MAS565 Numerical Analysis HW 1

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2.3. Fix any $\bar{x} \in [-1,1]$. First suppose that $\bar{x} \neq x_0$ and $\bar{x} \neq x_1$.

The interpolating polynomial $p(x)$ is

$$p(x) = f(x_0) \cdot \frac{x - x_1}{x_0 - x_1} + f(x_1) \cdot \frac{x - x_0}{x_1 - x_0}.$$

Consider a function $g : [-1,1] \to \mathbb{R}$ defined as

$$g(x) := f(x) - p(x) - \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)} (x - x_0)(x - x_1)$$

then clearly, $g(x_0) = g(x_1) = g(\bar{x}) = 0$. Since x_0, x_1, and $\bar{x}$ are all in $[-1,1]$, applying the Mean Value theorem there exist two points y_0 and y_1 in $[-1,1]$ such that $g'(y_0) = g'(y_1) = 0$, and again applying the Mean Value Theorem there exists some point $c \in [-1,1]$ such that $g''(c) = 0$. Meanwhile, $p(x)$ is linear, so a direct computation shows that

$$g''(x) = f''(x) - 2 \cdot \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)}$$

and, in turn, substituting $x = c$ we obtain

$$0 = g''(c) = f''(c) - 2 \cdot \frac{f(\bar{x}) - p(\bar{x})}{(\bar{x} - x_0)(\bar{x} - x_1)}$$

$$\Rightarrow \quad f(\bar{x}) - p(\bar{x}) = \frac{1}{2} f''(c) \cdot (\bar{x} - x_0)(\bar{x} - x_1). \quad \cdots \quad (*)$$

It is clear that $(*)$ holds also when $\bar{x} = x_0$ or $\bar{x} = x_1$, as both sides of $(*)$ become zero. Therefore

$$|f(x) - p(x)| \leq \frac{1}{2} \max_{\xi \in I} |f''(\xi)| \max_{x \in I} |(x-x_0)(x-x_1)|.$$
$$= \alpha,$$

for any $x \in I$. Therefore α is indeed the upper bound for the maximum absolute interpolation error.

To minimize α, we should minimize $\max_{x \in I} |(x-x_0)(x-x_1)|$. We claim that $\{x_0, x_1\} = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ is the minimizer. To show this claim, we show the stronger statement: we show that for any $a, b \in \mathbb{R}$, it holds that

$$\max_{x \in I} |(x-\tfrac{1}{\sqrt{2}})(x+\tfrac{1}{\sqrt{2}})| = \max_{x \in I} |x^2 - \tfrac{1}{2}| \leq \max_{x \in I} |x^2 + ax + b|.$$

Let $T(x) = x^2 - \frac{1}{2}$, $g(x) = x^2 + ax + b$. It is clear that $\max_{x \in I} |x^2 - \frac{1}{2}| = \frac{1}{2}$, so for the sake of contradiction suppose that $\max_{x \in I} |g(x)| < \frac{1}{2}$.

Then we would have

$$T(1) - g(1) = \tfrac{1}{2} - g(1) > 0,$$
$$T(0) - g(0) = -\tfrac{1}{2} - g(0) < 0,$$
$$T(-1) - g(-1) = \tfrac{1}{2} - g(-1) > 0$$

hence by the Intermediate Value Theorem, $T(x) - g(x)$ has at least two zeros in I. But $T(x) - g(x)$ is also a nonzero polynomial of degree at most 1, which is absurd. Therefore indeed $\{x_0, x_1\} = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ minimizes α.

Now, let $\theta = \cos^{-1} x$, then

$$\cos(2\cos^{-1} x) = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$$

so we have the relation, $(x-x_0)(x-x_1) = \frac{1}{2}\cos(2\cos^{-1} x)$ for x_0, x_1 minimizing α.

2.4. We know that the error of the polynomial interpolation is given by, for some $\xi \in [a,b]$,

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n).$$

Since $|f^{(n+1)}(\xi)| \le M$ for any $n \in \mathbb{N}$ and $\xi \in [a,b]$, we have

$$\begin{aligned}\|f(x) - P_n(x)\|_\infty &\le \frac{1}{(n+1)!}\max_{\xi\in[a,b]}\left|f^{(n+1)}(\xi)\right|\left|(x-x_0)\cdots(x-x_n)\right| \\ &\le \frac{M}{(n+1)!}|x-x_0|\cdots|x-x_n| \\ &\le \frac{M}{(n+1)!}(b-a)^{n+1}.\end{aligned}$$

As $b-a$ is a fixed constant we have $\lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!} = 0$, henceforth

$$\lim_{n\to\infty}\|f(x) - P_n(x)\|_\infty = 0.$$

That is, $P_n(x)$ converges uniformly on $[a,b]$ to $f(x)$ as $n\to\infty$.

2.16. The following lemma will be useful to solve this problem

Lemma. Let N be any positive integer, and for an integer j, $0 \le j \le N-1$, let $x_j = \frac{2\pi j}{N}$. Then

$$\sum_{j=0}^{N-1} \cos(kx_j) = \begin{cases} N & \text{if } k \equiv 0 \mod N \\ 0 & \text{if } k \not\equiv 0 \mod N, \end{cases}$$

$$\sum_{j=0}^{N-1} \sin(kx_j) = 0.$$

pf) If $k \equiv 0 \mod N$ then $kx_j = 2\pi j \cdot \frac{k}{N}$ is always an integer multiple, so the assertion follows. Hence we are left with the case where $k \not\equiv 0 \mod N$. If this is the case, $e^{2\pi k i/N} \neq 1$ but $\left(e^{2\pi k i/N}\right)^N = 1$, so we have

$$1 + e^{2\pi k i/N} + e^{4\pi k i/N} + \cdots + e^{2(N-1)\pi k i/N} = \frac{1 - e^{2N\pi k i/N}}{1 - e^{2\pi k i/N}} = 0. \quad \cdots (**)$$

Taking the real parts of (**) from both sides we get

$$\cos 0 + \cos\frac{2\pi k}{N} + \cdots + \cos\frac{2(N-1)\pi k}{N} = \sum_{j=0}^{N-1} \cos(kx_j) = 0$$

while taking the imaginary parts we get

$$\sin 0 + \sin\frac{2\pi k}{N} + \cdots + \sin\frac{2(N-1)\pi k}{N} = \sum_{j=0}^{N-1} \sin(kx_j) = 0. \quad \blacksquare$$

By theorem 2.3.1.12, under the assumption that $\Psi(x_k) = f(x_k) =: f_k$, we have the relations

$$\begin{cases} A_k = \frac{2}{N} \sum_{j=0}^{N-1} f_j \cos kx_j \\ B_k = \frac{2}{N} \sum_{j=0}^{N-1} f_j \sin kx_j \end{cases} \quad \text{for } k = 0, 1, \cdots, 2m.$$

By the absolute convergence we have

$$f_j = f(x_j) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx_j) + b_n \sin(nx_j)\right).$$

Hence if we set $N = 2m+1$, for $k = 0, 1, \cdots, m$ we get

$$\begin{aligned}
A_k &= \frac{2}{N}\sum_{j=0}^{N-1}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx_j) + b_n\sin(nx_j)\right)\right]\cos(kx_j) \\
&= \frac{2}{N}\sum_{j=0}^{N-1}\frac{a_0}{2}\cos(kx_j) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} a_n\cos(nx_j)\cos(kx_j)\right) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} b_n\sin(nx_j)\cos(kx_j)\right) \\
&= \sum_{j=0}^{N-1}\frac{a_0}{N}\cos(kx_j) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{a_n}{N}\left(\cos((n+k)x_j) + \cos((n-k)x_j)\right) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{b_n}{N}\left(\sin((n+k)x_j) + \sin((n-k)x_j)\right).
\end{aligned}$$

By the Lemma, the first term is a_0 only when $k=0$ and otherwise 0, the third term always vanishes, and the second term is nonzero only when $n \geq 1$ and $n \equiv \pm k \pmod{2m+1}$ and otherwise zero. Hence

$$A_k = a_k + \sum_{p=1}^{\infty}\left(a_{p(2m+1)+k} + a_{p(2m+1)-k}\right).$$

Similarly for $k = 1, \cdots, m$ we get

$$\begin{aligned}
B_k &= \frac{2}{N}\sum_{j=0}^{N-1}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx_j) + b_n\sin(nx_j)\right)\right]\sin(kx_j) \\
&= \frac{2}{N}\sum_{j=0}^{N-1}\frac{a_0}{2}\sin(kx_j) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} a_n\cos(nx_j)\sin(kx_j)\right) + \frac{2}{N}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{N-1} b_n\sin(nx_j)\sin(kx_j)\right) \\
&= \sum_{j=0}^{N-1}\frac{a_0}{N}\sin(kx_j) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{a_n}{N}\left(\sin((k+n)x_j) + \sin((k-n)x_j)\right) + \sum_{n=1}^{\infty}\sum_{j=0}^{N-1}\frac{b_n}{N}\left(\cos((n-k)x_j) - \cos((n+k)x_j)\right).
\end{aligned}$$

This time, by the Lemma, the first and second terms vanish. The inner sum of the third term is b_n if $n \equiv k \bmod 2m+1$, and $-b_N$ if $n \equiv -k \bmod 2m+1$. Hence

$$B_k = b_k + \sum_{p=1}^{\infty}\left(b_{p(2m+1)+k} - b_{p(2m+1)-k}\right).$$