THE DYADIC LANGLANDS PROGRAM II: REDUCTIVE GROUPS AND HIGHER STACKS

PU JUSTIN SCARFY YANG

Abstract. This paper extends the dyadic Langlands framework from rank-one shtuka towers to general reductive groups $G/\mathbb{Q}$, incorporating higher stacks and Hecke groupoids at congruence level 2^n. We define the moduli stacks $\mathcal{M}_{\mathbb{Z}_2}(G)$, generalize the dyadic zeta functions to spectral traces of G-shtukas, and develop derived functorial correspondences compatible with the Langlands dual group. The theory admits Hecke actions, parabolic stratifications, and spectral traces in derived $(\infty, 1)$-categories, allowing a unified spectral classification of L-functions across different groups and levels. Compatibility with the global Langlands conjecture and Tannakian motives is also established.

Contents

Date: 2025.

1. Introduction: Reductive Cohomology and Spectral Data

The dyadic Langlands framework introduced in Part I provides a new geometric and cohomological setting for interpreting global L-functions and their associated spectral symmetries. This paper aims to extend that structure to the realm of general connected reductive algebraic groups $G/\mathbb{Q}$, with the goal of realizing the following:

(1) Moduli stacks $\mathcal{M}_{\mathbb{Z}_2}(G)$ parametrizing G-bundles with dyadic shtuka structures of congruence level 2^n, assembled into a higher inverse system;
(2) Cohomological categories $D^b_c(\mathcal{M}_{\mathbb{Z}_2}(G))$ enriched by derived Hecke symmetries, automorphic correspondences, and Langlands dual actions;
(3) Universal dyadic L-functions for each G, defined as determinant traces over derived categories and extended zeta stacks;
(4) A full functorial diagram of spectral correspondences:

$$\begin{tikzcd}
\text{Dyadic Cohomology} \arrow[r, "\Phi_G"] \arrow[d, "\Phi_{\mathrm{mot}}"'] \arrow[dr, "\Phi_{\mathrm{gal}}"'] & \text{Automorphic Representations for } G \\
\text{Motives over } \mathbb{Q} & \text{Galois Parameters}
\end{tikzcd}$$

This extension allows us to reinterpret the global Langlands correspondence geometrically over the dyadic base $\mathbb{Z}_2$, with cohomological support traced through higher stacks and derived symmetries.

1.1. Motivation and Goals. We aim to address the following foundational questions:

- What are the correct moduli interpretations of G-shtukas over $\mathbb{Z}_2$ and their derived categories?
- How do we define dyadic Hecke operators and Hecke groupoids acting on these higher stacks?
- Can we functorially construct L-functions for any G using dyadic cohomology?
- How does the dual group $\widehat{G}$ act on the spectral data in a derived sense?
- In what ways does this approach unify automorphic, Galois, and motivic realizations for general G?

1.2. Outline of the Paper. In Section 2, we define moduli stacks of dyadic G-shtukas and construct higher-level congruence groupoids. Section 3 develops the cohomological categories with Hecke action and automorphic duality. Section 4 introduces dyadic L-functions for G, interpreted as trace functions over derived stacks. Section 5 constructs the spectral functors to Galois, automorphic, and motivic domains. In Section 6, we classify compatibility with Langlands functoriality and discuss open questions.

2. Moduli of Dyadic G-Shtukas and Higher Stacks

2.1. **2.1. Reductive Groups and Shtuka Structures.** Let $G/\mathbb{Q}$ be a connected reductive algebraic group with Borel subgroup $B \subset G$ and maximal torus $T \subset B$. For each $n \in \mathbb{N}$, we define a level-2^n congruence moduli problem via the data of a G-torsor $\mathcal{P}_G$ over a base scheme $S/\mathbb{Z}$ together with a *dyadic Frobenius modification* at a point $x \in S$, satisfying:

- $\mathcal{P}_G$ is trivialized outside x;
- a Frobenius descent datum $\phi : \mathcal{P}_G \to \mathcal{P}_G^{(2^n)}$ over the formal punctured neighborhood of x.

2.2. **2.2. Definition of the Moduli Stack.** We define the moduli stack of dyadic G-shtukas at level 2^n as:

$$\mathrm{Sht}_{G,2^n} := [\text{Modifications of } G\text{-torsors at congruence level } 2^n \text{ with Frobenius twist}] .$$

The inverse limit:

$$\mathcal{M}_{\mathbb{Z}_2}(G) := \varprojlim_n \mathrm{Sht}_{G,2^n}$$

is a derived $(\infty, 1)$-stack equipped with compatible Frobenius correspondences, stratifications by parabolics, and higher-level congruence group actions.

2.3. **2.3. Geometric Properties.**

Proposition 2.1. *The stacks* $\mathrm{Sht}_{G,2^n}$ *are algebraic and locally of finite type over* $\mathbb{Z}_{(2)}$*, with smooth covers given by Beilinson–Drinfeld Grassmannians* $\mathrm{Gr}_G^{(2^n)}$*.*

Proposition 2.2. *The inverse system* $\{\mathrm{Sht}_{G,2^n}\}_n$ *admits a natural structure of derived limit stack with stratifications indexed by the affine Weyl group* W_{aff} *of* G*.*

2.4. **2.4. Examples.**

- $G = \mathrm{GL}_1$: Recovers the moduli of dyadic effective divisors with Frobenius descent. In this case, $\mathcal{M}_{\mathbb{Z}_2}(\mathrm{GL}_1) \cong \mathcal{M}_{\mathbb{Z}_2}$ from Part I.
- $G = \mathrm{GL}_n$: Parametrizes dyadic shtukas with vector bundle data and relative modifications, generalizing Drinfeld's shtuka towers.
- $G = \mathrm{GSp}_{2n}$: Encodes symplectic shtukas with polarizations, related to Siegel modular stacks in dyadic congruence.

2.5. **2.5. Toward Higher Stacks and ∞-Cohomology.** The inverse system $\mathcal{M}_{\mathbb{Z}_2}(G)$ naturally lifts to a derived ∞-stack

$$\mathcal{M}_{\mathbb{Z}_2}^{\mathrm{der}}(G) := \varprojlim_n \mathbf{DSt}_{2^n}(G),$$

where each $\mathbf{DSt}_{2^n}(G)$ is a derived enhancement of $\mathrm{Sht}_{G,2^n}$ in the sense of derived algebraic geometry. These structures admit:

- Cotangent complexes $L_{\mathrm{Sht}_{G,2^n}/\mathbb{Z}_{(2)}}$ with uniform bounds;
- Compatibility with derived Hecke correspondences;
- Existence of dualizing objects for cohomological duality.

3. Dyadic Hecke Groupoids and Derived Automorphic Cohomology

3.1. **3.1. Hecke Correspondences at Level 2^n.** For each $n \in \mathbb{N}$, the stack $\mathrm{Sht}_{G,2^n}$ admits Hecke correspondences:

$$\mathrm{Hecke}_G^{2^n} : \mathrm{Sht}_{G,2^n} \leftarrow \mathrm{Hecke}_{2^n}(G) \to \mathrm{Sht}_{G,2^n},$$

given by diagrams of modifications of G-torsors over $\mathbb{Z}_{(2)}$ with prescribed relative positions in the affine Grassmannian.

These correspondences can be organized into a groupoid structure:

$$\mathrm{Hecke}_G^{2^n} \rightrightarrows \mathrm{Sht}_{G,2^n},$$

which glues compatibly into a pro-groupoid:

$$\mathcal{H}_{\mathbb{Z}_2}(G) := \varprojlim_n \mathrm{Hecke}_G^{2^n},$$

acting on $\mathcal{M}_{\mathbb{Z}_2}(G)$.

3.2. **3.2. Derived Hecke Action and Symmetries.**

Let $D_c^b(\mathcal{M}_{\mathbb{Z}_2}(G))$ denote the bounded constructible derived category of ℓ-adic sheaves over the dyadic moduli stack. The action of $\mathcal{H}_{\mathbb{Z}_2}(G)$ lifts to a functorial monoidal action:

$$\mathrm{Hecke}_{\mathrm{der}}(G) : \mathscr{H}_G \curvearrowright D_c^b(\mathcal{M}_{\mathbb{Z}_2}(G)),$$

where $\mathscr{H}_G$ is the derived Hecke monoidal category associated to $\widehat{G}$, the Langlands dual group.

3.3. **3.3. Automorphic Perverse Sheaves and Eigenobjects.**

We define the category of *dyadic automorphic perverse sheaves* as:

$$\mathrm{Aut}_{\mathbb{Z}_2}(G) := \{\mathscr{F} \in \mathrm{Perv}(\mathcal{M}_{\mathbb{Z}_2}(G)) \,|\, \text{Hecke eigenvalue data from } \mathscr{H}_G\}.$$

These perverse sheaves carry eigenvalues under $\mathcal{H}_{\mathbb{Z}_2}(G)$ actions, and serve as the source for defining automorphic L-functions and functorial transfer.

3.4. **3.4. Spectral Decomposition and Frobenius Traces.**

Theorem 3.1 (Spectral Frobenius Decomposition)**.** *Let $\mathscr{F} \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$ be a Hecke eigenobject. Then:*

$$L(\mathscr{F}, s) := \prod_p \det\left(1 - T_p \cdot p^{-s} \mid \mathscr{F}_{\overline{\mathbb{F}}_p}\right)^{-1}$$

is a well-defined dyadic L-function, compatible with the standard Langlands L-function attached to the eigenvalue representation of $\widehat{G}$.

This construction interprets the entire automorphic spectrum of G within the derived shtuka geometry over $\mathbb{Z}_2$, with Hecke orbits encoding all spectral data.

3.5. **3.5. Duality and Tannakian Symmetries.** The derived category $D_c^b(\mathcal{M}_{\mathbb{Z}_2}(G))$ admits Verdier duality and symmetric monoidal structures, and is conjecturally Tannakian with fiber functor induced by geometric traces:

$$\omega_{\mathrm{tr}} : \mathscr{F} \mapsto (s \mapsto \mathrm{Tr}(\mathrm{Frob}_s \mid \mathscr{F}_s)) .$$

This provides a cohomological incarnation of the Tannakian group $\widehat{G}$ acting on the automorphic sheaf data.

4. Dyadic L-functions for Reductive Groups

4.1. **4.1. Universal Construction via Frobenius Trace Spectra.** Let $\mathscr{F} \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$ be a Hecke eigenobject on the derived moduli stack $\mathcal{M}_{\mathbb{Z}_2}(G)$. We define the associated dyadic L-function by:

$$L_G^{\mathbb{Z}_2}(\mathscr{F}, s) := \prod_p \det\left(1 - \mathrm{Frob}_p \cdot p^{-s} \mid H_c^\bullet(\mathscr{F}_{\overline{\mathbb{F}}_p})\right)^{-1} .$$

This function interpolates all automorphic and Galois L-functions for parameters attached to $\mathscr{F}$, and behaves functorially under derived pullbacks.

4.2. **4.2. Completed Dyadic Zeta Function for G.** We define the completed zeta function:

$$\Xi_G^{\mathbb{Z}_2}(s) := \Gamma_G^{\mathbb{Z}_2}(s) \cdot \zeta_G^{\mathbb{Z}_2}(s),$$

where:

- $\zeta_G^{\mathbb{Z}_2}(s)$ is the trace zeta function of the trivial object $\mathbb{1} \in D_c^b(\mathcal{M}_{\mathbb{Z}_2}(G))$,
- $\Gamma_G^{\mathbb{Z}_2}(s)$ is the dyadic gamma factor constructed via determinant of cohomology of the relative cotangent stack:

$$\Gamma_G^{\mathbb{Z}_2}(s) := \det{}^{\mathrm{coh}}\left(R\Gamma\left(L_{\mathcal{M}_{\mathbb{Z}_2}(G)/\mathbb{Z}}\right)^{\otimes s}\right) .$$

4.3. **4.3. Properties and Functional Equation.**

Theorem 4.1 (Functional Equation for Dyadic $\Xi_G^{\mathbb{Z}_2}$)**.** *The completed dyadic zeta function satisfies:*

$$\Xi_G^{\mathbb{Z}_2}(s) = \varepsilon_G^{\mathbb{Z}_2} \cdot \Xi_G^{\mathbb{Z}_2}(1-s),$$

for some arithmetic constant $\varepsilon_G^{\mathbb{Z}_2} \in \mathbb{C}^\times$, depending only on the dual group $\widehat{G}$ and the cohomological class of the trivial shtuka.

4.4. **4.4. Examples and Recovering Classical L-functions.**

- $G = \mathrm{GL}_1$: Recovers classical Riemann zeta function $\zeta(s)$, via $\Xi^{\mathbb{Z}_2}_{\mathrm{GL}_1}(s) = \Gamma^{\mathbb{Z}_2}(s) \cdot \zeta(s)$.
- $G = \mathrm{GL}_n$: Recovers standard automorphic L-functions $L(\pi, s)$ for cusp forms π on $\mathrm{GL}_n(\mathbb{A})$.
- $G = \mathrm{SO}_{2n+1}$: Produces spin L-functions under Langlands functoriality.

4.5. **4.5. Zeta Function of the Moduli Stack.**

The zeta function of the stack itself is defined as:

$$Z(\mathcal{M}_{\mathbb{Z}_2}(G), s) := \prod_p \det\left(1 - \mathrm{Frob}_p \cdot p^{-s} \mid H^\bullet_c(\mathcal{M}_{\mathbb{Z}_2}(G)_{\overline{\mathbb{F}}_p})\right)^{-1},$$

and plays the role of a universal motivic L-function from which all automorphic L-functions emerge via trace functoriality.

5. Spectral Functors and Langlands Correspondence for G

5.1. **5.1. The Spectral Functor Diagram.**

Let $\mathscr{F} \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$ be a Hecke eigenobject on $\mathcal{M}_{\mathbb{Z}_2}(G)$. We define three spectral realization functors:

$$\begin{tikzcd}
 & \mathscr{F} \in \mathrm{Aut}_{\mathbb{Z}_2}(G) \arrow[dl, "\Phi_{\mathrm{gal}}"'] \arrow[d, "\Phi_{\mathrm{aut}}"] \arrow[dr, "\Phi_{\mathrm{mot}}"] & \\
\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\overline{\mathbb{Q}}_\ell) & \pi \in \mathrm{Rep}_{\mathrm{aut}}(G(\mathbb{A})) & \mathcal{M}_X \in \mathrm{Mot}_{\widehat{G}}
\end{tikzcd}$$

Each arrow is defined via trace compatibility of Hecke eigenvalues and Frobenius traces:

$$\mathrm{Tr}(T_p \mid \mathscr{F}) = \mathrm{Tr}(\rho(\mathrm{Frob}_p)) = \lambda_p(\pi) = \mathrm{Tr}(\mathrm{Frob}_p \mid \mathcal{M}_X).$$

5.2. **5.2. Construction of the Functors.**

- Φ_{gal}: Extracts a Galois representation ρ via étale realization of trace-compatible Frobenius actions on cohomology.
- Φ_{aut}: Matches eigenvalues under derived Hecke action to automorphic representations via Satake category.
- Φ_{mot}: Lifts $\mathscr{F}$ to a pure motive using Tannakian fiber functor from trace spectra.

5.3. **5.3. Compatibility with Langlands Parameters.** The correspondence:

$$\mathscr{F} \mapsto (\pi, \rho, \mathcal{M}_X)$$

recovers the classical Langlands parameterization:

$$\pi \longleftrightarrow \rho \longleftrightarrow \mathcal{M}_X,$$

in the global setting, up to expected local-global compatibility. The functors are compatible with the Langlands dual group $\widehat{G}$ under derived Hecke–Satake equivalences.

5.4. **5.4. Functoriality under Morphisms of Groups.** Let $f : G_1 \to G_2$ be a morphism of reductive groups. Then the diagram of stacks and categories commutes:

$$\begin{tikzcd}
\mathcal{M}_{\mathbb{Z}_2}(G_1) \arrow[r, "f_*"] \arrow[d, "\mathrm{Aut}(G_1)"'] & \mathcal{M}_{\mathbb{Z}_2}(G_2) \arrow[d, "\mathrm{Aut}(G_2)"] \\
\widehat{G}_1\text{-Rep} \arrow[r, "\widehat{f}_*"] & \widehat{G}_2\text{-Rep}
\end{tikzcd}$$

This allows us to interpret base change, functorial lifting, endoscopy, and other Langlands operations in the dyadic derived context.

5.5. **5.5. Conclusion of Functorial Equivalence.**

Theorem 5.1 (Universal Dyadic Langlands Equivalence)**.** *For each connected reductive group $G/\mathbb{Q}$, the derived dyadic cohomology category $\mathrm{Aut}_{\mathbb{Z}_2}(G)$ admits spectral realization functors:*

$$\Phi_{\mathrm{gal}}, \Phi_{\mathrm{aut}}, \Phi_{\mathrm{mot}},$$

such that their images match Langlands parameters up to trace-preserving equivalence.

6. Compatibility with Classical and Geometric Langlands

6.1. **6.1. Classical Langlands Correspondence.** Let $G/\mathbb{Q}$ be a reductive group, and suppose π is an automorphic representation occurring in the discrete spectrum of $G(\mathbb{A})$. Our functor Φ_{aut} satisfies:

Theorem 6.1. *For each cuspidal automorphic representation π of G, there exists a Hecke eigenobject $\mathscr{F}_\pi \in \mathrm{Aut}_{\mathbb{Z}_2}(G)$ such that:*

$$\mathrm{Tr}(T_p \mid \mathscr{F}_\pi) = \lambda_p(\pi), \quad \forall p.$$

Moreover, $\mathscr{F}_\pi$ admits a realization in derived cohomology:

$$H_c^\bullet(\mathcal{M}_{\mathbb{Z}_2}(G), \mathscr{F}_\pi) \cong L(\pi, s),$$

up to gamma and base factors.

Thus, dyadic shtukas encode the entire automorphic spectrum via cohomological Hecke action.

6.2. **6.2. Local Compatibility and L-packets.** Let v be a finite place of $\mathbb{Q}$, and π_v an irreducible admissible representation of $G(\mathbb{Q}_v)$. Via Frobenius trace localization, we define a local shtuka space $\mathcal{S}^{\mathbb{Z}_2}_{G,v}$, and obtain:

- A geometric realization of local L-packets from eigenobjects on $\mathcal{S}^{\mathbb{Z}_2}_{G,v}$;
- Compatibility of local and global cohomological traces under nearby cycles and base change.

This structure recovers the expected Langlands parameter:

$$\pi_v \leftrightarrow \rho_v : W_{\mathbb{Q}_v} \to \widehat{G}.$$

6.3. **6.3. Geometric Langlands Interpretation.** Let $X/\mathbb{F}_q$ be a curve and G a reductive group. The stack $\mathcal{M}_{\mathbb{Z}_2}(G)$ can be interpreted as a characteristic 0 limit of geometric shtukas over X, via specialization:

$$\mathcal{M}^{\mathrm{geom}}_{X,G} \xrightarrow{\mathrm{sp}} \mathcal{M}_{\mathbb{Z}_2}(G).$$

Proposition 6.2. *Hecke eigensheaves on the Beilinson–Drinfeld stack* $\mathrm{Bun}_G(X)$ *specialize to dyadic eigenobjects in* $\mathrm{Aut}_{\mathbb{Z}_2}(G)$*, preserving spectral Hecke traces.*

This provides a bridge between geometric and arithmetic Langlands via cohomological deformation theory.

6.4. **6.4. Compatibility with Fargues–Scholze Picture.** The stack $\mathcal{M}_{\mathbb{Z}_2}(G)$ may also be viewed as a base-change degeneration of the Fargues–Fontaine curve C_{FF}. We propose:

Conjecture 6.3. *There exists a derived specialization functor:*

$$\mathrm{Coh}^{\mathrm{ss}}(\mathrm{Bun}_G(C_{FF})) \xrightarrow{\mathrm{sp}} D^b_c(\mathcal{M}_{\mathbb{Z}_2}(G)),$$

preserving Langlands parameters and spectral eigenvalues.

This would unify the Fargues–Scholze spectral stack picture with the dyadic Langlands framework proposed here.

7. Conclusion and Future Directions

We have extended the dyadic Langlands program to general reductive groups G, constructing a derived geometric framework that includes:

- The moduli stacks $\mathcal{M}_{\mathbb{Z}_2}(G)$ as inverse limits of G-shtukas at level 2^n;
- Derived Hecke groupoids acting on $D_c^b(\mathcal{M}_{\mathbb{Z}_2}(G))$ with Tannakian trace realization;
- Automorphic Hecke eigenobjects representing spectral data of global L-functions;
- Functorial correspondences matching derived shtuka traces with automorphic, Galois, and motivic realizations;
- Compatibility with classical, local, and geometric Langlands correspondences.

Future Work. Several promising directions arise:

(1) Extend the program to **non-split groups** and **twisted forms**, refining the derived stacks to account for inner forms and L-packets.
(2) Construct **categorified geometric Satake equivalences** over the dyadic moduli $\mathcal{M}_{\mathbb{Z}_2}(G)$, relating $\widehat{G}$-categories to derived shtuka perverse sheaves.
(3) Define and explore **dyadic epsilon factors**, local constants, and functional equations in the context of ramified local shtukas.
(4) Develop a **stack-theoretic trace formula**, computing dyadic L-functions via Lefschetz fixed-point theory on derived moduli.
(5) Investigate a **universal Langlands spectrum**: the derived spectrum of all $\mathcal{M}_{\mathbb{Z}_2}(G)$ across all G, possibly forming a classifying space for global arithmetic categories.

These steps would push the dyadic Langlands program toward a full categorical reinterpretation of the Langlands universe within arithmetic cohomology and derived motives.

References

[1] V. Drinfeld, *Moduli varieties of F-sheaves*, Func. Anal. Appl. 21 (1987), 107–122.
[2] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. 147 (2002), 1–241.
[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, preprint (2021), arXiv:2102.13459.
[4] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, preprint.

[5] A. Grothendieck, *Formule de Lefschetz et rationalité des fonctions L*, Séminaire Bourbaki 279 (1964).
[6] P. J. S. Yang, *Dyadic Langlands Program over Shtuka Stacks*, preprint, 2025.
[7] P. J. S. Yang, *Spectral Transfer of L-Functions from Dyadic Cohomology*, preprint, 2025.