2021/8/25 처자석

#6.25. By the Geršgorin circle theorem, we know that three eigenvalues of A all lie on the union of three disks

$$D_1 = \{z : |z - 5.2| \le 2.8\},$$

$$D_2 = \{z : |z - 6.4| \le 1.1\},$$

$$D_3 = \{z : |z - 4.7| \le 2.7\}.$$

As A is symmetric all eigenvalues of A are real. The intersection of the real axis with the disks $D_1, D_2,$ and D_3 are $[2.4, 8]$, $[5.3, 7.5]$, and $[2, 7.4]$, respectively. Hence for any eigenvalue $\lambda(A)$ of A, we have the bounds $2 \le \lambda(A) \le 8$. Recall Exercise 8(a), from which we get

$$cond_2(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \le \frac{8}{2} = 4$$

since A being symmetric implies A being normal.

#6.26. (a) We know nothing about off-diagonal entries except that all off-diagonal entries have modulus less than or equal to $\frac{1}{4}$. Hence all we can do is to apply Geršgorin circle theorem directly. Once we note that for each row and column the sum of the moduli of off-diagonal entries are at most 1 thus each Geršgorin disks have radius bounded by 1, we conclude that the eigenvalues $\lambda_1, \cdots, \lambda_5$ of A with

$|\lambda_1| \geq \cdots \geq |\lambda_5|$ we have

$$|\lambda_1 - 21| \leq 1, \quad |\lambda_2 + 9| \leq 1, \quad |\lambda_3 - 4| \leq 1,$$

and since two Geršgorin disks $\{z : |z-1| \leq 1\}$ and $\{z : |z-0| \leq 1\}$ have nonempty intersection the best we can conclude is that

$$\{\lambda_4, \lambda_5\} \in \{z : |z-1| < 1 \text{ or } |z| < 1\}.$$

(b) Consider $D = \text{diag }\{d_1, d_2, d_3\}$ then for $A = \begin{bmatrix} 1 & 10^{-3} & 10^{-4} \\ 10^{-3} & 2 & 10^{-3} \\ 10^{-4} & 10^{-3} & 3 \end{bmatrix}$, the matrix $D^{-1}AD$ has the same spectrum with A and its (i,j)-entry is $\frac{a_{ij}d_j}{d_i}$ as long as $d_1 d_2 d_3 \neq 0$. Denote $\rho_i := \sum_{\substack{k=1 \\ k \neq i}}^{3} \left| \frac{a_{ik}d_k}{d_i} \right|$. Note that the diagonal entries are invariant under this similarity transformation. So the Geršgorin disks are always centered at 1, 2, and 3; hence, the Geršgorin disks always can be written as $D_i = \{z : |z-i| \leq \rho_i\}$ for $i = 1, 2, 3$.

Consider the special case where $d_1 = 1$, $d_2 = 10k$, $d_3 = k$ for positive real k. Then $\rho_1 = 10^{-2}k + 10^{-4}k$, $\rho_2 = 10^{-4}/k + 10^{-4}$, and $\rho_3 = 10^{-4}/k + 10^{-2}$. As $\rho_3 < \rho_2 + 1$, if $D_2 \cap D_1 = \emptyset$ then $D_3 \cap D_1 = \emptyset$ also. For $D_2 \cap D_1$ to be $\emptyset$ we must have $\rho_1 + \rho_2 < 1$. As $k \mapsto \rho_1 + \rho_2$ defines a convex function we have an interval that satisfies $\rho_1 + \rho_2 < 1$, and the infimum of k satisfying $\rho_1 + \rho_2 < 1$ is such that $\rho_1 + \rho_2 = 1$. Solving the equation $\rho_1 + \rho_2 = (10^{-2} + 10^{-4})k + \frac{10^{-4}}{k} + 10^{-4} = 1$ the root of this equation that is smaller is $\frac{9999 - \sqrt{99919597}}{202} < 1.0002 \times 10^{-4}$.

So ρ_1 can be made as small as 1.0102×10^{-6}.

Choosing $d_1 = k$, $d_2 = 10k$, and $d_3 = 1$ on the other hand now gives $\rho_1 = 10^{-4}/k + 10^{-2}$, $\rho_2 = 10^{-4}/k + 10^{-4}$, and $\rho_3 = 10^{-2}k + 10^{-4}k$, so the exact same logic in the previous paragraph except 1 and 3 exchanged applies, allowing

ρ_3 to be as small as 1.0102×10^{-6}. For each $i = 1, 2,$ and 3 as A is already a diagonally dominant matrix we have no ambiguity of letting λ_i to be the eigenvalue of A that is closest to i, then we get $|\lambda_1 - 1| < 1.0102 \times 10^{-6}$ and $|\lambda_3 - 3| < 1.0102 \times 10^{-6}$.

Finally to estimate λ_2, consider the special case where $\lambda_1 = k$, $\lambda_2 = 1$, and $\lambda_3 = k$ for some positive real k. Then we get $\rho_1 = \rho_3 = 10^{-3}/k + 10^{-4}$, and $\rho_2 = 2 \times 10^{-3} k$. Now the configuration of the disks become symmetric along the vertical line $\text{Re}(z) = 2$, so we get that three disks D_1, D_2, D_3 are disjoint if and only if $\rho_1 + \rho_2 < 1$. Again $k \mapsto \rho_1 + \rho_2$ is a convex function, so infimum of k occurs when $\rho_1 + \rho_2 = 1$. Solving this equation we get $k = \frac{9999 - \sqrt{99997201}}{40} < 1.0002 \times 10^{-3}$, so ρ_2 can be made as small as 2.0004×10^{-6}. Hence $|\lambda_2 - 2| < 2.0004 \times 10^{-6}$.

#6.27. (a) Let X be the block diagonal matrix $X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ then by the normality of H, from theorem 6.9.7, we get that for any eigenvalue $\lambda(X)$ of X there exists an eigenvalue $\lambda(H)$ of H such that

$$|\lambda(X) - \lambda(H)| \leq \text{lub}_2(X - H). \qquad \cdots (*)$$

Now as $\det(X - \mu I) = \det \begin{bmatrix} A - \mu I & 0 \\ 0 & B - \mu I \end{bmatrix} = \det(A - \mu I)\det(B - \mu I)$, any eigenvalue $\lambda(B)$ of B is also an eigenvalue of X. Thus for any eigenvalue $\lambda(B)$ of B, there exists an eigenvalue $\lambda(H)$ of H such that

$$|\lambda(B) - \lambda(H)| \leq \text{lub}_2(X - H).$$

If we can show that $lub_2(X-H) \leq \sqrt{lub_2(C^H C)}$ then we are done.

Observe that $X - H = \begin{bmatrix} 0 & -C \\ -C^H & 0 \end{bmatrix}$ so for vectors x and y of

suitable sizes and $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 = \|x\|^2 + \|y\|^2 = 1$, let σ_{max} be the largest

singular value of C then

$$(lub_2(X-H))^2 = \left(\max_{\|[^x_y]\|_2 = 1} \left\| \begin{bmatrix} 0 & C \\ C^H & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2 \right)^2$$

$$= \left(\max_{\|[^x_y]\|_2 = 1} \left\| \begin{bmatrix} Cy \\ C^H x \end{bmatrix} \right\|_2 \right)^2$$

$$= \max_{\|[^x_y]\|_2 = 1} \left\| \begin{bmatrix} Cy \\ C^H x \end{bmatrix} \right\|_2^2 \qquad \left(\because \left\| \begin{bmatrix} Cy \\ C^H x \end{bmatrix} \right\|_2 \geq 0 \right)$$

$$= \max_{\|x\|_2^2 + \|y\|_2^2 = 1} \|Cy\|_2^2 + \|C^H x\|_2^2$$

$$\leq \max_{\|x\|_2^2 + \|y\|_2^2 = 1} \sigma_{max}^2 \|y\|^2 + \sigma_{max}^2 \|x\|^2$$

$$= \sigma_{max}^2.$$

But we know that $\sigma_{max}^2 = lub_2(C^H C)$. Therefore $lub(X-H) \leq \sqrt{lub_2(C^H C)}$ indeed.

(b) Actually from (*) it follows that for any eigenvalue $\lambda(A)$ of

A there exists an eigenvalue $\lambda(H)$ of H such that

$$|\lambda(A) - \lambda(H)| \leq lub_2(X-H) \leq \sqrt{lub_2(C^H C)}.$$

Furthermore as X is also Hermitian hence normal, we can change the role

of X and H to conclude the reverse, that is, for any eigenvalue $\lambda(H)$ of

H, there exists an eigenvalue λ of either A or B such that

$$|\lambda(H) - \lambda| \leq lub_2(H-X) \leq \sqrt{lub_2(C^H C)}.$$

In the almost reducible tridiagonal case, $C^H C$ is a matrix with $(1,1)$-entry ε^2

and all other entries 0. Being diagonal, we have $\sqrt{lub_2(C^H C)} = \sqrt{\varepsilon^2} = \varepsilon$, Therefore,

all eigenvalues of H are in distance at most ε from some eigenvalue of either A or B.

#6.32. In order to apply Berdixon's theorem, define

$$H_1 = \frac{1}{2}(A + A^H) = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 15 & -2 \\ 0 & -2 & 19.5 \end{bmatrix} \quad \text{and} \quad H_2 = \frac{1}{2i}(A - A^H) = \begin{bmatrix} 0 & -0.1i & 0.1i \\ 0.1i & 0 & 0.1i \\ -0.1i & -0.1i & 0 \end{bmatrix}$$

then a direct application of the Geršgorin theorem on H_2 we get $-0.2 \le \lambda(H_2) \le 0.2$ for any eigenvalue $\lambda(H_2)$ of H_2, so it follows that $-0.2 \le \operatorname{Im} \lambda \le 0.2$ for any eigenvalue λ of A. Also applying Geršgorin theorem directly to H_1 we see that $2 \le \lambda_{min}(H_1) \le \lambda_{max}(H_1) \le 21.5$, but using similarity transformations as in #6.26(b) we can improve this bound. If we use a similarity transform $A \mapsto D^{-1}AD$ with an invertible diagonal matrix D then the diagonal entries do not change, and since A and H_1 have the same main diagonals. For convenience (with abuse of notation) we let D_1, D_2, and D_3 the Geršgorin disks each centered at 3, 15, and 19.5, and let their radii be ρ_1, ρ_2, and ρ_3, respectively. As, for any matrix A, A and A^T share the same spectrum, the Geršgorin disks can be constructed columnwise, that is, we may take ρ_i's to be the sum of the moduli of off-diagonal entries in the ith column. This will be the case in this solution. Consider $D^{-1}H_1 D$ with $D = \operatorname{diag}\{\lambda\mu, \mu, 1\}$, $\lambda > 0$, $\mu > 0$, then $\rho_1 = \lambda$, $\rho_2 = \frac{1}{\lambda} + 2\mu$, and $\rho_3 = \frac{2}{\mu}$. Finding an exact solution of a nonconvex minimization problem minimizing $\min\{3 - \rho_1, 15 - \rho_2, 19.5 - \rho_3\}$ is a difficult task, so we just take $\lambda = \frac{1}{11}, \mu = \frac{1}{2}$ so that $\min\{3 - \rho_1, 15 - \rho_2, 19.5 - \rho_3\} = 3 - \frac{1}{11}$. Similarly, maximizing $\max\{3 + \rho_1, 15 + \rho_2, 19.5 + \rho_3\}$ is also a difficult task, so we take $\mu = \frac{9}{4}$, $\lambda = \frac{9}{8}$ so that $\max\{3 + \rho_1, 15 + \rho_2, 19.5 + \rho_3\} = \frac{367}{18}$. From Berdixon theorem we have a rectangular region $R = \{z : \frac{32}{11} \le \operatorname{Re} z \le \frac{367}{18}, -0.2 \le \operatorname{Im} z \le 0.2\}$ which includes all eigenvalues of A. However we can reduce the region further

by applying Geršgorin theorem to $D^{-1}AD$ for a suitable diagonal matrix D.

Take $D = \text{diag}\{k, 1, 1\}$ for positive k. Then it becomes $\rho_1 = 0.9k + 0.1k = k$,

$\rho_2 = 1.9 + \frac{1.1}{k}$, and $\rho_3 = 2.1 + \frac{0.1}{k}$. For D_1 to be as small as possible while

not intersecting D_2 nor D_3, we need $\rho_1 + \rho_2 < 12$, and if $k < 5$ then

$\rho_2 > \rho_3$ hence $D_2 \cap D_1 = \emptyset$ will imply $D_3 \cap D_1 = \emptyset$. Using an argument similar

to #6.26(b) the optimal k is the one satisfying $\rho_1 + \rho_2 = k + 1.9 + \frac{1.1}{k} = 12$,

which is $k^* = \frac{1}{20}(101 - \sqrt{9761}) \approx 0.1101$, that is, the eigenvalue of A with the smallest

modulus is in the disk $D_1^* = \{z: |z - 3| \le \frac{1}{20}(101 - \sqrt{9761})\}$. On the other hand,

take $D = \text{diag}\{1, k, 1\}$, then it becomes $\rho_1 = \frac{0.9}{k} + 0.1$, $\rho_2 = 1.1k + 1.9k = 3k$,

and $\rho_3 = 0.1 + \frac{2.1}{k}$. For $D_1 \cap D_2 = \emptyset$ we need $\rho_1 + \rho_2 = \frac{0.9}{k} + 0.1 + 3k < 12$, which

gives the interval approximately $(0.07713, 3.8895)$. For $D_1 \cap D_3 = \emptyset$ we need:

$\rho_1 + \rho_3 = \frac{3}{k} + 0.2 < 16.5 \iff k > \frac{30}{163} \approx 0.1840$. That is, for D_1 not to intersect

$D_2 \cup D_3$ we need $\frac{30}{163} < k < 3.8875$. Now what we want is to have $D_2 \cup D_3$

as small as possible, but we already have an upper bound for $\text{Re}\,\lambda$

so we need a lower bound of $\{\text{Re}(z): z \in D_2 \cup D_3\}$. Then we have

to maximize $\min\{15 - \rho_2, 19.5 - \rho_3\} = \min\{15 - 3k, 19.4 - \frac{2.1}{k}\}$. As $15 - 3k$ is a

decreasing function while $19.4 - \frac{2.1}{k}$ is increasing in $k > 0$, the maximum is

when $15 - 3k = 19.4 - \frac{2.1}{k}$. Solving this for k we get $k^{**} = \frac{\sqrt{1114} - 22}{30} \approx 0.3792$,

and the maximum becomes $15 - 3k^{**} \approx 13.862$. Especially, this happens when

D_2 is inscribed in D_3, so $D_3^* = \{z: |z - 19.5| \le 0.1 + \frac{63}{\sqrt{1114} - 22}\}$ is the region

containing the remaining two eigenvalues of A.

In conclusion we obtain the region $R \cap (D_1^* \cup D_3^*)$ which contains all three

eigenvalues of A. This region is shaded in the complex plane as the following figure