EXPLICIT ENUMERATION OF NON-FIELD FONTAINE-STYLE RINGS

PU JUSTIN SCARFY YANG

Enumeration of Fontaine-Style Rings (Rings but not Fields)

Symbol	Name	Struc
	(A) Tilt/Witt/Kisin/Breuil Rings	
$\mathcal{O}C$	Integers in complete algebraically closed field C	Comp
Integers in the tilt	Perfect valuation ring, char p $A_{\text{inf}} := W(\mathcal{O}_{C^\flat})$	Witt
p-adically complete, perfect, char 0 $W(R)$	Witt vectors of perfect R	p-adic
Kisin ring	Frobenius lift, flat over $W(k)$ $S = \text{PD-Envelope}(\mathfrak{S})$	Breui
PD-thickening of $\mathfrak{S}$, p-adic $\mathfrak{S}^{\text{ur}}$	Unramified Kisin ring	Base
	(B) Crystalline/Semistable Rings	
A_{cris}	Crystalline period ring	PD-e
Semistable period ring	A_{cris} with N, φ $\mathcal{O}B$cris	Integ
Pre-fraction ring, stable under φ		
	(C) Positive Period Rings	
B_{dR}^+	de Rham positive ring	Comp
Hodge–Tate positive ring	Graded valuation ring $\mathcal{O}B$dR	Integ
t-adic completion of $\mathcal{O}_C[[t]]$		
	(D) Prismatic Cohomology Rings	
$\mathbb{A}$inf	Prismatic base ring	$= A$in
Crystalline prism ring	PD-envelope of $\mathbb{A}$inf $\mathbb{B}\text{inf}^+$	Fracti
Not a field; p-adic and filtered $\mathbb{B}_{\text{dR}}^+$	Prismatic de Rham positive ring	t-adic

tikz positioning, shapes.geometric, arrows.meta, decorations.pathreplacing

[node distance=1.8cm and 2.5cm, every node/.style=font=, ring/.style=draw, rounded rectangle, minimum width=2.7cm, minimum height=0.9cm, thick, align=center, tilt/.style=fill=blue!10, cryst/.style=fill=green!15, posper/.style=fill=orange!15, prism/.style=fill=purple!12, arrow/.style=-¿, thick, dashedarrow/.style=-¿, thick, dashed, every edge quotes/.style = midway, auto, font=]

Date: May 26, 2025.

[ring, tilt] (OCf)
OC$^\flat integers$; $[ring, tilt, below = of OCf](Ainf)$Ainf$vectors$; $[ring, tilt, left = of OCf](OC)\mathcal{O}_C inC$; $[ring, tilt, below left = 1.8cm and 2.4cm of Ainf](WR)W(R)ofperfectR$; $[ring, tilt, below right = 1.8cm and 2.2cm of Ainf](S)S = \mathrm{PD}(\mathfrak{S})ring$; $[ring, tilt, below = of S](Sur)\mathfrak{S}^{\mathrm{ur}}.Kisin$; $[ring, tilt, above = of S](S0)\mathfrak{S} = W(k)[[u]]ring$;
[ring, cryst, right=of Ainf] (Acris) $\mathrm{A}_{\mathrm{cris}} - envelope$; $[ring, cryst, right = of Acris](Ast)A_{\mathrm{st}}N, \varphi$; $[ring, cryst, below = of Acris](OBC)\mathcal{O}B\mathrm{cris}Crystalline$;
[ring, posper, right=3.3cm of Ast] (BdRp) $\mathrm{B}^+_{\mathrm{dR}}Rham+$; $[ring, posper, below = of BdRp](OBdR)\mathcal{O}B\mathrm{dR}deRham$; $[ring, posper, below = of OBdR](BHTp)B^+_{\mathrm{HT}}$–$Tate+$;
[ring, prism, below right=1.2cm and -1.1cm of OCf] (Ainfprism) Ainfbase; [ring, prism, below=of Ainfprism] (Acrisprism) Acriscrystalline; [ring, prism, below=of Acrisprism] (Binfp)
Binf^+prism; $[ring, prism, below = of Binfp](BdRprism)\mathbb{B}\mathrm{dR}^+deRham$;
[arrow] (OCf) – (Ainf) node[midway,left] W(-); [arrow] (OC) – (Ainf) node[midway, above right] tilt; [arrow] (WR) – (Ainf); [arrow] (S0) – (S); [arrow] (S) – (Sur); [arrow] (Ainf) – (Acris); [arrow] (Acris) – (Ast); [arrow] (Acris) – (OBC); [arrow] (Ast) – (BdRp);
[arrow] (BdRp) – (OBdR); [arrow] (OBdR) – (BHTp);
[arrow] (Ainfprism) – (Acrisprism); [arrow] (Acrisprism) – (Binfp); [arrow] (Binfp) – (BdRprism); [arrow, dashed] (Ainfprism) – (Ainf) node[midway, above left] same;
[decorate,decoration=brace,amplitude=6pt,xshift=-2pt,yshift=0pt] ([xshift=-1.5cm]OC.west) – ([xshift=-1.5cm]Sur.south west) node[midway,xshift=-1.0cm, rotate=90] **Tilt/Witt Family**;
[decorate,decoration=brace,amplitude=6pt,xshift=2pt,yshift=0pt] (Acris.north east) – (OBC.south east) node[midway,xshift=1.0cm, rotate=90] **Crystalline/Semistable**;
[decorate,decoration=brace,amplitude=6pt,xshift=2pt,yshift=0pt] (BdRp.north east) – (BHTp.south east) node[midway,xshift=1.0cm, rotate=90] **Positive Period**;
[decorate,decoration=brace,amplitude=6pt,xshift=2pt,yshift=0pt] (Ainfprism.north east) – (BdRprism.south east) node[midway,xshift=1.0cm, rotate=90] **Prismatic**;