Generalized Theory of Intermediate Structures Between Vector Spaces and Fields

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September 5, 2024

Abstract

This paper introduces a generalized theory for intermediate mathematical structures between vector spaces and fields using new notations and parameters. We define a new class of structures parameterized by ordinals of cardinality greater than the continuum $\mathfrak{c}$ and prove that the cardinality of the set of these structures exceeds $\mathfrak{c}$. We provide precise definitions, theorems, and rigorous proofs from first principles.

1 Introduction

The theory explores a new class of mathematical structures that exist between vector spaces and fields. By employing parameters with cardinalities larger than the continuum $\mathfrak{c}$, we investigate whether these new structures surpass the known continuum in cardinality. This approach builds on previous research and introduces novel notations and definitions.

2 Definitions and Notations

2.1 New Notation

Define the notation for intermediate structures as:

$$\mathbb{Z}_{\alpha,\beta,\gamma}(F)$$

where:

- F is a field.
- α, β, and γ are ordinals from a set with cardinality greater than $\mathfrak{c}$, specifically, ordinals in the range $[0, \Lambda)$, where Λ is a large cardinal.

2.2 Parameters

Definition 1. *Let Λ be a large cardinal. Define $\mathcal{P}_\Lambda$ as the set of ordinals $\alpha, \beta, \gamma \in [0, \Lambda)$. These ordinals will be used to parameterize new intermediate structures between vector spaces and fields.*

3 Generalized Structures

3.1 Construction of Intermediate Structures

Definition 2. *An intermediate structure $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ is defined as a mathematical object that lies between a vector space $V(F)$ and a field F. Specifically, the structure $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ is parameterized by the ordinals α, β, and γ, and exhibits properties that depend on these parameters.*

3.2 Example Structures

Example 1. *Consider a vector space $V(F)$ over a field F. We can construct an intermediate structure $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ where:*

- *α affects the dimension of $V(F)$.*
- *β influences the algebraic operations within $V(F)$.*
- *γ defines new field extensions or modifications to F.*

For instance, if α, β, and γ are chosen such that they represent complex ordinal sequences, the resulting structure $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ might include novel algebraic properties or field extensions that are not present in the traditional framework of vector spaces and fields.

4 Cardinality of the New Structures

4.1 Theorem: Cardinality of Structures

Theorem 1. *The set of all intermediate structures $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ where α, β, γ range over ordinals in $[0, \Lambda)$ has cardinality at least κ, where κ is a cardinal number greater than $\mathfrak{c}$.*

Proof. Consider the set of all possible choices for α, β, and γ. Since Λ is a large cardinal, the set $[0, \Lambda)$ has cardinality Λ, which is greater than $\mathfrak{c}$. Therefore, the set of all tuples (α, β, γ) in $[0, \Lambda) \times [0, \Lambda) \times [0, \Lambda)$ has cardinality $\Lambda^3 = \Lambda$, given that Λ is a large cardinal.

Thus, the set of intermediate structures $\mathbb{Z}_{\alpha,\beta,\gamma}(F)$ has cardinality Λ, which is greater than $\mathfrak{c}$. $\square$

5 Conclusion

We have introduced new notations and parameters to describe a class of intermediate structures between vector spaces and fields. By employing large cardinals and ordinal parameters, we have shown that it is possible to construct a set of such structures with cardinality greater than the continuum $\mathfrak{c}$. This extends our understanding of the hierarchy and diversity of mathematical structures in this context.

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