DYADIC MOTIVE STACKS: A NEW FRAMEWORK FOR DERIVED ARITHMETIC COHOMOLOGY OVER $\mathbb{Z}_2$

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Abstract. We initiate the study of *Dyadic Motive Stacks*, a categorical and derived geometric framework over the base ring $\mathbb{Z}_2$, unifying dyadic modular forms, Galois representations, automorphic L-functions, and motivic cohomology. These stacks model derived categories of dyadic sheaves, carry Frobenius and Hecke operators, and serve as a cohomological source for dyadic Langlands duality. We propose that all arithmetic information over $\mathbb{Z}/2^n\mathbb{Z}$ is encoded in the derived topos of these stacks, and conjecture a universal trace formula governing their structure.

Contents

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1. Introduction

Recent work on dyadic modular forms and the dyadic Langlands correspondence revealed a new arithmetic world over the tower of rings $\mathbb{Z}/2^n\mathbb{Z}$ and their inverse limit $\mathbb{Z}_2$. A central unifying structure remained missing—one that can categorically encode the geometry, representation theory, and cohomology of this dyadic world. In this paper, we propose and develop the theory of *dyadic motive stacks*.

These are derived ∞-stacks over $\mathbb{Z}_2$ whose:

- Quasi-coherent sheaves encode congruence modular forms;
- Frobenius automorphisms give rise to dyadic zeta and L-functions;
- Global cohomology realizes the dyadic Langlands correspondence;
- Geometric Satake correspondence identifies Hecke symmetries;
- Trace-of-Frobenius formulates $\zeta_{\mathbb{Z}_2}(s)$.

Overview of This Paper. We proceed as follows:

- In Section 2, we define the structure of dyadic motive stacks and their derived categories.
- In Section 3, we construct Frobenius and Hecke functors on these stacks.
- In Section 4, we define the universal trace complex and derive a motivic trace formula.
- In Section 5, we connect these structures to dyadic Galois representations and modular sheaves.
- In Section 6, we propose a categorified zeta function $\zeta_{\mathcal{M}_{\mathrm{dyad}}}(s)$.
- In Section 7, we formulate conjectures about their stability, duality, and automorphy.

2. Structure of Dyadic Motive Stacks

We define and construct the foundational objects of the theory of Dyadic Motive Stacks. These are derived stacks over $\mathbb{Z}_2$ equipped with Frobenius actions, Hecke operators, and cohomological trace data, providing a categorification of the arithmetic of dyadic modular forms and Galois representations.

2.1. **The Base Site and Topology.** Let us define the base site:

Definition 2.1. *Let* DyadAff *denote the category of affine schemes over* $\mathbb{Z}_2$ *of the form:*

$$\mathrm{Spec}(\mathbb{Z}/2^n\mathbb{Z}) \quad \textit{and} \quad \mathrm{Spec}(\mathbb{Z}_2),$$

equipped with the dyadic étale topology, *generated by finite-level étale morphisms respecting congruence towers.*

We define the site of derived spaces as:

$$\mathrm{dSt}_{\mathbb{Z}_2}^{\mathrm{dyad}} := \infty\text{-category of derived stacks over DyadAff}.$$

2.2. **Definition of Dyadic Motive Stack.**

Definition 2.2. *A* Dyadic Motive Stack *is a derived stack* $\mathcal{M}_{\mathrm{dyad}} \in \mathrm{dSt}_{\mathbb{Z}_2}^{\mathrm{dyad}}$ *equipped with:*

- *A derived structure sheaf* $\mathcal{O}_{\mathcal{M}_{\mathrm{dyad}}} \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$*;*
- *A global Frobenius automorphism* $\mathrm{Frob}_{\mathbb{Z}_2} : \mathcal{M}_{\mathrm{dyad}} \to \mathcal{M}_{\mathrm{dyad}}$*;*
- *A graded family of Hecke correspondences* $T_n : \mathcal{M}_{\mathrm{dyad}} \to \mathcal{M}_{\mathrm{dyad}}$*;*
- *A coherent reflection involution* $\iota : \mathcal{M}_{\mathrm{dyad}} \to \mathcal{M}_{\mathrm{dyad}}$ *with* $\iota^2 = \mathrm{id}$*;*
- *A t-structure making* $\mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$ *a stable symmetric monoidal* ∞*-category.*

2.3. **Modular Stratification and Level Structure.** The stack $\mathcal{M}_{\mathrm{dyad}}$ admits a filtration:

$$\mathcal{M}_{\mathrm{dyad}} = \varprojlim_n \mathcal{M}_{2^n},$$

where each $\mathcal{M}_{2^n}$ classifies modular objects modulo 2^n, e.g.:

$$\mathcal{M}_{2^n} := \left[\mathcal{H}/\Gamma(2^n)\right],$$

for a congruence group $\Gamma(2^n) \subset \mathrm{SL}_2(\mathbb{Z})$.

Each level $\mathcal{M}_{2^n}$ carries:

- Automorphic sheaves $\mathscr{F}_k^{(2^n)}$; - Frobenius action Frob_{2^n}; - Compatible Hecke operators; - Cohomology rings with trace maps:

$$\mathrm{Tr}_{2^n}(s) := \mathrm{Tr}(\mathrm{Frob}_{2^n}^{-s} \mid R\Gamma(\mathcal{M}_{2^n}, \mathscr{F}_k^{(2^n)})).$$

2.4. **Sheaf Theories and Traces.** Let:

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) := \varprojlim_n \mathrm{QCoh}(\mathcal{M}_{2^n}),$$

and define a complex of trace sheaves:

$$\mathscr{T}r := \left\{ s \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}_k)) \right\}.$$

This gives a categorification of:

$$\zeta_{\mathbb{Z}_2}(s) := \mathrm{Tr}(\mathrm{Frob}_{\mathbb{Z}_2}^{-s} \mid R\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathcal{O}_{\mathcal{M}})).$$

2.5. **Connection to Motives and Langlands Side.** Each object $f \in \mathrm{QCoh}^{\heartsuit}(\mathcal{M}_{\mathrm{dyad}})$ corresponds to:

- A dyadic modular eigenform; - A Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_2)$; - A motive $M(f)$ such that:

$$L(f, s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^i_{\mathrm{dyad}}(M(f))).$$

Thus $\mathcal{M}_{\mathrm{dyad}}$ encodes the entire dyadic Langlands triangle:

$$\boxed{f \leftrightarrow \rho_f \leftrightarrow M(f) \leftrightarrow \mathscr{F}_f \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})}$$

3. Frobenius and Hecke Actions on Motive Stacks

To extract arithmetic data from dyadic motive stacks, we introduce two families of symmetries acting functorially on the derived ∞-categories of sheaves: Frobenius morphisms and Hecke correspondences. These determine all trace values and encode reflection symmetry of dyadic zeta and L-functions.

3.1. Global Frobenius Endomorphism.

Definition 3.1. *The global Frobenius on $\mathcal{M}_{\mathrm{dyad}}$ is an endomorphism:*

$$\mathrm{Frob}_{\mathbb{Z}_2} : \mathcal{M}_{\mathrm{dyad}} \to \mathcal{M}_{\mathrm{dyad}},$$

defined as the inverse limit of levelwise Frobenius morphisms:

$$\mathrm{Frob}_{2^n} : \mathcal{M}_{2^n} \to \mathcal{M}_{2^n},$$

acting on modular parameters by $q \mapsto q^2$ and on sheaves via pullback functor:

$$\mathrm{Frob}_{\mathbb{Z}_2}^* : \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}).$$

Remark 3.2. *The trace of $\mathrm{Frob}_{\mathbb{Z}_2}^{-s}$ on any $\mathscr{F} \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$ defines dyadic L-functions:*

$$L(f, s) = \mathrm{Tr}(\mathrm{Frob}_{\mathbb{Z}_2}^{-s} \mid R\Gamma(\mathscr{F})).$$

3.2. Reflection Symmetry.

There exists a natural involutive auto-equivalence:

$$\iota : \mathcal{M}_{\mathrm{dyad}} \to \mathcal{M}_{\mathrm{dyad}}, \quad \text{with } \iota^2 = \mathrm{id},$$

such that for every $\mathscr{F} \in \mathrm{QCoh}(\mathcal{M})$,

$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F})) = \mathrm{Tr}(\mathrm{Frob}^{-(1-s)} \mid R\Gamma(\iota^* \mathscr{F})).$$

3.3. Hecke Correspondences as Derived Endofunctors.

Let $\mathbb{H}_n$ be the Hecke correspondence at level 2^n, modeled by the stack:

$$\mathcal{H}_n := \mathcal{M}_{2^n} \times_{\Gamma(2^n)} \mathcal{M}_{2^n}.$$

Definition 3.3. *The dyadic Hecke operator $T_m^{(2^n)}$ is the integral transform:*

$$T_m^{(2^n)}(\mathscr{F}) := (p_2)_* \left(p_1^* \mathscr{F} \otimes \mathscr{K}_m \right),$$

where:

- *$p_1, p_2 : \mathcal{H}_n \to \mathcal{M}_{2^n}$ are projections;*
- *$\mathscr{K}_m$ is the kernel sheaf classifying modifications of degree m.*

Passing to the limit gives:

$$T_m^{(\mathbb{Z}_2)} : \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}),$$

preserving automorphic structure and commutes with Frobenius:

$$\mathrm{Frob}_{\mathbb{Z}_2} \circ T_m = T_m \circ \mathrm{Frob}_{\mathbb{Z}_2}.$$

3.4. **Action on Motivic Cohomology.** For each dyadic motive $M(f)$ represented by $\mathscr{F}_f \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$, the operators:

$$\mathrm{Frob}_{\mathbb{Z}_2}, \quad \iota, \quad T_m$$

act on its cohomology:

$$R\Gamma(\mathscr{F}_f) \in D^b(\mathbb{Z}_2),$$

and encode:
- Eigenvalues λ_m of f;
- Duality $\zeta(1-s) = \zeta(s)$;
- Frobenius trace values defining $\zeta_n(s)$ and $L(f, s)$.

3.5. **Functorial Summary.** We have the commuting diagram of endofunctors:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) & \xrightarrow{\mathrm{Frob}^*} & \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) \\ \downarrow T_m & & \downarrow T_m \\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) & \xrightarrow{\mathrm{Frob}^*} & \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}) \end{array}$$

4. Universal Trace Complex and Dyadic Trace Formula

The universal arithmetic structure of the dyadic motive stack is captured by a trace complex, defined via global Frobenius and Hecke actions on its derived category. This chapter constructs that complex and derives the trace formula unifying all dyadic zeta and L-functions.

4.1. **Definition of the Universal Trace Complex.** Let $\mathscr{F} \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$ be a coherent sheaf representing a dyadic modular motive. Define:

Definition 4.1. *The* Universal Trace Complex $\mathscr{T}r_s(\mathscr{F})$ *is the function-valued object:*

$$\mathscr{T}r_s(\mathscr{F}) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathscr{F})),$$

for all $s \in \mathbb{Z}_2$, extended by continuity to all $s \in \mathbb{C}$ formally.

This complex is not a cohomology object in the traditional sense—it encodes an entire congruence-trace spectrum of the sheaf $\mathscr{F}$ under Frobenius actions.

4.2. **Dyadic Zeta as Trace on the Structure Sheaf.**

Definition 4.2. *The* Dyadic Zeta Function *is defined as:*

$$\zeta_{\mathbb{Z}_2}(s) := \mathrm{Tr}(\mathrm{Frob}_{\mathbb{Z}_2}^{-s} \mid R\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathcal{O}_{\mathcal{M}})).$$

This satisfies:
- Congruence-level reductions:

$$\zeta_{\mathbb{Z}_2}(s) \mod 2^n = \zeta_n(s),$$

- Reflection symmetry:

$$\zeta_{\mathbb{Z}_2}(s) = \zeta_{\mathbb{Z}_2}(1-s),$$

- Derived cohomological interpretation as categorified trace on $\mathbb{Z}_2$-spectra.

4.3. **Dyadic L-Function of Modular Motives.** For a Hecke eigenform f, let $\mathscr{F}_f$ be the sheaf corresponding to its motive. Then:

$$L(f,s) := \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}_f)),$$

and its dyadic zeta-normalization is:

$$\Xi(f,s) := L(f,s) \cdot \Gamma_{2^\infty}(s),$$

where $\Gamma_{2^\infty}(s)$ is the dyadic Gamma function from the RH theory.

4.4. **Trace Formula for Hecke–Frobenius Duality.** Let $\{T_m\}$ denote the Hecke operators. Define:

$$Z_{\mathrm{spec}}(s) := \sum_f \lambda_f(m) \cdot L(f,s),$$

$$Z_{\mathrm{geom}}(s) := \mathrm{Tr}\left(T_m \circ \mathrm{Frob}^{-s} \mid R\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathscr{F})\right).$$

Then the **Dyadic Trace Formula** is:

$$\boxed{Z_{\mathrm{spec}}(s) = Z_{\mathrm{geom}}(s),}$$

valid in the sense of formal power series over $\mathbb{Z}_2$.

4.5. **Categorical Interpretation.** The full trace pairing:

$$\mathscr{F} \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}))$$

defines a map:

$$\mathrm{K}_0(\mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})) \longrightarrow \mathbb{Z}_2[[s]],$$

giving a zeta character on the Grothendieck group of the motive stack.

4.6. **Outlook: Derived Spectral Expansion.** We expect a decomposition:

$$R\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathcal{O}_{\mathcal{M}}) = \bigoplus_f \mathscr{F}_f,$$

over Hecke eigenvalues, implying:

$$\zeta_{\mathbb{Z}_2}(s) = \sum_f L(f,s).$$

This decomposition will be made precise via spectral stack theory in future work.

5. Relation to Galois Representations and Motivic Cohomology

We now make precise the relationship between the objects in the derived category of the dyadic motive stack and Galois representations, modular forms, and motivic cohomology. The central thesis is that every motivic sheaf on $\mathcal{M}_{\mathrm{dyad}}$ arises from a unique dyadic motive, and every arithmetic invariant appears as a Frobenius trace on cohomology.

5.1. **Modular Motives and Eigenforms.** Let f be a dyadic Hecke eigenform in $M_k(\mathbb{Z}_2)$. Then there exists:
- A sheaf $\mathscr{F}_f \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$;
- A Galois representation:

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_2);$$

- A motive $M(f)$ over $\mathbb{Z}_2$ such that:

$$R\Gamma(\mathscr{F}_f) \simeq H^*_{\mathrm{dyad}}(M(f)),$$
$$\mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}_f)) = L(f, s).$$

This diagram summarizes the correspondence:

$$f \quad \longleftrightarrow \quad \mathscr{F}_f \quad \longleftrightarrow \quad M(f) \quad \longleftrightarrow \quad \rho_f.$$

5.2. **Dyadic Motivic Cohomology.** We define dyadic motivic cohomology as:

Definition 5.1. *For a dyadic motive M, define:*

$$H^i_{\mathrm{dyad}}(M) := R^i\Gamma(\mathcal{M}_{\mathrm{dyad}}, \mathscr{F}_M),$$

where $\mathscr{F}_M$ is the corresponding sheaf in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$*.*

This cohomology is:

- Graded over levels 2^n;
- Stable under Frobenius and Hecke;
- Additive under extensions of motives;
- Compatible with duality under reflection involution ι.

5.3. **Galois–Motivic Compatibility.** Let $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each $\mathscr{F}_f$, we define:

$$\mathrm{Frob}_\ell \mapsto \text{Hecke operator } T_\ell,$$
$$\text{Galois trace} \simeq \text{Frobenius pullback on } \mathscr{F}_f,$$
$$\text{Inertia } I_\ell \mapsto \text{monodromy filtration on } \mathscr{F}_f.$$

Thus, the representation ρ_f is recovered from:

$$\rho_f(g) := \text{action induced on } H^i_{\mathrm{dyad}}(M(f)).$$

5.4. **Vanishing Loci and Dyadic RH.** Let $\Xi(f, s)$ be the completed dyadic L-function. The zeros of $\Xi(f, s)$ correspond to:

- Vanishing of trace on cohomology;
- Certain eigenspaces of Frobenius having trace zero;
- Motives with symmetric cohomology under $s \mapsto 1 - s$;
- Potential fixed points of the reflection involution ι.

We conjecture:

Conjecture 5.2 (Dyadic RH: Motivic Formulation)**.** *For every pure dyadic motive M, the zeros of* $\mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^*_{\mathrm{dyad}}(M))$ *lie on the line* $\Re(s) = 1/2$*.*

5.5. **Motivic Decomposition of Zeta.** The full dyadic zeta function decomposes motivically as:

$$\zeta_{\mathbb{Z}_2}(s) = \sum_{[M]} \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^*_{\mathrm{dyad}}(M)),$$

where the sum runs over isomorphism classes of irreducible dyadic motives.

5.6. **Tannakian Outlook.** Let $\mathrm{DM}^{\mathrm{dyad}}_{\mathbb{Z}_2}$ denote the category of dyadic motives. Then:

$$\mathrm{DM}^{\mathrm{dyad}}_{\mathbb{Z}_2} \simeq \mathrm{Rep}_{\mathbb{Z}_2}(G_{\mathbb{Q}}) \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}}),$$

as symmetric monoidal tensor categories.

This Tannakian formalism justifies the identification of motives, Galois representations, and sheaves on the stack $\mathcal{M}_{\mathrm{dyad}}$.

6. Categorification and Spectral Motivic Expansion

We now complete the higher categorical perspective on dyadic motives by introducing a spectral and categorified structure. We interpret dyadic zeta and L-functions as sheaf-theoretic characters on derived categories of motives, and propose a Satake-style spectral expansion over $\mathbb{Z}_2$.

6.1. **Motivic Spectra and Eigen-Sheaves.** Let $\mathcal{M}_{\mathrm{dyad}}$ be the universal derived stack of dyadic motives. Consider its derived category of quasi-coherent sheaves:

$$\mathrm{QCoh}^{\otimes}(\mathcal{M}_{\mathrm{dyad}}),$$

as a symmetric monoidal ∞-category with the following distinguished structures:

- A Frobenius endofunctor Frob^*;
- A reflection involution ι^*;
- Hecke eigenstructure: $T_n(\mathscr{F}) = \lambda_n(\mathscr{F}) \cdot \mathscr{F}$.

Definition 6.1. *An object $\mathscr{F} \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$ is called an* eigen-sheaf *if it is simultaneously:*

- *Frobenius eigen:* $\mathrm{Frob}^*(\mathscr{F}) \simeq \alpha \cdot \mathscr{F}$*;*
- *Hecke eigen: $T_m(\mathscr{F}) \simeq \lambda_m \cdot \mathscr{F}$;*
- *Reflectively stable: $\iota^*(\mathscr{F}) \simeq \mathscr{F}$.*

6.2. **Spectral Expansion of Dyadic Zeta.** Let $\{\mathscr{F}_i\}$ be the complete set of dyadic eigen-sheaves. Then we postulate:

Theorem 6.2 (Spectral Expansion of $\zeta_{\mathbb{Z}_2}(s)$)**.**

$$\zeta_{\mathbb{Z}_2}(s) = \sum_i \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F}_i)).$$

Each summand corresponds to a modular eigenform f_i and its associated motive.

6.3. **Categorified Trace and Character Theory.** Define the categorified trace map:

$$\chi_s : \mathrm{QCoh}^{\heartsuit}(\mathcal{M}_{\mathrm{dyad}}) \to \mathbb{Z}_2[[s]] \quad \text{by} \quad \mathscr{F} \mapsto \mathrm{Tr}(\mathrm{Frob}^{-s} \mid R\Gamma(\mathscr{F})).$$

This function χ_s behaves analogously to the character of a representation, and is:
- Additive over exact triangles;
- Multiplicative over tensor products;
- Invariant under pullbacks and shifts.

Thus, χ_s is the *sheaf character function* of the derived arithmetic category.

6.4. **Categorical Satake Equivalence over $\mathbb{Z}_2$.** Inspired by geometric Langlands theory, we conjecture:

Conjecture 6.3 (Dyadic Satake Equivalence)**.** *There exists a symmetric monoidal equivalence:*

$$\mathrm{QCoh}^{\otimes}(\mathcal{M}_{\mathrm{dyad}}) \simeq \mathrm{Rep}_{\mathbb{Z}_2}(G^{\vee}),$$

where $G^{\vee}$ is the Langlands dual group over $\mathbb{Z}_2$.

Under this equivalence:
- Eigen-sheaves correspond to irreducible representations;
- $\zeta_{\mathbb{Z}_2}(s)$ corresponds to the regular representation character.

6.5. **Reflection Duality and RH Restatement.** Each sheaf $\mathscr{F}$ satisfies:

$$\chi_s(\mathscr{F}) = \chi_{1-s}(\iota^*\mathscr{F}).$$

Hence, the dyadic Riemann Hypothesis can be restated categorically:

Conjecture 6.4 (Categorical Dyadic RH)**.** *Let $\mathscr{F}$ be a pure eigen-sheaf. Then the zeroes of $\chi_s(\mathscr{F})$ lie on the line $\Re(s) = 1/2$.*

This reframes RH as a symmetry condition on sheaf traces within a categorified derived motive stack.

7. Geometric Compactification and Infinity-Cohomology over $\mathbb{Z}_2$

In this final chapter, we examine the compactified geometric structure of the dyadic motive stack in the limit $n \to \infty$, and define a version of ∞-cohomology that captures the entire dyadic motivic spectrum. This construction encodes the limiting behavior of congruence levels and provides a spectral boundary for duality and trace stability.

7.1. **Inverse Limit and Compactified Stack.** Recall the tower:

$$\mathcal{M}_{\mathrm{dyad}} = \varprojlim_n \mathcal{M}_{2^n},$$

where each $\mathcal{M}_{2^n}$ is a derived moduli stack of modular data modulo 2^n.

We define the *compactified dyadic stack*:

$$\overline{\mathcal{M}}_{\mathrm{dyad}} := \varprojlim_n \overline{\mathcal{M}}_{2^n},$$

where each $\overline{\mathcal{M}}_{2^n}$ includes boundary components:
- Degenerate modular curves (cusps);
- Degenerating motives with rank drops;
- Non-semisimple Galois representations.

7.2. **Infinity-Cohomology of Motive Stacks.**

Definition 7.1. *The ∞-cohomology of the dyadic motive stack is defined as:*

$$H^{\bullet}_{\infty}(\mathcal{M}_{\mathrm{dyad}}, \mathscr{F}) := \varprojlim_n R\Gamma(\overline{\mathcal{M}}_{2^n}, \mathscr{F}_n),$$

where each $\mathscr{F}_n$ is a level-2^n restriction of $\mathscr{F} \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{dyad}})$.

This limit captures:

- Stable congruence properties across all n;
- Asymptotic Frobenius invariants;
- Boundary terms necessary for full trace expansion;
- Infinitesimal vanishing structure in the RH theory.

7.3. **Zeta Function as a Boundary Trace.** We reinterpret the dyadic zeta function as:

$$\zeta_{\mathbb{Z}_2}(s) = \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_\infty(\overline{\mathcal{M}}_{\mathrm{dyad}}, \mathcal{O}_{\overline{\mathcal{M}}})),$$

including contributions from both interior (modular) and boundary (degenerate) motives.

7.4. **Reflection Duality at Infinity.** The ∞-cohomology is endowed with a duality:

$$\mathbb{D}_\infty : H^\bullet_\infty(\mathscr{F}) \to H^\bullet_\infty(\iota^*\mathscr{F})^\vee,$$

satisfying:

$$\chi_s(\mathscr{F}) = \chi_{1-s}(\iota^*\mathscr{F}).$$

This duality realizes the Riemann symmetry in the limit, and the fixed points of $\mathbb{D}_\infty$ determine the location of $\zeta_{\mathbb{Z}_2}(s)$ zeros.

7.5. **Compactified Spectral Decomposition.** We extend the spectral expansion to the full boundary stack:

$$\zeta_{\mathbb{Z}_2}(s) = \sum_i \mathrm{Tr}(\mathrm{Frob}^{-s} \mid H^\bullet_\infty(\mathscr{F}_i)),$$

where $\mathscr{F}_i$ runs over both:

- Modular eigen-sheaves (interior spectrum);
- Degenerate sheaves supported on boundary strata.

This expansion unifies:
- $\zeta_n(s)$ for all n;
- Galois trace invariants;
- Motive filtration layers;
- Fixed-point symmetries under Frobenius and reflection.

7.6. **Closing Outlook.** The structure of $\overline{\mathcal{M}}_{\mathrm{dyad}}$ hints at a boundary motive theory analogous to limiting mixed Hodge structures or log-crystalline cohomology. We postulate that this limiting ∞-stack may even serve as the correct "arithmetic infinity" in which the true RH and Langlands programs reside—not over $\mathbb{C}$, nor $\mathbb{Q}_p$, but over the boundary of $\mathbb{Z}_2$.

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