MAS565 Numerical Analysis HW3

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2.22 First thing to observe is that, for d_j in (2.4.2.6) it holds that

$$d_j = \frac{6}{h_j + h_{j+1}}\left(\frac{y_{j+1}-y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j}\right)$$

$$= \frac{6}{x_{j+1}-x_{j-1}}\left(\frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j} - \frac{f(x_j)-f(x_{j-1})}{x_j - x_{j-1}}\right)$$

$$= \frac{6}{x_{j+1}-x_{j-1}}\left(f[x_j, x_{j+1}] - f[x_{j-1}, x_j]\right)$$

$$= 6f[x_{j-1}, x_j, x_{j+1}].$$

Let $J_j = [x_{j-1}, x_{j+1}]$ then there exists $\xi_j \in J_j$ such that

$$\frac{d_j}{6} = f[x_{j-1}, x_j, x_{j+1}] = \frac{f''(\xi_j)}{2!}.$$

Similar phenomena happens for d_0 and d_n in (2.4.2.8); observe that

$$d_0 = \frac{6}{h_1}\left(\frac{y_1 - y_0}{h_1} - y_0'\right)$$

$$= \frac{6}{x_1 - x_0}\left(\frac{f(x_1)-f(x_0)}{x_1 - x_0} - f'(x_0)\right)$$

$$= \frac{6}{x_1 - x_0}\left(f[x_0, x_1] - f[x_0, x_0]\right)$$

$$= 6f[x_0, x_0, x_1]$$

so let $J_0 = [x_0, x_1]$ then there exists $\xi_0 \in J_0$ such that

$$\frac{d_0}{6} = f[x_0, x_0, x_1] = \frac{f''(\xi_0)}{2}$$

and analogously

$$
\begin{aligned}
d_n &= \frac{6}{h_n}\left(y_n' - \frac{y_n - y_{n-1}}{h_n} \right) \\
&= \frac{6}{x_n - x_{n-1}}\left(f'(x_n) - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \right) \\
&= \frac{6}{x_n - x_{n-1}}\left(f[x_n, x_n] - f[x_{n-1}, x_n] \right) \\
&= 6 f[x_{n-1}, x_n, x_n]
\end{aligned}
$$

so let $J_n = [x_{n-1}, x_n]$ then there exists $\xi_n \in J_n$ such that

$$
\frac{d_n}{6} = f[x_{n-1}, x_n, x_n] = \frac{f''(\xi_n)}{2!}.
$$

These observations, however show that if we only assume that $f \in \mathcal{K}^2(a,b)$ then the given statements may not hold. Indeed, $\xi_j \to x_j$ does not imply $f''(\xi_j) \to f''(x_j)$. For a concrete counterexample, consider a continuous function

$$
g(t) = \begin{cases} t^2 \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}
$$

and its antiderivative $G(x) = \int_0^t g(t)\,dt$ on $[-1,1]$. As we have

$$
G''(0) = g'(0) = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = 0
$$

while for $x \neq 0$ it holds that

$$
G''(x) = g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x^2}
$$

we have $|g'(x)| \leq |2x| |\sin \frac{1}{x}| + |\cos \frac{1}{x^2}| \leq 3$; that is, $g = G'$ is differentiable with its derivative bounded. Thus g is Lipschitz continuous, hence absolutely continuous. Furthermore $g' = G''$ is discontinuous only at $x = 0$, so $G'' \in L^2[-1,1]$. It follows that $G \in \mathcal{K}^2(-1,1)$. Now, consider Δ where for some $j \in \{1, 2, \cdots, n-1\}$ and $h > 0$ we have $x_{j-1} = -h$, $x_j = 0$, and $x_{j+1} = h$

Because g is an odd function, G being its antiderivative is even.

Put $f(x) = G(x) + x^2$, then $f \in \mathcal{R}^2(-1,1)$, f is even, but now $f''(0) = 2$.

As $f(x_{j+1}) = f(x_{j-1}) = f(h)$ and $f(x_j) = f(0) = G(0) + 0^2 = 0$, we get

$$\begin{aligned} d_j &= \frac{6}{h_j + h_{j+1}} \left(\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} - \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \right) \\ &= \frac{6}{2h} \cdot \frac{f(x_{j+1}) + f(x_{j-1})}{h} \\ &= \frac{6}{h^2} f(h) \\ &= \frac{6}{h^2} G(h) + 6 . \end{aligned}$$

When $\|\Delta\| \to 0$, consequently $h \to 0$, then by L'Hôpital's rule

$$\lim_{h\to 0} \frac{G(h)}{h^2} = \lim_{h \to 0} \frac{g(h)}{2h} = \lim_{h\to 0} \frac{1}{2} h \sin\frac{1}{h} = 0 .$$

That is, $d_j \to 6 \neq f''(0)$ as $\|\Delta\| \to 0$.

We need sufficient regularity conditions. For the first part assume that $f \in C^3(a,b)$, or, at least f is twice differentiable with f'' Lipschitz continuous. Then for some constant K, it holds that

$$\begin{aligned} |f''(\xi_j) - f''(x_j)| &\le K |\xi_j - x_j| \\ &\le K \cdot |J_j| \\ &\le 2K \cdot \|\Delta\| \end{aligned}$$

where $|J_j|$ denotes the length of the interval J_j. It follows that

$$f''(\xi_j) = f''(x_j) + O(\|\Delta\|),$$

and thus

$$d_j = 3f''(\xi_j) = 3f''(x_j) + O(\|\Delta\|).$$

Further, say that the knots are equidistant. Let h denote the distance between two neighboring knots. Then for $j=1,\cdots,n-1$, observe that d_j reduces into

$$d_j = \frac{6}{h+h}\left(\frac{f(x_{j+1})-f(x_j)}{h} - \frac{f(x_j)-f(x_{j-1})}{h}\right)$$
$$= \frac{3}{h^2}\left(f(x_{j+1}) - 2f(x_j) + f(x_{j-1})\right).$$

The regularity condition we consider here is that $f\in C^4[x_0,x_n]$. Then by Taylor's theorem, there exists $\varphi_j, \phi_j \in J_j$ such that

$$f(x_{j+1}) = f(x_j+h) = f(x_j) + hf'(x_j) + \frac{h^2}{2}f''(x_j) + \frac{h^3}{6}f^{(3)}(x_j) + \frac{h^4}{24}f^{(4)}(\varphi_j)$$
$$f(x_{j-1}) = f(x_j-h) = f(x_j) - hf'(x_j) + \frac{h^2}{2}f''(x_j) - \frac{h^3}{6}f^{(3)}(x_j) + \frac{h^4}{24}f^{(4)}(\phi_j)$$

hence adding the two equations above side by side we obtain

$$f(x_{j+1}) - f(x_{j-1}) = 2f(x_j) + h^2f''(x_j) + \frac{h^4}{24}\left(f^{(4)}(\varphi_j) + f^{(4)}(\phi_j)\right).$$

As $f^{(4)}$ is continuous, it is bounded on $[x_0,x_n]$, say $\|f^{(4)}\|_\infty \le M$, so for any $j=1,\cdots,n-1$ we get

$$\left|\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2} - f''(x_j)\right| = \frac{h^2}{24}\left|f^{(4)}(\varphi_j)+f^{(4)}(\phi_j)\right|$$
$$\le \frac{h^2}{12}M$$

Henceforth

$$\left|d_j - 3f''(x_j)\right| = 3\left|\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2} - f''(x_j)\right|$$
$$\le \frac{h^2}{4}M$$
$$= \frac{M}{4}\|\Delta\|^2.$$

We conclude that $d_j = 3f''(x_j) + O(\|\Delta\|^2)$.

2.31 It is unclear what "spline-like functions" are. We assume that $E_{\Delta,f}(x)$, given λ_i, is a function which satisfies:

(i) $f(x_i) = E_{\Delta,f}(x_i)$, $i = 0, 1, \cdots, N$.

(ii) $E_{\Delta f}(x) \in C^2[a,b]$

(iii) $E_{\Delta,f}(x_i)$ minimizes the given functional over all $\mathcal{K}^2(a,b)$

(iv) One of the boundary conditions in (2.4.1.2)

and from that cubic splines coincide with a cubic polynomial in each subinterval $[x_{j-1}, x_j]$, with that Theorem 2.4.1.4 implicitly assumes that the spline function S_Δ is a function in $\mathcal{K}^4(a,b)$, we impose the condition

(v) $E_{\Delta f}$ on the interval $[x_{j-1}, x_j]$ is a function in $\mathcal{K}^4(x_{j-1}, x_j)$, for $j = 1, \cdots, N$.

(a) There is an inconsistency in notation even within the given statement. We interpret the given as that $E_{\Delta f}(x)$ on each $[x_i, x_{i+1}]$ is defined as

$$E_{\Delta,f}(x) = \alpha_i + \beta_i x + \gamma_i \psi_i(x - x_i) + \delta_i \varphi_i(x - x_i) \qquad \cdots (*)$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are each some constants, and ψ_i, φ_i are functions defined as given. We proceed in several steps.

- **Lemma 1.** On each $[x_i, x_{i+1}]$, $E_{\Delta f}(x)$ as in $(*)$ is a solution to the ODE

$$y^{(4)} - \lambda_i^2 y'' = 0.$$

(pf) The auxiliary equation of the given ODE is $m^4 - \lambda_i^2 m^2 = 0$, of which the solutions are $m = \pm\lambda_i$ and $m = 0$ with multiplicity 2. Hence the solution of the ODE is generated by the basis

$$\{e^{0x}, xe^{0 \cdot x}, \cosh(\lambda_i x), \sinh(\lambda_i x)\} = \{1, x, \cosh(\lambda_i x), \sinh(\lambda_i x)\},$$

Meanwhile, since $\frac{d^4}{dx^4} y(x - x_i) - \lambda_i^2 \frac{d^2}{dx^2} y(x - x_i) = y^{(4)}(x - x_i) - \lambda_i^2 y''(x - x_i)$ the general solution of the given ODE can be written in the form

$$y = A_i + B_i(x - x_i) + C_i \cosh(\lambda_i(x - x_i)) + D_i \sinh(\lambda_i(x - x_i)).$$

By setting $A_i = \alpha_i - \frac{2\eta_i}{\lambda_i^2}$, $B_i = \beta_i - \frac{6\delta_i}{\lambda_i^2}$, $C_i = \frac{2\eta_i}{\lambda_i^2}$, and $D_i = \frac{6\delta_i}{\lambda_i^3}$ we recover $E_{\Delta,f}(x)$, showing the claim. ■

- <u>Lemma 2</u>. Define a seminorm of $f \in \mathcal{K}^2(a,b)$ as

$$\|f\|^2 = E[f] = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (f''(x))^2 + \lambda_i^2 (f'(x))^2 \, dx.$$

Let $E(x)$ be any interpolating spline of f, that is, satisfying conditions (i), (ii), and (v). Then it holds that

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2(f'-E')E''\Big|_a^b - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx. \quad \cdots (**)$$

pf) By simple algebraic manipulation we get

$$(f''-E'')^2 + \lambda_i^2 (f'-E')^2 = f'' + \lambda_i^2 (f')^2 + E'' + \lambda_i^2 (E')^2 - 2(f''E'' + \lambda_i^2 f'E')$$

$$= f'' + \lambda_i^2 (f')^2 - (E'' + \lambda_i^2 (E')^2) - 2((f''-E'')E'' + \lambda_i^2 (f'-E')E')$$

and here integration gives us

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f''-E'')E'' + \lambda_i^2 (f'-E')E' \, dx.$$

For each $i = 0, 1, \dots, N-1$, integration by parts gives us

$$\int_{x_i}^{x_{i+1}} (f''-E'')E''\,dx + \lambda_i^2 \int_{x_i}^{x_{i+1}} (f'-E')E'\,dx$$

$$= \Big[(f'-E')E'' + \lambda_i^2 (f-E)E'\Big]_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (f'-E')E'''\,dx - \lambda_i^2 \int_{x_i}^{x_{i+1}} (f-E)E''\,dx$$

$$= \Big[(f'-E')E'' + \lambda_i^2 (f-E)E' - (f-E)E'''\Big]_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx.$$

But by assumption, $f(x_{i+1}) - E(x_{i+1}) = f(x_i) - E(x_i) = 0$, and $(f'-E')E''$ is continuous, thus in total we get

$$\|f-E\|^2 = \|f\|^2 - \|E\|^2 - 2\sum_{i=0}^{N-1} (f'-E')E''\Big|_{x_i}^{x_{i+1}} - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx$$

$$= \|f\|^2 - \|E\|^2 - 2(f'-E')E''\Big|_a^b - 2\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (f-E)(E^{(4)} - \lambda_i^2 E'')\,dx. \quad ■$$

- <u>Theorem 3</u>. Let $E_{\Delta,f}(x)$ be an interpolating spline of f, i.e. satisfies the conditions (i), (ii), and (v), and is of the form (*) on each $[x_i, x_{i+1}]$. Then it holds that

$$\|f - E_{\Delta,f}\|^2 = \|f\|^2 - \|E_{\Delta,f}\|^2 - 2(f'-E')E''\Big|_a^b.$$

Furthermore if $E_{\sigma,f}$ satisfies the boundary condition (iv) then we have

$$\|f\|^2 - \|E_{\sigma,f}\|^2 = \|f - E_{\sigma,f}\|^2 \geq 0.$$

pf) The first part follows immediately from Lemma 1 and Lemma 2, as $E_{\sigma,f}^{(4)} - \lambda_i^2 E_{\sigma,f}''$ is identically 0 on (x_i, x_{i+1}), making the last term in (**) vanish. The second part is also immediate, as if the boundary condition is met then clearly

$$(f'(b) - E'(b))E''(b) - (f'(a) - E'(a))E''(a) = 0. \quad \blacksquare$$

• **Theorem 4.** Let $\mathcal{E}_{\sigma,f}(x)$ be an interpolating spline of f satisfying the conditions (i), (ii), (iv), and (v). Among such splines, let $E_{\sigma,f}(x)$ be the one that is of the form (*). Then $E_{\sigma,f}(x)$ also satisfies the condition (iii), that is, $E_{\sigma,f}(x)$ minimizes the given functional.

pf) From the given conditions, $E_{\sigma,f}$ can be seen as not only the interpolating spline of f but also an interpolating spline of $\mathcal{E}_{\sigma,f}$. Hence by Theorem 3, we have

$$\|\mathcal{E}_{\sigma,f}\|^2 \geq \|E_{\sigma,f}\|^2. \quad \blacksquare$$

By Theorem 4, $E_{\sigma,f}(x)$ should be in the form (*).

(b) By applying L'Hôpital's rule repeatedly, we get

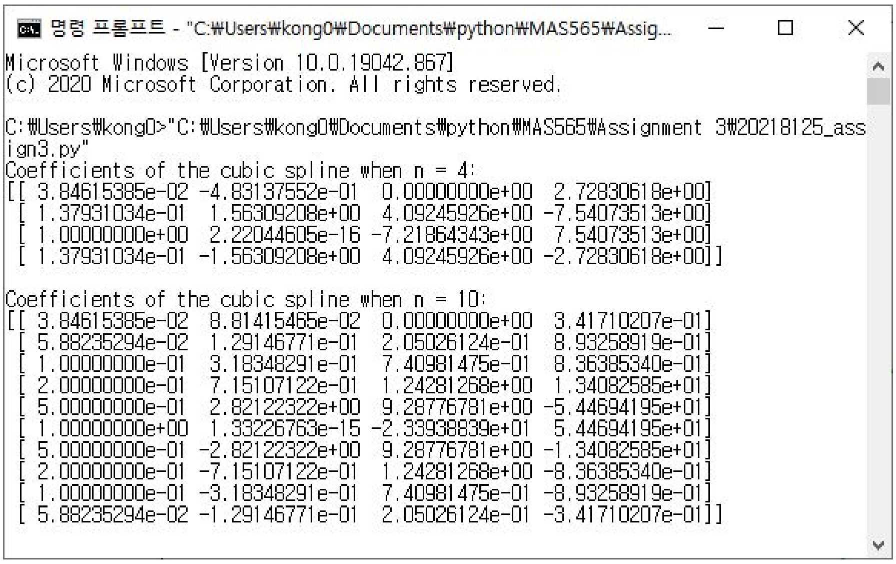
$$\begin{aligned}
\lim_{\lambda_i \to 0} \frac{6(\sinh(\lambda_i x) - \lambda_i x)}{\lambda_i^3} &= \lim_{\lambda_i \to 0} \frac{6(x\cosh(\lambda_i x) - x)}{3\lambda_i^2} \\
&= \lim_{\lambda_i \to 0} \frac{2(\cosh(\lambda_i x) - 1)}{\lambda_i^2} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{2x\sinh(\lambda_i x)}{2\lambda_i} \cdot x \\
&= \lim_{\lambda_i \to 0} \frac{x^2\cosh(\lambda_i x)}{1} \cdot x \quad = x^3
\end{aligned}$$

asserting that $\lim_{\lambda_i \to 0} \psi_i(x - x_i) = (x - x_i)^2$ and $\lim_{\lambda_i \to 0} \varphi_i(x - x_i) = (x - x_i)^3$. That is, when $\lambda_i \to 0$ the exponential spline reduces into a cubic spline.

Computer Assignment

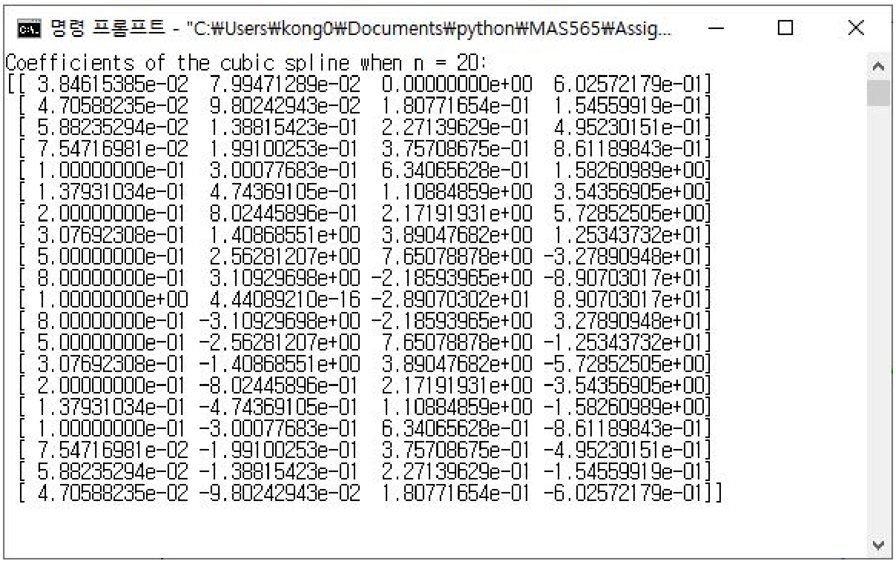
The program which does the required is submitted via KLMS along with this document. For various values of `n`, the program first computes the cubic spline $S(x)$ with knots $\{x_0, x_1, \ldots, x_n\}$. This is done by computing the moments using the system of linear equations learnt in class, then computing the coefficients of the cubic polynomial from the moments, ordinates, and knots. Then the program computes the error $f(x) - S(x)$ at the 41 equidistant points.

The computed cubic splines are printed out, in a matrix as the following figure.


```
명령 프롬프트 - "C:₩Users₩kong0₩Documents₩python₩MAS565₩Assig...
Microsoft Windows [Version 10.0.19042.867]
(c) 2020 Microsoft Corporation. All rights reserved.

C:₩Users₩kong0>"C:₩Users₩kong0₩Documents₩python₩MAS565₩Assignment 3₩20218125_assign3.py"
Coefficients of the cubic spline when n = 4:
[[ 3.84615385e-02 -4.83137552e-01  0.00000000e+00  2.72830618e+00]
 [ 1.37931034e-01  1.56309208e+00  4.09245926e+00 -7.54073513e+00]
 [ 1.00000000e+00  2.22044605e-16 -7.21864343e+00  7.54073513e+00]
 [ 1.37931034e-01 -1.56309208e+00  4.09245926e+00 -2.72830618e+00]]

Coefficients of the cubic spline when n = 10:
[[ 3.84615385e-02  8.81415465e-02  0.00000000e+00  3.41710207e-01]
 [ 5.88235294e-02  1.29146771e-01  2.05026124e-01  8.93258919e-01]
 [ 1.00000000e-01  3.18348291e-01  7.40981475e-01  8.36385340e-01]
 [ 2.00000000e-01  7.15107122e-01  1.24281268e+00  1.34082585e+01]
 [ 5.00000000e-01  2.82122322e+00  9.28776781e+00 -5.44694195e+01]
 [ 1.00000000e+00  1.33226763e-15 -2.33938839e+01  5.44694195e+01]
 [ 5.00000000e-01 -2.82122322e+00  9.28776781e+00 -1.34082585e+01]
 [ 2.00000000e-01 -7.15107122e-01  1.24281268e+00 -8.36385340e-01]
 [ 1.00000000e-01 -3.18348291e-01  7.40981475e-01 -8.93258919e-01]
 [ 5.88235294e-02 -1.29146771e-01  2.05026124e-01 -3.41710207e-01]]
```

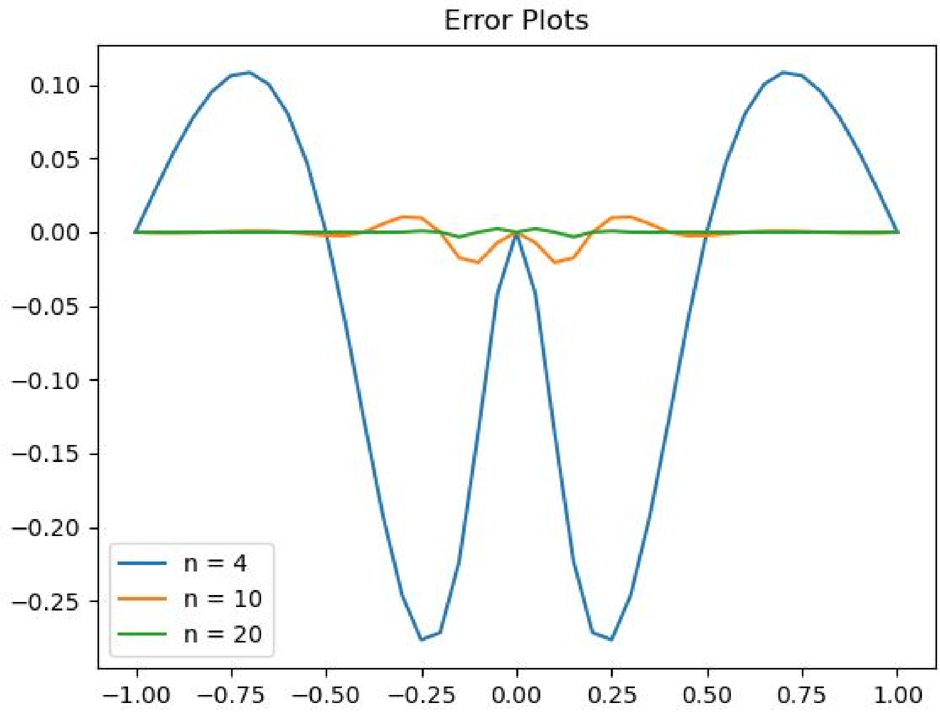

```
명령 프롬프트 - "C:₩Users₩kong0₩Documents₩python₩MAS565₩Assig...
Coefficients of the cubic spline when n = 20:
[[ 3.84615385e-02  7.99471289e-02  0.00000000e+00  6.02572179e-01]
 [ 4.70588235e-02  9.80242943e-02  1.80771654e-01  1.54559919e-01]
 [ 5.88235294e-02  1.38815423e-01  2.27139629e-01  4.95230151e-01]
 [ 7.54716981e-02  1.99100253e-01  3.75708675e-01  8.61189843e-01]
 [ 1.00000000e-01  3.00077683e-01  6.34065628e-01  1.58260989e+00]
 [ 1.37931034e-01  4.74369105e-01  1.10884859e+00  3.54356905e+00]
 [ 2.00000000e-01  8.02445896e-01  2.17191931e+00  5.72852505e+00]
 [ 3.07692308e-01  1.40868551e+00  3.89047682e+00  1.25343732e+01]
 [ 5.00000000e-01  2.56281207e+00  7.65078878e+00 -3.27890948e+01]
 [ 8.00000000e-01  3.10929698e+00 -2.18593965e+00 -8.90703017e+01]
 [ 1.00000000e+00  4.44089210e-16 -2.89070302e+01  8.90703017e+01]
 [ 8.00000000e-01 -3.10929698e+00 -2.18593965e+00  3.27890948e+01]
 [ 5.00000000e-01 -2.56281207e+00  7.65078878e+00 -1.25343732e+01]
 [ 3.07692308e-01 -1.40868551e+00  3.89047682e+00 -5.72852505e+00]
 [ 2.00000000e-01 -8.02445896e-01  2.17191931e+00 -3.54356905e+00]
 [ 1.37931034e-01 -4.74369105e-01  1.10884859e+00 -1.58260989e+00]
 [ 1.00000000e-01 -3.00077683e-01  6.34065628e-01 -8.61189843e-01]
 [ 7.54716981e-02 -1.99100253e-01  3.75708675e-01 -4.95230151e-01]
 [ 5.88235294e-02 -1.38815423e-01  2.27139629e-01 -1.54559919e-01]
 [ 4.70588235e-02 -9.80242943e-02  1.80771654e-01 -6.02572179e-01]]
```

The resulting matrix contains the list of coefficients of the cubic polynomial for each subintervals. For example, the first row of the first matrix should be interpreted as that the cubic spline when $n = 4$ on $[x_0, x_1] = [-1.0, -0.5]$ coincides with the cubic polynomial

$$0.03846 - 0.4831x + 0x^2 + 2.728x^3.$$

We have computed natural splines, so the coefficient of the quadratic term for the leftmost subinterval is always 0, as the results show.

The errors $f(x) - S(x)$ at 41 equidistant points are plotted, as the following figure, so that we can visualize and easily see how good the cubic splines are for various values of n.



The figure indicates that the interpolation accuracy increases with the number of knots.