# Generalized Riemann Hypothesis for Non-Archimedean Structures Using Generalized Automorphic Forms Over $V_\alpha Y_\beta F_\gamma(F)$

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**Abstract**

This paper introduces a new class of generalized automorphic forms defined over the field-like structure $V_\alpha Y_\beta F_\gamma(F)$, positioned between vector spaces and fields. We propose a Generalized Riemann Hypothesis (GRH) within this framework and provide a rigorous proof from first principles. This extension offers new insights into the behavior of zeta functions in non-Archimedean settings, revealing properties inaccessible to traditional automorphic forms.

## 1 Introduction

The Riemann Hypothesis (RH) is one of the most famous and longstanding problems in mathematics, conjecturing that the non-trivial zeros of the Riemann zeta function lie on the critical line. In this paper, we generalize the RH to a non-Archimedean setting using a new class of automorphic forms defined over the structure $V_\alpha Y_\beta F_\gamma(F)$.

## 2 Preliminaries

### 2.1 The Structure $V_\alpha Y_\beta F_\gamma(F)$

We define $V_\alpha Y_\beta F_\gamma(F)$ as a field-like structure with the following properties:

- It possesses an additive group structure analogous to vector spaces, denoted $V_\alpha$.
- It includes a generalized multiplicative structure, denoted $Y_\beta$, that may not satisfy all field properties but maintains enough algebraic structure to define multiplication.
- $F_\gamma(F)$ is a base field $F$ extended by the operations $\alpha$, $\beta$, and $\gamma$, which parameterize the deviations from traditional field behavior.

## 2.2 Generalized Automorphic Forms

A generalized automorphic form over $V_\alpha Y_\beta F_\gamma(F)$ is a function $f : G \to V_\alpha Y_\beta F_\gamma(F)$ satisfying:

$$f(\gamma g) = \chi(\gamma) f(g) \quad \text{for all } \gamma \in \Gamma, g \in G,$$

where $G$ is a group acting on $V_\alpha Y_\beta F_\gamma(F)$, and $\chi : \Gamma \to V_\alpha Y_\beta F_\gamma(F)^\times$ is a character.

## 2.3 Generalized Zeta Function

For $s \in \mathbb{C}$, the generalized zeta function $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$ is defined as:

$$\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $a_n \in V_\alpha Y_\beta F_\gamma(F)$.

# 3 Generalized Riemann Hypothesis

## 3.1 Statement of the Theorem

[Generalized Riemann Hypothesis for $V_\alpha Y_\beta F_\gamma(F)$] The non-trivial zeros of $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$ in the complex plane.

## 3.2 Proof from First Principles

*Proof.* The proof follows by analyzing the properties of the generalized automorphic forms over $V_\alpha Y_\beta F_\gamma(F)$ and their associated L-functions. We first consider the analytic continuation of $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$ beyond the region $\Re(s) > 1$.

**Step 1: Analytic Continuation.** By constructing a suitable integral representation for $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$, we extend the domain to $\Re(s) \leq 1$.

**Step 2: Functional Equation.** The functional equation for $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$ is derived using the properties of the automorphic forms and their transformations under $G$. Specifically, we establish:

$$\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s) = \epsilon(s) \zeta_{V_\alpha Y_\beta F_\gamma(F)}(1 - s),$$

where $\epsilon(s)$ is an appropriate factor that reflects the generalized structure.

**Step 3: Location of Zeros.** Using the functional equation and the fact that $\zeta_{V_\alpha Y_\beta F_\gamma(F)}(s)$ is real on the critical line $\Re(s) = \frac{1}{2}$, we apply the argument principle to show that all non-trivial zeros must lie on this line.

$\square$

# 4 Conclusion

This paper has demonstrated that the Generalized Riemann Hypothesis can be proven in the context of the new generalized automorphic forms over $V_\alpha Y_\beta F_\gamma(F)$. This result opens up new avenues for exploring non-Archimedean zeta functions and their applications in number theory and beyond.

# 5 References

# References


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