MAS565 Numerical Analysis HW 1

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2.3. Fix any $\bar{x} \in [-1,1]$. First suppose that $\bar{x} \neq x_0$ and $\bar{x} \neq x_1$.

The interpolating polynomial $p(x)$ is

$$p(x) = f(x_0) \cdot \frac{x - x_1}{x_0 - x_1} + f(x_1) \cdot \frac{x - x_0}{x_1 - x_0}.$$

Consider a function $g : [-1,1] \to \mathbb{R}$ defined as

$$g(x) := f(x) - p(x) - \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)} (x - x_0)(x - x_1)$$

then clearly, $g(x_0) = g(x_1) = g(\bar{x}) = 0$. Since x_0, x_1, and $\bar{x}$ are all in $[-1,1]$, applying the Mean Value theorem there exist two points y_0 and y_1 in $[-1,1]$ such that $g'(y_0) = g'(y_1) = 0$, and again applying the Mean Value Theorem there exists some point $c \in [-1,1]$ such that $g''(c) = 0$. Meanwhile, $p(x)$ is linear, so a direct computation shows that

$$g''(x) = f''(x) - 2 \cdot \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)}$$

and, in turn, substituting $x = c$ we obtain

$$0 = g''(c) = f''(c) - 2 \cdot \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)}$$

$$\Rightarrow \quad f(\bar{x}) - p(\bar{x}) = \frac{1}{2} f''(c) \cdot (\bar{x} - x_0)(\bar{x} - x_1). \quad \cdots \quad (*)$$

It is clear that $(*)$ holds also when $\bar{x} = x_0$ or $\bar{x} = x_1$, as both sides of $(*)$ become zero. Therefore

$$|f(x) - p(x)| \leq \frac{1}{2} \max_{\xi \in I} |f''(\xi)| \max_{x \in I} |(x-x_0)(x-x_1)| .$$
$$= \alpha,$$

for any $x \in I$. Therefore α is indeed the upper bound for the maximum absolute interpolation error.

To minimize α, we should minimize $\max_{x \in I} |(x-x_0)(x-x_1)|$. We claim that $\{x_0, x_1\} = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ is the minimizer. To show this claim, we show the stronger statement: we show that for any $a, b \in \mathbb{R}$, it holds that

$$\max_{x \in I} \left|(x-\tfrac{1}{\sqrt{2}})(x+\tfrac{1}{\sqrt{2}})\right| = \max_{x \in I} |x^2 - \tfrac{1}{2}| \leq \max_{x \in I} |x^2 + ax + b| .$$

Let $T(x) = x^2 - \frac{1}{2}$, $g(x) = x^2 + ax + b$. It is clear that $\max_{x \in I} |x^2 - \frac{1}{2}| = \frac{1}{2}$, so for the sake of contradiction suppose that $\max_{x \in I} |g(x)| < \frac{1}{2}$. Then we would have

$$T(1) - g(1) = \tfrac{1}{2} - g(1) > 0,$$
$$T(0) - g(0) = -\tfrac{1}{2} - g(0) < 0,$$
$$T(-1) - g(-1) = \tfrac{1}{2} - g(-1) > 0$$

hence by the Intermediate Value Theorem, $T(x) - g(x)$ has at least two zeros in I. But $T(x) - g(x)$ is also a nonzero polynomial of degree at most 1, which is absurd. Therefore indeed $\{x_0, x_1\} = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ minimizes α.

Now, let $\theta = \cos^{-1} x$, then

$$\cos(2\cos^{-1} x) = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$$

so we have the relation, $(x-x_0)(x-x_1) = \frac{1}{2}\cos(2\cos^{-1} x)$ for

x_0, x_1 minimizing α.

2.4. We know that the error of the polynomial interpolation is given by, for some $\xi \in [a,b]$,

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n).$$

Since $|f^{(n+1)}(\xi)| \leq M$ for any $n \in \mathbb{N}$ and $\xi \in [a,b]$, we have

$$\begin{aligned}
\|f(x) - P_n(x)\|_\infty &\leq \frac{1}{(n+1)!} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)| \, |(x-x_0)\cdots(x-x_n)| \\
&\leq \frac{M}{(n+1)!} |x-x_0| \cdots |x-x_n| \\
&\leq \frac{M}{(n+1)!}(b-a)^{n+1}.
\end{aligned}$$

As $b-a$ is a fixed constant we have $\lim_{n\to\infty} \frac{(b-a)^{n+1}}{(n+1)!} = 0$, henceforth

$$\lim_{n\to\infty} \|f(x) - P_n(x)\|_\infty = 0.$$

That is, $P_n(x)$ converges uniformly on $[a,b]$ to $f(x)$ as $n \to \infty$.

2.16. The following lemma will be useful to solve this problem

Lemma. Let N be any positive integer, and for an integer j, $0 \le j \le N-1$, let $\alpha_j = \frac{2\pi j}{N}$. Then

$$\sum_{j=0}^{N-1} \cos(k\alpha_j) = \begin{cases} N & \text{if } k \equiv 0 \mod N \\ 0 & \text{if } k \not\equiv 0 \mod N, \end{cases}$$

$$\sum_{j=0}^{N-1} \sin(k\alpha_j) = 0.$$

pf) If $k \equiv 0 \mod N$ then $k\alpha_j = 2\pi j \cdot \frac{k}{N}$ is always an integer multiple, so the assertion follows. Hence we are left with the case where $k \not\equiv 0 \mod N$. If this is the case, $e^{2\pi k i/N} \neq 1$ but $(e^{2\pi k i/N})^N = 1$, so we have

$$1 + e^{2\pi k i/N} + e^{4\pi k i/N} + \cdots + e^{2(N-1)\pi k i/N} = \frac{1 - e^{2N\pi k i/N}}{1 - e^{2\pi k i/N}} = 0. \quad \cdots (**)$$

Taking the real parts of $(**)$ from both sides we get

$$\cos 0 + \cos\frac{2\pi k}{N} + \cdots + \cos\frac{2(N-1)\pi k}{N} = \sum_{j=0}^{N-1} \cos(k\alpha_j) = 0$$

while taking the imaginary parts we get

$$\sin 0 + \sin\frac{2\pi k}{N} + \cdots + \sin\frac{2(N-1)\pi k}{N} = \sum_{j=0}^{N-1} \sin(k\alpha_j) = 0. \quad \blacksquare$$

By theorem 2.3.1.12, under the assumption that $\Psi(\alpha_k) = f(\alpha_k) =: f_k$, we have the relations

$$\begin{cases} A_k = \frac{2}{N}\sum_{j=0}^{N-1} f_j \cos k\alpha_j \\ B_k = \frac{2}{N}\sum_{j=0}^{N-1} f_j \sin k\alpha_j \end{cases} \quad \text{for } k = 0, 1, \cdots, 2m.$$

By the absolute convergence we have

$$f_j = f(x_j) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx_j) + b_n \sin(nx_j)\right).$$

Hence if we set $N = 2m+1$, for $k = 0, 1, \dots, m$ we get

$$\begin{aligned}
A_k &= \frac{2}{N}\sum_{j=0}^{N-1}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx_j) + b_n\sin(nx_j)\right)\right]\cos(kx_j) \\
&= \frac{2}{N}\sum_{j=0}^{N-1}\frac{a_0}{2}\cos(kx_j) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} a_n\cos(nx_j)\cos(kx_j)\right) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} b_n\sin(nx_j)\cos(kx_j)\right) \\
&= \sum_{j=0}^{N-1}\frac{a_0}{N}\cos(kx_j) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{a_n}{N}\left(\cos((n+k)x_j) + \cos((n-k)x_j)\right) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{b_n}{N}\left(\sin((n+k)x_j) + \sin((n-k)x_j)\right).
\end{aligned}$$

By the Lemma, the first term is a_0 only when $k=0$ and otherwise 0, the third term always vanishes, and the second term is nonzero only when $n \geq 1$ and $n \equiv \pm k \pmod{2m+1}$ and otherwise zero. Hence

$$A_k = a_k + \sum_{p=1}^{\infty}\left(a_{p(2m+1)+k} + a_{p(2m+1)-k}\right).$$

Similarly for $k = 1, \dots, m$ we get

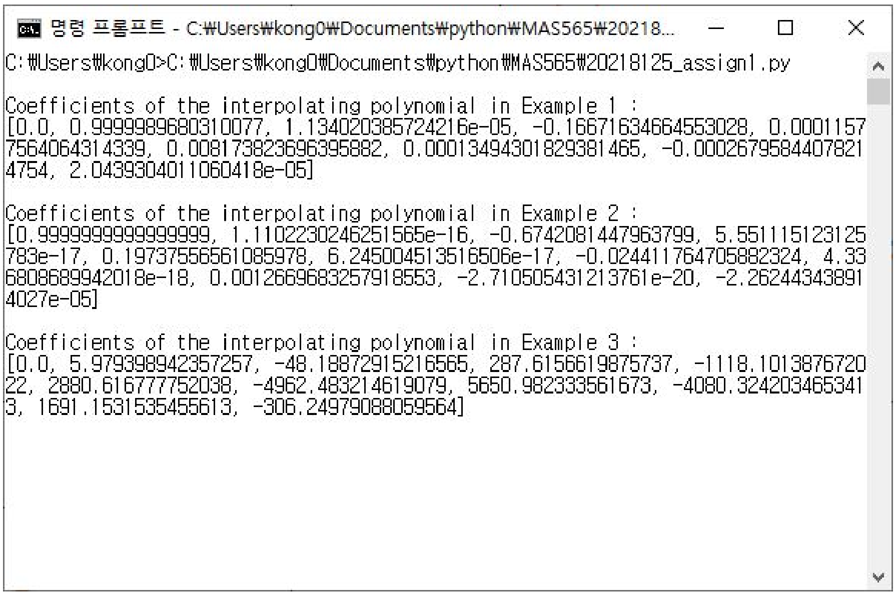
$$\begin{aligned}
B_k &= \frac{2}{N}\sum_{j=0}^{N-1}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx_j) + b_n\sin(nx_j)\right)\right]\sin(kx_j) \\
&= \frac{2}{N}\sum_{j=0}^{N-1}\frac{a_0}{2}\sin(kx_j) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} a_n\cos(nx_j)\sin(kx_j)\right) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} b_n\sin(nx_j)\sin(kx_j)\right) \\
&= \sum_{j=0}^{N-1}\frac{a_0}{N}\sin(kx_j) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{a_n}{N}\left(\sin((k+n)x_j) + \sin((k-n)x_j)\right) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{b_n}{N}\left(\cos((n-k)x_j) - \cos((n+k)x_j)\right).
\end{aligned}$$

This time, by the Lemma, the first and second terms vanish. The inner sum of the third term is b_n if $n \equiv k \mod 2m+1$, and $-b_N$ if $n \equiv -k \mod 2m+1$. Hence

$$B_k = b_k + \sum_{p=1}^{\infty}\left(b_{p(2m+1)+k} - b_{p(2m+1)-k}\right).$$

Computer Assignment

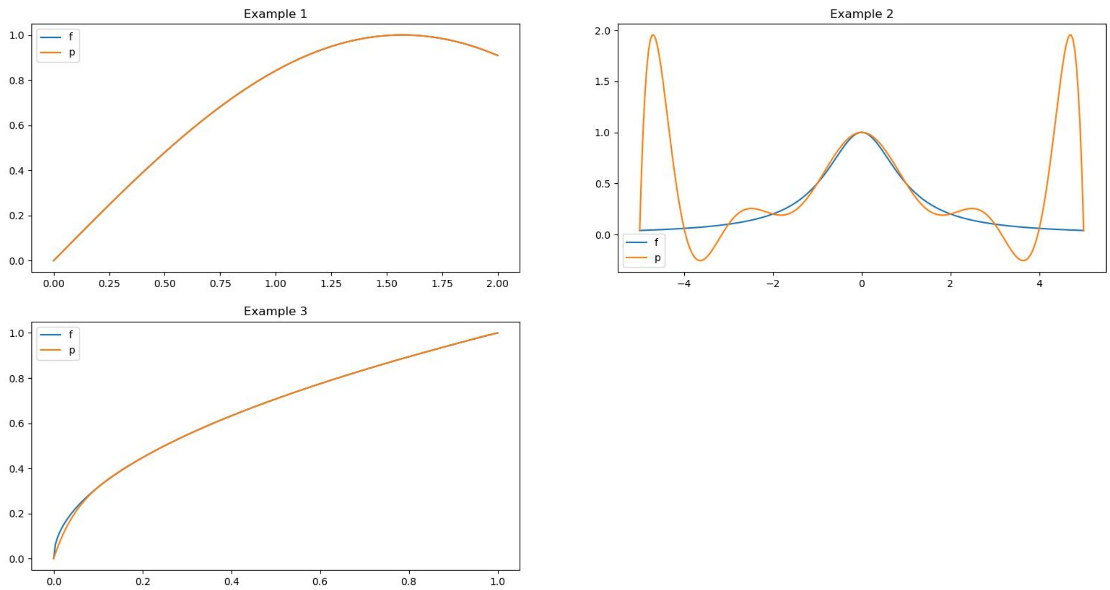
The program which does the required is submitted via KLMS along with this document. Given the two endpoints of the interval `a` and `b`, the step size `h`, and a function `f`, the program computes the interpolating polynomial using the divided difference scheme. The computed interpolating polynomials are printed out, as the following figure.



The results are the list of coefficients of the resulting polynomial, from the lowest degree term to the highest. For example the first result indicates that the interpolating polynomial is

$$p(x) = 0.0 + 1.000x + 1.134 \times 10^{-5}x^2 + \cdots + 2.044 \times 10^{-5}x^8.$$

The original function f and the interpolating polynomial p plotted together, for each example, is shown as in the following figure.



The first function $f(x) = \sin x$ looks well interpolated by the interpolating polynomial. Such a behavior is actually exactly as expected, because we have $\left\| f^{(n)}(x) \right\|_\infty \leq 1$ for any positive integer n, exactly the case discussed in Problem 2.4.

The second function, the Runge function $f(x) = \dfrac{1}{1+x^2}$, is badly interpolated, especially on the region near the end points of the given interval. It is indeed an example of a function where increasing the number of nodes does not increase the quality of the interpolation.

The third function $f(x) = \sqrt{x}$ is also not so well interpolated near $x = 0$. Indeed f has a vertical asymptote at $x = 0$, which is a trait a polynomial cannot have. As a result the error is even visible with our bare eyes.