The random sieve of Hawkins

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Abstract

This paper is concerned with the Hawkins random sieve which is a probabilistic analogue of the sieve of Eratosthenes. In the present paper the main part is based on Wunderlich's paper 'A probabilistic setting for prime number theory'. We are going to show that, almost surely, for the sequences generated by Hawkins random sieve 'the prime number theorem' holds. Finally, we give some additional results concerning Hawkins random sieve, as for example 'the Riemann Hypothesis' or the twin prime problem for random sequences.

Contents

1 Introduction

The random sieve of Hawkins [1] is a analog of the sieve of Eratosthenes. It produces random sequences which have many properties in common with the sequence of prime numbers. The Hawkins sieve is constructed recursively as follows:

Let $A_1 := \{2, 3, 4, ...\}$.

Step 1 : Put $X_1 = \min A_1$. From the set $A_1 \setminus \{X_1\}$ each member is deleted with probability $\frac{1}{X_1}$, i.e. not deleted with probability $1 - \frac{1}{X_1}$. The set of elements which remain is denoted by A_2.

Step n : Put $X_n = \min A_n$. From the set $A_n \setminus \{a_n\}$ each member is deleted with probability $\frac{1}{X_n}$ or not deleted with probability $1 - \frac{1}{X_n}$. The set of elements of $A_n \setminus \{X_n\}$ which remain is denoted by A_{n+1}.

Hawkins constructed a sequence of finite probability spaces $\Omega_2, \Omega_3, ...$ for which $\Omega_n = (B_n, \mathcal{F}_n, \mathbb{P}_n)$ where $B_n = \{2, 3, 4, ..., n\}$, $\mathcal{F}_n$ is the class of subsets of B_n which include 2 and $\mathbb{P}_n(B)$ is the probability that all numbers in $B \in \mathcal{F}_n$ survive the random sieve described above. Hawkins has shown that random sequences satisfy 'the prime number theorem' in the sence of the *weak* law of large numbers. Wunderlich [2] was able to show that the prime number theorem holds also in the sense of the *strong* law of large numbers. To do so, Wunderlich introduced a single probability space $\Omega = (X, \mathcal{F}, \mu)$ where X is the set of all sequences containing 2, $\mathcal{F}$ is contained in the power set 2^X and μ is defined so as to agree with $\mathcal{P}_n$ on $\mathcal{F}_n$. All the results obtained by Hawkins can be obtained in Ω. Details will be given in section 2 and 3, which are based on a paper of Wunderlich [2]. In section 4 we will recapture the main results which were obtained after that Hawkins introduced his random sieve. We will discuss a paper of W. Neudercker and D. Williams [3] in which they prove that 'the Riemann Hypothesis' holds almost surely. They use the probability space introduced by Wunderlich for proving 'the Riemann Hypothesis' but they could also give a less complicated method to prove 'the prime number theorem' in the sense of the *strong* law. Finally, we are going to show the results obtained by D. Forster and D. Williams [6].
If we use the above notation of Hawkins random sieve we can regard X_n as the nth 'random prime' and introduce the 'Mertens' product

$$Y_n = \prod_{1 \leq k \leq n} (1 - X_k^{-1})^{-1}.$$

As mentioned above, Wunderlich proved that, almost surely,

$$X_n \sim n \log(n), \; Y_n \sim \log(n) \quad (n \to \infty)$$

which is 'the prime number theorem' in the sense of the *strong* law of large numbers and 'Mertens theorem'. Neudecker and Williams proved that, almost surely, the

random limit
$$L = \lim_{n\to\infty} X_n \exp(-Y_n)$$
exists in $(0,\infty)$ and showed that

$$E_n = L\,\mathrm{li}(X_n/L) - n$$

should be regarded as the 'error term in the prime number theorem'. Then they established that, almost surely, the 'Riemann hypothesis':

$$E_n = \mathcal{O}(n^{\frac{1}{2}+\epsilon})$$

holds. Then D. Foster and D. Williams [5] extended the probability space introduced by Wunderlich so that they could define a Brownian Motion $B = \{B_t :\geq 0\}$ on it and proved that
$$E_n = B_n + \mathcal{O}\left(\frac{(n\log\log(n))^{\frac{1}{2}}}{\log(n)}\right).$$
In section 4, we are also going to talk about the twin prime problem for random sequences which was discussed by H. M. Bui and J. P. Keating [7].
I want to thank Dr. Paul-Olivier Dehaye for his great support and his many corrections while writing this thesis.

2 The probability space

Let $X = \{(a_i)_{i=1,\dots,n} \mid 1 \le n \le \infty; \forall i : a_i \in \mathbb{N} \setminus \{1\}, a_i < a_{i+1}\}$ denote the set of all increasing sequences of integers greater then 1. For $\alpha \in X$ we denote by α_n the set of elements of α which are less than n, i.e. $\alpha_n = \alpha \cap \{2, 3, 4, \dots, n-1\}$ and $\alpha^n = \alpha \setminus \alpha_n$.

Definition 2.1. *An element $E \in \Omega$ is called an* elementary set *if there exists a sequence $(a_1, \dots, a_k) \in X$ and an integer $n > a_k$ for which $\varepsilon \in E$ if and only if*

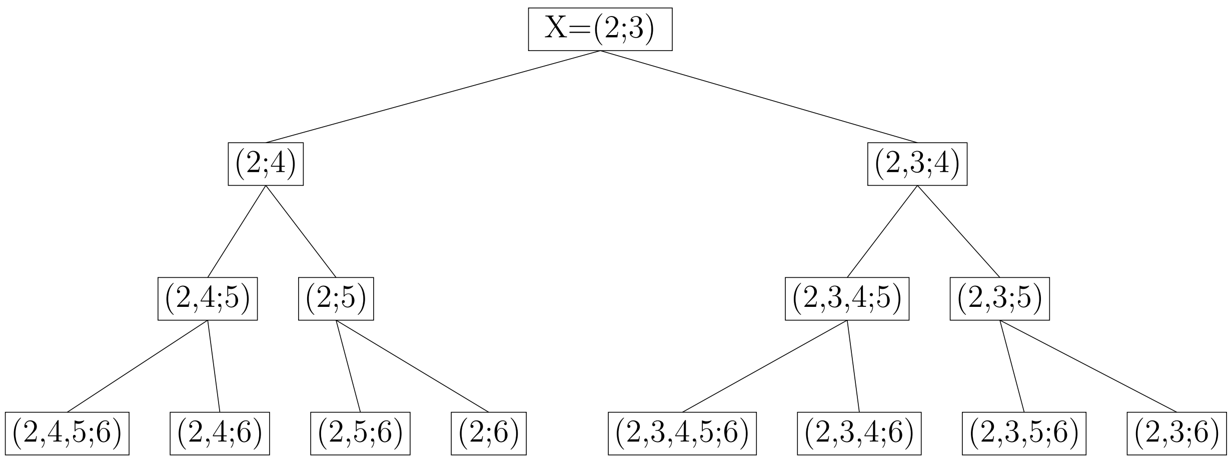
$$\varepsilon_n = (a_1, \dots, a_k).$$

E is denoted by $(a_1, \dots, a_k; n)$ or $(a; n)$ if a is used to denote the sequence $(a_1, \dots, a_k)$.

Remark 2.2. Each sequence α in an elementary set $E = (a_1, \dots, a_k; n)$ is of the form $\alpha = (a_1, \dots, a_k, a_{k+1}, a_{k+2} \dots)$ where $a_l \ge n$ for all $l \ge k+1$.

Definition 2.3. *For an elementary set $E = (a; n)$ we call n* the order of *E.*

Remark 2.4. Note that k could be zero in which case $(\ ; n)$ is the set of all sequences whose elements are not less than n. We will assume $n \ge 2$ since 1 is never an element of our sequences. Note also that $(\ ; 2) = X$. We denote by $\mathcal{E}$ the collection of all elementary sets.



Now we define recursively a set function on $\mathcal{E}$:

Definition 2.5. *Let $\mu : \mathcal{E} \to \mathbb{R}_{\ge 0}$ be the function defined by*

i. $\mu(\ ; 2) = 1$

ii. $\mu(a_1, \dots, a_k, n; n+1) = \mu(a_1, \dots, a_k; n) \prod_{1 \le i \le k} \left(1 - \frac{1}{a_i}\right)$

iii. $\mu(a_1, \dots, a_k; n+1) = \mu(a_1, \dots, a_k; n) \left(1 - \prod_{1 \le i \le k} \left(1 - \frac{1}{a_i}\right)\right)$

The following binary tree will be helpful for proving the next lemmas:

Remark 2.6. In order to speak of elementary sets we often speak of *nodes* since every elementary set corresponds to a node in the binary tree. If A is neither above or below B, $A \cap B = \emptyset$. Nodes on the same level have also the same order.

Remark 2.7. As we want to emulate the Hawkins random sieve, we have to consider how our set function μ is related to that sieve. Assume that we have constructed with the sieve of Hawkins a sequence $\alpha = (2, a_2, a_3, ...)$. For $n \in \mathbb{N} \setminus \{2\}$ let $k \in \mathbb{N}$ be the biggest integer such that $a_k < n$. By Hawkins random sieve, n will have been eliminated with the probabilities $1/2, 1/a_2, ..., 1/a_k$, thus the probability that n remains is

$$\prod_{i=1}^{k}\left(1 - \frac{1}{a_i}\right).$$

Now note that α belongs to the elementary set $(a_1, ..., a_k; n)$, i.e. the chance of obtaining α is at most $\mu(a_1, ..., a_k; n)$. By construction, the chance of obtaining sequences of the form $(a_1, ..., a_k, n, ...)$ is

$$\mu(a_1, ..., a_k, n; n+1)$$

which, by Definition 2.5, is equal to

$$\mu(a_1, ..., a_k; n) \prod_{1 \le i \le k} \left(1 - \frac{1}{a_i}\right).$$

The probability of obtaining sequences of the form $(a_1, ..., a_k, n+1, ...)$, i.e. sequences not containing n, is

$$\mu(a_1, ..., a_k; n+1) = \mu(a_1, ..., a_k; n)\left(1 - \prod_{1 \le i \le k} \left(1 - \frac{1}{a_i}\right)\right).$$

So we see that our set function from Def. 2.5 emulates the Hawkins random sieve.

Definition 2.8. *The* parent *of a node $B \in \mathcal{E}$ is the node A of order 1 less than the order of B such that $B \subset A$.*

Definition 2.9. *A node C is called a* subtree *of a node B if B is the parent of C. If two nodes B and C have the same parent, then we say that B is the* brother *of C and vice versa.*

Lemma 2.10. *Let B_1 and B_2 be disjoint elementary sets. The union $B_1 \cup B_2$ is an elementary set if and only if B_1 and B_2 are subtrees of the same node.*

Proof. Considering the binary tree it is clear that if B_1 and B_2 have the same parent, then $B_1 \cup B_2$ is equal to their parent, hence an elementary set. Thus it remains to show that if the union $B_1 \cup B_2$ is an elementary set, then B_1 and B_2 are subtrees of the same node. For the sake of contradiction assume that B_1 and B_2 are not subtrees of the same node and the union $B_1 \cup B_2$ is an elementary set. Let $A_i^{(1)}$ be respectively the parent of the node B_i for $i = 1, 2$. Then $A_1^{(1)}$ contains B_1 but it also contains sequences which are neither in B_1 nor in B_2, thus $A_1^{(1)}$ cannot be the union $B_1 \cup B_2$. The same holds for the parent $A_2^{(1)}$ of B_2. The next possible node containing B_1 is the parent of $A_1^{(1)}$, say $A_1^{(2)}$. If $A_1^{(2)} \neq X$ one can repeat the same argument as before to see that $A_1^{(2)}$ can not be equal to $B_1 \cup B_2$. Again, the same holds for the parent of $A_2^{(1)}$. Continuing in the same manner, after finitely many steps one of the parent of the parents, say $A_1^{(k)}$ will be equal to X. Clearly, X is only the union of its two subtrees, say C_1 and C_2. Thus $B_1 = C_1$ and $B_2 = C_2$ which is a contradiction since we have assumed that B_1 and B_2 are not subtrees of the same node. That is, the assumption that B_1 and B_2 are not subtrees of the same node led us to a contradiction. □

Lemma 2.11. *Let $A = B_1 \cup B_2 \cup ... \cup B_k$ such that $A, B_i \in \mathcal{E}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq k$, $k \geq 2$. Then*

$$\mu(A) = \sum_{1 \leq j \leq k} \mu(B_j),$$

that is μ is finitely additive on $\mathcal{E}$.

Proof. We prove this by induction on k. If $k = 2$ then, by Lemma 2.10, B_1 and B_2 must be subtrees of the same node A and the lemma follows from Definition 2.5. Assume now that the lemma holds for $k < m$ and $A = B_1 \cup B_2 \cup ... \cup B_m \in \mathcal{E}$.

<u>Claim :</u> Since A is the union of finitely many B_i's, at least two of them, say B_{m-1} and B_m must be subtrees of the same node, say B'.

Proof. Let B_m be a node having highest order. The parent of B_m, say A_m is by definition the smallest node ($\neq B_m$) containing B_m. Therefore we see that $A \supseteq A_m$, since $B_m \subseteq A$ and $A \in \mathcal{E}$ by assumption. But A_m contains also the brother of B_m, say B'_m. Since B_m has highest order, and B'_m is of the same order as B_m, one can see that B'_m can not be a union of nodes of the the set $\{B_1, ..., B_{m-1}\}$. Thus B'_m itself must be contained in $\{B_1, ..., B_{m-1}\}$ and we may assume that $B'_m = B_{m-1}$. □

By Claim 1 B_{m-1} and B_m have the same parent and since the union of two nodes having the same parent is equal to the parent itself, we can replace them by their parent, say C. So we can write $A = B_1 \cup ... \cup B_{m-2} \cup C$ which is a union of $n - 1$ nodes, hence our inductive assumption completes the proof. □

Definition 2.12. *For $R \subseteq X$ we say that R is* eventually arbitrary *if there exists a positive integer $n > 1$ such that the following holds: for any sequence $\rho \in R$ and for any sequence $\eta = (n_1, n_2, ...) \in X$ with $n_1 \geq n$ there exists a sequence $\tilde{\rho} \in R$ such that $\tilde{\rho}_n = \rho_n$ and $\tilde{\rho}^n = \eta$.*

Definition 2.13. *For an eventually arbitrary set R, the smallest $n \in \mathbb{N}$ satisfying the conditions in Definition 2.12 is called the* order *of R.*

Remark 2.14. Clearly, an elementary set $E = (\alpha_1, ..., \alpha_k; n)$ is eventually arbitrary since it contains by definition all sequences of the form $(\alpha_1, ..., \alpha_k, n_1, n_2, ...)$ for arbitrary $n_i \geq n$, $i \in \mathbb{N}$. Obviously any finite union of elementary sets having the same order m, is eventually arbitrary.

We can give an equivalent definition of eventually arbitrary sets as follows:

Definition 2.15. *$R \subseteq X$ is called* eventually arbitrary *if there exists $n > 1$ such that $\rho \in R$ if and only if the elementary set $(\rho_n; n)$ is contained in R.*

Remark 2.16. From the last definition one can see immediately that every eventually arbitrary set R is the union of finitely many elementary sets: let $\alpha_1, ..., \alpha_k$ be the set of all distinct finite sequences ρ_n, where ρ ranges over all of R. Then

$$\bigcup_{i=1}^{k}(\alpha_i; n) \subseteq R$$

by Definition 2.12. Now let δ be any sequence in R. Then δ_n is equal to some α_i, since ρ has ranged over all R, hence $\delta \in (\alpha_i; n)$. That is

$$R \subseteq \bigcup_{i=1}^{k}(\alpha_i; n)$$

and so we have shown that

$$R = \bigcup_{i=1}^{k}(\alpha_i; n).$$

We denote by $\mathcal{R}$ the collection of all eventually arbitrary sets. By Remark 2.14 the collection of elementary sets is contained in $\mathcal{R}$.

Lemma 2.17. *$\mathcal{R}$ is an algebra of subsets.*

Proof. Let $R \in \mathcal{R}$ be of order n. By Remark 2.16,

$$R = \bigcup_{i=1}^{k} E_i$$

where $E_i \in \mathcal{E}$. By Definition 2.15, any finite union of elementary sets is eventually arbitrary. Thus, $\mathcal{R}$ is closed under finite unions. To show that $\mathcal{R}$ is closed under differences it suffices to show that if A and B are in $\mathcal{E} \subseteq \mathcal{R}$, then $A - B$ is in $\mathcal{R}$. One can see from the binary tree representation that either $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$. Thus we need only consider the case $B \subseteq A$. Considering the binary tree one can see that A is the union of all elementary sets contained in A which have the same order as B. Thus $A - B$ is that same union minus B, hence again a union of elementary sets and consequently eventually arbitrary. Finally, X is clearly eventually arbitrary making $\mathcal{R}$ an algebra. $\square$

Now we can extend μ to $\mathcal{R}$:

Definition 2.18. *Let $R \in \mathcal{R}$ and let*

$$R = \bigcup_{1 \le i \le k} E_i$$

be its representation as a finite union of disjoint elementary sets. Then we define

$$\mu(R) := \sum_{1 \le i \le k} \mu(E_i).$$

Lemma 2.19. *The extension of μ to $\mathcal{R}$ is well defined.*

Proof. Let $R \in \mathcal{R}$ such that

$$R = \bigcup_{1 \le i \le k} E_i$$

and

$$R = \bigcup_{1 \le j \le l} F_j,$$

where $E_i, F_j \in \mathcal{E}$ with $E_{i_1} \cap E_{i_2} = \emptyset$ and $F_{j_1} \cap F_{j_2} = \emptyset$ for $1 \le i_1 \neq i_2 \le k$ and $1 \le j_1 \neq j_2 \le k$. We have to show that

$$(*) \qquad \sum_{1 \le i \le k} \mu(E_i) = \sum_{1 \le j \le l} \mu(F_j).$$

Now let $1 \le j_0 \le l$ and $\rho \in F_{j_0}$. Since

$$\bigcup_{1 \le j \le l} F_j = R = \bigcup_{1 \le i \le k} E_i$$

there exists $1 \le i_0 \le k$ such that $\rho \in E_{i_0}$, hence

$$E_{i_0} \cap F_{j_0} \neq \emptyset.$$

Since two elementary sets are either disjoint or one is contained in the other, we see that $F_{j_0} \subseteq E_{i_0}$ or $E_{i_0} \subseteq F_{j_0}$. Since j_0 was arbitrary we have for all $1 \le j_0 \le l$ there exists $1 \le i_0 \le k$ such that either $F_{j_0} \subseteq E_{i_0}$ or $E_{i_0} \subseteq F_{j_0}$. By the pigeonhole principle, for every $1 \le i \le k$ and for every $1 \le j \le l$ either there exists an index subset $I_j \subseteq \{1, 2, ..., k\}$ such that $i \in I_j$ and

$$F_j = \bigcup_{s \in I_j} E_s$$

or there exists an index subset $I_i \subseteq \{1, 2, ..., l\}$ such that $j \in I_i$ and

$$E_i = \bigcup_{s \in I_i} F_s.$$

Using Lemma 2.11 we can replace in the right hand side of $(*)$ each $\mu(F_j)$, which can be written as $F_j = \bigcup_{s \in I_j} E_s$, by

$$\sum_{s \in I_j} \mu(E_s).$$

Furthermore, the sum of those $\mu(F_j)$, for which there exists E_i with $E_i = \bigcup_{s \in I_i} F_s$ can be replaced by $\mu(E_i)$. Consequently, we have replaced the right hand side of $(*)$ by the left hand side. Thus our extension of μ is well defined. $\square$

Note that we have now a finitely additive set function μ defined on an algebra of sets $\mathcal{R}$ such that $\mu(X) = 1$. The next Lemma will be needed to show that μ is countably additive on $\mathcal{R}$.

Lemma 2.20. *For $i \in \mathbb{N}$ let $R_i \in \mathcal{R}$ such that $R_i \cap R_j = \emptyset$ if $i \neq j$. Then the union $\bigcup_{i=1}^{\infty} R_i$ is not in $\mathcal{R}$.*

Proof. For the sake of contradiction, assume that $R = \bigcup_{k=1}^{\infty} R_k$ for some $R \in \mathcal{R}$. By Lemma 2.17 every element in $\mathcal{R}$ can be written as a finite union of disjoint elementary sets, so

$$R = \bigcup_{i=1}^{n} E_i$$

and since each R_i is also an elementary set

$$R = \bigcup_{k=1}^{\infty} R_k = \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{n_k} E_{k,l}$$

where $E_{k,l}, E_i \in \mathcal{E}$ and $E_{k,l_1} \cap E_{k,l_2} = \emptyset$, $E_{i_1} \cap E_{i_2} = \emptyset$ for all $1 \leq k \leq \infty, 1 \leq l_1 \neq l_2 \leq n_k$, and $1 \leq i_1 \neq i_2 \leq n$. Since the E_i's are disjoint, there exists one which is the union of infinitely many elementary sets $E_{k,l}$, i.e.

$$E_i = \bigcup_{s=1}^{\infty} \tilde{E}_s$$

where $\tilde{E}_s \in \{E_{k,l}; 1 \leq k \leq \infty, 1 \leq l \leq n_k\}$ for all $s \in \mathbb{N}$. That is, E_i contains infinitely many disjoint elementary sets and is itself an elementary set. Now we construct a sequence $\alpha \in E_i$ which is not contained in $\bigcup_{s=1}^{\infty} \tilde{E}_s$, hence a contradiction.
Consider the binary tree having E_i as a root. Delete any vertex if it is a descendant of any vertex $\tilde{E}_s$. The remaining tree must contain an infinite path from the root downwards: assume that there doesn't exist an infinite path from the root downwards. Then the tree itself is finite and has the sequence $\tilde{E}_1, \tilde{E}_2, ...$ as terminal nodes which then are finite in number contrary to our assumptions. Hence there must be an infinite path.It is easy to see that the intersection of all nodes in this infinite path is a sequence α which is not in any of the $\tilde{E}_s$. $\square$

Definition 2.21. *For $i \in \mathbb{N}$ let $R_i \in \mathcal{R}$ such that $R_i \cap R_j = \emptyset$ if $i \neq j$. Then we define*

$$\mu\Big(\bigcup_{i=1}^{\infty} R_i\Big) := \sum_{i=1}^{\infty} \mu(R_i).$$

Note that by Lemma 2.20 this is well defined since an infinite union of disjoint sets in $\mathcal{R}$ is not in $\mathcal{R}$.
Since we have shown that μ is a countably additive set function on an algebra, we are allowed to use the measure extension theorem. That is, we can extend μ to the smallest σ-algebra containing $\mathcal{R}$. We denote this σ-algebra by $\mathcal{S}$.

3 Bertrand's postulate for random sequences

Bertrand's postulate for the sequence of prime number states that for every $n > 1$ there exists a prime number p such that $n < p < 2n$. The goal is to determine the measure of all random sequences satisfying Bertrand's postulate, i.e. the measure of

$$B := \{\alpha \in X | \forall n > 1 \ \exists \ a \in \alpha : n < a < 2n\}.$$

Note that since

$$B^c = \bigcup_{\substack{k \in \mathbb{N}, \\ (a_1,...,a_k) \in \mathbb{N}^k \\ 2 = a_1 < a_2 < ... < a_k}} (a_1, ..., a_k - 1; 2a_k)$$

we see that $B^c \in \mathcal{S}$, hence $B \in \mathcal{S}$, i.e. B is a measurable set. In this section we are going to determine the measure of a set which contains B, i.e. we will determine the measure of

$$C := \bigcup_{n=2}^{\infty} (2, 3, ..., n-1; 2n).$$

Note that this is a disjoint union and that B is contained in the complement C^c. Hence we calculate the measure of the following elementary set:

$$\mu((2, 3, 4, 5, ..., n-1; 2n) = \mu(2, 3, ..., n-1; 2n-1)\left(1 - \prod_{i=2}^{n-1}\left(1 - \frac{1}{i}\right)\right)$$

$$= \mu(2, 3, ..., n-1; 2n-2)\left(1 - \prod_{i=2}^{n-1}\left(1 - \frac{1}{i}\right)\right)^2 = ...$$

$$= \mu(2, 3, ..., n-1; n)\left(1 - \prod_{i=2}^{n-1}\left(1 - \frac{1}{i}\right)\right)^n$$

Now we obtain

$$\mu(2, 3, ..., n-1; n) = \mu(2, 3, ..., n-2; n-1)\prod_{i=2}^{n-2}\left(1 - \frac{1}{i}\right) = ...$$

$$= \frac{1}{2} \cdot \prod_{i=2}^{3}\left(1 - \frac{1}{i}\right) \cdot \prod_{i=2}^{4}\left(1 - \frac{1}{i}\right) \cdot ... \cdot \prod_{i=2}^{n-2}\left(1 - \frac{1}{i}\right) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot ... \cdot \frac{1}{n-2}$$

$$= \frac{1}{(n-2)!}$$

and by combining this with the following

$$\left(1 - \prod_{i=2}^{n-1}\left(1 - \frac{1}{i}\right)\right)^n = \left(1 - \frac{1}{n-1}\right)^n$$

we obtain

$$\mu((2,3,4,5,...,n-1;2n)) = \frac{1}{(n-2)!}\left(1-\frac{1}{n-1}\right)^n.$$

Thus, we can write

$$\mu(\bigcup_{n=2}^{\infty}(2,3,...,n-1;2n)) = \sum_{n=2}^{\infty}\mu((2,3,...,n-1;2n))$$

$$=\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\left(1-\frac{1}{n-1}\right)^n = \sum_{n=2}^{\infty}\frac{\left(1-\frac{1}{n-1}\right)^{n-1}}{(n-1)!}\cdot(n-2),$$

hence we have shown

$$\mu(C) = \sum_{n=2}^{\infty}\frac{\left(1-\frac{1}{n-1}\right)^{n-1}}{(n-1)!}\cdot(n-2).$$

As mentioned above, we have

$$\mu(B) \le \mu(C^c) = 1-\mu(C).$$

Hence a lower bound for $\mu(C)$ gives us an upper bound for $\mu(B)$, i.e. for the measure of all random sequences satisfying 'Bertrand's postulate'. In particular we have shown that $\mu(B) < 1$ since $\mu(C)$ is surely strictly bigger than 0. So it would be nice to determine the value of convergence of

$$\sum_{n=2}^{\infty}\frac{\left(1-\frac{1}{n-1}\right)^{n-1}}{(n-1)!}\cdot(n-2).$$

Note that for examlpe if one could show that

$$\sum_{n=2}^{\infty}\frac{\left(1-\frac{1}{n-1}\right)^{n-1}}{(n-1)!}\cdot(n-2) = 1$$

then we have proven that B is a set of measure zero.

4 Random sequences

Our next goal is to show that the sequences contained in our probability space behave like the primes, that is the number of elements less than x approximates $x/\log x$ in the sense of the weak law of large numbers. Most of the following results in this section were obtained by David Hawkins [4].

Definition 4.1. *Again we denote by X the set of all increasing sequences of integers greater then* 1*. For $n \geq 2$ and $\alpha \in X$ we define the following random variables*

$$x_n(\alpha) = \prod_{\substack{a\in\alpha \\ a<n}} \Big(1 - \frac{1}{a}\Big), \ x_2(\alpha) = 1,$$

$$h_n(\alpha) = \sum_{\substack{a\in\alpha \\ a<n}} 1.$$

$$z_n(\alpha) = \begin{cases} x_n(\alpha) & \text{if } n \in \alpha \\ 1 - x_n(\alpha) & \text{if } n \notin \alpha \end{cases}$$

Since x_n and h_n have constant values on any elementary set A_n of order n, we will often use the notation $x(A_n)$ and $h(A_n)$ instead of $x_n(\alpha)$ and $h_n(\alpha)$ for $\alpha \in A_n$. One should note that h_n is the analog of the prime number counting function π. That is, $h_n(\alpha)$ is the number of natural numbers greater than or equal to 2, which are contained in the random sequence α. One of our main goals will be to calculate the variance of the random variable h_n. As the next lemma will show, the random variables x_n and z_n will give us an explicit expression for the measure of any elementary set:

Lemma 4.2. *Let α be any sequence contained in an elementary set A_n of order n. Then*

$$\mu(A_n) = \prod_{j=3}^{n-1} z_j(\alpha).$$

Proof. We do this by induction on n. If $n = 4$, then $A_4 = (2, 3; 4)$ or $A_n = (2; 4)$ and the lemma follows immediately from the definition of μ. So assume that the lemma is true for all $l \leq n$ and let $\alpha = (a_1, ..., a_k; n + 1) \in A_{n+1}$. By definition, we have

$$\mu(a_1, ..., a_k; n+1) = \begin{cases} \mu(a_1, ..., a_{k-1}; n) \cdot \prod_{i=1}^{k-1}(1 - 1/a_i) & \text{if } a_k = n \\ \mu(a_1, ..., a_k; n) \cdot (1 - \prod_{i=1}^{k}(1 - 1/a_i)) & \text{if } a_k \neq n \end{cases}$$

$$= \prod_{i=3}^{n-1} z_i(\alpha) \times \begin{cases} x_n(\alpha) & \text{if } a_k = n \\ 1 - x_n(\alpha) & \text{if } a_k \neq n \end{cases}$$

$$= \prod_{i=3}^{n} z_i(\alpha)$$

where we have used the induction hypothesis for the second equality. □

The next proposition is a tool to derive sharp asymptotic formula:

Proposition 4.3. *Let $[a, b]$ be a real interval with $a < b$ and f a function on (a, b) with continuous derivative. Then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \int_a^b \{t\}\, f'(t)dt - f(b)\,\{b\} + f(a)\,\{a\}$$

where $\{t\}$ denotes the fractional part *of t, defined for any real t by $\{t\} = t - [t]$ and $[t]$ is the greatest integer not bigger than t.*

The proof is left to the reader or can be reconstructed in the book of Apostol [8].

Example 4.4. *We apply Proposition 4.3 to the harmonic series* $\sum_2^n \frac{1}{k}$. *Here, a=1, b=n and* $f(t) = 1/t$. *So we obtain*

$$\sum_2^n \frac{1}{k} = \int_1^n \frac{1}{t}dt - \int_1^n \frac{\{t\}}{t^2}dt = \log(n) - \int_1^n \frac{\{t\}}{t^2}dt.$$

Now obviously

$$\int_1^n \frac{\{t\}}{t^2}dt = \int_1^\infty \frac{\{t\}}{t^2} - \int_n^\infty \frac{\{t\}}{t^2}dt$$

and the last term is less than $\int_n^\infty \frac{1}{t^2}dt = \frac{1}{n}$. *By adding* 1 *on both sides we have proved*

$$\sum_{k=1}^n \frac{1}{k} = \log n + \gamma + \mathcal{O}(\frac{1}{n}),$$

where

$$\gamma := 1 - \int_1^\infty \frac{\{t\}}{t^2}dt$$

is known as the Euler-Mascheroni-constant.

Lemma 4.5. *Let* S_n *denote the set of all sequences in* X *containing* n, *then*

$$\mu(S_n) = \mathbb{E}[x_n] := \int_X x_n d\mu.$$

Proof. Let B_{n+1} denote any elementary set of order $n+1$ such that each sequence in B_{n+1} contains n and let $\mathcal{B}_{n+1}$ denote the set of all such elementary sets. Then

$$\mu(S_n) = \mu\Big(\bigcup_{\substack{B_{n+1}\in \\ \mathcal{B}_{n+1}}} B_{n+1}\Big) = \sum_{B_{n+1}\in\mathcal{B}_{n+1}} \mu(B_{n+1}) \overset{\text{Def. 2.5}}{=} \sum_{B_n} \mu(B_n)x(B_n)$$

where the last sum is extended over all elementary sets B_n of order n. This is exactly the expectation of x_n. $\square$

Remark 4.6. We frequently use the notation B_n for an elementary set of order n.

Lemma 4.7. *For any* $k \in \mathbb{Z}$ *we have the following recurrence relation*

$$\mathbb{E}[x_{n+1}^k] - \mathbb{E}[x_n^k] = \Big(\Big(1 - \frac{1}{n}\Big)^k - 1\Big)\mathbb{E}[x_n^{k+1}]$$

with the boundary condition $\mathbb{E}[x_2^k] = 1$.

Proof.

$$\mathbb{E}[x_{n+1}^k] := \sum_{B_{n+1}} \mu(B_{n+1}) x_{n+1}^k(B_{n+1})$$

$$= \sum_{\substack{B_{n+1}: \\ \alpha \in B_{n+1} \Rightarrow n \in \alpha}} \mu(B_{n+1}) x_{n+1}^k(B_{n+1}) + \sum_{\substack{B_{n+1}: \\ \alpha \in B_{n+1} \Rightarrow n \notin \alpha}} \mu(B_{n+1}) x_{n+1}^k(B_{n+1})$$

$$\overset{\text{Def. 2.5}}{=} \sum_{B_n} \mu(B_n) x_n(B_n) x_n^k(B_n) \Big(1 - \frac{1}{n}\Big)^k + \sum_{B_n} \mu(B_n)(1 - x_n(B_n)) x_n^k(B_n)$$

$$= \sum_{B_n} \mu(B_n) \Big(\Big(1 - \frac{1}{n}\Big)^k - 1\Big) x_n^{k+1}(B_n) + \mu(B_n) x_n^k(B_n)$$

$$= \Big(\Big(1 - \frac{1}{n}\Big)^k - 1\Big) \mathbb{E}[x_n^{k+1}] + \mathbb{E}[x_n^k]$$

□

Remark 4.8. *Note that by the last lemma we have*

$$\mathbb{E}[x_{n+1}^{-1}] - \mathbb{E}[x_n^{-1}] = \Big(1 - \frac{1}{n}\Big)^{-1} - 1 = \frac{1}{n-1}.$$

Lemma 4.9.

$$\mu(S_n) = \mathbb{E}[x_n] = \frac{1}{\log(n-1)} + \mathcal{O}\Big(\frac{1}{\log^2(n-1)}\Big)$$

Proof. Letting $k = -1$ in Lemma 4.7 and summing form 2 to $n-1$, we obtain

$$\mathbb{E}[x_n^{-1}] - 1 = \mathbb{E}[x_n^{-1}] - \mathbb{E}[x_2^{-1}] = \sum_{l=2}^{n-1} \mathbb{E}[x_{l+1}^{-1}] - \mathbb{E}[x_l^{-1}] \tag{1}$$

$$\overset{\text{Rem. 4.8}}{=} \sum_{l=2}^{n-1} \big((1 - \frac{1}{l})^{-1} - 1\big) \overset{\text{Rem. 4.8}}{=} 1 + \sum_{l=1}^{n-2} \frac{1}{l} = \log(n-2) + \gamma + 1 + \mathcal{O}(\frac{1}{n}).$$

$$\Longrightarrow \mathbb{E}[x_n^{-1}] = \log(n-2) + \gamma + 2 + \mathcal{O}(\frac{1}{n}).$$

Using Lemma 4.7 again but now for $k = 1$, we obtain

$$\frac{1}{\mathbb{E}[x_{n+1}]} - \frac{1}{\mathbb{E}[x_n]} = \frac{1}{-\frac{1}{n}\mathbb{E}[x_n^2] + \mathbb{E}[x_n]} - \frac{1}{\mathbb{E}[x_n]} = \frac{n}{n\mathbb{E}[x_n] - \mathbb{E}[x_n^2]} - \frac{1}{\mathbb{E}[x_n]} \tag{2}$$

$$= \frac{n\mathbb{E}[x_n] - n\mathbb{E}[x_n] + \mathbb{E}[x_n^2]}{\mathbb{E}[x_n](n\mathbb{E}[x_n] - \mathbb{E}[x_n^2])} = \left[\frac{n\mathbb{E}[x_n]^2}{\mathbb{E}[x_n^2]} - \mathbb{E}[x_n]\right]^{-1}.$$

By Jensen's inequality we have

$$\mathbb{E}[x_n^2] \geq \mathbb{E}^2[x_n]. \tag{3}$$

Using (2), one can see that

$$\mathbb{E}[x_{n+1}]^{-1} - \mathbb{E}[x_n]^{-1} \geq \frac{1}{n - \mathbb{E}[x_n]} \geq \frac{1}{n}.$$

Hence,

$$\text{(4)} \qquad \mathbb{E}[x_n]^{-1} - 1 = \sum_{l=2}^{n-1} \mathbb{E}[x_{l+1}]^{-1} - \mathbb{E}[x_l]^{-1} \geq \sum_{l=2}^{n-1} \frac{1}{k} \geq \log(n-1) + \gamma + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Applying Schwartz's inequality to the random variables x_n and x_n^{-1} we see that

$$\text{(5)} \qquad \mathbb{E}[x_n]\mathbb{E}[x_n^{-1}] \geq 1.$$

By (1),

$$\mathbb{E}[x_n] \geq \mathbb{E}[x_n^{-1}]^{-1} = \frac{1}{\log{(n-2)} + \gamma + 2 + \mathcal{O}(\frac{1}{n})} \geq \frac{1}{\log{(n-1)}} \cdot \frac{1}{(1 + \frac{C_1}{\log{(n-1)}})}$$

$$\geq \frac{1}{\log{(n-1)}} \cdot \Big(1 - \frac{C_1}{\log{(n-1)}}\Big) = \frac{1}{\log{(n-1)}} - C_1 \cdot \frac{1}{\log{(n-1)}^2}$$

and by (4) we obtain

$$\mathbb{E}[x_n] \leq \frac{1}{\log{(n-1)} + \gamma + 1 + \mathcal{O}(\frac{1}{n})} \leq \frac{1}{\log{(n-1)}} + \mathcal{O}(\frac{1}{n}).$$

Consequently we have

$$\mathbb{E}[x_n] = \frac{1}{\log(n-1)} + \mathcal{O}\Big(\frac{1}{\log^2(n-1)}\Big)$$

□

Lemma 4.10. *For any $k \in \mathbb{N}$ we have*

$$\mathbb{E}[x_n^{-k}] = \log^k(n-1) + \mathcal{O}(\log^{k-1}(n-1))$$

Proof. Using Lemma 4.7 we obtain

$$\mathbb{E}[x_{l+1}^{-k}] - \mathbb{E}[x_l^{-k}] = \Big(\frac{k}{l-1} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{-(k-1)}]$$

$$= \frac{k}{l-1} \cdot \log^{k-1}(l-1) + \mathcal{O}\Big(\frac{\log^{k-2}(l-1)}{l-1}\Big).$$

Summing this for $l = 2$ up to $n-1$ one obtains

$$\mathbb{E}[x_n^{-k}] - 1 = \sum_{l=2}^{n-1} \mathbb{E}[x_{l+1}^{-k}] - \mathbb{E}[x_l^{-k}] \leq \sum_{l=2}^{n-1} \frac{k}{l-1} \cdot \log^{k-1}(l-1) + C \cdot \frac{\log^{k-2}(l-1)}{l-1}$$

$$\leq \log(n-1)^{k-1}\sum_{l=2}^{n-1}\frac{1}{l-1} + C\log(n-1)^{k-1}\sum_{l=2}^{n-1}\frac{1}{l-1}$$

$$\overset{\text{Example 4.4}}{=} \log^k(n-1) + \mathcal{O}(\log^{k-1}(n-1)).$$

For the lower inequality we use the definition of the Riemann integral: Set $f(s) = \log(s)^k$. Then

$$\log(n-1)^k - \log(2)^k = f(n-1) - f(2) = \int_2^{n-1} f'(s)ds$$

$$= \inf_{\mathcal{P}}\sum_{l=2}^{m-1}(s_l - s_{l-1})\cdot \sup_{s_{l-1}<s<s_l} f'(s) \leq \sum_{l=2}^{n-1}((l)-(l-1))f'(l-1)$$

$$= \sum_{l=2}^{n-1}\frac{k}{l-1}\cdot\log^{k-1}(l-1)$$

where for the third equality we have used the definition of the Riemann integral and $\mathcal{P}$ denotes any partition of the interval $[2, n-1]$. The last inequality is obtained by drop monotonly. Consequently we have

$$\mathbb{E}[x_n^{-k}] \geq \log(n-1)^k \leq \sum_{l=2}^{n-1}\frac{k}{l-1}\cdot\log^{k-1}(l-1) + C\cdot\frac{\log^{k-2}(l-1)}{l-1}$$

$$\geq \log(n-1)^k + C\log(n-1)^{k-1}$$

$\square$

Lemma 4.11. *For any $k \in \mathbb{N}$ we have*

$$\mathbb{E}[x_n^k] = \frac{1}{\log(n-1)^k} + \mathcal{O}\Big(\frac{1}{\log(n-1)^{k+1}}\Big).$$

Proof. We prove this by induction on k. So for $k = 1$ the statement follows from Lemma 4.9. Now assume that the statement is true for $k-1$. Use Lemma 4.7 and write

$$\frac{1}{\mathbb{E}[x_{l+1}^k]} - \frac{1}{\mathbb{E}[x_l^k]} = \left[\frac{\mathbb{E}[x_{l+1}^k]\mathbb{E}[x_l^k]}{\mathbb{E}[x_l^k] - \mathbb{E}[x_{l+1}^k]}\right]^{-1}$$

$$\overset{\text{Lem. 4.7}}{=} \left[\frac{\Big(\mathbb{E}[x_l^k] - \Big(\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{k+1}]\Big)\cdot\mathbb{E}[x_l^k]}{\Big(\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{k+1}]}\right]^{-1}$$

$$= \left[\frac{\mathbb{E}^2[x_l^k] - \Big(\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{k+1}]\cdot\mathbb{E}[x_l^k]}{\Big(\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{k+1}]}\right]^{-1}$$

$$\geq \left[\frac{\mathbb{E}^2[x_l^k]}{\Big(\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)\Big)\mathbb{E}[x_l^{k+1}]}\right]^{-1} \qquad (*)$$

Applying Schwartz's inequality to $x_l^{\frac{k-1}{2}}$ and $x_l^{\frac{k+1}{2}}$ we obtain

$$\mathbb{E}[x_l^{k+1}]\mathbb{E}[x_l^{k-1}] \geq \mathbb{E}^2[x_l^k].$$

Using this we obtain that

$$(*) \geq \frac{\frac{k}{l} + \mathcal{O}\Big(\frac{1}{l^2}\Big)}{\mathbb{E}[x_l^{k-1}]} = \frac{k}{l}\log^{k-1}(l-1) + \mathcal{O}(\log^{k-2}(l-1)),$$

where we have used our inductive assumption for the last equality. Thus we conclude

$$\frac{1}{\mathbb{E}[x_{l+1}^k]} - \frac{1}{\mathbb{E}[x_l^k]} \geq \frac{k}{l}\log^{k-1}(l-1) + \mathcal{O}(\log^{k-2}(l-1)).$$

Summing this from $l = 2$ to $n-1$ yields

$$\mathbb{E}[x_n^k]^{-1} \geq \log^k(n) + \mathcal{O}(\log^{k-1}(n)),$$

hence

$$\mathbb{E}[x_n^k] \geq \frac{1}{\log^k(n) + \mathcal{O}(\log^{k-1}(n))} \geq \frac{1}{\log(n-1)^k} + \mathcal{O}\Big(\frac{1}{\log(n-1)^{k+1}}\Big).$$

Now using Lemma 4.10 and the fact that $\mathbb{E}[x_n^k]\mathbb{E}[x_n^{-k}] \geq 1$ (Schwartz), one can see that

$$\mathbb{E}[x_n^k] \leq \frac{1}{\log(n-1)^k} + \mathcal{O}\Big(\frac{1}{\log(n-1)^{k+1}}\Big)$$

and we are finished.

□

Lemma 4.12. *Define* $M_n := \mathbb{E}[x_n^{-1}]$ *and* $c_{k,n} := \mathbb{E}[(x_n^{-1} - M_n)^k]$*. Recall that by* (1) *in the proof of Lemma 4.7 we have*

$$M_n = 1 + \sum_{l=1}^{n-2} \frac{1}{l} = \mathcal{O}(\log n).$$

Then for every $k \in \mathbb{N}$ *the central moments of* x_n^{-1} *converge, i.e.*

$$\lim_{n\to\infty} c_{k,n} = c_k < \infty.$$

Proof. First, observe that

$$(6) \qquad c_{k,n} = \mathbb{E}[(x_n^{-1} - M_n)^k] = \sum_{s=0}^{k} (-1)^s \binom{k}{s} M_n^{k-s}\mathbb{E}[x_n^{-s}].$$

It suffices to show that for any $k \in \mathbb{N}$ we have $c_{k,n+1} - c_{k,n} = \mathcal{O}(1/n^{1+\epsilon})$ for some $\epsilon > 0$, since the series

$$\sum_{l=1}^{n} \frac{1}{l^{1+\epsilon}}$$

converges for all $\epsilon > 0$, $n \to \infty$. By Lemma 4.7, for any $q \geq 0$ we have

$$M_{n+1} = M_n + \frac{1}{n-1},$$

hence

$$M_{n+1}^q = \Big(M_n + \frac{1}{n-1}\Big)^q = \sum_{s=0}^{q} \binom{q}{s} M_n^{q-s} \Big(\frac{1}{n-1}\Big)^s$$

$$= M_n^q + \frac{q}{n-1} M_n^{q-1} + \mathcal{O}\Big(M_n^{q-2} \frac{1}{(n-1)^2}\Big) = M_n^q + \frac{q}{n-1} M_n^{q-1} + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

or equivalently,

$$(7) \qquad M_n^q = M_{n+1}^q - \frac{q}{n-1} M_n^{q-1} - \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big).$$

By Lemma 4.10 we have

$$\mathbb{E}[x_n^{-q}] = \mathcal{O}(\log(n)^q).$$

So we can write, using (6),

$$c_{k,n+1} - c_{k,n} \overset{(6)}{=} \sum_{s=0}^{k} (-1)^s \binom{k}{s} (M_{n+1}^{k-s} \mathbb{E}[x_{n+1}^{-s}] - M_n^{k-s} \mathbb{E}[x_n^{-s}])$$

$$\overset{(7)}{=} \underbrace{\sum_{s=0}^{k} (-1)^s \binom{k}{s} M_{n+1}^{k-s} (\mathbb{E}[x_{n+1}^{-s}] - \mathbb{E}[x_n^{-s}])}_{=:A}$$

$$+ \underbrace{\sum_{s=0}^{k-1} (-1)^s \binom{k}{s} \frac{k-s}{n-1} M_n^{k-s-1} \mathbb{E}[x_n^{-s}]}_{=:B} + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

where we have replaced M_n^{k-s} by (7) (for $q = k - s$) to obtain the second equality. Now we show that

$$A = -B + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big).$$

By Lemma 4.7 we have

$$A = \sum_{s=0}^{k} (-1)^s \binom{k}{s} M_{n+1}^{k-s} \Big(\Big(1 - \frac{1}{n}\Big)^s - 1\Big) \mathbb{E}[x_n^{-s+1}]$$

$$= \sum_{s=0}^{k} (-1)^s \binom{k}{s} M_{n+1}^{k-s} \Big(\frac{s}{n}\Big) \mathbb{E}[x_n^{-s+1}] + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

$$= \sum_{s=0}^{k} (-1)^s \binom{k}{s} \Big(M_n^{k-s} + \frac{k-s}{n-1} M_n^{k-s-1}\Big) \Big(\frac{s}{n}\Big) \mathbb{E}[x_n^{-s+1}] + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

$$= \sum_{s=0}^{k} (-1)^s \binom{k}{s} M_n^{k-s} \Big(\frac{s}{n-1}\Big) \mathbb{E}[x_n^{-s+1}] + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

$$= \sum_{s=0}^{k-1} (-1)^{s+1} \binom{k}{s+1} \Big(\frac{s+1}{n-1}\Big) M_n^{k-s-1} \mathbb{E}[x_n^{-s}] + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

$$= \sum_{s=0}^{k-1} (-1)^{s+1} \binom{k}{s} \Big(\frac{k-s}{n-1}\Big) M_n^{k-s-1} \mathbb{E}[x_n^{-s}] + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big)$$

$$= -B + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big).$$

Consequently, we have shown that

$$c_{k,n+1} - c_{k,n} = A + B + \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big) = \mathcal{O}\Big(\frac{1}{n^{1+\epsilon}}\Big).$$

□

Consider $M, x \in \mathbb{R} \setminus \{0\}$ such that $\mid \frac{x^{-1}-M}{M} \mid < 1$. Then, for any $s \in \mathbb{N}$ we have the following identity

$$Mx = \frac{1}{1 + \Big(\frac{x^{-1}-M}{M}\Big)} = \sum_{l=0}^{\infty} (-1)^l \Big(\frac{x^{-1}-M}{M}\Big)^l$$

$$= 1 - \frac{(x^{-1}-M)}{M} + \frac{(x^{-1}-M)^2}{M^2} - ... + (-1)^{s-1} \frac{(x^{-1}-M)^{s-1}}{M^{s-1}}$$
$$+ (-1)^s \sum_{l=s}^{\infty} (-1)^l \Big(\frac{x^{-1}-M}{M}\Big)^l$$

$$= 1 - \frac{(x^{-1}-M)}{M} + \frac{(x^{-1}-M)^2}{M^2} - ... + (-1)^{s-1} \frac{(x^{-1}-M)^{s-1}}{M^{s-1}}$$
$$+ (-1)^s \Big(\frac{x^{-1}-M}{M}\Big)^s \underbrace{\sum_{l=0}^{\infty} (-1)^l \Big(\frac{x^{-1}-M}{M}\Big)^l}_{Mx}$$

$$= 1 - \frac{(x^{-1}-M)}{M} + \frac{(x^{-1}-M)^2}{M^2} - ... + (-1)^{s-1} \frac{(x^{-1}-M)^{s-1}}{M^{s-1}}$$
$$+ (-1)^s \cdot \frac{x(x^{-1}-M)^s}{M^{s-1}}$$

or equivalently

$$x = \frac{1}{M} - \frac{(x^{-1}-M)}{M^2} + \frac{(x^{-1}-M)^2}{M^3} - ... \tag{8}$$
$$... + (-1)^{s-1} \frac{(x^{-1}-M)^{s-1}}{M^s} + (-1)^s \cdot \frac{x(x^{-1}-M)^s}{M^s}.$$

If we set $M = M_n := \mathbb{E}[x_n^{-1}]$, $x = x_n$ and integrate (8) over X we obtain

$$(9) \quad \mathbb{E}[x_n] = \frac{1}{M_n} + \frac{\mathbb{E}[(x_n^{-1} - M_n)^2]}{M_n^3} - ... + (-1)^{s-1} \cdot \frac{\mathbb{E}[(x_n^{-1} - M_n)^{s-1}]}{M_n^{s-1}}$$

$$+(-1)^s \cdot \frac{\mathbb{E}[x_n(x_n^{-1} - M_n)^s]}{M_n^s}.$$

Raising (8) to the k-th power before integrating produces

$$(10) \quad \mathbb{E}[x_n^k] = \frac{1}{M_n^k} + \binom{k+1}{2} \frac{\mathbb{E}[(x_n^{-1} - M_n)^2]}{M_n^{k+2}} - \binom{k+2}{3} \frac{\mathbb{E}[(x_n^{-1} - M_n)^3]}{M_n^{k+3}} +$$

$$+ ... + (-1)^{ks} \cdot \frac{\mathbb{E}[x_n^k(x_n^{-1} - M_n)^{ks}]}{M_n^{ks}}$$

Corollary 4.13. *For any integers $s, k \in \mathbb{N}$ there exist positive constants $c_{i,n}$ such that for each i there is a constant c_i such that*

$$\lim_{n\to\infty} c_{i,n} = c_i < \infty,$$

$$\mathbb{E}[x_n] = \frac{1}{M_n} + \frac{c_{2,n}}{M_n^3} - \frac{c_{3,n}}{M_n^4} + ... + \mathcal{O}\Big(\frac{1}{M_n^s}\Big)$$

and

$$\mathbb{E}[x_n^k] = \frac{1}{M_n^k} + \binom{k+1}{2} \frac{c_{2,n}}{M_n^{k+2}} - \binom{k+2}{3} \frac{c_{3,n}}{M_n^{k+3}} + ... + \mathcal{O}\Big(\frac{1}{M_n^{ks+1}}\Big).$$

Proof. Choose $c_{i,n} = \mathbb{E}[(x_n^{-1} - M_n)^i]$, then by Lemma 4.12 the limit

$$\lim_{n\to\infty} c_{i,n} =: c_i$$

exists and is finite. It remains only to prove that the last terms of (9) and (10) are finite, i.e. we have to show

$$\lim_{n\to\infty} \mathbb{E}[x_n(x_n^{-1} - M_n)^s] < \infty$$

and

$$\lim_{n\to\infty} \mathbb{E}[x_n^k(x_n^{-1} - M_n)^{ks}] < \infty.$$

This follows by applying Schwartz inequality, i.e.

$$\mathbb{E}[x_n(x_n^{-1} - M_n)^s]^2 \le \mathbb{E}[x_n^2]\mathbb{E}[(x_n^{-1}M_n)^{2s}]$$

and this expression converges by Lemma 4.11. Similarly for the last term of (10) we have

$$\mathbb{E}[x_n^k(x_n^{-1} - M_n)^{ks}]^2 \le \mathbb{E}[x_n^{2k}]\mathbb{E}[(x_n^{-1}M_n)^{2ks}]$$

and again use Lemma 4.11. □

Recall Chebyshev's inequality

Proposition 4.14. *Let h be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expected value μ and finite variance σ^2. Then for any $k \in \mathbb{R}$ we have*

$$\mathbb{P}\left(\frac{|\ h - \mu\ |}{\sigma} > k\right) \leq \frac{1}{k^2}$$

Remember that the random variable h_n is the analog of the prime number counting function π. We now turn our attention to the random variable h_n, that is we calculate its expected value and its variance so that we can use Chebyshev's inequality to formulate the prime number theorem for random sequences in the sense of the weak law of large numbers. First, we will obtain the recurrence relation

$$\mathbb{E}[h_{n+1}^2] - \mathbb{E}[h_n^2] = 2\mathbb{E}[x_n h_n] + \mathbb{E}[x_n].$$

Summing this relation from 2 up to $n-1$ we obtain that

$$\mathbb{E}[h_n^2] - 1 = \sum_{l=2}^{n-1} 2\mathbb{E}[x_l h_l] + \mathbb{E}[x_l].$$

So we see that if we know the asymptotic behavior of $\mathbb{E}[x_l h_l]$ and $\mathbb{E}[x_l]$ then we know the asymptotics of $\mathbb{E}[h_n^2]$ and that is exactly what we are going to do in Lemma 4.18. Similarly, we are going to calculate $\mathbb{E}[h_n]$ which is quite easier, since we have the recurrence relation

$$\mathbb{E}[h_{n+1}] - \mathbb{E}[h_n] = \mathbb{E}[x_n],$$

as we will see in Corollary 4.15. Consequently we are able to calculate the variance of h_n, namely

$$V[h_n] := \mathbb{E}[h_n^2] - \mathbb{E}[h_n]^2.$$

We use the notation $V[x]$ to denote the variance of a random variable x.

So let us now calculate the recurrence relation for h^k:

Corollary 4.15. *We use the same notation as in Lemma 4.7. For any $k \in \mathbb{N}$ we have*

i. $\mathbb{E}[h_{n+1}^k] = \sum_{B_n} \mu(B_n)(1 - x_n(B_n))h_n^k(B_n) + \mu(B_n)x_n(B_n)(h_n(B_n) + 1)^k$,
ii. $\mathbb{E}[h_{n+1}] - \mathbb{E}[h_n] = \mathbb{E}[x_n]$,
iii. $\mathbb{E}[h_{n+1}^2] - \mathbb{E}[h_n^2] = 2\mathbb{E}[x_n h_n] + \mathbb{E}[x_n]$,
iv. $\mathbb{E}[h_{n+1}x_{n+1}^k] - \mathbb{E}[h_n x_n^k] = \mathbb{E}[x_n^{k+1}] - \frac{k}{n}\mathbb{E}[h_n x_n^{k+1}] + \mathcal{O}\left(\frac{1}{n}\mathbb{E}[h_n x_n^{k+1}]\right)$

Proof. i.:

$$\mathbb{E}[h_{n+1}^k] = \sum_{\substack{B_{n+1} \\ \forall \alpha \in B_{n+1}: \\ n \in \alpha}} \mu(B_{n+1})h_{n+1}^k(B_{n+1}) + \sum_{\substack{B_{n+1} \\ \forall \alpha \in B_{n+1}: \\ n \notin \alpha}} \mu(B_{n+1})h_{n+1}^k(B_{n+1})$$

$$\overset{\text{Def.2.5}}{=} \sum_{B_n} \mu(B_n)x_n(B_n)(h_n(B_n) + 1)^k + \sum_{B_n} \mu(B_n)(1 - x_n(B_n))h_n^k(B_n).$$

ii. and iii.: follow immediately from $i.$.

iv.:

$$\mathbb{E}[h_{n+1}x_{n+1}^k] =$$

$$\sum_{\substack{B_{n+1}\\ \forall \alpha\in B_{n+1}:\\ n\in\alpha}} \mu(B_{n+1})h_{n+1}(B_{n+1})x_{n+1}^k(B_{n+1}) + \sum_{\substack{B_{n+1}\\ \forall \alpha\in B_{n+1}:\\ n\notin\alpha}} \mu(B_{n+1})h_{n+1}(B_{n+1})x_{n+1}^k(B_{n+1}) =$$

$$\sum_{B_n}\mu(B_n)x_n(B_n)(h_n(B_n)+1)x_n^k(B_n)\Big(1-\frac{1}{n}\Big)^k + \sum_{B_n}\mu(B_n)(1-x_n(B_n))h_n(B_n)x_n^k(B_n)$$

$$= \underbrace{\Big[\sum_{B_n}\mu(B_n)(h_n(B_n)+1)x_n^{k+1}(B_n)\Big]}_{\mathbb{E}[h_nx_n^{k+1}]+\mathbb{E}[x_n^{k+1}]}\Big(1-\frac{1}{n}\Big)^k + \underbrace{\sum_{B_n}\mu(B_n)(1-x_n(B_n))h_n(B_n)x_n^k(B_n)}_{\mathbb{E}[h_nx_n^k]-\mathbb{E}[h_nx_n^{k+1}]}$$

$$= \Big(\Big(1-\frac{1}{n}\Big)^k - 1\Big)\mathbb{E}[h_nx_n^{k+1}] + \Big(1-\frac{1}{n}\Big)^k\mathbb{E}[x_n^{k+1}] + \mathbb{E}[h_nx_n^k]$$

$$\Longrightarrow \mathbb{E}[h_{n+1}x_{n+1}^k] - \mathbb{E}[h_nx_n^k] = \Big(1-\frac{1}{n}\Big)^k\mathbb{E}[x_n^{k+1}] - \frac{k}{n}\mathbb{E}[h_nx_n^{k+1}] + \mathcal{O}\Big(\frac{1}{n^2}\mathbb{E}[h_nx_n^{k+1}]\Big)$$

$$= \mathbb{E}[x_n^{k+1}] - \frac{k}{n}\mathbb{E}[h_nx_n^{k+1}] + \mathcal{O}\Big(\frac{1}{n}\mathbb{E}[x_n^{k+1}]\Big)$$

where we have used that $\mathbb{E}[h_nx_n^{k+1}] = \mathcal{O}[n\mathbb{E}[x_n^{k+1}]]$ for the last equality. □

Lemma 4.16. *Let r, s and t be integers such that $s \geq 0$ and $0 < r < t$. Then*

$$(*) \qquad \sum_{k=2}^{n}\frac{k^s}{M_k^r} = \frac{1}{s+1}\cdot\frac{n^{s+1}}{M_n^r} + \frac{c(1,r,s)n^{s+1}}{M_n^{r+1}} +$$

$$\dots + \frac{c(t-r-1,r,s)n^{s+1}}{M_n^{r+(t-r-1)}} + \mathcal{O}\Big(\frac{n^{s+1}}{M_n^t}\Big)$$

where

$$c(1,r,s) = \frac{r}{(s+1)^2}$$

and

$$c(i,r,s) = \frac{r(r+1)\cdots(r+i-1)}{(s+1)^{i+1}}.$$

Proof. We prove this by reverse induction on r. So for $r = t-1 < t$ equality $(*)$ becomes

$$\sum_{k=2}^{n}\frac{k^s}{M_k^{t-1}} = \frac{1}{s+1}\cdot\frac{n^{s+1}}{M_n^{t-1}} + \mathcal{O}\Big(\frac{n^{s+1}}{M_n^t}\Big),$$

which follows by using

$$M_n = 1 + \sum_{l=2}^{n-2}\frac{1}{l} = \log(n-2) + \gamma + 2 + \mathcal{O}(\frac{1}{n})).$$

Assume that $(*)$ holds for $r+1$, $r<t-1$. Define $a_k = \frac{1}{M_k^r}$ and $b_k = k^s$ and use the partial summation formula

$$\sum_{k=2}^{n} a_k b_k = a_n b_n + \sum_{k=2}^{n-1}\Big(\sum_{l=2}^{k} b_l\Big)\cdot(a_k - a_{k+1})$$

to obtain

$$\sum_{k=2}^{n}\frac{k^s}{M_k^r} = \frac{n^s}{M_n^r} + \sum_{k=2}^{n-1}\underbrace{\Big(\sum_{l=2}^{k} l^s\Big)}_{(*)}\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big).$$

Now we replace $(*)$ by using the following equality

$$\sum_{l=1}^{k} l^s = \frac{(k+1)^{s+1}}{s+1} + \sum_{j=1}^{s}\frac{B_j}{s-j+1}\binom{s}{j}(k+1)^{s-j+1} = \frac{k^{s+1}}{s+1} + C\cdot k^s,$$

where the B_j's are the Bernoulli numbers and C is a constant independent of k. Consequently we obtain

$$\begin{aligned}
\sum_{k=2}^{n}\frac{k^s}{M_k^r} &= \frac{n^s}{M_n^r} + \sum_{k=2}^{n-1}\Big(\frac{k^{s+1}}{s+1} + C\cdot k^s\Big)\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big) \\
&\overset{n \text{ sufficiently large}}{\leq} \frac{1}{s+1}\frac{n^{s+1}}{M_n^r} + \sum_{k=2}^{n-1}\Big(\frac{k^{s+1}}{s+1} + C\cdot k^s\Big)\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big) \\
&= \frac{1}{s+1}\frac{n^{s+1}}{M_n^r} + \sum_{k=2}^{n-1}\frac{k^{s+1}}{s+1}\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big) + Cn^s
\end{aligned}$$

where, for the last equality, we have used that

$$\sum_{k=2}^{n-1} C\cdot k^s\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big) \leq Cn^s\sum_{k=2}^{n-1}\Big(\frac{1}{M_k^r} - \frac{1}{M_{k+1}^r}\Big) = Cn^s\Big(\frac{1}{M_2^r} - \frac{1}{M_n^r}\Big) \leq Cn^s.$$

Thus we have

$$\sum_{k=2}^{n}\frac{k^s}{M_k^r} = \frac{1}{s+1}\frac{n^{s+1}}{M_n^r} + \sum_{k=2}^{n-1}\frac{k^{s+1}}{s+1}\Big(\frac{M_{k+1}^r - M_k^r}{M_k^r M_{k+1}^r}\Big) + \mathcal{O}(n^s).$$

Since $M_{k+1} - M_k = \frac{1}{k-1}$ by Remark 4.8, we have for any $\epsilon > 0$

$$M_{k+1}^r = M_k^r + r\frac{M_k^{r-1}}{k-1} + \mathcal{O}(\frac{1}{k^{2-\epsilon}})$$

so that

$$\sum_{k=2}^{n}\frac{k^s}{M_k^r} = \frac{1}{s+1}\Big(\frac{n^{s+1}}{M_n^r} + r\sum_{k=2}^{n-1}\frac{k^s}{M_k^{r+1}} + \mathcal{O}(n^{s+\epsilon})\Big)$$

$$\overset{\text{induction hypothesis}}{=} \frac{1}{s+1}\Big(\frac{n^{s+1}}{M_n^r} + \frac{r}{s+1}\frac{n^{s+1}}{M_k^{r+1}} + \frac{r\cdot c(1,r+1,s)}{s+1}\frac{n^{s+1}}{M_k^{r+2}} + \\ \dots + \frac{r\cdot c(t-r,r+1,s)}{s+1}\frac{n^{s+1}}{M_k^{r+1+(t-r)}} + \mathcal{O}\Big(\frac{n^{s+1}}{M_k^t}\Big)\Big).$$

We see that

$$c(i,r,s) = \frac{c(i-1,r+1,s)\cdot r}{s+1}$$

and this substitution obtains $(*)$ of the lemma and finishes the reverse induction. $\square$

Corollary 4.17.

$$i.\quad \sum_{k=2}^{n}\frac{1}{M_k^4} = \frac{n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

$$ii.\quad \sum_{k=2}^{n}\frac{1}{M_n^3} = \frac{n}{M_n^3} + \frac{3n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

$$iii.\quad \sum_{k=2}^{n}\frac{1}{M_k^2} = \frac{n}{M_n^2} + \frac{2n}{M_n^3} + \frac{6n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

$$iv.\quad \sum_{k=2}^{n}\frac{1}{M_k} = \frac{n}{M_n} + \frac{n}{M_n^2} + \frac{2n}{M_n^3} + \frac{6n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

$$v.\quad \sum_{k=2}^{n}\frac{k}{M_k^4} = \frac{1}{2}\frac{n^2}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

$$vi.\quad \sum_{k=2}^{n}\frac{k}{M_k^3} = \frac{1}{2}\frac{n^2}{M_n^3} + \frac{3}{4}\frac{n^2}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

$$vii.\quad \sum_{k=2}^{n}\frac{k}{M_k^2} = \frac{1}{2}\frac{n^2}{M_n^2} + \frac{1}{2}\frac{n^2}{M_n^3} + \frac{3}{4}\frac{n^2}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

$$viii.\quad \sum_{k=2}^{n}\frac{k}{M_k} = \frac{1}{2}\frac{n^2}{M_n} + \frac{1}{4}\frac{n^2}{M_n^2} + \frac{1}{4}\frac{n^2}{M_n^3} + \frac{3}{8}\frac{n^2}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

Proof. Follows immediately from the last lemma. $\square$

Lemma 4.18.

$$i.\quad \mathbb{E}[h_n x_n^4] = \mathcal{O}(n\mathbb{E}[x_n^5])$$

$$ii.\quad \mathbb{E}[h_n x_n^3] = \frac{n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

$$iii.\quad \mathbb{E}[h_n x_n^2] = \frac{n}{M_n^3} + \frac{n}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}$$

$$iv.\quad \mathbb{E}[h_n x_n] = \frac{n}{M_n^2} + \frac{n}{M_n^3} + \frac{n(3C+2)}{M_n^4} + \mathcal{O}\Big(\frac{n}{M_n^5}\Big)$$

where $C := c_{2,n} = \lim_{n\to\infty}\mathbb{E}[(x_n^{-1} - M_n)^2]$.

Proof. i. : We need the following two equalities:

$$(*) \quad \mathbb{E}[h_{l+1}x_{l+1}^k] - \mathbb{E}[h_l x_l^k]$$

$$= \mathbb{E}[x_l^{k+1}] - \frac{k}{l}\mathbb{E}[h_l x_l^{k+1}] + \mathcal{O}\Big(\frac{1}{l}\mathbb{E}[h_l x_l^{k+1}]\Big) \text{ (see Corollary 4.15, } iv.)$$

and

$$\mathbb{E}[x_l^k] = \frac{1}{M_l^k} + \binom{k+1}{2}\frac{c_{2,l}}{M_l^{k+1}} + ... + \mathcal{O}\Big(\frac{1}{M_l^{ks+1}}\Big) \text{ (see Corollary 4.13).}$$

Consequently there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ we have

$$\mathbb{E}[h_{l+1}x_{l+1}^k] - \mathbb{E}[h_l x_l^k] \leq \mathbb{E}[x_l^{k+1}] - \frac{k}{l}\mathbb{E}[h_l x_l^{k+1}] + C \cdot \frac{1}{l}\mathbb{E}[h_l x_l^{k+1}]$$

and

$$\mathbb{E}[x_l^k] \leq \frac{1}{M_l^k} + \mathcal{O}\Big(\frac{1}{M_l^{k+1}}\Big)$$

Here we only show that

$$\mathbb{E}[h_n x_n^4] \leq Cn\mathbb{E}[x_n^5]$$

for some positive constant $C \in \mathbb{R}$, since the lower bound is obtained similarly. Setting $k = 4$ and summing $(*)$ from $l = 2$ up to $n-1$ we obtain

$$\mathbb{E}[h_n x_n^4] - 1 = \sum_{l=2}^{n-1} \mathbb{E}[h_{l+1}x_{l+1}^4] - \mathbb{E}[h_l x_l^4] \leq$$

$$\underbrace{\sum_{l=2}^{l_0-1} \mathbb{E}[h_{l+1}x_{l+1}^4] - \mathbb{E}[h_l x_l^4]}_{\leq \text{ const.}} + \sum_{l=l_0}^{n-1} \mathbb{E}[x_l^5] - \frac{k}{l}\mathbb{E}[h_l x_l^5] + C \cdot \frac{1}{l}\mathbb{E}[h_l x_l^5]$$

$$\overset{\text{Cor. 4.13}}{\leq} C(\sum_{l=2}^{n-1} \frac{1}{M_l^5} + \frac{1}{M_l^6}) \leq$$

$$\overset{\text{Lem. 4.16, with } s=0}{\leq} C(\frac{n}{M_n^5} + \frac{n}{M_n^6}).$$

Thus

$$\mathbb{E}[h_n x_n^4] = \mathcal{O}\Big(\frac{n}{M_n^5}\Big) \overset{\text{Cor. 4.13}}{=} \mathcal{O}(n\mathbb{E}[x_n^5]).$$

Similarly, *ii*) and *iii*) can be obtained by doing the same summation for $k = 3, 2$ and final that we use the estimations in Corollary 4.17 instead of Lemma 4.16. □

Proposition 4.19. *Let* $C := \lim_{n\to\infty} \mathbb{E}[(x_n^{-1} - M_n)^2]$. *Then*

$$i. \quad \mathbb{E}[h_n^2] = \frac{n^2}{M_n^2} + \frac{2n^2}{M_n^3} + \frac{n^3(3C+5)}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

$$ii. \quad \mathbb{E}[h_n] = \frac{n}{M_n} + \frac{n}{M_n^2} + \frac{n(C+2)}{M_n^3} + \mathcal{O}\Big(\frac{n}{M_n^4}\Big)$$

$$iii. \quad V[h_n] := \mathbb{E}[h_n^2] - \mathbb{E}[h_n]^2 = \frac{Cn^2}{M_n^4} + \mathcal{O}\Big(\frac{n^2}{M_n^5}\Big)$$

Proof. i. : Summing the expression in Corollary 4.15, *iii*) from 2 to $n-1$ we obtain

$$\mathbb{E}[h_n^2] - 1 = \sum_{l=2}^{n-1} \mathbb{E}[h_{l+1}^2] - \mathbb{E}[h_l^2] = \sum_{l=2}^{n-1} 2\mathbb{E}[x_l h_l] + \mathbb{E}[x_l].$$

Now one can use Lemma 4.18 and Corollary 4.17 to obtain the result.
ii. : Summing in the same way the expression in Corollary 4.15, *ii*) one obtains the result.
iii. : Obviously this follows directly from *i.* and *ii.*. □

Remark 4.20. It can be shown that the constant C appearing in Proposition 4.19 is

$$C = 1.2020569032... \,.$$

It is itself based on the constant appearing in Lemma 4.18.

Remark 4.21. Note that since $M_n = \mathcal{O}(\log(n))$ by Lemma 4.9, we have shown that there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N},\ n \geq n_0$ we have

$$V[h_n] \leq C \cdot \frac{n^2}{\log(n)^4}.$$

Now we formulate our final theorem which is due to D. Hawkins.

Theorem 4.22. *Let $\psi(n)$ be a function of n satisfying*

i. $\psi(n) = o(\log(n))$,
ii. $\psi(n) \to \infty$, *as* $n \to \infty$.

Then

$$\mu\Big(\Big\{\alpha \in X : \mid h_n(\alpha) - \mathbb{E}[h_n] \mid < \frac{n\psi(n)}{\log(n)^2}\Big\}\Big) > 1 - \mathcal{O}\Big(\frac{1}{\psi(n)^2}\Big) = 1 + o(n).$$

Proof. By Remark 4.21 the variance of h_n is finite, hence we are allowed to use Chebyshev's inequality and the result is immediately obtained. □

Remark 4.23. Theorem 4.22 shows that the prime number theorem hold in X in the sence of the weak law of large numbers.

Corollary 4.24. *Since $M_n = \mathcal{O}(\log(n))$ we obtain using Proposition 4.19, ii.*

$$\mathbb{E}[h_n] = \frac{n}{\log(n)} + \frac{n}{\log(n)^2} + \mathcal{O}\Big(\frac{n}{\log(n)^3}\Big)$$

and this implies that for any $\epsilon > 0$ we have

$$\mu\Bigg(\Bigg\{\alpha \in X : \left|\frac{h_n(\alpha) - \frac{n}{\log(n)}}{\frac{n}{\log(n)^{2-\epsilon}}}\right| < \frac{\psi(n)}{\log(n)^{\epsilon}}\Bigg\}\Bigg) > 1 - \frac{C}{\psi(n)^2} \geq 1 - \epsilon$$

for all $n \geq n_0$, $n_0 \in \mathbb{N}$ sufficiently large.

Proof. Follows immediately from Theorem 4.22. □

Remark 4.25. Corollary 4.24 shows that for any ϵ the error term is smaller than $n/\log(n)^{2-\epsilon}$ in the sense of the weak law.

5 Additional Results

In this section we are going to represent some results which were obtained after Hawkins introduced his random sieve. We first show Wunderlich's proof of the prime number theorem in the sense of the *strong law* of large numbers. To do so we use a method of Lévy [9], which was suggested to Wunderlich by Williams.

5.1 'Prime number theorem' in the sense of the strong law of large numbers

Theorem 5.1. *We consider the probability space $(X, \mathcal{S}, \mu)$, defined in Section 2. For almost all $\alpha \in X$ we have*

$$h_n(\alpha) \sim \frac{n}{\log(n)}.$$

Proof. Let $\delta \in \mathbb{R}_{>0}$ and define $l_n := [(1+\delta)]^n$. Letting $\psi(n) = \frac{\log(n)}{\log\log(n)}$ and $n = l_n$ in Theorem 4.22 we obtain

$$\mu\Bigg(\underbrace{\left\{\alpha \in \mathcal{S} : |h_{l_n}(\alpha) - \mathbb{E}[h_{l_n}]| > \frac{l_n}{\log(l_n)\log\log(l_n)}\right\}}_{=:A_n}\Bigg) < C \cdot \frac{(\log\log(l_n))^2}{\log(l_n)}$$

$$= \mathcal{O}\left(\frac{\log^2(n)}{n^2}\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{\log^2(n)}{n^2} < \infty$$

we have

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

and Borel-Cantellis lemma implies

$$\mu(A) = 0$$

where

$$A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i = \{A_n \text{ infinitely often}\}.$$

Hence, for almost all $\alpha \in X$ we have

$$(1) \qquad h_{l_n}(\alpha) \sim \mathbb{E}[h_{l_n}] \sim \frac{l_n}{\log(l_n)}.$$

Now fix $\alpha \in \mathcal{S}$. Obviously, $h_n(\alpha)$ is monotonic nondecreasing (in n) and for $i \in \mathbb{N}$ such that $l_n < i < l_{n+1}$ we have

$$h_{l_n}(\alpha) \leq h_i(\alpha) \leq h_{l_{n+1}}(\alpha)$$

or equivalently

$$\frac{h_{l_n}(\alpha)\log(i)}{i} \leq \frac{h_i(\alpha)\log(i)}{i} \leq \frac{h_{l_{n+1}}(\alpha)\log(i)}{i}$$

or

$$\frac{h_{l_n}(\alpha)\log(l_n)}{l_n} \leq \frac{h_i(\alpha)\log(i)}{i} \leq \frac{h_{l_{n+1}}(\alpha)\log(l_{n+1})}{l_n}.$$

Therefore (1) implies that for almost all $\alpha \in X$

$$(2) \qquad (1+o(1))\frac{1}{(1+\delta)} < \frac{h_i(\alpha)}{\frac{i}{\log(i)}} < 1+\delta+o(1)$$

for every $\delta > 0$. Now choose a sequence $\{\rho_i\}_{i\in\mathbb{N}}$ of positive numbers converging to zero and set

$$G_i = \left\{\alpha \in \mathcal{S} | \frac{1}{1+\rho_i} < \frac{h_j}{\frac{j}{log(j)}} < 1+\rho_i, \text{ for all but finitely many } j \in \mathbb{N}\right\}.$$

By (2) we have $\mu(G_i) = 1$ for each i and hence

$$\mu\left(\bigcap_{i=1}^{\infty} G_i\right) = 1.$$

But any sequence α contained in all the G_i must satisfy

$$h_n(\alpha) \sim \frac{n}{\log(n)}.$$

$\square$

Remark 5.2. Note that Theorem 5.1 corresponds to Theorem 3 of Wunderlich's paper [2]. Wunderlich also proved there that

$$x_n(\alpha) = \prod_{k<n,\ k\in\alpha}\left(1-\frac{1}{k}\right) \sim \frac{1}{\log(n)},$$

which is the probabilistic analog of Merten's theorem. To prove the probabilistic analog of Merten's theorem he first formulated another theorem (Theorem 2 in [2]), which allowed him also to obtain results of a number of other random variables and not just of x_n and h_n.

5.2 'Riemann Hypothesis' for random sequences generated by Hawkins sieve

We now come to a result obtained by W. Neudecker and D. Williams [3]. For $x > 1$ we define

$$\mathrm{li}(x) = \lim_{\delta\downarrow 0}\left(\int_0^{1-\delta} + \int_{1+\delta}^{x}\right)\frac{dz}{\log(z)} \sim \frac{x}{\log(x)}.$$

Recall that the Riemann Hypothesis is equivalent to the statement:

$$\mathrm{li}(p_n) = n + \mathcal{O}(n^{\frac{1}{2}+\epsilon})$$

where p_n denotes the nth prime. We use the same notation as in Section two and three for Hawkins random sieve. By Definition 4.1, we have for any $\alpha \in X$

$$Y_n(\alpha) := x_{n+1}^{-1}(\alpha) = \prod_{k \in \alpha, k \leq n} \left(1 - \frac{1}{k}\right)^{-1}.$$

Now let

$$X_n(\alpha) := n\text{th integer in } \alpha.$$

Note that if α is the sequence of the prime numbers, then $X_n(\alpha) =$ n-th prime number and $Y_n(\alpha)$ is the Euler product expansion (up to primes smaller than or equal to n) of the harmonic series. The following 'Riemann Hypothesis for the Hawkins Sieve' holds:

Theorem 5.3. *The random limit*

$$L = \lim_{n \to \infty} X_n \exp(-Y_n)$$

exists almost surely in $(0, \infty)$ *and*

$$L\,\mathrm{li}\left(\frac{1}{L} X_n\right) = n + \mathcal{O}(n^{\frac{1}{2}+\epsilon}) \ a.s.$$

Remark 5.4. Note that

$$E_n := L\,\mathrm{li}\left(\frac{1}{L} X_n\right) - n$$

should be regarded as 'the error term in the prime-number theorem', that is we have

$$E_n = \mathcal{O}(n^{\frac{1}{2}+\epsilon}).$$

To prove this theorem Neudercker and Williams used the fact that

$$\{(X_n, Y_n) | n \in \mathbb{N}\}$$

is a *Markov-process.* But it was Foster and Williams who understood this fact deeply and established the following result:

Theorem 5.5. *The sample space on which the Hawkins sieve is defined may be expanded so as to carry a Brownian motion* $B = \{B_t : t \geq 0\}$ *such that*

$$E_n = B_n + \mathcal{O}\left(\frac{(n \log\log(n))^{\frac{1}{2}}}{\log(n)}\right).$$

The main point in Theorem 5.5 is that it allows us to apply to the process E nearly all the known big theorems for Brownian motion, especially Strassen's [5] momentous improvement of the classical iterated-logarithm law:

Theorem 5.6. *Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of independent identically distributed random variables such that*

$$\mathbb{E}[Y_i] = 0$$

and

$$V[Y_i] = 1.$$

As usual, we denote by S_n the sum of the first n random variables of the $(Y_i)_{i\in\mathbb{N}}$. Then we have

$$\mathbb{P}\left(\limsup_{n\to\infty} n^{-1-(\frac{a}{2})}(2\log\log(n))^{-(\frac{a}{2})}\sum_{i=1}^{n}|S_i|^a = \frac{2(a+2)^{\frac{a}{2}-1}}{\left(\int_0^1 \frac{dt}{\sqrt{1-t^a}}\right)}\right) = 1$$

for any $a \geq 1$.

As already mentioned, Foster and Williams used the following facts: Write

$$\mathcal{F}_n = \sigma(\{(X_n, Y_n)|n \in \mathbb{N}\})$$

for the smallest σ-algebra with respect to which X_k and Y_k are measurable for every $k \leq n$. Then it is easily verified that the process

$$\{(X_n, Y_n)|n \in \mathbb{N}\}$$

is *Markovian* with

$$X_1 = Y_1 = 2$$
$$\mu(X_{n+1} - X_n = j|\mathcal{F}_n) = Y_n^{-1}(1 - Y_n^{-1})^{j-1}$$
$$Y_{n+1} = Y_n(1 - X_{n+1}^{-1})^{-1}$$

for all $j, n \in \mathbb{N}$. The following lemma follows immediately from elementary properties of the geometric distribution:

Lemma 5.7.

$$i.\ \mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] = Y_n$$
$$ii.\ \operatorname{Var}[X_{n+1} - X_n|\mathcal{F}_n] = Y_n^2 - Y_n$$

Now set

$$U_{n+1} = Y_n^{-1}(X_{n+1} - X_n)$$

and

$$S_n = \sum_{k=2}^{n}(U_k - 1).$$

Because of Lemma 5.7, $i.$ the process $\{S_n : n \in \mathbb{N}\}$ is a martingale. *Foster and William derive all results from the 'Strassen' properties of this martingale.*

5.3 The 'twin prime' problem for random sequences

Now we present another result concerning Hawkins random sieve which was obtained by H. M. Bui and J. P. Keating [6]. They established an asymptotic formula for the number of k-difference twin primes associated with the Hawkins random sieve. The formula for $k = 1$ was already established by Wunderlich [2].
For any $\alpha \in X$ the analogue of the k-difference twin prime counting function is defined as

$$\Pi_{X,\ X+k}(x; \alpha) = \# \{ j \leq x | j \in \alpha \text{ and } j + k \in \alpha \} .$$

Wunderlich [2] showed that $\Pi_{X,\ X+1}(x) \sim \frac{x}{\log^2(x)}$ almost surely, which is an analogue of Hardy and Littlewood's famous conjecture concerning the distribution of the twin primes [10]. Then Bui and Keating obtained the following theorems:

Theorem 5.8. *For any fixed integer k we have, as $x \to \infty$,*

$$\Pi_{X,\ X+k}(x) \sim \frac{x}{\log^2(x)} \text{ almost surely.}$$

Then Bui and Keating were able to extend Theorem 5.8 to l-tuples of Hawkins primes:

Theorem 5.9. *Let $0 < k_1 < k_2 < ... < k_{l-1}$ and denote by $\Pi_{X,X+k_1,...,X+k_{l-1}}(x; \alpha)$ the number of $m \leq x$ such that the set $\{m, m + k_1, ..., m + k_{l-1}\} \subset \alpha$. Then as $x \to \infty$,*

$$\Pi_{X,X+k_1,...,X+k_{l-1}}(x) \sim \frac{x}{\log^l(x)}.$$

An immediate corollary of this theorem is

Corollary 5.10. *For any positive integers d, l with $l \geq 2$ we have, as $x \to \infty$*

$$\Pi_{X,X+d,...,X+(l-1)d}(x) \sim \frac{x}{\log^l(x)}.$$

This is the analogue of a theorem of Green and Tao [11] on the existence of arbitrarily long arithmetic progressions in the primes.

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