SYMBOLIC DYNAMICS ON MULTIPLICATIVE FUNCTION SPACES: A DYNAMICAL SYSTEMS PERSPECTIVE ON DIRICHLET CONVOLUTION AND ARITHMETIC DIFFERENTIATION

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Abstract. We study the space of multiplicative arithmetic functions as symbolic sequences in a dynamical systems framework. By interpreting these functions as elements of shift-invariant subspaces over $\mathbb{N}$, we develop notions of topological dynamics, symbolic flows, entropy, and time evolution governed by the arithmetic derivative operator D. This framework offers new methods to analyze global patterns, chaotic behavior, and recurrence in number-theoretic contexts, establishing bridges to ergodic theory, automata theory, and arithmetic complexity.

Contents

1. Introduction and Motivation

Arithmetic functions such as the Möbius function $\mu(n)$, the Liouville function $\lambda(n)$, or the characteristic function of square-free integers, when viewed as infinite sequences $(f(n))_{n\in\mathbb{N}}$, exhibit remarkable structure and irregularity.

We explore these sequences from the viewpoint of symbolic dynamics. That is, we treat arithmetic functions as infinite words or configurations over an alphabet $\mathcal{A} \subseteq \mathbb{C}$, equipped with the shift operator σ, and study their orbits, recurrence, and entropy.

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2. Symbolic Function Spaces

Definition 2.1. Let $\mathcal{A} \subset \mathbb{C}$ be a finite or countable set. Define the space of arithmetic sequences:

$$\mathcal{F} := \mathcal{A}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathcal{A}\}.$$

We equip $\mathcal{F}$ with the product topology (Tychonoff topology).

Definition 2.2. The shift map $\sigma : \mathcal{F} \to \mathcal{F}$ is defined by:

$$(\sigma f)(n) := f(n+1).$$

A subspace $X \subset \mathcal{F}$ is shift-invariant if $\sigma(X) \subseteq X$.

Example 2.3. Let $\lambda(n) := (-1)^{\Omega(n)}$ be the Liouville function. Then $\lambda \in \{-1, +1\}^{\mathbb{N}}$, and its orbit under shift generates a symbolic dynamical system:

$$\mathcal{O}(\lambda) := \{\sigma^k \lambda \mid k \geq 0\}.$$

3. Topological Dynamics and Recurrence

Definition 3.1. A point $f \in \mathcal{F}$ is *recurrent* under σ if there exists $n_k \to \infty$ such that $\sigma^{n_k}(f) \to f$.

Definition 3.2. Define the orbit closure:

$$X_f := \overline{\{\sigma^k f : k \geq 0\}}.$$

The pair (X_f, σ) is a symbolic dynamical system.

Theorem 3.3. *If f is periodic with period q, then X_f is a finite cyclic shift system of size q, and entropy $h(X_f) = 0$.*

Proof. The orbit contains exactly q distinct elements:

$$\{f, \sigma f, \dots, \sigma^{q-1} f\}.$$

Since the space is finite and deterministic, the topological entropy defined via growth of distinct n-blocks is zero. $\square$

4. Entropy of Arithmetic Function Orbits

Definition 4.1. Let $f \in \mathcal{F}$, define the block complexity function:

$$P_f(n) := \#\{(f(k+1), f(k+2), \dots, f(k+n)) \mid k \in \mathbb{N}\}.$$

Topological entropy is defined by:

$$h(f) := \limsup_{n \to \infty} \frac{1}{n} \log P_f(n).$$

Proposition 4.2. *Let λ be the Liouville function. Then $h(\lambda) = \log 2$, assuming normality of λ.*

Sketch. Assuming statistical independence of $\lambda(n) \in \{\pm 1\}$, the number of length-n blocks is 2^n, yielding:

$$P_\lambda(n) = 2^n \Rightarrow h(\lambda) = \log 2.$$

$\square$

5. Interaction with the Arithmetic Derivative D

Definition 5.1. Define time evolution under arithmetic differentiation:

$$T_D(f)(n) := D(f)(n) = \log(n)f(n).$$

This defines a nonlinear transformation on symbolic sequence space $\mathcal{F}$.

Theorem 5.2. *Let $f \in \mathcal{F}$ be a bounded arithmetic function. Then $T_D(f) \in \mathcal{F}' \subseteq \mathcal{F}$, where:*

$$\mathcal{F}' := \{g \colon \mathbb{N} \to \mathbb{C} \mid g(n) = \log(n)f(n),\ f \in \mathcal{F}\}.$$

Remark 5.3. T_D acts as a "dilated shift" in symbolic space, modifying the symbol weights but preserving index structure. Studying orbits of $f \mapsto T_D^k(f)$ yields dynamical zeta phenomena.

6. Advanced Symbolic Dynamics in Arithmetic Function Spaces

6.1. Ergodic Properties of the Liouville Shift System.

Definition 6.1. Let $\lambda(n) := (-1)^{\Omega(n)}$, the Liouville function. Define $X_\lambda := \overline{\{\sigma^k\lambda : k \in \mathbb{N}\}}$, the orbit closure under shift. Equip X_λ with the uniform Bernoulli probability measure μ assigning equal weight to each ± 1.

Theorem 6.2. *If the Chowla conjecture holds, then (X_λ, σ, μ) is a Bernoulli shift and hence ergodic.*

Sketch. The Chowla conjecture asserts that the correlations

$$\mathbb{E}[\lambda(n)\lambda(n+h)] \to 0 \quad \text{as } n \to \infty,$$

for all $h > 0$. This implies the sequence has zero autocorrelation, which is characteristic of Bernoulli shifts. From classical ergodic theory, uncorrelated stationary sequences under shift generate ergodic systems. □

6.2. Symbolic Zeta Functions.

Definition 6.3. Let $X \subseteq \mathcal{A}^{\mathbb{N}}$ be a shift-invariant subspace. Define the symbolic zeta function:

$$\zeta_{\mathrm{sym}}(X, s) := \sum_{n=1}^{\infty} \frac{P_X(n)}{n^s},$$

where $P_X(n)$ is the number of distinct admissible n-blocks in X.

Proposition 6.4. *If X is the full binary shift, then:*

$$\zeta_{\mathrm{sym}}(X, s) = \sum_{n=1}^{\infty} \frac{2^n}{n^s}.$$

Proof. In the full binary shift, each n-block of $\{-1, +1\}^n$ is allowed. Thus, $P_X(n) = 2^n$. □

Remark 6.5. This symbolic zeta function diverges for all $s \leq \log_2 e$. Analytic continuation or regularization techniques may be required to study poles and functional identities.

6.3. **Symbolic Cocycle Dynamics over Multiplicative Groups.**

Definition 6.6. Let $G = (\mathbb{N}, \cdot)$, and let $\phi \colon \mathbb{N} \to \mathcal{A}$ be an arithmetic function. Define the cocycle:

$$\alpha(m, n) := \phi(mn).$$

We say α is multiplicative if $\alpha(m, n) = \phi(m)\phi(n)$.

Theorem 6.7. *If ϕ is completely multiplicative, then $\alpha(m, n) = \phi(mn)$ defines a 1-cocycle over G with values in $(\mathcal{A}, \cdot)$.*

Proof. We check the cocycle condition:

$$\alpha(m, n)\alpha(mn, k) = \phi(mn)\phi(mnk) = \phi(m)\phi(n)\phi(k) = \alpha(m, nk)\alpha(n, k),$$

satisfying the multiplicative cocycle identity. □

6.4. **Automata and Arithmetic Grammars.**

Definition 6.8. A finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ accepts an arithmetic function $f \colon \mathbb{N} \to \Sigma$ if the infinite word $f(1)f(2) \ldots \in \Sigma^\omega$ is accepted by $\mathcal{A}$.

Example 6.9. Let $\chi(n) = n \mod 3 \in \{0, 1, 2\}$. Then χ is recognized by a 3-state automaton cycling mod 3 transitions.

Proposition 6.10. *The Liouville function $\lambda(n)$ is not automatic.*

Sketch. Automatic sequences have bounded block complexity $P(n) = O(n)$, but assuming normality or Chowla, $\lambda(n)$ has exponential block complexity. Hence it cannot be automatic. □

Theorem 6.11. *Every eventually periodic arithmetic function is automatic.*

Proof. Let $f(n) = f(n+T)$ for all large n. Then a DFA with T states cycling through $f(n) \mod T$ suffices to generate f. □

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