cyclic motives over p-adic distribution algebras;
- a derived site of noncommutative prisms;
- trace-compatible period sheaves over quantum groups and enveloping algebras;
- a syntomic cohomology theory for Banach–analytic E_1-algebras.

Our framework combines ideas from prismatic cohomology, noncommutative motives, cyclic homology, and condensed solid analytic geometry.

2. Noncommutative Prisms and Derived Enveloping Algebras

To develop a noncommutative prismatic theory, we begin by generalizing the notion of prisms to the setting of p-adic E_1-algebras and derived completions of noncommutative rings such as universal enveloping algebras and Iwasawa-type distribution algebras. Our goal is to construct Frobenius-periodic structures, Nygaard-type filtrations, and trace-compatible period sheaves over these noncommutative bases.

2.1. **From Commutative to E_1 Prisms.** Recall that a commutative prism is a pair (A, I) where A is a p-adically complete δ-ring and $I \subset A$ is an ideal satisfying certain torsion and Frobenius conditions. In the noncommutative setting, we relax commutativity and use higher categorical structures.

Definition 2.1 (Noncommutative Prism). *Let A be a p-complete E_1-algebra over $\mathbb{Z}_p$ in the category of condensed spectra. A* noncommutative prism *is a pair (A, I) such that:*

(1) A admits a homotopical Frobenius lift $\varphi : A/p \to A/p$ in $\mathrm{CAlg}_{E_1}(\mathrm{Cond}(\mathbb{Z}_p/p))$;
(2) $I \subset A$ is a homotopically central ideal (i.e., commutes up to coherent homotopy) that generates a (p, I)-complete filtration on A;
(3) The associated graded $\mathrm{gr}_I^\bullet A$ admits a derived δ-structure compatible with φ.

Remark 2.2. We replace strict δ-structures by derived Frobenius lifts and filtrations, following approaches in spectral algebraic geometry and cyclotomic spectra.

2.2. **Completed Enveloping Algebras and Distribution Bases.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}_p$, and consider the p-adic completion of its universal enveloping algebra:

$$U_p(\mathfrak{g}) := \widehat{U(\mathfrak{g})}_{p\text{-adic}}.$$

This is a noncommutative p-adic Banach algebra and serves as a natural base for p-adic representations, quantum groups, and distribution theory.

Example 2.3. *Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{Q}_p)$. Then $U_p(\mathfrak{g})$ acts on locally analytic vectors in p-adic Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.*

We may also consider distribution algebras:

Definition 2.4. *Let G be a p-adic analytic group. The algebra $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$ of locally analytic distributions on G is defined as the continuous dual of $\mathcal{C}^{\mathrm{la}}(G, \mathbb{Q}_p)$ equipped with convolution product.*

Remark 2.5. $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$ is a noncommutative p-adic Banach algebra, often non-Noetherian and infinite dimensional. It naturally arises in p-adic Langlands theory.

2.3. **Derived Prismatization of Operator Algebras.** We now define the prismatic structure over $U_p(\mathfrak{g})$ or $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$ using derived categories of condensed modules.

Definition 2.6. *Let A be a p-adic E_1-algebra as above. The* derived prismatic envelope $\mathsf{Prism}_A^{\mathrm{nc}}$ *is a filtered complex in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$ equipped with:*

- *a filtered Frobenius semi-linear action φ;*
- *an operator trace map to $\mathrm{TC}(A)$;*
- *a Nygaard-type complete filtration induced by the derived I-adic structure.*

Theorem 2.7 (Existence of Noncommutative Prismatic Filtration)**.** *Let A be either $U_p(\mathfrak{g})$ or $\mathcal{D}^{\mathrm{la}}(G,\mathbb{Q}_p)$. Then $\mathsf{Prism}_A^{\mathrm{nc}}$ exists in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$ and admits a convergent filtration $\mathcal{N}^\bullet$ such that:*

$$\mathrm{gr}^i(\mathsf{Prism}_A^{\mathrm{nc}}) \simeq \mathrm{THH}_i(A)^{tC_p}.$$

Sketch. We use the framework of condensed cyclotomic spectra to define $\mathrm{THH}(A)$ and invoke the descent of the cyclotomic trace to define filtrations compatible with Frobenius. The key input is the solid analytic tensor product preserving convergence of p-adic completions. □

2.4. **Examples and Future Structures.**

Example 2.8. *Let $A = \mathcal{D}^{\mathrm{la}}(\mathrm{GL}_2(\mathbb{Q}_p),\mathbb{Q}_p)$. Then $\mathsf{Prism}_A^{\mathrm{nc}}$ controls the syntomic period sheaves attached to locally analytic p-adic automorphic forms.*

Example 2.9. *Let $A = U_p(\mathfrak{sl}_2)$. Then $\mathsf{Prism}_A^{\mathrm{nc}}$ captures traces of Frobenius in quantum group representations at p-adic roots of unity.*

3. Noncommutative Period Sheaves and Frobenius Trace Structures

Having established the notion of noncommutative prisms and their derived completions over p-adic operator algebras, we now construct noncommutative period sheaves, define Frobenius-trace structures, and interpret syntomic cohomology in this extended setting. Our constructions generalize classical period sheaves like $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{HT}}$, and their filtrations to modules and complexes over noncommutative p-adic E_1-algebras.

3.1. **Condensed Period Sheaves over E_1-Algebras.**

Let A be a p-complete E_1-algebra over $\mathbb{Z}_p$ in $\mathrm{Cond}(\mathsf{Ab})$. Assume A carries a homotopy Frobenius lift and a Nygaard-type filtration. We define period sheaves associated to $\mathsf{Prism}_A^{\mathrm{nc}}$.

Definition 3.1 (Noncommutative Period Sheaves)**.** *Let $\mathsf{Prism}_A^{\mathrm{nc}}$ be the noncommutative prismatic complex of A. Then define:*

$$\begin{aligned}
\mathbb{B}_{\mathrm{dR}}^{+,\mathrm{nc}}(A) &:= \widehat{\mathsf{Prism}_A}^{\mathrm{nc}} \\
\mathbb{B}_{\mathrm{HT}}^{\mathrm{nc}}(A) &:= \mathrm{gr}^\bullet(\mathbb{B}_{\mathrm{dR}}^{+,\mathrm{nc}}(A)) \\
\mathbb{B}_{\mathrm{crys}}^{\mathrm{nc}}(A) &:= (\mathsf{Prism}_A^{\mathrm{nc}})^{\varphi=1} \\
\mathbb{B}_{\mathrm{syn}}^{\mathrm{nc}}(A) &:= \left[(\mathsf{Prism}_A^{\mathrm{nc}})^{\varphi=p^i} \to \mathsf{Prism}_A^{\mathrm{nc}}\right]^{\mathrm{fib}}.
\end{aligned}$$

Remark 3.2. These sheaves live in the derived category $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$ and admit Frobenius, filtration, and trace compatibilities. Their grading is induced by the Nygaard filtration on $\mathsf{Prism}_A^{\mathrm{nc}}$.

3.2. **Frobenius Trace and Cyclotomic Fixed Points.** We extend the cyclotomic trace map $\mathrm{Tr}_{\mathrm{cycl}}$ from classical commutative rings to the noncommutative setting via the theory of condensed cyclotomic spectra.

Definition 3.3 (Frobenius Trace)**.** *Let A be a p-adic E_1-algebra. The* Frobenius trace *is the map:*

$$\mathrm{Tr}_\varphi : \mathrm{TC}(A) \longrightarrow \mathbb{B}_{\mathrm{syn}}^{\mathrm{nc}}(A)$$

defined by composing the cyclotomic trace $\mathrm{K}(A) \to \mathrm{TC}(A)$ *with the syntomic comparison via filtered prismatic cohomology.*

Theorem 3.4. *Let A be either $U_p(\mathfrak{g})$ or $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$. Then the Frobenius trace map* Tr_φ*:*

$$\mathrm{TC}(A) \longrightarrow \mathbb{B}_{\mathrm{syn}}^{\mathrm{nc}}(A)$$

is functorial in A and compatible with composition of p-adic operator extensions, preserving the action of derived Frobenius and solid completions.

Sketch. The map is constructed via the universal property of TC as a trace functor and the Frobenius-periodic structure on $\mathsf{Prism}_A^{\mathrm{nc}}$. Functoriality follows from the symmetric monoidal structure of the derived category $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$. □

3.3. **Noncommutative Syntomic Cohomology.** We now define the syntomic cohomology theory associated to a noncommutative prism.

Definition 3.5. *The* noncommutative syntomic cohomology *of A is:*

$$\mathrm{Syn}^{\mathrm{nc}}(A) := \mathrm{R}\Gamma_{\mathrm{syn}}(A) := \mathrm{fib}\left(\mathsf{Prism}_A^{\mathrm{nc},\varphi=p^i} \to \mathsf{Prism}_A^{\mathrm{nc}}\right).$$

Example 3.6. *Let $A = \mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$ for G compact p-adic Lie. Then* $\mathrm{Syn}^{\mathrm{nc}}(A)$ *governs the syntomic deformation theory of locally analytic representations of G.*

3.4. **Trace Filtrations and Quantum Period Dualities.** The filtration on $\mathsf{Prism}_A^{\mathrm{nc}}$ induces dualities between noncommutative period sheaves and representation-theoretic categories.

Conjecture 3.7 (Noncommutative Period Duality)**.** *There exists an equivalence:*

$$\mathbb{B}_{\mathrm{dR}}^{+,\mathrm{nc}}(A) \simeq \mathrm{RHom}_A(M, M)^{\mathrm{fil},\varphi}$$

where M is a perfect A-module with Frobenius-periodic filtration, and the right-hand side is computed in filtered derived ∞-categories over $\mathrm{Cond}(\mathbb{Z}_p)$.

This allows us to identify traces of p-adic quantum categories with syntomic and de Rham period sheaves.

In the next section, we define noncommutative Langlands stacks and explore applications of this framework to p-adic quantum moduli and categorical Frobenius representations.

4. Noncommutative Langlands Stacks and Quantum Prismatic Categories

In this final section, we synthesize the constructions of noncommutative prismatic motives, period sheaves, and syntomic cohomology to define a new framework: the **noncommutative Langlands stack**. This stack parametrizes categorical p-adic quantum data such as filtered Frobenius modules, topological cyclic traces, and prismatic structures over noncommutative operator algebras. It provides a new categorical moduli space for p-adic quantum groups and their Langlands-type dualities.

4.1. Derived Stack of Prismatic φ-Modules over E_1-Algebras.

Let A be a noncommutative p-adic E_1-algebra (e.g., $\mathcal{D}^{\mathrm{la}}(G, \mathbb{Q}_p)$). We define:

Definition 4.1. *The* stack of filtered prismatic Frobenius modules over *A is the derived prestack:*

$$\mathcal{M}^{\mathrm{nc}}_{\varphi,\mathrm{fil}}(A) := \left\{ (M, \varphi_M, \mathrm{Fil}^\bullet) \;\middle|\; \begin{array}{l} M \in \mathrm{Perf}(A),\ \varphi_M : M \to M \textit{ is a Frobenius lift,} \\ \mathrm{Fil}^\bullet \textit{ is a Nygaard-compatible filtration} \end{array} \right\}.$$

Remark 4.2. The moduli stack $\mathcal{M}^{\mathrm{nc}}_{\varphi,\mathrm{fil}}(A)$ is a derived algebraic stack in the ∞-topos of $\mathrm{Cond}(\mathbb{Z}_p)$-algebras, enhanced by a cyclotomic trace and period sheaf comparison.

4.2. Quantum Langlands Stack and Syntomic Realization.

We now extend the notion of Langlands parameters to a categorical and noncommutative context.

Definition 4.3. *The* noncommutative Langlands stack $\mathfrak{Lang}^{\mathrm{nc}}_p$ *is the moduli stack classifying:*

$$(\rho, \varphi, \mathbb{B}^{\mathrm{nc}}_{\mathrm{HT}}, \mathrm{Syn}^{\mathrm{nc}}(\rho))$$

where ρ is a categorical representation (e.g., a module category over A), φ is a Frobenius-compatible endofunctor, and $\mathbb{B}^{\mathrm{nc}}_{\mathrm{HT}}$ and $\mathrm{Syn}^{\mathrm{nc}}$ are the attached noncommutative period and syntomic invariants.

Example 4.4. *For $A = U_p(\mathfrak{sl}_2)$, objects of $\mathfrak{Lang}_p^{\mathrm{nc}}$ include quantum de Rham–crystalline categories attached to filtered Frobenius modules over U_p.*

Conjecture 4.5 (Prismatic–Langlands Correspondence)**.** *There exists a derived equivalence of stacks:*

$$\mathfrak{Lang}_p^{\mathrm{nc}} \simeq \mathcal{M}_{\varphi,\mathrm{fil}}^{\mathrm{nc}}(A)$$

realizing categorical quantum Langlands parameters as Frobenius-periodic prismatic motives over E_1-algebras.

4.3. **Quantum Cyclotomic Descent and Categorical Traces.** The syntomic realization admits a cyclotomic descent structure via THH and TC:

Theorem 4.6 (Cyclotomic Descent for Langlands Stacks)**.** *There is a fiber sequence in $\mathcal{D}(\mathrm{Cond}(\mathbb{Z}_p))$:*

$$\mathrm{K}(A) \xrightarrow{\mathrm{Tr}_{\mathrm{cycl}}} \mathrm{TC}(A) \xrightarrow{\mathrm{Tr}_{\varphi}} \mathrm{Syn}^{\mathrm{nc}}(A)$$

compatible with the filtered derived structures on $\mathfrak{Lang}_p^{\mathrm{nc}}$ and inducing a Frobenius trace on moduli points.

Remark 4.7. This yields a categorified and filtered interpretation of p-adic quantum zeta functions and trace formulas over noncommutative motives.

4.4. **Outlook: Noncommutative Syntomic Motives and Langlands Gravity.** Our theory suggests new directions in arithmetic geometry and representation theory:

- Prismatic cohomology for p-adic quantum groups, leading to nonabelian crystalline duality;
- Syntomic quantization of categorical Galois representations;
- Langlands–Frobenius flow stacks over filtered operator topoi;
- Traced monodromy in p-adic categorical field theories and entropy cohomology.

Conjecture 4.8 (Noncommutative Entropy Langlands Stack)**.** *There exists a categorified extension $\mathfrak{Lang}_p^{\mathrm{nc,ent}}$ incorporating entropy-filtered traces and quantum period flow sheaves, capturing spectral deformations of automorphic categories via noncommutative prismatic motives.*

This concludes our construction of noncommutative prismatic motives and their moduli. Future work will integrate motivic quantum entropy structures and polycategorical deformation fields across p-adic and analytic stacks.