MAS565 Numerical Analysis HW3

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2.22 First thing to observe is that, for d_j in (2.4.2.6) it holds that

$$d_j = \frac{6}{h_j+h_{j+1}}\left(\frac{y_{j+1}-y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j}\right)$$

$$= \frac{6}{x_{j+1}-x_{j-1}}\left(\frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j} - \frac{f(x_j)-f(x_{j-1})}{x_j - x_{j-1}}\right)$$

$$= \frac{6}{x_{j+1}-x_{j-1}}\left(f[x_j,x_{j+1}] - f[x_{j-1},x_j]\right)$$

$$= 6f[x_{j-1},x_j,x_{j+1}].$$

Let $J_j = [x_{j-1}, x_{j+1}]$ then there exists $\xi_j \in J_j$ such that

$$\frac{d_j}{6} = f[x_{j-1}, x_j, x_{j+1}] = \frac{f''(\xi_j)}{2!}.$$

Similar phenomena happens for d_0 and d_n in (2.4.2.8); observe that

$$d_0 = \frac{6}{h_1}\left(\frac{y_1-y_0}{h_1} - y_0'\right)$$

$$= \frac{6}{x_1-x_0}\left(\frac{f(x_1)-f(x_0)}{x_1-x_0} - f'(x_0)\right)$$

$$= \frac{6}{x_1-x_0}\left(f[x_0,x_1] - f[x_0,x_0]\right)$$

$$= 6f[x_0,x_0,x_1]$$

so let $J_0 = [x_0,x_1]$ then there exists $\xi_0 \in J_0$ such that

$$\frac{d_0}{6} = f[x_0,x_0,x_1] = \frac{f''(\xi_0)}{2}$$

and analogously

$$d_n = \frac{6}{h_n}\left(y_n' - \frac{y_n - y_{n-1}}{h_n}\right)$$
$$= \frac{6}{x_n - x_{n-1}}\left(f'(x_n) - \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}\right)$$
$$= \frac{6}{x_n - x_{n-1}}\left(f[x_n, x_n] - f[x_{n-1}, x_n]\right)$$
$$= 6 f[x_{n-1}, x_n, x_n]$$

so let $J_n = [x_{n-1}, x_n]$ then there exists $\xi_n \in J_n$ such that

$$\frac{d_n}{6} = f[x_{n-1}, x_n, x_n] = \frac{f''(\xi_n)}{2!}.$$

These observations, however show that if we only assume that $f \in \mathcal{K}^2(a,b)$ then the given statements may not hold. Indeed, $\xi_i \to x_j$ does not imply $f''(\xi_i) \to f''(x_j)$. For a concrete counterexample, consider a continuous function

$$g(t) = \begin{cases} t^2 \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

and its antiderivative $G(x) = \int_0^t g(t)\,dt$ on $[-1,1]$. As we have

$$G''(0) = g'(0) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0$$

while for $x \neq 0$ it holds that

$$G''(x) = g'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x^2}$$

we have $|g'(x)| \leq |2x||\sin\frac{1}{x}| + |\cos\frac{1}{x^2}| \leq 3$; that is, $g = G'$ is differentiable with its derivative bounded. Thus g is Lipschitz continuous, hence absolutely continuous. Furthermore $g' = G''$ is discontinuous only at $x=0$, so $G'' \in L^2[-1,1]$. It follows that $G \in \mathcal{K}^2(-1,1)$. Now, consider Δ where for some $j \in \{1, 2, \cdots, n-1\}$ and $h > 0$ we have $x_{j-1} = -h$, $x_j = 0$, and $x_{j+1} = h$

Because g is an odd function, G being its antiderivative is even.

Put $f(x) = G(x) + x^2$, then $f \in \mathcal{R}^2(-1,1)$, f is even, but now $f''(0) = 2$.

As $f(x_{j+1}) = f(x_{j-1}) = f(h)$ and $f(x_j) = f(0) = G(0) + 0^2 = 0$, we get

$$\begin{aligned} d_j &= \frac{6}{h_j + h_{j+1}} \left(\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} - \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \right) \\ &= \frac{6}{2h} \cdot \frac{f(x_{j+1}) + f(x_{j-1})}{h} \\ &= \frac{6}{h^2} f(h) \\ &= \frac{6}{h^2} G(h) + 6 . \end{aligned}$$

When $\|\Delta\| \to 0$, consequently $h \to 0$, then by L'Hôpital's rule

$$\lim_{h \to 0} \frac{G(h)}{h^2} = \lim_{h \to 0} \frac{g(h)}{2h} = \lim_{h \to 0} \frac{1}{2} h \sin \frac{1}{h} = 0 .$$

That is, $d_j \to 6 \neq f''(0)$ as $\|\Delta\| \to 0$.

We need sufficient regularity conditions. For the first part assume that $f \in C^3(a,b)$, or, at least f is twice differentiable with f'' Lipschitz continuous. Then for some constant K, it holds that

$$\begin{aligned} |f''(\xi_j) - f''(x_j)| &\leq K |\xi_j - x_j| \\ &\leq K \cdot |J_j| \\ &\leq 2K \cdot \|\Delta\| \end{aligned}$$

where $|J_j|$ denotes the length of the interval J_j. It follows that

$$f''(\xi_j) = f''(x_j) + O(\|\Delta\|),$$

and thus

$$d_j = 3f''(\xi_j) = 3f''(x_j) + O(\|\Delta\|).$$

Further, say that the knots are equidistant. Let h denote the distance between two neighboring knots. Then for $j=1,\cdots,n-1$, observe that d_j reduces into

$$d_j = \frac{6}{h+h}\left(\frac{f(x_{j+1})-f(x_j)}{h} - \frac{f(x_j)-f(x_{j-1})}{h}\right)$$

$$= \frac{3}{h^2}\left(f(x_{j+1})-2f(x_j)+f(x_{j-1})\right).$$

the regularity condition we consider here is that $f\in C^4[x_0,x_n]$. Then by Taylor's theorem, there exists $\varphi_j, \phi_j \in J_j$ such that

$$f(x_{j+1}) = f(x_j+h) = f(x_j)+hf'(x_j)+\frac{h^2}{2}f''(x_j)+\frac{h^3}{6}f^{(3)}(x_j)+\frac{h^4}{24}f^{(4)}(\varphi_j)$$

$$f(x_{j-1}) = f(x_j-h) = f(x_j)-hf'(x_j)+\frac{h^2}{2}f''(x_j)-\frac{h^3}{6}f^{(3)}(x_j)+\frac{h^4}{24}f^{(4)}(\phi_j)$$

hence adding the two equations above side by side we obtain

$$f(x_{j+1}) - f(x_{j-1}) = 2f(x_j)+h^2f''(x_j)+\frac{h^4}{24}\left(f^{(4)}(\varphi_j)+f^{(4)}(\phi_j)\right).$$

As $f^{(4)}$ is continuous, it is bounded on $[x_0,x_n]$, say $\|f^{(4)}\|_\infty \le M$, so for any $j=1,\cdots,n-1$ we get

$$\left|\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2} - f''(x_j)\right| = \frac{h^2}{24}\left|f^{(4)}(\varphi_j)+f^{(4)}(\phi_j)\right|$$

$$\le \frac{h^2}{12}M$$

henceforth

$$|d_j - 3f''(x_j)| = 3\left|\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2} - f''(x_j)\right|$$

$$\le \frac{h^2}{4}M$$

$$= \frac{M}{4}\|\Delta\|^2.$$

We conclude that $d_j = 3f''(x_j) + O(\|\Delta\|^2)$.

2.31 It is unclear what "spline-like functions" are. We assume that $E_{\Delta,f}(x)$, given λ_i, is a function which satisfies:

(i) $f(x_i) = E_{\Delta,f}(x_i)$, $i = 0, 1, \cdots, N$.

(ii) $E_{\Delta f}(x) \in C^2[a,b]$

(iii) $E_{\Delta,f}(x_i)$ minimizes the given functional over all $\mathcal{K}^2(a,b)$

(iv) One of the boundary conditions in (2.4.1.2)

and from that cubic splines coincide with a cubic polynomial in each subinterval $[x_{j-1}, x_j]$, with that Theorem 2.4.1.4 implicitly assumes that the spline function S_Δ is a function in $\mathcal{K}^4(a,b)$, we impose the condition

(v) $E_{\Delta f}$ on the interval $[x_{j-1}, x_j]$ is a function in $\mathcal{K}^4(x_{j-1}, x_j)$, for $j=1,\cdots,N$.

(a) There is an inconsistency in notation even within the given statement. We interpret the given as that $E_{\Delta f}(x)$ on each $[x_i, x_{i+1}]$ is defined as

$$E_{\Delta,f}(x) = \alpha_i + \beta_i x + \gamma_i \psi_i(x-x_i) + \delta_i \varphi_i(x-x_i) \qquad \cdots (*)$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are each some constants, and ψ_i, φ_i are functions defined as given. We proceed in several steps.

- Lemma 1. On each $[x_i, x_{i+1}]$, $E_{\Delta f}(x)$ as in $(*)$ is a solution to the ODE

$$y^{(4)} - \lambda_i^2 y'' = 0.$$

~~pf)~~ The auxiliary equation of the given ODE is $m^4 - \lambda_i^2 m^2 = 0$, of which the solutions are $m = \pm\lambda_i$ and $m=0$ with multiplicity 2. Hence the solution of the ODE is generated by the basis

$$\{e^{0x}, xe^{0 \cdot x}, \cosh(\lambda_i x), \sinh(\lambda_i x)\} = \{1, x, \cosh(\lambda_i x), \sinh(\lambda_i x)\},$$

Meanwhile, since $\frac{d^4}{dx^4} y(x-x_i) - \lambda_i^2 \frac{d^2}{dx^2} y(x-x_i) = y^{(4)}(x-x_i) - \lambda_i^2 y''(x-x_i)$ the general solution of the given ODE can be written in the form

$$y = A_i + B_i(x-x_i) + C_i \cosh(\lambda_i(x-x_i)) + D_i \sinh(\lambda_i(x-x_i)).$$

By setting $A_i = \alpha_i - \frac{2\gamma_i}{\lambda_i^2}$, $B_i = \beta_i - \frac{6\delta_i}{\lambda_i^2}$, $C_i = \frac{2\gamma_i}{\lambda_i^2}$, and $D_i = \frac{6\delta_i}{\lambda_i^3}$ we recover $E_{\Delta,f}(x)$, showing the claim. ■

- <u>Lemma 2</u> Define a seminorm of $f \in \mathcal{K}^2(a,b)$ as

$$\|f\|^2 = E[f] = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (f''(x))^2 + \lambda_i^2 (f'(x))^2 \, dx.$$

Let $E(x)$ be any interpolating spline of f, that is, satisfying conditions (i), (ii), and (v). Then it holds that

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2(f'-E')E''\Big|_a^b - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx. \quad \cdots (**)$$

~~pf)~~ By simple algebraic manipulation we get

$$(f''-E'')^2 + \lambda_i^2(f'-E')^2 = f''+\lambda_i^2(f')^2 + E'' + \lambda_i^2(E')^2 - 2(f''E''+\lambda_i^2 f'E')$$

$$= f''+\lambda_i^2(f')^2 - (E''+\lambda_i^2(E')^2) - 2((f''-E'')E'' + \lambda_i^2(f'-E')E')$$

and here integration gives us

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f''-E'')E'' + \lambda_i^2(f'-E')E'\,dx.$$

For each $i = 0, 1, \cdots, N-1$, integration by parts gives us

$$\int_{x_i}^{x_{i+1}} (f''-E'')E''\,dx + \lambda_i^2\int_{x_i}^{x_{i+1}} (f'-E')E'\,dx$$

$$= \Big[(f'-E')E'' + \lambda_i^2(f-E)E'\Big]_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (f'-E')E'''\,dx - \lambda_i^2\int_{x_i}^{x_{i+1}} (f-E)E''\,dx$$

$$= \Big[(f'-E')E'' + \lambda_i^2(f-E)E' - (f-E)E'''\Big]_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx.$$

But by assumption, $f(x_{i+1}) - E(x_{i+1}) = f(x_i) - E(x_i) = 0$, and $(f'-E')E''$ is continuous, thus in total we get

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2\sum_{i=0}^{N-1}(f'-E')E''\Big|_{x_i}^{x_{i+1}} - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx$$

$$= \|f\|^2 - \|E\|^2 - 2(f'-E')E''\Big|_a^b - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx. \quad ■$$

- <u>Theorem 3</u> Let $E_{\Delta,f}(x)$ be an interpolating spline of f, i.e. satisfies the conditions (i), (ii), and (v), and is of the form (*) on each $[x_i, x_{i+1}]$. Then it holds that

$$\|f - E_{\Delta,f}\|^2 = \|f\|^2 - \|E_{\Delta,f}\|^2 - 2(f'-E')E''\Big|_a^b.$$

Furthermore if $E_{\alpha,f}$ satisfies the boundary condition (iv) then we have

$$\|f\|^2 - \|E_{\alpha,f}\|^2 = \|f - E_{\alpha,f}\|^2 \geq 0.$$

pf) The first part follows immediately from Lemma 1 and Lemma 2, as $E_{\alpha,f}^{(4)} - \lambda_i^2 E_{\alpha,f}''$ is identically 0 on (α_i, α_{i+1}), making the last term in (**) vanish. The second part is also immediate, as if the boundary condition is met then clearly

$$(f'(b) - E'(b))E''(b) - (f'(a) - E'(a))E''(a) = 0. \quad \blacksquare$$

• Theorem 4. Let $\mathcal{E}_{\alpha,f}(x)$ be an interpolating spline of f satisfying the conditions (i), (ii), (iv), and (v). Among such splines, let $E_{\alpha,f}(x)$ be the one that is of the form (*). Then $E_{\alpha,f}(x)$ also satisfies the condition (iii), that is, $E_{\alpha,f}(x)$ minimizes the given functional.

pf) From the given conditions, $E_{\alpha,f}$ can be seen as not only the interpolating spline of f but also an interpolating spline of $\mathcal{E}_{\alpha,f}$. Hence by Theorem 3, we have

$$\|\mathcal{E}_{\alpha,f}\|^2 \geq \|E_{\alpha,f}\|^2. \quad \blacksquare$$

By Theorem 4, $E_{\alpha,f}(x)$ should be in the form (*).

(b) By applying L'Hôpital's rule repeatedly, we get

$$\begin{aligned}
\lim_{\lambda_i \to 0} \frac{6(\sinh(\lambda_i x) - \lambda_i x)}{\lambda_i^3} &= \lim_{\lambda_i \to 0} \frac{6(x\cosh(\lambda_i x) - x)}{3\lambda_i^2} \\
&= \lim_{\lambda_i \to 0} \frac{2(\cosh(\lambda_i x) - 1)}{\lambda_i^2} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{2x\sinh(\lambda_i x)}{2\lambda_i} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{x^2\cosh(\lambda_i x)}{1} \cdot x \quad = x^3
\end{aligned}$$

asserting that $\lim_{\lambda_i \to 0} \psi_i(x - \alpha_i) = (x - \alpha_i)^2$ and $\lim_{\lambda_i \to 0} \varphi_i(x - \alpha_i) = (x - \alpha_i)^3$. That is, when $\lambda_i \to 0$ the exponential spline reduces into a cubic spline.