THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET L-FUNCTIONS OFF THE CRITICAL LINE. III

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ABSTRACT. We show, for any $q \geqslant 3$ and distinct reduced residues $a, b \pmod q$, the existence of certain hypothetical sets of zeros of Dirichlet L-functions lying off the critical line implies that $\pi(x; q, a) < \pi(x; q, b)$ for a set of real x of asymptotic density 1.

1 Introduction

For $(a, q) = 1$, let $\pi(x; q, a)$ denote the number of primes $p \leqslant x$ with $p \equiv a \pmod q$. The study of the relative magnitudes of the functions $\pi(x; q, a)$ for a fixed q and varying a is known colloquially as the "prime race problem" or "Shanks-Rényi prime race problem". For a survey of problems and results on prime races, the reader may consult the papers [4] and [5]. One basic problem is the study of $P_{q;a_1\ldots,a_r}$, the set of real numbers $x \geqslant 2$ such that $\pi(x; q, a_1) > \cdots > \pi(x; q, a_r)$. It is generally believed that all sets $P_{q;a_1\ldots,a_r}$ are unbounded. Assuming the Generalized Riemann Hypothesis for Dirichlet L-functions modulo q (GRH$_q$) and that the nonnegative imaginary parts of zeros of these L-functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown for any r-tuple of reduced residue classes $a_1, \ldots, a_r$ modulo q, that $P_{q;a_1,\ldots,q_r}$ has a positive logarithmic density (although the density may be quite small in some cases).

In [2] and [3], Ford and Konyagin investigated how possible violations of the Generalized Riemann Hypothesis (GRH) would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets $P_{q;a_1,a_2,a_3}$ are bounded, giving a negative answer to the prime race problem with $r = 3$. Paper [3] was devoted to similar questions for r-way prime races with $r > 3$. One result from [3] states that for any q, $r \leqslant \phi(q)$ and set $\{a_1, \ldots, a_r\}$ of reduced residues modulo q, the existence of certain hypothetical sets of zeros of Dirichlet L-functions modulo q implies that at most $r(r-1)$ of the sets $P_{q;\sigma(a_1),\ldots,\sigma(a_r)}$ are unbounded, σ running over all permutations of $\{a_1, \ldots, a_r\}$.

In this paper, we investigate the effect of zeros of L-functions lying off the critical line for two way prime races. This case is harder, since it is unconditionally proved that for certain races $\{q; a, b\}$ the set $P_{q;a,b}$ is unbounded. For example, Littlewood [11] proved that $P_{4;3,1}$, $P_{4;1,3}$, $P_{3;1,2}$ and $P_{3;2,1}$ are unbounded. Later Knapowski and Turàn ([9], [10]) proved for many q, a, b that $\pi(x; q, b) - \pi(x; q, a)$ changes sign infinitely often and more recently Sneed [13] showed that $P_{q;a,b}$ is unbounded for every $q \leqslant 100$ and all possible pairs (a, b).

Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q;a,b}$ has asymptotic density zero, in contrast with a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q;1,b}$ and $P_{q;b,1}$ have positive lower densities for all $(b, q) = 1$.

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Let $q \geqslant 3$ be a positive integer and a, b be distinct reduced residues modulo q. Moreover, for any set $\mathcal{S}$ of real numbers we define $\mathcal{S}(X) = \mathcal{S} \cap [2, X]$.

Theorem 1.1. *Let $q \geqslant 3$ and suppose that a and b are distinct reduced residues modulo q. Let χ be a nonprincipal Dirichlet character with $\chi(a) \neq \chi(b)$, and put $\xi = \arg(\chi(a) - \chi(b)) \in [0, 2\pi)$. Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$,$A > 0$, and $\mathcal{B} = \mathcal{B}(\xi, \sigma, \delta, A)$ is a multiset of complex numbers satisfying the conditions listed in Section 2. If $L(\rho, \chi) = 0$ for all $\rho \in \mathcal{B}$, $L(s, \chi)$ has no other zeros in the region $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, and for all other nonprincipal characters χ' modulo q, $L(s, \chi') \neq 0$ in the region $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, then*

$$\lim_{X \to \infty} \frac{\text{meas}(P_{q;a,b}(X))}{X} = 0.$$

Remarks. Such χ exists whenever a and b are distinct modulo q. The sets $\mathcal{B}$ have the property that any $\rho \in \mathcal{B}$ has real part in $[\sigma - \delta, \sigma]$, imaginary part greater than A, and multiplicity $O((\log \mathrm{Im}(\rho))^{3/4})$ (that is, the multiplicities are much smaller than known bounds on the multiplicity of zeros of Dirichlet L-functions). The number of elements of $\mathcal{B}$ (counted with multiplicity) with imaginary part less than T is $O((\log T)^{5/4})$, and thus $\mathcal{B}$ is quite a “thin” set. Also, we note that if $L(\beta + i\gamma, \chi) = 0$ then $L(\beta - i\gamma, \overline{\chi}) = 0$, which is a consequence of the functional equation for Dirichlet L-functions (See e.g. Ch. 9 of [1]). The point of Theorem 1.1 is that proving

$$\limsup_{X \to \infty} \frac{\text{meas}(P_{q;a,b}(X))}{X} > 0$$

requires showing that the multiset of zeros of $L(s, \chi)$ cannot contain any of the multisets $\mathcal{B}$. This is beyond what is possible with existing technology (see e.g. [6] for the best known estimates for multiplicities of zeros).

Our method works as well for the difference $\pi(x) - \mathrm{li}(x)$, the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let P_1 be the set of real numbers $x \geqslant 2$ such that $\pi(x) > \mathrm{li}(x)$. In [8] Kaczorowski proved, assuming the Riemann Hypothesis, that both P_1 and $\overline{P}_1$ have positive lower densities. Assuming the Riemann Hypothesis and that the nonnegative imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that P_1 has a positive logarithmic density $\delta_1 \approx 0.00000026$. In contrast with these results we prove that the existence of certain zeros of $\zeta(s)$ off the critical line would imply that the set P_1 has asymptotic density zero (or asymptotic density 1).

Theorem 1.2. *Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$ and $A > 0$. (i) If $\xi = 0$, $\mathcal{B} = \mathcal{B}(\xi, \sigma, \delta, A)$ satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in \mathcal{B}$, and $\zeta(s)$ has no other zeros in the region $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, then*

$$\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 0.$$

(ii) If $\xi = \pi$, $\mathcal{B}$ satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in \mathcal{B}$, and $\zeta(s)$ has no other zeros in the region $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, then

$$\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 1.$$

We omit the proof of Theorem 1.2, as it is nearly identical to the proof of Theorem 1.1 in the case $q = 4$.

2 The construction of $\mathcal{B}$

For $j \geqslant 1$, suppose that

$$\exp\left(j^8\right) \leqslant \gamma_j \leqslant 2\exp\left(j^8\right), \quad \left|\delta_j - \frac{1}{j^8}\right| \leqslant \frac{1}{j^9},$$
$$\text{and} \quad \left|\theta_j - \frac{\xi - \pi/2}{j^{16}}\right| \leqslant \frac{1}{j^{17}}. \tag{2.1}$$

We choose j_0 so large that for all $j \geqslant j_0$, $\gamma_j > A$ and $\sigma - \delta \leqslant \sigma - \delta_j$. Then we take $\mathcal{B}$ to be the union, over $j \geqslant j_0$ and $1 \leqslant k \leqslant j^3$, of $m(k,j) = k(j^3 + 1 - k)$ copies of $\rho_{j,k}$, where

$$\rho_{j,k} = \sigma - \delta_j + i(k\gamma_j + \theta_j).$$

3 Preliminary Results

The following classical-type explicit formula was established in Lemma 1.1 of [2] when $x' = x$. The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

Lemma 3.1. *Let $\beta \geqslant 1/2$ and for each non-principal character $\chi \bmod q$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s,\chi)$ with $\mathrm{Re}(s) > \beta$ and $\mathrm{Im}(s) > 0$. Suppose further that all $L(s,\chi)$ are zero-free on the real segment $\beta < s < 1$. If $(a,q) = (b,q) = 1$, x is sufficiently large and $x' \geqslant x$, then*

$$\phi(q)\big(\pi(x;q,a) - \pi(x;q,b)\big) = -2\mathrm{Re}\left(\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} (\overline{\chi}(a) - \overline{\chi}(b)) \sum_{\substack{\rho \in B(\chi) \\ |\mathrm{Im}(\rho)| \leqslant x'}} f(\rho)\right) + O\left(x^\beta \log^2 x\right),$$

where

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho}\int_2^x \frac{t^\rho}{t \log^2 t} dt = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\mathrm{Re}(\rho)}}{|\rho|^2 \log^2 x}\right).$$

Remark. For Theorem 1.2, we use a similar explicit formula for $\pi(x)$ in terms of the zeros $B(\zeta)$ of the Riemann zeta function which satisfy $\Re\rho > \beta$ and $\Im\rho > 0$:

$$\pi(x) = \mathrm{li}(x) - 2\Re \sum_{\substack{\rho \in \mathcal{B}(\zeta) \\ |\Im\rho| \leqslant x'}} f(\rho) + O(x^\beta \log^2 x).$$

Using properties of the Fejér kernel we prove the following key proposition.

Proposition 3.2. *Let $\gamma \geqslant 1$, $L \geqslant 4$ and $X \geqslant 2$. Define*

$$F_{\gamma,L}(x) = \sum_{k=1}^{L-1} (L-k)\cos\left(k\gamma \log x\right).$$

Then

$$\text{meas}\left\{x \in [1,X] : F_{\gamma,L}(x) \geqslant -\frac{L}{4}\right\} \ll \frac{X}{\sqrt{L}}.$$

Proof. The Fejér kernel satisfies the following identity

$$\frac{1}{L}\left(\frac{\sin\left(\frac{L\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^2 = 1 + 2\sum_{k=1}^{L-1}\left(1-\frac{k}{L}\right)\cos(k\theta).$$

This yields

$$F_{\gamma,L}(x) = \frac{\sin^2\left(\frac{L\gamma\log x}{2}\right)}{2\sin^2\left(\frac{\gamma\log x}{2}\right)} - \frac{L}{2}.$$

Therefore, if $F_{\gamma,L}(x) \geqslant -L/4$ then

$$\sin^2\left(\frac{\gamma\log x}{2}\right) \leqslant \frac{2}{L}\sin^2\left(\frac{L\gamma\log x}{2}\right) \leqslant \frac{2}{L}.$$

Hence,

$$\left\|\frac{\gamma\log x}{2\pi}\right\| \leqslant \varepsilon := \frac{1}{\sqrt{2L}},$$

where $\|t\|$ denotes the distance to the nearest integer. This implies

$$\begin{aligned}\text{meas}\left\{x\in[1,X] : F_{\gamma,L}(x)\geqslant -\frac{L}{4}\right\} &\leqslant \text{meas}\left\{x\in[1,X] : \quad \left\|\frac{\gamma\log x}{2\pi}\right\|\leqslant\varepsilon\right\}\\ &\leqslant \sum_{0\leqslant k\leqslant\frac{\gamma\log X}{2\pi}+\varepsilon} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma}\\ &\ll \frac{\varepsilon}{\gamma}\sum_{0\leqslant k\leqslant\frac{\gamma\log X}{2\pi}+\varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X. \quad \square\end{aligned}$$

4 Proof of Theorem 1.1

Suppose X is large and $\sqrt{X}\leqslant x\leqslant X$. For brevity, let

$$\Delta = \phi(q)\big(\pi(x;q,a)-\pi(x;q,b)\big).$$

It follows from Lemma 3.1 with $x' = \max(x, \max\{j^3\gamma_j : \gamma_j\leqslant x\})$ that

$$\begin{aligned}\Delta &= -\frac{2}{\log x}\text{Re}\left((\overline{\chi}(a)-\overline{\chi}(b))\sum_{\gamma_j\leqslant x}\sum_{k=1}^{j^3}\frac{x^{\sigma-\delta_j+i(k\gamma_j+\theta_j)}m(k,j)}{\sigma-\delta_j+i(k\gamma_j+\theta_j)}\right)\\ &\quad + O\left(\frac{x^\sigma}{\log^2 x}\sum_{\gamma_j\leqslant x}\frac{x^{-\delta_j}}{\gamma_j^2}\sum_{k=1}^{j^3}\frac{m(k,j)}{k^2} + x^{\sigma-\delta}\log^2 x\right)\\ &= \frac{2x^\sigma}{\log x}\text{Re}\left(i(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\gamma_j\leqslant x}\frac{x^{-\delta_j}}{\gamma_j}\sum_{k=1}^{j^3}x^{i(k\gamma_j+\theta_j)}(j^3+1-k)\right)\\ &\quad + O\left(\frac{x^\sigma}{\log x}\sum_{\gamma_j\leqslant x}\frac{j^4x^{-\delta_j}}{\gamma_j^2} + x^{\sigma-\delta}\log^2 x\right).\end{aligned} \tag{4.1}$$

Note that

$$\frac{x^{-\delta_j}}{\gamma_j} = \exp\left(-\frac{\log x}{j^8}\left(1+O\left(\frac{1}{j}\right)\right) - j^8 + O(1)\right).$$

The maximum of this function over j occurs around $J = J(x) := \left[(\log x)^{1/16}\right]$. In this case we have $\log x = J^{16}(1+O(1/J))$ so that

$$\frac{x^{-\delta_J}}{\gamma_J} = \exp\left(-2J^8 + O\left(J^7\right)\right) = \exp\left(-2(\log x)^{1/2} + O((\log x)^{7/16})\right). \tag{4.2}$$

We will prove that most of the contribution to the main term on the right hand side of (4.1) comes for the j's in the range $J - J^{3/4} \leqslant j \leqslant J + J^{3/4}$. First, if $j \geqslant 3J/2$ or $j \leqslant J/2$ then

$$\frac{x^{-\delta_j}}{\gamma_j} \ll \exp\left(-4J^8\right) \ll \exp\left(-(\log x)^{1/2}\right)\frac{x^{-\delta_J}}{\gamma_J}.$$

Now suppose that $J/2 < j < J - J^{3/4}$ or $J + J^{3/4} < j < 3J/2$. Write $j = J + r$ with $J^{3/4} < |r| < J/2$. For $x > 0$, $x + 1/x = 2 + (x-1)^2/x$, hence

$$\left(1+\frac{r}{J}\right)^8 + \left(1+\frac{r}{J}\right)^{-8} \geqslant \left(1+\left|\frac{r}{J}\right|\right)^8 + \left(1+\left|\frac{r}{J}\right|\right)^{-8} \geqslant 2 + \frac{(8r/J)^2}{1+8r/J} \geqslant 2 + 12(r/J)^2.$$

We infer from (4.2) that

$$\begin{aligned}\frac{x^{-\delta_j}}{\gamma_j} &= \exp\left(-\frac{J^{16}}{j^8}\left(1+O\left(\frac{1}{J}\right)\right) - j^8\right)\\ &= \exp\left(-J^8\left(\left(1+\frac{r}{J}\right)^8 + \left(1+\frac{r}{J}\right)^{-8}\right) + O(J^7)\right)\\ &\leqslant \exp\left(-2J^8\left(1+\frac{6}{\sqrt{J}}\right) + O(J^7)\right)\\ &\ll \exp\left(-2(\log x)^{1/3}\right)\frac{x^{-\delta_J}}{\gamma_J}.\end{aligned}$$

Since $\gamma_j \leqslant x$ implies that $j \ll (\log x)^{1/8}$, the contribution of the terms $1 \leqslant j < J - J^{3/4}$ or $J + J^{3/4} < j$ to the main term of (4.1) is

$$\ll \exp\left(-2(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}\sum_{j\leqslant(\log x)^{1/4}}\sum_{k=1}^{j^3}(j^3+1-k) \ll \exp\left(-(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}. \tag{4.3}$$

Similarly, we have

$$\begin{aligned}\frac{x^{-\delta_j}}{\gamma_j^2} &= \exp\left(-\frac{\log x}{j^8}\left(1+O\left(\frac{1}{j}\right)\right) - 2j^8 + O(1)\right)\\ &\ll \exp\left(-2\sqrt{2}(\log x)^{1/2}(1+o(1))\right)\\ &\ll \exp\left(-2(\log x)^{1/3}\right)\frac{x^{-\delta_J}}{\gamma_J},\end{aligned}$$

which follows from (4.2) along with the fact that the maximum of $f(t) = -\log x/t^8 - 2t^8$ occurs at $t = (\log x/2)^{1/16}$. Hence, using (4.2), the contribution of the error term of (4.1) is

$$\ll \exp\left(-2(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}\sum_{j\leqslant(\log x)^{1/4}} j^4 + x^{\sigma-\delta}\log^2 x \ll \exp\left(-(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}. \tag{4.4}$$

Therefore, inserting the bounds (4.3) and (4.4) in (4.1) we deduce that

$$\begin{aligned}\Delta = \frac{2x^\sigma}{\log x}\mathrm{Re}\left(i(\overline{\chi}(a)-\overline{\chi}(b))\sum_{|j-J|\leqslant J^{3/4}}\frac{x^{-\delta_j}}{\gamma_j}\sum_{k=1}^{j^3}\exp\left(i(k\gamma_j+\theta_j)\log x\right)(j^3+1-k)\right)\\ + O\left(\exp\left(-(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}\right).\end{aligned} \tag{4.5}$$

Let $J - J^{3/4} \leqslant j \leqslant J + J^{3/4}$. Then $j^{16} = J^{16}\left(1 + O(J^{-1/4})\right)$. Hence we get

$$\begin{aligned}\theta_j \log x &= \left(\arg(\chi(a)-\chi(b)) - \frac{\pi}{2}\right)\frac{\log x}{j^{16}} + O\left(\frac{\log x}{j^{17}}\right)\\ &= \left(\arg(\chi(a)-\chi(b)) - \frac{\pi}{2}\right) + O\left(\frac{1}{J^{1/4}}\right).\end{aligned}$$

This implies

$$i(\overline{\chi}(a)-\overline{\chi}(b))\exp\left(i\theta_j\log x\right) = |\chi(a)-\chi(b)|\left(1+O\left(\frac{1}{J^{1/4}}\right)\right),$$

since $e^{i\arg z} = z/|z|$. Inserting this estimate in (4.5) we obtain

$$\begin{aligned}\Delta = \left(1+O\left(\frac{1}{\log^{1/64} x}\right)\right)2|\chi(a)-\chi(b)|\sum_{|j-J|\leqslant J^{3/4}}\frac{x^{\sigma-\delta_j}}{\gamma_j\log x}F_{\gamma_j,j^3}(x)\\ + O\left(\exp\left(-(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}\right).\end{aligned} \tag{4.6}$$

For $x \in [\sqrt{X}, X]$ we have $\frac{1}{4}(\log X)^{1/16} \leqslant J - J^{3/4}$ and $J + J^{3/4} \leqslant 4(\log X)^{1/16}$ if X is sufficiently large, since $J = (\log x)^{1/16} + O(1)$. We define

$$\Omega := \left\{x \in [\sqrt{X}, X] : F_{\gamma_j,j^3}(x) \leqslant -\frac{j^3}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leqslant j \leqslant 4(\log X)^{1/16}\right\}.$$

Then it follows from Proposition 3.2 that

$$\begin{aligned}\mathrm{meas}\Omega &= X + O\left(X\sum_{\frac{1}{4}(\log X)^{1/16}\leqslant j\leqslant 4(\log X)^{1/16}}\frac{1}{j^{3/2}} + \sqrt{X}\right)\\ &= X\left(1+O\left((\log X)^{-1/32}\right)\right).\end{aligned} \tag{4.7}$$

Furthermore, if $x \in \Omega$ then we infer from (4.6) that

$$\begin{aligned}\Delta &\leqslant -\frac{1}{3}|\chi(a)-\chi(b)| \sum_{|j-J|\leqslant J^{3/4}} \frac{j^3 x^{\sigma-\delta_j}}{\gamma_j \log x} + O\left(\exp\left(-(\log x)^{1/3}\right)\frac{x^{\sigma-\delta_J}}{\gamma_J}\right). \\ &\leqslant -\frac{1}{3}|\chi(a)-\chi(b)| \frac{J^3 x^{\sigma-\delta_J}}{\gamma_J \log x}\left(1+o(1)\right) < 0\end{aligned}$$

if X is sufficiently large, which completes the proof.

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