Cardinality and Construction of Intermediate Algebraic Structures Between Magmas and Groups

Pu Justin Scarfy Yang

September 5, 2024

Abstract

This paper investigates the cardinality of the set of intermediate algebraic structures that exist between well-defined algebraic systems such as magmas, monoids, semigroups, and groups. Specifically, we analyze the cardinality of the continuum, $\mathfrak{c}$, which arises when considering the number of possible structures in each transition from magmas to monoids, monoids to semigroups, and semigroups to groups, particularly when the cardinality of the underlying set n becomes infinite. We also provide explicit methods for constructing these intermediate structures, demonstrating the rich variety of possible algebraic systems. Furthermore, we explore a vast array of adjectives and ideas that can be applied to generalize and extend algebraic structures, leading to an infinite landscape of possible mathematical objects.

1 Introduction

Algebraic structures such as magmas, monoids, semigroups, and groups are fundamental in mathematics, each defined by specific properties related to associativity, identity elements, and inverses. The study of intermediate structures between these well-known systems provides insights into the complexity and richness of algebraic operations. This paper examines the number of such intermediate structures and demonstrates that, in the case of infinite sets, the cardinality of these structures is $\mathfrak{c}$, the cardinality of the continuum. Furthermore, we provide explicit methods for constructing these intermediate structures, highlighting the diversity of possible configurations. Additionally, we explore an infinite number of adjectives and ideas that can be used to generalize algebraic structures further, leading to a boundless exploration of mathematical possibilities.

2 Cardinality of Structures Between Magmas and Groups

To understand the full scope of intermediate structures, we consider the transitions from magmas to monoids, monoids to semigroups, and semigroups to groups. At each step, we introduce or relax certain properties such as associativity, the existence of an identity element, and the existence of inverses.

2.1 Magmas to Monoids

Magmas are sets M equipped with a binary operation $\cdot$ but with no additional properties such as associativity or identity elements.

Monoids are magmas with the additional property of having an identity element e and an associative operation.

The transition from magmas to monoids involves introducing associativity and identity elements, which can be defined partially, conditionally, or in layers. For an infinite set M, the number of ways to define these properties is vast, and the total number of possible intermediate structures between magmas and monoids is $\mathfrak{c}$.

2.2 Monoids to Semigroups

Monoids possess an identity element and associative operation.

Semigroups are defined by associativity alone, without necessarily having an identity element.

In this case, we may consider the introduction of partial identities or the removal of identities in certain subsets of the monoid. For an infinite set M, the number of configurations for introducing these properties leads to a space of structures with cardinality $\mathfrak{c}$.

2.3 Semigroups to Groups

Semigroups have associative operations but do not require the existence of identity elements or inverses.

Groups are semigroups where every element has both an identity element and an inverse.

Transitioning from semigroups to groups involves introducing identity elements (if not already present) and inverses for each element. The introduction of partial or layered inverses contributes to a continuum of possible structures, resulting in a cardinality of $\mathfrak{c}$ when n is infinite.

3 Explicit Construction of Intermediate Structures

In this section, we detail methods for explicitly constructing the intermediate algebraic structures between magmas, monoids, semigroups, and groups.

3.1 Constructing Intermediate Structures Between Magmas and Monoids

To construct intermediate structures between magmas and monoids, we focus on introducing associativity and identity elements in a gradual and controlled manner:

1. **Partial Associativity:** Define a subset $A \subseteq M \times M \times M$ where associativity holds. For example, define $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $(a, b, c) \in A$, but not necessarily for all triples in $M \times M \times M$.

2. **Layered Associativity:** Partition M into disjoint subsets $M_1, M_2, \ldots, M_k$, and define associativity within each layer M_i. Associativity might hold in M_1 but not in M_2, or it may vary between layers.

3. **Conditional Associativity:** Define associativity based on an external condition or parameter. For instance, associativity could hold only when a certain condition $C(a, b, c)$ is satisfied.

4. **Partial Identity:** Introduce a partial identity by defining a subset $I \subseteq M$ such that $e \cdot a = a \cdot e = a$ for all $e \in I$ and $a \in M$. This identity might only act as an identity for a subset of the elements in M.

3.2 Constructing Intermediate Structures Between Monoids and Semigroups

To construct structures between monoids and semigroups, we explore variations in the identity element:

1. **Partial Identity Removal:** In a monoid $(M, \cdot, e)$, gradually remove the identity property for certain elements. Define a subset $S \subseteq M$ where the identity element e does not act as an identity, i.e., $e \cdot s \neq s$ for some $s \in S$.

2. **Layered Identity:** Define different identity elements within different layers of M. For example, e_1 might be the identity for layer M_1, while e_2 is the identity for layer M_2. The structure as a whole may not have a global identity element.

3. **Conditional Identity:** Introduce an identity element that acts as such only under specific conditions. For instance, e could be an identity only when a condition $C(e, a)$ holds true.

3.3 Constructing Intermediate Structures Between Semigroups and Groups

Finally, to construct intermediate structures between semigroups and groups, we introduce partial or conditional inverses:

1. **Partial Inverses:** In a semigroup $(S, \cdot)$, define a subset $I \subseteq S$ where for each $a \in I$, there exists an element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, where e is the identity element. Not every element in S needs to have an inverse.

2. **Layered Inverses:** Partition S into layers, and define inverses within specific layers. For example, elements in S_1 might have inverses, while those in S_2 do not.

3. **Conditional Inverses:** Define inverses that only exist under certain conditions. For example, an element $a \in S$ might have an inverse a^{-1} only if a condition $C(a)$ is satisfied.

4 Exploring an Infinite Landscape of Adjectives and Ideas

In addition to "partial," "layered," and "conditional," there are infinitely many other adjectives and ideas that can be applied to generalize and create new algebraic structures. Here are some additional examples:

4.1 Additional Adjectives and Concepts

1. **Fractal:** Introduces self-similar or recursive properties within the structure. This can be used to study structures where operations are recursively defined or where similar patterns recur within substructures.

2. **Stochastic:** Operations or properties are defined probabilistically rather than deterministically, which is useful in modeling systems with inherent randomness or uncertainty.

3. **Fuzzy:** Elements and operations do not have sharply defined characteristics but instead possess degrees of membership or participation, relevant in fuzzy logic systems.

4. **Discontinuous:** Operations or properties hold only in certain parts of the structure or under specific conditions, leading to discontinuities in behavior.

5. **Hybrid:** Combines properties from different algebraic structures within the same framework, leading to a "hybrid" structure that inherits features from both.

6. **Adaptive:** The structure evolves over time or in response to external stimuli, relevant in dynamic systems and control theory.
7. **Hierarchical:** The structure is organized in layers or levels, with different properties or operations applying at each level, useful in multiscale modeling and network theory.
8. **Contextual:** Operations and properties depend on the context or environment in which the structure is considered, relevant in parametric systems.
9. **Modular:** The structure is divided into modules or components, each with its own internal algebraic properties, relevant in modular arithmetic and component-based design.
10. **Quantum:** Operations might involve quantum properties such as superposition or entanglement, important in quantum computing.
11. **Approximate:** Properties or operations are approximately defined rather than exact, useful in numerical methods and optimization.
12. **Fractional:** Elements or operations might involve fractional or non-integer values, relevant in fractional calculus and scaling systems.
13. **Non-commutative:** Operations are not commutative, leading to structures with richer or more complex behaviors, important in quantum mechanics.
14. **Tropical:** Operations follow the rules of tropical algebra, where usual addition and multiplication are replaced by operations based on maximum (or minimum) and addition.
15. **Non-associative:** Operations are not necessarily associative, leading to structures where the grouping of operations affects the outcome, relevant in the study of Lie algebras.
16. **Dual:** The structure is defined in terms of its dual, where the roles of elements and operations are reversed or otherwise transformed, important in algebraic geometry.
17. **Graded:** The structure is decomposed into "grades" or "levels," each of which might have different algebraic properties, relevant in homological algebra.
18. **Multivalued:** Operations might produce multiple outputs rather than a single result, leading to a multivalued algebraic structure, useful in multifunctions and uncertainty modeling.

4.2 Infinite Possibilities

The list above provides just a glimpse into the vast array of adjectives and ideas that can be used to explore different algebraic structures. Each adjective corresponds to a different way of modifying or generalizing existing structures, leading to an infinite number of possible structures and areas of study.

Many of these adjectives can be combined to create even more specialized and complex structures. For example, "partial quantum layered non-commutative structures" could describe an algebraic system that integrates several different concepts simultaneously. Beyond the traditional adjectives, researchers can invent new adjectives or concepts tailored to specific problems or applications, further expanding the landscape of algebraic structures.

5 Conclusion

The cardinality of the continuum $\mathfrak{c}$ emerges naturally when analyzing the number of intermediate algebraic structures between magmas and groups, particularly when the underlying set M is infinite. Each intermediate set of structures (magmas to monoids, monoids to semigroups, and semigroups to groups) contributes to this continuum number, reflecting the rich and complex landscape of algebraic systems. The introduction and variation of properties such as associativity, identity, and inverses across infinite sets lead to uncountably many configurations, affirming that the cardinality of the continuum is intrinsic to the study of these intermediate structures.

Moreover, the exploration of additional adjectives and ideas, such as fractal, stochastic, fuzzy, and others, opens up an infinite landscape of possible algebraic structures. This richness allows for the development of entirely new mathematical theories and applications, pushing the boundaries of what is possible in algebra and beyond.

References

[1] P. M. Cohn, *Basic Algebra: Groups, Rings, and Fields*. Springer, 2003.

[2] S. Lang, *Algebra*, 3rd ed. Addison-Wesley, 2002.

[3] G. Birkhoff, *Lattice Theory*, 3rd ed. American Mathematical Society, 1967.

[4] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed. Springer, 1998.

[5] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed. Springer, 1995.

[6] J. B. Fraleigh, *A First Course in Abstract Algebra*, 7th ed. Addison-Wesley, 2002.

[7] D. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, 3rd ed. Addison-Wesley, 1998.

[8] T. W. Hungerford, *Algebra*. Springer, 1974.

[9] P. A. Grillet, *Semigroups: An Introduction to the Structure Theory*. CRC Press, 2007.

[10] P. Schneider, *Noncommutative Geometry*, Springer, 1997.

[11] L. Fuchs, *Partially Ordered Algebraic Systems*. Pergamon Press, 1963.

[12] G.-C. Rota, *Indiscrete Thoughts*. Birkhäuser, 1996.

[13] G. Gierz, *Continuous Lattices and Domains*. Cambridge University Press, 2003.

[14] M. Hazewinkel, *Handbook of Algebra*. Elsevier, 2012.

[15] I. M. Gelfand, *Lectures on Linear Algebra*. Dover Publications, 1993.

[16] J. Baez, *Quantum Quandaries: A Category-Theoretic Perspective*. Springer, 2011.

[17] F. W. Lawvere and S. H. Schanuel, *Conceptual Mathematics: A First Introduction to Categories*. Cambridge University Press, 2003.

[18] M. Aigner, *A Course in Enumeration*. Springer, 2010.

[19] C. A. Weibel, *An Introduction to Homological Algebra*. Cambridge University Press, 1994.

[20] C. C. Pinter, *A Book of Abstract Algebra*. McGraw-Hill, 1990.

[21] A. L. Rosenberg, *The Number Theory of Function Fields*. Springer, 1995.

[22] M. Ziegler, *Lectures on Polytopes*. Springer, 1995.

[23] L. V. Ahlfors, *Complex Analysis*, 3rd ed. McGraw-Hill, 1979.