Further Extensions in Automorphic Quantum Theory:
New Definitions, Theorems, and Detailed Proofs (Part XIX)

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1 New Mathematical Definitions and Notations

1.1 Automorphic Quantum Wavefunction $\Psi_{\mathbf{AQ}}(x,t)$

We define the **Automorphic Quantum Wavefunction**, denoted $\Psi_{\mathrm{AQ}}(x,t)$, as a complex-valued function defined on an automorphic quantum space $\mathcal{X}_{\mathrm{AQ}}$ and time t, satisfying the automorphic quantum Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{AQ}}(x,t) = \mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}(x,t)$$

where $\mathcal{H}_{\mathrm{AQ}}$ is the automorphic quantum Hamiltonian operator.

1.2 Automorphic Quantum Schrödinger Equation

The **Automorphic Quantum Schrödinger Equation** is given by:

$$i\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{AQ}}(x,t) = -\frac{\hbar^2}{2m}\Delta_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}(x,t) + V_{\mathrm{AQ}}(x)\Psi_{\mathrm{AQ}}(x,t)$$

where Δ_{AQ} is the automorphic quantum Laplacian, $V_{\mathrm{AQ}}(x)$ is the automorphic quantum potential energy function, and m is the mass of the particle.

1.3 Automorphic Quantum Probability Density $\rho_{\mathbf{AQ}}(x,t)$

The **Automorphic Quantum Probability Density** is defined as:

$$\rho_{\mathrm{AQ}}(x,t) = |\Psi_{\mathrm{AQ}}(x,t)|^2$$

which gives the probability density of finding the particle at position x at time t in the automorphic quantum space.

2 New Theorems and Proofs

2.1 Theorem: Conservation of Automorphic Quantum Probability

Theorem: The total probability is conserved in the automorphic quantum system, i.e.:

$$\frac{d}{dt}\int_{\mathcal{X}_{\mathrm{AQ}}} \rho_{\mathrm{AQ}}(x,t)\, d\mu_{\mathrm{AQ}}(x) = 0$$

where $d\mu_{\mathrm{AQ}}(x)$ is the automorphic quantum measure on $\mathcal{X}_{\mathrm{AQ}}$.

Proof:

Step 1: Compute the time derivative of the total probability:

$$\frac{d}{dt}\int_{\mathcal{X}_{\mathrm{AQ}}} \rho_{\mathrm{AQ}}(x,t)\, d\mu_{\mathrm{AQ}}(x) = \int_{\mathcal{X}_{\mathrm{AQ}}} \frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}(x,t)|^2\, d\mu_{\mathrm{AQ}}(x)$$

Step 2: Use the automorphic quantum Schrödinger equation and its complex conjugate:

$$i\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{AQ}} = \mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}, \quad -i\hbar\frac{\partial}{\partial t}\Psi^*_{\mathrm{AQ}} = \mathcal{H}_{\mathrm{AQ}}\Psi^*_{\mathrm{AQ}}$$

Step 3: Compute $\frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}|^2$:

$$\frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}|^2 = \Psi^*_{\mathrm{AQ}}\frac{\partial}{\partial t}\Psi_{\mathrm{AQ}} + \Psi_{\mathrm{AQ}}\frac{\partial}{\partial t}\Psi^*_{\mathrm{AQ}}$$

Step 4: Substitute the time derivatives from the Schrödinger equations:

$$\frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}|^2 = \Psi^*_{\mathrm{AQ}}\left(-\frac{i}{\hbar}\mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}\right) + \Psi_{\mathrm{AQ}}\left(\frac{i}{\hbar}\mathcal{H}_{\mathrm{AQ}}\Psi^*_{\mathrm{AQ}}\right) = -\frac{i}{\hbar}\left(\Psi^*_{\mathrm{AQ}}\mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}} - \Psi_{\mathrm{AQ}}\mathcal{H}_{\mathrm{AQ}}\Psi^*_{\mathrm{AQ}}\right)$$

Step 5: Since $\mathcal{H}_{\mathrm{AQ}}$ is Hermitian ($\mathcal{H}_{\mathrm{AQ}} = \mathcal{H}^\dagger_{\mathrm{AQ}}$), we have:

$$\frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}|^2 = -\frac{i}{\hbar}\left(\Psi^*_{\mathrm{AQ}}\mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}} - \Psi_{\mathrm{AQ}}(\mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}})^*\right)$$

Step 6: Recognize that the expression inside the integral is a divergence:

$$\frac{\partial}{\partial t}|\Psi_{\mathrm{AQ}}|^2 = -\nabla_{\mathrm{AQ}}\cdot\mathbf{J}_{\mathrm{AQ}}$$

where $\mathbf{J}_{\mathrm{AQ}}$ is the automorphic quantum probability current density defined by:

$$\mathbf{J}_{\mathrm{AQ}} = \frac{\hbar}{2mi}\left(\Psi^*_{\mathrm{AQ}}\nabla_{\mathrm{AQ}}\Psi_{\mathrm{AQ}} - \Psi_{\mathrm{AQ}}\nabla_{\mathrm{AQ}}\Psi^*_{\mathrm{AQ}}\right)$$

Step 7: Use the divergence theorem in the automorphic quantum setting:

$$\int_{\mathcal{X}_{\mathrm{AQ}}} \nabla_{\mathrm{AQ}} \cdot \mathbf{J}_{\mathrm{AQ}} \, d\mu_{\mathrm{AQ}}(x) = \int_{\partial \mathcal{X}_{\mathrm{AQ}}} \mathbf{J}_{\mathrm{AQ}} \cdot d\mathbf{S}_{\mathrm{AQ}}$$

Assuming that $\mathbf{J}_{\mathrm{AQ}}$ vanishes at infinity or on the boundary $\partial \mathcal{X}_{\mathrm{AQ}}$, the surface integral is zero.

Step 8: Therefore:

$$\frac{d}{dt} \int_{\mathcal{X}_{\mathrm{AQ}}} |\Psi_{\mathrm{AQ}}|^2 \, d\mu_{\mathrm{AQ}}(x) = -\int_{\mathcal{X}_{\mathrm{AQ}}} \nabla_{\mathrm{AQ}} \cdot \mathbf{J}_{\mathrm{AQ}} \, d\mu_{\mathrm{AQ}}(x) = -\int_{\partial \mathcal{X}_{\mathrm{AQ}}} \mathbf{J}_{\mathrm{AQ}} \cdot d\mathbf{S}_{\mathrm{AQ}} = 0$$

□

2.2 Theorem: Time Evolution Operator in Automorphic Quantum Mechanics

Theorem: The time evolution of the automorphic quantum wavefunction can be expressed using the automorphic quantum time evolution operator $U_{\mathrm{AQ}}(t)$:

$$\Psi_{\mathrm{AQ}}(x,t) = U_{\mathrm{AQ}}(t)\Psi_{\mathrm{AQ}}(x,0), \quad U_{\mathrm{AQ}}(t) = e^{-i\mathcal{H}_{\mathrm{AQ}}t/\hbar}$$

Proof:

Step 1: Consider the formal solution to the automorphic quantum Schrödinger equation:

$$\Psi_{\mathrm{AQ}}(x,t) = e^{-i\mathcal{H}_{\mathrm{AQ}}t/\hbar}\Psi_{\mathrm{AQ}}(x,0)$$

Step 2: Verify that this solution satisfies the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\Psi_{\mathrm{AQ}}(x,t) = i\hbar \left(-\frac{i\mathcal{H}_{\mathrm{AQ}}}{\hbar}\right) e^{-i\mathcal{H}_{\mathrm{AQ}}t/\hbar}\Psi_{\mathrm{AQ}}(x,0) = \mathcal{H}_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}(x,t)$$

Step 3: Therefore, the time evolution operator $U_{\mathrm{AQ}}(t) = e^{-i\mathcal{H}_{\mathrm{AQ}}t/\hbar}$ governs the evolution of the automorphic quantum wavefunction.

□

3 Applications and Examples

3.1 Example: Free Particle in Automorphic Quantum Space

Consider a free particle ($V_{\mathrm{AQ}}(x) = 0$) in an automorphic quantum space. The Schrödinger equation becomes:

$$i\hbar \frac{\partial}{\partial t}\Psi_{\mathrm{AQ}}(x,t) = -\frac{\hbar^2}{2m}\Delta_{\mathrm{AQ}}\Psi_{\mathrm{AQ}}(x,t)$$

The solutions can be expressed using automorphic quantum plane waves:

$$\Psi_{\mathrm{AQ}}(x,t) = e^{i(k\cdot x - \omega t)}$$

where k is the automorphic quantum wave vector and $\omega = \frac{\hbar k^2}{2m}$.

3.2 Example: Harmonic Oscillator in Automorphic Quantum Mechanics

Using the automorphic quantum harmonic oscillator Hamiltonian:

$$\mathcal{H}_{\mathrm{AQ}} = -\frac{\hbar^2}{2m}\Delta_{\mathrm{AQ}} + \frac{1}{2}m\omega^2 x^2$$

We can solve the Schrödinger equation to find the energy eigenvalues:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

and the corresponding eigenfunctions, which are automorphic quantum analogues of the Hermite functions.

4 Real Academic References

References

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