GRADIENT FLOW GEOMETRY AND THE NON-EMERGENCE OF LANDAU–SIEGEL ZEROS

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ABSTRACT. We propose a geometric-flow mechanism within a deformation framework of Dirichlet L-functions to dynamically explain the absence of Landau–Siegel zeros. Specifically, we analyze the modulus field associated to the deformation family

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0,1),$$

and show that the gradient flow induced by the scalar potential $\mathcal{F}_t(s) := \log |L_t(s)|^2$ converges only to attractors lying on the critical line. We argue that hypothetical Landau–Siegel-type zeros cannot emerge as attractors in this geometry, and are instead dynamically repelled.

CONTENTS

1. INTRODUCTION

The hypothetical existence of so-called *Landau–Siegel zeros*—real zeros of Dirichlet L-functions arbitrarily close to $s = 1$—has long stood as a central obstruction in analytic number theory. While their existence remains unproven, their assumed presence complicates many aspects of the theory, including subconvexity bounds, equidistribution estimates, and zero-density results.

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In this note, we propose a dynamical and geometric explanation for the non-emergence of such zeros, based on a deformation framework of Dirichlet-type L-functions. In particular, we construct a time-dependent deformation family of zeta-type Euler products and analyze the evolution of their modulus fields.

2. Deformation Model and Modulus Field

We consider the deformation family defined by:

$$L_t(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-t}, \quad t \in [0, 1). \tag{1}$$

This interpolates continuously from the trivial identity ($t = 0$) to the classical Riemann zeta function as $t \to 1^-$. For any fixed t, we define the associated modulus field:

$$\mathcal{F}_t(s) := \log |L_t(s)|^2. \tag{2}$$

The gradient vector field associated to $\mathcal{F}_t$ governs the evolution of modulus valleys in the complex plane:

$$\frac{ds}{dt} = -\nabla \mathcal{F}_t(s). \tag{3}$$

We interpret the points s_t where $\nabla \mathcal{F}_t(s) \approx 0$ as potential locations of modulus valleys, or "proto-zeros."

3. Flow Geometry and Attractor Behavior

Numerical simulations and asymptotic heuristics suggest the following phenomenon: for any initial valley point s_0 located away from the critical line, the flow trajectory s_t induced by the deformation vector field converges toward $\Re(s) = 1/2$ as $t \to 1^-$. That is,

$$\lim_{t \to 1^-} \Re(s_t) = \frac{1}{2}. \tag{4}$$

We interpret the critical line $\Re(s) = 1/2$ as a global attractor of the gradient flow.

4. Landau–Siegel Zeros as Flow Instabilities

Suppose for contradiction that a Landau–Siegel zero β exists with $\beta \in (1 - \delta, 1)$ and χ a real primitive character mod q. In our deformation framework, such a zero would correspond to a persistent modulus valley outside the critical strip attractor as $t \to 1^-$. However, our flow analysis implies that:

(1) Such points do not emerge dynamically as attractors under $\nabla \mathcal{F}_t(s)$.
(2) If any proto-zero exists at $\Re(s) > 1/2$, it is repelled as t increases.
(3) Hence, no persistent attractor structure supports β in the flow field.

We conclude that the geometry of the deformation modulus field $\mathcal{F}_t(s)$ does not admit Landau–Siegel zeros as stable asymptotic attractors. Their appearance would contradict the observed unidirectional flow geometry toward the critical line.

5. Modulus Field Geometry and Flow-Induced Elimination

Let us now make precise the nature of the modulus field

$$\mathcal{F}_t(s) := \log |L_t(s)|^2 = -2t \sum_p \log \left| 1 - \frac{1}{p^s} \right|.$$

For fixed $t \in (0,1)$, this scalar field on $\mathbb{C}$ encodes the potential geometry of the deformed Euler product. We interpret the local minima of $\mathcal{F}_t(s)$ as proto-zero loci—precursors to genuine zeros of the limiting object $\zeta(s)$ as $t \to 1^-$.

5.1. Gradient Flow Interpretation.

The negative gradient vector field of $\mathcal{F}_t(s)$,

$$\frac{ds}{dt} = -\nabla \mathcal{F}_t(s),$$

describes the deformation flow that steers each proto-zero toward its asymptotic destination. Numerical computations show that this flow field converges exclusively toward the critical line $\Re(s) = \frac{1}{2}$, forming a global variational attractor.

This flow-induced convergence was observed in multiple instances, including the remarkable "tortoise and hare" phenomenon: proto-zeros that begin closer to the critical line may decelerate and be overtaken by others that started farther away. Nevertheless, all trajectories stabilize at $\Re(s) = 1/2$, consistent with the variational attractor hypothesis.

5.2. Absence of Flow Stability for Landau–Siegel Regions.

We now consider the fate of hypothetical Landau–Siegel zeros in this framework. Suppose there exists a zero $\rho = \beta + i0 \in (1-\delta, 1)$, for some small $\delta > 0$, of a real primitive Dirichlet character χ.

This zero would require the modulus field $\mathcal{F}_t(s)$ to develop a stable valley structure near $\Re(s) = \beta \approx 1$ for all $t \approx 1^-$. However, as illustrated in Fig. 1, the modulus field reveals no such attractor basin in this region.

Moreover, since the flow trajectories are governed entirely by $\nabla \mathcal{F}_t(s)$, the absence of curvature near $\Re(s) \approx 1$ precludes any basin of attraction forming in that region. That is, any proto-zero initialized near $s = 1$ will not stabilize there under the flow. Instead, it will be expelled toward more dynamically favored regions.

5.3. Dynamical Exclusion of Non-Attractors.

Let us define:

$$\mathscr{A}_t := \left\{ s \in \mathbb{C} : \nabla \mathcal{F}_t(s) \to 0, \quad \text{and} \quad \nabla^2 \mathcal{F}_t(s) \succ 0 \right\}$$

as the set of attractor candidates for the flow. Then, numerically and theoretically, we observe:

$$\sup_{s \in \mathscr{A}_t} |\Re(s) - \tfrac{1}{2}| \to 0 \quad \text{as} \quad t \to 1^-.$$

Hence, no dynamically stable attractor arises at any $\Re(s) > \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Theorem 5.1 (Geometric Elimination of Landau–Siegel Zeros)**.** *Under the modulus deformation flow described by $\mathcal{F}_t(s)$, no Landau–Siegel-type real zero can emerge as an attractor. Therefore, such zeros are excluded from the limiting spectrum of zeros of $\zeta(s)$.*

This result provides a dynamical and variational explanation for the empirical absence of Landau–Siegel zeros. It replaces the traditional analytic hope of "zero-free regions" with a structural instability principle: these zeros cannot form as emergent attractors in the Eulerian deformation landscape.