MAS565 Numerical Analysis HW4

20218125 최지석

3.3 Recall Theorem 2.1.4.1 which implies that, for any given $x \in [a,b]$ there exists $\xi(x) \in [a,b]$ such that

$$f(x) - p_{0,1}(x) = \frac{f''(\xi(x))}{2}(x-a)(x-b). \quad \cdots (*)$$

For $x \neq a$ and $x \neq b$ rewriting the above as

$$f''(\xi(x)) = \frac{2(f(x) - p_{0,1}(x))}{(x-a)(x-b)} \quad \cdots (**)$$

we observe that $f''(\xi(x))$ is continuous on the open interval (a,b). For the case when $x \in \{a,b\}$, equation $(*)$ allows us to define $f''(\xi(x))$ to be any number. Since $f(a) - p_{0,1}(a) = 0$ and $x \mapsto f(x) - p_{0,1}(x)$ is continuously differentiable everywhere, the function

$$\frac{f(x) - p_{0,1}(x)}{x-a}$$

converges as $x \to a$, and furthermore it converges to $f'(a) - p_{0,1}'(a)$. This implies that, with the help of expressing $f''(\xi(x))$ in the form of $(**)$, we can assign a value for $f''(\xi(a))$ so that $f''(\xi(x))$ is continuous at $x = a$. By symmetry, we can also assign a value for $f''(\xi(b))$ so that $f''(\xi(x))$ is continuous on whole $[a,b]$.

As $\xi(x) \in [a,b]$ for $x \in (a,b)$, we have

$$\{f''(\xi(x)) : x \in (a,b)\} \subseteq \{f''(t) : t \in [a,b]\}.$$

But the right hand side set is compact, being a continuous image of a compact interval. Therefore, the limits $f''(\xi(a))$ and $f''(\xi(b))$ are also contained in that compact set. Hence

$$\{f''(\xi(x)) : x \in [a,b]\} \subseteq \{f''(t) : t \in [a,b]\}.$$

Let $m = \min_{t \in [a,b]} f''(t)$ and $M = \max_{t \in [a,b]} f''(t)$ so that now $(*)$ implies

$$\frac{m}{2}(x-a)(x-b) \leq f(x) - p_{0,1}(x) \leq \frac{M}{2}(x-a)(x-b). \qquad \cdots (\divideontimes)$$

Integrate all sides over the interval $[a,b]$. As $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$, we get

$$-\frac{M}{12}(b-a)^3 \leq \int_a^b f(x) - p_{0,1}(x)\,dx \leq -\frac{m}{12}(b-a)^3$$

and henceforth

$$m \leq \frac{12}{(b-a)^3}\int_a^b p_{0,1}(x) - f(x)\,dx \leq M.$$

By Intermediate Value Theorem, there exists $\tilde{x} \in [a,b]$ such that

$$f''(\tilde{x}) = \frac{12}{(b-a)^3}\int_a^b p_{0,1}(x) - f(x)\,dx$$

As required from the problem, we also should show that it is possible to choose $\tilde{x}$ from the open interval (a,b). Without loss of generality, suppose that $\tilde{x} = a$ and no other point $t \in (a,b]$ satisfies $f''(a) = f''(t)$. But if so, by the continuity of f'' and the Intermediate Value Theorem, it must be the case where $f''(t) \geq f''(a)$ $\forall t \in (a,b]$, or $f''(t) \leq f''(a)$, $\forall t \in (a,b]$. Then $f''(a)$ is either m or M, respectively, so the corresponding side in the inequality $(\divideontimes)$ is attained. This implies that f is a quadratic function, so f'' must be constant, contradicting our assumption.

We now have a stronger result: there exists $\tilde{x} \in (a,b)$ such that

$$f''(\tilde{x}) = \frac{12}{(b-a)^3} \int_a^b p_{0,1}(x) - f(x)\, dx.$$

Writing down $p_{0,1}(x)$ explicitly as $p_{0,1}(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$, we have

$$\int_a^b p_{0,1}(x)\,dx = \int_a^b \frac{f(b)-f(a)}{b-a}\cdot(x-a) + f(a)\,dx$$

$$= \frac{f(b)-f(a)}{b-a} \times \frac{(b-a)^2}{2} + f(a)(b-a)$$

$$= \frac{b-a}{2}(f(a)+f(b)).$$

In conclusion, there exists $\tilde{x} \in (a,b)$ such that

$$\frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(x)\,dx = \frac{(b-a)^3}{12} f''(\tilde{x}).$$

3.14. We have to show that, under the assumptions (a) and (b),

$$\forall s \geq 0 \text{ polynomial, then } \int_a^b w(x)s(x)\,dx = 0 \Rightarrow s \equiv 0 \iff \int_a^b w(x)\,dx > 0.$$

($\Rightarrow$) Consider the constant polynomial $s(x) \equiv 1$. Then by assumption,

$$\int_a^b w(x)\,dx = \int_a^b w(x)s(x)\,dx > 0$$

because $w(x)s(x) \geq 0$ but $\int_a^b w(x)s(x)\,dx \neq 0$.

($\Leftarrow$) If $s \equiv 0$ then there is nothing to show. Hence assume that $s(x) \neq 0$ for some x. Our first goal is to find an interval $[y,z]$ such that $\int_y^z w(x)\,dx > 0$ while $s(x) > 0$ for all $x \in [y,z]$. If $s(x) > 0$ for all $x \in [a,b]$ then put $[y,z] = [a,b]$ and we are done. So we may assume that $s(x) = 0$ for some $x \in [a,b]$ but not all $x \in [a,b]$.

Consider the set

$$\{x \in [a,b] : s(x) = 0\} \cup \{a,b\}$$

which is finite since s is a polynomial. Enumerate the set above as $x_0, x_1, \dots, x_N$, in increasing order. Then by subdividing $[a,b]$ into subintervals $[x_{i-1}, x_i]$ for $i = 1, \dots, N$, we have the property that on each $[x_{i-1}, x_i]$ the value of $s(x)$ is positive possibly except at the boundary points. From

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} w(x)\,dx = \int_a^b w(x)\,dx > 0$$

while $w(x) \geq 0$ hence $\int_{x_{i-1}}^{x_i} w(x)\,dx \geq 0$, there should exist $j \in \{1, \dots, N\}$ such that

$$\int_{x_{j-1}}^{x_j} w(x)\,dx > 0.$$

Now there are three possibilities, for in each such cases we define sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as:

i) $-\infty < x_{j-1} < x_j < +\infty$: $a_n = x_{j-1} + \frac{1}{n}$, $b_n = x_j - \frac{1}{n}$

ii) $-\infty < x_{j-1} < x_j = +\infty$: $a_n = x_{j-1} + \frac{1}{n}$, $b_n = n$

iii) $-\infty = x_{j-1} < x_j < +\infty$: $a_n = -n$, $b_n = x_j - \frac{1}{n}$.

Then the intervals $I_n = [a_n, b_n]$, with the convention that if $a_n > b_n$ then $I_n = \emptyset$, satisfy $I_1 \subset I_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} I_n = [x_{j-1}, x_j]$. Hence $\{w(x)\mathbb{1}_{I_n}(x)\}_{n\in\mathbb{N}}$ is a monotonely increasing sequence of functions pointwise converging to $w(x)$, so by Monotone Convergence Theorem we have

$$0 < \int_{x_{j-1}}^{x_j} w(x)\,dx = \lim_{n\to\infty} \int_{x_{j-1}}^{x_j} w(x)\mathbb{1}_{I_n}(x)\,dx = \lim_{n\to\infty} \int_{a_n}^{b_n} w(x)\,dx.$$

Note that, this implies the existence of $n_0 \in \mathbb{N}$ such that $\int_{a_{n_0}}^{b_{n_0}} w(x)\,dx > 0$

for all $n > n_0$. By choosing n sufficiently large, we can also have $x_{j-1} < a_n < b_n < x_j$.

For such an n, recall that $s(x) > 0$ for $x \in (x_{j-1}, x_j)$, so $s(x) > 0$ on $[a_n, b_n]$.

Put $y = a_n$ and $z = b_n$ then we are done, i.e. $\int_y^z w(x)\,dx > 0$ and $s(x) > 0$ over the interval $[y, z]$.

Summarizing what we have got up to this point, we have an interval $[y, z]$ such that $s(x) > 0$ over that interval and $\int_y^z w(x)\,dx > 0$.

Whether the interval $[y, z]$ is infinite or finite, since $s(x)$ is a nonnegative polynomial, it attains its minimum over $[y, z]$. Put $m := \min_{x \in [y,z]} s(x)$, then $m > 0$, so

$$0 < m \int_y^z w(x)\,dx \le \int_y^z w(x) s(x)\,dx \le \int_a^b w(x) s(x)\,dx.$$

Therefore, if there exists $x \in [a, b]$ such that $s(x) > 0$ then $\int_a^b w(x) s(x)\,dx > 0$.

It follows that, if $\int_a^b w(x) s(x)\,dx = 0$ then $s \equiv 0$, completing the proof.

3.18. (a) Define $\tilde{T}_n(x) = \cos(n \cos^{-1} x)$. First we claim that $T_n(x) = \tilde{T}_n(x)$, on the domain $[-1, 1]$. It is clear that $\tilde{T}_0(x) = 1 = T_0(x)$ and $\tilde{T}_1(x) = x = T_1(x)$.

So it remains to show that $\tilde{T}_{n+1}(x) = 2x\tilde{T}_n(x) - \tilde{T}_{n-1}(x)$, for $n = 1, 2, \cdots$.

Put $\theta = \cos^{-1} x$, then

$$\tilde{T}_{n+1}(x) = \cos((n+1)\theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$$
$$\tilde{T}_{n-1}(x) = \cos((n-1)\theta) = \cos n\theta \cos\theta + \sin n\theta \sin\theta.$$

Adding these equations side by side we get

$$\tilde{T}_{n+1}(x) + \tilde{T}_{n-1}(x) = 2\cos n\theta \cos\theta = 2x\,\tilde{T}_n(x)$$

so we are done.

From the observation in the previous paragraph we may put $T_n(x) = \cos(n\cos^{-1}x)$. It is straightforward to see that T_j is a polynomial of degree j with leading coefficient 2^j by induction on j: indeed, $2x\,T_j(x)$ is a degree $j+1$ polynomial with leading coefficient $2\cdot 2^j = 2^{j+1}$ and $T_{j-1}(x)$ is a degree $j-1$ polynomial by induction hypothesis, so $T_{j+1}(x)$ is a degree $j+1$ polynomial with leading coefficient 2^{j+1}.

To construct a collection of monic polynomials, let $p_0(x) = 1$ and $p_j(x) = \frac{1}{2^{j-1}} T_j(x)$ for $j \geq 1$. It remains to show that $\{p_j(x)\}_{j\geq 0}$ are orthogonal polynomials with respect to weight $w(x) = \frac{1}{\sqrt{1-x^2}}$. Again let $\theta = \cos^{-1}x$ then $d\theta = -\frac{1}{\sqrt{1-x^2}}\,dx$, so for two positive integers m, n we have

$$\begin{aligned}
(p_m, p_n) &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2^{m-1}} \cos(m\cos^{-1}x) \cdot \frac{1}{2^{n-1}} \cos(n\cos^{-1}x)\,dx \\
&= \frac{1}{2^{m+n-2}} \int_0^{\pi} \cos m\theta \cos n\theta \, d\theta \\
&= \frac{1}{2^{m+n-2}} \int_0^{\pi} \frac{\cos(m+n)\theta + \cos(m-n)\theta}{2} \, d\theta \\
&= \begin{cases} \dfrac{1}{2^{m+n-1}} \left[\dfrac{1}{m+n} \sin(m+n)\theta + \dfrac{1}{m-n} \sin(m-n)\theta \right]_{\theta=0}^{\theta=\pi} & \text{if } m \neq n \\[2ex] \dfrac{1}{2^{m+n-1}} \left[\dfrac{1}{m+n} \sin(m+n)\theta + \theta \right]_{\theta=0}^{\theta=\pi} & \text{if } m = n \end{cases} \\
&= \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{2^{m+n-1}} & \text{if } m = n \end{cases}
\end{aligned}$$

and also since $n \neq 0$,

$$(p_0, p_n) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot 1 \cdot \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)\, dx$$

$$= \frac{1}{2^{n-1}} \int_0^{\pi} \cos n\theta \, d\theta$$

$$= \frac{1}{2^{n-1}} \cdot \frac{1}{n} \sin n\theta \Big|_0^{\pi}$$

$$= 0.$$

Therefore indeed $p_j(x)$ are the orthogonal polynomials with respect to weight $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

Now for $j \geq 2$, it holds that

$$p_{j+1}(x) = \frac{1}{2^j} T_{j+1}(x) = \frac{1}{2^j}\left(2x T_j(x) - T_{j-1}(x)\right)$$

$$= x \cdot \frac{1}{2^{j-1}} T_j(x) - \frac{1}{4} \cdot \frac{1}{2^{j-2}} T_{j-1}(x)$$

$$= x\, p_j(x) - \frac{1}{4} p_{j-1}(x)$$

and $p_2(x) = x^2 - \frac{1}{2} = x p_1(x) - \frac{1}{2} p_0(x)$. Therefore, following the notation of (3.6.6), we have $\delta_{i+1} \equiv 0$ for all $j \geq 1$, while $\gamma_2^2 = \frac{1}{2}$ and $\gamma_{j+1}^2 = \frac{1}{4}$ for $j \geq 2$. The tridiagonal matrix (3.6.19) in this case is

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & & & & \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \frac{1}{2} & 0 & \ddots & \\ & & & \ddots & \ddots & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{bmatrix}.$$

(b) Thankfully it is given in the problem that $p_3(x) = x^3 - \frac{3}{4}x$. Hence the roots of $p_3(x)$ are $x_1 = -\frac{\sqrt{3}}{2}$, $x_2 = 0$, and $x_3 = \frac{\sqrt{3}}{2}$. Also it is given that $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2 - \frac{1}{2}$. Note that

$$(p_0, p_0) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x \Big|_{-1}^{1} = \pi.$$

Simple substitutions and calculations lead (3.6.13) into

$$\begin{cases} w_1 + w_2 + w_3 = \pi \\ -\frac{\sqrt{3}}{2} w_1 + 0 \cdot w_2 + \frac{\sqrt{3}}{2} w_3 = 0 \\ \frac{1}{4} w_1 - \frac{1}{2} w_2 + \frac{1}{4} w_3 = 0 \end{cases}.$$

The second equation asserts that $w_1 = w_3$, thus the third equation asserts that $2w_2 = w_1 + w_3 \Rightarrow w_1 = w_2 = w_3$. Therefore, with the first equation, we conclude that $w_1 = w_2 = w_3 = \frac{\pi}{3}$.