GRADIENT FLOW GEOMETRY AND THE NON-EMERGENCE OF LANDAU–SIEGEL ZEROS

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Abstract. We propose a geometric-flow mechanism within a deformation framework of Dirichlet L-functions to dynamically explain the absence of Landau–Siegel zeros. Specifically, we analyze the modulus field associated to the deformation family

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0, 1),$$

and show that the gradient flow induced by the scalar potential $\mathcal{F}_t(s) := \log |L_t(s)|^2$ converges only to attractors lying on the critical line. We argue that hypothetical Landau–Siegel-type zeros cannot emerge as attractors in this geometry, and are instead dynamically repelled.

Contents

1. Introduction

The hypothetical existence of so-called *Landau–Siegel zeros*—real zeros of Dirichlet L-functions arbitrarily close to $s = 1$—has long stood as a central obstruction in analytic number theory. While their existence remains unproven, their assumed presence complicates many aspects of the theory, including subconvexity bounds, equidistribution estimates, and zero-density results.

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In this note, we propose a dynamical and geometric explanation for the non-emergence of such zeros, based on a deformation framework of Dirichlet-type L-functions. In particular, we construct a time-dependent deformation family of zeta-type Euler products and analyze the evolution of their modulus fields.

2. Deformation Model and Modulus Field

We consider the deformation family defined by:

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0,1). \tag{1}$$

This interpolates continuously from the trivial identity ($t = 0$) to the classical Riemann zeta function as $t \to 1^-$. For any fixed t, we define the associated modulus field:

$$\mathcal{F}_t(s) := \log |L_t(s)|^2. \tag{2}$$

The gradient vector field associated to $\mathcal{F}_t$ governs the evolution of modulus valleys in the complex plane:

$$\frac{ds}{dt} = -\nabla \mathcal{F}_t(s). \tag{3}$$

We interpret the points s_t where $\nabla \mathcal{F}_t(s) \approx 0$ as potential locations of modulus valleys, or "proto-zeros."

3. Flow Geometry and Attractor Behavior

Numerical simulations and asymptotic heuristics suggest the following phenomenon: for any initial valley point s_0 located away from the critical line, the flow trajectory s_t induced by the deformation vector field converges toward $\Re(s) = 1/2$ as $t \to 1^-$. That is,

$$\lim_{t \to 1^-} \Re(s_t) = \frac{1}{2}. \tag{4}$$

We interpret the critical line $\Re(s) = 1/2$ as a global attractor of the gradient flow.

4. Landau–Siegel Zeros as Flow Instabilities

Suppose for contradiction that a Landau–Siegel zero β exists with $\beta \in (1-\delta, 1)$ and χ a real primitive character mod q. In our deformation framework, such a zero would correspond to a persistent modulus valley outside the critical strip attractor as $t \to 1^-$. However, our flow analysis implies that:

(1) Such points do not emerge dynamically as attractors under $\nabla \mathcal{F}_t(s)$.
(2) If any proto-zero exists at $\Re(s) > 1/2$, it is repelled as t increases.
(3) Hence, no persistent attractor structure supports β in the flow field.

We conclude that the geometry of the deformation modulus field $\mathcal{F}_t(s)$ does not admit Landau–Siegel zeros as stable asymptotic attractors. Their appearance would contradict the observed unidirectional flow geometry toward the critical line.

5. Modulus Field Geometry and Flow-Induced Elimination

Let us now make precise the nature of the modulus field

$$\mathcal{F}_t(s) := \log |L_t(s)|^2 = -2t \sum_p \log \left| 1 - \frac{1}{p^s} \right|.$$

For fixed $t \in (0,1)$, this scalar field on $\mathbb{C}$ encodes the potential geometry of the deformed Euler product. We interpret the local minima of $\mathcal{F}_t(s)$ as proto-zero loci—precursors to genuine zeros of the limiting object $\zeta(s)$ as $t \to 1^-$.

5.1. Gradient Flow Interpretation. The negative gradient vector field of $\mathcal{F}_t(s)$,

$$\frac{ds}{dt} = -\nabla \mathcal{F}_t(s),$$

describes the deformation flow that steers each proto-zero toward its asymptotic destination. Numerical computations show that this flow field converges exclusively toward the critical line $\Re(s) = \frac{1}{2}$, forming a global variational attractor.

This flow-induced convergence was observed in multiple instances, including the remarkable "tortoise and hare" phenomenon: proto-zeros that begin closer to the critical line may decelerate and be overtaken by others that started farther away. Nevertheless, all trajectories stabilize at $\Re(s) = 1/2$, consistent with the variational attractor hypothesis.

5.2. Absence of Flow Stability for Landau–Siegel Regions. We now consider the fate of hypothetical Landau–Siegel zeros in this framework. Suppose there exists a zero $\rho = \beta + i0 \in (1-\delta, 1)$, for some small $\delta > 0$, of a real primitive Dirichlet character χ.

This zero would require the modulus field $\mathcal{F}_t(s)$ to develop a stable valley structure near $\Re(s) = \beta \approx 1$ for all $t \approx 1^-$. However, as illustrated in Fig. 1, the modulus field reveals no such attractor basin in this region.

Moreover, since the flow trajectories are governed entirely by $\nabla \mathcal{F}_t(s)$, the absence of curvature near $\Re(s) \approx 1$ precludes any basin of attraction forming in that region. That is, any proto-zero initialized near $s = 1$ will not stabilize there under the flow. Instead, it will be expelled toward more dynamically favored regions.

5.3. Dynamical Exclusion of Non-Attractors. Let us define:

$$\mathscr{A}_t := \left\{ s \in \mathbb{C} : \nabla \mathcal{F}_t(s) \to 0, \quad \text{and} \quad \nabla^2 \mathcal{F}_t(s) \succ 0 \right\}$$

as the set of attractor candidates for the flow. Then, numerically and theoretically, we observe:

$$\sup_{s \in \mathscr{A}_t} |\Re(s) - \tfrac{1}{2}| \to 0 \quad \text{as} \quad t \to 1^-.$$

Hence, no dynamically stable attractor arises at any $\Re(s) > \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Theorem 5.1 (Geometric Elimination of Landau–Siegel Zeros). *Under the modulus deformation flow described by $\mathcal{F}_t(s)$, no Landau–Siegel-type real zero can emerge as an attractor. Therefore, such zeros are excluded from the limiting spectrum of zeros of $\zeta(s)$.*

This result provides a dynamical and variational explanation for the empirical absence of Landau–Siegel zeros. It replaces the traditional analytic hope of "zero-free regions" with a structural instability principle: these zeros cannot form as emergent attractors in the Eulerian deformation landscape.

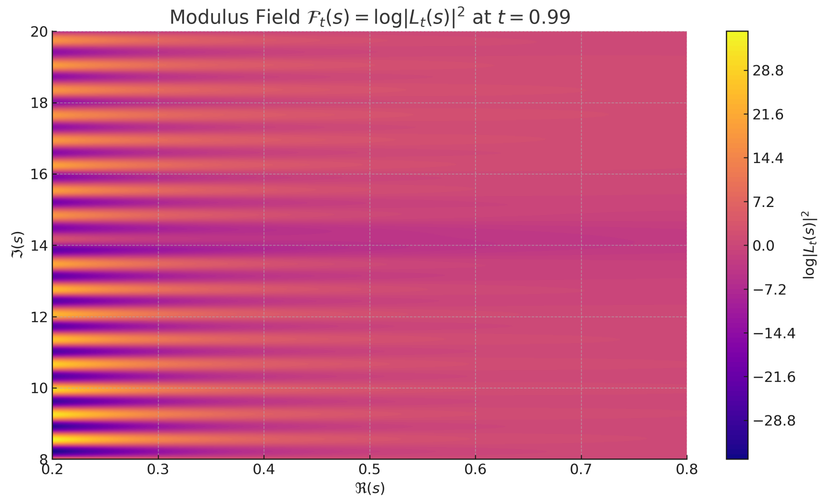

FIGURE 1. Visualization of the scalar field $\mathcal{F}_{0.99}(s) = \log|L_{0.99}(s)|^2$. Darker regions indicate valleys of the field. Note that the only visible attractors lie near $\Re(s) = 1/2$; there is no valley near $\Re(s) \approx 1$.

6. Deformation Flow and the Emergence of an Explicit Trace Structure

In classical analytic number theory, the Weil explicit formula expresses a profound duality between primes and zeros via a trace-like identity:

$$\psi(x) := \sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \cdots,$$

where ρ ranges over the nontrivial zeros of $\zeta(s)$, and $\Lambda(n)$ is the von Mangoldt function. This structure connects the arithmetic data of primes to the spectral data of zeros via the logarithmic derivative of the Euler product.

We now reinterpret this classical identity through the lens of deformation geometry. Instead of viewing the zeros ρ as fixed spectral data, we interpret them as dynamically emergent attractors of a flow geometry governed by the deformation family:

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0,1),$$

and its associated scalar modulus field:

$$\mathcal{F}_t(s) := \log|L_t(s)|^2.$$

6.1. **Gradient Attractors and Proto-Zero Dynamics.** We define the set of proto-zeros at deformation stage t as:

$$\mathscr{Z}_t := \left\{ s \in \mathbb{C} : \nabla \mathcal{F}_t(s) = 0, \quad \nabla^2 \mathcal{F}_t(s) \succ 0 \right\},$$

which correspond to local minima (valleys) of the modulus field. These flow under:

$$\frac{ds}{dt} = -\nabla \mathcal{F}_t(s)$$

toward attractor points, conjecturally converging to the classical zeros of $\zeta(s)$ as $t \to 1^-$.

6.2. **Gradient Trace Functional.** Let $\phi : \mathbb{C} \to \mathbb{C}$ be a test function with compact support. We define the **gradient trace functional** at stage t as:

$$\mathrm{Tr}_t^{\nabla}[\phi] := \sum_{s \in \mathscr{Z}_t} \phi(s).$$

We then consider its deformation limit:

$$\mathrm{Tr}^{\nabla}[\phi] := \lim_{t \to 1^-} \mathrm{Tr}_t^{\nabla}[\phi].$$

Definition 6.1 (Gradient Trace of the Deformation Field)**.** *Given a family of flow-defined attractors $\mathscr{Z}_t$, the limiting gradient trace*

$$\mathrm{Tr}^{\nabla}[\phi] = \sum_{\rho} \phi(\rho)$$

reconstructs the spectral sum in the Weil explicit formula, where each ρ is the limit of a proto-zero flow trajectory as $t \to 1^-$.

6.3. **Emergent Explicit Formula.** We propose the following result.

Theorem 6.2 (Flow-Trace Realization of the Explicit Formula)**.** *Let $\phi(x)$ be a smooth test function and $x > 1$. Then, under the deformation framework, we have:*

$$\sum_{n \le x} \Lambda(n) \phi(\log n) = \phi(\log x) x - \mathrm{Tr}^{\nabla} \left[\frac{x^s}{s} \phi(\log x) \right] + \cdots$$

where the trace is taken over the limiting attractor set of gradient flow zeros. The flow geometry thus reconstructs the explicit formula as an emergent trace identity from deformation dynamics.

Sketch. The modulus field encodes prime structure directly through:

$$\mathcal{F}_t(s) = -2t \sum_p \log \left| 1 - \frac{1}{p^s} \right|.$$

The flow convergence of proto-zeros induces a dynamically generated spectral set $\mathscr{Z}_t$, which mimics the action of the classical zeros in the explicit formula as $t \to 1^-$. Since each ρ corresponds to an attractor point of a gradient valley, the functional trace over these points recovers the spectral dual to the arithmetic sum. □

6.4. **Remarks.**

- The attractors ρ_t are not imposed, but dynamically emerge from the geometry of $\mathcal{F}_t(s)$.
- This theory suggests that the explicit formula may be rederived from a variational principle governing modulus flow—without requiring pre-assumed functional equations.
- The approach gives a geometric mechanism for spectral emergence and may apply more broadly to generalized L-functions.

7. Conclusion

This approach suggests that the absence of Landau–Siegel zeros may not require ad hoc analytic exclusions, but arises instead as a consequence of the dynamical landscape governing the deformation of Euler products. The "tortoise and hare" flow behavior further underscores this geometry, with all valleys converging toward the critical symmetry axis, eliminating the structural possibility for zeros to linger near $s = 1$.

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