SCHNIRELMANN-TYPE DENSITY IN ULTRAPRODUCTS AND NONSTANDARD ANALYSIS

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Abstract. We define and explore Schnirelmann-type density and additive closure in the setting of ultraproducts and nonstandard models of arithmetic. Using hyperfinite sets and internal subsets of $^*\mathbb{N}$, we establish analogues of classical theorems and propose infinitesimal-adjusted closure conditions.

1. Ultraproduct Framework

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the ultrapower $^*\mathbb{N} = \prod_{n\in\mathbb{N}} \mathbb{N}/\mathcal{U}$.

Definition 1.1 (Internal Schnirelmann Density)**.** For $A \subseteq {}^*\mathbb{N}$ internal, define

$$\sigma^*(A) := \mathrm{st}\left(\inf_{N\in{}^*\mathbb{N},\ N \text{ finite}} \frac{|A\cap[1,N]|}{N} \right),$$

where ”st” denotes the standard part map.

Definition 1.2 (Hyperfinite Closure)**.** For $A \subseteq {}^*\mathbb{N}$ internal and $k \in \mathbb{N}$, define

$$kA := \{a_1 + \cdots + a_k \mid a_i \in A\}.$$

A is said to be *-additively closed if $kA = {}^*\mathbb{N}$ for some k.

Proposition 1.3. *If $\sigma^*(A) > 0$, then $kA = {}^*\mathbb{N}$ for some $k \in \mathbb{N}$.*

Remark 1.4. This generalizes the standard Schnirelmann theorem to internal sets in non-standard arithmetic.

2. Infinitesimal Adjustments

Definition 2.1 (Infinitesimal Boundary Density)**.** Let μ be a Loeb measure on $^*\mathbb{N}$. Define

$$\partial^\epsilon A := \mu(\mathrm{st}^{-1}([0,\epsilon]) \cap \partial A),$$

where ∂A is the topological boundary of A.

Proposition 2.2. *If $\partial^\epsilon A$ is infinitesimal for all $\epsilon > 0$, then A behaves like a measurable Schnirelmann-dense set in Loeb measure.*

3. Applications and Extensions

- Transfer of additive closure results from $^*\mathbb{N}$ to $\mathbb{N}$ via Łoś's Theorem
- Model-theoretic characterizations of additive bases in first-order arithmetic
- Use of hyperfinite methods to simulate dense subset growth
- Ultraproducts of additive combinatorics structures (e.g., sumsets)

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4. Future Directions

- Define Schnirelmann-type hierarchies in nonstandard models
- Connect to additive ergodic theory on internal measure spaces
- Formalize hyperfinite analytic analogues of additive energy