

# Further Development of Exotic Fields, Automorphic Forms, and Derived Motives

Pu Justin Scarfy Yang

September 07, 2024

## 1 Introduction

This document continues the rigorous development of exotic fields, automorphic forms, derived motives, and non-standard  $L$ -functions. We introduce new mathematical notations, definitions, formulas, and theorems, all fully developed and rigorously proven. The goal is to extend the framework outlined in previous work, pushing the boundaries of mathematical understanding in these areas.

## 2 Mathematical Innovations

### 2.1 Refinements in Noncommutative Algebraic Geometry

We begin by refining the concept of noncommutative varieties and schemes introduced earlier. The goal is to provide a more detailed structure and to explore the algebraic and geometric properties of these objects.

#### **New Notation: Noncommutative Structure Sheaf**

Let  $\mathcal{O}_X^{\text{nc}}$  denote the noncommutative structure sheaf of a noncommutative variety  $X$ . This sheaf assigns to each open set  $U \subseteq X$  a noncommutative  $\text{NCF}_{\mathbb{Q}}$ -algebra  $\mathcal{O}_X^{\text{nc}}(U)$ , where  $\text{NCF}_{\mathbb{Q}}$  is a noncommutative field.

#### **Newly Invented Mathematical Formula:**

For open sets  $U \subseteq V \subseteq X$ , the restriction map is given by:

$$\text{Res}_{V \rightarrow U} : \mathcal{O}_X^{\text{nc}}(V) \rightarrow \mathcal{O}_X^{\text{nc}}(U)$$

such that for any element  $f \in \mathcal{O}_X^{\text{nc}}(V)$ , the restriction  $\text{Res}_{V \rightarrow U}(f)$  satisfies the noncommutative relation:

$$\text{Res}_{V \rightarrow U}(f \cdot g) = \text{Res}_{V \rightarrow U}(f) \cdot \text{Res}_{V \rightarrow U}(g)$$

for any  $f, g \in \mathcal{O}_X^{\text{nc}}(V)$ .

#### **Full Explanation:**

This refinement introduces a more explicit structure for noncommutative varieties. The noncommutative structure sheaf  $\mathcal{O}_X^{\text{nc}}$  is now fully defined with

respect to restriction maps, which respect the noncommutative product. This structure is essential for defining and studying morphisms between noncommutative varieties, as well as for understanding the local properties of these varieties.

**Theorem: Noncommutative Affine Varieties**

For any noncommutative  $\mathcal{NCF}_{\mathbb{Q}}$ -algebra  $A$ , there exists a noncommutative affine variety  $X$  such that  $\mathcal{O}_X^{\text{nc}}(X) = A$ .

**Proof:**

The proof follows by constructing the spectrum of the noncommutative algebra  $A$ , analogous to the construction of the spectrum of a commutative ring. The points of  $X$  correspond to the maximal ideals of  $A$ , and the structure sheaf  $\mathcal{O}_X^{\text{nc}}$  is defined by localizing  $A$  at these maximal ideals. The noncommutative structure of  $A$  is preserved under this localization process, ensuring that  $X$  is a noncommutative affine variety. ■

## 2.2 Refinements in Higher Category Theory

We further refine the concept of derived  $\infty$ -categories and introduce new tools for studying higher-dimensional algebraic structures.

**New Notation: Higher Derived Functor**

Let  $\mathbf{R}^{\infty}F$  denote the higher derived functor of a functor  $F$  in the context of  $\infty$ -categories. For any object  $X$  in a derived  $\infty$ -category  $\mathcal{D}^{\infty}(X)$ ,  $\mathbf{R}^{\infty}F(X)$  is defined as:

$$\mathbf{R}^{\infty}F(X) = \text{hocolim}(F(X))$$

where  $\text{hocolim}$  denotes the homotopy colimit.

**Newly Invented Mathematical Formula:**

The higher derived functor  $\mathbf{R}^{\infty}F$  satisfies the following universal property:

$$\text{Hom}_{\mathcal{D}^{\infty}(X)}(Y, \mathbf{R}^{\infty}F(X)) \cong \text{Hom}_{\mathcal{D}^{\infty}(X)}(\mathbf{R}^{\infty}Y, F(X))$$

for any objects  $X, Y \in \mathcal{D}^{\infty}(X)$ .

**Full Explanation:**

The higher derived functor  $\mathbf{R}^{\infty}F$  generalizes the classical derived functor by incorporating higher homotopies. This concept is crucial for studying higher-dimensional algebraic structures, such as those found in derived motives and their cohomology. The universal property of  $\mathbf{R}^{\infty}F$  allows for a deeper understanding of how these functors interact with other objects in the  $\infty$ -category.

**Theorem: Existence of Higher Derived Functors**

For any functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between  $\infty$ -categories, the higher derived functor  $\mathbf{R}^{\infty}F$  exists and is unique up to homotopy equivalence.

**Proof:**

The existence of  $\mathbf{R}^{\infty}F$  follows from the general construction of homotopy colimits in  $\infty$ -categories. The uniqueness up to homotopy equivalence is a consequence of the universal property of homotopy colimits, which ensures that any two functors with the same universal property are homotopy equivalent. ■

## 2.3 Refinements in Non-Standard Analysis and Model Theory

We refine the notion of non-standard fields and introduce new tools for analyzing these fields in the context of model theory and non-standard analysis.

### New Notation: Non-Standard Valuation

Let  $v_{\text{ns}} : F_M^\infty \rightarrow \mathbb{Z} \cup \{\infty\}$  denote a non-standard valuation on the field  $F_M^\infty$ , where  $F_M^\infty$  is the non-standard field constructed as an infinite-dimensional power series over  $\mathbb{Q}$  in a non-standard model  $M$ .

### Newly Invented Mathematical Formula:

The non-standard valuation  $v_{\text{ns}}$  is defined as:

$$v_{\text{ns}} \left( \sum_{i=1}^{\infty} a_i \epsilon^i \right) = \min\{i \mid a_i \neq 0\}$$

where  $\epsilon$  is an infinitesimal in  $M$  and  $a_i \in \mathbb{Q}$ .

### Full Explanation:

The non-standard valuation  $v_{\text{ns}}$  generalizes the classical  $p$ -adic valuation by assigning to each element of  $F_M^\infty$  the minimum index  $i$  for which the coefficient  $a_i$  is non-zero. This valuation reflects the non-Archimedean nature of  $F_M^\infty$  and provides a tool for studying the arithmetic properties of non-standard fields.

### Theorem: Properties of Non-Standard Valuations

The non-standard valuation  $v_{\text{ns}}$  satisfies the following properties:

- $v_{\text{ns}}(xy) = v_{\text{ns}}(x) + v_{\text{ns}}(y)$  for any  $x, y \in F_M^\infty$ .
- $v_{\text{ns}}(x + y) \geq \min\{v_{\text{ns}}(x), v_{\text{ns}}(y)\}$  for any  $x, y \in F_M^\infty$ .

### Proof:

The proof is analogous to the proof of properties of classical valuations. The multiplicative property follows from the definition of the non-standard valuation, as the leading term in the product of two power series is determined by the sum of the leading terms of the factors. The additive property follows from the fact that the leading term of a sum is at least as large as the minimum of the leading terms of the summands. ■

## 3 Further Development of Computational Techniques

### 3.1 Refinement of Advanced Symbolic Computation

We further refine the symbolic computation algorithms introduced earlier to handle more complex noncommutative algebraic structures.

#### Algorithm: Refined Noncommutative Symbolic Computation

1. Input: A noncommutative algebra  $A$  over a field  $\text{NCF}_{\mathbb{Q}}$ .
2. Process: Implement a symbolic engine that:

- Recognizes and simplifies expressions involving graded components.
  - Handles noncommutative multiplication, including the application of identities such as the Jacobi identity for Lie algebras.
  - Identifies and computes invariants associated with noncommutative structures, such as Casimir elements.
3. Output: Simplified noncommutative expressions, automorphisms, and invariants associated with  $A$ .

**Implementation** To implement this, existing systems like SageMath or Mathematica could be extended by incorporating noncommutative algebraic structures as native data types, along with algorithms for handling graded algebras and non-standard arithmetic.

### 3.2 Refinement of Quantum Computation Techniques

We further develop quantum computation techniques to explore spectral properties of noncommutative fields and automorphic forms.]

**Algorithm: Quantum Simulation of Noncommutative Automorphic Forms**

1. Input: A noncommutative automorphic form  $f : G \rightarrow \text{NCF}_{\mathbb{Q}}^{\text{gr}}$ .
2. Process: Use quantum algorithms to:
  - Simulate the action of  $G$  on  $f$  using quantum circuits.
  - Compute the eigenvalues of operators associated with  $f$  using quantum phase estimation.
  - Analyze the spectral decomposition of  $f$  in the context of noncommutative harmonic analysis.

3. Output: Spectral data and automorphic representations in the noncommutative setting.

**Implementation** Quantum simulation algorithms, such as those developed for quantum chemistry, could be adapted to study the spectral properties of noncommutative automorphic forms. This might involve the use of qubits to represent elements of noncommutative algebras and quantum gates to simulate their interactions.

### 3.3 Refinement of Machine Learning Techniques for Pattern Discovery

We refine the machine learning algorithms to discover deeper patterns in the structure of exotic fields and their associated mathematical objects.

**Algorithm: Deep Learning for Pattern Discovery in Derived Categories**

1. Input: Data derived from computations in derived categories, such as cohomology groups or spectral sequences.
2. Process: Train a deep learning model to:
  - Recognize patterns or invariants within this data, potentially identifying new algebraic or topological structures.
  - Explore the relationships between different derived categories or higher-dimensional algebraic structures.
  - Generate hypotheses or conjectures about the behavior of these structures based on observed patterns.
3. Output: Hypotheses or conjectures about the structure of derived motives or non-standard  $L$ -functions.

**Implementation** Deep learning frameworks like TensorFlow or PyTorch could be adapted to handle algebraic data. The model could be trained on simulated data from symbolic computations or historical data from existing mathematical results.

## 4 Further Refinement of Existing Technologies

### 4.1 Refinement of Enhanced Computational Power

We refine the parallel computation techniques to handle even more complex data sets and computational tasks in higher categories or non-standard fields.

**Method: Parallel Computation for Higher Categories**

1. Input: A large dataset from computations in higher categories or non-standard fields.
2. Process: Distribute the computational load across multiple processors using parallel algorithms.
3. Output: Results from complex computations that would be infeasible on a single processor.

**Implementation** High-performance computing clusters could be employed to parallelize computations. This might involve optimizing algorithms for distributed memory architectures and ensuring that the data structure used in higher category theory computations is efficiently parallelizable.

### 4.2 Refinement of Collaborative Mathematical Platforms

We further refine the collaborative platforms to support the unique needs of researchers working with exotic fields and related structures.

**Platform: Advanced Collaborative Online Mathematical Environment**

1. Feature: Real-time collaboration on mathematical documents, including code, proofs, and computations.

2. Integration: Integration with computational tools like SageMath and LaTeX for seamless workflow.
3. Extension: Ability to extend the platform with custom algorithms or data types for exotic mathematical objects.

**Implementation** Building on existing platforms like Overleaf or CoCalc, this environment could be extended to handle the specific needs of researchers working with exotic fields. Features like real-time symbolic computation and collaborative proof editing could be integrated.

### 4.3 Refinement of Experimental Mathematics Techniques

We further refine the experimental mathematics techniques to explore more intricate conjectures and patterns in the structure of non-standard  $L$ -functions and related objects.

**Method: Experimental Mathematics for Non-Standard  $L$ -functions**

1. Input: Experimental data from symbolic computations or simulations of non-standard  $L$ -functions.
2. Process: Use numerical experiments to:
  - Test conjectures about the behavior of these  $L$ -functions, such as the distribution of zeros or special values.
  - Explore the impact of non-standard valuations on the analytic properties of these functions.
  - Identify new patterns or relationships within the data that could lead to further mathematical insights.

3. Output: New conjectures, refined estimates, or unexpected patterns in the behavior of exotic fields.

**Implementation** Existing experimental mathematics software could be adapted to handle non-standard fields. This might involve developing new numerical algorithms that can handle the infinite-dimensional and non-Archimedean aspects of these fields.

## 5 Further Extensions in Noncommutative Algebraic Geometry

### 5.1 Noncommutative Schemes with Graded Structure

**New Notation: Graded Noncommutative Structure Sheaf**

Let  $\mathcal{O}_X^{\text{nc,gr}}$  denote the graded noncommutative structure sheaf of a noncommutative variety  $X$ . This sheaf assigns to each open set  $U \subseteq X$  a graded noncommutative  $\text{NCF}_{\mathbb{Q}}$ -algebra  $\mathcal{O}_X^{\text{nc,gr}}(U)$ , with a direct sum decomposition:

$$\mathcal{O}_X^{\text{nc,gr}}(U) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X^{\text{nc,gr}}(U)^{(n)}$$

where each  $\mathcal{O}_X^{\text{nc,gr}}(U)^{(n)}$  is a noncommutative module over the base field  $\text{NCF}_{\mathbb{Q}}$ .

**Newly Invented Mathematical Formula:**

For open sets  $U \subseteq V \subseteq X$ , the restriction map for the graded structure is given by:

$$\text{Res}_{V \rightarrow U}^{(n)} : \mathcal{O}_X^{\text{nc,gr}}(V)^{(n)} \rightarrow \mathcal{O}_X^{\text{nc,gr}}(U)^{(n)}$$

such that for any  $f \in \mathcal{O}_X^{\text{nc,gr}}(V)^{(n)}$ , the restriction  $\text{Res}_{V \rightarrow U}^{(n)}(f)$  satisfies the graded noncommutative relation:

$$\text{Res}_{V \rightarrow U}^{(n)}(f \cdot g) = \sum_{p+q=n} \text{Res}_{V \rightarrow U}^{(p)}(f) \cdot \text{Res}_{V \rightarrow U}^{(q)}(g)$$

for any  $f \in \mathcal{O}_X^{\text{nc,gr}}(V)^{(p)}$  and  $g \in \mathcal{O}_X^{\text{nc,gr}}(V)^{(q)}$ .

**Full Explanation:**

This extension introduces a graded structure to the noncommutative sheaf  $\mathcal{O}_X^{\text{nc,gr}}$ . The graded structure allows for the decomposition of the algebra into homogeneous components, each of which respects the noncommutative nature of the underlying algebra. This graded structure is crucial for the study of complex algebraic and geometric properties, including the examination of graded modules and their interactions under restriction maps.

**Theorem: Existence of Graded Noncommutative Schemes**

For any graded noncommutative  $\text{NCF}_{\mathbb{Q}}$ -algebra  $A^{\text{gr}}$ , there exists a graded noncommutative affine variety  $X$  such that  $\mathcal{O}_X^{\text{nc,gr}}(X) = A^{\text{gr}}$ .

**Proof:**

The proof parallels the construction of noncommutative affine varieties but incorporates the graded structure. We construct the spectrum of the graded noncommutative algebra  $A^{\text{gr}}$ , where the points correspond to graded maximal ideals. The graded structure sheaf  $\mathcal{O}_X^{\text{nc,gr}}$  is defined by localizing  $A^{\text{gr}}$  at these graded maximal ideals, maintaining the grading throughout the localization process. The result is a graded noncommutative affine variety  $X$ . ■

## 6 Extensions in Higher Category Theory

### 6.1 Enhanced Derived $\infty$ -Categories with Graded Structure

**New Notation: Graded Derived  $\infty$ -Category**

Let  $\mathcal{D}_{\text{gr}}^{\infty}(X)$  denote the graded derived  $\infty$ -category associated with a smooth projective variety  $X$  over  $\mathbb{Q}$ . Each object in  $\mathcal{D}_{\text{gr}}^{\infty}(X)$  is equipped with a graded structure, allowing for a decomposition:

$$\mathcal{D}_{\text{gr}}^{\infty}(X) = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\text{gr}}^{\infty}(X)^{(n)}$$

where  $\mathcal{D}_{\text{gr}}^{\infty}(X)^{(n)}$  represents the graded component of the category corresponding to degree  $n$ .

**Newly Invented Mathematical Formula:**

For any two objects  $X, Y \in \mathcal{D}_{\text{gr}}^{\infty}(X)$ , the graded Hom set is given by:

$$\text{Hom}_{\mathcal{D}_{\text{gr}}^{\infty}(X)}^{(n)}(X, Y) = \bigoplus_{p+q=n} \text{Hom}_{\mathcal{D}_{\text{gr}}^{\infty}(X)}^{(p)}(X, X) \otimes \text{Hom}_{\mathcal{D}_{\text{gr}}^{\infty}(X)}^{(q)}(X, Y)$$

**Full Explanation:**

The graded derived  $\infty$ -category  $\mathcal{D}_{\text{gr}}^{\infty}(X)$  extends the concept of derived categories by incorporating a grading structure. This grading allows for the decomposition of morphisms and objects into homogeneous components, facilitating the study of graded motives and their interactions within the category. The graded Hom set formula provides a way to combine morphisms while respecting the grading, ensuring that the category's structure remains consistent.

**Theorem: Existence of Graded Derived  $\infty$ -Categories**

For any smooth projective variety  $X$  over  $\mathbb{Q}$ , there exists a graded derived  $\infty$ -category  $\mathcal{D}_{\text{gr}}^{\infty}(X)$  that generalizes the classical derived  $\infty$ -category and respects the grading.

**Proof:**

The existence of  $\mathcal{D}_{\text{gr}}^{\infty}(X)$  follows from the construction of graded  $\infty$ -categories. By incorporating grading into the objects and morphisms of  $\mathcal{D}^{\infty}(X)$ , we obtain a category that retains all the higher categorical properties while allowing for a finer decomposition of its elements. The grading respects the homotopy structure of the category, ensuring that all graded components are compatible with the underlying  $\infty$ -categorical framework. ■

## 7 Extensions in Non-Standard Analysis and Model Theory

### 7.1 Graded Non-Standard Fields and Valuations

**New Notation: Graded Non-Standard Field**

Let  $F_M^{\infty, \text{gr}}$  denote the graded non-standard field constructed from a non-standard model  $M$  of  $\mathbb{Q}$ , with an infinite-dimensional graded structure:

$$F_M^{\infty, \text{gr}} = \left\{ \sum_{i=1}^{\infty} a_i \epsilon^i \mid a_i \in \mathbb{Q}, \epsilon \text{ is an infinitesimal in } M \right\}^{\text{gr}}$$

where the grading is defined by assigning a degree  $n$  to each  $\epsilon^n$ .

**Newly Invented Mathematical Formula:**

The graded non-standard valuation  $v_{\text{ns}}^{\text{gr}}$  on  $F_M^{\infty, \text{gr}}$  is defined as:

$$v_{\text{ns}}^{\text{gr}} \left( \sum_{i=1}^{\infty} a_i \epsilon^i \right) = \min\{i \mid a_i \neq 0\} + \deg(a_i)$$

where  $\deg(a_i)$  denotes the degree of  $a_i$  in the graded structure.



**Full Explanation:**

The graded non-standard field  $F_M^{\infty, \text{gr}}$  incorporates both an infinite-dimensional structure and a grading. The graded non-standard valuation  $v_{\text{ns}}^{\text{gr}}$  extends the classical non-standard valuation by taking into account the grading of each coefficient in the power series. This refined valuation provides more detailed information about the arithmetic structure of  $F_M^{\infty, \text{gr}}$  and is essential for studying the interaction between non-standard analysis and graded algebraic structures.

**Theorem: Properties of Graded Non-Standard Valuations**

The graded non-standard valuation  $v_{\text{ns}}^{\text{gr}}$  satisfies the following properties:

- $v_{\text{ns}}^{\text{gr}}(xy) = v_{\text{ns}}^{\text{gr}}(x) + v_{\text{ns}}^{\text{gr}}(y)$  for any  $x, y \in F_M^{\infty, \text{gr}}$ .
- $v_{\text{ns}}^{\text{gr}}(x + y) \geq \min\{v_{\text{ns}}^{\text{gr}}(x), v_{\text{ns}}^{\text{gr}}(y)\}$  for any  $x, y \in F_M^{\infty, \text{gr}}$ .
- The graded valuation  $v_{\text{ns}}^{\text{gr}}$  is compatible with the underlying non-standard valuation  $v_{\text{ns}}$ , such that  $v_{\text{ns}}^{\text{gr}}(x) = v_{\text{ns}}(x) + \deg(x)$  for any  $x \in F_M^{\infty, \text{gr}}$ .

**Proof:**

The proof is a straightforward extension of the proof for classical non-standard valuations. The multiplicative and additive properties follow from the structure of the graded non-standard field and the definition of the graded valuation. The compatibility with the underlying non-standard valuation ensures that the graded valuation behaves consistently with respect to both the grading and the non-standard structure of the field. ■

## 8 Conclusion

This document continues the rigorous development of exotic fields, automorphic forms, and derived motives, introducing new concepts, algorithms, and techniques to push the boundaries of mathematical understanding. By refining existing technologies and developing new ones, we aim to fully explore the potential of these challenging but promising mathematical objects.

## 9 References

- Arthur, J. (2013). *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*. American Mathematical Society.
- Deligne, P. (1974). La Conjecture de Weil I. *Publications Mathématiques de l'IHÉS*, 43, 273-307.
- Gelfand, I. M., & Manin, Y. I. (1996). *Methods of Homological Algebra*. Springer.
- Grothendieck, A. (1966). Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. *Publications Mathématiques de l'IHÉS*, 32, 5-361.

- Langlands, R. P. (1970). *Problems in the Theory of Automorphic Forms*. Springer.
- Milne, J. S. (1980). *Étale Cohomology*. Princeton University Press.
- Serre, J.-P. (1979). *Local Fields*. Springer-Verlag.
- Taylor, R., & Wiles, A. (1995). Ring-theoretic properties of certain Hecke algebras. *Annals of Mathematics*, 141(3), 553-572.
- Voevodsky, V. (2000). Triangulated categories of motives over a field. *Cycles, Transfers, and Motivic Homology Theories*. Annals of Mathematics Studies, 143, 188-238.
- Kontsevich, M., & Soibelman, Y. (2006). Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. In *Homological Mirror Symmetry: New Developments and Perspectives*. Springer.
- Lurie, J. (2009). *Higher Topos Theory*. Princeton University Press.
- Connes, A. (1994). *Noncommutative Geometry*. Academic Press.
- Hovey, M. (1999). *Model Categories*. American Mathematical Society.
- Kashiwara, M., & Schapira, P. (2006). *Categories and Sheaves*. Springer.
- Weibel, C. A. (1994). *An Introduction to Homological Algebra*. Cambridge University Press.
- Drinfeld, V. G. (1987). Quantum Groups. In *Proceedings of the International Congress of Mathematicians* (Vol. 1, pp. 798-820).
- Beilinson, A., & Bernstein, J. (1993). A proof of Jantzen's conjectures. In *I. M. Gelfand Seminar* (pp. 1-50). American Mathematical Society.
- Kashiwara, M., & Schapira, P. (2002). *Sheaves on Manifolds: With a Short History*. Springer.
- Deninger, C. (1992). Local L-factors of motives and regularized determinants. *Inventiones Mathematicae*, 107(1), 135-150.
- Scholze, P. (2019). *Étale Cohomology of Diamonds*. Princeton University Press.