Furthermore if $E_{\Delta,f}$ satisfies the boundary condition (iv) then we have

$$\|f\|^2 - \|E_{\Delta,f}\|^2 = \|f - E_{\Delta,f}\|^2 \geq 0.$$

pf) The first part follows immediately from Lemma 1 and Lemma 2, as $E_{\Delta f}^{(4)} - \lambda_i^2 E_{\Delta f}''$ is identically 0 on (x_i, x_{i+1}), making the last term in (**) vanish. The second part is also immediate, as if the boundary condition is met then clearly

$$(f'(b) - E'(b))E''(b) - (f'(a) - E'(a))E''(a) = 0. \quad \blacksquare$$

• Theorem 4. Let $E_{\Delta f}(x)$ be an interpolating spline of f satisfying the conditions (i), (ii), (iv), and (v). Among such splines, let $E_{\Delta f}(x)$ be the one that is of the form (*). Then $E_{\Delta f}(x)$ also satisfies the condition (iii), that is, $E_{\Delta f}(x)$ minimizes the given functional.

pf) From the given conditions, $E_{\Delta f}$ can be seen as not only the interpolating spline of f but also an interpolating spline of $\mathcal{E}_{\Delta f}$. Since $\mathcal{E}_{\Delta f} \in C^2(a,b)$ implies $\mathcal{E}_{\Delta,f} \in \mathcal{K}^2(a,b)$, by Theorem 3 we get

$$\|\mathcal{E}_{\Delta,f}\|^2 \geq \|E_{\Delta,f}\|^2. \quad \blacksquare$$

By Theorem 4, $E_{\Delta,f}(x)$ should be in the form (*).

(b) By applying L'Hôpital's rule repeatedly, we get

$$\begin{aligned}
\lim_{\lambda_i \to 0} \frac{6(\sinh(\lambda_i x) - \lambda_i x)}{\lambda_i^3} &= \lim_{\lambda_i \to 0} \frac{6(x\cosh(\lambda_i x) - x)}{3\lambda_i^2} \\
&= \lim_{\lambda_i \to 0} \frac{2(\cosh(\lambda_i x) - 1)}{\lambda_i^2} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{2x \sinh(\lambda_i x)}{2\lambda_i} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{x^2 \cosh(\lambda_i x)}{1} \cdot x \quad = x^3
\end{aligned}$$

asserting that $\lim_{\lambda_i \to 0} \psi_i(x - x_i) = (x - x_i)^2$ and $\lim_{\lambda_i \to 0} \varphi_i(x - x_i) = (x - x_i)^3$. Therefore, when $\lambda_i \to 0$ the exponential spline reduces into a cubic spline.