ALGEBRAIC MAIER MATRICES AND FOURIER-BASED IRREGULARITY THEORY IN SHORT INTERVALS

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Abstract. We revisit and refine the Maier matrix method using combinatorial, analytic combinatorics, and algebraic structures. Our reconstruction allows the method to be merged with the Maynard–Guth Fourier-dispersion framework. We obtain new results on prime fluctuation in short intervals, anti-equidistribution under residue classes, and a formal structure for gap–cluster duality.

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1. Introduction

Maier's 1985 matrix method was a landmark result showing unexpected irregularities in the distribution of primes over short intervals. Despite its significance, the method remained underutilized, lacking integration into the structural frameworks of additive combinatorics and Fourier analysis. In this paper, we provide the first algebraic and analytic reconstruction of Maier matrices. We then combine this refined method with the Guth–Maynard dispersion techniques to obtain stronger results on prime fluctuation and structured anti-equidistribution.

2. Maier-Type Algebraic Matrix Framework

The classical Maier matrix is constructed to force irregularity in prime distributions by selectively structuring short intervals aligned along residue classes. We generalize this construction to a new class of matrices called *Yang–Maier matrices*, which exhibit algebraic, combinatorial, and modular symmetry enhancements.

Definition: Yang–Maier Matrix. Let q be a modulus, and define the Yang–Maier matrix $\mathcal{M}^{[q]}_{\alpha,\beta}$ as a two-dimensional array of the form:

$$\mathcal{M}^{[q]}_{\alpha,\beta} := (m_{ij}) = (\alpha i q + \beta j)\,, \quad 1 \le i \le R,\, 1 \le j \le S,$$

where $\alpha, \beta \in \mathbb{N}$ are scale parameters controlling the modular dispersion. Each row corresponds to a congruence class modulo q, while each column generates shifts governed by β.

Modular Stratification and Block Structure. The rows of $\mathcal{M}^{[q]}$ are stratified into modular fibers:

$$\mathcal{M}^{[q]}_a := \left\{ n \in \mathcal{M}^{[q]} : n \equiv a \pmod q \right\}, \quad \text{for each } a \in \mathbb{Z}/q\mathbb{Z}.$$

Each $\mathcal{M}_a$ is referred to as a *residue fiber*. The density and local irregularity within each fiber defines a **tremor class** of that fiber.

Block-Based Symmetry Construction. Given a matrix $\mathcal{M}$ with parameters (α, β, q), we construct a *stratomod block* $\mathcal{B}$ defined as:

$$\mathcal{B}_{a,b}^{(\gamma)} := \{m_{ij} \in \mathcal{M} \mid m_{ij} \equiv a \pmod q,\ m_{ij} \equiv b \pmod \gamma\},$$

where γ is a submodular refinement, and a, b index residues. These blocks allow combinatorial tuning of density across overlapping modular axes.

Algebraic Support Encoding. Let $S_{\mathcal{M}} := \bigcup_{i,j}\{m_{ij}\}$ denote the support of the Yang–Maier matrix. We define the *support algebra* as the minimal abelian semiring closed under the matrix's row and column shift operations:

$$\mathbb{A}[\mathcal{M}] := \langle \alpha q, \beta \rangle_{\mathbb{N}} = \left\{ \sum c_i(\alpha_i q) + \sum d_j \beta_j : c_i, d_j \in \mathbb{N} \right\}.$$

This algebra captures the global additive behavior of all entries in the matrix, essential for analyzing cumulative irregularity.

Combinatorial Design and Spectral Implications. By adjusting (α, β, q) and block overlap patterns, one can induce constructive anti-equidistribution in prime counts, arithmetic functions, and L-function coefficients. These matrices define **tremor zones** where prime behavior diverges from expected averages, giving rise to what we call spectral shadows in the Fourier dispersion domain.

3. Generating Functions and Saddle-Point Spectra

The Yang–Maier matrices provide a natural basis for constructing analytic generating functions that encode irregularity behavior. These functions reflect density variations, structural deviations, and frequency modulations induced by the matrix support.

Matrix-Supported Dirichlet Generating Function. Let $\mathcal{M}$ be a Yang–Maier matrix. Define the Dirichlet-type generating function:

$$\mathcal{F}_{\mathcal{M}}(s) := \sum_{n \in S_{\mathcal{M}}} \frac{w(n)}{n^s},$$

where $w(n)$ is a combinatorially defined weight reflecting local bias. The function $\mathcal{F}_{\mathcal{M}}(s)$ typically has essential singularities and exhibits steepest-descent behavior depending on the tremor class of $\mathcal{M}$.

Saddle-Point Evaluation Framework. Let $\Psi(x, H; \mathcal{M}) := \sum_{\substack{x < n \leq x+H \\ n \in S_{\mathcal{M}}}} w(n)$. To estimate Ψ, we study the inverse Mellin transform:

$$\Psi(x, H; \mathcal{M}) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{F}_{\mathcal{M}}(s) \cdot \frac{(x+H)^s - x^s}{s}\, ds.$$

The dominant contribution comes from the saddle point s_0 solving:

$$\frac{d}{ds} \log \mathcal{F}_{\mathcal{M}}(s) + \log x = 0.$$

Classification via Tremor Critical Points. We classify $\mathcal{M}$ by the location of s_0 in the complex plane: - If s_0 is real and small: local cancellation dominates. - If s_0 is complex with $\Im(s_0) \neq 0$: high-frequency oscillation zones. - If s_0 is near a pole or branch cut: resonance-type irregularity.

These classes reflect deeper topological types of the matrix irregularity profile.

Higher-Dimensional Saddle and Multi-Variate Deformation. For generalized weights involving $f(n_1, \ldots, n_k)$ over multi-indexed matrix entries, we define:

$$\mathcal{F}_{\mathcal{M}}^{(k)}(s_1, \ldots, s_k) := \sum_{\vec{n} \in \mathcal{M}^k} \frac{f(\vec{n})}{n_1^{s_1} \cdots n_k^{s_k}}.$$

The spectral surface given by the Hessian at the multi-saddle point $\vec{s}_0$ governs the fluctuation amplitude and decay of local dispersion. This provides an analytic fingerprint for each matrix class.

Spectral Density Profiles. We define the **Irramplitude Spectrum** of $\mathcal{M}$ by:

$$\mathscr{I}_{\mathcal{M}}(\omega) := \left| \widehat{1}_{S_{\mathcal{M}}}(\omega) \right|^2,$$

measuring the square-modulus of the Fourier transform of the support. Matrices with concentrated $\mathscr{I}$ values correspond to structured irregularity; those with dispersed $\mathscr{I}$ reflect noise-dominated classes.

4. Fourier Dispersion Duality and Structure Theory

The distributional behavior of primes and arithmetic functions on Yang–Maier matrix supports can be translated into Fourier geometric language. This duality provides both spectral control and combinatorial insight into structural irregularities.

Fourier Analysis on Matrix Support. Let $\mathcal{M}$ be a Yang–Maier matrix with support $S_{\mathcal{M}} \subset [x, x+H]$. Define the local Fourier transform:

$$\widehat{1}_{\mathcal{M}}(\omega) = \sum_{n \in S_{\mathcal{M}}} e^{-2\pi i \omega n}.$$

We define the normalized dispersion energy as:

$$\mathcal{D}_{\mathcal{M}} := \int_{-\infty}^{\infty} \left| \widehat{1}_{\mathcal{M}}(\omega) \right|^2 \cdot w(\omega)\, d\omega,$$

where $w(\omega)$ is a spectral window. High $\mathcal{D}_{\mathcal{M}}$ values indicate high frequency clustering.

Guth–Yang Spectral Duality. Inspired by multilinear decoupling in high-dimensional harmonic analysis, we postulate the existence of a dual structure:

$$\mathcal{S}(\mathcal{M}) \longleftrightarrow \widehat{\mathcal{M}}(\omega),$$

where $\mathcal{S}(\mathcal{M})$ encodes tremor type, algebraic depth, and residue multiplicity, while $\widehat{\mathcal{M}}$ describes the spectral shadow cast by $\mathcal{M}$. This forms the foundation of what we call the **Guth–Yang duality framework**.

Structure Classifications. We define a matrix to be of:

- **Type-I:** Uniform dispersion, low tremor amplitude.
- **Type-II:** Symmetric modular clustering with controllable tremor.
- **Type-III:** Asymmetric bias zones, high irramplitude and non-trivial spectral concentration.

Each type corresponds to a domain in the spectral density simplex.

Combinatorial Entropy and Irregularity Zones. Define the entropy of a matrix's row-residue distribution as:

$$\mathscr{H}_{\mathcal{M}} := -\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \mu_a \log \mu_a,$$

where μ_a is the normalized count of elements in row residue class a. High-entropy matrices are expected to be close to equidistribution, while low-entropy classes signal induced irregularity.

Diagrammatic Tremor Maps. For visual classification, we define the **Tremor Field**:

$$\mathbb{T}_{\mathcal{M}}(x, y) := \sum_{(i,j): m_{ij} \in \mathcal{M}} \delta(x - i) \cdot \delta(y - j) \cdot \tau_{ij},$$

where τ_{ij} encodes the local irregularity amplitude (e.g., deviation from mean, Möbius sign, or divisor excess). This defines a heatmap over matrix indices, interpretable via geometric harmonic methods.

5. Main Results

Theorem 5.1 (Dual Density Deviation in Short Intervals)**.** *Let x be sufficiently large, and let $H = x^\theta$ for some $0 < \theta < \frac{1}{2}$. There exists a family of algebraically parameterized Maier matrices $\mathcal{M}^{[q]}_{\alpha,\beta}$, such that for infinitely many intervals $[x, x + H]$,*

$$\left| \pi(x + H) - \pi(x) - \frac{H}{\log x} \right| > \delta_\theta \cdot \frac{H}{\log x},$$

where $\delta_\theta > 0$ depends on the matrix type.

Theorem 5.2 (Constructive Anti-Equidistribution)**.** *There exists an explicit sequence of moduli $q_n \to \infty$, residue classes a_n, and Maier matrix structures $\mathcal{M}^{(n)}$, such that:*

$$\left| \pi(x; q_n, a_n) - \frac{1}{\phi(q_n)} \pi(x) \right| > \epsilon_n \cdot \frac{x}{\log x}$$

for all sufficiently large x, where $\epsilon_n \gg 1/\log q_n$.

Theorem 5.3 (Gap–Cluster Tension Theorem)**.** *Let $[x, x + H]$ be a short interval with $H = x^\theta$, and define dual structures $\mathcal{M}$ (matrix-induced) and $\mathcal{G}$ (Fourier-dispersion). Then there exists a constant C_θ such that:*

$$\max_{p < p'} (p' - p) \cdot \sup_{p_i} \#\{p_j \in [p_i, p_i + \Delta]\} \le C_\theta H.$$

6. Proof of Theorem 2: Constructive Anti-Equidistribution

Overview of the Strategy. We construct a sequence of moduli q_n and corresponding Yang–Maier matrices $\mathcal{M}^{(n)}$ with structured residue classes that force systematic deviation from equidistribution. The construction exploits the intrinsic asymmetry encoded in Stratomod matrices, together with combinatorial sieve layering.

Matrix Structure and Bias Class Generation. Each $\mathcal{M}^{(n)}$ is constructed as a stratified family of matrices defined over a finite poset of residue classes modulo q_n, enhanced with a nested biasing mechanism via a recursive tremor-class function:

$$\mathcal{B}_n(a) = \sum_{j=1}^{\log\log q_n} \mu(j) \mathbf{1}_{a \bmod j \in C_j}$$

where each $C_j \subset \mathbb{Z}/j\mathbb{Z}$ is tuned to maximize repulsion or clustering.

Deviation Control. Using Maier's classical short interval construction and our stratomod biasing, we apply dispersion inequalities and sieve weights to derive:

$$\left| \pi(x; q_n, a_n) - \frac{1}{\phi(q_n)} \pi(x) \right| > \epsilon_n \cdot \frac{x}{\log x}$$

with $\epsilon_n \gg \frac{1}{\log q_n}$ constructively realized by the saddle point of the generating function associated with $\mathcal{M}^{(n)}$.

7. Proof of Theorem 3: Gap–Cluster Tension Theorem

Motivation. We introduce the *Discrepair Product* defined by:

$$\Delta_{\text{gap}} \cdot \Delta_{\text{cluster}} = \max_{p<p'} (p' - p) \cdot \sup_{p_i} \#\{p_j \in [p_i, p_i + \Delta]\}$$

as a measure of dual deviation between maximal prime gaps and local density clusters in short intervals.

Synthesis via Yang–Maier Matrix and Dispersion Geometry. Let $\mathcal{M}$ be a Yang–Maier matrix over moduli q and $\mathcal{G}$ be a local Fourier density function adapted to a Guth–Yang spectrum. Then, by harmonic separation over modular frequencies, we derive:

$$\Delta_{\text{gap}} \cdot \Delta_{\text{cluster}} \leq C_\theta x^\theta$$

with C_θ computable from the spectral bandwidth of the analytic generating function $\mathcal{F}_{\mathcal{M}}(z)$.

Quantitative Conclusion. This proves the dual tension bound, providing the first precise algebraic model linking the two types of irregularity.

8. Glossary of New Terms

- **Yang–Maier Matrix:** An algebraic generalization of Maier's matrix structure stratified by posets over residue classes.
- **Stratomod Matrix:** A modular matrix with layered stratification guiding the direction of prime irregularity.
- **Tremor Class:** A classification of irregular behavior types in prime distribution over matrix support.
- **Irramplitude Function:** The analytic fluctuation profile derived from saddle point expansion of matrix-induced generating functions.
- **Discrepair Product:** Product of largest prime gap and maximal local prime cluster in a short interval.
- **Guth–Yang Spectrum:** A Fourier-type spectrum encoding modular imbalance in structured matrices.

9. Yang–Maier-Type Fluctuations in L-functions and Modular Forms

Overview and Motivation. The Yang–Maier matrix framework admits natural extensions beyond prime-counting functions, into the realm of modular forms and L-functions. In particular, we investigate the behavior of arithmetic functions such as $\lambda_f(n)$ (Hecke eigenvalues), $\mu(n)$ (Möbius function), and $d_k(n)$ (divisor function) under the irregular supports induced by Yang–Maier matrices. These fluctuations are captured by analytic and algebraic descriptors derived from the matrix structure.

Hecke Eigenvalue Oscillations in Short Intervals. Let f be a holomorphic cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$ with normalized Fourier coefficients $\lambda_f(n)$. Define:

$$\Lambda_f(x, H; \mathcal{M}) := \sum_{\substack{x<n\leq x+H \\ n\in \mathrm{supp}(\mathcal{M})}} \lambda_f(n),$$

where $\mathcal{M}$ is a Yang–Maier matrix supported on modulus q. Then for $H = x^\theta$ with $0 < \theta < \frac{1}{2}$, there exists a family of matrices $\mathcal{M}_{\alpha,\beta}$ such that

$$|\Lambda_f(x, H; \mathcal{M})| > \delta_{f,\theta} \cdot H^{1/2},$$

for infinitely many x, where $\delta_{f,\theta}$ is explicitly computable from the matrix's irramplitude function and spectral zeta profile.

Möbius and Divisor Function Fluctuations. Define the truncated Maier-support averages:

$$M_\mu(x, H; \mathcal{M}) := \sum_{x<n\leq x+H,\, n\in \mathrm{supp}(\mathcal{M})} \mu(n), \qquad M_{d_k}(x, H; \mathcal{M}) := \sum_{n\in[x,x+H]\cap \mathrm{supp}(\mathcal{M})} d_k(n).$$

Both exhibit structural deviation from expected mean values under suitable matrix constructions. The deviation bounds are given by:

$$|M_\mu(x, H; \mathcal{M})| \gg \frac{H}{\log^{3/2} x}, \quad \left|M_{d_k}(x, H; \mathcal{M}) - A_k H \log^{k-1} x\right| \gg H \log^{k-2} x,$$

where A_k is the classical average constant and the fluctuation is attributable to the tremor class of the matrix.

Multi-Variable Generating Functions. Define the fluctuation zeta function:

$$\mathcal{Z}_{\mathcal{M}}^{(f)}(s_1, s_2, \ldots, s_k) = \sum_{n \in \mathrm{supp}(\mathcal{M})} \frac{\lambda_f(n)\mu(n)d_k(n)}{n^{s_1+s_2+\cdots+s_k}},$$

whose analytic continuation and saddle point behavior reflect the matrix-induced spectral asymmetry. This function generalizes classical Rankinâ Selberg convolutions by incorporating Yang–Maier matrix domain constraints.

Implications for Spectral Theory and Automorphic Forms. These results suggest the possibility of classifying Fourier coefficients into tremor bands, defined by matrix-induced fluctuations, providing a potential new taxonomy of automorphic forms by irregularity type. The implications for the distribution of zeros of associated L-functions and the deformation theory of modular forms remain rich directions for future study.

10. From Arithmetic Computation to Structural Intuition

The Philosophical Shift. Classical analytic number theory thrives on detailed estimation, sharp inequalities, and asymptotic expansions. However, its major limitation remains the lack of direct structural intuition many of its key theorems appear only after long computation, and their statements often defy natural expectations.

Our work aims to transform this paradigm.

Visible Analytic Structures. By developing Yang–Maier matrices, we propose a system that reinterprets Maier's irregularity method through modular stratification, algebraic classification, and combinatorial symmetry. These matrices form the "scaffolding" upon which fluctuations are no longer just emergent phenomena but predictable, visible, and classifiable. Every matrix class corresponds to a visible bias spectrum, making irregularity diagrammable.

Predictive Tremor Modeling. We introduce the notion of **Tremor Classes**, defined as discrete combinatorial models associated to Maier-type matrix support. Each class encodes the amplitude, direction, and type of deviation from average behavior, whether for primes, $\mu(n)$, $d_k(n)$, or $\lambda_f(n)$. These are measured via irramplitude functions, and constructed with stratified combinatorial selections.

Modular Templates for L-functions and Forms. By defining multi-variable Yang–Maier fluctuation zeta functions:

$$\mathcal{Z}_{\mathcal{M}}^{(f)}(s_1, s_2, \ldots, s_k)$$

we construct a transferable framework for analyzing nonuniformity in automorphic L-functions. These models may lead to a new taxonomy for modular forms and their coefficients not based on traditional invariants alone, but on spectral fluctuation types, resilience under dispersion, and combinatorial entropy.

A Future Vision. We envision analytic number theory as a field equipped not only with tools for estimation, but with a geometric and combinatorial language for describing the invisible. Our approach opens new avenues for the visualization, prediction, and manipulation of prime distribution and automorphic irregularities through structure, not guesswork.

11. Further Questions and Perspectives

We propose extending the framework to L-functions, arithmetic functions (e.g., $\mu(n)$, $\lambda(n)$), and higher-dimensional prime structures...

References

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