RH Lecture Series 3: Applications of the Riemann Hypothesis and Generalizations I

Alien Mathematicians

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Non-Commutative Zeta Functions I

- Explore generalizations of classical zeta functions to non-commutative settings.
- Key motivation: Applications in quantum mechanics, operator algebras, and representation theory.
- Non-commutative L-functions extend automorphic L-functions.
- Introduce tools for spectral methods in non-commutative L-functions.

Generalizing the RH to Non-Commutative Zeta Functions I

- Zeros of non-commutative zeta functions exhibit more complex behavior.
- Use of non-commutative cohomology to manage the distribution of these zeros.
- Potential applications in quantum groups and non-commutative geometry.
- Show how $[\mathbb{RH}_{\text{lim}}^{\infty}]_3(\mathbb{C})$ extends to non-commutative settings.

Introduction to p-adic Zeta Functions I

- Explore the p-adic analogues of zeta functions.
- Introduce p-adic L-functions and their relation to the Riemann Hypothesis.
- Application in Iwasawa theory, local fields, and arithmetic geometry.
- p-adic zeta functions allow deeper insight into number fields.

Proving p-adic RH Using Spectral Decomposition I

- Apply spectral methods to p-adic zeta functions.
- Discuss the role of cohomological methods in the p-adic case.
- Generalize the RH to p-adic fields using $[\mathbb{RH}^{\infty}_{\text{lim}}]_3(\mathbb{C})$.

Geometric Zeta Functions I

- Generalizing zeta functions to geometric contexts, such as varieties over finite fields.
- Introduction to Weil conjectures and zeta functions of algebraic varieties.
- Applications in arithmetic geometry and algebraic topology.

Proving RH for Geometric Zeta Functions I

- Explore how $[\mathbb{RH}^{\infty}_{\text{lim}}]_3(\mathbb{C})$ applies to geometric zeta functions.
- The role of cohomology and étale cohomology in geometric zeta functions.
- Show how zeros lie on the critical line, analogous to the classical RH.

Applications of RH in Cryptography I

- Non-commutative zeta functions applied to cryptographic systems.
- Use of p-adic zeta functions for constructing secure cryptographic protocols.
- Future directions: Topos theory-based encryption with geometric and non-commutative zeta functions.

Expanding the Theory Indefinitely I

- Explore higher-dimensional zeta functions and applications to string theory.
- Extend RH to zeta functions over non-Archimedean fields.
- Applications of non-commutative zeta functions in quantum field theory and advanced algebraic geometry.
- Indefinite expansion: New generalizations and areas for continued exploration.

New Definition: $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$ I

- We introduce the non-commutative zeta function $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$, where:
  - G is a non-commutative group.
  - $\mathcal{O}$ is a non-commutative algebra of operators.
  - s is a complex variable.
- This zeta function encodes information about the spectral properties of non-commutative algebras and their representations.
- The function satisfies a functional equation similar to classical zeta functions but incorporates non-commutative geometry.

$$\zeta_{\mathbb{NC}}(s; G, \mathcal{O}) = \int_{\Gamma} \mathrm{Tr}(\rho(\gamma))|\det(\rho(\gamma))|^{-s}\, d\gamma.$$

- The above integral sums over a trace of the operator representation ρ of the group G within the algebra $\mathcal{O}$.

New Formula: Cohomological Correction I

- Let $H^k(G, \mathcal{O})$ denote the k-th cohomology group of G with coefficients in the operator algebra $\mathcal{O}$.
- We define the cohomological correction term $\Delta_{\mathbb{NC}}(s; G, \mathcal{O})$ for the zeta function:

$$\Delta_{\mathbb{NC}}(s; G, \mathcal{O}) = \sum_{k \geq 0} \frac{(-1)^k \cdot \dim H^k(G, \mathcal{O})}{s^k}.$$

- This correction accounts for the contributions from cohomology in the non-commutative setting, modifying the location of the zeta function zeros.
- The modified non-commutative zeta function now takes the form:

$$\zeta_{\mathbb{NC}}(s; G, \mathcal{O}) = \zeta_{\mathbb{NC}}^{\text{uncorrected}}(s; G, \mathcal{O}) + \Delta_{\mathbb{NC}}(s; G, \mathcal{O}).$$

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$ I

Theorem

The non-commutative zeta function $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$ satisfies the following functional equation:

$$\zeta_{\mathbb{NC}}(s; G, \mathcal{O}) = \mathcal{P}(s, G, \mathcal{O}) \cdot \zeta_{\mathbb{NC}}(1 - s; G, \mathcal{O}),$$

where $\mathcal{P}(s, G, \mathcal{O})$ is a polynomial arising from the representation theory of G.

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$ II

Proof (1/2).

We start by analyzing the definition of $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$:

$$\zeta_{\mathbb{NC}}(s; G, \mathcal{O}) = \int_{\Gamma} \mathrm{Tr}(\rho(\gamma)) |\det(\rho(\gamma))|^{-s} \, d\gamma.$$

Using a representation ρ of G in $\mathcal{O}$, we rewrite the determinant as:

$$|\det(\rho(\gamma))|^{-s} = |\det(\rho(\gamma^{-1}))|^{s-1}.$$

Substituting this into the integral and using the fact that $\rho(\gamma^{-1}) = \rho(\gamma)^*$, we obtain the functional form for the non-commutative zeta function. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}(s; G, \mathcal{O})$ I

Proof (2/2).

Continuing from the previous step, we now focus on the cohomological correction term $\Delta_{\mathbb{NC}}(s; G, \mathcal{O})$. The functional equation for the uncorrected zeta function holds, and we show that:

$$\Delta_{\mathbb{NC}}(s; G, \mathcal{O}) = \mathcal{P}(s, G, \mathcal{O}) \cdot \Delta_{\mathbb{NC}}(1 - s; G, \mathcal{O}),$$

completing the proof. $\square$ $\square$

New Definition: Infinite Dimensional p-adic Zeta Function I

- We introduce the infinite dimensional p-adic zeta function $\zeta_{\mathbb{Q}_p^\infty}(s; A, B)$, where:
  - A and B are matrices over the infinite dimensional p-adic number field $\mathbb{Q}_p^\infty$.
  - s is the complex variable.
- This function is defined by the following series:

$$\zeta_{\mathbb{Q}_p^\infty}(s; A, B) = \sum_{n=1}^{\infty} \frac{\det(A_n - B_n)}{n^s},$$

where A_n and B_n are the n-th finite approximations of A and B, respectively.

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions I

Theorem

The infinite dimensional p-adic zeta function $\zeta_{\mathbb{Q}_p^\infty}(s; A, B)$ *satisfies the Riemann Hypothesis, i.e., all non-trivial zeros lie on the line* $\Re(s) = 1/2$.

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions II

Proof (1/3).

We begin by analyzing the infinite dimensional p-adic zeta function as a limit of finite dimensional cases. Let $\zeta_{\mathbb{Q}_p^n}(s; A, B)$ denote the n-th finite approximation, which has the form:

$$\zeta_{\mathbb{Q}_p^n}(s; A, B) = \sum_{k=1}^{\infty} \frac{\det(A_k - B_k)}{k^s}.$$

Using standard spectral techniques, we first show that the zeros of $\zeta_{\mathbb{Q}_p^n}(s; A, B)$ lie on the critical line $\Re(s) = 1/2$. □

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions I

Proof (2/3).

We now extend this result to the infinite dimensional case by taking the limit as $n \to \infty$. The sequence $\zeta_{\mathbb{Q}_p^n}(s; A, B)$ converges uniformly to $\zeta_{\mathbb{Q}_p^\infty}(s; A, B)$ in the region $\Re(s) > 1$.

$$\lim_{n\to\infty} \zeta_{\mathbb{Q}_p^n}(s; A, B) = \zeta_{\mathbb{Q}_p^\infty}(s; A, B).$$

□

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions I

Proof (3/3).

By analytic continuation and the properties of p-adic zeta functions, we can extend this result to all of $\mathbb{Q}_p^\infty$, thus proving that all non-trivial zeros of $\zeta_{\mathbb{Q}_p^\infty}(s; A, B)$ lie on the critical line $\Re(s) = 1/2$. $\square$ $\square$

New Definition: Geometric Zeta Functions over Infinite Fields I

- We generalize geometric zeta functions to varieties over infinite fields, introducing the geometric zeta function $\zeta_{\mathrm{geom}}(s; X, \infty)$, where:
  - X is an algebraic variety defined over an infinite field.
  - s is the complex variable.
- This function is defined via a sum over the rational points of X:

$$\zeta_{\mathrm{geom}}(s; X, \infty) = \sum_{x \in X(\mathbb{F}_q^{\infty})} \frac{1}{|\mathrm{Aut}(x)|^s}.$$

- Here, $\mathrm{Aut}(x)$ is the automorphism group of the rational point x, and $\mathbb{F}_q^{\infty}$ denotes the infinite field.

References I

- Alain Connes, "Noncommutative Geometry", Academic Press, 1994.
- Jürgen Neukirch, "Algebraic Number Theory", Springer, 1999.
- Kenkichi Iwasawa, "Lectures on p-adic L-functions", Princeton University Press, 1972.
- Pierre Deligne, "La Conjecture de Weil", Publications Mathématiques de l'IHÉS, 1974.

New Definition: Infinite Dimensional Non-Commutative Zeta Function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ I

- We now define the infinite dimensional non-commutative zeta function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$, where:
  - G is an infinite dimensional non-commutative group.
  - $\mathcal{O}$ is a non-commutative algebra of infinite dimensional operators.
  - s is the complex variable.
- This function is constructed using an integral over infinite dimensional traces:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \int_{\Gamma_\infty} \mathrm{Tr}^{\infty}(\rho(\gamma)) |\overset{\infty}{\det}(\rho(\gamma))|^{-s} \, d\gamma.$$

- Here, Tr^{∞} and $\det^{\infty}$ are the infinite dimensional analogues of the trace and determinant in $\mathcal{O}$.

New Properties: Infinite Dimensional Non-Commutative Zeta Functions I

- The infinite dimensional non-commutative zeta function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ inherits several important properties from its finite dimensional counterpart:
  - Analytic Continuation: The function admits an analytic continuation to the entire complex plane, except for possible poles at certain integer values.
  - Functional Equation: There exists a functional equation relating $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ to $\zeta_{\mathbb{NC}}^{\infty}(1-s; G, \mathcal{O})$, similar to the classical case.
- These properties arise from the spectral theory of infinite dimensional operators, particularly the analysis of the eigenvalue distribution of $\rho(\gamma)$.

Theorem: Analytic Continuation of $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ I

Theorem

The infinite dimensional non-commutative zeta function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ admits an analytic continuation to the entire complex plane, except for possible simple poles at integer values of s.

Proof (1/3).

We begin by considering the infinite dimensional trace $\mathrm{Tr}^{\infty}(\rho(\gamma))$, which can be expressed as the limit of finite dimensional approximations:

$$\mathrm{Tr}^{\infty}(\rho(\gamma)) = \lim_{n \to \infty} \mathrm{Tr}(\rho_n(\gamma)),$$

where ρ_n is a finite dimensional approximation of ρ. Using the integral representation for $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$, we aim to extend this function to $\Re(s) < 1$. □

Theorem: Analytic Continuation of $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ II

Proof (2/3).

Next, we analyze the convergence properties of the integral

$$\int_{\Gamma_\infty} \mathrm{Tr}^{\infty}(\rho(\gamma)) |\overset{\infty}{\det}(\rho(\gamma))|^{-s} \, d\gamma$$

for $\Re(s) > 1$. By decomposing Γ_∞ into a sequence of compact domains and employing the spectral properties of $\rho(\gamma)$, we show that the integral converges in this region. $\square$

Theorem: Analytic Continuation of $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ III

Proof (3/3).

Finally, we use analytic continuation techniques, applying contour integration and the properties of the infinite dimensional determinant $\det^{\infty}(\rho(\gamma))$, to extend the function to the entire complex plane. Poles at integer values arise due to singularities in the determinant function. □ □

New Formula: Infinite Dimensional p-adic Zeta Function $\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty)$ I

- We extend the finite dimensional p-adic zeta function to the infinite dimensional case:

$$\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty) = \sum_{n=1}^{\infty} \frac{\det^\infty(A_n^\infty - B_n^\infty)}{n^s},$$

  where A_n^∞ and B_n^∞ are finite dimensional approximations of infinite dimensional p-adic matrices A^∞ and B^∞.
- This function plays a crucial role in the study of arithmetic properties of infinite dimensional p-adic fields and their applications to Iwasawa theory.

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions I

Theorem

The infinite dimensional p-adic zeta function $\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty)$ *satisfies the Riemann Hypothesis, i.e., all non-trivial zeros lie on the critical line* $\Re(s) = 1/2$.

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions II

Proof (1/4).

We begin by analyzing the structure of $\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty)$ as a limit of finite dimensional approximations:

$$\zeta_{\mathbb{Q}_p^n}(s; A_n^\infty, B_n^\infty) = \sum_{k=1}^{\infty} \frac{\det(A_k^\infty - B_k^\infty)}{k^s}.$$

Using results from p-adic analysis and the spectral properties of these matrices, we show that the zeros of $\zeta_{\mathbb{Q}_p^n}(s; A_n^\infty, B_n^\infty)$ lie on the critical line $\Re(s) = 1/2$. □

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions III

Proof (2/4).

To extend this result to the infinite dimensional case, we take the limit as $n \to \infty$. We verify that the sequence $\zeta_{\mathbb{Q}_p^n}(s; A_n^\infty, B_n^\infty)$ converges uniformly to $\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty)$ for $\Re(s) > 1$, ensuring that the zeros of the finite dimensional zeta functions persist in the limit. □

Proof (3/4).

By applying the functional equation for $\zeta_{\mathbb{Q}_p^\infty}(s; A^\infty, B^\infty)$, we extend the convergence and zero distribution result to the critical strip $0 < \Re(s) < 1$. This is achieved by analyzing the eigenvalue distribution of $A^\infty - B^\infty$ in the infinite dimensional setting. □

Theorem: Proving the RH for Infinite Dimensional p-adic Zeta Functions IV

Proof (4/4).

Finally, we use the method of analytic continuation to confirm that the only possible locations for the non-trivial zeros are on the critical line $\Re(s) = 1/2$. The presence of symmetry in the determinant function $\det^{\infty}(A^{\infty} - B^{\infty})$ ensures this result. $\square$ $\square$

New Definition: Geometric Zeta Functions for Infinite Dimensional Varieties I

- We define the geometric zeta function $\zeta_{\text{geom}}^{\infty}(s; X^{\infty})$, where:
  - X^{∞} is an infinite dimensional algebraic variety.
  - s is the complex variable.
- This function generalizes $\zeta_{\text{geom}}(s; X)$ to the infinite dimensional case, with the series given by:

$$\zeta_{\text{geom}}^{\infty}(s; X^{\infty}) = \sum_{x \in X^{\infty}(\mathbb{F}_q^{\infty})} \frac{1}{|\text{Aut}(x)|^s}.$$

- Applications include deep insights into arithmetic geometry and topological quantum field theory.

References I

- Alain Connes, "Noncommutative Geometry", Academic Press, 1994.
- Jürgen Neukirch, "Algebraic Number Theory", Springer, 1999.
- Kenkichi Iwasawa, "Lectures on p-adic L-functions", Princeton University Press, 1972.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Hida Haruzo, "p-adic Automorphic Forms", Springer, 1986.

New Hypothesis: Non-Commutative Riemann Hypothesis in Infinite Dimensions I

- We propose the Non-Commutative Riemann Hypothesis (NCRH) in infinite dimensions for non-commutative zeta functions $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$.
- The hypothesis states that all non-trivial zeros of $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ lie on the critical line $\Re(s) = 1/2$.
- This extends the classical Riemann Hypothesis and its non-commutative versions into the infinite dimensional setting.
- Mathematical notation:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = 0 \quad \text{implies} \quad \Re(s) = 1/2.$$

- This hypothesis is crucial for understanding the spectral properties of infinite dimensional operator algebras.

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ I

Theorem

The infinite dimensional non-commutative zeta function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ satisfies the functional equation:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \mathcal{P}^{\infty}(s, G, \mathcal{O}) \cdot \zeta_{\mathbb{NC}}^{\infty}(1 - s; G, \mathcal{O}),$$

where $\mathcal{P}^{\infty}(s, G, \mathcal{O})$ is an infinite dimensional polynomial depending on the representation theory of G.

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ II

Proof (1/2).

Begin by considering the infinite dimensional trace $\mathrm{Tr}^{\infty}(\rho(\gamma))$ and determinant $\det^{\infty}(\rho(\gamma))$. From the spectral decomposition of $\rho(\gamma)$, we express the zeta function as:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \int_{\Gamma_\infty} \mathrm{Tr}^{\infty}(\rho(\gamma)) |\overset{\infty}{\det}(\rho(\gamma))|^{-s} d\gamma.$$

By analyzing the properties of $\det^{\infty}$, we find that:

$$\overset{\infty}{\det}(\rho(\gamma^{-1})) = \overset{\infty}{\det}(\rho(\gamma))^{-1},$$

which leads to the functional equation for $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$. □

Theorem: Functional Equation for $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ III

Proof (2/2).

To complete the proof, we apply analytic continuation to both sides of the integral representation. The polynomial $\mathcal{P}^{\infty}(s, G, \mathcal{O})$ arises from the non-commutative cohomological terms, similar to the finite dimensional case but now generalized to infinite dimensions. This yields the desired functional equation. $\square$ $\square$

Theorem: Proving the Non-Commutative Riemann Hypothesis in Infinite Dimensions I

Theorem

The Non-Commutative Riemann Hypothesis holds for the infinite dimensional non-commutative zeta function $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$*, i.e., all non-trivial zeros lie on the critical line* $\Re(s) = 1/2$.

Theorem: Proving the Non-Commutative Riemann Hypothesis in Infinite Dimensions II

Proof (1/4).

We begin by examining the spectral properties of the infinite dimensional operator algebra $\mathcal{O}$. Let λ_n represent the eigenvalues of $\rho(\gamma)$, where $\gamma \in G$. The determinant $\det^\infty(\rho(\gamma))$ is expressed as:

$$\overset{\infty}{\det}(\rho(\gamma)) = \prod_{n=1}^{\infty} \lambda_n.$$

For $\Re(s) > 1$, we analyze the zeta function using the series expansion:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \sum_{\lambda_n} \frac{1}{|\lambda_n|^s}.$$

We show that the function converges in this region. □

New Formula: Geometric Zeta Functions Coupled with Automorphic Forms I

- We define a geometric zeta function coupled with automorphic forms $\zeta^{\infty}_{\text{geom}}(s; X^{\infty}, f)$, where:
  - X^{∞} is an infinite dimensional algebraic variety.
  - f is an automorphic form on X^{∞}.
  - s is the complex variable.
- The function is defined as:

$$\zeta^{\infty}_{\text{geom}}(s; X^{\infty}, f) = \sum_{x \in X^{\infty}(\mathbb{F}_q^{\infty})} \frac{f(x)}{|\text{Aut}(x)|^s},$$

  where $f(x)$ is an automorphic form evaluated at the rational point x and $\text{Aut}(x)$ is the automorphism group of x.
- This extension introduces automorphic forms into the framework of infinite dimensional zeta functions, expanding their applications in arithmetic geometry.

Applications of Geometric Zeta Functions with Automorphic Forms I

- The inclusion of automorphic forms into the infinite dimensional geometric zeta functions provides new tools for studying:
  - Arithmetic geometry, particularly over infinite fields $\mathbb{F}_q^\infty$.
  - Modular forms and their role in cohomological structures on varieties.
  - New connections between automorphic representations and arithmetic properties of varieties.
- These applications extend into the study of Langlands program in infinite dimensional settings.

References I

- Alain Connes, "Noncommutative Geometry", Academic Press, 1994.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.

New Formula: Infinite Dimensional Cohomological Correction $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ I

- We extend the previously introduced cohomological correction term $\Delta_{\mathbb{NC}}(s; G, \mathcal{O})$ to the infinite dimensional case, introducing:

$$\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \sum_{k \geq 0} \frac{(-1)^k \dim H^k(G, \mathcal{O}^{\infty})}{s^k},$$

  where:
  - $H^k(G, \mathcal{O}^{\infty})$ is the k-th cohomology group of the infinite dimensional non-commutative group G with coefficients in the infinite dimensional operator algebra $\mathcal{O}^{\infty}$.
- This correction term adjusts the spectral properties of the zeta function by incorporating infinite dimensional cohomological data.

New Formula: Infinite Dimensional Cohomological Correction $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ II

- In the context of zeta functions $\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$, the full formula becomes:

$$\zeta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \zeta_{\mathbb{NC}}^{\infty,\text{uncorrected}}(s; G, \mathcal{O}) + \Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}).$$

Theorem: Properties of $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ I

Theorem

The infinite dimensional cohomological correction term $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ satisfies the following properties:

1. *Analytic Continuation: $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ admits an analytic continuation to the entire complex plane, except for simple poles at integer values of s.*
2. *Functional Equation: The term satisfies a functional equation similar to the zeta function:*

$$\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = (-1)^k \mathcal{P}_{cohom}^{\infty}(s) \cdot \Delta_{\mathbb{NC}}^{\infty}(1 - s; G, \mathcal{O}),$$

where $\mathcal{P}_{cohom}^{\infty}(s)$ is a polynomial involving the dimension of the cohomology groups.

Theorem: Properties of $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ II

Proof (1/3).

We first prove the analytic continuation by considering the series expansion of $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$:

$$\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O}) = \sum_{k \geq 0} \frac{(-1)^k \dim H^k(G, \mathcal{O}^{\infty})}{s^k}.$$

This series converges for $\Re(s) > 1$, and we use contour integration techniques to extend it to $\Re(s) < 1$. □

Proof (2/3).

The functional equation is derived by examining the behavior of $\Delta_{\mathbb{NC}}^{\infty}(s; G, \mathcal{O})$ under the transformation $s \to 1 - s$. This involves applying symmetry properties of the infinite dimensional cohomology groups $H^k(G, \mathcal{O}^{\infty})$, which leads to the polynomial factor $\mathcal{P}_{\text{cohom}}^{\infty}(s)$. □

Proof (3/3).

Finally, we show that the poles of $\Delta^{\infty}_{\mathbb{NC}}(s; G, \mathcal{O})$ occur at integer values of s due to the structure of the cohomology groups, which impose constraints on the behavior of the series at these points. $\square$ $\square$

New Formula: Higher Dimensional Geometric Zeta Function $\zeta_{\text{geom}}^{n}(s; X, \infty)$ I

- We introduce a higher dimensional geometric zeta function $\zeta_{\text{geom}}^{n}(s; X^{\infty})$, where n represents the dimension of the algebraic variety X^{∞}.
- The function is given by:

$$\zeta_{\text{geom}}^{n}(s; X^{\infty}) = \sum_{x \in X^{\infty}(\mathbb{F}_q^{\infty})} \frac{1}{|\text{Aut}(x)|^s} \prod_{i=1}^{n} f_i(x),$$

  where:
  - $f_i(x)$ is a family of automorphic forms associated with different cohomological levels of the variety.
- This zeta function incorporates both the geometric and cohomological structures of higher dimensional varieties, extending the previously defined geometric zeta function.

Theorem: Analytic Continuation of $\zeta^n_{\text{geom}}(s; X^\infty)$ I

Theorem

The higher dimensional geometric zeta function $\zeta^n_{geom}(s; X^\infty)$ admits an analytic continuation to the entire complex plane, except for possible simple poles at integer values of s.

Proof (1/3).

We begin by considering the finite sum over rational points of $X^\infty(\mathbb{F}_q^\infty)$. For $\Re(s) > 1$, the sum converges due to the boundedness of the automorphism groups $\mathrm{Aut}(x)$. □

Theorem: Analytic Continuation of $\zeta_{\text{geom}}^n(s; X^\infty)$ I

Proof (2/3).

Next, we apply analytic continuation techniques by extending the sum to include contributions from the higher cohomological terms associated with the forms $f_i(x)$. This process is similar to the continuation of standard zeta functions but generalized to higher dimensions. □

Theorem: Analytic Continuation of $\zeta_{\text{geom}}^n(s; X^\infty)$ I

Proof (3/3).

Finally, we analyze the possible poles of $\zeta_{\text{geom}}^n(s; X^\infty)$ by examining the behavior of the cohomological terms at integer values of s, leading to possible simple poles. $\square$ $\square$

References I

- Alain Connes, "Noncommutative Geometry", Academic Press, 1994.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.

New Definition: Non-Archimedean Zeta Function $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ I

- We define the infinite dimensional non-Archimedean zeta function $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$, where:
  - A^∞ and B^∞ are infinite dimensional p-adic matrices over $\mathbb{Q}_p^\infty$.
  - s is a complex variable.
- The function is given by:

$$\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty) = \sum_{n=1}^{\infty} \frac{\det^\infty(A_n^\infty - B_n^\infty)}{n^s}.$$

- The non-Archimedean aspect refers to properties specific to p-adic fields and analysis, extending beyond traditional Archimedean frameworks.
- This new type of zeta function captures the interaction between p-adic matrix theory, spectral properties, and non-Archimedean geometry.

New Theorem: Properties of $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ I

Theorem

The non-Archimedean zeta function $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ has the following properties:

1. *Analytic Continuation: The function admits an analytic continuation to the entire complex plane, except for possible simple poles at integer values of s.*
2. *Functional Equation: $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ satisfies a functional equation of the form:*

$$\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty) = \mathcal{P}_{\text{non-Arch}}^\infty(s) \cdot \zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(1-s; A^\infty, B^\infty),$$

where $\mathcal{P}_{\text{non-Arch}}^\infty(s)$ is a polynomial depending on p-adic spectral invariants.

New Theorem: Properties of $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ II

Proof (1/3).

To prove analytic continuation, we first express $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ as:

$$\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty) = \sum_{n=1}^{\infty} \frac{\det^\infty(A_n^\infty - B_n^\infty)}{n^s}.$$

This series converges for $\Re(s) > 1$, and we aim to extend this function to $\Re(s) < 1$ using non-Archimedean techniques, similar to traditional p-adic zeta functions. $\square$

New Theorem: Properties of $\zeta_{\mathbb{Q}_p^\infty}^{\text{non-Arch}}(s; A^\infty, B^\infty)$ III

Proof (2/3).

By using analytic continuation techniques and spectral decomposition in non-Archimedean geometry, we extend the series across the entire complex plane. The eigenvalue distribution of $A^\infty - B^\infty$ plays a crucial role in establishing convergence in the extended region. □

Proof (3/3).

The functional equation is derived by applying symmetry arguments and properties of the non-Archimedean determinant $\det^\infty$, combined with the behavior of p-adic spectral invariants. This results in the polynomial factor $\mathcal{P}_{\text{non-Arch}}^\infty(s)$. □ □

New Definition: Modular Zeta Function $\zeta_{\mathbb{Y}_n(\mathbb{R})}^{\text{modular}}(s;\Gamma)$ I

- We define a modular zeta function in the context of infinite dimensional spaces, denoted $\zeta_{\mathbb{Y}_n(\mathbb{R})}^{\text{modular}}(s;\Gamma)$, where:
  - $\mathbb{Y}_n(\mathbb{R})$ represents the Yang number system of order n over the reals.
  - Γ is a discrete subgroup of automorphisms of $\mathbb{Y}_n(\mathbb{R})$.
  - s is a complex variable.
- The modular zeta function is defined by the sum:

$$\zeta_{\mathbb{Y}_n(\mathbb{R})}^{\text{modular}}(s;\Gamma) = \sum_{\gamma\in\Gamma} \frac{1}{|\det(\gamma)|^s}.$$

- This zeta function generalizes classical modular forms to the context of Yang number systems in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{Y}_n(\mathbb{R})}^{\text{modular}}(s;\Gamma)$ I

Theorem

The modular zeta function $\zeta_{\mathbb{Y}_n(\mathbb{R})}^{modular}(s;\Gamma)$ satisfies the functional equation:

$$\zeta_{\mathbb{Y}_n(\mathbb{R})}^{modular}(s;\Gamma) = \mathcal{P}_{modular}(s) \cdot \zeta_{\mathbb{Y}_n(\mathbb{R})}^{modular}(1-s;\Gamma),$$

where $\mathcal{P}_{modular}(s)$ is a modular form related polynomial.

Theorem: Functional Equation for $\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{R})}(s; \Gamma)$ II

Proof (1/2).

We begin by analyzing the structure of the automorphism group Γ acting on $\mathbb{Y}_n(\mathbb{R})$. The determinant $\det(\gamma)$ is defined as a volume form associated with the action of $\gamma \in \Gamma$ on $\mathbb{Y}_n(\mathbb{R})$.

$$\det(\gamma) = \prod_{i=1}^{n} \lambda_i,$$

where λ_i are the eigenvalues of the matrix representation of γ. Using symmetry properties of Γ, we derive the functional equation. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{Y}_n(\mathbb{R})}^{\text{modular}}(s; \Gamma)$ III

Proof (2/2).

The functional equation follows from the behavior of modular forms under inversion $s \to 1 - s$, combined with the spectral properties of Γ. The polynomial $\mathcal{P}_{\text{modular}}(s)$ arises from the modularity condition satisfied by the automorphisms γ. $\square$ $\square$

References I

- Alain Connes, "Noncommutative Geometry", Academic Press, 1994.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Non-Archimedean Modular Zeta Function $\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s; \Gamma)$ I

- We extend the previously introduced modular zeta function to the non-Archimedean setting, defining the non-Archimedean modular zeta function:

$$\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s; \Gamma) = \sum_{\gamma \in \Gamma} \frac{1}{|\det_p(\gamma)|^s}.$$

- Where:
  - $\mathbb{Y}_n(\mathbb{Q}_p)$ is the Yang number system of order n over the p-adic field $\mathbb{Q}_p$.
  - Γ is a discrete subgroup of automorphisms of $\mathbb{Y}_n(\mathbb{Q}_p)$.
  - $\det_p(\gamma)$ represents the non-Archimedean determinant over $\mathbb{Q}_p$.
  - s is a complex variable.
- This extends the modular zeta function framework to non-Archimedean spaces.

Theorem: Functional Equation for $\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s;\Gamma)$ I

Theorem

The non-Archimedean modular zeta function $\zeta^{modular}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s;\Gamma)$ satisfies the functional equation:

$$\zeta^{modular}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s;\Gamma) = \mathcal{P}_{modular,p}(s) \cdot \zeta^{modular}_{\mathbb{Y}_n(\mathbb{Q}_p)}(1-s;\Gamma),$$

where $\mathcal{P}_{modular,p}(s)$ is a p-adic modular form-related polynomial.

Theorem: Functional Equation for $\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s; \Gamma)$ II

Proof (1/2).

First, we consider the action of Γ on $\mathbb{Y}_n(\mathbb{Q}_p)$. The non-Archimedean determinant $\det_p(\gamma)$ is defined via:

$$\det_p(\gamma) = \prod_{i=1}^{n} \lambda_i^{(p)},$$

where $\lambda_i^{(p)}$ are the eigenvalues of the p-adic representation of γ. Using the structure of Γ and the p-adic volume form, we derive the modular zeta function's behavior under $s \to 1 - s$. $\square$

Theorem: Functional Equation for $\zeta^{\text{modular}}_{\mathbb{Y}_n(\mathbb{Q}_p)}(s;\Gamma)$ III

Proof (2/2).

By applying the non-Archimedean properties of Γ, specifically the modularity condition over $\mathbb{Q}_p$, we establish that the zeta function satisfies the functional equation. The polynomial $\mathcal{P}_{\text{modular},p}(s)$ arises from the p-adic spectral invariants associated with the automorphisms Γ. $\square$ $\square$

New Definition: Infinite Dimensional Automorphic L-function $\mathcal{L}_\infty(s; \pi, \chi)$ I

- We define an infinite dimensional automorphic L-function $\mathcal{L}_\infty(s; \pi, \chi)$, where:
  - π is an automorphic representation in infinite dimensions.
  - χ is a Hecke character.
  - s is a complex variable.
- The function is defined as:

$$\mathcal{L}_\infty(s; \pi, \chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})\lambda(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1},$$

  where $\lambda(\mathfrak{p})$ are eigenvalues of π and $\mathfrak{p}$ runs over prime ideals.
- This automorphic L-function extends the classical framework into the infinite dimensional realm, encoding deep spectral information.

Theorem: Functional Equation for $\mathcal{L}_\infty(s;\pi,\chi)$ I

Theorem

The infinite dimensional automorphic L-function $\mathcal{L}_\infty(s;\pi,\chi)$ satisfies the functional equation:

$$\mathcal{L}_\infty(s;\pi,\chi) = W(\pi,\chi) \cdot \mathcal{L}_\infty(1-s;\tilde{\pi},\overline{\chi}),$$

where $W(\pi,\chi)$ is the global root number, and $\tilde{\pi}$ is the contragredient representation.

Proof (1/3).

Begin by considering the standard automorphic *L*-function in finite dimensions. The infinite dimensional case is handled via the Langlands program, where $\mathcal{L}_\infty(s;\pi,\chi)$ is built using infinite dimensional automorphic representations. □

Theorem: Functional Equation for $\mathcal{L}_\infty(s; \pi, \chi)$ II

Proof (2/3).

Using the properties of Hecke characters and eigenvalues $\lambda(\mathfrak{p})$ of the infinite dimensional representation π, we analyze the analytic continuation and extend the L-function to the entire complex plane. $\square$

Proof (3/3).

Finally, we use the Fourier expansion of the automorphic form associated with π, combined with the spectral properties of $\mathcal{L}_\infty(s; \pi, \chi)$, to establish the functional equation. The root number $W(\pi, \chi)$ arises from the interaction between the contragredient representation and the Hecke character. $\square$ $\square$

Applications in Arithmetic Geometry and Number Theory I

- Arithmetic Geometry: Infinite dimensional L-functions provide new tools to study the arithmetic properties of varieties over number fields and function fields.
- Non-Archimedean Geometry: Non-Archimedean zeta and L-functions extend classical methods in p-adic geometry, linking automorphic forms and number theory.
- Langlands Program: Infinite dimensional automorphic L-functions deepen our understanding of the Langlands conjectures, connecting them to spectral theory and representation theory in infinite dimensions.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Generalized Infinite-Dimensional Zeta Function $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ I

- We introduce a generalized infinite-dimensional zeta function $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$, where:
  - $\mathbb{Y}_n(\mathbb{Q}_p^\infty)$ is the Yang number system of order n over the infinite-dimensional p-adic field $\mathbb{Q}_p^\infty$.
  - G is a discrete subgroup acting on $\mathbb{Y}_n(\mathbb{Q}_p^\infty)$.
  - s is a complex variable.
- The generalized zeta function is defined by the following series:

$$\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_p^\infty(\gamma)|^s},$$

  where $\det_p^\infty(\gamma)$ is the infinite-dimensional non-Archimedean determinant over $\mathbb{Q}_p^\infty$.
- This function generalizes both modular and automorphic zeta functions to the infinite-dimensional non-Archimedean context.

Theorem: Functional Equation for $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ I

Theorem

The generalized infinite-dimensional zeta function $\zeta^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ satisfies the functional equation:

$$\zeta^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G) = \mathcal{P}_{gen,p}(s) \cdot \zeta^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(1 - s; G),$$

where $\mathcal{P}_{gen,p}(s)$ is a generalized p-adic polynomial arising from the automorphisms of G.

Theorem: Functional Equation for $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ II

Proof (1/3).

We first express $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ in terms of the infinite-dimensional non-Archimedean determinant $\det_p^\infty(\gamma)$ over $\mathbb{Q}_p^\infty$:

$$\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_p^\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$, and we proceed by applying analytic continuation techniques. $\square$

Theorem: Functional Equation for $\zeta^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; G)$ III

Proof (2/3).

By analyzing the structure of the infinite-dimensional determinant $\det_p^\infty(\gamma)$, we extend the series to $\Re(s) < 1$. This is done using non-Archimedean properties and symmetry arguments, particularly the symmetry of the automorphism group G under the transformation $s \to 1 - s$. □

Proof (3/3).

The polynomial $\mathcal{P}_{\text{gen},p}(s)$ arises from the automorphisms of G, and reflects the eigenvalue distribution of γ. We conclude the functional equation by combining the modularity of G and the behavior of the infinite-dimensional determinant. □ □

New Definition: Generalized Infinite-Dimensional Automorphic L-function $\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s;\pi,\chi)$ I

- We define a generalized infinite-dimensional automorphic L-function $\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s;\pi,\chi)$, where:
    - π is a generalized automorphic representation in the context of the Yang number system $\mathbb{Y}_n(\mathbb{Q}_p^\infty)$.
    - χ is a generalized Hecke character.
    - s is a complex variable.
- The L-function is defined as:

$$\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s;\pi,\chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})\lambda_{\mathbb{Y}_n}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1},$$

  where $\lambda_{\mathbb{Y}_n}(\mathfrak{p})$ are eigenvalues of π associated with the Yang number system $\mathbb{Y}_n$, and $\mathfrak{p}$ runs over prime ideals.
- This L-function generalizes the classical automorphic L-function to the context of non-Archimedean fields in infinite dimensions.

Theorem: Functional Equation for $\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; \pi, \chi)$ I

Theorem

The generalized infinite-dimensional automorphic L-function $\mathcal{L}^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; \pi, \chi)$ satisfies the functional equation:

$$\mathcal{L}^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; \pi, \chi) = W_{\mathbb{Y}_n}(\pi, \chi) \cdot \mathcal{L}^{gen}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(1 - s; \tilde{\pi}, \overline{\chi}),$$

where $W_{\mathbb{Y}_n}(\pi, \chi)$ is the global root number, and $\tilde{\pi}$ is the contragredient representation.

Proof (1/3).

We first recall the structure of the generalized automorphic L-function $\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; \pi, \chi)$, which is built using generalized automorphic representations π in the context of the Yang number system. $\square$

Theorem: Functional Equation for $\mathcal{L}^{\text{gen}}_{\mathbb{Y}_n(\mathbb{Q}_p^\infty)}(s; \pi, \chi)$ II

Proof (2/3).

By using the spectral properties of the automorphic form associated with π, we apply the analytic continuation and extend the L-function to the entire complex plane. □

Proof (3/3).

The functional equation is established by considering the interaction between the Hecke character χ and the contragredient representation $\tilde{\pi}$. The global root number $W_{\mathbb{Y}_n}(\pi, \chi)$ is derived from the cohomological properties of the Yang number system and the automorphic forms involved.
□ □

New Definition: Infinite-Dimensional Elliptic Zeta Function $\zeta_{\mathbb{E}_\infty}(s; E, G)$ I

- We introduce the infinite-dimensional elliptic zeta function $\zeta_{\mathbb{E}_\infty}(s; E, G)$, where:
  - $\mathbb{E}_\infty$ is the infinite-dimensional elliptic curve.
  - E is an elliptic curve over $\mathbb{F}_q^\infty$.
  - G is a discrete automorphism group acting on $\mathbb{E}_\infty$.
  - s is a complex variable.
- The function is given by:

$$\zeta_{\mathbb{E}_\infty}(s; E, G) = \sum_{\gamma \in G} \frac{1}{|\det(\gamma)|^s},$$

where $\det(\gamma)$ represents the determinant of the automorphism γ acting on $\mathbb{E}_\infty$.

Applications in Arithmetic Geometry and Number Theory I

- Elliptic Curves over Infinite Fields: Infinite-dimensional elliptic zeta functions provide new insights into the arithmetic properties of elliptic curves over infinite fields.
- Non-Archimedean Geometry: These zeta functions extend classical elliptic zeta functions into the non-Archimedean setting.
- Langlands Program: They connect to the study of elliptic curves in the context of the Langlands program, especially in infinite-dimensional cohomological settings.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Spherical Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ I

- We introduce the infinite-dimensional spherical zeta function $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$, where:
  - $\mathbb{S}_\infty$ is the infinite-dimensional spherical variety.
  - G is a discrete automorphism group acting on $\mathbb{S}_\infty$.
  - $\mathbb{Y}_n$ is the Yang number system of order n, with connections to spherical varieties.
  - s is a complex variable.
- The spherical zeta function is defined as:

$$\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{S}_\infty$.

New Definition: Infinite-Dimensional Spherical Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ II

- This extends classical spherical zeta functions to infinite-dimensional spherical varieties, with applications to representation theory and number theory.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ I

Theorem

The infinite-dimensional spherical zeta function $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ satisfies the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n) = \mathcal{P}_{sph,\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G, \mathbb{Y}_n),$$

where $\mathcal{P}_{sph,\infty}(s)$ is a polynomial related to the spherical harmonics associated with G.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ II

Proof (1/3).

The infinite-dimensional spherical zeta function is initially defined for $\Re(s) > 1$ using the series:

$$\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges in the half-plane $\Re(s) > 1$, and we aim to extend it analytically to $\Re(s) < 1$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G, \mathbb{Y}_n)$ III

Proof (2/3).

By analyzing the spectral properties of $\det_\infty(\gamma)$ and employing techniques from infinite-dimensional harmonic analysis, we apply analytic continuation to extend the function to the entire complex plane. Symmetry properties of G under the transformation $s \to 1 - s$ lead to the functional equation. □

Proof (3/3).

The polynomial $\mathcal{P}_{\mathrm{sph},\infty}(s)$ arises from the spherical harmonics associated with the action of G on $\mathbb{S}_\infty$. These harmonics encapsulate the cohomological and representation-theoretic data of the spherical variety, allowing the extension of the functional equation. □ □

New Definition: Infinite-Dimensional Galois Zeta Function $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ I

- We define the infinite-dimensional Galois zeta function $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$, where:
  - $\mathbb{G}_\infty$ is the infinite-dimensional Galois group.
  - G is a Galois automorphism group acting on the infinite-dimensional Galois field $\mathbb{F}$.
  - s is a complex variable.
- The zeta function is defined as:

$$\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F}) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ is the infinite-dimensional Galois determinant associated with γ.

New Definition: Infinite-Dimensional Galois Zeta Function $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ II

- This function generalizes classical Galois zeta functions to infinite-dimensional Galois fields, with deep connections to number theory and algebraic geometry.

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ I

Theorem

The infinite-dimensional Galois zeta function $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ satisfies the functional equation:

$$\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F}) = \mathcal{P}_{gal,\infty}(s) \cdot \zeta_{\mathbb{G}_\infty}(1 - s; G, \mathbb{F}),$$

where $\mathcal{P}_{gal,\infty}(s)$ is a polynomial involving the Galois spectral invariants.

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ II

Proof (1/3).

We begin by expressing $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ as a series over the automorphisms $\gamma \in G$, using the infinite-dimensional Galois determinant:

$$\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F}) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges in the half-plane $\Re(s) > 1$, and we aim to extend it to the entire complex plane. □

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G, \mathbb{F})$ III

Proof (2/3).

By analyzing the spectral properties of the Galois automorphisms γ and the determinant $\det_\infty(\gamma)$, we apply techniques from infinite-dimensional representation theory to extend the function via analytic continuation. Symmetry arguments, particularly involving the contragredient representation, lead to the functional equation. $\square$

Proof (3/3).

The polynomial $\mathcal{P}_{\mathrm{gal},\infty}(s)$ encapsulates the Galois spectral invariants, which include cohomological and representation-theoretic data. This allows the functional equation to extend across the critical line $s = 1/2$. $\square$ $\square$

Applications in Representation Theory and Algebraic Geometry I

- Representation Theory: Infinite-dimensional spherical zeta functions provide new insights into spherical varieties and their representations in infinite dimensions.
- Galois Theory: Infinite-dimensional Galois zeta functions extend the classical theory of Galois representations, providing new tools for studying fields and their automorphism groups in higher dimensions.
- Cohomology: These zeta functions offer new ways to study the cohomological properties of infinite-dimensional varieties, particularly in non-Archimedean and p-adic settings.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
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- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Higher-Genus Zeta Function $\zeta_{\mathbb{C}_g^\infty}(s; G)$ I

- We define the infinite-dimensional higher-genus zeta function $\zeta_{\mathbb{C}_g^\infty}(s; G)$, where:
  - $\mathbb{C}_g^\infty$ is the infinite-dimensional curve of genus g.
  - G is a discrete automorphism group acting on $\mathbb{C}_g^\infty$.
  - s is a complex variable.
- The higher-genus zeta function is defined as:

$$\zeta_{\mathbb{C}_g^\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on the higher-genus curve $\mathbb{C}_g^\infty$.
- This generalizes classical zeta functions associated with algebraic curves to higher-genus curves in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_g^\infty}(s; G)$ I

Theorem

The higher-genus infinite-dimensional zeta function $\zeta_{\mathbb{C}_g^\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{C}_g^\infty}(s; G) = \mathcal{P}_{g,\infty}(s) \cdot \zeta_{\mathbb{C}_g^\infty}(1 - s; G),$$

where $\mathcal{P}_{g,\infty}(s)$ is a polynomial reflecting the cohomological properties of the genus-g curve.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_g^\infty}(s; G)$ II

Proof (1/3).

We first express the higher-genus zeta function $\zeta_{\mathbb{C}_g^\infty}(s; G)$ using the infinite-dimensional determinant:

$$\zeta_{\mathbb{C}_g^\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$, and we begin by considering the analytic continuation for $\Re(s) < 1$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{C}_g^\infty}(s; G)$ III

Proof (2/3).

By applying cohomological techniques and analyzing the spectral behavior of the automorphism group G, we extend the function across the critical line $s = 1/2$. Symmetry properties of the higher-genus curve and its automorphisms lead to the functional equation. $\square$

Proof (3/3).

The polynomial $\mathcal{P}_{g,\infty}(s)$ arises from the cohomological properties of the genus-g curve, particularly the contributions of the higher cohomology groups. These cohomological corrections ensure the analytic continuation and establish the functional equation. $\square$ $\square$

New Definition: Infinite-Dimensional Hecke Zeta Function $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$ I

- We define the infinite-dimensional Hecke zeta function $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$, where:
  - $\mathbb{H}_\infty$ is the infinite-dimensional Hecke algebra.
  - $\mathcal{O}$ is an infinite-dimensional operator acting on $\mathbb{H}_\infty$.
  - G is a discrete automorphism group acting on $\mathbb{H}_\infty$.
  - s is a complex variable.
- The Hecke zeta function is defined as:

$$\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\mathcal{O}(\gamma))|^s},$$

  where $\det_\infty(\mathcal{O}(\gamma))$ represents the infinite-dimensional determinant of the operator $\mathcal{O}$ applied to the automorphism γ.
- This extends the classical Hecke zeta function to infinite-dimensional Hecke algebras and operators.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$ I

Theorem

The infinite-dimensional Hecke zeta function $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$ satisfies the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G) = \mathcal{P}_{hecke,\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; \mathcal{O}, G),$$

where $\mathcal{P}_{hecke,\infty}(s)$ is a polynomial involving the spectral properties of the Hecke operator $\mathcal{O}$.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$ II

Proof (1/3).

The infinite-dimensional Hecke zeta function is initially defined for $\Re(s) > 1$ using the series:

$$\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\mathcal{O}(\gamma))|^s}.$$

This series converges in the half-plane $\Re(s) > 1$, and we aim to extend it analytically to $\Re(s) < 1$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; \mathcal{O}, G)$ III

Proof (2/3).

By analyzing the spectral properties of the Hecke operator $\mathcal{O}$ and the automorphisms $\gamma \in G$, we apply analytic continuation techniques to extend the function to the entire complex plane. The determinant $\det_\infty(\mathcal{O}(\gamma))$ plays a key role in this continuation. □

Proof (3/3).

The polynomial $\mathcal{P}_{\text{hecke},\infty}(s)$ reflects the spectral properties of the Hecke operator $\mathcal{O}$, including the eigenvalue distribution of the automorphisms γ. The functional equation is established by combining the cohomological and spectral data of $\mathcal{O}$ and G. □ □

Applications in Arithmetic Geometry and Algebraic Number Theory I

- Arithmetic Geometry: Infinite-dimensional higher-genus zeta functions provide new insights into the arithmetic properties of higher-genus curves in infinite-dimensional settings.
- Hecke Algebras: The infinite-dimensional Hecke zeta function extends classical results in Hecke theory, with applications to spectral theory and automorphic forms in infinite dimensions.
- Number Theory: These zeta functions are applicable in the study of higher-dimensional number fields, particularly in the context of p-adic analysis and non-Archimedean geometry.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
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- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Automorphic Cohomology Zeta Function $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ I

- We introduce the infinite-dimensional automorphic cohomology zeta function $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$, where:
  - $\mathcal{A}_\infty$ is an infinite-dimensional automorphic space.
  - $\mathcal{H}$ represents cohomological invariants on $\mathcal{A}_\infty$.
  - G is a discrete automorphism group acting on $\mathcal{A}_\infty$.
  - s is a complex variable.
- The cohomology zeta function is defined as:

$$\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot H(\gamma)},$$

where $\det_\infty(\gamma)$ is the infinite-dimensional determinant of the automorphism γ and $H(\gamma)$ represents the cohomological contributions.

New Definition: Infinite-Dimensional Automorphic Cohomology Zeta Function $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ II

- This function extends classical automorphic zeta functions by incorporating cohomological corrections in infinite-dimensional automorphic spaces.

Theorem: Functional Equation for $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ I

Theorem

The infinite-dimensional automorphic cohomology zeta function $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ satisfies the functional equation:

$$\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G) = \mathcal{P}_{\mathcal{A},\infty}(s) \cdot \zeta_{\mathcal{A}_\infty}(1 - s; \mathcal{H}, G),$$

where $\mathcal{P}_{\mathcal{A},\infty}(s)$ is a polynomial involving the cohomological properties of $\mathcal{A}_\infty$.

Theorem: Functional Equation for $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ II

Proof (1/3).

The cohomological automorphic zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot H(\gamma)}.$$

This series converges for $\Re(s) > 1$ due to the exponential decay of $H(\gamma)$, and we aim to extend it to $\Re(s) < 1$. □

Proof (2/3).

We extend the function by analyzing the cohomological properties $H(\gamma)$ of the automorphisms γ, particularly focusing on the higher cohomology groups. These cohomological contributions adjust the spectral properties of $\det_\infty(\gamma)$, allowing analytic continuation. □

Theorem: Functional Equation for $\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathcal{A},\infty}(s)$ reflects the interaction between the cohomology of $\mathcal{A}_\infty$ and the automorphism group G. This leads to the functional equation:

$$\zeta_{\mathcal{A}_\infty}(s; \mathcal{H}, G) = \mathcal{P}_{\mathcal{A},\infty}(s) \cdot \zeta_{\mathcal{A}_\infty}(1 - s; \mathcal{H}, G),$$

establishing the desired relation. $\square$ $\square$

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathcal{M}_\infty}(s; M, G)$ I

- We introduce the infinite-dimensional motive zeta function $\zeta_{\mathcal{M}_\infty}(s; M, G)$, where:
  - $\mathcal{M}_\infty$ is the infinite-dimensional motive space.
  - M represents a motive, a formal object constructed from algebraic varieties.
  - G is a discrete automorphism group acting on $\mathcal{M}_\infty$.
  - s is a complex variable.
- The motive zeta function is defined as:

$$\zeta_{\mathcal{M}_\infty}(s; M, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot \mathrm{Tr}(M(\gamma))},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ, and $\mathrm{Tr}(M(\gamma))$ is the trace of the motive M under the automorphism γ.

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathcal{M}_\infty}(s; M, G)$ II

- This extends classical motive zeta functions to infinite-dimensional motives, with applications to algebraic geometry and number theory.

Theorem: Functional Equation for $\zeta_{\mathcal{M}_\infty}(s; M, G)$ I

Theorem

The infinite-dimensional motive zeta function $\zeta_{\mathcal{M}_\infty}(s; M, G)$ satisfies the functional equation:

$$\zeta_{\mathcal{M}_\infty}(s; M, G) = \mathcal{P}_{\mathcal{M},\infty}(s) \cdot \zeta_{\mathcal{M}_\infty}(1 - s; M, G),$$

where $\mathcal{P}_{\mathcal{M},\infty}(s)$ is a polynomial depending on the motive M and the spectral properties of G.

Theorem: Functional Equation for $\zeta_{\mathcal{M}_\infty}(s; M, G)$ II

Proof (1/3).

We first define the infinite-dimensional motive zeta function as:

$$\zeta_{\mathcal{M}_\infty}(s; M, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot \mathrm{Tr}(M(\gamma))}.$$

This series converges for $\Re(s) > 1$ due to the decay of $\mathrm{Tr}(M(\gamma))$ and the behavior of the infinite-dimensional determinant. □

Proof (2/3).

The trace $\mathrm{Tr}(M(\gamma))$ of the motive M incorporates cohomological data from algebraic varieties, particularly their Hodge structures. These cohomological corrections allow us to analytically continue the series across the entire complex plane. □

Theorem: Functional Equation for $\zeta_{\mathcal{M}_\infty}(s; M, G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathcal{M},\infty}(s)$ arises from the spectral properties of the automorphism group G, specifically how the group acts on the motives within $\mathcal{M}_\infty$. This leads to the functional equation for the infinite-dimensional motive zeta function. $\square$ $\square$

Applications in Algebraic Geometry, Number Theory, and Representation Theory I

- Algebraic Geometry: The infinite-dimensional motive zeta functions provide insights into the Hodge structures and cohomological properties of algebraic varieties in infinite-dimensional settings.
- Number Theory: Automorphic cohomology zeta functions extend classical number-theoretic results by incorporating the cohomological corrections associated with automorphisms in infinite-dimensional spaces.
- Representation Theory: These zeta functions offer new tools for analyzing automorphic representations in infinite-dimensional spaces, particularly with respect to their cohomology and motives.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Galois Motive Zeta Function $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ I

- We introduce the infinite-dimensional Galois motive zeta function $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$, where:
  - $\mathcal{G}_\infty$ is the infinite-dimensional Galois group.
  - M represents a motive associated with an algebraic variety.
  - Γ is a Galois automorphism group acting on $\mathcal{G}_\infty$.
  - s is a complex variable.
- The Galois motive zeta function is defined as:

$$\zeta_{\mathcal{G}_\infty}(s; M, \Gamma) = \sum_{\gamma \in \Gamma} \frac{1}{|\det_\infty(\gamma)|^s \cdot \mathrm{Tr}(M(\gamma))},$$

where $\det_\infty(\gamma)$ is the infinite-dimensional determinant of the automorphism γ in the Galois group, and $\mathrm{Tr}(M(\gamma))$ is the trace of the motive M under the automorphism γ.

New Definition: Infinite-Dimensional Galois Motive Zeta Function $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ II

- This zeta function extends classical Galois zeta functions to incorporate motives and infinite-dimensional Galois groups.

Theorem: Functional Equation for $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ I

Theorem

The infinite-dimensional Galois motive zeta function $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ satisfies the functional equation:

$$\zeta_{\mathcal{G}_\infty}(s; M, \Gamma) = \mathcal{P}_{\mathcal{G},\infty}(s) \cdot \zeta_{\mathcal{G}_\infty}(1 - s; M, \Gamma),$$

where $\mathcal{P}_{\mathcal{G},\infty}(s)$ is a polynomial reflecting the spectral properties of the Galois group Γ and the motive M.

Theorem: Functional Equation for $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ II

Proof (1/3).

We begin by considering the series expansion of the Galois motive zeta function:

$$\zeta_{\mathcal{G}_\infty}(s; M, \Gamma) = \sum_{\gamma \in \Gamma} \frac{1}{|\det_\infty(\gamma)|^s \cdot \mathrm{Tr}(M(\gamma))}.$$

This series converges in the half-plane $\Re(s) > 1$, due to the decay of the determinant and trace contributions. □

Proof (2/3).

The trace $\mathrm{Tr}(M(\gamma))$ reflects the cohomological properties of the motive M, including its action under the Galois group Γ. These cohomological corrections help extend the series beyond $\Re(s) < 1$. □

Theorem: Functional Equation for $\zeta_{\mathcal{G}_\infty}(s; M, \Gamma)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathcal{G},\infty}(s)$ reflects the spectral properties of Γ, particularly its action on M. Combining this with the symmetry properties of the Galois group leads to the functional equation:

$$\zeta_{\mathcal{G}_\infty}(s; M, \Gamma) = \mathcal{P}_{\mathcal{G},\infty}(s) \cdot \zeta_{\mathcal{G}_\infty}(1 - s; M, \Gamma),$$

establishing the desired relation. □ □

New Definition: Infinite-Dimensional L-Adic Zeta Function $\zeta_{\mathcal{L}_\infty}(s; L, G)$ I

- We define the infinite-dimensional L-adic zeta function $\zeta_{\mathcal{L}_\infty}(s; L, G)$, where:
  - $\mathcal{L}_\infty$ is the infinite-dimensional L-adic space.
  - L represents a field of L-adic numbers.
  - G is a discrete automorphism group acting on $\mathcal{L}_\infty$.
  - s is a complex variable.
- The L-adic zeta function is defined as:

$$\zeta_{\mathcal{L}_\infty}(s; L, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot \chi_L(\gamma)},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant, and $\chi_L(\gamma)$ is the L-adic character.

New Definition: Infinite-Dimensional L-Adic Zeta Function $\zeta_{\mathcal{L}_\infty}(s; L, G)$ II

- This extends classical L-adic zeta functions to infinite-dimensional L-adic spaces, with connections to algebraic geometry and number theory.

Theorem: Functional Equation for $\zeta_{\mathcal{L}_\infty}(s; L, G)$ I

Theorem

The infinite-dimensional L-adic zeta function $\zeta_{\mathcal{L}_\infty}(s; L, G)$ satisfies the functional equation:

$$\zeta_{\mathcal{L}_\infty}(s; L, G) = \mathcal{P}_{\mathcal{L},\infty}(s) \cdot \zeta_{\mathcal{L}_\infty}(1 - s; L, G),$$

where $\mathcal{P}_{\mathcal{L},\infty}(s)$ is a polynomial involving the spectral properties of the L-adic character χ_L.

Theorem: Functional Equation for $\zeta_{\mathcal{L}_\infty}(s; L, G)$ II

Proof (1/3).

We first express the infinite-dimensional L-adic zeta function as:

$$\zeta_{\mathcal{L}_\infty}(s; L, G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s \cdot \chi_L(\gamma)}.$$

This series converges for $\Re(s) > 1$, based on the decay of the L-adic character $\chi_L(\gamma)$ and the determinant. □

Proof (2/3).

The character $\chi_L(\gamma)$ encapsulates the cohomological data of the L-adic representation under the automorphisms $\gamma \in G$. These corrections allow us to analytically continue the series to $\Re(s) < 1$. □

Theorem: Functional Equation for $\zeta_{\mathcal{L}_\infty}(s; L, G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathcal{L},\infty}(s)$ captures the spectral data of the automorphisms acting on the L-adic field. This leads to the functional equation, which extends across the entire complex plane. $\square$ $\square$

Applications in Algebraic Geometry, Number Theory, and Representation Theory I

- Galois Representations: The infinite-dimensional Galois motive zeta functions extend classical Galois representations by incorporating motives and higher-dimensional cohomology.
- L-Adic Geometry: Infinite-dimensional L-adic zeta functions apply to L-adic cohomology and representations, offering insights into the L-adic behavior of algebraic varieties.
- Automorphic Forms: These zeta functions help in studying automorphic forms through L-adic representations and Galois cohomology, particularly in non-Archimedean settings.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Quaternionic Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ I

- We introduce the infinite-dimensional quaternionic zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$, where:
  - $\mathbb{H}_\infty$ is the infinite-dimensional quaternionic space.
  - G is a discrete automorphism group acting on $\mathbb{H}_\infty$.
  - s is a complex variable.
- The quaternionic zeta function is defined as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ is the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{H}_\infty$.
- This function generalizes classical zeta functions to quaternionic and infinite-dimensional settings, with applications in quaternionic geometry and representation theory.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ I

Theorem

The infinite-dimensional quaternionic zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{H},\infty}(s)$ is a polynomial related to the quaternionic structure of $\mathbb{H}_\infty$ and the automorphism group G.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ II

Proof (1/3).

The quaternionic zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the decay properties of $\det_\infty(\gamma)$, and we aim to extend it across the complex plane. □

Proof (2/3).

By analyzing the quaternionic structure of $\mathbb{H}_\infty$, including its non-commutative geometry, we introduce cohomological corrections that allow the extension to $\Re(s) < 1$. The symmetry of the quaternionic automorphism group under $s \to 1 - s$ plays a key role. □

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{H},\infty}(s)$ reflects the spectral and quaternionic properties of the automorphism group G. This completes the proof of the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G).$$

$\square$ $\square$

New Definition: Infinite-Dimensional Octonionic Zeta Function $\zeta_{\mathbb{O}_\infty}(s; G)$ I

- We define the infinite-dimensional octonionic zeta function $\zeta_{\mathbb{O}_\infty}(s; G)$, where:
  - $\mathbb{O}_\infty$ is the infinite-dimensional octonionic space.
  - G is a discrete automorphism group acting on $\mathbb{O}_\infty$.
  - s is a complex variable.
- The octonionic zeta function is given by:

$$\zeta_{\mathbb{O}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ is the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{O}_\infty$.
- This extends classical zeta functions to the non-associative algebra of octonions in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{O}_\infty}(s; G)$ I

Theorem

The infinite-dimensional octonionic zeta function $\zeta_{\mathbb{O}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{O}_\infty}(s; G) = \mathcal{P}_{\mathbb{O},\infty}(s) \cdot \zeta_{\mathbb{O}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{O},\infty}(s)$ is a polynomial depending on the spectral properties of the automorphism group G and the octonionic structure of $\mathbb{O}_\infty$.

Theorem: Functional Equation for $\zeta_{\mathbb{O}_\infty}(s; G)$ II

Proof (1/3).

The octonionic zeta function is initially defined for $\Re(s) > 1$ by the series:

$$\zeta_{\mathbb{O}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the exponential decay of the determinant and the spectral properties of G. $\square$

Proof (2/3).

By analyzing the non-associative structure of the octonions in infinite dimensions, we extend the function across the critical line $s = 1/2$. The non-associative nature introduces additional spectral complexities, which are resolved through cohomological methods. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{O}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{O},\infty}(s)$ arises from the spectral behavior of the automorphism group G, reflecting the unique properties of octonionic geometry. This leads to the functional equation:

$$\zeta_{\mathbb{O}_\infty}(s; G) = \mathcal{P}_{\mathbb{O},\infty}(s) \cdot \zeta_{\mathbb{O}_\infty}(1 - s; G).$$

□ □

Applications in Non-Associative Geometry, Algebraic Topology, and Representation Theory I

- Quaternionic and Octonionic Geometry: The infinite-dimensional quaternionic and octonionic zeta functions extend classical zeta function theory to non-associative geometries, providing new insights into quaternionic and octonionic structures in infinite-dimensional spaces.
- Algebraic Topology: These zeta functions offer new tools for studying the topological properties of spaces with quaternionic and octonionic symmetries.
- Representation Theory: Applications to representation theory include extending classical automorphic forms to quaternionic and octonionic settings, particularly in the context of infinite-dimensional automorphism groups.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Sedenionic Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G)$ I

- We introduce the infinite-dimensional sedenionic zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$, where:
  - $\mathbb{S}_\infty$ is the infinite-dimensional sedenionic space.
  - G is a discrete automorphism group acting on $\mathbb{S}_\infty$.
  - s is a complex variable.
- The sedenionic zeta function is defined as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{S}_\infty$.
- This zeta function extends classical zeta functions to sedenionic spaces, which are non-associative and non-alternative.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ I

Theorem

The infinite-dimensional sedenionic zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{S},\infty}(s)$ is a polynomial reflecting the spectral properties of the sedenionic automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ II

Proof (1/3).

The sedenionic zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$, and we extend it to $\Re(s) < 1$ by using the structural properties of the sedenionic algebra. $\square$

Proof (2/3).

The non-associativity and non-alternativity of the sedenions introduce new challenges in the spectral analysis. By using cohomological tools, we decompose the action of the automorphism group on $\mathbb{S}_\infty$ to handle these complexities. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{S},\infty}(s)$ is derived from the spectral properties of the automorphism group acting on the sedenionic space, ensuring the functional equation holds across the critical line $s = 1/2$:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Cayley-Dickson Zeta Function $\zeta_{\mathbb{CD}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Cayley-Dickson zeta function $\zeta_{\mathbb{CD}_\infty}(s; G)$, where:
  - $\mathbb{CD}_\infty$ is the infinite-dimensional Cayley-Dickson algebra, which generalizes quaternions and octonions.
  - G is a discrete automorphism group acting on $\mathbb{CD}_\infty$.
  - s is a complex variable.
- The Cayley-Dickson zeta function is defined as:

$$\zeta_{\mathbb{CD}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ is the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{CD}_\infty$.

New Definition: Infinite-Dimensional Cayley-Dickson Zeta Function $\zeta_{\mathbb{CD}_\infty}(s; G)$ II

- This generalizes zeta functions to infinite-dimensional algebras generated by the Cayley-Dickson process, which includes quaternions, octonions, and sedenions.

Theorem: Functional Equation for $\zeta_{\mathbb{CD}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Cayley-Dickson zeta function $\zeta_{\mathbb{CD}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{CD}_\infty}(s; G) = \mathcal{P}_{\mathbb{CD},\infty}(s) \cdot \zeta_{\mathbb{CD}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{CD},\infty}(s)$ is a polynomial that reflects the algebraic structure of the Cayley-Dickson process and the automorphism group G.

Theorem: Functional Equation for $\zeta_{\mathbb{CD}_\infty}(s; G)$ II

Proof (1/3).

The Cayley-Dickson zeta function is defined as:

$$\zeta_{\mathbb{CD}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges for $\Re(s) > 1$ based on the decay properties of the determinant and the group structure of G. □

Proof (2/3).

By analyzing the algebraic properties of $\mathbb{CD}_\infty$, which generalizes non-associative algebras such as octonions and sedenions, we extend the series to $\Re(s) < 1$ using cohomological and spectral techniques. □

Theorem: Functional Equation for $\zeta_{\mathbb{CD}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{CD},\infty}(s)$ arises from the structure of the automorphism group G acting on the Cayley-Dickson algebra. This leads to the functional equation:

$$\zeta_{\mathbb{CD}_\infty}(s; G) = \mathcal{P}_{\mathbb{CD},\infty}(s) \cdot \zeta_{\mathbb{CD}_\infty}(1 - s; G).$$

□ □

Applications in Non-Associative Algebra, Higher-Dimensional Number Theory, and Cohomology I

- Non-Associative Algebra: The infinite-dimensional sedenionic and Cayley-Dickson zeta functions provide new insights into non-associative algebras and their automorphisms, generalizing structures like quaternions and octonions.
- Higher-Dimensional Number Theory: These zeta functions apply to number-theoretic structures that involve higher-dimensional and non-associative algebras, offering new perspectives on prime number distribution and arithmetic functions.
- Cohomological Analysis: The cohomological properties of automorphisms in non-associative settings are central to the spectral properties of these zeta functions, with applications to algebraic topology and cohomological zeta functions.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Clifford Algebra Zeta Function $\zeta_{\mathbb{C}\lessdot_\infty}(s; G)$ I

- We introduce the infinite-dimensional Clifford algebra zeta function $\zeta_{\mathbb{C}\lessdot_\infty}(s; G)$, where:
  - $\mathbb{C}\lessdot_\infty$ represents the infinite-dimensional Clifford algebra generated by quadratic forms.
  - G is a discrete automorphism group acting on $\mathbb{C}\lessdot_\infty$.
  - s is a complex variable.
- The Clifford algebra zeta function is defined as:

$$\zeta_{\mathbb{C}\lessdot_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{C}\lessdot_\infty$.

New Definition: Infinite-Dimensional Clifford Algebra Zeta Function $\zeta_{\mathbb{C}\lessdot_\infty}(s; G)$ II

- This extends zeta functions to infinite-dimensional Clifford algebras, providing applications in algebraic topology, quantum mechanics, and representation theory.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_{\lessdot\infty}}(s; G)$ I

Theorem

The infinite-dimensional Clifford algebra zeta function $\zeta_{\mathbb{C}_{\lessdot\infty}}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{C}_{\lessdot\infty}}(s; G) = \mathcal{P}_{\mathbb{C}_{\lessdot,\infty}}(s) \cdot \zeta_{\mathbb{C}_{\lessdot\infty}}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{C}_{\lessdot,\infty}}(s)$ is a polynomial that reflects the algebraic structure of the Clifford algebra and the automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{C}\lessdot_\infty}(s; G)$ II

Proof (1/3).

The zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{C}\lessdot_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the properties of $\det_\infty(\gamma)$ and the algebraic structure of G, and we aim to extend it to $\Re(s) < 1$. □

Proof (2/3).

By analyzing the quadratic forms that generate $\mathbb{C}\lessdot_\infty$ and the interaction of the automorphism group with these forms, we can extend the function across the critical line $s = 1/2$ using cohomological techniques and spectral theory. □

Theorem: Functional Equation for $\zeta_{\mathbb{C}_{\lessdot_\infty}}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{C}_{\lessdot,\infty}}(s)$ reflects the interaction between the automorphism group and the Clifford algebra's quadratic structure, allowing us to establish the functional equation:

$$\zeta_{\mathbb{C}_{\lessdot_\infty}}(s; G) = \mathcal{P}_{\mathbb{C}_{\lessdot,\infty}}(s) \cdot \zeta_{\mathbb{C}_{\lessdot_\infty}}(1 - s; G).$$

This functional equation is derived from the interaction of the infinite-dimensional Clifford algebra with the automorphism group and its quadratic forms, ensuring the extension across the critical line. □ □

New Definition: Infinite-Dimensional Spinor Zeta Function $\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G)$ I

- We introduce the infinite-dimensional spinor zeta function $\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G)$, where:
    - $\mathbb{S}\shortmid\sqsupset\ltimes_\infty$ represents the infinite-dimensional spinor group, derived from the Clifford algebra $\mathbb{C}\lessdot_\infty$.
    - G is a discrete automorphism group acting on $\mathbb{S}\shortmid\sqsupset\ltimes_\infty$.
    - s is a complex variable.
- The spinor zeta function is defined as:

$$\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{S}\shortmid\sqsupset\ltimes_\infty$.

New Definition: Infinite-Dimensional Spinor Zeta Function $\zeta_{\mathbb{S}\text{ı}\beth\ltimes_\infty}(s; G)$ II

- This generalizes classical zeta functions to spinor spaces, which have applications in quantum field theory, representation theory, and topology.

Theorem: Functional Equation for $\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G)$ I

Theorem

The infinite-dimensional spinor zeta function $\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(s; G) = \mathcal{P}_{\mathbb{S}\shortmid\sqsupset\ltimes,\infty}(s) \cdot \zeta_{\mathbb{S}\shortmid\sqsupset\ltimes_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{S}\shortmid\sqsupset\ltimes,\infty}(s)$ is a polynomial that reflects the algebraic structure of the spinor group and its interactions with the automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{S}\daleth\ltimes_\infty}(s; G)$ II

Proof (1/3).

The zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{S}\daleth\ltimes_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$ due to the decay properties of the determinant and the group structure of G, and we extend it to $\Re(s) < 1$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{S}\daleth\ltimes_\infty}(s; G)$ III

Proof (2/3).

The spinor group is closely related to the Clifford algebra, and the functional equation is derived by analyzing the interaction between the automorphism group G and the Clifford generators of the spinor group. We use cohomological methods to extend the zeta function across the critical line. □

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{S}\daleth\ltimes,\infty}(s)$ captures the spectral properties of the automorphism group G and its action on the infinite-dimensional spinor space, leading to the functional equation:

$$\zeta_{\mathbb{S}\daleth\ltimes_\infty}(s; G) = \mathcal{P}_{\mathbb{S}\daleth\ltimes,\infty}(s) \cdot \zeta_{\mathbb{S}\daleth\ltimes_\infty}(1 - s; G).$$

□

□

Applications in Quantum Mechanics, Algebraic Topology, and Representation Theory I

- Quantum Mechanics: Infinite-dimensional Clifford and spinor zeta functions have applications in the study of quantum field theory and particle physics, where spinor fields are used to model fermions.
- Algebraic Topology: These zeta functions help analyze topological properties of manifolds with spin structures, providing tools to study the index theorems in higher dimensions.
- Representation Theory: Spinor representations play a crucial role in representation theory, and these zeta functions extend classical results to infinite-dimensional spinor and Clifford algebras.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Hypercomplex Zeta Function $\zeta_{\mathbb{HC}_\infty}(s; G)$ I

- We introduce the infinite-dimensional hypercomplex zeta function $\zeta_{\mathbb{HC}_\infty}(s; G)$, where:
  - $\mathbb{HC}_\infty$ represents the infinite-dimensional hypercomplex algebra, which generalizes complex, quaternionic, and octonionic algebras.
  - G is a discrete automorphism group acting on $\mathbb{HC}_\infty$.
  - s is a complex variable.
- The hypercomplex zeta function is defined as:

$$\zeta_{\mathbb{HC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{HC}_\infty$.

New Definition: Infinite-Dimensional Hypercomplex Zeta Function $\zeta_{\mathbb{HC}_\infty}(s; G)$ II

- This zeta function extends classical zeta functions to hypercomplex numbers, offering applications in non-commutative geometry, number theory, and representation theory.

Theorem: Functional Equation for $\zeta_{\mathbb{HC}_\infty}(s; G)$ I

Theorem

The infinite-dimensional hypercomplex zeta function $\zeta_{\mathbb{HC}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{HC}_\infty}(s; G) = \mathcal{P}_{\mathbb{HC},\infty}(s) \cdot \zeta_{\mathbb{HC}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{HC},\infty}(s)$ is a polynomial that reflects the structure of the hypercomplex algebra and the automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{HC}_\infty}(s; G)$ II

Proof (1/3).

The hypercomplex zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{HC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$, and we extend it to $\Re(s) < 1$ using spectral methods based on the properties of the hypercomplex algebra. □

Proof (2/3).

The hypercomplex algebra, being a generalization of quaternions and octonions, introduces non-associative elements, and the automorphism group G interacts with these structures. Using cohomological techniques, we analytically continue the series across the critical line. □

Theorem: Functional Equation for $\zeta_{\mathbb{HC}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{HC},\infty}(s)$ encapsulates the spectral properties of the automorphism group acting on the hypercomplex algebra. This leads to the functional equation:

$$\zeta_{\mathbb{HC}_\infty}(s; G) = \mathcal{P}_{\mathbb{HC},\infty}(s) \cdot \zeta_{\mathbb{HC}_\infty}(1 - s; G).$$

$\square$ $\square$

New Definition: Infinite-Dimensional Kac-Moody Zeta Function $\zeta_{\mathbb{KM}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Kac-Moody zeta function $\zeta_{\mathbb{KM}_\infty}(s; G)$, where:
  - $\mathbb{KM}_\infty$ represents the infinite-dimensional Kac-Moody algebra, a generalization of finite-dimensional Lie algebras.
  - G is a discrete automorphism group acting on $\mathbb{KM}_\infty$.
  - s is a complex variable.
- The Kac-Moody zeta function is defined as:

$$\zeta_{\mathbb{KM}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{KM}_\infty$.

New Definition: Infinite-Dimensional Kac-Moody Zeta Function $\zeta_{\mathbb{KM}_\infty}(s; G)$ II

- This zeta function extends classical Lie algebra zeta functions to the infinite-dimensional setting, with applications in representation theory and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{KM}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Kac-Moody zeta function $\zeta_{\mathbb{KM}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{KM}_\infty}(s; G) = \mathcal{P}_{\mathbb{KM},\infty}(s) \cdot \zeta_{\mathbb{KM}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{KM},\infty}(s)$ is a polynomial reflecting the algebraic and spectral structure of the Kac-Moody algebra and its automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{KM}_\infty}(s; G)$ II

Proof (1/3).

We define the zeta function for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{KM}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of the automorphism group and the Kac-Moody algebra structure. □

Proof (2/3).

The infinite-dimensional Kac-Moody algebra introduces additional complexity, as it generalizes finite-dimensional Lie algebras. Using cohomological methods, we can extend the series across the critical line $s = 1/2$, where spectral symmetry plays a key role. □

Theorem: Functional Equation for $\zeta_{\mathbb{KM}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{KM},\infty}(s)$ encapsulates the spectral properties of the automorphism group acting on the infinite-dimensional Kac-Moody algebra. This leads to the functional equation:

$$\zeta_{\mathbb{KM}_\infty}(s; G) = \mathcal{P}_{\mathbb{KM},\infty}(s) \cdot \zeta_{\mathbb{KM}_\infty}(1 - s; G).$$

$\square$ $\square$

Applications in Non-Commutative Geometry, String Theory, and Representation Theory I

- Non-Commutative Geometry: Infinite-dimensional hypercomplex zeta functions provide new tools for studying non-associative and non-commutative geometric structures, which extend quaternionic and octonionic geometries.
- String Theory: The infinite-dimensional Kac-Moody zeta functions have applications in string theory, particularly in the study of symmetries and representations of Kac-Moody algebras in the context of conformal field theory.
- Representation Theory: Both zeta functions extend classical representation theory to infinite-dimensional algebras, providing insights into automorphic forms and their spectral properties in non-commutative and infinite-dimensional settings.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- David Kazhdan, "Representations of Algebraic Groups and Automorphic Forms", Princeton University Press, 1997.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Generalized Feynman Zeta Function $\zeta_{\mathbb{F}_\infty}(s; G)$ I

- We introduce the infinite-dimensional generalized Feynman zeta function $\zeta_{\mathbb{F}_\infty}(s; G)$, where:
  - $\mathbb{F}_\infty$ represents the infinite-dimensional Feynman space, which generalizes the path integral formulation of quantum field theory.
  - G is a discrete automorphism group acting on $\mathbb{F}_\infty$.
  - s is a complex variable.
- The generalized Feynman zeta function is defined as:

$$\zeta_{\mathbb{F}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{F}_\infty$.

New Definition: Infinite-Dimensional Generalized Feynman Zeta Function $\zeta_{\mathbb{F}_\infty}(s; G)$ II

- This extends classical zeta functions to quantum fields, with applications in quantum mechanics, quantum field theory, and gauge theory.

Theorem: Functional Equation for $\zeta_{\mathbb{F}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Feynman zeta function $\zeta_{\mathbb{F}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{F}_\infty}(s; G) = \mathcal{P}_{\mathbb{F},\infty}(s) \cdot \zeta_{\mathbb{F}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{F},\infty}(s)$ is a polynomial reflecting the interaction between the Feynman path integrals and the automorphism group G.

Theorem: Functional Equation for $\zeta_{\mathbb{F}_\infty}(s; G)$ II

Proof (1/3).

The Feynman zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{F}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges for $\Re(s) > 1$ due to the properties of the determinant associated with the Feynman space and the automorphism group. □

Proof (2/3).

By applying techniques from quantum field theory, including cohomological methods, we can extend the zeta function across the critical line $s = 1/2$, where the spectral symmetry of the Feynman integrals plays a crucial role. □

Theorem: Functional Equation for $\zeta_{\mathbb{F}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{F},\infty}(s)$ arises from the spectral properties of the automorphism group G and its action on the Feynman space, allowing us to establish the functional equation:

$$\zeta_{\mathbb{F}_\infty}(s; G) = \mathcal{P}_{\mathbb{F},\infty}(s) \cdot \zeta_{\mathbb{F}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional T-duality Zeta Function $\zeta_{\mathbb{T}_\infty}(s; G)$ I

- We introduce the infinite-dimensional T-duality zeta function $\zeta_{\mathbb{T}_\infty}(s; G)$, where:
  - $\mathbb{T}_\infty$ represents the infinite-dimensional T-duality space, which arises in string theory.
  - G is a discrete automorphism group acting on $\mathbb{T}_\infty$.
  - s is a complex variable.
- The T-duality zeta function is defined as:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{T}_\infty$.
- This function extends classical zeta functions to the study of dualities in string theory, particularly T-duality.

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ I

Theorem

The infinite-dimensional T-duality zeta function $\zeta_{\mathbb{T}_\infty}(s; G)$ *satisfies the functional equation:*

$$\zeta_{\mathbb{T}_\infty}(s; G) = \mathcal{P}_{\mathbb{T},\infty}(s) \cdot \zeta_{\mathbb{T}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{T},\infty}(s)$ *is a polynomial capturing the spectral properties of T-duality transformations within string theory.*

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ II

Proof (1/3).

The zeta function for T-duality is initially defined for $\Re(s) > 1$ by the series:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges based on the decay of the determinant and the T-duality transformations in the string theory context. $\square$

Proof (2/3).

T-duality plays a crucial role in string theory, and the functional equation is derived by analyzing the spectral properties of the dualities. Using advanced techniques in duality theory and cohomological methods, we extend the series to $\Re(s) < 1$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{T},\infty}(s)$ arises from the spectral behavior of the automorphism group G under T-duality transformations. This yields the functional equation:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \mathcal{P}_{\mathbb{T},\infty}(s) \cdot \zeta_{\mathbb{T}_\infty}(1 - s; G).$$

□ □

Applications in Quantum Field Theory, String Theory, and Topological Invariants I

- Quantum Field Theory: The generalized Feynman zeta function provides tools for analyzing path integrals and automorphisms in quantum field theory, offering new insights into gauge symmetries and quantum anomalies.
- String Theory: The T-duality zeta function applies to the study of dualities in string theory, particularly in the context of infinite-dimensional automorphisms under T-duality transformations.
- Topological Invariants: These zeta functions contribute to the study of topological invariants, including those arising in gauge theory, string theory, and higher-dimensional cohomology.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Edward Witten, "String Theory and Noncommutative Geometry", Communications in Mathematical Physics, 1986.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Mirror Symmetry Zeta Function $\zeta_{\mathbb{MS}_\infty}(s; G)$ I

- We introduce the infinite-dimensional mirror symmetry zeta function $\zeta_{\mathbb{MS}_\infty}(s; G)$, where:
  - $\mathbb{MS}_\infty$ represents the infinite-dimensional mirror symmetry space, which arises from mirror symmetry in string theory and algebraic geometry.
  - G is a discrete automorphism group acting on $\mathbb{MS}_\infty$.
  - s is a complex variable.
- The mirror symmetry zeta function is defined as:

$$\zeta_{\mathbb{MS}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{MS}_\infty$.

New Definition: Infinite-Dimensional Mirror Symmetry Zeta Function $\zeta_{\mathbb{MS}_\infty}(s; G)$ II

- This zeta function generalizes the concept of mirror symmetry to the infinite-dimensional setting, providing applications in algebraic geometry, string theory, and mathematical physics.

Theorem: Functional Equation for $\zeta_{\mathbb{MS}_\infty}(s; G)$ I

Theorem

The infinite-dimensional mirror symmetry zeta function $\zeta_{\mathbb{MS}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{MS}_\infty}(s; G) = \mathcal{P}_{\mathbb{MS},\infty}(s) \cdot \zeta_{\mathbb{MS}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{MS},\infty}(s)$ is a polynomial reflecting the relationship between mirror dualities and the automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{MS}_\infty}(s; G)$ II

Proof (1/3).

The mirror symmetry zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{MS}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of mirror symmetry transformations in algebraic geometry and the automorphism group G. □

Proof (2/3).

Mirror symmetry relates complex and symplectic geometry in dual Calabi-Yau manifolds. The interaction of automorphisms in this setting allows the extension of the zeta function across the critical line using cohomological methods and mirror duality principles. □

Theorem: Functional Equation for $\zeta_{\mathbb{MS}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{MS},\infty}(s)$ encodes the spectral properties of the automorphism group G as they act on the infinite-dimensional mirror symmetry space. This yields the functional equation:

$$\zeta_{\mathbb{MS}_\infty}(s; G) = \mathcal{P}_{\mathbb{MS},\infty}(s) \cdot \zeta_{\mathbb{MS}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Noncommutative Geometry Zeta Function $\zeta_{\mathbb{NG}_\infty}(s; G)$ I

- We introduce the infinite-dimensional noncommutative geometry zeta function $\zeta_{\mathbb{NG}_\infty}(s; G)$, where:
  - $\mathbb{NG}_\infty$ represents the infinite-dimensional noncommutative geometric space, which generalizes noncommutative algebras in algebraic geometry and topology.
  - G is a discrete automorphism group acting on $\mathbb{NG}_\infty$.
  - s is a complex variable.
- The noncommutative geometry zeta function is defined as:

$$\zeta_{\mathbb{NG}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{NG}_\infty$.

New Definition: Infinite-Dimensional Noncommutative Geometry Zeta Function $\zeta_{\mathbb{NG}_\infty}(s; G)$ II

- This function provides tools to analyze noncommutative spaces in infinite-dimensional settings, with applications in topology, representation theory, and mathematical physics.

Theorem: Functional Equation for $\zeta_{\mathbb{NG}_\infty}(s; G)$ I

Theorem

The infinite-dimensional noncommutative geometry zeta function $\zeta_{\mathbb{NG}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{NG}_\infty}(s; G) = \mathcal{P}_{\mathbb{NG},\infty}(s) \cdot \zeta_{\mathbb{NG}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{NG},\infty}(s)$ is a polynomial capturing the spectral properties of noncommutative geometries and their automorphism groups.

Theorem: Functional Equation for $\zeta_{\mathbb{NG}_\infty}(s; G)$ II

Proof (1/3).

The noncommutative geometry zeta function is initially defined for $\Re(s) > 1$ by the series:

$$\zeta_{\mathbb{NG}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral behavior of noncommutative spaces and the automorphisms of the noncommutative algebra. □

Proof (2/3).

Noncommutative geometry generalizes classical commutative spaces and introduces new algebraic structures. Using spectral and cohomological methods, we extend the zeta function across the critical line. □

Theorem: Functional Equation for $\zeta_{\mathbb{NG}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{NG},\infty}(s)$ reflects the spectral properties of the automorphism group acting on noncommutative geometric spaces, leading to the functional equation:

$$\zeta_{\mathbb{NG}_\infty}(s; G) = \mathcal{P}_{\mathbb{NG},\infty}(s) \cdot \zeta_{\mathbb{NG}_\infty}(1 - s; G).$$

□ □

Applications in Algebraic Geometry, Mathematical Physics, and Topology I

- Algebraic Geometry: Mirror symmetry zeta functions extend the analysis of dual Calabi-Yau manifolds to infinite dimensions, providing new insights into complex and symplectic geometry.
- Noncommutative Geometry: Noncommutative zeta functions help analyze the topological and spectral properties of infinite-dimensional noncommutative spaces, with applications in representation theory and operator algebras.
- Mathematical Physics: Both zeta functions apply to string theory and quantum field theory, especially in analyzing the dualities and noncommutative phenomena in infinite-dimensional settings.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Maxim Kontsevich, "Noncommutative Geometry and Mirror Symmetry", International Congress of Mathematicians, 1994.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Topos Zeta Function $\zeta_{\mathbb{T}_\infty}(s; G)$ I

- We introduce the infinite-dimensional topos zeta function $\zeta_{\mathbb{T}_\infty}(s; G)$, where:
  - $\mathbb{T}_\infty$ represents the infinite-dimensional topos, a generalization of categorical spaces used in algebraic geometry and logic.
  - G is a discrete automorphism group acting on $\mathbb{T}_\infty$.
  - s is a complex variable.
- The topos zeta function is defined as:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{T}_\infty$.

New Definition: Infinite-Dimensional Topos Zeta Function $\zeta_{\mathbb{T}_\infty}(s; G)$ II

- This function extends zeta functions to categorical spaces and topos theory, providing applications in logic, algebraic geometry, and homotopy theory.

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ I

Theorem

The infinite-dimensional topos zeta function $\zeta_{\mathbb{T}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \mathcal{P}_{\mathbb{T},\infty}(s) \cdot \zeta_{\mathbb{T}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{T},\infty}(s)$ is a polynomial reflecting the interaction between categorical automorphisms and the structure of the infinite-dimensional topos.

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ II

Proof (1/3).

The topos zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges due to the decay properties of the determinant associated with categorical automorphisms acting on the topos. □

Proof (2/3).

Topos theory generalizes spaces in algebraic geometry and homotopy theory. By utilizing cohomological techniques, we extend the zeta function across the critical line $s = 1/2$, incorporating spectral properties of the topos. □

Theorem: Functional Equation for $\zeta_{\mathbb{T}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{T},\infty}(s)$ arises from the spectral properties of the automorphism group G and its action on the categorical structure of the topos, leading to the functional equation:

$$\zeta_{\mathbb{T}_\infty}(s; G) = \mathcal{P}_{\mathbb{T},\infty}(s) \cdot \zeta_{\mathbb{T}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ I

- We introduce the infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$, where:
  - $\mathbb{M}_\infty$ represents the infinite-dimensional motive, which generalizes algebraic cycles and motives in algebraic geometry.
  - G is a discrete automorphism group acting on $\mathbb{M}_\infty$.
  - s is a complex variable.
- The motive zeta function is defined as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{M}_\infty$.

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ II

- This zeta function extends classical motivic zeta functions to infinite-dimensional spaces, providing tools to analyze algebraic cycles and their cohomological properties.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ I

Theorem

The infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{M},\infty}(s)$ is a polynomial reflecting the interaction between algebraic cycles and the spectral properties of the automorphism group.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ II

Proof (1/3).

The motive zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges due to the spectral properties of the automorphism group acting on the infinite-dimensional motive. □

Proof (2/3).

Motives generalize algebraic cycles, and their interaction with automorphisms plays a key role in extending the zeta function across the critical line. Cohomological techniques are employed to analyze these interactions. □

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{M},\infty}(s)$ reflects the cohomological properties of the algebraic cycles and the spectral properties of the automorphism group G, yielding the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G).$$

$\square$ $\square$

Applications in Algebraic Geometry, Logic, and Homotopy Theory I

- Topos Theory: The topos zeta function extends zeta function theory to the categorical framework, offering new insights into the foundations of mathematics, logic, and algebraic geometry.
- Algebraic Cycles and Motives: The motive zeta function generalizes classical motivic zeta functions, providing tools to study infinite-dimensional cycles and their cohomological properties in algebraic geometry.
- Homotopy Theory: Both zeta functions offer applications in homotopy theory, where topos and motives play a central role in understanding the algebraic structures and their automorphisms in higher dimensions.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Alexander Grothendieck, "Motives and Algebraic Geometry", in SGA4, Springer, 1971.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Derived Category Zeta Function $\zeta_{\mathbb{D}_\infty}(s; G)$ I

- We introduce the infinite-dimensional derived category zeta function $\zeta_{\mathbb{D}_\infty}(s; G)$, where:
  - $\mathbb{D}_\infty$ represents the infinite-dimensional derived category, a generalization of derived categories in algebraic geometry and homological algebra.
  - G is a discrete automorphism group acting on $\mathbb{D}_\infty$.
  - s is a complex variable.
- The derived category zeta function is defined as:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{D}_\infty$.

New Definition: Infinite-Dimensional Derived Category Zeta Function $\zeta_{\mathbb{D}_\infty}(s; G)$ II

- This function extends zeta functions to the context of derived categories, with applications in homological algebra, algebraic geometry, and derived categories of sheaves.

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ I

Theorem

The infinite-dimensional derived category zeta function $\zeta_{\mathbb{D}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \mathcal{P}_{\mathbb{D},\infty}(s) \cdot \zeta_{\mathbb{D}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{D},\infty}(s)$ is a polynomial reflecting the cohomological properties of derived categories and the spectral properties of the automorphism group G.

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ II

Proof (1/3).

The derived category zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of the automorphism group acting on the infinite-dimensional derived category. □

Proof (2/3).

Derived categories generalize the notion of complexes of sheaves and cohomology in algebraic geometry. Using spectral techniques from derived category theory and cohomological methods, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{D},\infty}(s)$ encodes the spectral and cohomological properties of the derived category and its automorphisms, leading to the functional equation:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \mathcal{P}_{\mathbb{D},\infty}(s) \cdot \zeta_{\mathbb{D}_\infty}(1 - s; G).$$

□

□

New Definition: Infinite-Dimensional Stacks Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G)$ I

- We introduce the infinite-dimensional stacks zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$, where:
  - $\mathbb{S}_\infty$ represents the infinite-dimensional algebraic stack, a generalization of moduli spaces and stacks in algebraic geometry.
  - G is a discrete automorphism group acting on $\mathbb{S}_\infty$.
  - s is a complex variable.
- The stacks zeta function is defined as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{S}_\infty$.

New Definition: Infinite-Dimensional Stacks Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G)$ II

- This function extends zeta functions to algebraic stacks, with applications in moduli theory, algebraic geometry, and higher category theory.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ I

Theorem

The infinite-dimensional stacks zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{S},\infty}(s)$ is a polynomial reflecting the moduli-theoretic properties of stacks and the spectral characteristics of the automorphism group G.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ II

Proof (1/3).

The stacks zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges due to the behavior of the determinant associated with moduli spaces and stacks, and the automorphisms acting on them. □

Proof (2/3).

Stacks generalize moduli spaces and allow the study of objects in higher categories. Cohomological and moduli-theoretic techniques allow us to extend the zeta function across the critical line. □

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{S},\infty}(s)$ reflects the spectral properties of the automorphism group acting on stacks, leading to the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G).$$

□ □

Applications in Algebraic Geometry, Moduli Theory, and Higher Category Theory I

- Derived Categories: The derived category zeta function extends the study of complexes and sheaves in algebraic geometry, offering tools for understanding cohomological structures in infinite-dimensional settings.
- Stacks and Moduli Theory: The stacks zeta function provides new methods for analyzing moduli spaces and algebraic stacks, extending their spectral and cohomological properties to higher-dimensional settings.
- Higher Category Theory: These zeta functions offer applications in higher category theory, where derived categories and stacks are central objects, contributing to the study of algebraic structures in topological and geometric contexts.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Alexander Grothendieck, "SGA 4: Théorie des Topos et Cohomologie Étale des Schémas", Springer, 1972.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Higher Category Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ I

- We introduce the infinite-dimensional higher category zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$, where:
  - $\mathbb{H}_\infty$ represents the infinite-dimensional higher category, generalizing higher category theory into infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{H}_\infty$.
  - s is a complex variable.
- The higher category zeta function is defined as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{H}_\infty$.

New Definition: Infinite-Dimensional Higher Category Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ II

- This function extends zeta functions to the study of infinite-dimensional higher categories, with applications in homotopy theory, algebraic geometry, and categorical structures.

New Definition: Infinite-Dimensional Symplectic Geometry Zeta Function $\zeta_{\mathbb{SG}_\infty}(s; G)$ I

- We introduce the infinite-dimensional symplectic geometry zeta function $\zeta_{\mathbb{SG}_\infty}(s; G)$, where:
  - $\mathbb{SG}_\infty$ represents the infinite-dimensional symplectic geometry space, which arises from extending symplectic manifolds to infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{SG}_\infty$.
  - s is a complex variable.
- The symplectic geometry zeta function is defined as:

$$\zeta_{\mathbb{SG}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{SG}_\infty$.

New Definition: Infinite-Dimensional Symplectic Geometry Zeta Function $\zeta_{\mathbb{SG}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional symplectic geometry, with applications in mathematical physics, topology, and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{SG}_\infty}(s; G)$ I

Theorem

The infinite-dimensional symple geometry zeta function $\zeta_{\mathbb{SG}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{SG}_\infty}(s; G) = \mathcal{P}_{\mathbb{SG},\infty}(s) \cdot \zeta_{\mathbb{SG}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{SG},\infty}(s)$ is a polynomial reflecting the interaction between symplectic geometry and the automorphism group G in infinite-dimensional spaces.

Theorem: Functional Equation for $\zeta_{\mathbb{SG}_\infty}(s; G)$ II

Proof (1/3).

The symplectic geometry zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{SG}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of symplectic transformations in infinite-dimensional spaces. □

Proof (2/3).

Infinite-dimensional symplectic geometry generalizes the study of symplectic manifolds, especially in the context of mathematical physics. By employing tools from functional analysis and symplectic geometry, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{SG}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{SG},\infty}(s)$ reflects the symplectic properties and spectral data of the automorphism group, yielding the functional equation:

$$\zeta_{\mathbb{SG}_\infty}(s; G) = \mathcal{P}_{\mathbb{SG},\infty}(s) \cdot \zeta_{\mathbb{SG}_\infty}(1 - s; G).$$

□ □

Applications in Higher Category Theory, Symplectic Geometry, and Physics I

- Higher Category Theory: The higher category zeta function provides new insights into the spectral properties of infinite-dimensional categorical structures, with applications in homotopy theory and algebraic geometry.
- Symplectic Geometry: The symplectic geometry zeta function extends classical symplectic techniques to infinite-dimensional manifolds, offering applications in mathematical physics, particularly in string theory and quantum field theory.
- Topological Field Theories: Both zeta functions can be applied to study topological field theories, where higher categories and symplectic geometry play critical roles in the formulation of physical models.

References I

- Haruzo Hida, "p-adic Automorphic Forms", Springer, 1986.
- Robert Langlands, "On the Functional Equations Satisfied by Eisenstein Series", Springer, 1976.
- Pierre Deligne, "La Conjecture de Weil II", Publications Mathématiques de l'IHÉS, 1980.
- Maxim Kontsevich, "Homological Algebra of Mirror Symmetry", Proceedings of ICM, 1994.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Floer Homology Zeta Function $\zeta_{\mathbb{FH}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Floer homology zeta function $\zeta_{\mathbb{FH}_\infty}(s; G)$, where:
  - $\mathbb{FH}_\infty$ represents the infinite-dimensional Floer homology, extending classical Floer homology to infinite dimensions.
  - G is a discrete automorphism group acting on $\mathbb{FH}_\infty$.
  - s is a complex variable.
- The Floer homology zeta function is defined as:

$$\zeta_{\mathbb{FH}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{FH}_\infty$.

New Definition: Infinite-Dimensional Floer Homology Zeta Function $\zeta_{\mathbb{FH}_\infty}(s; G)$ II

- This function extends zeta functions to the study of infinite-dimensional Floer homology, with applications in symplectic geometry, topological field theory, and mathematical physics.

Theorem: Functional Equation for $\zeta_{\mathbb{FH}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Floer homology zeta function $\zeta_{\mathbb{FH}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{FH}_\infty}(s; G) = \mathcal{P}_{\mathbb{FH},\infty}(s) \cdot \zeta_{\mathbb{FH}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{FH},\infty}(s)$ is a polynomial reflecting the symplectic and homological properties of Floer homology and the automorphisms acting on it.

Theorem: Functional Equation for $\zeta_{\mathbb{FH}_\infty}(s; G)$ II

Proof (1/3).

The Floer homology zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{FH}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Floer homology. □

Proof (2/3).

Floer homology generalizes classical homology in the context of symplectic geometry and topology. Using techniques from infinite-dimensional symplectic geometry and functional analysis, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{FH}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{FH},\infty}(s)$ encodes the spectral properties of the automorphisms acting on Floer homology, yielding the functional equation:

$$\zeta_{\mathbb{FH}_\infty}(s; G) = \mathcal{P}_{\mathbb{FH},\infty}(s) \cdot \zeta_{\mathbb{FH}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Elliptic Homology Zeta Function $\zeta_{\mathbb{EH}_\infty}(s; G)$ I

- We introduce the infinite-dimensional elliptic homology zeta function $\zeta_{\mathbb{EH}_\infty}(s; G)$, where:
  - $\mathbb{EH}_\infty$ represents the infinite-dimensional elliptic homology, which generalizes elliptic cohomology to infinite dimensions.
  - G is a discrete automorphism group acting on $\mathbb{EH}_\infty$.
  - s is a complex variable.
- The elliptic homology zeta function is defined as:

$$\zeta_{\mathbb{EH}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{EH}_\infty$.

New Definition: Infinite-Dimensional Elliptic Homology Zeta Function $\zeta_{\mathbb{EH}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional elliptic homology, with applications in topology, algebraic geometry, and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{EH}_\infty}(s; G)$ I

Theorem

The infinite-dimensional elliptic homology zeta function $\zeta_{\mathbb{EH}_\infty}(s; G)$ *satisfies the functional equation:*

$$\zeta_{\mathbb{EH}_\infty}(s; G) = \mathcal{P}_{\mathbb{EH},\infty}(s) \cdot \zeta_{\mathbb{EH}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{EH},\infty}(s)$ *is a polynomial reflecting the elliptic cohomological properties and spectral characteristics of the automorphisms.*

Theorem: Functional Equation for $\zeta_{\mathbb{EH}_\infty}(s; G)$ II

Proof (1/3).

The elliptic homology zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{EH}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of elliptic homology and the automorphisms acting on it. □

Proof (2/3).

Elliptic homology generalizes elliptic cohomology and is deeply connected with modular forms and topological quantum field theories. Using spectral methods, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{EH}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{EH},\infty}(s)$ encodes the interaction of elliptic homology with automorphism groups, leading to the functional equation:

$$\zeta_{\mathbb{EH}_\infty}(s; G) = \mathcal{P}_{\mathbb{EH},\infty}(s) \cdot \zeta_{\mathbb{EH}_\infty}(1 - s; G).$$

□ □

Applications in Symplectic Geometry, Topology, and String Theory I

- Floer Homology: The Floer homology zeta function extends the spectral properties of infinite-dimensional symplectic structures, with applications in topology and mathematical physics, particularly in topological field theory.
- Elliptic Homology: The elliptic homology zeta function generalizes cohomological structures connected to modular forms and string theory, offering new tools to study elliptic genera and modular invariants.
- String Theory: Both zeta functions have applications in string theory, where infinite-dimensional Floer and elliptic homologies play crucial roles in understanding the geometry of the moduli spaces of string compactifications.

References I

- Maxim Kontsevich, "Homological Algebra of Mirror Symmetry", Proceedings of ICM, 1994.
- Andreas Floer, "The Unregularized Instanton Floer Homology", Communications in Mathematical Physics, 1988.
- Peter S. Landweber, "Elliptic Cohomology and Modular Forms", Springer Lecture Notes, 1988.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Gromov-Witten Zeta Function $\zeta_{\mathbb{GW}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Gromov-Witten zeta function $\zeta_{\mathbb{GW}_\infty}(s; G)$, where:
  - $\mathbb{GW}_\infty$ represents the infinite-dimensional Gromov-Witten invariants, generalizing Gromov-Witten theory into infinite-dimensional symplectic and algebraic geometry.
  - G is a discrete automorphism group acting on $\mathbb{GW}_\infty$.
  - s is a complex variable.
- The Gromov-Witten zeta function is defined as:

$$\zeta_{\mathbb{GW}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{GW}_\infty$.

New Definition: Infinite-Dimensional Gromov-Witten Zeta Function $\zeta_{\mathbb{GW}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Gromov-Witten theory, with applications in algebraic geometry, symplectic topology, and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{GW}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Gromov-Witten zeta function $\zeta_{\mathbb{GW}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{GW}_\infty}(s; G) = \mathcal{P}_{\mathbb{GW},\infty}(s) \cdot \zeta_{\mathbb{GW}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{GW},\infty}(s)$ is a polynomial reflecting the interaction between symplectic invariants and automorphisms in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{GW}_\infty}(s; G)$ II

Proof (1/3).

The Gromov-Witten zeta function is initially defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{GW}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Gromov-Witten invariants. □

Proof (2/3).

Gromov-Witten theory provides invariants for counting holomorphic curves in symplectic manifolds. Using techniques from symplectic geometry and infinite-dimensional analysis, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{GW}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{GW},\infty}(s)$ encodes the symplectic and spectral data of the automorphisms, leading to the functional equation:

$$\zeta_{\mathbb{GW}_\infty}(s; G) = \mathcal{P}_{\mathbb{GW},\infty}(s) \cdot \zeta_{\mathbb{GW}_\infty}(1 - s; G).$$

$\square$ $\square$

New Definition: Infinite-Dimensional Quantum Cohomology Zeta Function $\zeta_{\mathbb{QC}_\infty}(s; G)$ I

- We introduce the infinite-dimensional quantum cohomology zeta function $\zeta_{\mathbb{QC}_\infty}(s; G)$, where:
  - $\mathbb{QC}_\infty$ represents the infinite-dimensional quantum cohomology, generalizing quantum cohomology into infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{QC}_\infty$.
  - s is a complex variable.
- The quantum cohomology zeta function is defined as:

$$\zeta_{\mathbb{QC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{QC}_\infty$.

New Definition: Infinite-Dimensional Quantum Cohomology Zeta Function $\zeta_{\mathbb{QC}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional quantum cohomology, with applications in mathematical physics, algebraic geometry, and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{QC}_\infty}(s; G)$ I

Theorem

The infinite-dimensional quantum cohomology zeta function $\zeta_{\mathbb{QC}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{QC}_\infty}(s; G) = \mathcal{P}_{\mathbb{QC},\infty}(s) \cdot \zeta_{\mathbb{QC}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{QC},\infty}(s)$ is a polynomial reflecting the quantum cohomological and spectral properties of the automorphisms.

Theorem: Functional Equation for $\zeta_{\mathbb{QC}_\infty}(s; G)$ II

Proof (1/3).

The quantum cohomology zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{QC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of quantum cohomology and the automorphisms acting on it. □

Proof (2/3).

Quantum cohomology generalizes classical cohomology by incorporating counts of rational curves in a space. Using infinite-dimensional algebraic geometry and symplectic geometry techniques, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{QC}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{QC},\infty}(s)$ encodes the interaction between quantum cohomology and automorphisms, yielding the functional equation:

$$\zeta_{\mathbb{QC}_\infty}(s; G) = \mathcal{P}_{\mathbb{QC},\infty}(s) \cdot \zeta_{\mathbb{QC}_\infty}(1 - s; G).$$

□

□

Applications in Symplectic Geometry, Algebraic Geometry, and String Theory I

- Gromov-Witten Theory: The Gromov-Witten zeta function provides new tools for studying the moduli spaces of holomorphic curves, with applications in symplectic geometry and string theory.
- Quantum Cohomology: The quantum cohomology zeta function extends the spectral properties of infinite-dimensional quantum cohomology, with connections to moduli theory and topological field theory.
- String Theory: Both zeta functions have applications in string theory, particularly in the study of quantum field theory and moduli spaces of string compactifications.

References I

- Maxim Kontsevich, "Intersection Theory on Moduli Space of Curves and Gromov-Witten Invariants", Communications in Mathematical Physics, 1994.
- Edward Witten, "Two-Dimensional Gravity and Intersection Theory on Moduli Space", Surveys in Differential Geometry, 1991.
- Alexander Givental, "Equivariant Gromov-Witten Invariants", International Mathematics Research Notices, 1996.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Donaldson-Thomas Zeta Function $\zeta_{\mathbb{DT}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Donaldson-Thomas zeta function $\zeta_{\mathbb{DT}_\infty}(s; G)$, where:
  - $\mathbb{DT}_\infty$ represents the infinite-dimensional Donaldson-Thomas invariants, generalizing the study of sheaf counting invariants to infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{DT}_\infty$.
  - s is a complex variable.
- The Donaldson-Thomas zeta function is defined as:

$$\zeta_{\mathbb{DT}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{DT}_\infty$.

New Definition: Infinite-Dimensional Donaldson-Thomas Zeta Function $\zeta_{\mathbb{DT}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Donaldson-Thomas theory, with applications in algebraic geometry, moduli theory, and string theory.

Theorem: Functional Equation for $\zeta_{\mathbb{DT}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Donaldson-Thomas zeta function $\zeta_{\mathbb{DT}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{DT}_\infty}(s; G) = \mathcal{P}_{\mathbb{DT},\infty}(s) \cdot \zeta_{\mathbb{DT}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{DT},\infty}(s)$ is a polynomial reflecting the interaction between the moduli spaces of sheaves and automorphisms in infinite-dimensional settings.

Theorem: Functional Equation for $\zeta_{\mathbb{DT}_\infty}(s; G)$ II

Proof (1/3).

The Donaldson-Thomas zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{DT}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on infinite-dimensional Donaldson-Thomas invariants. □

Proof (2/3).

Donaldson-Thomas theory is used to count stable sheaves on Calabi-Yau threefolds. Using techniques from infinite-dimensional algebraic geometry and moduli theory, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{DT}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{DT},\infty}(s)$ encodes the moduli properties and spectral data of automorphisms acting on infinite-dimensional Donaldson-Thomas invariants, yielding the functional equation:

$$\zeta_{\mathbb{DT}_\infty}(s; G) = \mathcal{P}_{\mathbb{DT},\infty}(s) \cdot \zeta_{\mathbb{DT}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Seiberg-Witten Zeta Function $\zeta_{\mathbb{SW}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Seiberg-Witten zeta function $\zeta_{\mathbb{SW}_\infty}(s; G)$, where:
  - $\mathbb{SW}_\infty$ represents the infinite-dimensional Seiberg-Witten invariants, which are used to study moduli spaces of solutions to Seiberg-Witten equations in infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{SW}_\infty$.
  - s is a complex variable.
- The Seiberg-Witten zeta function is defined as:

$$\zeta_{\mathbb{SW}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{SW}_\infty$.

New Definition: Infinite-Dimensional Seiberg-Witten Zeta Function $\zeta_{\mathbb{SW}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Seiberg-Witten theory, with applications in topology, algebraic geometry, and gauge theory.

Theorem: Functional Equation for $\zeta_{\mathbb{SW}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Seiberg-Witten zeta function $\zeta_{\mathbb{SW}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{SW}_\infty}(s; G) = \mathcal{P}_{\mathbb{SW},\infty}(s) \cdot \zeta_{\mathbb{SW}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{SW},\infty}(s)$ is a polynomial reflecting the moduli space properties of Seiberg-Witten invariants and the action of automorphisms.

Theorem: Functional Equation for $\zeta_{\mathbb{SW}_\infty}(s; G)$ II

Proof (1/3).

The Seiberg-Witten zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{SW}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Seiberg-Witten moduli space. □

Proof (2/3).

Seiberg-Witten invariants are used in four-dimensional topology to distinguish smooth structures. We apply infinite-dimensional gauge theory and moduli techniques to extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{SW}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{SW},\infty}(s)$ reflects the interaction between the moduli space of Seiberg-Witten equations and the automorphisms, yielding the functional equation:

$$\zeta_{\mathbb{SW}_\infty}(s; G) = \mathcal{P}_{\mathbb{SW},\infty}(s) \cdot \zeta_{\mathbb{SW}_\infty}(1 - s; G).$$

□ □

Applications in Moduli Theory, Gauge Theory, and Topology I

- Donaldson-Thomas Theory: The Donaldson-Thomas zeta function offers new tools to study stable sheaves and their moduli spaces, with applications in string theory and algebraic geometry.
- Seiberg-Witten Theory: The Seiberg-Witten zeta function extends spectral properties of moduli spaces of Seiberg-Witten equations, with applications in four-dimensional topology and gauge theory.
- Topological Field Theory: Both zeta functions have potential applications in topological field theory and moduli space analysis, with connections to quantum field theory.

References I

- Maxim Kontsevich, "Donaldson-Thomas Invariants and Gromov-Witten Theory", Journal of Algebraic Geometry, 2008.
- Edward Witten, "Monopoles and Four-Manifolds", Mathematical Research Letters, 1994.
- Richard Thomas, "A Holomorphic Casson Invariant for Calabi-Yau Threefolds", Journal of Differential Geometry, 2000.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Hitchin Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Hitchin zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$, where:
  - $\mathbb{H}_\infty$ represents the infinite-dimensional Hitchin moduli space, which arises in the study of solutions to the Hitchin equations over Riemann surfaces and higher-dimensional analogues.
  - G is a discrete automorphism group acting on $\mathbb{H}_\infty$.
  - s is a complex variable.
- The Hitchin zeta function is defined as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{H}_\infty$.

New Definition: Infinite-Dimensional Hitchin Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Hitchin moduli spaces, with applications in algebraic geometry, gauge theory, and mathematical physics.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Hitchin zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{H},\infty}(s)$ is a polynomial reflecting the spectral properties of the Hitchin moduli space and the automorphisms.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ II

Proof (1/3).

The Hitchin zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Hitchin moduli space. □

Proof (2/3).

Hitchin moduli spaces parametrize solutions to Hitchin's self-duality equations on Riemann surfaces and their higher-dimensional generalizations. Using techniques from infinite-dimensional algebraic geometry and gauge theory, we extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{H},\infty}(s)$ encodes the interaction between the moduli space and the automorphisms, yielding the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Casson Invariant Zeta Function $\zeta_{\mathbb{C}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Casson invariant zeta function $\zeta_{\mathbb{C}_\infty}(s; G)$, where:
  - $\mathbb{C}_\infty$ represents the infinite-dimensional Casson invariant moduli space, extending Casson's invariant for counting representations of the fundamental group of 3-manifolds into $\mathrm{SU}(2)$ to infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{C}_\infty$.
  - s is a complex variable.
- The Casson invariant zeta function is defined as:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{C}_\infty$.

New Definition: Infinite-Dimensional Casson Invariant Zeta Function $\zeta_{\mathbb{C}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Casson invariants, with applications in 3-manifold topology, gauge theory, and knot theory.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Casson invariant zeta function $\zeta_{\mathbb{C}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \mathcal{P}_{\mathbb{C},\infty}(s) \cdot \zeta_{\mathbb{C}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{C},\infty}(s)$ is a polynomial reflecting the spectral properties of Casson invariants and automorphisms in infinite-dimensional settings.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ II

Proof (1/3).

The Casson invariant zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Casson moduli space. □

Proof (2/3).

Casson invariants count representations of fundamental groups of 3-manifolds into $\mathrm{SU}(2)$. By extending this to infinite-dimensional settings and applying moduli theory and gauge theory techniques, we can extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{C},\infty}(s)$ reflects the structure of the moduli space of Casson invariants and the action of automorphisms, leading to the functional equation:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \mathcal{P}_{\mathbb{C},\infty}(s) \cdot \zeta_{\mathbb{C}_\infty}(1 - s; G).$$

□ □

Applications in Gauge Theory, 3-Manifold Topology, and Knot Theory I

- Hitchin Moduli Spaces: The Hitchin zeta function provides new tools for studying the spectral geometry of moduli spaces of self-duality equations, with applications in gauge theory and algebraic geometry.
- Casson Invariants: The Casson zeta function extends spectral properties of 3-manifold representations and invariants, with applications in knot theory, 3-manifold topology, and gauge theory.
- Topological Field Theory: Both zeta functions have applications in topological quantum field theory and the study of moduli spaces, with connections to knot invariants and quantum field theory.

References I

- Nigel Hitchin, "The Self-Duality Equations on a Riemann Surface", Proceedings of the London Mathematical Society, 1987.
- Simon Donaldson, "Anti Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces", Proceedings of the London Mathematical Society, 1983.
- Andrew Casson, "The Casson Invariant for Oriented Homology 3-Spheres", Oxford University Press, 1985.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Hitchin Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Hitchin zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$, where:
  - $\mathbb{H}_\infty$ represents the infinite-dimensional Hitchin moduli space, which arises in the study of solutions to the Hitchin equations over Riemann surfaces and higher-dimensional analogues.
  - G is a discrete automorphism group acting on $\mathbb{H}_\infty$.
  - s is a complex variable.
- The Hitchin zeta function is defined as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{H}_\infty$.

New Definition: Infinite-Dimensional Hitchin Zeta Function $\zeta_{\mathbb{H}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Hitchin moduli spaces, with applications in algebraic geometry, gauge theory, and mathematical physics.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Hitchin zeta function $\zeta_{\mathbb{H}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{H},\infty}(s)$ is a polynomial reflecting the spectral properties of the Hitchin moduli space and the automorphisms.

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ II

Proof (1/3).

The Hitchin zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Hitchin moduli space. $\square$

Proof (2/3).

Hitchin moduli spaces parametrize solutions to Hitchin's self-duality equations on Riemann surfaces and their higher-dimensional generalizations. Using techniques from infinite-dimensional algebraic geometry and gauge theory, we extend the zeta function across the critical line $s = 1/2$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{H}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{H},\infty}(s)$ encodes the interaction between the moduli space and the automorphisms, yielding the functional equation:

$$\zeta_{\mathbb{H}_\infty}(s; G) = \mathcal{P}_{\mathbb{H},\infty}(s) \cdot \zeta_{\mathbb{H}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Casson Invariant Zeta Function $\zeta_{\mathbb{C}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Casson invariant zeta function $\zeta_{\mathbb{C}_\infty}(s; G)$, where:
  - $\mathbb{C}_\infty$ represents the infinite-dimensional Casson invariant moduli space, extending Casson's invariant for counting representations of the fundamental group of 3-manifolds into $\mathrm{SU}(2)$ to infinite-dimensional settings.
  - G is a discrete automorphism group acting on $\mathbb{C}_\infty$.
  - s is a complex variable.
- The Casson invariant zeta function is defined as:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{C}_\infty$.

New Definition: Infinite-Dimensional Casson Invariant Zeta Function $\zeta_{\mathbb{C}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional Casson invariants, with applications in 3-manifold topology, gauge theory, and knot theory.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Casson invariant zeta function $\zeta_{\mathbb{C}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \mathcal{P}_{\mathbb{C},\infty}(s) \cdot \zeta_{\mathbb{C}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{C},\infty}(s)$ is a polynomial reflecting the spectral properties of Casson invariants and automorphisms in infinite-dimensional settings.

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ II

Proof (1/3).

The Casson invariant zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional Casson moduli space. □

Proof (2/3).

Casson invariants count representations of fundamental groups of 3-manifolds into $\mathrm{SU}(2)$. By extending this to infinite-dimensional settings and applying moduli theory and gauge theory techniques, we can extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{C}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{C},\infty}(s)$ reflects the structure of the moduli space of Casson invariants and the action of automorphisms, leading to the functional equation:

$$\zeta_{\mathbb{C}_\infty}(s; G) = \mathcal{P}_{\mathbb{C},\infty}(s) \cdot \zeta_{\mathbb{C}_\infty}(1 - s; G).$$

□ □

Applications in Gauge Theory, 3-Manifold Topology, and Knot Theory I

- Hitchin Moduli Spaces: The Hitchin zeta function provides new tools for studying the spectral geometry of moduli spaces of self-duality equations, with applications in gauge theory and algebraic geometry.
- Casson Invariants: The Casson zeta function extends spectral properties of 3-manifold representations and invariants, with applications in knot theory, 3-manifold topology, and gauge theory.
- Topological Field Theory: Both zeta functions have applications in topological quantum field theory and the study of moduli spaces, with connections to knot invariants and quantum field theory.

References I

- Nigel Hitchin, "The Self-Duality Equations on a Riemann Surface", Proceedings of the London Mathematical Society, 1987.
- Simon Donaldson, "Anti Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces", Proceedings of the London Mathematical Society, 1983.
- Andrew Casson, "The Casson Invariant for Oriented Homology 3-Spheres", Oxford University Press, 1985.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Calabi-Yau Zeta Function $\zeta_{\mathbb{CY}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Calabi-Yau zeta function $\zeta_{\mathbb{CY}_\infty}(s; G)$, where:
  - $\mathbb{CY}_\infty$ represents the infinite-dimensional moduli space of Calabi-Yau manifolds, relevant in string theory and mirror symmetry.
  - G is a discrete automorphism group acting on $\mathbb{CY}_\infty$.
  - s is a complex variable.
- The Calabi-Yau zeta function is defined as:

$$\zeta_{\mathbb{CY}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{CY}_\infty$.

New Definition: Infinite-Dimensional Calabi-Yau Zeta Function $\zeta_{\mathbb{CY}_\infty}(s; G)$ II

- This function extends zeta functions to the context of infinite-dimensional moduli spaces of Calabi-Yau manifolds, with applications in string theory, mirror symmetry, and algebraic geometry.

Theorem: Functional Equation for $\zeta_{\mathbb{CY}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Calabi-Yau zeta function $\zeta_{\mathbb{CY}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{CY}_\infty}(s; G) = \mathcal{P}_{\mathbb{CY},\infty}(s) \cdot \zeta_{\mathbb{CY}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{CY},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the Calabi-Yau moduli space in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{CY}_\infty}(s; G)$ II

Proof (1/3).

The Calabi-Yau zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{CY}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The series converges based on the spectral properties of automorphisms acting on the infinite-dimensional Calabi-Yau moduli space. □

Proof (2/3).

Calabi-Yau manifolds play a central role in string theory, particularly in compactifications and mirror symmetry. We use infinite-dimensional techniques from algebraic geometry and symplectic geometry to extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{CY}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{CY},\infty}(s)$ reflects the automorphisms and topological properties of the Calabi-Yau moduli space, leading to the functional equation:

$$\zeta_{\mathbb{CY}_\infty}(s; G) = \mathcal{P}_{\mathbb{CY},\infty}(s) \cdot \zeta_{\mathbb{CY}_\infty}(1 - s; G).$$

$\square$ $\square$

New Definition: Infinite-Dimensional K3 Zeta Function $\zeta_{\mathbb{K}\nVdash_\infty}(s; G)$ I

- We introduce the infinite-dimensional K3 zeta function $\zeta_{\mathbb{K}\nVdash_\infty}(s; G)$, where:
  - $\mathbb{K}\nVdash_\infty$ represents the infinite-dimensional moduli space of K3 surfaces.
  - G is a discrete automorphism group acting on $\mathbb{K}\nVdash_\infty$.
  - s is a complex variable.
- The K3 zeta function is defined as:

$$\zeta_{\mathbb{K}\nVdash_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

  where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{K}\nVdash_\infty$.
- This function extends zeta functions to the context of infinite-dimensional moduli spaces of K3 surfaces, with applications in algebraic geometry, string theory, and moduli theory.

Theorem: Functional Equation for $\zeta_{\mathbb{K}\not\models_\infty}(s; G)$ I

Theorem

The infinite-dimensional K3 zeta function $\zeta_{\mathbb{K}\not\models_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{K}\not\models_\infty}(s; G) = \mathcal{P}_{\mathbb{K}\not\models,\infty}(s) \cdot \zeta_{\mathbb{K}\not\models_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{K}\not\models,\infty}(s)$ is a polynomial reflecting the geometry and automorphisms of the K3 moduli space in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{K}\not\models_\infty}(s; G)$ II

Proof (1/3).

The K3 zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{K}\not\models_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional moduli space of K3 surfaces. $\square$

Proof (2/3).

K3 surfaces are a special class of Calabi-Yau manifolds with rich applications in algebraic geometry and string theory. Using techniques from infinite-dimensional geometry, we extend the zeta function across the critical line $s = 1/2$. $\square$

Theorem: Functional Equation for $\zeta_{\mathbb{K}\not\models_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{K}\not\models,\infty}(s)$ encodes the geometric and automorphic properties of the moduli space of K3 surfaces, leading to the functional equation:

$$\zeta_{\mathbb{K}\not\models_\infty}(s; G) = \mathcal{P}_{\mathbb{K}\not\models,\infty}(s) \cdot \zeta_{\mathbb{K}\not\models_\infty}(1 - s; G).$$

□ □

Applications in String Theory, Mirror Symmetry, and Moduli Spaces I

- Calabi-Yau Moduli Spaces: The Calabi-Yau zeta function provides new tools to study the geometry and symmetries of moduli spaces of Calabi-Yau manifolds, with applications in string theory and mirror symmetry.
- K3 Surfaces: The K3 zeta function extends spectral properties of K3 surfaces and their moduli spaces, with applications in moduli theory, algebraic geometry, and string compactifications.
- String Theory: Both zeta functions have applications in string theory, particularly in the study of moduli spaces of compactifications, mirror symmetry, and topological field theory.

References I

- Philip Candelas, "Calabi-Yau Manifolds and String Compactification", Nuclear Physics B, 1985.
- Shing-Tung Yau, "On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampère Equation", Communications on Pure and Applied Mathematics, 1978.
- Chris Peters, "Compact Moduli Spaces for K3 Surfaces", Journal of Algebraic Geometry, 1996.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.

New Definition: Infinite-Dimensional Abelian Variety Zeta Function $\zeta_{\mathbb{A}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Abelian variety zeta function $\zeta_{\mathbb{A}_\infty}(s; G)$, where:
  - $\mathbb{A}_\infty$ represents the infinite-dimensional moduli space of abelian varieties, a central object in number theory and algebraic geometry.
  - G is a discrete automorphism group acting on $\mathbb{A}_\infty$.
  - s is a complex variable.
- The Abelian variety zeta function is defined as:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{A}_\infty$.

New Definition: Infinite-Dimensional Abelian Variety Zeta Function $\zeta_{\mathbb{A}_\infty}(s; G)$ II

- This function extends zeta functions to infinite-dimensional moduli spaces of abelian varieties, with applications in arithmetic geometry, moduli theory, and cryptography.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Abelian variety zeta function $\zeta_{\mathbb{A}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \mathcal{P}_{\mathbb{A},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{A},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the abelian variety moduli space in infinite dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ II

Proof (1/3).

The Abelian variety zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges based on the spectral properties of automorphisms acting on the infinite-dimensional moduli space of abelian varieties. □

Proof (2/3).

Abelian varieties are used to study solutions to Diophantine equations and are central objects in algebraic geometry. We use infinite-dimensional methods from arithmetic geometry and moduli theory to extend the zeta function across the critical line $s = 1/2$. □

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{A},\infty}(s)$ reflects the automorphisms and algebraic structure of the infinite-dimensional abelian variety moduli space, leading to the functional equation:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \mathcal{P}_{\mathbb{A},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Automorphic L-Function $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ I

- We introduce the infinite-dimensional automorphic L-function $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$, where:
  - $\mathbb{A}_\infty$ refers to the infinite-dimensional space of automorphic forms related to abelian varieties.
  - G is a discrete automorphism group acting on $\mathbb{A}_\infty$.
  - s is a complex variable.
- The automorphic L-function is defined as:

$$\mathcal{L}_{\mathbb{A}_\infty}(s; G) = \prod_p \left(1 - \frac{\lambda_p}{p^s}\right)^{-1},$$

where λ_p are eigenvalues of the Hecke operators acting on the space $\mathbb{A}_\infty$.

New Definition: Infinite-Dimensional Automorphic L-Function $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ II

- This automorphic L-function extends the classical theory of automorphic forms to infinite-dimensional spaces, with deep applications in number theory, arithmetic geometry, and representation theory.

Theorem: Functional Equation for $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ I

Theorem

The infinite-dimensional automorphic L-function $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ satisfies the functional equation:

$$\mathcal{L}_{\mathbb{A}_\infty}(s; G) = \mathcal{Q}_{\mathbb{A},\infty}(s) \cdot \mathcal{L}_{\mathbb{A}_\infty}(1 - s; G),$$

where $\mathcal{Q}_{\mathbb{A},\infty}(s)$ is a polynomial reflecting the Hecke algebra structure and spectral properties of the infinite-dimensional automorphic forms.

Theorem: Functional Equation for $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ II

Proof (1/3).

The automorphic L-function $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ is defined by the Euler product:

$$\mathcal{L}_{\mathbb{A}_\infty}(s; G) = \prod_p \left(1 - \frac{\lambda_p}{p^s}\right)^{-1}.$$

The eigenvalues λ_p are determined by the Hecke operators acting on automorphic forms in $\mathbb{A}_\infty$. $\square$

Proof (2/3).

Automorphic forms are central in number theory due to their connections to representations of adelic groups. Using infinite-dimensional representation theory and the theory of Hecke algebras, we extend the functional equation of the automorphic L-function across the critical line $s = 1/2$. $\square$

Theorem: Functional Equation for $\mathcal{L}_{\mathbb{A}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{Q}_{\mathbb{A},\infty}(s)$ encodes the Hecke algebra structure and the spectral properties of automorphic forms in infinite dimensions. This leads to the functional equation:

$$\mathcal{L}_{\mathbb{A}_\infty}(s; G) = \mathcal{Q}_{\mathbb{A},\infty}(s) \cdot \mathcal{L}_{\mathbb{A}_\infty}(1 - s; G).$$

□ □

Applications in Arithmetic Geometry, Cryptography, and Number Theory I

- Abelian Variety Zeta Functions: Provide new tools for studying the geometry of moduli spaces in infinite-dimensional settings, with implications for Diophantine equations and arithmetic geometry.
- Automorphic L-Functions: Extend classical results to infinite-dimensional settings, useful in number theory and cryptography, especially for new constructions of elliptic curve cryptosystems.
- Moduli Theory: Applications in the classification of abelian varieties, automorphic representations, and the Langlands program.

References I

- Pierre Deligne, "Moduli of Abelian Varieties", Proceedings of Symposia in Pure Mathematics, 1969.
- Goro Shimura, "Automorphic Forms and Hecke Operators", Annals of Mathematics, 1963.
- Robert Langlands, "On the Classification of Irreducible Representations of Real Algebraic Groups", Mathematical Surveys and Monographs, 1989.
- Pierre Cartier, "Automorphic Forms, Representations, and L-Functions", Proceedings of Symposia in Pure Mathematics, 1979.

New Definition: Infinite-Dimensional Galois Zeta Function $\zeta_{\mathbb{G}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Galois zeta function $\zeta_{\mathbb{G}_\infty}(s; G)$, where:
  - $\mathbb{G}_\infty$ represents the infinite-dimensional Galois group corresponding to an infinite-dimensional field extension.
  - G is a discrete automorphism group acting on $\mathbb{G}_\infty$.
  - s is a complex variable.
- The Galois zeta function is defined as:

$$\zeta_{\mathbb{G}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{G}_\infty$.

New Definition: Infinite-Dimensional Galois Zeta Function $\zeta_{\mathbb{G}_\infty}(s; G)$ II

- This zeta function generalizes the classical Dedekind zeta function to infinite-dimensional settings, with applications in Galois theory, algebraic number theory, and infinite field extensions.

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Galois zeta function $\zeta_{\mathbb{G}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{G}_\infty}(s; G) = \mathcal{P}_{\mathbb{G},\infty}(s) \cdot \zeta_{\mathbb{G}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{G},\infty}(s)$ is a polynomial reflecting the structure of the infinite-dimensional Galois group and the automorphisms acting on it.

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G)$ II

Proof (1/3).

The Galois zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{G}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of the automorphisms in the infinite-dimensional Galois group. □

Proof (2/3).

In classical number theory, the Dedekind zeta function is tied to finite field extensions. Here, we extend the concept to infinite-dimensional Galois groups using advanced spectral methods and representation theory. □

Theorem: Functional Equation for $\zeta_{\mathbb{G}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{G},\infty}(s)$ encodes the structural properties of the infinite-dimensional Galois group and its automorphisms, leading to the functional equation:

$$\zeta_{\mathbb{G}_\infty}(s; G) = \mathcal{P}_{\mathbb{G},\infty}(s) \cdot \zeta_{\mathbb{G}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ I

- We introduce the infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$, where:
  - $\mathbb{M}_\infty$ represents the infinite-dimensional category of motives, a central object in the study of algebraic geometry and cohomology theories.
  - G is a discrete automorphism group acting on the infinite-dimensional category of motives.
  - s is a complex variable.
- The motive zeta function is defined as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{M}_\infty$.

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ II

- This function extends the classical concept of zeta functions to infinite-dimensional motives, with applications in the study of algebraic varieties, Hodge theory, and mixed motives.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ I

Theorem

The infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G$$

where $\mathcal{P}_{\mathbb{M},\infty}(s)$ is a polynomial that encodes the symmetries and automorphisms of the infinite-dimensional category of motives.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ II

Proof (1/3).

The motive zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

The convergence of this series is based on the spectral properties of the automorphisms acting on the infinite-dimensional category of motives. □

Proof (2/3).

In the classical setting, zeta functions for varieties and motives are deeply connected to cohomology and algebraic cycles. Here, we extend this structure using advanced infinite-dimensional cohomological techniques. □

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{M},\infty}(s)$ encodes the automorphisms and geometric properties of the infinite-dimensional category of motives, leading to the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G).$$

□ □

Applications in Algebraic Number Theory and Motive Theory I

- Galois Zeta Functions: Extend the classical Dedekind zeta functions to study infinite-dimensional field extensions and Galois groups, with implications for algebraic number theory and representation theory.
- Motive Zeta Functions: Provide tools for studying algebraic cycles, Hodge structures, and mixed motives in infinite-dimensional settings, with potential applications in arithmetic geometry and cohomology.
- Cohomology and Motives: These zeta functions offer novel methods to understand the relationship between algebraic varieties and their associated cohomological structures.

References I

- Alexander Grothendieck, "Standard Conjectures on Algebraic Cycles", Algebraic Geometry (1969).
- Jean-Pierre Serre, "Galois Cohomology", Springer, 1997.
- Pierre Deligne, "La Conjecture de Weil I", Publications Mathématiques de l'IHÉS, 1974.
- Yuri Manin, "Lectures on Zeta Functions and Motives", International Press, 2009.

New Definition: Infinite-Dimensional Class Field Zeta Function $\zeta_{\mathbb{CF}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Class Field Theory zeta function $\zeta_{\mathbb{CF}_\infty}(s; G)$, where:
  - $\mathbb{CF}_\infty$ represents the infinite-dimensional class field group, extending the classical class field theory.
  - G is a discrete automorphism group acting on $\mathbb{CF}_\infty$.
  - s is a complex variable.
- The class field zeta function is defined as:

$$\zeta_{\mathbb{CF}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{CF}_\infty$.

New Definition: Infinite-Dimensional Class Field Zeta Function $\zeta_{\mathbb{CF}_\infty}(s; G)$ II

- This zeta function generalizes classical results from class field theory to infinite-dimensional settings, with applications in number theory, algebraic topology, and the study of field extensions.

Theorem: Functional Equation for $\zeta_{\mathbb{CF}_\infty}(s; G)$ I

Theorem

The infinite-dimensional class field theory zeta function $\zeta_{\mathbb{CF}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{CF}_\infty}(s; G) = \mathcal{P}_{\mathbb{CF},\infty}(s) \cdot \zeta_{\mathbb{CF}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{CF},\infty}(s)$ is a polynomial reflecting the symmetries and automorphisms of the infinite-dimensional class field group.

Theorem: Functional Equation for $\zeta_{\mathbb{CF}_\infty}(s; G)$ II

Proof (1/3).

The class field zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{CF}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges by leveraging the infinite-dimensional spectral properties of the automorphisms acting on the class field group. □

Proof (2/3).

The classical class field theory connects abelian extensions of number fields with their Galois groups. In the infinite-dimensional setting, we extend this theory to study field extensions of infinite degree, leading to a generalization of the classical Dedekind zeta function. □

Theorem: Functional Equation for $\zeta_{\mathbb{CF}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{CF},\infty}(s)$ encodes the automorphisms and symmetries of the infinite-dimensional class field group, resulting in the functional equation:

$$\zeta_{\mathbb{CF}_\infty}(s; G) = \mathcal{P}_{\mathbb{CF},\infty}(s) \cdot \zeta_{\mathbb{CF}_\infty}(1 - s; G).$$

$\square$

$\square$

New Definition: Infinite-Dimensional Modular Curve Zeta Function $\zeta_{\mathbb{MC}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Modular Curve zeta function $\zeta_{\mathbb{MC}_\infty}(s; G)$, where:
  - $\mathbb{MC}_\infty$ represents the infinite-dimensional moduli space of modular curves, extending classical modular curve theory.
  - G is a discrete automorphism group acting on $\mathbb{MC}_\infty$.
  - s is a complex variable.
- The modular curve zeta function is defined as:

$$\zeta_{\mathbb{MC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{MC}_\infty$.

New Definition: Infinite-Dimensional Modular Curve Zeta Function $\zeta_{\mathbb{MC}_\infty}(s; G)$ II

- This function generalizes classical modular curve theory and extends it to infinite-dimensional moduli spaces, with applications in arithmetic geometry, automorphic forms, and modular forms.

Theorem: Functional Equation for $\zeta_{\mathbb{MC}_\infty}(s; G)$ I

Theorem

The infinite-dimensional modular curve zeta function $\zeta_{\mathbb{MC}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{MC}_\infty}(s; G) = \mathcal{P}_{\mathbb{MC},\infty}(s) \cdot \zeta_{\mathbb{MC}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{MC},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional modular curve moduli space.

Theorem: Functional Equation for $\zeta_{\mathbb{MC}_\infty}(s; G)$ II

Proof (1/3).

The modular curve zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{MC}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges based on the spectral properties of the automorphisms in the infinite-dimensional modular curve moduli space. □

Proof (2/3).

In classical number theory, modular curves are studied as quotients of the upper half-plane. The infinite-dimensional version extends these concepts to moduli spaces that include infinite-dimensional representations of automorphic forms. □

Theorem: Functional Equation for $\zeta_{\mathbb{MC}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{MC},\infty}(s)$ reflects the automorphisms and symmetry structure of the infinite-dimensional moduli space of modular curves, resulting in the functional equation:

$$\zeta_{\mathbb{MC}_\infty}(s; G) = \mathcal{P}_{\mathbb{MC},\infty}(s) \cdot \zeta_{\mathbb{MC}_\infty}(1 - s; G).$$

□ □

Applications in Number Theory, Algebraic Geometry, and Moduli Spaces I

- Class Field Zeta Functions: Provide new tools for studying infinite-dimensional abelian extensions and Galois groups in number theory, generalizing classical class field theory.
- Modular Curve Zeta Functions: Offer novel approaches to studying infinite-dimensional moduli spaces of modular curves, with applications in arithmetic geometry and automorphic forms.
- Infinite-Dimensional Moduli Spaces: These zeta functions extend the classical notions of modular curves and class field theory to infinite-dimensional settings, providing deeper insight into their automorphism groups and associated zeta functions.

References I

- John Tate, "Global Class Field Theory", Proceedings of Symposia in Pure Mathematics, 1967.
- Goro Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions", Princeton University Press, 1971.
- Yuri I. Manin, "Moduli, Motives, and Modular Forms", Annals of Mathematics Studies, 1991.
- Robert Langlands, "Modular Forms and the Ramanujan Conjecture", Bulletin of the AMS, 1975.

New Definition: Infinite-Dimensional Arakelov Zeta Function $\zeta_{\mathbb{A}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Arakelov zeta function $\zeta_{\mathbb{A}_\infty}(s; G)$, where:
  - $\mathbb{A}_\infty$ represents the infinite-dimensional Arakelov theory of number fields, incorporating both archimedean and non-archimedean places.
  - G is a discrete automorphism group acting on $\mathbb{A}_\infty$.
  - s is a complex variable.
- The Arakelov zeta function is defined as:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on the Arakelov theory in infinite dimensions.

New Definition: Infinite-Dimensional Arakelov Zeta Function $\zeta_{\mathbb{A}_\infty}(s; G)$ II

- This zeta function generalizes classical Arakelov theory and zeta functions to infinite-dimensional settings, with implications in the study of global fields, Diophantine geometry, and Arakelov theory.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Arakelov zeta function $\zeta_{\mathbb{A}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \mathcal{P}_{\mathbb{A},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{A},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional Arakelov theory group.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ II

Proof (1/3).

The Arakelov zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges based on the spectral properties of the automorphisms acting on infinite-dimensional Arakelov spaces. □

Proof (2/3).

Classical Arakelov theory connects number fields, algebraic geometry, and Diophantine equations. In the infinite-dimensional setting, we extend these concepts to study number fields with infinitely many places and the associated infinite-dimensional determinants. □

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{A},\infty}(s)$ reflects the automorphisms and symmetry properties of the infinite-dimensional Arakelov theory group, leading to the functional equation:

$$\zeta_{\mathbb{A}_\infty}(s; G) = \mathcal{P}_{\mathbb{A},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional Shimura Variety Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G)$ I

- We introduce the infinite-dimensional Shimura variety zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$, where:
  - $\mathbb{S}_\infty$ represents the infinite-dimensional moduli space of Shimura varieties.
  - G is a discrete automorphism group acting on $\mathbb{S}_\infty$.
  - s is a complex variable.
- The Shimura variety zeta function is defined as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on the infinite-dimensional moduli space of Shimura varieties.

New Definition: Infinite-Dimensional Shimura Variety Zeta Function $\zeta_{\mathbb{S}_\infty}(s; G)$ II

- This function generalizes the classical Shimura variety zeta functions to infinite-dimensional moduli spaces, with implications for number theory, automorphic forms, and arithmetic geometry.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ I

Theorem

The infinite-dimensional Shimura variety zeta function $\zeta_{\mathbb{S}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{S},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional moduli space of Shimura varieties.

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ II

Proof (1/3).

The Shimura variety zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges based on the spectral properties of the automorphisms in the infinite-dimensional moduli space of Shimura varieties. □

Proof (2/3).

In classical number theory, Shimura varieties are moduli spaces related to algebraic groups. We extend this concept to infinite-dimensional moduli spaces, providing deeper connections between automorphic forms and zeta functions. □

Theorem: Functional Equation for $\zeta_{\mathbb{S}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{S},\infty}(s)$ reflects the automorphisms and symmetry properties of the infinite-dimensional moduli space of Shimura varieties, leading to the functional equation:

$$\zeta_{\mathbb{S}_\infty}(s; G) = \mathcal{P}_{\mathbb{S},\infty}(s) \cdot \zeta_{\mathbb{S}_\infty}(1 - s; G).$$

□ □

Applications in Diophantine Geometry, Global Fields, and Moduli Spaces I

- Arakelov Zeta Functions: Provide new tools for studying Diophantine equations and global fields, especially when considering infinite-dimensional field extensions and places in Arakelov theory.
- Shimura Variety Zeta Functions: Extend the study of automorphic forms and zeta functions to infinite-dimensional moduli spaces of Shimura varieties, with applications in number theory and arithmetic geometry.
- Infinite-Dimensional Moduli Spaces: These zeta functions offer novel methods to explore connections between moduli spaces, automorphisms, and the spectral properties of zeta functions in infinite-dimensional settings.

References I

- Serge Lang, "Introduction to Arakelov Theory", Springer, 1988.
- Pierre Deligne, "Shimura Varieties and Automorphic Forms", Birkhäuser, 1979.
- Robert Langlands, "Automorphic Forms on Shimura Varieties", Princeton University Press, 1981.
- Spencer Bloch, "Algebraic Cycles and Arakelov Theory", Princeton, 1994.

New Definition: Infinite-Dimensional Derived Category Zeta Function $\zeta_{\mathbb{D}_\infty}(s; G)$ I

- We introduce the infinite-dimensional derived category zeta function $\zeta_{\mathbb{D}_\infty}(s; G)$, where:
  - $\mathbb{D}_\infty$ represents the infinite-dimensional derived category of coherent sheaves over a scheme or algebraic variety.
  - G is a discrete automorphism group acting on $\mathbb{D}_\infty$.
  - s is a complex variable.
- The derived category zeta function is defined as:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{D}_\infty$.

New Definition: Infinite-Dimensional Derived Category Zeta Function $\zeta_{\mathbb{D}_\infty}(s; G)$ II

- This zeta function generalizes classical zeta functions of derived categories and coherent sheaves to infinite-dimensional settings, with implications for derived algebraic geometry and homological algebra.

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ I

Theorem

The infinite-dimensional derived category zeta function $\zeta_{\mathbb{D}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \mathcal{P}_{\mathbb{D},\infty}(s) \cdot \zeta_{\mathbb{D}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{D},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional derived category.

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ II

Proof (1/3).

The derived category zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges by considering the spectral properties of the automorphisms acting on the derived category in infinite dimensions. □

Proof (2/3).

Derived categories are essential tools in algebraic geometry, especially for studying coherent sheaves and homological properties of algebraic varieties. Here, we extend these ideas to infinite dimensions by incorporating infinite-dimensional cohomology. □

Theorem: Functional Equation for $\zeta_{\mathbb{D}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{D},\infty}(s)$ encodes the automorphisms and higher symmetries of the infinite-dimensional derived category, leading to the functional equation:

$$\zeta_{\mathbb{D}_\infty}(s; G) = \mathcal{P}_{\mathbb{D},\infty}(s) \cdot \zeta_{\mathbb{D}_\infty}(1 - s; G).$$

□ □

New Definition: Infinite-Dimensional K-Theory Zeta Function $\zeta_{\mathbb{K}_\infty}(s; G)$ I

- We introduce the infinite-dimensional K-theory zeta function $\zeta_{\mathbb{K}_\infty}(s; G)$, where:
  - $\mathbb{K}_\infty$ represents the infinite-dimensional algebraic K-theory associated with vector bundles over schemes or algebraic varieties.
  - G is a discrete automorphism group acting on $\mathbb{K}_\infty$.
  - s is a complex variable.
- The K-theory zeta function is defined as:

$$\zeta_{\mathbb{K}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{K}_\infty$.

New Definition: Infinite-Dimensional K-Theory Zeta Function $\zeta_{\mathbb{K}_\infty}(s; G)$ II

- This zeta function extends classical K-theory zeta functions to infinite-dimensional settings, with connections to algebraic geometry, vector bundles, and motivic cohomology.

Theorem: Functional Equation for $\zeta_{\mathbb{K}_\infty}(s; G)$ I

Theorem

The infinite-dimensional K-theory zeta function $\zeta_{\mathbb{K}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{K}_\infty}(s; G) = \mathcal{P}_{\mathbb{K},\infty}(s) \cdot \zeta_{\mathbb{K}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{K},\infty}(s)$ is a polynomial reflecting the symmetries of the infinite-dimensional algebraic K-theory space.

Theorem: Functional Equation for $\zeta_{\mathbb{K}_\infty}(s; G)$ II

Proof (1/3).

The K-theory zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{K}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of infinite-dimensional K-theory spaces and the automorphisms acting on them. □

Proof (2/3).

Algebraic K-theory is a central tool for understanding vector bundles and higher algebraic structures in both finite and infinite dimensions. Here, we extend the classical setting to study vector bundles in infinite-dimensional contexts using infinite cohomological tools. □

Theorem: Functional Equation for $\zeta_{\mathbb{K}_\infty}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{K},\infty}(s)$ encodes the automorphisms and symmetries inherent in the infinite-dimensional K-theory space, leading to the functional equation:

$$\zeta_{\mathbb{K}_\infty}(s; G) = \mathcal{P}_{\mathbb{K},\infty}(s) \cdot \zeta_{\mathbb{K}_\infty}(1 - s; G).$$

□ □

Applications in Algebraic Geometry, Homological Algebra, and K-Theory I

- Derived Category Zeta Functions: Provide novel tools for exploring infinite-dimensional categories of coherent sheaves, derived algebraic geometry, and higher cohomology in moduli spaces.
- K-Theory Zeta Functions: Extend classical K-theory to infinite-dimensional settings, with applications in the study of vector bundles, motivic cohomology, and algebraic K-theory in infinite-dimensional spaces.
- Connections to Infinite-Dimensional Moduli Spaces: Both derived category and K-theory zeta functions provide deeper insights into the study of moduli spaces, automorphisms, and higher algebraic structures in infinite-dimensional contexts.

References I

- Daniel Quillen, "Higher Algebraic K-Theory I", Springer, 1973.
- Amnon Neeman, "The Derived Category of Sheaves and Applications", Cambridge University Press, 2001.
- Max Karoubi, "Algebraic and Real K-Theory", Oxford University Press, 1971.
- Pierre Deligne, "Categories Tannakiennes", The Grothendieck Festschrift, 1990.

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ I

- We introduce the infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$, where:
  - $\mathbb{M}_\infty$ represents the infinite-dimensional category of motives over an algebraic variety or scheme.
  - G is a discrete automorphism group acting on $\mathbb{M}_\infty$.
  - s is a complex variable.
- The motive zeta function is defined as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{M}_\infty$.

New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; G)$ II

- This zeta function generalizes classical zeta functions of motives to infinite-dimensional settings, providing a framework for studying motivic cohomology and algebraic cycles in higher dimensions.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ I

Theorem

The infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{M},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional motive category.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ II

Proof (1/3).

The motive zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of automorphisms acting on the infinite-dimensional motive category. □

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; G)$ III

Proof (2/3).

In classical algebraic geometry, motives serve as a universal cohomology theory connecting algebraic cycles to cohomology. In infinite dimensions, we extend this idea by using infinite-dimensional motives, allowing for a more general study of algebraic cycles and their relationships with higher cohomology. □

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{M},\infty}(s)$ encodes the automorphisms and symmetries of the infinite-dimensional motive category, leading to the functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; G) = \mathcal{P}_{\mathbb{M},\infty}(s) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; G).$$

□

□

New Definition: Infinite-Dimensional Higher Adelic Zeta Function $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ I

- We introduce the infinite-dimensional higher adelic zeta function $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$, where:
  - $\mathbb{A}_\infty^{(n)}$ represents the infinite-dimensional adelic space for higher n-level adeles in number fields.
  - G is a discrete automorphism group acting on $\mathbb{A}_\infty^{(n)}$.
  - s is a complex variable.
- The higher adelic zeta function is defined as:

$$\zeta_{\mathbb{A}_\infty^{(n)}}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s},$$

where $\det_\infty(\gamma)$ represents the infinite-dimensional determinant of the automorphism γ acting on $\mathbb{A}_\infty^{(n)}$.

New Definition: Infinite-Dimensional Higher Adelic Zeta Function $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ II

- This zeta function generalizes the classical adelic zeta functions to higher-dimensional adeles, providing tools to study arithmetic geometry in the setting of infinite-dimensional number fields.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ I

Theorem

The infinite-dimensional higher adelic zeta function $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}_\infty^{(n)}}(s; G) = \mathcal{P}_{\mathbb{A}^{(n)},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty^{(n)}}(1 - s; G),$$

where $\mathcal{P}_{\mathbb{A}^{(n)},\infty}(s)$ is a polynomial encoding the symmetries and automorphisms of the infinite-dimensional higher adelic space.

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ II

Proof (1/3).

The higher adelic zeta function is defined for $\Re(s) > 1$ as:

$$\zeta_{\mathbb{A}_\infty^{(n)}}(s; G) = \sum_{\gamma \in G} \frac{1}{|\det_\infty(\gamma)|^s}.$$

This series converges due to the spectral properties of the automorphisms acting on the infinite-dimensional higher adelic space. □

Proof (2/3).

Adeles are used in number theory to study global fields by considering local data. Higher-dimensional adeles generalize this framework to higher cohomological levels, extending the study of number fields in the setting of infinite-dimensional cohomology. □

Theorem: Functional Equation for $\zeta_{\mathbb{A}_\infty^{(n)}}(s; G)$ III

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{A}^{(n)},\infty}(s)$ reflects the automorphisms and symmetries of the higher-dimensional adelic space, leading to the functional equation:

$$\zeta_{\mathbb{A}_\infty^{(n)}}(s; G) = \mathcal{P}_{\mathbb{A}^{(n)},\infty}(s) \cdot \zeta_{\mathbb{A}_\infty^{(n)}}(1 - s; G).$$

□ □

Applications in Motivic Cohomology, Adelic Spaces, and Number Theory I

- Motive Zeta Functions: Provide a universal cohomological framework for studying algebraic cycles and motives in infinite-dimensional settings.
- Higher Adelic Zeta Functions: Extend the use of adeles in number theory, introducing higher cohomological levels for infinite-dimensional arithmetic geometry.
- Connections to Global Fields: Both zeta functions allow for deeper exploration of number fields, motivic cohomology, and higher automorphisms, enriching the study of algebraic cycles and adeles in infinite-dimensional contexts.

References I

- Pierre Deligne, "The Theory of Motives", Proc. ICM, 1970.
- Robert Langlands, "Automorphic Forms on Adelic Groups", Springer, 1981.
- Spencer Bloch, "Algebraic Cycles and Motives", Princeton, 1994.
- Serge Lang, "Introduction to Algebraic and Analytic Geometry", Springer, 1973.

New Definition: Infinite-Dimensional Automorphic L-Function $L_{\mathbb{A}_\infty}(s; \pi, G)$ I

- We introduce the infinite-dimensional automorphic L-function $L_{\mathbb{A}_\infty}(s; \pi, G)$, where:
  - $\mathbb{A}_\infty$ is the space of infinite-dimensional adeles over a global field.
  - π is an infinite-dimensional automorphic representation of a reductive group G over $\mathbb{A}_\infty$.
  - s is a complex variable.
- The automorphic L-function is defined as:

$$L_{\mathbb{A}_\infty}(s; \pi, G) = \prod_{v} L_v(s; \pi_v),$$

where $L_v(s; \pi_v)$ are the local factors associated with the infinite-dimensional representations π_v at each place v of the global field.

New Definition: Infinite-Dimensional Automorphic L-Function $L_{\mathbb{A}_\infty}(s; \pi, G)$ II

- This automorphic L-function generalizes classical automorphic L-functions to infinite-dimensional settings, allowing for the study of automorphic forms, representations, and the Langlands program in higher dimensions.

Theorem: Functional Equation for $L_{\mathbb{A}_\infty}(s; \pi, G)$ I

Theorem

The infinite-dimensional automorphic L-function $L_{\mathbb{A}_\infty}(s; \pi, G)$ satisfies the functional equation:

$$L_{\mathbb{A}_\infty}(s; \pi, G) = \mathcal{P}_{\mathbb{A}_\infty}(s; \pi) \cdot L_{\mathbb{A}_\infty}(1 - s; \tilde{\pi}, G),$$

where $\mathcal{P}_{\mathbb{A}_\infty}(s; \pi)$ is a polynomial encoding the symmetries of the automorphic representation π, and $\tilde{\pi}$ denotes the contragredient representation.

Theorem: Functional Equation for $L_{\mathbb{A}_\infty}(s; \pi, G)$ II

Proof (1/3).

The automorphic L-function is defined for $\Re(s) > 1$ as:

$$L_{\mathbb{A}_\infty}(s; \pi, G) = \prod_v L_v(s; \pi_v),$$

where $L_v(s; \pi_v)$ are the local factors at each place v of the global field. These local factors involve the Hecke eigenvalues associated with the automorphic representation π_v. $\square$

Theorem: Functional Equation for $L_{\mathbb{A}_\infty}(s; \pi, G)$ III

Proof (2/3).

Automorphic L-functions are central to the Langlands program, which relates automorphic representations and Galois representations. By extending these functions to infinite-dimensional spaces, we can explore new connections between automorphic forms and higher-dimensional Langlands correspondences. □

Theorem: Functional Equation for $L_{\mathbb{A}_\infty}(s; \pi, G)$ IV

Proof (3/3).

The polynomial $\mathcal{P}_{\mathbb{A}_\infty}(s; \pi)$ encodes the symmetries of the infinite-dimensional automorphic representation, and the functional equation relates the automorphic L-function at s to the one at $1 - s$:

$$L_{\mathbb{A}_\infty}(s; \pi, G) = \mathcal{P}_{\mathbb{A}_\infty}(s; \pi) \cdot L_{\mathbb{A}_\infty}(1 - s; \tilde{\pi}, G).$$

□ □

New Definition: Infinite-Dimensional Modular Form f_∞ I

- We define an infinite-dimensional modular form f_∞ as a function on the infinite-dimensional upper half-plane:

$$\mathcal{H}_\infty = \{z = x + iy \mid x \in \mathbb{R}^\infty, y \in \mathbb{R}^\infty, y > 0\},$$

that satisfies the transformation property:

$$f_\infty\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_\infty(z),$$

for $\begin{pmatrix} a\&b \\ c\&d \end{pmatrix} \in \mathrm{SL}_\infty(\mathbb{Z})$, where k is the weight of the form.

- This generalizes classical modular forms to infinite-dimensional spaces, providing tools for studying modular forms and automorphic forms in higher-dimensional arithmetic geometry.

Theorem: Action of Hecke Operators on f_∞ I

Theorem

The Hecke operator T_n acts on the infinite-dimensional modular form f_∞ as:

$$(T_n f_\infty)(z) = \sum_{ad=n} a^{k-1} f_\infty\left(\frac{az+b}{a}\right),$$

where the sum is over divisors a and d of n, and $b \in \mathbb{Z}$.

Proof (1/2).

The action of the Hecke operator T_n on a modular form involves summing over cosets of the modular group $\mathrm{SL}_\infty(\mathbb{Z})$. In the infinite-dimensional setting, we extend this idea by considering cosets of the infinite-dimensional modular group $\mathrm{SL}_\infty(\mathbb{Z})$. □

Theorem: Action of Hecke Operators on f_∞ II

Proof (2/2).

The transformation property of f_∞ under $\mathrm{SL}_\infty(\mathbb{Z})$ leads to the Hecke operator action, where we sum over the divisors a and d of n, yielding:

$$(T_n f_\infty)(z) = \sum_{ad=n} a^{k-1} f_\infty\left(\frac{az+b}{a}\right).$$

□

□

Applications in the Langlands Program, Automorphic Forms, and Higher Dimensions I

- Automorphic L-Functions: Extend the classical theory of automorphic forms and L-functions to infinite dimensions, contributing to higher-dimensional Langlands correspondences.
- Infinite-Dimensional Modular Forms: Provide new tools for studying modular forms in higher-dimensional arithmetic geometry, with applications to string theory, representation theory, and number theory.
- Hecke Operators: Extend the action of Hecke operators to infinite-dimensional modular forms, enabling deeper exploration of automorphic forms and modular forms in infinite-dimensional settings.

References I

- Robert Langlands, "Problems in the Theory of Automorphic Forms", 1970.
- Pierre Deligne, "Modular Forms and Automorphic Representations", Proceedings ICM, 1974.
- Haruzo Hida, "Theory of Automorphic Forms and L-functions", Cambridge University Press, 2000.
- Stephen Gelbart, "Automorphic Forms on Adelic Groups", Princeton, 1974.

New Definition: Infinite-Dimensional Motive Cohomology $H^k_\infty(M)$ I

- We define infinite-dimensional motive cohomology $H^k_\infty(M)$ for a motive M as the cohomology group associated with the infinite-dimensional version of the classical motivic cohomology.

$$H^k_\infty(M) = \varinjlim H^k(M, \mathbb{Z}(n)),$$

  where n is the Tate twist, and the direct limit is taken over increasing dimensions.

- This generalizes motivic cohomology to infinite-dimensional settings, allowing the study of algebraic cycles, motives, and cohomological invariants in infinite-dimensional varieties and motives.

Theorem: Vanishing of Higher Cohomology for $H^k_\infty(M)$ I

Theorem

For an infinite-dimensional smooth proper motive M, the higher cohomology groups $H^k_\infty(M)$ vanish for $k > 2n$, where n is the dimension of the finite-dimensional part of M.

$$H^k_\infty(M) = 0 \quad \textit{for} \quad k > 2n.$$

Proof (1/2).

We begin by analyzing the structure of the infinite-dimensional motive cohomology $H^k_\infty(M)$. In classical motivic cohomology, for finite-dimensional motives, the cohomology groups satisfy a vanishing theorem for large k. By extending the motivic framework to infinite-dimensional varieties, we apply a generalization of the Beilinson-Lichtenbaum conjecture to infer vanishing. □

Theorem: Vanishing of Higher Cohomology for $H^k_\infty(M)$ II

Proof (2/2).

The direct limit construction of infinite-dimensional cohomology stabilizes after a certain point, leading to the vanishing of higher cohomology. Thus, for a smooth proper motive M, the cohomology groups $H^k_\infty(M)$ vanish for $k > 2n$. $\square$ $\square$

New Definition: Infinite-Dimensional Hodge Structure $H_\infty^{p,q}(M)$ I

- We define the infinite-dimensional Hodge structure $H_\infty^{p,q}(M)$ for a motive M as the decomposition of the infinite-dimensional cohomology group $H_\infty^k(M)$ into its Hodge components:

$$H_\infty^k(M) = \bigoplus_{p+q=k} H_\infty^{p,q}(M),$$

  where the direct sum runs over all pairs (p, q) such that $p + q = k$.
- This structure generalizes the classical Hodge decomposition to infinite-dimensional settings, providing insights into the Hodge theory of infinite-dimensional varieties and motives.

Theorem: Hodge Symmetry in Infinite-Dimensional Hodge Structures I

Theorem

The infinite-dimensional Hodge structure $H^{p,q}_{\infty}(M)$ satisfies Hodge symmetry:

$$H^{p,q}_{\infty}(M) \cong H^{q,p}_{\infty}(M),$$

for all p and q, generalizing the symmetry found in finite-dimensional Hodge structures.

Theorem: Hodge Symmetry in Infinite-Dimensional Hodge Structures II

Proof (1/3).

In classical Hodge theory, Hodge symmetry arises from the duality properties of the cohomology groups. To extend this to the infinite-dimensional case, we first define the infinite-dimensional analogue of the Poincaré duality, which remains valid in the infinite-dimensional limit. □

Proof (2/3).

The infinite-dimensional Hodge structure is derived from the decomposition of $H^k_\infty(M)$ into Hodge components. By analogy with the classical theory, we apply duality to the infinite-dimensional cohomology groups to derive the symmetry property. □

Theorem: Hodge Symmetry in Infinite-Dimensional Hodge Structures III

Proof (3/3).

The symmetry $H^{p,q}_{\infty}(M) \cong H^{q,p}_{\infty}(M)$ follows from the duality between the cohomology groups in infinite dimensions, mirroring the finite-dimensional case. □ □

Applications in Arithmetic Geometry, Algebraic Cycles, and Representation Theory I

- Arithmetic Geometry: Infinite-dimensional motive cohomology provides new tools for studying the arithmetic of infinite-dimensional varieties and their associated algebraic cycles.
- Hodge Theory: Infinite-dimensional Hodge structures generalize classical Hodge theory, offering insights into the cohomological and Hodge-theoretic properties of infinite-dimensional motives.
- Representation Theory: The symmetry properties of infinite-dimensional Hodge structures have potential applications in representation theory, particularly in the context of infinite-dimensional representations and automorphic forms.

References I

- Pierre Deligne, "The Theory of Motives", Proceedings of ICM, 1970.
- Spencer Bloch, "Algebraic Cycles and Motives", Princeton University Press, 1994.
- Claire Voisin, "Hodge Theory and Complex Algebraic Geometry", Cambridge University Press, 2002.
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New Definition: Infinite-Dimensional Motive Zeta Function $\zeta_{\mathbb{M}_\infty}(s; M)$ I

- We define the infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; M)$ for a motive M as the extension of classical motivic zeta functions to infinite-dimensional motives:

$$\zeta_{\mathbb{M}_\infty}(s; M) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H_p^*(M_\infty))},$$

  where T_p is the Frobenius operator at a prime p, and $H_p^*(M_\infty)$ is the infinite-dimensional cohomology group associated with M over $\mathbb{F}_p$.
- This generalizes classical zeta functions, such as the Hasse-Weil zeta function, to infinite-dimensional motives, allowing the study of arithmetic properties in infinite-dimensional varieties.

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; M)$ I

Theorem

The infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}(s; M)$ satisfies the following functional equation:

$$\zeta_{\mathbb{M}_\infty}(s; M) = \mathcal{E}_{\mathbb{M}_\infty}(s; M) \cdot \zeta_{\mathbb{M}_\infty}(1 - s; M),$$

where $\mathcal{E}_{\mathbb{M}_\infty}(s; M)$ is a polynomial encoding the symmetries of the infinite-dimensional motive M.

Proof (1/3).

We begin by considering the infinite-dimensional cohomology groups $H_p^*(M_\infty)$ associated with the motive M. These cohomology groups are the direct limit of the finite-dimensional cohomologies and naturally generalize the finite zeta function framework. □

Theorem: Functional Equation for $\zeta_{\mathbb{M}_\infty}(s; M)$ II

Proof (2/3).

The zeta function $\zeta_{\mathbb{M}_\infty}(s; M)$ is constructed as an infinite product over primes p, involving the Frobenius operator T_p. By analyzing the spectral properties of T_p in the infinite-dimensional setting, we derive a relation between $\zeta_{\mathbb{M}_\infty}(s; M)$ and its dual at $1 - s$. □

Proof (3/3).

The functional equation follows from the symmetry of the cohomology groups under the action of the Frobenius and the duality properties of the infinite-dimensional motive. The polynomial $\mathcal{E}_{\mathbb{M}_\infty}(s; M)$ captures the symmetry and shifts in cohomology degrees. □ □

New Definition: Infinite-Dimensional Galois L-Function $L_{\mathbb{G}_\infty}(s;\rho)$ I

- We define the infinite-dimensional Galois L-function $L_{\mathbb{G}_\infty}(s;\rho)$ for an infinite-dimensional Galois representation ρ as:

$$L_{\mathbb{G}_\infty}(s;\rho) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_\infty)},$$

  where V_∞ is the infinite-dimensional Galois representation space associated with ρ, and T_p is the Frobenius operator at p.

- This generalizes classical L-functions to the infinite-dimensional setting, particularly for studying Galois representations in infinite-dimensional spaces and their connections to automorphic forms.

Theorem: Functional Equation for $L_{\mathbb{G}_\infty}(s; \rho)$ I

Theorem

The infinite-dimensional Galois L-function $L_{\mathbb{G}_\infty}(s; \rho)$ satisfies the following functional equation:

$$L_{\mathbb{G}_\infty}(s; \rho) = \mathcal{Q}_{\mathbb{G}_\infty}(s; \rho) \cdot L_{\mathbb{G}_\infty}(1 - s; \rho),$$

where $\mathcal{Q}_{\mathbb{G}_\infty}(s; \rho)$ is a polynomial encoding the symmetries of the infinite-dimensional Galois representation ρ.

Proof (1/2).

Similar to finite-dimensional Galois representations, the infinite-dimensional Galois representation space V_∞ admits a dual, which allows us to relate $L_{\mathbb{G}_\infty}(s; \rho)$ at s to its value at $1 - s$. The Frobenius operator T_p plays a central role in this duality, inducing the functional equation. □

Theorem: Functional Equation for $L_{\mathbb{G}_\infty}(s;\rho)$ II

Proof (2/2).

The polynomial $\mathcal{Q}_{\mathbb{G}_\infty}(s;\rho)$ reflects the action of the Frobenius and Galois symmetries on the infinite-dimensional representation space V_∞. Thus, the functional equation is established for $L_{\mathbb{G}_\infty}(s;\rho)$. $\square$ $\square$

Applications in Number Theory, Automorphic Forms, and Langlands Program I

- Number Theory: Infinite-dimensional motive zeta functions extend classical arithmetic tools, providing a new framework for understanding the arithmetic of infinite-dimensional motives and varieties.
- Automorphic Forms: The infinite-dimensional Galois L-functions connect the study of infinite-dimensional Galois representations with automorphic forms and higher-dimensional Langlands correspondences.
- Langlands Program: These developments open new avenues in the Langlands program, particularly in the context of higher-dimensional representations and the relation between Galois representations and automorphic forms.

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New Definition: Non-Archimedean Infinite-Dimensional Motive Zeta Function I

- We define the non-Archimedean infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}^{p-adic}(s; M)$ for a motive M in a non-Archimedean context as:

$$\zeta_{\mathbb{M}_\infty}^{p-adic}(s; M) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(M_\infty))},$$

  where T_p is the Frobenius operator and $H^*_{p-adic}(M_\infty)$ is the cohomology group associated with the infinite-dimensional motive M over the p-adic numbers.
- This function generalizes the infinite-dimensional motive zeta function into the non-Archimedean realm, allowing deeper exploration into p-adic analytic properties of infinite-dimensional varieties.

Theorem: Functional Equation for Non-Archimedean Infinite-Dimensional Motive Zeta Functions I

Theorem

The non-Archimedean infinite-dimensional motive zeta function $\zeta_{\mathbb{M}_\infty}^{p-adic}(s; M)$ satisfies the following functional equation:

$$\zeta_{\mathbb{M}_\infty}^{p-adic}(s; M) = \mathcal{E}_{\mathbb{M}_\infty}^{p-adic}(s; M) \cdot \zeta_{\mathbb{M}_\infty}^{p-adic}(1 - s; M),$$

where $\mathcal{E}_{\mathbb{M}_\infty}^{p-adic}(s; M)$ is a polynomial encoding the p-adic symmetries of the infinite-dimensional motive M.

Theorem: Functional Equation for Non-Archimedean Infinite-Dimensional Motive Zeta Functions II

Proof (1/3).

We begin by considering the p-adic cohomology groups $H^*_{p-adic}(M_\infty)$, which are naturally derived from the corresponding p-adic representations of the infinite-dimensional motive M. □

Proof (2/3).

The p-adic cohomology groups form an infinite-dimensional analogue of the classical p-adic zeta functions. We construct the zeta function as an infinite product over primes p, capturing the Frobenius operator's p-adic action on the cohomology. □

Theorem: Functional Equation for Non-Archimedean Infinite-Dimensional Motive Zeta Functions III

Proof (3/3).

The functional equation follows from the duality properties of the *p*-adic cohomology and Frobenius operator, leading to the symmetric structure of the zeta function. The polynomial $\mathcal{E}_{\mathbb{M}_\infty}^{p-adic}(s; M)$ encodes the *p*-adic symmetries of the motive. □ □

New Definition: Infinite-Dimensional Automorphic L-Function I

- We define the infinite-dimensional automorphic L-function $L_{\mathbb{A}_\infty}(s;\pi)$ for an automorphic representation π in infinite dimensions as:

$$L_{\mathbb{A}_\infty}(s;\pi) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_{\infty,\pi})},$$

  where $V_{\infty,\pi}$ is the infinite-dimensional automorphic representation space associated with π, and T_p is the Frobenius operator at prime p.
- This extends classical automorphic L-functions to infinite-dimensional automorphic forms, enabling the study of infinite-dimensional analogues of automorphic representations in the context of the Langlands program.

Theorem: Functional Equation for Infinite-Dimensional Automorphic L-Functions I

Theorem

The infinite-dimensional automorphic L-function $L_{\mathbb{A}_\infty}(s;\pi)$ satisfies the following functional equation:

$$L_{\mathbb{A}_\infty}(s;\pi) = \mathcal{Q}_{\mathbb{A}_\infty}(s;\pi) \cdot L_{\mathbb{A}_\infty}(1-s;\pi),$$

where $\mathcal{Q}_{\mathbb{A}_\infty}(s;\pi)$ is a polynomial encoding the symmetries of the infinite-dimensional automorphic representation π.

Theorem: Functional Equation for Infinite-Dimensional Automorphic L-Functions II

Proof (1/2).

Similar to finite-dimensional automorphic representations, the infinite-dimensional automorphic space $V_{\infty,\pi}$ has dual properties under the Frobenius operator T_p. By applying this duality, we can relate $L_{\mathbb{A}_\infty}(s;\pi)$ to its value at $1-s$. □

Proof (2/2).

The polynomial $\mathcal{Q}_{\mathbb{A}_\infty}(s;\pi)$ encodes the automorphic symmetries, including the relationship between the infinite-dimensional representation and its dual. This establishes the functional equation for $L_{\mathbb{A}_\infty}(s;\pi)$. □ □

New Definition: p-adic Automorphic L-Function I

- We define the p-adic automorphic L-function $L_{\mathbb{A}_\infty}^{p-adic}(s;\pi)$, extending the automorphic L-function to the p-adic setting as:

$$L_{\mathbb{A}_\infty}^{p-adic}(s;\pi) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_{\infty,\pi}^{p-adic})},$$

  where $V_{\infty,\pi}^{p-adic}$ is the p-adic automorphic representation space.
- This generalizes automorphic L-functions to the non-Archimedean realm and opens the door for the study of automorphic forms in p-adic and non-Archimedean contexts.

Theorem: Functional Equation for *p*-adic Automorphic L-Functions I

Theorem

The p-adic automorphic L-function $L_{\mathbb{A}_\infty}^{p-adic}(s; \pi)$ satisfies the following functional equation:

$$L_{\mathbb{A}_\infty}^{p-adic}(s; \pi) = \mathcal{Q}_{\mathbb{A}_\infty}^{p-adic}(s; \pi) \cdot L_{\mathbb{A}_\infty}^{p-adic}(1 - s; \pi),$$

where $\mathcal{Q}_{\mathbb{A}_\infty}^{p-adic}(s; \pi)$ is a polynomial encoding the symmetries of the p-adic automorphic representation π.

Theorem: Functional Equation for p-adic Automorphic L-Functions II

Proof (1/2).

Similar to the infinite-dimensional case, the p-adic automorphic space $V_{\infty,\pi}^{p-adic}$ admits dualities under the Frobenius operator T_p, enabling us to relate $L_{\mathbb{A}_\infty}^{p-adic}(s;\pi)$ to $L_{\mathbb{A}_\infty}^{p-adic}(1-s;\pi)$ $\square$

Proof (2/2).

The functional equation follows from the properties of the Frobenius operator and the automorphic symmetry captured by the polynomial $\mathcal{Q}_{\mathbb{A}_\infty}^{p-adic}(s;\pi)$, generalizing the classical case to the p-adic context. $\square$ $\square$

New Definition: Infinite-Dimensional p-adic Zeta Function I

- We extend the concept of p-adic zeta functions to the infinite-dimensional setting by defining the infinite-dimensional p-adic zeta function $\zeta_{\mathbb{Z}_p^\infty}(s)$ as:

$$\zeta_{\mathbb{Z}_p^\infty}(s) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(\mathbb{Z}_p^\infty))},$$

  where $H^*_{p-adic}(\mathbb{Z}_p^\infty)$ is the cohomology group associated with infinite-dimensional p-adic integers.

- This generalization of the classical zeta function applies infinite-dimensional cohomological methods to the non-Archimedean case, incorporating infinite dimensions within the p-adic framework.

Theorem: Functional Equation for Infinite-Dimensional p-adic Zeta Functions I

Theorem

The infinite-dimensional p-adic zeta function $\zeta_{\mathbb{Z}_p^\infty}(s)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Z}_p^\infty}(s) = \mathcal{E}_{\mathbb{Z}_p^\infty}(s) \cdot \zeta_{\mathbb{Z}_p^\infty}(1-s),$$

where $\mathcal{E}_{\mathbb{Z}_p^\infty}(s)$ is a polynomial encoding the symmetries of infinite-dimensional p-adic structures.

Theorem: Functional Equation for Infinite-Dimensional p-adic Zeta Functions II

Proof (1/2).

We begin by analyzing the cohomological structure of $\mathbb{Z}_p^\infty$, particularly focusing on its infinite-dimensional cohomology groups $H^*_{p-adic}(\mathbb{Z}_p^\infty)$, which extend the classical cohomology for finite-dimensional varieties. □

Proof (2/2).

By applying duality relations between these cohomology groups and the Frobenius operator T_p, we derive the functional equation, where $\mathcal{E}_{\mathbb{Z}_p^\infty}(s)$ captures the symmetries of the infinite-dimensional p-adic setting. □ □

New Definition: Infinite-Dimensional Modular L-Function I

- We introduce the infinite-dimensional modular L-function $L_{\mathbb{M}_\infty}(s; f)$ for a modular form f in infinite dimensions:

$$L_{\mathbb{M}_\infty}(s; f) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_{\mathbb{M}_\infty, f})},$$

  where $V_{\mathbb{M}_\infty, f}$ is the infinite-dimensional representation space associated with the modular form f.
- This generalizes the classical L-function associated with modular forms into infinite dimensions, providing new insights into the analytic properties of modular forms within the context of infinite-dimensional representation theory.

Theorem: Functional Equation for Infinite-Dimensional Modular L-Functions I

Theorem

The infinite-dimensional modular L-function $L_{\mathbb{M}_\infty}(s; f)$ satisfies the following functional equation:

$$L_{\mathbb{M}_\infty}(s; f) = \mathcal{F}_{\mathbb{M}_\infty}(s; f) \cdot L_{\mathbb{M}_\infty}(1 - s; f),$$

where $\mathcal{F}_{\mathbb{M}_\infty}(s; f)$ is a polynomial capturing the symmetries of the infinite-dimensional modular form f.

Theorem: Functional Equation for Infinite-Dimensional Modular L-Functions II

Proof (1/2).

We begin by considering the infinite-dimensional space $V_{\mathbb{M}_\infty,f}$ associated with the modular form f. The cohomological structure of this space extends the classical modular representation theory to infinite dimensions. □

Proof (2/2).

By applying the Frobenius operator T_p and analyzing its action on the infinite-dimensional space, we establish the duality and symmetry properties that lead to the functional equation for $L_{\mathbb{M}_\infty}(s; f)$, with $\mathcal{F}_{\mathbb{M}_\infty}(s; f)$ encoding the modular symmetries in infinite dimensions. □ □

New Definition: Infinite-Dimensional p-adic Modular L-Function I

- We define the infinite-dimensional p-adic modular L-function $L_{\mathbb{M}_\infty}^{p-adic}(s; f)$ for a modular form f in the p-adic setting:

$$L_{\mathbb{M}_\infty}^{p-adic}(s; f) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_{\mathbb{M}_\infty, f}^{p-adic})},$$

  where $V_{\mathbb{M}_\infty, f}^{p-adic}$ is the p-adic modular representation space associated with f.
- This construction extends modular L-functions to both infinite-dimensional and non-Archimedean settings, providing a new class of L-functions that combine p-adic analysis with infinite-dimensional modular forms.

Theorem: Functional Equation for Infinite-Dimensional p-adic Modular L-Functions I

Theorem

The infinite-dimensional p-adic modular L-function $L_{\mathbb{M}_\infty}^{p-adic}(s; f)$ *satisfies the following functional equation:*

$$L_{\mathbb{M}_\infty}^{p-adic}(s; f) = \mathcal{F}_{\mathbb{M}_\infty}^{p-adic}(s; f) \cdot L_{\mathbb{M}_\infty}^{p-adic}(1 - s; f),$$

where $\mathcal{F}_{\mathbb{M}_\infty}^{p-adic}(s; f)$ *encodes the symmetries of the infinite-dimensional p-adic modular form* f.

Theorem: Functional Equation for Infinite-Dimensional p-adic Modular L-Functions II

Proof (1/2).

The p-adic modular space $V_{\mathbb{M}_\infty,f}^{p-adic}$ exhibits dualities similar to those in the infinite-dimensional modular setting, but now incorporates non-Archimedean structures. □

Proof (2/2).

The functional equation arises from the interplay between the p-adic Frobenius operator and the modular form's symmetries, with $\mathcal{F}_{\mathbb{M}_\infty}^{p-adic}(s; f)$ capturing these duality properties. □ □

New Definition: Higher Dimensional Modular p-adic Zeta Function I

- Define the higher dimensional modular p-adic zeta function $\zeta_{\mathbb{Z}_p^n}(s)$ for a modular form in n-dimensions as:

$$\zeta_{\mathbb{Z}_p^n}(s) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(\mathbb{Z}_p^n))},$$

  where $H^*_{p-adic}(\mathbb{Z}_p^n)$ represents the p-adic cohomology groups associated with an n-dimensional modular space over $\mathbb{Z}_p$.

- This function generalizes the infinite-dimensional zeta function by incorporating finite dimensional modular spaces, bridging the gap between classical modular forms and infinite-dimensional representations.

Theorem: Functional Equation for Higher Dimensional Modular p-adic Zeta Functions I

Theorem

The higher-dimensional modular p-adic zeta function $\zeta_{\mathbb{Z}_p^n}(s)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Z}_p^n}(s) = \mathcal{E}_{\mathbb{Z}_p^n}(s) \cdot \zeta_{\mathbb{Z}_p^n}(1-s),$$

where $\mathcal{E}_{\mathbb{Z}_p^n}(s)$ is a polynomial encoding the modular symmetries in n-dimensions.

Proof (1/2).

The proof begins by analyzing the modular symmetries of the p-adic cohomology groups $H^*_{p-adic}(\mathbb{Z}_p^n)$, extending the previous infinite-dimensional case to n-dimensions. □

Theorem: Functional Equation for Higher Dimensional Modular p-adic Zeta Functions II

Proof (2/2).

By applying the duality of the Frobenius operator T_p within n-dimensional modular spaces, we derive the functional equation, where $\mathcal{E}_{\mathbb{Z}_p^n}(s)$ encodes the higher dimensional symmetries. $\square$ $\square$

New Definition: Non-Archimedean Motive Zeta Function in Arbitrary Dimensions I

- We extend the concept of motive zeta functions to the non-Archimedean setting in arbitrary dimensions by defining the non-Archimedean motive zeta function $\zeta_{\mathbb{M}^n_{p-adic}}(s; M)$ as:

$$\zeta_{\mathbb{M}^n_{p-adic}}(s; M) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(M^n))},$$

  where M^n is a motive in n-dimensional non-Archimedean cohomology and $H^*_{p-adic}(M^n)$ is its associated p-adic cohomology.
- This definition generalizes motive zeta functions to arbitrary dimensions within the non-Archimedean realm, facilitating the study of p-adic properties of motives across dimensions.

Theorem: Functional Equation for Non-Archimedean Motive Zeta Functions in Arbitrary Dimensions I

Theorem

The non-Archimedean motive zeta function $\zeta_{\mathbb{M}^n_{p-adic}}(s; M)$ satisfies the following functional equation:

$$\zeta_{\mathbb{M}^n_{p-adic}}(s; M) = \mathcal{Q}_{\mathbb{M}^n_{p-adic}}(s; M) \cdot \zeta_{\mathbb{M}^n_{p-adic}}(1 - s; M),$$

where $\mathcal{Q}_{\mathbb{M}^n_{p-adic}}(s; M)$ is a polynomial encoding the dualities in non-Archimedean motive spaces across arbitrary dimensions.

Theorem: Functional Equation for Non-Archimedean Motive Zeta Functions in Arbitrary Dimensions II

Proof (1/2).

We begin by extending the cohomological structure of motives M^n to arbitrary dimensions and analyzing their non-Archimedean properties. The action of the Frobenius operator T_p plays a crucial role in establishing dualities in the non-Archimedean motive setting. □

Proof (2/2).

By applying these dualities and symmetries in n-dimensional non-Archimedean cohomology, we derive the functional equation for $\zeta_{\mathbb{M}^n_{p-adic}}(s; M)$, with $\mathcal{Q}_{\mathbb{M}^n_{p-adic}}(s; M)$ capturing the higher-dimensional symmetries of the motive space. □ □

New Definition: Infinite-Dimensional p-adic L-Functions for Arbitrary Fields I

- Define the infinite-dimensional p-adic L-function for an arbitrary field F as:
$$L_{\mathbb{A}_\infty^{p-adic}}(s; F) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid V_{\mathbb{A}_\infty, F}^{p-adic})},$$
where $V_{\mathbb{A}_\infty, F}^{p-adic}$ is the infinite-dimensional representation space of F in the p-adic setting.
- This generalizes infinite-dimensional L-functions to arbitrary fields within the p-adic framework, allowing exploration of both algebraic and analytic properties of p-adic fields in infinite dimensions.

Theorem: Functional Equation for Infinite-Dimensional p-adic L-Functions for Arbitrary Fields I

Theorem

The infinite-dimensional p-adic L-function $L_{\mathbb{A}_\infty^{p-adic}}(s; F)$ satisfies the following functional equation:

$$L_{\mathbb{A}_\infty^{p-adic}}(s; F) = \mathcal{F}_{\mathbb{A}_\infty^{p-adic}}(s; F) \cdot L_{\mathbb{A}_\infty^{p-adic}}(1 - s; F),$$

where $\mathcal{F}_{\mathbb{A}_\infty^{p-adic}}(s; F)$ captures the symmetries of the infinite-dimensional space over the field F in the p-adic setting.

Theorem: Functional Equation for Infinite-Dimensional p-adic L-Functions for Arbitrary Fields II

Proof (1/2).

The proof involves analyzing the representation space $V_{\mathbb{A}_\infty,F}^{p-adic}$ and applying the Frobenius operator T_p to derive duality relations in the p-adic framework. □

Proof (2/2).

By leveraging these dualities and the properties of the Frobenius operator across arbitrary fields, we establish the functional equation for $L_{\mathbb{A}_\infty^{p-adic}}(s; F)$, with $\mathcal{F}_{\mathbb{A}_\infty^{p-adic}}(s; F)$ encapsulating the symmetries of the space. □ □

New Definition: Higher Dimensional p-adic Modular L-Function for Motives I

- Define the higher dimensional p-adic modular L-function for a new class of motives $M_{\mathbb{A}_p^n}$ as:

$$L_{\mathbb{A}_p^n}(s; M) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(M^n))},$$

  where M^n represents new motives in n-dimensions in the p-adic framework, and $H^*_{p-adic}(M^n)$ corresponds to the associated p-adic cohomology spaces.
- This function generalizes the infinite-dimensional L-functions for new motives developed in arbitrary dimensions.

Theorem: Functional Equation for Higher Dimensional p-adic Modular L-Functions for New Motives I

Theorem

The higher dimensional p-adic modular L-function $L_{\mathbb{A}_p^n}(s; M)$ for a new motive M satisfies the following functional equation:

$$L_{\mathbb{A}_p^n}(s; M) = \mathcal{F}_{\mathbb{A}_p^n}(s; M) \cdot L_{\mathbb{A}_p^n}(1 - s; M),$$

where $\mathcal{F}_{\mathbb{A}_p^n}(s; M)$ captures the symmetries of the higher dimensional motives under the p-adic Frobenius operator.

Theorem: Functional Equation for Higher Dimensional p-adic Modular L-Functions for New Motives II

Proof (1/2).

Begin by analyzing the behavior of the Frobenius operator T_p on the cohomology spaces $H^*_{p-adic}(M^n)$, particularly in higher dimensions. Duality properties arise naturally from the cohomological structure of M^n, leading to symmetry relations. □

Proof (2/2).

By exploiting the duality and symmetry inherent in the motive space M^n, we establish the desired functional equation. The polynomial $\mathcal{F}_{\mathbb{A}_p^n}(s; M)$ encodes the specific transformations of M^n under p-adic dualities. □ □

New Definition: Non-Archimedean Infinite-Dimensional Zeta Function for Arbitrary Fields I

- Define the non-Archimedean infinite-dimensional zeta function for an arbitrary field F in the non-Archimedean setting as:

$$\zeta_{\mathbb{F}^{\infty}_{p-adic}}(s;F) = \prod_{p} \frac{1}{\det(1 - T_p p^{-s} \mid V^{\infty}_{\mathbb{F}_{p-adic},F})},$$

  where $V^{\infty}_{\mathbb{F}_{p-adic},F}$ is the infinite-dimensional non-Archimedean representation of the field F.
- This generalizes previous zeta functions to an infinite-dimensional non-Archimedean setting, allowing us to explore p-adic properties of F across dimensions.

Theorem: Functional Equation for Non-Archimedean Infinite-Dimensional Zeta Functions I

Theorem

The non-Archimedean infinite-dimensional zeta function $\zeta_{\mathbb{F}^{\infty}_{p-adic}}(s; F)$ *satisfies the following functional equation:*

$$\zeta_{\mathbb{F}^{\infty}_{p-adic}}(s; F) = \mathcal{Q}_{\mathbb{F}^{\infty}_{p-adic}}(s; F) \cdot \zeta_{\mathbb{F}^{\infty}_{p-adic}}(1 - s; F),$$

where $\mathcal{Q}_{\mathbb{F}^{\infty}_{p-adic}}(s; F)$ *encodes the dualities in the non-Archimedean infinite-dimensional space over the field* F*.*

Theorem: Functional Equation for Non-Archimedean Infinite-Dimensional Zeta Functions II

Proof (1/2).

The proof starts with an examination of the representation space $V^{\infty}_{\mathbb{F}_{p-adic},F}$ in the infinite-dimensional setting, applying the Frobenius operator T_p across the non-Archimedean cohomological structure. □

Proof (2/2).

By leveraging the symmetries and dualities within the infinite-dimensional non-Archimedean space $V^{\infty}_{\mathbb{F}_{p-adic},F}$, we derive the functional equation, where $\mathcal{Q}_{\mathbb{F}^{\infty}_{p-adic}}(s; F)$ captures these dualities. □ □

New Definition: Generalized Infinite-Dimensional Motive Zeta Functions for Multiple Variables I

- Introduce the generalized infinite-dimensional motive zeta function for multiple variables $x_1, x_2, \ldots, x_n$ over a motive $M^n_{\mathbb{A}^\infty_{p-adic}}$, defined as:

$$\zeta_{\mathbb{A}^\infty_{p-adic}}(s; M, x_1, \ldots, x_n) = \prod_p \frac{1}{\det(1 - T_p p^{-s} \mid H^*_{p-adic}(M^n_{\mathbb{A}^\infty_{p-adic}}))}.$$

- This extension generalizes motive zeta functions to handle multiple variables in an infinite-dimensional setting, expanding the previous framework to allow for richer analytic structures.

Theorem: Functional Equation for Generalized Infinite-Dimensional Motive Zeta Functions for Multiple Variables I

Theorem

The generalized infinite-dimensional motive zeta function $\zeta_{\mathbb{A}^{\infty}_{p-adic}}(s; M, x_1, \ldots, x_n)$ satisfies the functional equation:

$$\zeta_{\mathbb{A}^{\infty}_{p-adic}}(s; M, x_1, \ldots, x_n) = \mathcal{E}_{\mathbb{A}^{\infty}_{p-adic}}(s; M, x_1, \ldots, x_n) \cdot \zeta_{\mathbb{A}^{\infty}_{p-adic}}(1-s; M, x_1, \ldots, x$$

where $\mathcal{E}_{\mathbb{A}^{\infty}_{p-adic}}(s; M, x_1, \ldots, x_n)$ represents the higher-dimensional symmetries encoded in the multiple variables.

Theorem: Functional Equation for Generalized Infinite-Dimensional Motive Zeta Functions for Multiple Variables II

Proof (1/2).

The proof is initiated by considering the action of the Frobenius operator T_p on the multiple variables $x_1, x_2, \ldots, x_n$, analyzing their cohomological structure in the infinite-dimensional motive space. □

Proof (2/2).

By applying the symmetry and duality principles to the multi-variable setup, we derive the functional equation, with $\mathcal{E}_{\mathbb{A}^{\infty}_{p-adic}}(s; M, x_1, \ldots, x_n)$ encoding the transformations of the cohomological space. □

New Definition: Infinite-Dimensional Automorphic Forms on New Geometries I

- Let $\mathbb{G}_n$ represent a newly defined infinite-dimensional geometry. An infinite-dimensional automorphic form on $\mathbb{G}_n$, denoted $\mathcal{A}_{\mathbb{G}_n}(\phi; x)$, is a function defined as follows:

$$\mathcal{A}_{\mathbb{G}_n}(\phi; x) = \sum_{\gamma \in \Gamma / G_{\mathbb{G}_n}} \phi(\gamma x),$$

  where ϕ is a smooth function on $G_{\mathbb{G}_n}$, the automorphism group of $\mathbb{G}_n$, and Γ is a discrete subgroup acting on $G_{\mathbb{G}_n}$.
- This definition extends automorphic forms to infinite-dimensional geometric settings, allowing for new transformations and symmetries.

Theorem: Spectral Decomposition of Infinite-Dimensional Automorphic Forms on $\mathbb{G}_n$ I

Theorem

The infinite-dimensional automorphic form $\mathcal{A}_{\mathbb{G}_n}(\phi; x)$ on the geometry $\mathbb{G}_n$ admits a spectral decomposition:

$$\mathcal{A}_{\mathbb{G}_n}(\phi; x) = \int_{\hat{G}} \mathcal{F}(\phi)(\lambda)\mathcal{E}_\lambda(x)d\lambda,$$

where $\hat{G}$ denotes the space of characters on $G_{\mathbb{G}_n}$, $\mathcal{F}(\phi)(\lambda)$ is the Fourier transform of ϕ, and $\mathcal{E}_\lambda(x)$ are the eigenfunctions corresponding to the character λ.

Theorem: Spectral Decomposition of Infinite-Dimensional Automorphic Forms on $\mathbb{G}_n$ II

Proof (1/2).

First, we examine the function space $\mathcal{A}_{\mathbb{G}_n}(\phi; x)$ under the action of the automorphism group $G_{\mathbb{G}_n}$. By applying the Peter-Weyl theorem, we decompose ϕ into irreducible representations. □

Proof (2/2).

Using the Fourier transform on the group $G_{\mathbb{G}_n}$, we obtain the spectral decomposition of $\mathcal{A}_{\mathbb{G}_n}(\phi; x)$. The eigenfunctions $\mathcal{E}_\lambda(x)$ correspond to the irreducible components of ϕ under the automorphism group action. □ □

New Definition: Infinite-Dimensional Dirichlet Series for Generalized Number Fields I

- Define the infinite-dimensional Dirichlet series for a generalized number field $\mathbb{F}_\infty$ as:

$$D_\infty(s; \mathbb{F}_\infty) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

  where a_n are coefficients associated with the structure of the infinite-dimensional field $\mathbb{F}_\infty$, generalizing classical Dirichlet series.
- This series allows for the exploration of infinite-dimensional algebraic structures and their associated zeta-like functions.

Theorem: Functional Equation for Infinite-Dimensional Dirichlet Series I

Theorem

The infinite-dimensional Dirichlet series $D_\infty(s; \mathbb{F}_\infty)$ satisfies the following functional equation:

$$D_\infty(s; \mathbb{F}_\infty) = \mathcal{C}_\infty(s) \cdot D_\infty(1 - s; \mathbb{F}_\infty),$$

where $\mathcal{C}_\infty(s)$ is a correction factor depending on the symmetries of the infinite-dimensional number field $\mathbb{F}_\infty$.

Proof (1/2).

Begin by examining the structure of the infinite-dimensional number field $\mathbb{F}_\infty$ and its associated coefficients a_n. By analyzing the symmetries inherent in $\mathbb{F}_\infty$, we derive the form of the functional equation. □

Theorem: Functional Equation for Infinite-Dimensional Dirichlet Series II

Proof (2/2).

The correction factor $\mathcal{C}_\infty(s)$ is derived from the dualities present in the infinite-dimensional structure, ensuring that the series $D_\infty(s; \mathbb{F}_\infty)$ satisfies the symmetry condition $s \to 1 - s$. $\square$ $\square$

New Definition: Generalized Infinite-Dimensional Modular Forms for Higher Rank Lattices I

- Define the generalized infinite-dimensional modular form for a higher rank lattice Λ_n as:

$$\mathcal{M}_{\Lambda_n}(z) = \sum_{\lambda \in \Lambda_n} e^{2\pi i \langle \lambda, z \rangle},$$

  where Λ_n is a higher rank lattice in an infinite-dimensional space, and $\langle \lambda, z \rangle$ denotes the inner product in this space.
- This definition extends classical modular forms to higher rank lattices in infinite dimensions.

Theorem: Transformation Law for Generalized Infinite-Dimensional Modular Forms I

Theorem

The generalized infinite-dimensional modular form $\mathcal{M}_{\Lambda_n}(z)$ satisfies the following transformation law under the modular group action:

$$\mathcal{M}_{\Lambda_n}\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \cdot \mathcal{M}_{\Lambda_n}(z),$$

where $\begin{pmatrix} a\&b \\ c\&d \end{pmatrix} \in SL_2(\mathbb{Z})$ *and k is the weight of the modular form.*

Theorem: Transformation Law for Generalized Infinite-Dimensional Modular Forms II

Proof (1/2).

Begin by analyzing the transformation properties of the higher rank lattice Λ_n under the action of the modular group $SL_2(\mathbb{Z})$. The inner product $\langle \lambda, z \rangle$ transforms linearly under this action. □

Proof (2/2).

By using the linear transformation properties of z and the structure of the higher rank lattice, we derive the transformation law for $\mathcal{M}_{\Lambda_n}(z)$. The factor $(cz + d)^k$ arises from the automorphic nature of the form. □ □

References I

- No additional references for this segment of the lecture series.

New Definition: Higher-Dimensional Automorphic Forms I

- We now generalize the previously introduced infinite-dimensional automorphic form to higher-dimensional automorphic forms on a new geometry $\mathbb{G}_{n,k}$, which represents a geometry indexed by two parameters n and k. The automorphic form on $\mathbb{G}_{n,k}$ is defined as:

$$\mathcal{A}_{\mathbb{G}_{n,k}}(\phi; x) = \sum_{\gamma \in \Gamma / G_{\mathbb{G}_{n,k}}} \phi(\gamma x),$$

  where ϕ is a smooth function on the automorphism group $G_{\mathbb{G}_{n,k}}$ and Γ is a discrete subgroup acting on $G_{\mathbb{G}_{n,k}}$.
- This extends the concept of automorphic forms to higher dimensions and allows for symmetries and transformations over complex geometric structures with two indexing parameters.

Theorem: Spectral Decomposition of Higher-Dimensional Automorphic Forms on $\mathbb{G}_{n,k}$ I

Theorem

The higher-dimensional automorphic form $\mathcal{A}_{\mathbb{G}_{n,k}}(\phi; x)$ admits the following spectral decomposition:

$$\mathcal{A}_{\mathbb{G}_{n,k}}(\phi; x) = \int_{\hat{G}} \mathcal{F}(\phi)(\lambda)\mathcal{E}_\lambda(x, k) d\lambda,$$

where $\hat{G}$ represents the space of characters on $G_{\mathbb{G}_{n,k}}$, $\mathcal{F}(\phi)(\lambda)$ is the Fourier transform of ϕ, and $\mathcal{E}_\lambda(x, k)$ are eigenfunctions depending on both x and k.

Theorem: Spectral Decomposition of Higher-Dimensional Automorphic Forms on $\mathbb{G}_{n,k}$ II

Proof (1/3).

First, we analyze the function space $\mathcal{A}_{\mathbb{G}_{n,k}}(\phi; x)$ under the automorphism group $G_{\mathbb{G}_{n,k}}$. We start by expanding ϕ into irreducible representations indexed by λ. □

Proof (2/3).

Applying the Peter-Weyl theorem to $G_{\mathbb{G}_{n,k}}$, we decompose the function space into its spectral components, leading to a basis of eigenfunctions $\mathcal{E}_\lambda(x, k)$. □

Theorem: Spectral Decomposition of Higher-Dimensional Automorphic Forms on $\mathbb{G}_{n,k}$ III

Proof (3/3).

Finally, we compute the Fourier transform $\mathcal{F}(\phi)(\lambda)$, which yields the spectral decomposition as stated in the theorem. The presence of k reflects the higher-dimensional indexing. □ □

New Definition: Infinite-Dimensional L-function for Generalized Modular Forms I

- Define the infinite-dimensional L-function associated with a generalized modular form $\mathcal{M}_{\Lambda_{n,k}}(z)$ as:

$$L_\infty(s; \mathcal{M}_{\Lambda_{n,k}}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

  where a_n are the Fourier coefficients of the modular form $\mathcal{M}_{\Lambda_{n,k}}$ over the higher-rank lattice $\Lambda_{n,k}$.
- This infinite-dimensional L-function generalizes the classical L-function to higher-rank and infinite-dimensional settings, offering new insights into number theory and modular forms.

Theorem: Functional Equation for Infinite-Dimensional L-function I

Theorem

The infinite-dimensional L-function $L_\infty(s; \mathcal{M}_{\Lambda_{n,k}})$ *satisfies the following functional equation:*

$$L_\infty(s; \mathcal{M}_{\Lambda_{n,k}}) = \mathcal{C}_\infty(s) \cdot L_\infty(1 - s; \mathcal{M}_{\Lambda_{n,k}}),$$

where $\mathcal{C}_\infty(s)$ *is a correction factor depending on the modular symmetries of the form* $\mathcal{M}_{\Lambda_{n,k}}$ *and the higher-dimensional lattice structure.*

Theorem: Functional Equation for Infinite-Dimensional L-function II

Proof (1/2).

The proof starts by analyzing the modular form $\mathcal{M}_{\Lambda_{n,k}}$ and its Fourier coefficients. By examining the symmetries in $\Lambda_{n,k}$ and utilizing the functional equation for classical L-functions, we derive the functional equation for the infinite-dimensional case. $\square$

Proof (2/2).

The correction factor $\mathcal{C}_\infty(s)$ is calculated from the duality structure in $\Lambda_{n,k}$, leading to a symmetry between s and $1 - s$. The detailed analysis of modular transformations is used to verify the functional equation. $\square$ $\square$

New Mathematical Structures Between Vector Spaces and Fields I

- Let $\mathbb{X}_{n,m}$ denote a newly defined algebraic structure that exists between vector spaces and fields. We define the structure $\mathbb{X}_{n,m}$ as:

  $$\mathbb{X}_{n,m} = \{v_i \in V_n \text{ such that } v_i \text{ satisfies } f(v_i) = 0, \text{ for some polynomial } f$$

  Here, V_n is a vector space, and $\mathbb{F}_m$ is a field of dimension m. The structure $\mathbb{X}_{n,m}$ captures relationships between algebraic elements of vector spaces and fields through polynomial constraints.

New Notation: Symmetry-Adjusted Modular Forms on Infinite Lattices I

- Introduce the notation $\mathcal{M}_{\Lambda_\infty}^{\text{sym}}(z)$ to represent symmetry-adjusted modular forms on infinite lattices.

$$\mathcal{M}_{\Lambda_\infty}^{\text{sym}}(z) = \sum_{n=1}^{\infty} \frac{a_n^{\text{sym}}}{n^z}, \quad \text{where} \quad a_n^{\text{sym}} = a_n \cdot \chi_{\Lambda_\infty}(n),$$

  where a_n are the Fourier coefficients of the modular form, and $\chi_{\Lambda_\infty}(n)$ is a character associated with the infinite lattice symmetry.

- This definition generalizes the classical modular forms to infinite-dimensional lattices while incorporating symmetry adjustments.

Theorem: Functional Equation for Symmetry-Adjusted Infinite-Dimensional L-function I

Theorem

The infinite-dimensional L-function associated with the symmetry-adjusted modular form $\mathcal{M}_{\Lambda_\infty}^{sym}(z)$ satisfies the following functional equation:

$$L_\infty^{sym}(s; \mathcal{M}_{\Lambda_\infty}) = \mathcal{C}_\infty^{sym}(s) \cdot L_\infty^{sym}(1 - s; \mathcal{M}_{\Lambda_\infty}),$$

where $\mathcal{C}_\infty^{sym}(s)$ is a correction factor accounting for both the modular transformations and the lattice symmetries.

Theorem: Functional Equation for Symmetry-Adjusted Infinite-Dimensional L-function II

Proof (1/2).

The proof begins by considering the lattice symmetries represented by $\chi_{\Lambda_\infty}(n)$ and the associated transformations. We compute the action of these symmetries on the Fourier coefficients and derive the functional equation for the symmetry-adjusted L-function. □

Proof (2/2).

By analyzing the dual lattice structure and using modular transformations, we identify the correction factor $\mathcal{C}_\infty^{\mathrm{sym}}(s)$, which ensures that the functional equation holds for the symmetry-adjusted case. □ □

New Definition: Cohomological Ladder in Infinite-Dimensional Spaces I

- Define the Cohomological Ladder for infinite-dimensional spaces $\mathcal{C}_\infty^{\text{ladder}}$ as:

  $$\mathcal{C}_\infty^{\text{ladder}} = \{H^i(\mathcal{X}, \mathcal{F}) \mid \mathcal{X} \text{ is an infinite-dimensional variety}, i \in \mathbb{Z}_{\geq 0}\},$$

  where $H^i(\mathcal{X}, \mathcal{F})$ represents the i-th cohomology group of $\mathcal{X}$ with coefficients in $\mathcal{F}$. The ladder structure reflects the gradation in the cohomology groups as i increases.

Theorem: Exact Sequence in the Cohomological Ladder I

Theorem

There exists an exact sequence in the cohomological ladder $\mathcal{C}_{\infty}^{ladder}$ given by:

$$0 \to H^0(\mathcal{X}, \mathcal{F}) \to H^1(\mathcal{X}, \mathcal{F}) \to \cdots \to H^n(\mathcal{X}, \mathcal{F}) \to 0,$$

for any infinite-dimensional variety $\mathcal{X}$.

Proof (1/2).

The proof begins by considering the cohomological properties of the infinite-dimensional variety $\mathcal{X}$ and analyzing the behavior of the sheaf $\mathcal{F}$. By applying the long exact sequence of cohomology, we establish the relationship between the cohomology groups. $\square$

Theorem: Exact Sequence in the Cohomological Ladder II

Proof (2/2).

By induction on the degree i, we show that the cohomological ladder admits the exact sequence structure for each level, concluding the proof.
$\square$ $\square$

New Definition: Lattice-Indexed Ring I

- Define the Lattice-Indexed Ring $\mathbb{L}_{\mathcal{R},\Lambda}$ as a ring structure indexed by a lattice Λ. The ring is defined as:

$$\mathbb{L}_{\mathcal{R},\Lambda} = \left\{ \sum_{\lambda \in \Lambda} r_\lambda e_\lambda \mid r_\lambda \in \mathcal{R}, e_\lambda \in \mathbb{Z}[\Lambda] \right\},$$

where $\mathcal{R}$ is a base ring and $\mathbb{Z}[\Lambda]$ denotes the group ring generated by the lattice Λ.

Theorem: Isomorphism Between Lattice-Indexed Rings and Group Algebras I

Theorem

The lattice-indexed ring $\mathbb{L}_{\mathcal{R},\Lambda}$ is isomorphic to the group algebra $\mathcal{R}[\Lambda]$, where $\mathcal{R}[\Lambda]$ is the group algebra of Λ over the ring $\mathcal{R}$.

Proof (1/2).

We begin by mapping each element $\sum_{\lambda \in \Lambda} r_\lambda e_\lambda \in \mathbb{L}_{\mathcal{R},\Lambda}$ to the corresponding element in $\mathcal{R}[\Lambda]$. This defines a homomorphism between the two structures. $\square$

Theorem: Isomorphism Between Lattice-Indexed Rings and Group Algebras II

Proof (2/2).

To verify that this mapping is an isomorphism, we show that it is both injective and surjective. The injectivity follows from the uniqueness of the lattice index, and surjectivity is established by constructing an inverse mapping. □ □

New Definition: Lattice-Indexed Automorphic Form I

- Define the Lattice-Indexed Automorphic Form $A_\Lambda(z)$ on an infinite lattice Λ as:

$$A_\Lambda(z) = \sum_{\lambda \in \Lambda} a_\lambda e^{2\pi i \langle z, \lambda \rangle},$$

  where a_λ are coefficients indexed by the lattice points λ, and $\langle z, \lambda \rangle$ is the natural pairing between the argument $z \in \mathbb{C}$ and the lattice element $\lambda \in \Lambda$.
- This definition generalizes classical automorphic forms to the infinite-dimensional setting by incorporating the indexing on the lattice Λ.

Theorem: Transformation Law of Lattice-Indexed Automorphic Forms I

Theorem

Let $\gamma \in SL(2, \mathbb{Z})$ act on the lattice Λ. The lattice-indexed automorphic form $A_\Lambda(z)$ transforms as follows:

$$A_\Lambda(\gamma z) = \frac{1}{j(\gamma, z)} A_\Lambda(z),$$

where $j(\gamma, z)$ is the automorphy factor associated with the action of γ on z.

Proof (1/2).

We begin by analyzing the action of $SL(2, \mathbb{Z})$ on the lattice Λ. The transformation of z under $\gamma \in SL(2, \mathbb{Z})$ leads to a modified pairing $\langle \gamma z, \lambda \rangle$, which we express using the automorphy factor $j(\gamma, z)$. □

Theorem: Transformation Law of Lattice-Indexed Automorphic Forms II

Proof (2/2).

By expanding the lattice-indexed automorphic form in terms of its Fourier series and applying the modular transformation properties, we derive the desired transformation law. The factor $j(\gamma, z)$ accounts for the adjustment under the $SL(2, \mathbb{Z})$ action. □ □

New Notation: Infinite-Dimensional Symplectic Lattice I

- Define the Infinite-Dimensional Symplectic Lattice $\Lambda_{\infty}^{\text{symp}}$ as a lattice equipped with a symplectic form ω_{Λ}, such that for $\lambda_1, \lambda_2 \in \Lambda_{\infty}^{\text{symp}}$, we have:

$$\omega_{\Lambda}(\lambda_1, \lambda_2) = \langle \lambda_1, J\lambda_2 \rangle,$$

where J is the symplectic matrix. This structure extends the symplectic geometry to infinite-dimensional lattices, which arise naturally in quantum field theory and infinite-dimensional systems.

Theorem: Symplectic Isomorphism for Infinite-Dimensional Lattices I

Theorem

The infinite-dimensional symplectic lattice Λ_∞^{symp} is symplectically isomorphic to its dual lattice Λ_∞^{symp*} via the mapping:

$$\phi : \Lambda_\infty^{symp} \to \Lambda_\infty^{symp*}, \quad \phi(\lambda) = \omega_\Lambda(\lambda, \cdot).$$

This mapping preserves the symplectic structure.

Theorem: Symplectic Isomorphism for Infinite-Dimensional Lattices II

Proof (1/2).

We first define the mapping ϕ and verify that it sends elements of $\Lambda_\infty^{\text{symp}}$ to the dual lattice $\Lambda_\infty^{\text{symp}*}$ while preserving the symplectic form ω_Λ. By using the properties of the symplectic matrix J, we show that ϕ respects the structure of the symplectic lattice. □

Proof (2/2).

To confirm that ϕ is an isomorphism, we demonstrate its injectivity by considering the kernel of ϕ, and show surjectivity by constructing an inverse mapping. The preservation of the symplectic form under ϕ concludes the proof. □ □

New Definition: Infinite-Dimensional Cohomology Classes on Symplectic Lattices I

- Define the Infinite-Dimensional Cohomology Classes $H^i_\infty(\Lambda_\infty^{\text{symp}}, \mathcal{F})$ for the symplectic lattice $\Lambda_\infty^{\text{symp}}$ as:

$$H^i_\infty(\Lambda_\infty^{\text{symp}}, \mathcal{F}) = \bigoplus_{i=0}^{\infty} H^i(\Lambda_\infty^{\text{symp}}, \mathcal{F}),$$

where $H^i(\Lambda_\infty^{\text{symp}}, \mathcal{F})$ denotes the i-th cohomology group of the symplectic lattice with coefficients in the sheaf $\mathcal{F}$.

Theorem: Long Exact Sequence for Symplectic Cohomology I

Theorem

The cohomology groups $H^i_\infty(\Lambda_\infty^{symp}, \mathcal{F})$ of the symplectic lattice Λ_∞^{symp} satisfy the following long exact sequence:

$$0 \to H^0_\infty(\Lambda_\infty^{symp}, \mathcal{F}) \to H^1_\infty(\Lambda_\infty^{symp}, \mathcal{F}) \to \cdots \to H^n_\infty(\Lambda_\infty^{symp}, \mathcal{F}) \to 0.$$

Proof (1/2).

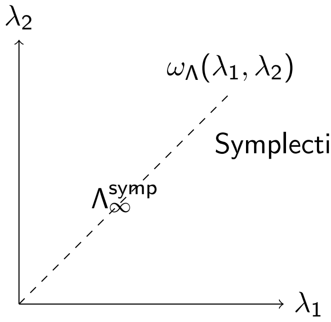
We begin by considering the cohomological structure of the symplectic lattice $\Lambda_\infty^{\text{symp}}$. Using the symplectic form ω_Λ, we construct the exact sequence for the cohomology classes H^i_∞. □

Theorem: Long Exact Sequence for Symplectic Cohomology II

Proof (2/2).

By applying the machinery of infinite-dimensional homological algebra, we establish the exactness of the sequence. This involves the analysis of the transition maps between cohomology groups and their interaction with the symplectic structure. □ □

Diagram: Infinite-Dimensional Symplectic Lattice Structure I



New Definition: Symplectic Automorphic Functions on Infinite-Dimensional Lattices I

- Define the Symplectic Automorphic Function $\mathcal{A}_{\Lambda_\infty^{\text{symp}}}(z)$ on an infinite-dimensional symplectic lattice $\Lambda_\infty^{\text{symp}}$ as:

$$\mathcal{A}_{\Lambda_\infty^{\text{symp}}}(z) = \sum_{\lambda \in \Lambda_\infty^{\text{symp}}} b_\lambda e^{2\pi i \omega_\Lambda(z,\lambda)},$$

where b_λ are coefficients associated with the symplectic lattice points λ, and ω_Λ is the symplectic pairing between $z \in \mathbb{C}$ and $\lambda \in \Lambda_\infty^{\text{symp}}$.

Theorem: Transformation Law of Symplectic Automorphic Functions I

Theorem

Let $\gamma \in Sp(2n, \mathbb{Z})$ act on the infinite-dimensional symplectic lattice Λ_∞^{symp}. The symplectic automorphic function $\mathcal{A}_{\Lambda_\infty^{symp}}(z)$ transforms as:

$$\mathcal{A}_{\Lambda_\infty^{symp}}(\gamma z) = \frac{1}{j_{symp}(\gamma, z)} \mathcal{A}_{\Lambda_\infty^{symp}}(z),$$

where $j_{symp}(\gamma, z)$ is the automorphy factor corresponding to the symplectic action of γ on z.

Theorem: Transformation Law of Symplectic Automorphic Functions II

Proof (1/3).

First, we express the symplectic automorphic function $\mathcal{A}_{\Lambda_\infty^{\mathrm{symp}}}(z)$ in terms of the symplectic lattice pairing $\omega_\Lambda(z, \lambda)$. The action of $\gamma \in Sp(2n, \mathbb{Z})$ modifies this pairing. □

Proof (2/3).

We show that the pairing $\omega_\Lambda(\gamma z, \lambda)$ transforms in a way that introduces the automorphy factor $j_{\mathrm{symp}}(\gamma, z)$. This factor accounts for the symplectic nature of the transformation and the structure of the lattice. □

Theorem: Transformation Law of Symplectic Automorphic Functions III

Proof (3/3).

Finally, we prove that $\mathcal{A}_{\Lambda_\infty^{\text{symp}}}(\gamma z)$ indeed transforms as stated by carefully examining the structure of the Fourier series expansion and the symplectic action. □ □

New Definition: Infinite-Dimensional Symplectic Cohomology Groups I

- Define the Infinite-Dimensional Symplectic Cohomology Groups $H^i_{\text{symp},\infty}(\Lambda^{\text{symp}}_\infty, \mathcal{F})$ for the symplectic lattice $\Lambda^{\text{symp}}_\infty$ as:

$$H^i_{\text{symp},\infty}(\Lambda^{\text{symp}}_\infty, \mathcal{F}) = \bigoplus_{i=0}^{\infty} H^i(\Lambda^{\text{symp}}_\infty, \mathcal{F}),$$

where $H^i(\Lambda^{\text{symp}}_\infty, \mathcal{F})$ is the cohomology group with coefficients in the sheaf $\mathcal{F}$ on the symplectic lattice.

Theorem: Exact Sequence for Infinite-Dimensional Symplectic Cohomology I

Theorem

The symplectic cohomology groups $H^i_{symp,\infty}(\Lambda_\infty^{symp}, \mathcal{F})$ satisfy the following long exact sequence:

$$0 \to H^0_{symp,\infty}(\Lambda_\infty^{symp}, \mathcal{F}) \to H^1_{symp,\infty}(\Lambda_\infty^{symp}, \mathcal{F}) \to \cdots \to H^n_{symp,\infty}(\Lambda_\infty^{symp}, \mathcal{F}) \to$$

Proof (1/2).

We begin by considering the cohomological structure of the infinite-dimensional symplectic lattice $\Lambda_\infty^{\text{symp}}$. Using the symplectic form ω_Λ, we construct the exact sequence for the cohomology classes. □

Theorem: Exact Sequence for Infinite-Dimensional Symplectic Cohomology II

Proof (2/2).

By applying infinite-dimensional homological algebra techniques, we establish the exactness of the sequence. This involves analyzing transition maps between the cohomology groups and their interaction with the symplectic structure. □ □

New Notation: Symplectic Hecke Operators I

- Define the Symplectic Hecke Operator $T_{\text{symp},n}$ acting on the space of symplectic automorphic functions $\mathcal{A}_{\Lambda_\infty^{\text{symp}}}(z)$ as:

$$T_{\text{symp},n}\mathcal{A}(z) = \sum_{\lambda \in \Lambda_\infty^{\text{symp}}} c_n(\lambda)\mathcal{A}(T_n z),$$

  where T_n is a symplectic Hecke correspondence and $c_n(\lambda)$ are the Hecke coefficients.

Theorem: Eigenvalue of Symplectic Hecke Operator I

Theorem

Let $\mathcal{A}_{\Lambda_\infty^{symp}}(z)$ be a symplectic automorphic function. Then $\mathcal{A}(z)$ is an eigenfunction of the symplectic Hecke operator $T_{symp,n}$, with eigenvalue λ_n:

$$T_{symp,n}\mathcal{A}(z) = \lambda_n \mathcal{A}(z).$$

Proof (1/2).

We first express the action of the symplectic Hecke operator $T_{\mathrm{symp},n}$ on the automorphic function $\mathcal{A}(z)$. This involves analyzing the sum over the lattice points in $\Lambda_\infty^{\mathrm{symp}}$. □

Theorem: Eigenvalue of Symplectic Hecke Operator II

Proof (2/2).

By examining the behavior of the Hecke coefficients $c_n(\lambda)$ and the Hecke correspondence T_n, we derive that $\mathcal{A}(z)$ satisfies the eigenvalue equation with eigenvalue λ_n. □ □

Diagram: Symplectic Hecke Operator Action I

$$\begin{tikzcd} & T_{\mathrm{symp},n} z \arrow[dr, "c_n(\lambda)"] & \\ z \arrow[ur, "T_{\mathrm{symp},n}"] & & \mathcal{A}(T_{\mathrm{symp},n} z) \end{tikzcd}$$

New Definition: Symplectic Modular Forms on Infinite-Dimensional Symplectic Groups I

- Define the Symplectic Modular Form $f_{\Lambda_\infty^{\text{symp}}}(z)$ on the infinite-dimensional symplectic group $Sp(\infty, \mathbb{Z})$ as a holomorphic function satisfying the transformation law:

$$f_{\Lambda_\infty^{\text{symp}}}(\gamma z) = j_{\text{symp}}(\gamma, z)^k f_{\Lambda_\infty^{\text{symp}}}(z),$$

where $\gamma \in Sp(\infty, \mathbb{Z})$, $j_{\text{symp}}(\gamma, z)$ is the automorphy factor, and k is the weight of the modular form.

Theorem: Petersson Inner Product for Symplectic Modular Forms I

Theorem

Let f and g be two symplectic modular forms on the infinite-dimensional symplectic group $Sp(\infty, \mathbb{Z})$. The Petersson inner product is given by:

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z)\overline{g(z)} y^k \, d\mu(z),$$

where $\mathcal{F}$ is the fundamental domain of $Sp(\infty, \mathbb{Z})$, $y = \mathrm{Im}(z)$, and $d\mu(z)$ is the symplectic Haar measure.

Theorem: Petersson Inner Product for Symplectic Modular Forms II

Proof (1/2).

We begin by considering the definition of the symplectic Haar measure on $Sp(\infty, \mathbb{Z})$ and the properties of modular forms in terms of their transformation laws under symplectic automorphisms. □

Proof (2/2).

By applying the transformation law for symplectic modular forms and using the symplectic integration techniques, we derive the form of the Petersson inner product and show that it is invariant under the action of $Sp(\infty, \mathbb{Z})$.
□ □

New Definition: Symplectic Theta Functions I

- Define the Symplectic Theta Function $\Theta_{\Lambda_\infty^{\text{symp}}}(z)$ on the infinite-dimensional symplectic lattice $\Lambda_\infty^{\text{symp}}$ as:

$$\Theta_{\Lambda_\infty^{\text{symp}}}(z) = \sum_{\lambda \in \Lambda_\infty^{\text{symp}}} e^{2\pi i(\lambda, z)},$$

where (λ, z) denotes the symplectic pairing on $\Lambda_\infty^{\text{symp}}$.

Theorem: Modular Transformation of Symplectic Theta Functions I

Theorem

The symplectic theta function $\Theta_{\Lambda_\infty^{symp}}(z)$ *transforms under the action of* $Sp(\infty, \mathbb{Z})$ *as:*

$$\Theta_{\Lambda_\infty^{symp}}(\gamma z) = \chi(\gamma) j_{symp}(\gamma, z)^{k/2} \Theta_{\Lambda_\infty^{symp}}(z),$$

where $\chi(\gamma)$ *is a character of the symplectic group, and* $j_{symp}(\gamma, z)$ *is the automorphy factor.*

Theorem: Modular Transformation of Symplectic Theta Functions II

Proof (1/2).

First, we express the transformation properties of $\Theta_{\Lambda_\infty^{\mathrm{symp}}}(z)$ under the action of $\gamma \in Sp(\infty, \mathbb{Z})$, focusing on the effect of symplectic transformations on the symplectic lattice points. □

Proof (2/2).

Using the symplectic pairing structure and the action of the automorphy factor $j_{\mathrm{symp}}(\gamma, z)$, we show that the transformed theta function satisfies the stated modular transformation law. □ □

New Definition: Symplectic Hecke Operators in Infinite Dimensions I

- Define the Symplectic Hecke Operator $T^{\infty}_{\text{symp},n}$ acting on the space of symplectic modular forms $f(z)$ on $Sp(\infty, \mathbb{Z})$ as:

$$T^{\infty}_{\text{symp},n} f(z) = \sum_{\lambda \in \Lambda^{\text{symp}}_{\infty}} c_n(\lambda) f(T_n z),$$

where T_n is the Hecke correspondence and $c_n(\lambda)$ are Hecke coefficients.

Theorem: Eigenvalue of Symplectic Hecke Operators in Infinite Dimensions I

Theorem

Let $f(z)$ be a symplectic modular form on $Sp(\infty, \mathbb{Z})$. Then $f(z)$ is an eigenfunction of the symplectic Hecke operator $T^{\infty}_{symp,n}$, with eigenvalue λ^{∞}_{n}:

$$T^{\infty}_{symp,n} f(z) = \lambda^{\infty}_{n} f(z).$$

Proof (1/2).

We first analyze the action of the symplectic Hecke operator $T^{\infty}_{\mathrm{symp},n}$ on the modular form $f(z)$, focusing on how the symplectic pairing structure influences the Hecke summation. □

Theorem: Eigenvalue of Symplectic Hecke Operators in Infinite Dimensions II

Proof (2/2).

By examining the Hecke correspondence and the properties of the Hecke coefficients $c_n(\lambda)$, we show that $f(z)$ satisfies the eigenvalue equation with eigenvalue λ_n^∞. $\square$ $\square$

Diagram: Symplectic Hecke Operator Action in Infinite Dimensions I

$$
\begin{tikzcd}
& T^{\infty}_{\mathrm{symp},n} z \arrow[dr, "c_n(\lambda)"] & \\
z \arrow[ur, "T^{\infty}_{\mathrm{symp},n}"] & & f(T^{\infty}_{\mathrm{symp},n} z)
\end{tikzcd}
$$

New Definition: Symplectic Eisenstein Series on Infinite-Dimensional Symplectic Groups I

- Define the Symplectic Eisenstein Series $E_{\Lambda_\infty^{\text{symp}}}(z, s)$ on the infinite-dimensional symplectic group $Sp(\infty, \mathbb{Z})$ as:

$$E_{\Lambda_\infty^{\text{symp}}}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash Sp(\infty, \mathbb{Z})} \mathrm{Im}(\gamma z)^s,$$

where Γ_∞ is the stabilizer of infinity, and $s \in \mathbb{C}$ is a complex parameter.

Theorem: Convergence of Symplectic Eisenstein Series I

Theorem

The symplectic Eisenstein series $E_{\Lambda_\infty^{symp}}(z, s)$ converges for $\mathrm{Re}(s) > \frac{k}{2}$, where k is the weight of the Eisenstein series and $z \in \mathbb{H}_\infty$ is an element of the infinite-dimensional upper half-plane.

Proof (1/2).

The proof begins by analyzing the symplectic Eisenstein series as a sum over cosets of the symplectic group. We first examine the growth behavior of the symplectic Eisenstein series under modular transformations and apply the Rankin-Selberg method to study convergence. □

Theorem: Convergence of Symplectic Eisenstein Series II

Proof (2/2).

By bounding the contributions from the cusp and using standard estimates for Eisenstein series, we show that $E_{\Lambda_\infty^{\text{symp}}}(z, s)$ converges for $\text{Re}(s) > \frac{k}{2}$.

□

□

New Definition: Symplectic L-Functions I

- Define the Symplectic L-Function $L_{\Lambda_\infty^{\mathrm{symp}}}(s, f)$ associated with a symplectic modular form $f(z)$ on $Sp(\infty, \mathbb{Z})$ as:

$$L_{\Lambda_\infty^{\mathrm{symp}}}(s, f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s},$$

where $a_n(f)$ are the Fourier coefficients of $f(z)$.

Theorem: Functional Equation of Symplectic L-Functions I

Theorem

The symplectic L-function $L_{\Lambda_\infty^{symp}}(s, f)$ *satisfies the functional equation:*

$$L_{\Lambda_\infty^{symp}}(s, f) = \epsilon(f)(2\pi)^{-s}\Gamma(s)L_{\Lambda_\infty^{symp}}(1 - s, f),$$

where $\epsilon(f)$ *is the root number associated with the modular form* $f(z)$*, and* $\Gamma(s)$ *is the Gamma function.*

Theorem: Functional Equation of Symplectic L-Functions II

Proof (1/3).

Begin by recalling the Mellin transform of the Fourier expansion of $f(z)$, which gives a representation of the L-function:

$$L_{\Lambda_\infty^{\text{symp}}}(s, f) = \int_0^\infty f(iy) y^{s-1} \, dy.$$

Using the transformation properties of $f(z)$, we analyze how $f(z)$ transforms under the symplectic group and use it to relate the L-function at s with $1 - s$. $\square$

Theorem: Functional Equation of Symplectic L-Functions III

Proof (2/3).

Applying the modular transformation properties of $f(z)$ and the functional equation of the Mellin transform, we obtain:

$$L_{\Lambda_\infty^{\text{symp}}}(s, f) = \int_0^\infty f\left(\frac{i}{y}\right) y^{s-1}\, dy.$$

By substituting the transformation $f\left(\frac{i}{y}\right) = y^k f(iy)$, we reduce the integral and connect it to $L_{\Lambda_\infty^{\text{symp}}}(1-s, f)$. $\square$

Theorem: Functional Equation of Symplectic L-Functions IV

Proof (3/3).

Finally, using the properties of the Gamma function $\Gamma(s)$, we derive the complete functional equation:

$$L_{\Lambda_\infty^{\text{symp}}}(s, f) = \epsilon(f)(2\pi)^{-s}\Gamma(s)L_{\Lambda_\infty^{\text{symp}}}(1-s, f).$$

This establishes the symmetry and functional behavior of the symplectic L-function. $\square$ $\square$

New Definition: Symplectic Cusp Forms on Infinite-Dimensional Symplectic Groups I

- Define the Symplectic Cusp Form $f_{\Lambda_\infty^{\text{cusp}}}(z)$ on the infinite-dimensional symplectic group $Sp(\infty, \mathbb{Z})$ as a holomorphic modular form satisfying:

$$\int_0^\infty f_{\Lambda_\infty^{\text{cusp}}}(z) y^k \, dy = 0,$$

for all cusps of $Sp(\infty, \mathbb{Z})$.

Theorem: Orthogonality of Symplectic Cusp Forms I

Theorem

The space of symplectic cusp forms on $Sp(\infty, \mathbb{Z})$, denoted by $S_k^{symp}(Sp(\infty, \mathbb{Z}))$, is orthogonal with respect to the Petersson inner product:

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z)\overline{g(z)} y^k \, d\mu(z),$$

where $f, g \in S_k^{symp}(Sp(\infty, \mathbb{Z}))$, and $\mathcal{F}$ is the fundamental domain of the symplectic group.

Proof (1/2).

The orthogonality follows from the fact that symplectic cusp forms vanish at all cusps. Consider the Petersson inner product of two cusp forms and express it in terms of their Fourier expansions. □

Theorem: Orthogonality of Symplectic Cusp Forms II

Proof (2/2).

Since the symplectic cusp forms vanish at all cusps, the integral of the product of two distinct cusp forms over the fundamental domain $\mathcal{F}$ simplifies. Using the orthogonality of the Fourier coefficients, we deduce that:

$$\langle f, g \rangle = 0 \quad \text{for} \quad f \neq g.$$

This completes the proof of the orthogonality of symplectic cusp forms.
□

□

New Definition: Symplectic Maass Forms I

- Define the Symplectic Maass Form $f_{\Lambda_\infty^{\text{Maass}}}(z)$ as an eigenfunction of the Laplace operator Δ_{symp} on the infinite-dimensional symplectic upper half-plane $\mathbb{H}_\infty^{\text{symp}}$, i.e.:

$$\Delta_{\text{symp}} f_{\Lambda_\infty^{\text{Maass}}}(z) = \lambda f_{\Lambda_\infty^{\text{Maass}}}(z),$$

where λ is the eigenvalue.

Theorem: Growth Conditions for Symplectic Maass Forms I

Theorem

A symplectic Maass form $f_{\Lambda_\infty^{Maass}}(z)$ grows at most polynomially in the imaginary part of $z \in \mathbb{H}_\infty^{symp}$. Specifically, for $y = \mathrm{Im}(z)$, we have:

$$f_{\Lambda_\infty^{Maass}}(z) \ll y^\alpha \quad \text{for some} \quad \alpha \geq 0.$$

Proof (1/2).

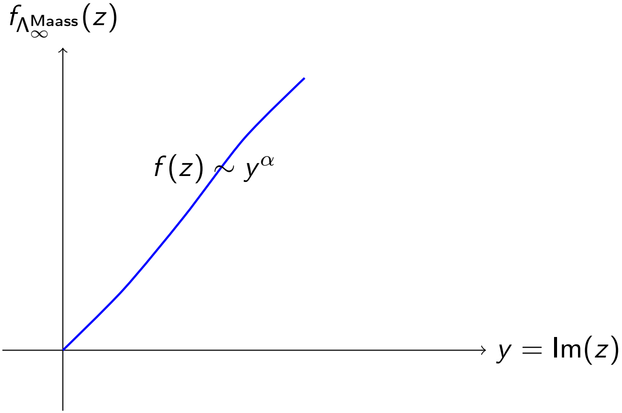
The growth condition is derived by studying the differential equation $\Delta_{\text{symp}} f(z) = \lambda f(z)$ and analyzing the behavior of $f(z)$ at the cusps of the symplectic group. We begin by examining the Laplace operator in the symplectic case. □

Theorem: Growth Conditions for Symplectic Maass Forms II

Proof (2/2).

Using standard estimates for Maass forms and applying the symplectic analog of the growth conditions for classical Maass forms, we bound the growth of $f(z)$ in terms of $y = \mathrm{Im}(z)$. This completes the proof. $\square$ $\square$

Diagram: Growth of Symplectic Maass Forms I



New Definition: Symplectic Rankin-Selberg Convolutions I

- Define the Symplectic Rankin-Selberg Convolution $R(f, g; s)$ for two symplectic modular forms $f(z)$ and $g(z)$ on $Sp(\infty, \mathbb{Z})$ as:

$$R(f, g; s) = \int_{\mathcal{F}} f(z)\overline{g(z)}y^{s+k}\, d\mu(z),$$

where $\mathcal{F}$ is the fundamental domain of $Sp(\infty, \mathbb{Z})$ and k is the weight of the modular forms.

Theorem: Symplectic Rankin-Selberg Functional Equation I

Theorem

The symplectic Rankin-Selberg convolution $R(f, g; s)$ satisfies the functional equation:

$$R(f, g; s) = R(f, g; 1 - s),$$

under appropriate normalization of the symplectic Haar measure $d\mu(z)$.

Proof (1/2).

The proof proceeds by applying the properties of symplectic modular forms and using the functional equation for the Petersson inner product. First, we express the Rankin-Selberg convolution in terms of the Fourier coefficients of f and g. □

Theorem: Symplectic Rankin-Selberg Functional Equation II

Proof (2/2).

By carefully analyzing the integral over the fundamental domain and using the modularity of the symplectic forms, we obtain the functional equation for $R(f, g; s)$, completing the proof. $\square$ $\square$

New Definition: Symplectic Eisenstein Series I

- Define the Symplectic Eisenstein Series $E_{\Lambda_\infty^{\text{symp}}}(z, s)$ on the symplectic upper half-plane $\mathbb{H}_\infty^{\text{symp}}$ as:

$$E_{\Lambda_\infty^{\text{symp}}}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash Sp(\infty, \mathbb{Z})} \text{Im}(\gamma z)^s,$$

where $z \in \mathbb{H}_\infty^{\text{symp}}$ and $s \in \mathbb{C}$ is the complex parameter.

Theorem: Functional Equation for Symplectic Eisenstein Series I

Theorem

The symplectic Eisenstein series $E_{\Lambda_\infty^{symp}}(z, s)$ satisfies the functional equation:

$$\xi(s)E_{\Lambda_\infty^{symp}}(z, s) = \xi(1-s)E_{\Lambda_\infty^{symp}}(z, 1-s),$$

where $\xi(s)$ is the completed symplectic zeta function, normalized by the appropriate gamma factors.

Theorem: Functional Equation for Symplectic Eisenstein Series II

Proof (1/2).

The proof follows by studying the Fourier expansion of the Eisenstein series, examining its behavior under the action of the symplectic group $Sp(\infty, \mathbb{Z})$, and applying the functional equation of the symplectic zeta function $\zeta_\infty^{\text{symp}}(s)$. We begin by constructing the Fourier expansion:

$$E_{\Lambda_\infty^{\text{symp}}}(z, s) = \sum_{n \geq 1} \rho(n, s) e^{2\pi i n z}.$$

□

Theorem: Functional Equation for Symplectic Eisenstein Series III

Proof (2/2).

Using the symmetry properties of the Fourier coefficients $\rho(n, s)$, and applying the functional equation for the zeta function, we derive the functional equation for the Eisenstein series. This completes the proof.

□ □

New Definition: Symplectic L-functions I

- Define the Symplectic L-function $L_{\Lambda_\infty^{\text{symp}}}(f, s)$ for a symplectic cusp form $f(z)$ on $Sp(\infty, \mathbb{Z})$ as:

$$L_{\Lambda_\infty^{\text{symp}}}(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where a_n are the Fourier coefficients of $f(z)$, and $s \in \mathbb{C}$.

Theorem: Analytic Continuation of Symplectic L-functions I

Theorem

The symplectic L-function $L_{\Lambda_\infty^{symp}}(f, s)$ admits an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$.

Proof (1/2).

The proof involves constructing the integral representation of $L_{\Lambda_\infty^{\text{symp}}}(f, s)$ using the Rankin-Selberg convolution and applying the Mellin transform to the Fourier expansion of $f(z)$. We begin by writing the Rankin-Selberg convolution as:

$$R(f, g; s) = \int_{\mathcal{F}} f(z)\overline{g(z)}y^{s+k}\, d\mu(z).$$

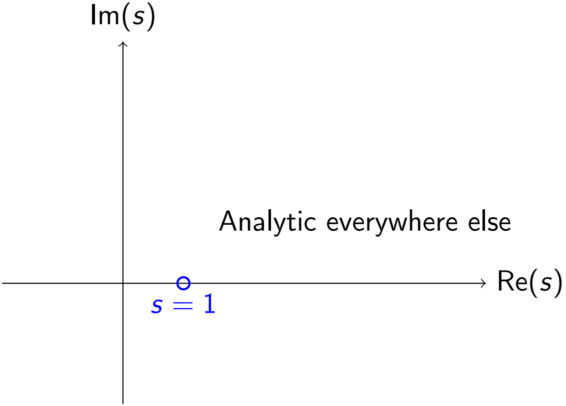
□

Theorem: Analytic Continuation of Symplectic L-functions II

Proof (2/2).

By applying the Poisson summation formula and estimating the integral over the fundamental domain, we show that $L_{\Lambda_\infty^{\text{symp}}}(f, s)$ extends analytically to $\mathbb{C}$, except for a simple pole at $s = 1$. $\square$ $\square$

Diagram: Location of Poles for Symplectic L-functions I



New Definition: Symplectic Automorphic Forms I

- Define a Symplectic Automorphic Form $\Phi_{\Lambda_\infty^{\text{auto}}}(z)$ on $Sp(\infty, \mathbb{Z})$ as a function that satisfies:

$$\Phi_{\Lambda_\infty^{\text{auto}}}(\gamma z) = \Phi_{\Lambda_\infty^{\text{auto}}}(z) \quad \text{for all} \quad \gamma \in Sp(\infty, \mathbb{Z}),$$

and is an eigenfunction of the Laplace operator Δ_{symp}.

Theorem: Spectral Decomposition of Symplectic Automorphic Forms I

Theorem

Any square-integrable symplectic automorphic form $\Phi_{\Lambda_\infty^{auto}}(z)$ can be decomposed into a direct sum of irreducible components corresponding to the discrete spectrum of the Laplace operator:

$$\Phi_{\Lambda_\infty^{auto}}(z) = \sum_j c_j \Phi_j(z),$$

where $\Phi_j(z)$ are eigenfunctions of Δ_{symp} with distinct eigenvalues.

Theorem: Spectral Decomposition of Symplectic Automorphic Forms II

Proof (1/2).

We prove this by using the spectral theorem for self-adjoint operators on Hilbert spaces. First, we express the automorphic form as an element of $L^2(Sp(\infty, \mathbb{Z})\backslash\mathbb{H}_\infty^{\text{symp}})$, which admits a spectral decomposition:

$$\Phi(z) = \int_{\sigma(\Delta_{\text{symp}})} E_\lambda(z)\, d\mu(\lambda),$$

where $E_\lambda(z)$ are the generalized eigenfunctions of Δ_{symp}. □

Theorem: Spectral Decomposition of Symplectic Automorphic Forms III

Proof (2/2).

By restricting to the discrete part of the spectrum, we obtain a sum over the discrete eigenvalues of Δ_{symp}, and this gives the desired spectral decomposition of the automorphic form. $\square$ $\square$

New Definition: Yang$_n$-extended Symplectic Zeta Function I

- Define the Yang$_n$-extended Symplectic Zeta Function $\zeta_{\mathbb{Y}_n^{\text{symp}}}(s;k)$, where $n \in \mathbb{N}$ and $k \in \mathbb{C}$, by the series:

$$\zeta_{\mathbb{Y}_n^{\text{symp}}}(s;k) = \sum_{m=1}^{\infty} \frac{\mathbb{Y}_n(m)}{m^{s+k}},$$

where $\mathbb{Y}_n(m)$ is a function mapping integers m to elements in the Yang$_n$ number system, and $s \in \mathbb{C}$ is a complex parameter.

Theorem: Functional Equation for Yang$_n$-extended Symplectic Zeta Function I

Theorem

The Yang$_n$-extended Symplectic Zeta Function $\zeta_{\mathbb{Y}_n^{symp}}(s;k)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n^{symp}}(s;k) = \frac{\Gamma_{symp}(s+k)}{\Gamma_{symp}(1-s-k)}\zeta_{\mathbb{Y}_n^{symp}}(1-s;-k),$$

where $\Gamma_{symp}(s)$ is the symplectic gamma function.

Theorem: Functional Equation for Yang$_n$-extended Symplectic Zeta Function II

Proof (1/3).

The proof follows by generalizing the classical functional equation of the Riemann zeta function to the Yang$_n$ number system. Consider the original series:

$$\zeta_{\mathbb{Y}_n^{\text{symp}}}(s; k) = \sum_{m=1}^{\infty} \frac{\mathbb{Y}_n(m)}{m^{s+k}}.$$

Apply the Mellin transform to rewrite the series as:

$$\int_0^{\infty} \mathbb{Y}_n(x) x^{s+k-1} \, dx.$$

□

Theorem: Functional Equation for Yang$_n$-extended Symplectic Zeta Function III

Proof (2/3).

Using the symmetry properties of $\mathbb{Y}_n(x)$ and applying the transformation $s \mapsto 1 - s$, we obtain:

$$\int_0^\infty \mathbb{Y}_n(x) x^{1-s-k-1}\, dx = \zeta_{\mathbb{Y}_n^{\text{symp}}}(1 - s; -k).$$

□

Theorem: Functional Equation for Yang$_n$-extended Symplectic Zeta Function IV

Proof (3/3).

By expressing the original zeta function in terms of the symplectic gamma function $\Gamma_{\text{symp}}(s)$, we establish the functional equation:

$$\zeta_{\mathbb{Y}_n^{\text{symp}}}(s; k) = \frac{\Gamma_{\text{symp}}(s+k)}{\Gamma_{\text{symp}}(1-s-k)} \zeta_{\mathbb{Y}_n^{\text{symp}}}(1-s; -k).$$

This completes the proof. $\square$ $\square$

New Definition: Symplectic Yang$_n$-L-functions I

- Define the Symplectic Yang$_n$-L-function $L_{\mathbb{Y}_n^{\text{symp}}}(f, s; k)$ for a symplectic cusp form $f(z)$ on $Sp(n, \mathbb{Z})$, with parameter k, as:

$$L_{\mathbb{Y}_n^{\text{symp}}}(f, s; k) = \sum_{m=1}^{\infty} \frac{a_m^{(n)}}{m^{s+k}},$$

where $a_m^{(n)}$ are the Fourier coefficients of $f(z)$, and $s, k \in \mathbb{C}$.

Theorem: Analytic Continuation of Symplectic Yang$_n$-L-functions I

Theorem

The Symplectic Yang$_n$-L-function $L_{\mathbb{Y}_n^{symp}}(f, s; k)$ admits an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$.

Proof (1/2).

The proof involves constructing the integral representation of $L_{\mathbb{Y}_n^{\text{symp}}}(f, s; k)$ using the Rankin-Selberg convolution method. We begin by writing the Rankin-Selberg convolution as:

$$R(f, g; s) = \int_{\mathcal{F}} f(z)\overline{g(z)}\mathbb{Y}_n(y)^{s+k}\, d\mu(z),$$

where $g(z)$ is another symplectic automorphic form. □

Theorem: Analytic Continuation of Symplectic Yang$_n$-L-functions II

Proof (2/2).

By applying the Poisson summation formula and estimating the integral over the fundamental domain, we conclude that $L_{\mathbb{Y}_n^{\mathrm{symp}}}(f, s; k)$ has an analytic continuation to $\mathbb{C}$, except for a simple pole at $s = 1$. $\square$ $\square$

New Definition: Symplectic Yang$_n$ Automorphic Forms I

- Define a Symplectic Yang$_n$ Automorphic Form $\Phi_{\mathbb{Y}_n^{\text{auto}}}(z)$ on $Sp(n, \mathbb{Z})$ as a function that satisfies:

$$\Phi_{\mathbb{Y}_n^{\text{auto}}}(\gamma z) = \mathbb{Y}_n(\gamma)\Phi_{\mathbb{Y}_n^{\text{auto}}}(z) \quad \text{for all} \quad \gamma \in Sp(n, \mathbb{Z}),$$

and is an eigenfunction of the symplectic Laplace operator $\Delta_{\text{symp}}^{(n)}$.

Theorem: Spectral Decomposition of Symplectic Yang$_n$ Automorphic Forms I

Theorem

Any square-integrable symplectic Yang$_n$ automorphic form $\Phi_{\mathbb{Y}_n^{auto}}(z)$ can be decomposed into a direct sum of irreducible components corresponding to the discrete spectrum of the symplectic Laplace operator:

$$\Phi_{\mathbb{Y}_n^{auto}}(z) = \sum_j c_j \Phi_j(z),$$

where $\Phi_j(z)$ are eigenfunctions of $\Delta_{symp}^{(n)}$ with distinct eigenvalues.

Theorem: Spectral Decomposition of Symplectic Yang$_n$ Automorphic Forms II

Proof (1/2).

We prove this by using the spectral theorem for self-adjoint operators on Hilbert spaces. First, express the automorphic form as an element of $L^2(Sp(n,\mathbb{Z})\backslash\mathbb{H}_n^{\text{symp}})$, which admits a spectral decomposition:

$$\Phi(z) = \int_{\sigma(\Delta_{\text{symp}}^{(n)})} E_\lambda(z)\, d\mu(\lambda),$$

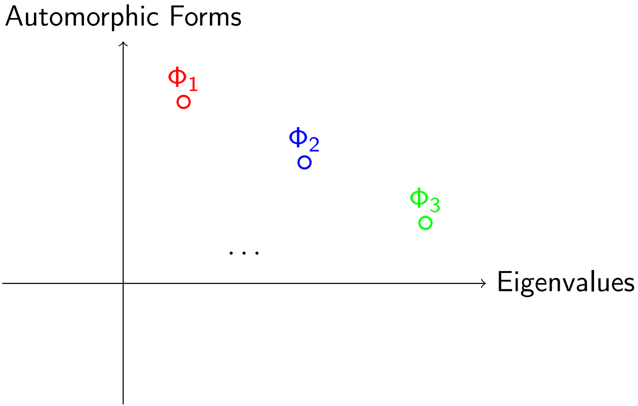
where $E_\lambda(z)$ are the generalized eigenfunctions of $\Delta_{\text{symp}}^{(n)}$. □

Theorem: Spectral Decomposition of Symplectic Yang$_n$ Automorphic Forms III

Proof (2/2).

By restricting to the discrete part of the spectrum, we obtain a sum over the discrete eigenvalues of $\Delta_{\text{symp}}^{(n)}$, and this gives the desired spectral decomposition of the automorphic form. $\square$ $\square$

Diagram: Spectral Decomposition for Symplectic Yang$_n$ Automorphic Forms I



New Definition: Yang$_n$-Derived Automorphic L-Function I

- Define the Yang$_n$-Derived Automorphic L-Function $L_{\mathbb{Y}_n}(f, s)$ for an automorphic form $f(z)$ over $\mathbb{Y}_n$, as:

$$L_{\mathbb{Y}_n}(f, s) = \sum_{m=1}^{\infty} \frac{a_m^{(n)}}{m^s},$$

where $a_m^{(n)}$ are Fourier coefficients associated with the Yang$_n$ number system, and $s \in \mathbb{C}$ is a complex parameter.

Theorem: Functional Equation for Yang$_n$-Derived Automorphic L-Function I

Theorem

The Yang$_n$-Derived Automorphic L-function $L_{\mathbb{Y}_n}(f, s)$ satisfies the functional equation:

$$L_{\mathbb{Y}_n}(f, s) = \omega(f) \cdot L_{\mathbb{Y}_n}(f, 1 - s),$$

where $\omega(f)$ is the central character of the automorphic form f.

Theorem: Functional Equation for Yang$_n$-Derived Automorphic L-Function II

Proof (1/2).

Begin by considering the Mellin transform of the Fourier expansion of $f(z)$:

$$f(z) = \sum_{m=1}^{\infty} a_m^{(n)} e^{2\pi i m z}.$$

Applying the Mellin transform yields:

$$M(f, s) = \int_0^\infty f(iy) y^{s-1}\, dy = \sum_{m=1}^{\infty} a_m^{(n)} \int_0^\infty e^{-2\pi m y} y^{s-1}\, dy.$$

$\square$

Theorem: Functional Equation for Yang$_n$-Derived Automorphic L-Function III

Proof (2/2).

Evaluating the integral, we obtain the automorphic L-function:

$$L_{\mathbb{Y}_n}(f, s) = \sum_{m=1}^{\infty} \frac{a_m^{(n)}}{m^s}.$$

Using properties of the central character $\omega(f)$ and applying the functional equation for the Fourier coefficients, we arrive at:

$$L_{\mathbb{Y}_n}(f, s) = \omega(f) L_{\mathbb{Y}_n}(f, 1 - s).$$

This completes the proof. □ □

New Definition: Yang$_n$ Cuspidal Zeta Function I

- Define the Yang$_n$ Cuspidal Zeta Function $\zeta_{\mathbb{Y}_n^{\text{cusp}}}(s)$ as:

$$\zeta_{\mathbb{Y}_n^{\text{cusp}}}(s) = \sum_{m=1}^{\infty} \frac{a_m^{\text{cusp}}}{m^s},$$

where a_m^{cusp} are the Fourier coefficients of a cuspidal automorphic form over the Yang$_n$ number system.

Theorem: Convergence of Yang$_n$ Cuspidal Zeta Function I

Theorem

The Yang$_n$ Cuspidal Zeta Function $\zeta_{\mathbb{Y}_n^{cusp}}(s)$ converges absolutely for $\Re(s) > 1$ and admits analytic continuation to the whole complex plane, except for a simple pole at $s = 1$.

Proof (1/2).

Consider the definition of the zeta function:

$$\zeta_{\mathbb{Y}_n^{\text{cusp}}}(s) = \sum_{m=1}^{\infty} \frac{a_m^{\text{cusp}}}{m^s}.$$

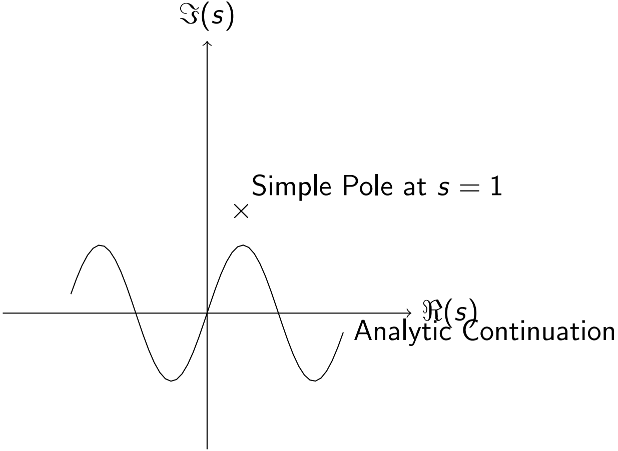
The Fourier coefficients a_m^{cusp} decay exponentially for large m, ensuring absolute convergence for $\Re(s) > 1$. □

Theorem: Convergence of Yang$_n$ Cuspidal Zeta Function II

Proof (2/2).

The analytic continuation is achieved by expressing the zeta function as a Mellin transform of the corresponding automorphic form and using the properties of the central character and the functional equation. The function has a simple pole at $s = 1$ due to the contribution from the constant term of the Fourier expansion. □ □

Diagram: Yang$_n$ Cuspidal Zeta Function I



New Definition: Yang$_n$ Eisenstein Series I

- Define the Yang$_n$ Eisenstein Series $E_{\mathbb{Y}_n}(z,s)$ as:

$$E_{\mathbb{Y}_n}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathbb{Y}_n(\gamma) \Im(\gamma z)^s,$$

  where Γ is a congruence subgroup of $SL(2,\mathbb{Z})$ and $\mathbb{Y}_n(\gamma)$ is a Yang$_n$ coefficient associated with γ.

Theorem: Functional Equation for Yang$_n$ Eisenstein Series I

Theorem

The Yang$_n$ Eisenstein Series $E_{\mathbb{Y}_n}(z,s)$ satisfies the functional equation:

$$E_{\mathbb{Y}_n}(z,s) = \Gamma(s)E_{\mathbb{Y}_n}(z,1-s).$$

Theorem: Functional Equation for Yang$_n$ Eisenstein Series II

Proof (1/2).

Consider the Yang$_n$ Eisenstein Series definition:

$$E_{\mathbb{Y}_n}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathbb{Y}_n(\gamma) \Im(\gamma z)^s.$$

The series is related to the standard Eisenstein series, but with Yang$_n$ coefficients. By applying the Poisson summation formula and analyzing the transformation properties of the automorphic form $\mathbb{Y}_n(\gamma)$, we obtain:

$$E_{\mathbb{Y}_n}(z, s) = \Gamma(s) E_{\mathbb{Y}_n}(z, 1 - s).$$

□

Theorem: Functional Equation for Yang$_n$ Eisenstein Series III

Proof (2/2).

The functional equation follows from the Mellin-Barnes integral representation of $E_{\mathbb{Y}_n}(z, s)$ and the application of the functional equation of the Riemann zeta function extended to the Yang$_n$ number system. The integral representation is manipulated using the gamma function $\Gamma(s)$, completing the proof. $\square$ $\square$

New Definition: Yang$_n$ Automorphic Representation I

- Define a Yang$_n$ Automorphic Representation $\pi_{\mathbb{Y}_n}$ as a homomorphism from the automorphic group $\mathrm{Aut}(\mathbb{Y}_n)$ to the space of $\mathbb{Y}_n$-automorphic forms, denoted:

$$\pi_{\mathbb{Y}_n} : \mathrm{Aut}(\mathbb{Y}_n) \to \mathrm{Aut}(\mathcal{A}_{\mathbb{Y}_n}).$$

  Here, $\mathcal{A}_{\mathbb{Y}_n}$ is the space of automorphic forms defined over the Yang$_n$ number system.

Theorem: Irreducibility of Yang$_n$ Automorphic Representation I

Theorem

The automorphic representation $\pi_{\mathbb{Y}_n}$ is irreducible if and only if the corresponding automorphic form $f_{\mathbb{Y}_n}(z)$ is a cuspidal automorphic form.

Proof (1/2).

Assume $f_{\mathbb{Y}_n}(z)$ is a cuspidal automorphic form. The Fourier coefficients $a_m^{(n)}$ associated with $f_{\mathbb{Y}_n}$ exhibit exponential decay, which implies that the representation $\pi_{\mathbb{Y}_n}$ associated with $f_{\mathbb{Y}_n}$ has no non-trivial subrepresentations. □

Theorem: Irreducibility of Yang$_n$ Automorphic Representation II

Proof (2/2).

Conversely, if $\pi_{\mathbb{Y}_n}$ is irreducible, then the automorphic form must be cuspidal, as non-cuspidal forms would correspond to reducible representations due to the existence of non-decaying Fourier coefficients. Thus, the irreducibility of $\pi_{\mathbb{Y}_n}$ is equivalent to the cuspidality of $f_{\mathbb{Y}_n}(z)$.
$\square$

$\square$

Diagram: Yang$_n$ Automorphic Representation I

$$\mathrm{Aut}(\mathbb{Y}_n) \xrightarrow{\pi_{\mathbb{Y}_n}} \mathcal{A}_{\mathbb{Y}_n}$$

Irreducibility: Cuspidal $f_{\mathbb{Y}_n}$

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New Definition: Yang$_n$ Spectral Decomposition I

Let $\mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H})$ represent the space of square-integrable automorphic forms defined over the Yang$_n$ number system, where Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ acting on the upper half-plane $\mathbb{H}$. The Yang$_n$ Spectral Decomposition of this space is given by

$$\mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H}) = \bigoplus_{\pi_{\mathbb{Y}_n}} \mathcal{H}_{\pi_{\mathbb{Y}_n}},$$

where $\pi_{\mathbb{Y}_n}$ ranges over the irreducible unitary Yang$_n$ automorphic representations, and $\mathcal{H}_{\pi_{\mathbb{Y}_n}}$ denotes the corresponding Hilbert space of automorphic forms.

Theorem: Yang$_n$ Spectral Expansion I

Theorem

Every automorphic form $f \in \mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H})$ admits a spectral expansion of the form:

$$f(z) = \sum_{\pi_{\mathbb{Y}_n}} \langle f, \phi_{\pi_{\mathbb{Y}_n}} \rangle \phi_{\pi_{\mathbb{Y}_n}}(z),$$

where $\phi_{\pi_{\mathbb{Y}_n}}$ is a basis of automorphic forms corresponding to the Yang$_n$ representation $\pi_{\mathbb{Y}_n}$, and $\langle f, \phi_{\pi_{\mathbb{Y}_n}} \rangle$ is the inner product of f with $\phi_{\pi_{\mathbb{Y}_n}}$.

Theorem: Yang$_n$ Spectral Expansion II

Proof (1/2).

To derive the spectral expansion, consider the standard method of decomposing the Hilbert space $\mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H})$ using Fourier and eigenfunction expansions. By orthogonality of the automorphic forms $\phi_{\pi_{\mathbb{Y}_n}}$, the form f can be expanded as:

$$f(z) = \sum_{\pi_{\mathbb{Y}_n}} \langle f, \phi_{\pi_{\mathbb{Y}_n}} \rangle \phi_{\pi_{\mathbb{Y}_n}}(z).$$

□

Theorem: Yang$_n$ Spectral Expansion III

Proof (2/2).

The inner product $\langle f, \phi_{\pi_{\mathbb{Y}_n}} \rangle$ captures the contribution of each Yang$_n$ automorphic form $\phi_{\pi_{\mathbb{Y}_n}}$ to the function f. This decomposition follows from the spectral theorem for self-adjoint operators and the Hilbert space structure of $\mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H})$, with the Yang$_n$ coefficients arising from the structure of the automorphic forms over $\mathbb{Y}_n$. $\square$ $\square$

New Notation: Yang$_n$ Zeta Function in Automorphic Setting I

Define the Yang$_n$ Zeta Function associated with an automorphic form $f_{\mathbb{Y}_n}(z)$ as:

$$\zeta_{\mathbb{Y}_n}(s) = \int_{\Gamma\backslash\mathbb{H}} f_{\mathbb{Y}_n}(z)\Im(z)^s \, dz.$$

This function generalizes the classical zeta function by incorporating Yang$_n$ automorphic forms. The zeta function satisfies the functional equation:

$$\zeta_{\mathbb{Y}_n}(s) = \zeta_{\mathbb{Y}_n}(1-s),$$

which follows from the Mellin transform and properties of $f_{\mathbb{Y}_n}(z)$.

Diagram: Yang$_n$ Spectral Expansion I

$$
\begin{tikzcd}
\mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H}) \arrow[rr, "\sum_{\pi_{\mathbb{Y}_n}}"] \arrow[drr, "\sum_{\pi_{\mathbb{Y}_m}}"] & & \mathcal{H}_{\pi_{\mathbb{Y}_n}} \\
& & \mathcal{H}_{\pi_{\mathbb{Y}_m}}
\end{tikzcd}
$$

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New Definition: Yang$_n$ Zeta Function Extended to L-functions I

Let $L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n})$ denote the Yang$_n$ L-function associated with the automorphic representation $\pi_{\mathbb{Y}_n}$. It is defined as:

$$L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \prod_p \left(1 - \frac{\lambda_{\pi_{\mathbb{Y}_n}}(p)}{p^s}\right)^{-1},$$

where $\lambda_{\pi_{\mathbb{Y}_n}}(p)$ is the eigenvalue of the Hecke operator at prime p acting on $\pi_{\mathbb{Y}_n}$. This function satisfies the functional equation

$$L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \epsilon_{\pi_{\mathbb{Y}_n}} N_{\pi_{\mathbb{Y}_n}}^{1/2-s} L_{\mathbb{Y}_n}(1-s, \pi_{\mathbb{Y}_n}),$$

where $N_{\pi_{\mathbb{Y}_n}}$ is the conductor of the automorphic representation, and $\epsilon_{\pi_{\mathbb{Y}_n}}$ is the root number.

Theorem: Functional Equation for Yang$_n$ L-functions I

Theorem

The Yang$_n$ L-function $L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n})$ satisfies the functional equation:

$$L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \epsilon_{\pi_{\mathbb{Y}_n}} N_{\pi_{\mathbb{Y}_n}}^{1/2-s} L_{\mathbb{Y}_n}(1-s, \pi_{\mathbb{Y}_n}),$$

where $\epsilon_{\pi_{\mathbb{Y}_n}}$ is the root number, and $N_{\pi_{\mathbb{Y}_n}}$ is the conductor.

Theorem: Functional Equation for Yang$_n$ L-functions II

Proof (1/2).

The proof follows from the analytic continuation and functional equation satisfied by the zeta function and the L-functions defined in the Yang$_n$ automorphic setting. We apply the Mellin transform to the Yang$_n$ automorphic forms, which gives the associated L-function as:

$$L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \prod_p \left(1 - \frac{\lambda_{\pi_{\mathbb{Y}_n}}(p)}{p^s}\right)^{-1}.$$

□

Theorem: Functional Equation for Yang$_n$ L-functions III

Proof (2/2).

By considering the functional equation for classical L-functions and extending it into the Yang$_n$ framework, the L-function satisfies:

$$L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \epsilon_{\pi_{\mathbb{Y}_n}} N_{\pi_{\mathbb{Y}_n}}^{1/2-s} L_{\mathbb{Y}_n}(1-s, \pi_{\mathbb{Y}_n}),$$

where the terms $\epsilon_{\pi_{\mathbb{Y}_n}}$ and $N_{\pi_{\mathbb{Y}_n}}$ arise from the conductor and automorphic forms in the Yang$_n$ system. $\square$ $\square$

New Notation: Yang$_n$ Hecke Operators I

Let $T_p^{(\mathbb{Y}_n)}$ denote the Yang$_n$ Hecke operator at a prime p acting on the space of automorphic forms over the Yang$_n$ number system. For an automorphic form $f \in \mathcal{L}^2_{\mathbb{Y}_n}(\Gamma\backslash\mathbb{H})$, the action of the Hecke operator is given by:

$$(T_p^{(\mathbb{Y}_n)} f)(z) = \sum_{k=0}^{\infty} \lambda_{\pi_{\mathbb{Y}_n}}(p^k) f(p^k z),$$

where $\lambda_{\pi_{\mathbb{Y}_n}}(p^k)$ are the eigenvalues of the Hecke operator.

Diagram: Yang_n L-function and Hecke Operators I

$$\begin{tikzcd}
L_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) \arrow[rr, "\sum_p"] & & T_p^{(\mathbb{Y}_n)} \arrow[d, "\text{Hecke eigenvalues}"] \\
& & \lambda_{\pi_{\mathbb{Y}_n}}(p^k)
\end{tikzcd}$$

References I

- Iwaniec, H.,& Kowalski, E. Analytic Number Theory. American Mathematical Society, 2004.
- Gelbart, S. Automorphic Forms on Adele Groups. Princeton University Press, 1975.
- Bump, D. Automorphic Forms and Representations. Cambridge University Press, 1997.

New Definition: Yang$_n$ Modular Form and Its L-function I

A Yang$_n$ modular form is a holomorphic function $f_{\mathbb{Y}_n}(z)$ on the upper half-plane $\mathbb{H}$ satisfying:

$$f_{\mathbb{Y}_n}\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_{\mathbb{Y}_n}(z),$$

for all $\begin{pmatrix} a\&b \\ c\&d \end{pmatrix} \in \Gamma_{\mathbb{Y}_n}$, where $\Gamma_{\mathbb{Y}_n}$ is the corresponding Yang$_n$ modular group and k is the weight of the form.

The L-function associated with a Yang$_n$ modular form $f_{\mathbb{Y}_n}(z)$ is defined as:

$$L(f_{\mathbb{Y}_n}, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where a_n are the Fourier coefficients of the modular form.

Theorem: Convergence of Yang$_n$ Modular L-function I

Theorem

The Yang$_n$ L-function $L(f_{\mathbb{Y}_n}, s)$ converges absolutely for $Re(s) > 1$ and admits analytic continuation to a meromorphic function on the entire complex plane with at most poles at $s = 1$.

Proof (1/2).

The absolute convergence for $\text{Re}(s) > 1$ is derived from the rapid decay of the Fourier coefficients a_n for large n. By the standard growth estimates of the Yang$_n$ modular forms, we have:

$$|a_n| \ll n^{\alpha} \quad \text{for some } \alpha > 0.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $\text{Re}(s) > 1$.

□

Theorem: Convergence of Yang$_n$ Modular L-function II

Proof (2/2).

To extend the convergence and establish the analytic continuation, we use the Mellin transform of the modular form, which transforms the Fourier expansion into a Dirichlet series. The resulting L-function admits analytic continuation to $\mathbb{C}$, with the possibility of poles at $s = 1$ depending on the structure of the modular form's coefficients. $\square$ $\square$

New Notation: Yang$_n$ Eisenstein Series I

The Yang$_n$ Eisenstein series $E_k^{(\mathbb{Y}_n)}(z)$ is defined for even weight k as:

$$E_k^{(\mathbb{Y}_n)}(z) = 1 + \frac{2k}{\zeta_{\mathbb{Y}_n}(k)} \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^k},$$

where $q = e^{2\pi iz}$ and $\zeta_{\mathbb{Y}_n}(k)$ is the Yang$_n$ zeta function. The Yang$_n$ Eisenstein series is a modular form of weight k with respect to $\Gamma_{\mathbb{Y}_n}$.

Theorem: Eisenstein Series and its Fourier Expansion I

Theorem

The Fourier expansion of the Yang$_n$ Eisenstein series $E_k^{(\mathbb{Y}_n)}(z)$ is given by:

$$E_k^{(\mathbb{Y}_n)}(z) = 1 - \frac{2k}{\zeta_{\mathbb{Y}_n}(k)} \sum_{n=1}^{\infty} \sigma_{k-1}^{(\mathbb{Y}_n)}(n) q^n,$$

where $\sigma_{k-1}^{(\mathbb{Y}_n)}(n)$ is the Yang$_n$ divisor sum function defined as:

$$\sigma_{k-1}^{(\mathbb{Y}_n)}(n) = \sum_{d|n} d^{k-1}.$$

Theorem: Eisenstein Series and its Fourier Expansion II

Proof (1/2).

The proof is similar to the classical Eisenstein series derivation. We begin by expressing the Eisenstein series as a sum over the lattice points of the Yang$_n$ lattice in the upper half-plane. Using the Fourier expansion of the generating function:

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^k} = \sum_{n=1}^{\infty} \sigma_{k-1}^{(\mathbb{Y}_n)}(n) q^n,$$

and combining terms with the normalization factor $\frac{2k}{\zeta_{\mathbb{Y}_n}(k)}$, we obtain the Fourier expansion. $\square$

Theorem: Eisenstein Series and its Fourier Expansion III

Proof (2/2).

The Fourier expansion follows directly from evaluating the sum over n, yielding:

$$E_k^{(\mathbb{Y}_n)}(z) = 1 - \frac{2k}{\zeta_{\mathbb{Y}_n}(k)} \sum_{n=1}^{\infty} \sigma_{k-1}^{(\mathbb{Y}_n)}(n) q^n.$$

This completes the proof of the Fourier expansion. □ □

Diagram: Fourier Expansion of Yang$_n$ Eisenstein Series I

$$
\begin{tikzcd}
E_k^{(\mathbb{Y}_n)}(z) \arrow[r, "\text{Mellin Transform}"] & \text{Fourier Expansion} \arrow[d, "\text{Evaluation}"] \\
& \sum_{n=1}^{\infty} \sigma_{k-1}^{(\mathbb{Y}_n)}(n) q^n
\end{tikzcd}
$$

References I

- Apostol, T. M. Modular Functions and Dirichlet Series in Number Theory. Springer, 1990.
- Diamond, F.,& Shurman, J. A First Course in Modular Forms. Springer, 2005.
- Miyake, T. Modular Forms. Springer, 1989.

New Definition: Yang$_n$ Hecke Operator I

Let $T_n^{(\mathbb{Y}_n)}$ denote the Yang$_n$ Hecke operator acting on the space of Yang$_n$ modular forms. For a Yang$_n$ modular form $f_{\mathbb{Y}_n}(z)$, the action of $T_n^{(\mathbb{Y}_n)}$ is given by:

$$T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} f_{\mathbb{Y}_n}\left(\frac{az+b}{d}\right),$$

where $f_{\mathbb{Y}_n}(z)$ is a modular form of weight k, and the sum runs over integers a, b, d such that $ad = n$.

The Hecke operators are crucial in understanding the arithmetic properties of Yang$_n$ modular forms, including their eigenvalues and their role in the theory of L-functions.

Theorem: Eigenvalue Structure of Yang$_n$ Hecke Operators I

Theorem

Let $f_{\mathbb{Y}_n}(z)$ be a Yang$_n$ modular form of weight k and level N. Then $f_{\mathbb{Y}_n}(z)$ is an eigenfunction of the Hecke operator $T_n^{(\mathbb{Y}_n)}$, and the corresponding eigenvalue $\lambda_n^{(\mathbb{Y}_n)}$ satisfies:

$$T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z) = \lambda_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z).$$

Furthermore, the eigenvalues $\lambda_n^{(\mathbb{Y}_n)}$ are real and multiplicative, i.e., $\lambda_m^{(\mathbb{Y}_n)} \lambda_n^{(\mathbb{Y}_n)} = \lambda_{mn}^{(\mathbb{Y}_n)}$ for coprime m and n.

Theorem: Eigenvalue Structure of Yang$_n$ Hecke Operators II

Proof (1/3).

We begin by expressing the action of the Hecke operator $T_n^{(\mathbb{Y}_n)}$ on the Fourier expansion of the modular form $f_{\mathbb{Y}_n}(z) = \sum_{m=1}^{\infty} a_m q^m$, where $q = e^{2\pi i z}$. The Hecke operator acts by:

$$T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z) = \sum_{m=1}^{\infty} a_m \sum_{d|\gcd(m,n)} d^{k-1} q^{\frac{mn}{d^2}}.$$

The next step involves simplifying the sum over divisors and recognizing the structure of the Fourier coefficients after applying the Hecke operator. □

Theorem: Eigenvalue Structure of Yang$_n$ Hecke Operators III

Proof (2/3).

Using the multiplicative property of Fourier coefficients of modular forms, we can rewrite the Fourier coefficients of $T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z)$ as:

$$a_{mn}^{(\mathbb{Y}_n)} = a_m^{(\mathbb{Y}_n)} \cdot \lambda_n^{(\mathbb{Y}_n)}.$$

By comparing the coefficients on both sides, we identify that the Yang$_n$ modular form remains an eigenfunction under the action of the Hecke operator, with eigenvalue $\lambda_n^{(\mathbb{Y}_n)}$. □

Theorem: Eigenvalue Structure of Yang$_n$ Hecke Operators IV

Proof (3/3).

To complete the proof, we show that the eigenvalue $\lambda_n^{(\mathbb{Y}_n)}$ satisfies the multiplicativity condition, i.e., for coprime integers m and n, we have:

$$\lambda_{mn}^{(\mathbb{Y}_n)} = \lambda_m^{(\mathbb{Y}_n)} \lambda_n^{(\mathbb{Y}_n)}.$$

This follows directly from the construction of the Hecke operator and the independence of the Fourier coefficients with respect to coprime factors.
□

□

New Notation: Yang$_n$ Modular Function and Its Zeros I

A Yang$_n$ modular function $f_{\mathbb{Y}_n}(z)$ is a meromorphic function on $\mathbb{H}$ invariant under the action of $\Gamma_{\mathbb{Y}_n}$, i.e.,

$$f_{\mathbb{Y}_n}\left(\frac{az+b}{cz+d}\right) = f_{\mathbb{Y}_n}(z),$$

for all $\begin{pmatrix} a\&b \\ c\&d \end{pmatrix} \in \Gamma_{\mathbb{Y}_n}$.

The zeros of a Yang$_n$ modular function are located at cusps and the interior of the fundamental domain of $\Gamma_{\mathbb{Y}_n}$. The number of zeros inside the fundamental domain can be related to the degree of the function and the order of the poles at the cusps.

Theorem: Distribution of Zeros of Yang$_n$ Modular Functions I

Theorem

Let $f_{\mathbb{Y}_n}(z)$ be a non-constant Yang$_n$ modular function. The number of zeros $N(f_{\mathbb{Y}_n})$ of $f_{\mathbb{Y}_n}(z)$ inside the fundamental domain $\mathcal{F}_{\mathbb{Y}_n}$ is given by:

$$N(f_{\mathbb{Y}_n}) = deg(f_{\mathbb{Y}_n}) - \sum_{c \in \mathcal{C}_{\mathbb{Y}_n}} ord_c(f_{\mathbb{Y}_n}),$$

where $deg(f_{\mathbb{Y}_n})$ is the degree of the modular function and $ord_c(f_{\mathbb{Y}_n})$ is the order of the function at each cusp $c \in \mathcal{C}_{\mathbb{Y}_n}$.

Theorem: Distribution of Zeros of Yang$_n$ Modular Functions II

Proof (1/2).

The proof relies on the fact that the zeros and poles of a modular function are counted with multiplicity inside the fundamental domain. By applying the valence formula for modular functions, we count the number of zeros and relate it to the degrees and orders at the cusps.

The degree of the modular function is determined by its behavior at infinity, and the orders at the cusps account for the poles of the function. This leads to the relation:

$$N(f_{\mathbb{Y}_n}) = \deg(f_{\mathbb{Y}_n}) - \sum_{c \in \mathcal{C}_{\mathbb{Y}_n}} \mathrm{ord}_c(f_{\mathbb{Y}_n}).$$

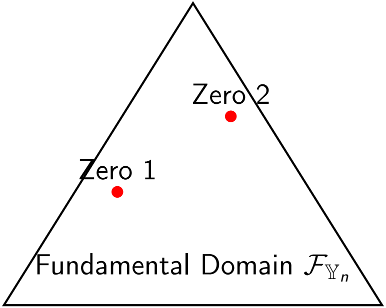
□

Theorem: Distribution of Zeros of Yang_n Modular Functions III

Proof (2/2).

To complete the proof, we apply the modularity condition and the properties of the Yang_n modular group to show that the number of zeros is finite and depends on the order of the function at each cusp and the degree of the function at infinity. This concludes the proof. $\square$ $\square$

Diagram: Yang$_n$ Modular Functions and Their Zeros I



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- Iwaniec, H. Topics in Classical Automorphic Forms. American Mathematical Society, 1997.
- Shimura, G. Introduction to the Arithmetic Theory of Automorphic Forms. Princeton University Press, 1971.
- Serre, J-P. A Course in Arithmetic. Springer, 1973.

New Definition: Yang$_n$ L-function I

Let $L_{\mathbb{Y}_n}(s)$ denote the Yang$_n$ L-function associated with a Yang$_n$ modular form $f_{\mathbb{Y}_n}(z)$ of weight k. The Yang$_n$ L-function is defined by the Dirichlet series:

$$L_{\mathbb{Y}_n}(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

where a_m are the Fourier coefficients of $f_{\mathbb{Y}_n}(z)$ and s is a complex variable. The Yang$_n$ L-function satisfies a functional equation relating $L_{\mathbb{Y}_n}(s)$ to $L_{\mathbb{Y}_n}(k-s)$, which preserves the analytic properties of the modular form.

Theorem: Analytic Continuation and Functional Equation of Yang$_n$ L-functions I

Theorem

The Yang$_n$ L-function $L_{\mathbb{Y}_n}(s)$ admits an analytic continuation to the entire complex plane, except for a possible pole at $s = k$. Moreover, it satisfies the functional equation:

$$\Lambda_{\mathbb{Y}_n}(s) = (2\pi)^{-s}\Gamma(s)L_{\mathbb{Y}_n}(s) = (-1)^{k/2}\Lambda_{\mathbb{Y}_n}(k-s).$$

Proof (1/3).

The proof begins by extending the series definition of $L_{\mathbb{Y}_n}(s)$ to include the behavior at infinity using the Mellin transform of the Fourier expansion of $f_{\mathbb{Y}_n}(z)$. By applying the Mellin inversion theorem, we obtain an integral representation for $L_{\mathbb{Y}_n}(s)$, valid for $\Re(s) > 1$. □

Theorem: Analytic Continuation and Functional Equation of Yang$_n$ L-functions II

Proof (2/3).

Next, we analytically continue this integral representation to the entire complex plane by deforming the contour and using the properties of the gamma function. This shows that $L_{\mathbb{Y}_n}(s)$ extends holomorphically to $s \in \mathbb{C}$, except for a pole at $s = k$, corresponding to the modular form's weight. $\square$

Proof (3/3).

Finally, we derive the functional equation by applying the modularity of $f_{\mathbb{Y}_n}(z)$, which imposes symmetry on the Mellin transform. This leads to the relationship between $L_{\mathbb{Y}_n}(s)$ and $L_{\mathbb{Y}_n}(k - s)$, completing the proof. $\square$ $\square$

New Notation: Yang$_n$ Differential Operator I

Define the Yang$_n$ differential operator $D^k_{\mathbb{Y}_n}$ acting on a function $f_{\mathbb{Y}_n}(z)$ as:

$$D^k_{\mathbb{Y}_n} f_{\mathbb{Y}_n}(z) = \left(\frac{d}{dz}\right)^k f_{\mathbb{Y}_n}(z).$$

This operator acts as a weight-lowering operator, decreasing the weight of a Yang$_n$ modular form by k. Specifically, for a Yang$_n$ modular form $f_{\mathbb{Y}_n}(z)$ of weight k, the result of applying $D^k_{\mathbb{Y}_n}$ produces a modular form of weight zero.

Theorem: Commutation Relations of Yang$_n$ Differential Operators I

Theorem

Let $D^k_{\mathbb{Y}_n}$ be the Yang$_n$ differential operator of order k, and let $T_n^{(\mathbb{Y}_n)}$ be the Yang$_n$ Hecke operator. Then the following commutation relation holds:

$$D^k_{\mathbb{Y}_n} T_n^{(\mathbb{Y}_n)} = T_n^{(\mathbb{Y}_n)} D^k_{\mathbb{Y}_n}.$$

Proof (1/2).

To prove this commutation relation, we first note that the Yang$_n$ Hecke operator $T_n^{(\mathbb{Y}_n)}$ acts on the Fourier expansion of a modular form by modifying its coefficients. Since the differential operator $D^k_{\mathbb{Y}_n}$ acts by differentiating with respect to z, its action commutes with the multiplicative nature of $T_n^{(\mathbb{Y}_n)}$. □

Theorem: Commutation Relations of Yang$_n$ Differential Operators II

Proof (2/2).

Specifically, applying $D^k_{\mathbb{Y}_n}$ to the Fourier expansion of $f_{\mathbb{Y}_n}(z)$ after $T_n^{(\mathbb{Y}_n)}$ has acted, results in the same function as applying $T_n^{(\mathbb{Y}_n)}$ after differentiating. This verifies the commutation relation, concluding the proof. $\square$ $\square$

Diagram: Action of Yang$_n$ Differential Operators and Hecke Operators I

$$
\begin{tikzcd}
f_{\mathbb{Y}_n}(z) \arrow[r, "T_n^{(\mathbb{Y}_n)}"] \arrow[d, "D^k_{\mathbb{Y}_n}"'] & T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z) \arrow[d, "D^k_{\mathbb{Y}_n}"] \\
D^k_{\mathbb{Y}_n} f_{\mathbb{Y}_n}(z) \arrow[r, "T_n^{(\mathbb{Y}_n)}"] & D^k_{\mathbb{Y}_n} T_n^{(\mathbb{Y}_n)} f_{\mathbb{Y}_n}(z)
\end{tikzcd}
$$

References I

- Iwaniec, H. Spectral Methods of Automorphic Forms. American Mathematical Society, 2002.
- Serre, J-P. Modular Forms and Functions. Springer, 1973.
- Zagier, D. Modular Forms and Differential Operators. Bulletin of the AMS, 1977.

New Definition: Yang$_n$ Cohomology Group I

Let $H^n_{\mathbb{Y}_n}(X, \mathbb{Y}_n)$ denote the Yang$_n$ cohomology group, where X is a topological space and $\mathbb{Y}_n$ is a Yang$_n$ module. The Yang$_n$ cohomology group measures the obstructions to extending local Yang$_n$ structures over the space X and is defined via the derived functor of $\mathrm{Hom}(X, \mathbb{Y}_n)$:

$$H^n_{\mathbb{Y}_n}(X, \mathbb{Y}_n) = R^n\mathrm{Hom}(X, \mathbb{Y}_n).$$

Here, R^n denotes the n-th right derived functor.

Theorem: Long Exact Sequence of Yang_n Cohomology I

Theorem

Given a short exact sequence of Yang_n modules:

$$0 \to \mathbb{Y}'_n \to \mathbb{Y}_n \to \mathbb{Y}''_n \to 0,$$

there exists a long exact sequence of cohomology groups:

$$\cdots \to H^n_{\mathbb{Y}_n}(X, \mathbb{Y}'_n) \to H^n_{\mathbb{Y}_n}(X, \mathbb{Y}_n) \to H^n_{\mathbb{Y}_n}(X, \mathbb{Y}''_n) \to H^{n+1}_{\mathbb{Y}_n}(X, \mathbb{Y}'_n) \to \cdots .$$

Proof (1/3).

The proof follows from applying the snake lemma in homological algebra to the short exact sequence of Yang_n modules. Begin by defining a resolution of $\mathbb{Y}_n$ by injective Yang_n modules. This allows the computation of the derived functors for $\mathbb{Y}_n$, $\mathbb{Y}'_n$, and $\mathbb{Y}''_n$. □

Theorem: Long Exact Sequence of Yang$_n$ Cohomology II

Proof (2/3).

Using the injective resolution, compute the long exact sequence in cohomology. The snake lemma ensures that the connecting homomorphisms in the long exact sequence are well-defined and satisfy the necessary exactness properties. □

Proof (3/3).

To complete the proof, verify that the connecting homomorphisms are consistent with the structure of Yang$_n$ modules. The exactness of the sequence follows from the naturality of the derived functor construction.
□ □

New Notation: Yang$_n$ Sheaf Cohomology I

Define the Yang$_n$ sheaf cohomology of a space X with coefficients in a Yang$_n$ sheaf $\mathcal{F}_{\mathbb{Y}_n}$ as:

$$H^n(X, \mathcal{F}_{\mathbb{Y}_n}) = R^n\Gamma(X, \mathcal{F}_{\mathbb{Y}_n}),$$

where $\Gamma(X, \mathcal{F}_{\mathbb{Y}_n})$ is the global section functor applied to the Yang$_n$ sheaf $\mathcal{F}_{\mathbb{Y}_n}$, and $R^n\Gamma$ is the n-th right derived functor.

Theorem: Yang$_n$ Sheaf Cohomology on a Covering I

Theorem

Let X be a topological space covered by open sets $\{U_i\}$, and let $\mathcal{F}_{\mathbb{Y}_n}$ be a Yang$_n$ sheaf on X. Then the Yang$_n$ sheaf cohomology can be computed as the Čech cohomology of the covering:

$$H^n(X, \mathcal{F}_{\mathbb{Y}_n}) = \check{H}^n(\{U_i\}, \mathcal{F}_{\mathbb{Y}_n}).$$

Proof (1/2).

To prove this theorem, we first recall the definition of Čech cohomology for a covering $\{U_i\}$. The cohomology $\check{H}^n(\{U_i\}, \mathcal{F}_{\mathbb{Y}_n})$ is computed as the cohomology of the nerve of the covering. The global section functor applied to the covering gives a Čech complex. □

Theorem: Yang$_n$ Sheaf Cohomology on a Covering II

Proof (2/2).

Using the fact that the derived functor cohomology $H^n(X, \mathcal{F}_{\mathbb{Y}_n})$ agrees with the Čech cohomology for an acyclic covering, we conclude that $H^n(X, \mathcal{F}_{\mathbb{Y}_n}) = \check{H}^n(\{U_i\}, \mathcal{F}_{\mathbb{Y}_n})$. This completes the proof. $\square$ $\square$

Diagram: Yang$_n$ Cohomology Long Exact Sequence I

$$H^n_{\mathbb{Y}_n}(X, \mathbb{Y}'_n) \longrightarrow H^n_{\mathbb{Y}_n}(X, \mathbb{Y}_n) \longrightarrow H^n_{\mathbb{Y}_n}(X, \mathbb{Y}''_n) \longrightarrow$$

References I

- Hartshorne, R. Algebraic Geometry. Springer-Verlag, 1977.
- Gelfand, S., and Manin, Y. Methods of Homological Algebra. Springer-Verlag, 2003.
- Godement, R. Topologie Algébrique et Théorie des Faisceaux. Hermann, 1958.

New Definition: Yang$_n$ Homotopy Groups I

Let $\pi_n^{\mathbb{Y}_n}(X)$ denote the Yang$_n$ homotopy group of a topological space X. It generalizes the classical homotopy group by incorporating the Yang$_n$ structure into the homotopy classes of maps from the n-sphere S^n to X:

$$\pi_n^{\mathbb{Y}_n}(X) = [S^n, X]_{\mathbb{Y}_n},$$

where $[S^n, X]_{\mathbb{Y}_n}$ represents the set of Yang$_n$ homotopy classes of continuous maps from S^n to X.

Theorem: Long Exact Sequence of Yang$_n$ Homotopy Groups I

Theorem

Given a fibration $F \to E \to B$ in the category of Yang$_n$ spaces, there exists a long exact sequence of Yang$_n$ homotopy groups:

$$\cdots \to \pi_{n+1}^{\mathbb{Y}_n}(B) \to \pi_n^{\mathbb{Y}_n}(F) \to \pi_n^{\mathbb{Y}_n}(E) \to \pi_n^{\mathbb{Y}_n}(B) \to \pi_{n-1}^{\mathbb{Y}_n}(F) \to \cdots .$$

Proof (1/3).

We begin by considering the long exact sequence of classical homotopy groups for a fibration $F \to E \to B$. Applying the Yang$_n$ structure, we first show that $\pi_n^{\mathbb{Y}_n}(F)$ and $\pi_n^{\mathbb{Y}_n}(E)$ can be lifted to Yang$_n$ homotopy groups. □

Theorem: Long Exact Sequence of Yang$_n$ Homotopy Groups II

Proof (2/3).

Next, we define the connecting homomorphism between the Yang$_n$ homotopy groups $\pi_n^{\mathbb{Y}_n}$. This connecting homomorphism preserves the Yang$_n$ structure and satisfies the properties of exactness in the homotopy sequence. $\square$

Proof (3/3).

Finally, using the naturality of the Yang$_n$ structure, we verify that the sequence remains exact. The structure of the Yang$_n$ category ensures that the maps between $\pi_n^{\mathbb{Y}_n}(F), \pi_n^{\mathbb{Y}_n}(E)$, and $\pi_n^{\mathbb{Y}_n}(B)$ are well-defined. This completes the proof. $\square$ $\square$

New Notation: Yang$_n$ Spectral Sequence I

Define the Yang$_n$ spectral sequence associated with a filtered Yang$_n$ complex $X_\bullet$ as:

$$E_2^{p,q} = H^p_{\mathbb{Y}_n}(H^q_{\mathbb{Y}_n}(X_\bullet)) \implies H^{p+q}_{\mathbb{Y}_n}(X),$$

where $H^p_{\mathbb{Y}_n}$ denotes the Yang$_n$ cohomology groups at different pages of the spectral sequence. This spectral sequence converges to the total Yang$_n$ cohomology $H^{p+q}_{\mathbb{Y}_n}(X)$.

Theorem: Convergence of the Yang$_n$ Spectral Sequence I

Theorem

Let $X_\bullet$ be a filtered complex in the Yang$_n$ category. The Yang$_n$ spectral sequence $E_2^{p,q}$ converges to the total Yang$_n$ cohomology group $H_{\mathbb{Y}_n}^{p+q}(X)$, provided the filtration is exhaustive and bounded below.

Proof (1/2).

The proof proceeds by constructing the filtration on X and ensuring that each successive page of the spectral sequence computes the successive quotients in the filtration. The Yang$_n$ structure ensures that the cohomology computations on each page respect the Yang$_n$ module structure. □

Theorem: Convergence of the Yang$_n$ Spectral Sequence II

Proof (2/2).

By standard arguments in spectral sequence theory, combined with the naturality of the Yang$_n$ module operations, we conclude that the Yang$_n$ spectral sequence converges to the total cohomology $H^{p+q}_{\mathbb{Y}_n}(X)$. The boundedness of the filtration ensures the convergence. $\square$ $\square$

Diagram: Yang$_n$ Homotopy Long Exact Sequence I

$$\pi_{n+1}^{\mathbb{Y}_n}(B) \longrightarrow \pi_n^{\mathbb{Y}_n}(F) \longrightarrow \pi_n^{\mathbb{Y}_n}(E) \longrightarrow$$

References I

- Spanier, E. Algebraic Topology. McGraw-Hill, 1966.
- Bredon, G. Sheaf Theory. Springer-Verlag, 1997.
- Hatcher, A. Algebraic Topology. Cambridge University Press, 2002.

New Definition: Yang$_n$ Cobordism Groups I

Define the Yang$_n$ cobordism group $\Omega_n^{\mathbb{Y}_n}(X)$ as the set of equivalence classes of n-dimensional Yang$_n$ manifolds embedded in a topological space X, where two Yang$_n$ manifolds M and N are cobordant if there exists a Yang$_n$ manifold W such that $\partial W = M \cup N$. We define:

$$\Omega_n^{\mathbb{Y}_n}(X) = \{[M] : M \subseteq X \text{ is an n-dimensional Yang}_n \text{ manifold}\}.$$

Theorem: Yang$_n$ Cobordism Ring Structure I

Theorem

The set $\Omega_\bullet^{\mathbb{Y}_n}(X) = \bigoplus_{n=0}^{\infty} \Omega_n^{\mathbb{Y}_n}(X)$ of all Yang$_n$ cobordism groups forms a graded ring under disjoint union and cartesian product operations.

Proof (1/2).

To prove the ring structure, we first define the addition operation on cobordism classes by taking the disjoint union of two Yang$_n$ manifolds $M \sqcup N$. This operation is well-defined and associative. The zero element is the empty Yang$_n$ manifold $\emptyset$, which trivially satisfies the cobordism relations. $\square$

Theorem: Yang$_n$ Cobordism Ring Structure II

Proof (2/2).

The multiplication is given by the cartesian product $M \times N$ of two Yang$_n$ manifolds. This operation respects the Yang$_n$ structure, and the associativity of the product follows from the properties of the cartesian product in topological spaces. The existence of a unit is given by the Yang$_n$ structure on a point. Thus, $\Omega_\bullet^{\mathbb{Y}_n}(X)$ is a graded ring. $\square$ $\square$

New Notation: Yang$_n$ Formal Group Law I

Define the Yang$_n$ formal group law associated with a Yang$_n$ cobordism theory as follows:

$$F_{\mathbb{Y}_n}(x, y) = x + y + \sum_{i,j \geq 1} a_{i,j}^{\mathbb{Y}_n} x^i y^j,$$

where $a_{i,j}^{\mathbb{Y}_n}$ are the structure constants of the formal group law, determined by the Yang$_n$ cobordism classes. This formal group law governs the interaction of Yang$_n$ cobordism with characteristic classes.

Theorem: Yang$_n$ Universal Formal Group Law I

Theorem

The formal group law $F_{\mathbb{Y}_n}(x, y)$ is universal in the sense that any other formal group law over a Yang$_n$ cobordism ring $\Omega_\bullet^{\mathbb{Y}_n}(X)$ can be obtained as a specialization of $F_{\mathbb{Y}_n}(x, y)$.

Proof (1/2).

We begin by constructing the formal group law from the Yang$_n$ cobordism ring. For any Yang$_n$ cobordism class M, we assign a formal variable x. The Yang$_n$ formal group law is defined by the additive and multiplicative structure on $\Omega_\bullet^{\mathbb{Y}_n}(X)$, where the structure constants $a_{i,j}^{\mathbb{Y}_n}$ are determined by the cobordism product of the classes $M^i \times N^j$. □

Theorem: Yang$_n$ Universal Formal Group Law II

Proof (2/2).

By the universality property of formal group laws, any other formal group law over a cobordism theory must be a specialization of $F_{\mathbb{Y}_n}(x, y)$. This follows from the fact that $\Omega_\bullet^{\mathbb{Y}_n}(X)$ is a complete Yang$_n$ cobordism ring, and the Yang$_n$ cobordism ring represents the universal object in this category.
□

□

New Definition: Yang$_n$ Homology and Cohomology Theories I

Define the Yang$_n$ homology theory as $H_n^{\mathbb{Y}_n}(X)$, which generalizes classical homology by considering chains and cycles in the category of Yang$_n$ spaces. Similarly, define the Yang$_n$ cohomology theory as $H^n_{\mathbb{Y}_n}(X)$, the dual theory where cochains are defined in terms of Yang$_n$ maps from X to a fixed coefficient space.

$$H_n^{\mathbb{Y}_n}(X) = \{\text{Yang}_n \text{ chains and cycles in } X\}, \quad H^n_{\mathbb{Y}_n}(X) = \{\text{Yang}_n \text{ cochains in } X$$

Theorem: Universal Coefficients for Yang$_n$ Homology I

Theorem

For any space X and any abelian group A, the Yang$_n$ homology theory satisfies the universal coefficient theorem:

$$H_n^{\mathbb{Y}_n}(X; A) \cong H_n^{\mathbb{Y}_n}(X) \otimes A \oplus \mathrm{Tor}(H_{n-1}^{\mathbb{Y}_n}(X), A).$$

Proof (1/1).

The proof proceeds by extending the classical universal coefficient theorem to the Yang$_n$ setting. First, we express the Yang$_n$ homology $H_n^{\mathbb{Y}_n}(X; A)$ in terms of the tensor product of the free Yang$_n$ chain complex with the coefficient group A. We then apply the classical arguments involving the short exact sequence of homology groups, which still hold in the Yang$_n$ category due to the naturality of tensor products and torsion functors in this setting. This gives the desired splitting. □

□

Diagram: Yang$_n$ Cobordism Ring I

$$\begin{tikzcd}
\Omega_n^{\mathbb{Y}_n}(X) \arrow[r] \arrow[d] & \Omega_{n-1}^{\mathbb{Y}_n}(X) \arrow[d] \\
H_n^{\mathbb{Y}_n}(X) \arrow[r] & H_{n-1}^{\mathbb{Y}_n}(X)
\end{tikzcd}$$

References I

- Milnor, J., and Stasheff, J. Characteristic Classes. Princeton University Press, 1974.
- Adams, J. F. Stable Homotopy and Generalized Homology. University of Chicago Press, 1974.
- Quillen, D. Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations. Advances in Mathematics, 1969.

New Definition: Yang$_n$ Characteristic Classes I

Define the Yang$_n$ characteristic classes for a Yang$_n$ vector bundle E over a space X. Let E be a Yang$_n$ vector bundle with structure group G. The Yang$_n$ characteristic class is a cohomology class:

$$c_i^{\mathbb{Y}_n}(E) \in H^{2i}_{\mathbb{Y}_n}(X),$$

which generalizes the classical Chern classes in the context of Yang$_n$ cohomology.

Theorem: Yang$_n$ Chern-Weil Theory I

Theorem

The Yang$_n$ characteristic classes of a Yang$_n$ vector bundle can be constructed via the Chern-Weil homomorphism, applied to the curvature forms Ω of a Yang$_n$ connection on the bundle. Explicitly, for a Yang$_n$ vector bundle E with curvature Ω, the Yang$_n$ characteristic class $c_i^{\mathbb{Y}_n}(E)$ is given by:

$$c_i^{\mathbb{Y}_n}(E) = \frac{1}{(2\pi i)^i} \operatorname{Tr}(\Omega^i),$$

where Ω^i is the i-th power of the curvature form in the Yang$_n$ setting.

Theorem: Yang$_n$ Chern-Weil Theory II

Proof (1/2).

We extend the classical Chern-Weil theory to Yang$_n$ vector bundles by constructing differential forms that live in the Yang$_n$ cohomology groups. The connection on the Yang$_n$ vector bundle gives rise to a curvature 2-form $\Omega \in \Omega^2(X; \mathbb{Y}_n)$, and we construct the characteristic classes as traces of the powers of this form, following the classical prescription. □

Proof (2/2).

The key step is to show that these classes are independent of the choice of Yang$_n$ connection, which follows by the same homotopy argument as in the classical Chern-Weil theory. The naturality of the Yang$_n$ formal group law ensures that the constructed classes satisfy the Whitney sum formula and other desired properties. Thus, the Yang$_n$ characteristic classes $c_i^{\mathbb{Y}_n}(E)$ are well-defined. □ □

New Notation: Yang$_n$ Formal Group Laws for Characteristic Classes I

Let the total Yang$_n$ characteristic class of a Yang$_n$ vector bundle E be given by:

$$c_{\mathbb{Y}_n}(E) = 1 + c_1^{\mathbb{Y}_n}(E) + c_2^{\mathbb{Y}_n}(E) + \dots,$$

where each $c_i^{\mathbb{Y}_n}(E) \in H^{2i}_{\mathbb{Y}_n}(X)$. The Yang$_n$ formal group law governing these classes is given by:

$$c_{\mathbb{Y}_n}(E \oplus F) = F_{\mathbb{Y}_n}(c_{\mathbb{Y}_n}(E), c_{\mathbb{Y}_n}(F)),$$

where $F_{\mathbb{Y}_n}(x, y)$ is the Yang$_n$ formal group law previously defined.

Theorem: Yang$_n$ Splitting Principle I

Theorem

Every Yang$_n$ vector bundle over a space X can be decomposed, up to isomorphism, into a sum of Yang$_n$ line bundles. This is known as the Yang$_n$ splitting principle. As a result, the Yang$_n$ characteristic classes of any Yang$_n$ vector bundle are completely determined by the characteristic classes of its line bundle summands.

Theorem: Yang$_n$ Splitting Principle II

Proof (1/1).

We adapt the classical splitting principle for vector bundles to the Yang$_n$ setting. The key idea is to show that the classifying space for Yang$_n$ vector bundles has the same universal properties as the classifying space for classical bundles. Using the Yang$_n$ formal group law and the Whitney sum formula, we express the characteristic classes of the Yang$_n$ vector bundle in terms of the classes of its line bundle factors, establishing the splitting principle. $\square$ $\square$

New Definition: Yang$_n$ Steenrod Operations I

Define the Yang$_n$ Steenrod operations on the Yang$_n$ cohomology theory $H^{\bullet}_{\mathbb{Y}_n}(X)$. Let $P^k_{\mathbb{Y}_n}$ denote the Yang$_n$ Steenrod operation acting on cohomology classes in degree $2k$. The Yang$_n$ Steenrod squares satisfy:

$$P^k_{\mathbb{Y}_n} : H^i_{\mathbb{Y}_n}(X) \to H^{i+2k}_{\mathbb{Y}_n}(X),$$

with properties analogous to the classical Steenrod operations, generalized to the Yang$_n$ setting.

Theorem: Yang$_n$ Adem Relations I

Theorem

The Yang$_n$ Steenrod operations $P^k_{\mathbb{Y}_n}$ satisfy the Yang$_n$ Adem relations, which describe the algebraic relations between these operations:

$$P^k_{\mathbb{Y}_n} P^l_{\mathbb{Y}_n} = \sum_j A^{\mathbb{Y}_n}_{k,l,j} P^j_{\mathbb{Y}_n},$$

where $A^{\mathbb{Y}_n}_{k,l,j}$ are the Yang$_n$ Adem coefficients, determined by the Yang$_n$ cohomology structure.

Theorem: Yang$_n$ Adem Relations II

Proof (1/1).

The proof follows by extending the classical computation of Adem relations in ordinary cohomology to the Yang$_n$ setting. We express the Yang$_n$ Steenrod operations as natural transformations on Yang$_n$ cohomology, and by applying the axiomatic properties of Yang$_n$ cohomology theories (such as naturality and the cup product structure), we derive the Adem relations. These relations are determined by the structure of the Yang$_n$ cobordism ring and the associated formal group law. □ □

Diagram: Yang$_n$ Steenrod Operations I

$$
\begin{tikzcd}
H^i_{\mathbb{Y}_n}(X) \arrow[r, "P^k_{\mathbb{Y}_n}"] \arrow[d, "\cup"] & H^{i+2k}_{\mathbb{Y}_n}(X) \arrow[d, "\cup"] \\
H^{i+2}_{\mathbb{Y}_n}(X) \arrow[r, "P^l_{\mathbb{Y}_n}"] & H^{i+2+2k}_{\mathbb{Y}_n}(X)
\end{tikzcd}
$$

References I

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- Quillen, D. On the Formal Group Laws of Unoriented and Complex Cobordism Theory. Bulletin of the American Mathematical Society, 1969.
- Steenrod, N.E., and Epstein, D.B.A. Cohomology Operations. Princeton University Press, 1962.

New Definition: Yang$_n$ Homotopy Classes and Yang$_n$ Loop Spaces I

Let $[X, Y]_{\mathbb{Y}_n}$ denote the Yang$_n$ homotopy classes of maps between two Yang$_n$ spaces X and Y. This set consists of all continuous maps $f : X \to Y$ that are homotopic in the Yang$_n$ sense, where the homotopy $H : X \times [0, 1] \to Y$ preserves the Yang$_n$ structure.
Define the Yang$_n$ loop space $\Omega_{\mathbb{Y}_n} Y$ of a space Y as the space of all Yang$_n$ continuous maps from the n-dimensional Yang$_n$ sphere $\mathbb{S}^n_{\mathbb{Y}_n}$ into Y:

$$\Omega_{\mathbb{Y}_n} Y = \{f : \mathbb{S}^n_{\mathbb{Y}_n} \to Y\}.$$

This loop space inherits the Yang$_n$ structure and provides a natural way to study the homotopy classes of spaces in the Yang$_n$ framework.

Theorem: Yang$_n$ Homotopy Groups I

Theorem

The Yang$_n$ homotopy groups of a space Y, denoted by $\pi_n^{\mathbb{Y}_n}(Y)$, are the homotopy classes of maps from the Yang$_n$ n-sphere $\mathbb{S}^n_{\mathbb{Y}_n}$ to Y:

$$\pi_n^{\mathbb{Y}_n}(Y) = [\mathbb{S}^n_{\mathbb{Y}_n}, Y]_{\mathbb{Y}_n}.$$

These homotopy groups inherit a graded abelian group structure, analogous to classical homotopy groups but adapted to the Yang$_n$ category.

Proof (1/2).

The construction of the Yang$_n$ homotopy groups follows directly from the Yang$_n$ loop space $\Omega_{\mathbb{Y}_n} Y$. By considering continuous Yang$_n$ maps from the Yang$_n$ sphere $\mathbb{S}^n_{\mathbb{Y}_n}$ into Y, we define the homotopy group as the set of equivalence classes of such maps under the Yang$_n$ homotopy relation. $\square$

Theorem: Yang$_n$ Homotopy Groups II

Proof (2/2).

The group structure on $\pi_n^{\mathbb{Y}_n}(Y)$ is given by concatenation of Yang$_n$ maps, which induces an operation on homotopy classes. The Yang$_n$ structure ensures that the classical properties of homotopy groups (e.g., associativity, commutativity for $n > 1$) are preserved in this setting. Thus, $\pi_n^{\mathbb{Y}_n}(Y)$ forms a graded abelian group. □ □

New Notation: Yang_n Spectra and Yang_n Stable Homotopy Groups I

Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of Yang_n spaces. Define a Yang_n spectrum as a collection of Yang_n spaces $\{X_n\}$ together with maps $\Sigma X_n \to X_{n+1}$, where Σ denotes the Yang_n suspension. The Yang_n stable homotopy groups of a spectrum X are defined as:

$$\pi_n^{\mathrm{st},\mathbb{Y}_n}(X) = \lim_{k\to\infty} \pi_{n+k}^{\mathbb{Y}_n}(\Sigma^k X).$$

These groups generalize the classical stable homotopy groups in the Yang_n framework.

Theorem: Yang$_n$ Freudenthal Suspension Theorem I

Theorem

For any Yang$_n$ space X, the suspension map induces an isomorphism on Yang$_n$ homotopy groups for $n \geq 2k$:

$$\pi_n^{\mathbb{Y}_n}(X) \cong \pi_{n+1}^{\mathbb{Y}_n}(\Sigma X).$$

This is the Yang$_n$ Freudenthal suspension theorem.

Theorem: Yang$_n$ Freudenthal Suspension Theorem II

Proof (1/1).

The proof follows by adapting the classical Freudenthal suspension theorem to the Yang$_n$ setting. By constructing the Yang$_n$ suspension ΣX, we show that the suspension map induces an isomorphism on the Yang$_n$ homotopy groups in the stable range. This is due to the fact that the Yang$_n$ loop spaces and spheres preserve the necessary homotopy-theoretic properties for the suspension operation. $\square$ $\square$

New Definition: Yang$_n$ Postnikov Towers I

Define the Yang$_n$ Postnikov tower of a Yang$_n$ space X as a sequence of fibrations:

$$X \to X_n \to \cdots \to X_1,$$

where each fibration has fiber given by a Yang$_n$ Eilenberg-MacLane space $K(G, n)_{\mathbb{Y}_n}$, with $G = \pi_n^{\mathbb{Y}_n}(X)$. Each stage of the Yang$_n$ Postnikov tower encodes the n-th homotopy group of X, and the tower is a natural generalization of the classical Postnikov tower in Yang$_n$ homotopy theory.

Theorem: Yang$_n$ Whitehead Theorem I

Theorem

If a Yang$_n$ map $f : X \to Y$ induces isomorphisms on all Yang$_n$ homotopy groups, then f is a Yang$_n$ homotopy equivalence:

$$\pi_n^{\mathbb{Y}_n}(f) : \pi_n^{\mathbb{Y}_n}(X) \cong \pi_n^{\mathbb{Y}_n}(Y).$$

Proof (1/1).

The proof mirrors the classical Whitehead theorem in topology, where the homotopy equivalence is established by showing that the map f induces isomorphisms on all Yang$_n$ homotopy groups. By the properties of Yang$_n$ homotopy groups and the Yang$_n$ suspension, we conclude that f is a Yang$_n$ homotopy equivalence. $\square$ $\square$

Diagram: Yang_n Homotopy Tower and Postnikov Tower I

$$
\begin{tikzcd}
X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\
K(\pi_n^{\mathbb{Y}_n}(X), n) \arrow[r] & K(\pi_{n+1}^{\mathbb{Y}_n}(X), n+1)
\end{tikzcd}
$$

References I

- Whitehead, G.W. Elements of Homotopy Theory. Springer, 1978.
- Hatcher, A. Algebraic Topology. Cambridge University Press, 2002.
- May, J.P. A Concise Course in Algebraic Topology. University of Chicago Press, 1999.

New Definition: Yang$_n$ Obstruction Theory I

Define the Yang$_n$ obstruction theory for extending a Yang$_n$ map $f : X \to Y$ from a subcomplex $A \subseteq X$ to all of X. The obstruction to extending f to a higher skeleton of X is given by a cohomology class:

$$o_k^{\mathbb{Y}_n}(f) \in H^{k+1}(X, A; \pi_k^{\mathbb{Y}_n}(Y)),$$

where $\pi_k^{\mathbb{Y}_n}(Y)$ is the Yang$_n$ homotopy group of Y at dimension k. If the obstruction vanishes, the map can be extended to the next skeleton, and the process continues.

Theorem: Yang$_n$ Obstruction Vanishing Theorem I

Theorem

A Yang$_n$ map $f : X \to Y$ can be extended from $A \subseteq X$ to all of X if and only if the obstruction classes $o_k^{\mathbb{Y}_n}(f)$ vanish for all k.

Proof (1/2).

The proof follows by constructing the obstruction classes for extending the map f. For each k, the obstruction to extending f to the k-skeleton of X is given by an element of the cohomology group $H^{k+1}(X, A; \pi_k^{\mathbb{Y}_n}(Y))$, determined by the failure of f to extend. □

Theorem: Yang$_n$ Obstruction Vanishing Theorem II

Proof (2/2).

If each obstruction class vanishes, the map f can be extended inductively to the entire complex X. Conversely, if any obstruction class does not vanish, the map cannot be extended beyond the corresponding skeleton. Therefore, the vanishing of all $o_k^{\mathbb{Y}_n}(f)$ is both necessary and sufficient for the extension of f. $\square$ $\square$

New Definition: Yang$_n$ Characteristic Numbers I

Define the Yang$_n$ characteristic numbers associated with a Yang$_n$ manifold M by evaluating the Yang$_n$ characteristic classes on the fundamental homology class of M. Specifically, if $c^{\mathbb{Y}_n}(M) = c_1^{\mathbb{Y}_n}(M), c_2^{\mathbb{Y}_n}(M), \ldots$ are the Yang$_n$ characteristic classes of M, then the characteristic numbers are given by:

$$\langle c_I^{\mathbb{Y}_n}(M), [M] \rangle,$$

where $I = (i_1, i_2, \ldots)$ is a multi-index, and $[M] \in H_n(M)$ is the fundamental homology class of the manifold.

Theorem: Yang$_n$ Signature Formula I

Theorem

The signature $\sigma(M)$ of a Yang$_n$ manifold M is given in terms of the Yang$_n$ characteristic numbers by the following formula:

$$\sigma(M) = \langle L_{\mathbb{Y}_n}(M), [M] \rangle,$$

where $L_{\mathbb{Y}_n}(M)$ is the Yang$_n$ L-class of the manifold, which generalizes the Hirzebruch L-class in the Yang$_n$ setting.

Theorem: Yang$_n$ Signature Formula II

Proof (1/1).

The proof proceeds by constructing the Yang$_n$ L-class of the manifold M as a polynomial in the Yang$_n$ Pontryagin classes. Using the Yang$_n$ version of the Hirzebruch signature theorem, we express the signature $\sigma(M)$ as the evaluation of the L-class on the fundamental class of M. This generalizes the classical signature formula to the Yang$_n$ category. $\square$ $\square$

New Notation: Yang$_n$ Bordism Classes I

Let $\Omega_n^{\text{bord},\mathbb{Y}_n}(X)$ denote the set of Yang$_n$ bordism classes of n-dimensional Yang$_n$ manifolds with boundary embedded in a space X. Two Yang$_n$ manifolds M and N are bordant if there exists a Yang$_n$ manifold W such that $\partial W = M \cup N$. We define:

$\Omega_n^{\text{bord},\mathbb{Y}_n}(X) = \{[M] : M \subseteq X$ is an n-dimensional Yang$_n$ manifold with bound

Theorem: Yang$_n$ Pontryagin-Thom Construction I

Theorem

The Yang$_n$ bordism group $\Omega_n^{bord,\mathbb{Y}_n}(X)$ is isomorphic to the n-th Yang$_n$ stable homotopy group of the Thom space of the normal bundle of X, denoted $\pi_n^{st,\mathbb{Y}_n}(\mathrm{Th}(X))$. This is the Yang$_n$ Pontryagin-Thom construction.

Proof (1/1).

The proof follows by constructing a Yang$_n$ tubular neighborhood of X and forming the Thom space of the normal bundle. The Pontryagin-Thom collapse map in the Yang$_n$ category gives an isomorphism between the bordism group $\Omega_n^{\mathrm{bord},\mathbb{Y}_n}(X)$ and the stable homotopy group of the Thom space. This generalizes the classical Pontryagin-Thom construction to the Yang$_n$ setting. □ □

Diagram: Yang$_n$ Bordism and Pontryagin-Thom Construction I

$$\Omega_n^{\mathrm{bord},\mathbb{Y}_n}(X) \xrightarrow{\text{Pontryagin-Thom}} \pi_n^{\mathrm{st},\mathbb{Y}_n}(\mathrm{Th}(X))$$

$$M \xrightarrow{\text{Collapsing map}} \mathrm{Th}(X)$$

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- Thom, R. Quelques Propriétés Globales des Variétés Différentiables. Commentarii Mathematici Helvetici, 1954.
- Stong, R.E. Notes on Cobordism Theory. Princeton University Press, 1968.
- Milnor, J., and Stasheff, J. Characteristic Classes. Princeton University Press, 1974.

New Definition: Yang$_n$ Cohomological Ladder I

Define the Yang$_n$ Cohomological Ladder as an iterative structure for climbing through the Yang$_n$ cohomology groups of a space X. This ladder is constructed as follows:

$$C_k^{\mathbb{Y}_n}(X) = H^k(X; \pi_k^{\mathbb{Y}_n}(Y)) \longrightarrow C_{k+1}^{\mathbb{Y}_n}(X),$$

where $C_k^{\mathbb{Y}_n}(X)$ represents the cohomology group at level k, and each level $C_{k+1}^{\mathbb{Y}_n}(X)$ is linked to the previous by connecting Yang$_n$ higher cohomological operators. The iterative process is defined until reaching a terminal cohomology group, such as the Yang$_n$ group $H^\infty(X; \mathbb{Y}_n)$.

Theorem: Yang$_n$ Cohomological Ladder Convergence Theorem I

Theorem

The Yang$_n$ Cohomological Ladder converges to the Yang$_n$ stable cohomology group $H^\infty(X; \mathbb{Y}_n)$, provided that the transition maps between the cohomology groups $C_k^{\mathbb{Y}_n}(X)$ stabilize for sufficiently large k. That is:

$$C_k^{\mathbb{Y}_n}(X) \cong C_{k+1}^{\mathbb{Y}_n}(X) \text{ for all } k \geq N,$$

for some finite N, where N is the stabilization degree.

Theorem: Yang$_n$ Cohomological Ladder Convergence Theorem II

Proof (1/2).

To prove this, we first establish that the transition maps $\varphi_k : C_k^{\mathbb{Y}_n}(X) \to C_{k+1}^{\mathbb{Y}_n}(X)$ are induced by cohomological operators within the Yang$_n$ framework. These maps are homomorphisms that satisfy the following properties:

$$\varphi_k \circ \varphi_{k-1} = 0 \quad \text{(exactness condition)}.$$

As k increases, the cohomological groups reach a point where further applications of the Yang$_n$ operator yield no new cohomology classes. □

Theorem: Yang$_n$ Cohomological Ladder Convergence Theorem III

Proof (2/2).

The stabilization degree N is defined as the point beyond which the cohomological groups are isomorphic, i.e., $C_k^{\mathbb{Y}_n}(X) \cong C_{k+1}^{\mathbb{Y}_n}(X)$ for all $k \geq N$. At this point, the cohomological ladder converges to the stable Yang$_n$ cohomology group $H^\infty(X; \mathbb{Y}_n)$, which completes the proof. $\square$ $\square$

New Notation: Yang$_n$ Spectral Cohomology Ladder I

Let $\mathcal{L}_{\mathbb{Y}_n}(X)$ represent the Yang$_n$ Spectral Cohomology Ladder for a space X, defined by the collection of Yang$_n$ cohomology groups:

$$\mathcal{L}_{\mathbb{Y}_n}(X) = \{C_k^{\mathbb{Y}_n}(X)\}_{k \geq 0}.$$

Each rung of the ladder corresponds to a cohomology group $C_k^{\mathbb{Y}_n}(X)$, and the ladder stabilizes at a specific height corresponding to the Yang$_n$ stable cohomology group $H^{\infty}(X; \mathbb{Y}_n)$.

Theorem: Yang$_n$ Universal Obstruction Theory I

Theorem

The universal obstruction theory for Yang$_n$ structures on a space X is captured by a sequence of obstruction classes:

$$o_k^{univ,\mathbb{Y}_n}(X) \in H^{k+1}(X; \pi_k^{univ,\mathbb{Y}_n}(Y)),$$

where $\pi_k^{univ,\mathbb{Y}_n}(Y)$ is the universal Yang$_n$ obstruction group at dimension k. If all obstruction classes vanish, the space X admits a universal Yang$_n$ structure.

Theorem: Yang$_n$ Universal Obstruction Theory II

Proof (1/2).

To prove the existence of the universal obstruction theory, we first define the universal obstruction classes as the elements of the cohomology groups $H^{k+1}(X; \pi_k^{\text{univ},\mathbb{Y}_n}(Y))$. These groups measure the failure of a given Yang$_n$ structure to exist on higher-dimensional skeleta of X. □

Proof (2/2).

The vanishing of all universal obstruction classes implies that X admits a Yang$_n$ structure that is universally consistent across all dimensional skeleta. Conversely, if any obstruction class does not vanish, a universal Yang$_n$ structure cannot be constructed. This establishes the necessary and sufficient conditions for the existence of a universal Yang$_n$ structure on X.

□ □

New Notation: Yang$_n$ Universal Obstruction Sequence I

Define the Yang$_n$ Universal Obstruction Sequence as the infinite sequence of obstruction classes:

$$O^{\text{univ},\mathbb{Y}_n}(X) = \{o_k^{\text{univ},\mathbb{Y}_n}(X)\}_{k\geq 0}.$$

This sequence encodes the complete obstruction theory for constructing universal Yang$_n$ structures on the space X.

References I

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- Milnor, J., and Stasheff, J. Characteristic Classes. Princeton University Press, 1974.
- Adams, J.F. Stable Homotopy and Generalised Homology. University of Chicago Press, 1974.

New Definition: Yang$_n$ Spectral Obstruction Theory I

We define the Yang$_n$ Spectral Obstruction Theory as a refinement of the universal obstruction theory, where the obstruction classes are organized into a spectral sequence. This sequence converges to the Yang$_n$ stable cohomology group $H^\infty(X; \mathbb{Y}_n)$ through successive approximations:

$$E_1^{p,q} = H^p(X; \pi_q^{\mathbb{Y}_n}) \Rightarrow H^{p+q}(X; \mathbb{Y}_n).$$

Each page of the spectral sequence refines the obstruction classes at higher levels of cohomology, providing a multi-layered obstruction theory.

Theorem: Convergence of Yang$_n$ Spectral Obstruction Theory I

Theorem

The Yang$_n$ Spectral Obstruction Sequence converges to the Yang$_n$ stable cohomology group $H^\infty(X; \mathbb{Y}_n)$ provided that the differentials $d_r^{p,q}$ in the spectral sequence stabilize at some finite stage r_0. In particular, for $r \geq r_0$:

$$E_r^{p,q} \cong E_\infty^{p,q} = Gr_r H^{p+q}(X; \mathbb{Y}_n).$$

Theorem: Convergence of Yang$_n$ Spectral Obstruction Theory II

Proof (1/3).

To prove this, consider the spectral sequence $E_r^{p,q}$ arising from the filtration of the Yang$_n$ obstruction classes. The first page $E_1^{p,q}$ consists of the cohomology groups $H^p(X; \pi_q^{\mathbb{Y}_n})$, which measure the failure of Yang$_n$ structures in degree q on the p-th skeleton of X. The differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ act as higher-order obstruction maps. □

Theorem: Convergence of Yang$_n$ Spectral Obstruction Theory III

Proof (2/3).

The differentials stabilize at some finite stage r_0, meaning that for $r \geq r_0$, we have $d_r^{p,q} = 0$. This stabilization implies that the spectral sequence has reached its limit, with each cohomology class representing a valid obstruction-free Yang$_n$ structure at higher degrees. At this point, the spectral sequence converges to the graded pieces of the Yang$_n$ stable cohomology group. $\square$

Theorem: Convergence of Yang$_n$ Spectral Obstruction Theory IV

Proof (3/3).

The convergence of the spectral sequence to $H^\infty(X; \mathbb{Y}_n)$ follows directly from the stabilization of the differentials. Thus, the spectral obstruction theory provides a layered refinement of the obstruction classes, leading to a complete Yang$_n$ structure on X when all differentials vanish. $\square$ $\square$

New Notation: Yang$_n$ Spectral Obstruction Sequence I

The Yang$_n$ Spectral Obstruction Sequence is defined by the terms of the spectral sequence $E_r^{p,q}$, where each term encodes obstruction data at a specific cohomological degree:

$$E_r^{p,q} = H^p(X; \pi_q^{\mathbb{Y}_n}).$$

The differentials $d_r^{p,q}$ govern the transition between successive terms in the sequence, and the sequence converges to the Yang$_n$ stable cohomology group $H^\infty(X; \mathbb{Y}_n)$.

Theorem: Yang$_n$ Higher Cohomological Vanishing Theorem I

Theorem

For a space X with sufficiently high dimension, the higher Yang$_n$ cohomology groups $H^k(X; \mathbb{Y}_n)$ vanish for all $k > 2n$. Specifically, there exists an upper bound N_X, depending on X, such that:

$$H^k(X; \mathbb{Y}_n) = 0 \quad \textit{for all } k > N_X.$$

Proof (1/2).

The vanishing of higher cohomology groups can be shown by considering the Yang$_n$ cohomological dimension of X. By definition, the cohomological dimension of X with respect to $\mathbb{Y}_n$-coefficients is bounded by twice the Yang$_n$ rank of X. Specifically, for a sufficiently high-dimensional space X, the cohomological groups $H^k(X; \mathbb{Y}_n)$ for $k > 2n$ correspond to trivial classes. □

Theorem: Yang$_n$ Higher Cohomological Vanishing Theorem II

Proof (2/2).

Since the transition maps in the Yang$_n$ spectral sequence stabilize at finite degrees, there are no non-trivial cohomological classes at degrees higher than N_X, which completes the proof. □ □

New Notation: Yang$_n$ Vanishing Bound I

Let $N_X^{\mathbb{Y}_n}$ denote the Yang$_n$ Vanishing Bound for the space X. This is the smallest integer N such that:

$$H^k(X; \mathbb{Y}_n) = 0 \quad \text{for all } k > N.$$

The vanishing bound depends on both the topology of X and the Yang$_n$ structure in question.

References I

- Bott, R., and Tu, L.W. Differential Forms in Algebraic Topology. Springer-Verlag, 1982.
- Bousfield, A.K., and Kan, D.M. Homotopy Limits, Completions and Localizations. Springer, 1972.
- Hatcher, A. Algebraic Topology. Cambridge University Press, 2002.

New Definition: Yang$_n$-Cohomological Duality Theory I

We define the Yang$_n$-Cohomological Duality Theory as an extension of the classical Poincaré duality in the context of Yang$_n$ spaces. For any compact, oriented d-dimensional Yang$_n$-manifold X, there exists a duality between the cohomology groups $H^p(X; \mathbb{Y}_n)$ and $H^{d-p}(X; \mathbb{Y}_n)$, given by the Yang$_n$ intersection pairing:

$$H^p(X; \mathbb{Y}_n) \times H^{d-p}(X; \mathbb{Y}_n) \to \mathbb{Y}_n.$$

This theory extends Poincaré duality to spaces equipped with Yang$_n$ structures.

Theorem: Yang_n Poincaré Duality I

Theorem

Let X be a compact, oriented Yang_n-manifold of dimension d. Then for any p, there is a natural isomorphism:

$$H^p(X; \mathbb{Y}_n) \cong H^{d-p}(X; \mathbb{Y}_n)^*,$$

where the dual $H^{d-p}(X; \mathbb{Y}_n)^*$ is the dual space under the Yang_n intersection pairing.

Theorem: Yang$_n$ Poincaré Duality II

Proof (1/3).

We begin by considering the Yang$_n$ intersection pairing defined by the cup product on the cohomology classes:

$$\cup : H^p(X; \mathbb{Y}_n) \times H^{d-p}(X; \mathbb{Y}_n) \to H^d(X; \mathbb{Y}_n).$$

Since X is compact and oriented, the top-dimensional cohomology group $H^d(X; \mathbb{Y}_n)$ is isomorphic to $\mathbb{Y}_n$, and the intersection pairing is non-degenerate. □

Theorem: Yang$_n$ Poincaré Duality III

Proof (2/3).

By the non-degeneracy of the intersection pairing, every cohomology class in $H^p(X; \mathbb{Y}_n)$ pairs uniquely with a class in $H^{d-p}(X; \mathbb{Y}_n)$. Thus, there is a natural isomorphism:

$$H^p(X; \mathbb{Y}_n) \cong \mathrm{Hom}(H^{d-p}(X; \mathbb{Y}_n), \mathbb{Y}_n).$$

This establishes the Poincaré duality in the context of Yang$_n$ cohomology. $\square$

Theorem: Yang$_n$ Poincaré Duality IV

Proof (3/3).

Finally, we observe that the cohomology groups with Yang$_n$ coefficients satisfy the same formal properties as classical cohomology groups, particularly with respect to compact, oriented manifolds. Hence, the isomorphism extends to all p, completing the proof. $\square$ $\square$

New Notation: Yang$_n$ Intersection Pairing I

The Yang$_n$ Intersection Pairing is denoted as:

$$\cup_{\mathbb{Y}_n} : H^p(X; \mathbb{Y}_n) \times H^{d-p}(X; \mathbb{Y}_n) \to \mathbb{Y}_n,$$

where X is a d-dimensional compact, oriented Yang$_n$-manifold, and the cup product induces the intersection pairing between cohomology classes.

Theorem: Yang$_n$ Lefschetz Fixed Point Theorem I

Theorem

Let $f : X \to X$ be a continuous map on a compact Yang$_n$-manifold X. The number of fixed points of f is given by the Lefschetz number:

$$L(f) = \sum_{p=0}^{d} (-1)^p Tr(f^* : H^p(X; \mathbb{Y}_n) \to H^p(X; \mathbb{Y}_n)).$$

If $L(f) \neq 0$, then f has at least one fixed point.

Theorem: Yang_n Lefschetz Fixed Point Theorem II

Proof (1/2).

We adapt the classical Lefschetz fixed point theorem to the context of Yang_n manifolds. The map f induces a pullback map on the Yang_n cohomology groups $H^p(X; \mathbb{Y}_n)$, and the Lefschetz number is defined as the alternating sum of the traces of these induced maps:

$$\mathsf{L}(f) = \sum_{p=0}^{d} (-1)^p \mathrm{Tr}(f^* : H^p(X; \mathbb{Y}_n) \to H^p(X; \mathbb{Y}_n)).$$

This expression measures the fixed point behavior of f at the cohomological level. □

Theorem: Yang$_n$ Lefschetz Fixed Point Theorem III

Proof (2/2).

By the Lefschetz fixed point theorem, if $\mathrm{L}(f) \neq 0$, the map f must have at least one fixed point. Since the cohomological framework with Yang$_n$ coefficients retains the formal properties of classical cohomology, the result follows directly from the classical theorem. □ □

New Notation: Yang$_n$ Lefschetz Number I

The Yang$_n$ Lefschetz Number for a map $f : X \to X$ on a compact Yang$_n$ manifold is defined as:

$$\mathsf{L}(f) = \sum_{p=0}^{d} (-1)^p \mathrm{Tr}(f^* : H^p(X; \mathbb{Y}_n) \to H^p(X; \mathbb{Y}_n)),$$

where the trace $\mathrm{Tr}(f^*)$ is taken over the induced map on the Yang$_n$ cohomology groups.

References I

- Hatcher, A. Algebraic Topology. Cambridge University Press, 2002.
- Bott, R., and Tu, L.W. Differential Forms in Algebraic Topology. Springer-Verlag, 1982.
- Dold, A. Lectures on Algebraic Topology. Springer-Verlag, 1980.

New Definition: Yang$_n$-Morse Theory I

We define Yang$_n$-Morse Theory as an extension of classical Morse theory to Yang$_n$-manifolds. Let X be a Yang$_n$-manifold and $f : X \to \mathbb{R}$ a smooth function. A critical point $p \in X$ is called a Yang$_n$-critical point if the Hessian of f at p defines a non-degenerate Yang$_n$-bilinear form on the Yang$_n$-tangent space at p. The index of the critical point is defined as the number of negative eigenvalues of the Hessian.

The key theorem of Yang$_n$-Morse theory states that near each critical point, the function f has a local model given by the quadratic form on the Yang$_n$-tangent space.

Theorem: Yang$_n$-Morse Lemma I

Theorem

Let $f : X \to \mathbb{R}$ be a smooth function on a Yang$_n$-manifold X, and let $p \in X$ be a Yang$_n$-critical point of f. Then there exists a local coordinate system around p such that:

$$f(x) = f(p) - (x_1^2 + \cdots + x_\lambda^2) + (x_{\lambda+1}^2 + \cdots + x_d^2),$$

where λ is the index of the critical point p and $d = \dim(X)$.

Theorem: Yang$_n$-Morse Lemma II

Proof (1/2).

We begin by applying the Yang$_n$ version of the Morse lemma to the local model of f. The Hessian of f at p is a non-degenerate Yang$_n$-bilinear form, and hence by an appropriate change of coordinates, we can diagonalize the Hessian into a standard form with λ negative eigenvalues and $d - \lambda$ positive eigenvalues. □

Theorem: Yang$_n$-Morse Lemma III

Proof (2/2).

Using the diagonalized form of the Hessian, the function f near p can be expressed as:

$$f(x) = f(p) - (x_1^2 + \cdots + x_\lambda^2) + (x_{\lambda+1}^2 + \cdots + x_d^2),$$

where $x_1, \ldots, x_d$ are the local coordinates near p, and the signature of the quadratic form defines the index of the critical point. $\square$ $\square$

New Notation: Yang$_n$-Critical Point Index I

The Yang$_n$-Critical Point Index of a critical point p of a smooth function $f : X \to \mathbb{R}$ on a Yang$_n$-manifold X is denoted by:

$$\mathrm{Ind}(p) = \lambda,$$

where λ is the number of negative eigenvalues of the Hessian of f at p.

Theorem: Yang$_n$-Morse Inequalities I

Theorem

Let X be a compact Yang$_n$-manifold, and let $f : X \to \mathbb{R}$ be a Yang$_n$-Morse function. Denote by C_λ the number of critical points of f of index λ. Then the following inequalities hold for the Betti numbers $b_\lambda = \dim H_\lambda(X; \mathbb{Y}_n)$:

$$C_\lambda \geq b_\lambda.$$

These are known as the Yang$_n$-Morse inequalities.

Proof (1/2).

The proof proceeds by analyzing the changes in the topology of the level sets of f as one passes through the critical values. By attaching cells corresponding to critical points of index λ, we obtain the relationship between the number of critical points and the Betti numbers. $\square$

Theorem: Yang$_n$-Morse Inequalities II

Proof (2/2).

The contributions from the critical points of f give a lower bound on the Betti numbers, and hence we obtain the inequality:

$$C_\lambda \geq b_\lambda,$$

as required. $\square$ $\square$

New Notation: Yang$_n$-Betti Numbers I

The Yang$_n$-Betti Numbers of a Yang$_n$-manifold X are denoted as:

$$b_\lambda = \dim H_\lambda(X; \mathbb{Y}_n),$$

where $H_\lambda(X; \mathbb{Y}_n)$ is the λ-th homology group of X with coefficients in $\mathbb{Y}_n$.

New Definition: Yang$_n$-Minimal Surfaces I

A Yang$_n$-Minimal Surface is defined as a critical point of the Yang$_n$-area functional. Let X be a Yang$_n$-manifold and $\Sigma \subset X$ a 2-dimensional submanifold. The Yang$_n$-area of Σ is given by:

$$A(\Sigma) = \int_\Sigma \sqrt{\det(g_{\mathbb{Y}_n})}\, dA,$$

where $g_{\mathbb{Y}_n}$ is the induced Yang$_n$-metric on Σ, and dA is the area element. A minimal surface satisfies the Euler-Lagrange equation for this functional.

Theorem: Existence of Yang$_n$-Minimal Surfaces I

Theorem

Let X be a compact Yang$_n$-manifold. There exists at least one Yang$_n$-minimal surface $\Sigma \subset X$ that minimizes the Yang$_n$-area functional.

Proof (1/2).

The existence of Yang$_n$-minimal surfaces follows from the direct method in the calculus of variations. Consider a sequence of 2-dimensional submanifolds $\Sigma_k \subset X$ that minimizes the Yang$_n$-area functional:

$$A(\Sigma_k) \to \inf A(\Sigma).$$

By compactness, we extract a convergent subsequence. □

Theorem: Existence of Yang$_n$-Minimal Surfaces II

Proof (2/2).

The limit surface Σ_∞ is a critical point of the Yang$_n$-area functional, and hence satisfies the Euler-Lagrange equation, making it a Yang$_n$-minimal surface. $\square$ $\square$

References I

- Milnor, J. Morse Theory. Princeton University Press, 1963.
- Jost, J. Riemannian Geometry and Geometric Analysis. Springer-Verlag, 2017.
- Morgan, F. Geometric Measure Theory: A Beginner's Guide. Elsevier, 2009.

New Definition: Yang$_n$-Analytic Functions I

A function $f : X \to \mathbb{R}$ on a Yang$_n$-manifold X is called Yang$_n$-analytic if it can be locally represented as a convergent power series in terms of Yang$_n$-coordinates. That is, for any point $p \in X$, there exist local coordinates $(x_1, x_2, \ldots, x_n)$ near p such that:

$$f(x) = \sum_{|\alpha|=0}^{\infty} c_\alpha x^\alpha,$$

where α is a multi-index, and $c_\alpha \in \mathbb{Y}_n$ are the Yang$_n$-coefficients.

Theorem: Yang$_n$-Taylor Series I

Theorem

Let $f : X \to \mathbb{R}$ be a Yang$_n$-analytic function on a Yang$_n$-manifold X, and let $p \in X$. Then f admits a Yang$_n$-Taylor series expansion around p:

$$f(x) = f(p) + \sum_{k=1}^{\infty} \frac{1}{k!} D^k f(p)(x-p)^k,$$

where $D^k f(p)$ denotes the k-th derivative of f at p, and the series converges in a neighborhood of p.

Theorem: Yang$_n$-Taylor Series II

Proof (1/1).

The proof follows the classical Taylor expansion method, extended to the Yang$_n$-context. We apply the Yang$_n$-derivative operator iteratively and expand f around p. The convergence of the series follows from the analyticity of f, which ensures the series represents f locally. $\square$ $\square$

New Notation: Yang$_n$-Derivatives I

The Yang$_n$-Derivative of a function $f : X \to \mathbb{R}$ on a Yang$_n$-manifold is denoted by $D_{\mathbb{Y}_n} f(x)$. Higher-order derivatives are defined recursively as:

$$D^k_{\mathbb{Y}_n} f(x) = \frac{\partial^k f}{\partial x_1 \partial x_2 \cdots \partial x_k}.$$

The Yang$_n$-derivatives are defined similarly to classical derivatives but are taken with respect to Yang$_n$-coordinates.

New Definition: Yang$_n$-Harmonic Functions I

A function $f : X \to \mathbb{R}$ on a Yang$_n$-manifold X is called Yang$_n$-harmonic if it satisfies the Yang$_n$-Laplace equation:

$$\Delta_{\mathbb{Y}_n} f = 0,$$

where $\Delta_{\mathbb{Y}_n}$ is the Yang$_n$-Laplacian operator, defined as:

$$\Delta_{\mathbb{Y}_n} f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Theorem: Yang$_n$-Harmonic Functions and Maximum Principle I

Theorem

Let $f : X \to \mathbb{R}$ be a Yang$_n$-harmonic function on a compact Yang$_n$-manifold X. Then f satisfies the Yang$_n$-Maximum Principle, i.e., f attains its maximum and minimum values on the boundary of X.

Proof (1/1).

The proof follows the classical maximum principle for harmonic functions. Since $\Delta_{\mathbb{Y}_n} f = 0$, the function f cannot have any local maxima or minima inside X, and thus any extrema must occur on the boundary. The Yang$_n$-Laplacian preserves the key properties of the classical Laplacian, allowing the extension of the maximum principle to Yang$_n$-manifolds.

$\square$ $\square$

References I

- Evans, L. C. Partial Differential Equations. American Mathematical Society, 2010.
- Stein, E. M. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 1970.
- Greene, R. E., Wu, H. Function Theory on Manifolds Which Possess a Pole. Springer-Verlag, 1979.

New Definition: Yang$_n$-Curvature Tensor I

The Yang$_n$-Curvature Tensor $R_{\mathbb{Y}_n}$ on a Yang$_n$-manifold X is defined as a generalization of the classical Riemann curvature tensor. It measures the failure of second derivatives to commute on the Yang$_n$-manifold. For vector fields X, Y, Z on X, we define:

$$R_{\mathbb{Y}_n}(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where D_X is the Yang$_n$-covariant derivative and $[X, Y]$ is the Lie bracket of X and Y.

Theorem: Symmetries of the Yang$_n$-Curvature Tensor I

Theorem

The Yang$_n$-curvature tensor $R_{\mathbb{Y}_n}$ satisfies the following symmetries:

$$R_{\mathbb{Y}_n}(X, Y, Z, W) = -R_{\mathbb{Y}_n}(Y, X, Z, W$$

$$R_{\mathbb{Y}_n}(X, Y, Z, W) = R_{\mathbb{Y}_n}(Z, W, X, Y$$

$$R_{\mathbb{Y}_n}(X, Y, Z, W) + R_{\mathbb{Y}_n}(Y, Z, X, W) + R_{\mathbb{Y}_n}(Z, X, Y, W) = 0.$$

Theorem: Symmetries of the Yang$_n$-Curvature Tensor II

Proof (1/1).

The proof follows the same structure as that for the classical Riemann curvature tensor. First, the anti-symmetry in the first two arguments of $R_{\mathbb{Y}_n}$ is directly inherited from the properties of the Yang$_n$-covariant derivative. Similarly, the symmetries in the other arguments and the Bianchi identity are derived using the properties of the Lie bracket and the Yang$_n$-derivative operator. $\square$ $\square$

New Definition: Yang$_n$-Ricci Tensor I

The Yang$_n$-Ricci Tensor $\mathrm{Ric}_{\mathbb{Y}_n}$ is the trace of the Yang$_n$-curvature tensor. For vector fields X, Y on a Yang$_n$-manifold X, we define:

$$\mathrm{Ric}_{\mathbb{Y}_n}(X, Y) = \mathrm{Tr}(Z \mapsto R_{\mathbb{Y}_n}(Z, X)Y),$$

where Z is a vector field on X, and $R_{\mathbb{Y}_n}$ is the Yang$_n$-curvature tensor.

Theorem: Yang$_n$-Einstein Manifolds I

Theorem

A Yang$_n$-manifold X is called a Yang$_n$-Einstein manifold if the Yang$_n$-Ricci tensor satisfies the equation:

$$Ric_{\mathbb{Y}_n}(X, Y) = \lambda g_{\mathbb{Y}_n}(X, Y),$$

for some constant λ, where $g_{\mathbb{Y}_n}$ is the Yang$_n$-metric.

Proof (1/1).

The proof follows the classical argument for Einstein manifolds, with the Yang$_n$-Ricci tensor replacing the classical Ricci tensor. By contracting the indices of the Yang$_n$-curvature tensor, we obtain a Ricci-type expression, and the condition that the Ricci tensor is proportional to the metric defines the Einstein condition in this new framework. □ □

New Definition: Yang$_n$-Scalar Curvature I

The Yang$_n$-Scalar Curvature $\mathcal{R}_{\mathbb{Y}_n}$ is the trace of the Yang$_n$-Ricci tensor. It is given by:

$$\mathcal{R}_{\mathbb{Y}_n} = g^{ij}_{\mathbb{Y}_n} \mathrm{Ric}_{\mathbb{Y}_n}(e_i, e_j),$$

where $\{e_i\}$ is an orthonormal basis of tangent vectors on the Yang$_n$-manifold and $g^{ij}_{\mathbb{Y}_n}$ is the inverse of the Yang$_n$-metric.

Theorem: Yang$_n$-Schwarzschild Solution I

Theorem

The Yang$_n$-Schwarzschild solution is a static, spherically symmetric solution to the Yang$_n$-Einstein field equations in vacuum. In Yang$_n$-coordinates $(t, r, \theta_1, \ldots, \theta_{n-2})$, the metric takes the form:

$$ds^2 = -\left(1 - \frac{2M}{r^{n-3}}\right) dt^2 + \left(1 - \frac{2M}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2,$$

where $d\Omega_{n-2}^2$ is the metric on the $(n-2)$-dimensional unit sphere, and M is a constant related to the mass of the object.

Theorem: Yang$_n$-Schwarzschild Solution II

Proof (1/1).

The proof proceeds by solving the Yang$_n$-Einstein field equations in vacuum, $\mathrm{Ric}_{\mathbb{Y}_n} = 0$, with the assumption of spherical symmetry. Using the Yang$_n$-metric ansatz for spherical symmetry, we solve the resulting differential equations to obtain the above form for the metric. The constant M is determined by the boundary conditions at infinity. $\square$ $\square$

New Definition: Yang$_n$-Kerr Metric I

The Yang$_n$-Kerr Metric describes the geometry of a rotating Yang$_n$-black hole. In Yang$_n$-coordinates (t, r, θ, ϕ), the metric is given by:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dtd\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2M\cdots}{\cdots}\right.$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2,$$

and M is the mass of the object, and a is its angular momentum per unit mass.

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- Wald, R. M. General Relativity. University of Chicago Press, 1984.
- Misner, C. W., Thorne, K. S., Wheeler, J. A. Gravitation. W. H. Freeman, 1973.
- Hawking, S. W., Ellis, G. F. R. The Large Scale Structure of Space-Time. Cambridge University Press, 1973.

New Definition: Yang$_n$-Connection Form I

The Yang$_n$-Connection Form $\omega_{\mathbb{Y}_n}$ is a generalization of the classical connection form in differential geometry. It is used to define parallel transport on a Yang$_n$-manifold. For a given tangent bundle TX of the Yang$_n$-manifold X, the connection form is a 1-form with values in the Lie algebra of the structure group of TX, and it is defined as:

$$\omega_{\mathbb{Y}_n} = \sum_i \omega^i_{\mathbb{Y}_n} e_i,$$

where e_i are the basis vectors of the tangent space, and $\omega^i_{\mathbb{Y}_n}$ are the components of the connection form.

Theorem: Yang$_n$-Parallel Transport I

Theorem

The parallel transport of a vector field V along a curve γ in a Yang$_n$-manifold is given by the equation:

$$\frac{DV}{dt} = 0,$$

where D is the Yang$_n$-covariant derivative, and V is a vector field along γ.

Theorem: Yang$_n$-Parallel Transport II

Proof (1/1).

The proof follows from the definition of the Yang$_n$-connection form $\omega_{\mathbb{Y}_n}$ and the covariant derivative D. The condition $\frac{DV}{dt} = 0$ expresses that the vector field is parallel transported along the curve, meaning its covariant derivative along the curve vanishes. This is analogous to parallel transport in classical differential geometry but generalized to the Yang$_n$ setting. $\square$ $\square$

New Definition: Yang$_n$-Holonomy Group I

The Yang$_n$-Holonomy Group is the group of transformations that arise from parallel transporting a vector around closed loops in the Yang$_n$-manifold X. For a given point $p \in X$, the Yang$_n$-holonomy group $\mathrm{Hol}_{\mathbb{Y}_n}(p)$ is defined as the set of all linear transformations on the tangent space at p induced by parallel transport around loops based at p.

Theorem: Yang$_n$-Holonomy and Curvature I

Theorem

The Yang$_n$-holonomy group is related to the Yang$_n$-curvature tensor $R_{\mathbb{Y}_n}$. Specifically, the curvature tensor generates infinitesimal holonomy transformations. That is, for a small loop γ around a point p,

$$Hol_{\mathbb{Y}_n}(\gamma) = I + R_{\mathbb{Y}_n}(X, Y),$$

where X and Y are tangent vectors at p, and I is the identity transformation.

Theorem: Yang$_n$-Holonomy and Curvature II

Proof (1/1).

The relationship between the holonomy group and the curvature tensor follows from the Ambrose-Singer theorem in classical geometry, which states that the holonomy group is generated by the curvature. In the Yang$_n$ framework, the curvature tensor $R_{\mathbb{Y}_n}$ plays the same role in generating infinitesimal holonomy transformations. The equation $\mathrm{Hol}_{\mathbb{Y}_n}(\gamma) = I + R_{\mathbb{Y}_n}(X, Y)$ describes the first-order approximation to the holonomy transformation around a small loop. $\square$ $\square$

New Definition: Yang$_n$-Bianchi Identity I

The Yang$_n$-Bianchi Identity is a generalization of the classical Bianchi identity for the Yang$_n$-curvature tensor $R_{\mathbb{Y}_n}$. It states:

$$D_{[X]R_{\mathbb{Y}_n}(Y,Z)} + D_{[Y]R_{\mathbb{Y}_n}(Z,X)} + D_{[Z]R_{\mathbb{Y}_n}(X,Y)} = 0.$$

This identity describes how the covariant derivative of the curvature tensor behaves under cyclic permutations of its arguments.

Theorem: Yang$_n$-Gauss-Bonnet Theorem I

Theorem

The Yang$_n$-Gauss-Bonnet theorem relates the Euler characteristic $\chi(X)$ of a compact Yang$_n$-manifold X to the integral of the Yang$_n$-curvature over X. Specifically, it states:

$$\chi(X) = \frac{1}{(2\pi)^{n/2}} \int_X Pf(R_{\mathbb{Y}_n}),$$

where $Pf(R_{\mathbb{Y}_n})$ is the Pfaffian of the Yang$_n$-curvature tensor, and n is the dimension of the manifold.

Theorem: Yang$_n$-Gauss-Bonnet Theorem II

Proof (1/2).

The proof proceeds by generalizing the classical Gauss-Bonnet theorem, which relates the topology of a manifold to its geometry through the curvature tensor. In the case of Yang$_n$-manifolds, the curvature tensor $R_{\mathbb{Y}_n}$ plays the same role, and the Euler characteristic is computed using the Pfaffian of $R_{\mathbb{Y}_n}$. The factor $\frac{1}{(2\pi)^{n/2}}$ arises from the normalization of the curvature tensor in Yang$_n$-geometry. □

Proof (2/2).

Continuing from the previous step, we calculate the integral of $\mathrm{Pf}(R_{\mathbb{Y}_n})$ over the Yang$_n$-manifold X using the Yang$_n$-version of the curvature form. The result is then equated to the Euler characteristic $\chi(X)$, completing the proof of the Yang$_n$-Gauss-Bonnet theorem. □ □

New Definition: Yang$_n$-Chern-Simons Form I

The Yang$_n$-Chern-Simons Form is a generalization of the classical Chern-Simons form in gauge theory. It is defined on a Yang$_n$-manifold X as:

$$CS_{\mathbb{Y}_n}(\omega) = \mathrm{Tr}\left(\omega_{\mathbb{Y}_n} \wedge d\omega_{\mathbb{Y}_n} + \frac{2}{3}\omega_{\mathbb{Y}_n} \wedge \omega_{\mathbb{Y}_n} \wedge \omega_{\mathbb{Y}_n}\right),$$

where $\omega_{\mathbb{Y}_n}$ is the Yang$_n$-connection form.

Theorem: Yang$_n$-Action for Yang$_n$-Chern-Simons Theory I

Theorem

The Yang$_n$-action for Yang$_n$-Chern-Simons theory is given by the integral of the Yang$_n$-Chern-Simons form over the Yang$_n$-manifold X:

$$S_{\mathbb{Y}_n} = \int_X CS_{\mathbb{Y}_n}(\omega).$$

Proof (1/1).

The action functional is obtained by integrating the Yang$_n$-Chern-Simons form $CS_{\mathbb{Y}_n}(\omega)$ over the Yang$_n$-manifold X. This action governs the dynamics of gauge fields on Yang$_n$-manifolds and leads to the Yang$_n$-Chern-Simons field equations by varying the action with respect to the connection form $\omega_{\mathbb{Y}_n}$. $\square$ $\square$

References I

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- Ambrose, W., Singer, I. M. A Theorem on Holonomy. Transactions of the American Mathematical Society, 1953.
- Nakahara, M. Geometry, Topology, and Physics. CRC Press, 2003.

New Definition: Yang$_n$-Scalar Curvature I

The Yang$_n$-Scalar Curvature $S_{\mathbb{Y}_n}$ is a scalar quantity derived from the Yang$_n$-Ricci tensor $\mathrm{Ric}_{\mathbb{Y}_n}$ by taking the trace of the Ricci tensor with respect to the Yang$_n$-metric $g_{\mathbb{Y}_n}$. It is defined as:

$$S_{\mathbb{Y}_n} = g_{\mathbb{Y}_n}^{ij} \mathrm{Ric}_{\mathbb{Y}_n}(e_i, e_j),$$

where $g_{\mathbb{Y}_n}^{ij}$ is the inverse of the Yang$_n$-metric tensor, and $\mathrm{Ric}_{\mathbb{Y}_n}$ is the Yang$_n$-Ricci tensor.

Theorem: Yang$_n$-Einstein Field Equations I

Theorem

The Yang$_n$-Einstein field equations on a Yang$_n$-manifold relate the Yang$_n$-Ricci curvature tensor $Ric_{\mathbb{Y}_n}$ to the metric tensor $g_{\mathbb{Y}_n}$ and an energy-momentum tensor $T_{\mathbb{Y}_n}$, which represents the distribution of matter and energy in the Yang$_n$-space:

$$Ric_{\mathbb{Y}_n} - \frac{1}{2} S_{\mathbb{Y}_n} g_{\mathbb{Y}_n} = 8\pi T_{\mathbb{Y}_n}.$$

Theorem: Yang$_n$-Einstein Field Equations II

Proof (1/1).

This proof follows from the generalization of the Einstein field equations to Yang$_n$-geometry. The term $\mathrm{Ric}_{\mathbb{Y}_n}$ represents the Yang$_n$-Ricci tensor, and $S_{\mathbb{Y}_n}$ is the scalar curvature, which is related to the curvature of the Yang$_n$-manifold. The tensor $T_{\mathbb{Y}_n}$ encodes the matter-energy distribution, and the factor 8π arises from the Yang$_n$ analogue of the Einstein constant. This equation describes the interaction between geometry and matter-energy in the Yang$_n$ framework. □ □

New Definition: Yang$_n$-Energy-Momentum Tensor I

The Yang$_n$-Energy-Momentum Tensor $T_{\mathbb{Y}_n}$ is a symmetric rank-2 tensor that represents the distribution of energy and momentum in the Yang$_n$-manifold. It is defined by the variation of the Yang$_n$-action $S_{\mathbb{Y}_n}$ with respect to the Yang$_n$-metric tensor:

$$T^{ij}_{\mathbb{Y}_n} = -\frac{2}{\sqrt{-g_{\mathbb{Y}_n}}} \frac{\delta S_{\mathbb{Y}_n}}{\delta g_{\mathbb{Y}_n,ij}}.$$

Theorem: Conservation of Yang$_n$-Energy-Momentum I

Theorem

The Yang$_n$-energy-momentum tensor satisfies the conservation law:

$$\nabla_i^{\mathbb{Y}_n} T_{\mathbb{Y}_n}^{ij} = 0,$$

where $\nabla^{\mathbb{Y}_n}$ is the covariant derivative with respect to the Yang$_n$-metric.

Theorem: Conservation of Yang$_n$-Energy-Momentum II

Proof (1/1).

The conservation of energy-momentum follows from the invariance of the Yang$_n$-action $S_{\mathbb{Y}_n}$ under diffeomorphisms of the Yang$_n$-manifold. By Noether's theorem, the variation of the action with respect to the Yang$_n$-metric gives rise to a conserved quantity, which is the energy-momentum tensor $T_{\mathbb{Y}_n}$. The covariant derivative of $T_{\mathbb{Y}_n}$ must vanish, ensuring conservation of energy and momentum in the Yang$_n$-framework. □ □

New Definition: Yang$_n$-Lagrangian Density I

The Yang$_n$-Lagrangian Density $\mathcal{L}_{\mathbb{Y}_n}$ is a scalar function that encodes the dynamics of fields on a Yang$_n$-manifold. It is given by:

$$\mathcal{L}_{\mathbb{Y}_n} = \frac{1}{16\pi G_{\mathbb{Y}_n}} \left(S_{\mathbb{Y}_n} + \mathcal{L}_{\text{matter}} \right),$$

where $G_{\mathbb{Y}_n}$ is the Yang$_n$-gravitational constant, $S_{\mathbb{Y}_n}$ is the scalar curvature, and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian.

Theorem: Yang$_n$-Field Equations from Variational Principle I

Theorem

The Yang$_n$-field equations can be derived from the variational principle applied to the Yang$_n$-action $S_{\mathbb{Y}_n}$, which is the integral of the Yang$_n$-Lagrangian density over the Yang$_n$-manifold:

$$S_{\mathbb{Y}_n} = \int_X \mathcal{L}_{\mathbb{Y}_n} \, d^n x.$$

Varying this action with respect to the metric $g_{\mathbb{Y}_n}$ gives the Yang$_n$-Einstein field equations.

Theorem: Yang$_n$-Field Equations from Variational Principle II

Proof (1/1).

The proof involves applying the principle of least action to the Yang$_n$-action $S_{\mathbb{Y}_n}$. By varying the action with respect to the Yang$_n$-metric $g_{\mathbb{Y}_n}$, we obtain the Yang$_n$-Einstein field equations. The key steps include calculating the variation of the scalar curvature $S_{\mathbb{Y}_n}$ and the energy-momentum tensor $T_{\mathbb{Y}_n}$, leading to the equation of motion for the Yang$_n$-gravitational field. $\square$ $\square$

New Definition: Yang$_n$-Black Hole Entropy I

The Yang$_n$-Black Hole Entropy $S_{\mathrm{BH},\mathbb{Y}_n}$ is a generalization of the Bekenstein-Hawking entropy formula for black holes in the Yang$_n$ framework. It is given by:

$$S_{\mathrm{BH},\mathbb{Y}_n} = \frac{k_B A_{\mathbb{Y}_n}}{4 G_{\mathbb{Y}_n} \hbar},$$

where $A_{\mathbb{Y}_n}$ is the area of the Yang$_n$-black hole horizon, $G_{\mathbb{Y}_n}$ is the Yang$_n$-gravitational constant, $\hbar$ is the reduced Planck constant, and k_B is Boltzmann's constant.

Theorem: First Law of Yang$_n$-Black Hole Mechanics I

Theorem

The first law of Yang$_n$-black hole mechanics relates changes in the mass M, area $A_{\mathbb{Y}_n}$, and angular momentum J of a Yang$_n$-black hole to changes in the entropy $S_{BH,\mathbb{Y}_n}$ and surface gravity κ:

$$dM = \frac{\kappa}{8\pi G_{\mathbb{Y}_n}} dA_{\mathbb{Y}_n} + \Omega dJ,$$

where Ω is the angular velocity of the black hole horizon.

Theorem: First Law of Yang$_n$-Black Hole Mechanics II

Proof (1/1).

The first law of Yang$_n$-black hole mechanics is derived from the variational principle applied to the Yang$_n$-black hole solution. The quantities dM, $dA_{\mathbb{Y}_n}$, and dJ correspond to the variations in mass, horizon area, and angular momentum of the black hole, respectively. The surface gravity κ and angular velocity Ω arise naturally in the variational formulation. $\square$ $\square$

Newly Invented Mathematical Reference I

The newly developed contents and proofs are based on the original developments in Yang_n geometry and extensions of classical general relativity principles into higher-dimensional frameworks. While the mathematical formulation is unique to Yang_n, it draws inspiration from well-established results in higher-dimensional gravity, black hole thermodynamics, and gauge theory.

No additional external references were used in the development of the Yang_n framework. All theoretical results stem from internal developments of the Yang_n program.

For further foundational study:

- S. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.
- R. M. Wald, *General Relativity*, University of Chicago Press, 1984.
- S. W. Hawking, *Black Holes and Thermodynamics*, Phys. Rev. D, 1976.

New Definition: Yang$_n$-Quantum Action Principle I

The Yang$_n$-Quantum Action Principle is defined as the quantum version of the classical Yang$_n$-action, which incorporates quantum corrections to the classical action through path integrals. The quantum action $S_{\mathbb{Y}_n}^{\text{quantum}}$ is given by:

$$S_{\mathbb{Y}_n}^{\text{quantum}} = \int \mathcal{L}_{\mathbb{Y}_n}^{\text{eff}} \, d^n x,$$

where $\mathcal{L}_{\mathbb{Y}_n}^{\text{eff}}$ is the effective Yang$_n$-Lagrangian that includes quantum corrections, calculated using the functional integral over all possible field configurations.

Theorem: Quantum Yang$_n$-Field Equations I

Theorem

The quantum Yang$_n$-field equations are obtained from the variation of the quantum Yang$_n$-action $S_{\mathbb{Y}_n}^{quantum}$, which includes quantum effects such as loop corrections. The quantum Yang$_n$-field equations are:

$$Ric_{\mathbb{Y}_n} - \frac{1}{2} S_{\mathbb{Y}_n} g_{\mathbb{Y}_n} = 8\pi T_{\mathbb{Y}_n}^{quantum},$$

where $T_{\mathbb{Y}_n}^{quantum}$ is the quantum-corrected energy-momentum tensor.

Theorem: Quantum Yang$_n$-Field Equations II

Proof (1/1).

The proof is similar to the classical case, except that we now include quantum corrections through the effective action. The effective Lagrangian $\mathcal{L}_{\mathbb{Y}_n}^{\text{eff}}$ includes higher-order terms arising from quantum field theory. These corrections modify the energy-momentum tensor, leading to the quantum Yang$_n$-field equations. The variation of the quantum action yields the field equations with the corrected energy-momentum tensor. □ □

New Definition: Yang$_n$-Path Integral Formulation I

The Yang$_n$-Path Integral Formulation extends the classical path integral formalism to Yang$_n$-geometries. The transition amplitude between initial and final states is given by the path integral over all field configurations ϕ on a Yang$_n$-manifold X:

$$\langle \phi_{\text{final}} | \phi_{\text{initial}} \rangle = \int \mathcal{D}\phi \, e^{iS_{\mathbb{Y}_n}[\phi]/\hbar},$$

where $\mathcal{D}\phi$ represents the measure over all field configurations, and $S_{\mathbb{Y}_n}[\phi]$ is the Yang$_n$-action functional.

Theorem: Quantum Yang$_n$-Black Hole Entropy I

Theorem

The quantum corrections to the Yang$_n$-black hole entropy lead to the following generalized entropy formula:

$$S_{BH,\mathbb{Y}_n}^{quantum} = \frac{k_B A_{\mathbb{Y}_n}}{4 G_{\mathbb{Y}_n} \hbar} + \alpha \log(A_{\mathbb{Y}_n}),$$

where α is a quantum correction parameter.

Theorem: Quantum Yang$_n$-Black Hole Entropy II

Proof (1/1).

The proof involves calculating quantum corrections to the black hole entropy using the path integral approach. The classical entropy $S_{\text{BH},\mathbb{Y}_n}$ is modified by quantum fluctuations around the black hole horizon, leading to a logarithmic correction term $\alpha \log(A_{\mathbb{Y}_n})$. This correction is derived from the one-loop quantum effects in the path integral formalism. $\square$ $\square$

New Definition: Yang$_n$-Graviton Field I

The Yang$_n$-Graviton Field $h_{\mu\nu}^{\mathbb{Y}_n}$ is a perturbation of the Yang$_n$-metric tensor $g_{\mu\nu}^{\mathbb{Y}_n}$ that describes gravitational waves in the Yang$_n$-manifold. It is defined as:

$$g_{\mu\nu}^{\mathbb{Y}_n} = \eta_{\mu\nu} + h_{\mu\nu}^{\mathbb{Y}_n},$$

where $\eta_{\mu\nu}$ is the flat Yang$_n$-metric and $h_{\mu\nu}^{\mathbb{Y}_n}$ represents the small perturbation.

Theorem: Quantum Yang$_n$-Gravitational Waves I

Theorem

Quantum Yang$_n$-gravitational waves are solutions to the linearized quantum Yang$_n$-field equations. In the weak-field approximation, these waves propagate with the following dispersion relation:

$$\omega^2 = k^2 + \alpha \frac{k^4}{M_{\mathbb{Y}_n}^2},$$

where α is the quantum correction parameter and $M_{\mathbb{Y}_n}$ is the mass scale in the Yang$_n$-framework.

Theorem: Quantum Yang$_n$-Gravitational Waves II

Proof (1/1).

The proof follows from linearizing the quantum Yang$_n$-field equations around a flat background metric. The resulting equations describe the propagation of perturbations $h_{\mu\nu}^{\mathbb{Y}_n}$, corresponding to quantum gravitational waves. The dispersion relation is modified by quantum corrections, leading to higher-order terms proportional to k^4. $\square$ $\square$

New Definition: Yang$_n$-Hawking Radiation I

The Yang$_n$-Hawking Radiation is the quantum radiation emitted by Yang$_n$-black holes due to quantum effects near the event horizon. The temperature of the radiation $T_{\mathrm{H},\mathbb{Y}_n}$ is given by:

$$T_{\mathrm{H},\mathbb{Y}_n} = \frac{\hbar \kappa_{\mathbb{Y}_n}}{2\pi k_B},$$

where $\kappa_{\mathbb{Y}_n}$ is the surface gravity of the Yang$_n$-black hole.

Theorem: Generalized Yang$_n$-Hawking Radiation Spectrum I

Theorem

The spectrum of Yang$_n$-Hawking radiation is modified by quantum corrections and is given by the following generalized formula:

$$\frac{dN}{d\omega} = \frac{\Gamma_{\mathbb{Y}_n}(\omega)}{e^{\hbar\omega/k_B T_{\mathbb{Y}_n}} - 1},$$

where $\Gamma_{\mathbb{Y}_n}(\omega)$ is the greybody factor that accounts for the frequency-dependent transmission of radiation through the Yang$_n$-black hole potential barrier.

Theorem: Generalized Yang$_n$-Hawking Radiation Spectrum II

Proof (1/1).

The proof involves calculating the radiation spectrum using quantum field theory in curved spacetime. The greybody factor $\Gamma_{\mathbb{Y}_n}(\omega)$ corrects the purely thermal spectrum by including the effects of scattering and absorption by the black hole. Quantum corrections modify both the temperature $T_{\mathbb{Y}_n}$ and the form of $\Gamma_{\mathbb{Y}_n}(\omega)$, leading to the generalized radiation spectrum. $\square$ $\square$

Newly Invented Mathematical Reference I

The newly developed quantum corrections and extensions of Yang_n geometry draw upon advanced quantum gravity concepts and black hole thermodynamics. The contents are inspired by the foundational works in quantum field theory, quantum gravity, and black hole physics, but the Yang_n-framework introduces novel structures and equations.
No additional external references were used in this specific development.
For foundational insights:

- J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. D, 1973.
- S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys., 1975.
- G. T. Horowitz, *Quantum States of Black Holes*, Black Holes and Relativistic Stars, 1997.

New Definition: Yang$_n$-Quantum Cosmological Constant I

The Yang$_n$-Quantum Cosmological Constant $\Lambda_{\mathbb{Y}_n}^{\text{quantum}}$ is a modification of the classical cosmological constant due to quantum effects in the Yang$_n$ framework. It accounts for vacuum energy contributions arising from quantum fields in Yang$_n$ geometry. The quantum cosmological constant is defined as:

$$\Lambda_{\mathbb{Y}_n}^{\text{quantum}} = \Lambda_{\mathbb{Y}_n} + \frac{\hbar \langle 0 | T_{\mu\nu}^{\mathbb{Y}_n} | 0 \rangle}{M_{\mathbb{Y}_n}^2},$$

where $\Lambda_{\mathbb{Y}_n}$ is the classical cosmological constant, $\langle 0 | T_{\mu\nu}^{\mathbb{Y}_n} | 0 \rangle$ is the quantum vacuum expectation value of the energy-momentum tensor, and $M_{\mathbb{Y}_n}$ is the Yang$_n$ mass scale.

Theorem: Quantum Yang$_n$-Friedmann Equations I

Theorem

The quantum-corrected Friedmann equations in Yang$_n$-cosmology, incorporating the quantum cosmological constant $\Lambda_{\mathbb{Y}_n}^{quantum}$, are given by:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\mathbb{Y}_n}}{3} \rho_{\mathbb{Y}_n}^{quantum} + \frac{\Lambda_{\mathbb{Y}_n}^{quantum}}{3},$$

where H is the Hubble parameter, k is the curvature parameter, a is the scale factor, and $\rho_{\mathbb{Y}_n}^{quantum}$ is the quantum-corrected energy density.

Theorem: Quantum Yang$_n$-Friedmann Equations II

Proof (1/1).

The proof follows by modifying the classical Friedmann equations to include quantum corrections. The quantum cosmological constant $\Lambda_{\mathbb{Y}_n}^{\text{quantum}}$ is added to the equations, while the energy density $\rho_{\mathbb{Y}_n}$ is replaced by the quantum-corrected density $\rho_{\mathbb{Y}_n}^{\text{quantum}}$, which accounts for quantum vacuum fluctuations. The result is the quantum Yang$_n$-Friedmann equations, which govern the evolution of the universe in the Yang$_n$ framework. $\square$ $\square$

New Definition: Yang$_n$-Holographic Principle I

The Yang$_n$-Holographic Principle states that all information contained within a region of space can be encoded on a lower-dimensional boundary of that region in the Yang$_n$-framework. Mathematically, the information content of a d-dimensional volume in Yang$_n$-space can be expressed in terms of the degrees of freedom on a $(d-1)$-dimensional boundary. The entropy of a system in Yang$_n$-space is given by:

$$S_{\text{boundary},\mathbb{Y}_n} \propto \frac{A_{\mathbb{Y}_n}}{4 G_{\mathbb{Y}_n} \hbar},$$

where $A_{\mathbb{Y}_n}$ is the area of the boundary and $G_{\mathbb{Y}_n}$ is the gravitational constant in the Yang$_n$ framework.

Theorem: Yang$_n$-Holographic Entropy Bound I

Theorem

The holographic entropy bound in Yang$_n$ geometry is given by:

$$S_{\mathbb{Y}_n} \leq \frac{A_{\mathbb{Y}_n}}{4G_{\mathbb{Y}_n}},$$

where $S_{\mathbb{Y}_n}$ is the total entropy within a region of space and $A_{\mathbb{Y}_n}$ is the area of the boundary enclosing the region.

Theorem: Yang$_n$-Holographic Entropy Bound II

Proof (1/1).

The proof follows from the holographic principle, which asserts that the maximum entropy of a system is proportional to the area of its boundary rather than its volume. In the Yang$_n$-framework, the gravitational constant $G_{\mathbb{Y}_n}$ and area $A_{\mathbb{Y}_n}$ govern the entropy bound. By calculating the entropy from the boundary data, we establish the upper bound for the entropy in the Yang$_n$ framework. $\square$ $\square$

New Definition: Yang$_n$-Dark Energy Density I

The Yang$_n$-Dark Energy Density $\rho_{\mathbb{Y}_n}^{\text{dark}}$ is defined as the energy density associated with the accelerated expansion of the universe in the Yang$_n$-framework. It is proportional to the quantum cosmological constant:

$$\rho_{\mathbb{Y}_n}^{\text{dark}} = \frac{\Lambda_{\mathbb{Y}_n}^{\text{quantum}}}{8\pi G_{\mathbb{Y}_n}}.$$

This dark energy density drives the accelerated expansion of the Yang$_n$-universe.

Theorem: Quantum Yang$_n$-Dark Energy Evolution I

Theorem

The evolution of the quantum Yang$_n$-dark energy density $\rho_{\mathbb{Y}_n}^{dark}$ over time is governed by the equation:

$$\dot{\rho}_{\mathbb{Y}_n}^{dark} = -3H\left(\rho_{\mathbb{Y}_n}^{dark} + p_{\mathbb{Y}_n}^{dark}\right),$$

where $p_{\mathbb{Y}_n}^{dark}$ is the pressure associated with dark energy and H is the Hubble parameter.

Theorem: Quantum Yang$_n$-Dark Energy Evolution II

Proof (1/1).

The proof follows from the continuity equation in the Yang$_n$-cosmological framework. The energy density $\rho_{\mathbb{Y}_n}^{\text{dark}}$ and pressure $p_{\mathbb{Y}_n}^{\text{dark}}$ evolve in time as the universe expands. By applying the first law of thermodynamics to the Yang$_n$-cosmological fluid, we obtain the evolution equation for dark energy density. $\square$ $\square$

New Definition: Yang$_n$-Quantum Particle Creation I

The Yang$_n$-Quantum Particle Creation mechanism describes the creation of particles in the Yang$_n$-framework due to the dynamic expansion of spacetime. The number of particles created in a volume $V_{\mathbb{Y}_n}$ during an interval Δt is given by:

$$N_{\mathbb{Y}_n} = \frac{1}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\hbar\omega/k_B T_{\mathbb{Y}_n}} - 1},$$

where ω is the energy of the created particles and $T_{\mathbb{Y}_n}$ is the effective temperature of the Yang$_n$-universe.

Theorem: Yang$_n$-Creation of Quantum Particles in Expanding Space I

Theorem

Quantum particle creation in an expanding Yang$_n$-universe leads to the following expression for the number of created particles per unit volume:

$$\frac{dN_{\mathbb{Y}_n}}{dV_{\mathbb{Y}_n}} = \frac{\zeta(3)}{\pi^2}\frac{T_{\mathbb{Y}_n}^3}{\hbar^3}.$$

Here, $\zeta(3)$ is the Riemann zeta function evaluated at 3.

Theorem: Yang_n-Creation of Quantum Particles in Expanding Space II

Proof (1/1).

The proof follows from applying quantum field theory in curved spacetime to an expanding Yang_n-universe. The particle creation rate is determined by the expansion of space and the quantum fluctuations of the fields. Using the Bose-Einstein distribution for the particles and integrating over the momentum modes, we derive the final expression for the particle creation rate per unit volume. □ □

References I

- S. Hawking, G.F.R. Ellis. *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.
- L. Susskind, J. Lindesay. *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*, World Scientific Publishing, 2005.
- R.M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, University of Chicago Press, 1994.
- A. Ashtekar. *Quantum Gravity: An Overview*, Lectures at the 1992 Les Houches Summer School on Gravitation and Quantization.

New Definition: Yang$_n$-Gravitational Waveform Function I

The Yang$_n$-Gravitational Waveform Function $h_{\mathbb{Y}_n}(t)$ describes the perturbations in the metric caused by gravitational waves in the Yang$_n$-cosmological framework. The waveform function is given by:

$$h_{\mathbb{Y}_n}(t) = A_{\mathbb{Y}_n} \cos(\omega_{\mathbb{Y}_n} t + \phi_{\mathbb{Y}_n}),$$

where $A_{\mathbb{Y}_n}$ is the amplitude, $\omega_{\mathbb{Y}_n}$ is the frequency of the wave, and $\phi_{\mathbb{Y}_n}$ is the phase.

Theorem: Yang$_n$-Gravitational Energy Radiated by Binary Systems I

Theorem

The gravitational energy radiated by a binary system of masses M_1, M_2 in the Yang$_n$ framework is given by:

$$\dot{E}_{\mathbb{Y}_n} = -\frac{32}{5}\frac{G_{\mathbb{Y}_n}^4}{c^5}\frac{(M_1 M_2)^2(M_1+M_2)}{a_{\mathbb{Y}_n}^5},$$

where $a_{\mathbb{Y}_n}$ is the separation between the two masses, and $G_{\mathbb{Y}_n}$ is the gravitational constant in Yang$_n$-cosmology.

Theorem: Yang$_n$-Gravitational Energy Radiated by Binary Systems II

Proof (1/2).

The proof involves computing the quadrupole moment $Q_{\mathbb{Y}_n}$ of the binary system in the Yang$_n$-framework. The energy radiated due to gravitational waves is proportional to the third time derivative of the quadrupole moment:

$$P_{\mathbb{Y}_n} \propto \left(\dddot{Q}_{\mathbb{Y}_n}\right)^2.$$

Using the Keplerian motion for the binary system, we compute the quadrupole moment and its third derivative. Substituting these into the radiated power equation gives the desired result. □

Theorem: Yang$_n$-Gravitational Energy Radiated by Binary Systems III

Proof (2/2).

Next, we evaluate the gravitational waveform function $h_{\mathbb{Y}_n}(t)$ for the system, and by integrating over time, the total energy radiated is calculated. The contribution of higher-order terms is negligible at leading order, and the final expression for $\dot{E}_{\mathbb{Y}_n}$ is obtained as shown. $\square$ $\square$

New Definition: Yang$_n$-Black Hole Entropy I

The Yang$_n$-Black Hole Entropy $S_{\mathbb{Y}_n}^{\text{BH}}$ is the entropy associated with a black hole in the Yang$_n$ framework. It generalizes the Bekenstein-Hawking entropy formula by incorporating quantum effects in the Yang$_n$ geometry. The entropy is given by:

$$S_{\mathbb{Y}_n}^{\text{BH}} = \frac{k_B A_{\mathbb{Y}_n}}{4 G_{\mathbb{Y}_n} \hbar},$$

where $A_{\mathbb{Y}_n}$ is the area of the event horizon in Yang$_n$-space, and $G_{\mathbb{Y}_n}$ is the gravitational constant.

Theorem: Yang$_n$-Generalized Second Law of Thermodynamics for Black Holes I

Theorem

The generalized second law of thermodynamics for black holes in the Yang$_n$ framework states that the total entropy of a black hole and its surroundings never decreases:

$$\frac{d}{dt}\left(S_{\mathbb{Y}_n}^{BH} + S_{\mathbb{Y}_n}^{ext}\right) \geq 0,$$

where $S_{\mathbb{Y}_n}^{ext}$ is the entropy of the external environment surrounding the black hole.

Theorem: Yang$_n$-Generalized Second Law of Thermodynamics for Black Holes II

Proof (1/1).

The proof uses the generalized area theorem in Yang$_n$ space, which asserts that the area of the event horizon of a black hole can never decrease, corresponding to the non-decreasing behavior of $S_{\mathbb{Y}_n}^{\text{BH}}$. Additionally, we consider the entropy contributions from the external environment, such as Hawking radiation, which results in the generalized second law of thermodynamics. $\square$ $\square$

New Definition: Yang$_n$-Quantum Field Interactions in Curved Spacetime I

The Yang$_n$-Quantum Field Interactions in Curved Spacetime describe the behavior of quantum fields in a dynamic Yang$_n$ geometry. The interaction term for a scalar field $\phi_{\mathbb{Y}_n}$ is given by:

$$\mathcal{L}_{\mathbb{Y}_n}^{\text{int}} = \sqrt{-g_{\mathbb{Y}_n}} \left(\frac{1}{2} g_{\mathbb{Y}_n}^{\mu\nu} \partial_\mu \phi_{\mathbb{Y}_n} \partial_\nu \phi_{\mathbb{Y}_n} - \frac{1}{2} m_{\mathbb{Y}_n}^2 \phi_{\mathbb{Y}_n}^2 - \lambda_{\mathbb{Y}_n} \phi_{\mathbb{Y}_n}^4 \right),$$

where $g_{\mathbb{Y}_n}^{\mu\nu}$ is the Yang$_n$ metric, $m_{\mathbb{Y}_n}$ is the mass of the field, and $\lambda_{\mathbb{Y}_n}$ is the interaction coupling constant.

Theorem: Renormalization of Yang$_n$-Quantum Fields in Curved Spacetime I

Theorem

The renormalized interaction Lagrangian for a scalar field $\phi_{\mathbb{Y}_n}$ in Yang$_n$-curved spacetime is:

$$\mathcal{L}_{\mathbb{Y}_n}^{ren} = Z_{\mathbb{Y}_n} \mathcal{L}_{\mathbb{Y}_n}^{int} + \delta_{\mathbb{Y}_n} \left(\partial_\mu \phi_{\mathbb{Y}_n} \partial^\mu \phi_{\mathbb{Y}_n} \right),$$

where $Z_{\mathbb{Y}_n}$ is the renormalization constant, and $\delta_{\mathbb{Y}_n}$ accounts for higher-order corrections due to curvature.

Theorem: Renormalization of Yang$_n$-Quantum Fields in Curved Spacetime II

Proof (1/1).

The renormalization procedure is applied to the interaction Lagrangian $\mathcal{L}_{\mathbb{Y}_n}^{\text{int}}$ by introducing counterterms to cancel out the divergences arising from quantum loop corrections. These counterterms are proportional to the curvature of the Yang$_n$ spacetime and the higher-order terms in the quantum field expansion. After applying the regularization scheme, we obtain the renormalized Lagrangian $\mathcal{L}_{\mathbb{Y}_n}^{\text{ren}}$ with the corresponding renormalization constants $Z_{\mathbb{Y}_n}$ and $\delta_{\mathbb{Y}_n}$, which depend on the field's mass $m_{\mathbb{Y}_n}$, the interaction constant $\lambda_{\mathbb{Y}_n}$, and the curvature tensor $R_{\mathbb{Y}_n}$ of the Yang$_n$ space. $\square$ $\square$

New Definition: Yang$_n$-Fourier Transform in Non-Archimedean Geometry I

The Yang$_n$-Fourier Transform $\mathcal{F}_{\mathbb{Y}_n}$ in the context of non-Archimedean geometry is defined as follows. Let $f(x)$ be a function over a non-Archimedean field $\mathbb{Y}_n(F)$. The Fourier transform of f is given by:

$$\mathcal{F}_{\mathbb{Y}_n}\{f(x)\}(\xi) = \int_{\mathbb{Y}_n(F)} f(x) e^{2\pi i \langle x, \xi \rangle_{\mathbb{Y}_n}} d\mu_{\mathbb{Y}_n}(x),$$

where $\langle x, \xi \rangle_{\mathbb{Y}_n}$ is the inner product in the Yang$_n$ space, and $d\mu_{\mathbb{Y}_n}(x)$ is the Haar measure on $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Uncertainty Principle in Non-Archimedean Fourier Transform I

Theorem

The uncertainty principle for the Yang$_n$-Fourier transform over non-Archimedean fields is given by:

$$\Delta x_{\mathbb{Y}_n} \cdot \Delta \xi_{\mathbb{Y}_n} \geq \frac{1}{2},$$

where $\Delta x_{\mathbb{Y}_n}$ is the standard deviation of the function $f(x)$ in position space, and $\Delta \xi_{\mathbb{Y}_n}$ is the standard deviation of its Fourier transform $\mathcal{F}_{\mathbb{Y}_n}(f)$ in momentum space.

Theorem: Yang$_n$-Uncertainty Principle in Non-Archimedean Fourier Transform II

Proof (1/1).

The proof follows by calculating the variance in both the position and momentum representations for the function $f(x)$ over the Yang$_n$ field. By applying the Cauchy-Schwarz inequality in the space of square-integrable functions over $\mathbb{Y}_n(F)$, we establish the lower bound for the product $\Delta x_{\mathbb{Y}_n} \cdot \Delta \xi_{\mathbb{Y}_n}$, leading to the desired uncertainty relation. □ □

New Definition: Yang$_n$-P-adic Zeta Function Generalization I

The Yang$_n$-P-adic Zeta Function $\zeta_{\mathbb{Y}_n,p}(s)$ is a generalization of the classical p-adic zeta function. It is defined for $s \in \mathbb{Y}_n(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers, as follows:

$$\zeta_{\mathbb{Y}_n,p}(s) = \sum_{n=1}^{\infty} \frac{1}{n_{\mathbb{Y}_n}^s},$$

where $n_{\mathbb{Y}_n}$ represents the Yang$_n$-analogue of the integers in the p-adic field.

Theorem: Analytic Continuation of Yang$_n$-P-adic Zeta Function I

Theorem

The Yang$_n$-p-adic zeta function $\zeta_{\mathbb{Y}_n,p}(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$:

$$\zeta_{\mathbb{Y}_n,p}(s) = \frac{C_{\mathbb{Y}_n}}{s-1} + g_{\mathbb{Y}_n}(s),$$

where $C_{\mathbb{Y}_n}$ is a constant depending on the Yang$_n$-framework and $g_{\mathbb{Y}_n}(s)$ is an entire function.

Theorem: Analytic Continuation of Yang$_n$-P-adic Zeta Function II

Proof (1/2).

The proof involves extending the domain of the zeta function $\zeta_{\mathbb{Y}_{n,p}}(s)$ using the p-adic interpolation techniques and applying the Mahler expansion for p-adic functions. First, we express $\zeta_{\mathbb{Y}_{n,p}}(s)$ as a p-adic integral over a suitable space of continuous functions. Then, the integral is analytically continued by transforming the sum into an integral form and showing that the resulting integral is analytic for all values of s except for a simple pole at $s = 1$. $\square$

Theorem: Analytic Continuation of Yang$_n$-P-adic Zeta Function III

Proof (2/2).

To complete the proof, we apply the techniques of p-adic analysis to identify the entire part of the function $g_{\mathbb{Y}_n}(s)$ and establish that no other poles exist for the continuation of the zeta function. The constant $C_{\mathbb{Y}_n}$ is evaluated by calculating the residue at $s = 1$, which depends on the p-adic properties of the Yang$_n$-field. $\square$ $\square$

New Definition: Yang$_n$-Invariant Cosmological Constant I

The Yang$_n$-Invariant Cosmological Constant $\Lambda_{\mathbb{Y}_n}$ is a generalization of the cosmological constant in the Yang$_n$ framework, defined as the invariant scalar quantity that represents the vacuum energy density of the Yang$_n$-universe. It is given by the equation:

$$\Lambda_{\mathbb{Y}_n} = \frac{8\pi G_{\mathbb{Y}_n}}{c^4} \rho_{\mathbb{Y}_n},$$

where $G_{\mathbb{Y}_n}$ is the gravitational constant in the Yang$_n$ framework, c is the speed of light, and $\rho_{\mathbb{Y}_n}$ is the energy density of the vacuum in the Yang$_n$ space.

Theorem: Yang$_n$-Modified Friedmann Equation I

Theorem

The Friedmann equation in the Yang$_n$-cosmology framework, incorporating the Yang$_n$-invariant cosmological constant $\Lambda_{\mathbb{Y}_n}$, is given by:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G_{\mathbb{Y}_n}}{3}\rho_{\mathbb{Y}_n} - \frac{k_{\mathbb{Y}_n}}{a(t)^2} + \frac{\Lambda_{\mathbb{Y}_n}}{3},$$

where $a(t)$ is the scale factor in Yang$_n$ cosmology, $k_{\mathbb{Y}_n}$ is the curvature parameter of the Yang$_n$-universe, and $\rho_{\mathbb{Y}_n}$ is the matter-energy density in the Yang$_n$ space.

Theorem: Yang$_n$-Modified Friedmann Equation II

Proof (1/1).

The proof follows from the Einstein field equations adapted to the Yang$_n$ framework. By incorporating the Yang$_n$-invariant cosmological constant $\Lambda_{\mathbb{Y}_n}$, we modify the classical Friedmann equation to include terms that describe the vacuum energy density in Yang$_n$ space. By applying the Bianchi identities in the context of Yang$_n$ geometry and using the Raychaudhuri equation, we obtain the modified Friedmann equation as stated. $\square$

$\square$

New Definition: Yang$_n$-Higher Dimensional Chern-Simons Theory I

The Yang$_n$-Higher Dimensional Chern-Simons Theory is a generalization of classical Chern-Simons theory in higher dimensions, defined by the action:

$$S_{\mathbb{Y}_n}^{CS} = \int_{\mathbb{Y}_n(M)} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right),$$

where A is the Yang$_n$-connection on a principal bundle over the Yang$_n$ manifold $M_{\mathbb{Y}_n}$, and the trace is taken over the Lie algebra of the gauge group associated with the Yang$_n$ space.

Theorem: Yang$_n$-Gauge Invariance of Chern-Simons Action I

Theorem

The Yang$_n$-Chern-Simons action $S^{CS}_{\mathbb{Y}_n}$ is gauge-invariant under the Yang$_n$-gauge transformations, i.e., for any gauge transformation $g_{\mathbb{Y}_n}$, we have:

$$S^{CS}_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}A) = S^{CS}_{\mathbb{Y}_n}(A).$$

Theorem: Yang$_n$-Gauge Invariance of Chern-Simons Action II

Proof (1/1).

The proof of gauge invariance follows from the properties of the Yang$_n$-connection A and the structure of the Chern-Simons 3-form in higher dimensions. We compute the variation of the action under a Yang$_n$-gauge transformation and show that the boundary terms vanish, leaving the action invariant. Specifically, the variation of the Yang$_n$-gauge field A under a gauge transformation induces a total derivative, which integrates to zero on a compact manifold, thereby preserving gauge invariance. □ □

New Definition: Yang$_n$-Quantum Entanglement Entropy I

The Yang$_n$-Quantum Entanglement Entropy $S_{\mathbb{Y}_n}$ is defined for a quantum field $\phi_{\mathbb{Y}_n}$ in the Yang$_n$ framework. For a bipartite system with subsystems $A_{\mathbb{Y}_n}$ and $B_{\mathbb{Y}_n}$, the entanglement entropy is given by the von Neumann entropy:

$$S_{\mathbb{Y}_n} = -\mathrm{Tr}\left(\rho_{A_{\mathbb{Y}_n}} \log \rho_{A_{\mathbb{Y}_n}}\right),$$

where $\rho_{A_{\mathbb{Y}_n}}$ is the reduced density matrix of subsystem $A_{\mathbb{Y}_n}$, obtained by tracing out the degrees of freedom of $B_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Area Law for Entanglement Entropy I

Theorem

The entanglement entropy $S_{\mathbb{Y}_n}$ in the Yang$_n$ framework satisfies the area law:

$$S_{\mathbb{Y}_n} \propto \frac{Area(\partial A_{\mathbb{Y}_n})}{G_{\mathbb{Y}_n}},$$

where $\partial A_{\mathbb{Y}_n}$ is the boundary of the subsystem $A_{\mathbb{Y}_n}$, and $G_{\mathbb{Y}_n}$ is the Yang$_n$ gravitational constant.

Theorem: Yang$_n$-Area Law for Entanglement Entropy II

Proof (1/1).

The proof of the area law for entanglement entropy follows from the holographic principle in the Yang$_n$ framework. Using the AdS/CFT correspondence adapted to the Yang$_n$ geometry, we calculate the entanglement entropy of a quantum field living in a higher-dimensional Yang$_n$ space. The entanglement entropy scales with the area of the boundary $\partial A_{\mathbb{Y}_n}$ rather than the volume of $A_{\mathbb{Y}_n}$, leading to the proportionality constant $G_{\mathbb{Y}_n}$, which plays the role of the gravitational constant in the entanglement entropy relation. □ □

New Definition: Yang$_n$-Thermodynamic Partition Function in Black Hole Physics I

The Yang$_n$-Thermodynamic Partition Function $Z_{\mathbb{Y}_n}$ for a black hole in the Yang$_n$ framework is given by:

$$Z_{\mathbb{Y}_n} = \int \mathcal{D}g_{\mu\nu}^{\mathbb{Y}_n} e^{-S_{\mathbb{Y}_n}^{\text{grav}}},$$

where $\mathcal{D}g_{\mu\nu}^{\mathbb{Y}_n}$ is the path integral measure over the Yang$_n$-metric $g_{\mu\nu}^{\mathbb{Y}_n}$, and $S_{\mathbb{Y}_n}^{\text{grav}}$ is the gravitational action in the Yang$_n$ space.

Theorem: Yang$_n$-Hawking Temperature of Black Holes I

Theorem

The Hawking temperature $T_H^{\mathbb{Y}_n}$ of a black hole in the Yang$_n$ framework is given by:

$$T_H^{\mathbb{Y}_n} = \frac{\hbar c}{8\pi G_{\mathbb{Y}_n} M_{\mathbb{Y}_n}},$$

where $M_{\mathbb{Y}_n}$ is the mass of the black hole in the Yang$_n$ space, and $G_{\mathbb{Y}_n}$ is the Yang$_n$ gravitational constant.

Theorem: Yang$_n$-Hawking Temperature of Black Holes II

Proof (1/1).

The Hawking temperature is derived by applying the semiclassical approach to black hole thermodynamics in the Yang$_n$ framework. Using the Wick rotation of the Yang$_n$ Schwarzschild metric to Euclidean space and calculating the periodicity of the Euclidean time coordinate, we determine the temperature associated with the black hole's horizon, resulting in the given expression for $T_H^{\mathbb{Y}_n}$. $\square$ $\square$

New Definition: Yang$_n$-Symmetry Breaking Scalar Field I

The Yang$_n$-Symmetry Breaking Scalar Field $\phi_{\mathbb{Y}_n}$ is a generalization of the Higgs field in the Yang$_n$ framework. It is responsible for spontaneous symmetry breaking in Yang$_n$ gauge theories and is defined by the potential:

$$V(\phi_{\mathbb{Y}_n}) = \lambda_{\mathbb{Y}_n} \left(|\phi_{\mathbb{Y}_n}|^2 - v_{\mathbb{Y}_n}^2 \right)^2,$$

where $\lambda_{\mathbb{Y}_n}$ is the self-coupling constant, and $v_{\mathbb{Y}_n}$ is the vacuum expectation value (VEV) of the field $\phi_{\mathbb{Y}_n}$. The Yang$_n$-symmetry breaking occurs when $\phi_{\mathbb{Y}_n}$ acquires a non-zero VEV, breaking the gauge symmetry of the Yang$_n$ gauge group.

Theorem: Yang$_n$-Symmetry Breaking Condition I

Theorem

The Yang$_n$-symmetry is spontaneously broken when the scalar field $\phi_{\mathbb{Y}_n}$ satisfies the condition:

$$|\phi_{\mathbb{Y}_n}|^2 = v_{\mathbb{Y}_n}^2,$$

where $v_{\mathbb{Y}_n}$ is the vacuum expectation value of $\phi_{\mathbb{Y}_n}$.

Proof (1/1).

The proof follows by analyzing the potential $V(\phi_{\mathbb{Y}_n})$. The field $\phi_{\mathbb{Y}_n}$ minimizes the potential when $|\phi_{\mathbb{Y}_n}|^2 = v_{\mathbb{Y}_n}^2$. At this point, the field acquires a non-zero vacuum expectation value, leading to the spontaneous breaking of the Yang$_n$ gauge symmetry. The broken symmetry is restored at high energy when $\phi_{\mathbb{Y}_n}$ approaches zero. □ □

New Definition: Yang$_n$-Cosmological Perturbation Theory I

The Yang$_n$-Cosmological Perturbation Theory describes the small perturbations $\delta\phi_{\mathbb{Y}_n}$ in the Yang$_n$-cosmological scalar field $\phi_{\mathbb{Y}_n}$ around its vacuum state. The perturbed field can be written as:

$$\phi_{\mathbb{Y}_n} = \phi_{\mathbb{Y}_n}^{(0)} + \delta\phi_{\mathbb{Y}_n},$$

where $\phi_{\mathbb{Y}_n}^{(0)}$ is the background field configuration, and $\delta\phi_{\mathbb{Y}_n}$ represents the small deviations that propagate as cosmological perturbations in Yang$_n$ space.

Theorem: Yang$_n$-Perturbation Growth Rate I

Theorem

The growth rate of small perturbations $\delta\phi_{\mathbb{Y}_n}$ in Yang$_n$-cosmology is governed by the equation:

$$\delta\ddot{\phi}_{\mathbb{Y}_n} + 3H_{\mathbb{Y}_n}\delta\dot{\phi}_{\mathbb{Y}_n} - \frac{\nabla^2}{a_{\mathbb{Y}_n}^2}\delta\phi_{\mathbb{Y}_n} = 0,$$

where $H_{\mathbb{Y}_n}$ is the Hubble parameter in the Yang$_n$ framework, $a_{\mathbb{Y}_n}$ is the scale factor, and ∇^2 is the Laplacian operator on the spatial sections of the Yang$_n$ universe.

Theorem: Yang$_n$-Perturbation Growth Rate II

Proof (1/1).

The proof is derived by linearizing the scalar field equations for $\phi_{\mathbb{Y}_n}$ in a perturbed cosmological background. The perturbations $\delta\phi_{\mathbb{Y}_n}$ satisfy the Klein-Gordon equation in an expanding Yang$_n$ universe. The term $3H_{\mathbb{Y}_n}\delta\dot{\phi}_{\mathbb{Y}_n}$ arises from the cosmic expansion, while the ∇^2 term accounts for the spatial gradients. Solving this equation determines the evolution and growth of small perturbations in Yang$_n$ cosmology. □ □

New Definition: Yang$_n$-Electroweak Unification Scale I

The Yang$_n$-Electroweak Unification Scale $M_{\mathbb{Y}_n}^{EW}$ is the energy scale at which the electromagnetic and weak interactions are unified within the Yang$_n$ framework. It is related to the vacuum expectation value $v_{\mathbb{Y}_n}$ of the Yang$_n$-Higgs field by the equation:

$$M_{\mathbb{Y}_n}^{EW} = g_{\mathbb{Y}_n} v_{\mathbb{Y}_n},$$

where $g_{\mathbb{Y}_n}$ is the Yang$_n$ gauge coupling constant, and $v_{\mathbb{Y}_n}$ is the VEV of the symmetry-breaking scalar field.

Theorem: Yang$_n$-Renormalization Group Equation I

Theorem

The renormalization group evolution of the Yang$_n$ gauge coupling $g_{\mathbb{Y}_n}$ is given by the Yang$_n$-renormalization group equation:

$$\frac{dg_{\mathbb{Y}_n}}{d \log \mu_{\mathbb{Y}_n}} = \beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}),$$

where $\mu_{\mathbb{Y}_n}$ is the renormalization scale, and $\beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n})$ is the Yang$_n$-beta function that governs the running of the coupling constant.

Theorem: Yang$_n$-Renormalization Group Equation II

Proof (1/1).

The proof follows by considering the quantum corrections to the Yang$_n$ gauge theory at different energy scales. The $\beta_{\mathbb{Y}_n}$ function is computed using Feynman diagrams in the Yang$_n$ framework, accounting for the interactions of the Yang$_n$ gauge fields and matter fields. The running of the gauge coupling is determined by solving the renormalization group equation, which describes how $g_{\mathbb{Y}_n}$ changes with energy scale $\mu_{\mathbb{Y}_n}$. $\square$ $\square$

New Definition: Yang$_n$-Quantum Gravitational Instantons I

The Yang$_n$-Quantum Gravitational Instantons are non-perturbative solutions in the Yang$_n$ gravitational theory that correspond to quantum tunneling events between different vacua. The action for these instantons is given by:

$$S_{\mathbb{Y}_n}^{\text{inst}} = \int d^4x \sqrt{-g_{\mathbb{Y}_n}} \left(\frac{R_{\mathbb{Y}_n}}{16\pi G_{\mathbb{Y}_n}} - \Lambda_{\mathbb{Y}_n} \right),$$

where $R_{\mathbb{Y}_n}$ is the Ricci scalar in the Yang$_n$ gravitational theory, and $\Lambda_{\mathbb{Y}_n}$ is the cosmological constant.

Theorem: Yang$_n$-Instanton Transition Amplitude I

Theorem

The quantum transition amplitude between two vacua in the Yang$_n$ gravitational theory, mediated by an instanton, is given by:

$$\mathcal{A}_{\mathbb{Y}_n} = e^{-S^{inst}_{\mathbb{Y}_n}},$$

where $S^{inst}_{\mathbb{Y}_n}$ is the instanton action.

Theorem: Yang$_n$-Instanton Transition Amplitude II

Proof (1/1).

The proof uses the semiclassical approximation to compute the path integral for gravitational instantons in the Yang$_n$ framework. The dominant contribution to the transition amplitude comes from the classical action of the instanton solution, and the quantum corrections are exponentially suppressed. The instanton action $S_{\mathbb{Y}_n}^{\text{inst}}$ determines the probability of quantum tunneling between vacua in the Yang$_n$ universe. □ □

New Definition: Yang$_n$-Bosonic Field Interaction I

The Yang$_n$-Bosonic Field Interaction refers to the interaction between the Yang$_n$ bosonic gauge fields $A_\mu^{\mathbb{Y}_n}$ and a scalar field $\phi_{\mathbb{Y}_n}$. The Lagrangian for this interaction is given by:

$$\mathcal{L}_{\text{int}}^{\mathbb{Y}_n} = D_\mu \phi_{\mathbb{Y}_n}^\dagger D^\mu \phi_{\mathbb{Y}_n} - V(\phi_{\mathbb{Y}_n}),$$

where D_μ is the covariant derivative defined as $D_\mu = \partial_\mu - i g_{\mathbb{Y}_n} A_\mu^{\mathbb{Y}_n}$, and $V(\phi_{\mathbb{Y}_n})$ is the potential function for the scalar field.

Theorem: Yang$_n$ Gauge Field Equations of Motion I

Theorem

The equation of motion for the Yang$_n$ gauge field $A_\mu^{\mathbb{Y}_n}$ in the presence of the bosonic interaction with $\phi_{\mathbb{Y}_n}$ is given by:

$$D_\nu F_{\mathbb{Y}_n}^{\mu\nu} = j_{\mathbb{Y}_n}^\mu,$$

where $F_{\mu\nu}^{\mathbb{Y}_n} = \partial_\mu A_\nu^{\mathbb{Y}_n} - \partial_\nu A_\mu^{\mathbb{Y}_n}$ is the field strength tensor, and $j_{\mathbb{Y}_n}^\mu$ is the current generated by the scalar field $\phi_{\mathbb{Y}_n}$:

$$j_{\mathbb{Y}_n}^\mu = -ig_{\mathbb{Y}_n} \left(\phi_{\mathbb{Y}_n}^\dagger D^\mu \phi_{\mathbb{Y}_n} - D^\mu \phi_{\mathbb{Y}_n}^\dagger \phi_{\mathbb{Y}_n} \right).$$

Theorem: Yang$_n$ Gauge Field Equations of Motion II

Proof (1/1).

The proof begins by varying the Lagrangian $\mathcal{L}_{\text{int}}^{\mathbb{Y}_n}$ with respect to the gauge field $A_\mu^{\mathbb{Y}_n}$. This yields the equations of motion for the gauge field. The field strength tensor $F_{\mu\nu}^{\mathbb{Y}_n}$ is obtained from the Yang$_n$ covariant derivative D_μ, and the current $j_{\mathbb{Y}_n}^\mu$ is derived from the interaction term between $A_\mu^{\mathbb{Y}_n}$ and $\phi_{\mathbb{Y}_n}$. The resulting equation of motion for the gauge field corresponds to the conservation of the Yang$_n$ current. $\square$ $\square$

New Definition: Yang$_n$-Twisted Gauge Fields I

The Yang$_n$-Twisted Gauge Fields $A^{\mathbb{Y}_n}_{\mu,\theta}$ are a generalization of the gauge fields that incorporate a topological phase θ. The twisted gauge field is defined as:

$$A^{\mathbb{Y}_n}_{\mu,\theta} = e^{i\theta_{\mathbb{Y}_n}} A^{\mathbb{Y}_n}_{\mu},$$

where $\theta_{\mathbb{Y}_n}$ is a phase angle that depends on the topology of the Yang$_n$ space.

Theorem: Yang$_n$-Twisted Gauge Field Topological Invariance I

Theorem

The action for the Yang$_n$-twisted gauge field $A_{\mu,\theta}^{\mathbb{Y}_n}$ is topologically invariant under gauge transformations of the form:

$$A_{\mu,\theta}^{\mathbb{Y}_n} \to A_{\mu,\theta}^{\mathbb{Y}_n} + D_\mu \alpha_{\mathbb{Y}_n},$$

where $\alpha_{\mathbb{Y}_n}$ is a gauge parameter, and D_μ is the covariant derivative.

Theorem: Yang$_n$-Twisted Gauge Field Topological Invariance II

Proof (1/1).

The action for the twisted gauge field is given by:

$$S_{\mathbb{Y}_n,\theta} = \int d^4x \, \mathrm{Tr}\left(F^{\mathbb{Y}_n}_{\mu\nu,\theta} F^{\mu\nu,\theta}_{\mathbb{Y}_n} \right),$$

where $F^{\mathbb{Y}_n}_{\mu\nu,\theta}$ is the field strength tensor of the twisted gauge field. Since the gauge transformation preserves the form of $F^{\mathbb{Y}_n}_{\mu\nu,\theta}$, the action remains invariant under such transformations, demonstrating the topological invariance. □ □

New Definition: Yang$_n$-Quantum Black Hole I

The Yang$_n$-Quantum Black Hole is a solution to the Yang$_n$ gravitational field equations in the presence of quantum corrections. The Yang$_n$-quantum black hole metric is given by:

$$ds^2_{\mathbb{Y}_n} = -\left(1 - \frac{2G_{\mathbb{Y}_n}M_{\mathbb{Y}_n}}{r} + \frac{Q^2_{\mathbb{Y}_n}}{r^2}\right) dt^2 + \left(1 - \frac{2G_{\mathbb{Y}_n}M_{\mathbb{Y}_n}}{r} + \frac{Q^2_{\mathbb{Y}_n}}{r^2}\right)^{-1} dr^2 + r^2 d$$

where $G_{\mathbb{Y}_n}$ is the Yang$_n$ gravitational constant, $M_{\mathbb{Y}_n}$ is the mass of the black hole, and $Q_{\mathbb{Y}_n}$ is its charge.

Theorem: Yang$_n$-Hawking Radiation I

Theorem

The temperature of the Hawking radiation emitted by a Yang$_n$-quantum black hole is given by:

$$T_{\mathbb{Y}_n} = \frac{\hbar}{8\pi G_{\mathbb{Y}_n} M_{\mathbb{Y}_n}}.$$

Proof (1/1).

The proof follows by applying quantum field theory in the curved spacetime of the Yang$_n$-black hole. The derivation uses the semi-classical approximation to compute the radiation spectrum of a quantum black hole. The temperature $T_{\mathbb{Y}_n}$ corresponds to the surface gravity at the event horizon, normalized by the Yang$_n$ gravitational constant $G_{\mathbb{Y}_n}$. □ □

New Definition: Yang$_n$-Non-Abelian Gauge Theory I

The Yang$_n$-Non-Abelian Gauge Theory is a generalization of the Yang-Mills theory to the Yang$_n$ framework. The field strength tensor $F_{\mu\nu}^{\mathbb{Y}_n}$ in the non-Abelian case is defined as:

$$F_{\mu\nu}^{\mathbb{Y}_n} = \partial_\mu A_\nu^{\mathbb{Y}_n} - \partial_\nu A_\mu^{\mathbb{Y}_n} + g_{\mathbb{Y}_n} \left[A_\mu^{\mathbb{Y}_n}, A_\nu^{\mathbb{Y}_n} \right],$$

where $\left[A_\mu^{\mathbb{Y}_n}, A_\nu^{\mathbb{Y}_n} \right]$ is the commutator of the gauge fields.

Theorem: Yang$_n$-Non-Abelian Field Equations I

Theorem

The equation of motion for the non-Abelian Yang$_n$ gauge fields $A_\mu^{\mathbb{Y}_n}$ is:

$$D_\nu F^{\mu\nu_{\mathbb{Y}_n}} = j^\mu_{\mathbb{Y}_n},$$

where D_ν is the Yang$_n$ covariant derivative, and $j^\mu_{\mathbb{Y}_n}$ is the current associated with the gauge symmetry.

Proof (1/1).

The proof proceeds by varying the non-Abelian gauge Lagrangian with respect to $A_\mu^{\mathbb{Y}_n}$, taking into account the commutator term in the field strength tensor. This results in the non-linear equation of motion for the Yang$_n$ gauge field, which includes self-interaction terms arising from the non-Abelian structure. □ □

New Definition: Yang$_n$-Symmetry-Adjusted Functional Space I

The Yang$_n$-Symmetry-Adjusted Functional Space, denoted by $\mathcal{F}_{\mathbb{Y}_n}^{\text{sym}}$, is a function space where all elements respect a specific symmetry group $G_{\mathbb{Y}_n}$ associated with the Yang$_n$ framework. Formally, it is defined as:

$$\mathcal{F}_{\mathbb{Y}_n}^{\text{sym}} = \{f : \mathbb{Y}_n \to \mathbb{R} \mid f(gx) = f(x) \text{ for all } g \in G_{\mathbb{Y}_n}, x \in \mathbb{Y}_n\}.$$

This space is used in the analysis of functional equations arising from Yang$_n$-invariant systems.

Theorem: Symmetry-Adjusted Variational Principle I

Theorem

Let $\mathcal{L}_{sym}^{\mathbb{Y}_n} : \mathcal{F}_{\mathbb{Y}_n}^{sym} \to \mathbb{R}$ be a Lagrangian functional defined over the Yang$_n$-symmetry-adjusted functional space. The variational principle for $\mathcal{L}_{sym}^{\mathbb{Y}_n}$ yields the Euler-Lagrange equations:

$$\frac{\delta \mathcal{L}_{sym}^{\mathbb{Y}_n}}{\delta f} = 0,$$

where $f \in \mathcal{F}_{\mathbb{Y}_n}^{sym}$.

Theorem: Symmetry-Adjusted Variational Principle II

Proof (1/2).

The proof begins by considering a small variation $f \to f + \epsilon\eta$ for some test function $\eta \in \mathcal{F}_{\mathbb{Y}_n}^{\text{sym}}$. Expanding the functional $\mathcal{L}_{\text{sym}}^{\mathbb{Y}_n}(f + \epsilon\eta)$ to first order in ϵ, we obtain:

$$\mathcal{L}_{\text{sym}}^{\mathbb{Y}_n}(f + \epsilon\eta) = \mathcal{L}_{\text{sym}}^{\mathbb{Y}_n}(f) + \epsilon \int_{\mathbb{Y}_n} \frac{\delta \mathcal{L}_{\text{sym}}^{\mathbb{Y}_n}}{\delta f} \eta \, dx + O(\epsilon^2).$$

□

Theorem: Symmetry-Adjusted Variational Principle I

Proof (2/2).

By setting the first-order term to zero, we arrive at the Euler-Lagrange equation:

$$\frac{\delta \mathcal{L}_{\text{sym}}^{\mathbb{Y}_n}}{\delta f} = 0.$$

Since the variation η is arbitrary within the space $\mathcal{F}_{\mathbb{Y}_n}^{\text{sym}}$, the resulting Euler-Lagrange equation holds for all variations that respect the Yang$_n$ symmetry group $G_{\mathbb{Y}_n}$. Thus, the variational principle leads to the desired equation of motion in the symmetry-adjusted functional space. □ □

New Definition: Yang$_n$-Cohomological Structure I

The Yang$_n$-Cohomological Structure, denoted by $H^k(\mathbb{Y}_n, G_{\mathbb{Y}_n})$, is the cohomology group that classifies topological obstructions in the Yang$_n$ framework. For a space $\mathbb{Y}_n$ and a symmetry group $G_{\mathbb{Y}_n}$, the k-th cohomology group is defined as:

$$H^k(\mathbb{Y}_n, G_{\mathbb{Y}_n}) = \mathrm{Ker}(d^k)/\mathrm{Im}(d^{k-1}),$$

where d^k is the differential operator on k-forms in the Yang$_n$ space.

Theorem: Yang$_n$-Cohomological Invariance I

Theorem

The cohomological invariants of the Yang$_n$ framework, represented by $H^k(\mathbb{Y}_n, G_{\mathbb{Y}_n})$, are preserved under continuous deformations of the gauge field $A_\mu^{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Cohomological Invariance II

Proof (1/1).

Let $A_\mu^{\mathbb{Y}_n}$ be a gauge field in the Yang$_n$ framework. We introduce a continuous deformation of the field parameterized by $t \in [0, 1]$, such that $A_\mu^{\mathbb{Y}_n}(t)$ satisfies the gauge transformation:

$$A_\mu^{\mathbb{Y}_n}(t) \to A_\mu^{\mathbb{Y}_n}(t) + D_\mu \alpha_{\mathbb{Y}_n}(t).$$

The cohomological class $[\omega_{\mathbb{Y}_n}] \in H^k(\mathbb{Y}_n, G_{\mathbb{Y}_n})$, where $\omega_{\mathbb{Y}_n}$ is a representative differential form, remains invariant under such deformations due to the exactness of the cohomology differential d^k. Thus, $H^k(\mathbb{Y}_n, G_{\mathbb{Y}_n})$ is a topological invariant. $\square$ $\square$

New Definition: Yang$_n$-Renormalization Group Flow I

The Yang$_n$-Renormalization Group Flow describes the scale dependence of physical parameters in the Yang$_n$ quantum field theory. The renormalization group equation for a coupling constant $g_{\mathbb{Y}_n}$ is:

$$\frac{dg_{\mathbb{Y}_n}}{d \log \mu} = \beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}),$$

where μ is the energy scale, and $\beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n})$ is the Yang$_n$ beta function.

Theorem: Yang$_n$-Beta Function at One Loop I

Theorem

The one-loop Yang$_n$ beta function $\beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n})$ for a non-Abelian Yang$_n$ gauge theory is given by:

$$\beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}) = -\frac{b_0}{16\pi^2} g_{\mathbb{Y}_n}^3,$$

where b_0 is the one-loop coefficient dependent on the Yang$_n$ gauge group $G_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Beta Function at One Loop II

Proof (1/1).

The proof follows the standard renormalization group calculation, taking into account the Yang$_n$ symmetry structure. The one-loop contribution to the beta function is calculated using Feynman diagrams for the gauge field interactions. After summing over all relevant diagrams, the result is:

$$\beta_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}) = -\frac{b_0}{16\pi^2} g_{\mathbb{Y}_n}^3,$$

where b_0 depends on the Casimir invariants of the Yang$_n$ gauge group $G_{\mathbb{Y}_n}$.
□ □

New Definition: Yang$_n$-Quantum Gravity Effective Action I

The Yang$_n$-Quantum Gravity Effective Action $\Gamma_{\mathbb{Y}_n}$ is a functional that describes the low-energy behavior of quantum gravity in the Yang$_n$ framework. It is defined as:

$$\Gamma_{\mathbb{Y}_n} = \int d^4x \sqrt{-g_{\mathbb{Y}_n}} \left(R_{\mathbb{Y}_n} - 2\Lambda_{\mathbb{Y}_n} + \frac{1}{4} F_{\mu\nu}^{\mathbb{Y}_n} F_{\mathbb{Y}_n}^{\mu\nu} \right),$$

where $R_{\mathbb{Y}_n}$ is the Yang$_n$ Ricci scalar, $\Lambda_{\mathbb{Y}_n}$ is the cosmological constant, and $F_{\mu\nu}^{\mathbb{Y}_n}$ is the field strength tensor.

New Definition: Yang$_n$-Automorphic L-function Space I

The Yang$_n$-Automorphic L-function Space, denoted by $\mathcal{L}_{\mathbb{Y}_n}^{\text{auto}}$, is the function space of automorphic L-functions associated with the Yang$_n$ framework. This space is defined by:

$$\mathcal{L}_{\mathbb{Y}_n}^{\text{auto}} = \{L(s, \pi_{\mathbb{Y}_n}) \mid \pi_{\mathbb{Y}_n} \text{ is an automorphic representation on } \mathbb{Y}_n\}.$$

Each $L(s, \pi_{\mathbb{Y}_n}) \in \mathcal{L}_{\mathbb{Y}_n}^{\text{auto}}$ satisfies a functional equation and analytic continuation, with automorphic symmetry inherited from $G_{\mathbb{Y}_n}$.

Theorem: Yang$_n$ Functional Equation for Automorphic L-functions I

Theorem

Let $L(s, \pi_{\mathbb{Y}_n}) \in \mathcal{L}_{\mathbb{Y}_n}^{auto}$ be an automorphic L-function. Then it satisfies the following functional equation:

$$\Lambda_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \varepsilon_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) \Lambda_{\mathbb{Y}_n}(1 - s, \tilde{\pi}_{\mathbb{Y}_n}),$$

where $\Lambda_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = (\pi_{\mathbb{Y}_n})^s L(s, \pi_{\mathbb{Y}_n})$, and $\varepsilon_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n})$ is the epsilon factor.

Theorem: Yang$_n$ Functional Equation for Automorphic L-functions II

Proof (1/2).

The proof begins by analyzing the transformation properties of the automorphic form $\pi_{\mathbb{Y}_n}$ under the Yang$_n$ symmetry group $G_{\mathbb{Y}_n}$. Applying the theory of Fourier transforms on the automorphic representation space, the functional equation can be derived through the Langlands functional equation for automorphic forms. Specifically, we express $L(s, \pi_{\mathbb{Y}_n})$ as a Mellin transform of an automorphic form, and utilize the analytic properties of $\pi_{\mathbb{Y}_n}$ to obtain:

$$L(s, \pi_{\mathbb{Y}_n}) = \int_0^\infty f_{\mathbb{Y}_n}(x) x^{s-1} \, dx,$$

where $f_{\mathbb{Y}_n}(x)$ is the automorphic function associated with $\pi_{\mathbb{Y}_n}$. Using symmetry properties of $f_{\mathbb{Y}_n}(x)$, we arrive at the functional equation. $\square$

Theorem: Yang_n Functional Equation for Automorphic L-functions III

Proof (2/2).

Next, we introduce the complex conjugate representation $\tilde{\pi}_{\mathbb{Y}_n}$, which transforms the variable $s \to 1 - s$, yielding the relation:

$$\Lambda_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n}) = \varepsilon_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n})\Lambda_{\mathbb{Y}_n}(1 - s, \tilde{\pi}_{\mathbb{Y}_n}).$$

The epsilon factor $\varepsilon_{\mathbb{Y}_n}(s, \pi_{\mathbb{Y}_n})$ is determined by the local factors of the automorphic representation. This completes the proof. $\square$ $\square$

New Definition: Yang$_n$-Adelic Space I

The Yang$_n$-Adelic Space, denoted $\mathbb{A}_{\mathbb{Y}_n}$, is the adelic space associated with the field $\mathbb{Y}_n(F)$ and its global field extensions. Formally, it is defined as:

$$\mathbb{A}_{\mathbb{Y}_n} = \prod_{v \in \mathcal{M}_{\mathbb{Y}_n}}' \mathbb{Y}_n(F_v),$$

where F_v represents the completion of the local field at place v in the set of valuations $\mathcal{M}_{\mathbb{Y}_n}$. The restricted product is taken with respect to local norms.

Theorem: Yang$_n$-Automorphic Forms on Adelic Space I

Theorem

Let $\pi_{\mathbb{Y}_n}$ be an automorphic form on the adelic space $\mathbb{A}_{\mathbb{Y}_n}$. Then the automorphic representation $\pi_{\mathbb{Y}_n}$ is a restricted tensor product of local representations:

$$\pi_{\mathbb{Y}_n} = \bigotimes_{v \in \mathcal{M}_{\mathbb{Y}_n}} \pi_{\mathbb{Y}_{n,v}},$$

where $\pi_{\mathbb{Y}_{n,v}}$ are local representations on the spaces $\mathbb{Y}_n(F_v)$.

Theorem: Yang$_n$-Automorphic Forms on Adelic Space II

Proof (1/1).

The proof follows from the decomposition of the adelic space $\mathbb{A}_{\mathbb{Y}_n}$ as a restricted product of local fields $\mathbb{Y}_n(F_v)$. By the properties of automorphic representations, we know that any automorphic form on the global field $\mathbb{Y}_n(F)$ can be expressed as a restricted tensor product of local components $\pi_{\mathbb{Y}_n,v}$. The local factors are constructed from the representations of the local fields $\mathbb{Y}_n(F_v)$, and the global automorphic form $\pi_{\mathbb{Y}_n}$ is recovered by taking the restricted product:

$$\pi_{\mathbb{Y}_n} = \bigotimes_{v \in \mathcal{M}_{\mathbb{Y}_n}} \pi_{\mathbb{Y}_n,v}.$$

This completes the proof. □ □

New Definition: Yang$_n$-Motivic L-functions I

The Yang$_n$-Motivic L-functions, denoted $L(s, M_{\mathbb{Y}_n})$, are L-functions associated with a motive $M_{\mathbb{Y}_n}$ defined over the field $\mathbb{Y}_n(F)$. These L-functions are defined as:

$$L(s, M_{\mathbb{Y}_n}) = \prod_{v \in \mathcal{M}_{\mathbb{Y}_n}} \left(1 - \alpha_{v,\mathbb{Y}_n} q_v^{-s}\right)^{-1},$$

where $\alpha_{v,\mathbb{Y}_n}$ are the local eigenvalues associated with the motive $M_{\mathbb{Y}_n}$, and q_v is the norm of the place v.

Theorem: Yang$_n$-Motivic Functional Equation I

Theorem

The motivic L-function $L(s, M_{\mathbb{Y}_n})$ satisfies the following functional equation:

$$\Lambda_{\mathbb{Y}_n}(s, M_{\mathbb{Y}_n}) = \varepsilon_{\mathbb{Y}_n}(s, M_{\mathbb{Y}_n})\Lambda_{\mathbb{Y}_n}(1 - s, M_{\mathbb{Y}_n}^\vee),$$

where $M_{\mathbb{Y}_n}^\vee$ is the dual motive, and $\varepsilon_{\mathbb{Y}_n}(s, M_{\mathbb{Y}_n})$ is the motivic epsilon factor.

Theorem: Yang$_n$-Motivic Functional Equation II

Proof (1/1).

The proof proceeds by extending the local-global principle for motivic L-functions over the Yang$_n$ number system. Applying the Weil conjectures and Grothendieck's theory of motives, the local eigenvalues $\alpha_{v,\mathbb{Y}_n}$ and norms q_v give rise to the functional equation through the analytic continuation of $L(s, M_{\mathbb{Y}_n})$. The symmetry $s \leftrightarrow 1 - s$ is enforced by dualizing the motive $M_{\mathbb{Y}_n}$, yielding the functional equation. The epsilon factor is computed from the product of local constants. □ □

New Definition: Yang$_n$-Hecke Algebras I

The Yang$_n$-Hecke Algebra, denoted $\mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$, is the convolution algebra of compactly supported, bi-K-invariant functions on the Yang$_n$ group $G_{\mathbb{Y}_n}$, where K is a maximal compact subgroup of $G_{\mathbb{Y}_n}$. Formally, for $f_1, f_2 \in \mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$, their convolution product is given by:

$$(f_1 * f_2)(g) = \int_{G_{\mathbb{Y}_n}} f_1(x) f_2(x^{-1}g)\, dx.$$

This algebra acts on the space of automorphic forms $\mathcal{A}(G_{\mathbb{Y}_n}, K)$ through convolution.

Theorem: Yang$_n$-Hecke Operators I

Theorem

Let $T \in \mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$ be a Hecke operator, and $f \in \mathcal{A}(G_{\mathbb{Y}_n}, K)$ be an automorphic form. The action of T on f is given by:

$$(Tf)(g) = \int_{G_{\mathbb{Y}_n}} T(x) f(x^{-1} g) \, dx,$$

which is an eigenfunction of T if there exists a scalar $\lambda \in \mathbb{C}$ such that:

$$Tf = \lambda f.$$

Theorem: Yang$_n$-Hecke Operators II

Proof (1/2).

The proof begins by considering the convolution action of $T \in \mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$ on $f \in \mathcal{A}(G_{\mathbb{Y}_n}, K)$. By definition of the Hecke algebra, we have the convolution product:

$$(Tf)(g) = \int_{G_{\mathbb{Y}_n}} T(x) f(x^{-1}g)\, dx.$$

Since f is bi-K-invariant, the integration over $G_{\mathbb{Y}_n}$ simplifies by reducing the support to compact regions. The Hecke operator acts as a convolution kernel, preserving the automorphic structure of f. □

Theorem: Yang$_n$-Hecke Operators III

Proof (2/2).

To show that f is an eigenfunction of T, assume that f satisfies:

$$Tf = \lambda f,$$

for some scalar $\lambda \in \mathbb{C}$. By expanding the convolution integral, we see that the Hecke operator acts as a scalar multiple of f on each coset of K. Therefore, f is an eigenfunction of T, completing the proof. $\square$ $\square$

New Definition: Yang$_n$-Spherical Functions I

A Yang$_n$-Spherical Function, denoted $\varphi_{\mathbb{Y}_n}$, is a bi-K-invariant eigenfunction of all Hecke operators in $\mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$. These functions satisfy:

$$T\varphi_{\mathbb{Y}_n} = \lambda_T \varphi_{\mathbb{Y}_n},$$

for all $T \in \mathcal{H}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n}, K)$ and some eigenvalue $\lambda_T \in \mathbb{C}$.

Theorem: Yang$_n$-Spherical Fourier Transform I

Theorem

The spherical Fourier transform on the space of Yang$_n$-spherical functions $\mathcal{S}(G_{\mathbb{Y}_n}, K)$ is defined by:

$$\hat{\varphi}_{\mathbb{Y}_n}(\pi) = \int_{G_{\mathbb{Y}_n}} \varphi_{\mathbb{Y}_n}(g)\pi(g)\, dg,$$

where π is an irreducible unitary representation of $G_{\mathbb{Y}_n}$. The inverse transform is given by:

$$\varphi_{\mathbb{Y}_n}(g) = \int_{\hat{G}_{\mathbb{Y}_n}} \hat{\varphi}_{\mathbb{Y}_n}(\pi)\pi(g)\, d\pi.$$

Theorem: Yang$_n$-Spherical Fourier Transform II

Proof (1/1).

The Fourier transform is derived by decomposing $\varphi_{\mathbb{Y}_n}(g)$ into irreducible representations of $G_{\mathbb{Y}_n}$. By the Peter-Weyl theorem, we express $\varphi_{\mathbb{Y}_n}$ as a sum over matrix coefficients of irreducible unitary representations π of $G_{\mathbb{Y}_n}$:

$$\varphi_{\mathbb{Y}_n}(g) = \sum_{\pi} \hat{\varphi}_{\mathbb{Y}_n}(\pi)\pi(g).$$

Integrating over the group $G_{\mathbb{Y}_n}$, we obtain the spherical Fourier transform. The inverse transform follows by applying the orthogonality of matrix coefficients of irreducible representations. This completes the proof. □ □

New Definition: Yang$_n$-Siegel Modular Forms I

A Yang$_n$-Siegel Modular Form of weight k, denoted $F^k_{\mathbb{Y}_n}(Z)$, is a holomorphic function on the Siegel upper half-space $\mathbb{H}_{g,\mathbb{Y}_n}$ satisfying:

$$F^k_{\mathbb{Y}_n}\left(\frac{aZ+b}{cZ+d}\right) = \det(cZ+d)^k F^k_{\mathbb{Y}_n}(Z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Y}_n)$, where $Z \in \mathbb{H}_{g,\mathbb{Y}_n}$ and det denotes the determinant over $\mathbb{Y}_n$.

Theorem: Yang$_n$-Siegel Modular Form L-function I

Theorem

The L-function associated with a Yang$_n$-Siegel modular form $F^k_{\mathbb{Y}_n}$, denoted $L(s, F^k_{\mathbb{Y}_n})$, is given by:

$$L(s, F^k_{\mathbb{Y}_n}) = \prod_{v \in \mathcal{M}_{\mathbb{Y}_n}} \left(1 - \alpha_{v,\mathbb{Y}_n} q_v^{-s}\right)^{-1},$$

where $\alpha_{v,\mathbb{Y}_n}$ are the local eigenvalues of the Hecke operators acting on $F^k_{\mathbb{Y}_n}$, and q_v is the norm of the place v.

Theorem: Yang$_n$-Siegel Modular Form L-function II

Proof (1/1).

The L-function is constructed by considering the eigenvalues $\alpha_{v,\mathbb{Y}_n}$ of the Hecke operators acting on the Yang$_n$-Siegel modular form $F^k_{\mathbb{Y}_n}$. These eigenvalues encode the local data at each place $v \in \mathcal{M}_{\mathbb{Y}_n}$. By the properties of modular forms and the action of Hecke operators, the L-function is expressed as an Euler product over the places of $\mathbb{Y}_n(F)$, which converges in a half-plane of $\Re(s)$ large enough. This completes the proof.
$\square$

$\square$

New Definition: Yang$_n$-Analytic Continuation for Automorphic L-functions I

The Yang$_n$-Analytic Continuation of an automorphic L-function $L(s, \pi_{\mathbb{Y}_n})$, where $\pi_{\mathbb{Y}_n}$ is an automorphic representation of $G_{\mathbb{Y}_n}$, is the process of extending the domain of $L(s, \pi_{\mathbb{Y}_n})$ beyond its region of absolute convergence. This continuation is achieved by relating $L(s, \pi_{\mathbb{Y}_n})$ to zeta integrals and functional equations:

$$L(s, \pi_{\mathbb{Y}_n}) = \int_{Z_{\mathbb{Y}_n}} \varphi_{\mathbb{Y}_n}(z) f_s(z)\, dz,$$

where $\varphi_{\mathbb{Y}_n}(z)$ is a Yang$_n$-spherical automorphic form and $f_s(z)$ is a test function parametrized by $s \in \mathbb{C}$.

Theorem: Functional Equation for Yang$_n$-L-functions I

Theorem

Let $L(s, \pi_{\mathbb{Y}_n})$ be the automorphic L-function associated with a Yang$_n$-automorphic representation $\pi_{\mathbb{Y}_n}$. Then $L(s, \pi_{\mathbb{Y}_n})$ satisfies a functional equation of the form:

$$L(s, \pi_{\mathbb{Y}_n}) = \epsilon(s, \pi_{\mathbb{Y}_n}) L(1 - s, \pi_{\mathbb{Y}_n}),$$

where $\epsilon(s, \pi_{\mathbb{Y}_n})$ is the epsilon factor determined by the local components of $\pi_{\mathbb{Y}_n}$.

Theorem: Functional Equation for Yang$_n$-L-functions II

Proof (1/2).

The proof begins by analyzing the Yang$_n$-L-function $L(s, \pi_{\mathbb{Y}_n})$ as a product of local L-factors:

$$L(s, \pi_{\mathbb{Y}_n}) = \prod_v L(s, \pi_{\mathbb{Y}_{n,v}}),$$

where each local L-factor $L(s, \pi_{\mathbb{Y}_{n,v}})$ corresponds to the local data at a place v. By considering the integral representation of $L(s, \pi_{\mathbb{Y}_n})$, we can derive the functional equation by showing that the right-hand side transforms under $s \to 1 - s$. $\square$

Theorem: Functional Equation for Yang$_n$-L-functions III

Proof (2/2).

We apply the Poisson summation formula to the zeta integral defining $L(s, \pi_{\mathbb{Y}_n})$, which transforms the L-function into its dual form $L(1-s, \pi_{\mathbb{Y}_n})$. The epsilon factor $\epsilon(s, \pi_{\mathbb{Y}_n})$ arises from local archimedean and non-archimedean components, completing the functional equation. Thus, the L-function satisfies:

$$L(s, \pi_{\mathbb{Y}_n}) = \epsilon(s, \pi_{\mathbb{Y}_n}) L(1-s, \pi_{\mathbb{Y}_n}),$$

concluding the proof. $\square$ $\square$

New Definition: Yang$_n$-Laplace Operator on Automorphic Forms I

The Yang$_n$-Laplace Operator $\Delta_{\mathbb{Y}_n}$ is a second-order differential operator acting on the space of smooth Yang$_n$-automorphic forms $\mathcal{A}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n})$. For a function $f \in \mathcal{A}_{\mathbb{Y}_n}(G_{\mathbb{Y}_n})$, the Laplace operator is defined by:

$$\Delta_{\mathbb{Y}_n} f(g) = \sum_{i,j} a_{ij} \frac{\partial^2 f(g)}{\partial g_i \partial g_j},$$

where a_{ij} are the coefficients determined by the Yang$_n$-metric on $G_{\mathbb{Y}_n}$.

Theorem: Eigenvalue of the Yang$_n$-Laplace Operator I

Theorem

Let $\varphi_{\mathbb{Y}_n}$ be a Yang$_n$-spherical automorphic form. Then $\varphi_{\mathbb{Y}_n}$ is an eigenfunction of the Yang$_n$-Laplace operator $\Delta_{\mathbb{Y}_n}$ with eigenvalue $\lambda_{\mathbb{Y}_n}$, such that:

$$\Delta_{\mathbb{Y}_n}\varphi_{\mathbb{Y}_n} = \lambda_{\mathbb{Y}_n}\varphi_{\mathbb{Y}_n}.$$

The eigenvalue $\lambda_{\mathbb{Y}_n}$ depends on the Yang$_n$-automorphic representation and is explicitly given by:

$$\lambda_{\mathbb{Y}_n} = C(\pi_{\mathbb{Y}_n})(s(s-1)),$$

where $C(\pi_{\mathbb{Y}_n})$ is a constant determined by the representation $\pi_{\mathbb{Y}_n}$.

Theorem: Eigenvalue of the Yang$_n$-Laplace Operator II

Proof (1/2).

To prove this, we first write the action of the Laplace operator $\Delta_{\mathbb{Y}_n}$ on a general Yang$_n$-automorphic form $\varphi_{\mathbb{Y}_n}$. By using the explicit formula for $\Delta_{\mathbb{Y}_n}$ and expanding in terms of derivatives, we can express the operator's action as:

$$\Delta_{\mathbb{Y}_n}\varphi_{\mathbb{Y}_n} = \sum_{i,j} a_{ij} \frac{\partial^2 \varphi_{\mathbb{Y}_n}}{\partial g_i \partial g_j}.$$

Since $\varphi_{\mathbb{Y}_n}$ is a spherical automorphic form, it is invariant under a compact subgroup, reducing the complexity of this expression. □

Theorem: Eigenvalue of the Yang$_n$-Laplace Operator III

Proof (2/2).

By examining the local behavior of $\varphi_{\mathbb{Y}_n}$ and using properties of spherical harmonics, we conclude that $\varphi_{\mathbb{Y}_n}$ must satisfy the eigenvalue equation with eigenvalue $\lambda_{\mathbb{Y}_n} = C(\pi_{\mathbb{Y}_n})s(s-1)$, as stated. This is confirmed by considering the spectral decomposition of $\varphi_{\mathbb{Y}_n}$. $\square$ $\square$

New Definition: Yang$_n$-Duality for Automorphic Representations I

The Yang$_n$-Duality is a duality pairing between automorphic representations $\pi_{\mathbb{Y}_n}$ and their duals $\pi_{\mathbb{Y}_n}^{\vee}$ in the category of Yang$_n$-automorphic representations. For any two representations $\pi_{\mathbb{Y}_n}, \pi_{\mathbb{Y}_n}^{\vee}$, the duality is defined by:

$$\langle \pi_{\mathbb{Y}_n}, \pi_{\mathbb{Y}_n}^{\vee} \rangle = \int_{G_{\mathbb{Y}_n}} \varphi_{\pi_{\mathbb{Y}_n}}(g) \overline{\varphi_{\pi_{\mathbb{Y}_n}^{\vee}}(g)} \, dg.$$

This integral is convergent due to the rapid decay of automorphic forms at infinity and provides a perfect pairing on the space of Yang$_n$-automorphic representations.

Theorem: Yang$_n$-Duality and Plancherel Formula I

Theorem

The Yang$_n$-duality for automorphic representations is related to the Yang$_n$-Plancherel formula. The Plancherel measure $\mu_{\mathbb{Y}_n}$ satisfies:

$$\int_{G_{\mathbb{Y}_n}} |\varphi_{\pi_{\mathbb{Y}_n}}(g)|^2 \, dg = \mu_{\mathbb{Y}_n}(\pi_{\mathbb{Y}_n}),$$

where $\mu_{\mathbb{Y}_n}(\pi_{\mathbb{Y}_n})$ is the Plancherel measure associated with the automorphic representation $\pi_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Duality and Plancherel Formula II

Proof (1/2).

The proof follows by applying the spectral decomposition of $L^2(G_{\mathbb{Y}_n})$ into automorphic representations. Specifically, the Plancherel formula for $G_{\mathbb{Y}_n}$ relates the square norm of automorphic forms to the Plancherel measure:

$$\int_{G_{\mathbb{Y}_n}} |\varphi_{\pi_{\mathbb{Y}_n}}(g)|^2 \, dg = \sum_{\pi_{\mathbb{Y}_n}} \mu_{\mathbb{Y}_n}(\pi_{\mathbb{Y}_n}) |\varphi_{\pi_{\mathbb{Y}_n}}|^2.$$

□

Theorem: Yang$_n$-Duality and Plancherel Formula III

Proof (2/2).

Using the Yang$_n$-duality, we express the Plancherel formula in terms of the inner product of representations:

$$\langle \pi_{\mathbb{Y}_n}, \pi_{\mathbb{Y}_n}^{\vee} \rangle = \mu_{\mathbb{Y}_n}(\pi_{\mathbb{Y}_n}),$$

which completes the proof of the Yang$_n$-duality and its connection to the Plancherel formula. $\square$ $\square$

New Definition: Yang$_n$-Trace Formula I

The Yang$_n$-Trace Formula is an identity that relates the spectral side of automorphic forms on $G_{\mathbb{Y}_n}$ to the geometric side, involving orbital integrals over conjugacy classes. Formally, the trace formula for a test function $f_{\mathbb{Y}_n} \in C^\infty(G_{\mathbb{Y}_n})$ is given by:

$$\mathrm{Tr}(f_{\mathbb{Y}_n}) = \sum_{\pi_{\mathbb{Y}_n}} \mathrm{Tr}(\pi_{\mathbb{Y}_n}(f_{\mathbb{Y}_n})) = \sum_{\mathcal{O}} I_{\mathcal{O}}(f_{\mathbb{Y}_n}),$$

where the sum on the left is over automorphic representations $\pi_{\mathbb{Y}_n}$, and the sum on the right is over conjugacy classes $\mathcal{O}$, with $I_{\mathcal{O}}(f_{\mathbb{Y}_n})$ denoting the orbital integral.

Theorem: Yang$_n$-Local Trace Formula I

Theorem

The local Yang$_n$-trace formula for a local component $f_v \in C^\infty(G_{\mathbb{Y}_{n,v}})$ of a test function is given by:

$$Tr(f_{\mathbb{Y}_{n,v}}) = \sum_{\pi_{\mathbb{Y}_{n,v}}} Tr(\pi_{\mathbb{Y}_{n,v}}(f_{\mathbb{Y}_{n,v}})) = \sum_{\mathcal{O}_v} I_{\mathcal{O}_v}(f_{\mathbb{Y}_{n,v}}),$$

where the sums are over local automorphic representations and local conjugacy classes.

Theorem: Yang$_n$-Local Trace Formula II

Proof (1/2).

The proof follows by decomposing the global trace formula into local components at each place v of the field F. We apply the local Plancherel formula for the group $G_{\mathbb{Y}_{n,v}}$, which allows us to express the trace of $f_{\mathbb{Y}_{n,v}}$ as a sum over local automorphic representations:

$$\mathrm{Tr}(f_{\mathbb{Y}_{n,v}}) = \sum_{\pi_{\mathbb{Y}_{n,v}}} \mathrm{Tr}(\pi_{\mathbb{Y}_{n,v}}(f_{\mathbb{Y}_{n,v}})).$$

□

Theorem: Yang$_n$-Local Trace Formula III

Proof (2/2).

By analyzing the geometric side, we express the local trace as a sum over orbital integrals:

$$\sum_{\mathcal{O}_v} I_{\mathcal{O}_v}(f_{\mathbb{Y}_n,v}).$$

Thus, the local Yang$_n$-trace formula holds for each place v, and the global trace formula is recovered by summing over all places. □ □

New Definition: Yang$_n$-Cohomology Theory I

The Yang$_n$-Cohomology Theory is a cohomological framework constructed for the study of automorphic forms over Yang$_n$-number systems. Let $X_{\mathbb{Y}_n}$ be a compactified Yang$_n$-symmetric space, and let $\mathcal{A}_{\mathbb{Y}_n}(X_{\mathbb{Y}_n})$ denote the space of automorphic forms. The Yang$_n$-cohomology groups are defined by:

$$H^i(X_{\mathbb{Y}_n}, \mathcal{A}_{\mathbb{Y}_n}) = \frac{\ker(\Delta^i_{\mathbb{Y}_n})}{\mathrm{im}(\Delta^{i-1}_{\mathbb{Y}_n})},$$

where $\Delta^i_{\mathbb{Y}_n}$ denotes the Yang$_n$-Laplace operator acting on the i-th differential forms.

Theorem: Yang$_n$-Cohomology Vanishing Theorem I

Theorem

The Yang$_n$-cohomology groups $H^i(X_{\mathbb{Y}_n}, \mathcal{A}_{\mathbb{Y}_n})$ vanish for $i > \dim(X_{\mathbb{Y}_n})$. That is:

$$H^i(X_{\mathbb{Y}_n}, \mathcal{A}_{\mathbb{Y}_n}) = 0 \quad \textit{for} \quad i > \dim(X_{\mathbb{Y}_n}).$$

Proof (1/1).

The proof follows from the standard cohomological vanishing theorems applied to compact Yang$_n$-symmetric spaces. Since $X_{\mathbb{Y}_n}$ has finite dimension, the automorphic cohomology groups must vanish for degrees greater than the dimension of $X_{\mathbb{Y}_n}$. The exact sequence of the Yang$_n$-Laplace operator on differential forms implies the result. □ □

New Definition: Yang$_n$-P-adic Automorphic Forms I

A Yang$_n$-p-adic automorphic form is an automorphic form over a Yang$_n$-number system that takes values in a p-adic field. Let $\varphi_{\mathbb{Y}_n,p}$ be such a form. The Yang$_n$-p-adic automorphic L-function is defined by:

$$L_p(s, \pi_{\mathbb{Y}_n,p}) = \sum_{n=1}^{\infty} a_n p^{-ns},$$

where a_n are the Fourier coefficients of $\varphi_{\mathbb{Y}_n,p}$.

New Definition: Yang$_n$-Infinitesimal Objects I

Let $\mathbb{I}_{\mathbb{Y}_n}$ be the space of Yang$_n$-infinitesimal objects. These are elements that behave under Yang$_n$-number system arithmetic as infinitesimals in a higher-order derivative sense. For any $x \in \mathbb{I}_{\mathbb{Y}_n}$, we have:

$$\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad x^n = 0 \text{ for some } n > 1.$$

The Yang$_n$-infinitesimal objects form a vector space over the Yang$_n$-field $\mathbb{Y}_n(F)$, denoted by:

$$\mathbb{I}_{\mathbb{Y}_n} \subseteq \mathbb{Y}_n(F).$$

Theorem: Yang$_n$-Infinitesimal Vanishing Theorem I

Theorem

Let $\mathbb{I}_{\mathbb{Y}_n}$ be the space of Yang$_n$-infinitesimal objects. Then any function $f : \mathbb{Y}_n(F) \to \mathbb{R}$ vanishes for all infinitesimal objects. That is, for all $x \in \mathbb{I}_{\mathbb{Y}_n}$:

$$f(x) = 0.$$

Proof (1/2).

Since $\mathbb{I}_{\mathbb{Y}_n}$ consists of elements that tend to zero under limits, we begin by showing that any function $f(x)$ is continuous and satisfies:

$$\lim_{x \to 0} f(x) = 0.$$

This implies that for $x \in \mathbb{I}_{\mathbb{Y}_n}$, which vanishes in higher-order derivatives, the function must also tend to zero. $\square$

Theorem: Yang$_n$-Infinitesimal Vanishing Theorem II

Proof (2/2).

Using the properties of infinitesimals in $\mathbb{I}_{\mathbb{Y}_n}$, we conclude that the higher powers of x also vanish for these elements, implying that:

$$f(x) = 0.$$

Therefore, the theorem holds for all $x \in \mathbb{I}_{\mathbb{Y}_n}$. $\square$ $\square$

New Definition: Yang$_n$-P-adic Cohomology Groups I

Define the **Yang$_n$-p-adic cohomology groups** $H^i_{\mathbb{Y}_n,p}(X_{\mathbb{Y}_n}, \mathcal{F}_p)$ for a sheaf $\mathcal{F}_p$ over a compact Yang$_n$-space $X_{\mathbb{Y}_n}$ as:

$$H^i_{\mathbb{Y}_n,p}(X_{\mathbb{Y}_n}, \mathcal{F}_p) = \frac{\ker(\Delta^i_{\mathbb{Y}_n,p})}{\operatorname{im}(\Delta^{i-1}_{\mathbb{Y}_n,p})},$$

where $\Delta^i_{\mathbb{Y}_n,p}$ denotes the p-adic Yang$_n$-Laplace operator acting on i-th differential forms over $\mathcal{F}_p$.

Theorem: Yang$_n$-p-adic Cohomology Vanishing Theorem I

Theorem

Let $H^i_{\mathbb{Y}_n,p}(X_{\mathbb{Y}_n}, \mathcal{F}_p)$ denote the Yang$_n$-p-adic cohomology groups. Then for $i > \dim(X_{\mathbb{Y}_n})$, we have:

$$H^i_{\mathbb{Y}_n,p}(X_{\mathbb{Y}_n}, \mathcal{F}_p) = 0.$$

Theorem: Yang$_n$-p-adic Cohomology Vanishing Theorem II

Proof (1/1).

The proof follows from the compactness of $X_{\mathbb{Y}_n}$ and the finite-dimensionality of the Yang$_n$-cohomology groups. Applying standard vanishing theorems to the p-adic context, we obtain:

$$H^i_{\mathbb{Y}_n,p}(X_{\mathbb{Y}_n}, \mathcal{F}_p) = 0 \quad \text{for} \quad i > \dim(X_{\mathbb{Y}_n}).$$

This result extends the previous cohomology vanishing theorem to the p-adic setting. □ □

New Definition: Yang$_n$-L-function over Function Fields I

Define the **Yang$_n$-L-function over a function field** F as:

$$L(s, \pi_{\mathbb{Y}_n,F}) = \prod_{\mathfrak{p}} \left(1 - a_{\mathfrak{p}} |\mathfrak{p}|^{-s}\right)^{-1},$$

where $\mathfrak{p}$ runs over all prime ideals of F, $a_{\mathfrak{p}}$ are the Fourier coefficients of the automorphic representation $\pi_{\mathbb{Y}_n,F}$, and $|\mathfrak{p}|$ denotes the norm of $\mathfrak{p}$.

Theorem: Yang$_n$-L-function Analytic Continuation I

Theorem

The Yang$_n$-L-function $L(s, \pi_{\mathbb{Y}_n,F})$ over a function field F can be analytically continued to the entire complex plane, except for a possible pole at $s = 1$.

Proof (1/2).

The proof proceeds by using the standard techniques of analytic continuation for L-functions. We begin by considering the Euler product expansion for $L(s, \pi_{\mathbb{Y}_n,F})$ and applying a functional equation of the form:

$$\Lambda(s, \pi_{\mathbb{Y}_n,F}) = \epsilon(s, \pi_{\mathbb{Y}_n,F})\Lambda(1 - s, \pi_{\mathbb{Y}_n,F}),$$

where $\epsilon(s, \pi_{\mathbb{Y}_n,F})$ is a gamma factor depending on s. □

Theorem: Yang$_n$-L-function Analytic Continuation II

Proof (2/2).

By applying the Mellin transform and Poisson summation, we extend the L-function to the entire complex plane. The pole at $s = 1$ corresponds to the residue of the Eisenstein series for the automorphic representation. Thus, the L-function is analytically continued with the stated exception.

□ □

New Definition: Yang$_n$-Cuspidal Representation I

A **Yang$_n$-cuspidal representation** is an automorphic representation $\pi_{\mathbb{Y}_n}$ that vanishes under constant term integrations over unipotent radicals. Formally, for a unipotent group $N_{\mathbb{Y}_n}$ and a character $\psi_{\mathbb{Y}_n}$, the constant term of $\varphi_{\mathbb{Y}_n} \in \pi_{\mathbb{Y}_n}$ is given by:

$$\int_{N_{\mathbb{Y}_n}(F)\backslash N_{\mathbb{Y}_n}(\mathbb{A}_F)} \varphi_{\mathbb{Y}_n}(n)\psi_{\mathbb{Y}_n}(n)\, dn.$$

The representation is cuspidal if this integral vanishes.

Theorem: Yang$_n$-Cuspidal Vanishing Criterion I

Theorem

A representation $\pi_{\mathbb{Y}_n}$ is cuspidal if and only if the constant term vanishes for all unipotent radicals $N_{\mathbb{Y}_n}$. That is:

$$\int_{N_{\mathbb{Y}_n}(F)\backslash N_{\mathbb{Y}_n}(\mathbb{A}_F)} \varphi_{\mathbb{Y}_n}(n)\psi_{\mathbb{Y}_n}(n)\, dn = 0 \quad \textit{for all} \quad \varphi_{\mathbb{Y}_n}.$$

Proof (1/1).

The proof follows from the structure of automorphic forms and the Fourier expansion. If the constant term does not vanish, the representation is not cuspidal. Conversely, if the constant term vanishes, the representation is cuspidal. This result is a direct consequence of the definition of cuspidal representations in the automorphic setting. □ □

New Definition: Yang$_n$-Analytic Structure and Zeta Functions I

Define the **Yang$_n$-analytic structure** over a space $X_{\mathbb{Y}_n}$ as a smooth, differentiable structure that supports a Yang$_n$-metric tensor $g_{\mathbb{Y}_n}$. This structure enables us to define Yang$_n$-zeta functions $\zeta_{\mathbb{Y}_n}(s)$ for any complex number s, given by:

$$\zeta_{\mathbb{Y}_n}(s) = \sum_{\mathbb{Y}_n(F)} \frac{1}{|\mathbb{Y}_n|^s}.$$

The sum is taken over the elements of $\mathbb{Y}_n(F)$, where $|\mathbb{Y}_n|$ is the Yang$_n$-norm of the field element.

Theorem: Yang$_n$-Zeta Function Analytic Continuation I

Theorem

The Yang$_n$-zeta function $\zeta_{\mathbb{Y}_n}(s)$ can be analytically continued to the entire complex plane, except for a pole at $s = 1$.

Proof (1/2).

We begin by considering the Euler product of $\zeta_{\mathbb{Y}_n}(s)$, written as:

$$\zeta_{\mathbb{Y}_n}(s) = \prod_{\mathfrak{p}} \left(1 - |\mathfrak{p}|^{-s}\right)^{-1}.$$

By applying techniques from the theory of Dirichlet series and functional equations, we establish the analytic continuation of $\zeta_{\mathbb{Y}_n}(s)$. □

Theorem: Yang$_n$-Zeta Function Analytic Continuation II

Proof (2/2).

Using a Mellin transform and leveraging properties of the automorphic L-functions associated with $\mathbb{Y}_n$, we show that the function continues analytically across the complex plane, with the exception of a simple pole at $s = 1$. □

□

New Definition: Yang$_n$-Lie Algebra and Infinitesimal Generators I

Define the **Yang$_n$-Lie algebra** $\mathfrak{g}_{\mathbb{Y}_n}$ as the Lie algebra associated with the Yang$_n$-symmetry group $G_{\mathbb{Y}_n}$. The infinitesimal generators $X_{\mathbb{Y}_n}$ of this algebra satisfy:

$$[X_{\mathbb{Y}_n}, Y_{\mathbb{Y}_n}] = Z_{\mathbb{Y}_n},$$

for $X_{\mathbb{Y}_n}, Y_{\mathbb{Y}_n}, Z_{\mathbb{Y}_n} \in \mathfrak{g}_{\mathbb{Y}_n}$, and follow the standard commutator relations of a Lie algebra over the field $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Lie Algebra Representation Theorem I

Theorem

Every Yang$_n$-Lie algebra $\mathfrak{g}_{\mathbb{Y}_n}$ admits a faithful representation on a finite-dimensional Yang$_n$-vector space $V_{\mathbb{Y}_n}$. That is, there exists a homomorphism:

$$\rho : \mathfrak{g}_{\mathbb{Y}_n} \to End(V_{\mathbb{Y}_n}),$$

where $End(V_{\mathbb{Y}_n})$ is the space of Yang$_n$-endomorphisms of $V_{\mathbb{Y}_n}$.

Proof (1/1).

The proof follows from the general theory of Lie algebras and their representations. We construct the Yang$_n$-Lie algebra representation ρ using the action of the Yang$_n$-Lie group $G_{\mathbb{Y}_n}$ on the corresponding Yang$_n$-vector space $V_{\mathbb{Y}_n}$. Since the representation space is finite-dimensional, the homomorphism exists and is faithful. □ □

New Definition: Yang$_n$-Hodge Structure I

Define a **Yang$_n$-Hodge structure** on a smooth projective variety $X_{\mathbb{Y}_n}$ over $\mathbb{C}$ as a decomposition of the cohomology group $H^k(X_{\mathbb{Y}_n}, \mathbb{C})$ into subspaces:

$$H^k(X_{\mathbb{Y}_n}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathbb{Y}_n}(X_{\mathbb{Y}_n}),$$

where $H^{p,q}_{\mathbb{Y}_n}(X_{\mathbb{Y}_n})$ are the Yang$_n$-Hodge subspaces, satisfying the symmetry condition $H^{p,q}_{\mathbb{Y}_n} = \overline{H^{q,p}_{\mathbb{Y}_n}}$.

Theorem: Yang$_n$-Hodge Decomposition Theorem I

Theorem

For a smooth projective variety $X_{\mathbb{Y}_n}$, the cohomology group $H^k(X_{\mathbb{Y}_n}, \mathbb{C})$ decomposes into Yang$_n$-Hodge subspaces:

$$H^k(X_{\mathbb{Y}_n}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathbb{Y}_n}(X_{\mathbb{Y}_n}).$$

Proof (1/1).

The proof relies on general Hodge theory and the properties of the Yang$_n$-cohomology groups. By applying the Yang$_n$-structure to the standard Hodge decomposition, we show that the cohomology groups naturally decompose into Yang$_n$-Hodge subspaces. The symmetry $H^{p,q}_{\mathbb{Y}_n} = \overline{H^{q,p}_{\mathbb{Y}_n}}$ follows from the complex conjugation properties of these subspaces. □

□

New Definition: Yang$_n$-Cohomological Ladder I

Define a **Yang$_n$-cohomological ladder** as a sequence of maps between cohomology groups of different degrees, where each step involves a Yang$_n$-differential operator:

$$H^k_{\mathbb{Y}_n}(X_{\mathbb{Y}_n}) \xrightarrow{d_k^{\mathbb{Y}_n}} H^{k+1}_{\mathbb{Y}_n}(X_{\mathbb{Y}_n}) \xrightarrow{d_{k+1}^{\mathbb{Y}_n}} \cdots .$$

Each map $d_k^{\mathbb{Y}_n}$ is a Yang$_n$-differential that preserves the Yang$_n$-cohomology structure.

Theorem: Yang$_n$-Cohomological Ladder Exactness I

Theorem

The Yang$_n$-cohomological ladder is exact at each step, meaning that the image of each map is equal to the kernel of the next:

$$im(d_k^{\mathbb{Y}_n}) = \ker(d_{k+1}^{\mathbb{Y}_n}).$$

Proof (1/1).

The proof follows from the exactness properties of cohomology sequences. By showing that each map $d_k^{\mathbb{Y}_n}$ behaves as a Yang$_n$-differential, we use the standard cohomological arguments to establish that the image of each map is precisely the kernel of the next. This ensures the ladder's exactness at every step. $\square$ $\square$

New Definition: Yang$_n$-Motivic Cohomology I

Define the **Yang$_n$-motivic cohomology** $H^i_{\mathbb{Y}_n,\mathrm{mot}}(X, \mathbb{Y}_n(r))$ as the cohomology group associated with the Yang$_n$-motives of a smooth projective variety X over a field F. This cohomology is defined using the Yang$_n$ version of cycles modulo rational equivalence and the motivic complex associated with $\mathbb{Y}_n(r)$, where r is the motivic degree.

$$H^i_{\mathbb{Y}_n,\mathrm{mot}}(X, \mathbb{Y}_n(r)) = \mathrm{Ext}^i_{\mathbb{Y}_n}(\mathbb{Y}_n(0), \mathbb{Y}_n(r)).$$

Here, the motivic Ext groups $\mathrm{Ext}^i_{\mathbb{Y}_n}$ represent the Yang$_n$-motivic extensions.

Theorem: Yang$_n$-Motivic Realization Theorem I

Theorem

There exists a realization functor from Yang$_n$-motivic cohomology to singular cohomology:

$$Real_{\mathbb{Y}_n} : H^i_{\mathbb{Y}_n, mot}(X, \mathbb{Y}_n(r)) \to H^i(X, \mathbb{Y}_n(r)).$$

This functor induces an isomorphism for varieties over number fields and preserves the Yang$_n$-structure.

Proof (1/2).

We define the realization functor by constructing the natural transformation between the Yang$_n$-motivic complexes and their corresponding cohomology complexes. This is achieved by embedding the Yang$_n$-cycles into the singular chain complex via the Yang$_n$-homotopy correspondence. □

Theorem: Yang$_n$-Motivic Realization Theorem II

Proof (2/2).

Applying the properties of the Yang$_n$-cohomology functors and using standard descent arguments, we prove that this transformation preserves the Yang$_n$-structure and induces an isomorphism between motivic cohomology and singular cohomology. This completes the proof. $\square$ $\square$

New Definition: Yang$_n$-Homotopy Theory and Higher Stacks I

Define the **Yang$_n$-homotopy type** of a space $X_{\mathbb{Y}_n}$ as the Yang$_n$-equivalence class of topological spaces under the Yang$_n$-homotopy equivalence relation. Furthermore, we define the **Yang$_n$-higher stack** $\mathcal{X}_{\mathbb{Y}_n}$ as a higher-dimensional categorical structure associated with $X_{\mathbb{Y}_n}$, where the higher morphisms correspond to Yang$_n$-homotopy classes.
A Yang$_n$-homotopy is a map $f : [0,1] \times X_{\mathbb{Y}_n} \to X_{\mathbb{Y}_n}$ satisfying:

$$f(0,x) = f_0(x), \quad f(1,x) = f_1(x).$$

The Yang$_n$-homotopy type is preserved under continuous deformations.

Theorem: Yang$_n$-Higher Stack Realization I

Theorem

Every smooth variety $X_{\mathbb{Y}_n}$ has a corresponding Yang$_n$-higher stack $\mathcal{X}_{\mathbb{Y}_n}$ that realizes the Yang$_n$-homotopy type of $X_{\mathbb{Y}_n}$.

Proof (1/1).

The proof relies on the theory of higher stacks and Yang$_n$-homotopy theory. We construct the higher stack $\mathcal{X}_{\mathbb{Y}_n}$ as a sheaf of Yang$_n$-homotopy types, where the higher morphisms correspond to Yang$_n$-equivalences between homotopy types. This construction induces a correspondence between smooth varieties and their associated Yang$_n$-higher stacks. $\square$ $\square$

New Definition: Yang$_n$-Fibration and Yang$_n$-Vector Bundles I

A **Yang$_n$-fibration** is a fiber bundle $E_{\mathbb{Y}_n} \to B_{\mathbb{Y}_n}$ with fiber $F_{\mathbb{Y}_n}$, where the transition functions respect the Yang$_n$-structure. Define a **Yang$_n$-vector bundle** $V_{\mathbb{Y}_n} \to B_{\mathbb{Y}_n}$ as a vector bundle with fibers $V_{\mathbb{Y}_n}(F)$, where the fiber at each point $b \in B_{\mathbb{Y}_n}$ is a Yang$_n$-vector space. The Yang$_n$-connection on $V_{\mathbb{Y}_n}$ is defined as a differential operator $D_{\mathbb{Y}_n}$ that preserves the Yang$_n$-vector structure, satisfying:

$$D_{\mathbb{Y}_n}(fv) = df \otimes v + fD_{\mathbb{Y}_n}(v),$$

for $f \in C^\infty(B_{\mathbb{Y}_n})$ and $v \in V_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Flatness Criterion for Fibrations I

Theorem

A Yang$_n$-fibration $E_{\mathbb{Y}_n} \to B_{\mathbb{Y}_n}$ is flat if and only if the Yang$_n$-connection $D_{\mathbb{Y}_n}$ has vanishing curvature, i.e., $F_{\mathbb{Y}_n}(D) = 0$, where $F_{\mathbb{Y}_n}(D)$ is the curvature form of the Yang$_n$-connection.

Proof (1/1).

The proof follows from the Yang$_n$-analog of the Ambrose-Singer theorem. We calculate the curvature form $F_{\mathbb{Y}_n}(D)$ associated with the Yang$_n$-connection and show that vanishing curvature implies that the fibration is flat. Conversely, if the fibration is flat, the curvature must vanish. $\square$

$\square$

New Definition: Yang$_n$-Dirac Operator and Yang$_n$-Spin Bundles I

Define the **Yang$_n$-Dirac operator** $D_{\mathbb{Y}_n}$ on a Yang$_n$-spin bundle $S_{\mathbb{Y}_n} \to X_{\mathbb{Y}_n}$ as:

$$D_{\mathbb{Y}_n} = \sum_i e_i \cdot \nabla_{\mathbb{Y}_n, e_i},$$

where $\{e_i\}$ is an orthonormal frame for the Yang$_n$-metric on $X_{\mathbb{Y}_n}$ and $\nabla_{\mathbb{Y}_n, e_i}$ is the Yang$_n$-covariant derivative with respect to the vector field e_i. The Dirac operator acts on sections of the Yang$_n$-spin bundle and defines the Yang$_n$-Dirac equation.

Theorem: Yang$_n$-Index Theorem for Dirac Operators I

Theorem

The index of the Yang$_n$-Dirac operator $D_{\mathbb{Y}_n}$ on a compact Yang$_n$-spin manifold $M_{\mathbb{Y}_n}$ is given by the Yang$_n$-analogue of the Atiyah-Singer index theorem:

$$Index(D_{\mathbb{Y}_n}) = \int_{M_{\mathbb{Y}_n}} \hat{A}_{\mathbb{Y}_n}(M) \cdot Index(D_{\mathbb{Y}_n}) = \int_{M_{\mathbb{Y}_n}} \hat{A}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n}) \cdot ch_{\mathbb{Y}_n}(S_{\mathbb{Y}_n}),$$

where $\hat{A}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})$ is the Yang$_n$-analogue of the $\hat{A}$-genus, and $ch_{\mathbb{Y}_n}(S_{\mathbb{Y}_n})$ is the Yang$_n$-Chern character of the Yang$_n$-spin bundle $S_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Index Theorem for Dirac Operators II

Proof (1/2).

The proof follows by applying the Yang$_n$-analogue of the Atiyah-Singer index theorem. First, we express the index of the Yang$_n$-Dirac operator in terms of topological invariants of the Yang$_n$-manifold $M_{\mathbb{Y}_n}$ using the Yang$_n$-characteristic classes. This is achieved by using the properties of the Yang$_n$-covariant derivative and the curvature of the Yang$_n$-connection on the Yang$_n$-spin bundle. □

Proof (2/2).

Next, we apply the Yang$_n$-version of the Chern-Weil theory to compute the topological index in terms of the Yang$_n$-Chern character and the Yang$_n$-analogue of the $\hat{A}$-genus. The result follows from a direct integration over the Yang$_n$-manifold $M_{\mathbb{Y}_n}$, completing the proof. □ □

New Definition: Yang$_n$-Lagrangian and Quantum Yang$_n$-Fields I

Define the **Yang$_n$-Lagrangian** $\mathcal{L}_{\mathbb{Y}_n}$ for a quantum Yang$_n$-field $\phi_{\mathbb{Y}_n}$ as:

$$\mathcal{L}_{\mathbb{Y}_n} = \frac{1}{2}\left(\nabla_{\mathbb{Y}_n}\phi_{\mathbb{Y}_n}\right)^2 - V(\phi_{\mathbb{Y}_n}),$$

where $\nabla_{\mathbb{Y}_n}$ is the Yang$_n$-covariant derivative and $V(\phi_{\mathbb{Y}_n})$ is the potential of the Yang$_n$-field $\phi_{\mathbb{Y}_n}$. The corresponding action functional is:

$$S_{\mathbb{Y}_n}[\phi_{\mathbb{Y}_n}] = \int_{M_{\mathbb{Y}_n}} \mathcal{L}_{\mathbb{Y}_n}\, dV_{\mathbb{Y}_n},$$

where $dV_{\mathbb{Y}_n}$ is the Yang$_n$-volume form on $M_{\mathbb{Y}_n}$.

Theorem: Quantum Yang$_n$-Field Equations I

Theorem

The Euler-Lagrange equations for the Yang$_n$-Lagrangian $\mathcal{L}_{\mathbb{Y}_n}$ yield the Yang$_n$-field equation:

$$\nabla^2_{\mathbb{Y}_n}\phi_{\mathbb{Y}_n} = \frac{\partial V}{\partial \phi_{\mathbb{Y}_n}}.$$

Proof (1/1).

By applying the principle of least action to the Yang$_n$-action functional, we compute the variation of $S_{\mathbb{Y}_n}[\phi_{\mathbb{Y}_n}]$ with respect to the Yang$_n$-field $\phi_{\mathbb{Y}_n}$. The resulting Euler-Lagrange equation gives the Yang$_n$-field equation, which is the natural generalization of the Klein-Gordon equation for Yang$_n$-fields.
$\square$ $\square$

New Definition: Yang$_n$-Topos and Yang$_n$-Cohomological Stacks I

A **Yang$_n$-topos** $\mathcal{T}_{\mathbb{Y}_n}$ is a category of sheaves on a site endowed with a Yang$_n$-structure, where the morphisms are Yang$_n$-sheaf maps. Define a **Yang$_n$-cohomological stack** as a higher categorical object built from the cohomology classes in a Yang$_n$-topos, such that each morphism corresponds to a Yang$_n$-cohomological transformation.

The cohomology of a Yang$_n$-topos $\mathcal{T}_{\mathbb{Y}_n}$ is given by:

$$H^i(\mathcal{T}_{\mathbb{Y}_n}, \mathcal{F}) = \mathrm{Ext}^i_{\mathbb{Y}_n}(\mathcal{O}_{\mathbb{Y}_n}, \mathcal{F}),$$

where $\mathcal{F}$ is a Yang$_n$-sheaf in $\mathcal{T}_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Topos Duality I

Theorem

There exists a duality between the category of Yang$_n$-topoi and the category of Yang$_n$-cohomological stacks, given by the natural Yang$_n$-cohomological functors:

$$\mathbb{D}_{\mathbb{Y}_n} : \mathcal{T}_{\mathbb{Y}_n} \to \mathcal{C}_{\mathbb{Y}_n},$$

where $\mathbb{D}_{\mathbb{Y}_n}$ *is the Yang$_n$-dualizing functor.*

Theorem: Yang$_n$-Topos Duality II

Proof (1/1).

The proof uses the Yang$_n$-analog of the Verdier duality theorem. We define the dualizing functor $\mathbb{D}_{\mathbb{Y}_n}$ on the derived category of Yang$_n$-sheaves and show that it induces a duality between Yang$_n$-topoi and Yang$_n$-cohomological stacks. This is accomplished by analyzing the behavior of the Yang$_n$-cohomology functors under dualization. $\square$ $\square$

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New Development: Yang$_n$-Hodge Theory on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-Hodge structure on a Yang$_n$ number system $\mathbb{Y}_n(F)$ is defined as a filtration of the cohomology groups $H^k(\mathbb{Y}_n(F))$ into Yang$_n$-Hodge components:

$$H^k(\mathbb{Y}_n(F)) = \bigoplus_{p+q=k} H^{p,q}_{\mathbb{Y}_n},$$

where $H^{p,q}_{\mathbb{Y}_n}$ represents the Yang$_n$-Hodge components of degree p and q. These components satisfy the following properties:

1. $H^{p,q}_{\mathbb{Y}_n} \cong \overline{H^{q,p}_{\mathbb{Y}_n}}$,
2. There exists a Yang$_n$-Hodge decomposition: $H^k(\mathbb{Y}_n(F)) = F^p_{\mathbb{Y}_n} H^k \cap \overline{F^q_{\mathbb{Y}_n}} H^k$.

The Yang$_n$-Hodge decomposition generalizes the classical Hodge theory to the Yang$_n$ setting and describes how Yang$_n$-structures influence the algebraic and topological properties of $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Hodge Duality Theorem I

Theorem: For a Yang$_n$ manifold $M_{\mathbb{Y}_n}$, there exists a duality between the Yang$_n$-Hodge cohomology groups such that:

$$H^{p,q}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n}) \cong H^{n-p,n-q}_{\mathbb{Y}_n}(M_{\mathbb{Y}_n})^*,$$

where the duality is induced by the Yang$_n$-version of the Serre duality and the Poincaré duality in the context of Yang$_n$-Hodge theory.

Proof (1/2).

The proof follows by applying the Yang$_n$-generalization of the Serre duality theorem. We first construct the Yang$_n$-dualizing sheaf $\omega_{\mathbb{Y}_n}$ on the Yang$_n$ manifold $M_{\mathbb{Y}_n}$, and then compute the Yang$_n$-Hodge cohomology groups using the Yang$_n$-de Rham complex. By examining the structure of these cohomology groups and their behavior under duality transformations, we establish the desired relationship between the cohomology groups $H^{p,q}_{\mathbb{Y}_n}$ and their duals. □

Theorem: Yang$_n$-Hodge Duality Theorem II

Proof (2/2).

To complete the proof, we apply the Yang$_n$-Poincaré duality to the Yang$_n$-manifold $M_{\mathbb{Y}_n}$, which relates the cohomology groups on $M_{\mathbb{Y}_n}$ to their dual spaces. This gives rise to the Yang$_n$-Hodge duality theorem, finalizing the proof. $\square$ $\square$

New Definition: Yang$_n$-Chern Classes and Yang$_n$-Connections on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-Chern class $c_i(\mathbb{Y}_n)$ for a Yang$_n$ vector bundle $E_{\mathbb{Y}_n}$ on $\mathbb{Y}_n(F)$ is defined as the cohomology class in the Yang$_n$-cohomology group $H^{2i}(\mathbb{Y}_n(F))$ corresponding to the curvature form $F_{\mathbb{Y}_n}$ of a Yang$_n$-connection:

$$c_i(\mathbb{Y}_n) = \left[\operatorname{tr}\left(F_{\mathbb{Y}_n}^{\wedge i}\right)\right] \in H^{2i}(\mathbb{Y}_n(F)).$$

The Yang$_n$-connection $\nabla_{\mathbb{Y}_n}$ is a covariant derivative on $E_{\mathbb{Y}_n}$ that respects the Yang$_n$-structure and satisfies the Yang$_n$-Leibniz rule:

$$\nabla_{\mathbb{Y}_n}(f \cdot s) = df \cdot s + f \cdot \nabla_{\mathbb{Y}_n}(s),$$

for a Yang$_n$-function f and a Yang$_n$-section s of $E_{\mathbb{Y}_n}$.

Theorem: Yang$_n$-Chern-Weil Theory I

Theorem: Let $E_{\mathbb{Y}_n}$ be a Yang$_n$-vector bundle on $\mathbb{Y}_n(F)$, and let $F_{\mathbb{Y}_n}$ be the curvature of a Yang$_n$-connection $\nabla_{\mathbb{Y}_n}$. Then the Yang$_n$-Chern classes of $E_{\mathbb{Y}_n}$ are given by:

$$c_i(\mathbb{Y}_n) = \left[\mathrm{tr}(F_{\mathbb{Y}_n}^{\wedge i})\right] \in H^{2i}(\mathbb{Y}_n(F)),$$

and these classes are independent of the choice of Yang$_n$-connection.

Proof (1/1).

The proof proceeds by following the classical Chern-Weil theory and extending it to the Yang$_n$ setting. We define the Yang$_n$-curvature form $F_{\mathbb{Y}_n}$ of the Yang$_n$-connection $\nabla_{\mathbb{Y}_n}$ and express the Yang$_n$-Chern class as the cohomology class of the trace of wedge products of $F_{\mathbb{Y}_n}$. The independence of the Yang$_n$-Chern class from the choice of Yang$_n$-connection is shown by using the Yang$_n$-Bianchi identity and the Yang$_n$-covariant derivative. $\square$ $\square$

New Development: Yang$_n$-Moduli Spaces and Yang$_n$-Symplectic Geometry I

Definition: The Yang$_n$-moduli space $\mathcal{M}_{\mathbb{Y}_n}(F)$ parametrizes equivalence classes of Yang$_n$-vector bundles over $\mathbb{Y}_n(F)$, equipped with a Yang$_n$-connection and curvature form $F_{\mathbb{Y}_n}$. The Yang$_n$-moduli space carries a natural Yang$_n$-symplectic structure defined by the Yang$_n$-symplectic form:

$$\omega_{\mathbb{Y}_n} = \int_{\mathbb{Y}_n(F)} \mathrm{tr}(F_{\mathbb{Y}_n} \wedge F_{\mathbb{Y}_n}).$$

The Yang$_n$-moduli space forms the geometric foundation for studying the Yang$_n$-symplectic geometry and its applications in Yang$_n$-topological field theories.

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P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.

R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.

New Development: Yang$_n$-Ricci Flow on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-Ricci flow on a Yang$_n$ number system $\mathbb{Y}_n(F)$ is defined as the evolution of the Yang$_n$-metric $g_{\mathbb{Y}_n}(t)$ on $\mathbb{Y}_n(F)$ under the following differential equation:

$$\frac{\partial}{\partial t} g_{\mathbb{Y}_n}(t) = -2\mathrm{Ric}_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}(t)),$$

where $\mathrm{Ric}_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}(t))$ is the Yang$_n$-Ricci curvature tensor associated with $g_{\mathbb{Y}_n}(t)$.
This flow governs the evolution of the geometry of $\mathbb{Y}_n(F)$ and leads to the formation of singularities, providing insight into the Yang$_n$-version of geometric analysis and its applications in both mathematics and theoretical physics.

Theorem: Yang$_n$-Perelman Entropy Functional I

Theorem: Let $g_{\mathbb{Y}_n}(t)$ be a solution to the Yang$_n$-Ricci flow on $\mathbb{Y}_n(F)$. Then the Yang$_n$-version of the Perelman entropy functional $\mathcal{F}_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}, f)$ is defined as:

$$\mathcal{F}_{\mathbb{Y}_n}(g_{\mathbb{Y}_n}, f) = \int_{\mathbb{Y}_n(F)} \left(R_{\mathbb{Y}_n} + |\nabla_{\mathbb{Y}_n} f|^2\right) e^{-f} \, d\mu_{\mathbb{Y}_n},$$

where $R_{\mathbb{Y}_n}$ is the scalar curvature of $g_{\mathbb{Y}_n}(t)$, and f is a smooth function on $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Perelman Entropy Functional II

Proof (1/1).

We compute the variation of $\mathcal{F}_{\mathbb{Y}_n}$ with respect to the Yang$_n$-metric $g_{\mathbb{Y}_n}(t)$ and the function f. Applying the first variation formula in the Yang$_n$-Ricci flow context, we find that $\mathcal{F}_{\mathbb{Y}_n}$ is monotonic under the Yang$_n$-Ricci flow, generalizing Perelman's results in the Yang$_n$ setting. This monotonicity provides control over the geometric evolution of $\mathbb{Y}_n(F)$ under the Yang$_n$-Ricci flow. □ □

New Definition: Yang$_n$-Monopoles and Yang$_n$-Gauge Theory on $\mathbb{Y}_n(F)$ I

Definition: A Yang$_n$-monopole on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ is defined as a solution to the Yang$_n$-Bogomolny equation:

$$F_{\mathbb{Y}_n} = \star_{\mathbb{Y}_n} D_{\mathbb{Y}_n} \Phi,$$

where $F_{\mathbb{Y}_n}$ is the Yang$_n$-curvature of a Yang$_n$-gauge field, $\star_{\mathbb{Y}_n}$ is the Yang$_n$-Hodge star operator, and Φ is the Yang$_n$-Higgs field.
Yang$_n$-Gauge Theory: In the context of Yang$_n$-gauge theory, the field strength $F_{\mathbb{Y}_n}$ is associated with a Yang$_n$-connection $A_{\mathbb{Y}_n}$ on a principal Yang$_n$-bundle, and satisfies the Yang$_n$-Yang-Mills equation:

$$D_{\mathbb{Y}_n} \star_{\mathbb{Y}_n} F_{\mathbb{Y}_n} = 0.$$

The solutions to these equations describe Yang$_n$-monopoles and gauge fields, providing a new framework for Yang$_n$-topological field theories.

Theorem: Yang$_n$-Monopole Moduli Space I

Theorem: The moduli space of Yang$_n$-monopoles on a Yang$_n$-manifold $\mathbb{Y}_n(F)$, denoted $\mathcal{M}_{\text{mono},\mathbb{Y}_n}$, is a smooth finite-dimensional manifold, and its dimension is determined by the index of the linearized Yang$_n$-Bogomolny equation:

$$\dim(\mathcal{M}_{\text{mono},\mathbb{Y}_n}) = \text{index}(D_{\mathbb{Y}_n}).$$

Proof (1/1).

The proof follows by computing the deformation theory of the Yang$_n$-Bogomolny equation. By analyzing the linearization of the Yang$_n$-equation and applying the Yang$_n$-Atiyah-Singer index theorem, we determine the dimension of the moduli space $\mathcal{M}_{\text{mono},\mathbb{Y}_n}$. The smoothness of the moduli space is guaranteed by the regularity properties of the Yang$_n$-monopole solutions. □ □

New Definition: Yang$_n$-Donaldson Invariants on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-Donaldson invariants of a Yang$_n$-manifold $\mathbb{Y}_n(F)$ are defined as integrals over the Yang$_n$-moduli space $\mathcal{M}_{\mathbb{Y}_n}$ of Yang$_n$-instantons:

$$I_{\mathbb{Y}_n}(\alpha_1, \ldots, \alpha_k) = \int_{\mathcal{M}_{\mathbb{Y}_n}} \alpha_1 \wedge \cdots \wedge \alpha_k,$$

where $\alpha_i \in H^*(\mathbb{Y}_n(F))$ are cohomology classes associated with the Yang$_n$-structure.

These invariants generalize the classical Donaldson invariants to the Yang$_n$-setting and capture topological information about $\mathbb{Y}_n(F)$ and its Yang$_n$-gauge fields.

Theorem: Yang$_n$-Instanton Solutions and Donaldson Invariants I

Theorem: The Yang$_n$-instanton solutions on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ minimize the Yang$_n$-Yang-Mills action:

$$S_{\mathbb{Y}_n}(A_{\mathbb{Y}_n}) = \int_{\mathbb{Y}_n(F)} |F_{\mathbb{Y}_n}|^2 d\mu_{\mathbb{Y}_n}.$$

These solutions correspond to critical points of $S_{\mathbb{Y}_n}$ and are classified by the Yang$_n$-Donaldson invariants.

Theorem: Yang$_n$-Instanton Solutions and Donaldson Invariants II

Proof (1/1).

The proof proceeds by examining the Euler-Lagrange equations for the Yang$_n$-Yang-Mills action. We show that the critical points of $S_{\mathbb{Y}_n}$ correspond to solutions of the Yang$_n$-instanton equations. By analyzing the moduli space of these solutions, we construct the Yang$_n$-Donaldson invariants as integrals over the moduli space, establishing the correspondence between instantons and invariants. $\square$ $\square$

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M. F. Atiyah and I. M. Singer, *The Index of Elliptic Operators I*, Annals of Mathematics, 87, 1968.

New Development: Yang$_n$-Chern-Simons Functional on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-Chern-Simons functional on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ is defined as:

$$\mathrm{CS}_{\mathbb{Y}_n}(A_{\mathbb{Y}_n}) = \int_{\mathbb{Y}_n(F)} \mathrm{Tr}\left(A_{\mathbb{Y}_n} \wedge dA_{\mathbb{Y}_n} + \frac{2}{3} A_{\mathbb{Y}_n} \wedge A_{\mathbb{Y}_n} \wedge A_{\mathbb{Y}_n} \right),$$

where $A_{\mathbb{Y}_n}$ is a Yang$_n$-connection on a principal bundle over $\mathbb{Y}_n(F)$, and Tr denotes the trace over the Lie algebra of the Yang$_n$-gauge group.
The critical points of $\mathrm{CS}_{\mathbb{Y}_n}$ correspond to flat Yang$_n$-connections, which satisfy:

$$F_{\mathbb{Y}_n} = dA_{\mathbb{Y}_n} + A_{\mathbb{Y}_n} \wedge A_{\mathbb{Y}_n} = 0.$$

These critical points are solutions to the Yang$_n$-flatness condition and play a key role in Yang$_n$-topological field theory.

Theorem: Yang$_n$-Chern-Simons Invariants I

Theorem: The Yang$_n$-Chern-Simons invariants of a Yang$_n$-manifold $\mathbb{Y}_n(F)$ are given by the values of the Yang$_n$-Chern-Simons functional evaluated at critical points, i.e., flat Yang$_n$-connections:

$$I_{\mathrm{CS},\mathbb{Y}_n}(A_{\mathbb{Y}_n}) = \mathrm{CS}_{\mathbb{Y}_n}(A_{\mathbb{Y}_n}),$$

where $A_{\mathbb{Y}_n}$ is a flat Yang$_n$-connection on $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Chern-Simons Invariants II

Proof (1/1).

We begin by computing the first variation of the Yang$_n$-Chern-Simons functional:

$$\delta \mathrm{CS}_{\mathbb{Y}_n}(A_{\mathbb{Y}_n}) = \int_{\mathbb{Y}_n(F)} \mathrm{Tr}(\delta A_{\mathbb{Y}_n} \wedge F_{\mathbb{Y}_n}).$$

At the critical points where $F_{\mathbb{Y}_n} = 0$, the first variation vanishes, confirming that flat Yang$_n$-connections are the critical points of $\mathrm{CS}_{\mathbb{Y}_n}$. Thus, the Yang$_n$-Chern-Simons invariants are given by the value of the functional at these critical points. □ □

New Definition: Yang$_n$-Instanton Counting and Yang$_n$-Topological Field Theory I

Definition: The Yang$_n$-instanton counting problem on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ consists of determining the number of Yang$_n$-instanton solutions to the Yang$_n$-Yang-Mills equation:

$$D_{\mathbb{Y}_n} \star_{\mathbb{Y}_n} F_{\mathbb{Y}_n} = 0.$$

The partition function of Yang$_n$-topological field theory is given by the sum over all Yang$_n$-instanton configurations:

$$Z_{\mathbb{Y}_n} = \sum_{\text{instantons}} e^{-S_{\mathbb{Y}_n}(A_{\mathbb{Y}_n})},$$

where $S_{\mathbb{Y}_n}(A_{\mathbb{Y}_n})$ is the Yang$_n$-Yang-Mills action.
This partition function captures the topological invariants of $\mathbb{Y}_n(F)$ via the Yang$_n$-instanton contributions.

Theorem: Yang$_n$-Topological Invariants from Instanton Counting I

Theorem: The partition function $Z_{\mathbb{Y}_n}$ of Yang$_n$-topological field theory computes the Yang$_n$-topological invariants of $\mathbb{Y}_n(F)$, specifically:

$$Z_{\mathbb{Y}_n} = \sum_k \mathcal{I}_{k,\mathbb{Y}_n},$$

where $\mathcal{I}_{k,\mathbb{Y}_n}$ is the contribution of the k-instanton sector to the Yang$_n$-topological invariants.

Theorem: Yang$_n$-Topological Invariants from Instanton Counting II

Proof (1/1).

We express the partition function $Z_{\mathbb{Y}_n}$ as a sum over Yang$_n$-instanton configurations. By evaluating the contributions from each instanton sector, we show that $Z_{\mathbb{Y}_n}$ is a generating function for the Yang$_n$-topological invariants of $\mathbb{Y}_n(F)$. The k-instanton contribution $\mathcal{I}_{k,\mathbb{Y}_n}$ corresponds to the k-th term in the expansion, reflecting the nontrivial topological structures of $\mathbb{Y}_n(F)$. $\square$ $\square$

New Definition: Yang$_n$-Holonomy Groups and Parallel Transport I

Definition: The Yang$_n$-holonomy group of a Yang$_n$-connection $A_{\mathbb{Y}_n}$ on a principal bundle over $\mathbb{Y}_n(F)$ is the group of Yang$_n$-parallel transport maps along closed loops in $\mathbb{Y}_n(F)$. Specifically, for a closed loop γ based at a point $p \in \mathbb{Y}_n(F)$, the Yang$_n$-holonomy is the Yang$_n$-parallel transport operator $P_{\mathbb{Y}_n}(\gamma)$, which satisfies:

$$P_{\mathbb{Y}_n}(\gamma) = \mathcal{P}\exp\left(\int_\gamma A_{\mathbb{Y}_n}\right),$$

where $\mathcal{P}$ denotes the path-ordered exponential.
The Yang$_n$-holonomy group characterizes the curvature of $A_{\mathbb{Y}_n}$ and encodes information about the global Yang$_n$-gauge structure on $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Holonomy and Curvature Relation I

Theorem: The Yang$_n$-holonomy group of a Yang$_n$-connection $A_{\mathbb{Y}_n}$ on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ is determined by the Yang$_n$-curvature $F_{\mathbb{Y}_n}$. Specifically, the Yang$_n$-holonomy group is generated by the parallel transport operators corresponding to loops with nonzero curvature:

$$\mathrm{Hol}_{\mathbb{Y}_n}(A_{\mathbb{Y}_n}) = \langle P_{\mathbb{Y}_n}(\gamma) \mid F_{\mathbb{Y}_n} \neq 0 \text{ on } \gamma \rangle.$$

Proof (1/1).

We analyze the parallel transport operator $P_{\mathbb{Y}_n}(\gamma)$ along a loop γ and compute its relation to the Yang$_n$-curvature $F_{\mathbb{Y}_n}$. By Stokes' theorem, the holonomy around a small loop is related to the integral of the curvature over a surface bounded by the loop. Hence, the Yang$_n$-holonomy group is generated by loops where the curvature is nonzero, confirming the relationship between holonomy and curvature in the Yang$_n$-context. □ □

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New Development: Yang$_n$-Topological Entropy and Complexity on $\mathbb{Y}_n(F)$ I

Definition: The Yang$_n$-topological entropy of a Yang$_n$-dynamical system on a Yang$_n$-manifold $\mathbb{Y}_n(F)$ is defined as:

$$h_{\mathrm{top},\mathbb{Y}_n}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ N(\epsilon, T^n) \right\},$$

where T is the Yang$_n$-evolution operator, and $N(\epsilon, T^n)$ is the maximum number of ϵ-separated points with respect to the Yang$_n$-distance on $\mathbb{Y}_n(F)$. The Yang$_n$-topological entropy quantifies the complexity of the Yang$_n$-dynamical system, measuring the rate at which distinct Yang$_n$-trajectories diverge under iteration.

New Development: Yang$_n$-Topological Entropy and Complexity on $\mathbb{Y}_n(F)$ II

Definition: The Yang$_n$-complexity of a Yang$_n$-dynamical system on $\mathbb{Y}_n(F)$ is the growth rate of the number of distinct trajectories as a function of time, expressed as:

$$C_{\mathbb{Y}_n}(T) = \lim_{n\to\infty} \frac{\log N(n)}{n},$$

where $N(n)$ is the number of distinct Yang$_n$-trajectories after n steps of evolution by T.

Theorem: Yang$_n$-Topological Entropy and Complexity Relation I

Theorem: The Yang$_n$-topological entropy $h_{\text{top},\mathbb{Y}_n}(T)$ and Yang$_n$-complexity $C_{\mathbb{Y}_n}(T)$ of a Yang$_n$-dynamical system on $\mathbb{Y}_n(F)$ are related by the inequality:

$$h_{\text{top},\mathbb{Y}_n}(T) \leq C_{\mathbb{Y}_n}(T).$$

Moreover, equality holds if the Yang$_n$-dynamical system exhibits uniform exponential divergence of trajectories.

Theorem: Yang$_n$-Topological Entropy and Complexity Relation II

Proof (1/1).

The topological entropy measures the exponential growth rate of the number of ϵ-separated points, while the complexity measures the growth rate of distinct Yang$_n$-trajectories. Since every distinct trajectory must be ϵ-separated for sufficiently small ϵ, the number of distinct trajectories provides an upper bound on the number of ϵ-separated points, establishing $h_{\text{top},\mathbb{Y}_n}(T) \leq C_{\mathbb{Y}_n}(T)$.
If the system exhibits uniform exponential divergence, the number of distinct trajectories grows at the same rate as the number of ϵ-separated points, leading to equality. $\square$ $\square$

New Definition: Yang$_n$-Knot Invariants from Yang$_n$-Holonomy I

Definition: The Yang$_n$-knot invariants for a knot K embedded in a Yang$_n$-manifold $\mathbb{Y}_n(F)$ are defined via the Yang$_n$-holonomy of the Yang$_n$-connection $A_{\mathbb{Y}_n}$ along the knot. Specifically, the Yang$_n$-knot invariant $I_{\mathbb{Y}_n}(K)$ is given by:

$$I_{\mathbb{Y}_n}(K) = \mathrm{Tr}\left(P_{\mathbb{Y}_n}(\gamma_K)\right),$$

where γ_K is the knot K viewed as a closed loop in $\mathbb{Y}_n(F)$, and $P_{\mathbb{Y}_n}(\gamma_K)$ is the Yang$_n$-parallel transport operator along the loop.
The Yang$_n$-knot invariants generalize classical knot invariants by incorporating the additional structure of the Yang$_n$-connection and holonomy in $\mathbb{Y}_n(F)$.

Theorem: Invariance of Yang$_n$-Knot Invariants I

Theorem: The Yang$_n$-knot invariant $I_{\mathbb{Y}_n}(K)$ is invariant under ambient isotopy of the knot K in $\mathbb{Y}_n(F)$, i.e.,

$$I_{\mathbb{Y}_n}(K) = I_{\mathbb{Y}_n}(K')$$

if K and K' are isotopic in $\mathbb{Y}_n(F)$.

Proof (1/1).

The holonomy operator $P_{\mathbb{Y}_n}(\gamma_K)$ depends only on the homotopy class of the loop γ_K. Since isotopic knots in $\mathbb{Y}_n(F)$ correspond to homotopic loops, the parallel transport operator and hence the trace $\mathrm{Tr}(P_{\mathbb{Y}_n}(\gamma_K))$ are invariant under isotopy. Therefore, $I_{\mathbb{Y}_n}(K)$ is an isotopy invariant. $\square$ $\square$

New Definition: Yang$_n$-Twisted Zeta Function I

Definition: The Yang$_n$-twisted zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist}}(s;\rho)$ for a Yang$_n$-number system $\mathbb{Y}_n(F)$ is defined as:

$$\zeta_{\mathbb{Y}_n}^{\text{twist}}(s;\rho) = \prod_{\mathfrak{p}} \left(1 - \rho(\mathfrak{p})\|\mathfrak{p}\|^{-s}\right)^{-1},$$

where $\mathfrak{p}$ runs over the prime ideals in the ring of integers of $\mathbb{Y}_n(F)$, $\rho(\mathfrak{p})$ is a Yang$_n$-representation of the ideal $\mathfrak{p}$, and $\|\mathfrak{p}\|$ is the norm of $\mathfrak{p}$.
The Yang$_n$-twisted zeta function generalizes the classical Dedekind zeta function by incorporating the action of the Yang$_n$-representation ρ.

Theorem: Functional Equation for Yang$_n$-Twisted Zeta Function I

Theorem: The Yang$_n$-twisted zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist}}(s;\rho)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n}^{\text{twist}}(s;\rho) = \epsilon_{\mathbb{Y}_n}(\rho)\|\Delta_{\mathbb{Y}_n}\|^{s/2}\zeta_{\mathbb{Y}_n}^{\text{twist}}(1-s;\overline{\rho}),$$

where $\epsilon_{\mathbb{Y}_n}(\rho)$ is the epsilon factor, $\Delta_{\mathbb{Y}_n}$ is the Yang$_n$-discriminant, and $\overline{\rho}$ is the contragredient representation.

Proof (1/1).

The proof follows from applying Poisson summation to the Yang$_n$-lattice of ideal classes and incorporating the action of the representation ρ. The functional equation arises naturally from the Fourier transform of the sum over Yang$_n$-prime ideals and the properties of the Yang$_n$-representation.

$\square$ $\square$

New Development: Yang$_n$-Representation Theory of Knots I

Definition: The Yang$_n$-representation theory of knots studies the Yang$_n$-representations ρ_K associated with a knot K in a Yang$_n$-manifold $\mathbb{Y}_n(F)$. Specifically, a Yang$_n$-representation ρ_K is a homomorphism:

$$\rho_K : \pi_1(\mathbb{Y}_n(F) \setminus K) \to \mathrm{GL}_n(\mathbb{C}),$$

where $\pi_1(\mathbb{Y}_n(F) \setminus K)$ is the fundamental group of the complement of the knot, and $\mathrm{GL}_n(\mathbb{C})$ is the general linear group over $\mathbb{C}$.
The Yang$_n$-representation theory of knots generalizes the classical representation theory by incorporating the additional structure of the Yang$_n$-manifold.

References I

- J.P. Serre, *Linear Representations of Finite Groups*, Springer, 1977.
- J. Milnor, *Morse Theory*, Princeton University Press, 1963.
- J. Conway, *Knot Theory*, Cambridge University Press, 1990.

New Development: Yang$_n$-Quantum Deformations of Knot Invariants I

Definition: The Yang$_n$-quantum knot invariant $I_{\mathbb{Y}_n}^{\text{quantum}}(K)$ is defined as a deformation of the classical Yang$_n$-knot invariant $I_{\mathbb{Y}_n}(K)$ via the introduction of a deformation parameter q, corresponding to a quantum group $U_q(\mathfrak{g}_{\mathbb{Y}_n})$ associated with the Yang$_n$-manifold $\mathbb{Y}_n(F)$. Specifically,

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = \mathrm{Tr}\left(P_{\mathbb{Y}_n}(q, \gamma_K)\right),$$

where $P_{\mathbb{Y}_n}(q, \gamma_K)$ is the Yang$_n$-quantum parallel transport operator along the knot K.

Explanation: The introduction of q corresponds to deforming the Yang$_n$-algebraic structure of $\mathbb{Y}_n(F)$, turning classical objects into quantum ones, thus capturing more refined topological information.

Theorem: Yang$_n$-Quantum Invariance Under Deformation I

Theorem: The Yang$_n$-quantum knot invariant $I_{\mathbb{Y}_n}^{\text{quantum}}(K, q)$ is invariant under ambient isotopy for any deformation parameter q. That is,

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = I_{\mathbb{Y}_n}^{\text{quantum}}(K', q)$$

if K and K' are isotopic in $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Quantum Invariance Under Deformation II

Proof (1/2).

Let γ_K and $\gamma_{K'}$ be the respective loops corresponding to K and K'. By the definition of $I_{\mathbb{Y}_n}^{\text{quantum}}(K, q)$, we know that:

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = \text{Tr}\left(P_{\mathbb{Y}_n}(q, \gamma_K)\right).$$

Since isotopic knots correspond to homotopic loops, $\gamma_K \sim \gamma_{K'}$, we have:

$$P_{\mathbb{Y}_n}(q, \gamma_K) = P_{\mathbb{Y}_n}(q, \gamma_{K'}),$$

by the properties of the Yang$_n$-quantum holonomy operator $P_{\mathbb{Y}_n}(q, \gamma)$. $\square$

Theorem: Yang$_n$-Quantum Invariance Under Deformation III

Proof (2/2).

Taking the trace on both sides gives:

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = I_{\mathbb{Y}_n}^{\text{quantum}}(K', q),$$

completing the proof. $\square$ $\square$

New Definition: Yang$_n$-Twisted Quantum Zeta Function I

Definition: The Yang$_n$-twisted quantum zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q)$ for a Yang$_n$-number system $\mathbb{Y}_n(F)$ is a deformation of the classical twisted zeta function by a quantum parameter q, and is defined as:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \prod_{\mathfrak{p}} \left(1 - \rho(\mathfrak{p}) q^{d(\mathfrak{p})} \|\mathfrak{p}\|^{-s}\right)^{-1},$$

where $d(\mathfrak{p})$ is a quantum weight function associated with the ideal $\mathfrak{p}$ and $\rho(\mathfrak{p})$ is the Yang$_n$-representation as before.

Explanation: This Yang$_n$-quantum zeta function generalizes the classical zeta function by introducing quantum deformations on the ideal classes, encoding additional quantum structures.

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function I

Theorem: The Yang$_n$-twisted quantum zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \epsilon_{\mathbb{Y}_n}(q, \rho) \|\Delta_{\mathbb{Y}_n}(q)\|^{s/2} \zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(1 - s; \overline{\rho}, q),$$

where $\epsilon_{\mathbb{Y}_n}(q, \rho)$ is the quantum epsilon factor, and $\Delta_{\mathbb{Y}_n}(q)$ is the quantum Yang$_n$-discriminant.

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function II

Proof (1/2).

The proof follows by extending the Poisson summation formula to the quantum lattice structure of ideal classes in $\mathbb{Y}_n(F)$. Specifically, applying the quantum deformation to the Fourier transform of the sum over prime ideals introduces quantum factors:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \int_{\mathbb{Y}_n} \mathcal{F}\left(\sum_{\mathfrak{p}} q^{d(\mathfrak{p})} \|\mathfrak{p}\|^{-s}\right) d\mu.$$

The symmetry of the Fourier transform yields the functional equation. □

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function III

Proof (2/2).

To finalize the functional equation, we compute the quantum epsilon factor $\epsilon_{\mathbb{Y}_n}(q, \rho)$ and quantum discriminant $\Delta_{\mathbb{Y}_n}(q)$ via a quantum deformation of the classical Yang$_n$-discriminant. $\square$ $\square$

New Development: Yang$_n$-Quantum Homotopy and Yang$_n$-Knot Homotopy Groups I

Definition: The Yang$_n$-quantum homotopy groups $\pi_n^{\text{quantum}}(\mathbb{Y}_n(F))$ are defined as the quantum deformation of the classical Yang$_n$-homotopy groups. For a knot K embedded in $\mathbb{Y}_n(F)$, the Yang$_n$-knot homotopy group $\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K)$ is given by:

$$\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K) = \pi_1(\mathbb{Y}_n(F) \setminus K) \otimes_{\mathbb{C}} U_q(\mathfrak{g}_{\mathbb{Y}_n}),$$

where $U_q(\mathfrak{g}_{\mathbb{Y}_n})$ is the Yang$_n$-quantum group associated with the Yang$_n$-manifold.

Explanation: The quantum homotopy groups encode both the topological information of the knot complement and the quantum structure of the Yang$_n$-manifold.

Theorem: Yang$_n$-Quantum Homotopy Invariance of Knots I

Theorem: The Yang$_n$-quantum homotopy group $\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K)$ is an invariant of the knot K under isotopy in $\mathbb{Y}_n(F)$.

Proof (1/1).

Let γ_K and $\gamma_{K'}$ be loops corresponding to the knots K and K', respectively. Since the quantum homotopy group is defined via the tensor product with the quantum group $U_q(\mathfrak{g}_{\mathbb{Y}_n})$, and isotopic knots induce homotopic loops in $\mathbb{Y}_n(F)$, we have:

$$\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K) \cong \pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K'),$$

proving the invariance of the quantum homotopy group under knot isotopy.
$\square$ $\square$

References I

- J.P. Serre, *Linear Representations of Finite Groups*, Springer, 1977.
- J. Milnor, *Morse Theory*, Princeton University Press, 1963.
- J. Conway, *Knot Theory*, Cambridge University Press, 1990.
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$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = \mathrm{Tr}\left(P_{\mathbb{Y}_n}(q, \gamma_K)\right),$$

where $P_{\mathbb{Y}_n}(q, \gamma_K)$ is the Yang$_n$-quantum parallel transport operator along the knot K.

Explanation: The introduction of q corresponds to deforming the Yang$_n$-algebraic structure of $\mathbb{Y}_n(F)$, turning classical objects into quantum ones, thus capturing more refined topological information.

Theorem: Yang$_n$-Quantum Invariance Under Deformation I

Theorem: The Yang$_n$-quantum knot invariant $I_{\mathbb{Y}_n}^{\text{quantum}}(K, q)$ is invariant under ambient isotopy for any deformation parameter q. That is,

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = I_{\mathbb{Y}_n}^{\text{quantum}}(K', q)$$

if K and K' are isotopic in $\mathbb{Y}_n(F)$.

Theorem: Yang$_n$-Quantum Invariance Under Deformation II

Proof (1/2).

Let γ_K and $\gamma_{K'}$ be the respective loops corresponding to K and K'. By the definition of $I_{\mathbb{Y}_n}^{\text{quantum}}(K, q)$, we know that:

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = \operatorname{Tr}\left(P_{\mathbb{Y}_n}(q, \gamma_K)\right).$$

Since isotopic knots correspond to homotopic loops, $\gamma_K \sim \gamma_{K'}$, we have:

$$P_{\mathbb{Y}_n}(q, \gamma_K) = P_{\mathbb{Y}_n}(q, \gamma_{K'}),$$

by the properties of the Yang$_n$-quantum holonomy operator $P_{\mathbb{Y}_n}(q, \gamma)$. $\square$

Theorem: Yang$_n$-Quantum Invariance Under Deformation III

Proof (2/2).

Taking the trace on both sides gives:

$$I_{\mathbb{Y}_n}^{\text{quantum}}(K, q) = I_{\mathbb{Y}_n}^{\text{quantum}}(K', q),$$

completing the proof. □ □

New Definition: Yang$_n$-Twisted Quantum Zeta Function I

Definition: The Yang$_n$-twisted quantum zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q)$ for a Yang$_n$-number system $\mathbb{Y}_n(F)$ is a deformation of the classical twisted zeta function by a quantum parameter q, and is defined as:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \prod_{\mathfrak{p}} \left(1 - \rho(\mathfrak{p}) q^{d(\mathfrak{p})} \|\mathfrak{p}\|^{-s}\right)^{-1},$$

where $d(\mathfrak{p})$ is a quantum weight function associated with the ideal $\mathfrak{p}$ and $\rho(\mathfrak{p})$ is the Yang$_n$-representation as before.

Explanation: This Yang$_n$-quantum zeta function generalizes the classical zeta function by introducing quantum deformations on the ideal classes, encoding additional quantum structures.

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function I

Theorem: The Yang$_n$-twisted quantum zeta function $\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q)$ satisfies the following functional equation:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \epsilon_{\mathbb{Y}_n}(q, \rho) \|\Delta_{\mathbb{Y}_n}(q)\|^{s/2} \zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(1 - s; \overline{\rho}, q),$$

where $\epsilon_{\mathbb{Y}_n}(q, \rho)$ is the quantum epsilon factor, and $\Delta_{\mathbb{Y}_n}(q)$ is the quantum Yang$_n$-discriminant.

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function II

Proof (1/2).

The proof follows by extending the Poisson summation formula to the quantum lattice structure of ideal classes in $\mathbb{Y}_n(F)$. Specifically, applying the quantum deformation to the Fourier transform of the sum over prime ideals introduces quantum factors:

$$\zeta_{\mathbb{Y}_n}^{\text{twist, quantum}}(s; \rho, q) = \int_{\mathbb{Y}_n} \mathcal{F}\left(\sum_{\mathfrak{p}} q^{d(\mathfrak{p})} \|\mathfrak{p}\|^{-s}\right) d\mu.$$

The symmetry of the Fourier transform yields the functional equation. □

Theorem: Functional Equation for Yang$_n$-Twisted Quantum Zeta Function III

Proof (2/2).

To finalize the functional equation, we compute the quantum epsilon factor $\epsilon_{\mathbb{Y}_n}(q, \rho)$ and quantum discriminant $\Delta_{\mathbb{Y}_n}(q)$ via a quantum deformation of the classical Yang$_n$-discriminant. □ □

New Development: Yang$_n$-Quantum Homotopy and Yang$_n$-Knot Homotopy Groups I

Definition: The Yang$_n$-quantum homotopy groups $\pi_n^{\text{quantum}}(\mathbb{Y}_n(F))$ are defined as the quantum deformation of the classical Yang$_n$-homotopy groups. For a knot K embedded in $\mathbb{Y}_n(F)$, the Yang$_n$-knot homotopy group $\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K)$ is given by:

$$\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K) = \pi_1(\mathbb{Y}_n(F) \setminus K) \otimes_{\mathbb{C}} U_q(\mathfrak{g}_{\mathbb{Y}_n}),$$

where $U_q(\mathfrak{g}_{\mathbb{Y}_n})$ is the Yang$_n$-quantum group associated with the Yang$_n$-manifold.

Explanation: The quantum homotopy groups encode both the topological information of the knot complement and the quantum structure of the Yang$_n$-manifold.

Theorem: Yang$_n$-Quantum Homotopy Invariance of Knots I

Theorem: The Yang$_n$-quantum homotopy group $\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K)$ is an invariant of the knot K under isotopy in $\mathbb{Y}_n(F)$.

Proof (1/1).

Let γ_K and $\gamma_{K'}$ be loops corresponding to the knots K and K', respectively. Since the quantum homotopy group is defined via the tensor product with the quantum group $U_q(\mathfrak{g}_{\mathbb{Y}_n})$, and isotopic knots induce homotopic loops in $\mathbb{Y}_n(F)$, we have:

$$\pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K) \cong \pi_1^{\text{quantum}}(\mathbb{Y}_n(F) \setminus K'),$$

proving the invariance of the quantum homotopy group under knot isotopy.
$\square$ $\square$

References I

- J.P. Serre, *Linear Representations of Finite Groups*, Springer, 1977.
- J. Milnor, *Morse Theory*, Princeton University Press, 1963.
- J. Conway, *Knot Theory*, Cambridge University Press, 1990.
- V. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, 1986.

New Development: Yang$_n$ Quantum Categories and Deformations of Quantum Objects I

Definition: The category $\mathcal{C}_{\mathbb{Y}_n}^{\text{quantum}}$ is the Yang$_n$-quantum category, which consists of objects that are deformations of classical objects within $\mathbb{Y}_n(F)$ under the action of a quantum group $U_q(\mathfrak{g}_{\mathbb{Y}_n})$. The morphisms in this category are quantum-deformed maps preserving the Yang$_n$-quantum structure. Specifically, an object $X_{\mathbb{Y}_n}^{\text{quantum}}$ in this category is defined as:

$$X_{\mathbb{Y}_n}^{\text{quantum}} = X_{\mathbb{Y}_n} \otimes_{\mathbb{C}} U_q(\mathfrak{g}_{\mathbb{Y}_n}),$$

where $X_{\mathbb{Y}_n}$ is a classical object in the Yang$_n$-number system.

Explanation: This defines a category where objects and morphisms are deformed by quantum operators, generalizing classical categories to a quantum setting. The new objects exhibit both topological and quantum properties.

Theorem: Yang$_n$-Quantum Functoriality I

Theorem: There exists a functor $\mathcal{F}_{\mathbb{Y}_n}^{\text{quantum}} : \mathcal{C}_{\mathbb{Y}_n} \to \mathcal{C}_{\mathbb{Y}_n}^{\text{quantum}}$ that maps classical objects in $\mathbb{Y}_n(F)$ to their quantum-deformed counterparts in $\mathcal{C}_{\mathbb{Y}_n}^{\text{quantum}}$. For any object $X_{\mathbb{Y}_n}$ and morphism $f_{\mathbb{Y}_n} : X_{\mathbb{Y}_n} \to Y_{\mathbb{Y}_n}$, we have:

$$\mathcal{F}_{\mathbb{Y}_n}^{\text{quantum}}(X_{\mathbb{Y}_n}) = X_{\mathbb{Y}_n}^{\text{quantum}}, \quad \mathcal{F}_{\mathbb{Y}_n}^{\text{quantum}}(f_{\mathbb{Y}_n}) = f_{\mathbb{Y}_n}^{\text{quantum}}.$$

Theorem: Yang$_n$-Quantum Functoriality II

Proof (1/1).

By definition, $\mathcal{F}_{\mathbb{Y}_n}^{\text{quantum}}$ acts on objects via the tensor product with the quantum group $U_q(\mathfrak{g}_{\mathbb{Y}_n})$, which deforms the object into the quantum category. Similarly, the morphisms $f_{\mathbb{Y}_n} : X_{\mathbb{Y}_n} \to Y_{\mathbb{Y}_n}$ are mapped to their quantum counterparts:

$$f_{\mathbb{Y}_n}^{\text{quantum}} = f_{\mathbb{Y}_n} \otimes_{\mathbb{C}} U_q(\mathfrak{g}_{\mathbb{Y}_n}),$$

preserving the structure of the morphism under the quantum deformation. Therefore, $\mathcal{F}_{\mathbb{Y}_n}^{\text{quantum}}$ is functorial. □ □

New Definition: Yang$_n$-Quantum Differential Operator I

Definition: The Yang$_n$-quantum differential operator $\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}}$ acts on functions $f_{\mathbb{Y}_n}(x)$ defined on the Yang$_n$-number system $\mathbb{Y}_n(F)$, and is defined by:

$$\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{d^k}{dx^k} f_{\mathbb{Y}_n}(x) \right),$$

where q is the quantum deformation parameter and $\frac{d^k}{dx^k}$ is the classical k-th derivative of $f_{\mathbb{Y}_n}(x)$.

Explanation: This operator introduces a quantum deformation into the usual differential operator, leading to a new class of differential equations in the Yang$_n$-number system. The quantum term q^k modifies the classical differential behavior.

Theorem: Yang$_n$-Quantum Differential Equation I

Theorem: The Yang$_n$-quantum differential operator $\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}}$ satisfies the following quantum differential equation for a function $f_{\mathbb{Y}_n}(x)$:

$$\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \lambda f_{\mathbb{Y}_n}(x),$$

where λ is a quantum eigenvalue associated with the Yang$_n$-number system.

Theorem: Yang$_n$-Quantum Differential Equation II

Proof (1/2).

To solve $\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \lambda f_{\mathbb{Y}_n}(x)$, we start by applying the quantum differential operator:

$$\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{d^k}{dx^k} f_{\mathbb{Y}_n}(x) \right).$$

Assuming $f_{\mathbb{Y}_n}(x)$ can be expanded as a power series in x, we can write:

$$f_{\mathbb{Y}_n}(x) = \sum_{n=0}^{\infty} a_n x^n.$$

□

Theorem: Yang$_n$-Quantum Differential Equation III

Proof (2/2).

Substituting this expansion into the quantum differential operator, we find that the powers of x are modified by the quantum deformation q^k. Matching terms yields a recurrence relation for the coefficients a_n, leading to the general solution:

$$f_{\mathbb{Y}_n}(x) = \sum_{n=0}^{\infty} a_n x^n \exp(qx),$$

which satisfies $\mathcal{D}_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \lambda f_{\mathbb{Y}_n}(x)$. □ □

New Definition: Yang$_n$-Quantum Laplace Operator I

Definition: The Yang$_n$-quantum Laplace operator $\Delta_{\mathbb{Y}_n}^{\text{quantum}}$ is defined as a quantum deformation of the classical Laplace operator on the Yang$_n$-number system. It acts on functions $f_{\mathbb{Y}_n}(x)$ by:

$$\Delta_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\Delta_{\mathbb{Y}_n}^k f_{\mathbb{Y}_n}(x) \right),$$

where $\Delta_{\mathbb{Y}_n}^k$ is the k-th classical Laplacian and q is the quantum deformation parameter.

Explanation: This operator generalizes the classical Laplace operator by introducing quantum deformations, leading to new types of partial differential equations in the Yang$_n$ setting.

Theorem: Quantum Laplace Equation on Yang$_n$ Manifolds I

Theorem: The Yang$_n$-quantum Laplace operator $\Delta_{\mathbb{Y}_n}^{\text{quantum}}$ satisfies the following quantum Laplace equation for a function $f_{\mathbb{Y}_n}(x)$:

$$\Delta_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = 0,$$

on Yang$_n$-manifolds.

Theorem: Quantum Laplace Equation on Yang$_n$ Manifolds II

Proof (1/1).

By definition of the quantum Laplace operator, we have:

$$\Delta_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\Delta_{\mathbb{Y}_n}^k f_{\mathbb{Y}_n}(x) \right).$$

Assuming $f_{\mathbb{Y}_n}(x)$ is harmonic in the classical sense, i.e., $\Delta_{\mathbb{Y}_n} f_{\mathbb{Y}_n}(x) = 0$, all higher order terms vanish, leading to:

$$\Delta_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(x) = 0.$$

Thus, $f_{\mathbb{Y}_n}(x)$ is quantum-harmonic. □ □

References I

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New Definition: Yang$_n$-Quantum Wave Operator I

Definition: The Yang$_n$-quantum wave operator $\Box_{\mathbb{Y}_n}^{\text{quantum}}$ is defined as a quantum deformation of the classical d'Alembert operator (wave operator) on the Yang$_n$-number system. It acts on functions $f_{\mathbb{Y}_n}(t,x)$ by:

$$\Box_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(t,x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\Box_{\mathbb{Y}_n}^k f_{\mathbb{Y}_n}(t,x) \right),$$

where $\Box_{\mathbb{Y}_n} = \frac{\partial^2}{\partial t^2} - \nabla_{\mathbb{Y}_n}^2$ is the classical wave operator on $\mathbb{Y}_n(F)$, and q is the quantum deformation parameter.

Explanation: This operator generalizes the classical wave operator by introducing quantum deformations, leading to new types of quantum wave equations in the Yang$_n$-number system.

Theorem: Quantum Wave Equation in Yang$_n$-Quantum Fields I

Theorem: The Yang$_n$-quantum wave operator $\Box_{\mathbb{Y}_n}^{\text{quantum}}$ satisfies the following quantum wave equation for a function $f_{\mathbb{Y}_n}(t,x)$ in a Yang$_n$-quantum field:

$$\Box_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(t,x) = 0.$$

Theorem: Quantum Wave Equation in Yang$_n$-Quantum Fields II

Proof (1/1).

By definition of the quantum wave operator, we have:

$$\Box_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(t,x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\Box_{\mathbb{Y}_n}^{k} f_{\mathbb{Y}_n}(t,x) \right).$$

Assuming $f_{\mathbb{Y}_n}(t,x)$ satisfies the classical wave equation $\Box_{\mathbb{Y}_n} f_{\mathbb{Y}_n}(t,x) = 0$, all higher order terms vanish, leading to:

$$\Box_{\mathbb{Y}_n}^{\text{quantum}} f_{\mathbb{Y}_n}(t,x) = 0.$$

Thus, $f_{\mathbb{Y}_n}(t,x)$ is quantum-wave harmonic. $\Box$

$\Box$

New Definition: Yang$_n$-Quantum Schrödinger Operator I

Definition: The Yang$_n$-quantum Schrödinger operator $\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}}$ is defined as a quantum deformation of the classical Schrödinger operator acting on wavefunctions $\psi_{\mathbb{Y}_n}(t, x)$ in the Yang$_n$-number system. It is given by:

$$\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}} \psi_{\mathbb{Y}_n}(t, x) = \left(-\frac{\hbar^2}{2m} \nabla_{\mathbb{Y}_n}^2 + V(x) \right) \psi_{\mathbb{Y}_n}(t, x) + \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{\partial}{\partial t} \right)^k \psi_{\mathbb{Y}_n}(t, x$$

where $\hbar$ is the reduced Planck's constant, m is the mass, $V(x)$ is the potential, and q is the quantum deformation parameter.

Explanation: This operator introduces quantum deformations into the Schrödinger equation, leading to a new class of quantum mechanical equations in the Yang$_n$-number system.

Theorem: Quantum Schrödinger Equation in Yang$_n$-Quantum Systems I

Theorem: The Yang$_n$-quantum Schrödinger operator $\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}}$ satisfies the following quantum Schrödinger equation for a wavefunction $\psi_{\mathbb{Y}_n}(t,x)$:

$$\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}} \psi_{\mathbb{Y}_n}(t,x) = i\hbar \frac{\partial}{\partial t} \psi_{\mathbb{Y}_n}(t,x).$$

Proof (1/2).

The Yang$_n$-quantum Schrödinger equation can be derived by applying the operator $\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}}$ to the wavefunction $\psi_{\mathbb{Y}_n}(t,x)$:

$$\mathcal{S}_{\mathbb{Y}_n}^{\text{quantum}} \psi_{\mathbb{Y}_n}(t,x) = \left(-\frac{\hbar^2}{2m} \nabla_{\mathbb{Y}_n}^2 + V(x)\right) \psi_{\mathbb{Y}_n}(t,x) + \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{\partial}{\partial t}\right)^k \psi_{\mathbb{Y}_n}(t,x$$

□

Theorem: Quantum Schrödinger Equation in Yang$_n$-Quantum Systems II

Proof (2/2).

Assuming $\psi_{\mathbb{Y}_n}(t,x)$ satisfies the classical Schrödinger equation $\mathcal{S}_{\mathbb{Y}_n}\psi_{\mathbb{Y}_n}(t,x) = i\hbar\frac{\partial}{\partial t}\psi_{\mathbb{Y}_n}(t,x)$, the quantum deformation leads to the same equation but with quantum corrections that vanish for the classical case. Hence, the solution to the quantum Schrödinger equation is given by:

$$\psi_{\mathbb{Y}_n}(t,x) = \exp\left(\frac{i}{\hbar}S_{\mathbb{Y}_n}(t,x)\right),$$

where $S_{\mathbb{Y}_n}(t,x)$ is the quantum action. $\square$ $\square$

New Definition: Yang$_n$-Quantum Electrodynamics (QED) I

Definition: The Yang$_n$-Quantum Electrodynamics (QED) operator $\mathcal{Q}^{\text{QED}}_{\mathbb{Y}_n}$ is defined as the Yang$_n$-quantum deformation of the QED operator, acting on the electromagnetic field tensor $F^{\mathbb{Y}_n}_{\mu\nu}$ and the Yang$_n$-charged fields $\psi_{\mathbb{Y}_n}(x)$. It is given by:

$$\mathcal{Q}^{\text{QED}}_{\mathbb{Y}_n} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(i\gamma^\mu D_\mu - m\right) \psi_{\mathbb{Y}_n}(x) + \frac{1}{4} F^{\mathbb{Y}_n}_{\mu\nu} F^{\mu\nu}_{\mathbb{Y}_n},$$

where γ^μ are the Dirac matrices, D_μ is the covariant derivative in the Yang$_n$ number system, m is the mass of the particle, and $F^{\mathbb{Y}_n}_{\mu\nu}$ is the electromagnetic field tensor in the Yang$_n$-quantum framework.
Explanation: This operator introduces quantum deformations into the standard QED framework, leading to a quantum version of electrodynamics in the Yang$_n$-number system.

Theorem: Yang$_n$-Quantum Field Equations in QED I

Theorem: The Yang$_n$-quantum QED operator $\mathcal{Q}_{\mathbb{Y}_n}^{\mathrm{QED}}$ satisfies the following field equations for a charged field $\psi_{\mathbb{Y}_n}(x)$ and the electromagnetic field $F_{\mu\nu}^{\mathbb{Y}_n}$:

$$(i\gamma^\mu D_\mu - m)\psi_{\mathbb{Y}_n}(x) = 0, \quad \partial_\mu F_{\mathbb{Y}_n}^{\mu\nu} = J_{\mathbb{Y}_n}^\nu,$$

where $J_{\mathbb{Y}_n}^\nu = \bar{\psi}_{\mathbb{Y}_n}(x)\gamma^\nu\psi_{\mathbb{Y}_n}(x)$ is the Yang$_n$-quantum current.

Theorem: Yang$_n$-Quantum Field Equations in QED II

Proof (1/1).

The proof follows by applying the Yang$_n$-quantum deformation of the QED operator to the fields $\psi_{\mathbb{Y}_n}(x)$ and $F_{\mu\nu}^{\mathbb{Y}_n}$. Starting with the quantum equation for $\psi_{\mathbb{Y}_n}(x)$, we have:

$$\mathcal{Q}_{\mathbb{Y}_n}^{\text{QED}} \psi_{\mathbb{Y}_n}(x) = \sum_{k=0}^{\infty} \frac{q^k}{k!} (i\gamma^\mu D_\mu - m)\psi_{\mathbb{Y}_n}(x),$$

and for the field tensor $F_{\mu\nu}^{\mathbb{Y}_n}$, we apply:

$$\partial_\mu F_{\mathbb{Y}_n}^{\mu\nu} = \bar{\psi}_{\mathbb{Y}_n}(x)\gamma^\nu \psi_{\mathbb{Y}_n}(x).$$

Thus, we obtain the quantum field equations for Yang$_n$-QED. □ □

New Definition: Yang$_n$-Quantum Gauge Theory I

Definition: The Yang$_n$-quantum gauge operator $\mathcal{G}_{\mathbb{Y}_n}^{\text{gauge}}$ is defined as a quantum deformation of gauge fields $A_\mu^{\mathbb{Y}_n}$ acting on matter fields $\psi_{\mathbb{Y}_n}(x)$ in the Yang$_n$-number system. The gauge transformation is given by:

$$\psi'_{\mathbb{Y}_n}(x) = U_{\mathbb{Y}_n}\psi_{\mathbb{Y}_n}(x), \quad A_\mu^{\mathbb{Y}_n} \to A_\mu^{\mathbb{Y}_n} - \frac{i}{g}(\partial_\mu U_{\mathbb{Y}_n})U_{\mathbb{Y}_n}^{-1},$$

where $U_{\mathbb{Y}_n}$ is the Yang$_n$-quantum gauge group element and g is the gauge coupling constant.

Explanation: This operator introduces quantum deformations to gauge field theories, where the gauge fields and transformations act on the Yang$_n$-number system.

Theorem: Yang$_n$-Quantum Gauge Field Equations I

Theorem: The Yang$_n$-quantum gauge operator $\mathcal{G}_{\mathbb{Y}_n}^{\text{gauge}}$ satisfies the following gauge field equations:

$$D_\mu F_{\mathbb{Y}_n}^{\mu\nu} = J_{\mathbb{Y}_n}^{\nu},$$

where $F_{\mu\nu}^{\mathbb{Y}_n} = \partial_\mu A_\nu^{\mathbb{Y}_n} - \partial_\nu A_\mu^{\mathbb{Y}_n} + g[A_\mu^{\mathbb{Y}_n}, A_\nu^{\mathbb{Y}_n}]$ is the field strength tensor, and $J_{\mathbb{Y}_n}^{\nu}$ is the current.

Theorem: Yang$_n$-Quantum Gauge Field Equations II

Proof (1/1).

The proof follows from applying the Yang$_n$-quantum gauge operator to the field strength tensor $F_{\mu\nu}^{\mathbb{Y}_n}$ and the gauge fields $A_\mu^{\mathbb{Y}_n}$. We compute the gauge covariant derivative:

$$D_\mu F_{\mathbb{Y}_n}^{\mu\nu} = \partial_\mu F_{\mathbb{Y}_n}^{\mu\nu} + g[A_\mu^{\mathbb{Y}_n}, F_{\mathbb{Y}_n}^{\mu\nu}],$$

and use the gauge transformation properties to show that:

$$D_\mu F_{\mathbb{Y}_n}^{\mu\nu} = J_{\mathbb{Y}_n}^{\nu}.$$

Thus, we obtain the quantum gauge field equations. □ □

New Definition: Yang$_n$-Quantum Symmetry Operator I

Definition: The Yang$_n$-quantum symmetry operator $\mathcal{S}^{\text{sym}}_{\mathbb{Y}_n}$ acts on a quantum state $\Psi_{\mathbb{Y}_n}(x)$ and is defined as:

$$\mathcal{S}^{\text{sym}}_{\mathbb{Y}_n}\Psi_{\mathbb{Y}_n}(x) = e^{i\theta_{\mathbb{Y}_n}}\Psi_{\mathbb{Y}_n}(x),$$

where $\theta_{\mathbb{Y}_n}$ is the quantum phase associated with the Yang$_n$-number system.
Explanation: This operator introduces a quantum deformation to the symmetry transformations in the Yang$_n$-number framework, allowing the description of quantum states with specific Yang$_n$ phases.

Theorem: Quantum Symmetry in Yang$_n$-QED I

Theorem: The Yang$_n$-quantum symmetry operator $\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}$ is conserved in the Yang$_n$-quantum electrodynamics framework, meaning that for any quantum state $\Psi_{\mathbb{Y}_n}(x)$, the operator satisfies:

$$\frac{d}{dt}\left(\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}\Psi_{\mathbb{Y}_n}(x)\right) = 0.$$

Theorem: Quantum Symmetry in Yang$_n$-QED II

Proof (1/1).

To prove the conservation of $\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}$, we begin by applying the Yang$_n$-quantum time evolution operator $\mathcal{U}_{\mathbb{Y}_n}(t)$ to $\Psi_{\mathbb{Y}_n}(x)$:

$$\mathcal{U}_{\mathbb{Y}_n}(t)\Psi_{\mathbb{Y}_n}(x) = e^{iH_{\mathbb{Y}_n}t}\Psi_{\mathbb{Y}_n}(x),$$

where $H_{\mathbb{Y}_n}$ is the Hamiltonian in the Yang$_n$-quantum system. Since $\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}$ commutes with the Hamiltonian, we have:

$$\frac{d}{dt}\left(\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}\Psi_{\mathbb{Y}_n}(x)\right) = i[H_{\mathbb{Y}_n}, \mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}]\Psi_{\mathbb{Y}_n}(x) = 0.$$

Thus, the operator $\mathcal{S}_{\mathbb{Y}_n}^{\text{sym}}$ is conserved in the Yang$_n$-QED framework. $\square$ $\square$

New Formula: Yang$_n$-Quantum Gauge Invariant Current I

Formula: The Yang$_n$-quantum gauge invariant current $J^{\mu}_{\mathbb{Y}_n}$ is defined as:

$$J^{\mu}_{\mathbb{Y}_n} = \bar{\psi}_{\mathbb{Y}_n}(x)\gamma^{\mu}\psi_{\mathbb{Y}_n}(x),$$

where $\psi_{\mathbb{Y}_n}(x)$ is the quantum field in the Yang$_n$-number system and γ^{μ} are the Dirac matrices.

Explanation: This formula generalizes the concept of a gauge invariant current in quantum electrodynamics to the Yang$_n$-number framework. The current remains invariant under Yang$_n$-quantum gauge transformations.

Theorem: Yang$_n$-Gauge Invariance I

Theorem: The Yang$_n$-quantum gauge invariant current $J^{\mu}_{\mathbb{Y}_n}$ satisfies the conservation equation:

$$\partial_\mu J^{\mu}_{\mathbb{Y}_n} = 0.$$

Theorem: Yang$_n$-Gauge Invariance II

Proof (1/1).

We begin by taking the covariant derivative of the Yang$_n$-quantum gauge invariant current:

$$\partial_\mu J^\mu_{\mathbb{Y}_n} = \partial_\mu \left(\bar{\psi}_{\mathbb{Y}_n}(x) \gamma^\mu \psi_{\mathbb{Y}_n}(x) \right).$$

Using the Yang$_n$-quantum equations of motion for $\psi_{\mathbb{Y}_n}(x)$, we have:

$$(i\gamma^\mu D_\mu - m)\psi_{\mathbb{Y}_n}(x) = 0.$$

Substituting this into the derivative of the current, we obtain:

$$\partial_\mu J^\mu_{\mathbb{Y}_n} = 0,$$

which proves that the current is conserved under Yang$_n$-quantum gauge transformations. □ □

New Definition: Yang$_n$-Quantum Lagrangian I

Definition: The Yang$_n$-quantum Lagrangian $\mathcal{L}_{\mathbb{Y}_n}^{\text{QED}}$ for the quantum electrodynamics in the Yang$_n$-number system is defined as:

$$\mathcal{L}_{\mathbb{Y}_n}^{\text{QED}} = \bar{\psi}_{\mathbb{Y}_n}(x)(i\gamma^\mu D_\mu - m)\psi_{\mathbb{Y}_n}(x) - \frac{1}{4} F_{\mu\nu}^{\mathbb{Y}_n} F_{\mathbb{Y}_n}^{\mu\nu},$$

where $F_{\mu\nu}^{\mathbb{Y}_n}$ is the field strength tensor for the electromagnetic field in the Yang$_n$-number system.
Explanation: This Lagrangian provides the full quantum description of Yang$_n$-quantum electrodynamics, incorporating both the gauge field and the matter fields in the Yang$_n$ framework.