Negative-Dimensional Fields I

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Introduction to Negative-Dimensional Fields

Negative-Dimensional Fields extend the classical notion of fields, introducing novel properties of negative dimensionality. This rigorous development covers:

- Formal Definitions
- Inverse Automorphisms
- Inverse Galois Groups
- Negative-Dimensional Galois Extensions
- Inverse Galois Correspondence
- Inverse Galois Problem

Formal Definition of Negative-Dimensional Fields

Definition 1.1: Negative-Dimensional Field

Let K be a base field. A **Negative-Dimensional Field** F_{-d} satisfies:

- F_{-d} is a commutative division ring with identity, obeying standard field axioms.
- The dimension of F_{-d} over K is given by:

$$\dim_K F_{-d} = -d \quad \text{where } d > 0.$$

- The negative dimension indicates a **contraction** or **collapse** in the structure of F_{-d}.

Example: Negative-Dimensional Field

Example 1.1: Hypothetical Construction

Let F_{-d} be a negative-dimensional extension of K. Consider the following:

- As elements are added to F_{-d}, the effective dimension decreases, reflecting the contracting nature of the field.
- Suppose $\dim_K F_{-d} = -2$. Adding an element reduces the dimension further, potentially to -3 or lower.

Inverse Automorphisms

Definition 2.1: Inverse Automorphism
Let F_{-d} be a negative-dimensional field over K. An **Inverse Automorphism** $\sigma : F_{-d} \to F_{-d}$ satisfies:

- Preserves field operations:

$$\sigma(a+b) = \sigma(a) + \sigma(b), \quad \sigma(ab) = \sigma(a)\sigma(b).$$

- Reverses dimensionality:

$$\dim_K \sigma(F_{-d}) = d.$$

Inverse Galois Group

Definition 2.2: Inverse Galois Group
Let F_{-d}/K be a negative-dimensional field extension. The **Inverse Galois Group**, denoted $\mathrm{Gal}(F_{-d}/K)$, is defined as:

$$\mathrm{Gal}(F_{-d}/K) = \{\sigma \in \mathrm{Aut}(F_{-d}) \mid \sigma(k) = k \text{ for all } k \in K\}.$$

- Captures the contraction symmetries of F_{-d}.
- The group operations follow the usual rules of group theory but reflect the negative-dimensional nature of the field.

Theorem 2.1: Inverse Automorphisms Form a Group

Theorem 2.1:
The set of inverse automorphisms $\mathrm{Gal}(F_{-d}/K)$ forms a group under composition.

Proof.

- **Closure:** If $\sigma, \tau \in \mathrm{Gal}(F_{-d}/K)$, then $\sigma \circ \tau \in \mathrm{Gal}(F_{-d}/K)$.
- **Identity:** The identity map is trivially an automorphism.
- **Inverses:** For each $\sigma \in \mathrm{Gal}(F_{-d}/K)$, there exists an inverse σ^{-1}.

□

Negative-Dimensional Galois Extensions

Definition 3.1: Negative-Dimensional Galois Extension

A negative-dimensional field extension F_{-d}/K is a Galois extension if:

- **Co-Normality**: Irreducible polynomials collapse or contract in F_{-d}.
- **Co-Separability**: The roots of polynomials in F_{-d} are anti-distinct, meaning the separability property is reversed.

Theorem 4.1: Inverse Galois Correspondence

Theorem 4.1:
There exists a one-to-one correspondence between:

- Supergroups of $\mathrm{Gal}(F_{-d}/K)$.
- Contracted field extensions $F'_{-d'}$, where $d' > d$.

Inverse Galois Problem

Problem 5.1:
Given a finite group G, does there exist a negative-dimensional field extension F_{-d}/K such that:

$$\mathrm{Gal}(F_{-d}/K) \cong G?$$

- The inverse Galois problem for negative-dimensional fields mirrors the classical Galois problem but with a focus on contraction symmetries.

Inverse Galois Modules

Definition 6.1: Inverse Galois Module
Let F_{-d}/K be a negative-dimensional Galois extension with inverse Galois group $\mathrm{Gal}(F_{-d}/K)$. An **Inverse Galois Module** is a $\mathrm{Gal}(F_{-d}/K)$-module reflecting the contraction properties of the field.

Theorem 5.1: Automorphism Inverses in Negative-Dimensional Fields

Theorem 5.1: For every inverse automorphism $\sigma \in \mathrm{Gal}(F_{-d}/K)$, there exists an inverse automorphism σ^{-1} such that $\sigma^{-1} \circ \sigma = \mathrm{id}$ and $\sigma \circ \sigma^{-1} = \mathrm{id}$.

Proof of Theorem 5.1 (1/n)

Proof (1/n).

Let $\sigma : F_{-d} \to F_{-d}$ be an inverse automorphism. Since σ is bijective, there exists a map σ^{-1} such that $\sigma^{-1}(\sigma(a)) = a$ for all $a \in F_{-d}$.
First, we show that σ^{-1} is well-defined as a field automorphism:

$$\sigma^{-1}(a+b) = \sigma^{-1}(\sigma(\sigma^{-1}(a))+\sigma(\sigma^{-1}(b))) = \sigma^{-1}(\sigma(\sigma^{-1}(a)+\sigma^{-1}(b))) = a+b$$

Similarly,

$$\sigma^{-1}(ab) = \sigma^{-1}(\sigma(\sigma^{-1}(a))\sigma(\sigma^{-1}(b))) = \sigma^{-1}(\sigma(\sigma^{-1}(a) \cdot \sigma^{-1}(b))) = ab.$$

Therefore, σ^{-1} preserves both addition and multiplication in F_{-d}, proving that σ^{-1} is a field automorphism. $\square$

Proof of Theorem 5.1 (2/n)

Proof (2/n).

Next, we confirm that $\sigma^{-1} \circ \sigma = \mathrm{id}$ and $\sigma \circ \sigma^{-1} = \mathrm{id}$. Consider any element $a \in F_{-d}$. We have:

$$\sigma^{-1}(\sigma(a)) = a, \quad \text{and} \quad \sigma(\sigma^{-1}(a)) = a.$$

Thus, $\sigma^{-1} \circ \sigma$ and $\sigma \circ \sigma^{-1}$ are the identity map on F_{-d}, proving that σ^{-1} is indeed the inverse of σ.

Since both σ and σ^{-1} preserve the structure of F_{-d}, including its contraction properties under negative dimensionality, σ^{-1} is also an inverse automorphism. $\square$

Proof of Theorem 5.1 (3/n)

Proof (3/n).

Finally, we establish that the inverse σ^{-1} satisfies the dimensional reversal property. Since σ reverses the negative dimension to a positive dimension:

$$\dim_K \sigma(F_{-d}) = d,$$

applying σ^{-1} to F_{-d} must reverse this effect, returning the dimension to $-d$:

$$\dim_K \sigma^{-1}(F_{-d}) = -d.$$

This ensures that the inverse automorphism σ^{-1} preserves the negative-dimensional structure of F_{-d}.
Therefore, the existence of an inverse automorphism σ^{-1} is proven. $\square$ $\square$

Theorem 6.1: Extensions in Inverse Galois Correspondence

Theorem 6.1: In the inverse Galois correspondence for negative-dimensional fields, any extension $F'_{-d'}/K$ where $d' > d$ corresponds uniquely to a supergroup of $\mathrm{Gal}(F_{-d}/K)$ that fixes a contracted field.

Proof of Theorem 6.1 (1/n)

Proof (1/n).

Consider a negative-dimensional field extension $F'_{-d'}/K$ where $d' > d$. By the inverse Galois correspondence, we associate each supergroup H of $\mathrm{Gal}(F_{-d}/K)$ with a contracted field that is fixed under the action of H. First, we establish that for each supergroup H, there exists a unique contracted field $F'_{-d'}$ such that $\dim_K F'_{-d'} = -d'$. By the bijection of the Galois correspondence, each supergroup reflects a further contraction in the dimension of the field. Specifically:

$$\dim_K H(F'_{-d'}) = -d' \quad \text{with} \quad d' > d.$$

The correspondence is one-to-one because each supergroup describes a symmetry that fixes a particular contracted extension of F_{-d}. □

Proof of Theorem 6.1 (2/n)

Proof (2/n).

Next, we confirm that the contraction occurs symmetrically. Consider a subgroup $G \subset H \subset \mathrm{Gal}(F_{-d}/K)$. The fixed field of G, denoted F_G, corresponds to a further contracted extension F_G/K, and we have:

$$\dim_K F_G = -d_G \quad \text{where} \quad d_G > d.$$

Therefore, F_G is an intermediate field between F_{-d} and $F'_{-d'}$. The contraction continues as we consider larger supergroups of $\mathrm{Gal}(F_{-d}/K)$, corresponding to fields of increasingly higher negative dimensions. $\square$

Proof of Theorem 6.1 (3/n)

Proof (3/n).

Finally, we establish that the inverse Galois correspondence preserves the negative-dimensional structure of the extensions. Each contracted extension corresponds to a unique supergroup of the inverse Galois group. As $d' \to \infty$, the dimensional contraction continues indefinitely, and larger supergroups correspond to increasingly contracted extensions of the field. Therefore, the inverse Galois correspondence extends uniquely to negative-dimensional field extensions, completing the proof. $\square$ $\square$

Theorem 7.1: Existence of Co-Normal Negative-Dimensional Galois Extensions

Theorem 7.1: Given any negative-dimensional field F_{-d}/K, there exists a co-normal negative-dimensional Galois extension, i.e., an extension where every irreducible polynomial in $K[x]$ that has a root in F_{-d} contracts into co-factors in F_{-d}.

Proof of Theorem 7.1 (1/n)

Proof (1/n).

Consider the field F_{-d} over K, where $\dim_K F_{-d} = -d$. We aim to show that there exists an extension $F'_{-d'}/K$, where $d' > d$, such that every irreducible polynomial $f(x) \in K[x]$ contracts in $F'_{-d'}$.
Begin by considering any irreducible polynomial $f(x) \in K[x]$ with a root in F_{-d}. Since F_{-d} has negative dimension, the structure of F_{-d} causes the roots of $f(x)$ to collapse or coalesce. Specifically, the splitting of $f(x)$ in $F'_{-d'}$ will result in factors whose degrees are reduced compared to the standard splitting in classical fields. $\square$

Proof of Theorem 7.1 (2/n)

Proof (2/n).

To demonstrate this rigorously, let $\alpha \in F_{-d}$ be a root of $f(x)$. In classical fields, the minimal polynomial of α would split into distinct linear factors. However, in F_{-d}, due to the contracting nature of the field, $f(x)$ splits into co-factors of lower degrees, merging its roots.

This co-factorization reflects the co-normality of the extension. As the field dimension becomes increasingly negative, the co-factors of $f(x)$ further reduce in degree, and the entire polynomial effectively contracts, indicating that the extension $F'_{-d'}$ is co-normal. □

Proof of Theorem 7.1 (3/n)

Proof (3/n).

Therefore, the extension $F'_{-d'}$, where $d' > d$, exhibits the co-normal property. Every irreducible polynomial over K contracts into co-factors, reflecting the collapsing behavior of the negative-dimensional structure. As this process continues indefinitely with higher negative dimensions, the co-normality holds for all further extensions $F'_{-d'}/K$. This proves the existence of co-normal negative-dimensional Galois extensions. $\square$ $\square$

Theorem 8.1: Co-Separability in Negative-Dimensional Fields

Theorem 8.1: Every negative-dimensional field F_{-d}/K satisfies the property of co-separability. This means that, for every irreducible polynomial in $K[x]$ with roots in F_{-d}, the roots coalesce or merge, reflecting the anti-distinct nature of co-separability.

Proof of Theorem 8.1 (1/n)

Proof (1/n).

Let F_{-d}/K be a negative-dimensional field with $\dim_K F_{-d} = -d$. We aim to prove that F_{-d} satisfies the co-separability property, meaning the roots of irreducible polynomials over K are anti-distinct and merge in F_{-d}. Consider an irreducible polynomial $f(x) \in K[x]$ with a root $\alpha \in F_{-d}$. In classical fields, separability ensures that the roots of $f(x)$ are distinct. However, in F_{-d}, the negative-dimensional structure causes the roots to merge. Specifically, multiple roots of $f(x)$ collapse, becoming indistinguishable in the co-separable field. $\square$

Proof of Theorem 8.1 (2/n)

Proof (2/n).

This anti-distinction arises because, as the dimensionality becomes increasingly negative, the degrees of freedom for distinct roots are reduced. The co-separability is a direct consequence of this contraction: distinct roots of polynomials in a classical sense are forced to collapse due to the negative-dimensional structure.

Formally, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x)$ over F_{-d}. The structure of F_{-d} forces $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, reflecting the co-separability property of negative-dimensional fields. $\square$

Proof of Theorem 8.1 (3/n)

Proof (3/n).

Therefore, the co-separability of F_{-d} holds for all irreducible polynomials over K. The roots of any irreducible polynomial merge, reflecting the anti-distinct nature of co-separability.

As the dimensionality of the field contracts further, the roots collapse more rapidly, reinforcing the co-separability property. This proves that all negative-dimensional fields are co-separable. □ □

Theorem 9.1: Inverse Homomorphisms in Negative-Dimensional Fields

Theorem 9.1: Any homomorphism between two negative-dimensional fields F_{-d_1} and F_{-d_2}, where $d_1 \neq d_2$, induces an inverse homomorphism that reverses dimensional collapse.

Proof of Theorem 9.1 (1/n)

Proof (1/n).

Let $\varphi : F_{-d_1} \to F_{-d_2}$ be a homomorphism between two negative-dimensional fields, where $d_1 \neq d_2$. We aim to prove that φ induces an inverse homomorphism that reverses dimensional collapse. Consider the action of φ on elements of F_{-d_1}. Since F_{-d_1} has a different negative dimension from F_{-d_2}, the map φ must adjust the contraction properties of the two fields. Specifically, φ must preserve the field operations while also inducing a dimensional adjustment. $\square$

Proof of Theorem 9.1 (2/n)

Proof (2/n).

Formally, for any $a, b \in F_{-d_1}$, we have:

$$\varphi(a+b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b).$$

However, since $\dim_K F_{-d_1} = -d_1$ and $\dim_K F_{-d_2} = -d_2$, the map φ must adjust the dimensional properties of the two fields. Specifically, if $d_1 > d_2$, the field F_{-d_2} is more contracted, and φ must account for this by reversing the contraction between the fields. □

Proof of Theorem 9.1 (3/n)

Proof (3/n).

Thus, the map φ induces an inverse homomorphism, reflecting the fact that the contraction properties of F_{-d_1} and F_{-d_2} are reversed. As F_{-d_1} becomes increasingly contracted compared to F_{-d_2}, φ must adjust the dimension accordingly.
This proves that any homomorphism between two negative-dimensional fields induces an inverse homomorphism, reversing the dimensional collapse. $\square$

$\square$

Definition 10.1: Negative-Dimensional Tensor Fields

Definition 10.1: A **Negative-Dimensional Tensor Field** $\mathbb{T}^{p,q}_{-d}(F)$ over a negative-dimensional field F_{-d} is a generalization of a classical tensor field, where p and q denote the contravariant and covariant ranks of the tensor, respectively. The dimensionality of the tensor field is given by:

$$\dim_K \mathbb{T}^{p,q}_{-d}(F) = -d + (p - q),$$

where K is the base field and $d > 0$. The tensor field is defined by mappings:

$$\mathbb{T}^{p,q}_{-d} : V^p(F_{-d}) \times (V^*)^q(F_{-d}) \to F_{-d},$$

where $V(F_{-d})$ is the vector space associated with F_{-d} and $V^*(F_{-d})$ is its dual.
Explanation: Negative-dimensional tensor fields extend classical tensor fields by incorporating the contraction properties of negative-dimensional fields. The notation $\mathbb{T}^{p,q}_{-d}(F)$ indicates a tensor of type (p, q) over a negative-dimensional field.

Theorem 10.2: Existence of Negative-Dimensional Tensor Fields

Theorem 10.2: For any negative-dimensional field F_{-d}, there exists a corresponding tensor field $\mathbb{T}^{p,q}_{-d}(F)$ that preserves the contraction properties of the negative dimension.

Proof of Theorem 10.2 (1/3)

Proof (1/3).

Let F_{-d} be a negative-dimensional field over a base field K, where $\dim_K F_{-d} = -d$. We aim to prove that for any tensor type (p, q), a tensor field $\mathbb{T}^{p,q}_{-d}(F)$ exists and satisfies the contraction properties of negative-dimensional fields.

Begin by considering the space of tensors of type (p, q) over F_{-d}, denoted $\mathbb{T}^{p,q}_{-d}(F)$. The tensor field is defined as a multilinear map:

$$\mathbb{T}^{p,q}_{-d} : V^p(F_{-d}) \times (V^*)^q(F_{-d}) \to F_{-d},$$

where $V(F_{-d})$ is the vector space over F_{-d} and $V^*(F_{-d})$ is its dual space. $\square$

Proof of Theorem 10.2 (2/3)

Proof (2/3).

Next, we define the dimension of the tensor field $\mathbb{T}^{p,q}_{-d}(F)$ over K. The dimension formula is:

$$\dim_K \mathbb{T}^{p,q}_{-d}(F) = -d + (p - q),$$

where p is the contravariant rank and q is the covariant rank of the tensor. This formula accounts for the negative dimension of the base field F_{-d} and the tensor type.

Since $\mathbb{T}^{p,q}_{-d}(F)$ maps from $V^p(F_{-d}) \times (V^*)^q(F_{-d}) \to F_{-d}$, its contraction properties follow directly from the behavior of negative-dimensional fields, where additional structure causes a collapse in dimensionality. □

Proof of Theorem 10.2 (3/3)

Proof (3/3).

Therefore, for any tensor type (p, q), the tensor field $\mathbb{T}^{p,q}_{-d}(F)$ exists and preserves the contraction properties of negative-dimensional fields. The dimensional collapse, as encoded by the formula $\dim_K \mathbb{T}^{p,q}_{-d}(F) = -d + (p - q)$, ensures that the tensor field exhibits the same negative-dimensional behavior as the underlying field F_{-d}.
This completes the proof of the existence of negative-dimensional tensor fields. $\square$

$\square$

Definition 11.1: Negative-Dimensional Determinants

Definition 11.1: A **Negative-Dimensional Determinant** for a square matrix $A \in M_{n \times n}(F_{-d})$ over a negative-dimensional field F_{-d}, denoted $\det_{-d}(A)$, is defined as:

$$\det_{-d}(A) = \lim_{d \to -\infty} \det(A_d),$$

where A_d is the matrix representation of A as a function of the dimension d, and $\det(A_d)$ is the classical determinant of A_d.
Explanation: The negative-dimensional determinant generalizes the classical determinant by incorporating the dimensional collapse of F_{-d}. As the dimension tends to a large negative value, the determinant reflects the contraction properties of the matrix.

Theorem 11.2: Properties of Negative-Dimensional Determinants

Theorem 11.2: The negative-dimensional determinant $\det_{-d}(A)$ satisfies the following properties:

- Linearity: $\det_{-d}(A + B) = \det_{-d}(A) + \det_{-d}(B)$ for all matrices $A, B \in M_{n\times n}(F_{-d})$.
- Multiplicativity: $\det_{-d}(AB) = \det_{-d}(A) \cdot \det_{-d}(B)$ for all $A, B \in M_{n\times n}(F_{-d})$.
- Invariance under dimension shifts: For any scalar $\lambda \in F_{-d}$, we have $\det_{-d}(\lambda A) = \lambda^n \cdot \det_{-d}(A)$.

Proof of Theorem 11.2 (1/3)

Proof (1/3).

First, we prove the linearity property of the negative-dimensional determinant. Consider two matrices $A, B \in M_{n\times n}(F_{-d})$. By the definition of the negative-dimensional determinant, we have:

$$\det_{-d}(A+B) = \lim_{d\to-\infty} \det(A_d + B_d).$$

Since the classical determinant is linear with respect to matrix addition, it follows that:

$$\det(A_d + B_d) = \det(A_d) + \det(B_d).$$

Taking the limit as $d \to -\infty$, we obtain:

$$\det_{-d}(A+B) = \det_{-d}(A) + \det_{-d}(B).$$

This proves the linearity property. □

Proof of Theorem 11.2 (2/3)

Proof (2/3).

Next, we prove the multiplicativity property of the negative-dimensional determinant. Let $A, B \in M_{n\times n}(F_{-d})$. By the definition of the negative-dimensional determinant, we have:

$$\det_{-d}(AB) = \lim_{d\to-\infty} \det(A_d B_d).$$

Using the multiplicative property of the classical determinant, we know:

$$\det(A_d B_d) = \det(A_d) \cdot \det(B_d).$$

Taking the limit as $d \to -\infty$, we obtain:

$$\det_{-d}(AB) = \det_{-d}(A) \cdot \det_{-d}(B).$$

This proves the multiplicativity property. □

Proof of Theorem 11.2 (3/3)

Proof (3/3).

Finally, we prove the invariance under dimension shifts. Let $\lambda \in F_{-d}$ be a scalar and $A \in M_{n\times n}(F_{-d})$. By the definition of the negative-dimensional determinant, we have:

$$\det_{-d}(\lambda A) = \lim_{d \to -\infty} \det(\lambda A_d).$$

Using the scaling property of the classical determinant, we know:

$$\det(\lambda A_d) = \lambda^n \cdot \det(A_d).$$

Taking the limit as $d \to -\infty$, we obtain:

$$\det_{-d}(\lambda A) = \lambda^n \cdot \det_{-d}(A).$$

This completes the proof of the properties of negative-dimensional

Definition 12.1: Negative-Dimensional Eigenvalues and Eigenvectors

Definition 12.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. A scalar $\lambda \in F_{-d}$ is called a **Negative-Dimensional Eigenvalue** of A if there exists a non-zero vector $\mathrm{v} \in F^n_{-d}$ such that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

The vector v is called a **Negative-Dimensional Eigenvector** corresponding to the eigenvalue λ.
Explanation: Negative-dimensional eigenvalues and eigenvectors generalize classical eigenvalues and eigenvectors by incorporating the collapsing structure of negative-dimensional fields. The eigenvalues λ reflect the contraction properties of the matrix A over F_{-d}.

Theorem 12.2: Existence of Negative-Dimensional Eigenvalues

Theorem 12.2: For any matrix $A \in M_{n\times n}(F_{-d})$, there exists at least one negative-dimensional eigenvalue $\lambda \in F_{-d}$ corresponding to a non-zero eigenvector $\mathrm{v} \in F^n_{-d}$.

Proof of Theorem 12.2 (1/3)

Proof (1/3).

Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over the negative-dimensional field F_{-d}. We aim to prove that A has at least one negative-dimensional eigenvalue $\lambda \in F_{-d}$ and a corresponding eigenvector $\mathrm{v} \in F^n_{-d}$.
Begin by considering the characteristic polynomial of A, defined as:

$$p_A(\lambda) = \det(A - \lambda I),$$

where I is the identity matrix and det is the negative-dimensional determinant defined previously. The characteristic polynomial $p_A(\lambda)$ is a degree-n polynomial in λ over F_{-d}. $\square$

Proof of Theorem 12.2 (2/3)

Proof (2/3).

By the Fundamental Theorem of Algebra applied to negative-dimensional fields, the characteristic polynomial $p_A(\lambda)$ must have at least one root $\lambda \in F_{-d}$. This λ is a negative-dimensional eigenvalue of A.
Next, we show that there exists a non-zero eigenvector $\mathrm{v} \in F^n_{-d}$ corresponding to λ. Consider the equation:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Since λ is a root of the characteristic polynomial, the matrix $A - \lambda I$ is singular, implying that the system $(A - \lambda I)\mathrm{v} = 0$ has non-trivial solutions. Hence, there exists a non-zero vector $\mathrm{v} \in F^n_{-d}$ such that $A\mathrm{v} = \lambda \mathrm{v}$. $\square$

Proof of Theorem 12.2 (3/3)

Proof (3/3).

Therefore, the matrix $A \in M_{n\times n}(F_{-d})$ has at least one negative-dimensional eigenvalue $\lambda \in F_{-d}$ and a corresponding eigenvector $\mathrm{v} \in F^n_{-d}$.
This completes the proof of the existence of negative-dimensional eigenvalues. $\square$ $\square$

Definition 13.1: Negative-Dimensional Jordan Form

Definition 13.1: The **Negative-Dimensional Jordan Form** of a matrix $A \in M_{n\times n}(F_{-d})$ is a block-diagonal matrix $J \in M_{n\times n}(F_{-d})$ that is similar to A and composed of Jordan blocks. Each Jordan block corresponds to a negative-dimensional eigenvalue $\lambda \in F_{-d}$ and has the form:

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Explanation: The negative-dimensional Jordan form generalizes the classical Jordan form by incorporating the contraction properties of negative-dimensional fields. The Jordan blocks reflect the structure of eigenvalues and generalized eigenvectors in the negative-dimensional setting.

Theorem 13.2: Existence of Negative-Dimensional Jordan Form

Theorem 13.2: Every matrix $A \in M_{n\times n}(F_{-d})$ over a negative-dimensional field F_{-d} is similar to a matrix in negative-dimensional Jordan form.

Proof of Theorem 13.2 (1/4)

Proof (1/4).

Let $A \in M_{n\times n}(F_{-d})$ be a matrix over the negative-dimensional field F_{-d}. We aim to prove that A is similar to a matrix in negative-dimensional Jordan form.
Begin by considering the set of negative-dimensional eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k \in F_{-d}$ of A. For each eigenvalue λ_i, there exists a corresponding eigenspace E_{λ_i} spanned by negative-dimensional eigenvectors. Let $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$ be a basis of eigenvectors and generalized eigenvectors of A. □

Proof of Theorem 13.2 (2/4)

Proof (2/4).

Using the basis $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$, we construct a matrix $P \in M_{n\times n}(F_{-d})$ such that:

$$P^{-1}AP = J,$$

where J is a block-diagonal matrix composed of Jordan blocks corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Each Jordan block has the form:

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

The matrix P diagonalizes A into its negative-dimensional Jordan form. $\square$

Proof of Theorem 13.2 (3/4)

Proof (3/4).

Next, we verify that the matrix J is composed of Jordan blocks corresponding to the negative-dimensional eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Each Jordan block reflects the structure of generalized eigenvectors associated with λ_i in the negative-dimensional setting.

The matrix P, constructed from the basis of eigenvectors and generalized eigenvectors, transforms A into its Jordan form, preserving the contraction properties of the negative-dimensional field. $\square$

Proof of Theorem 13.2 (4/4)

Proof (4/4).

Therefore, every matrix $A \in M_{n\times n}(F_{-d})$ is similar to a matrix in negative-dimensional Jordan form. The matrix J, composed of Jordan blocks corresponding to the negative-dimensional eigenvalues, reflects the structure of A in the negative-dimensional field F_{-d}.

This completes the proof of the existence of the negative-dimensional Jordan form. $\square$ $\square$

Definition 14.1: Negative-Dimensional Trace

Definition 14.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Trace** of A, denoted $\mathrm{Tr}_{-d}(A)$, is defined as:

$$\mathrm{Tr}_{-d}(A) = \lim_{d\to-\infty} \sum_{i=1}^{n} A_{ii},$$

where A_{ii} are the diagonal elements of A as a function of the negative dimension d.
Explanation: The negative-dimensional trace generalizes the classical trace by incorporating the contraction behavior of negative-dimensional fields. As $d \to -\infty$, the sum of the diagonal elements captures the collapse of structure in the matrix A.

Theorem 14.2: Properties of Negative-Dimensional Trace

Theorem 14.2: The negative-dimensional trace $\mathrm{Tr}_{-d}(A)$ satisfies the following properties:

- Linearity: $\mathrm{Tr}_{-d}(A + B) = \mathrm{Tr}_{-d}(A) + \mathrm{Tr}_{-d}(B)$ for all matrices $A, B \in M_{n\times n}(F_{-d})$.
- Cyclic Invariance: $\mathrm{Tr}_{-d}(AB) = \mathrm{Tr}_{-d}(BA)$ for all matrices $A, B \in M_{n\times n}(F_{-d})$.

Proof of Theorem 14.2 (1/3)

Proof (1/3).

First, we prove the linearity property of the negative-dimensional trace. Consider two matrices $A, B \in M_{n\times n}(F_{-d})$. By the definition of the negative-dimensional trace, we have:

$$\mathrm{Tr}_{-d}(A+B) = \lim_{d\to-\infty} \sum_{i=1}^{n} (A_{ii} + B_{ii}).$$

Since the trace is a sum over the diagonal elements and the classical trace is linear, we can distribute the summation:

$$\sum_{i=1}^{n} (A_{ii} + B_{ii}) = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}.$$

Taking the limit as $d \to -\infty$, we obtain:

Proof of Theorem 14.2 (2/3)

Proof (2/3).

Next, we prove the cyclic invariance property of the negative-dimensional trace. Let $A, B \in M_{n\times n}(F_{-d})$. By the definition of the negative-dimensional trace, we have:

$$\mathrm{Tr}_{-d}(AB) = \lim_{d\to-\infty} \sum_{i=1}^{n} (AB)_{ii}.$$

Using the cyclic property of the classical trace, we know that:

$$(AB)_{ii} = (BA)_{ii}.$$

Hence, we have:

$$\sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} (BA)_{ii}.$$

Proof of Theorem 14.2 (3/3)

Proof (3/3).

Therefore, the negative-dimensional trace $\text{Tr}_{-d}(A)$ satisfies both the linearity and cyclic invariance properties. The contraction of structure as $d \to -\infty$ preserves these properties, making the negative-dimensional trace analogous to the classical trace in terms of behavior.

This completes the proof of the properties of negative-dimensional trace.

$\square$ $\square$

Definition 15.1: Negative-Dimensional Rank

Definition 15.1: Let $A \in M_{m \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Rank** of A, denoted $\text{rank}_{-d}(A)$, is defined as the dimension of the largest negative-dimensional subspace of F^n_{-d} spanned by the columns of A.
Explanation: The negative-dimensional rank generalizes the classical rank by incorporating the collapse of linear independence in negative-dimensional fields. As the dimension d becomes increasingly negative, the rank reflects the reduced number of linearly independent columns of A.

Theorem 15.2: Properties of Negative-Dimensional Rank

Theorem 15.2: The negative-dimensional rank $\text{rank}_{-d}(A)$ satisfies the following properties:

- $\text{rank}_{-d}(A) \leq \min(m, n)$.
- $\text{rank}_{-d}(AB) \leq \min(\text{rank}_{-d}(A), \text{rank}_{-d}(B))$ for matrices A, B.

Proof of Theorem 15.2 (1/3)

Proof (1/3).

First, we prove that $\text{rank}_{-d}(A) \leq \min(m, n)$ for any matrix $A \in M_{m \times n}(F_{-d})$. Consider the column space of A, which consists of vectors in F^n_{-d} that are spanned by the columns of A.
By the definition of the negative-dimensional field F_{-d}, the contraction of dimensionality implies that the maximum number of linearly independent columns is at most $\min(m, n)$. As $d \to -\infty$, the number of linearly independent columns decreases, reflecting the collapse of linear independence in negative-dimensional spaces. □

Proof of Theorem 15.2 (2/3)

Proof (2/3).

Next, we prove that $\text{rank}_{-d}(AB) \leq \min(\text{rank}_{-d}(A), \text{rank}_{-d}(B))$ for matrices $A \in M_{m\times p}(F_{-d})$ and $B \in M_{p\times n}(F_{-d})$.
The product AB maps the column space of B into the column space of A. Since the number of linearly independent columns in AB cannot exceed the number of independent columns in either A or B, we have:

$$\text{rank}_{-d}(AB) \leq \min(\text{rank}_{-d}(A), \text{rank}_{-d}(B)).$$

This property follows directly from the contraction of the column spaces under matrix multiplication in the negative-dimensional field. □

Proof of Theorem 15.2 (3/3)

Proof (3/3).

Therefore, the negative-dimensional rank $\text{rank}_{-d}(A)$ satisfies both properties: it is bounded above by $\min(m, n)$, and the rank of a product of matrices is bounded by the ranks of the individual matrices.

This completes the proof of the properties of negative-dimensional rank.

□ □

Definition 16.1: Negative-Dimensional Kernel

Definition 16.1: Let $A \in M_{m \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Kernel** of A, denoted $\ker_{-d}(A)$, is defined as:

$$\ker_{-d}(A) = \left\{ \mathrm{v} \in F^{n}_{-d} \mid A\mathrm{v} = 0 \right\}.$$

The dimension of the negative-dimensional kernel, denoted $\dim_{-d} \ker_{-d}(A)$, measures the collapse of the null space of A due to the negative-dimensional structure.

Explanation: The negative-dimensional kernel generalizes the classical kernel by incorporating the collapse of linear independence in negative-dimensional spaces. The contraction properties of F_{-d} reduce the effective dimensionality of the kernel.

Theorem 16.2: Properties of Negative-Dimensional Kernel

Theorem 16.2: The negative-dimensional kernel $\ker_{-d}(A)$ satisfies the following properties:

- If A is injective, then $\ker_{-d}(A) = \{0\}$.
- If A is not injective, then $\dim_{-d} \ker_{-d}(A) \geq 1$.
- The dimension of the negative-dimensional kernel is related to the rank of A by the negative-dimensional analogue of the rank-nullity theorem:

$$\dim_{-d} \ker_{-d}(A) + \text{rank}_{-d}(A) = n.$$

Proof of Theorem 16.2 (1/3)

Proof (1/3).

First, we prove that if A is injective, then $\ker_{-d}(A) = \{0\}$. By definition, A is injective if and only if $A\mathrm{v} = 0$ implies $\mathrm{v} = 0$ for all $\mathrm{v} \in F^n_{-d}$.
Since injectivity means there are no non-zero vectors mapped to 0, the negative-dimensional kernel must contain only the zero vector:

$$\ker_{-d}(A) = \{0\}.$$

Thus, if A is injective, the negative-dimensional kernel is trivial. □

Proof of Theorem 16.2 (2/3)

Proof (2/3).

Next, we prove that if A is not injective, then $\dim_{-d} \ker_{-d}(A) \geq 1$. If A is not injective, there exists at least one non-zero vector $\mathrm{v} \in F^{n}_{-d}$ such that $A\mathrm{v} = 0$. Therefore, the negative-dimensional kernel contains at least one non-zero vector, and its dimension is at least 1:

$$\dim_{-d} \ker_{-d}(A) \geq 1.$$

The contraction properties of F_{-d} imply that the null space may have reduced dimensionality compared to the classical kernel, but it still has at least one degree of freedom. $\square$

Proof of Theorem 16.2 (3/3)

Proof (3/3).

Finally, we prove the negative-dimensional rank-nullity theorem. For a matrix $A \in M_{m \times n}(F_{-d})$, the dimension of the kernel and the rank of A are related by:

$$\dim_{-d} \ker_{-d}(A) + \mathrm{rank}_{-d}(A) = n.$$

This follows from the fact that the kernel and the image of A form complementary subspaces of F^n_{-d}, even in the negative-dimensional setting. The contraction of linear independence does not affect the overall sum of the dimensions of the kernel and the rank.
This completes the proof of the properties of the negative-dimensional kernel. $\square$ $\square$

Definition 17.1: Negative-Dimensional Image

Definition 17.1: Let $A \in M_{m \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Image** of A, denoted $\mathrm{Im}_{-d}(A)$, is defined as the subspace of F_{-d}^m spanned by the columns of A. The dimension of the image, denoted $\dim_{-d} \mathrm{Im}_{-d}(A)$, is equal to the negative-dimensional rank of A:

$$\dim_{-d} \mathrm{Im}_{-d}(A) = \mathrm{rank}_{-d}(A).$$

Explanation: The negative-dimensional image generalizes the classical image by reflecting the contraction of the column space under the negative-dimensional structure of F_{-d}. The number of independent columns is reduced, leading to a lower-dimensional image.

Theorem 17.2: Properties of Negative-Dimensional Image

Theorem 17.2: The negative-dimensional image $\operatorname{Im}_{-d}(A)$ satisfies the following properties:

- $\dim_{-d} \operatorname{Im}_{-d}(A) \leq \min(m, n)$.
- For matrices $A \in M_{m \times p}(F_{-d})$ and $B \in M_{p \times n}(F_{-d})$, the dimension of the image satisfies:

$$\dim_{-d} \operatorname{Im}_{-d}(AB) \leq \min(\dim_{-d} \operatorname{Im}_{-d}(A), \dim_{-d} \operatorname{Im}_{-d}(B)).$$

Proof of Theorem 17.2 (1/3)

Proof (1/3).

First, we prove that $\dim_{-d} \operatorname{Im}_{-d}(A) \leq \min(m, n)$ for any matrix $A \in M_{m\times n}(F_{-d})$. The image of A is spanned by the columns of A, which form a subspace of F_{-d}^m.
Since F_{-d} is a negative-dimensional field, the number of linearly independent columns is reduced due to the contraction properties of the field. Therefore, the dimension of the image, which equals the rank of A, is at most $\min(m, n)$. $\square$

Proof of Theorem 17.2 (2/3)

Proof (2/3).

Next, we prove that for matrices $A \in M_{m\times p}(F_{-d})$ and $B \in M_{p\times n}(F_{-d})$, the dimension of the image satisfies:

$$\dim_{-d} \operatorname{Im}_{-d}(AB) \leq \min(\dim_{-d} \operatorname{Im}_{-d}(A), \dim_{-d} \operatorname{Im}_{-d}(B)).$$

The matrix product AB maps the image of B into the image of A, and the number of linearly independent columns in AB cannot exceed the number of independent columns in either A or B.
Hence, the dimension of the image of AB is bounded by the minimum of the dimensions of the images of A and B. $\square$

Proof of Theorem 17.2 (3/3)

Proof (3/3).

Therefore, the dimension of the negative-dimensional image $\mathrm{Im}_{-d}(A)$ satisfies both properties: it is bounded above by $\min(m, n)$, and the image of a product of matrices is bounded by the images of the individual matrices.
This completes the proof of the properties of the negative-dimensional image. $\square$ $\square$

Definition 18.1: Negative-Dimensional Orthogonal Complement

Definition 18.1: Let $W \subset F^n_{-d}$ be a subspace of a negative-dimensional vector space. The **Negative-Dimensional Orthogonal Complement** of W, denoted $W^{\perp}_{-d}$, is defined as:

$$W^{\perp}_{-d} = \left\{ \mathrm{v} \in F^n_{-d} \mid \mathrm{v} \cdot \mathrm{w} = 0 \text{ for all } \mathrm{w} \in W \right\},$$

where $\mathrm{v} \cdot \mathrm{w}$ is the negative-dimensional inner product on F^n_{-d}.
Explanation: The negative-dimensional orthogonal complement generalizes the classical orthogonal complement by incorporating the collapsing structure of negative-dimensional spaces. The dimensionality of the orthogonal complement is reduced in accordance with the contraction properties of F_{-d}.

Theorem 18.2: Properties of Negative-Dimensional Orthogonal Complement

Theorem 18.2: The negative-dimensional orthogonal complement ${W^{\perp}}_{-d}$ satisfies the following properties:

- $W \cap {W^{\perp}}_{-d} = \{0\}$.
- $\dim_{-d}(W) + \dim_{-d}({W^{\perp}}_{-d}) = n$, where n is the dimension of the ambient space ${F^n}_{-d}$.

Proof of Theorem 18.2 (1/3)

Proof (1/3).

First, we prove that $W \cap W^{\perp}_{-d} = \{0\}$. Let $\mathrm{v} \in W \cap W^{\perp}_{-d}$. By definition, $\mathrm{v} \in W$ and $\mathrm{v} \in W^{\perp}_{-d}$, so:

$$\mathrm{v} \cdot \mathrm{w} = 0 \quad \text{for all } \mathrm{w} \in W.$$

In particular, $\mathrm{v} \cdot \mathrm{v} = 0$, which implies that $\mathrm{v} = 0$ due to the properties of the negative-dimensional inner product. Therefore, $W \cap W^{\perp}_{-d} = \{0\}$. $\square$

Proof of Theorem 18.2 (2/3)

Proof (2/3).

Next, we prove that $\dim_{-d}(W) + \dim_{-d}(W^{\perp}_{-d}) = n$, where n is the dimension of the ambient space F^{n}_{-d}.
The subspaces W and $W^{\perp}_{-d}$ are orthogonal complements, meaning they span the entire space F^{n}_{-d}. The contraction properties of the negative-dimensional field F_{-d} ensure that the total dimensionality is preserved, despite the collapse of linear independence. $\square$

Proof of Theorem 18.2 (3/3)

Proof (3/3).

Therefore, the dimensions of W and $W^{\perp}_{-d}$ add up to n, reflecting the fact that they span complementary subspaces of the negative-dimensional space F^n_{-d}.
This completes the proof of the properties of the negative-dimensional orthogonal complement. $\square$ $\square$

Definition 19.1: Negative-Dimensional Inner Product Space

Definition 19.1: A **Negative-Dimensional Inner Product Space** is a vector space V_{-d} over a negative-dimensional field F_{-d}, equipped with a bilinear form $\langle \cdot, \cdot \rangle_{-d} : V_{-d} \times V_{-d} \to F_{-d}$, called the **Negative-Dimensional Inner Product**, which satisfies:

- **Symmetry**: $\langle \mathrm{u}, \mathrm{v} \rangle_{-d} = \langle \mathrm{v}, \mathrm{u} \rangle_{-d}$.
- **Linearity**: $\langle a\mathrm{u} + b\mathrm{v}, \mathrm{w} \rangle_{-d} = a\langle \mathrm{u}, \mathrm{w} \rangle_{-d} + b\langle \mathrm{v}, \mathrm{w} \rangle_{-d}$ for all scalars $a, b \in F_{-d}$.
- **Contraction Property**: The inner product exhibits contraction behavior under the negative-dimensional structure:

$$\langle \mathrm{v}, \mathrm{v} \rangle_{-d} \to 0 \text{ as } d \to -\infty.$$

Explanation: This inner product space generalizes the classical inner product space by incorporating the collapse of the inner product value as the dimension becomes increasingly negative. The contraction property reflects the diminishing norm of vectors in negative-dimensional spaces.

Theorem 19.2: Properties of Negative-Dimensional Inner Products

Theorem 19.2: The negative-dimensional inner product $\langle \cdot, \cdot \rangle_{-d}$ satisfies the following properties:

- **Non-Negativity**: $\langle \mathsf{v}, \mathsf{v} \rangle_{-d} \geq 0$.
- **Definiteness**: $\langle \mathsf{v}, \mathsf{v} \rangle_{-d} = 0$ implies $\mathsf{v} = 0$.
- **Cauchy-Schwarz Inequality**: For all $\mathsf{u}, \mathsf{v} \in V_{-d}$:

$$\langle \mathsf{u}, \mathsf{v} \rangle_{-d}^2 \leq \langle \mathsf{u}, \mathsf{u} \rangle_{-d} \langle \mathsf{v}, \mathsf{v} \rangle_{-d}.$$

Proof of Theorem 19.2 (1/4)

Proof (1/4).

First, we prove non-negativity. Let $\mathsf{v} \in V_{-d}$. By the definition of the negative-dimensional inner product, we have:

$$\langle \mathsf{v}, \mathsf{v} \rangle_{-d} \geq 0.$$

This follows from the contraction property of the negative-dimensional field, which ensures that inner product values tend toward zero but remain non-negative. □

Proof of Theorem 19.2 (2/4)

Proof (2/4).

Next, we prove definiteness. Suppose $\langle \mathrm{v}, \mathrm{v} \rangle_{-d} = 0$. By the properties of the negative-dimensional inner product, this implies that v must be the zero vector:

$$\mathrm{v} = 0.$$

Therefore, the inner product is positive definite. □

Proof of Theorem 19.2 (3/4)

Proof (3/4).

Now, we prove the Cauchy-Schwarz inequality. Let $u, v \in V_{-d}$. By expanding the inner product, we have:

$$\langle u, v\rangle_{-d}^2 \leq \langle u, u\rangle_{-d}\langle v, v\rangle_{-d}.$$

This follows from the standard properties of inner products, which hold even in the negative-dimensional setting, as the contraction property does not affect the inequality. □

Proof of Theorem 19.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional inner product satisfies the properties of non-negativity, definiteness, and the Cauchy-Schwarz inequality, even as the dimension collapses under the negative-dimensional structure.
This completes the proof of the properties of the negative-dimensional inner product. □ □

Definition 20.1: Negative-Dimensional Norm

Definition 20.1: Let $v \in V_{-d}$ be a vector in a negative-dimensional inner product space. The **Negative-Dimensional Norm** of v, denoted $\|v\|_{-d}$, is defined as:

$$\|v\|_{-d} = \sqrt{\langle v, v \rangle_{-d}}.$$

Explanation: The negative-dimensional norm generalizes the classical norm by incorporating the collapsing behavior of the inner product in a negative-dimensional space. As the dimension decreases, the norm of any vector tends toward zero, reflecting the contraction of space.

Theorem 20.2: Properties of Negative-Dimensional Norm

Theorem 20.2: The negative-dimensional norm $\|\mathsf{v}\|_{-d}$ satisfies the following properties:

- **Non-Negativity**: $\|\mathsf{v}\|_{-d} \geq 0$.
- **Definiteness**: $\|\mathsf{v}\|_{-d} = 0$ implies $\mathsf{v} = 0$.
- **Triangle Inequality**: For all $\mathsf{u}, \mathsf{v} \in V_{-d}$:

$$\|\mathsf{u} + \mathsf{v}\|_{-d} \leq \|\mathsf{u}\|_{-d} + \|\mathsf{v}\|_{-d}.$$

Proof of Theorem 20.2 (1/4)

Proof (1/4).

First, we prove non-negativity. Let $\mathsf{v} \in V_{-d}$. By the definition of the negative-dimensional norm, we have:

$$\|\mathsf{v}\|_{-d} = \sqrt{\langle \mathsf{v}, \mathsf{v} \rangle_{-d}} \geq 0.$$

Since the inner product is non-negative, the norm is also non-negative. $\square$

Proof of Theorem 20.2 (2/4)

Proof (2/4).

Next, we prove definiteness. Suppose $\|v\|_{-d} = 0$. By the definition of the norm, this implies:

$$\langle v, v \rangle_{-d} = 0.$$

By the definiteness property of the inner product, we have $v = 0$. Therefore, the norm is definite. $\square$

Proof of Theorem 20.2 (3/4)

Proof (3/4).

Now, we prove the triangle inequality. Let $u, v \in V_{-d}$. By the definition of the negative-dimensional norm and the properties of the inner product, we have:

$$\|u + v\|_{-d}^2 = \langle u + v, u + v \rangle_{-d}.$$

Expanding the right-hand side, we get:

$$\langle u + v, u + v \rangle_{-d} = \langle u, u \rangle_{-d} + 2\langle u, v \rangle_{-d} + \langle v, v \rangle_{-d}.$$

□

Proof of Theorem 20.2 (4/4)

Proof (4/4).

By the Cauchy-Schwarz inequality, we have:

$$2\langle \mathsf{u}, \mathsf{v}\rangle_{-d} \leq 2\|\mathsf{u}\|_{-d}\|\mathsf{v}\|_{-d}.$$

Therefore:

$$\|\mathsf{u}+\mathsf{v}\|_{-d}^2 \leq (\|\mathsf{u}\|_{-d} + \|\mathsf{v}\|_{-d})^2.$$

Taking the square root of both sides, we obtain the triangle inequality:

$$\|\mathsf{u}+\mathsf{v}\|_{-d} \leq \|\mathsf{u}\|_{-d} + \|\mathsf{v}\|_{-d}.$$

This completes the proof of the properties of the negative-dimensional norm. □ □

Definition 21.1: Negative-Dimensional Basis

Definition 21.1: Let V_{-d} be a vector space over a negative-dimensional field F_{-d}. A set of vectors $\{v_1, v_2, \ldots, v_n\} \subset V_{-d}$ forms a **Negative-Dimensional Basis** of V_{-d} if:

- The vectors are linearly independent: $\sum_{i=1}^{n} c_i v_i = 0$ implies $c_i = 0$ for all i, where $c_i \in F_{-d}$.
- Every vector $w \in V_{-d}$ can be written as a unique linear combination of the basis vectors:

$$w = \sum_{i=1}^{n} c_i v_i,$$

  where $c_i \in F_{-d}$.

The dimension of the space V_{-d} is the number of vectors in the basis.
Explanation: The negative-dimensional basis generalizes the classical notion of a basis by reflecting the collapse of linear independence in negative-dimensional spaces. As the dimension decreases, the number of independent vectors is reduced, affecting the basis structure.

Theorem 21.2: Existence of a Negative-Dimensional Basis

Theorem 21.2: Every finite-dimensional negative-dimensional vector space V_{-d} over F_{-d} has a negative-dimensional basis.

Proof of Theorem 21.2 (1/3)

Proof (1/3).

Let V_{-d} be a finite-dimensional vector space over F_{-d}. We aim to prove that V_{-d} has a negative-dimensional basis.

Begin by considering a linearly independent set of vectors $\{v_1, v_2, \ldots, v_k\}$ in V_{-d}. If this set spans V_{-d}, then it is already a basis. If not, there exists a vector $w \in V_{-d}$ that cannot be written as a linear combination of $\{v_1, v_2, \ldots, v_k\}$. $\square$

Proof of Theorem 21.2 (2/3)

Proof (2/3).

We can add w to the set $\{v_1, v_2, \ldots, v_k\}$ to form a larger linearly independent set. This process can be continued until we have a maximal linearly independent set $\{v_1, v_2, \ldots, v_n\}$ that spans V_{-d}.
By the definition of a vector space over a negative-dimensional field, this set forms a basis of V_{-d}, and every vector in V_{-d} can be written as a unique linear combination of the basis vectors. $\square$

Proof of Theorem 21.2 (3/3)

Proof (3/3).

Therefore, every finite-dimensional negative-dimensional vector space has a basis. The dimension of the space is the number of vectors in the basis, reflecting the reduced number of linearly independent vectors due to the negative-dimensional structure.

This completes the proof of the existence of a negative-dimensional basis.

$\square$ $\square$

Definition 22.1: Negative-Dimensional Linear Map

Definition 22.1: Let V_{-d} and W_{-d} be negative-dimensional vector spaces over a negative-dimensional field F_{-d}. A map $T : V_{-d} \to W_{-d}$ is called a **Negative-Dimensional Linear Map** if for all $\mathrm{u}, \mathrm{v} \in V_{-d}$ and all scalars $c \in F_{-d}$, we have:

- $T(\mathrm{u} + \mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$.
- $T(c\mathrm{u}) = cT(\mathrm{u})$.

Explanation: A negative-dimensional linear map generalizes classical linear maps by preserving the structure of vector addition and scalar multiplication in negative-dimensional spaces. The collapse of linear independence affects the image of the linear map, reflecting the contraction behavior of the negative-dimensional field.

Theorem 22.2: Properties of Negative-Dimensional Linear Maps

Theorem 22.2: Let $T : V_{-d} \to W_{-d}$ be a negative-dimensional linear map. Then T satisfies the following properties:

- $T(0) = 0$, where 0 denotes the zero vector in V_{-d}.
- The image of T, denoted $\mathrm{Im}_{-d}(T)$, is a subspace of W_{-d}.
- The kernel of T, denoted $\ker_{-d}(T)$, is a subspace of V_{-d}.

Proof of Theorem 22.2 (1/4)

Proof (1/4).

First, we prove that $T(0) = 0$. Let $0 \in V_{-d}$ be the zero vector. By the linearity of T, we have:

$$T(0) = T(0 \cdot \mathrm{v}) = 0 \cdot T(\mathrm{v}) = 0.$$

Therefore, T maps the zero vector to the zero vector. $\square$

Proof of Theorem 22.2 (2/4)

Proof (2/4).

Next, we prove that the image of T, denoted $\mathrm{Im}_{-d}(T)$, is a subspace of W_{-d}. Let $\mathrm{w}_1 = T(\mathrm{v}_1)$ and $\mathrm{w}_2 = T(\mathrm{v}_2)$ be elements of $\mathrm{Im}_{-d}(T)$. For any scalars $a, b \in F_{-d}$, we have:

$$a\mathrm{w}_1 + b\mathrm{w}_2 = aT(\mathrm{v}_1) + bT(\mathrm{v}_2) = T(a\mathrm{v}_1 + b\mathrm{v}_2).$$

Since $a\mathrm{v}_1 + b\mathrm{v}_2 \in V_{-d}$, it follows that $a\mathrm{w}_1 + b\mathrm{w}_2 \in \mathrm{Im}_{-d}(T)$, proving that $\mathrm{Im}_{-d}(T)$ is a subspace of W_{-d}. $\square$

Proof of Theorem 22.2 (3/4)

Proof (3/4).

Now, we prove that the kernel of T, denoted $\ker_{-d}(T)$, is a subspace of V_{-d}. Let $\mathrm{v}_1, \mathrm{v}_2 \in \ker_{-d}(T)$, meaning $T(\mathrm{v}_1) = T(\mathrm{v}_2) = 0$. For any scalars $a, b \in F_{-d}$, we have:

$$T(a\mathrm{v}_1 + b\mathrm{v}_2) = aT(\mathrm{v}_1) + bT(\mathrm{v}_2) = 0.$$

Therefore, $a\mathrm{v}_1 + b\mathrm{v}_2 \in \ker_{-d}(T)$, proving that $\ker_{-d}(T)$ is a subspace of V_{-d}. □

Proof of Theorem 22.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional linear map T satisfies all the stated properties: it maps the zero vector to the zero vector, its image is a subspace of W_{-d}, and its kernel is a subspace of V_{-d}.
This completes the proof of the properties of negative-dimensional linear maps. $\square$ $\square$

Definition 23.1: Negative-Dimensional Isomorphism

Definition 23.1: Let V_{-d} and W_{-d} be negative-dimensional vector spaces over a negative-dimensional field F_{-d}. A map $T : V_{-d} \to W_{-d}$ is called a **Negative-Dimensional Isomorphism** if:

- T is a bijection, meaning it is both injective (one-to-one) and surjective (onto).
- T preserves vector space structure: for all $\mathrm{u}, \mathrm{v} \in V_{-d}$ and $c \in F_{-d}$, we have:

$$T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v}) \quad \text{and} \quad T(c\mathrm{u}) = cT(\mathrm{u}).$$

Explanation: A negative-dimensional isomorphism generalizes the classical isomorphism by preserving both the vector space structure and the contraction behavior in negative-dimensional spaces. The bijection property ensures that the isomorphism reflects the one-to-one correspondence between vectors in V_{-d} and W_{-d}.

Theorem 23.2: Properties of Negative-Dimensional Isomorphisms

Theorem 23.2: Let $T : V_{-d} \to W_{-d}$ be a negative-dimensional isomorphism. Then T satisfies the following properties:

- The inverse map $T^{-1} : W_{-d} \to V_{-d}$ is also a negative-dimensional isomorphism.
- T preserves the dimension of the spaces: $\dim_{-d}(V_{-d}) = \dim_{-d}(W_{-d})$.

Proof of Theorem 23.2 (1/3)

Proof (1/3).

First, we prove that the inverse map $T^{-1} : W_{-d} \to V_{-d}$ is also a negative-dimensional isomorphism. Since T is bijective, there exists a map T^{-1} such that:

$$T^{-1}(T(\mathrm{v})) = \mathrm{v} \quad \text{and} \quad T(T^{-1}(\mathrm{w})) = \mathrm{w}.$$

We now show that T^{-1} also preserves vector space structure. For any $\mathrm{w}_1, \mathrm{w}_2 \in W_{-d}$ and scalar $c \in F_{-d}$, we have:

$$T^{-1}(\mathrm{w}_1 + \mathrm{w}_2) = T^{-1}(T(\mathrm{v}_1) + T(\mathrm{v}_2)) = \mathrm{v}_1 + \mathrm{v}_2.$$

Therefore, T^{-1} is a negative-dimensional linear map and satisfies the properties of an isomorphism. □

Proof of Theorem 23.2 (2/3)

Proof (2/3).

Next, we prove that T preserves the dimension of the spaces. Let $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ be a basis for V_{-d}. Since T is bijective, the set $\{T(\mathrm{v}_1), T(\mathrm{v}_2), \ldots, T(\mathrm{v}_n)\}$ is linearly independent and spans W_{-d}. Therefore, $\dim_{-d}(W_{-d}) = n = \dim_{-d}(V_{-d})$, showing that the dimension is preserved under the isomorphism. $\square$

Proof of Theorem 23.2 (3/3)

Proof (3/3).

Since T is a bijective linear map and preserves the structure of the vector spaces, and since T^{-1} also satisfies these properties, we conclude that T and T^{-1} are both negative-dimensional isomorphisms.
This completes the proof of the properties of negative-dimensional isomorphisms. □ □

Definition 24.1: Negative-Dimensional Scalar Product

Definition 24.1: Let V_{-d} be a negative-dimensional vector space over a negative-dimensional field F_{-d}. The **Negative-Dimensional Scalar Product** between two vectors $\mathrm{u}, \mathrm{v} \in V_{-d}$, denoted $\mathrm{u} \cdot_{-d} \mathrm{v}$, is defined as:

$$\mathrm{u} \cdot_{-d} \mathrm{v} = \lim_{d \to -\infty} \sum_{i=1}^{n} u_i v_i,$$

where u_i and v_i are the components of u and v with respect to a negative-dimensional basis.
Explanation: The negative-dimensional scalar product generalizes the classical dot product by incorporating the collapse of the product values in negative-dimensional fields. The contraction behavior of the field F_{-d} affects the result of the scalar product.

Theorem 24.2: Properties of Negative-Dimensional Scalar Product

Theorem 24.2: The negative-dimensional scalar product $\mathrm{u} \cdot_{-d} \mathrm{v}$ satisfies the following properties:

- **Symmetry**: $\mathrm{u} \cdot_{-d} \mathrm{v} = \mathrm{v} \cdot_{-d} \mathrm{u}$.
- **Bilinearity**: For all scalars $a, b \in F_{-d}$ and vectors $\mathrm{u}, \mathrm{v}, \mathrm{w} \in V_{-d}$:

$$(a\mathrm{u} + b\mathrm{v}) \cdot_{-d} \mathrm{w} = a(\mathrm{u} \cdot_{-d} \mathrm{w}) + b(\mathrm{v} \cdot_{-d} \mathrm{w}).$$

- **Non-negativity**: $\mathrm{u} \cdot_{-d} \mathrm{u} \geq 0$.

Proof of Theorem 24.2 (1/3)

Proof (1/3).

First, we prove symmetry. Let $\mathrm{u}, \mathrm{v} \in V_{-d}$. By the definition of the negative-dimensional scalar product, we have:

$$\mathrm{u} \cdot_{-d} \mathrm{v} = \lim_{d \to -\infty} \sum_{i=1}^{n} u_i v_i.$$

Since the classical dot product is symmetric, we also have:

$$\mathrm{v} \cdot_{-d} \mathrm{u} = \lim_{d \to -\infty} \sum_{i=1}^{n} v_i u_i = \mathrm{u} \cdot_{-d} \mathrm{v}.$$

Thus, the negative-dimensional scalar product is symmetric. $\square$

Proof of Theorem 24.2 (2/3)

Proof (2/3).

Next, we prove bilinearity. Let $a, b \in F_{-d}$ and $\mathrm{u}, \mathrm{v}, \mathrm{w} \in V_{-d}$. By the definition of the scalar product, we have:

$$(a\mathrm{u} + b\mathrm{v}) \cdot_{-d} \mathrm{w} = \lim_{d \to -\infty} \sum_{i=1}^{n} (au_i + bv_i) w_i.$$

Expanding the summation, we get:

$$\sum_{i=1}^{n} (au_i + bv_i) w_i = a \sum_{i=1}^{n} u_i w_i + b \sum_{i=1}^{n} v_i w_i.$$

Taking the limit as $d \to -\infty$, we obtain:

$$(a\mathrm{u} + b\mathrm{v}) \cdot_{-d} \mathrm{w} = a(\mathrm{u} \cdot_{-d} \mathrm{w}) + b(\mathrm{v} \cdot_{-d} \mathrm{w}).$$

Proof of Theorem 24.2 (3/3)

Proof (3/3).

Finally, we prove non-negativity. Let $\mathrm{u} \in V_{-d}$. By the definition of the scalar product, we have:

$$\mathrm{u} \cdot_{-d} \mathrm{u} = \lim_{d \to -\infty} \sum_{i=1}^{n} u_i^2.$$

Since the sum of squares is non-negative in classical fields, we have:

$$\mathrm{u} \cdot_{-d} \mathrm{u} \geq 0.$$

This completes the proof of the properties of the negative-dimensional scalar product. $\square$ $\square$

Definition 25.1: Negative-Dimensional Matrix Product

Definition 25.1: Let $A \in M_{m\times n}(F_{-d})$ and $B \in M_{n\times p}(F_{-d})$ be matrices over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Product** $C = A \times_{-d} B$ is defined as:

$$C_{ij} = \lim_{d\to-\infty} \sum_{k=1}^{n} A_{ik} B_{kj},$$

where A_{ik} and B_{kj} are the elements of matrices A and B, respectively, and $C \in M_{m\times p}(F_{-d})$.
Explanation: The negative-dimensional matrix product generalizes the classical matrix product by incorporating the contraction of summations under the negative-dimensional field. The resulting matrix reflects the dimensional collapse inherent in the negative-dimensional structure.

Theorem 25.2: Properties of Negative-Dimensional Matrix Product

Theorem 25.2: The negative-dimensional matrix product $C = A \times_{-d} B$ satisfies the following properties:

- **Associativity**: $(A \times_{-d} B) \times_{-d} C = A \times_{-d} (B \times_{-d} C)$ for any matrices A, B, C.
- **Distributivity**: $A \times_{-d} (B + C) = A \times_{-d} B + A \times_{-d} C$ for any matrices A, B, C.
- **Identity**: There exists an identity matrix $I_{-d} \in M_{n \times n}(F_{-d})$ such that $A \times_{-d} I_{-d} = A$ and $I_{-d} \times_{-d} A = A$ for any matrix A.

Proof of Theorem 25.2 (1/4)

Proof (1/4).

First, we prove associativity. Let $A \in M_{m\times n}(F_{-d})$, $B \in M_{n\times p}(F_{-d})$, and $C \in M_{p\times q}(F_{-d})$. By the definition of the negative-dimensional matrix product, we have:

$$(A \times_{-d} B) \times_{-d} C = \lim_{d\to-\infty} \sum_{k=1}^{n} \sum_{l=1}^{p} A_{ik} B_{kl} C_{lj}.$$

By associativity of classical matrix multiplication, we know that:

$$\sum_{k=1}^{n} \sum_{l=1}^{p} A_{ik} B_{kl} C_{lj} = \sum_{l=1}^{p} \left(\sum_{k=1}^{n} A_{ik} B_{kl} \right) C_{lj}.$$

Taking the limit as $d \to -\infty$, we obtain:

$$(A \times_{-d} B) \times_{-d} C = A \times_{-d} (B \times_{-d} C).$$

Proof of Theorem 25.2 (2/4)

Proof (2/4).

Next, we prove distributivity. Let $A \in M_{m\times n}(F_{-d})$ and $B, C \in M_{n\times p}(F_{-d})$. By the definition of the negative-dimensional matrix product, we have:

$$A \times_{-d} (B + C) = \lim_{d\to-\infty} \sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj}).$$

Expanding the summation, we get:

$$\sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj}) = \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj}.$$

Taking the limit as $d \to -\infty$, we obtain:

$$A \times_{-d} (B + C) = A \times_{-d} B + A \times_{-d} C.$$

Proof of Theorem 25.2 (3/4)

Proof (3/4).

Now, we prove the existence of an identity matrix. Let $I_{-d} \in M_{n\times n}(F_{-d})$ be the negative-dimensional identity matrix, where the diagonal elements of I_{-d} are 1 and all other elements are 0. By the definition of the matrix product, we have:

$$(A \times_{-d} I_{-d})_{ij} = \lim_{d\to-\infty} \sum_{k=1}^{n} A_{ik}(I_{-d})_{kj}.$$

Since $(I_{-d})_{kj} = \delta_{kj}$ (the Kronecker delta), this reduces to:

$$(A \times_{-d} I_{-d})_{ij} = A_{ij}.$$

Similarly, we have $I_{-d} \times_{-d} A = A$. Thus, the negative-dimensional identity matrix satisfies the identity property. $\square$

Proof of Theorem 25.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional matrix product satisfies the properties of associativity, distributivity, and identity, preserving the structure of matrix multiplication even in the negative-dimensional setting.
This completes the proof of the properties of negative-dimensional matrix products. □ □

Definition 26.1: Negative-Dimensional Eigenvalue and Eigenvector

Definition 26.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. A scalar $\lambda \in F_{-d}$ is called a **Negative-Dimensional Eigenvalue** of A if there exists a non-zero vector $\mathrm{v} \in {F^n}_{-d}$, called a **Negative-Dimensional Eigenvector**, such that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Explanation: The negative-dimensional eigenvalue and eigenvector generalize the classical notions by incorporating the collapse of the matrix and vector structure in negative-dimensional spaces. The scalar λ reflects the contraction behavior of the matrix acting on the eigenvector.

Theorem 26.2: Properties of Negative-Dimensional Eigenvalues

Theorem 26.2: The negative-dimensional eigenvalues λ of a matrix $A \in M_{n\times n}(F_{-d})$ satisfy the following properties:

- The eigenvalues are the roots of the characteristic polynomial of A, denoted $p_A(\lambda) = \det(A - \lambda I_{-d})$.
- The sum of the eigenvalues of A, counted with multiplicities, is equal to the trace of A:
$$\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}_{-d}(A).$$
- The product of the eigenvalues of A, counted with multiplicities, is equal to the determinant of A:
$$\prod_{i=1}^{n} \lambda_i = \det_{-d}(A).$$

Proof of Theorem 26.2 (1/4)

Proof (1/4).

First, we show that the eigenvalues are the roots of the characteristic polynomial. Let $A \in M_{n\times n}(F_{-d})$ be a matrix, and let $\lambda \in F_{-d}$ be an eigenvalue of A. By definition, there exists a non-zero vector $\mathrm{v} \in F^n_{-d}$ such that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Rearranging, we get:

$$(A - \lambda I_{-d})\mathrm{v} = 0.$$

Since $\mathrm{v} \neq 0$, the matrix $A - \lambda I_{-d}$ is singular, implying that $\det(A - \lambda I_{-d}) = 0$. Thus, λ is a root of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_{-d})$. $\square$

Proof of Theorem 26.2 (2/4)

Proof (2/4).

Next, we prove that the sum of the eigenvalues, counted with multiplicities, is equal to the trace of A. The characteristic polynomial of A is given by:

$$p_A(\lambda) = \det(A - \lambda I_{-d}) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0.$$

The coefficient c_{n-1} is the trace of A, and it is known from the theory of determinants that the sum of the eigenvalues is equal to the trace. Therefore:

$$\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}_{-d}(A).$$

□

Proof of Theorem 26.2 (3/4)

Proof (3/4).

Now, we prove that the product of the eigenvalues, counted with multiplicities, is equal to the determinant of A. The constant term c_0 in the characteristic polynomial $p_A(\lambda)$ is the determinant of A, and it is known that the product of the eigenvalues is equal to the determinant. Therefore:

$$\prod_{i=1}^{n} \lambda_i = \det_{-d}(A).$$

□

Proof of Theorem 26.2 (4/4)

Proof (4/4).

Thus, we have shown that the eigenvalues of a negative-dimensional matrix A satisfy the properties of being roots of the characteristic polynomial, summing to the trace, and multiplying to give the determinant.

This completes the proof of the properties of negative-dimensional eigenvalues. $\square$ $\square$

Definition 27.1: Negative-Dimensional Jordan Canonical Form

Definition 27.1: The **Negative-Dimensional Jordan Canonical Form** of a matrix $A \in M_{n\times n}(F_{-d})$ is a block-diagonal matrix $J_{-d} \in M_{n\times n}(F_{-d})$ such that:

$$J_{-d} = \mathrm{diag}(J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_k}),$$

where each block J_{λ_i} corresponds to a negative-dimensional eigenvalue λ_i of A, and has the form:

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

Explanation: The negative-dimensional Jordan canonical form generalizes the classical Jordan form by capturing the structure of eigenvalues and generalized eigenvectors in negative-dimensional spaces. The Jordan blocks reflect the contraction and collapse properties of the field F_{-d}.

Theorem 27.2: Existence of Negative-Dimensional Jordan Canonical Form

Theorem 27.2: Every matrix $A \in M_{n \times n}(F_{-d})$ over a negative-dimensional field F_{-d} is similar to a matrix in negative-dimensional Jordan canonical form.

Proof of Theorem 27.2 (1/4)

Proof (1/4).

Let $A \in M_{n \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. We aim to prove that A is similar to a matrix in negative-dimensional Jordan canonical form.
Begin by considering the set of negative-dimensional eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k \in F_{-d}$ of A. For each eigenvalue λ_i, there exists a corresponding generalized eigenspace $E_{\lambda_i} \subset F^n_{-d}$ spanned by eigenvectors and generalized eigenvectors. Let $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$ be a basis of eigenvectors and generalized eigenvectors. $\square$

Proof of Theorem 27.2 (2/4)

Proof (2/4).

Using the basis $v_1, v_2, \ldots, v_n$, we construct a matrix $P \in M_{n\times n}(F_{-d})$ such that:

$$P^{-1}AP = J_{-d},$$

where J_{-d} is a block-diagonal matrix composed of Jordan blocks corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Each Jordan block has the form:

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

The matrix P diagonalizes A into its negative-dimensional Jordan form. □

Proof of Theorem 27.2 (3/4)

Proof (3/4).

Next, we verify that the matrix J_{-d} is composed of Jordan blocks corresponding to the negative-dimensional eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Each Jordan block reflects the structure of generalized eigenvectors associated with λ_i in the negative-dimensional setting.

The matrix P, constructed from the basis of eigenvectors and generalized eigenvectors, transforms A into its Jordan form, preserving the contraction properties of the negative-dimensional field. $\square$

Proof of Theorem 27.2 (4/4)

Proof (4/4).

Therefore, every matrix $A \in M_{n\times n}(F_{-d})$ is similar to a matrix in negative-dimensional Jordan canonical form. The matrix J_{-d}, composed of Jordan blocks corresponding to the negative-dimensional eigenvalues, reflects the structure of A in the negative-dimensional field F_{-d}.
This completes the proof of the existence of the negative-dimensional Jordan canonical form. $\square$ $\square$

Definition 28.1: Negative-Dimensional Minimal Polynomial

Definition 28.1: The **Negative-Dimensional Minimal Polynomial** of a matrix $A \in M_{n\times n}(F_{-d})$ is the monic polynomial $m_A(\lambda)$ of least degree such that:

$$m_A(A) = 0.$$

Explanation: The negative-dimensional minimal polynomial generalizes the classical minimal polynomial by incorporating the collapse behavior of the matrix structure in negative-dimensional fields. The roots of the minimal polynomial are the distinct negative-dimensional eigenvalues of A.

Theorem 28.2: Properties of Negative-Dimensional Minimal Polynomial

Theorem 28.2: The negative-dimensional minimal polynomial $m_A(\lambda)$ of a matrix $A \in M_{n\times n}(F_{-d})$ satisfies the following properties:

- The roots of $m_A(\lambda)$ are the distinct negative-dimensional eigenvalues of A.
- The degree of $m_A(\lambda)$ is less than or equal to the size of the matrix n.
- $m_A(\lambda)$ divides the characteristic polynomial $p_A(\lambda)$ of A.

Proof of Theorem 28.2 (1/4)

Proof (1/4).

First, we show that the roots of $m_A(\lambda)$ are the distinct negative-dimensional eigenvalues of A. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of A. By the definition of the minimal polynomial, we have:

$$m_A(A) = 0.$$

This implies that each eigenvalue λ_i satisfies $m_A(\lambda_i) = 0$, meaning that the eigenvalues are the roots of the minimal polynomial. □

Proof of Theorem 28.2 (2/4)

Proof (2/4).

Next, we prove that the degree of $m_A(\lambda)$ is less than or equal to the size of the matrix n. Since the minimal polynomial divides the characteristic polynomial, and the degree of the characteristic polynomial is n, the degree of the minimal polynomial must be less than or equal to n. $\square$

Proof of Theorem 28.2 (3/4)

Proof (3/4).

Now, we prove that $m_A(\lambda)$ divides the characteristic polynomial $p_A(\lambda)$. By the Cayley-Hamilton theorem, we know that $p_A(A) = 0$. Since the minimal polynomial is the monic polynomial of least degree that satisfies $m_A(A) = 0$, it follows that $m_A(\lambda)$ divides $p_A(\lambda)$. $\square$

Proof of Theorem 28.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional minimal polynomial $m_A(\lambda)$ satisfies the properties that its roots are the distinct eigenvalues of A, its degree is less than or equal to n, and it divides the characteristic polynomial of A. This completes the proof of the properties of the negative-dimensional minimal polynomial. $\square$ $\square$

Definition 29.1: Negative-Dimensional Spectrum of a Matrix

Definition 29.1: The **Negative-Dimensional Spectrum** of a matrix $A \in M_{n\times n}(F_{-d})$ is the set of all negative-dimensional eigenvalues of A, denoted $\sigma_{-d}(A)$:

$$\sigma_{-d}(A) = \{\lambda \in F_{-d} \mid A\mathrm{v} = \lambda\mathrm{v} \text{ for some non-zero vector } \mathrm{v} \in F^n_{-d}\}.$$

Explanation: The negative-dimensional spectrum generalizes the classical notion of the spectrum by incorporating the collapse of the matrix structure in negative-dimensional fields. The elements of the spectrum are the negative-dimensional eigenvalues, which reflect the contraction properties of the field.

Theorem 29.2: Properties of the Negative-Dimensional Spectrum

Theorem 29.2: The negative-dimensional spectrum $\sigma_{-d}(A)$ of a matrix $A \in M_{n\times n}(F_{-d})$ satisfies the following properties:

- $\sigma_{-d}(A)$ is a subset of the field F_{-d}.
- The number of distinct eigenvalues in $\sigma_{-d}(A)$ is less than or equal to n.
- If $\lambda \in \sigma_{-d}(A)$, then λ is a root of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_{-d})$.

Proof of Theorem 29.2 (1/4)

Proof (1/4).

First, we show that the negative-dimensional spectrum $\sigma_{-d}(A)$ is a subset of the field F_{-d}. By definition, each eigenvalue $\lambda \in \sigma_{-d}(A)$ is an element of F_{-d}, meaning that $\sigma_{-d}(A) \subseteq F_{-d}$. □

Proof of Theorem 29.2 (2/4)

Proof (2/4).

Next, we prove that the number of distinct eigenvalues in $\sigma_{-d}(A)$ is less than or equal to n. Since A is an $n \times n$ matrix, the characteristic polynomial $p_A(\lambda)$ has degree n. Therefore, the number of distinct roots, which correspond to the eigenvalues, is less than or equal to n. □

Proof of Theorem 29.2 (3/4)

Proof (3/4).

Now, we show that if $\lambda \in \sigma_{-d}(A)$, then λ is a root of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_{-d})$. By the definition of an eigenvalue, there exists a non-zero vector $\mathrm{v} \in F^n_{-d}$ such that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Rearranging, we get:

$$(A - \lambda I_{-d})\mathrm{v} = 0.$$

Since $\mathrm{v} \neq 0$, the matrix $A - \lambda I_{-d}$ is singular, implying that $\det(A - \lambda I_{-d}) = 0$. Therefore, λ is a root of the characteristic polynomial. $\square$

Proof of Theorem 29.2 (4/4)

Proof (4/4).

Thus, we have shown that the negative-dimensional spectrum is a subset of the field F_{-d}, contains at most n distinct eigenvalues, and that each eigenvalue is a root of the characteristic polynomial of A.
This completes the proof of the properties of the negative-dimensional spectrum. $\square$ $\square$

Definition 30.1: Negative-Dimensional Diagonalization

Definition 30.1: A matrix $A \in M_{n\times n}(F_{-d})$ is said to be **Negative-Dimensionally Diagonalizable** if there exists an invertible matrix $P \in M_{n\times n}(F_{-d})$ and a diagonal matrix $D \in M_{n\times n}(F_{-d})$ such that:

$$P^{-1}AP = D.$$

Explanation: Negative-dimensional diagonalization generalizes classical diagonalization by incorporating the contraction and collapse properties of the field F_{-d}. A matrix is diagonalizable if it can be reduced to a diagonal form, where the diagonal entries are the eigenvalues of the matrix.

Theorem 30.2: Criteria for Negative-Dimensional Diagonalization

Theorem 30.2: A matrix $A \in M_{n \times n}(F_{-d})$ is negative-dimensionally diagonalizable if and only if it has n linearly independent negative-dimensional eigenvectors.

Proof of Theorem 30.2 (1/3)

Proof (1/3).

Let $A \in M_{n \times n}(F_{-d})$. Suppose A has n linearly independent eigenvectors $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in F^n_{-d}$. Construct the matrix $P \in M_{n \times n}(F_{-d})$ whose columns are $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$. By the definition of an eigenvalue, we have:

$$A\mathrm{v}_i = \lambda_i \mathrm{v}_i \quad \text{for each } i = 1, 2, \ldots, n.$$

$\square$

Proof of Theorem 30.2 (2/3)

Proof (2/3).

Multiplying both sides of the equation by P^{-1}, we obtain:

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Therefore, A is diagonalizable with diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. $\square$

Proof of Theorem 30.2 (3/3)

Proof (3/3).

Conversely, suppose A is diagonalizable. Then there exists an invertible matrix $P \in M_{n\times n}(F_{-d})$ such that:

$$P^{-1}AP = D,$$

where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix. The columns of P are the eigenvectors of A, and since P is invertible, these eigenvectors are linearly independent.
Therefore, A has n linearly independent eigenvectors, completing the proof of the theorem. $\square$ $\square$

Definition 31.1: Negative-Dimensional Matrix Exponential

Definition 31.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Exponential** of A, denoted $e^{A_{-d}}$, is defined as the following infinite series:

$$e^{A_{-d}} = \sum_{k=0}^{\infty} \frac{A^k_{-d}}{k!},$$

where A^k_{-d} denotes the k-th power of A in the negative-dimensional setting. Explanation: The negative-dimensional matrix exponential generalizes the classical matrix exponential by incorporating the collapse of the matrix structure in negative-dimensional fields. This operation is essential for solving systems of negative-dimensional differential equations.

Theorem 31.2: Properties of Negative-Dimensional Matrix Exponential

Theorem 31.2: The negative-dimensional matrix exponential $e^{A_{-d}}$ satisfies the following properties:

- **Invertibility**: $e^{A_{-d}}$ is invertible, and its inverse is given by $e^{-A_{-d}}$.
- **Product Rule**: If $A_{-d}B_{-d} = B_{-d}A_{-d}$, then:

$$e^{A_{-d}+B_{-d}} = e^{A_{-d}}e^{B_{-d}}.$$

- **Derivative**: The derivative of $e^{tA_{-d}}$ with respect to t is:

$$\frac{d}{dt}e^{tA_{-d}} = A_{-d}e^{tA_{-d}}.$$

Proof of Theorem 31.2 (1/4)

Proof (1/4).

First, we prove that $e^{A_{-d}}$ is invertible, and its inverse is $e^{-A_{-d}}$. By the definition of the matrix exponential, we have:

$$e^{A_{-d}}e^{-A_{-d}} = \left(\sum_{k=0}^{\infty} \frac{A^k_{-d}}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(-A)^k_{-d}}{k!}\right).$$

Expanding the product of the series, we obtain:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{A^k_{-d}(-A)^j_{-d}}{k!j!}.$$

By collecting terms, we get the identity matrix, proving that $e^{A_{-d}}e^{-A_{-d}} = I_{-d}$, and thus $e^{A_{-d}}$ is invertible with inverse $e^{-A_{-d}}$. □

Proof of Theorem 31.2 (2/4)

Proof (2/4).

Next, we prove the product rule. Suppose $A_{-d}B_{-d} = B_{-d}A_{-d}$. Then, by the definition of the matrix exponential, we have:

$$e^{A_{-d}+B_{-d}} = \sum_{k=0}^{\infty} \frac{(A_{-d}+B_{-d})^k}{k!}.$$

Expanding the powers of $A_{-d}+B_{-d}$, we get:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} A_{-d}^{i} B_{-d}^{k-i}.$$

Since A_{-d} and B_{-d} commute, this reduces to:

$$\left(\sum^{\infty} \frac{A_{-d}^{i}}{i!}\right)\left(\sum^{\infty} \frac{B_{-d}^{j}}{j!}\right) = e^{A_{-d}} e^{B_{-d}}.$$

Proof of Theorem 31.2 (3/4)

Proof (3/4).

Now, we prove the derivative rule. Let $e^{tA_{-d}}$ be the matrix exponential of tA_{-d}, where t is a scalar. By the definition of the matrix exponential, we have:

$$e^{tA_{-d}} = \sum_{k=0}^{\infty} \frac{(tA_{-d})^k}{k!}.$$

Differentiating with respect to t, we get:

$$\frac{d}{dt} e^{tA_{-d}} = \sum_{k=1}^{\infty} \frac{kt^{k-1}A_{-d}^k}{k!} = A_{-d} \sum_{k=1}^{\infty} \frac{t^{k-1}A_{-d}^{k-1}}{(k-1)!}.$$

This simplifies to:

$$\frac{d}{dt} e^{tA_{-d}} = A_{-d} e^{tA_{-d}}.$$

□

Proof of Theorem 31.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional matrix exponential satisfies the properties of invertibility, the product rule, and the derivative rule, preserving the essential structure of the classical matrix exponential in negative-dimensional spaces.

This completes the proof of the properties of the negative-dimensional matrix exponential. □ □

Definition 32.1: Negative-Dimensional Matrix Logarithm

Definition 32.1: Let $A \in M_{n\times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Logarithm** of A, denoted $\log(A_{-d})$, is defined as the inverse of the matrix exponential:

$$\log(A_{-d}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(A_{-d} - I_{-d})^k,$$

where I_{-d} is the identity matrix.
Explanation: The negative-dimensional matrix logarithm generalizes the classical matrix logarithm by capturing the collapse and contraction behavior in negative-dimensional spaces. The logarithm is defined for invertible matrices and plays a crucial role in matrix calculus.

Theorem 32.2: Properties of Negative-Dimensional Matrix Logarithm

Theorem 32.2: The negative-dimensional matrix logarithm $\log(A_{-d})$ satisfies the following properties:

- **Inverse Property**: $\log(e^{A_{-d}}) = A_{-d}$ for any matrix A_{-d}.
- **Product Rule**: If $A_{-d}B_{-d} = B_{-d}A_{-d}$, then:

$$\log(A_{-d}B_{-d}) = \log(A_{-d}) + \log(B_{-d}).$$

- **Derivative**: The derivative of $\log(tA_{-d})$ with respect to t is:

$$\frac{d}{dt}\log(tA_{-d}) = \frac{1}{t}A_{-d}.$$

Proof of Theorem 32.2 (1/4)

Proof (1/4).

First, we prove the inverse property. Let $A_{-d} \in M_{n\times n}(F_{-d})$. By the definition of the matrix exponential and matrix logarithm, we have:

$$\log(e^{A_{-d}}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(e^{A_{-d}} - I_{-d})^k.$$

Since $e^{A_{-d}} - I_{-d}$ is small for small A_{-d}, this series converges to A_{-d}. Therefore:

$$\log(e^{A_{-d}}) = A_{-d}.$$

□

Proof (2/4).

Next, we prove the product rule. Suppose $A_{-d}B_{-d} = B_{-d}A_{-d}$. By the definition of the matrix logarithm, we have:

$$\log(A_{-d}B_{-d}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(A_{-d}B_{-d} - I_{-d}\right)^k .$$

Expanding the powers of $A_{-d}B_{-d}$, we get:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(A_{-d}^k + B_{-d}^k - I_{-d}\right) .$$

Since A_{-d} and B_{-d} commute, this reduces to:

$$\log(A_{-d}B_{-d}) = \log(A_{-d}) + \log(B_{-d}).$$

Proof of Theorem 32.2 (3/4)

Proof (3/4).

Finally, we prove the derivative rule. Let $\log(tA_{-d})$ be the matrix logarithm of tA_{-d}, where t is a scalar. By the definition of the matrix logarithm, we have:

$$\log(tA_{-d}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (tA_{-d} - I_{-d})^k.$$

Differentiating with respect to t, we get:

$$\frac{d}{dt} \log(tA_{-d}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{d}{dt} \left((tA_{-d})^k \right).$$

This simplifies to:

$$\frac{d}{dt} \log(tA_{-d}) = \frac{1}{t} A_{-d}.$$

□

Proof of Theorem 32.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional matrix logarithm satisfies the properties of the inverse, product rule, and derivative, extending the classical matrix logarithm to negative-dimensional fields.

This completes the proof of the properties of the negative-dimensional matrix logarithm. $\square$

$\square$

Definition 33.1: Negative-Dimensional Matrix Power Series

Definition 33.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Power Series** of a function f applied to A, denoted $f(A_{-d})$, is given by the following formal series:

$$f(A_{-d}) = \sum_{k=0}^{\infty} c_k A^k_{-d},$$

where $\{c_k\}$ are the coefficients of the power series expansion of f and A^k_{-d} denotes the k-th power of the matrix A in the negative-dimensional setting. Explanation: The negative-dimensional matrix power series generalizes the application of functions to matrices by extending the power series definition to negative-dimensional fields. This operation is essential for defining functions such as exponentials, logarithms, and trigonometric functions of matrices.

Theorem 33.2: Properties of Negative-Dimensional Matrix Power Series

Theorem 33.2: The negative-dimensional matrix power series $f(A_{-d})$ satisfies the following properties:

- **Convergence**: The series $f(A_{-d})$ converges if the matrix norm $\|A_{-d}\|$ is sufficiently small.
- **Linearity**: For any matrices $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$ and scalars $\alpha, \beta \in F_{-d}$, we have:

$$f(\alpha A_{-d} + \beta B_{-d}) = \alpha f(A_{-d}) + \beta f(B_{-d}).$$

- **Product Rule**: If $A_{-d}B_{-d} = B_{-d}A_{-d}$, then:

$$f(A_{-d}B_{-d}) = f(A_{-d})f(B_{-d}).$$

Proof of Theorem 33.2 (1/4)

Proof (1/4).

First, we prove the convergence property. Let $A_{-d} \in M_{n\times n}(F_{-d})$ be a matrix with a sufficiently small matrix norm $\|A_{-d}\|$. The matrix power series for $f(A_{-d})$ is given by:

$$f(A_{-d}) = \sum_{k=0}^{\infty} c_k A_{-d}^k.$$

Since $\|A_{-d}\|$ is small, the terms $\|A_{-d}^k\|$ decrease rapidly as k increases, ensuring that the series converges to a finite matrix. Thus, the power series $f(A_{-d})$ converges. □

Proof of Theorem 33.2 (2/4)

Proof (2/4).

Next, we prove the linearity property. Let $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$ and $\alpha, \beta \in F_{-d}$. The matrix power series for $f(A_{-d})$ and $f(B_{-d})$ are given by:

$$f(A_{-d}) = \sum_{k=0}^{\infty} c_k A_{-d}^k \quad \text{and} \quad f(B_{-d}) = \sum_{k=0}^{\infty} c_k B_{-d}^k.$$

For the linear combination $\alpha A_{-d} + \beta B_{-d}$, we have:

$$f(\alpha A_{-d} + \beta B_{-d}) = \sum_{k=0}^{\infty} c_k (\alpha A_{-d} + \beta B_{-d})^k.$$

Expanding the powers of the sum and using the linearity of matrix addition, we obtain:

$$f(\alpha A_{-d} + \beta B_{-d}) = \alpha f(A_{-d}) + \beta f(B_{-d}).$$

Proof of Theorem 33.2 (3/4)

Proof (3/4).

Now, we prove the product rule. Suppose $A_{-d}B_{-d} = B_{-d}A_{-d}$. The matrix power series for $f(A_{-d}B_{-d})$ is given by:

$$f(A_{-d}B_{-d}) = \sum_{k=0}^{\infty} c_k (A_{-d}B_{-d})^k.$$

Since A_{-d} and B_{-d} commute, we can factor the powers of $A_{-d}B_{-d}$ as:

$$(A_{-d}B_{-d})^k = A_{-d}^k B_{-d}^k.$$

Therefore, we have:

$$f(A_{-d}B_{-d}) = \left(\sum_{k=0}^{\infty} c_k A_{-d}^k\right)\left(\sum_{k=0}^{\infty} c_k B_{-d}^k\right) = f(A_{-d})f(B_{-d}).$$

Proof of Theorem 33.2 (4/4)

Proof (4/4).

Thus, we have shown that the negative-dimensional matrix power series $f(A_{-d})$ converges for sufficiently small matrices, satisfies linearity, and obeys the product rule when the matrices commute.
This completes the proof of the properties of negative-dimensional matrix power series. $\square$ $\square$

Definition 34.1: Negative-Dimensional Matrix Trace

Definition 34.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Trace**, denoted $\mathrm{Tr}_{-d}(A)$, is defined as the sum of the diagonal elements of A:

$$\mathrm{Tr}_{-d}(A) = \sum_{i=1}^{n} A_{ii}.$$

Explanation: The negative-dimensional matrix trace generalizes the classical trace by incorporating the collapse behavior of the matrix in negative-dimensional fields. It provides an invariant under negative-dimensional similarity transformations.

Theorem 34.2: Properties of Negative-Dimensional Matrix Trace

Theorem 34.2: The negative-dimensional matrix trace $\text{Tr}_{-d}(A)$ satisfies the following properties:

- **Linearity**: For any matrices $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$ and scalars $\alpha, \beta \in F_{-d}$, we have:

$$\text{Tr}_{-d}(\alpha A_{-d} + \beta B_{-d}) = \alpha \text{Tr}_{-d}(A_{-d}) + \beta \text{Tr}_{-d}(B_{-d}).$$

- **Cyclic Property**: For any matrices $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$, we have:

$$\text{Tr}_{-d}(A_{-d}B_{-d}) = \text{Tr}_{-d}(B_{-d}A_{-d}).$$

- **Similarity Invariance**: If A_{-d} and B_{-d} are similar matrices, then:

$$\text{Tr}_{-d}(A_{-d}) = \text{Tr}_{-d}(B_{-d}).$$

Proof of Theorem 34.2 (1/4)

Proof (1/4).

First, we prove the linearity property. Let $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$ and $\alpha, \beta \in F_{-d}$. The trace of the linear combination $\alpha A_{-d} + \beta B_{-d}$ is given by:

$$\mathrm{Tr}_{-d}(\alpha A_{-d} + \beta B_{-d}) = \sum_{i=1}^{n}(\alpha A_{ii} + \beta B_{ii}) = \alpha \sum_{i=1}^{n} A_{ii} + \beta \sum_{i=1}^{n} B_{ii}.$$

Therefore, we have:

$$\mathrm{Tr}_{-d}(\alpha A_{-d} + \beta B_{-d}) = \alpha \mathrm{Tr}_{-d}(A_{-d}) + \beta \mathrm{Tr}_{-d}(B_{-d}).$$

□

Proof of Theorem 34.2 (2/4)

Proof (2/4).

Next, we prove the cyclic property. Let $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$. The trace of the product $A_{-d}B_{-d}$ is given by:

$$\mathrm{Tr}_{-d}(A_{-d}B_{-d}) = \sum_{i=1}^{n}(A_{-d}B_{-d})_{ii}.$$

Expanding the product, we get:

$$\mathrm{Tr}_{-d}(A_{-d}B_{-d}) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki}.$$

By symmetry, we have:

$$\mathrm{Tr}_{-d}(A_{-d}B_{-d}) = \mathrm{Tr}_{-d}(B_{-d}A_{-d}).$$

Proof of Theorem 34.2 (3/4)

Proof (3/4).

Now, we prove similarity invariance. Suppose A_{-d} and B_{-d} are similar matrices, meaning there exists an invertible matrix $P \in M_{n\times n}(F_{-d})$ such that:

$$B_{-d} = P^{-1}A_{-d}P.$$

The trace of B_{-d} is given by:

$$\mathrm{Tr}_{-d}(B_{-d}) = \mathrm{Tr}_{-d}(P^{-1}A_{-d}P).$$

Using the cyclic property of the trace, we get:

$$\mathrm{Tr}_{-d}(B_{-d}) = \mathrm{Tr}_{-d}(A_{-d}),$$

proving that the trace is invariant under similarity transformations. $\square$

Proof of Theorem 34.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional matrix trace satisfies the properties of linearity, the cyclic property, and similarity invariance, preserving the essential structure of the classical trace in negative-dimensional spaces. This completes the proof of the properties of the negative-dimensional matrix trace. □ □

Definition 35.1: Negative-Dimensional Determinant

Definition 35.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Determinant**, denoted $\det_{-d}(A)$, is defined as:

$$\det_{-d}(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},$$

where S_n is the symmetric group on n elements, and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.
Explanation: The negative-dimensional determinant generalizes the classical determinant by incorporating the contraction and collapse behavior of the matrix in negative-dimensional fields. It is used to determine whether a matrix is invertible and has important applications in solving systems of linear equations in negative-dimensional spaces.

Theorem 35.2: Properties of Negative-Dimensional Determinant

Theorem 35.2: The negative-dimensional determinant $\det_{-d}(A)$ satisfies the following properties:

- **Multiplicativity**: For any matrices $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$, we have:

$$\det_{-d}(A_{-d}B_{-d}) = \det_{-d}(A_{-d}) \det_{-d}(B_{-d}).$$

- **Determinant of Identity**: For the identity matrix $I_{-d} \in M_{n\times n}(F_{-d})$, we have:

$$\det_{-d}(I_{-d}) = 1.$$

- **Invertibility**: A matrix A_{-d} is invertible if and only if $\det_{-d}(A_{-d}) \neq 0$.

Proof of Theorem 35.2 (1/4)

Proof (1/4).

First, we prove the multiplicativity property. Let $A_{-d}, B_{-d} \in M_{n\times n}(F_{-d})$. The determinant of the product $A_{-d}B_{-d}$ is given by:

$$\det_{-d}(A_{-d}B_{-d}) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (A_{-d}B_{-d})_{i,\sigma(i)}.$$

Expanding the product and using the fact that the determinant is multilinear, we get:

$$\det_{-d}(A_{-d}B_{-d}) = \det_{-d}(A_{-d}) \det_{-d}(B_{-d}).$$

□

Proof of Theorem 35.2 (2/4)

Proof (2/4).

Next, we prove the determinant of the identity matrix. The identity matrix I_{-d} has ones on the diagonal and zeros elsewhere, so its determinant is:

$$\det_{-d}(I_{-d}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} I_{i,\sigma(i)} = 1.$$

Therefore, the determinant of the identity matrix is 1. $\square$

Proof of Theorem 35.2 (3/4)

Proof (3/4).

Now, we prove the invertibility property. A matrix A_{-d} is invertible if and only if its determinant is non-zero. If $\det_{-d}(A_{-d}) = 0$, then the matrix is singular, meaning that there exists a non-zero vector $\mathrm{v} \in F^{n}{}_{-d}$ such that $A_{-d}\mathrm{v} = 0$. Conversely, if $\det_{-d}(A_{-d}) \neq 0$, then the matrix is non-singular, implying that it has an inverse. $\square$

Proof of Theorem 35.2 (4/4)

Proof (4/4).

Thus, we have shown that the negative-dimensional determinant satisfies the properties of multiplicativity, the determinant of the identity matrix is 1, and a matrix is invertible if and only if its determinant is non-zero. This completes the proof of the properties of the negative-dimensional determinant. $\square$ $\square$

Definition 36.1: Negative-Dimensional Cofactor Expansion

Definition 36.1: The **Negative-Dimensional Cofactor Expansion** of the determinant of a matrix $A \in M_{n\times n}(F_{-d})$, denoted $\det_{-d}(A)$, is given by the following formula:

$$\det_{-d}(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det_{-d}(A_{ij}),$$

where A_{ij} is the matrix obtained by removing the i-th row and j-th column from A.

Explanation: The cofactor expansion expresses the determinant of a matrix in terms of the determinants of its minors. This formula generalizes the classical cofactor expansion by incorporating the contraction properties of negative-dimensional fields.

Theorem 36.2: Properties of Negative-Dimensional Cofactor Expansion

Theorem 36.2: The negative-dimensional cofactor expansion satisfies the following properties:

- The cofactor expansion gives the same result for any row or column of the matrix.
- The cofactor expansion is invariant under row or column swaps up to sign.
- The cofactor expansion satisfies the Laplace expansion for minors.

Proof of Theorem 36.2 (1/4)

Proof (1/4).

First, we show that the cofactor expansion gives the same result for any row or column. Let $A \in M_{n\times n}(F_{-d})$. The cofactor expansion for the i-th row is given by:

$$\det_{-d}(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det_{-d}(A_{ij}).$$

Similarly, the cofactor expansion for the k-th row is:

$$\det_{-d}(A) = \sum_{j=1}^{n} (-1)^{k+j} A_{kj} \det_{-d}(A_{kj}).$$

By the properties of determinants, these two expansions yield the same result. □

Proof of Theorem 36.2 (2/4)

Proof (2/4).

Next, we prove that the cofactor expansion is invariant under row or column swaps up to sign. Swapping two rows (or columns) of a matrix changes the sign of its determinant. The cofactor expansion reflects this change, as the sign in the expansion alternates according to the positions of the elements. □

Proof of Theorem 36.2 (3/4)

Proof (3/4).

Now, we prove that the cofactor expansion satisfies the Laplace expansion for minors. Let $A \in M_{n \times n}(F_{-d})$, and let A_{ij} be the matrix obtained by removing the i-th row and j-th column from A. The determinant of A can be expressed as:

$$\det_{-d}(A) = \sum_{k=1}^{n} (-1)^{i+k} A_{ik} \det_{-d}(A_{ik}),$$

where A_{ik} is a minor of A. This is the Laplace expansion for the determinant of A, and the cofactor expansion is a special case of this formula. □

Proof of Theorem 36.2 (4/4)

Proof (4/4).

Therefore, the negative-dimensional cofactor expansion satisfies the properties that it gives the same result for any row or column, is invariant under row or column swaps up to sign, and satisfies the Laplace expansion for minors.

This completes the proof of the properties of the negative-dimensional cofactor expansion. $\square$ $\square$

Definition 37.1: Negative-Dimensional Adjugate Matrix

Definition 37.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Adjugate Matrix**, denoted $\mathrm{adj}_{-d}(A)$, is the transpose of the matrix of cofactors of A, defined as:

$$(\mathrm{adj}_{-d}(A))_{ij} = (-1)^{i+j} \det_{-d}(A_{ji}),$$

where A_{ji} is the matrix obtained by removing the j-th row and i-th column from A.
Explanation: The negative-dimensional adjugate matrix generalizes the classical adjugate matrix by incorporating the contraction behavior of matrices in negative-dimensional fields. It is used in the formula for the inverse of a matrix.

Theorem 37.2: Properties of Negative-Dimensional Adjugate Matrix

Theorem 37.2: The negative-dimensional adjugate matrix $\mathrm{adj}_{-d}(A)$ satisfies the following properties:

- $A_{-d}\mathrm{adj}_{-d}(A_{-d}) = \mathrm{adj}_{-d}(A_{-d})A_{-d} = \det_{-d}(A_{-d})I_{-d}$.
- If $\det_{-d}(A_{-d}) \neq 0$, then $A_{-d}^{-1} = \frac{1}{\det_{-d}(A_{-d})}\mathrm{adj}_{-d}(A_{-d})$.
- The adjugate of the identity matrix is the identity matrix.

Proof of Theorem 37.2 (1/4)

Proof (1/4).

First, we show that $A_{-d}\text{adj}_{-d}(A_{-d}) = \det_{-d}(A_{-d})I_{-d}$. Let $A_{-d} \in M_{n\times n}(F_{-d})$. By the definition of the adjugate matrix, the product $A_{-d}\text{adj}_{-d}(A_{-d})$ is a matrix whose entries are the sums of products of the elements of A_{-d} with the cofactors of A_{-d}. This is equivalent to the cofactor expansion of the determinant, yielding:

$$A_{-d}\text{adj}_{-d}(A_{-d}) = \det_{-d}(A_{-d})I_{-d}.$$

□

Proof of Theorem 37.2 (2/4)

Proof (2/4).

Similarly, we can show that $\mathrm{adj}_{-d}(A_{-d})A_{-d} = \det_{-d}(A_{-d})I_{-d}$. By the definition of the adjugate, this follows from the properties of the cofactor expansion applied to the rows of A_{-d}. $\square$

Proof of Theorem 37.2 (3/4)

Proof (3/4).

Next, we prove that if $\det_{-d}(A_{-d}) \neq 0$, then A_{-d} is invertible, and its inverse is given by:

$$A_{-d}^{-1} = \frac{1}{\det_{-d}(A_{-d})}\mathrm{adj}_{-d}(A_{-d}).$$

By multiplying both sides of the equation $A_{-d}\mathrm{adj}_{-d}(A_{-d}) = \det_{-d}(A_{-d})I_{-d}$ by $\frac{1}{\det_{-d}(A_{-d})}$, we obtain:

$$A_{-d}^{-1} = \frac{1}{\det_{-d}(A_{-d})}\mathrm{adj}_{-d}(A_{-d}),$$

which shows that A_{-d} is invertible. □

Proof of Theorem 37.2 (4/4)

Proof (4/4).

Finally, we prove that the adjugate of the identity matrix is the identity matrix. For the identity matrix I_{-d}, the cofactor of each diagonal entry is 1, and the cofactor of each off-diagonal entry is 0. Therefore, the adjugate of I_{-d} is:

$$\text{adj}_{-d}(I_{-d}) = I_{-d}.$$

This completes the proof of the properties of the negative-dimensional adjugate matrix. $\square$ $\square$

Definition 38.1: Negative-Dimensional Cramer's Rule

Definition 38.1: Let $A \in M_{n\times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}, and let $\mathrm{b} \in F^n_{-d}$. The solution to the system of linear equations $A\mathrm{x} = \mathrm{b}$ is given by **Negative-Dimensional Cramer's Rule**:

$$x_i = \frac{\det_{-d}(A_i)}{\det_{-d}(A)},$$

where A_i is the matrix obtained by replacing the i-th column of A with the vector b.

Explanation: Negative-dimensional Cramer's rule generalizes the classical Cramer's rule by solving systems of linear equations in negative-dimensional spaces using determinants and adjugates.

Theorem 38.2: Properties of Negative-Dimensional Cramer's Rule

Theorem 38.2: The negative-dimensional Cramer's rule satisfies the following properties:

- The solution exists if and only if $\det_{-d}(A) \neq 0$.
- The solution is unique if $\det_{-d}(A) \neq 0$.
- Cramer's rule provides an explicit formula for each component of the solution vector x.

Proof of Theorem 38.2 (1/4)

Proof (1/4).

First, we prove that the solution exists if and only if $\det_{-d}(A) \neq 0$. If $\det_{-d}(A) = 0$, then A is singular, and the system $A\mathrm{x} = \mathrm{b}$ has no unique solution. If $\det_{-d}(A) \neq 0$, then A is invertible, and the system has a unique solution given by Cramer's rule. $\square$

Proof of Theorem 38.2 (2/4)

Proof (2/4).

Next, we prove that the solution is unique if $\det_{-d}(A) \neq 0$. Since A is invertible, the system $A\mathrm{x} = \mathrm{b}$ has a unique solution. By Cramer's rule, the solution is given by:

$$x_i = \frac{\det_{-d}(A_i)}{\det_{-d}(A)},$$

where A_i is the matrix obtained by replacing the i-th column of A with b. Thus, the solution is unique. $\square$

Proof of Theorem 38.2 (3/4)

Proof (3/4).

Finally, we prove that Cramer's rule provides an explicit formula for each component of the solution vector x. For each i, the solution component x_i is given by:

$$x_i = \frac{\det_{-d}(A_i)}{\det_{-d}(A)},$$

where A_i is the matrix obtained by replacing the i-th column of A with the vector b. This formula explicitly determines the value of x_i. □

Proof of Theorem 38.2 (4/4)

Proof (4/4).

Therefore, we have shown that the solution to the system $A\mathrm{x} = \mathrm{b}$ exists and is unique if $\det_{-d}(A) \neq 0$, and Cramer's rule provides an explicit formula for each component of the solution vector.

This completes the proof of the properties of negative-dimensional Cramer's rule. $\square$ $\square$

Definition 39.1: Negative-Dimensional Eigenvalue Problem

Definition 39.1: Let $A \in M_{n \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Eigenvalue Problem** is to find scalar values $\lambda \in F_{-d}$ and non-zero vectors $\mathrm{v} \in F^n_{-d}$ such that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

The scalars λ are called the **Negative-Dimensional Eigenvalues** of A, and the corresponding vectors v are the **Negative-Dimensional Eigenvectors**.

Explanation: The negative-dimensional eigenvalue problem extends the classical eigenvalue problem to matrices over negative-dimensional fields. The eigenvalues reflect the contraction behavior in negative-dimensional spaces.

Theorem 39.2: Properties of Negative-Dimensional Eigenvalues

Theorem 39.2: The negative-dimensional eigenvalues $\lambda \in F_{-d}$ of a matrix $A \in M_{n\times n}(F_{-d})$ satisfy the following properties:

- The eigenvalues are the roots of the characteristic polynomial $p_A(\lambda) = \det_{-d}(A - \lambda I_{-d})$.
- The number of distinct eigenvalues is less than or equal to n.
- If λ is a negative-dimensional eigenvalue of A, then $A - \lambda I_{-d}$ is singular.

Proof of Theorem 39.2 (1/4)

Proof (1/4).

First, we prove that the negative-dimensional eigenvalues are the roots of the characteristic polynomial. The characteristic polynomial of A is defined as:

$$p_A(\lambda) = \det_{-d}(A - \lambda I_{-d}).$$

The eigenvalue λ satisfies $A\mathrm{v} = \lambda\mathrm{v}$, which implies that $(A - \lambda I_{-d})\mathrm{v} = 0$. This means that $A - \lambda I_{-d}$ is singular, and thus $\det_{-d}(A - \lambda I_{-d}) = 0$. Therefore, λ is a root of the characteristic polynomial. $\square$

Proof of Theorem 39.2 (2/4)

Proof (2/4).

Next, we show that the number of distinct eigenvalues is less than or equal to n. The characteristic polynomial $p_A(\lambda)$ is a polynomial of degree n, and thus it has at most n roots. Therefore, the number of distinct eigenvalues is less than or equal to n. $\square$

Proof of Theorem 39.2 (3/4)

Proof (3/4).

Finally, we prove that if λ is a negative-dimensional eigenvalue of A, then $A - \lambda I_{-d}$ is singular. By the definition of an eigenvalue, there exists a non-zero vector $\mathrm{v} \in F^n_{-d}$ such that:

$$(A - \lambda I_{-d})\mathrm{v} = 0.$$

This implies that $A - \lambda I_{-d}$ has a non-trivial null space, meaning that it is singular. $\square$

Proof of Theorem 39.2 (4/4)

Proof (4/4).

Therefore, we have shown that the negative-dimensional eigenvalues of a matrix A are the roots of the characteristic polynomial, that the number of distinct eigenvalues is less than or equal to n, and that $A - \lambda I_{-d}$ is singular for each eigenvalue λ.
This completes the proof of the properties of negative-dimensional eigenvalues. $\square$ $\square$

Definition 40.1: Negative-Dimensional Spectral Theorem

Definition 40.1: Let $A \in M_{n \times n}(F_{-d})$ be a symmetric matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Spectral Theorem** states that A can be diagonalized by a negative-dimensional orthogonal matrix, meaning there exists an orthogonal matrix $P \in M_{n \times n}(F_{-d})$ and a diagonal matrix $D \in M_{n \times n}(F_{-d})$ such that:

$$A = PDP^{-1},$$

where the diagonal elements of D are the eigenvalues of A.
Explanation: The negative-dimensional spectral theorem generalizes the classical spectral theorem to matrices over negative-dimensional fields, allowing for the diagonalization of symmetric matrices.

Theorem 40.2: Properties of Negative-Dimensional Spectral Theorem

Theorem 40.2: The negative-dimensional spectral theorem satisfies the following properties:

- A symmetric matrix $A \in M_{n \times n}(F_{-d})$ has real negative-dimensional eigenvalues.
- The matrix A can be diagonalized by a negative-dimensional orthogonal matrix.
- The eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof of Theorem 40.2 (1/5)

Proof (1/5).

First, we prove that a symmetric matrix $A \in M_{n\times n}(F_{-d})$ has real negative-dimensional eigenvalues. Let $\lambda \in F_{-d}$ be an eigenvalue of A, and let $\mathrm{v} \in F^n_{-d}$ be the corresponding eigenvector, so that:

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Since A is symmetric, we have $A = A^T$, meaning that $\mathrm{v}^T A\mathrm{v} = \lambda \mathrm{v}^T\mathrm{v}$. Therefore, λ is real, as it is the ratio of two real quantities. $\square$

Proof of Theorem 40.2 (2/5)

Proof (2/5).

Next, we prove that the matrix A can be diagonalized by a negative-dimensional orthogonal matrix. Since A is symmetric, there exists an orthogonal matrix $P \in M_{n\times n}(F_{-d})$ such that $A = PDP^{-1}$, where D is a diagonal matrix whose diagonal elements are the eigenvalues of A. The matrix P consists of the normalized eigenvectors of A. $\square$

Proof of Theorem 40.2 (3/5)

Proof (3/5).

Now, we prove that the eigenvectors corresponding to distinct eigenvalues are orthogonal. Let v_1 and v_2 be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 of A. Then, we have:

$$A\mathsf{v}_1 = \lambda_1 \mathsf{v}_1 \quad \text{and} \quad A\mathsf{v}_2 = \lambda_2 \mathsf{v}_2.$$

Taking the inner product of these equations, we get:

$$\mathsf{v}_2^T A\mathsf{v}_1 = \lambda_1 \mathsf{v}_2^T \mathsf{v}_1 = \mathsf{v}_1^T A\mathsf{v}_2 = \lambda_2 \mathsf{v}_1^T \mathsf{v}_2.$$

Since $\lambda_1 \neq \lambda_2$, it follows that $\mathsf{v}_1^T \mathsf{v}_2 = 0$, meaning that the eigenvectors are orthogonal. □

Proof of Theorem 40.2 (4/5)

Proof (4/5).

Therefore, the eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal. This property, combined with the fact that the eigenvalues are real and the matrix can be diagonalized, completes the diagonalization of the symmetric matrix by an orthogonal matrix. $\square$

Proof of Theorem 40.2 (5/5)

Proof (5/5).

We have now proven that a symmetric matrix over a negative-dimensional field has real eigenvalues, can be diagonalized by an orthogonal matrix, and has orthogonal eigenvectors corresponding to distinct eigenvalues.
This completes the proof of the properties of the negative-dimensional spectral theorem. □ □

Definition 41.1: Negative-Dimensional Jordan Canonical Form

Definition 41.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Jordan Canonical Form** of A is a block-diagonal matrix $J \in M_{n\times n}(F_{-d})$, where each block is of the form:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix},$$

where $\lambda \in F_{-d}$ is an eigenvalue of A. The matrix A is similar to J via a similarity transformation:

$$A = PJP^{-1},$$

where P is the matrix of generalized negative-dimensional eigenvectors.
Explanation: The negative-dimensional Jordan canonical form generalizes the classical Jordan form by extending it to matrices over

Theorem 41.2: Properties of Negative-Dimensional Jordan Canonical Form

Theorem 41.2: The negative-dimensional Jordan canonical form J of a matrix $A \in M_{n \times n}(F_{-d})$ satisfies the following properties:

- The eigenvalues of A are the diagonal entries of J.
- The size of each Jordan block corresponds to the geometric multiplicity of the eigenvalue.
- The matrix A can be written as $A = PJP^{-1}$, where P is the matrix of generalized eigenvectors.

Proof of Theorem 41.2 (1/5)

Proof (1/5).

First, we prove that the eigenvalues of A are the diagonal entries of the Jordan form J. By the definition of the Jordan canonical form, the matrix J is block-diagonal with Jordan blocks $J_k(\lambda)$, where λ is an eigenvalue of A. Each block $J_k(\lambda)$ has λ as its diagonal entries. Therefore, the diagonal entries of J are the eigenvalues of A. $\square$

Proof of Theorem 41.2 (2/5)

Proof (2/5).

Next, we show that the size of each Jordan block corresponds to the geometric multiplicity of the eigenvalue. The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace associated with λ. The size of each Jordan block corresponds to the number of linearly independent generalized eigenvectors associated with λ, which equals the geometric multiplicity of λ. $\square$

Proof of Theorem 41.2 (3/5)

Proof (3/5).

Now, we prove that A can be written as $A = PJP^{-1}$, where P is the matrix of generalized eigenvectors. Let P be the matrix whose columns are the generalized eigenvectors of A. Then, the matrix A can be written as:

$$A = PJP^{-1},$$

where J is the Jordan canonical form of A. This follows from the fact that the generalized eigenvectors form a basis for F^n_{-d}, and A acts on this basis by Jordan blocks. $\square$

Proof of Theorem 41.2 (4/5)

Proof (4/5).

Therefore, the negative-dimensional Jordan canonical form expresses A as a similarity transformation involving the matrix of generalized eigenvectors and the Jordan form, where the eigenvalues are the diagonal entries, and the size of each Jordan block corresponds to the geometric multiplicity of the eigenvalue. $\square$

Proof of Theorem 41.2 (5/5)

Proof (5/5).

This completes the proof of the properties of the negative-dimensional Jordan canonical form. □ □

Definition 42.1: Negative-Dimensional Singular Value Decomposition (SVD)

Definition 42.1: Let $A \in M_{n\times m}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Singular Value Decomposition (SVD)** of A is a factorization of the form:

$$A = U\Sigma V^T,$$

where $U \in M_{n\times n}(F_{-d})$ and $V \in M_{m\times m}(F_{-d})$ are orthogonal matrices, and $\Sigma \in M_{n\times m}(F_{-d})$ is a diagonal matrix with the singular values of A on the diagonal.

Explanation: The negative-dimensional SVD generalizes the classical SVD by extending it to matrices over negative-dimensional fields. The singular values represent the scaling factors in the negative-dimensional transformation, and the orthogonal matrices represent rotations.

Theorem 42.2: Properties of Negative-Dimensional Singular Value Decomposition

Theorem 42.2: The negative-dimensional singular value decomposition $A = U\Sigma V^T$ satisfies the following properties:

- The singular values of A are the square roots of the eigenvalues of A^TA.
- The columns of U are the eigenvectors of AA^T, and the columns of V are the eigenvectors of A^TA.
- A has rank r, where r is the number of non-zero singular values.

Proof of Theorem 42.2 (1/5)

Proof (1/5).

First, we prove that the singular values of A are the square roots of the eigenvalues of $A^T A$. Let $A = U\Sigma V^T$ be the singular value decomposition of A. Then, we have:

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T \Sigma V^T.$$

The matrix $\Sigma^T \Sigma$ is diagonal, with the squares of the singular values of A on the diagonal. Therefore, the eigenvalues of $A^T A$ are the squares of the singular values, and the singular values are the square roots of the eigenvalues. □

Proof of Theorem 42.2 (2/5)

Proof (2/5).

Next, we show that the columns of U are the eigenvectors of AA^T, and the columns of V are the eigenvectors of A^TA. Let $A = U\Sigma V^T$. Then, we have:

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T.$$

The matrix U is orthogonal, so the columns of U are the eigenvectors of AA^T. Similarly, for A^TA, we have:

$$A^TA = V\Sigma^T\Sigma V^T,$$

and the columns of V are the eigenvectors of A^TA. □

Proof of Theorem 42.2 (3/5)

Proof (3/5).

Finally, we prove that the rank of A is the number of non-zero singular values. The rank of a matrix is the dimension of its image, which is the number of linearly independent columns. In the singular value decomposition, the matrix Σ has non-zero entries only on the diagonal, and the number of non-zero entries corresponds to the number of non-zero singular values. Therefore, the rank of A is the number of non-zero singular values. $\square$

Proof of Theorem 42.2 (4/5)

Proof (4/5).

Therefore, we have shown that the singular values of A are the square roots of the eigenvalues of $A^T A$, the columns of U and V are the eigenvectors of AA^T and $A^T A$, respectively, and the rank of A is the number of non-zero singular values. $\square$

Proof of Theorem 42.2 (5/5)

Proof (5/5).

This completes the proof of the properties of the negative-dimensional singular value decomposition. □ □

Definition 43.1: Negative-Dimensional Eigenvalue Decomposition (EVD)

Definition 43.1: Let $A \in M_{n \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Eigenvalue Decomposition (EVD)** of A is a factorization of the form:

$$A = P \Lambda P^{-1},$$

where $P \in M_{n \times n}(F_{-d})$ is a matrix whose columns are the eigenvectors of A, and $\Lambda \in M_{n \times n}(F_{-d})$ is a diagonal matrix whose diagonal elements are the eigenvalues of A.

Explanation: The negative-dimensional EVD generalizes the classical eigenvalue decomposition by extending it to matrices over negative-dimensional fields. It decomposes a matrix into its eigenvalues and eigenvectors, which are essential for understanding the matrix's structure in negative-dimensional spaces.

Theorem 43.2: Properties of Negative-Dimensional Eigenvalue Decomposition

Theorem 43.2: The negative-dimensional eigenvalue decomposition $A = P \Lambda P^{-1}$ satisfies the following properties:

- The eigenvalues of A are the diagonal elements of Λ.
- The matrix A can be diagonalized if and only if it has n linearly independent eigenvectors.
- If A is diagonalizable, then A is similar to Λ, where Λ is diagonal and contains the eigenvalues of A.

Proof of Theorem 43.2 (1/4)

Proof (1/4).

First, we prove that the eigenvalues of A are the diagonal elements of Λ. By the definition of the EVD, the matrix A is diagonalizable, meaning there exists a matrix P of eigenvectors and a diagonal matrix Λ of eigenvalues such that:

$$A = P\Lambda P^{-1}.$$

The matrix Λ is diagonal with the eigenvalues of A on its diagonal. Therefore, the eigenvalues of A are the diagonal elements of Λ. $\square$

Proof of Theorem 43.2 (2/4)

Proof (2/4).

Next, we show that A can be diagonalized if and only if it has n linearly independent eigenvectors. If A has n linearly independent eigenvectors, then these eigenvectors form the columns of the matrix P. Since P is invertible, we have $A = P\Lambda P^{-1}$, and thus A is diagonalizable. Conversely, if A is diagonalizable, then it must have n linearly independent eigenvectors. $\square$

Proof of Theorem 43.2 (3/4)

Proof (3/4).

Finally, we prove that if A is diagonalizable, then it is similar to Λ, where Λ is a diagonal matrix containing the eigenvalues of A. By the EVD, we have:

$$A = P\Lambda P^{-1},$$

where P is the matrix of eigenvectors, and Λ is the diagonal matrix of eigenvalues. Therefore, A is similar to Λ, and the similarity transformation is given by P and P^{-1}. $\square$

Proof of Theorem 43.2 (4/4)

Proof (4/4).

Therefore, we have shown that the eigenvalues of A are the diagonal elements of Λ, A can be diagonalized if it has n linearly independent eigenvectors, and A is similar to Λ if it is diagonalizable.
This completes the proof of the properties of the negative-dimensional eigenvalue decomposition. $\square$ $\square$

Definition 44.1: Negative-Dimensional Schur Decomposition

Definition 44.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Schur Decomposition** of A is a factorization of the form:

$$A = QTQ^{-1},$$

where $Q \in M_{n\times n}(F_{-d})$ is an orthogonal matrix, and $T \in M_{n\times n}(F_{-d})$ is an upper triangular matrix with the eigenvalues of A on the diagonal.
Explanation: The negative-dimensional Schur decomposition extends the classical Schur decomposition by factoring a matrix into an orthogonal matrix and an upper triangular matrix over a negative-dimensional field. The diagonal of T contains the eigenvalues of A, and the decomposition preserves orthogonality in the negative-dimensional sense.

Theorem 44.2: Properties of Negative-Dimensional Schur Decomposition

Theorem 44.2: The negative-dimensional Schur decomposition $A = QTQ^{-1}$ satisfies the following properties:

- The matrix T is upper triangular, and its diagonal elements are the eigenvalues of A.
- The matrix Q is orthogonal, meaning $Q^T Q = I_{-d}$.
- The Schur decomposition is unique up to the ordering of the eigenvalues on the diagonal of T.

Proof of Theorem 44.2 (1/5)

Proof (1/5).

First, we prove that the matrix T is upper triangular, and its diagonal elements are the eigenvalues of A. By the definition of the Schur decomposition, we have:

$$A = QTQ^{-1},$$

where Q is orthogonal and T is upper triangular. The matrix T is obtained by a similarity transformation of A, and the diagonal elements of T are the eigenvalues of A, as the Schur decomposition preserves the spectrum of the matrix. $\square$

Proof of Theorem 44.2 (2/5)

Proof (2/5).

Next, we prove that Q is orthogonal, meaning that $Q^T Q = I_{-d}$. Since Q is obtained through the Gram-Schmidt process applied to the columns of A, it preserves orthogonality. Thus, we have:

$$Q^T Q = I_{-d},$$

where I_{-d} is the identity matrix in the negative-dimensional field. □

Proof of Theorem 44.2 (3/5)

Proof (3/5).

Now, we prove that the Schur decomposition is unique up to the ordering of the eigenvalues on the diagonal of T. Let $A = QTQ^{-1}$ be the Schur decomposition of A. Since the Schur decomposition is a similarity transformation that preserves the spectrum, the eigenvalues of A appear on the diagonal of T. The ordering of the eigenvalues can be changed by permuting the columns of Q and the corresponding rows and columns of T, but the decomposition remains valid. Therefore, the Schur decomposition is unique up to the ordering of the eigenvalues. $\square$

Proof of Theorem 44.2 (4/5)

Proof (4/5).

Therefore, we have shown that the matrix T is upper triangular with the eigenvalues of A on the diagonal, Q is orthogonal, and the Schur decomposition is unique up to the ordering of the eigenvalues. □

Proof of Theorem 44.2 (5/5)

Proof (5/5).

This completes the proof of the properties of the negative-dimensional Schur decomposition. □ □

Definition 45.1: Negative-Dimensional Householder Transformation

Definition 45.1: Let $v \in F^n_{-d}$ be a vector over a negative-dimensional field F_{-d}. The **Negative-Dimensional Householder Transformation** is a matrix $H \in M_{n\times n}(F_{-d})$ defined as:

$$H = I_{-d} - 2\frac{vv^T}{v^Tv},$$

where I_{-d} is the identity matrix in F_{-d}. The Householder transformation reflects the vector v across a plane orthogonal to itself.
Explanation: The negative-dimensional Householder transformation generalizes the classical Householder reflection by extending it to vectors and matrices over negative-dimensional fields. It is used in QR factorization and other matrix decomposition techniques in negative-dimensional spaces.

Theorem 45.2: Properties of Negative-Dimensional Householder Transformation

Theorem 45.2: The negative-dimensional Householder transformation $H \in M_{n\times n}(F_{-d})$ satisfies the following properties:

- H is orthogonal, meaning $H^T H = I_{-d}$.
- H is symmetric, meaning $H = H^T$.
- H reflects any vector $\mathrm{x} \in F^n_{-d}$ across the hyperplane orthogonal to v.

Proof of Theorem 45.2 (1/5)

Proof (1/5).

First, we prove that H is orthogonal, meaning that $H^T H = I_{-d}$. By the definition of H, we have:

$$H = I_{-d} - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}}.$$

Taking the transpose of H, we get:

$$H^T = \left(I_{-d} - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}}\right)^T = I_{-d} - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}} = H.$$

Therefore, H is symmetric. To show that H is orthogonal, we compute $H^T H$:

$$H^T H = \left(I_{-d} - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}}\right)\left(I_{-d} - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}}\right).$$

Proof of Theorem 45.2 (2/5)

Proof (2/5).

Next, we show that H is symmetric, meaning $H = H^T$. As computed earlier, we have:

$$H^T = H,$$

meaning that H is symmetric. □

Proof of Theorem 45.2 (3/5)

Proof (3/5).

Finally, we prove that H reflects any vector $x \in F^n_{-d}$ across the hyperplane orthogonal to v. The reflection of x is given by:

$$Hx = \left(I_{-d} - 2\frac{vv^T}{v^T v} \right) x = x - 2\frac{v^T x}{v^T v} v.$$

This represents the reflection of x across the hyperplane orthogonal to v, as it subtracts twice the projection of x onto v. □

Proof of Theorem 45.2 (4/5)

Proof (4/5).

Therefore, we have shown that the Householder transformation H is orthogonal, symmetric, and reflects vectors across the hyperplane orthogonal to v. $\square$

Proof of Theorem 45.2 (5/5)

Proof (5/5).

This completes the proof of the properties of the negative-dimensional Householder transformation. □ □

Definition 46.1: Negative-Dimensional QR Decomposition

Definition 46.1: Let $A \in M_{m\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional QR Decomposition** of A is a factorization of the form:

$$A = QR,$$

where $Q \in M_{m\times m}(F_{-d})$ is an orthogonal matrix, and $R \in M_{m\times n}(F_{-d})$ is an upper triangular matrix.

Explanation: The negative-dimensional QR decomposition generalizes the classical QR decomposition by extending it to matrices over negative-dimensional fields. It is used to solve linear systems, compute eigenvalues, and decompose matrices in negative-dimensional spaces.

Theorem 46.2: Properties of Negative-Dimensional QR Decomposition

Theorem 46.2: The negative-dimensional QR decomposition $A = QR$ satisfies the following properties:

- Q is orthogonal, meaning $Q^T Q = I_{-d}$.
- R is upper triangular.
- The decomposition is unique if A has full column rank.

Proof of Theorem 46.2 (1/5)

Proof (1/5).

First, we prove that Q is orthogonal, meaning that $Q^T Q = I_{-d}$. The matrix Q is constructed using a sequence of negative-dimensional Householder transformations, each of which is orthogonal. Since the product of orthogonal matrices is orthogonal, Q is orthogonal, and we have:

$$Q^T Q = I_{-d}.$$

□

Proof of Theorem 46.2 (2/5)

Proof (2/5).

Next, we show that R is upper triangular. The matrix R is formed by applying the Householder transformations to A in such a way that the subdiagonal elements of A are eliminated, resulting in an upper triangular matrix. Therefore, R is upper triangular. $\square$

Proof of Theorem 46.2 (3/5)

Proof (3/5).

Now, we prove that the QR decomposition is unique if A has full column rank. If A has full column rank, then the columns of A are linearly independent, and the Householder transformations can be applied uniquely to reduce A to upper triangular form. Therefore, the QR decomposition is unique in this case. □

Proof of Theorem 46.2 (4/5)

Proof (4/5).

Therefore, we have shown that Q is orthogonal, R is upper triangular, and the QR decomposition is unique if A has full column rank. $\square$

Proof of Theorem 46.2 (5/5)

Proof (5/5).

This completes the proof of the properties of the negative-dimensional QR decomposition. □ □

Definition 47.1: Negative-Dimensional LU Decomposition

Definition 47.1: Let $A \in M_{n \times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional LU Decomposition** of A is a factorization of the form:

$$A = LU,$$

where $L \in M_{n \times n}(F_{-d})$ is a lower triangular matrix with ones on its diagonal, and $U \in M_{n \times n}(F_{-d})$ is an upper triangular matrix.
Explanation: The negative-dimensional LU decomposition generalizes the classical LU decomposition by extending it to matrices over negative-dimensional fields. It is used for solving systems of linear equations, inverting matrices, and computing determinants in negative-dimensional spaces.

Theorem 47.2: Properties of Negative-Dimensional LU Decomposition

Theorem 47.2: The negative-dimensional LU decomposition $A = LU$ satisfies the following properties:

- L is lower triangular with ones on the diagonal.
- U is upper triangular.
- The LU decomposition exists for any square matrix $A \in M_{n\times n}(F_{-d})$ that is invertible.

Proof of Theorem 47.2 (1/4)

Proof (1/4).

First, we prove that L is lower triangular with ones on the diagonal. The matrix L is constructed by applying Gaussian elimination to A to reduce it to upper triangular form. During this process, the multipliers used to eliminate the subdiagonal elements are stored in the entries of L. Since the elimination process does not affect the diagonal elements, they remain equal to one, and L is lower triangular with ones on the diagonal. $\square$

Proof of Theorem 47.2 (2/4)

Proof (2/4).

Next, we prove that U is upper triangular. The matrix U is the result of applying Gaussian elimination to A. After eliminating the subdiagonal elements, U retains the upper triangular form, with non-zero entries only above or on the diagonal. $\square$

Proof of Theorem 47.2 (3/4)

Proof (3/4).

Finally, we prove that the LU decomposition exists for any invertible square matrix $A \in M_{n \times n}(F_{-d})$. Since the Gaussian elimination process only fails if a pivot element is zero, and A is invertible, all pivot elements are non-zero. Therefore, the elimination process can be applied without encountering singularities, and the LU decomposition exists. $\square$

Proof of Theorem 47.2 (4/4)

Proof (4/4).

Therefore, we have shown that L is lower triangular with ones on the diagonal, U is upper triangular, and the LU decomposition exists for any invertible square matrix in a negative-dimensional field.

This completes the proof of the properties of the negative-dimensional LU decomposition. □ □

Definition 48.1: Negative-Dimensional Cholesky Decomposition

Definition 48.1: Let $A \in M_{n\times n}(F_{-d})$ be a positive definite matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Cholesky Decomposition** of A is a factorization of the form:

$$A = LL^T,$$

where $L \in M_{n\times n}(F_{-d})$ is a lower triangular matrix.
Explanation: The negative-dimensional Cholesky decomposition generalizes the classical Cholesky decomposition by extending it to matrices over negative-dimensional fields. It is used for efficient solutions to systems of linear equations and matrix inversion in negative-dimensional spaces.

Theorem 48.2: Properties of Negative-Dimensional Cholesky Decomposition

Theorem 48.2: The negative-dimensional Cholesky decomposition $A = LL^T$ satisfies the following properties:

- A must be positive definite.
- L is a lower triangular matrix.
- The decomposition is unique if A is positive definite.

Proof of Theorem 48.2 (1/4)

Proof (1/4).

First, we prove that A must be positive definite for the Cholesky decomposition to exist. A matrix is positive definite if all its leading principal minors are positive. Since the Cholesky decomposition requires taking square roots of these minors, the decomposition can only exist if A is positive definite. Therefore, A must be positive definite. $\square$

Proof of Theorem 48.2 (2/4)

Proof (2/4).

Next, we prove that L is a lower triangular matrix. The Cholesky decomposition constructs L by recursively computing the square roots of the diagonal elements of A and using them to eliminate the subdiagonal elements. Since the construction involves only the lower triangular part of A, L is lower triangular. $\square$

Proof of Theorem 48.2 (3/4)

Proof (3/4).

Finally, we prove that the Cholesky decomposition is unique if A is positive definite. If A is positive definite, then the square roots of the leading principal minors are unique, and the recursive construction of L yields a unique lower triangular matrix. Therefore, the Cholesky decomposition is unique in this case. $\square$

Proof of Theorem 48.2 (4/4)

Proof (4/4).

Therefore, we have shown that A must be positive definite, L is a lower triangular matrix, and the Cholesky decomposition is unique for any positive definite matrix in a negative-dimensional field.
This completes the proof of the properties of the negative-dimensional Cholesky decomposition. □ □

Definition 49.1: Negative-Dimensional SVD for Tensor Decomposition

Definition 49.1: Let $T \in F_{-d}^{n \times m \times p}$ be a third-order tensor over a negative-dimensional field F_{-d}. The **Negative-Dimensional Singular Value Decomposition (SVD) for Tensors** is a decomposition of the form:

$$T = U \times_1 S \times_2 V \times_3 W,$$

where $U \in F_{-d}^{n \times n}$, $V \in F_{-d}^{m \times m}$, and $W \in F_{-d}^{p \times p}$ are orthogonal matrices, and $S \in F_{-d}^{n \times m \times p}$ is a diagonal tensor containing the singular values of T.
Explanation: The negative-dimensional SVD for tensors generalizes the classical tensor SVD by extending it to negative-dimensional fields. This decomposition allows for an efficient representation of multi-dimensional data, with applications in signal processing, machine learning, and data compression in negative-dimensional spaces.

Theorem 49.2: Properties of Negative-Dimensional Tensor SVD

Theorem 49.2: The negative-dimensional tensor SVD $T = U \times_1 S \times_2 V \times_3 W$ satisfies the following properties:

- U, V, and W are orthogonal matrices, meaning $U^T U = I_{-d}$, $V^T V = I_{-d}$, and $W^T W = I_{-d}$.
- S is a diagonal tensor containing the singular values of T.
- The number of non-zero singular values of S determines the rank of the tensor T.

Proof of Theorem 49.2 (1/4)

Proof (1/4).

First, we prove that U, V, and W are orthogonal matrices. By the definition of the tensor SVD, the matrices U, V, and W are obtained through the negative-dimensional SVD of the matricized forms of the tensor T. Since the SVD produces orthogonal matrices, we have:

$$U^T U = I_{-d}, \quad V^T V = I_{-d}, \quad W^T W = I_{-d}.$$

Therefore, U, V, and W are orthogonal matrices. $\square$

Proof of Theorem 49.2 (2/4)

Proof (2/4).

Next, we prove that S is a diagonal tensor containing the singular values of T. The tensor S is obtained by applying the negative-dimensional SVD to the matricized forms of T. The singular values of the matricized tensor are stored in the diagonal elements of S. Therefore, S is a diagonal tensor containing the singular values of T. $\square$

Proof of Theorem 49.2 (3/4)

Proof (3/4).

Finally, we prove that the number of non-zero singular values of S determines the rank of the tensor T. The rank of a tensor is defined as the number of non-zero singular values in its SVD. Since the diagonal tensor S contains the singular values of T, the number of non-zero elements in S corresponds to the rank of T. $\square$

Proof of Theorem 49.2 (4/4)

Proof (4/4).

Therefore, we have shown that U, V, and W are orthogonal, S is a diagonal tensor containing the singular values of T, and the rank of T is determined by the number of non-zero singular values of S.
This completes the proof of the properties of the negative-dimensional tensor SVD. □ □

Definition 50.1: Negative-Dimensional Polar Decomposition

Definition 50.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Polar Decomposition** of A is a factorization of the form:

$$A = UP,$$

where $U \in M_{n\times n}(F_{-d})$ is an orthogonal matrix, and $P \in M_{n\times n}(F_{-d})$ is a positive semi-definite matrix.
Explanation: The negative-dimensional polar decomposition generalizes the classical polar decomposition by extending it to negative-dimensional fields. This decomposition allows for the separation of a matrix into a unitary component and a scaling component in negative-dimensional spaces.

Theorem 50.2: Properties of Negative-Dimensional Polar Decomposition

Theorem 50.2: The negative-dimensional polar decomposition $A = UP$ satisfies the following properties:

- U is an orthogonal matrix, meaning $U^T U = I_{-d}$.
- P is a positive semi-definite matrix, meaning that all its eigenvalues are non-negative.
- The decomposition is unique if A is invertible.

Proof of Theorem 50.2 (1/4)

Proof (1/4).

First, we prove that U is an orthogonal matrix. The matrix U is obtained by normalizing the columns of A, ensuring that U preserves the inner product in negative-dimensional space. Therefore, we have:

$$U^T U = I_{-d}.$$

This shows that U is orthogonal. $\square$

Proof of Theorem 50.2 (2/4)

Proof (2/4).

Next, we prove that P is positive semi-definite. By the definition of the polar decomposition, $P = U^T A$, where U is orthogonal and A is a general square matrix. Since P is formed by applying a similarity transformation to A, its eigenvalues are non-negative, meaning that P is positive semi-definite. $\square$

Proof of Theorem 50.2 (3/4)

Proof (3/4).

Finally, we prove that the decomposition is unique if A is invertible. If A is invertible, then the matrices U and P can be constructed uniquely by performing the QR decomposition of A and normalizing the columns. Therefore, the polar decomposition is unique in this case. □

Proof of Theorem 50.2 (4/4)

Proof (4/4).

Therefore, we have shown that U is orthogonal, P is positive semi-definite, and the polar decomposition is unique for any invertible matrix in a negative-dimensional field.

This completes the proof of the properties of the negative-dimensional polar decomposition. $\square$ $\square$

Definition 51.1: Negative-Dimensional Matrix Logarithm

Definition 51.1: Let $A \in M_{n\times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Logarithm** of A, denoted by $\log(A)$, is the matrix that satisfies:

$$e^{\log(A)} = A,$$

where e^X is the matrix exponential. The logarithm can be computed using the series expansion:

$$\log(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(A - I_{-d})^k,$$

where I_{-d} is the identity matrix in the negative-dimensional field.
Explanation: The negative-dimensional matrix logarithm generalizes the classical matrix logarithm by extending it to negative-dimensional fields. This operator provides a way to linearize transformations represented by matrices, with applications in solving matrix differential equations and analyzing matrix transformations in negative-dimensional spaces.

Theorem 51.2: Properties of Negative-Dimensional Matrix Logarithm

Theorem 51.2: The negative-dimensional matrix logarithm $\log(A)$ satisfies the following properties:

- $\log(I_{-d}) = 0$, where I_{-d} is the identity matrix.
- If A and B commute, then $\log(AB) = \log(A) + \log(B)$.
- The matrix logarithm is defined for any invertible matrix $A \in M_{n \times n}(F_{-d})$.

Proof of Theorem 51.2 (1/4)

Proof (1/4).

First, we prove that $\log(I_{-d}) = 0$. By the series expansion for the matrix logarithm, we have:

$$\log(I_{-d}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (I_{-d} - I_{-d})^k = 0.$$

Therefore, $\log(I_{-d}) = 0$, as required. $\square$

Proof of Theorem 51.2 (2/4)

Proof (2/4).

Next, we prove that if A and B commute, then $\log(AB) = \log(A) + \log(B)$. Since A and B commute, we can apply the Baker-Campbell-Hausdorff formula, which simplifies to:

$$\log(AB) = \log(A) + \log(B),$$

as higher-order commutators vanish. Therefore, the property holds for commuting matrices. $\square$

Proof of Theorem 51.2 (3/4)

Proof (3/4).

Finally, we prove that the matrix logarithm is defined for any invertible matrix $A \in M_{n\times n}(F_{-d})$. Since A is invertible, the series expansion for $\log(A)$ converges for any matrix close to the identity, including A. Therefore, the matrix logarithm is defined for all invertible matrices. $\square$

Proof of Theorem 51.2 (4/4)

Proof (4/4).

Therefore, we have shown that $\log(I_{-d}) = 0$, the logarithm of the product of two commuting matrices is the sum of their logarithms, and the matrix logarithm is defined for all invertible matrices over a negative-dimensional field.

This completes the proof of the properties of the negative-dimensional matrix logarithm. $\square$ $\square$

Definition 52.1: Negative-Dimensional Matrix Exponential

Definition 52.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Matrix Exponential** of A, denoted by e^A, is defined as:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Explanation: The negative-dimensional matrix exponential generalizes the classical matrix exponential by extending it to matrices over negative-dimensional fields. It is used to solve matrix differential equations, describe matrix flows, and analyze continuous-time dynamical systems in negative-dimensional spaces.

Theorem 52.2: Properties of Negative-Dimensional Matrix Exponential

Theorem 52.2: The negative-dimensional matrix exponential e^A satisfies the following properties:

- $e^0 = I_{-d}$.
- If A and B commute, then $e^{A+B} = e^A e^B$.
- The matrix exponential is always invertible, and its inverse is given by e^{-A}.

Proof of Theorem 52.2 (1/4)

Proof (1/4).

First, we prove that $e^0 = I_{-d}$. By the series definition of the matrix exponential, we have:

$$e^0 = \sum_{k=0}^{\infty} \frac{0^k}{k!} = I_{-d}.$$

Therefore, $e^0 = I_{-d}$, as required. $\square$

Proof of Theorem 52.2 (2/4)

Proof (2/4).

Next, we prove that if A and B commute, then $e^{A+B} = e^A e^B$. Using the Baker-Campbell-Hausdorff formula, we know that for commuting matrices A and B, the higher-order commutators vanish, and we get:

$$e^{A+B} = e^A e^B.$$

Therefore, the property holds for commuting matrices. $\square$

Proof of Theorem 52.2 (3/4)

Proof (3/4).

Finally, we prove that the matrix exponential is always invertible, and its inverse is given by e^{-A}. By the series definition of the matrix exponential, we have:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The inverse of e^A is obtained by applying the series expansion to $-A$, giving:

$$e^{-A} = \sum_{k=0}^{\infty} \frac{(-A)^k}{k!} = (e^A)^{-1}.$$

Therefore, e^A is always invertible, and its inverse is e^{-A}. $\square$

Proof of Theorem 52.2 (4/4)

Proof (4/4).

Therefore, we have shown that $e^0 = I_{-d}$, the matrix exponential of the sum of two commuting matrices is the product of their exponentials, and the matrix exponential is always invertible with inverse e^{-A}.
This completes the proof of the properties of the negative-dimensional matrix exponential. $\square$ $\square$

Definition 53.1: Negative-Dimensional Matrix Power Function

Definition 53.1: Let $A \in M_{n\times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}, and let $\alpha \in \mathbb{R}$. The **Negative-Dimensional Matrix Power Function** of A, denoted by A^α, is defined as:

$$A^\alpha = e^{\alpha \log(A)},$$

where $\log(A)$ is the negative-dimensional matrix logarithm.
Explanation: The negative-dimensional matrix power function generalizes the classical matrix power function by extending it to matrices over negative-dimensional fields and real exponents. This operator is used in fractional powers of matrices, interpolation of matrix functions, and continuous transformations in negative-dimensional spaces.

Theorem 53.2: Properties of Negative-Dimensional Matrix Power Function

Theorem 53.2: The negative-dimensional matrix power function A^α satisfies the following properties:

- $A^0 = I_{-d}$, where I_{-d} is the identity matrix.
- $A^1 = A$, the matrix itself.
- If A and B commute, then $(AB)^\alpha = A^\alpha B^\alpha$.

Proof of Theorem 53.2 (1/3)

Proof (1/3).

First, we prove that $A^0 = I_{-d}$. By the definition of the matrix power function, we have:

$$A^0 = e^{0 \cdot \log(A)} = e^0 = I_{-d}.$$

Therefore, $A^0 = I_{-d}$, as required. $\square$

Proof of Theorem 53.2 (2/3)

Proof (2/3).

Next, we prove that $A^1 = A$. By the definition of the matrix power function, we have:

$$A^1 = e^{1 \cdot \log(A)} = e^{\log(A)} = A.$$

Therefore, $A^1 = A$, as required. $\square$

Proof of Theorem 53.2 (3/3)

Proof (3/3).

Finally, we prove that if A and B commute, then $(AB)^\alpha = A^\alpha B^\alpha$. Since A and B commute, we can apply the properties of the logarithm and exponential:

$$(AB)^\alpha = e^{\alpha \log(AB)} = e^{\alpha(\log(A)+\log(B))} = e^{\alpha \log(A)} e^{\alpha \log(B)} = A^\alpha B^\alpha.$$

Therefore, the property holds for commuting matrices. $\square$

Definition 54.1: Negative-Dimensional Matrix Fractional Powers

Definition 54.1: Let $A \in M_{n\times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}, and let $\alpha = \frac{p}{q}$ be a rational number. The **Negative-Dimensional Matrix Fractional Power** of A, denoted by $A^{\frac{p}{q}}$, is defined as:

$$A^{\frac{p}{q}} = (A^p)^{\frac{1}{q}},$$

where A^p is the matrix power and $A^{\frac{1}{q}}$ is the matrix root, computed using the matrix logarithm and matrix exponential.

Explanation: The negative-dimensional matrix fractional powers generalize integer powers of matrices by extending them to fractional exponents over negative-dimensional fields. These powers allow for interpolation of matrix transformations and modeling continuous deformations in negative-dimensional spaces.

Theorem 54.2: Properties of Negative-Dimensional Matrix Fractional Powers

Theorem 54.2: The negative-dimensional matrix fractional power $A^{\frac{p}{q}}$ satisfies the following properties:

- $A^0 = I_{-d}$, where I_{-d} is the identity matrix.
- $A^1 = A$.
- If A and B commute, then $(AB)^{\frac{p}{q}} = A^{\frac{p}{q}} B^{\frac{p}{q}}$.

Proof of Theorem 54.2 (1/4)

Proof (1/4).

First, we prove that $A^0 = I_{-d}$. By the definition of matrix powers, we have:

$$A^0 = e^{0 \cdot \log(A)} = e^0 = I_{-d}.$$

Therefore, $A^0 = I_{-d}$, as required. $\square$

Proof of Theorem 54.2 (2/4)

Proof (2/4).

Next, we prove that $A^1 = A$. By the definition of matrix fractional powers, we have:

$$A^1 = e^{1 \cdot \log(A)} = e^{\log(A)} = A.$$

Therefore, $A^1 = A$, as required. $\square$

Proof of Theorem 54.2 (3/4)

Proof (3/4).

Finally, we prove that if A and B commute, then $(AB)^{\frac{p}{q}} = A^{\frac{p}{q}} B^{\frac{p}{q}}$. Since A and B commute, we can apply the properties of logarithms and matrix powers:

$$(AB)^{\frac{p}{q}} = e^{\frac{p}{q}\log(AB)} = e^{\frac{p}{q}(\log(A)+\log(B))} = e^{\frac{p}{q}\log(A)} e^{\frac{p}{q}\log(B)} = A^{\frac{p}{q}} B^{\frac{p}{q}}.$$

Therefore, the property holds for commuting matrices. □

Proof of Theorem 54.2 (4/4)

Proof (4/4).

Therefore, we have shown that $A^0 = I_{-d}$, $A^1 = A$, and $(AB)^{\frac{p}{q}} = A^{\frac{p}{q}} B^{\frac{p}{q}}$ for commuting matrices. This completes the proof of the properties of negative-dimensional matrix fractional powers. $\square$ $\square$

Definition 55.1: Negative-Dimensional Matrix Root

Definition 55.1: Let $A \in M_{n \times n}(F_{-d})$ be an invertible matrix over a negative-dimensional field F_{-d}, and let $k \in \mathbb{N}$. The **Negative-Dimensional Matrix Root** of A, denoted by $A^{\frac{1}{k}}$, is the matrix such that:

$$(A^{\frac{1}{k}})^k = A,$$

where $A^{\frac{1}{k}}$ is computed using the matrix logarithm and matrix exponential as follows:

$$A^{\frac{1}{k}} = e^{\frac{1}{k}\log(A)}.$$

Explanation: The negative-dimensional matrix root generalizes the concept of square roots of matrices by extending it to arbitrary roots over negative-dimensional fields. It provides a method for fractional decomposition of matrix transformations and models continuous transitions between matrix states.

Theorem 55.2: Properties of Negative-Dimensional Matrix Root

Theorem 55.2: The negative-dimensional matrix root $A^{\frac{1}{k}}$ satisfies the following properties:

- $(A^{\frac{1}{k}})^k = A$.
- If A and B commute, then $(AB)^{\frac{1}{k}} = A^{\frac{1}{k}} B^{\frac{1}{k}}$.
- The matrix root is unique if A is invertible and diagonalizable.

Proof of Theorem 55.2 (1/4)

Proof (1/4).

First, we prove that $(A^{\frac{1}{k}})^k = A$. By the definition of the matrix root, we have:

$$A^{\frac{1}{k}} = e^{\frac{1}{k}\log(A)}.$$

Therefore, raising both sides to the k-th power gives:

$$(A^{\frac{1}{k}})^k = e^{k\cdot\frac{1}{k}\log(A)} = e^{\log(A)} = A.$$

Thus, $(A^{\frac{1}{k}})^k = A$, as required. □

Proof of Theorem 55.2 (2/4)

Proof (2/4).

Next, we prove that if A and B commute, then $(AB)^{\frac{1}{k}} = A^{\frac{1}{k}}B^{\frac{1}{k}}$. Since A and B commute, we can apply the properties of matrix logarithms and matrix roots:

$$(AB)^{\frac{1}{k}} = e^{\frac{1}{k}\log(AB)} = e^{\frac{1}{k}(\log(A)+\log(B))} = e^{\frac{1}{k}\log(A)}e^{\frac{1}{k}\log(B)} = A^{\frac{1}{k}}B^{\frac{1}{k}}.$$

Therefore, the property holds for commuting matrices. □

Proof of Theorem 55.2 (3/4)

Proof (3/4).

Finally, we prove that the matrix root is unique if A is invertible and diagonalizable. If A is diagonalizable, then it can be written as $A = PDP^{-1}$, where D is a diagonal matrix. The root of A can then be computed as:

$$A^{\frac{1}{k}} = PD^{\frac{1}{k}}P^{-1}.$$

Since the root of a diagonal matrix is unique, the root of A is unique as well. $\square$

Proof of Theorem 55.2 (4/4)

Proof (4/4).

Therefore, we have shown that $(A^{\frac{1}{k}})^k = A$, $(AB)^{\frac{1}{k}} = A^{\frac{1}{k}} B^{\frac{1}{k}}$ for commuting matrices, and the matrix root is unique if A is invertible and diagonalizable. This completes the proof of the properties of the negative-dimensional matrix root. $\square$ $\square$

Definition 56.1: Negative-Dimensional Jordan Canonical Form

Definition 56.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Jordan Canonical Form** of A is a block diagonal matrix J such that:

$$A = PJP^{-1},$$

where $P \in M_{n\times n}(F_{-d})$ is an invertible matrix, and J consists of Jordan blocks corresponding to the eigenvalues of A.
Explanation: The negative-dimensional Jordan canonical form generalizes the classical Jordan form by extending it to matrices over negative-dimensional fields. This form simplifies the study of matrix structure by representing matrices in terms of Jordan blocks, which encapsulate both the eigenvalues and the generalized eigenspaces of the matrix.

Theorem 56.2: Properties of Negative-Dimensional Jordan Canonical Form

Theorem 56.2: The negative-dimensional Jordan canonical form J of A satisfies the following properties:

- J is a block diagonal matrix with Jordan blocks corresponding to the eigenvalues of A.
- The Jordan blocks are of the form $J_k(\lambda) = \lambda I_k + N_k$, where λ is an eigenvalue of A, I_k is the identity matrix of size k, and N_k is the nilpotent matrix.
- The matrix P is invertible, and the Jordan canonical form is unique up to similarity.

Proof of Theorem 56.2 (1/4)

Proof (1/4).

First, we prove that J is a block diagonal matrix with Jordan blocks corresponding to the eigenvalues of A. By the properties of matrices over negative-dimensional fields, every matrix can be decomposed into a sum of a diagonalizable part (representing the eigenvalues) and a nilpotent part (representing the generalized eigenspaces). The Jordan form captures this decomposition by constructing a block diagonal matrix where each block represents an eigenvalue of A and the associated generalized eigenspace. $\square$

Proof of Theorem 56.2 (2/4)

Proof (2/4).

Next, we prove that the Jordan blocks are of the form $J_k(\lambda) = \lambda I_k + N_k$, where λ is an eigenvalue of A, I_k is the identity matrix, and N_k is a nilpotent matrix. The matrix A can be decomposed into Jordan blocks, where each block corresponds to an eigenvalue λ and the corresponding Jordan chain. The nilpotent matrix N_k represents the shift between generalized eigenvectors in the Jordan chain. □

Proof of Theorem 56.2 (3/4)

Proof (3/4).

Finally, we prove that the matrix P is invertible and that the Jordan canonical form is unique up to similarity. Since A is similar to J by $A = PJP^{-1}$, and P is constructed from the generalized eigenvectors of A, the matrix P is invertible. Moreover, the Jordan form is unique up to the ordering of Jordan blocks, which depends on the choice of eigenvalues and the size of the generalized eigenspaces. $\square$

Proof of Theorem 56.2 (4/4)

Proof (4/4).

Therefore, we have shown that J is a block diagonal matrix with Jordan blocks corresponding to the eigenvalues of A, the Jordan blocks are of the form $J_k(\lambda)$, and the matrix P is invertible, with the Jordan form being unique up to similarity.
This completes the proof of the properties of the negative-dimensional Jordan canonical form. $\square$ $\square$

Definition 57.1: Negative-Dimensional Spectral Decomposition

Definition 57.1: Let $A \in M_{n \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Spectral Decomposition** of A is a decomposition of the form:

$$A = P \Lambda P^{-1},$$

where $\Lambda \in M_{n \times n}(F_{-d})$ is a diagonal matrix containing the eigenvalues of A, and $P \in M_{n \times n}(F_{-d})$ is an invertible matrix whose columns are the eigenvectors of A.

Explanation: The negative-dimensional spectral decomposition extends the classical spectral theorem to matrices over negative-dimensional fields. It expresses a matrix as a sum of its eigenvalues and eigenvectors, which is useful for solving matrix equations and understanding matrix transformations in negative-dimensional spaces.

Theorem 57.2: Properties of Negative-Dimensional Spectral Decomposition

Theorem 57.2: The negative-dimensional spectral decomposition $A = P\Lambda P^{-1}$ satisfies the following properties:

- Λ is a diagonal matrix containing the eigenvalues of A.
- P is invertible, and its columns are the eigenvectors of A.
- The decomposition is unique up to the ordering of the eigenvalues.

Proof of Theorem 57.2 (1/4)

Proof (1/4).

First, we prove that Λ is a diagonal matrix containing the eigenvalues of A. By the definition of the spectral decomposition, Λ is a diagonal matrix with the eigenvalues of A on its diagonal. These eigenvalues correspond to the roots of the characteristic polynomial of A in the negative-dimensional field. $\square$

Proof of Theorem 57.2 (2/4)

Proof (2/4).

Next, we prove that P is invertible and its columns are the eigenvectors of A. The matrix P is constructed from the eigenvectors of A, which form a linearly independent set. Therefore, P is invertible, and the columns of P correspond to the eigenvectors of A. □

Proof of Theorem 57.2 (3/4)

Proof (3/4).

Finally, we prove that the spectral decomposition is unique up to the ordering of the eigenvalues. Since the eigenvectors of A are unique up to scaling, the ordering of the eigenvalues in Λ can be permuted without affecting the decomposition. Thus, the spectral decomposition is unique up to the ordering of the eigenvalues. $\square$

Proof of Theorem 57.2 (4/4)

Proof (4/4).

Therefore, we have shown that Λ is a diagonal matrix containing the eigenvalues of A, P is invertible with its columns being the eigenvectors of A, and the spectral decomposition is unique up to the ordering of the eigenvalues.
This completes the proof of the properties of the negative-dimensional spectral decomposition. $\square$ $\square$

Definition 58.1: Negative-Dimensional Eigenvalue Decomposition

Definition 58.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Eigenvalue Decomposition** of A is a factorization of the form:

$$A = V\Lambda V^{-1},$$

where $\Lambda \in M_{n\times n}(F_{-d})$ is a diagonal matrix containing the eigenvalues of A, and $V \in M_{n\times n}(F_{-d})$ is an invertible matrix whose columns are the eigenvectors of A.
Explanation: The negative-dimensional eigenvalue decomposition is an extension of the classical eigenvalue decomposition to matrices over negative-dimensional fields. It provides a direct representation of matrix transformations in terms of their eigenvalues and eigenvectors, with applications in solving linear systems and matrix functions in negative-dimensional spaces.

Theorem 58.2: Properties of Negative-Dimensional Eigenvalue Decomposition

Theorem 58.2: The negative-dimensional eigenvalue decomposition $A = V\Lambda V^{-1}$ satisfies the following properties:

- Λ is a diagonal matrix containing the eigenvalues of A.
- V is invertible and its columns are the eigenvectors of A.
- The decomposition is unique up to the ordering of the eigenvalues in Λ.

Proof of Theorem 58.2 (1/4)

Proof (1/4).

First, we prove that Λ is a diagonal matrix containing the eigenvalues of A. By the properties of matrices over negative-dimensional fields, the eigenvalues of A are the roots of the characteristic polynomial of A. The matrix Λ is constructed by placing these eigenvalues on the diagonal, making Λ a diagonal matrix. $\square$

Proof of Theorem 58.2 (2/4)

Proof (2/4).

Next, we prove that V is invertible and its columns are the eigenvectors of A. Since the eigenvectors of A form a linearly independent set, the matrix V is invertible, and its columns correspond to the eigenvectors of A. Therefore, $V^{-1}AV = \Lambda$ holds by construction. □

Proof of Theorem 58.2 (3/4)

Proof (3/4).

Finally, we prove that the eigenvalue decomposition is unique up to the ordering of the eigenvalues. The eigenvectors corresponding to distinct eigenvalues are uniquely determined up to scalar multiples. Therefore, while the ordering of the eigenvalues in Λ can be permuted, the decomposition remains valid as long as the corresponding eigenvectors in V are permuted accordingly. $\square$

Proof of Theorem 58.2 (4/4)

Proof (4/4).

Therefore, we have shown that Λ is a diagonal matrix containing the eigenvalues of A, V is invertible with its columns corresponding to the eigenvectors of A, and the decomposition is unique up to the ordering of the eigenvalues.

This completes the proof of the properties of the negative-dimensional eigenvalue decomposition. $\square$ $\square$

Definition 59.1: Negative-Dimensional Singular Value Decomposition (SVD)

Definition 59.1: Let $A \in M_{n\times m}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Singular Value Decomposition (SVD)** of A is a factorization of the form:

$$A = U\Sigma V^T,$$

where $U \in M_{n\times n}(F_{-d})$ and $V \in M_{m\times m}(F_{-d})$ are orthogonal matrices, and $\Sigma \in M_{n\times m}(F_{-d})$ is a diagonal matrix containing the singular values of A.
Explanation: The negative-dimensional SVD generalizes the classical singular value decomposition by extending it to matrices over negative-dimensional fields. It provides a way to decompose a matrix into orthogonal components, with applications in data analysis, signal processing, and optimization in negative-dimensional spaces.

Theorem 59.2: Properties of Negative-Dimensional Singular Value Decomposition

Theorem 59.2: The negative-dimensional SVD $A = U\Sigma V^T$ satisfies the following properties:

- U and V are orthogonal matrices, meaning $U^T U = I_{-d}$ and $V^T V = I_{-d}$.
- Σ is a diagonal matrix containing the singular values of A.
- The rank of A is the number of non-zero singular values in Σ.

Proof of Theorem 59.2 (1/4)

Proof (1/4).

First, we prove that U and V are orthogonal matrices. By the definition of the singular value decomposition, the matrices U and V are obtained from the left and right singular vectors of A, respectively. Since the singular vectors form an orthonormal set, we have:

$$U^T U = I_{-d}, \quad V^T V = I_{-d}.$$

Therefore, U and V are orthogonal matrices. $\square$

Proof of Theorem 59.2 (2/4)

Proof (2/4).

Next, we prove that Σ is a diagonal matrix containing the singular values of A. The matrix Σ is constructed from the singular values, which are the square roots of the eigenvalues of $A^T A$. These singular values are placed on the diagonal of Σ, making it a diagonal matrix. $\square$

Proof of Theorem 59.2 (3/4)

Proof (3/4).

Finally, we prove that the rank of A is the number of non-zero singular values in Σ. Since the rank of a matrix is defined as the dimension of the subspace spanned by its singular vectors corresponding to non-zero singular values, the rank of A is precisely the number of non-zero entries in Σ. $\square$

Proof of Theorem 59.2 (4/4)

Proof (4/4).

Therefore, we have shown that U and V are orthogonal matrices, Σ is a diagonal matrix containing the singular values of A, and the rank of A is the number of non-zero singular values in Σ.
This completes the proof of the properties of the negative-dimensional singular value decomposition. □ □

Definition 60.1: Negative-Dimensional QR Decomposition

Definition 60.1: Let $A \in M_{n \times m}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional QR Decomposition** of A is a factorization of the form:

$$A = QR,$$

where $Q \in M_{n \times n}(F_{-d})$ is an orthogonal matrix (i.e., $Q^T Q = I_{-d}$) and $R \in M_{n \times m}(F_{-d})$ is an upper triangular matrix.
Explanation: The negative-dimensional QR decomposition generalizes the classical QR decomposition by extending it to matrices over negative-dimensional fields. It provides an efficient method for solving linear systems, least-squares problems, and eigenvalue computations in negative-dimensional spaces.

Theorem 60.2: Properties of Negative-Dimensional QR Decomposition

Theorem 60.2: The negative-dimensional QR decomposition $A = QR$ satisfies the following properties:

- Q is an orthogonal matrix, meaning $Q^T Q = I_{-d}$.
- R is an upper triangular matrix.
- If A is full rank, then the decomposition is unique.

Proof of Theorem 60.2 (1/3)

Proof (1/3).

First, we prove that Q is an orthogonal matrix. By the construction of the QR decomposition, the matrix Q is obtained from the Gram-Schmidt orthogonalization process applied to the columns of A. This ensures that the columns of Q form an orthonormal set, and hence $Q^T Q = I_{-d}$. $\square$

Proof of Theorem 60.2 (2/3)

Proof (2/3).

Next, we prove that R is an upper triangular matrix. The matrix R is constructed during the Gram-Schmidt process as the coefficients of the projections of the columns of A onto the orthogonal basis formed by Q. Since these projections are computed sequentially, R is an upper triangular matrix. □

Proof of Theorem 60.2 (3/3)

Proof (3/3).

Finally, we prove that if A is full rank, then the decomposition is unique. If A has full rank, then the Gram-Schmidt process produces a unique orthogonal matrix Q, and the resulting matrix R is uniquely determined by the projections. Therefore, the QR decomposition is unique when A is full rank. □

Definition 61.1: Negative-Dimensional LU Decomposition

Definition 61.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional LU Decomposition** of A is a factorization of the form:

$$A = LU,$$

where $L \in M_{n\times n}(F_{-d})$ is a lower triangular matrix with ones on the diagonal, and $U \in M_{n\times n}(F_{-d})$ is an upper triangular matrix.
Explanation: The negative-dimensional LU decomposition generalizes the classical LU decomposition by extending it to matrices over negative-dimensional fields. It is used for solving linear systems, inverting matrices, and computing determinants in negative-dimensional spaces.

Theorem 61.2: Properties of Negative-Dimensional LU Decomposition

Theorem 61.2: The negative-dimensional LU decomposition $A = LU$ satisfies the following properties:

- L is a lower triangular matrix with ones on the diagonal.
- U is an upper triangular matrix.
- If A is nonsingular, the decomposition is unique.

Proof of Theorem 61.2 (1/3)

Proof (1/3).

First, we prove that L is a lower triangular matrix with ones on the diagonal. By the construction of the LU decomposition, the matrix L is formed by eliminating the entries above the diagonal using row operations. Since these row operations do not affect the diagonal entries, L remains a lower triangular matrix with ones on the diagonal. $\square$

Proof of Theorem 61.2 (2/3)

Proof (2/3).

Next, we prove that U is an upper triangular matrix. The matrix U is constructed by performing Gaussian elimination on A, resulting in an upper triangular matrix. Therefore, by construction, U is upper triangular. $\square$

Proof of Theorem 61.2 (3/3)

Proof (3/3).

Finally, we prove that if A is nonsingular, the decomposition is unique. If A is nonsingular, then the row operations used to compute the LU decomposition are uniquely determined. This ensures that both L and U are uniquely determined, making the LU decomposition unique when A is nonsingular. □

Definition 62.1: Negative-Dimensional Cholesky Decomposition

Definition 62.1: Let $A \in M_{n\times n}(F_{-d})$ be a positive definite matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Cholesky Decomposition** of A is a factorization of the form:

$$A = LL^T,$$

where $L \in M_{n\times n}(F_{-d})$ is a lower triangular matrix with positive diagonal entries.
Explanation: The negative-dimensional Cholesky decomposition generalizes the classical Cholesky decomposition by extending it to matrices over negative-dimensional fields. It provides a numerically stable method for solving linear systems, especially when the matrix is positive definite.

Theorem 62.2: Properties of Negative-Dimensional Cholesky Decomposition

Theorem 62.2: The negative-dimensional Cholesky decomposition $A = LL^T$ satisfies the following properties:

- L is a lower triangular matrix with positive diagonal entries.
- The decomposition exists if and only if A is positive definite.
- The decomposition is unique.

Proof of Theorem 62.2 (1/3)

Proof (1/3).

First, we prove that L is a lower triangular matrix with positive diagonal entries. The Cholesky decomposition is constructed by recursively factoring the matrix A into a product of lower triangular matrices. At each step of the recursion, the diagonal entries of L are computed as square roots of the leading principal minors of A, ensuring that they are positive. $\square$

Proof of Theorem 62.2 (2/3)

Proof (2/3).

Next, we prove that the decomposition exists if and only if A is positive definite. The positive definiteness of A ensures that all leading principal minors are positive, which guarantees the existence of the Cholesky decomposition. Conversely, if the Cholesky decomposition exists, then A must be positive definite, as this property is preserved by the decomposition process. □

Proof of Theorem 62.2 (3/3)

Proof (3/3).

Finally, we prove that the decomposition is unique. Since the Cholesky decomposition involves computing the square roots of the diagonal entries, and these entries are constrained to be positive, the factorization is unique. This completes the proof of the properties of the negative-dimensional Cholesky decomposition. □ □

Definition 63.1: Negative-Dimensional Schur Decomposition

Definition 63.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Schur Decomposition** of A is a factorization of the form:

$$A = QTQ^T,$$

where $Q \in M_{n\times n}(F_{-d})$ is an orthogonal matrix (i.e., $Q^T Q = I_{-d}$) and $T \in M_{n\times n}(F_{-d})$ is an upper triangular matrix.
Explanation: The negative-dimensional Schur decomposition generalizes the classical Schur decomposition by extending it to matrices over negative-dimensional fields. It represents the matrix in terms of orthogonal and triangular components, providing an efficient way to compute eigenvalues and solve matrix equations in negative-dimensional spaces.

Theorem 63.2: Properties of Negative-Dimensional Schur Decomposition

Theorem 63.2: The negative-dimensional Schur decomposition $A = QTQ^T$ satisfies the following properties:

- Q is an orthogonal matrix, meaning $Q^T Q = I_{-d}$.
- T is an upper triangular matrix whose diagonal entries are the eigenvalues of A.
- The decomposition is always possible for any square matrix A, regardless of whether A is diagonalizable.

Proof of Theorem 63.2 (1/3)

Proof (1/3).

First, we prove that Q is an orthogonal matrix. By the construction of the Schur decomposition, the matrix Q is obtained by orthogonally diagonalizing A in a similar way to the QR decomposition. This ensures that $Q^T Q = I_{-d}$, meaning Q is orthogonal. $\square$

Proof of Theorem 63.2 (2/3)

Proof (2/3).

Next, we prove that T is an upper triangular matrix whose diagonal entries are the eigenvalues of A. The matrix T is constructed during the Schur decomposition process, where it is brought into upper triangular form. The diagonal entries of T are the eigenvalues of A, which can be directly read from the diagonal of T. $\square$

Proof of Theorem 63.2 (3/3)

Proof (3/3).

Finally, we prove that the Schur decomposition is always possible for any square matrix A, regardless of whether A is diagonalizable. The Schur decomposition does not require A to be diagonalizable, as it produces an upper triangular matrix T even when the matrix does not have a full set of linearly independent eigenvectors. This guarantees the existence of the Schur decomposition for all square matrices. □

Definition 64.1: Negative-Dimensional Block Matrix Decomposition

Definition 64.1: Let $A \in M_{n \times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Block Matrix Decomposition** of A is a representation of A in terms of smaller block matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are matrices over F_{-d}.
Explanation: The negative-dimensional block matrix decomposition generalizes the classical block matrix decomposition by extending it to matrices over negative-dimensional fields. This decomposition is useful in analyzing large-scale systems, parallel computations, and solving matrix equations by breaking the matrix into smaller, more manageable submatrices.

Theorem 64.2: Properties of Negative-Dimensional Block Matrix Decomposition

Theorem 64.2: The negative-dimensional block matrix decomposition $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ satisfies the following properties:

- The submatrices $A_{11}, A_{12}, A_{21}, A_{22}$ retain the structure of the original matrix.
- The determinant of A can be computed recursively using the block matrices.
- Block matrix operations such as inversion, multiplication, and addition can be performed using the submatrices.

Proof of Theorem 64.2 (1/3)

Proof (1/3).

First, we prove that the submatrices $A_{11}, A_{12}, A_{21}, A_{22}$ retain the structure of the original matrix. Since the block matrix decomposition divides the matrix into smaller submatrices without altering the underlying structure, each submatrix inherits the properties of the original matrix A. For example, if A is symmetric, then A_{11}, A_{22} are symmetric, and $A_{12} = A_{21}^T$. $\square$

Proof of Theorem 64.2 (2/3)

Proof (2/3).

Next, we prove that the determinant of A can be computed recursively using the block matrices. By the properties of block matrices, the determinant of A can be expressed in terms of the determinants of the submatrices. Specifically, for a block matrix of the form:

$$\det(A) = \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12}).$$

This recursive relation allows for efficient computation of the determinant using the block matrix decomposition. $\square$

Proof of Theorem 64.2 (3/3)

Proof (3/3).

Finally, we prove that block matrix operations such as inversion, multiplication, and addition can be performed using the submatrices. For example, the inverse of a block matrix can be computed using the following formula:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} S^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S^{-1} \\ -S^{-1} A_{21} A_{11}^{-1} & S^{-1} \end{bmatrix},$$

where $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$. This demonstrates that block matrix operations can be efficiently performed using the submatrices.
This completes the proof of the properties of the negative-dimensional block matrix decomposition. □ □

Definition 65.1: Negative-Dimensional Tensor Decomposition

Definition 65.1: Let $T \in \mathcal{T}_{n\times m\times p}(F_{-d})$ be a tensor of rank 3 over a negative-dimensional field F_{-d}. The **Negative-Dimensional Tensor Decomposition** is a factorization of T into a sum of outer products of vectors:

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i,$$

where $u_i, v_i, w_i \in F^n_{-d}$ are vectors, and r is the rank of the tensor.
Explanation: The negative-dimensional tensor decomposition extends the classical tensor decomposition by considering tensors over negative-dimensional fields. It is used in higher-order generalizations of matrix decompositions and finds applications in multi-dimensional data analysis, quantum computing, and physics in negative-dimensional spaces.

Theorem 65.2: Properties of Negative-Dimensional Tensor Decomposition

Theorem 65.2: The negative-dimensional tensor decomposition $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$ satisfies the following properties:

- The rank r of the tensor is the minimal number of terms required for the decomposition.
- The vectors u_i, v_i, w_i are linearly independent.
- The decomposition is unique up to scaling and reordering of the terms.

Proof of Theorem 65.2 (1/3)

Proof (1/3).

First, we prove that the rank r of the tensor is the minimal number of terms required for the decomposition. By definition, the rank of a tensor is the minimum number of outer product terms required to represent the tensor. If more terms are used than the rank, the tensor decomposition would be overcomplete, contradicting the minimality of r. □

Proof of Theorem 65.2 (2/3)

Proof (2/3).

Next, we prove that the vectors u_i, v_i, w_i are linearly independent. The independence of these vectors is necessary for the decomposition to fully capture the structure of the tensor. If any of the vectors were linearly dependent, the tensor could be written as a sum of fewer terms, which would contradict the definition of the rank. $\square$

Proof of Theorem 65.2 (3/3)

Proof (3/3).

Finally, we prove that the decomposition is unique up to scaling and reordering of the terms. Since the tensor rank is minimal and the vectors u_i, v_i, w_i are independent, the decomposition is determined uniquely except for possible scalar multiplication of the vectors and reordering of the terms. This completes the proof of the properties of the negative-dimensional tensor decomposition. □ □

Definition 66.1: Negative-Dimensional Convex Optimization

Definition 66.1: A **Negative-Dimensional Convex Optimization Problem** is an optimization problem over a negative-dimensional field F_{-d}, where the objective function $f : F^n_{-d} \to F_{-d}$ is convex, i.e.,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in F^n_{-d}$ and $\lambda \in [0,1]_{-d}$.
Explanation: Negative-dimensional convex optimization extends the framework of convex optimization to fields with negative dimensions, allowing for the development of optimization techniques in exotic algebraic structures. This framework is particularly relevant in settings where classical optimization is insufficient, such as negative-curvature spaces and quantum systems.

Theorem 66.2: Existence and Uniqueness in Negative-Dimensional Convex Optimization

Theorem 66.2: Let $f : F^n_{-d} \to F_{-d}$ be a convex function. A solution $x^* \in F^n_{-d}$ to the convex optimization problem exists and is unique if f is strictly convex, i.e., if:

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$$

for all distinct $x, y \in F^n_{-d}$ and $\lambda \in (0,1)_{-d}$.

Proof of Theorem 66.2 (1/2)

Proof (1/2).

First, we prove the existence of a solution. Since f is convex and defined over a negative-dimensional field F_{-d}, the lower semicontinuity of f guarantees that the infimum of f over a closed and bounded set $S \subseteq F^n_{-d}$ is attained at some point $x^* \in S$. Hence, a solution exists.
Next, we prove the uniqueness of the solution. If f is strictly convex, then for any distinct points $x, y \in F^n_{-d}$ and any $\lambda \in (0, 1)_{-d}$, we have:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

This implies that the objective function has a unique minimizer. $\square$

Proof of Theorem 66.2 (2/2)

Proof (2/2).

Therefore, the strict convexity of f ensures that the solution x^* is unique. This completes the proof of the existence and uniqueness of solutions in negative-dimensional convex optimization. $\square$ $\square$

Definition 67.1: Negative-Dimensional Gradient Descent

Definition 67.1: Let $f : F^n_{-d} \to F_{-d}$ be a differentiable convex function. The **Negative-Dimensional Gradient Descent** algorithm is an iterative optimization method defined by the update rule:

$$x^{k+1} = x^k - \alpha \nabla f(x^k),$$

where $\alpha > 0$ is the learning rate, and $\nabla f(x^k)$ is the gradient of f at x^k, computed over the negative-dimensional field F_{-d}.
Explanation: Negative-dimensional gradient descent extends the classical gradient descent method to optimization problems over negative-dimensional fields. It allows for iterative optimization in spaces with negative curvature or other non-standard geometrical properties, making it a powerful tool in exotic optimization settings.

Theorem 67.2: Convergence of Negative-Dimensional Gradient Descent

Theorem 67.2: Let $f : F^n_{-d} \to F_{-d}$ be a differentiable convex function. The negative-dimensional gradient descent algorithm converges to a global minimum $x^* \in F^n_{-d}$ if the learning rate α is chosen sufficiently small and f is Lipschitz smooth, i.e., there exists $L > 0$ such that:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in F^n_{-d}.$$

Proof of Theorem 67.2 (1/2)

Proof (1/2).

First, we prove the convergence of the negative-dimensional gradient descent algorithm. Since f is convex and Lipschitz smooth, the gradient of f provides a direction of steepest descent. By choosing a sufficiently small learning rate α, the descent step $x^{k+1} = x^k - \alpha \nabla f(x^k)$ ensures that the objective function $f(x^{k+1})$ decreases monotonically at each iteration.
We now establish that the algorithm converges to a global minimum. Since f is convex, every local minimum is a global minimum. $\square$

Proof of Theorem 67.2 (2/2)

Proof (2/2).

Finally, by the continuity and differentiability of f in the negative-dimensional field F_{-d}, the sequence $\{x^k\}$ generated by the gradient descent algorithm converges to a global minimizer $x^* \in F^n_{-d}$. This completes the proof of the convergence of negative-dimensional gradient descent. $\square$ $\square$

Definition 68.1: Negative-Dimensional Newton's Method

Definition 68.1: Let $f : F^n_{-d} \to F_{-d}$ be a twice-differentiable convex function. The **Negative-Dimensional Newton's Method** is an iterative optimization algorithm defined by the update rule:

$$x^{k+1} = x^k - H_f(x^k)^{-1} \nabla f(x^k),$$

where $H_f(x^k)$ is the Hessian matrix of f at x^k computed over the negative-dimensional field F_{-d}, and $\nabla f(x^k)$ is the gradient of f at x^k.
Explanation: Negative-dimensional Newton's method extends the classical Newton's method to negative-dimensional fields. It accelerates convergence to the minimum by incorporating second-order information, making it more efficient than first-order methods like gradient descent in certain exotic optimization problems.

Theorem 68.2: Convergence of Negative-Dimensional Newton's Method

Theorem 68.2: Let $f : F^n_{-d} \to F_{-d}$ be a twice-differentiable strictly convex function. If the initial point $x^0 \in F^n_{-d}$ is sufficiently close to the global minimizer x^*, then the negative-dimensional Newton's method converges quadratically to x^*.

Proof of Theorem 68.2 (1/2)

Proof (1/2).

First, we prove that the negative-dimensional Newton's method converges quadratically to the global minimizer x^*. Since f is twice-differentiable and strictly convex, the Hessian $H_f(x)$ is positive definite for all $x \in F^n_{-d}$. This guarantees that the Newton update step moves x^{k+1} closer to the minimizer.

Let $e^k = x^k - x^*$ denote the error at iteration k. By Taylor expansion of $\nabla f(x)$ around x^*, we have:

$$\nabla f(x^k) = H_f(x^*)e^k + \mathcal{O}((e^k)^2).$$

Substituting this into the Newton update equation gives:

$$e^{k+1} = -H_f(x^*)^{-1}\nabla f(x^k) = \mathcal{O}((e^k)^2),$$

which shows that the error decreases quadratically. □

Proof of Theorem 68.2 (2/2)

Proof (2/2).

Therefore, the quadratic convergence of the error e^k implies that the Newton iterates converge quadratically to the global minimizer x^*. This completes the proof of the convergence of negative-dimensional Newton's method. □ □

Definition 69.1: Negative-Dimensional Lagrangian Multipliers

Definition 69.1: Let $f : F^n_{-d} \to F_{-d}$ be the objective function, and let $g_i : F^n_{-d} \to F_{-d}$ for $i = 1, \ldots, m$ be constraint functions. The **Negative-Dimensional Lagrangian** for the constrained optimization problem is defined as:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in F^m_{-d}$ are the Lagrange multipliers.
Explanation: Negative-dimensional Lagrange multipliers extend the classical method of Lagrange multipliers to negative-dimensional fields, providing a technique for solving constrained optimization problems in these exotic spaces.

Theorem 69.2: Existence of Negative-Dimensional Lagrange Multipliers

Theorem 69.2: Let $f : F^n_{-d} \to F_{-d}$ and $g_i : F^n_{-d} \to F_{-d}$ be continuously differentiable functions. If $x^* \in F^n_{-d}$ is a local minimizer of f subject to the constraints $g_i(x) = 0$ for $i = 1, \ldots, m$, then there exist Lagrange multipliers $\lambda^* \in F^m_{-d}$ such that:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \quad \text{and} \quad g_i(x^*) = 0 \quad \text{for all } i.$$

Proof of Theorem 69.2 (1/2)

Proof (1/2).

First, we prove the existence of Lagrange multipliers λ^*. By the method of Lagrange multipliers, the necessary condition for x^* to be a local minimizer subject to the constraints $g_i(x^*) = 0$ is that the gradient of the Lagrangian $\mathcal{L}(x, \lambda)$ with respect to x vanishes at x^*:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0.$$

Since f and g_i are continuously differentiable and defined over the negative-dimensional field F_{-d}, the gradient equation is well-defined, and a solution λ^* exists. $\square$

Proof of Theorem 69.2 (2/2)

Proof (2/2).

Furthermore, the constraints $g_i(x^*) = 0$ must hold at the solution. This gives the system of equations:

$$g_i(x^*) = 0 \quad \text{for all } i = 1, \ldots, m,$$

ensuring that the constraints are satisfied. This completes the proof of the existence of negative-dimensional Lagrange multipliers. □ □

Definition 70.1: Negative-Dimensional Karush-Kuhn-Tucker (KKT) Conditions

Definition 70.1: The **Negative-Dimensional Karush-Kuhn-Tucker (KKT) Conditions** for the optimization problem:

$$\min f(x) \quad \text{subject to } g_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m,$$

where $f : F^n_{-d} \to F_{-d}$ and $g_i : F^n_{-d} \to F_{-d}$, are the following:

- Primal feasibility: $g_i(x^*) \leq 0$ for all i.
- Dual feasibility: $\lambda_i^* \geq 0$ for all i.
- Stationarity: $\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$.
- Complementary slackness: $\lambda_i^* g_i(x^*) = 0$ for all i.

Explanation: The negative-dimensional KKT conditions extend the classical KKT conditions to optimization problems over negative-dimensional fields. These conditions are necessary for optimality in constrained optimization problems.

Definition 71.1: Negative-Dimensional Singular Value Decomposition (SVD)

Definition 71.1: Let $A \in M_{n\times m}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Singular Value Decomposition (SVD)** is a factorization of A into the product of three matrices:

$$A = U\Sigma V^T,$$

where:

- $U \in M_{n\times n}(F_{-d})$ is an orthogonal matrix (i.e., $U^T U = I_{-d}$),
- $\Sigma \in M_{n\times m}(F_{-d})$ is a diagonal matrix whose diagonal entries are the singular values of A,
- $V \in M_{m\times m}(F_{-d})$ is an orthogonal matrix (i.e., $V^T V = I_{-d}$).

Explanation: The negative-dimensional SVD extends the classical singular value decomposition to matrices over negative-dimensional fields. This factorization is crucial in data compression, dimensionality reduction, and solving inverse problems in exotic algebraic structures.

Theorem 71.2: Properties of Negative-Dimensional SVD

Theorem 71.2: The negative-dimensional SVD $A = U\Sigma V^T$ satisfies the following properties:

- The singular values in Σ are non-negative, real, and correspond to the square roots of the eigenvalues of $A^T A$ over F_{-d}.
- The columns of U are the left singular vectors of A, and the columns of V are the right singular vectors of A.
- The decomposition always exists, even if A is not square.

Proof of Theorem 71.2 (1/3)

Proof (1/3).

First, we prove that the singular values in Σ are non-negative and real. The singular values of A are defined as the square roots of the eigenvalues of $A^T A$. Since $A^T A$ is symmetric and positive semi-definite, its eigenvalues are non-negative, and thus the singular values are non-negative and real, even in negative-dimensional fields. $\square$

Proof of Theorem 71.2 (2/3)

Proof (2/3).

Next, we prove that the columns of U are the left singular vectors and the columns of V are the right singular vectors. By construction, the matrix U contains the eigenvectors of AA^T, and the matrix V contains the eigenvectors of A^TA. These eigenvectors form the singular vectors of A, corresponding to the non-zero singular values in Σ. $\square$

Proof of Theorem 71.2 (3/3)

Proof (3/3).

Finally, we prove that the negative-dimensional SVD exists for all matrices A, even if A is not square. The factorization $A = U\Sigma V^T$ holds for any matrix $A \in M_{n\times m}(F_{-d})$ because the matrices U and V can always be orthogonalized, and the singular values are the square roots of the eigenvalues of $A^T A$ and AA^T, which always exist in negative-dimensional fields. This completes the proof of the properties of negative-dimensional SVD. □ □

Definition 72.1: Negative-Dimensional QR Decomposition

Definition 72.1: Let $A \in M_{n\times m}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional QR Decomposition** is a factorization of A into the product of two matrices:

$$A = QR,$$

where:

- $Q \in M_{n\times n}(F_{-d})$ is an orthogonal matrix (i.e., $Q^T Q = I_{-d}$),
- $R \in M_{n\times m}(F_{-d})$ is an upper triangular matrix.

Explanation: The negative-dimensional QR decomposition generalizes the classical QR decomposition to matrices over negative-dimensional fields. This decomposition is useful in solving linear systems, least-squares problems, and eigenvalue problems in exotic algebraic structures.

Theorem 72.2: Properties of Negative-Dimensional QR Decomposition

Theorem 72.2: The negative-dimensional QR decomposition $A = QR$ satisfies the following properties:

- The matrix Q is orthogonal, meaning $Q^T Q = I_{-d}$.
- The matrix R is upper triangular.
- The decomposition always exists for any full-rank matrix $A \in M_{n \times m}(F_{-d})$.

Proof of Theorem 72.2 (1/2)

Proof (1/2).

First, we prove that Q is orthogonal, meaning $Q^T Q = I_{-d}$. By the construction of the QR decomposition, the matrix Q is formed by applying the Gram-Schmidt process to the columns of A, ensuring that Q is orthogonal. This property holds in negative-dimensional fields as well, since orthogonality is a geometric property that generalizes to negative-dimensional spaces. □

Proof of Theorem 72.2 (2/2)

Proof (2/2).

Next, we prove that R is an upper triangular matrix. The matrix R is formed by projecting the columns of A onto the orthogonal basis defined by Q, resulting in an upper triangular structure. Finally, we prove that the decomposition exists for any full-rank matrix $A \in M_{n \times m}(F_{-d})$, as the Gram-Schmidt process always produces an orthogonal matrix Q and an upper triangular matrix R, completing the QR decomposition. □ □

Definition 73.1: Negative-Dimensional Eigenvalue Problem

Definition 73.1: Let $A \in M_{n\times n}(F_{-d})$ be a square matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Eigenvalue Problem** is the problem of finding scalars $\lambda \in F_{-d}$ and non-zero vectors $v \in F^n_{-d}$ such that:

$$Av = \lambda v.$$

Explanation: The negative-dimensional eigenvalue problem generalizes the classical eigenvalue problem to matrices over negative-dimensional fields. Solving this problem provides critical information about the behavior of linear transformations in negative-dimensional spaces, including stability, spectral decomposition, and dynamic properties of systems.

Definition 74.1: Negative-Dimensional Eigenvalue Decomposition

Definition 74.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Eigenvalue Decomposition** is a factorization of A into the form:

$$A = V\Lambda V^{-1},$$

where:

- $V \in M_{n\times n}(F_{-d})$ is a matrix whose columns are the eigenvectors of A,
- $\Lambda \in M_{n\times n}(F_{-d})$ is a diagonal matrix whose diagonal entries are the eigenvalues of A.

Explanation: The negative-dimensional eigenvalue decomposition extends the classical eigenvalue decomposition to matrices over negative-dimensional fields. This factorization is crucial in analyzing linear transformations and dynamical systems in spaces with negative dimensions.

Theorem 74.2: Properties of Negative-Dimensional Eigenvalue Decomposition

Theorem 74.2: The negative-dimensional eigenvalue decomposition $A = V\Lambda V^{-1}$ satisfies the following properties:

- The eigenvalues in Λ are the roots of the characteristic polynomial $\det(A - \lambda I_{-d}) = 0$.
- The eigenvectors in V are linearly independent.
- The decomposition exists if A is diagonalizable over F_{-d}.

Proof of Theorem 74.2 (1/2)

Proof (1/2).

First, we prove that the eigenvalues in Λ are the roots of the characteristic polynomial. The characteristic polynomial of A is defined as:

$$p(\lambda) = \det(A - \lambda I_{-d}),$$

where I_{-d} is the identity matrix in the negative-dimensional field F_{-d}. The roots of this polynomial are the eigenvalues of A, which form the diagonal entries of Λ. $\square$

Proof of Theorem 74.2 (2/2)

Proof (2/2).

Next, we prove that the eigenvectors in V are linearly independent. Since the eigenvalues are distinct (if A is diagonalizable), the corresponding eigenvectors are linearly independent. Finally, we prove that the eigenvalue decomposition exists if A is diagonalizable. If A has n linearly independent eigenvectors, it can be diagonalized as $A = V\Lambda V^{-1}$, completing the proof of the properties of negative-dimensional eigenvalue decomposition. □ □

Definition 75.1: Negative-Dimensional Jordan Canonical Form

Definition 75.1: Let $A \in M_{n\times n}(F_{-d})$ be a matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Jordan Canonical Form** of A is a decomposition of A into the form:

$$A = PJP^{-1},$$

where:

- $P \in M_{n\times n}(F_{-d})$ is an invertible matrix,
- $J \in M_{n\times n}(F_{-d})$ is a block diagonal matrix, with each block being a Jordan block corresponding to an eigenvalue of A.

Explanation: The negative-dimensional Jordan canonical form extends the classical Jordan canonical form to negative-dimensional fields. This decomposition is useful in studying the structure of linear operators in exotic algebraic systems, especially when A is not diagonalizable.

Theorem 75.2: Properties of Negative-Dimensional Jordan Canonical Form

Theorem 75.2: The negative-dimensional Jordan canonical form $A = PJP^{-1}$ satisfies the following properties:

- The diagonal entries of J are the eigenvalues of A.
- The size of each Jordan block corresponds to the geometric multiplicity of the eigenvalue.
- The decomposition exists for any matrix $A \in M_{n \times n}(F_{-d})$.

Proof of Theorem 75.2 (1/2)

Proof (1/2).

First, we prove that the diagonal entries of J are the eigenvalues of A. The matrix J is block diagonal, with each block corresponding to an eigenvalue of A, and thus the diagonal entries of J are precisely the eigenvalues. Next, we prove that the size of each Jordan block corresponds to the geometric multiplicity of the eigenvalue. The Jordan blocks account for the generalized eigenvectors associated with each eigenvalue, and their size corresponds to the geometric multiplicity of the eigenvalue. $\square$

Proof of Theorem 75.2 (2/2)

Proof (2/2).

Finally, we prove that the Jordan canonical form exists for any matrix $A \in M_{n\times n}(F_{-d})$. The Jordan form generalizes the diagonalization process to cases where A is not diagonalizable. Since every square matrix over a field has a Jordan canonical form, this holds in negative-dimensional fields as well. This completes the proof of the properties of negative-dimensional Jordan canonical form. $\square$ $\square$

Definition 76.1: Negative-Dimensional Spectral Theorem

Definition 76.1: Let $A \in M_{n\times n}(F_{-d})$ be a symmetric matrix over a negative-dimensional field F_{-d}. The **Negative-Dimensional Spectral Theorem** states that A can be diagonalized as:

$$A = V\Lambda V^T,$$

where:

- $V \in M_{n\times n}(F_{-d})$ is an orthogonal matrix whose columns are the eigenvectors of A,
- $\Lambda \in M_{n\times n}(F_{-d})$ is a diagonal matrix whose diagonal entries are the eigenvalues of A.

Explanation: The negative-dimensional spectral theorem extends the classical spectral theorem to symmetric matrices over negative-dimensional fields, allowing for the diagonalization of these matrices and providing insights into the spectral properties of linear operators in exotic algebraic systems.

Theorem 76.2: Properties of Negative-Dimensional Spectral Theorem

Theorem 76.2: The negative-dimensional spectral theorem $A = V\Lambda V^T$ satisfies the following properties:

- The matrix A is diagonalizable if and only if it is symmetric.
- The eigenvalues in Λ are real.
- The eigenvectors in V form an orthonormal basis of F^n_{-d}.

Definition 77.1: Negative-Dimensional Fourier Transform

Definition 77.1: Let $f : F^n_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier Transform** of f is defined as:

$$\hat{f}(\xi) = \int_{F^n_{-d}} f(x) e^{-2\pi i \langle x, \xi \rangle_{-d}} \, dx,$$

where $\xi \in F^n_{-d}$ and $\langle x, \xi \rangle_{-d}$ denotes the negative-dimensional inner product.

Explanation: The negative-dimensional Fourier transform generalizes the classical Fourier transform to functions over negative-dimensional fields. It is a tool for analyzing the frequency components of signals or functions in negative-dimensional spaces.

Theorem 77.2: Inversion Formula for Negative-Dimensional Fourier Transform

Theorem 77.2: Let $\hat{f}$ be the negative-dimensional Fourier transform of f. The inverse Fourier transform is given by:

$$f(x) = \int_{F^n_{-d}} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle_{-d}} \, d\xi,$$

where $x \in F^n{}_{-d}$.
Explanation: The inversion formula allows the recovery of the original function f from its Fourier transform $\hat{f}$ in negative-dimensional fields, preserving the duality between time and frequency domains.

Proof of Theorem 77.2 (1/2)

Proof (1/2).

First, we prove the inversion formula for the negative-dimensional Fourier transform by computing the inverse of the transform. Using the Fourier inversion theorem, we start by expressing $f(x)$ as:

$$f(x) = \int_{F^n_{-d}} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle_{-d}} \, d\xi.$$

We then substitute the expression for $\hat{f}(\xi)$ into the equation:

$$\hat{f}(\xi) = \int_{F^n_{-d}} f(x) e^{-2\pi i \langle x, \xi \rangle_{-d}} \, dx.$$

□

Proof of Theorem 77.2 (2/2)

Proof (2/2).

Substituting $\hat{f}(\xi)$ into the inversion formula, we get:

$$f(x) = \int_{F^n_{-d}} \left(\int_{F^n_{-d}} f(x') e^{-2\pi i \langle x', \xi \rangle_{-d}} \, dx' \right) e^{2\pi i \langle x, \xi \rangle_{-d}} \, d\xi.$$

Applying Fubini's theorem and simplifying the exponentials yields the Dirac delta function $\delta(x - x')$, which gives $f(x) = f(x')$, proving the inversion formula. □ □

Definition 78.1: Negative-Dimensional Convolution Theorem

Definition 78.1: Let $f, g : F^n_{-d} \to F_{-d}$ be functions over a negative-dimensional field F_{-d}. The **Negative-Dimensional Convolution Theorem** states that the Fourier transform of the convolution of f and g is the product of their Fourier transforms:

$$\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi),$$

where $*$ denotes the convolution operation.
Explanation: The negative-dimensional convolution theorem extends the classical convolution theorem to negative-dimensional fields, providing a powerful tool for solving differential equations and analyzing systems in exotic spaces.

Theorem 78.2: Convolution Formula for Negative-Dimensional Fields

Theorem 78.2: Let $f, g : F^n_{-d} \to F_{-d}$ be functions. The convolution of f and g in negative-dimensional fields is given by:

$$(f * g)(x) = \int_{F^n_{-d}} f(x')g(x - x')\, dx'.$$

Explanation: This formula defines the convolution operation in negative-dimensional fields, preserving the same structure as in classical convolution, but adapted to exotic algebraic structures.

Proof of Theorem 78.2 (1/2)

Proof (1/2).

We begin by proving the convolution formula in negative-dimensional fields. By definition, the convolution of f and g is:

$$(f * g)(x) = \int_{F^n_{-d}} f(x')g(x - x')\, dx'.$$

To show that this satisfies the convolution theorem, we take the Fourier transform of both sides:

$$\mathcal{F}(f * g)(\xi) = \int_{F^n_{-d}} \left(\int_{F^n_{-d}} f(x')g(x - x')\, dx' \right) e^{-2\pi i \langle x, \xi \rangle_{-d}}\, dx.$$

□

Proof of Theorem 78.2 (2/2)

Proof (2/2).

Applying Fubini's theorem and using the properties of the negative-dimensional Fourier transform, we simplify the convolution integral:

$$\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi),$$

which proves the convolution theorem in negative-dimensional fields. □ □

Definition 79.1: Negative-Dimensional Parseval's Theorem

Definition 79.1: Let $f, g : {F^n}_{-d} \to F_{-d}$ be square-integrable functions over a negative-dimensional field F_{-d}. **Negative-Dimensional Parseval's Theorem** states that:

$$\int_{{F^n}_{-d}} f(x)\overline{g(x)}\, dx = \int_{{F^n}_{-d}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi,$$

where $\hat{f}$ and $\hat{g}$ are the Fourier transforms of f and g, respectively. Explanation: Parseval's theorem in negative-dimensional fields extends the classical result, providing a connection between the inner product of functions in the time domain and their corresponding Fourier transforms in the frequency domain.

Definition 80.1: Negative-Dimensional Laplace Transform

Definition 80.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Laplace Transform** of f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

where $t, s \in F_{-d}$.
Explanation: The negative-dimensional Laplace transform extends the classical Laplace transform to functions over negative-dimensional fields. It is commonly used to solve differential equations and analyze systems in negative-dimensional spaces.

Theorem 80.2: Inversion Formula for Negative-Dimensional Laplace Transform

Theorem 80.2: The inverse Laplace transform is given by:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st}\, ds,$$

where $\gamma \in F_{-d}$ is a real number chosen such that the contour of integration lies to the right of all singularities of $F(s)$.
Explanation: This inversion formula allows the recovery of the original function $f(t)$ from its Laplace transform in negative-dimensional fields.

Proof of Theorem 80.2 (1/3)

Proof (1/3).

To prove the inversion formula, we start by expressing the Laplace transform $F(s)$ of a function $f(t)$ as:

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt.$$

Now, to recover $f(t)$, we apply the Bromwich integral, defined as:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}\, ds.$$

□

Proof of Theorem 80.2 (2/3)

Proof (2/3).

Substituting the definition of $F(s)$ into the Bromwich integral, we get:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left(\int_0^\infty f(\tau) e^{-s\tau} \, d\tau \right) e^{st} \, ds.$$

By applying Fubini's theorem to switch the order of integration, we obtain:

$$f(t) = \int_0^\infty f(\tau) \left(\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{s(t-\tau)} \, ds \right) d\tau.$$

□

Proof of Theorem 80.2 (3/3)

Proof (3/3).

The integral inside the parentheses is the inverse Laplace transform of $e^{s(t-\tau)}$, which results in a Dirac delta function $\delta(t-\tau)$. Thus, the expression simplifies to:

$$f(t) = \int_0^\infty f(\tau)\delta(t-\tau)\, d\tau = f(t),$$

which completes the proof. $\square$ $\square$

Definition 81.1: Negative-Dimensional Z-Transform

Definition 81.1: Let $f[n] : F_{-d} \to F_{-d}$ be a discrete-time function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Z-Transform** of f is defined as:

$$\mathcal{Z}\{f[n]\} = F(z) = \sum_{n=0}^{\infty} f[n] z^{-n},$$

where $z \in F_{-d}$.
Explanation: The negative-dimensional Z-transform generalizes the classical Z-transform to functions over negative-dimensional fields, allowing for the analysis of discrete-time systems and signals in these spaces.

Theorem 81.2: Inversion Formula for Negative-Dimensional Z-Transform

Theorem 81.2: The inverse Z-transform is given by:

$$f[n] = \frac{1}{2\pi i} \oint_C F(z) z^{n-1} \, dz,$$

where C is a closed contour that encircles all singularities of $F(z)$.
Explanation: This formula allows the recovery of the original discrete-time function $f[n]$ from its Z-transform in negative-dimensional fields.

Proof of Theorem 81.2 (1/2)

Proof (1/2).

To prove the inversion formula, we start by expressing the Z-transform $F(z)$ of $f[n]$ as:

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}.$$

Now, to recover $f[n]$, we apply the inverse Z-transform formula:

$$f[n] = \frac{1}{2\pi i} \oint_C F(z)z^{n-1}\, dz.$$

□

Proof of Theorem 81.2 (2/2)

Proof (2/2).

Substituting the expression for $F(z)$ into the contour integral, we obtain:

$$f[n] = \frac{1}{2\pi i} \oint_C \left(\sum_{k=0}^{\infty} f[k] z^{-k} \right) z^{n-1} \, dz.$$

By applying the residue theorem and recognizing the contour integral as selecting the z^n-term from the series, we recover the original sequence $f[n]$, completing the proof. □ □

Definition 82.1: Negative-Dimensional Green's Function

Definition 82.1: Let L be a linear differential operator acting on a function space over a negative-dimensional field F_{-d}. The **Negative-Dimensional Green's Function** $G(x, x')$ is a function that satisfies:

$$LG(x, x') = \delta(x - x'),$$

where $\delta(x - x')$ is the Dirac delta function in negative-dimensional fields. Explanation: The negative-dimensional Green's function extends the classical Green's function to negative-dimensional fields, used to solve inhomogeneous differential equations in these spaces.

Definition 83.1: Negative-Dimensional Wavelet Transform

Definition 83.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let ψ be a wavelet function. The **Negative-Dimensional Wavelet Transform** of f is defined as:

$$\mathcal{W}_{\psi}\{f(t)\}(a, b) = \int_{F_{-d}} f(t) \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t-b}{a}\right) dt,$$

where $a, b \in F_{-d}$ are the scale and translation parameters, respectively. Explanation: The negative-dimensional wavelet transform extends the classical wavelet transform to negative-dimensional fields. It is used to decompose functions into scaled and translated versions of a mother wavelet, providing time-frequency analysis in negative-dimensional spaces.

Theorem 83.2: Inversion Formula for Negative-Dimensional Wavelet Transform

Theorem 83.2: The inverse wavelet transform is given by:

$$f(t) = \frac{1}{C_\psi} \int_{F_{-d}} \int_{F_{-d}} \mathcal{W}_\psi\{f\}(a, b) \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t-b}{a}\right) \frac{da\,db}{a^2},$$

where C_ψ is the wavelet admissibility constant.
Explanation: This inversion formula allows the reconstruction of the original function f from its wavelet transform in negative-dimensional fields.

Proof of Theorem 83.2 (1/3)

Proof (1/3).

To prove the inversion formula for the negative-dimensional wavelet transform, we start by writing the wavelet transform of f as:

$$\mathcal{W}_\psi\{f(t)\}(a,b) = \int_{F_{-d}} f(t') \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t'-b}{a}\right) dt'.$$

The goal is to invert this transform and express $f(t)$ in terms of its wavelet coefficients. Using the inverse wavelet transform formula, we have:

$$f(t) = \frac{1}{C_\psi} \int_{F_{-d}} \int_{F_{-d}} \mathcal{W}_\psi\{f(t')\}(a,b) \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t-b}{a}\right) \frac{da\,db}{a^2}.$$

□

Proof of Theorem 83.2 (2/3)

Proof (2/3).

Substituting the expression for $\mathcal{W}_\psi\{f(t')\}(a, b)$ into the inversion formula, we get:

$$f(t) = \frac{1}{C_\psi} \int_{F_{-d}} \int_{F_{-d}} \left(\int_{F_{-d}} f(t') \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t'-b}{a}\right) dt' \right) \frac{1}{\sqrt{|a|_{-d}}} \psi\left(\frac{t-b}{a}\right.$$

Applying Fubini's theorem to switch the order of integration, we simplify the equation. □

Proof of Theorem 83.2 (3/3)

Proof (3/3).

The inner integral can be reduced using the orthogonality of the wavelets ψ, leading to the expression:

$$f(t) = \int_{F_{-d}} f(t')\delta(t - t')\, dt',$$

where $\delta(t - t')$ is the Dirac delta function in negative-dimensional fields. This simplifies to $f(t) = f(t)$, completing the proof. $\square$ $\square$

Definition 84.1: Negative-Dimensional Hilbert Transform

Definition 84.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Hilbert Transform** of f is defined as:

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} \, d\tau,$$

where the integral is understood as a Cauchy principal value.
Explanation: The negative-dimensional Hilbert transform extends the classical Hilbert transform to negative-dimensional fields, providing a method to extract the analytical signal and analyze phase and amplitude relationships.

Theorem 84.2: Properties of Negative-Dimensional Hilbert Transform

Theorem 84.2: The negative-dimensional Hilbert transform $\mathcal{H}\{f(t)\}$ satisfies the following properties:

- Linearity: $\mathcal{H}\{af + bg\} = a\mathcal{H}\{f\} + b\mathcal{H}\{g\}$ for $a, b \in F_{-d}$.
- Involution: $\mathcal{H}\{\mathcal{H}\{f(t)\}\} = -f(t)$.
- Frequency domain: $\mathcal{H}\{\cos(\omega t)\} = \sin(\omega t)$.

Proof of Theorem 84.2 (1/2)

Proof (1/2).

We begin by proving the linearity property. For any functions $f, g : F_{-d} \to F_{-d}$ and constants $a, b \in F_{-d}$, we have:

$$\mathcal{H}\{af + bg\}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{af(\tau) + bg(\tau)}{t - \tau} \, d\tau.$$

By the linearity of integration, this becomes:

$$\mathcal{H}\{af + bg\}(t) = a\mathcal{H}\{f\}(t) + b\mathcal{H}\{g\}(t).$$

$\square$

Proof of Theorem 84.2 (2/2)

Proof (2/2).

To prove the involution property, we apply the Hilbert transform twice:

$$\mathcal{H}\{\mathcal{H}\{f(t)\}\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{H}\{f(\tau)\}}{t - \tau} \, d\tau.$$

Using the known result for the Hilbert transform of a Hilbert transform, we find:

$$\mathcal{H}\{\mathcal{H}\{f(t)\}\} = -f(t),$$

proving the involution property. □ □

Definition 85.1: Negative-Dimensional Mellin Transform

Definition 85.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Mellin Transform** of f is defined as:

$$\mathcal{M}\{f(t)\}(s) = \int_0^\infty t^{s-1} f(t)\, dt,$$

where $s \in F_{-d}$.
Explanation: The negative-dimensional Mellin transform extends the classical Mellin transform to negative-dimensional fields. It is often used in the analysis of scale-invariant systems and in solving differential equations.

Theorem 85.2: Inversion Formula for Negative-Dimensional Mellin Transform

Theorem 85.2: The inverse Mellin transform is given by:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{f(t)\}(s) t^{-s} \, ds,$$

where c is a real number such that the contour of integration lies in the region of convergence of $\mathcal{M}\{f(t)\}(s)$.
Explanation: This formula allows the recovery of the original function $f(t)$ from its Mellin transform in negative-dimensional fields.

Definition 86.1: Negative-Dimensional Fourier-Bessel Transform

Definition 86.1: Let $f : F_{-d} \to F_{-d}$ be a radially symmetric function over a negative-dimensional field F_{-d}, and let J_ν denote the Bessel function of the first kind of order ν. The **Negative-Dimensional Fourier-Bessel Transform** of f is defined as:

$$\mathcal{F}_B\{f(r)\}(k) = \int_0^\infty f(r) J_\nu(kr) r \, dr,$$

where $r, k \in F_{-d}$ and $\nu \in \mathbb{R}$.
Explanation: The negative-dimensional Fourier-Bessel transform generalizes the classical Fourier-Bessel transform to functions defined over negative-dimensional fields, often used in problems involving radial symmetry.

Theorem 86.2: Inversion Formula for Negative-Dimensional Fourier-Bessel Transform

Theorem 86.2: The inverse Fourier-Bessel transform is given by:

$$f(r) = \int_0^\infty \mathcal{F}_B\{f(k)\} J_\nu(kr) k \, dk,$$

where J_ν is the Bessel function of the first kind of order ν.
Explanation: This inversion formula allows the recovery of the original function $f(r)$ from its Fourier-Bessel transform in negative-dimensional fields.

Proof of Theorem 86.2 (1/2)

Proof (1/2).

The Fourier-Bessel transform of $f(r)$ is given by:

$$\mathcal{F}_B\{f(r)\}(k) = \int_0^\infty f(r) J_\nu(kr) r \, dr.$$

To invert the transform, we substitute $\mathcal{F}_B\{f(k)\}$ into the inversion formula:

$$f(r) = \int_0^\infty \left(\int_0^\infty f(t) J_\nu(kt) t \, dt \right) J_\nu(kr) k \, dk.$$

By switching the order of integration using Fubini's theorem, we get:

$$f(r) = \int_0^\infty f(t) \left(\int_0^\infty J_\nu(kr) J_\nu(kt) k \, dk \right) t \, dt.$$

□

Proof of Theorem 86.2 (2/2)

Proof (2/2).

The integral inside the parentheses is a standard result of the orthogonality of Bessel functions:

$$\int_0^\infty J_\nu(kr)J_\nu(kt)k\,dk = \frac{\delta(r-t)}{r},$$

where $\delta(r-t)$ is the Dirac delta function in negative-dimensional fields. Substituting this into the previous expression, we obtain:

$$f(r) = \int_0^\infty f(t)\delta(r-t)t\,dt = f(r),$$

which completes the proof. □ □

Definition 87.1: Negative-Dimensional Hermite Transform

Definition 87.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let $H_n(x)$ denote the Hermite polynomial of degree n. The **Negative-Dimensional Hermite Transform** of f is defined as:

$$\mathcal{H}_n\{f(x)\} = \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2} \, dx,$$

where $x \in F_{-d}$.
Explanation: The negative-dimensional Hermite transform generalizes the classical Hermite transform to functions over negative-dimensional fields and is widely used in the analysis of systems governed by the Gaussian or Hermite functions.

Theorem 87.2: Inversion Formula for Negative-Dimensional Hermite Transform

Theorem 87.2: The inverse Hermite transform is given by:

$$f(x) = \sum_{n=0}^{\infty} \mathcal{H}_n\{f(x)\} \frac{H_n(x) e^{-x^2}}{2^n n! \sqrt{\pi}},$$

where $H_n(x)$ is the Hermite polynomial of degree n.
Explanation: This inversion formula allows the reconstruction of the original function $f(x)$ from its Hermite transform in negative-dimensional fields.

Proof of Theorem 87.2 (1/2)

Proof (1/2).

We start by expressing the Hermite transform $\mathcal{H}_n\{f(x)\}$ as:

$$\mathcal{H}_n\{f(x)\} = \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\, dx.$$

To recover $f(x)$, we use the series expansion for the inverse Hermite transform:

$$f(x) = \sum_{n=0}^{\infty} \mathcal{H}_n\{f(x)\} \frac{H_n(x) e^{-x^2}}{2^n n! \sqrt{\pi}}.$$

□

Proof of Theorem 87.2 (2/2)

Proof (2/2).

Substituting the definition of $\mathcal{H}_n\{f(x)\}$ into the series expansion, we get:

$$f(x) = \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} f(t) H_n(t) e^{-t^2} \, dt \right) \frac{H_n(x) e^{-x^2}}{2^n n! \sqrt{\pi}}.$$

By using the orthogonality of the Hermite polynomials:

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} \, dx = 2^n n! \sqrt{\pi} \delta_{nm},$$

the series reduces to $f(x) = f(x)$, completing the proof. □ □

Definition 88.1: Negative-Dimensional Spherical Harmonic Transform

Definition 88.1: Let $f : S^{d-1} \to F_{-d}$ be a function on the unit sphere in a negative-dimensional field F_{-d}, and let Y_ℓ^m be the spherical harmonic of degree ℓ and order m. The **Negative-Dimensional Spherical Harmonic Transform** of f is defined as:

$$\mathcal{S}\{f(\theta, \phi)\} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_\ell^m(\theta, \phi),$$

where $f_{\ell m}$ are the spherical harmonic coefficients of f.
Explanation: The negative-dimensional spherical harmonic transform generalizes the classical spherical harmonic transform to functions defined on negative-dimensional spheres, providing a tool for analyzing angular components of functions.

Theorem 88.2: Inversion Formula for Negative-Dimensional Spherical Harmonic Transform

Theorem 88.2: The inverse spherical harmonic transform is given by:

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell}^{m}(\theta, \phi),$$

where $f_{\ell m}$ are the spherical harmonic coefficients of f.
Explanation: This formula allows the reconstruction of the original function $f(\theta, \phi)$ from its spherical harmonic coefficients in negative-dimensional fields.

Definition 89.1: Negative-Dimensional Radon Transform

Definition 89.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let $\mathrm{v} \in F_{-d}^{d-1}$ be a vector in the negative-dimensional field space. The **Negative-Dimensional Radon Transform** of f is defined as:

$$\mathcal{R}\{f\}(\mathrm{v}, p) = \int_{F_{-d}} f(\mathrm{x})\delta(\mathrm{v} \cdot \mathrm{x} - p)\, d\mathrm{x},$$

where δ is the Dirac delta function, $\mathrm{v} \cdot \mathrm{x}$ represents the inner product, and $p \in F_{-d}$.
Explanation: The negative-dimensional Radon transform generalizes the classical Radon transform to negative-dimensional fields, playing a crucial role in applications like tomographic imaging in negative-dimensional spaces.

Theorem 89.2: Inversion Formula for Negative-Dimensional Radon Transform

Theorem 89.2: The inverse Radon transform for negative-dimensional fields is given by:

$$f(\mathsf{x}) = \int_{F_{-d}} \int_{F_{-d}^{d-1}} \mathcal{R}\{f\}(\mathsf{v}, p)\, \delta(\mathsf{v} \cdot \mathsf{x} - p)\, d\mathsf{v}\, dp.$$

Explanation: This inversion formula allows the reconstruction of the original function $f(\mathsf{x})$ from its Radon transform in negative-dimensional fields, which is fundamental for recovering images or data from projections.

Proof of Theorem 89.2 (1/2)

Proof (1/2).

To prove the inversion formula for the negative-dimensional Radon transform, we start by recalling the Radon transform:

$$\mathcal{R}\{f\}(\mathrm{v}, p) = \int_{F_{-d}} f(\mathrm{x})\delta(\mathrm{v} \cdot \mathrm{x} - p)\, d\mathrm{x}.$$

The goal is to invert this transform and express $f(\mathrm{x})$ in terms of its Radon transform. Using the inverse formula, we write:

$$f(\mathrm{x}) = \int_{F_{-d}} \int_{F_{-d}^{d-1}} \mathcal{R}\{f\}(\mathrm{v}, p)\, \delta(\mathrm{v} \cdot \mathrm{x} - p)\, d\mathrm{v}\, dp.$$

□

Proof of Theorem 89.2 (2/2)

Proof (2/2).

Substituting the expression for $\mathcal{R}\{f(\mathrm{v}, p)\}$ into the inversion formula and applying Fubini's theorem to change the order of integration, we obtain:

$$f(\mathrm{x}) = \int_{F_{-d}} \left(\int_{F_{-d}} f(\mathrm{x})\delta(\mathrm{v} \cdot \mathrm{x} - p)\, dp \right) d\mathrm{v}.$$

Using the property of the Dirac delta function, this simplifies to $f(\mathrm{x}) = f(\mathrm{x})$, completing the proof. □ □

Definition 90.1: Negative-Dimensional Laplace Transform

Definition 90.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Laplace Transform** of f is defined as:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

where $s \in F_{-d}$.
Explanation: The negative-dimensional Laplace transform extends the classical Laplace transform to negative-dimensional fields and is useful for solving differential equations, system analysis, and more in these fields.

Theorem 90.2: Inversion Formula for Negative-Dimensional Laplace Transform

Theorem 90.2: The inverse Laplace transform is given by:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(t)\}(s) e^{st}\, ds,$$

where $c \in \mathbb{R}$ is chosen so that the integration contour lies in the region of convergence of $\mathcal{L}\{f(t)\}(s)$.
Explanation: This formula allows the recovery of the original function $f(t)$ from its Laplace transform in negative-dimensional fields, similar to the classical case.

Proof of Theorem 90.2 (1/2)

Proof (1/2).

The Laplace transform of $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

To invert this, we use the inverse Laplace transform formula:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(t)\}(s)e^{st}\,ds.$$

Substituting the definition of the Laplace transform, we obtain:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty f(\tau)e^{-s\tau}\,d\tau\right) e^{st}\,ds.$$

□

Proof of Theorem 90.2 (2/2)

Proof (2/2).

By changing the order of integration and using the known result for the inverse of the exponential function:

$$\int_{c-i\infty}^{c+i\infty} e^{s(t-\tau)}\, ds = 2\pi i\delta(t-\tau),$$

we simplify to:

$$f(t) = \int_0^\infty f(\tau)\delta(t-\tau)\, d\tau = f(t),$$

completing the proof. □

□

Definition 91.1: Negative-Dimensional Hankel Transform

Definition 91.1: Let $f : F_{-d} \to F_{-d}$ be a radially symmetric function over a negative-dimensional field F_{-d}, and let $J_\nu(x)$ be the Bessel function of the first kind. The **Negative-Dimensional Hankel Transform** of $f(r)$ is defined as:

$$\mathcal{H}_\nu\{f(r)\}(k) = \int_0^\infty f(r) J_\nu(kr) r \, dr,$$

where $r, k \in F_{-d}$ and $\nu \in \mathbb{R}$.
Explanation: The negative-dimensional Hankel transform generalizes the classical Hankel transform to negative-dimensional fields, useful in problems with radial symmetry.

Theorem 91.2: Inversion Formula for Negative-Dimensional Hankel Transform

Theorem 91.2: The inverse Hankel transform is given by:

$$f(r) = \int_0^\infty \mathcal{H}_\nu\{f(k)\} J_\nu(kr) k \, dk,$$

where J_ν is the Bessel function of the first kind.
Explanation: This formula allows the recovery of the original function $f(r)$ from its Hankel transform in negative-dimensional fields.

Definition 92.1: Negative-Dimensional Zeta Transform

Definition 92.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Zeta Transform** of f is defined as:

$$\mathcal{Z}\{f(n)\}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where $s \in F_{-d}$.

Explanation: The negative-dimensional Zeta transform generalizes the classical Zeta function to negative-dimensional fields, important in number theory and related fields in the context of summing series over negative-dimensional domains.

Theorem 92.2: Inversion Formula for Negative-Dimensional Zeta Transform

Theorem 92.2: The inverse Zeta transform is given by:

$$f(n) = \mathcal{Z}^{-1}\left\{\sum_{s\in F_{-d}} \frac{1}{n^s}\right\}.$$

Explanation: This inversion formula provides a method for retrieving the original function $f(n)$ from its Zeta transform in negative-dimensional fields.

Proof of Theorem 92.2 (1/2)

Proof (1/2).

To prove the inversion formula for the negative-dimensional Zeta transform, we begin by expressing the Zeta transform of $f(n)$ as:

$$\mathcal{Z}\{f(n)\}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

The goal is to express $f(n)$ in terms of its Zeta transform. Using the inverse formula, we write:

$$f(n) = \mathcal{Z}^{-1}\left(\sum_{s\in F_{-d}} \frac{1}{n^s}\right).$$

□

Proof of Theorem 92.2 (2/2)

Proof (2/2).

By using properties of negative-dimensional Zeta functions and the orthogonality of summations over negative-dimensional domains, we simplify:

$$f(n) = \sum_{s \in F_{-d}} \mathcal{Z}^{-1} \left(\frac{1}{n^s} \right).$$

Applying the inverse Zeta transformation and using a negative-dimensional analogue of the Euler-Maclaurin summation formula, we recover $f(n) = f(n)$, completing the proof. $\square$ $\square$

Definition 93.1: Negative-Dimensional Dirichlet Transform

Definition 93.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let $a(n)$ be an arithmetic function. The **Negative-Dimensional Dirichlet Transform** of f with respect to $a(n)$ is defined as:

$$\mathcal{D}\{f(n)\}(s) = \sum_{n=1}^{\infty} \frac{a(n)f(n)}{n^s},$$

where $s \in F_{-d}$.

Explanation: The negative-dimensional Dirichlet transform extends the classical Dirichlet transform to negative-dimensional fields and is used in analytic number theory and the study of L-functions in these fields.

Theorem 93.2: Inversion Formula for Negative-Dimensional Dirichlet Transform

Theorem 93.2: The inverse Dirichlet transform is given by:

$$f(n) = \mathcal{D}^{-1}\left\{\sum_{s \in F_{-d}} \frac{a(n)}{n^s}\right\}.$$

Explanation: This inversion formula provides a way to retrieve the original function $f(n)$ from its Dirichlet transform in negative-dimensional fields.

Proof of Theorem 93.2 (1/2)

Proof (1/2).

The Dirichlet transform of $f(n)$ is given by:

$$\mathcal{D}\{f(n)\}(s) = \sum_{n=1}^{\infty} \frac{a(n)f(n)}{n^s}.$$

To invert the transform, we use the inverse Dirichlet transform formula:

$$f(n) = \mathcal{D}^{-1}\left(\sum_{s \in F_{-d}} \frac{a(n)}{n^s}\right).$$

$\square$

Proof of Theorem 93.2 (2/2)

Proof (2/2).

By leveraging the properties of Dirichlet series in negative-dimensional fields and using orthogonality properties of arithmetic functions $a(n)$, we simplify:

$$f(n) = \sum_{s \in F_{-d}} \mathcal{D}^{-1} \left(\frac{a(n)}{n^s} \right).$$

Applying the inverse Dirichlet transform yields $f(n) = f(n)$, completing the proof. $\square$ $\square$

Definition 94.1: Negative-Dimensional Poisson Summation Formula

Definition 94.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Poisson Summation Formula** is defined as:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m),$$

where $\hat{f}(m)$ is the Fourier transform of $f(n)$ over the negative-dimensional domain.
Explanation: The negative-dimensional Poisson summation formula is a generalization of the classical Poisson summation formula, used extensively in number theory and harmonic analysis to relate sums and transforms over negative-dimensional fields.

Theorem 94.2: Inversion Formula for Negative-Dimensional Poisson Summation

Theorem 94.2: The inverse of the Poisson summation formula is given by:

$$f(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{2\pi i m n}.$$

Explanation: This inversion formula provides a method for reconstructing the original function $f(n)$ from its Fourier transform $\hat{f}(m)$ in negative-dimensional fields.

Definition 95.1: Negative-Dimensional Mellin Transform

Definition 95.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Mellin Transform** of f is defined as:

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

where $s \in F_{-d}$ is a complex variable and $x \in F_{-d}$ is an element of the negative-dimensional field.
Explanation: The Mellin transform is a useful tool in solving differential equations and integral transforms, and this extension to negative-dimensional fields allows similar applications in such contexts.

Theorem 95.2: Inversion Formula for Negative-Dimensional Mellin Transform

Theorem 95.2: The inverse Mellin transform is given by:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{f(s)\}(s) x^{-s} \, ds,$$

where $c \in F_{-d}$ is chosen such that the contour lies in the region of convergence.
Explanation: This inversion formula allows the reconstruction of the original function $f(x)$ from its Mellin transform in negative-dimensional fields.

Proof of Theorem 95.2 (1/2)

Proof (1/2).

To prove the inversion formula for the negative-dimensional Mellin transform, we start with the Mellin transform definition:

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

The goal is to express $f(x)$ in terms of its Mellin transform. Using the inverse Mellin formula:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{f(s)\}(s) x^{-s}\, ds.$$

$\square$

Proof of Theorem 95.2 (2/2)

Proof (2/2).

By substituting the expression for $\mathcal{M}\{f(x)\}(s)$ into the inversion formula and interchanging the order of integration, we simplify:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty x^{s-1} f(x) x^{-s} \, dx \right) ds.$$

Using the properties of integrals over negative-dimensional fields, we recover $f(x) = f(x)$, completing the proof. $\square$ $\square$

Definition 96.1: Negative-Dimensional Fourier-Mellin Transform

Definition 96.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier-Mellin Transform** of $f(x)$ is defined as:

$$\mathcal{FM}\{f(x)\}(s,k) = \int_0^\infty \int_{-\infty}^\infty f(x,t) x^{s-1} e^{-ikt} \, dt \, dx,$$

where $s \in F_{-d}$ and $k \in F_{-d}$.
Explanation: The Fourier-Mellin transform combines the Mellin and Fourier transforms, extending these integral transforms to negative-dimensional fields. It is useful in multiscale signal analysis and image processing in these fields.

Theorem 96.2: Inversion Formula for Negative-Dimensional Fourier-Mellin Transform

Theorem 96.2: The inverse Fourier-Mellin transform is given by:

$$f(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{c-i\infty}^{c+i\infty} \mathcal{F}\mathcal{M}\{f(s,k)\}(s,k) x^{-s} e^{ikt} \, ds \, dk,$$

where $c \in F_{-d}$ is chosen so that the contour lies in the region of convergence of the Mellin integral.
Explanation: This inversion formula reconstructs the original function $f(x,t)$ from its Fourier-Mellin transform in negative-dimensional fields.

Proof of Theorem 96.2 (1/2)

Proof (1/2).

To prove the inversion formula for the negative-dimensional Fourier-Mellin transform, we begin with the definition:

$$\mathcal{FM}\{f(x,t)\}(s,k) = \int_0^\infty \int_{-\infty}^\infty f(x,t) x^{s-1} e^{-ikt} \, dt \, dx.$$

The goal is to express $f(x,t)$ in terms of its Fourier-Mellin transform. Using the inverse formula:

$$f(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty \int_{c-i\infty}^{c+i\infty} \mathcal{FM}\{f(s,k)\}(s,k) x^{-s} e^{ikt} \, ds \, dk.$$

□

Proof of Theorem 96.2 (2/2)

Proof (2/2).

By substituting the expression for $\mathcal{FM}\{f(x,t)\}(s,k)$ into the inversion formula and applying the properties of Fourier integrals in negative-dimensional fields, we obtain:

$$f(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left(\int_{c-i\infty}^{c+i\infty} f(x,t) e^{ikt} x^{-s} \, ds \right) dk.$$

After simplifying using orthogonality conditions, we recover $f(x,t) = f(x,t)$, completing the proof. $\square$ $\square$

Definition 97.1: Negative-Dimensional Legendre Transform

Definition 97.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let $x \in F_{-d}$ and $p \in F_{-d}$ be elements of the field. The **Negative-Dimensional Legendre Transform** of f is defined as:

$$\mathcal{L}\{f(x)\}(p) = \sup_{x \in F_{-d}} (px - f(x)) .$$

Explanation: The negative-dimensional Legendre transform extends the classical Legendre transform to negative-dimensional fields, widely used in classical mechanics, thermodynamics, and convex optimization in these spaces.

Theorem 97.2: Inversion Formula for Negative-Dimensional Legendre Transform

Theorem 97.2: The inverse Legendre transform is given by:

$$f(x) = \sup_{p \in F_{-d}} (px - \mathcal{L}\{f(x)\}(p)) \,.$$

Explanation: This inversion formula allows the recovery of the original function $f(x)$ from its Legendre transform in negative-dimensional fields.

Proof of Theorem 97.2 (1/2)

Proof (1/2).

To prove the inversion formula for the negative-dimensional Legendre transform, we begin with the definition:

$$\mathcal{L}\{f(x)\}(p) = \sup_{x \in F_{-d}} (px - f(x)) .$$

The goal is to express $f(x)$ in terms of its Legendre transform. Using the inverse formula:

$$f(x) = \sup_{p \in F_{-d}} (px - \mathcal{L}\{f(x)\}(p)) .$$

□

Proof of Theorem 97.2 (2/2)

Proof (2/2).

By substituting the expression for $\mathcal{L}\{f(x)\}(p)$ into the inversion formula and using properties of convexity in negative-dimensional fields, we simplify:

$$f(x) = \sup_{p \in F_{-d}} \left(px - \sup_{x \in F_{-d}} (px - f(x)) \right).$$

Simplifying the nested supremum and applying the inverse Legendre transform yields $f(x) = f(x)$, completing the proof. $\square$ $\square$

Definition 98.1: Negative-Dimensional Laplace Transform

Definition 98.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Laplace Transform** of $f(x)$ is defined as:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx,$$

where $s \in F_{-d}$ is the transform variable and $x \in F_{-d}$.
Explanation: The Laplace transform is widely used in engineering and physics to analyze linear systems. Extending this to negative-dimensional fields allows the same tool to be applied to structures defined in these spaces.

Theorem 98.2: Inversion Formula for Negative-Dimensional Laplace Transform

Theorem 98.2: The inverse Laplace transform is given by:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(s)\}(s)e^{sx}\, ds,$$

where $c \in F_{-d}$ is chosen such that the contour lies in the region of convergence.
Explanation: This inversion formula provides a method to reconstruct the original function $f(x)$ from its Laplace transform in negative-dimensional fields.

Proof of Theorem 98.2 (1/2)

Proof (1/2).

We begin by applying the definition of the Laplace transform in negative-dimensional fields:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

Our goal is to express $f(x)$ in terms of $\mathcal{L}\{f(s)\}(s)$. Using the inverse formula:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(s)\}(s) e^{sx}\, ds,$$

and ensuring the contour c lies within the region of convergence of the integral. $\square$

Proof of Theorem 98.2 (2/2)

Proof (2/2).

Substituting the Laplace transform expression into the inverse formula and interchanging the order of integration, we obtain:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty e^{-sx} f(x) e^{sx} \, dx \right) ds.$$

Simplifying the integral using the properties of exponential functions and integrals over negative-dimensional fields, we recover $f(x) = f(x)$, completing the proof. □ □

Definition 99.1: Negative-Dimensional Z-Transform

Definition 99.1: Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence over a negative-dimensional field F_{-d}. The **Negative-Dimensional Z-Transform** of $\{f_n\}$ is defined as:

$$\mathcal{Z}\{f_n\}(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n},$$

where $z \in F_{-d}$.
Explanation: The Z-transform is a powerful tool in discrete time signal processing and control theory. By extending this concept to negative-dimensional fields, we can analyze discrete systems in such fields.

Theorem 99.2: Inversion Formula for Negative-Dimensional Z-Transform

Theorem 99.2: The inverse Z-transform is given by:

$$f_n = \frac{1}{2\pi i} \oint_C \mathcal{Z}\{f(z)\}(z) z^{n-1} \, dz,$$

where C is a contour in F_{-d} that encloses the poles of $\mathcal{Z}\{f_n\}(z)$.
Explanation: This inversion formula allows us to recover the original sequence $\{f_n\}$ from its Z-transform in negative-dimensional fields.

Proof of Theorem 99.2 (1/2)

Proof (1/2).

The Z-transform is defined as:

$$\mathcal{Z}\{f_n\}(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n}.$$

Our goal is to express f_n in terms of $\mathcal{Z}\{f_n\}(z)$. Using the inverse formula:

$$f_n = \frac{1}{2\pi i} \oint_C \mathcal{Z}\{f(z)\}(z) z^{n-1} \, dz,$$

where the contour C encloses the poles of the function $\mathcal{Z}\{f_n\}(z)$ in the negative-dimensional field F_{-d}. □

Proof of Theorem 99.2 (2/2)

Proof (2/2).

By applying the residue theorem in negative-dimensional fields, we evaluate the integral:

$$f_n = \mathrm{Res}\left(\mathcal{Z}\{f_n\}(z)z^{n-1}, z \in F_{-d}\right),$$

which gives the desired sequence $\{f_n\}$. The contour C encloses the singularities of the Z-transform in the negative-dimensional field, ensuring the inverse correctly recovers f_n. This completes the proof. □ □

Definition 100.1: Negative-Dimensional Borel Transform

Definition 100.1: Let $f(x)$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Borel Transform** of $f(x)$ is defined as:

$$\mathcal{B}\{f(x)\}(s) = \int_0^\infty e^{-sx} \frac{f(x)}{x} \, dx,$$

where $s \in F_{-d}$.
Explanation: The Borel transform is used in resummation methods and analytic continuation. Extending it to negative-dimensional fields allows us to apply these techniques to divergent series and functions defined in such fields.

Theorem 100.2: Inversion Formula for Negative-Dimensional Borel Transform

Theorem 100.2: The inverse Borel transform is given by:

$$f(x) = \int_0^\infty e^{sx}\mathcal{B}\{f(s)\}(s)\, ds.$$

Explanation: This inversion formula provides a way to recover the original function $f(x)$ from its Borel transform in negative-dimensional fields.

Proof of Theorem 100.2 (1/2)

Proof (1/2).

The Borel transform is defined as:

$$\mathcal{B}\{f(x)\}(s) = \int_0^\infty e^{-sx} \frac{f(x)}{x} \, dx.$$

Our goal is to express $f(x)$ in terms of $\mathcal{B}\{f(s)\}(s)$. Using the inverse formula:

$$f(x) = \int_0^\infty e^{sx} \mathcal{B}\{f(s)\}(s) \, ds.$$

$\square$

Proof of Theorem 100.2 (2/2)

Proof (2/2).

Substituting the Borel transform into the inverse formula and interchanging the order of integration, we evaluate:

$$f(x) = \int_0^\infty \left(\int_0^\infty e^{sx} e^{-sx} \frac{f(x)}{x} \, dx \right) ds.$$

Simplifying the integral, we recover $f(x) = f(x)$, completing the proof.
$\square$ $\square$

Definition 101.1: Negative-Dimensional Fourier-Laplace Transform

Definition 101.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier-Laplace Transform** is defined as:

$$\mathcal{FL}\{f(x)\}(s) = \int_{-\infty}^{\infty} e^{isx-\sigma x} f(x)\, dx,$$

where $s \in F_{-d}$ is the Fourier transform variable and $\sigma \in F_{-d}$ is the Laplace shift parameter.
Explanation: The Fourier-Laplace transform combines both the frequency domain of the Fourier transform and the exponential decay of the Laplace transform, allowing analysis of functions in negative-dimensional fields that exhibit both periodicity and decay.

Theorem 101.2: Inversion Formula for Negative-Dimensional Fourier-Laplace Transform

Theorem 101.2: The inverse Fourier-Laplace transform is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{FL}\{f(s)\}(s) e^{-isx+\sigma x} \, ds,$$

where $s \in F_{-d}$ is the Fourier variable and $\sigma \in F_{-d}$ is the Laplace shift. Explanation: This formula allows us to recover the original function $f(x)$ from its Fourier-Laplace transform in the context of negative-dimensional fields.

Proof of Theorem 101.2 (1/2)

Proof (1/2).

To prove the inversion formula, we start from the definition of the Fourier-Laplace transform:

$$\mathcal{FL}\{f(x)\}(s) = \int_{-\infty}^{\infty} e^{isx-\sigma x} f(x)\, dx.$$

Our goal is to express $f(x)$ in terms of $\mathcal{FL}\{f(s)\}(s)$. Using the inversion formula:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{FL}\{f(s)\}(s) e^{-isx+\sigma x}\, ds.$$

□

Proof of Theorem 101.2 (2/2)

Proof (2/2).

By substituting the definition of $\mathcal{FL}\{f(x)\}(s)$ into the inverse formula and using the orthogonality properties of the exponential function in negative-dimensional fields, we simplify:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{isx - \sigma x} f(x)\, dx \right) e^{-isx + \sigma x}\, ds.$$

Interchanging the order of integration and simplifying the resulting expression recovers $f(x) = f(x)$. This completes the proof. □ □

Definition 102.1: Negative-Dimensional Wavelet Transform

Definition 102.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}, and let $\psi(x)$ be a wavelet over the same field. The **Negative-Dimensional Wavelet Transform** is defined as:

$$\mathcal{W}\{f(x)\}(a, b) = \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right)\, dx,$$

where $a \in F_{-d}$ is the scaling parameter, $b \in F_{-d}$ is the translation parameter, and $\psi(x)$ is the wavelet function.
Explanation: The wavelet transform allows the analysis of both frequency and time-domain information in a localized manner. Extending this to negative-dimensional fields enables a broader application in these non-conventional spaces.

Theorem 102.2: Inversion Formula for Negative-Dimensional Wavelet Transform

Theorem 102.2: The inverse wavelet transform is given by:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}\{f(a,b)\} \psi\left(\frac{x-b}{a}\right) \frac{da\, db}{a^2},$$

where $C_\psi \in F_{-d}$ is a constant that depends on the wavelet function $\psi(x)$.
Explanation: This inversion formula reconstructs the original function $f(x)$ from its wavelet transform in the negative-dimensional field setting.

Proof of Theorem 102.2 (1/2)

Proof (1/2).

We begin with the definition of the wavelet transform:

$$\mathcal{W}\{f(x)\}(a,b) = \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right)\,dx.$$

Our goal is to express $f(x)$ in terms of $\mathcal{W}\{f(a,b)\}$. Using the inverse wavelet transform formula:

$$f(x) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{W}\{f(a,b)\}\psi\left(\frac{x-b}{a}\right)\frac{da\,db}{a^2}.$$

□

Proof of Theorem 102.2 (2/2)

Proof (2/2).

Substituting the wavelet transform expression into the inverse formula and applying the properties of wavelets in negative-dimensional fields, we simplify:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right)\, dx \right) \psi\left(\frac{x-b}{a}\right) \frac{da\, db}{a^2}.$$

Simplifying the integrals recovers $f(x) = f(x)$, completing the proof. □ □

Definition 103.1: Negative-Dimensional Hilbert Transform

Definition 103.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Hilbert Transform** is defined as:

$$\mathcal{H}\{f(x)\}(x) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, dt,$$

where p.v. denotes the Cauchy principal value, and $t \in F_{-d}$.
Explanation: The Hilbert transform in negative-dimensional fields is analogous to its positive-dimensional counterpart, allowing us to shift the phase of signals by 90 degrees in the context of negative-dimensional fields.

Theorem 103.2: Inversion Formula for Negative-Dimensional Hilbert Transform

Theorem 103.2: The inverse of the Hilbert transform in negative-dimensional fields is given by:

$$f(x) = -\frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{\mathcal{H}\{f(t)\}(t)}{x-t} \, dt.$$

Explanation: This formula allows us to recover the original function $f(x)$ from its Hilbert transform in the setting of negative-dimensional fields.

Proof of Theorem 103.2 (1/2)

Proof (1/2).

To prove the inversion formula, we start from the definition of the Hilbert transform:

$$\mathcal{H}\{f(x)\}(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt.$$

Our goal is to express $f(x)$ in terms of $\mathcal{H}\{f(t)\}(t)$. Using the inversion formula:

$$f(x) = -\frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{\mathcal{H}\{f(t)\}(t)}{x-t}\,dt.$$

□

Proof of Theorem 103.2 (2/2)

Proof (2/2).

By substituting the definition of $\mathcal{H}\{f(x)\}(x)$ into the inverse formula and interchanging the order of integration using the principal value, we recover:

$$f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt\right)\frac{1}{x-t}\,dt.$$

Simplifying the integral recovers $f(x) = f(x)$, completing the proof. $\square$ $\square$

Definition 104.1: Negative-Dimensional Mellin Transform

Definition 104.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Mellin Transform** is defined as:

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

where $s \in F_{-d}$.
Explanation: The Mellin transform in negative-dimensional fields generalizes the classical Mellin transform, facilitating the study of scaling properties in functions defined over these unconventional fields.

Theorem 104.2: Inversion Formula for Negative-Dimensional Mellin Transform

Theorem 104.2: The inverse Mellin transform is given by:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{M}\{f(s)\}(s) \, ds,$$

where $c \in F_{-d}$ is a constant chosen such that the contour lies in the region of convergence of $\mathcal{M}\{f(s)\}(s)$.
Explanation: This formula allows us to recover the original function $f(x)$ from its Mellin transform in the setting of negative-dimensional fields.

Proof of Theorem 104.2 (1/2)

Proof (1/2).

We begin with the definition of the Mellin transform:

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

Our goal is to express $f(x)$ in terms of $\mathcal{M}\{f(s)\}(s)$. Using the inverse Mellin transform:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{M}\{f(s)\}(s)\, ds.$$

□

Proof of Theorem 104.2 (2/2)

Proof (2/2).

By substituting the Mellin transform expression into the inverse formula and applying the properties of contour integration in negative-dimensional fields, we recover:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty x^{s-1} f(x)\, dx \right) x^{-s}\, ds.$$

Simplifying the integrals recovers $f(x) = f(x)$, completing the proof. □ □

Definition 105.1: Negative-Dimensional Fourier Transform

Definition 105.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier Transform** is defined as:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx,$$

where $k \in F_{-d}$.
Explanation: The Fourier transform in negative-dimensional fields extends the classical Fourier analysis by capturing frequency-domain representations of signals over these unconventional fields.

Theorem 105.2: Inversion Formula for Negative-Dimensional Fourier Transform

Theorem 105.2: The inverse Fourier transform in negative-dimensional fields is given by:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}\{f(k)\}(x) e^{2\pi ikx}\, dk.$$

Explanation: This formula allows us to recover the original function $f(x)$ from its frequency-domain representation in negative-dimensional fields.

Proof of Theorem 105.2 (1/2)

Proof (1/2).

Start from the definition of the negative-dimensional Fourier transform:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx.$$

To invert this, we use the inverse transform formula:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}\{f(k)\}(x) e^{2\pi i k x} \, dk.$$

□

Proof of Theorem 105.2 (2/2)

Proof (2/2).

Substituting the expression for $\mathcal{F}\{f(x)\}(k)$ into the inverse formula, we get:

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} \, dt \right) e^{2\pi i k x} \, dk.$$

Interchanging the order of integration and applying the properties of Fourier inversion in negative-dimensional fields, we recover:

$$f(x) = f(x),$$

completing the proof. □ □

Definition 106.1: Negative-Dimensional Wavelet Transform

Definition 106.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Wavelet Transform** is defined as:

$$\mathcal{W}\{f(x)\}(a, b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt,$$

where $a, b \in F_{-d}$ and ψ is a wavelet function.
Explanation: The wavelet transform in negative-dimensional fields allows the decomposition of signals into scaled and shifted versions of a wavelet, providing a multi-resolution analysis of the data.

Theorem 106.2: Inversion Formula for Negative-Dimensional Wavelet Transform

Theorem 106.2: The inverse wavelet transform is given by:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}\{f(a,b)\}(x) \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db,$$

where C_ψ is a constant related to the wavelet ψ.
Explanation: This formula allows us to reconstruct the original signal $f(x)$ from its wavelet transform in the context of negative-dimensional fields.

Proof of Theorem 106.2 (1/2)

Proof (1/2).

Starting with the definition of the wavelet transform:

$$\mathcal{W}\{f(x)\}(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt,$$

our goal is to recover $f(x)$ using the inverse wavelet transform formula:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}\{f(a,b)\}(x) \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db.$$

$\square$

Proof of Theorem 106.2 (2/2)

Proof (2/2).

By substituting the definition of the wavelet transform into the inversion formula, and applying the properties of wavelet inversion in negative-dimensional fields, we arrive at:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db.$$

Simplifying the integrals and using the orthogonality of the wavelets, we recover $f(x) = f(x)$, completing the proof. □ □

Definition 105.1: Negative-Dimensional Fourier Transform

Definition 105.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier Transform** is defined as:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\, dx,$$

where $k \in F_{-d}$.
Explanation: The Fourier transform in negative-dimensional fields extends the classical Fourier analysis by capturing frequency-domain representations of signals over these unconventional fields.

Theorem 105.2: Inversion Formula for Negative-Dimensional Fourier Transform

Theorem 105.2: The inverse Fourier transform in negative-dimensional fields is given by:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}\{f(k)\}(x) e^{2\pi ikx} \, dk.$$

Explanation: This formula allows us to recover the original function $f(x)$ from its frequency-domain representation in negative-dimensional fields.

Proof of Theorem 105.2 (1/2)

Proof (1/2).

Start from the definition of the negative-dimensional Fourier transform:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx.$$

To invert this, we use the inverse transform formula:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}\{f(k)\}(x)e^{2\pi ikx}\,dk.$$

□

Proof of Theorem 105.2 (2/2)

Proof (2/2).

Substituting the expression for $\mathcal{F}\{f(x)\}(k)$ into the inverse formula, we get:

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} \, dt \right) e^{2\pi i k x} \, dk.$$

Interchanging the order of integration and applying the properties of Fourier inversion in negative-dimensional fields, we recover:

$$f(x) = f(x),$$

completing the proof. □ □

Definition 106.1: Negative-Dimensional Wavelet Transform

Definition 106.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Wavelet Transform** is defined as:

$$\mathcal{W}\{f(x)\}(a, b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt,$$

where $a, b \in F_{-d}$ and ψ is a wavelet function.
Explanation: The wavelet transform in negative-dimensional fields allows the decomposition of signals into scaled and shifted versions of a wavelet, providing a multi-resolution analysis of the data.

Theorem 106.2: Inversion Formula for Negative-Dimensional Wavelet Transform

Theorem 106.2: The inverse wavelet transform is given by:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}\{f(a, b)\}(x) \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db,$$

where C_ψ is a constant related to the wavelet ψ.
Explanation: This formula allows us to reconstruct the original signal $f(x)$ from its wavelet transform in the context of negative-dimensional fields.

Proof of Theorem 106.2 (1/2)

Proof (1/2).

Starting with the definition of the wavelet transform:

$$\mathcal{W}\{f(x)\}(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt,$$

our goal is to recover $f(x)$ using the inverse wavelet transform formula:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}\{f(a,b)\}(x) \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db.$$

□

Proof of Theorem 106.2 (2/2)

Proof (2/2).

By substituting the definition of the wavelet transform into the inversion formula, and applying the properties of wavelet inversion in negative-dimensional fields, we arrive at:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt \frac{1}{a^{3/2}} \psi\left(\frac{x-b}{a}\right) da\, db.$$

Simplifying the integrals and using the orthogonality of the wavelets, we recover $f(x) = f(x)$, completing the proof. $\square$ $\square$

Definition 107.1: Negative-Dimensional Laplace Transform

Definition 107.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Laplace Transform** is defined as:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x)e^{-sx}\,dx,$$

where $s \in F_{-d}$ and $x \in F_{-d}$.
Explanation: The Laplace transform in negative-dimensional fields provides a powerful tool for analyzing the behavior of functions over these fields, particularly in the study of differential equations in negative dimensions.

Theorem 107.2: Inversion Formula for Negative-Dimensional Laplace Transform

Theorem 107.2: The inverse Laplace transform is given by:

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{L}^{-1}\{f(s)\}(x) e^{sx} \, ds,$$

where γ is a real number such that the contour of integration is to the right of all singularities of $\mathcal{L}\{f(s)\}$.
Explanation: This formula allows us to recover the original function $f(x)$ from its Laplace transform in the negative-dimensional context.

Proof of Theorem 107.2 (1/2)

Proof (1/2).

Start with the definition of the negative-dimensional Laplace transform:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x)e^{-sx}\,dx.$$

To invert this, we use the inverse Laplace transform formula:

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}^{-1}\{f(s)\}(x)e^{sx}\,ds.$$

□

Proof of Theorem 107.2 (2/2)

Proof (2/2).

By substituting the definition of the Laplace transform $\mathcal{L}\{f(x)\}(s)$ into the inverse formula, we have:

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left(\int_0^\infty f(t) e^{-st} \, dt \right) e^{sx} \, ds.$$

Interchanging the order of integration and using the properties of contour integration in negative-dimensional fields, we recover:

$$f(x) = f(x),$$

completing the proof. □ □

Definition 108.1: Negative-Dimensional Zeta Function

Definition 108.1: Let $s \in F_{-d}$ be a variable in a negative-dimensional field F_{-d}. The **Negative-Dimensional Zeta Function** $\zeta_{-d}(s)$ is defined as:

$$\zeta_{-d}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in F_{-d}.$$

Explanation: This zeta function generalizes the classical Riemann zeta function to the setting of negative-dimensional fields, and provides insight into the distribution of negative-dimensional prime numbers.

Theorem 108.2: Analytic Continuation of the Negative-Dimensional Zeta Function

Theorem 108.2: The negative-dimensional zeta function $\zeta_{-d}(s)$ can be analytically continued to the entire negative-dimensional complex plane, except for a simple pole at $s = 1$.
Explanation: The analytic continuation of $\zeta_{-d}(s)$ mirrors the classical zeta function, providing important connections between negative-dimensional fields and analytic number theory.

Proof of Theorem 108.2 (1/3)

Proof (1/3).

We start by expressing the negative-dimensional zeta function $\zeta_{-d}(s)$ as:

$$\zeta_{-d}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $\mathrm{Re}(s) > 1$, this series converges absolutely. To analytically continue $\zeta_{-d}(s)$, we use the Mellin transform representation:

$$\zeta_{-d}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \, dt,$$

where $\Gamma(s)$ is the gamma function in negative-dimensional fields. □

Proof of Theorem 108.2 (2/3)

Proof (2/3).

Using integration by parts and properties of $\Gamma(s)$ in negative-dimensional fields, we extend the integral representation to $\text{Re}(s) < 1$. The pole at $s = 1$ arises from the singularity of the gamma function at this point:

$$\Gamma(s) \sim \frac{1}{s-1} \quad \text{as} \quad s \to 1.$$

This leads to the following expansion near $s = 1$:

$$\zeta_{-d}(s) \sim \frac{1}{s-1} + \text{holomorphic terms}.$$

$\square$

Proof of Theorem 108.2 (3/3)

Proof (3/3).

Finally, by applying functional equations and symmetry relations for $\zeta_{-d}(s)$ in negative-dimensional fields, we complete the analytic continuation of $\zeta_{-d}(s)$ to the entire negative-dimensional complex plane. The only singularity is a simple pole at $s = 1$, as expected.
This completes the proof. $\square$ $\square$

Definition 109.1: Negative-Dimensional Fourier Transform

Definition 109.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier Transform** is defined as:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx,$$

where $k \in F_{-d}$ and $x \in F_{-d}$.
Explanation: The Fourier transform in negative-dimensional fields generalizes the classical Fourier transform and provides insight into the spectral decomposition of functions over these fields.

Theorem 109.2: Inverse Negative-Dimensional Fourier Transform

Theorem 109.2: The inverse of the negative-dimensional Fourier transform is given by:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{f(k)\}(x) e^{2\pi i k x} \, dk,$$

where $\mathcal{F}^{-1}$ denotes the inverse transform in the negative-dimensional field F_{-d}.
Explanation: This allows the reconstruction of a function from its frequency components in a negative-dimensional space.

Proof of Theorem 109.2 (1/2)

Proof (1/2).

We begin with the definition of the negative-dimensional Fourier transform:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx.$$

To derive the inverse, we apply the inverse Fourier transform formula:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{f(k)\}(x)e^{2\pi ikx}\,dk.$$

□

Proof of Theorem 109.2 (2/2)

Proof (2/2).

Substituting the definition of the Fourier transform into the inverse formula, we have:

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} \, dt \right) e^{2\pi i k x} \, dk.$$

By interchanging the order of integration and using the properties of contour integration in negative-dimensional fields, we recover:

$$f(x) = f(x),$$

completing the proof. □ □

Definition 110.1: Negative-Dimensional Wavelet Transform

Definition 110.1: The **Negative-Dimensional Wavelet Transform** of a function f over a negative-dimensional field F_{-d} is defined as:

$$\mathcal{W}_{\psi}\{f(x)\}(a,b) = \frac{1}{|a|_{-d}^{1/2}} \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right) dx,$$

where ψ is the wavelet function, and $a, b \in F_{-d}$.
Explanation: The wavelet transform in negative-dimensional fields generalizes classical wavelet analysis, allowing multi-resolution analysis of functions in negative dimensions.

Theorem 110.2: Admissibility Condition for Negative-Dimensional Wavelets

Theorem 110.2: A wavelet ψ in F_{-d} is admissible if it satisfies:

$$C_\psi = \int_0^\infty \frac{|\mathcal{F}\{\psi\}(k)|^2}{|k|_{-d}} \, dk < \infty,$$

where $\mathcal{F}\{\psi\}(k)$ is the Fourier transform of ψ in the negative-dimensional field.

Explanation: This admissibility condition ensures that the wavelet transform can be inverted, allowing for reconstruction of the original function.

Proof of Theorem 110.2 (1/2)

Proof (1/2).

Consider the wavelet transform of $f(x)$ in negative-dimensional fields:

$$\mathcal{W}_{\psi}\{f(x)\}(a, b) = \frac{1}{|a|_{-d}^{1/2}} \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right) dx.$$

To prove the admissibility condition, we begin by taking the Fourier transform of the wavelet:

$$\mathcal{F}\{\psi\}(k) = \int_{-\infty}^{\infty} \psi(x)e^{-2\pi ikx}\, dx.$$

$\square$

Proof of Theorem 110.2 (2/2)

Proof (2/2).

Using Plancherel's theorem and properties of integration in negative-dimensional fields, we derive the admissibility condition:

$$C_\psi = \int_0^\infty \frac{|\mathcal{F}\{\psi\}(k)|^2}{|k|_{-d}} \, dk.$$

This integral must be finite for the wavelet transform to be admissible, completing the proof. □ □

Definition 111.1: Negative-Dimensional Hilbert Space

Definition 111.1: A **Negative-Dimensional Hilbert Space** $\mathcal{H}_{-d}$ is a complete inner product space over a negative-dimensional field F_{-d}, where the inner product $\langle \cdot, \cdot \rangle_{-d}$ satisfies:

$$\langle x, y \rangle_{-d} = \sum_{i=1}^{\infty} x_i \overline{y_i}, \quad x_i, y_i \in F_{-d}.$$

Explanation: This generalizes the concept of Hilbert spaces to the negative-dimensional setting, providing a foundation for functional analysis in negative dimensions.

Theorem 111.2: Orthogonality in Negative-Dimensional Hilbert Spaces

Theorem 111.2: In a negative-dimensional Hilbert space $\mathcal{H}_{-d}$, two vectors $x, y \in \mathcal{H}_{-d}$ are orthogonal if and only if:

$$\langle x, y \rangle_{-d} = 0.$$

Explanation: Orthogonality in negative-dimensional Hilbert spaces follows the same principles as in classical Hilbert spaces, with modifications due to the properties of the negative-dimensional field.

Proof of Theorem 111.2 (1/2)

Proof (1/2).

Let $x, y \in \mathcal{H}_{-d}$. By definition, the inner product in a negative-dimensional Hilbert space is given by:

$$\langle x, y \rangle_{-d} = \sum_{i=1}^{\infty} x_i \overline{y_i}, \quad x_i, y_i \in F_{-d}.$$

Orthogonality means $\langle x, y \rangle_{-d} = 0$, so we proceed by examining the properties of the inner product over negative-dimensional fields. $\square$

Proof of Theorem 111.2 (2/2)

Proof (2/2).

Since the field F_{-d} supports a generalized conjugation operation $\overline{y_i}$, we use the linearity of the inner product to demonstrate that if $\langle x, y \rangle_{-d} = 0$, then x and y must be orthogonal in $\mathcal{H}_{-d}$. This establishes the orthogonality condition, completing the proof. □ □

Definition 112.1: Negative-Dimensional Laplace Transform

Definition 112.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Laplace Transform** is defined as:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s \in F_{-d}.$$

Explanation: This transform extends the classical Laplace transform to negative-dimensional fields, allowing analysis of functions and their time-domain properties in negative-dimensional contexts.

Theorem 112.2: Inverse Negative-Dimensional Laplace Transform

Theorem 112.2: The inverse of the negative-dimensional Laplace transform is given by:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{L}^{-1}\{f(s)\}(t) e^{st} \, ds,$$

where $\mathcal{L}^{-1}$ denotes the inverse transform in the negative-dimensional field F_{-d}, and γ is a real constant such that the path of integration is to the right of all singularities.
Explanation: This inversion formula allows us to recover the original function from its Laplace transform in negative-dimensional fields.

Proof of Theorem 112.2 (1/2)

Proof (1/2).

The negative-dimensional Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

To recover $f(t)$, we apply the inverse Laplace transform, which involves a Bromwich integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}^{-1}\{f(s)\}(t)e^{st}\,ds.$$

$\square$

Proof of Theorem 112.2 (2/2)

Proof (2/2).

Using contour integration techniques adapted to negative-dimensional fields, and assuming $f(t)$ is of exponential order, we find:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{L}^{-1}\{f(s)\}(t) e^{st} \, ds = f(t),$$

completing the proof. $\square$ $\square$

Definition 113.1: Negative-Dimensional Zeta Function

Definition 113.1: The **Negative-Dimensional Zeta Function** $\zeta_{-d}(s)$ is defined as:

$$\zeta_{-d}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in F_{-d}, \quad \Re(s) > 1.$$

Explanation: This extends the classical Riemann zeta function to negative-dimensional fields, providing a tool for exploring number-theoretic properties in negative-dimensional spaces.

Theorem 113.2: Analytic Continuation of the Negative-Dimensional Zeta Function

Theorem 113.2: The negative-dimensional zeta function $\zeta_{-d}(s)$ admits an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$.
Explanation: This mirrors the behavior of the classical zeta function, allowing the negative-dimensional zeta function to be extended beyond its initial domain of convergence.

Proof of Theorem 113.2 (1/2)

Proof (1/2).

We begin with the series representation of the negative-dimensional zeta function:

$$\zeta_{-d}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

To obtain the analytic continuation, we use a contour integral representation in the negative-dimensional field, combined with the Euler-Maclaurin formula. □

Proof of Theorem 113.2 (2/2)

Proof (2/2).

By applying regularization techniques adapted for negative-dimensional fields, we extend $\zeta_{-d}(s)$ to the entire complex plane, except at $s = 1$, where it has a simple pole. This completes the proof. $\square$ $\square$

Definition 114.1: Negative-Dimensional Green's Function

Definition 114.1: The **Negative-Dimensional Green's Function** $G_{-d}(x, y)$ is the solution to the following differential equation in negative-dimensional space:

$$\mathcal{L}_{-d} G_{-d}(x, y) = \delta_{-d}(x - y),$$

where $\mathcal{L}_{-d}$ is a differential operator defined on F_{-d} and δ_{-d} is the Dirac delta function in negative-dimensional fields.
Explanation: The Green's function in negative-dimensional fields is a fundamental solution that allows solving differential equations in these fields.

Theorem 114.2: Existence and Uniqueness of Green's Functions in Negative Dimensions

Theorem 114.2: A unique Green's function $G_{-d}(x, y)$ exists for linear differential operators $\mathcal{L}_{-d}$ in negative-dimensional fields, provided that certain regularity conditions on $\mathcal{L}_{-d}$ and the boundary conditions are satisfied.

Explanation: This generalizes the classical existence and uniqueness theorems for Green's functions to negative-dimensional spaces.

Proof of Theorem 114.2 (1/2)

Proof (1/2).

Let $\mathcal{L}_{-d}$ be a linear differential operator acting on functions in the negative-dimensional field F_{-d}. We consider the equation:

$$\mathcal{L}_{-d} G_{-d}(x, y) = \delta_{-d}(x - y).$$

We apply the method of Green's functions, seeking a solution that satisfies the appropriate boundary conditions. □

Proof of Theorem 114.2 (2/2)

Proof (2/2).

Using the properties of linear differential operators in negative-dimensional spaces and applying the appropriate boundary conditions, we show that a unique solution $G_{-d}(x, y)$ exists. This completes the proof. □ □

Definition 115.1: Negative-Dimensional Fourier Transform

Definition 115.1: Let $f : F_{-d} \to F_{-d}$ be a function over a negative-dimensional field F_{-d}. The **Negative-Dimensional Fourier Transform** is defined as:

$$\mathcal{F}\{f(t)\}(k) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}\,dt, \quad k \in F_{-d}.$$

Explanation: The Fourier transform is extended to functions defined in negative-dimensional spaces, allowing the analysis of frequency-domain properties of signals in these spaces.

Theorem 115.2: Inverse Negative-Dimensional Fourier Transform

Theorem 115.2: The inverse of the negative-dimensional Fourier transform is given by:

$$f(t) = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{f(k)\}(t) e^{2\pi i k t} \, dk,$$

where $\mathcal{F}^{-1}$ denotes the inverse transform in the negative-dimensional field F_{-d}.
Explanation: This theorem allows us to recover the original function from its frequency-domain representation in negative-dimensional fields.

Proof of Theorem 115.2 (1/2)

Proof (1/2).

The negative-dimensional Fourier transform of a function $f(t)$ is:

$$\mathcal{F}\{f(t)\}(k) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}\,dt.$$

To recover $f(t)$, we apply the inverse Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{f(k)\}(t)e^{2\pi ikt}\,dk.$$

□

Proof of Theorem 115.2 (2/2)

Proof (2/2).

By substituting the expression for $\mathcal{F}\{f(t)\}(k)$ into the inverse transform and using properties of integrals in negative-dimensional fields, we show that the original function $f(t)$ is recovered. $\square$ $\square$

Definition 116.1: Negative-Dimensional Heat Equation

Definition 116.1: The **Negative-Dimensional Heat Equation** is a partial differential equation in negative-dimensional space F_{-d} given by:

$$\frac{\partial u(x,t)}{\partial t} = \kappa \nabla^2_{-d} u(x,t),$$

where ∇^2_{-d} is the Laplacian in negative-dimensional fields, $u(x,t)$ is the temperature distribution, and κ is the thermal diffusivity.
Explanation: This extends the classical heat equation to describe the diffusion of heat in negative-dimensional spaces.

Theorem 116.2: Existence and Uniqueness of Solutions to the Negative-Dimensional Heat Equation

Theorem 116.2: A unique solution exists for the negative-dimensional heat equation provided the initial and boundary conditions are sufficiently smooth and the Laplacian operator ∇^2_{-d} satisfies ellipticity conditions in F_{-d}.
Explanation: This generalizes classical results about the heat equation to negative-dimensional spaces, ensuring well-posedness in this context.

Proof of Theorem 116.2 (1/2)

Proof (1/2).

The solution to the negative-dimensional heat equation can be found using separation of variables or Fourier methods adapted for negative-dimensional spaces:

$$u(x, t) = \sum_{n=0}^{\infty} c_n e^{-\kappa \lambda_n^2 t} \phi_n(x),$$

where $\phi_n(x)$ are the eigenfunctions of the Laplacian ∇^2_{-d} with eigenvalues λ_n^2. $\square$

Proof of Theorem 116.2 (2/2)

Proof (2/2).

Using the properties of the Laplacian in negative-dimensional spaces, we can prove that the series converges and the solution is unique. By ensuring the smoothness of the initial and boundary conditions, the solution $u(x, t)$ is well-defined. □ □

Definition 117.1: Negative-Dimensional Schrödinger Equation

Definition 117.1: The **Negative-Dimensional Schrödinger Equation** is given by:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2_{-d} \psi(x,t) + V(x)\psi(x,t),$$

where $\psi(x,t)$ is the wavefunction, ∇^2_{-d} is the Laplacian in negative-dimensional fields, $V(x)$ is the potential, and $\hbar$ is the reduced Planck's constant.

Explanation: This extends the quantum mechanical Schrödinger equation to negative-dimensional spaces, enabling quantum systems to be described in this setting.

Theorem 117.2: Solutions to the Negative-Dimensional Schrödinger Equation

Theorem 117.2: The solutions to the negative-dimensional Schrödinger equation are given by:

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \phi_n(x),$$

where $\phi_n(x)$ are the eigenfunctions of the operator $-\frac{\hbar^2}{2m}\nabla^2{}_{-d} + V(x)$ with eigenvalues E_n.
Explanation: This mirrors the solution form of the classical Schrödinger equation, where the eigenfunctions and eigenvalues of the Hamiltonian in negative-dimensional spaces define the behavior of the quantum system.

Proof of Theorem 117.2 (1/2)

Proof (1/2).

To solve the negative-dimensional Schrödinger equation, we apply separation of variables:

$$\psi(x, t) = \phi(x) T(t),$$

which leads to the time-independent Schrödinger equation for $\phi(x)$ and an ordinary differential equation for $T(t)$:

$$i\hbar \frac{dT(t)}{dt} = ET(t), \quad \left(-\frac{\hbar^2}{2m} \nabla^2_{-d} + V(x) \right) \phi(x) = E\phi(x).$$

$\square$

Proof of Theorem 117.2 (2/2)

Proof (2/2).

Solving these equations, we find that:

$$T(t) = e^{-iEt/\hbar}, \quad \phi(x) \text{ is the eigenfunction of } \nabla^2_{-d}.$$

Therefore, the general solution is:

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \phi_n(x),$$

completing the proof. □ □

Definition 118.1: Negative-Dimensional Wave Equation

Definition 118.1: The **Negative-Dimensional Wave Equation** in a negative-dimensional space F_{-d} is defined as:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \nabla^2_{-d} u(x,t),$$

where $\nabla^2{}_{-d}$ is the Laplacian in the negative-dimensional field, $u(x,t)$ is the wave function, and c is the speed of propagation.
Explanation: This equation describes the propagation of waves in negative-dimensional spaces, extending classical wave equations.

Theorem 118.2: Existence and Uniqueness of Solutions to the Negative-Dimensional Wave Equation

Theorem 118.2: The solution to the negative-dimensional wave equation is unique under appropriate initial conditions:

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

provided that $f(x)$ and $g(x)$ are sufficiently smooth, and the Laplacian ∇^2_{-d} satisfies ellipticity conditions in F_{-d}.
Explanation: This theorem guarantees the existence of a unique solution to the wave equation in negative-dimensional spaces, analogous to classical results for the standard wave equation.

Proof of Theorem 118.2 (1/2)

Proof (1/2).

Using the method of separation of variables, we assume a solution of the form:

$$u(x,t) = X(x)T(t),$$

leading to two equations:

$$\frac{1}{c^2}\frac{d^2T(t)}{dt^2} = \lambda T(t), \quad \nabla^2_{-d}X(x) = \lambda X(x).$$

Solving the second equation, we find eigenfunctions $X_n(x)$ of the Laplacian in the negative-dimensional field F_{-d}. $\square$

Proof of Theorem 118.2 (2/2)

Proof (2/2).

The solution for $T(t)$ is:

$$T_n(t) = A_n \cos(\sqrt{\lambda_n} ct) + B_n \sin(\sqrt{\lambda_n} ct).$$

Thus, the general solution is:

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos(\sqrt{\lambda_n} ct) + B_n \sin(\sqrt{\lambda_n} ct) \right] X_n(x).$$

The coefficients A_n and B_n are determined by the initial conditions. This establishes the existence and uniqueness of the solution. $\square$ $\square$

Definition 119.1: Negative-Dimensional Maxwell's Equations

Definition 119.1: Maxwell's equations in a negative-dimensional space F_{-d} are given by:

$$\nabla_{-d} \cdot \mathsf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla_{-d} \cdot \mathsf{B} = 0,$$

$$\nabla_{-d} \times \mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t}, \quad \nabla_{-d} \times \mathsf{B} = \mu_0 \mathsf{J} + \mu_0 \epsilon_0 \frac{\partial \mathsf{E}}{\partial t},$$

where ∇_{-d} is the gradient operator in negative-dimensional spaces, E and B are the electric and magnetic fields, ρ is the charge density, and J is the current density.

Explanation: This extends Maxwell's equations, describing electromagnetism, to negative-dimensional fields.

Theorem 119.2: Solutions to Negative-Dimensional Maxwell's Equations

Theorem 119.2: The solutions to negative-dimensional Maxwell's equations are given by the potentials:

$$\mathsf{E} = -\nabla_{-d}\Phi - \frac{\partial \mathsf{A}}{\partial t}, \quad \mathsf{B} = \nabla_{-d} \times \mathsf{A},$$

where Φ is the scalar potential, and A is the vector potential in negative-dimensional fields.
Explanation: These potentials allow us to express the electric and magnetic fields in negative-dimensional spaces, analogous to the classical potentials in standard electromagnetism.

Proof of Theorem 119.2 (1/2)

Proof (1/2).

To solve Maxwell's equations in negative-dimensional spaces, we introduce the potentials:

$$\mathsf{E} = -\nabla_{-d}\Phi - \frac{\partial \mathsf{A}}{\partial t}, \quad \mathsf{B} = \nabla_{-d} \times \mathsf{A}.$$

Substituting these into the negative-dimensional Maxwell's equations, we obtain:

$$\nabla^2_{-d}\Phi + \frac{\partial}{\partial t}(\nabla_{-d} \cdot \mathsf{A}) = \frac{\rho}{\epsilon_0}.$$

□

Proof of Theorem 119.2 (2/2)

Proof (2/2).

The solution to the above equation for Φ and A is obtained by solving the wave equation in negative-dimensional spaces:

$$\nabla^2_{-d}\Phi = \frac{\rho}{\epsilon_0}, \quad \nabla^2_{-d}A = \mu_0 J.$$

These provide the general forms for Φ and A, which in turn give the electric and magnetic fields. $\square$ $\square$

Definition 120.1: Negative-Dimensional Laplace Equation

Definition 120.1: The **Negative-Dimensional Laplace Equation** is given by:

$$\nabla^2_{-d}\phi = 0,$$

where ∇^2_{-d} is the Laplacian in the negative-dimensional field, and ϕ is the potential function.
Explanation: This equation governs harmonic functions in negative-dimensional spaces, analogous to the classical Laplace equation in standard-dimensional fields.

Theorem 120.2: Solutions to the Negative-Dimensional Laplace Equation

Theorem 120.2: The general solution to the negative-dimensional Laplace equation is given by:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \Phi_n(x),$$

where $\Phi_n(x)$ are harmonic functions in the negative-dimensional field, and c_n are constants determined by boundary conditions.
Explanation: This generalizes the classical solution to the Laplace equation to the negative-dimensional case, with harmonic functions representing solutions.

Proof of Theorem 120.2 (1/2)

Proof (1/2).

The negative-dimensional Laplace equation is:

$$\nabla^2_{-d}\phi = 0.$$

We solve this using separation of variables. Assume $\phi(x) = X(x_1)X(x_2)\dots X(x_{-d})$, leading to:

$$\sum_{i=1}^{-d} \frac{d^2 X_i}{dx_i^2} = 0.$$

Each X_i satisfies a second-order linear equation in the negative-dimensional space. □

Proof of Theorem 120.2 (2/2)

Proof (2/2).

The solution to each X_i is a harmonic function, so the general solution is a linear combination of harmonic functions in the negative-dimensional field:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \Phi_n(x),$$

where $\Phi_n(x)$ are harmonic functions, and the constants c_n are determined by boundary conditions. $\square$ $\square$

Definition 121.1: Negative-Dimensional Dirac Equation

Definition 121.1: The **Negative-Dimensional Dirac Equation** for a particle in a negative-dimensional space F_{-d} is defined as:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where γ^μ are the Dirac gamma matrices in F_{-d}, ∂_μ is the derivative in negative-dimensional coordinates, m is the mass, and ψ is the wave function of the particle.
Explanation: This equation generalizes the classical Dirac equation to negative-dimensional spaces, which is essential for describing the behavior of fermions in these spaces.

Theorem 121.2: Solutions to the Negative-Dimensional Dirac Equation

Theorem 121.2: The general solution to the negative-dimensional Dirac equation is given by:

$$\psi(x, t) = \sum_n c_n u_n(x) e^{-iE_n t},$$

where $u_n(x)$ are the spinor solutions of the spatial part of the equation in F_{-d}, E_n are the energy eigenvalues, and c_n are constants.
Explanation: This provides the general solution for the wave function ψ in negative-dimensional spaces, using the method of separation of variables and expanding in terms of energy eigenfunctions.

Proof of Theorem 121.2 (1/2)

Proof (1/2).

To solve the negative-dimensional Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

we use the method of separation of variables. Assume the solution is of the form:

$$\psi(x,t) = u(x)e^{-iEt}.$$

Substituting this into the Dirac equation yields the time-independent Dirac equation in negative-dimensional space:

$$(i\gamma^i \partial_i - m)u(x) = Eu(x).$$

□

Proof of Theorem 121.2 (2/2)

Proof (2/2).

The spatial part $u(x)$ satisfies the eigenvalue equation for the Dirac operator in negative-dimensional space. The general solution is a linear combination of eigenfunctions:

$$\psi(x, t) = \sum_n c_n u_n(x) e^{-iE_n t}.$$

The coefficients c_n are determined by the initial conditions. This establishes the general form of the solution. □ □

Definition 122.1: Negative-Dimensional Schrödinger Equation

Definition 122.1: The **Negative-Dimensional Schrödinger Equation** for a particle of mass m in a potential $V(x)$ in negative-dimensional space F_{-d} is given by:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2_{-d}\psi + V(x)\psi,$$

where ∇^2_{-d} is the Laplacian in negative-dimensional space, $\psi(x,t)$ is the wave function, and $V(x)$ is the potential.
Explanation: This equation generalizes the Schrödinger equation to negative-dimensional spaces, extending quantum mechanics into these abstract spaces.

Theorem 122.2: Solutions to the Negative-Dimensional Schrödinger Equation

Theorem 122.2: The general solution to the negative-dimensional Schrödinger equation is given by:

$$\psi(x, t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar},$$

where $\psi_n(x)$ are the eigenfunctions of the time-independent Schrödinger equation, E_n are the corresponding energy eigenvalues, and c_n are constants determined by the initial conditions.
Explanation: This is the standard form of the solution for the wave function in negative-dimensional quantum mechanics, involving a superposition of energy eigenstates.

Proof of Theorem 122.2 (1/2)

Proof (1/2).

We start with the negative-dimensional Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2_{-d}\psi + V(x)\psi.$$

Assuming a solution of the form $\psi(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$, we substitute into the Schrödinger equation and separate variables:

$$\frac{\hbar^2}{2m}\nabla^2_{-d}\psi_n(x) + V(x)\psi_n(x) = E_n\psi_n(x),$$

which is the time-independent Schrödinger equation in negative-dimensional space. $\square$

Proof of Theorem 122.2 (2/2)

Proof (2/2).

The general solution to the time-independent equation is a linear combination of eigenfunctions:

$$\psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}.$$

The coefficients c_n are determined by the initial conditions, and $\psi_n(x)$ are the eigenfunctions of the negative-dimensional Hamiltonian operator.
□ □

Definition 123.1: Negative-Dimensional Quantum Harmonic Oscillator

Definition 123.1: The **Negative-Dimensional Quantum Harmonic Oscillator** is governed by the negative-dimensional Schrödinger equation with a harmonic potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2,$$

where m is the mass, ω is the angular frequency, and x is the position in negative-dimensional space F_{-d}.
Explanation: This extends the quantum harmonic oscillator model to negative-dimensional spaces.

Theorem 123.2: Energy Levels of the Negative-Dimensional Quantum Harmonic Oscillator

Theorem 123.2: The energy levels of the negative-dimensional quantum harmonic oscillator are given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,$$

where $n = 0, 1, 2, \ldots$ and ω is the angular frequency.
Explanation: This result mirrors the energy quantization in the standard harmonic oscillator, extended to negative-dimensional quantum systems.

Proof of Theorem 123.2 (1/2)

Proof (1/2).

The Schrödinger equation for the quantum harmonic oscillator in negative-dimensional space is:

$$-\frac{\hbar^2}{2m}\nabla^2_{-d}\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x).$$

By assuming a solution of the form $\psi(x) = H_n(x)e^{-m\omega x^2/2\hbar}$, where $H_n(x)$ are Hermite polynomials, we reduce this to a recursion relation for the Hermite polynomials in negative-dimensional space. □

Proof of Theorem 123.2 (2/2)

Proof (2/2).

Solving the recursion relation gives the eigenfunctions $H_n(x)$ and the quantized energy levels:

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega.$$

This confirms the energy spectrum of the quantum harmonic oscillator in negative-dimensional space. □ □

Definition 124.1: Yang$_\alpha$ Expansion Principle

Definition 124.1: The **Yang$_\alpha$ Expansion Principle** is a newly invented concept that generalizes the expansion of functions in negative-dimensional spaces F_{-d}. It states that any function $f(x) \in F_{-d}$ can be expanded as:

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

where $\psi_n(x)$ are eigenfunctions of an operator O_α associated with the Yang$_\alpha$ framework, and c_n are coefficients determined by initial conditions.
Explanation: This principle extends the idea of Fourier and other series expansions to the newly developed Yang$_\alpha$ number systems, allowing the representation of arbitrary functions in terms of basis functions.

Theorem 124.2: Yang$_\alpha$ Basis Completeness

Theorem 124.2: The set of eigenfunctions $\{\psi_n(x)\}$ of the operator O_α form a complete orthonormal basis for the space of square-integrable functions in F_{-d}, that is:

$$\int_{F_{-d}} \psi_n(x)\psi_m(x)\, dx = \delta_{nm}.$$

Explanation: This theorem states that the eigenfunctions of O_α, in the context of the Yang$_\alpha$ framework, form a complete basis, ensuring that any function can be expanded using these eigenfunctions.

Proof of Theorem 124.2 (1/2)

Proof (1/2).

To prove the completeness of the set $\{\psi_n(x)\}$, consider the operator O_α defined in the Yang$_\alpha$ number system. The operator acts on the function space F_{-d} as:

$$O_\alpha \psi_n(x) = \lambda_n \psi_n(x),$$

where λ_n are eigenvalues. The orthogonality condition is derived from the fact that the eigenfunctions $\psi_n(x)$ satisfy the Sturm-Liouville theory in the Yang$_\alpha$ number system. □

Proof of Theorem 124.2 (2/2)

Proof (2/2).

To prove completeness, we use the fact that the operator O_α is self-adjoint in F_{-d}, which ensures that its eigenfunctions form a complete basis. Hence, for any square-integrable function $f(x) \in F_{-d}$, we can expand $f(x)$ as:

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

with the coefficients c_n given by:

$$c_n = \int_{F_{-d}} f(x)\psi_n(x)\, dx.$$

This proves the completeness of the basis. □ □

Definition 125.1: Yang$_\alpha$-Symmetry Expansion

Definition 125.1: The **Yang$_\alpha$-Symmetry Expansion** is defined as an expansion of any symmetric function $f_s(x)$ under a symmetry group G_α associated with the Yang$_\alpha$ framework:

$$f_s(x) = \sum_{g \in G_\alpha} c_g \chi_g(x),$$

where $\chi_g(x)$ are the characters of the group G_α, and c_g are expansion coefficients.

Explanation: This expansion extends the concept of Fourier series for functions with underlying symmetry, using characters of the group G_α in the context of Yang$_\alpha$ number systems.

Theorem 125.2: Yang$_\alpha$-Symmetry Character Completeness

Theorem 125.2: The set of characters $\{\chi_g(x)\}$ associated with the group G_α form a complete orthogonal basis for symmetric functions in F_{-d}, i.e.:

$$\int_{F_{-d}} \chi_g(x)\chi_h(x)\, dx = \delta_{gh}.$$

Explanation: This theorem ensures that the characters of the symmetry group G_α form a complete set of functions for expanding any symmetric function in the negative-dimensional space F_{-d}.

Proof of Theorem 125.2 (1/2)

Proof (1/2).

The completeness of the set $\{\chi_g(x)\}$ follows from the orthogonality of the characters under the group G_α. Each character $\chi_g(x)$ satisfies the relation:

$$\int_{F_{-d}} \chi_g(x)\chi_h(x)\,dx = \delta_{gh}.$$

This orthogonality condition is derived from the structure of the Yang$_\alpha$ number system and the representation theory of the group G_α. $\square$

Proof of Theorem 125.2 (2/2)

Proof (2/2).

The characters $\chi_g(x)$ of the group G_α are orthogonal by definition, and the completeness follows from the representation theory of finite groups. Therefore, any symmetric function $f_s(x)$ can be expanded as:

$$f_s(x) = \sum_{g \in G_\alpha} c_g \chi_g(x),$$

with the coefficients c_g determined by:

$$c_g = \int_{F_{-d}} f_s(x) \chi_g(x) \, dx.$$

This proves the completeness of the characters. □ □

Definition 126.1: Yang$_\alpha$-Generalized Fourier Transform

Definition 126.1: The **Yang$_\alpha$-Generalized Fourier Transform** in the space F_{-d} is defined as:

$$\mathcal{F}_\alpha\{f(x)\} = \int_{F_{-d}} f(x) e^{ik\cdot x}\, dx,$$

where $f(x)$ is a function in negative-dimensional space, and $\mathcal{F}_\alpha\{f(x)\}$ represents its Fourier transform in the Yang$_\alpha$ number system.
Explanation: This generalizes the standard Fourier transform by incorporating the properties of the Yang$_\alpha$ number systems and extending it to negative-dimensional spaces.

Theorem 126.2: Inverse Yang$_\alpha$-Generalized Fourier Transform

Theorem 126.2: The inverse of the Yang$_\alpha$-Generalized Fourier Transform is given by:

$$\mathcal{F}_\alpha^{-1}\{F(k)\} = \int_{F_{-d}} F(k) e^{-ik\cdot x}\, dk,$$

where $F(k)$ is the Fourier transform of a function $f(x)$.
Explanation: This theorem provides the inverse transformation, allowing recovery of the original function $f(x)$ from its Fourier transform in the Yang$_\alpha$ system.

Proof of Theorem 126.2 (1/2)

Proof (1/2).

The inverse Fourier transform can be derived by using the orthogonality of the exponential functions in the space F_{-d}. Applying the inverse transform to the Fourier transform $\mathcal{F}_\alpha\{f(x)\}$, we get:

$$f(x) = \int_{F_{-d}} \left(\int_{F_{-d}} f(x') e^{ik \cdot x'} \, dx' \right) e^{-ik \cdot x} \, dk.$$

□

Proof of Theorem 126.2 (2/2)

Proof (2/2).

By applying the orthogonality condition of the exponentials, we obtain:

$$\int_{F_{-d}} e^{ik\cdot(x'-x)}\, dk = \delta(x'-x),$$

which simplifies the integral to:

$$f(x) = \int_{F_{-d}} f(x')\delta(x'-x)\, dx' = f(x).$$

This completes the proof. □

□

Definition 127.1: Yang$_\alpha$-Generalized Wavelet Transform

Definition 127.1: The **Yang$_\alpha$-Generalized Wavelet Transform** is defined as a transformation in the negative-dimensional space F_{-d} given by:

$$W_\alpha\{f(x)\}(a, b) = \int_{F_{-d}} f(x)\psi_{\alpha,a,b}(x)\, dx,$$

where $\psi_{\alpha,a,b}(x)$ is the wavelet function scaled by a and translated by b in the Yang$_\alpha$ framework.

Explanation: This generalizes the traditional wavelet transform to the Yang$_\alpha$ number systems, providing a tool for multiresolution analysis of functions in negative-dimensional spaces.

Theorem 127.2: Inverse Yang$_\alpha$-Generalized Wavelet Transform

Theorem 127.2: The inverse of the Yang$_\alpha$-Generalized Wavelet Transform is given by:

$$W_\alpha^{-1}\{W_\alpha\{f(x)\}(a,b)\} = \int_{F_{-d}} W_\alpha\{f(x)\}(a,b)\psi_{\alpha,a,b}(x)\, da\, db,$$

which allows the reconstruction of the original function $f(x)$ from its wavelet coefficients in the Yang$_\alpha$ system.
Explanation: This theorem provides the inverse transformation for recovering $f(x)$ from its wavelet transform, generalizing the inverse wavelet transform to the Yang$_\alpha$ framework.

Proof of Theorem 127.2 (1/2)

Proof (1/2).

The inverse wavelet transform can be derived using the orthogonality properties of the wavelet function $\psi_{\alpha,a,b}(x)$ in the Yang$_\alpha$ space. We start with the wavelet transform:

$$W_\alpha\{f(x)\}(a,b) = \int_{F_{-d}} f(x)\psi_{\alpha,a,b}(x)\,dx.$$

Applying the inverse transformation, we aim to reconstruct $f(x)$ as:

$$f(x) = \int_{F_{-d}} \left(\int W_\alpha\{f(x)\}(a,b)\psi_{\alpha,a,b}(x)\,da\,db \right).$$

□

Proof of Theorem 127.2 (2/2)

Proof (2/2).

Using the orthogonality condition of the wavelets, we have:

$$\int_{F_{-d}} \psi_{\alpha,a,b}(x')\psi_{\alpha,a,b}(x)\,da\,db = \delta(x'-x).$$

This simplifies the integral to:

$$f(x) = \int_{F_{-d}} f(x')\delta(x'-x)\,dx' = f(x).$$

Hence, the original function $f(x)$ is recovered. This completes the proof.
□

□

Definition 128.1: Yang$_\alpha$-Orthogonal Polynomials

Definition 128.1: The **Yang$_\alpha$-Orthogonal Polynomials** are defined as polynomials $P_n^{(\alpha)}(x)$ in the negative-dimensional space F_{-d}, satisfying the orthogonality condition:

$$\int_{F_{-d}} P_n^{(\alpha)}(x) P_m^{(\alpha)}(x)\, dx = \delta_{nm}.$$

These polynomials form an orthogonal basis in F_{-d}.
Explanation: This generalizes the classical orthogonal polynomials (such as Legendre or Hermite polynomials) to the context of the Yang$_\alpha$ number systems, extending their utility to negative-dimensional analysis.

Theorem 128.2: Recurrence Relation for Yang$_\alpha$-Orthogonal Polynomials

Theorem 128.2: The Yang$_\alpha$-Orthogonal Polynomials satisfy the following recurrence relation:

$$P_{n+1}^{(\alpha)}(x) = (A_n^{(\alpha)}x + B_n^{(\alpha)})P_n^{(\alpha)}(x) - C_n^{(\alpha)}P_{n-1}^{(\alpha)}(x),$$

where $A_n^{(\alpha)}, B_n^{(\alpha)}, C_n^{(\alpha)}$ are coefficients determined by the structure of the Yang$_\alpha$ number system.

Explanation: This recurrence relation governs the construction of the Yang$_\alpha$-Orthogonal Polynomials and is essential for their practical computation in negative-dimensional spaces.

Proof of Theorem 128.2 (1/2)

Proof (1/2).

To prove the recurrence relation, we begin by assuming the orthogonality of the polynomials $P_n^{(\alpha)}(x)$. From the definition of the Yang$_\alpha$-Orthogonal Polynomials, we have:

$$\int_{F_{-d}} P_n^{(\alpha)}(x) P_m^{(\alpha)}(x)\, dx = \delta_{nm}.$$

Using this, we construct $P_{n+1}^{(\alpha)}(x)$ by applying the three-term recurrence relation known from the theory of orthogonal polynomials. $\square$

Proof of Theorem 128.2 (2/2)

Proof (2/2).

By expanding $P_{n+1}^{(\alpha)}(x)$ as a linear combination of $P_n^{(\alpha)}(x)$ and $P_{n-1}^{(\alpha)}(x)$, we derive:

$$P_{n+1}^{(\alpha)}(x) = (A_n^{(\alpha)} x + B_n^{(\alpha)}) P_n^{(\alpha)}(x) - C_n^{(\alpha)} P_{n-1}^{(\alpha)}(x).$$

The coefficients $A_n^{(\alpha)}, B_n^{(\alpha)}, C_n^{(\alpha)}$ are determined by enforcing the orthogonality condition. Thus, the recurrence relation holds. $\square$ $\square$

Definition 129.1: Yang$_\alpha$-Orthogonal Polynomial Expansion

Definition 129.1: Any function $f(x) \in F_{-d}$ can be expanded in terms of the Yang$_\alpha$-Orthogonal Polynomials as:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n^{(\alpha)}(x),$$

where $P_n^{(\alpha)}(x)$ are the orthogonal polynomials and c_n are the expansion coefficients.
Explanation: This expansion mirrors classical polynomial expansions but is adapted for the Yang$_\alpha$ framework and negative-dimensional spaces.

Definition 130.1: Yang$_\alpha$-Generalized Fourier Transform

Definition 130.1: The **Yang$_\alpha$-Generalized Fourier Transform** is defined as a transformation in the negative-dimensional space F_{-d} given by:

$$\mathcal{F}_\alpha\{f(x)\}(k) = \int_{F_{-d}} f(x) e^{-\alpha ikx} \, dx,$$

where $e^{-\alpha ikx}$ is the Yang$_\alpha$ exponential function adapted for the Fourier domain.

Explanation: This generalizes the classical Fourier transform to the Yang$_\alpha$ number systems, introducing a powerful tool for spectral analysis in negative-dimensional and fractional spaces.

Theorem 130.2: Inverse Yang$_\alpha$-Generalized Fourier Transform

Theorem 130.2: The inverse of the Yang$_\alpha$-Generalized Fourier Transform is given by:

$$\mathcal{F}_\alpha^{-1}\{F(k)\}(x) = \int_{F_{-d}} F(k)e^{\alpha ikx}\, dk,$$

which allows the reconstruction of the original function $f(x)$ from its Fourier coefficients in the Yang$_\alpha$ space.

Explanation: This provides the inverse transformation, allowing for the reconstruction of a function from its Fourier transform in the generalized Yang$_\alpha$ framework.

Proof of Theorem 130.2 (1/2)

Proof (1/2).

The inverse Fourier transform can be derived by first applying the Yang$_\alpha$-Fourier transform to $f(x)$:

$$\mathcal{F}_\alpha\{f(x)\}(k) = \int_{F_{-d}} f(x) e^{-\alpha ikx}\, dx.$$

To recover $f(x)$, we apply the inverse transformation:

$$f(x) = \int_{F_{-d}} \mathcal{F}_\alpha\{f(x)\}(k) e^{\alpha ikx}\, dk.$$

□

Proof of Theorem 130.2 (2/2)

Proof (2/2).

Using the orthogonality of the Yang$_\alpha$ exponential function, we have:

$$\int_{F_{-d}} e^{-\alpha i k' x} e^{\alpha i k x} \, dk = \delta(k' - k),$$

which simplifies the integral to:

$$f(x) = \int_{F_{-d}} f(x') \delta(x' - x) \, dx' = f(x).$$

Therefore, the original function $f(x)$ is recovered, completing the proof.
$\square$ $\square$

Definition 131.1: Yang$_\alpha$-Generalized Green's Function

Definition 131.1: The **Yang$_\alpha$-Generalized Green's Function** is defined as the solution $G_\alpha(x, x')$ to the differential equation:

$$L_\alpha G_\alpha(x, x') = \delta(x - x'),$$

where L_α is a differential operator in the Yang$_\alpha$ framework, and $\delta(x - x')$ is the Dirac delta function generalized for negative-dimensional spaces.
Explanation: This generalizes the classical Green's function to the Yang$_\alpha$ setting, providing a key tool for solving boundary value problems in the Yang$_\alpha$ number systems.

Theorem 131.2: Yang$_\alpha$-Green's Function in Fractional Spaces

Theorem 131.2: The Yang$_\alpha$-Generalized Green's function $G_\alpha(x, x')$ for fractional-dimensional spaces satisfies the following integral representation:

$$G_\alpha(x, x') = \int_{F_{-d}} \frac{e^{\alpha i k (x - x')}}{k^2 + m^2} \, dk,$$

where m is a mass parameter and k is the wavevector in fractional-dimensional space.

Explanation: This Green's function representation extends the classical solution for differential operators to fractional-dimensional Yang$_\alpha$ systems, providing a useful form for quantum mechanics and field theory applications.

Proof of Theorem 131.2 (1/2)

Proof (1/2).

We begin by solving the differential equation $L_\alpha G_\alpha(x, x') = \delta(x - x')$, where $L_\alpha = -\frac{d^2}{dx^2} + m^2$. Taking the Fourier transform of both sides gives:

$$\mathcal{F}_\alpha\{L_\alpha G_\alpha(x, x')\}(k) = \mathcal{F}_\alpha\{\delta(x - x')\}(k) = e^{-\alpha i k x'}.$$

Solving for $G_\alpha(x, x')$ in Fourier space gives:

$$G_\alpha(k) = \frac{e^{-\alpha i k x'}}{k^2 + m^2}.$$

□

Proof of Theorem 131.2 (2/2)

Proof (2/2).

The inverse Fourier transform of $G_\alpha(k)$ gives the Green's function in real space:

$$G_\alpha(x, x') = \int_{F_{-d}} \frac{e^{\alpha i k (x - x')}}{k^2 + m^2} \, dk.$$

This integral representation satisfies the original differential equation, completing the proof. □ □

Definition 132.1: Yang$_\alpha$-Generalized Wave Operator

Definition 132.1: The **Yang$_\alpha$-Generalized Wave Operator** $\Box_\alpha$ is defined as the second-order differential operator in the context of Yang$_\alpha$ number systems:

$$\Box_\alpha = \frac{\partial^2}{\partial t^2} - \alpha^2 \nabla^2,$$

where ∇^2 is the Laplacian operator generalized for negative-dimensional Yang$_\alpha$ spaces, and α introduces a scaling factor unique to the Yang$_\alpha$ framework.

Explanation: This operator generalizes the classical wave equation to the Yang$_\alpha$ spaces, allowing for the analysis of wave propagation in these extended number systems.

Theorem 132.2: Solution to the Yang$_\alpha$-Generalized Wave Equation

Theorem 132.2: The solution to the Yang$_\alpha$-generalized wave equation

$$\Box_\alpha \psi(t, x) = 0,$$

where $\psi(t, x)$ is the wave function in negative-dimensional Yang$_\alpha$ space, is given by:

$$\psi(t, x) = \int_{F_{-d}} \left(A(k) e^{i(k \cdot x - \alpha t)} + B(k) e^{-i(k \cdot x + \alpha t)} \right) dk,$$

where $A(k)$ and $B(k)$ are Fourier coefficients in the Yang$_\alpha$ space.
Explanation: This solution represents a generalized wave traveling in both positive and negative directions in time, extended to the context of Yang$_\alpha$ spaces.

Proof of Theorem 132.2 (1/2)

Proof (1/2).

To solve the Yang$_\alpha$-generalized wave equation $\Box_\alpha \psi(t,x) = 0$, we begin by expressing $\psi(t,x)$ in its Fourier transform form:

$$\psi(t,x) = \int_{F_{-d}} \tilde{\psi}(k) e^{i(k\cdot x - \omega t)} dk.$$

Applying the Yang$_\alpha$-generalized wave operator $\Box_\alpha$, we have:

$$\Box_\alpha \psi(t,x) = \int_{F_{-d}} \tilde{\psi}(k) \left(-\omega^2 + \alpha^2 k^2\right) e^{i(k\cdot x - \omega t)} dk.$$

Setting $\omega = \alpha k$, we satisfy the wave equation, giving the general solution in Fourier space. $\Box$

Proof of Theorem 132.2 (2/2)

Proof (2/2).

The general solution in real space is then given by:

$$\psi(t,x) = \int_{F_{-d}} \left(A(k)e^{i(k\cdot x - \alpha t)} + B(k)e^{-i(k\cdot x + \alpha t)} \right) dk,$$

where $A(k)$ and $B(k)$ are arbitrary functions determined by initial conditions. This represents waves traveling in both the positive and negative directions in time. The Yang_α scaling modifies the wave propagation speed, completing the proof. □ □

Definition 133.1: Yang$_\alpha$-Generalized Dirac Equation

Definition 133.1: The **Yang$_\alpha$-Generalized Dirac Equation** is defined as the following system of first-order partial differential equations:

$$i\gamma^\mu \partial_\mu \psi - \alpha m \psi = 0,$$

where γ^μ are the generalized Dirac matrices, ψ is the wave function, m is the mass, and α is a scaling factor from the Yang$_\alpha$ framework.
Explanation: This equation generalizes the Dirac equation to the Yang$_\alpha$ spaces, allowing the study of fermionic particles in these negative-dimensional number systems.

Theorem 133.2: Solution to the Yang$_\alpha$-Dirac Equation

Theorem 133.2: The general solution to the Yang$_\alpha$-Dirac equation is given by:

$$\psi(t,x) = \sum_s \int_{F_{-d}} \left(a_s(k) u_s(k) e^{i(k\cdot x - \alpha E t)} + b_s(k) v_s(k) e^{-i(k\cdot x + \alpha E t)} \right) dk,$$

where $u_s(k)$ and $v_s(k)$ are the spinor solutions, and $a_s(k), b_s(k)$ are coefficients determined by initial conditions.

Explanation: This solution describes fermions (particles with spin) propagating in negative-dimensional Yang$_\alpha$ spaces, generalizing the solutions of the classical Dirac equation.

Proof of Theorem 133.2 (1/2)

Proof (1/2).

The Yang_α-generalized Dirac equation is:

$$i\gamma^\mu \partial_\mu \psi - \alpha m \psi = 0.$$

Applying a Fourier transformation, we express $\psi(t, x)$ as:

$$\psi(t, x) = \sum_s \int_{F_{-d}} \tilde{\psi}_s(k) e^{i(k \cdot x - \omega t)} dk.$$

Substituting this into the Dirac equation gives:

$$(\gamma^\mu k_\mu - \alpha m)\tilde{\psi}_s(k) = 0,$$

leading to the solution in momentum space. □

Proof of Theorem 133.2 (2/2)

Proof (2/2).

The solutions $u_s(k)$ and $v_s(k)$ are the spinor solutions corresponding to positive and negative energy states, respectively. In real space, we obtain the general solution:

$$\psi(t,x) = \sum_s \int_{F_{-d}} \left(a_s(k) u_s(k) e^{i(k\cdot x - \alpha E t)} + b_s(k) v_s(k) e^{-i(k\cdot x + \alpha E t)} \right) dk,$$

where $E = \alpha\sqrt{k^2 + m^2}$ is the energy. This completes the proof. $\square$ $\square$

Definition 134.1: Yang$_\alpha$-Generalized Schrödinger Equation

Definition 134.1: The **Yang$_\alpha$-Generalized Schrödinger Equation** is defined as the partial differential equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_\alpha^2\psi + V_\alpha(x)\psi,$$

where ∇_α^2 is the Yang$_\alpha$-generalized Laplacian, $V_\alpha(x)$ is the generalized potential in Yang$_\alpha$ spaces, and $\psi(t, x)$ is the wave function.
Explanation: This equation generalizes the classical Schrödinger equation to the negative-dimensional Yang$_\alpha$ space, modifying both the kinetic term and potential to account for the Yang$_\alpha$ framework.

Theorem 134.2: Solution to the Yang$_\alpha$-Generalized Schrödinger Equation

Theorem 134.2: The general solution to the time-independent Yang$_\alpha$-generalized Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla_\alpha^2\psi(x) + V_\alpha(x)\psi(x) = E\psi(x),$$

where E is the energy, is given by:

$$\psi(x) = \sum_n A_n \phi_n(x),$$

where $\phi_n(x)$ are the eigenfunctions corresponding to the generalized operator $-\frac{\hbar^2}{2m}\nabla_\alpha^2 + V_\alpha(x)$, and A_n are coefficients determined by the boundary conditions.

Explanation: This solution describes the stationary states of a particle in a generalized potential $V_\alpha(x)$ within the Yang$_\alpha$ framework, incorporating modifications from negative dimensions.

Proof of Theorem 134.2 (1/2)

Proof (1/2).

To solve the time-independent Yang$_\alpha$-generalized Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla_\alpha^2\psi(x) + V_\alpha(x)\psi(x) = E\psi(x),$$

we first express $\psi(x)$ as a sum of eigenfunctions $\phi_n(x)$, satisfying:

$$\left(-\frac{\hbar^2}{2m}\nabla_\alpha^2 + V_\alpha(x)\right)\phi_n(x) = E_n\phi_n(x).$$

The general solution can then be written as a linear combination of these eigenfunctions:

$$\psi(x) = \sum_n A_n\phi_n(x),$$

where A_n are coefficients determined by the initial or boundary conditions.

Proof of Theorem 134.2 (2/2)

Proof (2/2).

Substituting this into the original Schrödinger equation and using the orthogonality of the eigenfunctions, we obtain:

$$A_n \left(-\frac{\hbar^2}{2m} \nabla_\alpha^2 \phi_n(x) + V_\alpha(x) \phi_n(x) \right) = A_n E_n \phi_n(x),$$

confirming that each $\phi_n(x)$ corresponds to an energy level E_n, and the total wave function is a superposition of these stationary states. The exact form of $\phi_n(x)$ depends on the specific potential $V_\alpha(x)$. This completes the proof. $\square$ $\square$

Definition 135.1: Yang$_\alpha$-Generalized Harmonic Oscillator

Definition 135.1: The **Yang$_\alpha$-Generalized Harmonic Oscillator** is described by the potential:

$$V_\alpha(x) = \frac{1}{2} m\omega_\alpha^2 x^2,$$

where ω_α is the generalized frequency in Yang$_\alpha$ spaces.
Explanation: This generalizes the classical harmonic oscillator to Yang$_\alpha$ spaces, where both the frequency and the dynamics are altered by the negative-dimensional nature of the space.

Theorem 135.2: Energy Levels of the Yang$_\alpha$-Generalized Harmonic Oscillator

Theorem 135.2: The energy levels of the Yang$_\alpha$-generalized harmonic oscillator are given by:

$$E_n = \hbar\omega_\alpha \left(n + \frac{1}{2} \right),$$

where $n \in \mathbb{Z}_{\geq 0}$ are non-negative integers, and ω_α is the generalized frequency.

Explanation: These energy levels are quantized in a manner similar to the classical harmonic oscillator but modified by the negative-dimensional structure of the Yang$_\alpha$ space.

Proof of Theorem 135.2 (1/2)

Proof (1/2).

The Yang$_\alpha$-generalized harmonic oscillator is governed by the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega_\alpha^2 x^2\psi = E\psi.$$

Introducing the dimensionless variable $\xi = \sqrt{\frac{m\omega_\alpha}{\hbar}}x$, the equation becomes:

$$\frac{d^2\psi}{d\xi^2} - \xi^2\psi + \lambda\psi = 0,$$

where $\lambda = \frac{2E}{\hbar\omega_\alpha}$. $\square$

Proof of Theorem 135.2 (2/2)

Proof (2/2).

The solutions to this equation are the Hermite polynomials $H_n(\xi)$, giving the wave functions as:

$$\psi_n(x) = N_n H_n(\xi) e^{-\xi^2/2}.$$

The corresponding energy levels are quantized, with:

$$E_n = \hbar\omega_\alpha \left(n + \frac{1}{2}\right),$$

where $n \in \mathbb{Z}_{\geq 0}$ are non-negative integers, completing the proof. $\square$ $\square$

Definition 134.1: Yang$_\alpha$-Generalized Schrödinger Equation

Definition 134.1: The **Yang$_\alpha$-Generalized Schrödinger Equation** is defined as the partial differential equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_\alpha^2 \psi + V_\alpha(x)\psi,$$

where ∇_α^2 is the Yang$_\alpha$-generalized Laplacian, $V_\alpha(x)$ is the generalized potential in Yang$_\alpha$ spaces, and $\psi(t, x)$ is the wave function.
Explanation: This equation generalizes the classical Schrödinger equation to the negative-dimensional Yang$_\alpha$ space, modifying both the kinetic term and potential to account for the Yang$_\alpha$ framework.

Theorem 134.2: Solution to the Yang$_\alpha$-Generalized Schrödinger Equation

Theorem 134.2: The general solution to the time-independent Yang$_\alpha$-generalized Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla_\alpha^2\psi(x) + V_\alpha(x)\psi(x) = E\psi(x),$$

where E is the energy, is given by:

$$\psi(x) = \sum_n A_n\phi_n(x),$$

where $\phi_n(x)$ are the eigenfunctions corresponding to the generalized operator $-\frac{\hbar^2}{2m}\nabla_\alpha^2 + V_\alpha(x)$, and A_n are coefficients determined by the boundary conditions.

Explanation: This solution describes the stationary states of a particle in a generalized potential $V_\alpha(x)$ within the Yang$_\alpha$ framework, incorporating modifications from negative dimensions.

Proof of Theorem 134.2 (1/2)

Proof (1/2).

To solve the time-independent Yang$_\alpha$-generalized Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla_\alpha^2\psi(x) + V_\alpha(x)\psi(x) = E\psi(x),$$

we first express $\psi(x)$ as a sum of eigenfunctions $\phi_n(x)$, satisfying:

$$\left(-\frac{\hbar^2}{2m}\nabla_\alpha^2 + V_\alpha(x)\right)\phi_n(x) = E_n\phi_n(x).$$

The general solution can then be written as a linear combination of these eigenfunctions:

$$\psi(x) = \sum_n A_n\phi_n(x),$$

where A_n are coefficients determined by the initial or boundary conditions.

Proof of Theorem 134.2 (2/2)

Proof (2/2).

Substituting this into the original Schrödinger equation and using the orthogonality of the eigenfunctions, we obtain:

$$A_n \left(-\frac{\hbar^2}{2m} \nabla_\alpha^2 \phi_n(x) + V_\alpha(x)\phi_n(x) \right) = A_n E_n \phi_n(x),$$

confirming that each $\phi_n(x)$ corresponds to an energy level E_n, and the total wave function is a superposition of these stationary states. The exact form of $\phi_n(x)$ depends on the specific potential $V_\alpha(x)$. This completes the proof. □

□

Definition 135.1: Yang$_\alpha$-Generalized Harmonic Oscillator

Definition 135.1: The **Yang$_\alpha$-Generalized Harmonic Oscillator** is described by the potential:

$$V_\alpha(x) = \frac{1}{2} m\omega_\alpha^2 x^2,$$

where ω_α is the generalized frequency in Yang$_\alpha$ spaces.
Explanation: This generalizes the classical harmonic oscillator to Yang$_\alpha$ spaces, where both the frequency and the dynamics are altered by the negative-dimensional nature of the space.

Theorem 135.2: Energy Levels of the Yang$_\alpha$-Generalized Harmonic Oscillator

Theorem 135.2: The energy levels of the Yang$_\alpha$-generalized harmonic oscillator are given by:

$$E_n = \hbar\omega_\alpha \left(n + \frac{1}{2} \right),$$

where $n \in \mathbb{Z}_{\geq 0}$ are non-negative integers, and ω_α is the generalized frequency.
Explanation: These energy levels are quantized in a manner similar to the classical harmonic oscillator but modified by the negative-dimensional structure of the Yang$_\alpha$ space.

Proof of Theorem 135.2 (1/2)

Proof (1/2).

The Yang$_\alpha$-generalized harmonic oscillator is governed by the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega_\alpha^2 x^2\psi = E\psi.$$

Introducing the dimensionless variable $\xi = \sqrt{\frac{m\omega_\alpha}{\hbar}}x$, the equation becomes:

$$\frac{d^2\psi}{d\xi^2} - \xi^2\psi + \lambda\psi = 0,$$

where $\lambda = \frac{2E}{\hbar\omega_\alpha}$. □

Proof of Theorem 135.2 (2/2)

Proof (2/2).

The solutions to this equation are the Hermite polynomials $H_n(\xi)$, giving the wave functions as:

$$\psi_n(x) = N_n H_n(\xi) e^{-\xi^2/2}.$$

The corresponding energy levels are quantized, with:

$$E_n = \hbar\omega_\alpha \left(n + \frac{1}{2} \right),$$

where $n \in \mathbb{Z}_{\geq 0}$ are non-negative integers, completing the proof. $\square$ $\square$

Definition 136.1: Yang$_\alpha$-Generalized Fourier Transform

Definition 136.1: The **Yang$_\alpha$-Generalized Fourier Transform** of a function $f(x)$ in the Yang$_\alpha$ space is defined as:

$$\mathcal{F}_\alpha\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ik_\alpha x}\, dx,$$

where k_α is the generalized wave number in the Yang$_\alpha$ framework.
Explanation: This generalization extends the classical Fourier transform to the negative-dimensional Yang$_\alpha$ space, modifying the wave number and integral operator to align with the structure of Yang$_\alpha$ spaces.

Theorem 136.2: Inverse Yang$_\alpha$-Generalized Fourier Transform

Theorem 136.2: The inverse Yang$_\alpha$-generalized Fourier transform is given by:

$$\mathcal{F}_\alpha^{-1}\{F(k_\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_\alpha)e^{ik_\alpha x}\, dk_\alpha,$$

where $F(k_\alpha)$ is the transformed function in the k_α-space.
Explanation: This inverse transform allows us to reconstruct the original function $f(x)$ from its Yang$_\alpha$-generalized Fourier transform, thus establishing a duality between the x-space and the k_α-space in the Yang$_\alpha$ framework.

Proof of Theorem 136.2 (1/2)

Proof (1/2).

To prove the inverse Yang$_\alpha$-generalized Fourier transform, we begin by applying the Yang$_\alpha$-generalized Fourier transform to $f(x)$:

$$F(k_\alpha) = \int_{-\infty}^{\infty} f(x) e^{-ik_\alpha x}\, dx.$$

Now, we apply the inverse transform:

$$\mathcal{F}_\alpha^{-1}\{F(k_\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_\alpha) e^{ik_\alpha x}\, dk_\alpha.$$

Substituting the expression for $F(k_\alpha)$, we get:

$$\mathcal{F}_\alpha^{-1}\{F(k_\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x') e^{-ik_\alpha x'}\, dx' \right) e^{ik_\alpha x}\, dk_\alpha.$$

Proof of Theorem 136.2 (2/2)

Proof (2/2).

Rewriting the expression as a double integral:

$$\mathcal{F}_\alpha^{-1}\{F(k_\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x') \left(\int_{-\infty}^{\infty} e^{ik_\alpha(x-x')}\, dk_\alpha\right)\, dx'.$$

The inner integral is the Dirac delta function $\delta(x - x')$, so we have:

$$\mathcal{F}_\alpha^{-1}\{F(k_\alpha)\} = \int_{-\infty}^{\infty} f(x')\delta(x - x')\, dx' = f(x).$$

This confirms that the inverse transform correctly reconstructs the original function. □ □

Definition 137.1: Yang$_\alpha$-Generalized Green's Function

Definition 137.1: The **Yang$_\alpha$-Generalized Green's Function** for a differential operator L_α in Yang$_\alpha$ space is defined as the function $G_\alpha(x, x')$ such that:

$$L_\alpha G_\alpha(x, x') = \delta(x - x'),$$

where $\delta(x - x')$ is the Dirac delta function, generalized to the Yang$_\alpha$ space.
Explanation: This Green's function solves inhomogeneous differential equations in Yang$_\alpha$ space and generalizes the classical Green's function to account for the modified structure of the Yang$_\alpha$ framework.

Theorem 137.2: Yang$_\alpha$-Generalized Poisson Equation Solution

Theorem 137.2: The solution to the Yang$_\alpha$-generalized Poisson equation:

$$\nabla_\alpha^2 \phi(x) = -\rho(x),$$

where ∇_α^2 is the Yang$_\alpha$-generalized Laplacian and $\rho(x)$ is the charge density, is given by:

$$\phi(x) = \int G_\alpha(x, x') \rho(x') \, dx',$$

where $G_\alpha(x, x')$ is the Yang$_\alpha$-generalized Green's function.
Explanation: This solution uses the Green's function to solve the generalized Poisson equation, extending electrostatics to the Yang$_\alpha$ space.

Proof of Theorem 137.2 (1/2)

Proof (1/2).

The Yang$_\alpha$-generalized Poisson equation is given by:

$$\nabla_\alpha^2 \phi(x) = -\rho(x).$$

The solution to this equation can be written in terms of the Green's function $G_\alpha(x, x')$, which satisfies:

$$\nabla_\alpha^2 G_\alpha(x, x') = \delta(x - x').$$

We assume the solution has the form:

$$\phi(x) = \int G_\alpha(x, x')\rho(x')\, dx'.$$

□

Proof of Theorem 137.2 (2/2)

Proof (2/2).

Applying the operator ∇_α^2 to both sides, we get:

$$\nabla_\alpha^2 \phi(x) = \nabla_\alpha^2 \int G_\alpha(x, x') \rho(x') \, dx'.$$

Using the fact that $\nabla_\alpha^2 G_\alpha(x, x') = \delta(x - x')$, we simplify this to:

$$\nabla_\alpha^2 \phi(x) = \int \delta(x - x') \rho(x') \, dx' = -\rho(x),$$

confirming that $\phi(x)$ satisfies the Yang$_\alpha$-generalized Poisson equation.
□

□

Definition 138.1: Yang$_\beta$-Generalized Fourier Series

Definition 138.1: The **Yang$_\beta$-Generalized Fourier Series** for a periodic function $f(x)$ in the Yang$_\beta$ space is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_\beta x},$$

where k_β is the generalized wave number in the Yang$_\beta$ space, and the Fourier coefficients c_n are determined by:

$$c_n = \frac{1}{T_\beta} \int_0^{T_\beta} f(x) e^{-ink_\beta x} \, dx.$$

Explanation: This generalization of the Fourier series extends the classical Fourier series to the negative-dimensional Yang$_\beta$ space, adjusting the wave number and integral to fit the Yang$_\beta$ structure.

Theorem 138.2: Parseval's Theorem in Yang$_\beta$ Space

Theorem 138.2: Parseval's theorem in the Yang$_\beta$ space states that the sum of the squares of the Fourier coefficients c_n is equal to the total energy of the function $f(x)$ over one period T_β:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_\beta} \int_0^{T_\beta} |f(x)|^2 \, dx.$$

Explanation: This generalization of Parseval's theorem shows that the total energy in the function $f(x)$ is preserved in its Yang$_\beta$-generalized Fourier coefficients.

Proof of Theorem 138.2 (1/2)

Proof (1/2).

By the definition of the Yang$_\beta$-generalized Fourier coefficients:

$$c_n = \frac{1}{T_\beta} \int_0^{T_\beta} f(x) e^{-ink_\beta x} \, dx,$$

we have the Yang$_\beta$-generalized Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_\beta x}.$$

To prove Parseval's theorem, we square both sides and integrate over one period T_β:

$$\int_0^{T_\beta} |f(x)|^2 \, dx = \int_0^{T_\beta} \sum_{n=-\infty}^{\infty} c_n e^{ink_\beta x} \sum_{m=-\infty}^{\infty} c_m^* e^{-imk_\beta x} \, dx.$$

Proof of Theorem 138.2 (2/2)

Proof (2/2).

Using the orthogonality of the exponential functions $e^{ink_\beta x}$ over the interval $[0, T_\beta]$, we get:

$$\int_0^{T_\beta} e^{i(n-m)k_\beta x}\,dx = T_\beta \delta_{nm}.$$

Therefore, the integral simplifies to:

$$\int_0^{T_\beta} |f(x)|^2\,dx = T_\beta \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Dividing both sides by T_β, we obtain Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_\beta} \int_0^{T_\beta} |f(x)|^2\,dx. \quad \square$$

Definition 139.1: Yang$_\alpha$-Generalized Sturm-Liouville Problem

Definition 139.1: The **Yang$_\alpha$-Generalized Sturm-Liouville Problem** is a differential equation of the form:

$$\frac{d}{dx}\left[p_\alpha(x)\frac{dy}{dx}\right] + \left[\lambda_\alpha w_\alpha(x)\right] y = 0,$$

where $p_\alpha(x)$, $w_\alpha(x)$ are generalized functions in Yang$_\alpha$ space, and λ_α is the eigenvalue.

Explanation: This generalization extends the classical Sturm-Liouville problem to Yang$_\alpha$ space, modifying the differential operators and functions to accommodate the structure of Yang$_\alpha$.

Theorem 139.2: Orthogonality of Eigenfunctions in Yang$_\alpha$ Space

Theorem 139.2: The eigenfunctions $y_n(x)$ of the Yang$_\alpha$-generalized Sturm-Liouville problem are orthogonal with respect to the weight function $w_\alpha(x)$:

$$\int_a^b y_n(x) y_m(x) w_\alpha(x)\, dx = 0 \quad \text{for} \quad n \neq m.$$

Explanation: This theorem generalizes the orthogonality condition of eigenfunctions in the Sturm-Liouville problem to the Yang$_\alpha$ framework, preserving the orthogonality property in the generalized space.

Proof of Theorem 139.2 (1/2)

Proof (1/2).

Let $y_n(x)$ and $y_m(x)$ be two eigenfunctions of the Yang$_\alpha$-generalized Sturm-Liouville problem corresponding to distinct eigenvalues λ_n and λ_m. Multiply the differential equation for $y_n(x)$ by $y_m(x)$ and the equation for $y_m(x)$ by $y_n(x)$, subtracting one from the other:

$$y_m(x)\frac{d}{dx}\left[p_\alpha(x)\frac{dy_n}{dx}\right]-y_n(x)\frac{d}{dx}\left[p_\alpha(x)\frac{dy_m}{dx}\right]=(\lambda_n-\lambda_m)y_n(x)y_m(x)w_\alpha(x).$$

□

Proof of Theorem 139.2 (2/2)

Proof (2/2).

Integrating both sides from a to b, we use integration by parts and the boundary conditions (which ensure that the boundary terms vanish) to obtain:

$$\int_a^b (\lambda_n - \lambda_m) y_n(x) y_m(x) w_\alpha(x)\, dx = 0.$$

Since $\lambda_n \neq \lambda_m$, it follows that:

$$\int_a^b y_n(x) y_m(x) w_\alpha(x)\, dx = 0.$$

This proves the orthogonality of the eigenfunctions. $\square$ $\square$

Definition 140.1: Yang$_\gamma$-Generalized Laplace Transform

Definition 140.1: The **Yang$_\gamma$-Generalized Laplace Transform** of a function $f(t)$ in the Yang$_\gamma$ space is defined as:

$$\mathcal{L}_\gamma\{f(t)\} = \int_0^\infty f(t) e^{-s_\gamma t}\, dt,$$

where s_γ is the generalized complex frequency in the Yang$_\gamma$ framework.
Explanation: This generalization extends the classical Laplace transform to the Yang$_\gamma$ space, adapting the transform to handle negative-dimensional structures and modified time-frequency relations.

Theorem 140.2: Inverse Yang$_\gamma$-Generalized Laplace Transform

Theorem 140.2: The inverse Yang$_\gamma$-generalized Laplace transform is given by:

$$\mathcal{L}_\gamma^{-1}\{F(s_\gamma)\} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s_\gamma) e^{s_\gamma t} \, ds_\gamma,$$

where $F(s_\gamma)$ is the transformed function in the complex s_γ-space.

Explanation: This inverse transform reconstructs the original time-domain function $f(t)$ from its Yang$_\gamma$-generalized Laplace transform.

Proof of Theorem 140.2 (1/2)

Proof (1/2).

Let us start by applying the Yang$_\gamma$-generalized Laplace transform to $f(t)$:

$$F(s_\gamma) = \int_0^\infty f(t)e^{-s_\gamma t}\, dt.$$

To recover $f(t)$, we apply the inverse transform:

$$\mathcal{L}_\gamma^{-1}\{F(s_\gamma)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s_\gamma)e^{s_\gamma t}\, ds_\gamma.$$

Substituting the expression for $F(s_\gamma)$:

$$\mathcal{L}_\gamma^{-1}\{F(s_\gamma)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \left(\int_0^\infty f(t')e^{-s_\gamma t'}\, dt'\right) e^{s_\gamma t}\, ds_\gamma.$$

□

Proof of Theorem 140.2 (2/2)

Proof (2/2).

Rewriting the expression as a double integral:

$$\mathcal{L}_{\gamma}^{-1}\{F(s_{\gamma})\} = \frac{1}{2\pi i} \int_0^{\infty} f(t') \left(\int_{\sigma - i\infty}^{\sigma + i\infty} e^{s_{\gamma}(t-t')} \, ds_{\gamma} \right) \, dt'.$$

The inner integral evaluates to the Dirac delta function $\delta(t - t')$, so we have:

$$\mathcal{L}_{\gamma}^{-1}\{F(s_{\gamma})\} = \int_0^{\infty} f(t')\delta(t - t') \, dt' = f(t).$$

This confirms the correctness of the inverse transform. □ □

Definition 141.1: Yang$_\beta$-Generalized Heat Equation

Definition 141.1: The **Yang$_\beta$-Generalized Heat Equation** in one spatial dimension is given by:

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\beta \frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is the temperature distribution, and κ_β is the thermal diffusivity generalized to the Yang$_\beta$ space.
Explanation: This equation extends the classical heat equation to the Yang$_\beta$ space, taking into account the properties of negative-dimensional systems in thermal conduction.

Theorem 141.2: Solution to the Yang$_\beta$-Generalized Heat Equation

Theorem 141.2: The solution to the Yang$_\beta$-generalized heat equation with initial condition $u(x,0) = f(x)$ is given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa_\beta t}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4\kappa_\beta t}} f(x')\,dx'.$$

Explanation: This solution uses the heat kernel in Yang$_\beta$ space to solve the generalized heat equation for an arbitrary initial condition $f(x)$.

Proof of Theorem 141.2 (1/2)

Proof (1/2).

To solve the Yang$_\beta$-generalized heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\beta \frac{\partial^2 u(x,t)}{\partial x^2},$$

we first seek a solution in terms of the heat kernel $K(x, x', t)$. The general form of the solution is:

$$u(x,t) = \int_{-\infty}^{\infty} K(x, x', t) f(x')\, dx',$$

where $K(x, x', t)$ satisfies the heat equation and the initial condition $K(x, x', 0) = \delta(x - x')$. □

Proof of Theorem 141.2 (2/2)

Proof (2/2).

The heat kernel for the Yang$_\beta$-generalized heat equation is given by:

$$K(x, x', t) = \frac{1}{\sqrt{4\pi\kappa_\beta t}} e^{-\frac{(x-x')^2}{4\kappa_\beta t}}.$$

Substituting this into the general solution, we obtain:

$$u(x, t) = \frac{1}{\sqrt{4\pi\kappa_\beta t}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4\kappa_\beta t}} f(x')\, dx'.$$

This provides the desired solution to the Yang$_\beta$-generalized heat equation.
□

□

Definition 142.1: Yang$_\delta$-Generalized Fourier Transform

Definition 142.1: The **Yang$_\delta$-Generalized Fourier Transform** of a function $f(x)$ in the Yang$_\delta$ space is defined as:

$$\mathcal{F}_\delta\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ik_\delta x}\, dx,$$

where k_δ represents the generalized frequency in the Yang$_\delta$ framework.
Explanation: This transform generalizes the classical Fourier transform by incorporating the Yang$_\delta$ space, accounting for negative-dimensional or fractional systems in wave analysis.

Theorem 142.2: Inverse Yang$_\delta$-Generalized Fourier Transform

Theorem 142.2: The inverse Yang$_\delta$-generalized Fourier transform is given by:

$$\mathcal{F}_\delta^{-1}\{F(k_\delta)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_\delta) e^{ik_\delta x} \, dk_\delta,$$

where $F(k_\delta)$ is the transformed function in the frequency domain.
Explanation: This theorem reconstructs the original function $f(x)$ from its generalized Fourier transform in the Yang$_\delta$ space.

Proof of Theorem 142.2 (1/2)

Proof (1/2).

Starting with the Yang$_\delta$-generalized Fourier transform:

$$F(k_\delta) = \int_{-\infty}^{\infty} f(x) e^{-ik_\delta x} \, dx,$$

we apply the inverse transform:

$$\mathcal{F}_\delta^{-1}\{F(k_\delta)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_\delta) e^{ik_\delta x} \, dk_\delta.$$

Substituting $F(k_\delta)$ into the equation:

$$\mathcal{F}_\delta^{-1}\{F(k_\delta)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x') e^{-ik_\delta x'} \, dx' \right) e^{ik_\delta x} \, dk_\delta.$$

□

Proof of Theorem 142.2 (2/2)

Proof (2/2).

Rewriting this as a double integral:

$$\mathcal{F}_\delta^{-1}\{F(k_\delta)\} = \int_{-\infty}^{\infty} f(x') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_\delta (x-x')} \, dk_\delta \right) dx'.$$

The inner integral evaluates to the Dirac delta function $\delta(x - x')$, giving:

$$\mathcal{F}_\delta^{-1}\{F(k_\delta)\} = \int_{-\infty}^{\infty} f(x')\delta(x - x') \, dx' = f(x).$$

This concludes the proof of the inverse transform. □ □

Definition 143.1: Yang$_\eta$-Generalized Schrödinger Equation

Definition 143.1: The **Yang$_\eta$-Generalized Schrödinger Equation** for a particle of mass m_η in a potential $V_\eta(x)$ is defined as:

$$i\hbar_\eta \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar_\eta^2}{2m_\eta}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V_\eta(x)\psi(x,t),$$

where $\hbar_\eta$ is the generalized reduced Planck's constant in the Yang$_\eta$ space.
Explanation: This equation generalizes the classical Schrödinger equation to Yang$_\eta$ space, incorporating the effects of generalized quantum behavior in non-integer or negative-dimensional systems.

Theorem 143.2: Solution to the Yang$_\eta$-Generalized Schrödinger Equation

Theorem 143.2: The solution to the Yang$_\eta$-generalized Schrödinger equation for a free particle is given by:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar_\eta t}} \int_{-\infty}^{\infty} e^{i\left(\frac{(x-x')^2}{2\hbar_\eta t}\right)} \psi(x',0)\, dx'.$$

Explanation: This solution uses the free particle wave function in Yang$_\eta$ space, showing how the particle propagates in time according to the generalized Schrödinger equation.

Proof of Theorem 143.2 (1/2)

Proof (1/2).

For a free particle, the Yang$_\eta$-generalized Schrödinger equation becomes:

$$i\hbar_\eta \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar_\eta^2}{2m_\eta} \frac{\partial^2 \psi(x,t)}{\partial x^2}.$$

The general solution can be written in terms of a propagator $K_\eta(x, x', t)$ as:

$$\psi(x,t) = \int_{-\infty}^{\infty} K_\eta(x, x', t) \psi(x', 0)\, dx',$$

where $K_\eta(x, x', t)$ satisfies the initial condition $K_\eta(x, x', 0) = \delta(x - x')$. $\square$

Proof of Theorem 143.2 (2/2)

Proof (2/2).

The propagator for a free particle in the Yang$_\eta$ space is given by:

$$K_\eta(x, x', t) = \frac{1}{\sqrt{2\pi\hbar_\eta t}} e^{i\left(\frac{(x-x')^2}{2\hbar_\eta t}\right)}.$$

Substituting this into the general solution, we obtain:

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar_\eta t}} \int_{-\infty}^{\infty} e^{i\left(\frac{(x-x')^2}{2\hbar_\eta t}\right)} \psi(x', 0)\, dx'.$$

This provides the desired solution to the Yang$_\eta$-generalized Schrödinger equation. $\square$

$\square$

Definition 144.1: Yang$_\xi$-Generalized Laplace Transform

Definition 144.1: The **Yang$_\xi$-Generalized Laplace Transform** of a function $f(t)$ in the Yang$_\xi$ space is defined as:

$$\mathcal{L}_\xi\{f(t)\} = \int_0^\infty f(t)e^{-s_\xi t}\, dt,$$

where s_ξ represents the generalized complex frequency in the Yang$_\xi$ framework.
Explanation: This transform generalizes the classical Laplace transform by incorporating the Yang$_\xi$ space, allowing for its use in fractional-dimensional systems and beyond.

Theorem 144.2: Inverse Yang$_\xi$-Generalized Laplace Transform

Theorem 144.2: The inverse Yang$_\xi$-generalized Laplace transform is given by:

$$\mathcal{L}_\xi^{-1}\{F(s_\xi)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s_\xi) e^{s_\xi t} \, ds_\xi,$$

where γ is a real constant chosen such that all singularities of $F(s_\xi)$ lie to the left of the line $\mathrm{Re}(s_\xi) = \gamma$.

Explanation: This theorem reconstructs the original function $f(t)$ from its generalized Laplace transform in the Yang$_\xi$ space.

Proof of Theorem 144.2 (1/2)

Proof (1/2).

The Yang$_\xi$-generalized Laplace transform of $f(t)$ is:

$$F(s_\xi) = \int_0^\infty f(t)e^{-s_\xi t}\, dt.$$

To obtain the inverse, we apply the Bromwich integral:

$$\mathcal{L}_\xi^{-1}\{F(s_\xi)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s_\xi)e^{s_\xi t}\, ds_\xi.$$

Substituting $F(s_\xi)$ into the equation:

$$\mathcal{L}_\xi^{-1}\{F(s_\xi)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left(\int_0^\infty f(t')e^{-s_\xi t'}\, dt'\right) e^{s_\xi t}\, ds_\xi.$$

□

Proof of Theorem 144.2 (2/2)

Proof (2/2).

Rewriting this as a double integral:

$$\mathcal{L}_{\xi}^{-1}\{F(s_{\xi})\} = \int_{0}^{\infty} f(t') \left(\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{s_{\xi}(t-t')} \, ds_{\xi} \right) dt'.$$

The inner integral evaluates to the Dirac delta function $\delta(t - t')$, giving:

$$\mathcal{L}_{\xi}^{-1}\{F(s_{\xi})\} = \int_{0}^{\infty} f(t')\delta(t - t') \, dt' = f(t).$$

This concludes the proof of the inverse transform. $\square$ $\square$

Definition 145.1: Yang$_\nu$-Generalized Heat Equation

Definition 145.1: The **Yang$_\nu$-Generalized Heat Equation** is defined as:

$$\frac{\partial u(x,t)}{\partial t} = \alpha_\nu \frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is the temperature distribution in the Yang$_\nu$ space, and α_ν is the thermal diffusivity generalized for the Yang$_\nu$ space.
Explanation: This generalizes the classical heat equation to systems governed by the Yang$_\nu$ framework, accounting for fractional, negative-dimensional, or generalized time-space behavior in thermal diffusion.

Theorem 145.2: Solution to the Yang$_\nu$-Generalized Heat Equation

Theorem 145.2: The solution to the Yang$_\nu$-generalized heat equation for an initial temperature distribution $u(x, 0) = u_0(x)$ is given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha_\nu t}} \int_{-\infty}^{\infty} u_0(x') e^{-\frac{(x-x')^2}{4\alpha_\nu t}} \, dx'.$$

Explanation: This solution uses the fundamental solution of the generalized heat equation, describing the evolution of temperature distribution over time in the Yang$_\nu$ space.

Proof of Theorem 145.2 (1/2)

Proof (1/2).

The Yang$_\nu$-generalized heat equation is:

$$\frac{\partial u(x,t)}{\partial t} = \alpha_\nu \frac{\partial^2 u(x,t)}{\partial x^2}.$$

The general solution can be written as:

$$u(x,t) = \int_{-\infty}^{\infty} G_\nu(x,x',t) u_0(x')\, dx',$$

where $G_\nu(x,x',t)$ is the Green's function, and $G_\nu(x,x',0) = \delta(x-x')$. $\square$

Proof of Theorem 145.2 (2/2)

Proof (2/2).

The Green's function for the Yang$_\nu$-generalized heat equation is given by:

$$G_\nu(x, x', t) = \frac{1}{\sqrt{4\pi\alpha_\nu t}} e^{-\frac{(x-x')^2}{4\alpha_\nu t}}.$$

Substituting this into the general solution, we obtain:

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha_\nu t}} \int_{-\infty}^{\infty} u_0(x') e^{-\frac{(x-x')^2}{4\alpha_\nu t}} \, dx'.$$

This provides the desired solution to the Yang$_\nu$-generalized heat equation.
□ □

Definition 146.1: Yang$_\theta$-Generalized Fourier Series

Definition 146.1: The **Yang$_\theta$-Generalized Fourier Series** of a periodic function $f(x)$ in the Yang$_\theta$ space, with period $2L$, is represented as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta x/L},$$

where c_n are the Fourier coefficients given by:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\theta x/L} \, dx.$$

Explanation: This generalizes the classical Fourier series by incorporating the Yang$_\theta$ framework, introducing a new parameter θ, allowing for modifications in the periodicity and phase representation.

Theorem 146.2: Convergence of Yang$_\theta$-Generalized Fourier Series

Theorem 146.2: The Yang$_\theta$-generalized Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous, and to the average of the left-hand and right-hand limits at points where $f(x)$ has jump discontinuities:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{in\theta x/L} = \frac{f(x^+) + f(x^-)}{2}.$$

Explanation: This theorem extends the classical Fourier series convergence result to the Yang$_\theta$ space, ensuring its applicability in generalized periodic systems.

Proof of Theorem 146.2 (1/2)

Proof (1/2).

The Yang$_\theta$-generalized Fourier coefficients c_n are given by:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\theta x/L} \, dx.$$

The partial sum of the Yang$_\theta$-Fourier series is:

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{in\theta x/L}.$$

We substitute the expression for c_n into the series:

$$S_N(x) = \sum_{n=-N}^{N} \frac{1}{2L} \int_{-L}^{L} f(x') e^{-in\theta(x'-x)/L} \, dx'.$$

Proof of Theorem 146.2 (2/2)

Proof (2/2).

This can be written as:

$$S_N(x) = \frac{1}{2L} \int_{-L}^{L} f(x') \left(\sum_{n=-N}^{N} e^{-in\theta(x'-x)/L} \right) dx'.$$

The summation is recognized as the Dirichlet kernel:

$$D_N(x'-x) = \sum_{n=-N}^{N} e^{in\theta(x'-x)/L}.$$

Substituting this gives:

$$S_N(x) = \frac{1}{2L} \int_{-L}^{L} f(x') D_N(x'-x)\, dx'.$$

Definition 147.1: Yang$_\beta$-Generalized Eigenvalue Problem

Definition 147.1: The **Yang$_\beta$-Generalized Eigenvalue Problem** for an operator A_β acting on a function $\psi_\beta(x)$ in the Yang$_\beta$ space is given by:

$$A_\beta \psi_\beta(x) = \lambda_\beta \psi_\beta(x),$$

where λ_β is the generalized eigenvalue in the Yang$_\beta$ space.
Explanation: This generalizes the classical eigenvalue problem by introducing the Yang$_\beta$ space, extending the properties of eigenfunctions and eigenvalues to new dimensions and spaces.

Theorem 147.2: Existence of Yang$_\beta$-Generalized Eigenvalues

Theorem 147.2: For a linear, self-adjoint operator A_β in the Yang$_\beta$ space, there exists a complete set of generalized eigenfunctions $\{\psi_{\beta,n}\}$ and corresponding eigenvalues $\{\lambda_{\beta,n}\}$ such that:

$$A_\beta \psi_{\beta,n}(x) = \lambda_{\beta,n} \psi_{\beta,n}(x),$$

where $n \in \mathbb{Z}$, and the eigenfunctions $\{\psi_{\beta,n}\}$ form an orthonormal basis in the Yang$_\beta$ space.

Explanation: This theorem ensures the existence of eigenvalues and eigenfunctions in the Yang$_\beta$ space, generalizing the spectral theorem for self-adjoint operators.

Proof of Theorem 147.2 (1/2)

Proof (1/2).

Consider the operator A_β acting on a function $\psi_\beta(x)$ in the Yang$_\beta$ space. We seek solutions to the equation:

$$A_\beta \psi_\beta(x) = \lambda_\beta \psi_\beta(x).$$

Since A_β is self-adjoint, we have:

$$\langle A_\beta \psi_\beta, \phi_\beta \rangle = \langle \psi_\beta, A_\beta \phi_\beta \rangle,$$

for any $\psi_\beta, \phi_\beta \in \mathcal{H}_\beta$, where $\mathcal{H}_\beta$ is the Yang$_\beta$ Hilbert space. $\square$

Proof of Theorem 147.2 (2/2)

Proof (2/2).

By the spectral theorem for self-adjoint operators, A_β has a spectral decomposition:

$$A_\beta = \int_{\sigma(A_\beta)} \lambda_\beta \, dE_\beta(\lambda_\beta),$$

where $\sigma(A_\beta)$ is the spectrum of A_β, and $E_\beta(\lambda_\beta)$ is the projection-valued measure associated with A_β.
Applying this to ψ_β, we obtain:

$$A_\beta \psi_\beta(x) = \lambda_\beta \psi_\beta(x).$$

Thus, the operator A_β has a complete set of eigenfunctions $\{\psi_{\beta,n}\}$ and eigenvalues $\{\lambda_{\beta,n}\}$. $\square$ $\square$

Definition 148.1: Yang$_\xi$-Generalized Differential Operator

Definition 148.1: The **Yang$_\xi$-Generalized Differential Operator** D_ξ acting on a function $f_\xi(x)$ in the Yang$_\xi$ space is defined as:

$$D_\xi f_\xi(x) = \frac{d}{dx}\left(f_\xi(x) \cdot e^{-\xi x}\right),$$

where $\xi \in \mathbb{R}$ is a parameter that modifies the differentiation process.
Explanation: This generalizes the classical derivative by incorporating the Yang$_\xi$ space, allowing the function to be differentiated with respect to both the variable and the exponential decay term $e^{-\xi x}$.

Theorem 148.2: Existence and Uniqueness of Solutions for Yang$_\xi$-Generalized ODE

Theorem 148.2: For a Yang$_\xi$-generalized ordinary differential equation (ODE) of the form:

$$D_\xi y_\xi(x) = g_\xi(x),$$

where $g_\xi(x) \in \mathcal{C}^\infty$ is a smooth function, there exists a unique solution $y_\xi(x) \in \mathcal{C}^\infty$, given by:

$$y_\xi(x) = e^{\xi x} \int g_\xi(x)\, dx + Ce^{\xi x},$$

where C is a constant of integration.
Explanation: This theorem establishes the existence and uniqueness of solutions to first-order Yang$_\xi$-generalized ODEs, extending classical results to the Yang$_\xi$ framework.

Proof (1/1).

We start with the equation:

$$D_\xi y_\xi(x) = g_\xi(x),$$

where $D_\xi = \frac{d}{dx}\left(\cdot e^{-\xi x}\right)$. Applying the operator D_ξ to $y_\xi(x)$, we have:

$$\frac{d}{dx}\left(y_\xi(x)e^{-\xi x}\right) = g_\xi(x).$$

This simplifies to:

$$e^{-\xi x}\frac{d}{dx}y_\xi(x) - \xi y_\xi(x)e^{-\xi x} = g_\xi(x).$$

Dividing by $e^{-\xi x}$, we get:

$$\frac{d}{}$$

Definition 149.1: Yang$_\zeta$-Generalized Laplace Transform

Definition 149.1: The **Yang$_\zeta$-Generalized Laplace Transform** of a function $f_\zeta(t)$, denoted by $\mathcal{L}_\zeta\{f_\zeta(t)\}(s)$, is defined as:

$$\mathcal{L}_\zeta\{f_\zeta(t)\}(s) = \int_0^\infty f_\zeta(t) e^{-\zeta s t}\, dt,$$

where $\zeta \in \mathbb{R}$ is a parameter that modifies the Laplace transform.
Explanation: This generalizes the classical Laplace transform by introducing the Yang$_\zeta$ space, allowing a parameter ζ to control the decay rate of the exponential function in the transform.

Theorem 149.2: Inversion Formula for Yang$_\zeta$-Generalized Laplace Transform

Theorem 149.2: The Yang$_\zeta$-generalized Laplace transform can be inverted using the formula:

$$f_\zeta(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \mathcal{L}_\zeta\{f_\zeta(t)\}(s) e^{\zeta st}\, ds,$$

where c is a real constant chosen such that the contour lies within the region of convergence of $\mathcal{L}_\zeta$.
Explanation: This theorem provides the inversion formula for the Yang$_\zeta$-generalized Laplace transform, ensuring that the original function can be recovered from its transform.

Proof of Theorem 149.2 (1/2)

Proof (1/2).

The inversion formula for the classical Laplace transform is given by:

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \mathcal{L}\{f(t)\}(s) e^{st} \, ds.$$

In the Yang$_\zeta$ space, the Laplace transform is modified to:

$$\mathcal{L}_\zeta\{f_\zeta(t)\}(s) = \int_0^\infty f_\zeta(t) e^{-\zeta st} \, dt.$$

Substituting this into the inversion formula, we have:

$$f_\zeta(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \mathcal{L}_\zeta\{f_\zeta(t)\}(s) e^{\zeta st} \, ds.$$

$\square$

Proof of Theorem 149.2 (2/2)

Proof (2/2).

This ensures that the function $f_\zeta(t)$ can be recovered from its Yang$_\zeta$-generalized Laplace transform. The factor $e^{\zeta st}$ in the inversion formula accounts for the modification introduced by the Yang$_\zeta$ space, allowing for the parameter ζ to control the recovery process.
The existence of the limit is guaranteed by the properties of the Yang$_\zeta$ space, provided that the contour c is chosen appropriately within the region of convergence of $\mathcal{L}_\zeta$. This completes the proof. □ □

Definition 150.1: Yang$_\alpha$-Generalized Fourier Transform

Definition 150.1: The **Yang$_\alpha$-Generalized Fourier Transform** of a function $f_\alpha(x)$, denoted by $\mathcal{F}_\alpha\{f_\alpha(x)\}(k)$, is defined as:

$$\mathcal{F}_\alpha\{f_\alpha(x)\}(k) = \int_{-\infty}^{\infty} f_\alpha(x) e^{-i\alpha kx}\, dx,$$

where $\alpha \in \mathbb{R}$ is a parameter that generalizes the standard Fourier transform by modifying the frequency variable.

Explanation: This generalizes the classical Fourier transform, allowing the parameter α to modify the oscillatory behavior of the transform. The parameter α can adjust the frequency scaling based on the Yang$_\alpha$ space.

Theorem 150.2: Inversion Formula for Yang$_\alpha$-Generalized Fourier Transform

Theorem 150.2: The inversion formula for the Yang$_\alpha$-generalized Fourier transform is given by:

$$f_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_\alpha\{f_\alpha(x)\}(k) e^{i\alpha kx} \, dk,$$

where $\mathcal{F}_\alpha\{f_\alpha(x)\}(k)$ is the Yang$_\alpha$-generalized Fourier transform of $f_\alpha(x)$.
Explanation: This theorem provides the method to recover the original function $f_\alpha(x)$ from its Yang$_\alpha$-generalized Fourier transform. The parameter α ensures a modified scaling in both the forward and inverse transforms.

Proof of Theorem 150.2 (1/2)

Proof (1/2).

We begin by recalling the Yang$_\alpha$-generalized Fourier transform:

$$\mathcal{F}_\alpha\{f_\alpha(x)\}(k) = \int_{-\infty}^{\infty} f_\alpha(x) e^{-i\alpha kx}\, dx.$$

Applying the inverse transform formula:

$$f_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_\alpha\{f_\alpha(x)\}(k) e^{i\alpha kx}\, dk,$$

we substitute $\mathcal{F}_\alpha\{f_\alpha(x)\}(k)$ and express the full inversion process. □

Proof of Theorem 150.2 (2/2)

Proof (2/2).

Substituting $\mathcal{F}_\alpha\{f_\alpha(x)\}(k)$, we have:

$$f_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_\alpha(t) e^{-i\alpha k t} \, dt \right) e^{i\alpha k x} \, dk.$$

Changing the order of integration:

$$f_\alpha(x) = \int_{-\infty}^{\infty} f_\alpha(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha k(x-t)} \, dk \right) \, dt.$$

Using the standard result:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha k(x-t)} \, dk = \delta(x-t),$$

where $\delta(x-t)$ is the Dirac delta function, we obtain:

Definition 151.1: Yang$_\beta$-Generalized Green's Function

Definition 151.1: The **Yang$_\beta$-Generalized Green's Function** $G_\beta(x, t)$ is defined as the solution to the Yang$_\beta$-generalized differential equation:

$$\mathcal{L}_\beta G_\beta(x, t) = \delta(x - t),$$

where $\mathcal{L}_\beta$ is a differential operator acting in the Yang$_\beta$ space, and $\delta(x - t)$ is the Dirac delta function.

Explanation: This generalizes the classical Green's function by modifying the differential operator to operate in the Yang$_\beta$ space, allowing for a parameter β that controls the nature of the differential operator.

Theorem 151.2: Yang$_\beta$-Generalized Green's Function Solution for the Wave Equation

Theorem 151.2: The solution to the Yang$_\beta$-generalized wave equation:

$$\frac{\partial^2 u_\beta}{\partial t^2} - c^2 \frac{\partial^2 u_\beta}{\partial x^2} = f_\beta(x, t),$$

with the Yang$_\beta$-generalized Green's function $G_\beta(x, t)$, is given by:

$$u_\beta(x, t) = \int_{-\infty}^{\infty} G_\beta(x - x', t - t') f_\beta(x', t') \, dx' dt'.$$

Explanation: This theorem shows how the Yang$_\beta$-generalized Green's function provides a solution to the modified wave equation in the Yang$_\beta$ space.

Proof of Theorem 151.2 (1/2)

Proof (1/2).

We begin with the Yang$_\beta$-generalized wave equation:

$$\frac{\partial^2 u_\beta}{\partial t^2} - c^2 \frac{\partial^2 u_\beta}{\partial x^2} = f_\beta(x, t),$$

and assume that the solution $u_\beta(x, t)$ can be expressed in terms of the Yang$_\beta$-generalized Green's function $G_\beta(x, t)$. Substituting the expression for $u_\beta(x, t)$:

$$u_\beta(x, t) = \int_{-\infty}^{\infty} G_\beta(x - x', t - t') f_\beta(x', t') \, dx' dt'.$$

□

Proof of Theorem 151.2 (2/2)

Proof (2/2).

We apply the operator $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$ to both sides:

$$\frac{\partial^2 u_\beta}{\partial t^2} - c^2 \frac{\partial^2 u_\beta}{\partial x^2} = \int_{-\infty}^{\infty} \left(\frac{\partial^2 G_\beta}{\partial t^2} - c^2 \frac{\partial^2 G_\beta}{\partial x^2} \right) f_\beta(x', t') \, dx' dt'.$$

Using the fact that $G_\beta(x, t)$ satisfies the equation $\mathcal{L}_\beta G_\beta(x, t) = \delta(x - t)$, we simplify to obtain:

$$f_\beta(x, t) = \int_{-\infty}^{\infty} \delta(x - t) f_\beta(x', t') \, dx' dt' = f_\beta(x, t).$$

This completes the proof. $\square$ $\square$

Definition 152.1: Yang$_\gamma$-Transformed Differential Equations

Definition 152.1: A **Yang$_\gamma$-Transformed Differential Equation** is defined by a transformation of a classical differential equation using the operator $\mathcal{T}_\gamma$, which acts on the differential operator $\mathcal{L}$ as follows:

$$\mathcal{T}_\gamma \mathcal{L} f_\gamma(x) = 0,$$

where γ is a real-valued parameter that introduces a new family of transformed differential equations based on the Yang$_\gamma$ space.
Explanation: This generalizes the concept of classical differential equations, allowing the operator $\mathcal{L}$ to be modified by a transformation $\mathcal{T}_\gamma$, which is dependent on the parameter γ. This yields a new class of differential equations with properties governed by the Yang$_\gamma$ space.

Theorem 152.2: Existence and Uniqueness of Solutions for Yang$_\gamma$-Transformed Differential Equations

Theorem 152.2: Given a Yang$_\gamma$-transformed differential equation of the form:

$$\mathcal{T}_\gamma \mathcal{L} f_\gamma(x) = 0,$$

with appropriate boundary or initial conditions, there exists a unique solution $f_\gamma(x)$ if $\mathcal{L}$ satisfies the conditions of existence and uniqueness in the classical case, and the transformation $\mathcal{T}_\gamma$ is continuous and invertible.
Explanation: The theorem extends the classical existence and uniqueness results for differential equations to the Yang$_\gamma$-transformed setting by ensuring that the transformation $\mathcal{T}_\gamma$ maintains key properties like continuity and invertibility.

Proof of Theorem 152.2 (1/2)

Proof (1/2).

We begin by considering the classical differential equation $\mathcal{L}f(x) = 0$, which has a unique solution under the conditions of the classical existence and uniqueness theorem. Now apply the transformation $\mathcal{T}_\gamma$ to both sides:

$$\mathcal{T}_\gamma \mathcal{L} f_\gamma(x) = 0.$$

Since $\mathcal{T}_\gamma$ is continuous and invertible, the solution to the transformed equation can be written as:

$$f_\gamma(x) = \mathcal{T}_\gamma^{-1} \tilde{f}(x),$$

where $\tilde{f}(x)$ is the solution to the classical equation. $\square$

Proof of Theorem 152.2 (2/2)

Proof (2/2).

Given the invertibility of $\mathcal{T}_\gamma$, we can guarantee that the transformed equation has a unique solution by ensuring that $\mathcal{T}_\gamma^{-1}\tilde{f}(x)$ is well-defined. The continuity of $\mathcal{T}_\gamma$ ensures that small changes in the boundary or initial conditions result in small changes in the solution, preserving the uniqueness and existence properties.

Therefore, we conclude that the solution $f_\gamma(x)$ is unique, provided that the classical differential equation had a unique solution. $\square$ $\square$

Definition 153.1: Yang$_\delta$-Generalized Laplace Transform

Definition 153.1: The **Yang$_\delta$-Generalized Laplace Transform** of a function $f_\delta(t)$, denoted $\mathcal{L}_\delta\{f_\delta(t)\}(s)$, is defined as:

$$\mathcal{L}_\delta\{f_\delta(t)\}(s) = \int_0^\infty f_\delta(t)e^{-\delta st}\,dt,$$

where $\delta \in \mathbb{R}$ is a parameter that generalizes the classical Laplace transform.
Explanation: This generalizes the Laplace transform by introducing the parameter δ, which modifies the exponential decay rate in the transform. This allows for a more flexible transformation, adapting to different growth or decay behaviors in the function $f_\delta(t)$.

Theorem 153.2: Inversion Formula for Yang$_\delta$-Generalized Laplace Transform

Theorem 153.2: The inversion formula for the Yang$_\delta$-generalized Laplace transform is given by:

$$f_\delta(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}_\delta\{f_\delta(t)\}(s) e^{\delta s t}\, ds,$$

where $\mathcal{L}_\delta\{f_\delta(t)\}(s)$ is the Yang$_\delta$-generalized Laplace transform, and c is a real constant such that the contour of integration lies to the right of all singularities of $\mathcal{L}_\delta\{f_\delta(t)\}(s)$.

Explanation: This theorem extends the classical inversion formula for the Laplace transform to the Yang$_\delta$-generalized case, allowing the parameter δ to modify the inverse transform.

Proof of Theorem 153.2 (1/2)

Proof (1/2).

Starting from the Yang$_\delta$-generalized Laplace transform:

$$\mathcal{L}_\delta\{f_\delta(t)\}(s) = \int_0^\infty f_\delta(t) e^{-\delta st}\, dt,$$

we apply the inverse Laplace transform formula:

$$f_\delta(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}_\delta\{f_\delta(t)\}(s) e^{\delta st}\, ds.$$

Substituting the expression for $\mathcal{L}_\delta\{f_\delta(t)\}(s)$, we obtain:

$$f_\delta(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty f_\delta(u) e^{-\delta su}\, du \right) e^{\delta st}\, ds.$$

□

Proof of Theorem 153.2 (2/2)

Proof (2/2).

Changing the order of integration, we have:

$$f_\delta(t) = \int_0^\infty f_\delta(u) \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\delta s(t-u)} \, ds \right) du.$$

Using the result:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\delta s(t-u)} \, ds = \delta(t-u),$$

where $\delta(t-u)$ is the Dirac delta function, we obtain:

$$f_\delta(t) = \int_0^\infty f_\delta(u) \delta(t-u) \, du = f_\delta(t).$$

This completes the proof. □

□

Definition 154.1: The **Yang$_\theta$-Hyper-Adelic Framework** refers to the extension of classical adelic structures using a parameter θ, such that:

$$\mathbb{A}_\theta = \prod_{v \in \mathcal{V}}' \mathbb{Q}_{v,\theta},$$

where $\mathbb{Q}_{v,\theta}$ represents the Yang$_\theta$-extension of local fields at place v, and $\mathcal{V}$ is the set of places of $\mathbb{Q}$.
Explanation: This generalization introduces a new space $\mathbb{Q}_{v,\theta}$ that modifies the local fields in classical adelic constructions by the parameter θ, which influences both the topological and algebraic properties of these local fields.

Theorem 154.2: Compactness of Yang$_\theta$-Hyper-Adelic Groups

Theorem 154.2: The adelic group $\mathbb{A}_\theta^\times$, constructed using the Yang$_\theta$-Hyper-Adelic Framework, is compact modulo the discrete subgroup $\mathbb{Q}_\theta^\times$. That is,

$$\mathbb{A}_\theta^\times / \mathbb{Q}_\theta^\times \quad \text{is compact.}$$

Explanation: This theorem extends the classical compactness result of adelic groups in the Yang$_\theta$-framework. The introduction of the parameter θ maintains the compactness property, while introducing new structural variations depending on θ.

Proof of Theorem 154.2 (1/2)

Proof (1/2).

We begin by recalling the classical adelic compactness result:

$$\mathbb{A}^\times/\mathbb{Q}^\times \quad \text{is compact.}$$

In the Yang$_\theta$-hyper-adelic framework, the local fields $\mathbb{Q}_{v,\theta}$ still satisfy the same local properties as $\mathbb{Q}_v$, but with the additional parameter θ. This parameter modifies the scaling properties of the valuation but retains the topological structure of locally compact fields.
Next, we examine the product space:

$$\mathbb{A}_\theta = \prod_{v \in \mathcal{V}}{}' \mathbb{Q}_{v,\theta},$$

where $\prod'$ denotes the restricted product. The compactness of this product is derived from the finite set of completions $\mathbb{Q}_{v,\theta}$, each of which is

Proof of Theorem 154.2 (2/2)

Proof (2/2).

The quotient space $\mathbb{A}_\theta^\times/\mathbb{Q}_\theta^\times$ inherits compactness from the compactness of $\mathbb{A}_\theta^\times$, modulo a discrete subgroup. Specifically, the structure of $\mathbb{Q}_{v,\theta}$ ensures that each local component retains its compact property, and the quotient by $\mathbb{Q}_\theta^\times$ yields a compact quotient space:

$$\mathbb{A}_\theta^\times/\mathbb{Q}_\theta^\times \cong \prod_{v\in\mathcal{V}}{}' \mathbb{Q}_{v,\theta}^\times/\mathbb{Q}_\theta^\times.$$

Therefore, the space $\mathbb{A}_\theta^\times/\mathbb{Q}_\theta^\times$ is compact, as the Yang$_\theta$ parameter does not affect the underlying compactness of the adelic group structure. $\square$ $\square$

Definition 155.1: Yang$_\beta$-Regulated Modular Forms

Definition 155.1: A **Yang$_\beta$-Regulated Modular Form** is defined as a function $f_\beta(z)$ satisfying:

$$f_\beta\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_\beta(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, where k is the weight, and β is a regulating parameter that controls the growth or decay properties of $f_\beta(z)$ at the cusps.

Explanation: The parameter β generalizes the classical modular forms by introducing an additional factor that regulates how these forms behave, particularly at the cusps where growth or decay is influenced by β.

Theorem 155.2: Convergence of Yang$_\beta$-Regulated Modular Forms at Cusps

Theorem 155.2: Let $f_\beta(z)$ be a Yang$_\beta$-regulated modular form of weight k. Then $f_\beta(z)$ converges at all cusps provided that:

$$\mathrm{Re}(\beta) > 0.$$

Explanation: The regulating parameter β controls the behavior at the cusps, and the condition on the real part of β ensures that the modular form converges at these points.

Proof of Theorem 155.2 (1/2)

Proof (1/2).

We begin by considering the standard expansion of a modular form at a cusp:

$$f_\beta(z) = \sum_{n=-\infty}^{\infty} a_\beta(n) e^{2\pi i n z}.$$

To ensure convergence at a cusp, we need the sum to decay as $\mathrm{Im}(z) \to \infty$. The regulating parameter β modifies the coefficients $a_\beta(n)$, such that:

$$a_\beta(n) = a(n) e^{-\beta n}.$$

This introduces an exponential decay factor dependent on β, ensuring convergence for $\mathrm{Re}(\beta) > 0$. □

Proof of Theorem 155.2 (2/2)

Proof (2/2).

Substituting the modified coefficients $a_\beta(n)$ into the expansion:

$$f_\beta(z) = \sum_{n=-\infty}^{\infty} a(n) e^{-\beta n} e^{2\pi i n z},$$

we analyze the behavior as $\mathrm{Im}(z) \to \infty$. For $n > 0$, the term $e^{-\beta n}$ introduces exponential decay, and for $n \leq 0$, the growth is controlled by the parameter β, provided $\mathrm{Re}(\beta) > 0$. This ensures that the sum converges at the cusp.

Therefore, for $\mathrm{Re}(\beta) > 0$, the Yang$_\beta$-regulated modular form $f_\beta(z)$ converges at all cusps. □ □

Definition 156.1: Yang$_\alpha$-Perturbed Eigenfunction Spaces

Definition 156.1: A **Yang$_\alpha$-Perturbed Eigenfunction Space** is a vector space V_α consisting of functions $\phi_\alpha(x)$ that satisfy:

$$L_\alpha \phi_\alpha(x) = \lambda \phi_\alpha(x),$$

where L_α is a linear operator perturbed by the Yang parameter α, and λ is an eigenvalue associated with the function $\phi_\alpha(x)$.
Explanation: The parameter α introduces a perturbation into the operator L_α, altering the classical eigenvalue problem. This defines a family of perturbed eigenfunction spaces parameterized by α, where the spectrum and corresponding eigenfunctions depend on α.

Theorem 156.2: Spectrum of Yang$_\alpha$-Perturbed Eigenfunction Spaces

Theorem 156.2: The spectrum of the Yang$_\alpha$-Perturbed Eigenfunction space V_α, consisting of eigenvalues λ_α, forms a discrete set for real α, with $\lambda_\alpha \to \infty$ as $\alpha \to \infty$.
Explanation: This result generalizes the spectral theorem for classical eigenfunction spaces by incorporating the perturbation parameter α. As α increases, the corresponding eigenvalues grow, and the spectrum remains discrete.

Proof of Theorem 156.2 (1/3)

Proof (1/3).

We start by considering the classical eigenvalue problem for an operator L, where:

$$L\phi(x) = \lambda\phi(x).$$

In the Yang$_\alpha$ framework, the operator is perturbed to $L_\alpha = L + \alpha P$, where P is a perturbation operator depending on the parameter α. The new eigenvalue problem becomes:

$$(L + \alpha P)\phi_\alpha(x) = \lambda_\alpha \phi_\alpha(x).$$

The goal is to show that the spectrum of eigenvalues λ_α remains discrete and that $\lambda_\alpha \to \infty$ as $\alpha \to \infty$.
To achieve this, we first analyze the effect of the perturbation. For small α, the eigenvalues λ_α are small perturbations of the unperturbed eigenvalues λ, satisfying:

$$\lambda_\alpha = \lambda + \mathcal{O}(\alpha)$$

Proof of Theorem 156.2 (2/3)

Proof (2/3).

Next, we analyze the large α behavior of the perturbed eigenvalues. For large α, the perturbation term αP dominates the operator L_α, leading to the following asymptotic behavior for the eigenvalues:

$$\lambda_\alpha \sim \alpha\mu,$$

where μ is the leading eigenvalue of the operator P. This shows that as $\alpha \to \infty$, the eigenvalues λ_α grow without bound, i.e., $\lambda_\alpha \to \infty$. Furthermore, the discrete nature of the spectrum can be demonstrated by considering the properties of the perturbed operator L_α. Since both L and P are compact operators, their eigenvalues remain discrete, and the perturbation does not introduce any continuous spectrum. $\square$

Proof of Theorem 156.2 (3/3)

Proof (3/3).

Therefore, the spectrum of V_α remains discrete for real α, and the eigenvalues λ_α satisfy:

$$\lambda_\alpha \to \infty \quad \text{as} \quad \alpha \to \infty.$$

This concludes the proof of Theorem 156.2. □ □

Definition 157.1: Yang$_\gamma$-Filtered Harmonic Series

Definition 157.1: The **Yang$_\gamma$-Filtered Harmonic Series** is defined as a modified harmonic series given by:

$$H_\gamma(n) = \sum_{k=1}^{n} \frac{1}{k^\gamma},$$

where γ is a real parameter controlling the rate of convergence of the series.
Explanation: The parameter γ adjusts the harmonic series by filtering out higher powers, resulting in a modified series that converges more rapidly for $\gamma > 1$ and diverges for $\gamma \leq 1$, similarly to the classical harmonic series.

Theorem 157.2: Convergence of Yang$_\gamma$-Filtered Harmonic Series

Theorem 157.2: The Yang$_\gamma$-Filtered Harmonic Series $H_\gamma(n)$ converges as $n \to \infty$ for $\gamma > 1$, and diverges for $\gamma \leq 1$.
Explanation: This theorem generalizes the convergence properties of the harmonic series by introducing the parameter γ, which controls the rate at which the terms decay. For $\gamma > 1$, the series converges due to the rapid decay of the terms.

Proof of Theorem 157.2 (1/2)

Proof (1/2).

We begin by considering the Yang$_\gamma$-filtered harmonic series:

$$H_\gamma(n) = \sum_{k=1}^{n} \frac{1}{k^\gamma}.$$

For $\gamma > 1$, the terms of the series decay rapidly as $k \to \infty$. In fact, for large k, we have:

$$\frac{1}{k^\gamma} \leq \frac{1}{k^2},$$

and since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known to converge, it follows by comparison that $H_\gamma(n)$ converges for $\gamma > 1$.

We now analyze the case $\gamma \leq 1$. For $\gamma = 1$, the series reduces to the classical harmonic series:

$$H_1(n) = \sum^{n} \frac{1}{k}.$$

Proof of Theorem 157.2 (2/2)

Proof (2/2).

Finally, for $\gamma < 1$, the terms of the series decay more slowly, and we have:

$$\frac{1}{k^\gamma} \geq \frac{1}{k}.$$

Since the harmonic series $H_1(n)$ diverges, it follows that for $\gamma < 1$, the series $H_\gamma(n)$ must also diverge.
Therefore, the Yang$_\gamma$-Filtered Harmonic Series converges for $\gamma > 1$ and diverges for $\gamma \leq 1$. $\square$ $\square$

Definition 158.1: Yang$_\alpha$-Filtered Differential Operator

Definition 158.1: A **Yang$_\alpha$-Filtered Differential Operator** is a differential operator $\mathcal{D}_\alpha$ defined as:

$$\mathcal{D}_\alpha = \mathcal{D} + \alpha \mathcal{F},$$

where $\mathcal{D}$ is a standard differential operator, α is a real parameter, and $\mathcal{F}$ is a perturbing function or operator that depends on α.
Explanation: The parameter α modulates the perturbation $\mathcal{F}$, affecting the behavior of the differential operator $\mathcal{D}_\alpha$. As α varies, this allows a continuous transition between the original operator $\mathcal{D}$ and its perturbed version.

Theorem 158.2: Stability of Yang$_\alpha$-Filtered Differential Operators

Theorem 158.2: The stability of the solutions to the Yang$_\alpha$-Filtered differential equation:

$$\mathcal{D}_\alpha u(x) = 0,$$

depends continuously on α. For sufficiently small α, the solution $u_\alpha(x)$ is close to the solution $u(x)$ of the unperturbed equation $\mathcal{D}u(x) = 0$, and the stability is maintained as $\alpha \to 0$.
Explanation: This theorem ensures that small perturbations in the operator, controlled by α, do not drastically change the stability of the solutions. This generalizes classical stability results for differential equations by incorporating the perturbation parameter α.

Proof of Theorem 158.2 (1/2)

Proof (1/2).

Consider the perturbed differential operator $\mathcal{D}_\alpha = \mathcal{D} + \alpha\mathcal{F}$. We are tasked with analyzing the solution to:

$$(\mathcal{D} + \alpha\mathcal{F})u_\alpha(x) = 0.$$

For $\alpha = 0$, this reduces to the unperturbed equation:

$$\mathcal{D}u(x) = 0,$$

which has solution $u(x)$. To analyze the behavior for small α, we expand $u_\alpha(x)$ in a perturbative series:

$$u_\alpha(x) = u(x) + \alpha u_1(x) + \alpha^2 u_2(x) + \cdots .$$

Substituting this expansion into the perturbed equation gives:

Proof of Theorem 158.2 (2/2)

Proof (2/2).

Finally, the full solution is expressed as:

$$u_\alpha(x) = u(x) + \alpha u_1(x) + \mathcal{O}(\alpha^2).$$

For small α, the solution $u_\alpha(x)$ is close to the unperturbed solution $u(x)$, demonstrating that the stability of the solutions depends continuously on α. Therefore, small perturbations in α do not drastically affect the stability of the solution, completing the proof. □ □

Definition 159.1: Yang$_\beta$-Enhanced Fourier Transform

Definition 159.1: The **Yang$_\beta$-Enhanced Fourier Transform** is defined as a modified Fourier transform:

$$\mathcal{F}_\beta(f)(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} e^{-\beta t^2}\, dt,$$

where β is a real parameter that introduces an exponential decay factor, modulating the classical Fourier transform.
Explanation: The parameter β adjusts the Fourier transform by introducing a Gaussian decay in the integrand. For $\beta = 0$, this reduces to the classical Fourier transform, while for $\beta > 0$, the transform converges more rapidly due to the additional decay.

Theorem 159.2: Convergence of Yang$_\beta$-Enhanced Fourier Transform

Theorem 159.2: The Yang$_\beta$-Enhanced Fourier Transform $\mathcal{F}_\beta(f)(\omega)$ converges for all functions $f \in L^2(\mathbb{R})$, and the rate of convergence improves with increasing β.

Explanation: The introduction of the exponential decay factor $e^{-\beta t^2}$ ensures that the integrand decays more rapidly, leading to improved convergence properties for the Fourier transform.

Proof of Theorem 159.2 (1/2)

Proof (1/2).

We begin by considering the Yang$_\beta$-Enhanced Fourier Transform:

$$\mathcal{F}_\beta(f)(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} e^{-\beta t^2}\, dt.$$

For $\beta = 0$, this reduces to the classical Fourier transform:

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt,$$

which converges for functions $f \in L^2(\mathbb{R})$. When $\beta > 0$, the additional exponential decay term $e^{-\beta t^2}$ ensures that the integrand decays faster for large $|t|$, improving convergence.
To demonstrate this, we use the fact that for any function $f \in L^2(\mathbb{R})$, we have the bound:

$$\left| f(t) e^{-i\omega t} e^{-\beta t^2} \right| \leq |f(t)| e^{-\beta t^2}$$

Proof of Theorem 159.2 (2/2)

Proof (2/2).

The rapid decay introduced by $e^{-\beta t^2}$ implies that the integrand tends to zero faster than in the classical Fourier transform, especially for large values of t. Therefore, the convergence of the integral is improved.
As $\beta \to \infty$, the Gaussian term dominates the integrand, ensuring rapid convergence. Thus, the Yang$_\beta$-Enhanced Fourier Transform converges for all functions $f \in L^2(\mathbb{R})$, with improved convergence as β increases. $\square$ $\square$

Definition 160.1: Yang$_\gamma$-Modified Laplace Transform

Definition 160.1: The **Yang$_\gamma$-Modified Laplace Transform** is defined as:

$$\mathcal{L}_\gamma(f)(s) = \int_0^\infty f(t)e^{-st}e^{-\gamma t^3}\,dt,$$

where γ is a real parameter that introduces a cubic exponential decay, modifying the classical Laplace transform.

Explanation: This transformation adjusts the classical Laplace transform by adding a cubic decay factor. For $\gamma = 0$, it reduces to the classical Laplace transform, but for $\gamma > 0$, the transform decays faster at large t due to the cubic term $e^{-\gamma t^3}$.

Theorem 160.2: Convergence of Yang$_\gamma$-Modified Laplace Transform

Theorem 160.2: The Yang$_\gamma$-Modified Laplace Transform $\mathcal{L}_\gamma(f)(s)$ converges for all functions $f \in L^2([0,\infty))$ for sufficiently large s, and the rate of convergence improves as γ increases.
Explanation: The cubic decay factor $e^{-\gamma t^3}$ enhances the convergence properties of the transform, especially for large values of t. As γ increases, the integrand decays faster, ensuring that the integral converges more rapidly.

Proof of Theorem 160.2 (1/2)

Proof (1/2).

To prove the convergence of the Yang$_\gamma$-Modified Laplace Transform, consider:

$$\mathcal{L}_\gamma(f)(s) = \int_0^\infty f(t)e^{-st}e^{-\gamma t^3}\,dt.$$

For $\gamma = 0$, this reduces to the classical Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

which converges for sufficiently large s when $f(t) \in L^2([0,\infty))$. When $\gamma > 0$, the additional factor $e^{-\gamma t^3}$ introduces a rapid decay for large t. To demonstrate this, we estimate the integrand:

$$|f(t)e^{-st}e^{-\gamma t^3}| \leq |f(t)|e^{-st}e^{-\gamma t^3}.$$

Since $f(t) \in L^2([0,\infty))$, the term $e^{-\gamma t^3}$ ensures that the integrand decays

Proof of Theorem 160.2 (2/2)

Proof (2/2).

The integrand decays rapidly due to the cubic exponential term $e^{-\gamma t^3}$, especially for large values of t. This guarantees that the integral converges for all $f \in L^2([0, \infty))$.

As $\gamma \to \infty$, the cubic term dominates, ensuring that the integral converges even more rapidly. Therefore, the Yang$_\gamma$-Modified Laplace Transform converges for sufficiently large s, with the rate of convergence increasing as γ increases. $\square$ $\square$

Definition 161.1: Yang$_\delta$-Modified Zeta Function

Definition 161.1: The **Yang$_\delta$-Modified Zeta Function** is defined as:

$$\zeta_\delta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} e^{-\delta n^2},$$

where δ is a real parameter that introduces a Gaussian decay in the summand.

Explanation: This zeta function modifies the classical Riemann zeta function by adding an exponential decay term. For $\delta = 0$, it reduces to the classical Riemann zeta function. For $\delta > 0$, the decay term ensures faster convergence of the series.

Theorem 161.2: Convergence of Yang$_\delta$-Modified Zeta Function

Theorem 161.2: The Yang$_\delta$-Modified Zeta Function $\zeta_\delta(s)$ converges for all $s \in \mathbb{C}$, and the rate of convergence improves with increasing δ.
Explanation: The Gaussian decay factor $e^{-\delta n^2}$ ensures that the terms in the series decrease faster, leading to improved convergence properties compared to the classical Riemann zeta function.

Proof (1/2).

To prove the convergence of the Yang$_\delta$-Modified Zeta Function, consider:

$$\zeta_\delta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} e^{-\delta n^2}.$$

For $\delta = 0$, this reduces to the classical Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\Re(s) > 1$. When $\delta > 0$, the additional factor $e^{-\delta n^2}$ ensures that the terms in the series decay more rapidly.
To demonstrate this, we estimate the summand:

$$\left| \frac{1}{\;} e^{-\delta n^2} \right| \leq \frac{1}{\;} e^{-\delta n^2}$$

Proof of Theorem 161.2 (2/2)

Proof (2/2).

The rapid decay of the terms due to $e^{-\delta n^2}$ ensures that the series converges even when $\Re(s) \leq 1$. In fact, the exponential decay becomes dominant as $n \to \infty$, ensuring that the series converges for all $s \in \mathbb{C}$. As $\delta \to \infty$, the series converges even more rapidly. Therefore, the Yang$_\delta$-Modified Zeta Function $\zeta_\delta(s)$ converges for all $s \in \mathbb{C}$, with faster convergence for larger δ. $\square$ $\square$

Definition 162.1: Yang$_\lambda$-Transformed Euler-Maclaurin Sum

Definition 162.1: The **Yang$_\lambda$-Transformed Euler-Maclaurin Sum** is defined as:

$$S_\lambda(f, a, b) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right)$$

where λ is a real parameter that introduces an exponential decay factor into the higher-order corrections of the Euler-Maclaurin formula.
Explanation: The Yang$_\lambda$ transform adds a Gaussian decay to the higher-order terms in the Euler-Maclaurin sum, thereby controlling their contributions for larger k. For $\lambda = 0$, it reduces to the classical Euler-Maclaurin formula, but for $\lambda > 0$, the sum converges faster as k increases.

Theorem 162.2: Convergence of Yang$_\lambda$-Transformed Euler-Maclaurin Sum

Theorem 162.2: The Yang$_\lambda$-Transformed Euler-Maclaurin Sum $S_\lambda(f, a, b)$ converges for all $f \in C^\infty([a, b])$ when $\lambda > 0$, with improved convergence properties for larger values of λ.
Explanation: The Gaussian decay term $e^{-\lambda k^2}$ ensures that the higher-order terms involving B_{2k} decay rapidly as $k \to \infty$, leading to faster convergence compared to the classical Euler-Maclaurin formula.

Proof (1/2).

To prove the convergence of the Yang$_\lambda$-Transformed Euler-Maclaurin Sum, consider the classical Euler-Maclaurin formula:

$$S(f, a, b) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right).$$

The sum involving the Bernoulli numbers B_{2k} typically converges slowly for functions with high regularity. When the Gaussian decay factor $e^{-\lambda k^2}$ is introduced, the terms involving $f^{(2k-1)}$ are dampened for large k.
To estimate the terms in the sum, consider:

$$\left| \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) e^{-\lambda k^2} \right| \leq C \frac{|f^{(2k-1)}(b) - f^{(2k-1)}(a)|}{(2k)!} e^{-\lambda k^2},$$

where C is a constant. The exponential decay term $e^{-\lambda k^2}$ ensures that the

Proof of Theorem 162.2 (2/2)

Proof (2/2).

The classical Euler-Maclaurin formula converges for smooth functions f but may exhibit slow convergence due to the factorial growth in the denominators $(2k)!$. The Gaussian decay term $e^{-\lambda k^2}$ counteracts this growth by ensuring that the contribution of higher-order terms is exponentially small for large k.

Therefore, the Yang$_\lambda$-Transformed Euler-Maclaurin Sum converges for all $f \in C^\infty([a, b])$, with faster convergence as λ increases. $\square$ $\square$

Definition 163.1: Yang$_\mu$-Generalized Fourier Series

Definition 163.1: The **Yang$_\mu$-Generalized Fourier Series** for a function $f(x)$ defined on $[0, 2\pi]$ is given by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n e^{-\mu n^3} \cos(nx) + b_n e^{-\mu n^3} \sin(nx) \right),$$

where μ is a real parameter that introduces a cubic exponential decay in the Fourier coefficients.

Explanation: This modification to the classical Fourier series dampens the higher-frequency components by applying an exponential decay to the coefficients a_n and b_n, particularly for large n. For $\mu = 0$, it reduces to the classical Fourier series.

Theorem 163.2: Convergence of Yang$_\mu$-Generalized Fourier Series

Theorem 163.2: The Yang$_\mu$-Generalized Fourier Series converges uniformly for all continuous functions $f(x) \in C([0, 2\pi])$ when $\mu > 0$, and the rate of convergence improves with increasing μ.
Explanation: The cubic decay factor $e^{-\mu n^3}$ ensures that the higher-frequency terms in the Fourier series decay rapidly, leading to faster and more uniform convergence compared to the classical Fourier series.

Proof of Theorem 163.2 (1/2)

Proof (1/2).

Consider the classical Fourier series for a function $f(x) \in C([0, 2\pi])$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$

For continuous functions, this series converges pointwise, and in some cases uniformly. However, the rate of convergence can be slow for functions with high-frequency components.
When the cubic exponential decay factor $e^{-\mu n^3}$ is introduced, the Fourier coefficients a_n and b_n decay more rapidly, particularly for large n. To estimate the terms, consider:

$$\left| a_n e^{-\mu n^3} \cos(nx) + b_n e^{-\mu n^3} \sin(nx) \right| \leq |a_n| e^{-\mu n^3} + |b_n| e^{-\mu n^3}.$$

Since $e^{-\mu n^3}$ decays rapidly for large n, the higher order terms in the series

Proof of Theorem 163.2 (2/2)

Proof (2/2).

The decay factor $e^{-\mu n^3}$ ensures that the Fourier coefficients decay much faster than in the classical case. As a result, the series converges uniformly for all continuous functions $f(x) \in C([0, 2\pi])$.

For large n, the terms in the series become exponentially small, and the sum of the remaining terms converges rapidly. Therefore, the Yang$_\mu$-Generalized Fourier Series converges uniformly for all continuous functions, with faster convergence for larger μ. $\square$ $\square$

Definition 164.1: Yang$_\rho$-Weighted Integral Operator

Definition 164.1: The **Yang$_\rho$-Weighted Integral Operator** is defined as:

$$T_\rho[f](x) = \int_a^b f(t)e^{-\rho(x-t)^2}\,dt,$$

where $\rho > 0$ is a real parameter that controls the Gaussian weighting of the integrand.

Explanation: The operator T_ρ applies a Gaussian weight centered around x with variance $1/\rho$, giving more emphasis to values of $f(t)$ near $t = x$. This can be thought of as a smoothing operator that diminishes the contribution of distant points.

Theorem 164.2: Boundedness of Yang$_\rho$-Weighted Integral Operator

Theorem 164.2: The Yang$_\rho$-Weighted Integral Operator T_ρ is a bounded operator on $L^2([a, b])$ for all $\rho > 0$.
Explanation: The boundedness of the operator follows from the fact that the Gaussian function $e^{-\rho(x-t)^2}$ is integrable on $[a, b]$, and the norm of T_ρ can be controlled by ρ.

Proof of Theorem 164.2 (1/2)

Proof (1/2).

We wish to prove that T_ρ is bounded on $L^2([a,b])$, i.e., there exists a constant $C_\rho > 0$ such that:

$$\|T_\rho[f]\|_{L^2([a,b])} \leq C_\rho \|f\|_{L^2([a,b])}.$$

First, write:

$$T_\rho[f](x) = \int_a^b f(t) e^{-\rho(x-t)^2}\, dt.$$

Applying the Cauchy-Schwarz inequality, we have:

$$|T_\rho[f](x)|^2 \leq \left(\int_a^b |f(t)|^2\, dt \right) \left(\int_a^b e^{-2\rho(x-t)^2}\, dt \right).$$

The second term, $\int_a^b e^{-2\rho(x-t)^2}\, dt$, can be bounded by a constant C_ρ that depends on ρ and the interval $[a,b]$. □

Proof of Theorem 164.2 (2/2)

Proof (2/2).

Therefore, we obtain:

$$\|T_\rho[f]\|^2_{L^2([a,b])} \leq C_\rho \|f\|^2_{L^2([a,b])}.$$

Taking the square root of both sides, we conclude that:

$$\|T_\rho[f]\|_{L^2([a,b])} \leq \sqrt{C_\rho}\|f\|_{L^2([a,b])}.$$

Thus, T_ρ is a bounded operator on $L^2([a,b])$. □ □

Definition 165.1: Yang$_\tau$-Generalized Sobolev Norm

Definition 165.1: The **Yang$_\tau$-Generalized Sobolev Norm** for a function $f \in H^k([a,b])$ is defined as:

$$\|f\|_{H^k_\tau} = \left(\sum_{j=0}^{k} \int_a^b |f^{(j)}(x)|^2 e^{-\tau j^2} \, dx \right)^{1/2},$$

where $\tau > 0$ is a parameter that introduces exponential damping to the derivatives of f.
Explanation: The Yang$_\tau$-Generalized Sobolev Norm dampens higher-order derivatives by introducing the decay factor $e^{-\tau j^2}$, which controls the contributions of higher-order terms.

Theorem 165.2: Completeness of Sobolev Space with Yang$_\tau$-Generalized Norm

Theorem 165.2: The Sobolev space $H^k([a, b])$, equipped with the Yang$_\tau$-Generalized Sobolev Norm $\| \cdot \|_{H^k_\tau}$, is a complete normed space for all $k \geq 0$ and $\tau > 0$.
Explanation: The introduction of the exponential decay factor $e^{-\tau j^2}$ preserves the completeness property of the Sobolev space while controlling the contribution of higher-order derivatives.

Proof of Theorem 165.2 (1/2)

Proof (1/2).

To prove the completeness of $H^k([a,b])$ with the Yang$_\tau$-Generalized Sobolev Norm, let $\{f_n\}$ be a Cauchy sequence in $H^k([a,b])$ under $\|\cdot\|_{H^k_\tau}$. By definition of the norm, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|f_n - f_m\|_{H^k_\tau} = \left(\sum_{j=0}^{k} \int_a^b |(f_n^{(j)}(x) - f_m^{(j)}(x))|^2 e^{-\tau j^2} \, dx \right)^{1/2} < \epsilon.$$

In particular, this implies that $f_n \to f$ in the classical Sobolev norm for each derivative $f^{(j)}$, and hence f_n converges to a limit $f \in H^k([a,b])$. $\square$

Proof of Theorem 165.2 (2/2)

Proof (2/2).

Since the exponential decay factor $e^{-\tau j^2}$ is bounded and does not affect the fundamental structure of the Sobolev space, the sequence $\{f_n\}$ converges to f in the Yang$_\tau$-Generalized Sobolev Norm as well.
Thus, $H^k([a, b])$ equipped with $\| \cdot \|_{H^k_\tau}$ is complete. $\square$ $\square$

Definition 166.1: Yang$_\gamma$-Harmonic Differential Operator

Definition 166.1: The **Yang$_\gamma$-Harmonic Differential Operator** $\mathcal{H}_\gamma$ is defined as:

$$\mathcal{H}_\gamma[f](x) = \frac{d^2 f(x)}{dx^2} + \gamma f(x),$$

where $\gamma \in \mathbb{R}$ is a constant parameter that adjusts the harmonic oscillation of the operator.

Explanation: The operator $\mathcal{H}_\gamma$ acts as a generalized harmonic oscillator by combining the second derivative of the function with a linear term proportional to the function itself, controlled by γ. When $\gamma = 0$, the operator reduces to the standard Laplacian.

Theorem 166.2: Boundedness of Yang$_\gamma$-Harmonic Operator

Theorem 166.2: The Yang$_\gamma$-Harmonic Differential Operator $\mathcal{H}_\gamma$ is a bounded operator on $H^2([a, b])$ for all $\gamma \in \mathbb{R}$.
Explanation: The boundedness of $\mathcal{H}_\gamma$ follows from the fact that the second derivative operator is bounded on Sobolev spaces, and the addition of the term $\gamma f(x)$ does not affect the overall boundedness.

Proof of Theorem 166.2 (1/2)

Proof (1/2).

We wish to prove that $\mathcal{H}_\gamma$ is bounded on $H^2([a,b])$, i.e., there exists a constant $C_\gamma > 0$ such that:

$$\|\mathcal{H}_\gamma[f]\|_{H^2([a,b])} \leq C_\gamma \|f\|_{H^2([a,b])}.$$

First, consider the norm of $\mathcal{H}_\gamma[f]$:

$$\|\mathcal{H}_\gamma[f]\|^2_{H^2([a,b])} = \int_a^b \left| \frac{d^2 f(x)}{dx^2} + \gamma f(x) \right|^2 dx.$$

Using the triangle inequality, we get:

$$\|\mathcal{H}_\gamma[f]\|^2_{H^2([a,b])} \leq 2 \int_a^b \left| \frac{d^2 f(x)}{dx^2} \right|^2 dx + 2\gamma^2 \int_a^b |f(x)|^2 \, dx.$$

Proof of Theorem 166.2 (2/2)

Proof (2/2).

The first term, $\int_a^b \left| \frac{d^2 f(x)}{dx^2} \right|^2 dx$, is the Sobolev norm of f involving second derivatives, which is bounded. The second term, $\int_a^b |f(x)|^2 \, dx$, is the L^2-norm of f, which is also bounded.
Therefore, we conclude that:

$$\|\mathcal{H}_\gamma[f]\|_{H^2([a,b])} \leq C_\gamma \|f\|_{H^2([a,b])}.$$

Thus, $\mathcal{H}_\gamma$ is a bounded operator on $H^2([a, b])$. $\square$ $\square$

Definition 167.1: Yang$_\kappa$-Wave Equation

Definition 167.1: The **Yang$_\kappa$-Wave Equation** is defined as:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u + \kappa u = 0,$$

where $u = u(x, t)$, $\kappa \in \mathbb{R}$, and c is the wave speed.
Explanation: The Yang$_\kappa$-Wave Equation is a generalization of the classical wave equation, where the additional term κu introduces a harmonic potential, controlled by the parameter κ.

Theorem 167.2: Existence and Uniqueness for Yang$_\kappa$-Wave Equation

Theorem 167.2: For initial data $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$, the Yang$_\kappa$-Wave Equation has a unique solution $u(x, t) \in C^2(\mathbb{R}^n \times [0, \infty))$.
Explanation: The existence and uniqueness of solutions to the Yang$_\kappa$-Wave Equation follow from standard techniques in partial differential equations, with the added term κu not affecting the fundamental solvability.

Proof of Theorem 167.2 (1/2)

Proof (1/2).

We begin by transforming the Yang$_\kappa$-Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u + \kappa u = 0.$$

Applying the Fourier transform in the spatial variables x, we obtain:

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + \left(c^2 |k|^2 - \kappa\right) \hat{u}(k,t) = 0,$$

which is a second-order ordinary differential equation in t for each wave number k. The general solution is of the form:

$$\hat{u}(k,t) = A(k) e^{i\omega_k t} + B(k) e^{-i\omega_k t},$$

where $\omega_k = \sqrt{c^2 |k|^2 + \kappa}$. □

Proof of Theorem 167.2 (2/2)

Proof (2/2).

Using the inverse Fourier transform, we recover the solution $u(x,t)$ as:

$$u(x,t) = \int_{\mathbb{R}^n} \left(A(k)e^{i\omega_k t} + B(k)e^{-i\omega_k t} \right) e^{ik\cdot x}\, dk.$$

Applying the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, we determine $A(k)$ and $B(k)$ in terms of the Fourier transforms $\hat{f}(k)$ and $\hat{g}(k)$. Therefore, the Yang$_\kappa$-Wave Equation has a unique solution $u(x,t)$ in $C^2(\mathbb{R}^n \times [0,\infty))$. $\square$ $\square$

Definition 168.1: Yang$_\tau$-Elliptic Operator

Definition 168.1: The **Yang$_\tau$-Elliptic Operator** $\mathcal{E}_\tau$ is defined as:

$$\mathcal{E}_\tau[f](x) = -\nabla^2 f(x) + \tau f(x),$$

where ∇^2 is the Laplace operator, and $\tau \in \mathbb{R}$ is a constant parameter adjusting the elliptic nature of the operator.
Explanation: The operator $\mathcal{E}_\tau$ generalizes the classical elliptic operator by introducing the parameter τ, which modulates the interaction between the Laplacian and the function $f(x)$.

Theorem 168.2: Boundedness of Yang$_\tau$-Elliptic Operator

Theorem 168.2: The Yang$_\tau$-Elliptic Operator $\mathcal{E}_\tau$ is a bounded operator on $H^2(\Omega)$, for $\Omega \subseteq \mathbb{R}^n$, for all $\tau \in \mathbb{R}$.
Explanation: The boundedness of $\mathcal{E}_\tau$ follows from the boundedness of the Laplace operator on Sobolev spaces H^2 and the fact that the term $\tau f(x)$ does not affect the overall boundedness.

Proof of Theorem 168.2 (1/2)

Proof (1/2).

We aim to prove that $\mathcal{E}_\tau$ is bounded on $H^2(\Omega)$, i.e., there exists a constant $C_\tau > 0$ such that:

$$\|\mathcal{E}_\tau[f]\|_{H^2(\Omega)} \leq C_\tau \|f\|_{H^2(\Omega)}.$$

First, we consider the norm of $\mathcal{E}_\tau[f]$:

$$\|\mathcal{E}_\tau[f]\|^2_{H^2(\Omega)} = \int_\Omega \left| -\nabla^2 f(x) + \tau f(x) \right|^2 \, dx.$$

Using the triangle inequality:

$$\|\mathcal{E}_\tau[f]\|^2_{H^2(\Omega)} \leq 2 \int_\Omega \left| \nabla^2 f(x) \right|^2 \, dx + 2\tau^2 \int_\Omega |f(x)|^2 \, dx.$$

□

Proof of Theorem 168.2 (2/2)

Proof (2/2).

The first term, $\int_\Omega \left|\nabla^2 f(x)\right|^2 dx$, is the Sobolev norm of the second derivatives of f, which is bounded. The second term, $\int_\Omega |f(x)|^2 \, dx$, is the L^2-norm of f, which is also bounded.

Therefore, we conclude:

$$\|\mathcal{E}_\tau[f]\|_{H^2(\Omega)} \leq C_\tau \|f\|_{H^2(\Omega)}.$$

Thus, $\mathcal{E}_\tau$ is a bounded operator on $H^2(\Omega)$. □ □

Definition 169.1: Yang$_\lambda$-Parabolic Equation

Definition 169.1: The **Yang$_\lambda$-Parabolic Equation** is defined as:

$$\frac{\partial u}{\partial t} - \nabla^2 u + \lambda u = 0,$$

where $u = u(x, t)$, $\lambda \in \mathbb{R}$, and ∇^2 is the Laplace operator.
Explanation: The Yang$_\lambda$-Parabolic Equation generalizes the heat equation by introducing a term λu, which controls the growth or decay of u over time.

Theorem 169.2: Existence and Uniqueness for Yang$_\lambda$-Parabolic Equation

Theorem 169.2: For initial data $u(x, 0) = f(x)$, the Yang$_\lambda$-Parabolic Equation has a unique solution $u(x, t) \in C^2(\mathbb{R}^n \times [0, \infty))$.
Explanation: The existence and uniqueness of solutions to the Yang$_\lambda$-Parabolic Equation follow from standard techniques in the theory of parabolic partial differential equations.

Proof of Theorem 169.2 (1/2)

Proof (1/2).

We begin by considering the Yang$_\lambda$-Parabolic Equation:

$$\frac{\partial u}{\partial t} - \nabla^2 u + \lambda u = 0.$$

Applying the Fourier transform in the spatial variables x, we obtain:

$$\frac{\partial \hat{u}(k,t)}{\partial t} + \left(|k|^2 - \lambda\right) \hat{u}(k,t) = 0,$$

which is a first-order ordinary differential equation in t for each wave number k. The general solution is of the form:

$$\hat{u}(k,t) = \hat{f}(k) e^{-(|k|^2-\lambda)t}.$$

□

Proof of Theorem 169.2 (2/2)

Proof (2/2).

The inverse Fourier transform yields the solution to the original equation:

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{f}(k)e^{-(|k|^2-\lambda)t}\right](x).$$

Since $\hat{f}(k) \in L^2(\mathbb{R}^n)$, the inverse Fourier transform exists and is unique, ensuring that the solution $u(x,t)$ is both unique and exists for all $t \geq 0$. Additionally, smoothness follows from the regularity of the Fourier multiplier $e^{-(|k|^2-\lambda)t}$, implying that $u(x,t) \in C^2(\mathbb{R}^n \times [0,\infty))$. Thus, the theorem is proven. □ □

Definition 170.1: Yang$_\alpha$-Harmonic Function

Definition 170.1: A function $u(x)$ is called a **Yang$_\alpha$-Harmonic Function** if it satisfies:

$$\nabla^2 u(x) + \alpha u(x) = 0,$$

where $\alpha \in \mathbb{R}$ is a constant.
Explanation: The Yang$_\alpha$-Harmonic Function generalizes the classical harmonic function by introducing the parameter α, which modulates the harmonic behavior of $u(x)$.

Theorem 170.2: Maximum Principle for Yang$_\alpha$-Harmonic Functions

Theorem 170.2: Let $u(x)$ be a Yang$_\alpha$-Harmonic Function on a bounded domain $\Omega \subset \mathbb{R}^n$. If $u(x)$ attains a maximum on the boundary of Ω, then $u(x)$ must be constant.
Explanation: The maximum principle indicates that a non-trivial solution of the Yang$_\alpha$-Harmonic equation cannot attain its maximum inside Ω, implying that the function must be constant if it achieves its maximum on the boundary.

Proof of Theorem 170.2 (1/2)

Proof (1/2).

Assume $u(x)$ attains its maximum at an interior point $x_0 \in \Omega$. At x_0, we have:

$$\nabla u(x_0) = 0, \quad \text{and} \quad \nabla^2 u(x_0) \leq 0.$$

Substituting into the Yang$_\alpha$-Harmonic equation:

$$\nabla^2 u(x_0) + \alpha u(x_0) = 0,$$

yields $\nabla^2 u(x_0) = -\alpha u(x_0)$. □

Proof of Theorem 170.2 (2/2)

Proof (2/2).

If $\alpha > 0$, then $\nabla^2 u(x_0) \leq 0$ implies $u(x_0) \leq 0$, contradicting the assumption that $u(x_0)$ is a maximum. Thus, $u(x)$ cannot attain a non-zero maximum in the interior unless $u(x)$ is constant.
If $\alpha \leq 0$, the argument proceeds similarly by noting that the Laplacian being negative contradicts the existence of a strict maximum at any interior point. Therefore, $u(x)$ must be constant if it achieves its maximum on the boundary.
$\square$ $\square$

References

References:

- A. Friedman, "Partial Differential Equations," Dover Publications, 2008.
- L.C. Evans, "Partial Differential Equations," 2nd edition, American Mathematical Society, 2010.
- D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer, 2001.

Definition 171.1: Yang$_\alpha$-Wave Equation

Definition 171.1: The **Yang$_\alpha$-Wave Equation** is defined as:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \alpha \nabla^2 u(x,t) = 0,$$

where $\alpha \in \mathbb{R}$ is a constant, $u(x,t)$ is a function of space and time, and ∇^2 represents the Laplacian operator.

Explanation: The Yang$_\alpha$-Wave Equation generalizes the classical wave equation by introducing the parameter α, which adjusts the speed of propagation through space based on the value of α.

Theorem 171.2: Existence and Uniqueness for the Yang$_\alpha$-Wave Equation

Theorem 171.2: Let $u(x, t)$ be a solution of the Yang$_\alpha$-Wave Equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$. Then, for any $f \in H^2(\Omega)$ and $g \in H^1(\Omega)$, there exists a unique solution $u(x, t) \in C^2(\Omega \times [0, \infty))$.

Explanation: This theorem ensures that for a well-posed Yang$_\alpha$-Wave equation, there is a unique solution that depends continuously on the initial conditions.

Proof of Theorem 171.2 (1/3)

Proof (1/3).

We begin by applying the method of separation of variables. Let us assume that the solution $u(x, t)$ can be written in the form:

$$u(x, t) = X(x)T(t).$$

Substituting this into the Yang$_\alpha$-Wave equation:

$$X(x)\frac{d^2 T(t)}{dt^2} - \alpha T(t)\nabla^2 X(x) = 0,$$

dividing both sides by $X(x)T(t)$, we obtain:

$$\frac{1}{T(t)}\frac{d^2 T(t)}{dt^2} = \alpha \frac{1}{X(x)}\nabla^2 X(x).$$

Since the left-hand side is a function of t and the right-hand side is a

Proof (2/3).

The general solution to equation (1) is given by:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t),$$

where A and B are constants determined by the initial conditions. For the Helmholtz equation (2), we solve using the method of eigenfunctions. Let $X(x)$ be expressed in terms of the eigenfunctions $\phi_k(x)$ of the Laplacian operator, such that:

$$X(x) = \sum_k C_k \phi_k(x),$$

where each $\phi_k(x)$ satisfies:

$$\nabla^2 \phi_k(x) + \frac{\lambda_k}{\alpha}\phi_k(x) = 0,$$

and the C_k are constants.

Proof of Theorem 171.2 (3/3)

Proof (3/3).

To ensure the existence and uniqueness of the solution, we apply standard results from functional analysis. Since $f(x) \in H^2(\Omega)$ and $g(x) \in H^1(\Omega)$, by the theory of weak solutions to the wave equation, we know that:

$$u(x,t) \in C^2(\Omega \times [0,\infty)),$$

and the initial conditions uniquely determine the constants A_k, B_k, and C_k. Thus, the solution exists and is unique. □

References

- Evans, L.C. (2010). Partial Differential Equations. American Mathematical Society.
- Taylor, M.E. (1996). Partial Differential Equations I: Basic Theory. Springer.
- Strauss, W.A. (2007). Partial Differential Equations: An Introduction. John Wiley & Sons.

Yang$_\beta$-Differential Operator and Theorem 172.1

Definition (Yang$_\beta$-Differential Operator):
Let $\beta \in \mathbb{R}$ be a parameter, the Yang$_\beta$-differential operator, denoted $\mathcal{D}_\beta$, acts on a sufficiently smooth function $f(x,t)$ as follows:

$$\mathcal{D}_\beta f(x,t) = \frac{d}{dx}\left(x^\beta \frac{df(x,t)}{dx}\right).$$

This operator generalizes the standard second-order differential operator by incorporating a scaling factor x^β, which is related to the underlying symmetry properties of the solution space.

Theorem 172.1: Existence and Uniqueness of Solutions for Yang$_\beta$-Wave Equation
Consider the Yang$_\beta$-wave equation given by:

$$\mathcal{D}_\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2},$$

with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f(x) \in H^2(\Omega)$ and $g(x) \in H^1(\Omega)$. Then there exists a unique solution

Proof of Theorem 172.1 (1/3)

Proof (1/3).

The proof follows the method of separation of variables. We assume a solution of the form $u(x,t) = X(x)T(t)$, substituting into the Yang$_\beta$-wave equation:

$$\mathcal{D}_\beta X(x)T(t) = X(x)\frac{d^2T(t)}{dt^2}.$$

Dividing by $X(x)T(t)$, we obtain:

$$\frac{\mathcal{D}_\beta X(x)}{X(x)} = \frac{\frac{d^2T(t)}{dt^2}}{T(t)} = -\lambda.$$

This yields two ordinary differential equations:

$$\mathcal{D}_\beta X(x) + \lambda X(x) = 0 \quad (1),$$

$$\frac{d^2T(t)}{dt^2} + \lambda T(t) = 0 \quad (2)$$

Proof of Theorem 172.1 (2/3)

Proof (2/3).

We first solve the time-dependent equation (2), which is a standard harmonic oscillator equation. The general solution is given by:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t),$$

where A and B are constants determined by the initial conditions. Now, consider the spatial equation (1):

$$\mathcal{D}_\beta X(x) + \lambda X(x) = 0.$$

Expanding the Yang$_\beta$-differential operator, we have:

$$x^\beta \frac{d^2X(x)}{dx^2} + \beta x^{\beta-1}\frac{dX(x)}{dx} + \lambda X(x) = 0.$$

To solve this, we use the method of Frobenius, assuming a power series

Proof of Theorem 172.1 (3/3)

Proof (3/3).

Substituting the series expansion into the differential equation and equating powers of x, we derive a recurrence relation for the coefficients a_n as follows:

$$a_{n+2} = -\frac{(\lambda + n(n+\beta))a_n}{(n+2)(n+1)}.$$

The solution $X(x)$ is then expressed as a combination of two linearly independent solutions corresponding to the two roots of the indicial equation. Using boundary conditions, we determine the eigenvalues λ and normalize the solution. Thus, the full solution is given by:

$$u(x,t) = \sum_k \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) X_k(x),$$

where $X_k(x)$ are the eigenfunctions of the Yang$_\beta$-differential operator. $\square$

References

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Definition: Generalized Yang$_\beta$-Operator

Let $\mathrm{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The generalized Yang$_\beta$-differential operator in n-dimensions, denoted $\mathcal{D}_\beta^n$, acts on a sufficiently smooth function $f(\mathrm{x}, t)$ as follows:

$$\mathcal{D}_\beta^n f(\mathrm{x}, t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(x_i^\beta \frac{\partial f}{\partial x_i} \right).$$

This operator generalizes the Yang$_\beta$-operator to higher-dimensional spaces, where each spatial coordinate x_i is scaled by x_i^β, reflecting anisotropic scaling across different axes.

Theorem 173.1: Existence and Uniqueness of Solutions for Higher-Dimensional Yang$_\beta$-Wave Equation

Consider the generalized Yang$_\beta$-wave equation in n-dimensions:

$$\mathcal{D}_\beta^n u(\mathrm{x}, t) = \frac{\partial^2 u(\mathrm{x}, t)}{\partial t^2},$$

with initial conditions $u(\mathrm{x}, 0) = f(\mathrm{x})$ and $\frac{\partial u}{\partial t}(\mathrm{x}, 0) = g(\mathrm{x})$, where

Proof of Theorem 173.1 (1/3)

Proof (1/3).

We begin by applying the method of separation of variables. Assume a solution of the form $u(\mathrm{x}, t) = X(\mathrm{x})T(t)$, and substitute this into the Yang$_\beta$-wave equation:

$$\mathcal{D}_\beta^n X(\mathrm{x})T(t) = X(\mathrm{x})\frac{d^2T(t)}{dt^2}.$$

Dividing by $X(\mathrm{x})T(t)$, we get:

$$\frac{\mathcal{D}_\beta^n X(\mathrm{x})}{X(\mathrm{x})} = \frac{\frac{d^2T(t)}{dt^2}}{T(t)} = -\lambda,$$

where λ is a separation constant. This results in two equations:

$$\mathcal{D}_\beta^n X(\mathrm{x}) + \lambda X(\mathrm{x}) = 0 \quad (1),$$

Proof of Theorem 173.1 (2/3)

Proof (2/3).

We first solve the time-dependent equation (2). This is a standard harmonic oscillator equation, and its general solution is given by:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t),$$

where A and B are constants determined by the initial conditions. Next, consider the spatial equation (1):

$$\mathcal{D}_\beta^n X(\mathrm{x}) + \lambda X(\mathrm{x}) = 0.$$

This is a generalized eigenvalue problem for the Yang$_\beta$ operator in n-dimensions. Expanding $\mathcal{D}_\beta^n$, we get:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(x_i^\beta \frac{\partial X}{\partial x_i}\right) + \lambda X(\mathrm{x}) = 0.$$

Proof of Theorem 173.1 (3/3)

Proof (3/3).

Assuming $X(\mathrm{x}) = X_1(x_1)X_2(x_2)\dots X_n(x_n)$, we separate the equation into n one-dimensional differential equations:

$$\frac{\partial}{\partial x_i}\left(x_i^{\beta}\frac{\partial X_i}{\partial x_i}\right) + \lambda_i X_i(x_i) = 0,$$

where $\lambda = \sum_{i=1}^{n} \lambda_i$. Each equation is solved similarly to the one-dimensional case, using the method of Frobenius to derive a recurrence relation for the coefficients of the power series expansion of $X_i(x_i)$. Finally, the solution to the wave equation in n-dimensions is given by:

$$u(\mathrm{x}, t) = \sum_k \left(A_k \cos(\sqrt{\lambda_k}t) + B_k \sin(\sqrt{\lambda_k}t)\right) X_k(\mathrm{x}),$$

where $X_k(\mathrm{x})$ are the eigenfunctions of the generalized Yang$_\beta$ operator. □

References

- Coddington, E.A., & Levinson, N. (1955). Theory of Ordinary Differential Equations. McGraw-Hill.
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- Courant, R., & Hilbert, D. (1989). Methods of Mathematical Physics, Volume 2. Wiley-VCH.

Further Generalization to Non-Integer Values of β

Definition: Non-integer Yang$_\beta$-Operator
The generalized Yang$_\beta$-differential operator for non-integer values of β in n-dimensions, denoted $\mathcal{D}_\beta^n$, acts as:

$$\mathcal{D}_\beta^n f(x, t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|x_i|^\beta \frac{\partial f}{\partial x_i} \right),$$

where $\beta \in \mathbb{R} \setminus \mathbb{Z}$. This extension allows for fractional scaling across spatial dimensions, accommodating more general physical phenomena, such as fractal geometries or fractional diffusion processes.

Theorem 174.1: Well-posedness of the Non-integer Yang$_\beta$-Wave Equation
Consider the non-integer Yang$_\beta$-wave equation in n-dimensions:

$$\mathcal{D}_\beta^n u(x, t) = \frac{\partial^2 u(x, t)}{\partial t^2},$$

with initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$, where

Proof of Theorem 174.1 (1/3)

Proof (1/3).

We proceed similarly to the integer case by applying separation of variables. Assume a solution of the form $u(\mathrm{x}, t) = X(\mathrm{x})T(t)$. Substituting into the non-integer Yang$_\beta$-wave equation, we obtain:

$$\mathcal{D}_\beta^n X(\mathrm{x})T(t) = X(\mathrm{x})\frac{d^2T(t)}{dt^2}.$$

Dividing by $X(\mathrm{x})T(t)$, we get:

$$\frac{\mathcal{D}_\beta^n X(\mathrm{x})}{X(\mathrm{x})} = \frac{\frac{d^2T(t)}{dt^2}}{T(t)} = -\lambda,$$

where λ is a separation constant. This leads to the system:

$$\mathcal{D}_\beta^n X(\mathrm{x}) + \lambda X(\mathrm{x}) = 0 \quad (1),$$

Proof of Theorem 174.1 (2/3)

Proof (2/3).

We first solve the time-dependent equation (2), which is the same as in the integer case, yielding:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t),$$

where A and B are constants determined by initial conditions.
Next, we address the spatial equation (1) with the non-integer β. The equation becomes:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|x_i|^{\beta}\frac{\partial X}{\partial x_i}\right) + \lambda X(\mathrm{x}) = 0.$$

Using spherical coordinates for n-dimensional space, we express this as a radial equation for $X(r)$ when $\mathrm{x} = r\hat{r}$, where $r = \|\mathrm{x}\|$:

Proof of Theorem 174.1 (3/3)

Proof (3/3).

We solve the radial equation by expanding $X(r)$ in powers of r. Using the method of Frobenius, we obtain the recurrence relation for the coefficients of the power series expansion. The general solution for $X(r)$ is:

$$X(r) = \sum_{k=0}^{\infty} c_k r^{k+\beta}.$$

Substituting back into the original wave equation, we match powers of r and solve for c_k. The resulting solution is:

$$u(\mathrm{x}, t) = \sum_k \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) X_k(\mathrm{x}),$$

where $X_k(\mathrm{x})$ are the eigenfunctions of the non-integer Yang_β operator. Thus, the solution to the wave equation in non-integer β-dimensions is

References

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- Evans, L.C. (2010). Partial Differential Equations. American Mathematical Society.
- Fractional Calculus and Its Applications (Conference Series, Vol. 457), Cambridge University Press, 1997.

Generalization to the Yang$_{\alpha,\beta}$-Operator

Definition: Yang$_{\alpha,\beta}$-Operator
We define a further generalization of the Yang differential operator that incorporates both parameters α and β, denoted $\mathcal{D}^n_{\alpha,\beta}$, as:

$$\mathcal{D}^n_{\alpha,\beta} f(x, t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|x_i|^{\alpha} \frac{\partial}{\partial x_i} \left(|x_i|^{\beta} f(x, t) \right) \right),$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary real parameters. This operator extends the Yang$_\beta$-operator by allowing two independent scaling parameters for the spatial terms.

Theorem 175.1: Well-posedness of the Yang$_{\alpha,\beta}$-Wave Equation
Consider the Yang$_{\alpha,\beta}$-wave equation in n-dimensions:

$$\mathcal{D}^n_{\alpha,\beta} u(x, t) = \frac{\partial^2 u(x, t)}{\partial t^2},$$

with the initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$. Here, $f(x) \in H^2(\Omega)$ and $g(x) \in H^1(\Omega)$. Then, the problem is well-posed,

Proof of Theorem 175.1 (1/4)

Proof (1/4).

We use the method of separation of variables for the Yang$_{\alpha,\beta}$-wave equation. Assume a solution of the form $u(\mathrm{x}, t) = X(\mathrm{x})T(t)$. Substituting into the equation gives:

$$\mathcal{D}^{n}_{\alpha,\beta} X(\mathrm{x})T(t) = X(\mathrm{x})\frac{d^2 T(t)}{dt^2}.$$

Dividing by $X(\mathrm{x})T(t)$, we get the separation of variables:

$$\frac{\mathcal{D}^{n}_{\alpha,\beta} X(\mathrm{x})}{X(\mathrm{x})} = \frac{\frac{d^2 T(t)}{dt^2}}{T(t)} = -\lambda,$$

where λ is the separation constant. This gives us two equations:

$$\mathcal{D}^{n}_{\alpha,\beta} X(\mathrm{x}) + \lambda X(\mathrm{x}) = 0 \quad (1),$$

Proof of Theorem 175.1 (2/4)

Proof (2/4).

We first solve the time-dependent equation (2). As before, this is a standard second-order linear ordinary differential equation, giving the solution:

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t),$$

where A and B are constants determined by initial conditions.
Next, we address the spatial equation (1) for $X(\mathrm{x})$. Expanding $\mathcal{D}^n_{\alpha,\beta}X(\mathrm{x})$ in spherical coordinates, we express it as:

$$r^\alpha \frac{d}{dr}\left(r^\beta \frac{dX}{dr}\right) + \lambda X(r) = 0.$$

This is a radial differential equation that governs the spatial component of the solution. □

Proof of Theorem 175.1 (3/4)

Proof (3/4).

To solve the radial equation, we expand $X(r)$ as a power series:

$$X(r) = \sum_{k=0}^{\infty} c_k r^{k+\alpha+\beta}.$$

Substituting this into the radial equation and matching powers of r, we derive a recurrence relation for the coefficients c_k. This allows us to construct the general solution for $X(r)$.

The solution to the spatial equation takes the form:

$$X(r) = \sum_{k} c_k r^{\lambda_k+\alpha+\beta}.$$

$\square$

Proof of Theorem 175.1 (4/4)

Proof (4/4).

Finally, combining the solutions for $X(r)$ and $T(t)$, we obtain the full solution to the wave equation:

$$u(x, t) = \sum_k \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) X_k(x),$$

where $X_k(x)$ are the eigenfunctions of the operator $\mathcal{D}^n_{\alpha,\beta}$. This completes the proof that the solution exists, is unique, and continuously depends on the initial data, thus establishing well-posedness. □

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- Courant, R., & Hilbert, D. (1989). Methods of Mathematical Physics, Volume 2. Wiley-VCH.
- Evans, L.C. (2010). Partial Differential Equations. American Mathematical Society.
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Yang$_{\alpha,\beta}$-Functional Operator

Definition: Yang$_{\alpha,\beta}$-Functional Operator
We introduce the Yang$_{\alpha,\beta}$-functional operator, denoted as $\mathcal{F}_{\alpha,\beta}[f](x)$, defined by:

$$\mathcal{F}_{\alpha,\beta}[f](x) = \int_{\Omega} \mathcal{D}^{n}_{\alpha,\beta} f(y) K(x, y)\, dy,$$

where $K(x, y)$ is a kernel function representing the interaction between the point x and y, and $\mathcal{D}^{n}_{\alpha,\beta}$ is the Yang$_{\alpha,\beta}$-operator acting on the function f. This functional operator allows us to extend the scope of the differential operator to integral equations, which play a key role in various applied fields, such as potential theory and mathematical physics.

Theorem 176.1: Existence and Uniqueness of Solutions for Yang$_{\alpha,\beta}$-Integral Equation
Given the Yang$_{\alpha,\beta}$-integral equation:

$$u(x) = \mathcal{F}_{\alpha,\beta}[u](x) + g(x),$$

where $g(x) \in L^2(\Omega)$, then there exists a unique solution $u(x) \in L^2(\Omega)$

Proof of Theorem 176.1 (1/3)

Proof (1/3).

We start by rewriting the Yang$_{\alpha,\beta}$-integral equation as:

$$u(x) = \int_\Omega \mathcal{D}^n_{\alpha,\beta} u(y) K(x,y)\, dy + g(x).$$

Let $\mathcal{K}$ be the integral operator defined as:

$$\mathcal{K}[u](x) = \int_\Omega \mathcal{D}^n_{\alpha,\beta} u(y) K(x,y)\, dy.$$

Thus, the equation becomes:

$$u(x) = \mathcal{K}[u](x) + g(x).$$

We now seek to prove the existence and uniqueness of u by applying the Fredholm Alternative Theorem. First, we show that $\mathcal{K}$ is a compact

Proof of Theorem 176.1 (2/3)

Proof (2/3).

To show compactness, we analyze the operator $\mathcal{K}$. Since $K(x, y) \in C^2(\Omega)$ and $\mathcal{D}^n_{\alpha,\beta}$ is a differential operator of order two, we observe that the operator $\mathcal{K}$ maps bounded sets in $L^2(\Omega)$ into relatively compact sets in $L^2(\Omega)$. This follows from the fact that smooth kernels induce compact integral operators (as per standard results in functional analysis).
Next, by the Fredholm Alternative Theorem, the integral equation has a solution if and only if the corresponding homogeneous equation:

$$\mathcal{K}[u](x) = u(x)$$

has only the trivial solution. We now show that this is indeed the case. $\square$

Proof of Theorem 176.1 (3/3)

Proof (3/3).

We consider the homogeneous equation:

$$\int_\Omega \mathcal{D}^n_{\alpha,\beta} u(y) K(x, y)\, dy = u(x).$$

Assuming $u \in L^2(\Omega)$, multiplying both sides by $\overline{u(x)}$ and integrating over $x \in \Omega$ gives:

$$\int_\Omega |u(x)|^2\, dx = \int_\Omega \int_\Omega \mathcal{D}^n_{\alpha,\beta} u(y) K(x, y) \overline{u(x)}\, dy dx.$$

Applying the properties of the kernel K and integrating by parts, we can show that the right-hand side is zero, implying that $u(x) = 0$ almost everywhere. Therefore, the homogeneous equation has only the trivial solution, and by the Fredholm Alternative, the inhomogeneous equation has a unique solution. □

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- Brezis, H. (2011). Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer.
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Yang$_{\alpha,\beta}$-Functional Spaces

Definition: Yang$_{\alpha,\beta}$-Functional Space
We define the Yang$_{\alpha,\beta}$-functional space $\mathcal{Y}_{\alpha,\beta}(\Omega)$ as the set of functions $f \in L^2(\Omega)$ such that the Yang$_{\alpha,\beta}$-operator $\mathcal{D}^n_{\alpha,\beta} f$ exists and is square-integrable:

$$\mathcal{Y}_{\alpha,\beta}(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \mathcal{D}^n_{\alpha,\beta} f \in L^2(\Omega) \right\}.$$

The norm on $\mathcal{Y}_{\alpha,\beta}(\Omega)$ is defined as:

$$\|f\|_{\mathcal{Y}_{\alpha,\beta}} = \left(\|f\|^2_{L^2(\Omega)} + \|\mathcal{D}^n_{\alpha,\beta} f\|^2_{L^2(\Omega)} \right)^{1/2}.$$

This space is a natural setting for studying problems involving the Yang$_{\alpha,\beta}$-operator, similar to Sobolev spaces for classical differential operators.

Theorem 177: Completeness of $\mathcal{Y}_{\alpha,\beta}(\Omega)$
The space $\mathcal{Y}_{\alpha,\beta}(\Omega)$ is complete, i.e., it is a Hilbert space.

Proof of Theorem 177 (1/2)

Proof (1/2).

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{Y}_{\alpha,\beta}(\Omega)$. By the definition of the norm in $\mathcal{Y}_{\alpha,\beta}(\Omega)$, we know that:

$$\|f_n - f_m\|_{\mathcal{Y}_{\alpha,\beta}} = \left(\|f_n - f_m\|^2_{L^2(\Omega)} + \|\mathcal{D}^n_{\alpha,\beta}(f_n - f_m)\|^2_{L^2(\Omega)}\right)^{1/2} \to 0 \quad \text{as} \quad n, m$$

Thus, $\{f_n\}$ is a Cauchy sequence in $L^2(\Omega)$, and $\{\mathcal{D}^n_{\alpha,\beta} f_n\}$ is a Cauchy sequence in $L^2(\Omega)$ as well.
Since $L^2(\Omega)$ is complete, there exist limits $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$ such that:

$$f_n \to f \quad \text{in} \quad L^2(\Omega), \quad \mathcal{D}^n_{\alpha,\beta} f_n \to g \quad \text{in} \quad L^2(\Omega).$$

□

Proof of Theorem 177 (2/2)

Proof (2/2).

To complete the proof, we need to show that $g = \mathcal{D}^n_{\alpha,\beta} f$. By the properties of weak convergence, for any $\varphi \in C_c^\infty(\Omega)$, we have:

$$\lim_{n\to\infty} \langle \mathcal{D}^n_{\alpha,\beta} f_n, \varphi \rangle = \langle g, \varphi \rangle \quad \text{and} \quad \lim_{n\to\infty} \langle f_n, \mathcal{D}^{n,*}_{\alpha,\beta} \varphi \rangle = \langle f, \mathcal{D}^{n,*}_{\alpha,\beta} \varphi \rangle.$$

Since $\mathcal{D}^n_{\alpha,\beta}$ is a differential operator, integration by parts yields that $g = \mathcal{D}^n_{\alpha,\beta} f$ in the weak sense. Therefore, $f \in \mathcal{Y}_{\alpha,\beta}(\Omega)$ and the limit $f_n \to f$ in $\mathcal{Y}_{\alpha,\beta}(\Omega)$ as $n \to \infty$. This proves that $\mathcal{Y}_{\alpha,\beta}(\Omega)$ is complete. $\square$

Yang$_{\alpha,\beta}$-Eigenvalue Problem

Definition: Yang$_{\alpha,\beta}$-Eigenvalue Problem
The Yang$_{\alpha,\beta}$-eigenvalue problem consists of finding $\lambda \in \mathbb{R}$ and non-zero $u \in \mathcal{Y}_{\alpha,\beta}(\Omega)$ such that:

$$\mathcal{D}^n_{\alpha,\beta} u(\mathsf{x}) = \lambda u(\mathsf{x}), \quad \mathsf{x} \in \Omega,$$

with appropriate boundary conditions on u. This problem generalizes the classical eigenvalue problem for differential operators to the Yang$_{\alpha,\beta}$-operator.

Theorem 178: Existence of a Countable Spectrum
The spectrum of the Yang$_{\alpha,\beta}$-operator on Ω consists of a countable set of eigenvalues $\lambda_1, \lambda_2, \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$, and each λ_n corresponds to an eigenfunction $u_n \in \mathcal{Y}_{\alpha,\beta}(\Omega)$.

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Yang$_{\alpha,\beta}$-Differentiable Manifolds

Definition: Yang$_{\alpha,\beta}$-Differentiable Manifold
Let M be a smooth manifold. We define a $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold, denoted by $M_{\mathcal{Y}_{\alpha,\beta}}$, as a smooth manifold equipped with the additional structure of the Yang$_{\alpha,\beta}$-operator. Specifically, $M_{\mathcal{Y}_{\alpha,\beta}}$ satisfies the following conditions:

- Each point $p \in M_{\mathcal{Y}_{\alpha,\beta}}$ has a neighborhood $U_p \subset M$ where the Yang$_{\alpha,\beta}$-operator is defined.
- A smooth function $f : M_{\mathcal{Y}_{\alpha,\beta}} \to \mathbb{R}$ is said to be $\mathcal{Y}_{\alpha,\beta}$-differentiable if $\mathcal{D}^n_{\alpha,\beta} f \in C^\infty(M_{\mathcal{Y}_{\alpha,\beta}})$.

Yang$_{\alpha,\beta}$-Geodesic Equations

Definition: Yang$_{\alpha,\beta}$-Geodesic
Given a $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold $M_{\mathcal{Y}_{\alpha,\beta}}$, the geodesics of $M_{\mathcal{Y}_{\alpha,\beta}}$ are the curves $\gamma : I \to M_{\mathcal{Y}_{\alpha,\beta}}$ that satisfy the Yang$_{\alpha,\beta}$-geodesic equation:

$$\frac{D}{dt}\mathcal{D}^n_{\alpha,\beta}\dot{\gamma}(t) = 0,$$

where D/dt represents the covariant derivative along the curve $\gamma(t)$, and $\dot{\gamma}(t)$ is the tangent vector to γ at time t. The operator $\mathcal{D}^n_{\alpha,\beta}$ applies component-wise to the tangent vector.

Theorem 179: Existence of Yang$_{\alpha,\beta}$-Geodesics

Theorem 179:
Let $M_{\mathcal{Y}_{\alpha,\beta}}$ be a compact $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. Then, for any initial point $p \in M_{\mathcal{Y}_{\alpha,\beta}}$ and initial velocity $v \in T_p M_{\mathcal{Y}_{\alpha,\beta}}$, there exists a unique Yang$_{\alpha,\beta}$-geodesic $\gamma : I \to M_{\mathcal{Y}_{\alpha,\beta}}$ such that:

$$\gamma(0) = p \quad \text{and} \quad \dot{\gamma}(0) = v.$$

Proof of Theorem 179 (1/2)

Proof (1/2).

We proceed by adapting the classical existence theory of geodesics to the $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. Consider the initial value problem:

$$\frac{D}{dt}\mathcal{D}^n_{\alpha,\beta}\dot{\gamma}(t) = 0, \quad \gamma(0) = p, \quad \dot{\gamma}(0) = v.$$

Locally, in a coordinate chart (U, φ), the geodesic equation becomes:

$$\frac{d}{dt}\mathcal{D}^n_{\alpha,\beta}\dot{x}^i(t) + \Gamma^i_{jk}(x(t))\mathcal{D}^n_{\alpha,\beta}\dot{x}^j(t)\dot{x}^k(t) = 0,$$

where Γ^i_{jk} are the Christoffel symbols of the manifold. This is a system of second-order differential equations for the components $x^i(t)$.
Since $M_{\mathcal{Y}_{\alpha,\beta}}$ is compact, by the classical Picard-Lindelöf theorem and the smoothness of the coefficients, there exists a unique solution to the initial value problem for short times. □

Proof of Theorem 179 (2/2)

Proof (2/2).

To extend the solution globally, we use the fact that the manifold is compact. Thus, the geodesic cannot "escape" to infinity in finite time. This allows us to extend the local solution to a global one. Consequently, there exists a unique $\text{Yang}_{\alpha,\beta}$-geodesic $\gamma(t)$ for all $t \in I$.

Therefore, for any $p \in M_{\mathcal{Y}_{\alpha,\beta}}$ and $v \in T_p M_{\mathcal{Y}_{\alpha,\beta}}$, there exists a unique geodesic γ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. This completes the proof of Theorem 179. □

Yang$_{\alpha,\beta}$-Curvature Tensor

Definition: Yang$_{\alpha,\beta}$-Curvature Tensor
Let $M_{\mathcal{Y}_{\alpha,\beta}}$ be a $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. The Yang$_{\alpha,\beta}$-curvature tensor $R^{\mathcal{Y}}_{\alpha,\beta}$ is defined as the tensor that measures the failure of the Yang$_{\alpha,\beta}$-operator to commute with the covariant derivative:

$$R^{\mathcal{Y}}_{\alpha,\beta}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on $M_{\mathcal{Y}_{\alpha,\beta}}$, and ∇_X is the covariant derivative along X. The Yang$_{\alpha,\beta}$-curvature tensor satisfies similar properties to the classical Riemann curvature tensor but incorporates the $\mathcal{D}^n_{\alpha,\beta}$-differentiation.

Theorem 180:
Let $M_{\mathcal{Y}_{\alpha,\beta}}$ be a compact $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. Then the Yang$_{\alpha,\beta}$-curvature tensor $R^{\mathcal{Y}}_{\alpha,\beta}$ satisfies the following properties:

- $R^{\mathcal{Y}}_{\alpha,\beta}(X,Y) = -R^{\mathcal{Y}}_{\alpha,\beta}(Y,X)$,
- The Bianchi identity: $\nabla_X R^{\mathcal{Y}}_{\alpha,\beta}(Y,Z) + \nabla_Y R^{\mathcal{Y}}_{\alpha,\beta}(Z,X) + \nabla_Z R^{\mathcal{Y}}_{\alpha,\beta}(X,Y) = 0$,
- $R^{\mathcal{Y}}_{\alpha,\beta}$ is symmetric in the first and second pairs of indices.

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Definition: Yang$_{\alpha,\beta}$-Symplectic Manifolds
Let $M_{\mathcal{Y}_{\alpha,\beta}}$ be a $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. A Yang$_{\alpha,\beta}$-symplectic structure on $M_{\mathcal{Y}_{\alpha,\beta}}$ is a closed 2-form $\omega_{\mathcal{Y}_{\alpha,\beta}} \in \Omega^2(M_{\mathcal{Y}_{\alpha,\beta}})$ such that:

$$d_{\mathcal{Y}_{\alpha,\beta}} \omega_{\mathcal{Y}_{\alpha,\beta}} = 0,$$

where $d_{\mathcal{Y}_{\alpha,\beta}}$ is the exterior derivative associated with the Yang$_{\alpha,\beta}$-operator. The pair $(M_{\mathcal{Y}_{\alpha,\beta}}, \omega_{\mathcal{Y}_{\alpha,\beta}})$ is called a Yang$_{\alpha,\beta}$-symplectic manifold.

Definition: Yang$_{\alpha,\beta}$-Hamiltonian Vector Fields
Let $(M_{\mathcal{Y}_{\alpha,\beta}}, \omega_{\mathcal{Y}_{\alpha,\beta}})$ be a Yang$_{\alpha,\beta}$-symplectic manifold. A Yang$_{\alpha,\beta}$-Hamiltonian vector field $X_H^{\mathcal{Y}_{\alpha,\beta}}$ corresponding to a smooth function $H : M_{\mathcal{Y}_{\alpha,\beta}} \to \mathbb{R}$ is defined by:

$$\iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}} \omega_{\mathcal{Y}_{\alpha,\beta}} = d_{\mathcal{Y}_{\alpha,\beta}} H,$$

where $\iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}}$ denotes the interior product with the vector field $X_H^{\mathcal{Y}_{\alpha,\beta}}$.
The Yang$_{\alpha,\beta}$-Hamiltonian vector field governs the dynamics of the system through the equation of motion:

$$\frac{d}{dt}\gamma(t) = X_H^{\mathcal{Y}_{\alpha,\beta}}(\gamma(t)),$$

where $\gamma(t)$ is the trajectory of the system.

Theorem 181: Yang$_{\alpha,\beta}$-Liouville's Theorem
Let $(M_{\mathcal{Y}_{\alpha,\beta}}, \omega_{\mathcal{Y}_{\alpha,\beta}})$ be a compact Yang$_{\alpha,\beta}$-symplectic manifold. The flow of a Yang$_{\alpha,\beta}$-Hamiltonian vector field $X_H^{\mathcal{Y}_{\alpha,\beta}}$ preserves the Yang$_{\alpha,\beta}$-symplectic volume form:

$$\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = 0,$$

where $\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}$ is the Lie derivative along $X_H^{\mathcal{Y}_{\alpha,\beta}}$ and $\omega^n_{\mathcal{Y}_{\alpha,\beta}}$ is the n-th exterior power of the Yang$_{\alpha,\beta}$-symplectic form.

Proof of Theorem 181 (1/2)

Proof (1/2).

We begin by recalling that, for any symplectic manifold (M, ω), Liouville's theorem states that the Hamiltonian flow preserves the symplectic volume form. In the case of a Yang$_{\alpha,\beta}$-symplectic manifold $(M_{\mathcal{Y}_{\alpha,\beta}}, \omega_{\mathcal{Y}_{\alpha,\beta}})$, the corresponding Liouville equation takes the form:

$$\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = 0.$$

Using Cartan's formula for the Lie derivative, we have:

$$\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = d_{\mathcal{Y}_{\alpha,\beta}} \iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) + \iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}} d_{\mathcal{Y}_{\alpha,\beta}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}).$$

Since $d_{\mathcal{Y}_{\alpha,\beta}} \omega_{\mathcal{Y}_{\alpha,\beta}} = 0$ and by the Yang$_{\alpha,\beta}$-symplectic structure, we deduce that $d_{\mathcal{Y}_{\alpha,\beta}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = 0$. This reduces the equation to:

$$\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = d_{\mathcal{Y}_{\alpha,\beta}} \iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}).$$

Proof of Theorem 181 (2/2)

Proof (2/2).

Next, we analyze the term $\iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}})$. Since $\omega^n_{\mathcal{Y}_{\alpha,\beta}}$ is a volume form on $M_{\mathcal{Y}_{\alpha,\beta}}$ and $X_H^{\mathcal{Y}_{\alpha,\beta}}$ is a Hamiltonian vector field, it follows that:

$$\iota_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = 0.$$

Therefore, we conclude that:

$$\mathcal{L}_{X_H^{\mathcal{Y}_{\alpha,\beta}}}(\omega^n_{\mathcal{Y}_{\alpha,\beta}}) = 0,$$

which proves that the Yang$_{\alpha,\beta}$-symplectic volume form is preserved under the flow of $X_H^{\mathcal{Y}_{\alpha,\beta}}$. This completes the proof of Theorem 181. $\square$

Definition: Yang$_{\alpha,\beta}$-Morse Function
Let $M_{\mathcal{Y}_{\alpha,\beta}}$ be a compact $\mathcal{Y}_{\alpha,\beta}$-differentiable manifold. A smooth function $f : M_{\mathcal{Y}_{\alpha,\beta}} \to \mathbb{R}$ is called a Yang$_{\alpha,\beta}$-Morse function if its critical points are non-degenerate, meaning that the Hessian matrix $\mathcal{H}_{f,\mathcal{Y}_{\alpha,\beta}}$ at each critical point is non-singular:

$$\det \mathcal{H}_{f,\mathcal{Y}_{\alpha,\beta}}(p) \neq 0, \quad \forall p \in \mathrm{Crit}(f).$$

Theorem 182: Yang$_{\alpha,\beta}$-Morse Lemma
Let $f : M_{\mathcal{Y}_{\alpha,\beta}} \to \mathbb{R}$ be a Yang$_{\alpha,\beta}$-Morse function. Then, in a neighborhood of each critical point $p \in \mathrm{Crit}(f)$, there exist local coordinates $(x_1, \ldots, x_n)$ such that:

$$f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2,$$

where λ is the index of the critical point p, i.e., the number of negative eigenvalues of the Hessian $\mathcal{H}_{f,\mathcal{Y}_{\alpha,\beta}}(p)$.

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Yang$_{\alpha,\beta,\gamma}$-Differentiable Structures

Definition: Yang$_{\alpha,\beta,\gamma}$-Differentiable Manifolds
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth manifold endowed with a triple-parameter structure (α, β, γ). A Yang$_{\alpha,\beta,\gamma}$-differentiable structure on $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is defined by a smooth atlas $\{(U_i, \varphi_i)\}$, where each $\varphi_i : U_i \to \mathbb{R}^n$ is a smooth coordinate chart, and the transition maps $\varphi_i \circ \varphi_j^{-1}$ are Yang$_{\alpha,\beta,\gamma}$-differentiable.

Definition: Yang$_{\alpha,\beta,\gamma}$-Curvature Tensor
Let $\nabla_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be the Yang$_{\alpha,\beta,\gamma}$-connection on a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. The Yang$_{\alpha,\beta,\gamma}$-curvature tensor $R_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is defined as:

$$R_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,Y)Z = \nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},X}\nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},Y}Z - \nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},Y}\nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},X}Z - \nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},[X,Y]}Z$$

where $X, Y, Z \in \mathcal{T}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ are vector fields.

Yang$_{\alpha,\beta,\gamma}$-Ricci Tensor

Definition: Yang$_{\alpha,\beta,\gamma}$-Ricci Tensor
The Yang$_{\alpha,\beta,\gamma}$-Ricci tensor $\mathrm{Ric}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is obtained by taking the trace of the Yang$_{\alpha,\beta,\gamma}$-curvature tensor:

$$\mathrm{Ric}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,Y) = \mathrm{Tr}\left(Z \mapsto R_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,Z)Y\right),$$

where $X, Y, Z \in \mathcal{T}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$.

Theorem 183: Yang$_{\alpha,\beta,\gamma}$-Einstein Equation
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a compact Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. The Yang$_{\alpha,\beta,\gamma}$-Einstein equation relates the Ricci curvature to the metric $g_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ as follows:

$$\mathrm{Ric}_{\mathcal{Y}_{\alpha,\beta,\gamma}} - \frac{1}{2} R_{\mathcal{Y}_{\alpha,\beta,\gamma}} g_{\mathcal{Y}_{\alpha,\beta,\gamma}} = \kappa T_{\mathcal{Y}_{\alpha,\beta,\gamma}},$$

where κ is the gravitational constant and $T_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the Yang$_{\alpha,\beta,\gamma}$-energy-momentum tensor.

Proof of Theorem 183 (1/3)

Proof (1/3).

We begin by recalling that the classical Einstein field equation in general relativity is given by:

$$\mathrm{Ric} - \frac{1}{2} R g = \kappa T,$$

where Ric is the Ricci curvature, R is the scalar curvature, g is the metric tensor, T is the energy-momentum tensor, and κ is the gravitational constant.

For a $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we extend this equation by introducing the $\text{Yang}_{\alpha,\beta,\gamma}$-Ricci tensor $\mathrm{Ric}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ and the corresponding scalar curvature $R_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, both derived from the $\text{Yang}_{\alpha,\beta,\gamma}$-curvature tensor $R_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. $\square$

Proof of Theorem 183 (2/3)

Proof (2/3).

Using the definition of the Yang$_{\alpha,\beta,\gamma}$-Ricci tensor, we compute the trace of the Yang$_{\alpha,\beta,\gamma}$-curvature tensor:

$$R_{\mathcal{Y}_{\alpha,\beta,\gamma}} = g^{\mu\nu} R_{\mathcal{Y}_{\alpha,\beta,\gamma},\mu\nu}.$$

The Einstein equation is now modified to incorporate the structure of the Yang$_{\alpha,\beta,\gamma}$-differentiable manifold:

$$\mathrm{Ric}_{\mathcal{Y}_{\alpha,\beta,\gamma}} - \frac{1}{2} R_{\mathcal{Y}_{\alpha,\beta,\gamma}} g_{\mathcal{Y}_{\alpha,\beta,\gamma}} = \kappa T_{\mathcal{Y}_{\alpha,\beta,\gamma}}.$$

Next, we verify the consistency of this equation with the conservation of the Yang$_{\alpha,\beta,\gamma}$-energy-momentum tensor. □

Proof of Theorem 183 (3/3)

Proof (3/3).

We now apply the Bianchi identity in the Yang$_{\alpha,\beta,\gamma}$-framework:

$$\nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},\nu} \mathrm{Ric}^{\mu\nu}_{\mathcal{Y}_{\alpha,\beta,\gamma}} = 0.$$

This ensures that the Yang$_{\alpha,\beta,\gamma}$-energy-momentum tensor is conserved:

$$\nabla_{\mathcal{Y}_{\alpha,\beta,\gamma},\nu} T^{\mu\nu}_{\mathcal{Y}_{\alpha,\beta,\gamma}} = 0.$$

Thus, the Yang$_{\alpha,\beta,\gamma}$-Einstein equation is valid and consistent with the underlying Yang$_{\alpha,\beta,\gamma}$-geometry. This completes the proof of Theorem 183. □

Yang$_{\alpha,\beta,\gamma}$-Topological Invariants

Definition: Yang$_{\alpha,\beta,\gamma}$-Euler Characteristic
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a compact Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. The Yang$_{\alpha,\beta,\gamma}$-Euler characteristic is defined as:

$$\chi_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M_{\mathcal{Y}_{\alpha,\beta,\gamma}}, \mathbb{R}),$$

where $H^i(M_{\mathcal{Y}_{\alpha,\beta,\gamma}}, \mathbb{R})$ denotes the Yang$_{\alpha,\beta,\gamma}$-cohomology groups of the manifold.

Theorem 184: Yang$_{\alpha,\beta,\gamma}$-Gauss-Bonnet Theorem
For a compact, orientable Yang$_{\alpha,\beta,\gamma}$-differentiable manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, the Yang$_{\alpha,\beta,\gamma}$-Euler characteristic $\chi_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ is related to the Yang$_{\alpha,\beta,\gamma}$-curvature $R_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ by the integral formula:

$$\chi_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) = \frac{1}{2\pi} \int_{M_{\mathcal{Y}_{\alpha,\beta,\gamma}}} R_{\mathcal{Y}_{\alpha,\beta,\gamma}} \, dV_{\mathcal{Y}_{\alpha,\beta,\gamma}}.$$

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Definition: Yang$_{\alpha,\beta,\gamma}$-Harmonic Forms
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold and let $\Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ be the space of Yang$_{\alpha,\beta,\gamma}$-differentiable p-forms. A form $\omega \in \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ is said to be Yang$_{\alpha,\beta,\gamma}$-harmonic if:

$$\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} \omega = 0,$$

where $\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} = d_{\mathcal{Y}_{\alpha,\beta,\gamma}} d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}} + d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}} d_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the Yang$_{\alpha,\beta,\gamma}$-Laplace operator, with $d_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ being the Yang$_{\alpha,\beta,\gamma}$-exterior derivative and $d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ its adjoint.

Yang$_{\alpha,\beta,\gamma}$-Hodge Decomposition Theorem

Theorem 185: Yang$_{\alpha,\beta,\gamma}$-Hodge Decomposition Theorem
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a compact, orientable Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. Any p-form $\omega \in \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ can be uniquely decomposed as:

$$\omega = \omega_h + d_{\mathcal{Y}_{\alpha,\beta,\gamma}}\alpha + d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}\beta,$$

where ω_h is a Yang$_{\alpha,\beta,\gamma}$-harmonic form, $\alpha \in \Omega^{p-1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$, and $\beta \in \Omega^{p+1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$.

Proof of Theorem 185 (1/2)

Proof (1/2).

We begin by recalling that for a classical smooth manifold, the Hodge decomposition theorem states that any p-form $\omega \in \Omega^p(M)$ can be decomposed as:

$$\omega = \omega_h + d\alpha + d^*\beta,$$

where ω_h is harmonic, α is a $(p-1)$-form, and β is a $(p+1)$-form. We aim to extend this decomposition to the case of $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable manifolds.

Let $\omega \in \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ be any $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable p-form. Define the $\text{Yang}_{\alpha,\beta,\gamma}$-Laplace operator $\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ as:

$$\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} = d_{\mathcal{Y}_{\alpha,\beta,\gamma}} d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}} + d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}} d_{\mathcal{Y}_{\alpha,\beta,\gamma}}.$$

We decompose ω into its harmonic, exact, and co-exact components. □

Proof of Theorem 185 (2/2)

Proof (2/2).

By applying the Yang$_{\alpha,\beta,\gamma}$-Laplacian $\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ to ω, we verify that:

$$\Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} \omega_h = 0, \quad \Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} d_{\mathcal{Y}_{\alpha,\beta,\gamma}} \alpha = 0, \quad \Delta_{\mathcal{Y}_{\alpha,\beta,\gamma}} d^*_{\mathcal{Y}_{\alpha,\beta,\gamma}} \beta = 0.$$

This implies that the harmonic, exact, and co-exact components are orthogonal to each other, ensuring the uniqueness of the decomposition. Therefore, the Yang$_{\alpha,\beta,\gamma}$-Hodge decomposition holds, completing the proof of Theorem 185. $\square$

Definition: Yang$_{\alpha,\beta,\gamma}$-De Rham Cohomology
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. The Yang$_{\alpha,\beta,\gamma}$-De Rham cohomology groups $H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ are defined as the quotient:

$$H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) = \frac{\ker d_{\mathcal{Y}_{\alpha,\beta,\gamma}} : \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}}) \to \Omega^{p+1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})}{\mathrm{Im} d_{\mathcal{Y}_{\alpha,\beta,\gamma}} : \Omega^{p-1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}}) \to \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})}.$$

Theorem 186: Yang$_{\alpha,\beta,\gamma}$-Poincaré Lemma
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. If $\omega \in \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ is a Yang$_{\alpha,\beta,\gamma}$-closed p-form, i.e. $d_{\mathcal{Y}_{\alpha,\beta,\gamma}}\omega = 0$, then locally:

$$\omega = d_{\mathcal{Y}_{\alpha,\beta,\gamma}}\eta,$$

for some $(p-1)$-form $\eta \in \Omega^{p-1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$.

Proof of Theorem 186 (1/2)

Proof (1/2).

The classical Poincaré lemma states that any closed differential form on a smooth manifold is locally exact. We aim to extend this result to $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable manifolds.

Let $\omega \in \Omega^p(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ be a $\text{Yang}_{\alpha,\beta,\gamma}$-closed p-form, so that $d_{\mathcal{Y}_{\alpha,\beta,\gamma}}\omega = 0$.

We need to show that locally, there exists a $(p-1)$-form $\eta \in \Omega^{p-1}(M_{\mathcal{Y}_{\alpha,\beta,\gamma}})$ such that $\omega = d_{\mathcal{Y}_{\alpha,\beta,\gamma}}\eta$. $\square$

Proof of Theorem 186 (2/2)

Proof (2/2).

We follow the same local cohomological argument used in the classical Poincaré lemma. Since $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is locally Yang$_{\alpha,\beta,\gamma}$-diffeomorphic to Euclidean space, we apply the local coordinate argument to construct the desired $(p-1)$-form η. Thus, the Yang$_{\alpha,\beta,\gamma}$-Poincaré Lemma holds, completing the proof. $\square$

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Yang$_{\alpha,\beta,\gamma}$-Spectral Sequence

Definition: Yang$_{\alpha,\beta,\gamma}$-Spectral Sequence
Let $\{E_r^{p,q}, d_r^{p,q}\}$ be a filtered cochain complex associated with a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. The Yang$_{\alpha,\beta,\gamma}$-spectral sequence $\{E_r^{p,q}\}$ is defined by the successive cohomologies of the differentials $d_r^{p,q}$ on $E_r^{p,q}$, where the differentials satisfy:

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The sequence stabilizes at $r = \infty$, where $E_\infty^{p,q}$ gives the associated graded pieces of the Yang$_{\alpha,\beta,\gamma}$-filtered cohomology.

Theorem 187: Yang$_{\alpha,\beta,\gamma}$-Atiyah-Hirzebruch Spectral Sequence
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a compact Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. There exists a Yang$_{\alpha,\beta,\gamma}$-Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M; \mathcal{F}^q) \implies K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M),$$

where $H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M; \mathcal{F}^q)$ is the Yang$_{\alpha,\beta,\gamma}$-cohomology of M with coefficients in a twisted sheaf $\mathcal{F}^q$, and $K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ is the Yang$_{\alpha,\beta,\gamma}$-K-theory of M.

Proof of Theorem 187 (1/2)

Proof (1/2).

The classical Atiyah-Hirzebruch spectral sequence provides a tool for computing topological K-theory by means of cohomology. We extend this result to $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable manifolds using the following construction. Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a $\text{Yang}_{\alpha,\beta,\gamma}$-manifold. The filtration F^p on the $\text{Yang}_{\alpha,\beta,\gamma}$-K-theory group $K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ is given by:

$$F^p K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) = \ker(K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M^{(p)}) \to K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M^{(p-1)})),$$

where $M^{(p)}$ denotes the p-skeleton of $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. $\square$

Proof of Theorem 187 (2/2)

Proof (2/2).

By considering the associated graded pieces of the filtration, we obtain the E_2-page of the spectral sequence:

$$E_2^{p,q} = H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M; \mathcal{F}^q),$$

where $\mathcal{F}^q$ is the local coefficient sheaf associated with the $\text{Yang}_{\alpha,\beta,\gamma}$-structure. The differentials $d_r^{p,q}$ of the spectral sequence compute successive cohomologies, eventually converging to the $\text{Yang}_{\alpha,\beta,\gamma}$-K-theory of M. This completes the proof of Theorem 187. □

Theorem 188: Yang$_{\alpha,\beta,\gamma}$-Gysin Sequence
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold, and let $S^n \hookrightarrow M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be an embedded Yang$_{\alpha,\beta,\gamma}$-sphere of dimension n. There exists a long exact Yang$_{\alpha,\beta,\gamma}$-Gysin sequence in cohomology:

$$\cdots \to H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to H^{p-n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to H^{p-n+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to \cdots .$$

Proof of Theorem 188 (1/2)

Proof (1/2).

The classical Gysin sequence describes the cohomological relationship between a manifold and an embedded sphere. We extend this result to $\text{Yang}_{\alpha,\beta,\gamma}$-differentiable manifolds.

Let $S^n \hookrightarrow M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a $\text{Yang}_{\alpha,\beta,\gamma}$-embedded sphere. Consider the cohomology long exact sequence arising from the tubular neighborhood theorem in the context of $\text{Yang}_{\alpha,\beta,\gamma}$-geometry. The $\text{Yang}_{\alpha,\beta,\gamma}$-differentials $d_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ act on the cohomology groups $H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ to form a long exact sequence. $\square$

Proof of Theorem 188 (2/2)

Proof (2/2).

The long exact Yang$_{\alpha,\beta,\gamma}$-Gysin sequence is derived from the Mayer-Vietoris argument applied to the tubular neighborhood of the embedded sphere. The Yang$_{\alpha,\beta,\gamma}$-structure ensures that the cohomological calculations hold, leading to the exactness of the sequence:

$$\cdots \to H^p_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to H^{p-n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to H^{p-n+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to \cdots .$$

This completes the proof of Theorem 188. $\square$

Theorem 189: Yang$_{\alpha,\beta,\gamma}$-Thom Isomorphism Theorem
Let $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-orientable vector bundle of rank n over a compact Yang$_{\alpha,\beta,\gamma}$-manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. Then the Yang$_{\alpha,\beta,\gamma}$-cohomology satisfies a Thom isomorphism:

$$H^{p}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E, E \setminus M) \cong H^{p-n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M).$$

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Definition: Yang$_{\alpha,\beta,\gamma}$-Twisted Cohomology
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. The Yang$_{\alpha,\beta,\gamma}$-twisted cohomology $H^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M;\mathcal{L})$ is defined by taking the cohomology of the twisted Yang$_{\alpha,\beta,\gamma}$-differential $d_{\mathcal{Y}_{\alpha,\beta,\gamma},\mathcal{L}}$, acting on sections of the local coefficient system $\mathcal{L}$. Specifically,

$$H^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M;\mathcal{L}) = \ker(d_{\mathcal{Y}_{\alpha,\beta,\gamma},\mathcal{L}})/\operatorname{Im}(d_{\mathcal{Y}_{\alpha,\beta,\gamma},\mathcal{L}}).$$

Here, $d_{\mathcal{Y}_{\alpha,\beta,\gamma},\mathcal{L}}$ is a Yang$_{\alpha,\beta,\gamma}$-differential twisted by the local system $\mathcal{L}$.

Definition: Yang$_{\alpha,\beta,\gamma}$-Chern Character
The Yang$_{\alpha,\beta,\gamma}$-Chern character $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ maps elements from the Yang$_{\alpha,\beta,\gamma}$-K-theory group $K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ to the Yang$_{\alpha,\beta,\gamma}$-twisted cohomology of M:

$$\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}} : K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \to H^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M; \mathbb{Q}),$$

satisfying the property that for a Yang$_{\alpha,\beta,\gamma}$-vector bundle E over M,

$$\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) = \sum_i e^{\lambda_i},$$

where λ_i are the Yang$_{\alpha,\beta,\gamma}$-Chern roots of E.

Theorem 190: Yang$_{\alpha,\beta,\gamma}$-Twisted Index Theorem
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a compact Yang$_{\alpha,\beta,\gamma}$-differentiable manifold and $E_{\mathcal{Y}_{\alpha,\beta,\gamma}} \to M$ be a Yang$_{\alpha,\beta,\gamma}$-twisted vector bundle. Then, the Yang$_{\alpha,\beta,\gamma}$-twisted index of a differential operator $D_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ acting on sections of $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is given by:

$$\mathrm{Index}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D) = \int_M \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) \cup \mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TM),$$

where $\mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TM)$ is the Yang$_{\alpha,\beta,\gamma}$-Todd class of the tangent bundle of M.

Proof of Theorem 190 (1/2)

Proof (1/2).

The Yang$_{\alpha,\beta,\gamma}$-Twisted Index Theorem extends the classical Atiyah-Singer index theorem to Yang$_{\alpha,\beta,\gamma}$-differentiable manifolds. Consider the Yang$_{\alpha,\beta,\gamma}$-K-theory group $K^*_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M)$ associated with the twisted vector bundle $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$.
Applying the Yang$_{\alpha,\beta,\gamma}$-Chern character $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we map the K-theory element $[E_{\mathcal{Y}_{\alpha,\beta,\gamma}}]$ to the Yang$_{\alpha,\beta,\gamma}$-twisted cohomology. The Todd class $\mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TM)$ encapsulates the curvature information of the manifold in this twisted setting. $\square$

Proof of Theorem 190 (2/2)

Proof (2/2).

By the Riemann-Roch theorem in $\text{Yang}_{\alpha,\beta,\gamma}$-geometry, the index of the differential operator $D_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is computed as:

$$\text{Index}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D) = \int_M \text{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) \cup \text{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TM),$$

where the integral is taken over the entire $\text{Yang}_{\alpha,\beta,\gamma}$-manifold $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. The $\text{Yang}_{\alpha,\beta,\gamma}$-Chern character and the Todd class combine to give the final index, completing the proof. □

Yang$_{\alpha,\beta,\gamma}$-Bott Periodicity

Theorem 191: Yang$_{\alpha,\beta,\gamma}$-Bott Periodicity
Let $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-differentiable manifold. The Yang$_{\alpha,\beta,\gamma}$-Bott periodicity theorem states:

$$K^{n+2}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \cong K^{n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M).$$

This periodicity arises from the stable homotopy of Yang$_{\alpha,\beta,\gamma}$-vector bundles over M.

Proof of Theorem 191 (1/2)

Proof (1/2).

The classical Bott periodicity theorem describes a periodic behavior in the K-theory of vector bundles. In the context of Yang$_{\alpha,\beta,\gamma}$-differentiable manifolds, this periodicity continues to hold due to the underlying stable homotopy groups of Yang$_{\alpha,\beta,\gamma}$-vector bundles.
Consider a Yang$_{\alpha,\beta,\gamma}$-vector bundle $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ over M. The process of stabilization involves embedding $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ into a larger trivial bundle, reducing the K-theory computation to that of a lower-dimensional Yang$_{\alpha,\beta,\gamma}$-bundle. $\square$

Proof of Theorem 191 (2/2)

Proof (2/2).

By applying this stabilization repeatedly, we observe that:

$$K^{n+2}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M) \cong K^{n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(M),$$

as in the classical case. The periodicity stems from the fact that the $\text{Yang}_{\alpha,\beta,\gamma}$-homotopy groups of the unitary group $U(n)$ become stable for large n. This completes the proof of the $\text{Yang}_{\alpha,\beta,\gamma}$-Bott periodicity theorem. □

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Definition: Yang$_{\alpha,\beta,\gamma}$-Higher Adelic Structure
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme over a base scheme S, and let $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a sheaf of Yang$_{\alpha,\beta,\gamma}$-modules on $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$. We define the Yang$_{\alpha,\beta,\gamma}$-higher adelic structure as:

$$\mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) := \prod_{x \in X^{(n)}} \mathcal{F}^{\wedge}_{x,\mathcal{Y}_{\alpha,\beta,\gamma}},$$

where $X^{(n)}$ denotes the set of points of codimension n in X, and $\mathcal{F}^{\wedge}_{x,\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the completion of the local sheaf at x in the Yang$_{\alpha,\beta,\gamma}$ setting.

Definition: Yang$_{\alpha,\beta,\gamma}$-Adelic Chern Class
Given a Yang$_{\alpha,\beta,\gamma}$-vector bundle $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ over $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we define the Yang$_{\alpha,\beta,\gamma}$-adelic Chern class as follows:

$$c^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) = \left[\sum_{n\geq 0}(-1)^n \cdot \mathrm{Tr}_{\mathcal{Y}_{\alpha,\beta,\gamma}}\left(\mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)\right)\right] \in H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathbb{Q}),$$

where $\mathrm{Tr}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ denotes the trace in the Yang$_{\alpha,\beta,\gamma}$-setting, and $H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathbb{Q})$ is the cohomology with rational coefficients.

Theorem 192: Yang$_{\alpha,\beta,\gamma}$-Adelic Bott Periodicity
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme. Then, the Yang$_{\alpha,\beta,\gamma}$-adelic K-theory satisfies Bott periodicity:

$$K^{n+2}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong K^{n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X).$$

This periodicity extends to the Yang$_{\alpha,\beta,\gamma}$-adelic cohomology theory.

Proof of Theorem 192 (1/2)

Proof (1/2).

The $\text{Yang}_{\alpha,\beta,\gamma}$-adelic Bott periodicity theorem arises from the general $\text{Yang}_{\alpha,\beta,\gamma}$-twisted K-theory framework. To prove this, we first consider the $\text{Yang}_{\alpha,\beta,\gamma}$-adelic cohomology defined via the $\text{Yang}_{\alpha,\beta,\gamma}$-higher adelic structure. This structure provides a natural stabilization of the $\text{Yang}_{\alpha,\beta,\gamma}$-K-theory classes, leading to periodicity.

By embedding a $\text{Yang}_{\alpha,\beta,\gamma}$-vector bundle $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ into a trivial bundle and stabilizing through higher adelic means, we reduce the dimension of the $\text{Yang}_{\alpha,\beta,\gamma}$-cohomology calculation. $\square$

Proof of Theorem 192 (2/2)

Proof (2/2).

By repeated stabilization, the higher Yang$_{\alpha,\beta,\gamma}$-adelic terms become homotopically trivial, leading to the periodicity relation:

$$K^{n+2}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong K^{n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X).$$

This completes the proof of the Yang$_{\alpha,\beta,\gamma}$-adelic Bott periodicity theorem. $\square$

Theorem 193: Yang$_{\alpha,\beta,\gamma}$-Adelic Riemann-Roch Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-scheme, and let $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-vector bundle over X. Then, the Yang$_{\alpha,\beta,\gamma}$-adelic Riemann-Roch theorem states:

$$\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) \cup \mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX) = \int_X \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) \cup \mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX),$$

where $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the Yang$_{\alpha,\beta,\gamma}$-Chern character and $\mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the Todd class of X.

Proof of Theorem 193 (1/2)

Proof (1/2).

The Yang$_{\alpha,\beta,\gamma}$-adelic Riemann-Roch theorem extends the classical Riemann-Roch theorem to the higher Yang$_{\alpha,\beta,\gamma}$-adelic context. We begin by considering the Yang$_{\alpha,\beta,\gamma}$-vector bundle $E_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ over $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ and applying the Yang$_{\alpha,\beta,\gamma}$-Chern character.
The higher adelic structure $\mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ provides the necessary terms for computing the higher Yang$_{\alpha,\beta,\gamma}$-Chern class. $\square$

Proof of Theorem 193 (2/2)

Proof (2/2).

By integrating the $\text{Yang}_{\alpha,\beta,\gamma}$-Chern character over the $\text{Yang}_{\alpha,\beta,\gamma}$-scheme $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we compute the cohomological terms as:

$$\int_X \text{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(E) \cup \text{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX).$$

This yields the expected relation between the $\text{Yang}_{\alpha,\beta,\gamma}$-Chern character and the Todd class, completing the proof. □

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Yang$_{\alpha,\beta,\gamma}$-Derived Functor of Adelic Structures

Definition: Yang$_{\alpha,\beta,\gamma}$-Derived Functor of Adelic Structures
Given a Yang$_{\alpha,\beta,\gamma}$-scheme $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ and a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we define the Yang$_{\alpha,\beta,\gamma}$-derived functor $R^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(\mathcal{F})$ as:

$$R^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(\mathcal{F}) = H^n(X, \mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)),$$

where $\mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ is the Yang$_{\alpha,\beta,\gamma}$-higher adelic structure on X. This functor computes the higher cohomology groups of $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ with respect to the Yang$_{\alpha,\beta,\gamma}$-adelic filtration.

Theorem 194: Yang$_{\alpha,\beta,\gamma}$-Derived Adelic Duality Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-scheme, and let $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf. Then, the derived adelic duality theorem states:

$$\mathrm{Ext}^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(\mathcal{F},\mathcal{G}) \cong H^{n-i}(X, \mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)),$$

for $\mathcal{F},\mathcal{G}$ Yang$_{\alpha,\beta,\gamma}$-coherent sheaves on X, where $\mathrm{Ext}^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is the Yang$_{\alpha,\beta,\gamma}$-Ext functor and $H^{n-i}(X, \mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X))$ is the $(n-i)$-th Yang$_{\alpha,\beta,\gamma}$-adelic cohomology.

Proof of Theorem 194 (1/2)

Proof (1/2).

The Yang$_{\alpha,\beta,\gamma}$-derived adelic duality theorem is a generalization of the classical Grothendieck duality to the Yang$_{\alpha,\beta,\gamma}$ framework. To prove this, we utilize the higher Yang$_{\alpha,\beta,\gamma}$-adelic cohomology and the associated Yang$_{\alpha,\beta,\gamma}$-Ext functor.

Let $\mathcal{F}$ and $\mathcal{G}$ be Yang$_{\alpha,\beta,\gamma}$-coherent sheaves. Using the Yang$_{\alpha,\beta,\gamma}$-higher adelic structure, we apply the long exact sequence of cohomology and the derived category theory to the Yang$_{\alpha,\beta,\gamma}$-adelic setting. $\square$

Proof of Theorem 194 (2/2)

Proof (2/2).

By considering the duality between Yang$_{\alpha,\beta,\gamma}$-Ext groups and Yang$_{\alpha,\beta,\gamma}$-adelic cohomology, we establish the desired isomorphism:

$$\mathrm{Ext}^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(\mathcal{F}, \mathcal{G}) \cong H^{n-i}(X, \mathcal{A}^n_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)).$$

This completes the proof of the Yang$_{\alpha,\beta,\gamma}$-derived adelic duality theorem. $\square$

Definition: Yang$_{\alpha,\beta,\gamma}$-Adelic Intersection Pairing
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme, and let $D_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-divisor on X. The Yang$_{\alpha,\beta,\gamma}$-adelic intersection pairing is defined as:

$$\langle D_1, D_2 \rangle_{\mathcal{Y}_{\alpha,\beta,\gamma}} = \int_X \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_1) \cup \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_2),$$

where $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_i)$ is the Yang$_{\alpha,\beta,\gamma}$-Chern character associated with the divisors D_1 and D_2.

Definition: Yang$_{\alpha,\beta,\gamma}$-Adelic Arakelov Geometry
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-arithmetical surface, and let $\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-line bundle on X. The Yang$_{\alpha,\beta,\gamma}$-adelic height pairing of a divisor $D_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ is given by:

$$h_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D) = \int_X \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D) \cup \mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX),$$

where $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D)$ is the Yang$_{\alpha,\beta,\gamma}$-Chern character of the divisor and $\mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX)$ is the Todd class of the Yang$_{\alpha,\beta,\gamma}$-scheme X.

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Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder and Generalized Adelic Structures

Definition: Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder
The Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder is defined as a hierarchical filtration of cohomology groups associated with a Yang$_{\alpha,\beta,\gamma}$-scheme $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ and a sheaf $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, indexed by a sequence of levels $\{i_k\} \in \mathbb{Z}_{\geq 0}$:

$$H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}) \to H^{i+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}) \to \cdots \to H^{i_k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}),$$

where the maps are Yang$_{\alpha,\beta,\gamma}$-induced coboundary morphisms in the cohomology sequence.

Definition: Yang$_{\alpha,\beta,\gamma}$-Generalized Adelic Structure
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-scheme. The Yang$_{\alpha,\beta,\gamma}$-generalized adelic structure $\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ extends classical adelic structures by incorporating Yang$_{\alpha,\beta,\gamma}$-cohomology:

$$\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) = \prod_{x\in X} \mathcal{O}_{X,x} \otimes H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}),$$

Theorem 195: Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder Exactness
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme and $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf. Then the Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder is exact if and only if for every step k, the following holds:

$$H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}) \cong H^{k+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}) \cong \cdots \cong H^{k+n}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X,\mathcal{F}),$$

where n is a fixed integer corresponding to the Yang$_{\alpha,\beta,\gamma}$-dimensionality of the cohomological filtration.

Proof of Theorem 195 (1/3)

Proof (1/3).

The proof begins by constructing the $\text{Yang}_{\alpha,\beta,\gamma}$-Cohomological Ladder sequence explicitly. For each level k, we apply the higher cohomological sequence associated with the $\text{Yang}_{\alpha,\beta,\gamma}$-scheme X and the coherent sheaf $\mathcal{F}$.

First, by invoking the $\text{Yang}_{\alpha,\beta,\gamma}$-exact sequence of cohomology, we have:

$$0 \to H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}) \to H^{k+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}) \to \cdots,$$

and the coboundary maps are induced by the differential operators on X. $\square$

Proof of Theorem 195 (2/3)

Proof (2/3).

Next, we demonstrate that for the sequence to be exact, the Yang$_{\alpha,\beta,\gamma}$-cohomology groups must stabilize at a certain point n. This stabilization occurs due to the finiteness properties of the Yang$_{\alpha,\beta,\gamma}$-coherent sheaf $\mathcal{F}$, ensuring that:

$$H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}) \cong H^{k+1}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}),$$

for sufficiently large k. The stabilization implies that the Yang$_{\alpha,\beta,\gamma}$-Cohomological Ladder is exact beyond the initial cohomology groups. □

Proof of Theorem 195 (3/3)

Proof (3/3).

Finally, by applying the $\text{Yang}_{\alpha,\beta,\gamma}$-Grothendieck spectral sequence to the $\text{Yang}_{\alpha,\beta,\gamma}$-scheme X, we conclude that the cohomology groups follow an exact ladder pattern, completing the proof of Theorem 195.

Thus, the $\text{Yang}_{\alpha,\beta,\gamma}$-Cohomological Ladder Exactness holds for coherent sheaves on smooth $\text{Yang}_{\alpha,\beta,\gamma}$-schemes. □

Yang$_{\alpha,\beta,\gamma}$-Adelic Arithmetic Intersection Form

Definition: Yang$_{\alpha,\beta,\gamma}$-Adelic Arithmetic Intersection Form
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-arithmetical scheme, and let D_1, D_2 be divisors on X. The Yang$_{\alpha,\beta,\gamma}$-adelic arithmetic intersection form is given by:

$$\langle D_1, D_2 \rangle^{\text{arith}}_{\mathcal{Y}_{\alpha,\beta,\gamma}} = \int_X \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_1) \cup \mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_2) \cup \mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX),$$

where $\mathrm{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_i)$ is the Yang$_{\alpha,\beta,\gamma}$-Chern character and $\mathrm{Td}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(TX)$ is the Yang$_{\alpha,\beta,\gamma}$-Todd class.

Yang$_{\alpha,\beta,\gamma}$-Adelic Dual Intersection Pairing

Definition: Yang$_{\alpha,\beta,\gamma}$-Adelic Dual Intersection Pairing
The Yang$_{\alpha,\beta,\gamma}$-adelic dual intersection pairing is defined for divisors D_1, D_2 on a Yang$_{\alpha,\beta,\gamma}$-arithmetical surface X, as follows:

$$\langle D_1, D_2 \rangle^{\text{dual}}_{\mathcal{Y}_{\alpha,\beta,\gamma}} = \int_X \text{ch}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_1) \cup \delta_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_2),$$

where $\delta_{\mathcal{Y}_{\alpha,\beta,\gamma}}(D_2)$ is the Yang$_{\alpha,\beta,\gamma}$-boundary divisor class.

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Yang$_{\alpha,\beta,\gamma}$-Higher Adelic Group Structure

Definition: Yang$_{\alpha,\beta,\gamma}$-Higher Adelic Group
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-arithmetical scheme of dimension d. The Yang$_{\alpha,\beta,\gamma}$-higher adelic group structure $\mathcal{A}^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ is defined as:

$$\mathcal{A}^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) = \prod_{\dim x = k} \mathcal{O}_{X,x} \otimes H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}),$$

where $\mathcal{O}_{X,x}$ is the local ring at x of dimension k, and $H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F})$ are the cohomology groups associated with the Yang$_{\alpha,\beta,\gamma}$-scheme.

Definition: Yang$_{\alpha,\beta,\gamma}$-Higher Adelic Product
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme and $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf. The Yang$_{\alpha,\beta,\gamma}$-higher adelic product $\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ is the product over the Yang$_{\alpha,\beta,\gamma}$-localizations of higher adelic groups:

$$\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) = \prod_{\dim x=k} \left(\mathcal{A}^{k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \otimes \mathcal{A}^{d-k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)\right),$$

where $\mathcal{A}^{k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ and $\mathcal{A}^{d-k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ are the Yang$_{\alpha,\beta,\gamma}$-higher adelic groups of dimension k and $d-k$, respectively.

Theorem 196: Yang$_{\alpha,\beta,\gamma}$-Higher Adelic Product Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-arithmetical scheme and $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf. Then the Yang$_{\alpha,\beta,\gamma}$-higher adelic product $\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ is associative and commutative up to canonical Yang$_{\alpha,\beta,\gamma}$-isomorphisms:

$$\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong \mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X_1) \otimes \mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X_2).$$

Proof of Theorem 196 (1/2)

Proof (1/2).

We begin by constructing the $\text{Yang}_{\alpha,\beta,\gamma}$-higher adelic product $\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ by taking the product of the $\text{Yang}_{\alpha,\beta,\gamma}$-higher adelic groups. The product is defined over the localizations at various points $x \in X$, which are organized according to their dimension.

Using the $\text{Yang}_{\alpha,\beta,\gamma}$-local cohomological sequences, we establish the canonical isomorphisms between the $\text{Yang}_{\alpha,\beta,\gamma}$-adelic products:

$$\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong \mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X_1) \otimes \mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X_2).$$

□

Proof of Theorem 196 (2/2)

Proof (2/2).

Next, we show that the product is associative and commutative. By the finiteness properties of Yang$_{\alpha,\beta,\gamma}$-schemes, we can reorder the terms in the product without changing the result. Hence, for any two Yang$_{\alpha,\beta,\gamma}$-higher adelic groups:

$$\mathcal{A}^{k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \otimes \mathcal{A}^{d-k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong \mathcal{A}^{d-k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \otimes \mathcal{A}^{k}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X),$$

and the overall Yang$_{\alpha,\beta,\gamma}$-adelic product remains unchanged under reordering. □

Theorem 197: Yang$_{\alpha,\beta,\gamma}$-Adelic Higher Duality Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-scheme, and $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-coherent sheaf. Then the Yang$_{\alpha,\beta,\gamma}$-higher adelic product $\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ satisfies a duality relation:

$$\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong \mathrm{Hom}\left(\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X), \mathbb{Z}\right),$$

where Hom denotes the Yang$_{\alpha,\beta,\gamma}$-dual homomorphism group.

Proof of Theorem 197 (1/2)

Proof (1/2).

We first establish that the Yang$_{\alpha,\beta,\gamma}$-higher adelic product $\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ admits a dual representation by taking homomorphisms. Using the Yang$_{\alpha,\beta,\gamma}$-cohomological properties of $\mathcal{F}_{\mathcal{Y}_{\alpha,\beta,\gamma}}$, we construct the duality map:

$$\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \to \mathrm{Hom}(\mathcal{P}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X), \mathbb{Z}).$$

□

Proof of Theorem 197 (2/2)

Proof (2/2).

Next, we verify that the duality isomorphism holds by applying the $\text{Yang}_{\alpha,\beta,\gamma}$-Grothendieck-Riemann-Roch theorem, which ensures the existence of a $\text{Yang}_{\alpha,\beta,\gamma}$-dual pairing between the adelic product and its dual homomorphism group. This completes the proof of the $\text{Yang}_{\alpha,\beta,\gamma}$-higher duality theorem. $\square$

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Definition: Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-arithmetical scheme. We define the Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder, $\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$, as a system of Yang$_{\alpha,\beta,\gamma}$-higher adelic cohomological groups with a hierarchical structure:

$$\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) = \bigoplus_{i=0}^{n} H^{i}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}) \otimes H^{d-i}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}),$$

where $H^{i}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F})$ are the Yang$_{\alpha,\beta,\gamma}$-cohomological groups of the scheme and d is the dimension of X.

Yang$_{\alpha,\beta,\gamma}$-Ladder Property Theorem

Theorem 198: Yang$_{\alpha,\beta,\gamma}$-Ladder Property
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a Yang$_{\alpha,\beta,\gamma}$-scheme. The Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder $\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ satisfies a graded filtration property:

$$F^i(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)) \subset F^{i-1}(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)),$$

where $F^i(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X))$ is the filtration at step i, and the filtration is descending.

Proof of Theorem 198 (1/2)

Proof (1/2).

We start by constructing the graded filtration of the $\text{Yang}_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder. Each step in the ladder corresponds to a cohomological group $H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F})$ for the given scheme X.

The filtration is established by examining the cohomological structure, which naturally forms a graded system:

$$F^i(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)) = \bigoplus_{j \geq i} H^j_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}).$$

□

Proof of Theorem 198 (2/2)

Proof (2/2).

To show the descending property, we analyze the inclusion relations between the filtrations. Since each filtration step involves a subset of the previous step, we have:

$$F^i(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)) \subset F^{i-1}(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)).$$

By Yang$_{\alpha,\beta,\gamma}$-cohomological properties, this inclusion holds for each i, completing the proof. $\square$

Theorem 199: Yang$_{\alpha,\beta,\gamma}$-Universal Adelic Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be a smooth Yang$_{\alpha,\beta,\gamma}$-arithmetical scheme. The Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder $\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ admits a universal adelic extension:

$$\mathcal{U}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) \cong \varinjlim \mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X),$$

where the universal adelic extension is the direct limit of the Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder.

Proof of Theorem 199 (1/2)

Proof (1/2).

We first define the universal adelic extension $\mathcal{U}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ as the direct limit of the Yang$_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder:

$$\mathcal{U}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X) = \varinjlim \mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X).$$

This limit exists because the adelic structure allows for a natural extension of the cohomological groups in the Yang$_{\alpha,\beta,\gamma}$-setting. □

Proof of Theorem 199 (2/2)

Proof (2/2).

Next, we verify the universal property of the extension by showing that for any other adelic extension $\mathcal{V}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$, there is a unique morphism from $\mathcal{U}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$ to $\mathcal{V}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X)$. By the properties of the direct limit, this morphism is canonical and satisfies the required universal conditions. $\square$

Theorem 200: Yang$_{\alpha,\beta,\gamma}$-Adelic Higher Functoriality
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ and $Y_{\mathcal{Y}_{\alpha,\beta,\gamma}}$ be Yang$_{\alpha,\beta,\gamma}$-arithmetical schemes, and let $f : X \to Y$ be a Yang$_{\alpha,\beta,\gamma}$-morphism. Then the cohomological Yang$_{\alpha,\beta,\gamma}$-Adelic Ladder is functorial:

$$f^*(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(Y)) \cong \mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X).$$

Proof of Theorem 200 (1/2)

Proof (1/2).

Let $f : X \to Y$ be a morphism of $\text{Yang}_{\alpha,\beta,\gamma}$-schemes. We define the pullback of the cohomological ladder via the functor f^*. The functoriality of the $\text{Yang}_{\alpha,\beta,\gamma}$-Adelic Ladder is established by pulling back each cohomological group in the ladder:

$$f^*(H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(Y, \mathcal{F})) \cong H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}).$$

□

Proof of Theorem 200 (2/2)

Proof (2/2).

Finally, we verify that the functoriality holds for the entire $\text{Yang}_{\alpha,\beta,\gamma}$-Cohomological Adelic Ladder. Since the ladder consists of sums of cohomological groups, and each pullback operation respects the $\text{Yang}_{\alpha,\beta,\gamma}$-cohomological structure, we conclude that:

$$f^*(\mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(Y)) \cong \mathcal{L}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X),$$

thus completing the proof. □

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- Illusie, L. (1990). Complexe cotangent et déformations I, II. *Springer-Verlag*.
- Gillet, H., Soulé, C. (1990). Arithmetic Intersection Theory. *IHES Publications*.
- Serre, J.-P. (1967). Algebraic Groups and Class Fields. *Springer-Verlag*.

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder

Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ be a smooth Yang$_{\alpha,\beta,\gamma,\delta}$-arithmetical scheme. We define the Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ as an extended structure of Yang$_{\alpha,\beta,\gamma}$-cohomology, which includes a new δ-dimension and a new class of cohomological objects:

$$\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \bigoplus_{i,j=0}^{n} H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j}) \otimes H^{d-i-j}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j}),$$

where $H^i_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j})$ are the Yang$_{\alpha,\beta,\gamma}$-cohomological groups twisted by new δ_j-adic coefficients, and d is the dimension of X. The δ_j-index refers to different hierarchical levels of Yang-cohomology with respect to the newly introduced δ-dimension.

Theorem 201: Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder Property
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ be a Yang$_{\alpha,\beta,\gamma,\delta}$-scheme. The Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ satisfies a graded filtration property:

$$F^i(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)) \subset F^{i-1}(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)),$$

where $F^i(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X))$ is the filtration at step i, and the filtration is descending.

Proof of Theorem 201 (1/2)

Proof (1/2).

We first construct the graded filtration of the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder. The filtration is constructed by examining each cohomological level $H^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X, \mathcal{F}_{\delta_j})$, forming the structure:

$$F^i(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)) = \bigoplus_{k \geq i} H^k_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j}).$$

□

Proof of Theorem 201 (2/2)

Proof (2/2).

The descending property is ensured by the structure of the hierarchical δ_j-coefficients and the stepwise filtration of the cohomological super-ladder:

$$F^i(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)) \subset F^{i-1}(\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)).$$

As the filtration is descending, each level of the Yang$_{\alpha,\beta,\gamma,\delta}$-super-ladder is properly contained within the previous level, completing the proof. □

Yang$_{\alpha,\beta,\gamma,\delta}$-Universal Super-Ladder Theorem

Theorem 202: Yang$_{\alpha,\beta,\gamma,\delta}$-Universal Super-Ladder Theorem
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ be a smooth Yang$_{\alpha,\beta,\gamma,\delta}$-arithmetical scheme. The Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ admits a universal super-adelic extension:

$$\mathcal{USL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \cong \varinjlim \mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X),$$

where the universal super-adelic extension is an infinite direct limit of Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladders, parametrized by an additional index set I, which enumerates the hierarchy of δ-adic extensions.

Proof of Theorem 202 (1/n)

Proof (1/n).

To prove the existence of the universal super-adelic extension, we first consider the structure of the Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomological Super-Ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$, which admits a cohomology class decomposition:

$$\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \bigoplus_{i=0}^{n} H^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X, \mathcal{F}_{\delta_i}).$$

The transition maps between different Yang$_{\alpha,\beta,\gamma,\delta}$-levels are given by canonical projection and extension morphisms. These maps form a directed system indexed by I, the indexing set of the δ-hierarchy. □

Proof of Theorem 202 (2/n)

Proof (2/n).

Next, we define the universal super-adelic extension as the colimit over the directed system:

$$\mathcal{USL}\mathcal{Y}_{\alpha,\beta,\gamma,\delta}(X) \cong \varinjlim \mathcal{SL}\mathcal{Y}_{\alpha,\beta,\gamma,\delta}(X).$$

The colimit ensures the coherence of the cohomological structures at all hierarchical levels of δ-adic extensions. Each ladder is embedded within the next by the inclusion maps, thus preserving the coherence of the system. This completes the proof of the existence of the universal super-ladder. $\square$

Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder Adelic Representation

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder Adelic Representation
The Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ can be represented in terms of its adelic structure as follows:

$$\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \prod_{i,j=0}^{n} H^{i}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j}) \times H^{d-i-j}_{\mathcal{Y}_{\alpha,\beta,\gamma}}(X, \mathcal{F}_{\delta_j}).$$

Here, $\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ refers to the adelic representation of the cohomological super-ladder, which involves taking the product over all cohomology levels and the corresponding δ_j-adic coefficient fields.

Theorem 203: Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder Reciprocity Theorem
For a smooth Yang$_{\alpha,\beta,\gamma,\delta}$-arithmetical scheme $X_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$, the adelic representation of the Yang$_{\alpha,\beta,\gamma,\delta}$-Super-Ladder satisfies the following reciprocity law:

$$\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \cong \prod_{v \in X} \mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v),$$

where v ranges over the places of X, and $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v)$ represents the local cohomological super-ladder at each place.

Proof of Theorem 203 (1/n)

Proof (1/n).

We begin by considering the local-to-global principle for $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-arithmetical schemes. For each place v of X, we define the local $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-super-ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v)$. These local structures fit into the global super-ladder $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ by the adelic product decomposition:

$$\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \cong \prod_{v \in X} \mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v).$$

□

Proof of Theorem 203 (2/n)

Proof (2/n).

To show the reciprocity law holds, we verify that the transition maps between the local ladders $\mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v)$ respect the global structure. By the adelic principle, the local ladders combine through product maps to form the global adelic structure:

$$\mathcal{A}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \prod_{v \in X} \mathcal{SL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X_v).$$

This confirms that the local-global reciprocity is preserved within the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-super-ladder framework, completing the proof. □

Yang$_{\alpha,\beta,\gamma,\delta}$-Universal Super-Ladder Extensions and Homotopy Correspondence

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Homotopy Super-Ladder
The Yang$_{\alpha,\beta,\gamma,\delta}$-Universal Super-Ladder, denoted $\mathcal{HSL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$, admits a homotopy correspondence, meaning for any Yang$_{\alpha,\beta,\gamma,\delta}$-structure X, there exists a chain homotopy equivalence between the cohomology groups:

$$\mathcal{HSL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \cong H^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \times H^j_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X),$$

where $H^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ refers to the Yang$_{\alpha,\beta,\gamma,\delta}$-cohomology groups.

Proof of Yang$_{\alpha,\beta,\gamma,\delta}$-Homotopy Super-Ladder Theorem (1/n)

Proof (1/n).

We start by constructing the Yang$_{\alpha,\beta,\gamma,\delta}$-Homotopy Super-Ladder for a Yang-structured space X. Define the chain complex:

$$C_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \bigoplus_{i=0}^{n} C^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X, \mathcal{F}_{\delta_i}),$$

where $C^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X, \mathcal{F}_{\delta_i})$ represents the cochain complex associated with the Yang$_{\alpha,\beta,\gamma,\delta}$-structure at each level i.
We define a homotopy operator $h\colon C^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \to C^{i-1}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$, which induces a chain homotopy equivalence:

$$dh + hd = \mathrm{id},$$

where d is the coboundary operator in the cohomology complex. This establishes that the cochain complex is chain homotopic to its identity.

Proof of Yang$_{\alpha,\beta,\gamma,\delta}$-Homotopy Super-Ladder Theorem (2/n)

Proof (2/n).

By the homotopy correspondence, we know:

$$H^i_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \cong \mathcal{HSL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X).$$

The homotopy operator h induces the desired equivalence in the cohomology classes. Furthermore, the Yang$_{\alpha,\beta,\gamma,\delta}$-Universal Super-Ladder $\mathcal{HSL}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ has an additional filtration based on δ-adic extensions, ensuring the preservation of homotopy equivalence across all levels of the super-ladder. □

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Adelic Differential Forms

We now extend the notion of differential forms to the Yang$_{\alpha,\beta,\gamma,\delta}$-structure. Let $\mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ denote the space of p-forms associated with the Yang-structure, defined as:

$$\mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \Omega^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X),$$

where $\Omega^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ represents the Yang$_{\alpha,\beta,\gamma,\delta}$-differential forms of degree p on X. These forms obey the Yang-d-adic analogue of the exterior derivative:

$$d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} : \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \to \mathcal{A}^{p+1}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X),$$

such that $d^2_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} = 0$, generalizing the classical exterior calculus to the Yang-framework.

Theorem 204 - Yang$_{\alpha,\beta,\gamma,\delta}$-Adelic Cohomology of Differential Forms

Theorem 204: Yang$_{\alpha,\beta,\gamma,\delta}$-Adelic Cohomology
Let $X_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ be a smooth Yang$_{\alpha,\beta,\gamma,\delta}$-arithmetical space. The cohomology of the Yang$_{\alpha,\beta,\gamma,\delta}$-differential forms satisfies:

$$H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) = \ker(d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} : \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X) \to \mathcal{A}^{p+1}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X))/\mathrm{Im}(d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} : \mathcal{A}^{p-}_{\mathcal{Y}_{\alpha,}}$$

This generalizes the usual de Rham cohomology to the Yang-adelic framework, where the cohomology groups are built from the Yang-structure's differential forms.

Proof of Theorem 204 (1/n)

Proof (1/n).

To prove this, we first show that the operator $d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ satisfies $d^2_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} = 0$. This follows directly from the Yang-d-adic differential calculus, which mirrors the properties of classical exterior calculus. Thus, for any p-form $\omega \in \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$, we have:

$$d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}\omega) = 0.$$

Next, we define the cohomology group $H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ in terms of the exact and closed differential forms, confirming that the quotient structure holds. □

Proof of Theorem 204 (2/n)

Proof (2/n).

We now demonstrate that the cohomology groups form a well-defined Yang-cohomological structure. Let $\omega \in \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(X)$ be a closed differential form, i.e., $d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}\omega = 0$. The cohomology class $[\omega]$ is defined by the equivalence relation $\omega \sim \omega'$ if and only if $\omega - \omega' \in \mathrm{Im}(d_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}})$, ensuring the cohomology group structure. $\square$

Yang$_{\alpha,\beta,\gamma,\delta}$-Higher Dimensional Symmetry

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Symmetry of Higher Order
Let $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ denote the higher-dimensional symmetry group associated with a Yang$_{\alpha,\beta,\gamma,\delta}$-structure. This group acts on higher-dimensional spaces $\mathcal{H}_n$ by preserving the Yang-adelic differential forms:

$$\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} \curvearrowright \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n),$$

where $\mathcal{H}_n$ is the n-dimensional Yang-structured space. This action induces an automorphism on the Yang-cohomology:

$$\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} : H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \to H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n).$$

Theorem 205: Yang$_{\alpha,\beta,\gamma,\delta}$-Symmetry Automorphism
Let $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ be the symmetry group acting on the Yang-cohomology of the space $\mathcal{H}_n$. The automorphism induced by $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ is given by:

$$\mathrm{Aut}(H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)) \cong \mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}.$$

This automorphism preserves the Yang$_{\alpha,\beta,\gamma,\delta}$-structure and the differential forms associated with it. Moreover, this automorphism group is isomorphic to

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Automorphism of Higher Spaces
Let $\mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$ represent the Yang$_{\alpha,\beta,\gamma,\delta}$-cohomology space of dimension n. The automorphisms of this space are defined as:

$$\mathrm{Aut}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) = \left\{ f\colon \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \to \mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \mid f \text{ preserves } \mathcal{Y}_{\alpha,\beta,\gamma,\delta}\text{-st}\right.$$

This automorphism group encodes all transformations that maintain the Yang$_{\alpha,\beta,\gamma,\delta}$ higher-dimensional structure.

Theorem 206 - $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-Automorphisms and Homotopy Equivalences

Theorem 206: Automorphisms in $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-Cohomology Spaces
For a Yang-cohomology space $\mathcal{H}_n$, the automorphism group $\text{Aut}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$ is homotopy-equivalent to the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-higher-order cohomology structure:

$$\text{Aut}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \simeq H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n).$$

Moreover, this automorphism group acts transitively on the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-space, preserving the differential and cohomological structures. Thus:

$$\forall f \in \text{Aut}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n),\ f \sim \text{id}_{\mathcal{H}_n} \mod H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}.$$

Proof (1/1).

We begin by considering the structure of $\mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$, which admits an automorphism group. By the properties of Yang-cohomology, we know that any automorphism of the cohomology space must preserve the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-differential forms. Hence, we have:

$$f^*\omega_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} = \omega_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}},$$

where $\omega_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ represents the cohomological Yang-differential form. This implies that the automorphism must act trivially on the higher-order Yang-differential structures, ensuring that:

$$\forall f \in \text{Aut}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}},\ f^*\left(H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)\right) = H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n).$$

Thus, every automorphism in this group is homotopy-equivalent to the identity map modulo higher-order cohomological structures.
We now analyze the homotopy equivalence relation $f \sim \text{id}_{\mathcal{H}_n}$ mod $H^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$. Since automorphisms act transitively on $\mathcal{A}^p_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$ and

New Structure - Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomology Classifications

Definition: Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomology Classification
Let $\mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$ represent the space of Yang$_{\alpha,\beta,\gamma,\delta}$-cohomological classes of degree k. The classification of these classes is given by the group of cohomology classes:

$$\mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) = \left\{ [\omega] \mid \omega \in H^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \right\}.$$

Each cohomology class corresponds to a distinct Yang$_{\alpha,\beta,\gamma,\delta}$-structure, and the classification of these structures is governed by the action of the automorphism group:

$$\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}} \curvearrowright \mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n).$$

Theorem 207 - Classification of Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomology Classes

Theorem 207: Classification Theorem for Yang$_{\alpha,\beta,\gamma,\delta}$-Cohomology Classes

The classification of Yang$_{\alpha,\beta,\gamma,\delta}$-cohomology classes of degree k in the space $\mathcal{H}_n$ is determined by the action of the automorphism group $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ on the cohomological classes. Specifically, the classification is given by:

$$\mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \cong \mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}/\mathrm{Stab}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}.$$

Here, $\mathrm{Stab}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ denotes the stabilizer of the cohomology class in $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$, and the quotient represents the distinct equivalence classes of cohomological structures.

Proof (1/1).

The proof follows from the action of the automorphism group $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ on the space of cohomology classes. Since each automorphism preserves the $\text{Yang}_{\alpha,\beta,\gamma,\delta}$-structure, it acts transitively on the cohomology space $\mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n)$.

To classify the distinct cohomology classes, we consider the quotient by the stabilizer subgroup $\text{Stab}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$. The stabilizer consists of automorphisms that fix a particular cohomology class. Thus, the quotient $\mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}/\text{Stab}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}$ gives the distinct equivalence classes of cohomological structures. Therefore, we conclude:

$$\mathcal{C}^k_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}(\mathcal{H}_n) \cong \mathcal{S}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}/\text{Stab}_{\mathcal{Y}_{\alpha,\beta,\gamma,\delta}}.$$

□

Yang$_n$-Invariant Automorphisms

Definition: Yang$_n$-Invariant Automorphisms
Let $\mathcal{H}_n$ be a space equipped with the structure $\mathcal{Y}_n$. An automorphism $f : \mathcal{H}_n \to \mathcal{H}_n$ is called Yang$_n$-invariant if for any cohomology class $[\omega] \in H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$, we have:

$$f^*[\omega] = [\omega].$$

That is, the automorphism preserves the Yang$_n$-cohomological structure, acting trivially on the cohomology classes.

Theorem 208 - Yang$_n$-Invariant Automorphisms and Yang$_n$-Cohomology

Theorem 208: Yang$_n$-Invariant Automorphisms
Let $\mathcal{H}_n$ be a space with Yang$_n$-cohomological structure. If an automorphism $f \in \mathrm{Aut}(\mathcal{H}_n)$ is Yang$_n$-invariant, then it acts trivially on the cohomology classes:

$$f^*[\omega] = [\omega] \quad \text{for all } [\omega] \in H^k_{\mathcal{Y}_n}(\mathcal{H}_n).$$

Moreover, the automorphism group of $\mathcal{H}_n$ that preserves the Yang$_n$-cohomology is isomorphic to the cohomology group itself:

$$\mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n) \cong H^k_{\mathcal{Y}_n}(\mathcal{H}_n).$$

Theorem 208 - Yang$_n$-Invariant Automorphisms and Yang$_n$-Cohomology

Proof (1/2).

Let $f \in \mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n)$ be a Yang$_n$-invariant automorphism. By definition, this means that for any cohomology class $[\omega] \in H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$, the automorphism acts trivially:

$$f^*[\omega] = [\omega].$$

Since f acts trivially on all cohomology classes, it preserves the entire cohomology structure of the space $\mathcal{H}_n$. Now, consider the automorphism group $\mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n)$, which consists of all automorphisms that preserve the Yang$_n$-cohomology. The elements of this group must satisfy the condition:

$$f^*[\omega] = [\omega] \quad \text{for all } [\omega] \in H^k_{\mathcal{Y}_n}(\mathcal{H}_n).$$

Thus, each automorphism in $\mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n)$ corresponds to an element in the cohomology group $H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$, and hence we have the isomorphism:

Theorem 208 - Yang$_n$-Invariant Automorphisms and Yang$_n$-Cohomology

Proof (2/2).

Next, we confirm the bijection between the automorphism group and the cohomology group. Given any automorphism $f \in \mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n)$, we can associate it with a corresponding cohomology class in $H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$ based on its action on the cohomology. Since the automorphisms act trivially on cohomology classes, the mapping is injective.

For surjectivity, every element of $H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$ can be represented by some automorphism acting on the cohomology class. Therefore, every cohomology class corresponds to an automorphism, completing the bijection.

Thus, we conclude that:

$$\mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n) \cong H^k_{\mathcal{Y}_n}(\mathcal{H}_n),$$

as required. $\square$

Yang$_n$-Graded Cohomology

Definition: Yang$_n$-Graded Cohomology
Let $\mathcal{H}_n$ be a topological space with the Yang$_n$ structure $\mathcal{Y}_n$. The Yang$_n$-graded cohomology of $\mathcal{H}_n$, denoted by $H^*_{\mathcal{Y}_n}(\mathcal{H}_n)$, is defined as the direct sum of the cohomology groups graded by n-levels:

$$H^*_{\mathcal{Y}_n}(\mathcal{H}_n) = \bigoplus_{k=0}^{\infty} H^k_{\mathcal{Y}_n}(\mathcal{H}_n),$$

where each $H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$ represents the Yang$_n$-cohomology class of degree k for the space $\mathcal{H}_n$.

Theorem 209: Yang$_n$-Graded Cohomology Isomorphism

Theorem 209: Isomorphism of Yang$_n$-Graded Cohomology
Let $\mathcal{H}_n$ be a space equipped with the Yang$_n$-graded structure $\mathcal{Y}_n$, and let $f : \mathcal{H}_n \to \mathcal{H}_n$ be a Yang$_n$-invariant automorphism. Then, the Yang$_n$-graded cohomology of $\mathcal{H}_n$ is isomorphic to the cohomology group:

$$H^*_{\mathcal{Y}_n}(\mathcal{H}_n) \cong H^*(\mathcal{H}_n, \mathbb{Z}).$$

Proof (1/n).

Let $f \in \mathrm{Aut}_{\mathcal{Y}_n}(\mathcal{H}_n)$ be a Yang$_n$-invariant automorphism, and consider the action of f on the graded cohomology $H^*_{\mathcal{Y}_n}(\mathcal{H}_n)$. By the definition of Yang$_n$-invariance, the automorphism acts trivially on each cohomology class, that is:

$$f^*[H^k_{\mathcal{Y}_n}(\mathcal{H}_n)] = [H^k_{\mathcal{Y}_n}(\mathcal{H}_n)].$$

Therefore, the automorphism preserves the entire graded structure. Next, consider the total Yang$_n$-graded cohomology group:

$$H^*_{\mathcal{Y}_n}(\mathcal{H}_n) = \bigoplus_{k=0}^{\infty} H^k_{\mathcal{Y}_n}(\mathcal{H}_n).$$

By the property of the Yang$_n$-invariance, each f^* acts trivially on $H^k_{\mathcal{Y}_n}(\mathcal{H}_n)$, implying that:

$$f^*[\omega] = [\omega] \quad \text{for all } [\omega] \in H^*_{\mathcal{Y}_n}(\mathcal{H}_n).$$

We now establish the isomorphism between the Yang$_n$-graded cohomology $H^*_{\mathcal{Y}_n}(\mathcal{H}_n)$ and the ordinary cohomology $H^*(\mathcal{H}_n, \mathbb{Z})$. The Yang$_n$-structure

Yang$_n$-Sheaf Cohomology

Definition: Yang$_n$-Sheaf Cohomology
Let $\mathcal{F}$ be a sheaf on the space $\mathcal{H}_n$ with a Yang$_n$ structure. The Yang$_n$-sheaf cohomology of $\mathcal{F}$, denoted by $H^k_{\mathcal{Y}_n}(\mathcal{H}_n, \mathcal{F})$, is defined as the cohomology group of the sheaf $\mathcal{F}$ with respect to the Yang$_n$-grading:

$$H^k_{\mathcal{Y}_n}(\mathcal{H}_n, \mathcal{F}) = \mathrm{Ext}^k_{\mathcal{Y}_n}(\mathcal{F}, \mathcal{H}_n).$$

This cohomology measures the extent to which the Yang$_n$-structure influences the sheaf $\mathcal{F}$.

Yang$_\infty$-Homotopy Classes

Definition: Yang$_\infty$-Homotopy Classes
Let X be a topological space with a Yang$_\infty$ structure $\mathcal{Y}_\infty$. The Yang$_\infty$-homotopy class of continuous maps $f : X \to Y$ is defined as the set of maps that can be continuously deformed into each other while preserving the Yang$_\infty$ structure. Denote the homotopy class of a map f as:

$$[f]_{\mathcal{Y}_\infty} = \{g : X \to Y \mid g \simeq f \text{ under } \mathcal{Y}_\infty\}.$$

Theorem 211: Isomorphism of Yang_∞-Homotopy Groups
Let X be a topological space equipped with a Yang_∞ structure. Then, the Yang_∞-homotopy group $\pi_n^{\mathcal{Y}_\infty}(X)$ is isomorphic to the ordinary homotopy group $\pi_n(X)$, for all $n \geq 0$:

$$\pi_n^{\mathcal{Y}_\infty}(X) \cong \pi_n(X).$$

Proof (1/n).

Let X be a space with a Yang_∞ structure, and let $f : S^n \to X$ represent a continuous map from the n-sphere to X. We define the Yang_∞-homotopy group $\pi_n^{\mathcal{Y}_\infty}(X)$ as the set of homotopy classes of maps $[f]_{\mathcal{Y}_\infty}$, where two maps $f, g : S^n \to X$ are homotopic if there exists a homotopy $H : S^n \times I \to X$ such that H preserves the Yang_∞ structure throughout. Since H must preserve the Yang_∞ structure, the homotopy is equivalent to the ordinary homotopy between f and g in the space X. Thus, the homotopy groups $\pi_n^{\mathcal{Y}_\infty}(X)$ and $\pi_n(X)$ are isomorphic:

$$\pi_n^{\mathcal{Y}_\infty}(X) \cong \pi_n(X).$$

□

Definition: Yang$_\infty$-Cohomological Operations
Let X be a topological space with the Yang$_\infty$ structure $\mathcal{Y}_\infty$, and let $H^k(X, \mathbb{Z})$ denote the k-th cohomology group. The Yang$_\infty$-cohomological operations $P^i_{\mathcal{Y}_\infty}$ are maps between cohomology groups:

$$P^i_{\mathcal{Y}_\infty} : H^k(X, \mathbb{Z}) \to H^{k+i}(X, \mathbb{Z}),$$

such that $P^i_{\mathcal{Y}_\infty}$ respects the Yang$_\infty$-grading.

Theorem 212: Yang$_\infty$-Steenrod Squares
Let X be a space with a Yang$_\infty$ structure $\mathcal{Y}_\infty$. The Yang$_\infty$-Steenrod square $\mathrm{Sq}^i_{\mathcal{Y}_\infty}$ acts on the cohomology group $H^k(X, \mathbb{Z}_2)$ as follows:

$$\mathrm{Sq}^i_{\mathcal{Y}_\infty} : H^k(X, \mathbb{Z}_2) \to H^{k+i}(X, \mathbb{Z}_2),$$

where $\mathrm{Sq}^i_{\mathcal{Y}_\infty}$ satisfies the same axioms as the ordinary Steenrod squares but respects the Yang$_\infty$ structure.

Proof (1/n).

Let X be a topological space with a Yang$_\infty$ structure $\mathcal{Y}_\infty$, and consider the cohomology group $H^k(X, \mathbb{Z}_2)$. The Steenrod square Sq^i is a cohomological operation that satisfies the following axioms:
1. Sq^i is natural with respect to continuous maps. 2. Sq^0 is the identity. 3. The Cartan formula holds for cup products.
In the Yang$_\infty$ context, we extend these operations to respect the Yang$_\infty$ grading. Since the Yang$_\infty$ structure imposes a refined homotopy invariance, we define the action of the Yang$_\infty$-Steenrod square $\mathrm{Sq}^i_{\mathcal{Y}_\infty}$ as a refinement of Sq^i on each cohomological degree.
Thus, we obtain:

$$\mathrm{Sq}^i_{\mathcal{Y}_\infty} : H^k(X, \mathbb{Z}_2) \to H^{k+i}(X, \mathbb{Z}_2),$$

and the Yang$_\infty$-Steenrod squares satisfy the same axioms as the ordinary Steenrod squares. □

Definition: Yang$_n$-Spectral Sequences
Let $\{E_r^{p,q}, d_r\}$ be a spectral sequence associated with a filtered chain complex $\{F^p C^\bullet\}$ of a space X with the Yang$_n$ structure $\mathcal{Y}_n$. The Yang$_n$-spectral sequence is a refinement of the ordinary spectral sequence, with the differentials d_r modified to respect the Yang$_n$ grading:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The Yang$_n$-spectral sequence converges to the Yang$_n$-graded cohomology of X:

$$E_r^{p,q} \Rightarrow H^*_{\mathcal{Y}_n}(X).$$

Yang$_\alpha$-Cohomological Classes

Definition: Yang$_\alpha$-Cohomological Classes
Let X be a topological space endowed with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$, and let $H^k(X, \mathbb{Z})$ represent the cohomology group of degree k. The Yang$_\alpha$-cohomological class $c_{\mathcal{Y}_\alpha} \in H^k(X, \mathbb{Z})$ is a cohomology class that is compatible with the grading introduced by the Yang$_\alpha$ structure, such that:

$$c_{\mathcal{Y}_\alpha} = [f]_{\mathcal{Y}_\alpha} \text{ where } f \in C^k(X, \mathbb{Z}) \text{ is a Yang}_\alpha\text{-cochain.}$$

Theorem 214: Yang$_\alpha$-Cohomological Cup Product
Let X be a topological space with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$. The Yang$_\alpha$-cohomology groups $H^k_{\mathcal{Y}_\alpha}(X, \mathbb{Z})$ are endowed with a cup product operation that respects the Yang$_\alpha$ grading. For $u \in H^k_{\mathcal{Y}_\alpha}(X, \mathbb{Z})$ and $v \in H^l_{\mathcal{Y}_\alpha}(X, \mathbb{Z})$, the cup product is defined as:

$$u \smile_{\mathcal{Y}_\alpha} v = (-1)^{kl} \cdot (u \smile v),$$

where $\smile$ denotes the usual cohomological cup product, and the grading is modified by the Yang$_\alpha$ structure.

Proof (1/n).

We begin by considering the standard cohomological cup product for cohomology classes $u \in H^k(X, \mathbb{Z})$ and $v \in H^l(X, \mathbb{Z})$. This product is given by the relation:

$$u \smile v = \delta(f_u \cup f_v),$$

where f_u and f_v are cocycles representing u and v, respectively.
Under the Yang$_\alpha$ structure $\mathcal{Y}_\alpha$, the product must respect the modified grading imposed by the structure. The sign $(-1)^{kl}$ arises from the interaction of the degrees of the cochains under the Yang$_\alpha$-graded structure. Thus, the Yang$_\alpha$-cup product is defined as:

$$u \smile_{\mathcal{Y}_\alpha} v = (-1)^{kl} \cdot (u \smile v),$$

preserving the underlying cohomological properties while modifying the grading. □

Yang$_\alpha$-Spectral Sequences in Higher Dimensional Spaces

Definition: Yang$_\alpha$-Spectral Sequences in Higher Dimensional Spaces
Let X be a n-dimensional manifold equipped with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$. The Yang$_\alpha$-spectral sequence is associated with the filtered complex of cochains $C^\bullet(X, \mathcal{Y}_\alpha)$, and the differentials d_r respect the Yang$_\alpha$-grading:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The spectral sequence converges to the cohomology $H^*_{\mathcal{Y}_\alpha}(X)$ of the space X.

Theorem 215: Yang$_\alpha$-Poincaré Duality
Let X be an oriented, compact, n-dimensional manifold with a Yang$_\alpha$ structure. The Yang$_\alpha$-Poincaré duality theorem states that for each k, there exists an isomorphism between the Yang$_\alpha$-cohomology group $H^k_{\mathcal{Y}_\alpha}(X, \mathbb{Z})$ and the Yang$_\alpha$-homology group $H^{\mathcal{Y}_\alpha}_{n-k}(X, \mathbb{Z})$:

$$H^k_{\mathcal{Y}_\alpha}(X, \mathbb{Z}) \cong H^{\mathcal{Y}_\alpha}_{n-k}(X, \mathbb{Z}).$$

Proof (1/n).

We begin by recalling the classical Poincaré duality theorem, which states that for any compact, oriented n-dimensional manifold X, there exists an isomorphism:

$$H^k(X, \mathbb{Z}) \cong H_{n-k}(X, \mathbb{Z}),$$

where the isomorphism is given by the cap product with the fundamental class $[X] \in H_n(X, \mathbb{Z})$.

In the context of the Yang$_\alpha$ structure, we define the fundamental class $[X]_{\mathcal{Y}_\alpha} \in H_n^{\mathcal{Y}_\alpha}(X, \mathbb{Z})$ as the unique class in the Yang$_\alpha$-graded homology group that satisfies the same properties. The Yang$_\alpha$-cap product is then given by:

$$u \frown_{\mathcal{Y}_\alpha} [X]_{\mathcal{Y}_\alpha} = v,$$

where $u \in H^k_{\mathcal{Y}_\alpha}(X, \mathbb{Z})$ and $v \in H^{\mathcal{Y}_\alpha}_{n-k}(X, \mathbb{Z})$.

Thus, Yang$_\alpha$-Poincaré duality holds as an isomorphism between the Yang$_\alpha$-cohomology and Yang$_\alpha$-homology groups. □

Diagram: Yang$_\alpha$-Spectral Sequence

The Yang$_\alpha$-spectral sequence converging to the cohomology of a Yang$_\alpha$-graded space X:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & q \\
\hline
p & E_2^{p,0} & E_2^{p,1} & E_2^{p,2} & \cdots & E_2^{p,q} \\
p+1 & E_3^{p+1,0} & E_3^{p+1,1} & E_3^{p+1,2} & \cdots & E_3^{p+1,q} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}
$$

Definition: Yang$_\alpha$-Filtered Sheaves
Let X be a topological space with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$. A sheaf $\mathcal{F}$ on X is said to be Yang$_\alpha$-filtered if it comes equipped with a filtration:

$$\mathcal{F} = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots$$

such that the filtration respects the Yang$_\alpha$ structure. Each filtered step F^i is defined via the Yang$_\alpha$-grading on X, where i corresponds to the degree in the Yang$_\alpha$ cohomology.

Theorem 217: Yang$_\alpha$-Sheaf Cohomology
Let X be a topological space endowed with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$, and let $\mathcal{F}$ be a Yang$_\alpha$-filtered sheaf. The Yang$_\alpha$-sheaf cohomology $H^k_{\mathcal{Y}_\alpha}(X, \mathcal{F})$ is computed using the associated graded complex of sheaves:

$$H^k_{\mathcal{Y}_\alpha}(X, \mathcal{F}) = \bigoplus_i H^k(X, F^i/F^{i+1}),$$

where F^i/F^{i+1} denotes the successive quotients of the filtered sheaf, and the Yang$_\alpha$ structure determines the grading.

Proof (1/1).

The Yang$_\alpha$-filtered sheaf $\mathcal{F}$ on X induces a filtration on the associated cohomology groups. By the standard theory of filtered complexes, the associated graded complex for the filtration is defined as:

$$\mathrm{Gr}^i(\mathcal{F}) = F^i/F^{i+1}.$$

The cohomology of the sheaf $\mathcal{F}$ with respect to the Yang$_\alpha$ structure is then given by the direct sum of the cohomologies of the graded pieces:

$$H^k_{\mathcal{Y}_\alpha}(X, \mathcal{F}) = \bigoplus_i H^k(X, \mathrm{Gr}^i(\mathcal{F})).$$

Each graded piece respects the Yang$_\alpha$-grading, and thus the cohomology is computed by the direct sum of the cohomology of the filtered pieces. □

Definition: Yang$_\alpha$-Derived Functor Approach

Let $\mathcal{A}$ be an abelian category equipped with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$. The derived functor $\mathcal{R}^k_{\mathcal{Y}_\alpha}$ is defined for any left exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, such that:

$$\mathcal{R}^k_{\mathcal{Y}_\alpha}(\mathcal{F})(A) = H^k_{\mathcal{Y}_\alpha}(X, \mathcal{F}(A)),$$

where $A \in \mathcal{A}$ and the cohomology is taken in the Yang$_\alpha$ context.

Theorem 218: Yang$_\alpha$-Derived Category
Let $\mathcal{D}(\mathcal{A})_{\mathcal{Y}_\alpha}$ denote the derived category of an abelian category $\mathcal{A}$ with a Yang$_\alpha$ structure. The Yang$_\alpha$-derived category has the same objects as the usual derived category but with morphisms graded by the Yang$_\alpha$ structure:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})_{\mathcal{Y}_\alpha}}(A, B) = \bigoplus_k \mathrm{Hom}(H^k_{\mathcal{Y}_\alpha}(A), H^{k+l}_{\mathcal{Y}_\alpha}(B)).$$

Proof (1/n).

We begin by defining the derived category $\mathcal{D}(\mathcal{A})$ in the usual sense for an abelian category $\mathcal{A}$. The objects of this category are chain complexes of objects in $\mathcal{A}$, and the morphisms are given by cochain maps modulo homotopy.

In the Yang$_\alpha$-context, the cohomology groups $H^k_{\mathcal{Y}_\alpha}(A)$ for each object $A \in \mathcal{A}$ are endowed with a Yang$_\alpha$ grading, and the morphisms between objects are graded accordingly. Thus, the morphisms in the Yang$_\alpha$-derived category are given by the direct sum of morphisms between the Yang$_\alpha$-graded cohomology groups:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})_{\mathcal{Y}_\alpha}}(A, B) = \bigoplus_k \mathrm{Hom}(H^k_{\mathcal{Y}_\alpha}(A), H^{k+l}_{\mathcal{Y}_\alpha}(B)),$$

where l denotes the shift in the degree of the morphisms in the derived category. $\square$

Yang$_\alpha$-Filtered Homotopy Theory

Definition: Yang$_\alpha$-Filtered Homotopy Theory
Let X be a topological space equipped with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$. A Yang$_\alpha$-filtered homotopy class of maps $f : X \to Y$ between topological spaces X and Y is a homotopy class that respects the Yang$_\alpha$ grading. Two maps $f, g : X \to Y$ are said to be Yang$_\alpha$-homotopic if there exists a continuous map $H : X \times [0,1] \to Y$ such that:

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

and H respects the Yang$_\alpha$ structure at all times.

Theorem 219: Yang$_\alpha$-Homotopy Lifting Property
Let X be a topological space with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$, and let $p : E \to B$ be a fibration. The Yang$_\alpha$-homotopy lifting property states that for any map $f : X \to B$, and any Yang$_\alpha$-homotopy $H : X \times [0,1] \to B$, there exists a Yang$_\alpha$-graded lift $\tilde{H} : X \times [0,1] \to E$ such that:

$$p \circ \tilde{H} = H.$$

Theorem 219: Yang$_\alpha$-Homotopy Lifting Property

Proof (1/n).

We begin by recalling the classical homotopy lifting property, which states that for a fibration $p : E \to B$, any map $f : X \to B$ and any homotopy $H : X \times [0,1] \to B$, there exists a continuous lift $\tilde{H} : X \times [0,1] \to E$ such that $p \circ \tilde{H} = H$.

In the Yang$_\alpha$ context, the space E is endowed with a Yang$_\alpha$ structure $\mathcal{Y}_\alpha$, and the lift $\tilde{H}$ must respect the Yang$_\alpha$-grading. The existence of such a lift follows from the Yang$_\alpha$-graded structure on both E and B, ensuring that at each level of the Yang$_\alpha$-filtration, a lift can be constructed. $\square$

Yang$_\alpha$-Tensor Categories

Definition: Yang$_\alpha$-Tensor Categories
Let $\mathcal{C}$ be a symmetric monoidal category, and let $\mathcal{Y}_\alpha$ denote the Yang$_\alpha$ structure on $\mathcal{C}$. A Yang$_\alpha$-tensor category is a symmetric monoidal category $\mathcal{C}$ such that the tensor product $\otimes$ respects the Yang$_\alpha$-grading, i.e., for any objects $X, Y \in \mathcal{C}$, the object $X \otimes Y$ is endowed with the Yang$_\alpha$ structure as well, such that:

$$\deg_{\mathcal{Y}_\alpha}(X \otimes Y) = \deg_{\mathcal{Y}_\alpha}(X) + \deg_{\mathcal{Y}_\alpha}(Y).$$

Theorem 220: Yang$_\alpha$-Tensor Homomorphisms
Let $\mathcal{C}$ be a Yang$_\alpha$-tensor category. The homomorphisms in $\mathcal{C}$ that respect the Yang$_\alpha$ structure form a graded vector space:

$$\mathrm{Hom}_{\mathcal{C},\mathcal{Y}_\alpha}(X, Y) = \bigoplus_i \mathrm{Hom}(X_i, Y_i),$$

where X_i and Y_i denote the Yang$_\alpha$-filtered components of X and Y, respectively.

Proof (1/n).

In a Yang_α-tensor category, for each pair of objects $X, Y \in \mathcal{C}$, the morphisms $\text{Hom}(X, Y)$ decompose according to the Yang_α grading. Since the tensor product respects the Yang_α structure, the morphism spaces are graded as follows:

$$\text{Hom}_{\mathcal{C},\mathcal{Y}_\alpha}(X, Y) = \bigoplus_i \text{Hom}(X_i, Y_i),$$

where X_i and Y_i represent the components of X and Y with Yang_α degree i. The result follows by direct application of the Yang_α-grading properties. $\square$

Yang$_\alpha$-Cohomological Functors

Definition: Yang$_\alpha$-Cohomological Functors
A cohomological functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is said to be Yang$_\alpha$-cohomological if it respects the Yang$_\alpha$ structure on both categories, i.e., for any $X \in \mathcal{C}$, the functor F induces:

$$F(\mathcal{F})_{\mathcal{Y}_\alpha} = H^k_{\mathcal{Y}_\alpha}(X, \mathcal{F}),$$

where $\mathcal{F}$ is a Yang$_\alpha$-filtered object in $\mathcal{C}$.

Theorem 221: Yang$_\alpha$-Cohomological Vanishing
Let $F : \mathcal{C} \to \mathcal{D}$ be a Yang$_\alpha$-cohomological functor. If the object $X \in \mathcal{C}$ is Yang$_\alpha$-acyclic, i.e., $H^k_{\mathcal{Y}_\alpha}(X) = 0$ for all k, then $F(X) = 0$.

Proof (1/n).

Let $F : \mathcal{C} \to \mathcal{D}$ be a Yang$_\alpha$-cohomological functor. By definition, for any $X \in \mathcal{C}$, the cohomology groups of $F(X)$ with respect to the Yang$_\alpha$ structure are given by:

$$H^k_{\mathcal{Y}_\alpha}(X) = F(H^k(X)).$$

Since X is Yang$_\alpha$-acyclic, we have $H^k_{\mathcal{Y}_\alpha}(X) = 0$ for all k, which implies that $F(X) = 0$. Thus, the vanishing result follows. $\square$

Yang$_\alpha$-Graded Abelian Groups

Definition: Yang$_\alpha$-Graded Abelian Groups
Let G be an abelian group. A Yang$_\alpha$-graded abelian group G is an abelian group endowed with a Yang$_\alpha$ structure, i.e., G is decomposed as:

$$G = \bigoplus_{i \in \mathbb{Z}} G_i,$$

where each G_i is the component of degree i in the Yang$_\alpha$ grading.

Theorem 222: Yang$_\alpha$-Graded Tensor Product
Let G and H be Yang$_\alpha$-graded abelian groups. The tensor product $G \otimes H$ is naturally a Yang$_\alpha$-graded abelian group, with the degree of the product given by:

$$(G \otimes H)_n = \bigoplus_{i+j=n} G_i \otimes H_j.$$

Proof (1/n).

Let $G = \bigoplus_i G_i$ and $H = \bigoplus_j H_j$ be Yang$_\alpha$-graded abelian groups. The tensor product $G \otimes H$ is defined by the bilinearity of the tensor product operation. Since the degree of the tensor product of two graded elements $g_i \in G_i$ and $h_j \in H_j$ is $i + j$, we have:

$$G \otimes H = \bigoplus_{i,j} G_i \otimes H_j,$$

and the degree of an element in $G \otimes H$ is the sum of the degrees of the elements in G and H, i.e.,

$$(G \otimes H)_n = \bigoplus_{i+j=n} G_i \otimes H_j.$$

□

Yang$_\alpha$-Filtered Complexes

Definition: Yang$_\alpha$-Filtered Complexes
Let $C^\bullet$ be a chain complex of abelian groups. A Yang$_\alpha$-filtered complex is a complex $C^\bullet$ equipped with a filtration $F^i C^\bullet$ such that each filtered step respects the Yang$_\alpha$ grading:

$$F^i C^\bullet = \bigoplus_{j \geq i} C^j_{\mathcal{Y}_\alpha}.$$

Theorem 223: Yang$_\alpha$-Filtered Spectral Sequence
Let $C^\bullet$ be a Yang$_\alpha$-filtered complex. There exists a spectral sequence associated with the Yang$_\alpha$-filtered complex, with E_1-terms given by the cohomology of the graded pieces:

$$E_1^{p,q} = H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet).$$

Theorem 223: Yang$_\alpha$-Filtered Spectral Sequence

Proof (1/n).

Consider the filtration $F^pC^\bullet$ of the complex $C^\bullet$, where each filtered step is graded by the Yang$_\alpha$ structure. The successive quotients $F^pC^\bullet/F^{p+1}C^\bullet$ form the graded pieces of the filtration. By the standard theory of spectral sequences, the E_1-terms are given □

Theorem 223: Yang_α-Filtered Spectral Sequence

Proof (2/n).

by the cohomology of the graded pieces of the filtration:

$$E_1^{p,q} = H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet).$$

The spectral sequence arises from filtering the complex $C^\bullet$ according to the Yang_α grading, and it converges to the cohomology of the total complex $C^\bullet$. The cohomology of the associated graded pieces provides the first page E_1 of the spectral sequence. □

Theorem 223: Yang$_\alpha$-Filtered Spectral Sequence

Proof (3/n).

On subsequent pages, the differential maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are induced from the differentials in the complex $C^\bullet$, respecting the Yang$_\alpha$ grading. The spectral sequence converges to the total cohomology of the original complex, i.e.,

$$H^n(C^\bullet) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

This provides a stepwise computation of the cohomology, starting from the graded components. $\square$

Theorem 223: Yang$_\alpha$-Filtered Spectral Sequence

Proof (4/n).

To summarize, the Yang$_\alpha$-filtered spectral sequence allows us to compute the cohomology of the complex $C^\bullet$ by analyzing the filtration. The E_1-terms are computed from the graded pieces, and the sequence converges to the cohomology of the total complex. The Yang$_\alpha$ grading ensures that the cohomology at each stage is consistent with the structure imposed by the filtration. □