MULTIPLICOID GEOMETRY: GENERALIZING PERFECTOID STRUCTURES BEYOND ADDITIVITY

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ABSTRACT. We introduce *multiplicoid spaces* as a natural generalization of perfectoid spaces, extending the notion of infinite-level congruence towers from the additive to the multiplicative regime. This new framework establishes a coherent structure for studying weight–monodromy phenomena, torsor filtrations, and motivic realization over multiplicative congruence bases, without invoking any dyadic assumptions. We outline the foundational algebraic, geometric, and cohomological consequences of such spaces, and propose a hierarchy of generalized period geometries built upon multiplicative growth.

CONTENTS

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1. Introduction and Motivation

The theory of perfectoid spaces, introduced by Peter Scholze, has revolutionized p-adic Hodge theory and arithmetic geometry by enabling systematic passage between characteristic zero and characteristic p in towers of ramified covers. These structures crucially rely on *additive congruence filtrations*, built from successive π-adic divisions in the ring of integers of p-adic fields.

In this paper, we initiate a program to transcend the additive and linear constraints of perfectoid geometry by introducing a new class of geometric spaces we call **multiplicoid spaces**. These spaces are organized around *multiplicative congruence filtrations*, in which the fundamental growth structure arises from repeated multiplication rather than addition or valuation.

Our primary goal is to generalize the perfectoid philosophy—infinite-level congruence descent, tilting equivalence, and cohomological control—to settings governed by multiplicative laws of growth. Instead of relying on valuation-theoretic completeness, we build our theory upon *congruence towers of multiplicative depth*, such as:

$$A \longrightarrow A/(\times 2)^n\mathbb{Z}, \quad \text{or more generally,} \quad A/(\prod_{i=1}^{n} m_i)\mathbb{Z}$$

for increasing multiplicative sequences $\{m_i\}$. Such systems exhibit structural coherence without invoking any p-adic valuation, dyadic base, or Archimedean comparison.

This generalization is motivated by several overlapping paradigms:

- In motivic cohomology and regulator theory, multiplicative polylogarithmic towers appear naturally, particularly in the study of K-theory and special values of L-functions.
- In arithmetic dynamics and automorphic forms, multiplicative scaling symmetries (e.g., Hecke operators) often dominate the spectral geometry of moduli stacks.

- From a categorical perspective, multiplicative filtrations better encode the tensorial nature of objects such as line bundles, torsors, and gerbes under scaling equivalence classes.
- Finally, our construction serves as a launching point for further trans-recursive generalizations to exponential, hyper-exponential, and higher Knuth-arrow based geometries, which will be developed in Volumes I and II.

The theory of multiplicoid spaces presented here is fully independent of dyadic or p-adic assumptions. Nonetheless, in the sequel to this paper, we shall also develop a dyadic-supported version that embeds these multiplicative towers within the broader framework of dyadic period geometry and ε-stratified motivic stacks.

The multiplicoid framework is meant not merely as an extension of perfectoid techniques, but as the first member of a new hierarchy of *growth-structured geometries*, each indexed by a class of recursion-theoretic or arithmetic functions (such as $n \mapsto \exp(n)$, $n \mapsto a \uparrow^k n$, etc.), reflecting the increasing depth of arithmetic and cohomological complexity.

Meta-Philosophical Interlude. The space-theoretic reality explored in this work is grounded not in valuation or measure, but in *growth stratification*—a logic of levels, thresholds, and actions that proceeds from the multiplicative intuition of duplication, scaling, and factorization.

The traditional view of geometry, rooted in the continuity of real or p-adic numbers, assumes that space is essentially *infinitesimally approximable*, and that coherence emerges from local-to-global constructions under an additive, topological microscope.

By contrast, multiplicoid geometry opens a conceptual shift: space becomes a carrier of *stratified multiplicities*, where the hierarchy of structure arises from multiplicative descent and recursive actions, rather than local neighborhoods. Instead of approximating space via infinitesimals, we organize it via modular multiplicative resolutions, capturing the dynamics of systems that do not admit linear flattening or Archimedean compression.

This philosophical shift invites a new type of ontological commitment: *spaces indexed not by points, but by the growth behaviors of their automorphisms.* Under this view, a "multiplicoid space" is not a set equipped with structure, but a stratified arena in which multiplicative towers act as geometric beings—entities whose coherence is internal to the growth operations themselves.

The foundational question we pursue is therefore not "what is a space?", but rather: *what class of growth operations generate stable, recursive, cohomologically meaningful geometries?* The answer begins with multiplicoid towers.

2. Foundations of Multiplicoid Towers

2.1. **Multiplicoid Base Rings and Congruence Structures.** We begin by defining the fundamental algebraic objects over which multiplicoid spaces are constructed. These are not equipped with additive topologies or valuations, but instead with congruence relations governed by multiplicative growth.

Definition 2.1 (Multiplicative Congruence System). *Let A be a commutative ring. A* multiplicative congruence system *on A is a descending family of ideals $\{I_n\}_{n\geq 0}$ satisfying:*

(1) $I_0 = A$, and $I_n \supseteq I_{n+1}$ for all n;
(2) Each I_n is of the form $I_n = (f_n)$, where f_n is a product of multiplicative generators:

$$f_n := \prod_{i=1}^{n} m_i, \quad \textit{with } m_i \in A \textit{ non-units};$$

(3) The sequence $\{m_i\}$ satisfies $m_i \mid m_{i+1}$, i.e., the congruence structure is multiplicatively nested.

The simplest example is given by $A = \mathbb{Z}$ with $m_i = 2$, yielding the tower $I_n = (2^n)$. More generally, one can consider geometric or arithmetic sequences such as $m_i = q^i$ for fixed $q \in \mathbb{N}$, or factorial growth $m_i = i!$.

Definition 2.2 (Multiplicoid Base Ring). *A ring A equipped with a multiplicative congruence system $\{I_n\}$ is called a* multiplicoid base ring*, denoted $A_\times$. We define the inverse system of quotient rings:*

$$A_\times^\infty := \varprojlim_n A/I_n$$

and call it the multiplicoid completion *of A.*

Example 2.3. Let $A = \mathbb{Z}$, and $I_n = (2^n)$. Then $A_\times^\infty = \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$ is the 2-adic completion $\mathbb{Z}_2$—but viewed not as a 2-adic valuation ring, but as a multiplicoid base with respect to the congruence tower.

This perspective frees us from valuation-theoretic dependence and allows multiplicoid constructions over arbitrary bases.

2.2. **Multiplicoid Filtrations and Sheaves.** We now define filtrations compatible with multiplicoid base structure.

Definition 2.4 (Multiplicoid Filtration). *Let M be an A-module over a multiplicoid base ring $A_\times$. A* multiplicoid filtration *on M is a descending chain of submodules $\{F^n M\}_{n\geq 0}$ such that*

$$F^n M := \ker(M \to M/I_n M)$$

for each n, with respect to the multiplicative congruence system $\{I_n\}$ on A.

These filtrations are not indexed by integers additively, but by multiplicative depth. One can visualize them as forming a tower:

$$M = F^0M \supseteq F^1M \supseteq F^2M \supseteq \cdots$$

where F^nM consists of sections of M congruent to 0 modulo $(m_1 \cdots m_n)$.

Definition 2.5 (Multiplicoid Sheaf)**.** *Let $X = \mathrm{Spec}(A_\times)$ be the spectrum of a multiplicoid base ring. A* multiplicoid sheaf *over X is a sheaf of $A_\times$-modules equipped with a compatible multiplicoid filtration as above.*

We denote the category of multiplicoid sheaves over X by $\mathbf{Sh}^\times(X)$.

2.3. **Definition of Multiplicoid Spaces.**

Definition 2.6 (Multiplicoid Space)**.** *A* multiplicoid space *is a locally ringed space $(X, \mathcal{O}_X)$ such that:*

(1) For every open affine $U \subseteq X$, $\mathcal{O}_X(U)$ is a multiplicoid base ring;
(2) The structure sheaf $\mathcal{O}_X$ admits a multiplicoid filtration compatible with the congruence system on each stalk;
(3) The gluing maps preserve multiplicoid congruence towers.

We denote the category of such spaces by **Mult**.

2.4. **Basic Properties and Examples.**

Example 2.7 (Multiplicative Completion of $\mathbb{A}^1$). Let $A = \mathbb{Z}[x]$ and define $I_n = ((x+1)(x+2)\cdots(x+n))$. The multiplicoid completion $A_\times^\infty$ defines a formal neighborhood of ∞ under multiplicative translation.

Proposition 2.8. *Let $A_\times$ be a multiplicoid base ring, and let $X = \mathrm{Spec}(A_\times)$. Then:*

(1) $\mathbf{Sh}^\times(X)$ is an abelian category;
(2) The filtration functor $F^n(-)$ is exact on flat modules;
(3) **Mult** *admits fiber products and a natural site structure.*

Sketch. The filtration conditions are inherited from the projective system defining $A_\times^\infty$; exactness follows from the compatible inverse limits; site structure is induced via congruence base changes. □

Ontological Interpretation of Multiplicoid Spaces. From an ontological perspective, the nature of a *space* depends not merely on its points, topology, or charts, but on the generative logic by which its structure is layered and made perceptible. In classical schemes, this generative logic is additive, local, and infinitesimal. In perfectoid spaces, it is valuation-theoretic, descending from additive norm filtration indexed by powers of a uniformizer.

Multiplicoid spaces introduce a shift: they are not defined through infinitesimal approximation, but rather through multiplicative descent. In such spaces, the fundamental organizing principle is not "closeness" in a metric or valuation sense, but "depth" of congruence under multiplicative growth. That is:

Proximity is replaced by congruence collapse under iteration of multiplication.

This suggests that a multiplicoid space is not a space in which one moves linearly, but a space in which one *folds, scales, and stratifies* via multiplication.

Each multiplicoid filtration level F^n should be viewed not merely as a layer of vanishing, but as a *modal stage* of existence—an ontological region accessible only after descending through n multiplicative thresholds. These thresholds are not numeric levels; they are stages of being, governed by *operations* rather than *coordinates*.

In this light, a multiplicoid space is an *operation-indexed object*: a structure stratified by the complexity of the multiplicative acts required to resolve its local or global data. Its sheaves are not merely functions or sections, but *growth-conditioned entities*—objects whose coherence emerges through their transformation laws under iterated scaling.

This opens the door to a broader project: to define a new class of geometries whose fundamental units are not points, but *processes*—geometries whose categories are enriched not over sets, but over sequences of structural operations. Multiplicoid spaces are thus the first layer in an ontological ladder of growth-indexed spaces.

Diagrammatic Motivation: Growth-Based Geometric Hierarchy. To orient the reader, we now present a schematic diagram showing how multiplicoid geometry emerges as the first extension of perfectoid space theory in the direction of operation-based stratification.

$$
\begin{tikzcd}
\textbf{Classical Geometry} \arrow[r, "Additive Completion"] \arrow[d, dashed, "Functor of Points"'] & \textbf{Perfectoid Geometry} \arrow[r, "Growth Shift: Multiplicative"] \arrow[d, dashed, "Tilting/Prismatic"] & \textbf{Multiplicoid Geometry} \arrow[r, "\exp(-)"] & \textbf{Exponentoid Geometry} \arrow[r, "\uparrow^n"] & \textbf{Knuthoid Geometry} \arrow[r, "Ontology"] & \textbf{Ontoid Geometry} \\
\text{Schemes over } \mathbb{Z} \arrow[r] & \text{Adic / Diamond Spaces} \arrow[r] & \text{Operation-Based Towers}
\end{tikzcd}
$$

This diagram encapsulates the direction of generalization:

- We begin with schemes and classical topology;
- Perfectoid geometry introduces additive infinite-level towers;
- Multiplicoid geometry replaces additive depth with multiplicative congruence depth;
- Further operations—exponential, iterated exponentiation, or Knuth-arrow growth—define new spatial hierarchies;

- Ultimately, geometry becomes indexed not by numbers, but by *growth functions*, and stratified by *ontological processes*.

In this progression, multiplicoid spaces are the critical *first transcendence step* beyond the perfectoid world, retaining coherence while freeing the framework from valuation, base fields, and additive metrics. They offer a geometric arena for the exploration of motivic congruence, torsor dynamics, and structural recursion under non-linear growth regimes.

This motivates the next section: to construct the cohomological backbone of multiplicoid geometry, and to interpret these spaces through their period sheaves and ε-structured realizations.

3. Cohomology and Period Structures of Multiplicoid Spaces

3.1. Multiplicoid Period Rings and Sheaves.

Just as perfectoid geometry gives rise to de Rham and crystalline period rings via inverse systems over additive uniformizers, multiplicoid geometry naturally generates new types of period rings based on multiplicative congruence towers.

Definition 3.1 (Multiplicoid Period Ring)**.** *Let $A_\times$ be a multiplicoid base ring with multiplicative congruence system $\{I_n\}$. Define the* multiplicoid period ring *as the inverse limit*

$$B_{\mathit{mult,dR}}(A) := \varprojlim_n A/I_n \otimes_{\mathbb{Z}} \mathbb{Q},$$

equipped with a canonical multiplicoid filtration

$$F^n B_{\mathit{mult,dR}}(A) := \ker \left(B_{\mathit{mult,dR}}(A) \to A/I_n \otimes \mathbb{Q} \right).$$

This filtration is multiplicative in nature and grows not linearly but exponentially in ideal size. One may view $B_{\mathrm{mult},dR}$ as the ambient period ring capturing multiplicative-deformation classes of cohomology.

Definition 3.2 (Multiplicoid Period Sheaf)**.** *Let X be a multiplicoid space. A* multiplicoid period sheaf *over X is a sheaf of $B_{\mathit{mult,dR}}$-modules equipped with:*

- *A multiplicoid filtration $F^\bullet$ as above;*
- *A flat connection compatible with the multiplicative stratification;*
- *Transition morphisms respecting torsor action under $\times n$ scaling maps.*

We denote the category of such sheaves by $\mathbf{Sh}^{\nabla}_{\mathrm{mult}}(X)$.

3.2. Torsors and ε-Filtration Towers.

Just as perfectoid cohomology admits interpretation via torsors over Galois or Frobenius groups, multiplicoid cohomology admits *multiplicative torsors*, which encode the levels of filtration via multiplicative action.

Definition 3.3 (Multiplicoid Torsor). *Let $G = \mathbb{Z}_{>0}$ act on a space X via multiplicative scaling $x \mapsto m \cdot x$. A* multiplicoid torsor *is a principal G-bundle $\mathcal{T} \to X$ such that:*

- *For each n, the level-n fiber $\mathcal{T}_n$ corresponds to $F^n\mathcal{O}_X$;*
- *The action satisfies scaling compatibility: $g \cdot \mathcal{T}_n \simeq \mathcal{T}_{gn}$.*

Definition 3.4 (ε-Filtration Tower). *Let E be a sheaf over a multiplicoid space X. An* ε-filtration tower *is a sequence of subobjects $\{F^{\varepsilon^n}E\}$ satisfying:*

$$F^{\varepsilon^n}E := \ker\left(E \to E/I_nE\right), \quad \text{with } I_n = (\prod_{i=1}^{n} m_i).$$

Here, ε^n acts as an abstract growth operator, encoding congruence depth stratification.

This ε-filtration system gives rise to new types of cohomology.

3.3. **Multiplicoid Cohomology.** We now define the cohomology theory intrinsic to multiplicoid geometry.

Definition 3.5 (Multiplicoid Cohomology). *Let X be a multiplicoid space, and $\mathcal{F}$ a multiplicoid sheaf. The* multiplicoid cohomology groups *are defined as:*

$$H^i_{mult}(X, \mathcal{F}) := \varprojlim_n H^i(X, \mathcal{F}/F^n\mathcal{F}),$$

where $F^n\mathcal{F}$ denotes the n-th level of the multiplicoid filtration.

Remark 3.6. This cohomology captures the stabilization of local data under multiplicative congruence descent. It mirrors syntomic and crystalline cohomology in the additive world, but with fundamentally different growth behavior and torsor action.

3.4. **Regulators and Period Morphisms.**

Definition 3.7 (Multiplicoid Regulator Map). *Let K be a field and X/K a smooth multiplicoid space. Then the* multiplicoid regulator map *is a natural transformation:*

$$r_{mult} : K_n(X) \longrightarrow H^n_{mult}(X, \mathbb{Q}(n)),$$

constructed via classes of multiplicoid ε-filtration torsors.

This regulator behaves compatibly with multiplicative period growth and can be used to study special values of multiplicative L-functions and motivic ε-heights.

3.5. **Examples and Future Directions.**

Example 3.8. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$ with multiplicative tower $I_n = ((x+1)\cdots(x+n))$. The sheaf $\mathcal{F} = \mathcal{O}_X$ admits a filtration via vanishing modulo multiplicative products. The associated cohomology $H^1_{\mathrm{mult}}(X, \mathcal{F})$ detects relations in polynomial growth towers.

Example 3.9. Let X be the moduli stack of multiplicatively-scaled elliptic curves. The period sheaf over X admits ε-filtration indexed by the multiplicative conductor, and the cohomology H^2_{mult} captures automorphic growth.

These structures prepare the foundation for developing *Exponentoid* and *Knuthoid* spaces in subsequent volumes, where the operators $x \mapsto x^n$, $x \mapsto \exp^n(x)$, and $x \mapsto a \uparrow^k x$ serve as generators of stratified geometry.

4. Tilting and Universal Structures

4.1. **Toward a Tilting Theory for Multiplicoid Spaces.** Tilting equivalences in perfectoid geometry—via characteristic p–0 correspondences—enable deep cohomological comparisons and facilitate the study of p-adic period spaces through characteristic p models. In the multiplicoid setting, we instead seek to define a new form of "tilting" not across characteristics, but across *multiplicative congruence towers.*

Definition 4.1 (Multiplicoid Tilting System)**.** *Let $A_\times$ be a multiplicoid base ring. A* multiplicoid tilting system *consists of:*

- *A projective system $\{A_n\}_{n\geq 0}$, where $A_n := A/I_n$ with $I_n = (m_1 \cdots m_n)$;*
- *A compatible family of multiplicative lifts $\varphi_n : A_{n+1} \to A_n$ satisfying:*

$$\varphi_n(x \bmod I_{n+1}) = x^k \bmod I_n$$

  for some fixed $k \in \mathbb{Z}_{>0}$ (interpreted as the tilting exponent);
- *A limit algebra $A^\flat_\times := \varprojlim_{\varphi_n} A_n$, called the* multiplicoid tilt *of $A_\times$.*

This multiplicoid tilt captures a recursively structured version of the base ring, and enables the transport of multiplicoid sheaves and cohomology across congruence scales.

Remark 4.2. Unlike perfectoid tilting, which involves passage between mixed and equal characteristics, multiplicoid tilting is indexed by operation classes (e.g. $\times n$, $\exp(n)$), and produces a self-similar structure internal to the multiplicoid filtration.

4.2. **Torsor Functoriality and Universal ε-Gerbes.** Given the tower of torsors $\{\mathcal{T}_n\}$ over a multiplicoid space X, we now organize these into a universal stack.

Definition 4.3 (Universal Multiplicoid Torsor Stack)**.** *Define the stack $\mathbb{T}^{[\times]}$ over the site of multiplicoid spaces as the fibered category whose objects over X are sequences of torsors $\mathcal{T}_n \to X$ satisfying:*

1. *For each n, $\mathcal{T}_n$ is a principal $(\mathbb{Z}/I_n)$-torsor;*
2. *The action maps are compatible: $\mathcal{T}_{n+1} \to \mathcal{T}_n$ respect multiplicative scaling;*
3. *There exists a universal object $\mathcal{T}_\infty := \varprojlim \mathcal{T}_n$.*

This stack represents the moduli of multiplicoid descent structures, and governs the ε-filtration levels across different multiplicoid cohomological theories.

Definition 4.4 (Universal ε-Gerbe)**.** *Let X be a multiplicoid space. A* universal ε-gerbe *over X is a gerbe $\mathcal{G}_\varepsilon$ banded by a profinite abelian group G such that:*

- *G acts on each $\mathcal{T}_n$ through congruence level mod I_n;*
- *There exists a morphism of stacks $\mathbb{T}^{[\times]} \to B\mathcal{G}_\varepsilon$;*
- *The cohomology $H^2_{mult}(X, \mathcal{G}_\varepsilon)$ classifies filtered multiplicoid torsor deformations.*

This ε-gerbe plays a role analogous to B^+_{dR}-torsors in perfectoid Hodge theory, encoding obstruction classes and period morphisms under multiplicative scaling.

4.3. **Universal Period Towers.**

Definition 4.5 (Universal Multiplicoid Period Tower)**.** *Let X be a multiplicoid space. The* universal multiplicoid period tower *is the projective system of period rings:*

$$\mathrm{Per}^\infty_{mult}(X) := \left\{ B^{(n)}_{mult,dR}(X) := \mathcal{O}_X / I_n \otimes \mathbb{Q} \right\}_{n \in \mathbb{Z}_{\geq 0}}.$$

This tower comes equipped with:

- *Morphisms $B^{(n+1)} \to B^{(n)}$ induced by congruence reduction;*
- *Filtration structure $F^n := \ker(B^{(\infty)} \to B^{(n)})$;*
- *Period realizations via ε-torsors: $\mathrm{Per}^\infty_{mult} \to \mathbb{T}^{[\times]}$.*

4.4. **Functoriality and Universal Realization Functor.**

Theorem 4.6 (Universal Period Realization)**.** *Let* **Mult** *be the category of multiplicoid spaces, and let* **MultCoh** *be the associated category of filtered multiplicoid sheaves. Then there exists a realization functor*

$$\mathscr{R}_{mult} : \mathbf{MultCoh} \longrightarrow \mathbb{T}^{[\times]} \times_{\mathbb{Z}} \mathrm{Per}^\infty_{mult}$$

that respects filtrations, torsor morphisms, and cohomological classes.

Sketch. The functor is constructed by associating to each filtered sheaf its image in the torsor tower, then evaluating sections over the period tower. The transition maps ensure compatibility. □

5. Motivic Realization and Applications

5.1. Multiplicoid Motives and Realization Functors. The theory of motives aims to unify cohomological realizations of algebraic varieties across various contexts—de Rham, étale, crystalline, syntomic, etc.—via a universal motivic object. In the context of multiplicoid geometry, we propose a new type of realization: a *multiplicoid motivic realization*, reflecting congruence-depth stratification rather than valuation-based descent.

Definition 5.1 (Multiplicoid Motive)**.** *Let X be a smooth multiplicoid space. A* multiplicoid motive $M^{[\times]}(X)$ *is an object in a tensor-triangulated category $\mathcal{M}_{mult}$ equipped with:*

(1) A multiplicoid filtration $F^n M$ induced by congruence collapse modulo I_n;
(2) Realization functors to cohomology:

$$real_{dR}^{mult} : M^{[\times]}(X) \to H^{\bullet}_{mult}(X, B_{mult,dR})$$

and similarly to $\mathbb{T}^{[\times]}$ and $\mathrm{Per}^{\infty}_{mult}$;
(3) Compatibility with the regulator map and torsor-tower action.

We conjecture the existence of a universal multiplicoid motivic category $\mathcal{M}_{\mathrm{mult}}$ extending Voevodsky's $\mathbf{DM}_{\mathrm{gm}}$ through congruence-indexed ε-stratifications.

5.2. **Multiplicoid Regulators and ε-Pairings.** We revisit the multiplicoid regulator map:

$$r_{\mathrm{mult}} : K_n(X) \longrightarrow H^n_{\mathrm{mult}}(X, \mathbb{Q}(n)),$$

constructed through the period realization of multiplicoid motives. This map respects torsor stratifications and may be thought of as a generalized "multiplicative Borel regulator".

Definition 5.2 (ε-Pairing)**.** *Let $M^{[\times]}(X)$ be a multiplicoid motive with torsor tower $\mathbb{T}^{[\times]}$. The* ε-pairing *is a map:*

$$\langle -, - \rangle_{\varepsilon^n} : F^{\varepsilon^n} M \otimes F^{\varepsilon^n} M^{\vee} \longrightarrow \mathbb{Q}$$

satisfying compatibility with the period filtration and collapsing as $n \to \infty$.

This pairing captures motivic height-type information encoded in ε-stratified congruence levels.

5.3. **Special Values and Multiplicoid L-Functions.** Inspired by Beilinson's conjectures and higher polylogarithmic regulators, we define a new class of arithmetic functions that interpolate multiplicoid cohomological invariants.

Definition 5.3 (Multiplicoid L-Function)**.** *Let $M^{[\times]}$ be a multiplicoid motive over X. Its associated multiplicoid L-function is:*

$$L_{mult}(M, s) := \prod_{n \geq 0} \det \left(1 - (\mathrm{Frob}_n \cdot p^{-s}) \mid F^n M\right)^{-1},$$

where Frob_n *denotes the multiplicative action on the n-th filtration layer.*

This function reflects the interaction between torsor depth and period realizability. In the dyadic-supported version of this theory, the primes p may be replaced by powers of 2, leading to logarithmic or polylogarithmic interpolation.

5.4. **Applications to Polylogarithmic Stacks and Automorphic Periods.** Multiplicoid geometry naturally applies to polylogarithmic motives, especially those appearing in higher K-theory and modular symbols.

Example 5.4. Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and consider the multiplicoid motive generated by iterated logarithms:

$$\log^n(x) \in F^{\varepsilon^n} M^{[\times]}(X),$$

whose periods correspond to multiple zeta values filtered by congruence growth. The associated multiplicoid L-function reflects the asymptotic depth of polylogarithmic torsors.

Example 5.5. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties. Its cohomology admits a multiplicoid stratification via multiplicative conductor. The torsor towers classify Hecke eigenforms under multiplicative growth. Multiplicoid regulators provide new invariants beyond classical Eichler–Shimura theory.

5.5. **Further Directions.** The multiplicoid realization functor admits natural extensions:

- To exponentoid geometry: motives with exponential depth filtrations;
- To knuthoid geometry: with towers indexed by $\uparrow^n$ operators;
- To epsilon-gerbe stacks: defining arithmetic heights in terms of ε-torsor classes;
- To motivic sheaf-to-space equivalences: realizing ε-motives as functors into ontological geometries.

6. Future Directions and Foundational Implications

6.1. **From Multiplicoid to Exponentoid Structures.** The development of multiplicoid geometry has revealed a new principle: that geometric structure need not arise from topology, valuation, or infinitesimal analysis alone—but may instead be generated from *congruence growth under structural operations.*

This realization opens the door to further generalizations. If multiplicoid spaces correspond to congruence towers indexed by multiplicative depth $n \mapsto \prod_{i=1}^n m_i$, then it is natural to define *exponentoid spaces*, based on:

$$n \mapsto \exp^k(n), \quad \text{where } \exp^k(n) = \underbrace{\exp(\cdots \exp(n) \cdots)}_{k \text{ times}}.$$

These geometries stratify space not merely by uniform congruence levels, but by rapidly expanding depth layers governed by recursion.

6.2. **Knuthoid Geometry and Hyperstratified Arithmetic.** Pushing further, one arrives at *Knuthoid geometry*, indexed by hyper-operations such as $a \uparrow^k b$. In this regime, space is no longer an object with points, but a categorically indexed web of operation-driven torsor classes.

Filtrations become hyper-filtrations, towers become hyper-towers, and periods become functions of trans-recursive interaction. The categorical complexity of such spaces may surpass current model-theoretic formalizations, demanding a new logical foundation for geometry itself.

6.3. **Ontoid Geometry and the Logic of Space.** Ultimately, the highest layer of this progression is *ontoid geometry*—a vision of space constructed not from numbers, but from *operations as first-order entities*. In this world, the "structure sheaf" $\mathcal{O}_{\mathrm{Ont}}$ is not a sheaf of functions, but a stratified class of transformations indexed by the growth laws governing the space.

Under this paradigm:

- Points disappear, replaced by procedural strata;
- Filtrations become logical levels of realization;
- Cohomology becomes the study of coherence under transfinite recursion;
- Geometry becomes not a container of objects, but a process in itself.

6.4. **Foundational Summary.** The theory initiated in this volume stands at the beginning of a new program:

Hyperstratified Geometry: A Recursion-Theoretic Foundation of Arithmetic Space

Its core tenets are:

(1) Filtration structures are derived from recursive growth, not local topology;
(2) Torsors are the primary geometric entities, organizing operation-based descent;
(3) Periods, regulators, and cohomology reflect structural depth rather than metric magnitude;
(4) The geometry of the future is a stratified ontology of procedures—not merely an extension of classical form.

Outlook. Volumes I (Exponentoid Geometry) and II (Hyper-Filtration Theory) will pursue this progression in full, while Volumes III–V develop applications to motivic arithmetic, logical geometry, and the space-theoretic ontology of mathematics.

The ultimate goal is a theory of space which reflects not what exists—but what emerges through the logic of generation itself.

7. Comparison with Additive and Valuation-Based Geometries

7.1. **From Valuations to Congruence Filtrations.** In classical arithmetic geometry, structures are often built atop additive or valuation-based topologies. Perfectoid geometry, for instance, depends critically on the existence of a non-archimedean valuation $|\cdot|$ and a compatible Frobenius-lifted tower.

In multiplicoid geometry, however, no valuation or additive topology is assumed. Instead, congruence is stratified multiplicatively, and structure is encoded not in distance or approximation, but in multiplicative divisibility depth:

$$x \in F^n \iff x \equiv 0 \mod \prod_{i=1}^{n} m_i.$$

Additive/Valuation-Based	**Multiplicoid (Congruence-Based)**
Valuation $v(x)$	Congruence depth n (via multiplicative tower)
Topology from norm	No topology, only arithmetic strata
Neighborhood basis	Layered ε-filtration
Perfectoid space	Multiplicoid space
Frobenius tilting	Multiplicoid recursive tilting
p-adic Hodge theory	$\times$-based Period Stratification

TABLE 1. Conceptual comparison of foundational geometries

7.2. **Behavior of Filtration Towers.** The additive filtration $F^n_{\mathrm{add}} := p^n\mathcal{O}$ satisfies exponential decay, while multiplicoid filtration $F^n_{\mathrm{mult}} := \ker(\mathcal{O} \to \mathcal{O}/I_n)$ grows in complexity by multiplicative nesting. They differ in direction, intensity, and stratification mechanism.

- **Additive:** Linear base, exponential scaling, valuation-based convergence.
- **Multiplicoid:** Multiplicative base, recursive nesting, congruence collapse without limit topology.

Yet both produce filtrations of period rings, regulate cohomology, and support motivic realization functors. The crucial difference is that multiplicoid towers are *arithmetic*, not topological.

7.3. **Comparison of Realization Functors.** Let us consider:
- $\mathscr{R}_{\mathrm{perf}}$: perfectoid realization (via Frobenius and tilting),
- $\mathscr{R}_{\mathrm{mult}}$: multiplicoid realization (via torsor and congruence depth).

While the former realizes structure through compatible topological lifts, the latter proceeds via stratified descent through torsor towers:

$$\mathscr{R}_{\mathrm{mult}}(\mathcal{F}) = \{F^n\mathcal{F}\} \to \mathrm{Per}^{\infty}_{\mathrm{mult}} \times \mathbb{T}^{[\times]}.$$

7.4. **Axiomatic Divergence.**

Valuation: $\exists$ a nonzero map $\nu : A \to \Gamma \cup \{\infty\}$ satisfying triangle inequality.
Multiplicoid: $\exists$ a congruence sequence $\{I_n\}$ such that $I_n \supset I_{n+1}$ with
$I_n = (m_1 \cdots m_n)$.

The former axiomatizes local approximation, the latter recursive divisibility.

7.5. **Interoperability and Hybrid Filtrations.** It is possible to combine both paradigms:

$$F^{(n)} := (p^n \cap I_n)\,, \quad \text{mixed syntomic–congruence tower.}$$

This leads to a new class of filtrations indexed by both valuation depth and arithmetic congruence. Such constructions may find application in ε-syntomic cohomology, or arithmetic Hodge theory with congruence control.

7.6. **Conclusion.** This section shows that multiplicoid and perfectoid geometries are not mutually exclusive, but operate on orthogonal axes: one via topological convergence, the other via stratified divisibility. Their comparison reveals a broader categorical structure: filtration geometry, where convergence is just one special case of recursive growth.

8. Exponentoid and Knuthoid Transition Models

8.1. **Beyond Multiplicative Growth.** While multiplicoid geometry replaces additive proximity with multiplicative congruence, there remains a vast hierarchy of operational growth rates. For instance, exponential growth $x \mapsto a^x$ and hyperoperations like $x \mapsto a \uparrow^k x$ represent natural generalizations of recursive depth.

We now define models to transition from multiplicoid filtrations to those governed by higher operations.

8.2. **Exponentoid Stratification.** Let $\exp(n)$ denote a base-e or base-b exponential growth function. We define the following:

Definition 8.1 (Exponentoid Tower)**.** *Let* $I_n := (\exp(n)) = (\underbrace{a^{a^{\cdots^{a}}}}_{n\ times})$. *The* exponentoid filtration *is given by:*

$$F^{\exp(n)}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/I_n\mathcal{F}).$$

Such a tower generalizes the multiplicoid tower F^n by replacing $\prod m_i$ with $\exp^n(a)$. This induces *accelerated congruence descent*, corresponding to period sheaves with exponentially faster torsor collapsing.

Definition 8.2 (Exponentoid Period Ring)**.** *Define*

$$B_{\exp,dR} := \varprojlim_n A/\exp(n) \otimes \mathbb{Q},$$

equipped with exp-*indexed* ε*-filtration.*

8.3. **Knuthoid Models.** Knuth's up-arrow notation defines towers of iterated exponentials:

$$a \uparrow^k b = k\text{-times iterated exponential.}$$

We may then define:

Definition 8.3 (Knuthoid Filtration). *For fixed $a > 1$, let $I_n := (a \uparrow^k n)$, and define:*

$$F^{\uparrow^k n}\mathcal{F} := \ker(\mathcal{F} \to \mathcal{F}/I_n\mathcal{F}).$$

This yields the Knuthoid filtration tower *of $\mathcal{F}$.*

These filtrations induce trans-recursive period structures. The associated cohomologies stabilize extremely slowly but stratify enormous computational depth in torsor descent.

8.4. **Transition Maps and Functoriality.** Between multiplicoid and exponentoid towers, we propose transition morphisms:

$$F^n \longrightarrow F^{\exp(n)}, \quad F^{\exp(n)} \longrightarrow F^{\uparrow^2 n},$$

given by functional lifts and divisibility domination. These define natural transformations of period rings and torsor stacks.

Let $\Phi_n^{\exp} : \mathbb{T}_n^{[\times]} \to \mathbb{T}_n^{[\exp]}$ be the induced torsor morphism collapsing multiplicative layers into exponential equivalence classes.

8.5. **Towards Filtration Category Towers.** We may now organize all filtration types into a filtered category:

$$\mathsf{Filt}_\infty := \left\{\text{objects } F^*\mathcal{F}, \text{ morphisms } F^{f(n)} \to F^{g(n)} \text{ for } f \prec g\right\}.$$

This category allows comparison, pushforward, and pullback of period sheaves under growth morphisms. In future volumes, this becomes the foundational language for arithmetic sheaves of infinite generative order.

8.6. **Conclusion.** Exponentoid and Knuthoid geometries form the first two higher-order generalizations of multiplicoid space. They suggest that what we call "space" may actually be an emergent effect of growth law stratification—and that by indexing these laws hierarchically, we may define infinitely generative geometric universes.

9. Ontological Layering of Stratified Growth

9.1. **From Geometric Spaces to Generative Processes.** In traditional geometry, a space is often conceived as a set endowed with topological or algebraic structure. In multiplicoid, exponentoid, and knuthoid geometries, however, space emerges from the stratification of growth operations. This demands a shift in perspective: from *point-based ontology* to *process-based ontology.*

> The identity of a geometric object is no longer a location, but a position within a growth process.

9.2. **Stratified Ontology and Layered Existence.** Let $\mathbb{Y}_\infty$ denote a hypothetical object whose structure is governed entirely by filtration dynamics. Define:

Definition 9.1 (Growth Layer)**.** *A* growth layer L_n *is an equivalence class of sections of* $\mathcal{F}$ *modulo* $F^{g(n)}\mathcal{F}$*, for some growth function* $g(n)$ *(e.g.,* n*,* $\exp(n)$*,* $\uparrow^k n$*). The collection* $\{L_n\}$ *determines an* ontological tower *of existence.*

Each such layer represents not a set-theoretic subset, but a manifestation of mathematical being at a given level of generative depth. The filtration F^* becomes a logic of emergence.

9.3. **Functorial Ontologies.** We now define an ontology-valued presheaf:

$$\mathcal{O}nt : \mathsf{Filt}_\infty^{\mathrm{op}} \to \mathbf{Cat}, \quad F^{f(n)} \mapsto \text{Category of growth-sheaves over } F^{f(n)}.$$

This presheaf encodes the reality of a filtered space as a categorical stack over growth laws, not coordinates.

9.4. **Torsors as Ontological Agents.** Each torsor $\mathcal{T}_n$ is no longer a "bundle over a base", but an automorphism group acting on a growth layer. In this picture:
- **Objects:** Layers of $\mathcal{F}$.
- **Morphisms:** Transition maps between filtration layers.
- **Torsors:** Auto-equivalence structures (e.g. under $\mathbb{Z}/N\mathbb{Z}$) governing stability at that level.

Definition 9.2 (Ontoid Structure)**.** *An* ontoid structure *is a functor* $\mathcal{S} : \mathbb{N} \to$ **Topos** *such that*

$$\mathcal{S}(n) = \textit{Sheaf category over } F^{g(n)}\mathcal{F}, \quad g(n) \in \{\textit{growth functions}\}.$$

The topos $\mathcal{S}(n)$ represents "reality at level n", where mathematical properties exist under the constraints of stratified filtration.

9.5. **Existence via Growth Constraints.** This perspective suggests:
- The notion of a *mathematical object* is dynamic;
- Existence is indexed by growth capacity;
- Identity is relational, defined via stratified equivalence.

Mathematics becomes a *philosophy of expansion*—objects exist not because they are constructed, but because they persist under generative rules.

9.6. **Toward Meta-Geometry.** Ultimately, stratified growth provides a foundational alternative to set-theoretic axioms. It defines a meta-geometric framework where:
- Filtrations are logic;
- Torsors are agents of self-similarity;
- Cohomology is the study of internal persistence across existence levels.

$$\boxed{\text{Space} := \text{Sheaf of Ontologies over Growth-Induced Filtration Categories.}}$$

This leads naturally to the final section: a synthesis of multiplicoid geometry and a proposal for trans-recursive foundational axioms.

10. Concluding Synthesis and Infinite-Generation Conjectures

10.1. **Summary of Multiplicoid Geometry.** In this volume, we introduced the framework of multiplicoid geometry as a generalization of perfectoid and p-adic geometric theories, built upon multiplicative congruence towers rather than additive or valuation-theoretic approximations.

Key foundational principles include:

- **Filtration as Structure:** The primary geometric data arises from congruence depth, not topological closeness.
- **Torsors as Generators:** Spaces are stratified via auto-equivalences of congruence actions rather than local trivializations.
- **Period Rings as Cohomological Media:** Each filtration layer corresponds to a realization of period information under recursive descent.
- **Motivic and Ontological Unification:** Filtrations, torsors, and regulators are synthesized through a common ε-stratified motivic realization.

10.2. **Transition to Higher Geometries.** Multiplicoid theory serves as the foundational level of a more general hierarchy of geometries:

$$\text{Additive} \to \text{Multiplicoid} \to \text{Exponentoid} \to \text{Knuthoid} \to \text{Ontoid}.$$

Each level corresponds to a deeper level of stratification, indexed by increasingly powerful generative operations:
- Additive: $x + n$
- Multiplicative: $x \cdot n$
- Exponential: x^n
- Hyper-Exponential: $a \uparrow^k n$
- Ontological: indexed logic or meta-generators

10.3. **Infinite-Generation Conjectures.** We now propose a class of conjectures describing how the structure of space may emerge from recursively layered stratification:

Conjecture 10.1 (Recursive Stratification of Geometry)**.** *Every cohomological realization functor is induced by a stratified tower of filtration categories* Filt_∞*, whose growth functions are strictly increasing under recursive composition.*

Conjecture 10.2 (Existence via Growth Law)**.** *The existence of a geometric object is determined by its persistence across infinitely many filtration levels. That is,*

$$X \text{ exists } \iff \forall n,\, F^n\mathcal{F}(X) \neq 0.$$

Conjecture 10.3 (Trans-Recursive Periodicity)**.** *Let* $\mathcal{T}_{\uparrow^k}$ *be the Knuthoid torsor tower. Then there exists a stabilization functor*

$$\lim_{\uparrow^k \to \infty} \mathcal{T}_{\uparrow^k} \cong \mathcal{T}_\infty^{Ont},$$

that classifies all growth-based space torsors as images of an ontological stack.

10.4. **Philosophical Implication.** At its heart, this theory asks:

> What is the structure of space if it arises not from position, but from recursion?

The answer, we argue, lies in the infinite generation of torsor-based, filtration-indexed growth. In this vision, geometry is not static—but dynamically layered, generative, and recursive.

10.5. **Outlook.** Future volumes will explore:
- Exponentoid filtrations over exponential congruence towers (Volume I);
- ε-stratified motivic cohomology in exponential and transfinite regimes (Volume II);
- Generalized Weight–Monodromy theories and Knuth-level dynamics (Volume III);
- Ontoid reconstructions of space and arithmetic ontology (Volume IV);
- Categorical arithmetic and logic over growth-generated sheaf-theoretic stacks (Volume V).

Each volume pushes further into the recursion-indexed universe of stratified geometric generation—toward a meta-geometry whose foundations are laws of expansion rather than sets of points.

End of Volume I' (Non-Dyadic Supported)

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