THE META-DISCRIMINANT AS A DERIVED DETERMINANT IN ARITHMETIC GEOMETRY

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ABSTRACT. We define a derived-categorical refinement of the discriminant ideal—termed the meta-discriminant—as the determinant of the derived trace pairing complex. While the classical discriminant compresses global ramification data into an ideal in the base ring, the meta-discriminant captures local and derived structure via determinant functors. This formalism extends to moduli-theoretic settings, Galois stacks, and entropy sheaves.

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1. Introduction

The discriminant $\Delta_{L/K}$ of a finite separable extension of number fields L/K is a central invariant in number theory. It is both an ideal in $\mathcal{O}_K$ and a global measure of ramification. Algebraically, it is defined as the determinant of the trace pairing on a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$.

From the modern perspective of derived algebraic geometry, the trace pairing becomes a morphism of complexes in the derived category. This raises the natural question: *Can we define a "derived determinant" of this pairing?*

In this paper, we define the *meta-discriminant* as the determinant of the derived trace complex, capturing both classical and higher-order homological ramification phenomena. This derived refinement extends the arithmetic discriminant to a geometric and categorical setting, applicable to arithmetic stacks, derived Galois categories, and entropy sheaves.

We explore its properties, functoriality, behavior under base change, and connections with the theory of determinant line bundles and Deligne's determinant functors.

2. The Classical Discriminant and Determinant Constructions

2.1. The Discriminant of a Field Extension.

Let L/K be a finite separable extension of number fields, with $[L : K] = n$, and let $\mathcal{O}_L$, $\mathcal{O}_K$ denote their respective rings of integers. If $\{x_1, \ldots, x_n\}$ is an $\mathcal{O}_K$-basis of $\mathcal{O}_L$, then the trace pairing:

$$(x_i, x_j) \mapsto \mathrm{Tr}_{L/K}(x_i x_j)$$

defines an $n \times n$ symmetric matrix $T = (\mathrm{Tr}(x_i x_j))$. The discriminant is defined as:

$$\Delta_{L/K} := \det(T).$$

This is independent of basis up to square of determinant of change-of-basis, and defines an ideal in $\mathcal{O}_K$.

2.2. Interpretation via Determinants of Modules.

For any $\mathcal{O}_K$-algebra A of rank n, one defines its determinant line as:

$$\det_{\mathcal{O}_K}(A) := \bigwedge\nolimits^n_{\mathcal{O}_K} A.$$

Then the trace pairing is a symmetric bilinear form:

$$\mathrm{Tr} : A \otimes_{\mathcal{O}_K} A \to \mathcal{O}_K,$$

which induces a bilinear form on $\det_{\mathcal{O}_K}(A)$ by:

$$(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n) \mapsto \det(\mathrm{Tr}(x_i y_j)).$$

This defines a canonical quadratic form:

$$q : \det_{\mathcal{O}_K}(A) \otimes \det_{\mathcal{O}_K}(A) \to \mathcal{O}_K,$$

whose value on a basis element is the discriminant. This reveals the discriminant as a section of the line bundle:

$$\det_{\mathcal{O}_K}(A)^{\otimes 2}.$$

2.3. **Determinant Functors and Line Bundles.** In the framework of Deligne and Knudsen–Mumford, one may consider the determinant of perfect complexes:

$$\det : D^{\mathrm{perf}}(R) \longrightarrow \mathrm{Pic}(R),$$

mapping bounded complexes of finitely generated projective R-modules to the Picard group of line bundles.

This functor is multiplicative on exact triangles:

$$\det(A^\bullet) \otimes \det(C^\bullet) \simeq \det(B^\bullet)$$

whenever $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle.

This motivates the next section: defining the meta-discriminant via the determinant of the derived trace pairing complex.

Next Section Preview:

We next define the meta-discriminant via derived determinant functors, and analyze its localization, functoriality, and behavior under base change and field extensions.

3. The Meta-Discriminant via Derived Determinant

Let $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$ be the finite morphism associated to a separable extension L/K of number fields. As in the previous article, we consider the derived trace pairing:

$$\mathrm{Tr}^\bullet : \mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_K} \mathcal{O}_L \longrightarrow \mathcal{O}_K$$

as a morphism of perfect complexes in $D^{\mathrm{perf}}(\mathcal{O}_K)$.

Definition 3.1. The *meta-discriminant* $\Delta^{\mathrm{meta}}_{L/K}$ is defined as the derived determinant:

$$\Delta^{\mathrm{meta}}_{L/K} := \det\left(\mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_K} \mathcal{O}_L \xrightarrow{\mathrm{Tr}^\bullet} \mathcal{O}_K\right) \in \mathrm{Pic}(\mathcal{O}_K).$$

That is, $\Delta_{L/K}^{\mathrm{meta}}$ is a virtual line bundle in $\mathrm{Pic}(\mathcal{O}_K)$ associated to the cone of the trace pairing. When this cone is concentrated in degree zero, this recovers the classical discriminant ideal.

Remark 3.2. *As an object in the determinant groupoid, $\Delta_{L/K}^{\mathrm{meta}}$ retains both the ideal-theoretic class of the discriminant and additional homological data measuring ramification depth and complexity.*

3.1. **Explicit Models.** Let $\{x_1, \ldots, x_n\}$ be an $\mathcal{O}_K$-basis of $\mathcal{O}_L$. Then the classical trace matrix $T = (\mathrm{Tr}_{L/K}(x_i x_j))$ gives a presentation of the derived trace pairing as a complex of free modules:

$$\left[\mathcal{O}_K^n \xrightarrow{T} \mathcal{O}_K^n\right],$$

with determinant $\det(T)$ corresponding to the classical discriminant.

The meta-discriminant is then defined as the determinant of this complex:

$$\Delta_{L/K}^{\mathrm{meta}} := \det\left[\mathcal{O}_K^n \xrightarrow{T} \mathcal{O}_K^n\right] = \det(T) \cdot \mathcal{O}_K.$$

However, in general, the complex may not be concentrated in degree zero, and the determinant must be interpreted via the formalism of Deligne–Knudsen–Mumford.

3.2. **Relation to Meta-Different.** We recall from Article 1 that the meta-different is defined as:

$$\mathbb{D}_{L/K}^{\mathrm{meta}} := \mathrm{cone}(\mathrm{Tr}^{\bullet})[-1].$$

Then, by multiplicativity of the determinant functor, we have:

$$\Delta_{L/K}^{\mathrm{meta}} = \det(\mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_K} \mathcal{O}_L) \otimes \det(\mathcal{O}_K)^{-1}.$$

Alternatively, using the triangle:

$$\mathbb{D}_{L/K}^{\mathrm{meta}} \to \mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_K \to,$$

we deduce:

$$\det(\mathbb{D}_{L/K}^{\mathrm{meta}}) \simeq \Delta_{L/K}^{\mathrm{meta}} \otimes \mathcal{O}_K^{-1},$$

so the meta-discriminant is, up to twist, the determinant of the meta-different complex.

4. Functoriality and Arithmetic Behavior

4.1. **Localization at Primes.**

Proposition 4.1. *Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime. Then:*

$$\Delta_{L/K}^{\mathrm{meta}} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{p}} \simeq \Delta_{L_{\mathfrak{q}}/K_{\mathfrak{p}}}^{\mathrm{meta}}$$

where $\mathfrak{q} \mid \mathfrak{p}$ runs over primes of $\mathcal{O}_L$ lying above $\mathfrak{p}$.

Proof. This follows from the compatibility of determinant functors with localization and base change in the derived category. ☐

4.2. **Field Extension Towers.** Let $K \subset L \subset M$ be finite separable extensions. Then we have:

Proposition 4.2. *There exists a functorial isomorphism:*

$$\Delta^{\mathrm{meta}}_{M/K} \simeq \Delta^{\mathrm{meta}}_{M/L} \otimes \mathrm{Nm}_{L/K}(\Delta^{\mathrm{meta}}_{L/K}).$$

Proof. This follows from the multiplicativity of trace maps and the behavior of determinant line bundles under tensor compositions:

$$\det(\mathrm{Tr}_{M/K}) = \det(\mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{M/L}) = \det(\mathrm{Tr}_{M/L}) \otimes \mathrm{Nm}_{L/K}(\det(\mathrm{Tr}_{L/K})).$$

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4.3. **Refined Arithmetic Ramification.** We interpret $\Delta^{\mathrm{meta}}_{L/K}$ as an arithmetic refinement of the discriminant ideal:
- It reflects not only the valuation of discriminants at ramified primes but also higher-order obstructions encoded in cohomological torsion;
- The support of Δ^{meta} in $\mathrm{Spec}(\mathcal{O}_K)$ defines a derived ramification divisor, generalizing the classical one;
- In moduli contexts, it descends to a determinant line bundle over arithmetic stacks of representations or extensions.

5. Geometric Realization and Moduli Interpretation

5.1. **Discriminant Line Bundles on Moduli of Extensions.** Let $\mathcal{M}_{\mathrm{FE}}$ denote the moduli stack of finite étale (or more generally finite locally free) extensions of a fixed base ring R. Over this stack, one can define a universal extension $\mathcal{A}/\mathcal{R}$ and the derived trace pairing:

$$\mathrm{Tr}^\bullet : \mathcal{A} \overset{L}{\otimes}_{\mathcal{R}} \mathcal{A} \longrightarrow \mathcal{R}.$$

Definition 5.1. The *universal meta-discriminant line bundle* is:

$$\mathcal{L}_{\Delta^{\mathrm{meta}}} := \det\left(\mathcal{A} \overset{L}{\otimes}_{\mathcal{R}} \mathcal{A} \xrightarrow{\mathrm{Tr}^\bullet} \mathcal{R}\right) \in \mathrm{Pic}(\mathcal{M}_{\mathrm{FE}}).$$

This construction descends to any base scheme S and is compatible with base change and pushforward along representable morphisms of stacks. It generalizes the classical discriminant divisor on the Hurwitz stack.

5.2. **Derived Ramification Locus.** Let $f : X \to Y$ be a finite flat morphism of derived schemes. Then the classical ramification divisor corresponds to the vanishing locus of the discriminant section.

In the derived setting, the meta-discriminant defines a derived line bundle whose vanishing locus is the *singular support* of the failure of f to be étale:

$$\mathrm{SS}(f) = \mathrm{Supp}\left(\mathrm{cone}\left(\mathrm{Tr}^{\bullet}\right)\right).$$

This leads to the interpretation of Δ^{meta} as the structure sheaf of the derived ramification locus. In particular:

- Over tame points, Δ^{meta} is locally trivial;
- Over wildly ramified points, Δ^{meta} encodes higher differentials and torsion;
- Its pushforward defines a class in K-theory or motivic cohomology of Y.

5.3. **Categorified Discriminants over Arithmetic Stacks.** Given a stack $\mathcal{X}$ parametrizing Galois representations or algebra extensions, the meta-discriminant line bundle:

$$\mathcal{L}_{\Delta^{\mathrm{meta}}} \in \mathrm{Pic}(\mathcal{X})$$

classifies the trace failure locus globally. One may interpret this as a categorified obstruction to unramifiedness or as a secondary conductor.

This provides a bridge between:
- Algebraic number theory;
- Moduli of Galois representations;
- Derived algebraic geometry.

6. Entropy Sheaves and Categorified Zeta Theory

6.1. **Entropy Trace Sheaves.** Following the entropy sheaf formalism in spectral cohomology theory, we can consider:

$$\mathcal{E}_{L/K} := \mathrm{cone}(\mathrm{Tr}^{\bullet})$$

as a categorified entropy sheaf measuring failure of symmetry in field extension structure.

Proposition 6.1. *The entropy sheaf $\mathcal{E}_{L/K}$ is a perfect complex whose determinant is the meta-discriminant line bundle:*

$$\det(\mathcal{E}_{L/K}) = \Delta^{\mathrm{meta}}_{L/K}.$$

Proof. Follows directly from the triangle defining $\mathcal{E}_{L/K}$ and the multiplicativity of the determinant functor. ∎

6.2. **Categorified Zeta Functions and Discriminant Flows.** We define the *categorified zeta complex* as:

$$\mathbb{Z}_K^{\mathrm{cat}} := \bigoplus_{L/K \text{ finite}} \mathbb{D}_{L/K}^{\mathrm{meta}},$$

where $\mathbb{D}_{L/K}^{\mathrm{meta}}$ is the meta-different complex from Article 1.

Then the derived zeta-sheaf:

$$\mathcal{Z}_K := \bigoplus_{L/K} \det\left(\mathbb{D}_{L/K}^{\mathrm{meta}}\right) = \bigoplus_{L/K} \Delta_{L/K}^{\mathrm{meta}}$$

can be viewed as a categorical deformation of the classical $\zeta_K(s)$.

6.3. **Future Directions.** The theory of meta-discriminants invites future developments in:

- Derived class field theory and higher global conductors;
- Moduli-theoretic dualities and discriminant gerbes;
- Quantized and entropy-deformed arithmetic sites;
- Connections to categorical Hodge theory and trace stacks.

We conjecture that there exists a categorified Artin reciprocity map:

$$\mathrm{Gal}^{\mathrm{ab}}(K) \longrightarrow \mathrm{Pic}^{\mathrm{meta}}(\mathcal{X}_K),$$

assigning to abelian Galois data a meta-discriminant line bundle in the categorified arithmetic site $\mathcal{X}_K$.

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