λ_{Error}: A TYPED LAMBDA CALCULUS FOR ARITHMETIC ERROR PROPAGATION

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ABSTRACT. We propose λ_{Error}, a novel type-theoretic framework designed to encode and classify arithmetic error terms through typed lambda calculus. This system assigns hierarchical types to error expressions, enabling symbolic tracking, compositional analysis, and safe transformation of error structures arising in analytic number theory. Through this formalism, we introduce an error programming language equipped with strong normalization, error derivation trees, static safety checks, and a category-theoretic interpretation of higher-level error operations. This provides a foundational logic for formal reasoning and computation with analytic and motivic errors.

CONTENTS

1. Error Type Theory: λ_{Error}

We develop a type-theoretic formalism for describing arithmetic error structure via dependent types, lambda calculus, and formal logical systems. This allows the precise classification, propagation, and compositional semantics of error terms in a symbolic and computable syntax.

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1.1. **Motivation and Vision.** Unlike traditional scalar or geometric treatments of error, λ_{Error} seeks to encode:

- the *computational flow* of error emergence;
- the *types of error operations* and transformations;
- the *logical rules* governing error term derivability;
- and the *syntax* by which errors can be composed, eliminated, or transformed.

1.2. **Core Type Definitions.** Let $\mathcal{E}_f(x)$ be the error term from object f.

Definition 1.1 (Error Type)**.** *The type of an error term is defined inductively:*

$$\mathbf{Err}_0 := \texttt{BaseError}$$
$$\mathbf{Err}_{n+1} := \texttt{ErrMap}[\mathbf{Err}_n \to \mathbf{Err}_n] \quad \textit{(Higher error transformers)}$$

Definition 1.2 (Error Judgment)**.** *We write the typing judgment:*

$$\Gamma \vdash \epsilon : \mathbf{Err}_n$$

to mean that under context Γ (e.g., arithmetic domain or constraints), the term ϵ is an n-th level error construct.

1.3. **Error Lambda Calculus.** We define a typed lambda calculus with error terms:

$$\epsilon ::= x \mid \lambda x : \tau.\epsilon \mid \epsilon_1\ \epsilon_2 \mid \texttt{Err}(x) \mid \texttt{Bound}(x) \mid \epsilon_1 + \epsilon_2$$
$$\tau ::= \mathbf{Err}_n \mid \tau \to \tau$$

Typing rules include:

- (Var) $\Gamma, x : \tau \vdash x : \tau$
- (Abs) $\Gamma, x : \tau_1 \vdash \epsilon : \tau_2 \Rightarrow \Gamma \vdash \lambda x : \tau_1.\epsilon : \tau_1 \to \tau_2$
- (App) $\Gamma \vdash \epsilon_1 : \tau_1 \to \tau_2,\ \Gamma \vdash \epsilon_2 : \tau_1 \Rightarrow \Gamma \vdash \epsilon_1\epsilon_2 : \tau_2$
- (Err) $\Gamma \vdash x : \tau \Rightarrow \Gamma \vdash \texttt{Err}(x) : \mathbf{Err}_0$

1.4. **Error Reduction and Normalization.**

Theorem 1.3 (Strong Normalization for λ_{Error})**.** *Every well-typed term in λ_{Error} reduces to a unique normal form via finite steps under beta-reduction.*

Proof. Follows from the strong normalization of simply typed lambda calculus extended with total constructors for $\mathbf{Err}_n$ and closed error operators. □

1.5. **Applications and Perspectives.**

- **Error Derivation Tracking:** Every computed error term includes a full symbolic derivation tree.
- **Error Transformation Compiler:** Rewrite engines can perform safe optimization of error compositions.
- **Proof-Carrying Arithmetic:** Each step in a prime-counting estimate can be tagged with error witnesses.
- **Type-Based Error Control:** Higher-type error constructors can detect unsafe error combinations statically.

2. Results and Analytical Consequences of λ_{Error}

The type-theoretic treatment of arithmetic error terms yields new insights into their symbolic structure, propagation behavior, and compositional invariants. We now explore several key outcomes derived from the λ_{Error} system.

2.1. **Result I: Syntax-Encoded Error Derivations.**

Proposition 2.1. *Every error term ϵ well-typed in λ_{Error} admits a unique derivation tree with nodes labeled by typing rules and leaf nodes corresponding to base errors.*

Proof. This follows from the inductive structure of the typing rules and the determinism of lambda term construction. □

2.2. **Result II: Compositional Predictability of Error Growth.**

Theorem 2.2. *Let $\epsilon_1 : \mathbf{Err}_n$, $\epsilon_2 : \mathbf{Err}_n$. Then their composition $\lambda x.\epsilon_1(\epsilon_2(x))$ has type $\mathbf{Err}_n \to \mathbf{Err}_n$ and preserves the growth class of its constituents under beta-reduction.*

Proof. By closure of the type system under function application and abstraction. The size and depth of terms remain within definable bounds. □

2.3. **Result III: Error Transformation Closure.**

Proposition 2.3. *If T is a transformation function of type $\mathbf{Err}_n \to \mathbf{Err}_n$ and $\epsilon : \mathbf{Err}_n$, then $T(\epsilon)$ is well-typed and normalizable.*

Proof. This is a direct consequence of strong normalization and the well-formedness of λ_{Error} types. □

2.4. **Result IV: Static Detection of Illegitimate Error Forms.**

Theorem 2.4. *There exists a type-checking algorithm $\text{Check}_{\lambda_{\text{Error}}}$ that detects whether an error term composition violates structural consistency, i.e., whether:*

$$\Gamma \nvdash \epsilon : \mathbf{Err}_n$$

Proof. By constructing a finite type-derivation tree, either a derivation is found or a type error is detected. This is decidable due to the finite rule set. □

2.5. **Phenomenological Consequences.**

- **Compositional Traceability:** Each error now carries an explicit symbolic trace.
- **Error Logic Programming:** One may encode rewrite rules for error propagation as logic programs.
- **Static Error Classification:** Errors can be ranked and detected via type depth and kind.
- **Error Category Semantics:** Each type $\mathbf{Err}_n$ corresponds to an object in an internal error category with morphisms as transformations.

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