MAS565 Numerical Analysis HW4

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3.3 Recall Theorem 2.1.4.1 which implies that, for any given $x \in [a,b]$ there exists $\xi(x) \in [a,b]$ such that

$$f(x) - p_{0,1}(x) = \frac{f''(\xi(x))}{2}(x-a)(x-b). \quad \cdots (*)$$

For $x \neq a$ and $x \neq b$ rewriting the above as

$$f''(\xi(x)) = \frac{2(f(x) - p_{0,1}(x))}{(x-a)(x-b)} \quad \cdots (**)$$

we observe that $f''(\xi(x))$ is continuous on the open interval (a,b). For the case when $x \in \{a,b\}$, equation $(*)$ allows us to define $f''(\xi(x))$ to be any number. Since $f(a) - p_{0,1}(a) = 0$ and $x \mapsto f(x) - p_{0,1}(x)$ is continuously differentiable everywhere, the function

$$\frac{f(x) - p_{0,1}(x)}{x-a}$$

converges as $x \to a$, and furthermore it converges to $f'(a) - p_{0,1}'(a)$. This implies that, with the help of expressing $f''(\xi(x))$ in the form of $(**)$, we can assign a value for $f''(\xi(a))$ so that $f''(\xi(x))$ is continuous at $x=a$. By symmetry, we can also assign a value for $f''(\xi(b))$ so that $f''(\xi(x))$ is continuous on whole $[a,b]$.

As $\xi(x) \in [a,b]$ for $x \in (a,b)$, we have

$$\{f''(\xi(x)) : x \in (a,b)\} \subseteq \{f''(t) : t \in [a,b]\}.$$

But the right hand side set is compact, being a continuous image of a compact interval. Therefore the limits $f''(\xi(a))$ and $f''(\xi(b))$ are also contained in that compact set. Hence

$$\{f''(\xi(x)) : x \in [a,b]\} \subseteq \{f''(t) : t \in [a,b]\}.$$

Let $m = \min_{t \in [a,b]} f''(t)$ and $M = \max_{t \in [a,b]} f''(t)$ so that now $(*)$ implies

$$\frac{m}{2}(x-a)(x-b) \le f(x) - p_{0,1}(x) \le \frac{M}{2}(x-a)(x-b). \qquad \cdots (\ast\ast)$$

Integrate all sides over the interval $[a,b]$. As $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$, we get

$$-\frac{M}{12}(b-a)^3 \le \int_a^b f(x) - p_{0,1}(x)\,dx \le -\frac{m}{12}(b-a)^3$$

and henceforth

$$m \le \frac{12}{(b-a)^3}\int_a^b p_{0,1}(x) - f(x)\,dx \le M.$$

By Intermediate Value Theorem, there exists $\tilde{x} \in [a,b]$ such that

$$f''(\tilde{x}) = \frac{12}{(b-a)^3}\int_a^b p_{0,1}(x) - f(x)\,dx.$$

As required from the problem, we also should show that it is possible to choose $\tilde{x}$ from the open interval (a,b). Without loss of generality, suppose that $\tilde{x} = a$ and no other point $t \in (a,b]$ satisfies $f''(a) = f''(t)$. But if so, by the continuity of f'' and the Intermediate Value Theorem, it must be the case where $f''(t) \ge f''(a)$ $\forall t \in (a,b]$, or $f''(t) \le f''(a)$, $\forall t \in (a,b]$. Then $f''(a)$ is either m or M, respectively, so the corresponding side in the inequality $(\ast\ast)$ is attained. This implies that f is a quadratic function, so f'' must be constant, contradicting our assumption.

We now have a stronger result: there exists $\tilde{x} \in (a,b)$ such that

$$f''(\tilde{x}) = \frac{12}{(b-a)^3} \int_a^b p_{0,1}(x) - f(x) \, dx.$$

Writing down $p_{0,1}(x)$ explicitly as $p_{0,1}(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$, we have

$$\begin{aligned} \int_a^b p_{0,1}(x) \, dx &= \int_a^b \frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a) \, dx \\ &= \frac{f(b)-f(a)}{b-a} \times \frac{(b-a)^2}{2} + f(a)(b-a) \\ &= \frac{b-a}{2}(f(a)+f(b)). \end{aligned}$$

In conclusion, there exists $\tilde{x} \in (a,b)$ such that

$$\frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(x) \, dx = \frac{(b-a)^3}{12} f''(\tilde{x}).$$

3.14. We have to show that, under the assumptions (a) and (b),

$$\forall s \geq 0 \text{ polynomial, then } \int_a^b w(x)s(x) \, dx = 0 \Rightarrow s \equiv 0 \quad \Longleftrightarrow \quad \int_a^b w(x) \, dx > 0.$$

($\Rightarrow$) Consider the constant polynomial $s(x) \equiv 1$. Then by assumption,

$$\int_a^b w(x) \, dx = \int_a^b w(x) s(x) \, dx > 0$$

because $w(x)s(x) \geq 0$ but $\int_a^b w(x) s(x) \, dx \neq 0$.

($\Leftarrow$) If $s \equiv 0$ then there is nothing to show. Hence assume that $s(x) \neq 0$ for some x. Our first goal is to find an interval $[y, z]$ such that $\int_y^z w(x) \, dx > 0$ while $s(x) > 0$ for all $x \in [y, z]$. If $s(x) > 0$ for all $x \in [a, b]$ then put $[y, z] = [a, b]$ and we are done. So we may assume that $s(x) = 0$ for some $x \in [a, b]$ but not all $x \in [a, b]$.

Consider the set

$$\{x \in [a,b] : s(x) = 0\} \cup \{a, b\}$$

which is finite since s is a polynomial. Enumerate the set above as $x_0, x_1, \cdots, x_N$, in increasing order. Then by subdividing $[a,b]$ into subintervals $[x_{i-1}, x_i]$ for $i = 1, \cdots, N$, we have the property that on each $[x_{i-1}, x_i]$ the value of $s(x)$ is positive possibly except at the boundary points. From

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} w(x)\,dx = \int_a^b w(x)\,dx > 0$$

while $w(x) \geq 0$ hence $\int_{x_{i-1}}^{x_i} w(x)\,dx \geq 0$, there should exist $j \in \{1, \cdots, N\}$ such that

$$\int_{x_{j-1}}^{x_j} w(x)\,dx > 0.$$

Now there are three possibilities, for in each such cases we define sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as:

i) $-\infty < x_{j-1} < x_j < +\infty$: $a_n = x_{j-1} + \frac{1}{n}$, $b_n = x_j - \frac{1}{n}$

ii) $-\infty < x_{j-1} < x_j = +\infty$: $a_n = x_{j-1} + \frac{1}{n}$, $b_n = n$

iii) $-\infty = x_{j-1} < x_j < +\infty$: $a_n = -n$, $b_n = x_j - \frac{1}{n}$.

Then the intervals $I_n = [a_n, b_n]$, with the convention that if $a_n > b_n$ then $I_n = \emptyset$, satisfy $I_1 \subset I_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} I_n = [x_{j-1}, x_j]$. Hence $\{w(x)\mathbb{1}_{I_n}(x)\}_{n\in\mathbb{N}}$ is a monotonely increasing sequence of functions pointwise converging to $w(x)$, so by Monotone Convergence Theorem we have

$$0 < \int_{x_{j-1}}^{x_j} w(x)\,dx = \lim_{n\to\infty} \int_{x_{j-1}}^{x_j} w(x)\mathbb{1}_{I_n}(x)\,dx = \lim_{n\to\infty} \int_{a_n}^{b_n} w(x)\,dx.$$

Note that, this implies the existence of $n_0 \in \mathbb{N}$ such that $\int_{a_{n_0}}^{b_{n_0}} w(x)\,dx > 0$

for all $n > n_0$. By choosing n sufficiently large, we can also have $x_{j-1} < a_n < b_n < x_j$.

For such an n, recall that $s(x) > 0$ for $x \in (x_{j-1}, x_j)$, so $s(x) > 0$ on $[a_n, b_n]$.

Put $y = a_n$ and $z = b_n$ then we are done, i.e. $\int_y^z w(x)\,dx > 0$ and $s(x) > 0$ over the interval $[y, z]$.

Summarizing what we have got up to this point, we have an interval $[y, z]$ such that $s(x) > 0$ over that interval and $\int_y^z w(x)\,dx > 0$.

Whether the interval $[y, z]$ is infinite or finite, since $s(x)$ is a nonnegative polynomial, it attains its minimum over $[y, z]$. Put $m := \min_{x \in [y,z]} s(x)$, then $m > 0$, so

$$0 < m \int_y^z w(x)\,dx \le \int_y^z w(x) s(x)\,dx \le \int_a^b w(x) s(x)\,dx.$$

Therefore, if there exists $x \in [a, b]$ such that $s(x) > 0$ then $\int_a^b w(x) s(x)\,dx > 0$.

It follows that, if $\int_a^b w(x) s(x)\,dx = 0$ then $s \equiv 0$, completing the proof.

3.18. (a) Define $\tilde{T}_n(x) = \cos(n \cos^{-1} x)$. First we claim that $T_n(x) = \tilde{T}_n(x)$, on the domain $[-1, 1]$. It is clear that $\tilde{T}_0(x) = 1 = T_0(x)$ and $\tilde{T}_1(x) = x = T_1(x)$.

So it remains to show that $\tilde{T}_{n+1}(x) = 2x\tilde{T}_n(x) - \tilde{T}_{n-1}(x)$, for $n = 1, 2, \cdots$.

Put $\theta = \cos^{-1} x$, then

$$\tilde{T}_{n+1}(x) = \cos((n+1)\theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$$
$$\tilde{T}_{n-1}(x) = \cos((n-1)\theta) = \cos n\theta \cos\theta + \sin n\theta \sin\theta.$$

Adding these equations side by side we get

$$\tilde{T}_{n+1}(x) + \tilde{T}_{n-1}(x) = 2\cos n\theta \cos\theta = 2x\,\tilde{T}_n(x)$$

so we are done.

From the observation in the previous paragraph we may put $T_n(x) = \cos(n\cos^{-1}x)$. It is straightforward to see that T_j is a polynomial of degree j with leading coefficient 2^j by induction on j: indeed, $2x\,T_j(x)$ is a degree $j+1$ polynomial with leading coefficient $2 \cdot 2^j = 2^{j+1}$ and $T_{j-1}(x)$ is a degree $j-1$ polynomial by induction hypothesis, so $T_{j+1}(x)$ is a degree $j+1$ polynomial with leading coefficient 2^{j+1}.

To construct a collection of monic polynomials, let $p_0(x) = 1$ and $p_j(x) = \frac{1}{2^{j-1}} T_j(x)$ for $j \geq 1$. It remains to show that $\{p_j(x)\}_{j\geq 0}$ are orthogonal polynomials with respect to weight $w(x) = \frac{1}{\sqrt{1-x^2}}$. Again let $\theta = \cos^{-1}x$ then $d\theta = -\frac{1}{\sqrt{1-x^2}}dx$, so for two positive integers m, n we have

$$\begin{aligned}
(p_m, p_n) &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2^{m-1}} \cos(m\cos^{-1}x) \cdot \frac{1}{2^{n-1}} \cos(n\cos^{-1}x)\,dx \\
&= \frac{1}{2^{m+n-2}} \int_0^{\pi} \cos m\theta \cos n\theta \, d\theta \\
&= \frac{1}{2^{m+n-2}} \int_0^{\pi} \frac{\cos(m+n)\theta + \cos(m-n)\theta}{2}\, d\theta \\
&= \begin{cases} \dfrac{1}{2^{m+n-1}} \left[\dfrac{1}{m+n} \sin(m+n)\theta + \dfrac{1}{m-n} \sin(m-n)\theta \right]_{\theta=0}^{\theta=\pi} & \text{if } m \neq n \\[2ex] \dfrac{1}{2^{m+n-1}} \left[\dfrac{1}{m+n} \sin(m+n)\theta + \theta \right]_{\theta=0}^{\theta=\pi} & \text{if } m = n \end{cases} \\
&= \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{2^{m+n-1}} & \text{if } m = n \end{cases}
\end{aligned}$$

and also since $n \neq 0$,

$$(p_0, p_n) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot 1 \cdot \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)\, dx$$

$$= \frac{1}{2^{n-1}} \int_0^{\pi} \cos n\theta \, d\theta$$

$$= \frac{1}{2^{n-1}} \cdot \frac{1}{n} \sin n\theta \Big|_0^{\pi}$$

$$= 0.$$

Therefore indeed $p_j(x)$ are the orthogonal polynomials with respect to weight $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

Now for $j \geq 2$, it holds that

$$p_{j+1}(x) = \frac{1}{2^j} T_{j+1}(x) = \frac{1}{2^j}\left(2x T_j(x) - T_{j-1}(x)\right)$$

$$= x \cdot \frac{1}{2^{j-1}} T_j(x) - \frac{1}{4} \cdot \frac{1}{2^{j-2}} T_{j-1}(x)$$

$$= x\, p_j(x) - \frac{1}{4} p_{j-1}(x)$$

and $p_2(x) = x^2 - \frac{1}{2} = x p_1(x) - \frac{1}{2} p_0(x)$. Therefore following the notation of (3.6.6), we have $\delta_{j+1} \equiv 0$ for all $j \geq 1$, while $\gamma_2^2 = \frac{1}{2}$ and $\gamma_{j+1}^2 = \frac{1}{4}$ for $j \geq 2$. The tridiagonal matrix (3.6.19) in this case is

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & & & & \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \frac{1}{2} & 0 & \ddots & \\ & & & \ddots & \ddots & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{bmatrix}.$$

(b) Thankfully it is given in the problem that $p_3(x) = x^3 - \frac{3}{4}x$. Hence the roots of $p_3(x)$ are $x_1 = -\frac{\sqrt{3}}{2}$, $x_2 = 0$, and $x_3 = \frac{\sqrt{3}}{2}$. Also it is given that $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2 - \frac{1}{2}$. Note that

$$(p_0, p_0) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x \Big|_{-1}^{1} = \pi.$$

Simple substitutions and calculations lead (3.6.13) into

$$\begin{cases} w_1 + w_2 + w_3 = \pi \\ -\frac{\sqrt{3}}{2} w_1 + 0 \cdot w_2 + \frac{\sqrt{3}}{2} w_3 = 0 \\ \frac{1}{4} w_1 - \frac{1}{2} w_2 + \frac{1}{4} w_3 = 0 \end{cases}.$$

The second equation asserts that $w_1 = w_3$, thus the third equation asserts that $2w_2 = w_1 + w_3 \Rightarrow w_1 = w_2 = w_3$. Therefore, with the first equation, we conclude that $w_1 = w_2 = w_3 = \frac{\pi}{3}$.

Computer Assignment

The program which does the required is submitted via KLMS along with this document. Given a function `f` with the domain of the integral [`a`, `b`], we desire to use Gaussian quadrature in order to approximate the integral $\int_a^b f(x)\,dx$. Using the change of variables $x = a + \frac{(t+1)(b-a)}{2}$ and letting $g(t) = f(x)$ we can transform the domain of intergation into $[-1, 1]$ as

$$\int_a^b f(x)dx = \int_{-1}^{1} \frac{b-a}{2} f\left(a + \frac{(t+1)(b-a)}{2}\right) dt = \frac{b-a}{2} \int_{-1}^{1} g(t)dt.$$

Now the last integral is from -1 to 1 thus we can apply the Gaussian quadrature method we have learnt. To compute the weights w_i and nodes x_i,with the help of Theorems 3.6.20 and 3.6.21, it suffices to compute the eigenvalue decomposition of the tridiagonal matrix

$$J_n := \begin{bmatrix} \delta_1 & \gamma_2 & & & \\ \gamma_2 & \delta_2 & \gamma_3 & & \\ & \gamma_3 & \delta_3 & \ddots & \\ & & \ddots & \ddots & \gamma_n \\ & & & \gamma_n & \delta_n \end{bmatrix}.$$

The nontrivial fact is that Legendre polynomials $P_n(x)$ satisfy the recurrence relation

$$P_{n+1}(x) = xP_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x), \quad n = 1, 2, \cdots$$

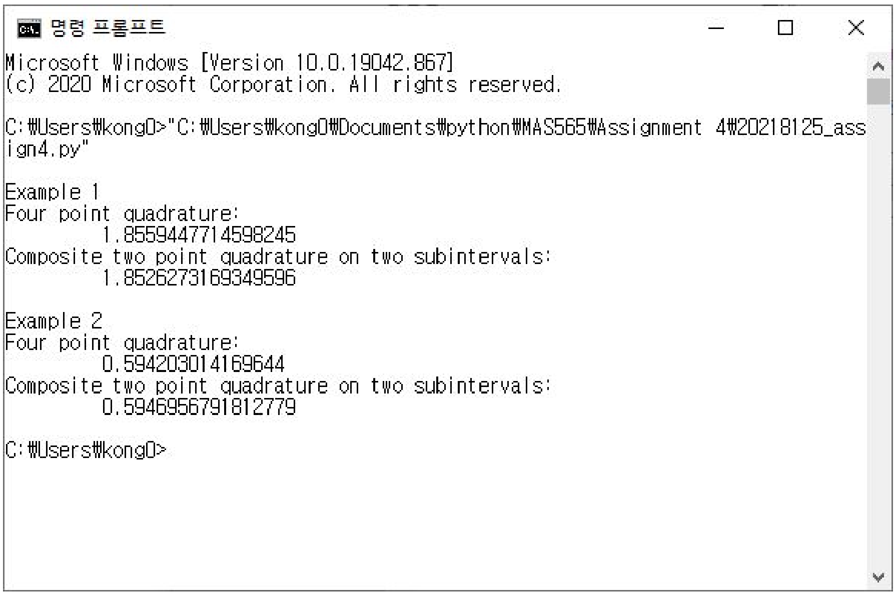
which tells us that $\delta_n = 0$ and $\gamma_{n+1} = \frac{n}{\sqrt{4n^2-1}}$. The recurrence relation above is proved in the appendix.

The four point Gaussian quadrature is now ready to be computed. For the two point composite Gaussian quadrature on two subintervals, we break the integral into

$$\int_{-1}^{1} g(t)\,dt = \int_{-1}^{0} g(t)\,dt + \int_{0}^{1} g(t)\,dt$$

and apply the integration procedure on the two integrals on the right hand side, starting over from changing the variables again.

The computed approximations of the integrals are printed out, as the following screenshot of the console window.



WolframAlpha tells us that the given integrals are approximately

$$
\begin{aligned}
I_1 &= \int_{-1}^{1} e^{x^2} \ln(2-x)\, dx \;\approx\; 1.8557244913526205, \\
I_2 &= \int_{1}^{3} \frac{dx}{\sqrt{x^4+1}} \;\approx\; 0.59411342601921545.
\end{aligned}
$$

We can see, in both cases, the four point Gaussian quadrature produces a better approximation of the integral than the two point composite Gaussian quadrature on two subintervals.

Appendix

As promised, we show that the Legendre polynomials $P_n(x)$ satisfy the recurrence relation

$$
P_{n+1}(x) = xP_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x), \quad n = 1, 2, \cdots .
$$

Lemma 1. *For any positive integer n, it holds that*

$$
\frac{d^n}{dx^n} x(x^2-1)^n = x \frac{d^n}{dx^n}(x^2-1)^n + n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n.
$$

Proof. The general Leibniz rule states that for any n times differentiable functions f and g it holds that

$$
(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.
$$

Applying the rule on $f(x) = x$ and $g(x) = (x^2-1)^n$, the derivatives of order higher than the second derivative of $f(x)$ is now zero, so the given equation immediately follows. □

Now recall the formula (3.6.18) which states that

$$
P_n(x) = \frac{n!}{(2n)!} \frac{d^n}{dx^n}(x^2-1)^n.
$$

Lemma 2. *It holds that*

$$
xP_n(x) = \frac{n!}{(2n)!} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\big((n+1)x^2 - (n-1)\big).
$$

Proof. With the help of **Lemma 1** we get

$$
\begin{aligned}
xP_n(x) &= \frac{n!}{(2n)!} \cdot x \frac{d^n}{dx^n}(x^2-1)^n \\
&= \frac{n!}{(2n)!} \left(\frac{d^n}{dx^n} x(x^2-1)^n - n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \right).
\end{aligned}
$$

Simple calculation leads to

$$
\begin{aligned}
\frac{(2n)!}{n!} \cdot xP_n(x) &= \frac{d^n}{dx^n} x(x^2-1)^n - n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \\
&= \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dx} x(x^2-1)^n - n(x^2-1)^n \right) \\
&= \frac{d^{n-1}}{dx^{n-1}} \big((x^2-1)^n + 2nx^2(x^2-1)^{n-1} - n(x^2-1)^n \big) \\
&= \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^{n-1}\big((n+1)x^2 + n - 1\big)
\end{aligned}
$$

and the result follows. □

Lemma 3. *The Legendre polynomials can also be represented as*

$$P_{n+1}(x) = \frac{(n+1)!}{(2n+1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\big((2n+1)x^2-1\big).$$

Proof. As we have

$$\begin{aligned}\frac{d^2}{dx^2}(x^2-1)^{n+1} &= 2(n+1)\frac{d}{dx}\,x(x^2-1)^n \\ &= 2(n+1)(x^2-1)^n + 2(n+1)x\cdot 2nx(x^2-1)^{n-1} \\ &= (2n+2)(x^2-1)^n + 4n(n+1)x^2(x^2-1)^{n-1} \\ &= (2n+2)(x^2-1)^{n-1}\big((2n+1)x^2-1\big)\end{aligned}$$

it follows that

$$\begin{aligned}P_{n+1}(x) &= \frac{(n+1)!}{(2n+2)!}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{d^2}{dx^2}(x^2-1)^{n+1}\right) \\ &= \frac{(n+1)!}{(2n+1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\big((2n+1)x^2-1\big)\end{aligned}$$

which is exactly the claimed. $\square$

Finally we are ready to show the following.

Theorem 1. *The Legendre polynomials $P_n(x)$ satisfy the recurrence relation*

$$P_{n+1}(x) = xP_n(x) - \frac{n^2}{4n^2-1}P_{n-1}(x), \quad n = 1, 2, \cdots.$$

Proof. From **Lemma 3** and equation (3.6.18) in the textbook, we have

$$\begin{aligned}\frac{(2n+1)!}{(n+1)!}P_{n+1}(x) &= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\big((2n+1)x^2-1\big), \\ \frac{(2n-2)!}{(n-1)!}P_{n-1}(x) &= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\end{aligned}$$

Adding these two equations side by side and dividing both sides by $2n+1$ we get

$$\frac{d^{n-1}}{dx^{n-1}}\,x^2(x^2-1)^{n-1} = \frac{(2n)!}{(n+1)!}P_{n+1}(x) + \frac{(2n-2)!}{(n-1)!(2n+1)}P_{n-1}(x).$$

For $n \geq 2$, by **Lemma 2** it follows that

$$\begin{aligned}\frac{(2n)!}{n!}\,xP_n(x) &= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\big((n+1)x^2+n-1\big) \\ &= (n+1)\left(\frac{(2n)!}{(n+1)!}P_{n+1}(x) + \frac{(2n-2)!}{(n-1)!(2n+1)}P_{n-1}(x)\right) + (n-1)\frac{(2n-2)!}{(n-1)!}P_{n-1}(x) \\ &= \frac{(2n)!}{n!}P_{n+1}(x) + \left(\frac{(2n-2)!(n+1)}{(n-1)!(2n+1)} + \frac{(2n-2!)}{(n-2)!}\right)P_{n-1}(x).\end{aligned}$$

Therefore

$$\begin{aligned}xP_n(x) &= P_{n+1}(x) + \left(\frac{n(n+1)}{2n(2n-1)(2n+1)} + \frac{n(n-1)}{2n(2n-1)}\right)P_{n-1}(x) \\ &= P_{n+1}(x) + \frac{n^2}{(2n-1)(2n+1)}P_{n-1}(x)\end{aligned}$$

hence the given recurrence relation. By noting that $P_2(x) = x^2 - \frac{1}{3}$, $P_1(x) = x$, and $P_0(x) = 1$ we see that the recurrence relation also holds for $n = 1$. $\square$