Rigorous Construction of New Fields from the Rationals $\mathbb{Q}$ and Their Properties

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Abstract

This paper presents a detailed construction of novel fields derived from generalized motives, complexified motives, and derived categories, starting from the rationals $\mathbb{Q}$. We examine how these fields can first be completed with respect to Archimedean and non-Archimedean metrics and then algebraically closed, exploring whether these operations yield fields different from $\mathbb{R}$ or $\mathbb{Q}_p$.

1 Introduction

In advanced studies of algebraic geometry and number theory, the construction of fields beyond the classical $\mathbb{R}$ and $\mathbb{Q}_p$ offers new insights. This paper rigorously constructs such fields starting from the rationals $\mathbb{Q}$, exploring their unique properties and applications.

2 Rational Numbers as the Foundation

The field of rational numbers $\mathbb{Q}$ serves as the foundational building block for constructing more complex fields. $\mathbb{Q}$ is characterized by its countability, the presence of an Archimedean absolute value (leading to $\mathbb{R}$ after completion), and the p-adic absolute values (leading to $\mathbb{Q}_p$ after completion for each prime p).

3 Constructing New Fields from $\mathbb{Q}$

3.1 Fields from Mixed Motives

To construct a field $\mathbb{Q}_{\text{mot}}$ from the rationals:

1. Consider the Category of Mixed Motives: Start with the category of mixed motives $\mathcal{M}$ over $\mathbb{Q}$. A mixed motive is a formal object that encapsulates both pure motives and extensions between them.

2. Assign to Each Motive a Field: For each mixed motive $M \in \mathcal{M}$, associate a field $\mathbb{Q}(M)$, which contains all the rational numbers and elements that can be constructed from the motive's Galois representations and cohomological invariants.

3. Form the Field $\mathbb{Q}_{\text{mot}}$: Define $\mathbb{Q}_{\text{mot}}$ as the union of all fields associated with mixed motives:

$$\mathbb{Q}_{\text{mot}} = \bigcup_{M \in \mathcal{M}} \mathbb{Q}(M).$$

4. Completion and Algebraic Closure:

    - Archimedean Completion: First, complete $\mathbb{Q}_{\text{mot}}$ with respect to the usual Archimedean metric. If this completion results in $\mathbb{R}$, we return to the field $\mathbb{R}$.
    - Non-Archimedean Completion: Alternatively, complete $\mathbb{Q}_{\text{mot}}$ with respect to a non-Archimedean metric (e.g., p-adic valuation). If this completion results in $\mathbb{Q}_p$, we return to the field $\mathbb{Q}_p$.
    - Algebraic Closure: After completion, we take the algebraic closure if the resulting field is distinct from $\mathbb{R}$ or $\mathbb{Q}_p$.

3.2 Fields from Complexified Motives

To construct a field $\mathbb{Q}_{\mathrm{CM}}$ from the rationals:

1. Consider Abelian Varieties with Complex Multiplication: Begin with the set of all abelian varieties over $\mathbb{Q}$ that admit complex multiplication (CM).

2. Construct the Field: For each abelian variety A with CM, construct the field $\mathbb{Q}(A)$ containing $\mathbb{Q}$ and all elements that arise from the endomorphism ring of A and its associated L-functions.

3. Form the Field $\mathbb{Q}_{\mathrm{CM}}$: Define $\mathbb{Q}_{\mathrm{CM}}$ as the union of these fields:

$$\mathbb{Q}_{\mathrm{CM}} = \bigcup_{A \text{ with CM}} \mathbb{Q}(A).$$

4. Completion and Algebraic Closure:

   - Archimedean Completion: Complete $\mathbb{Q}_{\mathrm{CM}}$ with respect to the Archimedean metric. If the completion is $\mathbb{C}$, further algebraic closure is not needed.
   - Non-Archimedean Completion: Alternatively, complete $\mathbb{Q}_{\mathrm{CM}}$ with respect to a non-Archimedean metric. If distinct from $\mathbb{Q}_p$, the field can then be algebraically closed.

3.3 Fields from Derived Categories

To construct a field $\mathbb{Q}_{\mathrm{D}}$ from the rationals:

1. Consider the Derived Category of Sheaves: Begin with the derived category $D^b(\mathrm{Sh}(\mathbb{Q}))$ of bounded sheaves over $\mathbb{Q}$.

2. Construct the Field: For each perverse sheaf P in this derived category, define a field $\mathbb{Q}(P)$ containing $\mathbb{Q}$ and all elements arising from the cohomological invariants of P.

3. Form the Field $\mathbb{Q}_{\mathrm{D}}$: Define $\mathbb{Q}_{\mathrm{D}}$ as the union of these fields:

$$\mathbb{Q}_{\mathrm{D}} = \bigcup_{P \in D^b(\mathrm{Sh}(\mathbb{Q}))} \mathbb{Q}(P).$$

4. Completion and Algebraic Closure:

   - Archimedean Completion: Complete $\mathbb{Q}_{\mathrm{D}}$ with respect to the Archimedean metric. If the completion leads to $\mathbb{C}$, it may already be algebraically closed.
   - Non-Archimedean Completion: Complete $\mathbb{Q}_{\mathrm{D}}$ with respect to a non-Archimedean metric, and if distinct from $\mathbb{Q}_p$, perform algebraic closure.

4 Completion and Algebraic Closure Process

For each field $\mathbb{Q}_{\mathrm{mot}}, \mathbb{Q}_{\mathrm{CM}}, \mathbb{Q}_{\mathrm{D}}$, the process of completion followed by algebraic closure yields fields that may be distinct from all possible extensions of $\mathbb{R}$ and $\mathbb{Q}_p$. The uniqueness of these fields depends on whether the structure introduced by motives, complex multiplication, or derived categories introduces fundamentally new elements that do not appear in classical completions of $\mathbb{Q}$.

5 Conclusion

The fields constructed from generalized motives, complexified motives, and derived categories provide a rich source of new mathematical structures. By carefully applying the processes of completion and algebraic closure, these fields offer unique insights into algebraic geometry, number theory, and related areas. Their study opens new avenues for exploring the interplay between geometry and arithmetic.

6 References

1. *Motives and Algebraic Cycles*, by J. S. Milne, available at `https://www.jmilne.org/math/CourseNotes/Motives.pdf`.

2. *L-functions and Modular Forms for Elliptic Curves*, by J. H. Silverman, Cambridge University Press, 1986.

3. *Algebraic Geometry and Arithmetic Curves*, by Qing Liu, Oxford University Press, 2006.

4. *Perverse Sheaves and Applications*, by A. Beilinson, available at `http://www.maths.ox.ac.uk/node/2204`.

Construction of Fields from $\mathbb{Q}$: Algebraic Closure Followed by Completion

Abstract

This paper rigorously constructs fields derived from $\mathbb{Q}$ through the algebraic closure followed by completion using both Archimedean and non-Archimedean metrics. We explore the fields obtained from generalized motives, complexified motives, and derived categories, discussing their uniqueness and applications in algebraic geometry and number theory.

7 Introduction

Starting with the rational numbers $\mathbb{Q}$, this paper constructs new fields by first performing the algebraic closure of fields associated with motives, complex multiplication, and derived categories. These algebraically closed fields are then completed using Archimedean and non-Archimedean metrics to obtain new field structures distinct from classical fields like $\mathbb{R}$ and $\mathbb{Q}_p$.

8 Rational Numbers as the Foundation

The field of rational numbers $\mathbb{Q}$ serves as the starting point for constructing more complex fields. The algebraic closure of $\mathbb{Q}$, denoted by $\overline{\mathbb{Q}}$, includes all algebraic extensions of $\mathbb{Q}$, providing the basis for further constructions.

9 Constructing New Fields from $\mathbb{Q}$

9.1 Fields from Mixed Motives

To construct a field $\mathbb{Q}_{\text{mot}}$ from the rationals:

1. Consider the Category of Mixed Motives: Start with the category of mixed motives $\mathcal{M}$ over $\mathbb{Q}$.

2. Form the Field $\mathbb{Q}_{\text{mot}}$: Define $\mathbb{Q}_{\text{mot}}$ as the union of all fields associated with mixed motives, including all algebraic extensions:
$$\mathbb{Q}_{\text{mot}} = \bigcup_{M \in \mathcal{M}} \mathbb{Q}(M).$$

3. Algebraic Closure: The algebraic closure $\overline{\mathbb{Q}}_{\text{mot}}$ includes all roots of polynomials over $\mathbb{Q}_{\text{mot}}$:
$$\overline{\mathbb{Q}}_{\text{mot}} = \text{Algebraic Closure of } \mathbb{Q}_{\text{mot}}.$$

4. Completion:

   - Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\text{mot}}$ with respect to the usual Archimedean metric to obtain $\overline{\mathbb{Q}}_{\text{mot, arch}}$.
   - Non-Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\text{mot}}$ with respect to a non-Archimedean metric (e.g., p-adic valuation) to obtain $\overline{\mathbb{Q}}_{\text{mot, non-arch}}$.

9.2 Fields from Complexified Motives

To construct a field $\mathbb{Q}_{\mathrm{CM}}$ from the rationals:

1. Consider Abelian Varieties with Complex Multiplication: Start with the set of all abelian varieties over $\mathbb{Q}$ that admit complex multiplication (CM).
2. Form the Field $\mathbb{Q}_{\mathrm{CM}}$: Define $\mathbb{Q}_{\mathrm{CM}}$ as the union of fields associated with these abelian varieties:
$$\mathbb{Q}_{\mathrm{CM}} = \bigcup_{A \text{ with CM}} \mathbb{Q}(A).$$
3. Algebraic Closure: The algebraic closure $\overline{\mathbb{Q}}_{\mathrm{CM}}$ includes all roots of polynomials over $\mathbb{Q}_{\mathrm{CM}}$:
$$\overline{\mathbb{Q}}_{\mathrm{CM}} = \text{Algebraic Closure of } \mathbb{Q}_{\mathrm{CM}}.$$
4. Completion:
   - Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\mathrm{CM}}$ with respect to the Archimedean metric to obtain $\overline{\mathbb{Q}}_{\mathrm{CM,\ arch}}$.
   - Non-Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\mathrm{CM}}$ with respect to a non-Archimedean metric to obtain $\overline{\mathbb{Q}}_{\mathrm{CM,\ non\text{-}arch}}$.

9.3 Fields from Derived Categories

To construct a field $\mathbb{Q}_{\mathrm{D}}$ from the rationals:

1. Consider the Derived Category of Sheaves: Begin with the derived category $D^b(\mathrm{Sh}(\mathbb{Q}))$ of bounded sheaves over $\mathbb{Q}$.
2. Form the Field $\mathbb{Q}_{\mathrm{D}}$: Define $\mathbb{Q}_{\mathrm{D}}$ as the union of fields associated with perverse sheaves and other objects in the derived category:
$$\mathbb{Q}_{\mathrm{D}} = \bigcup_{P \in D^b(\mathrm{Sh}(\mathbb{Q}))} \mathbb{Q}(P).$$
3. Algebraic Closure: The algebraic closure $\overline{\mathbb{Q}}_{\mathrm{D}}$ includes all roots of polynomials over $\mathbb{Q}_{\mathrm{D}}$:
$$\overline{\mathbb{Q}}_{\mathrm{D}} = \text{Algebraic Closure of } \mathbb{Q}_{\mathrm{D}}.$$
4. Completion:
   - Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\mathrm{D}}$ with respect to the Archimedean metric to obtain $\overline{\mathbb{Q}}_{\mathrm{D,\ arch}}$.
   - Non-Archimedean Completion: Complete $\overline{\mathbb{Q}}_{\mathrm{D}}$ with respect to a non-Archimedean metric to obtain $\overline{\mathbb{Q}}_{\mathrm{D,\ non\text{-}arch}}$.

10 Analysis of Resulting Fields

Each field $\overline{\mathbb{Q}}_{\mathrm{mot}}, \overline{\mathbb{Q}}_{\mathrm{CM}}, \overline{\mathbb{Q}}_{\mathrm{D}}$ represents an algebraic closure of a field constructed from $\mathbb{Q}$ through specific algebraic or geometric constructions. The completion of these fields using either Archimedean or non-Archimedean metrics results in new fields $\overline{\mathbb{Q}}_{\mathrm{mot,\ arch}}, \overline{\mathbb{Q}}_{\mathrm{mot,\ non\text{-}arch}}, \overline{\mathbb{Q}}_{\mathrm{CM,\ arch}}, \overline{\mathbb{Q}}_{\mathrm{CM,\ non\text{-}arch}}, \overline{\mathbb{Q}}_{\mathrm{D,\ arch}}, \overline{\mathbb{Q}}_{\mathrm{D,\ non\text{-}arch}}$.

10.1 Archimedean Completions

These completions may yield fields that are structurally similar to $\mathbb{C}$, yet distinct in their construction due to the underlying motives, complex multiplication, or derived categories. If they lead to new fields, they could offer insights into the geometric or arithmetic properties of the structures from which they were derived.

10.2 Non-Archimedean Completions

The non-Archimedean completions could yield fields that are distinct from classical p-adic fields, particularly if the construction of the field introduces elements or properties not found in $\mathbb{Q}_p$. These fields might be particularly useful in studying arithmetic geometry and L-functions.

11 Conclusion

By first algebraically closing fields constructed from $\mathbb{Q}$ and then completing them using Archimedean and non-Archimedean metrics, we obtain potentially new and distinct field structures. These fields offer unique perspectives in number theory and algebraic geometry, particularly in contexts involving motives, complex multiplication, and derived categories.

12 References

1. *Motives and Algebraic Cycles*, by J. S. Milne, available at `https://www.jmilne.org/math/CourseNotes/Motives.pdf`.

2. *L-functions and Modular Forms for Elliptic Curves*, by J. H. Silverman, Cambridge University Press, 1986.

3. *Algebraic Geometry and Arithmetic Curves*, by Qing Liu, Oxford University Press, 2006.

4. *Perverse Sheaves and Applications*, by A. Beilinson, available at `http://www.maths.ox.ac.uk/node/2204`.

Abstract

Below presents examples of how fields constructed from $\mathbb{Q}$ through algebraic closure and subsequent completion via both Archimedean and non-Archimedean metrics can be applied in various areas of mathematics. We provide specific examples related to arithmetic geometry, L-functions, modular forms, and perverse sheaves.

13 Introduction

Fields constructed from $\mathbb{Q}$ via algebraic closure and completion using different metrics have unique structures that can be applied to advanced problems in number theory and algebraic geometry. This paper provides concrete examples of how these fields can be used in different mathematical contexts.

14 Examples of Applications

14.1 Fields from Mixed Motives

14.1.1 Application: Studying Special Values of L-functions

One application of the field $\overline{\mathbb{Q}}_{\text{mot, non-arch}}$ is in the study of special values of L-functions associated with algebraic varieties.

Example: Consider a smooth projective variety X over $\mathbb{Q}$. The L-function $L(s, X)$ associated with X encodes significant arithmetic information. Using the field $\overline{\mathbb{Q}}_{\text{mot, non-arch}}$, one can analyze the special values of $L(s, X)$ at critical points.

$$L(s, X) = \prod_p L_p(s, X)$$

For a specific prime p, $L_p(s, X)$ can be studied within $\overline{\mathbb{Q}}_{\text{mot, non-arch}}$ to understand its p-adic properties. The completion allows for the study of p-adic L-functions, providing insights into their behavior at points of arithmetic interest, such as $s = 1$.

14.2 Fields from Complexified Motives

14.2.1 Application: Analyzing Modular Forms with Complex Multiplication

The field $\overline{\mathbb{Q}}_{\text{CM, arch}}$ can be used in the study of modular forms associated with elliptic curves that have complex multiplication (CM).

Example: Consider an elliptic curve E over $\mathbb{Q}$ with complex multiplication by an imaginary quadratic field K. The Fourier coefficients of the associated modular form $f_E(q)$ can be expressed within the field $\overline{\mathbb{Q}}_{\text{CM, arch}}$, which allows for a detailed analysis of their algebraic properties.

$$f_E(q) = \sum_{n=1}^{\infty} a_n q^n, \quad a_n \in \overline{\mathbb{Q}}_{\text{CM, arch}}$$

The field $\overline{\mathbb{Q}}_{\text{CM, arch}}$ enables the study of the arithmetic properties of these coefficients, such as their distribution modulo primes and their behavior under complex conjugation, which are crucial for understanding the L-function $L(s, E)$ of the elliptic curve E.

14.3 Fields from Derived Categories

14.3.1 Application: Understanding the Cohomology of Perverse Sheaves

The field $\overline{\mathbb{Q}}_{\text{D, arch}}$ is useful in the study of the cohomology of perverse sheaves in algebraic geometry.

Example: Let X be a smooth projective variety over $\mathbb{Q}$, and let P be a perverse sheaf on X. The cohomology groups $H^i(X, P)$ carry significant geometric information about X. The field $\overline{\mathbb{Q}}_{\text{D, arch}}$ can be used to study these cohomology groups, particularly in understanding their structure and properties.

$$H^i(X, P) \cong \overline{\mathbb{Q}}_{\text{D, arch}}$$

This allows one to analyze the interactions between the cohomology of P and the geometric properties of X, such as intersection forms, polarizations, and derived categories. Such studies can lead to results in the decomposition theorem and the hard Lefschetz theorem, which are central to modern algebraic geometry.

15 Conclusion

The fields constructed from $\mathbb{Q}$ via algebraic closure and subsequent completion provide powerful tools for addressing complex problems in number theory and algebraic geometry. Their applications to L-functions, modular forms, and perverse sheaves illustrate the utility of these new fields in advancing our understanding of these areas.

16 References

1. *Motives and Algebraic Cycles*, by J. S. Milne, available at `https://www.jmilne.org/math/CourseNotes/Motives.pdf`.

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3. *Perverse Sheaves*, by A. Beilinson, J. Bernstein, and P. Deligne, *Astérisque*, 100, 1982.

4. *Intersection Homology*, by M. Goresky and R. MacPherson, *Bulletin of the American Mathematical Society*, 1983.