MAS565 Numerical Analysis HW5

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3.15. When f and g are polynomials of degree less than n then $f(x)g(x)$ is a polynomial of degree less than $2n-1$. We would like to show that

$$\int_{-1}^{1} f(x)g(x)\,dx = (f,g) = \sum_{i=1}^{n} \gamma_i f(x_i) g(x_i)$$

holds. If $\gamma_i = w_i$ where w_i are the weights with respect to the weight function $\omega(x)=1$ then by Theorem 3.6.12 the result is immediate. Our goal is to show that $\gamma_i = w_i$ indeed is true.

To do so, it suffices to show that, for Legendre polynomials p_k,

$$\sum_{i=1}^{n} p_k(x_i)\gamma_i = \begin{cases} (p_0, p_0) & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

holds, as we already know that this system of linear equations of γ_i's has a unique solution $\gamma_i = w_i$, $i=1,\cdots,n$. Let h be any polynomial of degree less than n. Then, Lagrange interpolation of h at abscissae $x_1,\cdots,x_n$ gives us the interpolation

$$h(x) = \sum_{i=1}^{n} h(x_i) L_i(x)$$

which is exact due to the degree of h. Using this fact we get

$$\begin{aligned} \int_{-1}^{1} h(x)\,dx &= \int_{-1}^{1} \sum_{i=1}^{n} h(x_i) L_i(x)\,dx \\ &= \sum_{i=1}^{n} h(x_i) \int_{-1}^{1} L_i(x)\,dx \\ &= \sum_{i=1}^{n} h(x_i)\gamma_i . \end{aligned}$$

Now consider the special cases where $h = p_k$, for each $k = 0, 1, \cdots, n-1$. From that $p_0(x) = 1$ and $w(x) = 1$ we can rewrite the above into

$$(p_0, p_k) = \int_{-1}^{1} p_k(x)\,dx = \sum_{i=1}^{n} p_k(x_i)\, d_i$$

where $k = 0, 1, \cdots, n-1$. We already know that $(p_0, p_k) = 0$ whenever $k \neq 0$, so indeed we have the relations

$$\sum_{i=1}^{n} p_k(x_i)\, d_i = (p_0, p_k) = \begin{cases} (p_0, p_0) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we are done.

5.4. Using calculus, we observe that p has a local extrema at points where $p'(x) = 3x^2 - 2x - 1 = 0$, i.e. at $x = -\frac{1}{3}$ and $x = 1$. As f is a monic cubic, $x = -\frac{1}{3}$ is the local maximum with $p(-\frac{1}{3}) = -\frac{22}{27} < 0$, so the given root ξ is the only real root of p.

We claim that $\Phi(x) = \sqrt[3]{x^2 + x + 1}$ is a suitable iteration function. That Φ has ξ as its fixed point is immediate, since $x \mapsto \sqrt[3]{x}$ is a bijection from $\mathbb{R}$ onto itself hence

$$\begin{aligned} p(\xi) = 0 &\Leftrightarrow \xi^3 = \xi^2 + \xi + 1 \\ &\Leftrightarrow \xi = \sqrt[3]{\xi^2 + \xi + 1}. \end{aligned}$$

It remains to show that the iteration converges to ξ for any starting point.

We claim that $\Phi(x)$ is a contractive mapping on $\mathbb{R}$. Noting that $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$, we see that $\Phi(x)$ is twice continuously differentiable with

$$\Phi'(x) = \frac{2x+1}{3(x^2+x+1)^{2/3}}, \quad \Phi''(x) = -\frac{2(x^2+x-2)}{9(x^2+x+1)^{5/3}}.$$

Hence $\Phi''(x) = 0$ if and only if $x = -2$ or $x = 1$. One can also observe that

Φ'' is negative on $(-\infty, -2) \cup (1, \infty)$ and positive on $(-2, 1)$. As $\lim_{x\to-\infty} \Phi'(x) = 0$, $\Phi'(-2) = -\frac{1}{3^{2/3}}$, $\Phi'(1) = \frac{1}{3^{2/3}}$ and $\lim_{x\to\infty} \Phi'(x) = 0$, it follows that $|\Phi'(x)| \leq \frac{1}{3^{2/3}}$ for all $x \in \mathbb{R}$. Now, fix any $x, y \in \mathbb{R}$ with $x \neq y$ then by the Mean Value Theorem there exists some c between x and y with

$$\left| \frac{\Phi(x) - \Phi(y)}{x - y} \right| = |\Phi'(c)| \leq \frac{1}{3^{2/3}}.$$

It is now immediate that the inequality

$$|\Phi(x) - \Phi(y)| \leq \frac{1}{3^{2/3}} |x - y|$$

holds for all $x, y \in \mathbb{R}$, but $\frac{1}{3^{2/3}} < 1$, therefore indeed, Φ is a contractive mapping on $\mathbb{R}$.

For any given x_0, let $r = |\xi - x_0| + 1$ so that $x_0 \in \{z : |z - \xi| < r\}$. Then Theorem 5.2.2 ensures that the sequence $\{x_i\}_{i=0}^{\infty}$ generated by iterating Φ, i.e. $x_{i+1} = \Phi(x_i)$, converges at least linearly to ξ. Therefore the iteration process converges to ξ regardless of the starting point x_0.

5.7. Let $\varepsilon = 10^{-4}$, $p(x) = p_0(x)|_{a_4 = 16}$, and $p_\varepsilon(x) = p(x) + \varepsilon = p_0(x)|_{a_4 = 16 + 10^{-4}}$. It is told in the problem that $\xi = 2$ is a quadruple root of $p(x)$. Let h be a number such that $p_\varepsilon(\xi + h) = 0$. By Taylor expansion we get

$$0 = p_\varepsilon(\xi + h) = p_\varepsilon(\xi) + h p_\varepsilon'(\xi) + \frac{h^2}{2!} p_\varepsilon''(\xi) + \frac{h^3}{3!} p_\varepsilon'''(\xi) + \frac{h^4}{4!} p_\varepsilon^{(4)}(\xi) + \cdots.$$

Since $p(x)$ and $p_\varepsilon(x)$ differ only by a constant, the derivatives of $p(x)$ and $p_\varepsilon(x)$ coincide. Also, as $p_\varepsilon(\xi) = p(\xi) + \varepsilon = \varepsilon$

$p'(\xi) = p''(\xi) = p'''(\xi) = 0$, and $p^{(4)}(\xi) = 4!$ we get

$$0 = p_\varepsilon(\xi) + \frac{h^4}{4!} p^{(4)}(\xi) + \cdots = \varepsilon + h^4 + \cdots.$$

To first approximation, we disregard all terms behind the quartic term (as the quartic term is the first nontrivial nonzero term). Doing so gives the approximation

$$0 \approx \varepsilon + h^4 \quad \Rightarrow \quad h \approx (-\varepsilon)^{1/4}.$$

Note that up to this point we never relied on the actual value of ε. That is, we will get the same approximation even if we started with $\varepsilon = -10^{-4}$. Therefore, to first approximation, the roots of $x^4 - 8x^3 + 24x^2 - 32x + 16 \pm 10^{-4}$ is at

$$\begin{aligned} \xi + h &\approx 2 + (\mp 10^{-4})^{1/4} \\ &= 2 + 0.1(\mp 1)^{1/4} \end{aligned}$$

where the $1/4$-th power is taken in the complex exponential sense so that it is four-valued. In other words,

$$\text{if } a_4 = 16 + 10^{-4} \Rightarrow \text{roots approximately } 2 + 0.1e^{\frac{2k+1}{8}\pi i}, \ k \in \mathbb{Z}$$

$$\text{if } a_4 = 16 - 10^{-4} \Rightarrow \text{roots approximately } 2 - 0.1e^{\frac{k\pi i}{4}}, \ k \in \mathbb{Z}.$$

Computer Assignment

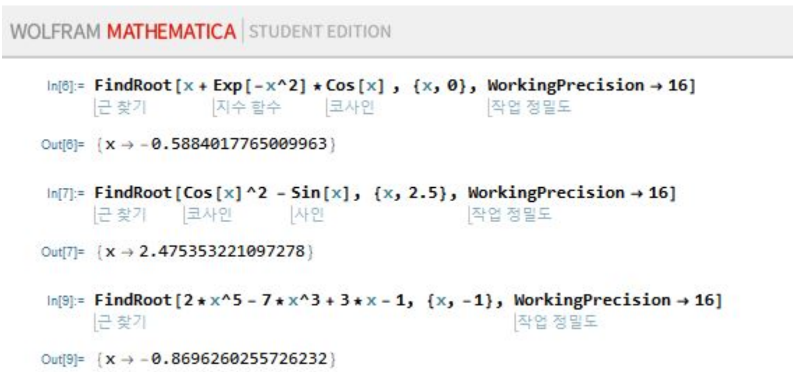
The program which does the required is submitted via KLMS along with this document. It is in general desirable to approximate the location of the root we are finding for, for example by using bisection method, before applying the Newton's method because Newton's method guarantees only local convergence. However, since it is also our goal to observe how Newton's method behaves according to the choice of the initial point, we make an approach the other way around; that is, we first apply Newton's method on various sample points inside the given interval, and then search for a result that falls into that interval.

The stop condition is chosen so that the relative error $\dfrac{|x_{i+1}-x_i|}{|x_{i+1}|}$ is less than the given tolerance $\epsilon = 10^{-6}$. Other choices of the stop condition may not lead to convergence of the iteration. We may have $|x_{i+1}-x_i| \to 0$ even when the sequence $\{x_i\}_{i\in\mathbb{N}}$ diverges, for example the harmonic numbers. Also if f has more than one root then $f(x_i) \to 0$ and $|f(x_{i+1}) - f(x_i)| \to 0$ may both happen even when $\{x_i\}_{i\in\mathbb{N}}$ oscillates between the neighborhoods of distinct roots. In terms of measuring the errors, relative errors have apparent advantages over absolute errors, as absolute errors will cause trouble for very small and very large values of $|x_{i+1}|$. It is also natural to consider relative errors once we realize that we are dealing with floating point numbers; significand is much more important than the exponent.

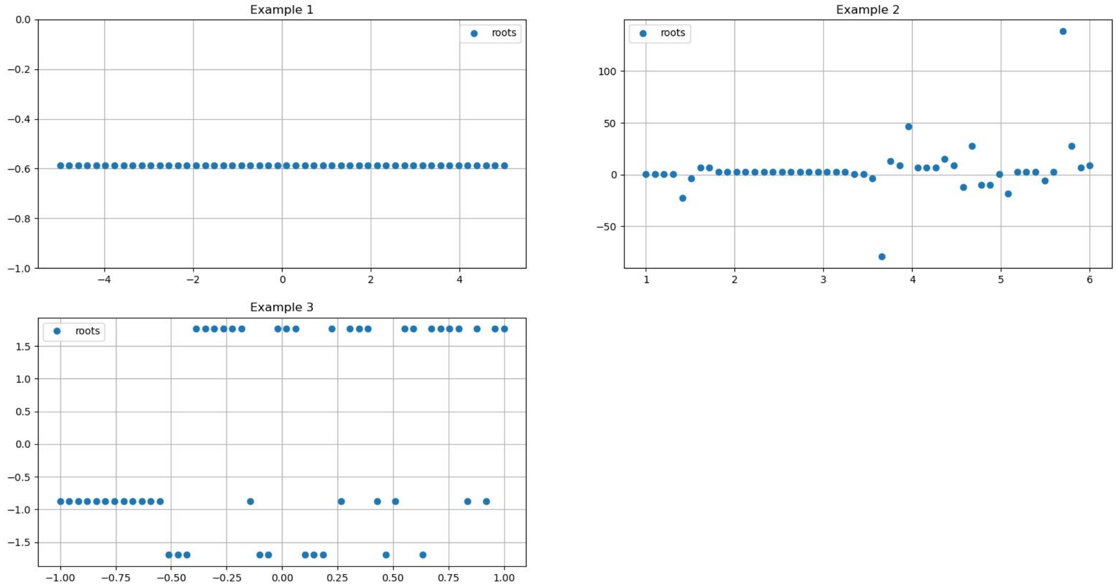
There are also some other technical considerations implemented in the code for Newton's method, for instance the special treatment of the case $|x_{i+1}| \ll 1$. This is needed because if x_{i+1} is very close to 0 then the computation of the relative error becomes problematic. Other considerations are notated as comments in the `.py` file.

The roots found are printed out, as we see the following figure.

The code works well, as we can see from that same results are obtained by using Wolfram Mathematica.



As mentioned before we are also interested on how the result changes according to the initial point. The following figure shows the scatter plot where the x-axis is the initial point, and the corresponding value along the y-axis is the result of the Newton's method.



The first figure shows that, regardless of the initial point Newton's method produces the same (unique) root of $f(x) = x + e^{-x^2} \cos x$. Since f has a unique root in $\mathbb{R}$, that the successful results are all equal is of less importance; what we should note is that the Newton's method seems to give global convergence. However it is pure luck that such a phenomenon has occured. For example there exist points such that $f'(x) = 0$, so in theory if those points were in the sample then the Newton's method would not have converged. However there is a plausible explanation why the iteration tends to not diverge. With some simple calculus and algebraic manipulation we can express the iteration function explicitly as

$$\Phi(x) = \frac{-e^{-x^2}(\cos x + x \sin x + 2x^2 \cos x)}{1 - e^{-x^2}(\sin x + 2x \cos x)}.$$

Even if at some point in the interation process we got far away from the root, the rapid decay of e^{-x^2} sends us back to a point near, often extremely close to, 0. After that the iterations remain sufficiently close to the root, eventually converging.

The second and third examples show much more interesting results. Although the root is unique inside the given interval, the results show that Newton's method may converge to a different root, outside of the given interval. The third example $2x^5 - 7x^3 + 3x - 1 = 0$ has only three distinct real roots, ergo there are three different results. The second example is more extreme; there are even points that converge to very distant roots such as -79.21 and 138.9. However for initial points that are sufficiently close to the desired roots, as learnt, Newton's method indeed converges to the desired root. These two examples clearly indicate that the initial points must be judiciously chosen.