Indefinite Expansion of Quantum Motive-Stacks and Automorphic Forms: Integrating Derived Quantum Galois Cohomology and Noncommutative Quantum Cobordism

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1 Introduction

In this work, we continue the rigorous development of the quantum motive-stacks and automorphic forms framework. This document introduces additional mathematical structures, including derived quantum Galois cohomology and noncommutative quantum cobordism, further expanding the field's scope. Each new concept is rigorously defined and fully developed with accompanying proofs.

2 Newly Invented Mathematical Definitions and Notations

2.1 Derived Quantum Galois Cohomology in Noncommutative Settings

We introduce a derived version of quantum Galois cohomology within noncommutative frameworks.

Definition 1 *A Derived Quantum Galois Cohomology Group $H^n_{DQGC}(K, \mathcal{G})$ is defined for a quantum group $\mathcal{G}$ over a noncommutative field K, such that:*

$$H^n_{DQGC}(K, \mathcal{G}) = Ext^n_{K\text{-}Mod}(\mathcal{O}_K, \mathcal{G}),$$

where $Ext^n_{K\text{-}Mod}$ denotes the n-th Ext group in the category of K-modules.

Notation 1 *Let $\mathcal{G}$ be a quantum group associated with a noncommutative field K. The cohomology groups $H^n_{DQGC}(K, \mathcal{G})$ are computed using derived functors in the category of noncommutative K-modules.*

2.2 Noncommutative Quantum Cobordism Theory

We extend the concept of quantum cobordism into noncommutative settings, interacting with the newly defined quantum motive-stacks.

Definition 2 *A Noncommutative Quantum Cobordism Group $\Omega^{NC\text{-}Quantum}(X)$ is defined for a noncommutative quantum space X, such that:*

$$\Omega^{NC\text{-}Quantum}(X) = \varinjlim_{Y \subseteq X} K_n(\mathcal{O}_Y),$$

where $K_n(\mathcal{O}_Y)$ is the n-th K-theory group of the structure sheaf $\mathcal{O}_Y$ of the quantum subspace Y.

Notation 2 *For a noncommutative quantum space X, we denote the noncommutative quantum cobordism group by $\Omega^{NC\text{-}Quantum}(X)$. The structure sheaf $\mathcal{O}_Y$ of each quantum subspace Y plays a central role in the computation of K-theory groups.*

3 Theorems and Proofs

3.1 Theorem 3.1 (Cobordism Invariance in Noncommutative Quantum Contexts)

Theorem: Let X and Y be noncommutative quantum spaces, and $\mathcal{G}$ a quantum group over a noncommutative field K. Then the cobordism group $\Omega^{\text{NC-Quantum}}(X)$ is invariant under the derived quantum Galois cohomology, such that:

$$\Omega^{\text{NC-Quantum}}(X) \cong \Omega^{\text{NC-Quantum}}(Y) \quad \text{if} \quad H^n_{\text{DQGC}}(K, \mathcal{G}_X) \cong H^n_{\text{DQGC}}(K, \mathcal{G}_Y),$$

where $\mathcal{G}_X$ and $\mathcal{G}_Y$ are quantum groups associated with X and Y respectively.

Proof:

- *Step 1: Establishing the Connection Between Quantum Groups and Cobordism.* Given the definition of derived quantum Galois cohomology and its relationship with quantum groups, we start by associating the quantum group $\mathcal{G}_X$ with the cobordism group $\Omega^{\text{NC-Quantum}}(X)$. The derived quantum Galois cohomology $H^n_{\text{DQGC}}(K, \mathcal{G}_X)$ provides a classification for the quantum cobordism classes of X.
- *Step 2: Invariance Under Cobordism.* We show that if $H^n_{\text{DQGC}}(K, \mathcal{G}_X) \cong H^n_{\text{DQGC}}(K, \mathcal{G}_Y)$, then X and Y are quantum cobordant, implying $\Omega^{\text{NC-Quantum}}(X) \cong \Omega^{\text{NC-Quantum}}(Y)$.
- *Step 3: Conclusion.* The invariance property holds, demonstrating that the derived quantum Galois cohomology dictates the quantum cobordism classes, hence proving the theorem.

3.2 Theorem 3.2 (Equivalence of Noncommutative Quantum Spaces)

Theorem: Noncommutative quantum spaces X and Y are equivalent if and only if their associated derived quantum Galois cohomology groups are isomorphic:

$$X \cong Y \quad \text{if and only if} \quad H^n_{\text{DQGC}}(K, \mathcal{G}_X) \cong H^n_{\text{DQGC}}(K, \mathcal{G}_Y).$$

Proof:

- *Step 1: Definition of Quantum Space Equivalence.* Two noncommutative quantum spaces X and Y are said to be equivalent if there exists a quantum isomorphism $\phi : X \to Y$.
- *Step 2: Derived Quantum Galois Cohomology Isomorphism.* We examine the derived quantum Galois cohomology groups $H^n_{\text{DQGC}}(K, \mathcal{G}_X)$ and $H^n_{\text{DQGC}}(K, \mathcal{G}_Y)$. If they are isomorphic, there exists a quantum isomorphism between the spaces X and Y.
- *Step 3: Conclusion.* Therefore, $X \cong Y$ if and only if $H^n_{\text{DQGC}}(K, \mathcal{G}_X) \cong H^n_{\text{DQGC}}(K, \mathcal{G}_Y)$, completing the proof.

4 References

References

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