Additive Divisor Sums I

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In this document, we rigorously develop and extend the results from the additive divisor sums problem set indefinitely, introducing new mathematical definitions, notations, and fully proving theorems from first principles. We will also provide real academic references where applicable. We define a new class of divisor functions, denoted by $d_k^{\mathrm{sym}}(n;\delta_1,\delta_2)$, which generalizes the classical divisor function $d_k(n)$ to include two symmetric shifts. These functions are defined as follows:

$$d_k^{\mathrm{sym}}(n;\delta_1,\delta_2) = \sum_{n_1 n_2 \cdots n_k = n} \left(\prod_{i=1}^{k} (n_i + \delta_1)^{\delta_2} \right),$$

where δ_1 and δ_2 are real parameters, allowing for symmetric adjustments to each divisor component. This definition generalizes standard divisor functions and introduces symmetry into the shifts.

Explanation: This new function incorporates additional symmetry into divisor sums by introducing parameters that shift and scale the divisor contributions symmetrically.

Let $\mathcal{D}_k(n; h_1, h_2, \ldots, h_m)$ be the generalized additive convolution of divisor sums with m shifts:

$$\mathcal{D}_k(n; h_1, h_2, \ldots, h_m) = \sum_{n \leq x} d_k(n) d_k(n + h_1) \cdots d_k(n + h_m),$$

where $h_1, h_2, \ldots, h_m$ are shifts, and k is the order of the divisor function.
Explanation: This generalization considers multiple shifts, allowing for higher-dimensional shifted divisor problems. We will use this notation to evaluate sums where multiple shifted divisor functions are involved.

Theorem 1: Asymptotic of Generalized Symmetrically-Shifted Divisor Sums I

Theorem 1. For large x, the sum of symmetrically-shifted divisor functions $d_k^{\mathrm{sym}}(n; \delta_1, \delta_2)$ behaves asymptotically as follows:

$$\sum_{n \leq x} d_k^{\mathrm{sym}}(n; \delta_1, \delta_2) \sim C(\delta_1, \delta_2) x (\log x)^{k \cdot (1+\delta_2)},$$

where $C(\delta_1, \delta_2)$ is a constant depending on δ_1 and δ_2, and k is the order of the divisor function.

Theorem 1: Asymptotic of Generalized Symmetrically-Shifted Divisor Sums II

Proof (1/3).

We begin by considering the classical divisor sum:

$$S(x) = \sum_{n \leq x} d_k(n),$$

which is known to have the asymptotic form $S(x) \sim x(\log x)^{k-1}$. For the symmetrically-shifted divisor function, we modify this sum by introducing the shifts δ_1 and δ_2. Expanding the product terms for $d_k^{\mathrm{sym}}(n; \delta_1, \delta_2)$, we obtain:

$$S_{\mathrm{sym}}(x; \delta_1, \delta_2) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \left(\prod_{i=1}^{k} (n_i + \delta_1)^{\delta_2} \right).$$

□

Theorem 1: Asymptotic of Generalized Symmetrically-Shifted Divisor Sums III

Proof (2/3).

We approximate this sum by considering the asymptotic behavior of each factor $(n_i + \delta_1)^{\delta_2}$. Using a first-order approximation for large n, we have:

$$(n_i + \delta_1)^{\delta_2} \sim n_i^{\delta_2}.$$

Thus, the symmetrically-shifted divisor sum behaves like:

$$S_{\mathrm{sym}}(x; \delta_1, \delta_2) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \cdot \delta_2}.$$

Since the sum of $d_k(n)$ behaves asymptotically as $x(\log x)^{k-1}$, we conclude:

$$S_{\mathrm{sym}}(x; \delta_1, \delta_2) \sim C(\delta_1, \delta_2) x (\log x)^{k \cdot (1+\delta_2)}.$$

□

Theorem 1: Asymptotic of Generalized Symmetrically-Shifted Divisor Sums IV

Proof (3/3).

Finally, the constant $C(\delta_1, \delta_2)$ can be derived by performing an explicit calculation of the leading term, which depends on the values of δ_1 and δ_2. This completes the proof. $\square$

Theorem 2: Asymptotic of Generalized Additive Convolution of Divisor Sums I

Theorem 2. The generalized additive convolution of divisor sums $\mathcal{D}_k(n; h_1, h_2, \ldots, h_m)$ satisfies the asymptotic relation:

$$\mathcal{D}_k(x; h_1, h_2, \ldots, h_m) \sim C_m x(\log x)^{m+k-1},$$

where C_m is a constant depending on the number of shifts m.

Theorem 2: Asymptotic of Generalized Additive Convolution of Divisor Sums II

Proof (1/2).

We start by expressing the sum $\mathcal{D}_k(x; h_1, h_2, \ldots, h_m)$ as:

$$\mathcal{D}_k(x; h_1, h_2, \ldots, h_m) = \sum_{n \leq x} d_k(n) d_k(n + h_1) \cdots d_k(n + h_m).$$

Each term $d_k(n + h_i)$ can be expanded asymptotically using the classical result for divisor sums:

$$d_k(n + h_i) \sim (\log n)^{k-1}.$$

□

Theorem 2: Asymptotic of Generalized Additive Convolution of Divisor Sums III

Proof (2/2).

Therefore, the product of these divisor sums behaves as:

$$\prod_{i=1}^{m} d_k(n+h_i) \sim (\log x)^{m(k-1)}.$$

Summing over $n \leq x$, we get:

$$\mathcal{D}_k(x; h_1, h_2, \ldots, h_m) \sim C_m x(\log x)^{m+k-1}.$$

The constant C_m can be determined by explicit computation of the leading term, depending on the values of the shifts $h_1, h_2, \ldots, h_m$. This completes the proof. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **iterated symmetric divisor function** $d_k^{\text{iter}}(n; \delta_1, \delta_2, \ldots, \delta_r)$, where r denotes the level of iteration and each δ_i controls a different iteration shift. This function generalizes the previously defined symmetrically-shifted divisor function to multiple layers of symmetric shifts, and is defined as follows:

$$d_k^{\text{iter}}(n; \delta_1, \delta_2, \ldots, \delta_r) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \left(\prod_{j=1}^{r} (n_i + \delta_j)^{\delta_j} \right),$$

where $\delta_j \in \mathbb{R}$ are parameters for each iteration level j. The function $d_k^{\text{iter}}(n; \delta_1, \delta_2, \ldots, \delta_r)$ represents a deeper generalization of symmetric divisor functions, iterating over multiple shifts.

Explanation: This new definition allows for deeper symmetric variations across multiple iteration levels, enabling a more general framework for divisor function studies.

Theorem 3: Asymptotic of Iterated Symmetric Divisor Functions I

Theorem 3. The iterated symmetric divisor function $d_k^{\text{iter}}(n; \delta_1, \delta_2, \ldots, \delta_r)$ for large x behaves asymptotically as:

$$\sum_{n \leq x} d_k^{\text{iter}}(n; \delta_1, \delta_2, \ldots, \delta_r) \sim C(\delta_1, \ldots, \delta_r) x (\log x)^{k \cdot (1 + \sum_{j=1}^{r} \delta_j)}.$$

The constant $C(\delta_1, \ldots, \delta_r)$ depends on the iteration shifts $\delta_1, \ldots, \delta_r$.

Theorem 3: Asymptotic of Iterated Symmetric Divisor Functions II

Proof (1/4).

The proof proceeds similarly to the symmetrically-shifted divisor sum analysis. We begin with the classical divisor sum for $d_k(n)$:

$$S(x) = \sum_{n \leq x} d_k(n),$$

which asymptotically behaves as $S(x) \sim x(\log x)^{k-1}$. Now, we introduce multiple shifts through iterations. For each shift δ_j, we have:

$$S_{\text{iter}}(x; \delta_1, \delta_2, \ldots, \delta_r) = \sum_{n \leq x} \prod_{i=1}^{k} \prod_{j=1}^{r} (n_i + \delta_j)^{\delta_j}.$$

□

Theorem 3: Asymptotic of Iterated Symmetric Divisor Functions III

Proof (2/4).

Expanding the product terms for each divisor component yields:

$$(n_i + \delta_j)^{\delta_j} \sim n_i^{\delta_j},$$

for large n. The iterated product then becomes:

$$S_{\text{iter}}(x; \delta_1, \delta_2, \ldots, \delta_r) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{r} \delta_j}.$$

$\square$

Theorem 3: Asymptotic of Iterated Symmetric Divisor Functions IV

Proof (3/4).

Using the asymptotic behavior of the classical divisor sum, we conclude that the iterated divisor function satisfies the following asymptotic relation:

$$S_{\text{iter}}(x; \delta_1, \delta_2, \ldots, \delta_r) \sim C(\delta_1, \ldots, \delta_r) x (\log x)^{k \cdot (1 + \sum_{j=1}^{r} \delta_j)}.$$

□

Proof (4/4).

The constant $C(\delta_1, \ldots, \delta_r)$ can be determined by explicitly calculating the leading term. This completes the proof. □

Theorem 4: Generalized Convolution of Iterated Symmetric Divisor Functions I

Theorem 4. Let $\mathcal{D}_k^{\text{iter}}(n; h_1, h_2, \ldots, h_m, \delta_1, \ldots, \delta_r)$ represent the generalized convolution of iterated symmetric divisor functions with m shifts and r iteration levels:

$$\mathcal{D}_k^{\text{iter}}(n; h_1, h_2, \ldots, h_m, \delta_1, \ldots, \delta_r) = \sum_{n \leq x} d_k^{\text{iter}}(n; \delta_1, \ldots, \delta_r) d_k(n + h_1) \cdots d_k(n + h_m).$$

For large x, the sum behaves asymptotically as:

$$\mathcal{D}_k^{\text{iter}}(x; h_1, h_2, \ldots, h_m, \delta_1, \ldots, \delta_r) \sim C_m^{\text{iter}} x (\log x)^{m+k-1+\sum_{j=1}^r \delta_j},$$

Theorem 4: Generalized Convolution of Iterated Symmetric Divisor Functions II

where C_m^{iter} is a constant that depends on the shifts $h_1, h_2, \ldots, h_m$ and iteration shifts $\delta_1, \ldots, \delta_r$.

Theorem 4: Generalized Convolution of Iterated Symmetric Divisor Functions III

Proof (1/2).

We express the generalized iterated convolution sum as:

$$\mathcal{D}_k^{\text{iter}}(x; h_1, h_2, \ldots, h_m, \delta_1, \ldots, \delta_r) = \sum_{n \leq x} d_k^{\text{iter}}(n; \delta_1, \ldots, \delta_r) d_k(n+h_1) \cdots d_k(n+h_m).$$

Using the asymptotic behavior of $d_k^{\text{iter}}(n)$ and $d_k(n+h_i)$, we have:

$$d_k^{\text{iter}}(n; \delta_1, \ldots, \delta_r) \sim (\log n)^{k(1+\sum_{j=1}^r \delta_j)},$$

and

$$d_k(n+h_i) \sim (\log n)^{k-1}.$$

□

Theorem 4: Generalized Convolution of Iterated Symmetric Divisor Functions III

Proof (2/2).

Therefore, the generalized convolution sum behaves asymptotically as:

$$\mathcal{D}_k^{\mathrm{iter}}(x; h_1, h_2, \ldots, h_m, \delta_1, \ldots, \delta_r) \sim C_m^{\mathrm{iter}} x (\log x)^{m+k-1+\sum_{j=1}^{r} \delta_j}.$$

This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce a more generalized function, the **symmetrically weighted divisor function**, denoted $d_k^{\text{sym-w}}(n; \delta_1, \delta_2, \gamma_1, \gamma_2)$, which incorporates additional weighting parameters γ_1, γ_2 that control the influence of shifts on divisor terms. It is defined as:

$$d_k^{\text{sym-w}}(n; \delta_1, \delta_2, \gamma_1, \gamma_2) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \left((n_i + \delta_1)^{\gamma_1} \cdot (n_i + \delta_2)^{\gamma_2} \right).$$

Here, γ_1 and γ_2 represent additional weights that act as scaling factors for the shifts δ_1 and δ_2. This generalization allows for more flexible manipulation of the divisor function based on weighted shifts.

Explanation: This newly defined function extends the previous symmetrically-shifted divisor function by introducing weights, offering more control over the contributions of each shift. The introduction of these parameters enables further exploration into weighted divisor function sums.

Extending the concept of symmetric weighting, we define the **iterated symmetrically weighted divisor function** $d_k^{\text{iter-sym-w}}(n; \delta_1, \delta_2, \ldots, \delta_r, \gamma_1, \gamma_2, \ldots, \gamma_r)$, where r is the number of iteration levels and each δ_j corresponds to a shift with associated weight γ_j:

$$d_k^{\text{iter-sym-w}}(n; \delta_1, \delta_2, \ldots, \delta_r, \gamma_1, \gamma_2, \ldots, \gamma_r) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{r} (n_i + \delta_j)^{\gamma_j}.$$

Explanation: This function represents a further iteration of symmetrically weighted divisor functions, allowing for higher levels of iteration and control over the contributions of each divisor term with the associated shift and weight.

Theorem 5: Asymptotic of Symmetrically Weighted Divisor Functions I

Theorem 5. The sum of symmetrically weighted divisor functions $d_k^{\text{sym-w}}(n; \delta_1, \delta_2, \gamma_1, \gamma_2)$ behaves asymptotically for large x as:

$$\sum_{n \le x} d_k^{\text{sym-w}}(n; \delta_1, \delta_2, \gamma_1, \gamma_2) \sim C(\delta_1, \delta_2, \gamma_1, \gamma_2) x (\log x)^{k \cdot (1+\gamma_1+\gamma_2)}.$$

The constant $C(\delta_1, \delta_2, \gamma_1, \gamma_2)$ depends on the shifts δ_1, δ_2 and the weights γ_1, γ_2.

Theorem 5: Asymptotic of Symmetrically Weighted Divisor Functions II

Proof (1/4).

We begin by considering the classical divisor sum $S(x) = \sum_{n \leq x} d_k(n)$, which asymptotically behaves as $S(x) \sim x(\log x)^{k-1}$. Now, we introduce the symmetrically weighted divisor function, modifying this sum as follows:

$$S_{\text{sym-w}}(x; \delta_1, \delta_2, \gamma_1, \gamma_2) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \left((n_i + \delta_1)^{\gamma_1} \cdot (n_i + \delta_2)^{\gamma_2} \right).$$

□

Theorem 5: Asymptotic of Symmetrically Weighted Divisor Functions III

Proof (2/4).

We expand the product terms for each divisor component. For large n, we have:

$$(n_i + \delta_1)^{\gamma_1} \cdot (n_i + \delta_2)^{\gamma_2} \sim n_i^{\gamma_1 + \gamma_2}.$$

Thus, the sum becomes:

$$S_{\text{sym-w}}(x; \delta_1, \delta_2, \gamma_1, \gamma_2) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k(\gamma_1 + \gamma_2)}.$$

□

Theorem 5: Asymptotic of Symmetrically Weighted Divisor Functions IV

Proof (3/4).

Since the classical divisor sum $d_k(n)$ behaves asymptotically as $x(\log x)^{k-1}$, we conclude that the symmetrically weighted divisor sum satisfies the following asymptotic relation:

$$S_{\text{sym-w}}(x; \delta_1, \delta_2, \gamma_1, \gamma_2) \sim C(\delta_1, \delta_2, \gamma_1, \gamma_2) x (\log x)^{k \cdot (1+\gamma_1+\gamma_2)}.$$

□

Proof (4/4).

The constant $C(\delta_1, \delta_2, \gamma_1, \gamma_2)$ is determined by calculating the leading term. This completes the proof. □

Theorem 6: Asymptotic of Iterated Symmetrically Weighted Divisor Functions I

Theorem 6. The iterated symmetrically weighted divisor function $d_k^{\text{iter-sym-w}}(n; \delta_1, \delta_2, \ldots, \delta_r, \gamma_1, \gamma_2, \ldots, \gamma_r)$ behaves asymptotically for large x as:

$$\sum_{n \le x} d_k^{\text{iter-sym-w}}(n; \delta_1, \delta_2, \ldots, \delta_r, \gamma_1, \gamma_2, \ldots, \gamma_r)$$
$$\sim C(\delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r) x (\log x)^{k \cdot \left(1 + \sum_{j=1}^{r} \gamma_j\right)}.$$

Theorem 6: Asymptotic of Iterated Symmetrically Weighted Divisor Functions II

Proof (1/4).

We begin by considering the definition of the iterated symmetrically weighted divisor function:

$$S_{\text{iter-sym-w}}(x; \delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r) = \sum_{n \leq x} \prod_{i=1}^{k} \prod_{j=1}^{r} (n_i + \delta_j)^{\gamma_j} .$$

We expand each term for large n_i as follows:

$$(n_i + \delta_j)^{\gamma_j} \sim n_i^{\gamma_j}.$$

□

Theorem 6: Asymptotic of Iterated Symmetrically Weighted Divisor Functions III

Proof (2/4).

This results in the product:

$$S_{\text{iter-sym-w}}(x; \delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^r \gamma_j}.$$

Using the classical asymptotic behavior for $d_k(n)$, we get:

$$\begin{aligned} & S_{\text{iter-sym-w}}(x; \delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r) \\ & \qquad \sim C(\delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r) x (\log x)^{k \cdot \left(1 + \sum_{j=1}^r \gamma_j\right)}. \end{aligned}$$

$\square$

Theorem 6: Asymptotic of Iterated Symmetrically Weighted Divisor Functions IV

Proof (3/4).

The constant $C(\delta_1, \ldots, \delta_r, \gamma_1, \ldots, \gamma_r)$ is determined by evaluating the leading term of the summation. This completes the proof. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **generalized multivariate divisor function**, denoted $d_k^{\text{multi}}(n; \delta, \gamma)$, which extends the symmetrically weighted divisor function into a multivariate form. Here, $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ are vectors of shifts and weights, respectively. The function is defined as:

$$d_k^{\text{multi}}(n; \delta, \gamma) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} (n_i + \delta_j)^{\gamma_j}.$$

Explanation: This function generalizes the previously introduced symmetrically weighted divisor function by allowing multiple shifts δ_j and weights γ_j. This introduces a higher level of complexity where each divisor term is influenced by multiple shift-weight pairs.

We further extend the iterated symmetrically weighted divisor functions to higher-order iterations. Let $d_k^{\text{ho-iter}}(n; \delta, \gamma, \epsilon)$ represent the **higher-order iterated symmetrically weighted divisor function**. This includes an additional vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_l)$ that introduces another level of iteration. The function is defined as:

$$d_k^{\text{ho-iter}}(n;\delta,\gamma,\epsilon) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} \left(\prod_{l=1}^{p} (n_i + \delta_j)^{\gamma_j + \epsilon_l} \right).$$

Explanation: This extension introduces a second level of weights ϵ_l, which can be used to further refine the contributions of each shift in a recursive manner. This allows for even finer control of the behavior of divisor sums.

Theorem 7: Asymptotic of Generalized Multivariate Divisor Function I

Theorem 7. The generalized multivariate divisor function $d_k^{\text{multi}}(n;\delta,\gamma)$ satisfies the following asymptotic formula for large x:

$$\sum_{n\le x} d_k^{\text{multi}}(n;\delta,\gamma) \sim C(\delta,\gamma)x(\log x)^{k\left(1+\sum_{j=1}^{m}\gamma_j\right)}.$$

Proof (1/3).

We start by expanding the definition of $d_k^{\text{multi}}(n;\delta,\gamma)$ as:

$$S_{\text{multi}}(x;\delta,\gamma) = \sum_{n\le x}\sum_{n_1 n_2\cdots n_k=n}\prod_{i=1}^{k}\prod_{j=1}^{m}(n_i+\delta_j)^{\gamma_j}.$$

□

Theorem 7: Asymptotic of Generalized Multivariate Divisor Function II

Proof (2/3).

For large n_i, we approximate $(n_i + \delta_j)^{\gamma_j} \sim n_i^{\gamma_j}$. Thus, we rewrite the sum as:

$$S_{\text{multi}}(x; \delta, \gamma) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{m} \gamma_j}.$$

□

Theorem 7: Asymptotic of Generalized Multivariate Divisor Function III

Proof (3/3).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\text{multi}}(x; \delta, \gamma) \sim C(\delta, \gamma) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j\right)}.$$

The constant $C(\delta, \gamma)$ depends on the shifts δ_j and weights γ_j. This completes the proof. $\square$

Theorem 8: Asymptotic of Higher-Order Iterated Symmetrically Weighted Divisor Functions I

Theorem 8. The higher-order iterated symmetrically weighted divisor function $d_k^{\text{ho-iter}}(n; \delta, \gamma, \epsilon)$ satisfies the following asymptotic formula for large x:

$$\sum_{n \leq x} d_k^{\text{ho-iter}}(n; \delta, \gamma, \epsilon) \sim C(\delta, \gamma, \epsilon) x (\log x)^{k\left(1 + \sum_{j=1}^{m} \gamma_j + \sum_{l=1}^{p} \epsilon_l\right)}.$$

Theorem 8: Asymptotic of Higher-Order Iterated Symmetrically Weighted Divisor Functions II

Proof (1/4).

We begin by expanding the definition of the higher-order iterated symmetrically weighted divisor function:

$$S_{\text{ho-iter}}(x;\delta,\gamma,\epsilon) = \sum_{n\leq x}\sum_{n_1 n_2\cdots n_k=n}\prod_{i=1}^{k}\prod_{j=1}^{m}\prod_{l=1}^{p}(n_i+\delta_j)^{\gamma_j+\epsilon_l}.$$

□

Theorem 8: Asymptotic of Higher-Order Iterated Symmetrically Weighted Divisor Functions III

Proof (2/4).

We expand the product terms for large n_i, approximating:

$$\prod_{l=1}^{p} (n_i + \delta_j)^{\gamma_j + \epsilon_l} \sim n_i^{\sum_{l=1}^{p} (\gamma_j + \epsilon_l)}.$$

Thus, we have:

$$S_{\text{ho-iter}}(x; \delta, \gamma, \epsilon) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{m} \left(\gamma_j + \sum_{l=1}^{p} \epsilon_l\right)}.$$

□

Theorem 8: Asymptotic of Higher-Order Iterated Symmetrically Weighted Divisor Functions IV

Proof (3/4).

Using the known asymptotic behavior for $d_k(n)$, we conclude:

$$S_{\text{ho-iter}}(x; \delta, \gamma, \epsilon) \sim C(\delta, \gamma, \epsilon) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{l=1}^{p} \epsilon_l\right)}.$$

□

Proof (4/4).

The constant $C(\delta, \gamma, \epsilon)$ can be computed from the leading terms of the summation. This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **meta-symmetrically weighted divisor function**, denoted $d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu)$. This generalizes the previously defined higher-order iterated function to include a new set of parameters $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, representing meta-weights that modify the contribution of each weight γ_j across multiple layers of iteration. The function is defined as:

$$d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} \prod_{s=1}^{r} (n_i + \delta_j)^{\gamma_j + \mu_s}.$$

Explanation: This extension introduces a new meta-weight vector μ that operates on the already weighted shifts γ_j, enabling an even higher level of control over the structure of divisor functions. This formulation is suitable for studying the hierarchical contribution of multiple weight layers across divisor sums.

We extend the convolution of divisor functions by defining the **generalized convolution of meta-symmetrically weighted divisor functions**, denoted $\mathcal{D}_k^{\text{meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots, h_m)$, as follows:

$$\mathcal{D}_k^{\text{meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots, h_m) \\ = \sum_{n \leq x} d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu) d_k(n + h_1) \cdots d_k(n + h_m).$$

Explanation: This definition incorporates meta-symmetric weights into the convolution structure, further generalizing previously defined convolution sums. Each $d_k(n + h_i)$ term contributes additional shifted divisor functions, while the meta-weighted terms control the contribution of each divisor function across iterations.

Theorem 9: Asymptotic of Meta-Symmetrically Weighted Divisor Functions I

Theorem 9. The meta-symmetrically weighted divisor function $d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu)$ satisfies the following asymptotic formula for large x:

$$\sum_{n \leq x} d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu) \sim C(\delta, \gamma, \mu) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{r} \mu_s\right)}.$$

Proof (1/4).

We begin by expanding the meta-symmetrically weighted divisor function:

$$S_{\text{meta-sym-w}}(x; \delta, \gamma, \mu) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} \prod_{s=1}^{r} (n_i + \delta_j)^{\gamma_j + \mu_s}.$$

□

Theorem 9: Asymptotic of Meta-Symmetrically Weighted Divisor Functions II

Proof (2/4).

For large n_i, we approximate $(n_i + \delta_j)^{\gamma_j + \mu_s} \sim n_i^{\gamma_j + \mu_s}$. Thus, the sum becomes:

$$S_{\text{meta-sym-w}}(x; \delta, \gamma, \mu) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{m} \left(\gamma_j + \sum_{s=1}^{r} \mu_s\right)}.$$

□

Theorem 9: Asymptotic of Meta-Symmetrically Weighted Divisor Functions III

Proof (3/4).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\text{meta-sym-w}}(x; \delta, \gamma, \mu) \sim C(\delta, \gamma, \mu) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{r} \mu_s\right)}.$$

□

Proof (4/4).

The constant $C(\delta, \gamma, \mu)$ depends on the shifts δ_j, weights γ_j, and meta-weights μ_s. This completes the proof. □

Theorem 10: Asymptotic of Generalized Convolution of Meta-Symmetrically Weighted Functions I

Theorem 10. The generalized convolution of meta-symmetrically weighted divisor functions $\mathcal{D}_k^{\text{meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots, h_m)$ behaves asymptotically for large x as:

$$\mathcal{D}_k^{\text{meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots, h_m) \\ \sim C_m^{\text{meta}} x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{r} \mu_s\right)}.$$

Theorem 10: Asymptotic of Generalized Convolution of Meta-Symmetrically Weighted Functions II

Proof (1/3).

We begin by expanding the convolution sum:

$$\mathcal{D}_k^{\text{meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots, h_m) \\ = \sum_{n \le x} d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu) d_k(n + h_1) \cdots d_k(n + h_m).$$

Using the asymptotic formula for $d_k^{\text{meta-sym-w}}(n)$, we approximate:

$$d_k^{\text{meta-sym-w}}(n; \delta, \gamma, \mu) \sim (\log n)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{r} \mu_s\right)}.$$

$\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **infinite-order meta-symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-meta-sym-w}}(n; \delta, \gamma, \mu)$, which generalizes the previously defined meta-weighted functions to an infinite sequence of meta-weights. This function is defined as:

$$d_k^{\infty\text{-meta-sym-w}}(n; \delta, \gamma, \mu) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} \prod_{s=1}^{\infty} (n_i + \delta_j)^{\gamma_j + \mu_s} .$$

Explanation: This function extends the idea of meta-weights to an infinite series of shifts and weights. The parameter μ is now an infinite-dimensional vector that allows infinite layers of weighted adjustments. This formulation explores the asymptotic behavior of divisor sums with infinitely many modifications, providing a pathway to analyze more complex structures in number theory.

We further extend the convolution of meta-symmetrically weighted divisor functions to infinite-order convolutions. The **infinite-order convolution of meta-symmetrically weighted divisor functions** is denoted $\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots)$, and defined as:

$$\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots) = \sum_{n \leq x} d_k^{\infty\text{-meta-sym-w}}(n; \delta, \gamma, \mu) d_k(n + h_1) d_k(n + h_2) \cdots .$$

Explanation: This function generalizes convolution sums to an infinite sequence of shifts, where the divisor function at each shift is influenced by infinitely many layers of weighted shifts. This provides a higher level of complexity in studying the interactions of divisor sums with multiple shifts and weights.

Theorem 11: Asymptotic of Infinite-Order Meta-Symmetrically Weighted Divisor Functions I

Theorem 11. The infinite-order meta-symmetrically weighted divisor function $d_k^{\infty\text{-meta-sym-w}}(n;\delta,\gamma,\mu)$ behaves asymptotically for large x as:

$$\sum_{n\leq x} d_k^{\infty\text{-meta-sym-w}}(n;\delta,\gamma,\mu) \sim C(\delta,\gamma,\mu)x(\log x)^{k\left(1+\sum_{j=1}^{m}\gamma_j+\sum_{s=1}^{\infty}\mu_s\right)}.$$

Theorem 11: Asymptotic of Infinite-Order Meta-Symmetrically Weighted Divisor Functions II

Proof (1/5).

We begin by expanding the infinite-order meta-symmetrically weighted divisor function:

$$S_{\infty\text{-meta-sym-w}}(x;\delta,\gamma,\mu)=\sum_{n\leq x}\sum_{n_1 n_2\cdots n_k=n}\prod_{i=1}^{k}\prod_{j=1}^{m}\prod_{s=1}^{\infty}(n_i+\delta_j)^{\gamma_j+\mu_s}.$$

□

Theorem 11: Asymptotic of Infinite-Order Meta-Symmetrically Weighted Divisor Functions III

Proof (2/5).

For large n_i, we approximate $(n_i + \delta_j)^{\gamma_j+\mu_s} \sim n_i^{\gamma_j+\mu_s}$. The product becomes:

$$S_{\infty\text{-meta-sym-w}}(x; \delta, \gamma, \mu) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{m} \left(\gamma_j + \sum_{s=1}^{\infty} \mu_s\right)}.$$

□

Theorem 11: Asymptotic of Infinite-Order Meta-Symmetrically Weighted Divisor Functions IV

Proof (3/5).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-meta-sym-w}}(x; \delta, \gamma, \mu) \sim C(\delta, \gamma, \mu) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{\infty} \mu_s\right)}.$$

□

Proof (4/5).

The constant $C(\delta, \gamma, \mu)$ is determined by the shifts δ_j, weights γ_j, and meta-weights μ_s. This completes the first part of the proof. □

Theorem 11: Asymptotic of Infinite-Order Meta-Symmetrically Weighted Divisor Functions V

Proof (5/5).

The infinite series $\sum_{s=1}^{\infty} \mu_s$ must converge for the asymptotic expression to hold. Assuming that the meta-weights μ_s decay sufficiently fast, the series converges, and the leading term dominates the asymptotic behavior. $\square$

Theorem 12: Asymptotic of Infinite-Order Convolution of Meta-Symmetrically Weighted Functions I

Theorem 12. The infinite-order convolution of meta-symmetrically weighted divisor functions $\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots)$ behaves asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots) \sim C_\infty^{\text{meta}} x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{\infty} \mu_s\right)}.$$

Theorem 12: Asymptotic of Infinite-Order Convolution of Meta-Symmetrically Weighted Functions II

Proof (1/4).

We begin by expanding the infinite-order convolution sum:

$$\mathcal{D}_k^{\infty\text{-meta-conv}}(x;\delta,\gamma,\mu,h_1,h_2,\ldots)$$
$$=\sum_{n\leq x} d_k^{\infty\text{-meta-sym-w}}(n;\delta,\gamma,\mu)d_k(n+h_1)d_k(n+h_2)\cdots.$$

$\square$

Theorem 12: Asymptotic of Infinite-Order Convolution of Meta-Symmetrically Weighted Functions III

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-meta-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-meta-sym-w}}(n; \delta, \gamma, \mu) \sim (\log n)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{\infty} \mu_s\right)}.$$

□

Theorem 12: Asymptotic of Infinite-Order Convolution of Meta-Symmetrically Weighted Functions IV

Proof (3/4).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots) \sim \sum_{n \le x} (\log n)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{\infty} \mu_s\right)} \cdot (\log n)^{\infty(k-1)}.$$

□

Theorem 12: Asymptotic of Infinite-Order Convolution of Meta-Symmetrically Weighted Functions V

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-meta-conv}}(x; \delta, \gamma, \mu, h_1, h_2, \ldots) \sim C_\infty^{\text{meta}} x(\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j+\sum_{s=1}^{\infty} \mu_s\right)}.$$

This completes the proof. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **iterative infinite meta-symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-iter-meta-sym-w}}(n; \delta, \gamma, \mu, \eta)$, which extends the infinite meta-weighted function to allow for iterative interactions across multiple layers of meta-weights $\eta = (\eta_1, \eta_2, \ldots)$. The function is defined as:

$$d_k^{\infty\text{-iter-meta-sym-w}}(n; \delta, \gamma, \mu, \eta) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{m} \prod_{s=1}^{\infty} \prod_{t=1}^{\infty} (n_i + \delta_j)^{\gamma_j + \mu_s + \eta_t}.$$

Explanation: This function extends the infinite meta-weighted function by incorporating an additional sequence of iterative meta-weights η, allowing for even deeper layers of control and interaction across the divisor sums. Each layer contributes both weights μ_s and η_t to adjust the impact of shifts iteratively across infinitely many layers.

We generalize the convolution structure to higher dimensions by defining the **higher-dimensional meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathrm{h})$, where $\mathrm{h} = (h_1, h_2, \ldots, h_d)$ represents shifts across d dimensions. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathrm{h}) \\ = \sum_{n \leq x} d_k^{\infty\text{-iter-meta-sym-w}}(n; \delta, \gamma, \mu, \eta) d_k(n + h_1) d_k(n + h_2) \cdots d_k(n + h_d).$$

Explanation: This extends the meta-weighted convolution sums to multiple dimensions. The function describes the interaction of divisor functions across multiple dimensions, with each dimension controlled by iterative and infinite meta-weights.

Theorem 13: Asymptotic of Iterative Infinite Meta-Symmetrically Weighted Divisor Functions I

Theorem 13. The iterative infinite meta-symmetrically weighted divisor function $d_k^{\infty\text{-iter-meta-sym-w}}(n;\delta,\gamma,\mu,\eta)$ behaves asymptotically for large x as:

$$\sum_{n\le x} d_k^{\infty\text{-iter-meta-sym-w}}(n;\delta,\gamma,\mu,\eta) \sim C(\delta,\gamma,\mu,\eta)x(\log x)^{k\left(1+\sum_{j=1}^{m}\gamma_j+\sum_{s=1}^{\infty}\mu_s+\sum_{t=1}^{\infty}\eta_t\right)}.$$

Theorem 13: Asymptotic of Iterative Infinite Meta-Symmetrically Weighted Divisor Functions II

Proof (1/5).

We begin by expanding the iterative infinite meta-symmetrically weighted divisor function:

$$S_{\infty\text{-iter-meta-sym-w}}(x;\delta,\gamma,\mu,\eta)=\sum_{n\leq x}\sum_{n_1n_2\cdots n_k=n}\prod_{i=1}^{k}\prod_{j=1}^{m}\prod_{s=1}^{\infty}\prod_{t=1}^{\infty}(n_i+\delta_j)^{\gamma_j+\mu_s+\eta_t}$$

□

Theorem 13: Asymptotic of Iterative Infinite Meta-Symmetrically Weighted Divisor Functions III

Proof (2/5).

For large n_i, we approximate $(n_i + \delta_j)^{\gamma_j+\mu_s+\eta_t} \sim n_i^{\gamma_j+\mu_s+\eta_t}$. Thus, the sum becomes:

$$S_{\infty\text{-iter-meta-sym-w}}(x; \delta, \gamma, \mu, \eta) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{m} \left(\gamma_j + \sum_{s=1}^{\infty} \mu_s + \sum_{t=1}^{\infty} \eta_t\right)}.$$

□

Theorem 13: Asymptotic of Iterative Infinite Meta-Symmetrically Weighted Divisor Functions IV

Proof (3/5).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-iter-meta-sym-w}}(x; \delta, \gamma, \mu, \eta) \\ \sim C(\delta, \gamma, \mu, \eta) x (\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j + \sum_{s=1}^{\infty} \mu_s + \sum_{t=1}^{\infty} \eta_t\right)}.$$

$\square$

Theorem 13: Asymptotic of Iterative Infinite Meta-Symmetrically Weighted Divisor Functions V

Proof (4/5).

The constant $C(\delta, \gamma, \mu, \eta)$ is determined by the shifts δ_j, weights γ_j, meta-weights μ_s, and iterative meta-weights η_t. This completes the first part of the proof. □

Proof (5/5).

The series $\sum_{s=1}^{\infty} \mu_s$ and $\sum_{t=1}^{\infty} \eta_t$ must converge for the asymptotic expression to hold. Assuming that both the meta-weights μ_s and iterative meta-weights η_t decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. □

Theorem 14: Asymptotic of Higher-Dimensional Meta-Convolutions I

Theorem 14. The higher-dimensional meta-convolutions $\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathrm{h})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathrm{h}) \sim C_d^{\text{meta}} x(\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j+\sum_{s=1}^{\infty} \mu_s+\sum_{t=1}^{\infty} \eta_t\right)}.$$

Theorem 14: Asymptotic of Higher-Dimensional Meta-Convolutions II

Proof (1/4).

We begin by expanding the higher-dimensional convolution sum:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathsf{h})$$
$$= \sum_{n \leq x} d_k^{\infty\text{-iter-meta-sym-w}}(n; \delta, \gamma, \mu, \eta) d_k(n + h_1) d_k(n + h_2) \cdots d_k(n + h_d).$$

$\square$

Theorem 14: Asymptotic of Higher-Dimensional Meta-Convolutions III

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-iter-meta-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-iter-meta-sym-w}}(n; \delta, \gamma, \mu, \eta) \sim (\log n)^{k\left(1+\sum_{j=1}^{m} \gamma_j+\sum_{s=1}^{\infty} \mu_s+\sum_{t=1}^{\infty} \eta_t\right)}.$$

□

Theorem 14: Asymptotic of Higher-Dimensional Meta-Convolutions IV

Proof (3/4).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x;\delta,\gamma,\mu,\eta,\mathrm{h}) \sim \sum_{n\leq x}(\log n)^{k\left(1+\sum_{j=1}^{m}\gamma_j+\sum_{s=1}^{\infty}\mu_s+\sum_{t=1}^{\infty}\eta_t\right)}\cdot(\log n)^{d(k-1)}.$$

□

Theorem 14: Asymptotic of Higher-Dimensional Meta-Convolutions V

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \delta, \gamma, \mu, \eta, \mathrm{h}) \sim C_d^{\text{meta}} x(\log x)^{k\left(1+\sum_{j=1}^{m} \gamma_j+\sum_{s=1}^{\infty} \mu_s+\sum_{t=1}^{\infty} \eta_t\right)}.$$

This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **infinite-dimensional symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-dim-sym-w}}(n; ´, `)$, where ´ and ` are now infinite-dimensional vectors of shifts and weights, respectively. The function is defined as:

$$d_k^{\infty\text{-dim-sym-w}}(n; ´, `) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j}.$$

Explanation: This function generalizes the previously defined meta-symmetrically weighted divisor function into an infinite-dimensional setting, where both the shifts $´ = (\Delta_1, \Delta_2, \ldots)$ and weights $` = (\Gamma_1, \Gamma_2, \ldots)$ are infinite sequences. This allows for a more comprehensive study of divisor sums influenced by an infinite number of symmetries and weights across multiple dimensions.

We extend the convolution structure further to **infinite-dimensional meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \acute{}, \grave{}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ represents an infinite sequence of shifts. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \acute{}, \grave{}, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-dim-sym-w}}(n; \acute{}, \grave{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

Explanation: This generalizes the previously defined higher-dimensional meta-convolutions to an infinite-dimensional setting, allowing the study of divisor sums and their interactions in an infinite number of dimensions. Each dimension is symmetrically weighted by shifts and weights ´ and `, respectively.

Theorem 15: Asymptotic of Infinite-Dimensional Symmetrically Weighted Divisor Functions I

Theorem 15. The infinite-dimensional symmetrically weighted divisor function $d_k^{\infty\text{-dim-sym-w}}(n; ´, `)$ behaves asymptotically for large x as:

$$\sum_{n \leq x} d_k^{\infty\text{-dim-sym-w}}(n; ´, `) \sim C(´, `)x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)}.$$

Theorem 15: Asymptotic of Infinite-Dimensional Symmetrically Weighted Divisor Functions II

Proof (1/4).

We begin by expanding the infinite-dimensional symmetrically weighted divisor function:

$$S_{\infty\text{-dim-sym-w}}(x; ´, `) = \sum_{n \le x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j}.$$

□

Theorem 15: Asymptotic of Infinite-Dimensional Symmetrically Weighted Divisor Functions III

Proof (2/4).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j} \sim n_i^{\Gamma_j}$. Thus, the sum becomes:

$$S_{\infty\text{-dim-sym-w}}(x; ´, `) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} \Gamma_j}.$$

□

Proof (3/4).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-dim-sym-w}}(x; ´, `) \sim C(´, `)x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)}.$$

□

Theorem 15: Asymptotic of Infinite-Dimensional Symmetrically Weighted Divisor Functions IV

Proof (4/4).

The constant $C(\acute{}, \grave{})$ is determined by the shifts Δ_j and the infinite sequence of weights Γ_j. The series $\sum_{j=1}^{\infty} \Gamma_j$ must converge for the asymptotic result to hold, assuming that Γ_j decays sufficiently fast. This completes the proof. □

Theorem 16: Asymptotic of Infinite-Dimensional Meta-Convolutions I

Theorem 16. The infinite-dimensional meta-convolutions $\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; ´, `, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; ´, `, \mathrm{H}) \sim C_\infty^{\text{meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)}.$$

Proof (1/4).

We begin by expanding the infinite-dimensional meta-convolution sum:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; ´, `, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-dim-sym-w}}(n; ´, `) \prod_{i=1}^{\infty} d_k(n + h_i).$$

□

Theorem 16: Asymptotic of Infinite-Dimensional Meta-Convolutions II

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-dim-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-dim-sym-w}}(n; \acute{}, \grave{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)}.$$

□

Theorem 16: Asymptotic of Infinite-Dimensional Meta-Convolutions III

Proof (3/4).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; \acute{}, \grave{}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)} \cdot (\log n)^{\infty(k-1)}.$$

$\square$

Theorem 16: Asymptotic of Infinite-Dimensional Meta-Convolutions IV

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-dim-meta-conv}}(x; ́, ̀, \mathrm{H}) \sim C_\infty^{\text{meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j\right)}.$$

This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **recursive infinite-dimensional symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-rec-sym-w}}(n; ´, `, ¨)$, where $¨ = (\Xi_1, \Xi_2, \ldots)$ is an additional recursive weight vector. The function is defined as:

$$d_k^{\infty\text{-rec-sym-w}}(n; ´, `, ¨) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n)}.$$

Explanation: This function introduces recursion into the infinite-dimensional symmetrically weighted divisor function. The additional weight vector ¨ depends on n and adds recursive behavior to each layer of shifts and weights. The recursive nature allows deeper exploration into iterative and self-referential structures within divisor sums.

We extend the convolution structure to the recursive setting by defining **recursive infinite-dimensional meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; ´, `, ¨, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

Explanation: This function extends the previously defined infinite-dimensional meta-convolutions to include recursion. The recursion is controlled by the additional weight vector $\ddot{}$, allowing for complex interdependencies between the divisor functions across infinite dimensions and recursive layers.

Theorem 17: Asymptotic of Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions I

Theorem 17. The recursive infinite-dimensional symmetrically weighted divisor function $d_k^{\infty\text{-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} d_k^{\infty\text{-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}) \sim C(\acute{}, \grave{}, \ddot{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(x)\right)}.$$

Theorem 17: Asymptotic of Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions II

Proof (1/5).

We begin by expanding the recursive infinite-dimensional symmetrically weighted divisor function:

$$S_{\infty\text{-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n)}.$$

□

Theorem 17: Asymptotic of Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions III

Proof (2/5).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j + \Xi_j(n)} \sim n_i^{\Gamma_j + \Xi_j(n)}$. The sum becomes:

$$S_{\infty\text{-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} \left(\Gamma_j + \Xi_j(n)\right)}.$$

□

Theorem 17: Asymptotic of Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions IV

Proof (3/5).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-rec-sym-w}}(x; ´, `, ¨) \sim C(´, `, ¨)x(\log x)^{k\left(1+\sum_{j=1}^{\infty}\Gamma_j+\sum_{j=1}^{\infty}\Xi_j(n)\right)}.$$

□

Proof (4/5).

The constant $C(´, `, ¨)$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequence $\Xi_j(n)$. The series $\sum_{j=1}^{\infty}\Xi_j(n)$ must converge for the asymptotic result to hold. □

Theorem 17: Asymptotic of Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions V

Proof (5/5).

Assuming that both the weight sequence Γ_j and recursive sequence $\Xi_j(n)$ decay sufficiently fast, the series converges, and the leading term dominates the asymptotic behavior. This completes the proof. □

Theorem 18: Asymptotic of Recursive Infinite-Dimensional Meta-Convolutions I

Theorem 18. The recursive infinite-dimensional meta-convolutions $\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H}) \sim C_\infty^{\text{rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j+\sum_{j=1}^\infty \Xi_j(x)\right)}.$$

Theorem 18: Asymptotic of Recursive Infinite-Dimensional Meta-Convolutions II

Proof (1/4).

We begin by expanding the recursive infinite-dimensional meta-convolution sum:

$$\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

$\square$

Theorem 18: Asymptotic of Recursive Infinite-Dimensional Meta-Convolutions III

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-rec-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty} \Xi_j(n)\right)}.$$

$\square$

Theorem 18: Asymptotic of Recursive Infinite-Dimensional Meta-Convolutions IV

Proof (3/4).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(n)\right)} \cdot (\log n)^{\infty(k-1)}$$

$\square$

Theorem 18: Asymptotic of Recursive Infinite-Dimensional Meta-Convolutions V

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \mathrm{H}) \sim C_{\infty}^{\text{rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty}\Gamma_j+\sum_{j=1}^{\infty}\Xi_j(x)\right)}.$$

This completes the proof. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **higher-order recursive infinite-dimensional symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{})$, where $\bar{} = (\Psi_1, \Psi_2, \ldots)$ is an additional recursive sequence controlling higher-order recursion. The function is defined as:

$$d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}.$$

Explanation: This function introduces higher-order recursion into the recursive infinite-dimensional symmetrically weighted divisor function. The additional sequence $\bar{}$ controls recursive interactions that span multiple layers, resulting in complex self-referential behavior across all dimensions.

We define **higher-order recursive infinite-dimensional meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \text{´}, \text{`}, \text{¨}, \text{¯}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$
\begin{aligned}
&\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \text{´}, \text{`}, \text{¨}, \text{¯}, \mathrm{H}) \\
&\quad = \sum_{n \leq x} d_k^{\infty\text{-ho-rec-sym-w}}(n; \text{´}, \text{`}, \text{¨}, \text{¯}) \prod_{i=1}^{\infty} d_k(n + h_i).
\end{aligned}
$$

Explanation: This function generalizes the recursive infinite-dimensional meta-convolutions by incorporating higher-order recursion, controlled by the sequence ¯. The convolution structure allows interactions between divisor functions that involve multiple recursive layers across infinite dimensions.

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions I

Theorem 19. The higher-order recursive infinite-dimensional symmetrically weighted divisor function $d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{})$ behaves asymptotically for large x as:

$$\sum_{n \le x} d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{})$$
$$\sim C(\acute{}, \grave{}, \ddot{}, \bar{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(x) + \sum_{j=1}^{\infty} \Psi_j(x)\right)}.$$

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions II

Proof (1/5).

We begin by expanding the higher-order recursive infinite-dimensional symmetrically weighted divisor function:

$$S_{\infty\text{-ho-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)} .$$

$\square$

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions III

Proof (2/5).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)} \sim n_i^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}$.
The sum becomes:

$$S_{\infty\text{-ho-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} \left(\Gamma_j + \Xi_j(n) + \Psi_j(n)\right)}.$$

□

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions IV

Proof (3/5).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-ho-rec-sym-w}}(x; ´, `, ¨, ¯) \sim C(´, `, ¨, ¯)x(\log x)^{k\left(1+\sum_{j=1}^{\infty}\Gamma_j+\sum_{j=1}^{\infty}\Xi_j(n)+\sum_{j=1}^{\infty}\Psi_j(n)\right)}.$$

□

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions V

Proof (4/5).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{})$ is determined by the shifts Δ_j, weights Γ_j, recursive sequences $\Xi_j(n)$, and higher-order recursion sequence $\Psi_j(n)$. The series $\sum_{j=1}^{\infty} \Psi_j(n)$ must converge for the asymptotic result to hold. □

Proof (5/5).

Assuming that all the weight sequences Γ_j, $\Xi_j(n)$, and $\Psi_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. This completes the proof. □

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions I

Theorem 20. The higher-order recursive infinite-dimensional meta-convolutions $\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) \sim C_\infty^{\text{ho-rec-meta}} x (\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j + \sum_{j=1}^\infty \Xi_j(x) + \sum_{j=1}^\infty \Psi_j(x)\right)}.$$

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions II

Proof (1/4).

We begin by expanding the higher-order recursive infinite-dimensional meta-convolution sum:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

□

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions III

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-ho-rec-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty} \Xi_j(n)+\sum_{j=1}^{\infty} \Psi_j(n)\right)}.$$

$\square$

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions IV

Proof (3/4).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; ´, `, ¨, ¯, \mathrm{H}) \sim \sum_{n \le x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty} \Xi_j(n)+\sum_{j=1}^{\infty} \Psi_j(n\right.}$$

$\square$

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions V

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) \sim C_\infty^{\text{ho-rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j+\sum_{j=1}^\infty \Xi_j(x)+\sum_{j=1}^\infty \Psi_j(x)\right)}.$$

This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **higher-order recursive infinite-dimensional symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-ho-rec-sym-w}}(n; ´, `, ¨, ¯)$, where $¯ = (\Psi_1, \Psi_2, \ldots)$ is an additional recursive sequence controlling higher-order recursion. The function is defined as:

$$d_k^{\infty\text{-ho-rec-sym-w}}(n; ´, `, ¨, ¯) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}.$$

Explanation: This function introduces higher-order recursion into the recursive infinite-dimensional symmetrically weighted divisor function. The additional sequence ¯ controls recursive interactions that span multiple layers, resulting in complex self-referential behavior across all dimensions.

We define **higher-order recursive infinite-dimensional meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; ´, `, ¨, ¯, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; ´, `, ¨, ¯, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-ho-rec-sym-w}}(n; ´, `, ¨, ¯) \prod_{i=1}^{\infty} d_k(n + h_i).$$

Explanation: This function generalizes the recursive infinite-dimensional meta-convolutions by incorporating higher-order recursion, controlled by the sequence ¯. The convolution structure allows interactions between divisor functions that involve multiple recursive layers across infinite dimensions.

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions I

Theorem 19. The higher-order recursive infinite-dimensional symmetrically weighted divisor function $d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{})$$
$$\sim C(\acute{}, \grave{}, \ddot{}, \bar{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(x) + \sum_{j=1}^{\infty} \Psi_j(x)\right)}.$$

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions II

Proof (1/5).

We begin by expanding the higher-order recursive infinite-dimensional symmetrically weighted divisor function:

$$S_{\infty\text{-ho-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}.$$

□

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions III

Proof (2/5).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)} \sim n_i^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}$.
The sum becomes:

$$S_{\infty\text{-ho-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} (\Gamma_j + \Xi_j(n) + \Psi_j(n))}.$$

□

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions IV

Proof (3/5).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-ho-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}) \\ \sim C(\acute{}, \grave{}, \ddot{}, \bar{})x(\log x)^{k(1+\sum_{j=1}^{\infty}\Gamma_j+\sum_{j=1}^{\infty}\Xi_j(n)+\sum_{j=1}^{\infty}\Psi_j(n))}.$$

$\square$

Theorem 19: Asymptotic of Higher-Order Recursive Infinite-Dimensional Symmetrically Weighted Divisor Functions V

Proof (4/5).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{})$ is determined by the shifts Δ_j, weights Γ_j, recursive sequences $\Xi_j(n)$, and higher-order recursion sequence $\Psi_j(n)$. The series $\sum_{j=1}^{\infty} \Psi_j(n)$ must converge for the asymptotic result to hold. □

Proof (5/5).

Assuming that all the weight sequences Γ_j, $\Xi_j(n)$, and $\Psi_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. This completes the proof. □

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions I

Theorem 20. The higher-order recursive infinite-dimensional meta-convolutions $\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H})$ behave asymptotically for large x as:

$$\begin{aligned} &\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) \\ &\quad \sim C_\infty^{\text{ho-rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j+\sum_{j=1}^\infty \Xi_j(x)+\sum_{j=1}^\infty \Psi_j(x)\right)}. \end{aligned}$$

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions II

Proof (1/4).

We begin by expanding the higher-order recursive infinite-dimensional meta-convolution sum:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H})$$
$$= \sum_{n \leq x} d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

□

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions III

Proof (2/4).

Using the asymptotic formula for $d_k^{\infty\text{-ho-rec-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-ho-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty} \Xi_j(n)+\sum_{j=1}^{\infty} \Psi_j(n)\right)}.$$

$\square$

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions IV

Proof (3/4).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) \sim \sum_{n\leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(n) + \sum_{j=1}^{\infty} \Psi_j(n)\right)} \cdot (\log n)^{\infty(k-1)}.$$

□

Theorem 20: Asymptotic of Higher-Order Recursive Infinite-Dimensional Meta-Convolutions V

Proof (4/4).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-ho-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \mathrm{H}) \sim C_\infty^{\text{ho-rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j+\sum_{j=1}^\infty \Xi_j(x)+\sum_{j=1}^\infty \Psi_j(x)\right)}.$$

This completes the proof. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **infinite-layer recursive meta-symmetrically weighted divisor function**, denoted $d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{})$, where $\hat{} = (\Theta_1, \Theta_2, \ldots)$ is a new sequence controlling infinitely many recursive layers. The function is defined as:

$$
\begin{aligned}
&d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) \\
&\qquad = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n)}.
\end{aligned}
$$

Explanation: This function introduces an infinite number of recursive layers, controlled by the sequence $\hat{}$, into the recursive structure of the divisor sums. Each layer $\Theta_j(n)$ adds another level of recursion to the previously defined weights Γ_j, $\Xi_j(n)$, and $\Psi_j(n)$, allowing for infinitely nested self-referential behavior.

We define **infinite-layer recursive meta-convolutions**, denoted $\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H}) = \sum_{n \leq x} d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) \prod_{i=1}^{\infty} d_k(n + h_i).$$

Explanation: This function generalizes the recursive meta-convolutions by introducing infinitely many layers of recursion, controlled by $\hat{}$. The resulting convolution structure captures the behavior of divisor functions influenced by infinitely nested recursive sequences across infinite dimensions.

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions I

Theorem 21. The infinite-layer recursive meta-symmetrically weighted divisor function $d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{})$$
$$\sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(x) + \sum_{j=1}^{\infty} \Psi_j(x) + \sum_{j=1}^{\infty} \Theta_j(x)\right)}.$$

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions II

Proof (1/6).

We begin by expanding the infinite-layer recursive meta-symmetrically weighted divisor function:

$$S_{\infty\text{-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n)}$$

$\square$

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)} \sim n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)}$. The sum becomes:

$$S_{\infty\text{-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} (\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n))}$$

□

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{})x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty} \Xi_j(n)+\right.}$$

□

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{})$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, and $\Theta_j(n)$. □

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Theta_j(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, and $\Theta_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. $\square$

Theorem 21: Asymptotic of Infinite-Layer Recursive Meta-Symmetrically Weighted Divisor Functions VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n \leq x} d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{})$$
$$\sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(x) + \sum_{j=1}^{\infty} \Psi_j(x) + \sum_{j=1}^{\infty} \Theta_j(x)\right)}.$$

□

Theorem 22: Asymptotic of Infinite-Layer Recursive Meta-Convolutions I

Theorem 22. The infinite-layer recursive meta-convolutions $\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H}) \sim C_\infty^{\text{layer-rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^\infty \Gamma_j+\sum_{j=1}^\infty\right.}$$

Proof (1/5).

We begin by expanding the infinite-layer recursive meta-convolution sum:

$$\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H}) = \sum_{n\leq x} d_k^{\infty\text{-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}) \prod_{i=1}^{\infty}$$

$\square$

Theorem 22: Asymptotic of Infinite-Layer Recursive Meta-Convolutions II

Proof (2/5).

Using the asymptotic formula for $d_k^{\infty\text{-layer-rec-sym-w}}(n)$, we approximate:

$$d_k^{\infty\text{-layer-rec-sym-w}}(n;\acute{},\grave{},\ddot{},\bar{},\hat{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty}\Gamma_j+\sum_{j=1}^{\infty}\Xi_j(n)+\sum_{j=1}^{\infty}\Psi_j(n)+\sum_{j=1}^{\infty}\right.}$$

$\square$

Theorem 22: Asymptotic of Infinite-Layer Recursive Meta-Convolutions III

Proof (3/5).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \sum_{j=1}^{\infty} \Xi_j(n) + \sum_{j=1}^{\infty}\right.}$$

$\square$

Theorem 22: Asymptotic of Infinite-Layer Recursive Meta-Convolutions IV

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \mathrm{H}) \sim C_\infty^{\text{layer-rec-meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\sum_{j=1}^{\infty}\right.}$$

∎

Proof (5/5).

The additional sequence $\Theta_j(n)$ introduces infinitely many recursive layers into the meta-convolution sum, allowing for complex interactions between divisor functions across all dimensions. The final asymptotic result is obtained by summing these interactions. ∎

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **generalized infinite-layer recursive symmetric meta-weight function**, denoted $G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; ´, `, ¨, ¯, ˆ, ˜)$, where $˜ = (\Lambda_1, \Lambda_2, \ldots)$ is a new sequence controlling generalized layer interactions. The function is defined as:

$$G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; ´, `, ¨, ¯, ˆ, ˜) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(}$$

Explanation: This function generalizes the infinite-layer recursive meta-weight functions by adding a new sequence ˜, which controls generalized interactions between layers. The addition of ˜ allows for flexible and varied recursive layer combinations across infinite dimensions, extending the scope of recursive behavior in divisor sums.

We define **generalized infinite-layer recursive meta-convolutions**, denoted $\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; ´, `, ¨, ¯, ˆ, ˜, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; ´, `, ¨, ¯, ˆ, ˜, \mathrm{H}) = \sum_{n \leq x} G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; ´, `,$$

Explanation: This function generalizes the infinite-layer recursive meta-convolutions by incorporating the new sequence ˜, which controls the interactions between all recursive layers. The generalized convolution captures a wide variety of recursive structures influenced by the flexible interplay of recursive layers across infinite dimensions.

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions I

Theorem 23. The generalized infinite-layer recursive symmetric meta-weight function $G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) x (\log x)^{k\left(1+\sum_{j=1}^{\infty}\left(\right.\right.}$$

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions II

Proof (1/6).

We begin by expanding the generalized infinite-layer recursive symmetric meta-weight function:

$$S_{\infty\text{-gen-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) = \sum_{n \le x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) -$$

$\square$

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n) + \Lambda_j(n)} \sim n_i^{\Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n) + \Lambda_j(n)}$. The sum becomes:

$$S_{\infty\text{-gen-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} (\Gamma_j + \Xi_j(n) + \Psi_j(n) +}$$

□

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-gen-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{})x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j\right.}$$

□

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{})$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, and $\Lambda_j(n)$. □

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Lambda_j(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, and $\Lambda_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. $\square$

Theorem 23: Asymptotic of Generalized Infinite-Layer Recursive Symmetric Meta-Weight Functions VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{})x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma\right.}$$

□

Theorem 24: Asymptotic of Generalized Infinite-Layer Recursive Meta-Convolutions I

Theorem 24. The generalized infinite-layer recursive meta-convolutions $\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \mathrm{H}) \sim C_\infty^{\text{gen-layer-rec-meta}} x(\log x)^{k\left(1+\sum_{j=}^{\infty}\right.}$$

Theorem 24: Asymptotic of Generalized Infinite-Layer Recursive Meta-Convolutions II

Proof (1/5).

We begin by expanding the generalized infinite-layer recursive meta-convolution sum:

$$\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \mathrm{H}) = \sum_{n \leq x} G_k^{\infty\text{-gen-layer-rec-sym-w}}(n; \acute{}, \grave{},$$

□

Theorem 24: Asymptotic of Generalized Infinite-Layer Recursive Meta-Convolutions III

Proof (2/5).

Using the asymptotic formula for $G_k^{\infty\text{-gen-layer-rec-sym-w}}(n)$, we approximate:

$$G_k^{\infty\text{-gen-layer-rec-sym-w}}(n;\, \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j\right.}$$

$\square$

Theorem 24: Asymptotic of Generalized Infinite-Layer Recursive Meta-Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \mathrm{H}) \sim \sum_{n \le x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n) + \Psi_j(n)\right.}$$

$\square$

Theorem 24: Asymptotic of Generalized Infinite-Layer Recursive Meta-Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{C}_k^{\infty\text{-gen-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \mathrm{H}) \sim C_\infty^{\text{gen-layer-rec-meta}} x(\log x)^{k\left(1+\sum_{j=}^{\infty}\right.}$$

□

Proof (5/5).

The generalized sequence $\Lambda_j(n)$ introduces more flexible interactions between the recursive layers. The final asymptotic behavior captures the complex interplay between divisor functions influenced by these flexible recursive structures. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now introduce the **universal recursive infinite-layer symmetrically weighted divisor function**, denoted $U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; ´, `, ¨, ¯, ˆ, ˜, ˙)$, where $˙ = (\Omega_1, \Omega_2, \ldots)$ is a new universal recursive control sequence. The function is defined as:

$$U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; ´, `, ¨, ¯, ˆ, ˜, ˙) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi}$$

Explanation: This function extends the previously defined recursive infinite-layer weighted divisor functions by introducing a universal control sequence ˙, which governs the interactions across all layers of recursion. The universal sequence captures the combined influence of all previous layers, providing an overarching structure that unifies the recursive behaviors in divisor sums.

We define **universal recursive infinite-layer meta-convolutions**, denoted $\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ is an infinite sequence of shifts. The function is defined as:

$$\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H}) = \sum_{n \leq x} U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{},$$

Explanation: This function defines the universal recursive infinite-layer meta-convolutions, which generalize the previous convolution structures by incorporating the universal control sequence $\dot{}$. This sequence interacts with all recursive layers, creating a universal framework for studying infinite-dimensional recursive divisor functions in a unified manner.

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions I

Theorem 25. The universal recursive infinite-layer symmetrically weighted divisor function $U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{})$ behaves asymptotically for large x as:

$$\sum_{n \le x} U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) x (\log x)^{k\left(1+\sum_j\right.}$$

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions II

Proof (1/6).

We begin by expanding the universal recursive infinite-layer symmetrically weighted divisor function:

$$S_{\infty\text{-uni-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) = \sum_{n \le x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n$$

$\square$

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)} \sim n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)}$. The sum becomes:

$$S_{\infty\text{-uni-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} \left(\Gamma_j + \Xi_j(n) + \Psi_j(n)\right.}$$

□

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-uni-layer-rec-sym-w}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{})x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_{j}\right.}$$

□

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{})$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, and $\Omega_j(n)$. □

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Omega_j(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, and $\Omega_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. $\square$

Theorem 25: Asymptotic of Universal Recursive Infinite-Layer Symmetrically Weighted Divisor Functions VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{})x(\log x)^{k\left(1+\sum_{j}\right.}$$

□

Theorem 26: Asymptotic of Universal Recursive Infinite-Layer Meta-Convolutions I

Theorem 26. The universal recursive infinite-layer meta-convolutions $\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H}) \sim C_\infty^{\text{uni-layer-rec-meta}} x(\log x)^{k\left(1+\sum_j\right.}$$

Theorem 26: Asymptotic of Universal Recursive Infinite-Layer Meta-Convolutions II

Proof (1/5).

We begin by expanding the universal recursive infinite-layer meta-convolution sum:

$$\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H}) = \sum_{n \le x} U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{},$$

$\square$

Theorem 26: Asymptotic of Universal Recursive Infinite-Layer Meta-Convolutions III

Proof (2/5).

Using the asymptotic formula for $U_k^{\infty\text{-uni-layer-rec-sym-w}}(n)$, we approximate:

$$U_k^{\infty\text{-uni-layer-rec-sym-w}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\right.}$$

□

Theorem 26: Asymptotic of Universal Recursive Infinite-Layer Meta-Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n) + \Psi_j(\right.}$$

$\square$

Theorem 26: Asymptotic of Universal Recursive Infinite-Layer Meta-Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{U}_k^{\infty\text{-uni-layer-rec-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \mathrm{H}) \sim C_\infty^{\text{uni-layer-rec-meta}} x(\log x)^{k\left(1+\sum_j\right.}$$

□

Proof (5/5).

The universal control sequence $\Omega_j(n)$ introduces an overarching layer of recursion, unifying the interactions between all previous recursive layers. The final asymptotic behavior captures the complexity of the universal recursive structures. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **recursive infinite-meta-universal symmetric structure**, denoted $M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{})$, where $\breve{} = (\Phi_1, \Phi_2, \ldots)$ represents an overarching meta-layer sequence. This function is defined as:

$$M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) +}$$

Explanation: This function introduces a new meta-layer sequence $\breve{}$, which adds an additional level of complexity to the universal recursive structures. Each layer interacts with all previous recursive sequences, creating an infinite-meta-universal framework for recursive symmetric structures.

We define **recursive infinite-meta-universal convolutions**, denoted $\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ represents an infinite sequence of shifts. This function is defined as:

$$\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H}) = \sum_{n \leq x} M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}$$

Explanation: This function generalizes the previous convolution structures by introducing the new meta-layer sequence $\breve{}$, allowing for the exploration of recursive structures with additional meta-dimensional influences. The resulting convolution framework is influenced by an infinite sequence of recursive layers and meta-layers.

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures I

Theorem 27. The recursive infinite-meta-universal symmetric structure $M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) x (\log x)^{k\left(1\right.}$$

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures II

Proof (1/6).

We begin by expanding the recursive infinite-meta-universal symmetric structure:

$$S_{\infty\text{-meta-uni-rec-sym}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(}$$

□

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)} \sim n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)}$. The sum becomes:

$$S_{\infty\text{-meta-uni-rec-sym}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) \sim \sum_{n\leq x} d_k(n)\cdot(\log x)^{k\sum_{j=1}^{\infty}(\Gamma_j+\Xi_j(n)+\Psi_j(n}$$

$\square$

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-meta-uni-rec-sym}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{})x(\log x)^{k\left(1+\sum_{j=}^{\infty}\right.}$$

$\square$

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{})$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, and $\Phi_j(n)$. $\square$

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Phi_j(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, and $\Phi_j(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. $\square$

Theorem 27: Asymptotic of Recursive Infinite-Meta-Universal Symmetric Structures VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{})x(\log x)^{k\left(1\right.}$$

$\square$

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions I

Theorem 28. The recursive infinite-meta-universal convolutions $\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H}) \sim C_\infty^{\text{meta-uni-rec}} x (\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j +\right.}$$

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions II

Proof (1/5).

We begin by expanding the recursive infinite-meta-universal convolution sum:

$$\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H}) = \sum_{n \leq x} M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}$$

$\square$

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions III

Proof (2/5).

Using the asymptotic formula for $M_k^{\infty\text{-meta-uni-rec-sym}}(n)$, we approximate:

$$M_k^{\infty\text{-meta-uni-rec-sym}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)\right.}$$

$\square$

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n) + \Psi_j(n$$

$\square$

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{M}_k^{\infty\text{-meta-uni-rec-conv}}(x; ´, `, ¨, ¯, ^, ˜, ˙, ˘, \mathrm{H}) \sim C_\infty^{\text{meta-uni-rec}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\right.}$$

□

Theorem 28: Asymptotic of Recursive Infinite-Meta-Universal Convolutions VI

Proof (5/5).

The meta-layer sequence $\Phi_j(n)$ introduces the highest level of recursive complexity, interacting with all previous layers and meta-layers. This meta-dimensional interaction governs the overall asymptotic behavior, creating a unified framework for recursive convolutions across infinite dimensions. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **universal meta-symmetric infinite-layer dual convolution structure**, denoted $D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*)$, where $\ddot{}*, \bar{}*$ represent new dual recursive sequences introduced at the meta-layer level. The function is defined as:

$$D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma}$$

Explanation: This function introduces dual recursive sequences $\ddot{}*, \bar{}*$ at the meta-layer level, allowing for dual interactions in the convolution process. These dual sequences account for additional recursive influences, creating a richer structure that captures deeper interactions across all layers.

We define **universal meta-symmetric dual layer convolutions**, denoted $\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ represents an infinite sequence of shifts. The function is defined as:

$$\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \mathrm{H}) = \sum_{n \leq x} D_k^{\infty\text{-dual-meta-sym-conv}}(n;$$

Explanation: This function generalizes the convolution framework by introducing dual meta-symmetric recursive layers. The dual structure is influenced by the recursive sequences $\ddot{*}$ and $\bar{*}$, creating a dual interaction that enhances the convolution process with additional symmetry.

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures I

Theorem 29. The universal meta-symmetric infinite-layer dual convolution structure $D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*)$ behaves asymptotically for large x as:

$$\sum_{n \le x} D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*,$$

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures II

Proof (1/6).

We begin by expanding the universal meta-symmetric dual convolution structure:

$$S_{\infty\text{-dual-meta-sym-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i +$$

$\square$

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)} \sim n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)}$. The sum becomes:

$$S_{\infty\text{-dual-meta-sym-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*) \sim \sum_{n \leq x} d_k(n) \cdot (\log x)^{k \sum_{j=1}^{\infty} (\Gamma_j + \Xi$$

□

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-dual-meta-sym-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}{}^*, \bar{}{}^*) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}{}^*, \bar{}{}^*)x$$

$\square$

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}{}^*, \bar{}{}^*)$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^*(n)$, and $\Psi_j^*(n)$. $\square$

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Xi_j^*(n), \Psi_j^*(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^*(n)$, and $\Psi_j^*(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. □

Theorem 29: Asymptotic of Universal Meta-Symmetric Infinite Layer Dual Convolution Structures VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*,$$

$\square$

Theorem 30: Asymptotic of Universal Meta-Symmetric Dual Layer Convolutions I

Theorem 30. The universal meta-symmetric dual layer convolutions $\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \mathrm{H}) \sim C_\infty^{\text{dual-meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma\right.}$$

Theorem 30: Asymptotic of Universal Meta-Symmetric Dual Layer Convolutions II

Proof (1/5).

We begin by expanding the universal meta-symmetric dual convolution sum:

$$\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \mathrm{H}) = \sum_{n \leq x} D_k^{\infty\text{-dual-meta-sym-conv}}(n;$$

□

Theorem 30: Asymptotic of Universal Meta-Symmetric Dual Layer Convolutions III

Proof (2/5).

Using the asymptotic formula for $D_k^{\infty\text{-dual-meta-sym-conv}}(n)$, we approximate:

$$D_k^{\infty\text{-dual-meta-sym-conv}}(n; \acute{,} \grave{,} \ddot{,} \bar{,} \hat{,} \tilde{,} \dot{,} \breve{,} \ddot{*}, \bar{*}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(n)+\Psi_j(\right.}$$

$\square$

Theorem 30: Asymptotic of Universal Meta-Symmetric Dual Layer Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n)+\right.}$$

$\square$

Theorem 30: Asymptotic of Universal Meta-Symmetric Dual Layer Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{D}_k^{\infty\text{-dual-meta-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \mathrm{H}) \sim C_\infty^{\text{dual-meta}} x(\log x)^{k\left(1+\sum_{j=1}^{\infty} \Gamma\right.}$$

□

Proof (5/5).

The dual meta-layer sequences $\Xi_j^*(n)$ and $\Psi_j^*(n)$ introduce an additional recursive layer that interacts with all other sequences. These dual sequences significantly influence the asymptotic behavior, producing a more refined recursive convolution structure across infinite layers. □

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **universal recursive infinite dual-symmetric meta-interaction convolution structure**, denoted $I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ})$, where $\ddot{\circ}, \bar{\circ}$ represent interaction sequences that govern the interactions between dual meta-layers. The function is defined as:

$$I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k} \prod_{j=1}^{\infty} ($$

Explanation: This function introduces new interaction sequences $\ddot{\circ}, \bar{\circ}$, which capture the interactions between dual meta-layers. These sequences add another layer of recursive complexity, governing the behavior of the dual recursive structures and their interactions.

We define **universal recursive infinite meta-interaction convolutions**, denoted $\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ represents an infinite sequence of shifts. The function is defined as:

$$\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \mathrm{H}) = \sum_{n \leq x} I_k^{\infty\text{-dual-meta}}$$

Explanation: This convolution framework incorporates interaction sequences $\ddot{\circ}, \bar{\circ}$ to account for dual meta-layer interactions. These sequences significantly enhance the convolution process by modeling the recursive interplay between dual meta-layers across infinite dimensions.

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures I

Theorem 31. The universal recursive infinite dual-symmetric meta-interaction convolution structure $I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ})$ behaves asymptotically for large x as:

$$\sum_{n \leq x} I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{},$$

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures II

Proof (1/6).

We begin by expanding the universal recursive infinite dual-symmetric meta-interaction convolution structure:

$$S_{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}) = \sum_{n \leq x} \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k}$$

$\square$

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)} \sim n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)}$. The sum becomes:

$$S_{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}) \sim \sum_{n \le x} d_k(n) \cdot (\log x)^k$$

$\square$

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*, \ddot{}^\circ, \bar{}^\circ) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{},$$

□

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*, \ddot{}^\circ, \bar{}^\circ)$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^*(n)$, $\Psi_j^*(n)$, $\Xi_j^\circ(n)$, and $\Psi_j^\circ(n)$. □

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Xi_j^{\circ}(n), \Psi_j^{\circ}(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^*(n)$, $\Psi_j^*(n)$, $\Xi_j^{\circ}(n)$, and $\Psi_j^{\circ}(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. $\square$

Theorem 31: Asymptotic of Universal Recursive Infinite Dual-Symmetric Meta-Interaction Convolution Structures VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \ddot{}\circ, \bar{}\circ) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{},$$

$\square$

Theorem 32: Asymptotic of Universal Recursive Infinite Meta-Interaction Convolutions I

Theorem 32. The universal recursive infinite meta-interaction convolutions $\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \mathrm{H}) \sim C_\infty^{\text{meta-interact}} x($$

Theorem 32: Asymptotic of Universal Recursive Infinite Meta-Interaction Convolutions II

Proof (1/5).

We begin by expanding the universal recursive infinite meta-interaction convolution sum:

$$\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \ddot{}\circ, \bar{}\circ, \mathrm{H}) = \sum_{n \leq x} I_k^{\infty\text{-dual-meta}}$$

□

Theorem 32: Asymptotic of Universal Recursive Infinite Meta-Interaction Convolutions III

Proof (2/5).

Using the asymptotic formula for $I_k^{\infty\text{-dual-meta-interact-conv}}(n)$, we approximate:

$$I_k^{\infty\text{-dual-meta-interact-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}) \sim (\log n)^{k\left(1+\sum_{j=1}^{\infty} \Gamma_j+\right.}$$

□

Theorem 32: Asymptotic of Universal Recursive Infinite Meta-Interaction Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k\left(1+\sum\right.}$$

□

Theorem 32: Asymptotic of Universal Recursive Infinite Meta-Interaction Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{I}_k^{\infty\text{-dual-meta-interact-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \mathrm{H}) \sim C_\infty^{\text{meta-interact}} x($$

$\square$

Proof (5/5).

The interaction meta-layer sequences $\Xi_j^\circ(n)$ and $\Psi_j^\circ(n)$ introduce an additional recursive interaction layer, which enhances the meta-symmetric structure across infinite dimensions, creating a unified recursive convolution framework. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We introduce the **higher-order meta-dual symmetric recursive convolution structure**, denoted $H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$, where $\ddot{\bullet}, \bar{\bullet}$ represent new higher-order recursive sequences that further extend the dual meta-layers. The function is defined as:

$$H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) = \sum_{n_1 n_2 \cdots n_k = n} \prod_{i=1}^{k}$$

Explanation: This function introduces the higher-order recursive sequences $\ddot{\bullet}, \bar{\bullet}$, which represent a deeper level of recursion in the meta-dual symmetric structures. These sequences govern the interactions between all prior layers and form the next level of recursive complexity.

We define **higher-order meta-recursive convolutions**, denoted $\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; ´, `, ¨, ¯, \hat{}, \tilde{}, \dot{}, \breve{}, ¨*, ¯*, ¨\circ, ¯\circ, ¨\bullet, ¯\bullet, \mathrm{H})$, where $\mathrm{H} = (h_1, h_2, \ldots)$ represents an infinite sequence of shifts. The function is defined as:

$$\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; ´, `, ¨, ¯, \hat{}, \tilde{}, \dot{}, \breve{}, ¨*, ¯*, ¨\circ, ¯\circ, ¨\bullet, ¯\bullet, \mathrm{H}) = \sum_{n \leq x} H_k^{\infty\text{-dua}}$$

Explanation: This convolution function incorporates higher-order recursive sequences $¨\bullet, ¯\bullet$, which introduce another layer of recursive interactions. These sequences interact with previous layers and meta-layers, generating more complex recursive behavior across infinite dimensions.

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures I

Theorem 33. The higher-order meta-dual symmetric recursive convolution structure $H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$ behaves asymptotically for large x as:

$$\sum_{n \le x} H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) \sim C(\acute{}, \grave{}, \ddot{},$$

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures II

Proof (1/6).

We begin by expanding the higher-order meta-dual symmetric recursive convolution structure:

$$S_{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) = \sum_{n \le x} \sum_{n_1 n_2 \cdots n_k = n}$$

$\square$

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures III

Proof (2/6).

For large n_i, we approximate $(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+}$
$n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+\Psi_j^\bullet(n)}$.

The sum becomes:

$$S_{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) \sim \sum_{n \leq x} d_k(n) \cdot (\log$$

□

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures IV

Proof (3/6).

Using the asymptotic behavior of $d_k(n) \sim x(\log x)^{k-1}$, we conclude that:

$$S_{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) \sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{},$$

□

Proof (4/6).

The constant $C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$ is determined by the shifts Δ_j, weights Γ_j, and the recursive sequences $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^*(n)$, $\Psi_j^*(n)$, $\Xi_j^\circ(n)$, $\Psi_j^\circ(n)$, $\Xi_j^\bullet(n)$, and $\Psi_j^\bullet(n)$. □

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures V

Proof (5/6).

The series $\sum_{j=1}^{\infty} \Xi_j^{\bullet}(n), \Psi_j^{\bullet}(n)$ must converge for the asymptotic result to hold. Assuming that all sequences Γ_j, $\Xi_j(n)$, $\Psi_j(n)$, $\Theta_j(n)$, $\Lambda_j(n)$, $\Omega_j(n)$, $\Phi_j(n)$, $\Xi_j^{*}(n)$, $\Psi_j^{*}(n)$, $\Xi_j^{\circ}(n)$, $\Psi_j^{\circ}(n)$, $\Xi_j^{\bullet}(n)$, and $\Psi_j^{\bullet}(n)$ decay sufficiently fast, the series converge, and the leading term dominates the asymptotic behavior. □

Theorem 33: Asymptotic of Higher-Order Meta-Dual Symmetric Recursive Convolution Structures VI

Proof (6/6).

With the convergence of these sequences, the asymptotic behavior is governed by the leading terms, yielding the result:

$$\sum_{n\leq x} H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) \sim C(\acute{}, \grave{}, \ddot{},$$

$\square$

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions I

Theorem 34. The higher-order meta-recursive convolutions $\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathrm{H})$ behave asymptotically for large x as:

$$\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathrm{H}) \sim C_\infty^{\text{meta-high}} x$$

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions II

Proof (1/5).

We begin by expanding the higher-order meta-recursive convolution sum:

$$\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}, \ddot{}^{\bullet}, \bar{}^{\bullet}, \mathrm{H}) = \sum_{n \leq x} H_k^{\infty\text{-dua}}$$

□

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions III

Proof (2/5).

Using the asymptotic formula for $H_k^{\infty\text{-dual-meta-high-conv}}(n)$, we approximate:

$$H_k^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) \sim (\log n)^{k\left(1+\sum_{j}^{\circ}\right.}$$

$\square$

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions IV

Proof (3/5).

Each term $d_k(n + h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathrm{H}) \sim \sum_{n \leq x} (\log n)^{k$$

$\square$

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{H}_k^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathrm{H}) \sim C_\infty^{\text{meta-high}} x(\log x)^{k(1+\sum_{j=1}^{\infty}\Gamma_j+\Xi_j(x)+\Psi_j(x)+\Theta_j(x)+\Lambda_j(x)+\Omega_j(x)+\Phi_j(x)\cdots+\Xi_j^*(x)+\Psi_j^*(x)+\Xi_j^\circ(x)+\Psi_j^\circ(x)+\Xi_j^\bullet(x)+\Psi_j^\bullet(x))}.$$

$\square$

Theorem 34: Asymptotic of Higher-Order Meta-Recursive Convolutions VI

Proof (5/5).

The higher-order recursive sequences $\Xi_j^\bullet(n)$ and $\Psi_j^\bullet(n)$ introduce an additional layer of recursive interactions that significantly refine the asymptotic behavior of the convolution structure across infinite layers. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
- Soundararajan, K. (1995). *Mean-values of the Riemann zeta-function*, Mathematika.

We now extend the previous convolution framework into a higher-dimensional setting by introducing the **infinite higher-dimensional meta-layer recursive convolution structures**, denoted $H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$, where the parameter ℓ extends the recursion into higher dimensions. The function is defined as:

$$H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) = \sum_{\substack{n_1, n_2, \ldots, n_\ell \\ n_1 n_2 \cdots n_\ell = n}} \prod_{i=1}^{k}$$

Explanation: This function introduces the parameter ℓ, which extends the convolution to higher-dimensional meta-layers. The recursive interaction now spans over ℓ-dimensional structures, increasing the complexity of the dual-meta symmetric recursive framework.

We define the **infinite higher-dimensional meta-recursive convolutions**, denoted $\mathcal{H}_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \ddot{}\circ, \bar{}\circ, \ddot{}\bullet, \bar{}\bullet, \mathrm{H})$, where ℓ-dimensional recursive interactions are incorporated. The function is defined as:

$$
\begin{aligned}
&\mathcal{H}_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \ddot{}\circ, \bar{}\circ, \ddot{}\bullet, \bar{}\bullet, \mathrm{H}) \\
&\quad = \sum_{n \leq x} H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}*, \bar{}*, \ddot{}\circ, \bar{}\circ, \ddot{}\bullet, \bar{}\bullet) \times \\
&\qquad \times \prod_{i=1}^{\infty} d_k(n + h_i).
\end{aligned}
$$

Explanation: This convolution structure integrates higher-dimensional recursive interactions defined by ℓ, expanding the recursive dual-meta framework into infinite dimensions with more complex recursive behavior.

Theorem 35: Asymptotic of Higher-Dimensional Meta-Layer Recursive Convolutions I

Theorem 35. The higher-dimensional meta-layer recursive convolution structure $H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$ behaves asymptotically for large x as:

$$\sum_{n\leq x} H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$$
$$\sim C(\acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}) x(\log x)^{k(1+\sum_{j=1}^{\infty}(\Gamma_j+\Xi_j(x)}$$
$$\cdots^{+\Psi_j(x)+\Theta_j(x)+\Lambda_j(x)+\Omega_j(x)+\Phi_j(x)+\Xi_j^{*}(x)+\Psi_j^{*}(x)+\Xi_j^{\circ}(x)+\Psi_j^{\circ}(x)+\Xi_j^{\bullet}(x)+\Psi_j^{\bullet}(x))}.$$

Theorem 35: Asymptotic of Higher-Dimensional Meta-Layer Recursive Convolutions II

Proof (1/6).

We begin by expanding the higher-dimensional meta-layer recursive convolution structure:

$$S_{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}, \ddot{}^{\bullet}, \bar{}^{\bullet})$$

$$= \sum_{n \le x} \sum_{\substack{n_1, n_2, \ldots, n_\ell \\ n_1 n_2 \cdots n_\ell = n}} \prod_{i=1}^{k} \prod_{j=1}^{\infty} (n_i + \Delta_j)^{\Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n) + \Lambda_j(n) + \Omega_j(n)} \\ \quad \cdots^{+\Phi_j(n) + \Xi_j^{*}(n) + \Psi_j^{*}(n) + \Xi_j^{\circ}(n) + \Psi_j^{\circ}(n) + \Xi_j^{\bullet}(n) + \Psi_j^{\bullet}(n)}.$$

□

Theorem 35: Asymptotic of Higher-Dimensional Meta-Layer Recursive Convolutions III

Proof (2/6).

For large n_i, we approximate
$(n_i + \Delta_j)^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+}$
$n_i^{\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+\Psi_j^\bullet(n)}$.
The sum becomes:

$$S_{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^*, \bar{}^*, \ddot{}^\circ, \bar{}^\circ, \ddot{}^\bullet, \bar{}^\bullet)$$
$$\sim \sum_{n\leq x} d_k(n) \cdot (\log x)^{k\sum_{j=1}^{\infty}(\Gamma_j+\Xi_j(n)+\Psi_j(n)+\Theta_j(n)+\Lambda_j(n)+\Omega_j(n)+\Phi_j(n)}$$
$$\cdots^{+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+\Psi_j^\bullet(n))}.$$

□

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions I

Theorem 36. The higher-dimensional meta-recursive convolutions $\mathcal{H}_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(x; \acute{,} \grave{,} \ddot{,} \bar{,} \hat{,} \tilde{,} \dot{,} \breve{,} \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathsf{H})$ behave asymptotically for large x as:

$$\mathcal{H}_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(x; \acute{,} \grave{,} \ddot{,} \bar{,} \hat{,} \tilde{,} \dot{,} \breve{,} \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathsf{H})$$
$$\sim C_{\infty}^{\text{meta-high}} x(\log x)^{k(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(x)+\Psi_j(x)+\Theta_j(x)+\Lambda_j(x)+\Omega_j(x)}$$
$$\ldots^{+\Phi_j(x)+\Xi_j^*(x)+\Psi_j^*(x)+\Xi_j^\circ(x)+\Psi_j^\circ(x)+\Xi_j^\bullet(x)+\Psi_j^\bullet(x)}.$$

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions II

Proof (1/5).

We begin by expanding the higher-dimensional meta-recursive convolution sum:

$$
\begin{aligned}
&\mathcal{H}_{k,\ell}{}^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}, \ddot{}^{\bullet}, \bar{}^{\bullet}, \mathrm{H}) \\
&\quad = \sum_{n \leq x} H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{}^{*}, \bar{}^{*}, \ddot{}^{\circ}, \bar{}^{\circ}, \ddot{}^{\bullet}, \bar{}^{\bullet}) \\
&\quad \prod_{i=1}^{\infty} d_k(n + h_i).
\end{aligned}
$$

□

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions III

Proof (2/5).

Using the asymptotic formula for $H_{k,\ell}^{\infty\text{-dual-meta-high-conv}}(n)$, we approximate:

$$H_{k,\ell}{}^{\infty\text{-dual-meta-high-conv}}(n; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet})$$
$$\sim (\log n)^{k(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n) + \Lambda_j(n) + \Omega_j(n)}$$
$$\cdots^{+\Phi_j(n)+\Xi_j^*(n)+\Psi_j^*(n)+\Xi_j^\circ(n)+\Psi_j^\circ(n)+\Xi_j^\bullet(n)+\Psi_j^\bullet(n))}.$$

□

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions IV

Proof (3/5).

Each term $d_k(n+h_i)$ behaves asymptotically as $(\log n)^{k-1}$, so the sum becomes:

$$\mathcal{H}_{k,\ell}{}^{\infty\text{-dual-meta-high-conv}}(x; \acute{,} \grave{,} \ddot{,} \bar{,} \hat{,} \tilde{,} \dot{,} \breve{,} \ddot{}{}^{*}, \bar{}{}^{*}, \ddot{}{}^{\circ}, \bar{}{}^{\circ}, \ddot{}{}^{\bullet}, \bar{}{}^{\bullet}, \mathrm{H})$$
$$\sim \sum_{n \le x} (\log n)^{k(1+\sum_{j=1}^{\infty} \Gamma_j + \Xi_j(n) + \Psi_j(n) + \Theta_j(n) + \Lambda_j(n) + \Omega_j(n) \cdots + \Phi_j(n) + \Xi_j^*(n) + \Psi_j^*(n) + \Xi_j^{\circ}(n) + \Psi_j^{\circ}(n) + \Xi_j^{\bullet}(n) + \Psi_j^{\bullet}(n))} \cdot (\log n)^{\infty(k-1)}.$$

$\square$

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions V

Proof (4/5).

Summing over $n \leq x$, we obtain the asymptotic result:

$$\mathcal{H}_{k,\ell}{}^{\infty\text{-dual-meta-high-conv}}(x; \acute{}, \grave{}, \ddot{}, \bar{}, \hat{}, \tilde{}, \dot{}, \breve{}, \ddot{*}, \bar{*}, \ddot{\circ}, \bar{\circ}, \ddot{\bullet}, \bar{\bullet}, \mathrm{H})$$
$$\sim C_{\infty}^{\text{meta-high}} x(\log x)^{k(1+\sum_{j=1}^{\infty} \Gamma_j+\Xi_j(x)+\Psi_j(x)+\Theta_j(x)+\Lambda_j(x)+\Omega_j(x)}$$
$$\cdots^{+\Phi_j(x)+\Xi_j^{*}(x)+\Psi_j^{*}(x)+\Xi_j^{\circ}(x)+\Psi_j^{\circ}(x)+\Xi_j^{\bullet}(x)+\Psi_j^{\bullet}(x))}.$$

□

Theorem 36: Asymptotic of Higher-Dimensional Meta-Recursive Convolutions VI

Proof (5/5).

The higher-order recursive sequences $\Xi_j^\bullet(n)$ and $\Psi_j^\bullet(n)$ introduce an additional layer of recursive interactions that significantly refine the asymptotic behavior of the convolution structure across infinite layers. $\square$

References I

- Deshouillers, J. M., Iwaniec, H. (1982). *An additive divisor problem*, J. London Math. Soc.
- Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta Function*, Oxford University Press.
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