Atomic physics : Basics of quantum optics/light matter interactions

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The goal of this class (and the next ones) is to introduce basic concepts of **light - matter interactions** on the atomic level = **Quantum optics**. Basically, we will see how to use lasers to manipulate quantum states of atoms.

- Book suggestions:
  - "*Photons and Atoms: Introduction to Quantum Electrodynamics*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Atom-Photon Interactions: Basic Process and Appilcations*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Quantum Optics* " D. F. Walls, G. J. Milburn (accessible and complete)
  - "*Quantum Computation and Quantum Information*", *M. A. Nielsen, I. Chuang* (standard text for quantum information theory, has also a good part on master equations, available online for free)
  - "*The Quantum World of Ultra-Cold Atoms and Light Book I*", P. Zoller and C. Gardiner (also "Quantum Noise") (very advanced and concise)

- Outline (may vary)

1. Atom-field interaction Hamiltonian (RWA)/Rabi Problem and AC Stark shift.

2. The Bloch sphere for two-level atoms and perturbation theory for multi-level atoms. **This time**

3. Three level systems/STIRAP and field quantization.

4. Atom decay, density matrix and master equations.

Last time

- We looked at the problem of a classical oscillating (optical frequency) electric field coupled to an atom in the dipole approximation:

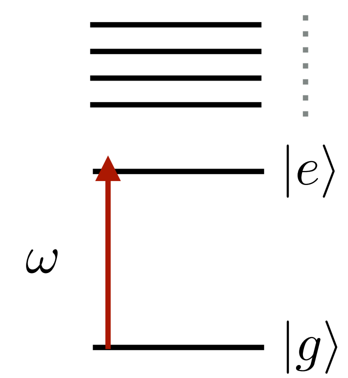
$$\hat{H}_{AF} = \hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}_{\text{cl.}} \qquad\qquad i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{AF} |\psi(t)\rangle$$

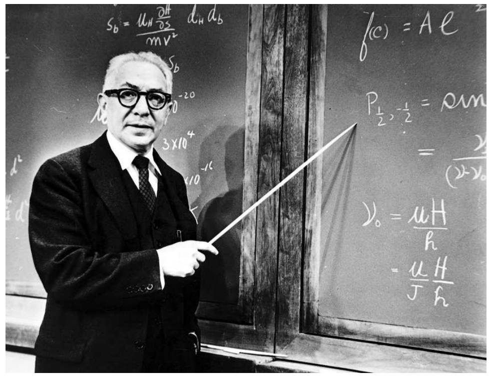
- The problem can be transformed to a rotating frame (at an optical frequency). Then, “counter-rotating” terms can be eliminated in perturbation theory: **Rotating wave approximation (RWA)**.

- In the case where a laser is tuned close to an internal transition of the atom, also other atomic level can be eliminated in perturbation theory (more on the details on this theory later).

- Then we arrive at the “Rabi-problem” for a two-level system

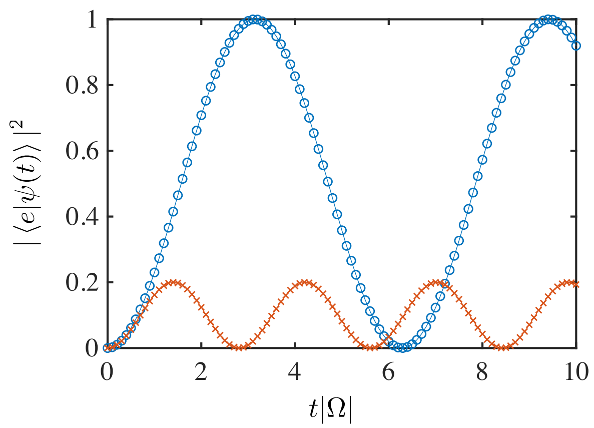
$$\hat{H} = -\Delta |e\rangle \langle e| - \boldsymbol{\mu}_{eg} \cdot \boldsymbol{E}^{-}(t) |e\rangle \langle g| - \boldsymbol{\mu}_{ge} \cdot \boldsymbol{E}^{+}(t) |g\rangle \langle e|$$





- We solved the Rabi-Problem for the case were initially the atom is in the ground state. Then, the key results are **Rabi oscillations** at the frequency:

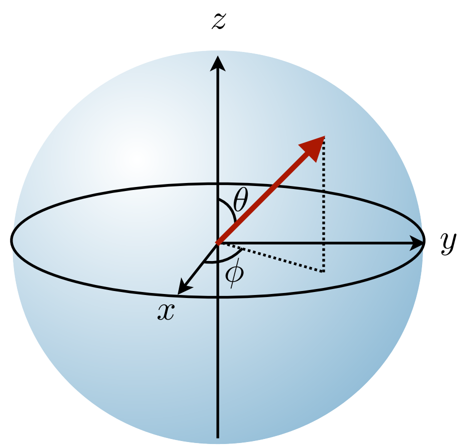
$$\Omega_{\text{eff}} \equiv \sqrt{\Delta^2 + |\Omega|^2}$$



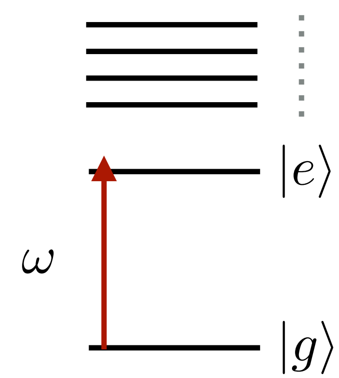
- In the far-detuned case we find level (AC Stark) shifts which can be used for atom trapping (optical lattices).

This time - Outline

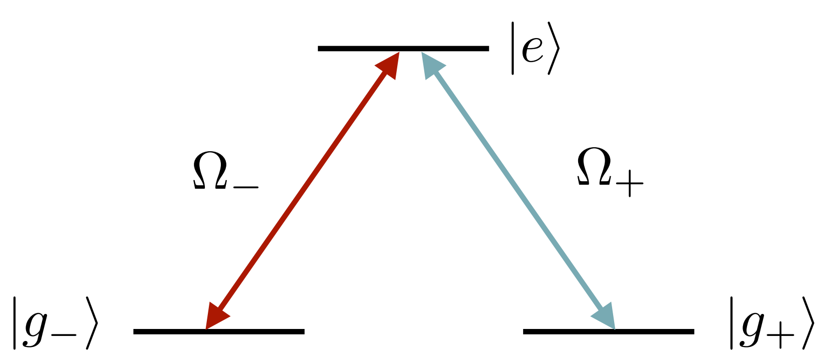
- 2.1 — We want to first analyze how we can describe the full dynamics of the two-level system, in particular in terms of a so-called Bloch sphere.



- 2.2 —Then we will perform a proper perturbation theory for the full N-level atom. This will help us to justify the conditions when the two-level approximation and the RWA is valid.

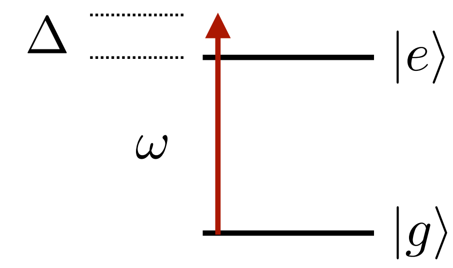


- 2.3 — Lastly, we will start looking at a possible application of our formalisms for three-level systems.



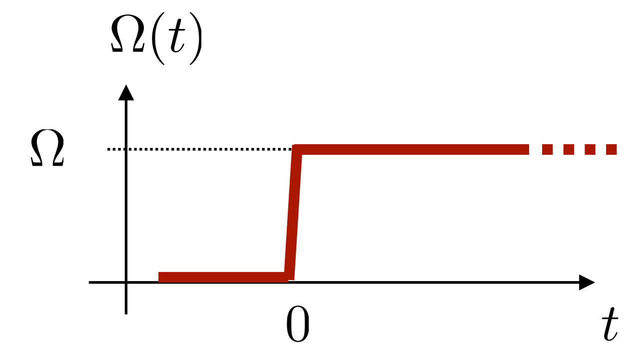
2.1 - General time-evolution in the Rabi-Problem

- This time, we first want to understand better how the general evolution of a two-level system coupled to a laser can be described.



- **Remember:**

$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix} \qquad \Omega = 2(\boldsymbol{\mu}_{eg} \cdot \boldsymbol{\epsilon})\mathcal{E}$$



- Diagonalization: $\tan(\theta) = -\dfrac{|\Omega|}{\Delta}$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \qquad |+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$

- The general solution will be given by the time-evolution matrix (propagator):

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}t\hat{H}}\,|\psi(0)\rangle \equiv \hat{U}\,|\psi(0)\rangle$$

$$U_{eg} = \langle e|\,\hat{U}\,|g\rangle$$

$$\hat{U} = \mathrm{e}^{-\mathrm{i}t\hat{H}} = \mathrm{e}^{-\mathrm{i}tE_+}\,|+\rangle\langle +| + \mathrm{e}^{-\mathrm{i}tE_-}\,|-\rangle\langle -| = \begin{pmatrix} U_{ee} & U_{eg} \\ U_{ge} & U_{gg} \end{pmatrix}$$

“Probability amplitude to go from g to e”

2.1 - General time-evolution in the Rabi-Problem

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \qquad |+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$

- Last time we already showed ($\phi = 0$)

$$\boxed{U_{eg}} = \langle e|\,\hat{U}\,|g\rangle = \langle e|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}tE_+}\langle e|+\rangle\langle +|g\rangle + \mathrm{e}^{-\mathrm{i}tE_-}\langle e|-\rangle\langle -|g\rangle$$

$$= \mathrm{e}^{-\mathrm{i}tE_+}\cos(\theta/2)\sin(\theta/2) - \mathrm{e}^{-\mathrm{i}tE_-}\cos(\theta/2)\sin(\theta/2)$$

$$= \frac{1}{2}\sin(\theta)\left(\mathrm{e}^{-\mathrm{i}tE_+} - \mathrm{e}^{-\mathrm{i}tE_-}\right)$$

$$\sin(\alpha) = 2\sin(\alpha/2)\cos(\alpha/2)$$

$$\boxed{= \mathrm{i}\sin(\theta)\mathrm{e}^{-\mathrm{i}t\Delta/2}\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right)}$$

$$\Omega_{\mathrm{eff}} \equiv \sqrt{\Delta^2 + |\Omega|^2}$$

- From this, we can see quickly:

$$\boxed{U_{ge}} = \mathrm{e}^{-\mathrm{i}tE_+}\langle g|+\rangle\langle +|e\rangle + \mathrm{e}^{-\mathrm{i}tE_-}\langle g|-\rangle\langle -|e\rangle = \mathrm{e}^{-\mathrm{i}tE_+}\cos(\theta/2)\sin(\theta/2) - \mathrm{e}^{-\mathrm{i}tE_-}\cos(\theta/2)\sin(\theta/2)$$

$$\boxed{= \mathrm{i}\sin(\theta)\mathrm{e}^{-\mathrm{i}t\Delta/2}\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right)} = U_{ge}$$

2.1 - General time-evolution in the Rabi-Problem

$$E_{\pm} = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \qquad |+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$

- Furthermore:

$$U_{ee} = \mathrm{e}^{-\mathrm{i}tE_+} \langle e|+\rangle \langle +|e\rangle + \mathrm{e}^{-\mathrm{i}tE_-} \langle e|-\rangle \langle -|e\rangle \quad = \mathrm{e}^{-\mathrm{i}tE_+} \cos^2(\theta/2) + \mathrm{e}^{-\mathrm{i}tE_-} \sin^2(\theta/2)$$

$$\sin^2(\alpha) = \frac{1}{2}\left[1 - \cos(2\alpha)\right]$$

$$\cos^2(\alpha) = \frac{1}{2}\left[1 + \cos(2\alpha)\right]$$

$$= \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}tE_+} + \mathrm{e}^{-\mathrm{i}tE_-}\right] + \frac{1}{2}\cos(\theta)\left[\mathrm{e}^{-\mathrm{i}tE_+} - \mathrm{e}^{-\mathrm{i}tE_-}\right]$$

…from before

$$= \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}tE_+} + \mathrm{e}^{-\mathrm{i}tE_-}\right] + \mathrm{i}\mathrm{e}^{-\mathrm{i}t\Delta/2}\cos(\theta)\sin(t\frac{1}{2}\Omega_{\mathrm{eff}})$$

$$\cos(\alpha) = \frac{1}{2}\left(\mathrm{e}^{+\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha}\right)$$

$$= \mathrm{e}^{-\mathrm{i}t\Delta/2}\cos\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) + \mathrm{i}\mathrm{e}^{-\mathrm{i}t\Delta/2}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right)$$

- **In total:**

$$\hat{U} = \begin{pmatrix} U_{ee} & U_{eg} \\ U_{ge} & U_{gg} \end{pmatrix} = \mathrm{e}^{-\mathrm{i}t\Delta/2} \begin{pmatrix} \cos\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) + \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) & \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) \\ \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) & \cos\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) - \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\mathrm{eff}}\right) \end{pmatrix}$$

U_{gg}

2.1 - General time-evolution in the Rabi-Problem - $\pi/2$ pulse

$$\hat{U} = \mathrm{e}^{-\mathrm{i}t\Delta/2} \begin{pmatrix} \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) + \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \\ \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) - \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \end{pmatrix}$$

$$\Omega_{\text{eff}} \equiv \sqrt{\Delta^2 + |\Omega|^2} \qquad \tan(\theta) = -\frac{|\Omega|}{\Delta}$$

- Example … let's look for evolution in the resonant case $\Delta = 0 \rightarrow \theta = \pi/2 \rightarrow \sin(\theta) = 1$, $\cos(\theta) = 0$

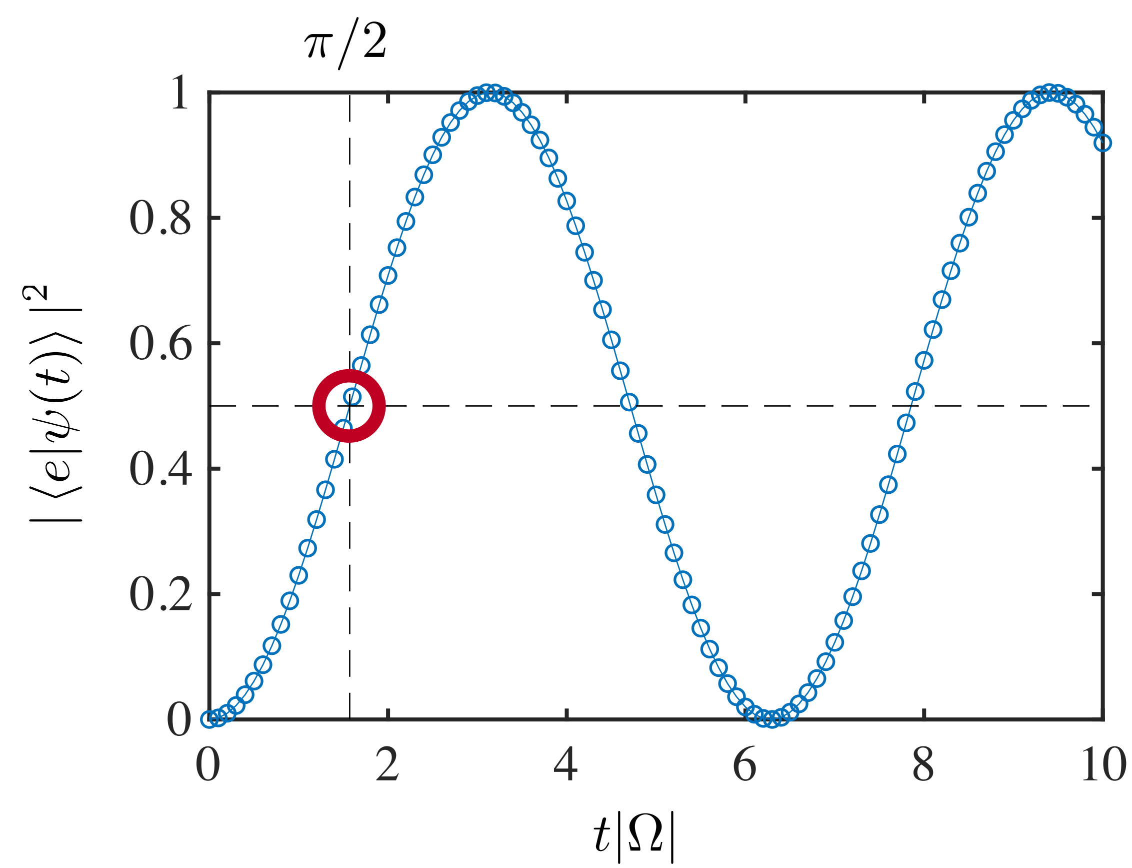
… and for a time $t_0 = \frac{\pi}{2}\Omega^{-1} \rightarrow \cos\left(t_0 \frac{1}{2}\Omega\right) = \sin\left(t_0 \frac{1}{2}\Omega\right) = \frac{1}{\sqrt{2}}$

$$\boxed{\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}}$$

- Then using this "pulse" on an atom initialized in the ground-state:

$$\hat{U}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|g\rangle + \mathrm{i}\,|e\rangle\right)$$

- This pulse is called a ***π*/2 pulse**. It creates a "coherent" super-position state of excited and ground-state.



2.1 - General time-evolution in the Rabi-Problem - $\pi/2$ pulse

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

- Applying the same pulse again:

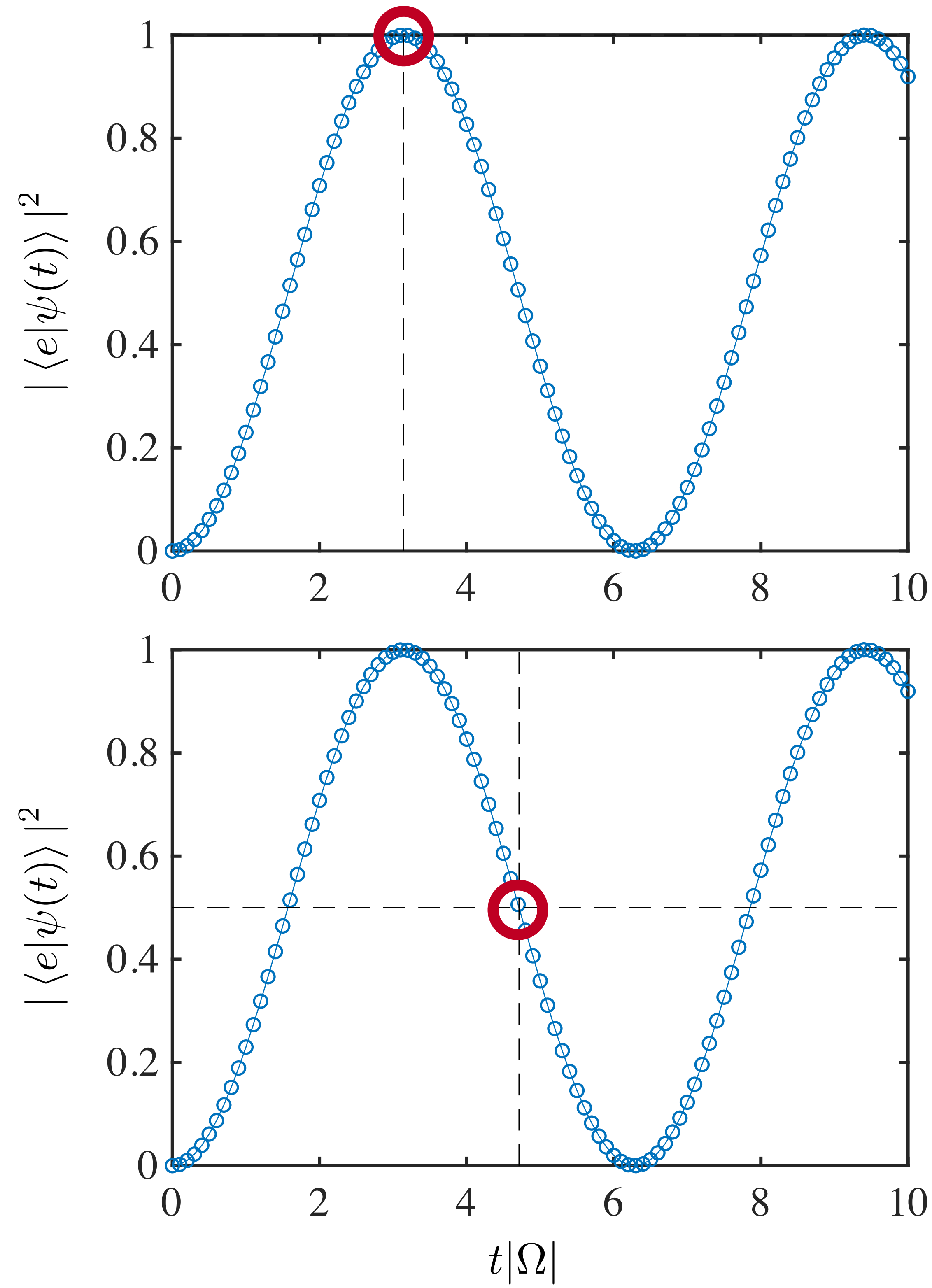
$$\hat{U}\hat{U}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{U}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{i}+\mathrm{i} \\ -1+1 \end{pmatrix} = \begin{pmatrix} \mathrm{i} \\ 0 \end{pmatrix} = |e\rangle$$

- As expected, this is now a π **pulse** which flips the state.

- … and again

$$\hat{U}\hat{U}\hat{U}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{U}\begin{pmatrix} \mathrm{i} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|g\rangle - \mathrm{i}\,|e\rangle\right)$$

- We have again a super-position, now however with a different relative phase.



2.1 - Visualization on the "Bloch-Sphere"

- There is an elegant way, to visualize the evolution of a two-level system

- A general two-level state: $|\psi\rangle = c_g\,|g\rangle + c_e\,|e\rangle$

  … can be parametrized because of the **normalization** condition

$$1 = \langle\psi|\psi\rangle = |c_g|^2 + |c_e|^2$$

Since furthermore only a relative phase matters, we can write

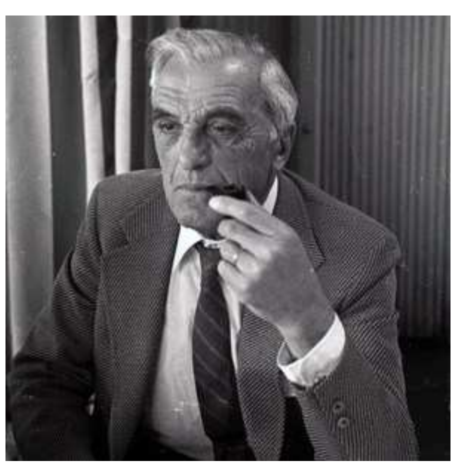
$$\boxed{c_e = \cos(\theta/2)} \qquad \boxed{c_g = \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi}}$$

$$|c_g|^2 + |c_e|^2 = \sin^2(\theta/2) + \cos^2(\theta/2) = 1$$

$\theta \in [0, \pi]$ → $\cos(\theta/2) \in [1, 0]$, $\sin(\theta/2) \in [0, 1]$

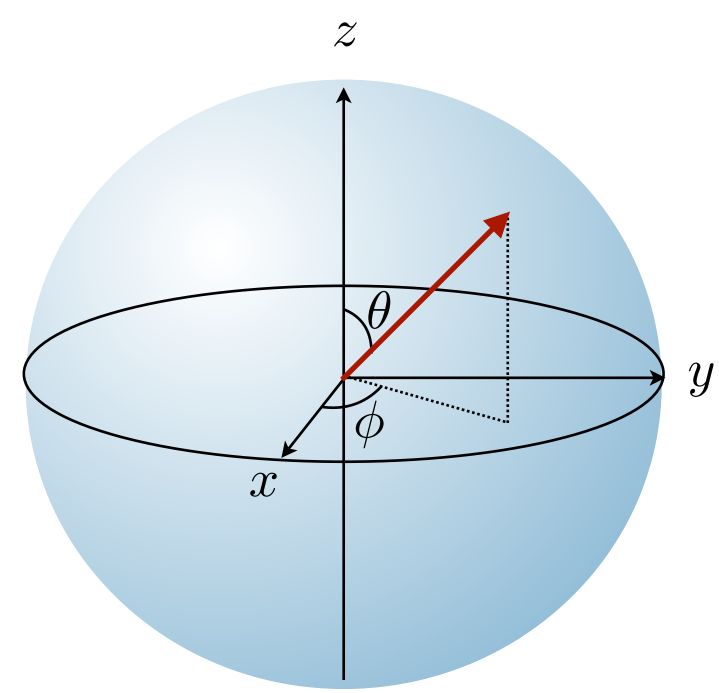
$\phi \in [0, 2\pi]$

- **Any (pure) state can be written in terms of two angles = any state corresponds to a point on unit-sphere.**



Felix Bloch, Swiss physicist (1905-1983)

This representation is called the **Bloch sphere**. Any quantum state can be described by a **Bloch vector.**

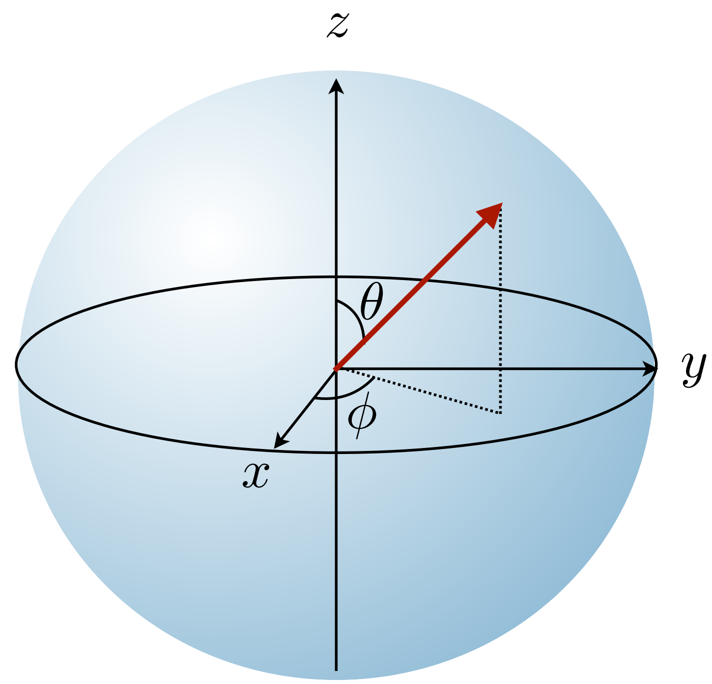


2.1 - Visualization on the "Bloch-Sphere"

$$|\psi\rangle = c_g\,|g\rangle + c_e\,|e\rangle$$

$c_e = \cos(\theta/2)$ $\qquad$ $c_g = \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi}$

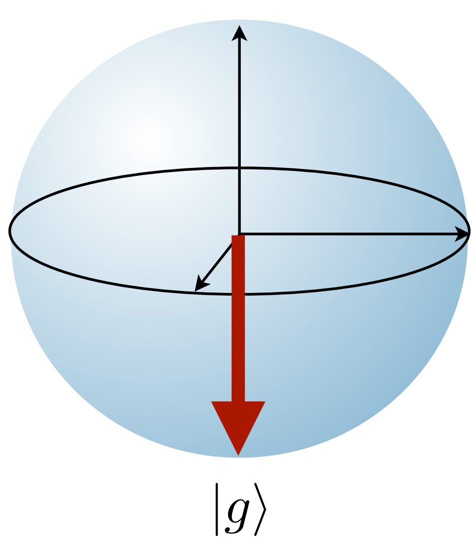
- Points on a (unit) sphere in 3D:

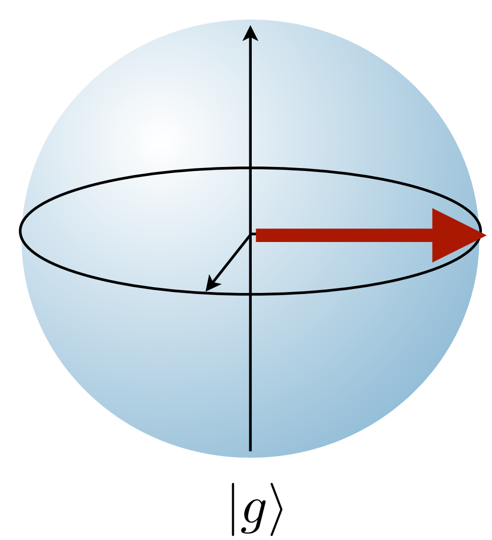


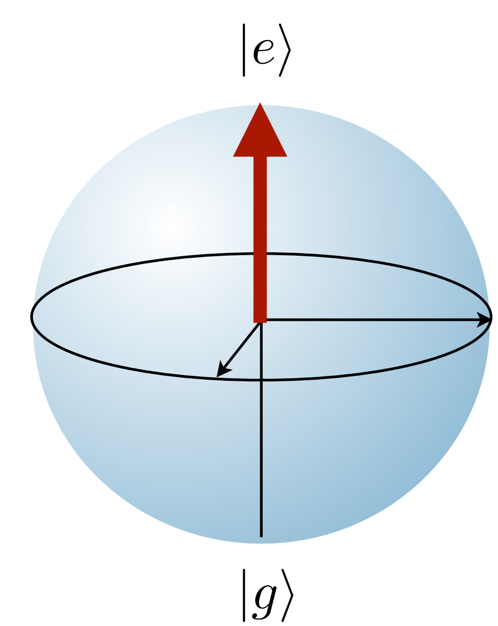
$$x = r\sin(\theta/2)\cos(\phi)$$
$$y = r\sin(\theta/2)\sin(\phi) \qquad r \equiv 1$$
$$z = r\cos(\theta/2)$$

- Examples (for our laser pulses)

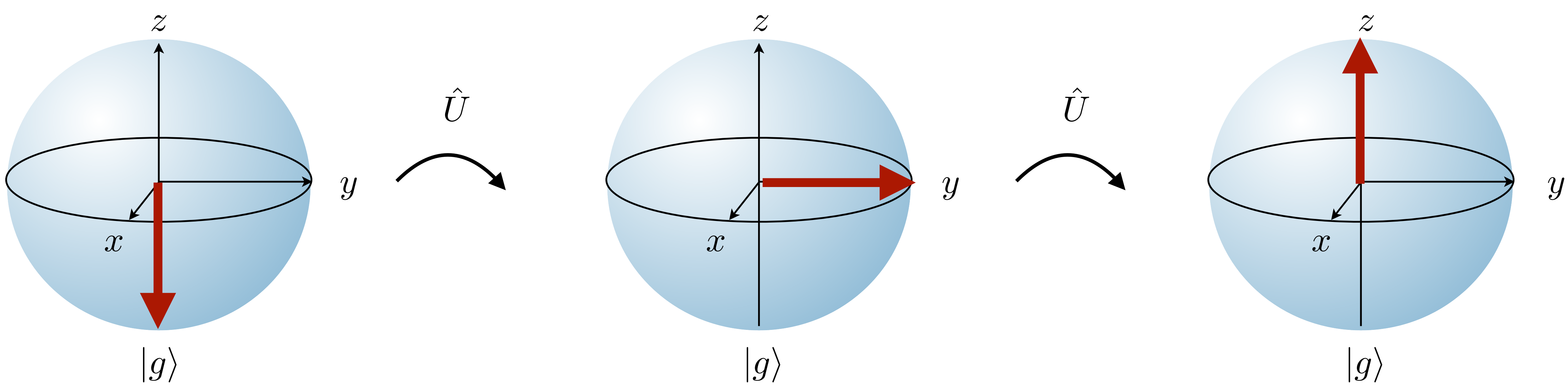
$\lvert g\rangle$	$\hat{U}$	$\frac{1}{\sqrt{2}}(\lvert g\rangle + \mathrm{i}\,\lvert e\rangle)$	$\hat{U}$	$\lvert e\rangle$
$c_e = 0 \quad c_g = 1$		$c_g = 1/\sqrt{2} = c_e$		$c_e = 1 \quad c_g = 0$
$\theta = \pi \quad \phi = \text{arb.}$		$\theta = \pi/2 \quad \phi = \frac{\pi}{2}$		$\theta = 0 \quad \phi = \text{arb.}$







2.1 - Visualization on the "Bloch-Sphere"



- Our $\pi/2$ pulse is a counter-clockwise rotation of the Bloch vector around the x-axis by $\pi/2$.
- Since any state corresponds to a point on a Bloch-sphere:

> Any unitary evolution of the two-level system corresponds to a rotation of the Bloch vector.

- **Remark:** Mathematically this follows from group theory:

$SU(2)$	*"isomorphic"* $\longleftrightarrow$	$SO(3)$
"special unitary" group = group of 2x2 unitary matrices		*"special orthogonal" group = group of rotations in 3D*

In simplified terms, this means that each Pauli matrix can be associated with a rotation.

2.1 - Visualization on the “Bloch-Sphere”

Any unitary evolution of the two-level system corresponds to a rotation of the Bloch vector.

$$SU(2) \xleftrightarrow{\text{“isomorphic”}} SO(3)$$

In simplified terms, this means that each Pauli matrix can be associated with a rotation.

- **Remember:**

$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix} = -\frac{\Delta}{2}\hat{\sigma}^z - \frac{1}{2}\Omega\hat{\sigma}^- - \frac{1}{2}\Omega^*\hat{\sigma}^+$$

$$\hat{\sigma}^z = |e\rangle\langle e| - |g\rangle\langle g| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}^- = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}^+ = (\hat{\sigma}^-)^\dagger = |e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- **Define also:**

$$\hat{\sigma}^x = \hat{\sigma}^+ + \hat{\sigma}^- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}^y = -\mathrm{i}(\hat{\sigma}^+ - \hat{\sigma}^-) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

$$\hat{H} = -\frac{\Delta}{2}\hat{\sigma}^z - \frac{1}{2}\mathrm{Re}(\Omega)\hat{\sigma}^x - \frac{1}{2}\mathrm{Im}(\Omega)\hat{\sigma}^y$$

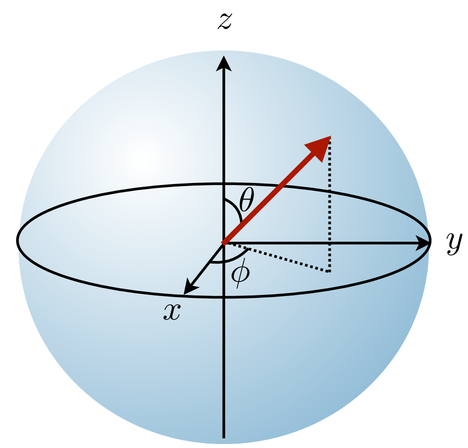
- From this, it is clear that in the case of no de-tuning and real Omega, we get a rotation around x. Generally one can show that the evolution under this Hamiltonian is a rotation around:

$$\vec{r} = -\begin{pmatrix} \mathrm{Re}(\Omega) & \mathrm{Im}(\Omega) & \Delta \end{pmatrix}$$

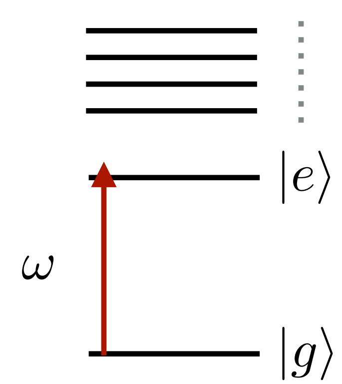
- **Remark:** This is the same as for a spin-1/2 rotating in various magnetic fields.

This time - Outline

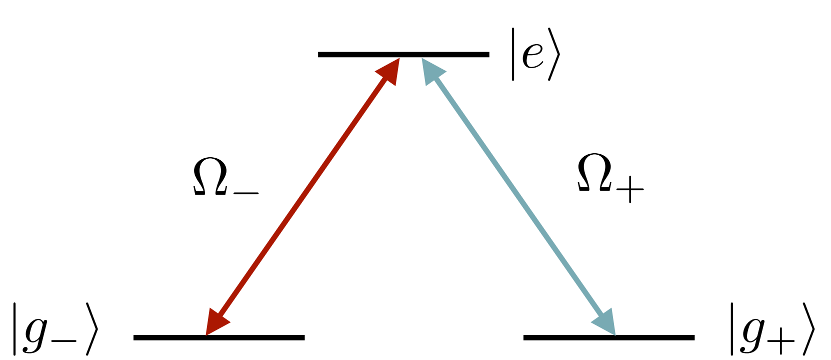
- 2.1 — We want to first analyze how we can describe the full dynamics of the two-level system, in particular in terms of a so-called Bloch sphere.



- 2.2 —Then we will perform a proper perturbation theory for the full N-level atom. This will help us to justify the conditions when the two-level approximation and the RWA is valid.



- 2.3 — Lastly, we will start looking at a possible application of our formalisms for three-level systems.



2.2 - Back to the full problem with many levels

- **Reminder:**

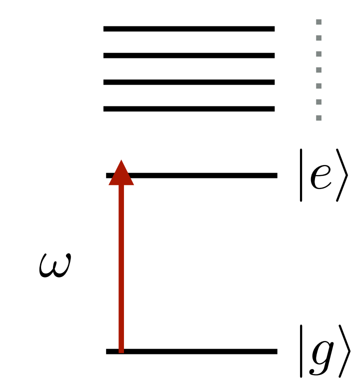
1. Two-level System (TLS) + Rotating Wave approximation (RWA): In a first step we solve the Schrödinger equation for the resonantly coupled TLS (consisting of "g" and "e"). We keep only the terms in the Hamiltonian which correspond to resonant couplings (RWA).

$$\hat{H} = \hbar\omega_e \left|e\right\rangle\left\langle e\right| - \boldsymbol{\mu}_{eg} \cdot \boldsymbol{E}^{-}(t) \left|e\right\rangle\left\langle g\right| - \boldsymbol{\mu}_{ge} \cdot \boldsymbol{E}^{+}(t) \left|g\right\rangle\left\langle e\right|$$

2. Non-resonant couplings: We treat the remaining "non-resonant" interactions in perturbation theory. This will both justify the RWA and the fact that other levels can be ignored. Furthermore, it gives rise to "AC Stark shifts" of atomic levels.

- **There is no such thing as a two-level atoms.** We will therefore now solve the full N-level problem in time-dependent **perturbation theory:**

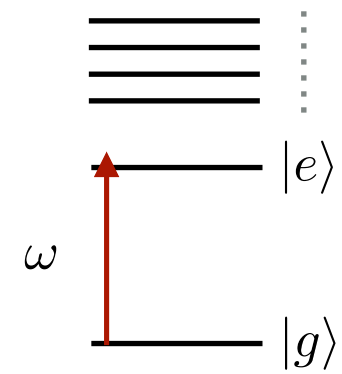
$$\mathrm{i}\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \left[\hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}(t)\right]\left|\psi(t)\right\rangle$$



2.2 - Laser coupled N-level atom in perturbation theory

$$\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \left[\hat{H}_{0A} - \hat{\boldsymbol{\mu}}\cdot\boldsymbol{E}(t)\right]\left|\psi(t)\right\rangle$$

$$\hat{H}_{0A} = \sum_n \omega_n \left|n\right\rangle\left\langle n\right|$$



- **Ansatz:** Expansion into bare atomic states

$$\left|\psi(t=0)\right\rangle = \left|g\right\rangle \qquad \left|\psi(t)\right\rangle = \sum_n a_n(t)\left|n\right\rangle \qquad \boldsymbol{\mu}_{nk} = \left\langle n\right|\hat{\boldsymbol{\mu}}\left|k\right\rangle$$

- Plugging this into the Schrödinger equation gives:

$$\frac{d}{dt}a_n(t) = -\mathrm{i}\omega_n a_n(t) + \mathrm{i}\sum_k \boldsymbol{\mu}_{nk}\cdot\boldsymbol{E}(t)a_k(t)$$

… with the initial condition: $a_n(t=0) = \delta_{ng}$

- We want to go to the “interaction picture”, i.e. we want to do a basis-change and go into a frame, where the bare-level oscillations are transformed away:

$$|\tilde{\psi}\rangle = \hat{U}\left|\psi\right\rangle$$

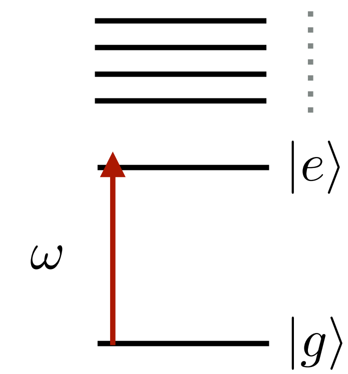
- This is a transformation with the unitary diagonal matrix $\hat{U} = \sum_n \mathrm{e}^{\mathrm{i}\omega_n t}\left|n\right\rangle\left\langle n\right| = \mathrm{e}^{\mathrm{i}\hat{H}_{0A}t}$ $\qquad \hat{U}^\dagger\hat{U} = \mathbb{1}$

2.2 - Laser coupled N-level atom in perturbation theory

$$\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \left[\hat{H}_{0A} - \hat{\boldsymbol{\mu}}\cdot\boldsymbol{E}(t)\right]\left|\psi(t)\right\rangle$$

$$|\tilde{\psi}\rangle = \hat{U}\,|\psi\rangle$$

$$\hat{U} = \sum_n \mathrm{e}^{\mathrm{i}\omega_n t}\,|n\rangle\langle n| = \mathrm{e}^{\mathrm{i}\hat{H}_{0A}t} \qquad \hat{U}^\dagger\hat{U} = \mathbb{1}$$



- **Reminder:** General interaction picture transformation

$$\mathrm{i}\frac{d}{dt}\left(\hat{U}\,|\psi\rangle\right) = \mathrm{i}\left(\frac{d}{dt}\hat{U}\right)|\psi\rangle + \mathrm{i}\hat{U}\left(\frac{d}{dt}|\psi\rangle\right) = \mathrm{i}(\mathrm{i}\hat{H}_{0A})\hat{U}\,|\psi\rangle + \mathrm{i}\hat{U}\left(\frac{d}{dt}|\psi\rangle\right)$$

$$\mathrm{i}\hat{U}\left(\frac{d}{dt}|\psi\rangle\right) = \mathrm{i}\frac{d}{dt}\left(\hat{U}\,|\psi\rangle\right) + \hat{H}_{0A}\hat{U}\,|\psi\rangle = \mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle + \hat{H}_{0A}|\tilde{\psi}\rangle$$

- Then transforming the Schrödinger equation:

$$\mathrm{i}\hat{U}\left(\frac{d}{dt}|\psi\rangle\right) = \hat{U}\hat{H}\,|\psi\rangle = \hat{U}\hat{H}\hat{U}^\dagger\hat{U}\,|\psi\rangle = \hat{U}\hat{H}\hat{U}^\dagger\,|\tilde{\psi}\rangle = \mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle + \hat{H}_{0A}|\tilde{\psi}\rangle$$

Interaction picture:

$$\mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle = \hat{U}\hat{H}\hat{U}^\dagger\,|\tilde{\psi}\rangle - \hat{H}_{0A}|\tilde{\psi}\rangle = -\hat{U}\hat{\boldsymbol{\mu}}\cdot\boldsymbol{E}(t)\hat{U}^\dagger\,|\tilde{\psi}\rangle$$

$$\left(\hat{U}\hat{H}_{0A}\hat{U}^\dagger = \hat{H}_{0A}\right)$$

- Since: $\hat{U}\hat{\boldsymbol{\mu}}\hat{U}^\dagger = \sum_{n,k}\hat{U}\,|n\rangle\langle k|\,\hat{U}^\dagger\boldsymbol{\mu}_{nk} = \sum_{n,k}\mathrm{e}^{\mathrm{i}(\omega_n-\omega_k)t}\boldsymbol{\mu}_{nk}\,|n\rangle\langle k|$

- The Schrödinger equation for the transformed state amplitudes reads:

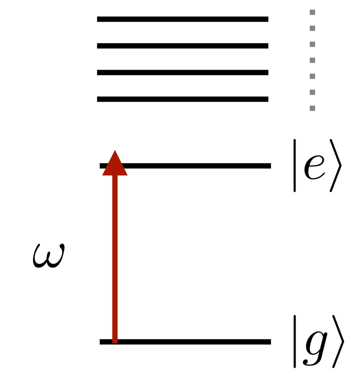
$$\frac{d}{dt}\tilde{a}_n(t) = \mathrm{i}\sum_k \boldsymbol{\mu}_{nk}\cdot\boldsymbol{E}(t)\tilde{a}_k(t)\mathrm{e}^{\mathrm{i}(\omega_n-\omega_k)t}$$

2.2 - Laser coupled N-level atom in perturbation theory

$$\frac{d}{dt}\tilde{a}_n(t) = \mathrm{i}\sum_k \boldsymbol{\mu}_{nk}\cdot\boldsymbol{E}(t)\tilde{a}_k(t)\mathrm{e}^{\mathrm{i}(\omega_n-\omega_k)t}$$

… with the initial condition:

$$\tilde{a}_n(t=0) = \delta_{ng}$$



- We convert this to an integral equation:

$$\tilde{a}_n(t) = \delta_{ng} + \mathrm{i}\sum_k \int_0^t d\tau\,\mathrm{e}^{\mathrm{i}(\omega_n-\omega_k)\tau}\boldsymbol{\mu}_{nk}\cdot\boldsymbol{E}(\tau)\tilde{a}_k(\tau)$$

- As usual, in time-dependent perturbation theory, we will re-insert the amplitudes on the RHS
- In zero order (for very short times), all amplitudes are zero, except for the ground-state, the first order iteration reads:

$$\tilde{a}_{n\neq g}(t) = \mathrm{i}\int_0^t d\tau\,\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g)\tau}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{E}(\tau)\tilde{a}_g(\tau)$$

- … and let's remember our E-field parametrization:

$$\boldsymbol{E}(t) = \left[\boldsymbol{E}(t)^+ + \boldsymbol{E}(t)^-\right]$$

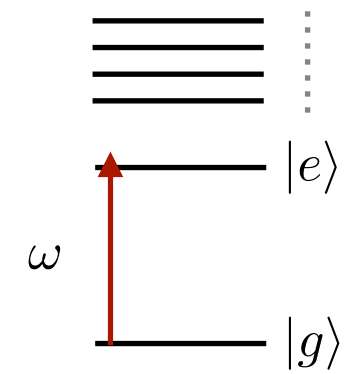
$$\boldsymbol{E}(t)^- = \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\boldsymbol{E}(t)^+ = \mathcal{E}(t)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}\omega t}$$

$$\tilde{a}_{n\neq g}(t) = \mathrm{i}\int_0^t d\tau\,\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g)\tau}\boldsymbol{\mu}_{ng}\cdot\left[\mathcal{E}(\tau)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega\tau} + \mathcal{E}(\tau)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}\omega\tau}\right]\tilde{a}_g(\tau)$$

2.2 - Laser coupled N-level atom in perturbation theory

$$\tilde{a}_{n\neq g}(t) = \mathrm{i}\int_0^t d\tau\, \mathrm{e}^{\mathrm{i}(\omega_n-\omega_g)\tau}\boldsymbol{\mu}_{ng}\cdot\left[\mathcal{E}(\tau)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega\tau}+\mathcal{E}(\tau)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}\omega\tau}\right]\tilde{a}_g(\tau)$$



- As last time, we consider $\frac{d}{dt}|\mathcal{E}|/\mathcal{E} \ll \omega$ $\qquad \frac{d}{dt}|\mathcal{E}|/\mathcal{E} \ll \omega_n - \omega_g$

… which means again that we can make an “adiabatic” approximation. On a coarse-grained integration time-scale, the laser switch-on time/system dynamics is much slower than an “optical oscillation” and can be assumed constant. This means, we can integrate the optical frequency parts only to arrive at

$$\tilde{a}_{n\neq g}(t) \approx \left(\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g-\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)}{(\omega_n-\omega_g-\omega)}+\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g+\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}^*\mathcal{E}(t)^*}{(\omega_n-\omega_g+\omega)}\right)\tilde{a}_g(t)$$

$$\int_0^t d\tau\, \mathrm{e}^{\mathrm{i}\alpha\tau} = \frac{\mathrm{e}^{\mathrm{i}\alpha t}}{\mathrm{i}\alpha}$$

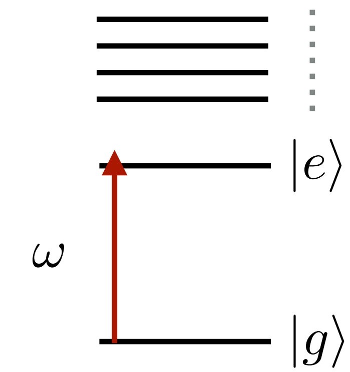
- And the N-level wavefunction in this first order is: $\tilde{a}_g(t) \approx 1$

$$|\tilde{\psi}(t)\rangle = |g\rangle + |e\rangle\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_e-\omega_g-\omega)t}\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)}{(\omega_e-\omega_g-\omega)}+\frac{\mathrm{e}^{\mathrm{i}(\omega_e-\omega_g+\omega)t}\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon}^*\mathcal{E}(t)^*}{(\omega_e-\omega_g+\omega)}\right)$$

$$+\sum_{n\neq g,e}|n\rangle\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g-\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)}{(\omega_n-\omega_g-\omega)}+\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega_g+\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}^*\mathcal{E}(t)^*}{(\omega_n-\omega_g+\omega)}\right)$$

- Note that this wave-function is only approximately normalized (and technically all population is in the ground-state)

2.2 - Laser coupled N-level atom in perturbation theory



$$|\tilde{\psi}(t)\rangle = |g\rangle + |e\rangle \left(\frac{e^{i(\omega_e-\omega_g-\omega)t}\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)}{(\omega_e-\omega_g-\omega)} + \frac{e^{i(\omega_e-\omega_g+\omega)t}\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon}^*\mathcal{E}(t)^*}{(\omega_e-\omega_g+\omega)} \right)$$

$$+ \sum_{n\neq g,e} |n\rangle \left(\frac{e^{i(\omega_n-\omega_g-\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)}{(\omega_n-\omega_g-\omega)} + \frac{e^{i(\omega_n-\omega_g+\omega)t}\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}^*\mathcal{E}(t)^*}{(\omega_n-\omega_g+\omega)} \right)$$

- Or: $|\tilde{\psi}(t)\rangle = |g\rangle\, e^{i\omega_g t} + |e\rangle \frac{1}{2} \left(\frac{e^{i(\omega_e-\omega)t}\Omega_{eg}(t)}{(\omega_e-\omega_g-\omega)} + \frac{e^{i(\omega_e+\omega)t}\Omega_{eg}(t)^*}{(\omega_e-\omega_g+\omega)} \right)$ … with: $\Omega_{ng} \equiv 2\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)$

$$+ \sum_{n\neq g,e} |n\rangle \frac{1}{2} \left(\frac{e^{i(\omega_n-\omega)t}\Omega_{ng}(t)}{(\omega_n-\omega_g-\omega)} + \frac{e^{i(\omega_n+\omega)t}\Omega_{ng}(t)^*}{(\omega_n-\omega_g+\omega)} \right)$$

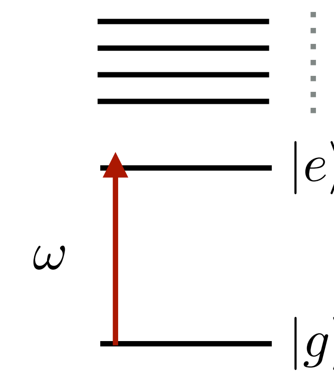
- Near resonant situation: $\Delta = \omega - \omega_e - \omega_g \approx 0$

- Then, the sum is dominated by one term $|\tilde{\psi}(t)\rangle = |g\rangle\, e^{i\omega_g t} + |e\rangle \frac{e^{i(\omega_e-\omega)t}\Omega_{eg}(t)}{2\Delta}$

- **Note:** the denominator diverges for zero de-tuning, which is of course an artifact of the perturbation theory, which brakes down in this limit. Still this is the same state as computed earlier for the Rabi-Problem in the "far-detuned limit" (i.e. the AC-Stark shifted state).

- **This part shouldn't be treated perturbatively in the near resonant case** (and we solved it non-perturbatively earlier)

2.2 - Laser coupled N-level atom in perturbation theory



$$|\tilde{\psi}(t)\rangle = |g\rangle\, \mathrm{e}^{\mathrm{i}\omega_g t} + |e\rangle \frac{1}{2}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_e-\omega)t}\Omega_{eg}(t)}{(\omega_e-\omega_g-\omega)} + \frac{\mathrm{e}^{\mathrm{i}(\omega_e+\omega)t}\Omega_{eg}(t)^*}{(\omega_e-\omega_g+\omega)}\right)$$

$$+ \sum_{n\neq g,e} |n\rangle \frac{1}{2}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega)t}\Omega_{ng}(t)}{(\omega_n-\omega_g-\omega)} + \frac{\mathrm{e}^{\mathrm{i}(\omega_n+\omega)t}\Omega_{ng}(t)^*}{(\omega_n-\omega_g+\omega)}\right)$$

$$\Omega_{ng} \equiv 2\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)$$

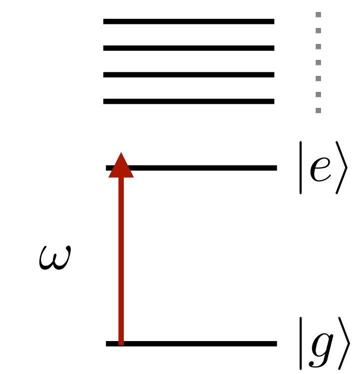
- **Counterrotating terms:**
- For all transitions $\omega_n - \omega_g > 0$

  Therefore all these terms are divided by an optical frequency: $\propto \frac{1}{\omega + \text{“} > 0''}$

- **Neglecting them is the RWA.** We can now specify the condition for the validity of the RWA:

$$|\Omega_{ng}(t)|/2 \ll \omega_n - \omega_g + \omega$$

2.2 - Laser coupled N-level atom in perturbation theory



$$|\tilde{\psi}(t)\rangle = |g\rangle\, \mathrm{e}^{\mathrm{i}\omega_g t} + |e\rangle\, \frac{1}{2}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_e-\omega)t}\Omega_{eg}(t)}{(\omega_e-\omega_g-\omega)} + \frac{\mathrm{e}^{\mathrm{i}(\omega_e+\omega)t}\Omega_{eg}(t)^*}{(\omega_e-\omega_g+\omega)}\right)$$

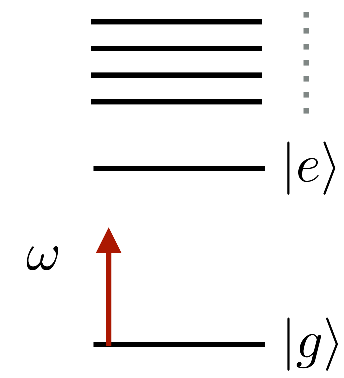
$$\Omega_{ng} \equiv 2\boldsymbol{\mu}_{ng}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)$$

$$+\sum_{n\neq g,e}|n\rangle\,\frac{1}{2}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_n-\omega)t}\Omega_{ng}(t)}{(\omega_n-\omega_g-\omega)} + \frac{\mathrm{e}^{\mathrm{i}(\omega_n+\omega)t}\Omega_{ng}(t)^*}{(\omega_n-\omega_g+\omega)}\right)$$

- **Off-resonant terms of other levels**
- Defining a general detuning for all states $\Delta_n = \omega - \omega_n - \omega_g$

… these terms are $\propto \dfrac{1}{|\Delta_n|}$ … and generally any term with large de-tuning can be treated perturbatively.

2.2 - Laser coupled N-level atom in perturbation theory



$$\Omega_{ng} \equiv 2\boldsymbol{\mu}_{ng} \cdot \boldsymbol{\epsilon}\mathcal{E}(t)$$

- **Let's now take an example where the laser is far-detuned from all levels** … the perturbation theory is valid.

- The Schrödinger equation for the GS amplitude was

$$\frac{d}{dt}\tilde{a}_g(t) = \mathrm{i}\sum_{k\neq g} \boldsymbol{\mu}_{gk} \cdot \boldsymbol{E}(t)\tilde{a}_k(t)\mathrm{e}^{\mathrm{i}(\omega_g-\omega_k)} = \mathrm{i}\sum_{k\neq g}\frac{1}{2}\left(\Omega_{gk}(t)\mathrm{e}^{\mathrm{i}(\omega_g-\omega_k-\omega)t} + \Omega_{gk}(t)^*\mathrm{e}^{\mathrm{i}(\omega_g-\omega_k+\omega)t}\right)\tilde{a}_k(t)$$

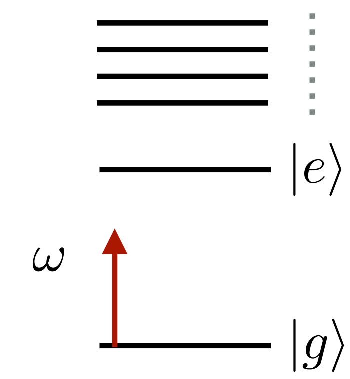
… and in perturbation theory we had
$$\tilde{a}_{k\neq g}(t) \approx \frac{1}{2}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega_k-\omega_g-\omega)t}\Omega_{kg}(t)}{(\omega_k-\omega_g-\omega)} + \frac{\mathrm{e}^{\mathrm{i}(\omega_k-\omega_g+\omega)t}\Omega_{kg}(t)^*}{(\omega_k-\omega_g+\omega)}\right)\tilde{a}_g(t)$$

- Which gives
$$\frac{d}{dt}\tilde{a}_g(t) = \mathrm{i}\sum_{k\neq g}\frac{1}{4}\left(\frac{|\Omega_{gk}(t)|^2}{\omega_k-\omega_g+\omega} + \frac{|\Omega_{gk}(t)|^2}{\omega_k-\omega_g-\omega} + \cancel{\frac{|\Omega_{gk}(t)|^2\mathrm{e}^{-2\mathrm{i}\omega_t}}{\omega_k-\omega_g-\omega}} + \cancel{\frac{|\Omega_{gk}(t)|^2\mathrm{e}^{2\mathrm{i}\omega_t}}{\omega_k-\omega_g+\omega}}\right)\tilde{a}_g(t)$$

- And in RWA
$$\frac{d}{dt}\tilde{a}_g(t) = \mathrm{i}\sum_{k\neq g}\frac{1}{2}\left(\frac{(\omega_k-\omega_g)|\Omega_{gk}(t)|^2}{(\omega_k-\omega_g)^2-\omega^2}\right)\tilde{a}_g(t)$$

$$\frac{1}{a+b} + \frac{1}{a-b} = \frac{2a}{a^2-b^2}$$

2.2 - Laser coupled N-level atom in perturbation theory



$$\Omega_{ng} \equiv 2\boldsymbol{\mu}_{ng} \cdot \boldsymbol{\epsilon}\mathcal{E}(t)$$

- **Let's now take an example where the laser is far-detuned from all levels …** the perturbation theory is valid.

$$\frac{d}{dt}\tilde{a}_g(t) = \mathrm{i}\sum_{k\neq g}\frac{1}{2}\left(\frac{(\omega_k-\omega_g)|\Omega_{gk}(t)|^2}{(\omega_k-\omega_g)^2-\omega^2}\right)\tilde{a}_g(t)$$

- This means that while all population remains in the ground-state, the effective Hamiltonian for the ground-state is:

$$\boxed{\hat{\tilde{H}}_{\text{eff}} = \delta\omega_g(t)\,|g\rangle\langle g|}$$

$$\boxed{\delta\omega_g = \sum_{k\neq g}\frac{1}{2}\left(\frac{(\omega_k-\omega_g)|\Omega_{gk}(t)|^2}{(\omega_k-\omega_g)^2-\omega^2}\right) \equiv \frac{1}{2}\alpha(\omega)|\mathcal{E}|^2}$$

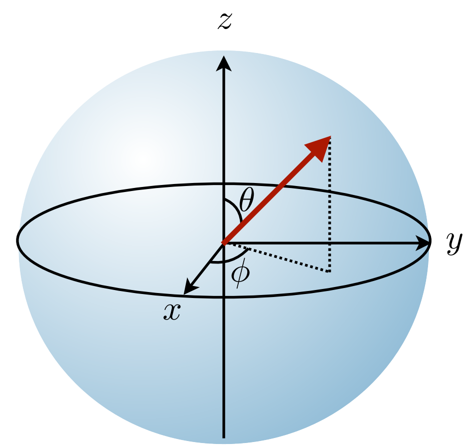
- As a function of time, the ground-state just acquires a phase, due to a energy shift coming from each level
- **Remark:** In the last line we have defined the GS energy as "polarizability" of the atom times the intensity.
- **Remark:** This formula is always true as long as the laser is detuned from all transitions. Note that in the case that we tune to one particular transition, e.g. $\Delta = \omega - \omega_e - \omega_g \approx 0$

  … then again one term dominates the sum and we recover the limits of a single two-level system in the (still) far detuned regime:

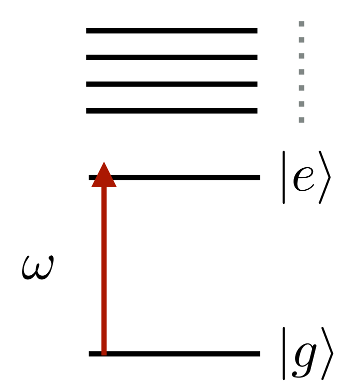
$$\delta\omega_g = \frac{1}{4}\frac{|\Omega_{eg}|^2}{\Delta}$$

This time - Outline

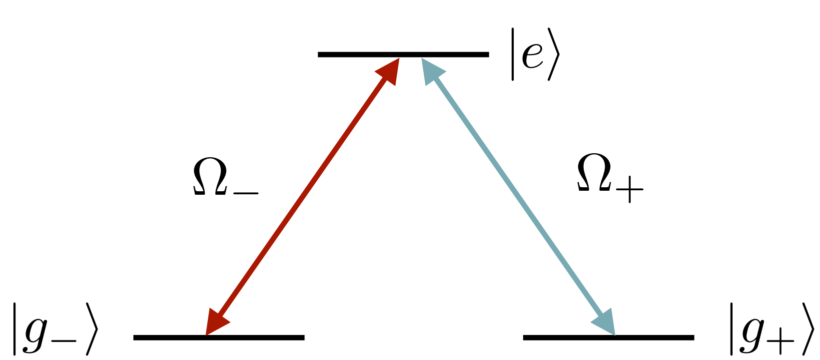
- 2.1 — We want to first analyze how we can describe the full dynamics of the two-level system, in particular in terms of a so-called Bloch sphere.



- 2.2 —Then we will perform a proper perturbation theory for the full N-level atom. This will help us to justify the conditions when the two-level approximation and the RWA is valid.



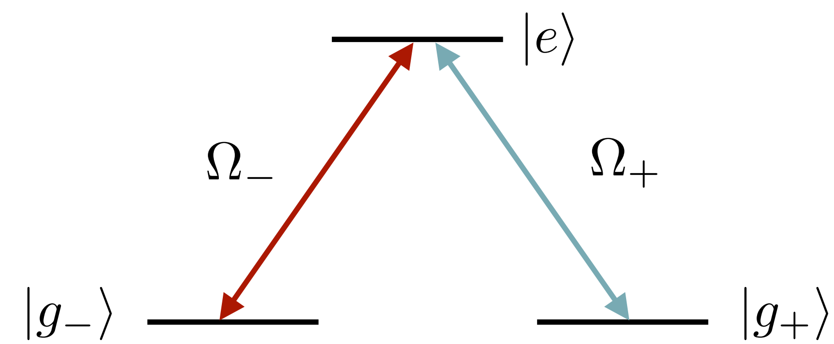
- 2.3 — Lastly, we will start looking at a possible application of our formalisms for three-level systems.



3.1 - Three level systems

We have treated a **general isolated N-level atom** coupled to a **classical electric (laser field).** We understand how and when we can treat it as effective **two-level system**. We will now look at a specific application of the tools we have developed for **three-level** systems.

- In the following let's consider a so-called "Lambda" configuration: We have **two lasers** tuned to **two transitions** to the **same excited state.**

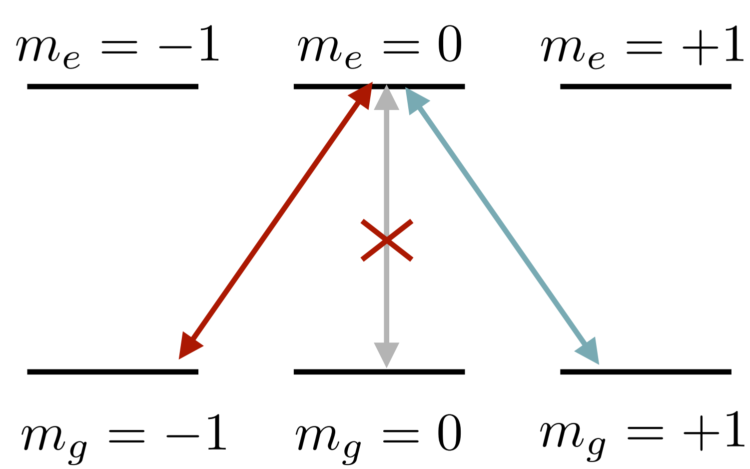


Λ

- This is an important scheme that is routinely realized in atom-physics experiments. For example, consider an atom with ground and excited states and a laser which is in a superposition of circularly polarized light

$|J_e = 1, m_e = -1, 0, 1\rangle$

$|J_g = 1, m_g = -1, 0, 1\rangle$

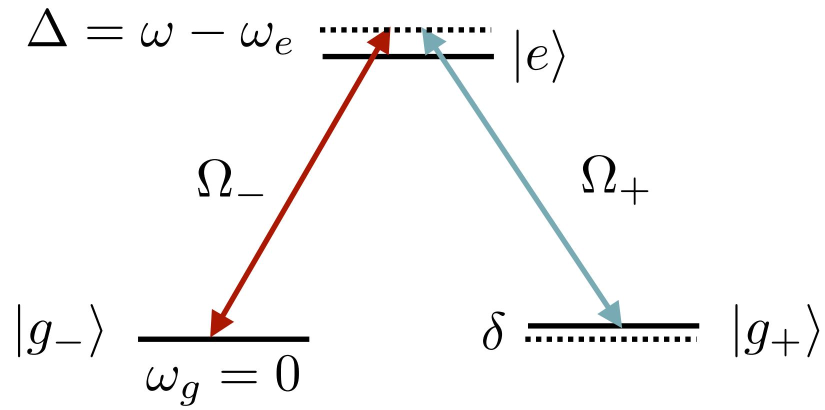


$$\boldsymbol{E}(t) = \left(\boldsymbol{\epsilon}_{-1}\mathcal{E}_- + \boldsymbol{\epsilon}_{+1}\mathcal{E}_+\right) \mathrm{e}^{-\mathrm{i}\omega t}$$

- Angular momentum has to be conserved and thus we have dynamics only in the Lambda sub-system.

For this system we are interested in a mechanism called **STIRAP** (**STI**mulated **R**aman **A**diabatic **P**assage). This is a popular experimental method to "coherently" transfer population from one ground-state to the other one.

3.1 - Lambda system



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- Hamiltonian in RWA

$$\hat{H}_\Lambda = \omega_e \left|e\right\rangle \left\langle e\right| + \delta \left|g_+\right\rangle \left\langle g_+\right| - \frac{1}{2}\left(\Omega_- \mathrm{e}^{-\mathrm{i}\omega t} \left|e\right\rangle \left\langle g_-\right| + \mathrm{h.c.}\right) - \frac{1}{2}\left(\Omega_+ \mathrm{e}^{-\mathrm{i}\omega t} \left|e\right\rangle \left\langle g_+\right| + \mathrm{h.c.}\right)$$

- We know the routine by now … unitary transformation into the laser frame via the ansatz:

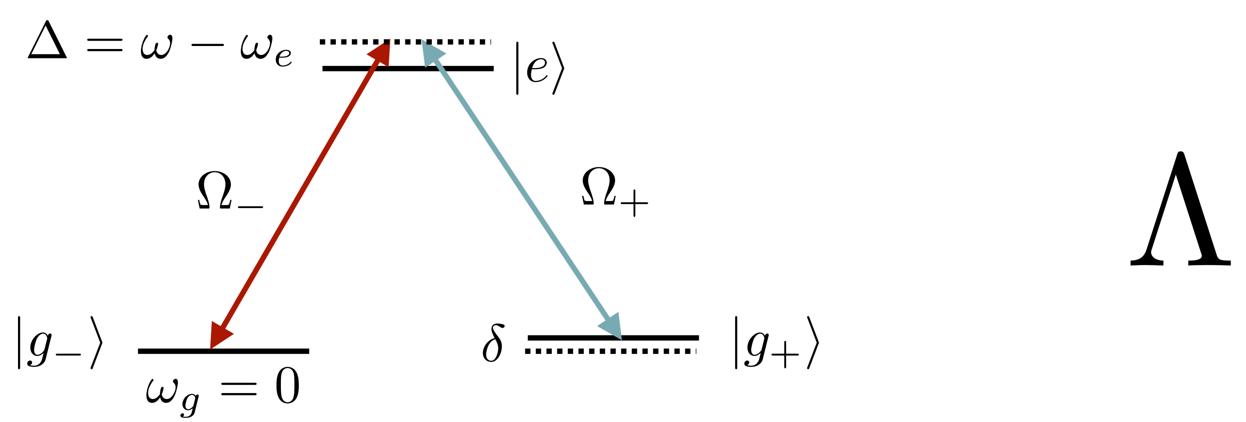
$$\left|\psi(t)\right\rangle = a_-(t)\left|g_-\right\rangle + a_0(t)\mathrm{e}^{-\mathrm{i}\omega t}\left|e\right\rangle + a_+(t)\left|g_+\right\rangle$$

- Then the optical frequency is transformed away and we have the Schrödinger equation:

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_-(t) & 0 \\ -\frac{1}{2}\Omega_-(t) & -\Delta - \mathrm{i}\frac{1}{2}\Gamma & -\frac{1}{2}\Omega_+(t) \\ 0 & -\frac{1}{2}\Omega_+(t) & \delta \end{pmatrix}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix}$$

- **Remark:** Here we have added an imaginary part to the excited level to mimic the finite life-time of this state. This is a common (but not really a proper) way to include the decay of an excited atomic state … we will come back to this later.

3.1 - Lambda system



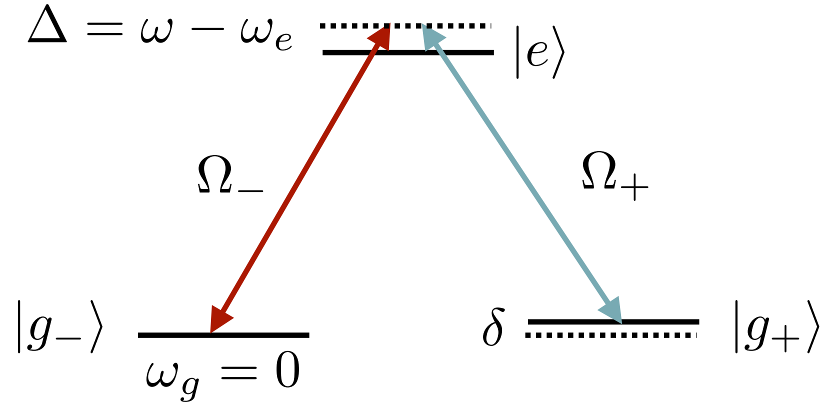
$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_-(t) & 0 \\ -\frac{1}{2}\Omega_-(t) & -\Delta - \mathrm{i}\frac{1}{2}\Gamma & -\frac{1}{2}\Omega_+(t) \\ 0 & -\frac{1}{2}\Omega_+(t) & \delta \end{pmatrix}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix}$$

- We will now solve this problem in the case of a large de-tuning, i.e. for $\Delta \gg \Omega_+, \Omega_-, \Gamma$
- Take the evolution for the excited state amplitude:

$$\mathrm{i}\frac{d}{dt}a_0 = -\frac{1}{2}\Omega_+(t)a_+ - \frac{1}{2}\Omega_-(t)a_- - \left(\Delta + \mathrm{i}\frac{\Gamma}{2}\right)a_0$$

- What we will do now is a so-called **adiabatic elimination of the excited state.**
- … which means we assume $\dot{a}_0 \approx 0$

3.1 - Lambda system



Λ

$$\mathrm{i}\frac{d}{dt}a_0 = -\frac{1}{2}\Omega_+(t)a_+ - \frac{1}{2}\Omega_-(t)a_- - \left(\Delta + \mathrm{i}\frac{\Gamma}{2}\right)a_0$$

- Now we assume: $\dot{a}_0 \approx 0$
- **Why/when is this a good approximation?** Good question! Mathematically rigorously justifying it is complicated.
- Simple argument: Excited state stays effectively in a "steady state"

$\frac{1}{\Omega_\pm}$ is our time-scale of interest. It is very long compared to $\frac{1}{\Delta}$

… so let's consider that the excited state always reaches its steady state at each instant.

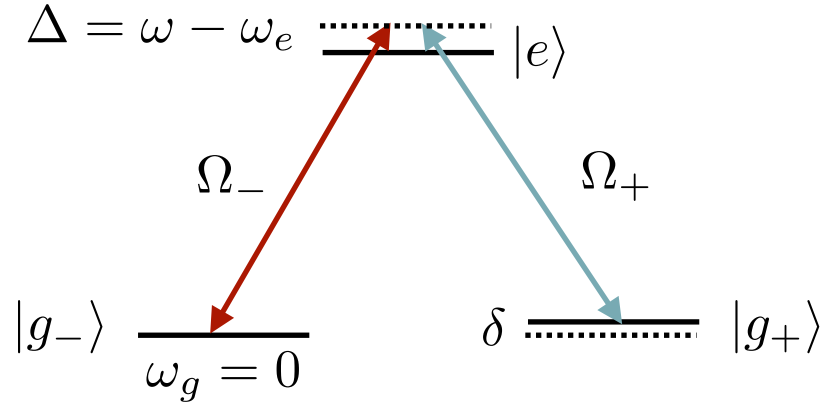
- Then:

$$0 \approx -\frac{1}{2}\Omega_+(t)a_+ - \frac{1}{2}\Omega_-(t)a_- - \left(\Delta + \mathrm{i}\frac{\Gamma}{2}\right)a_0$$

$$a_0 \approx -\frac{\Omega_+(t)}{2\Delta + \mathrm{i}\Gamma}a_+ - \frac{\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}a_-$$

- **Remark:** Also large decay can be used to argue for the validity of this approximation.

3.1 - Lambda system



Λ

$$\dot{a}_0 \approx 0$$

$$a_0 \approx -\frac{\Omega_+(t)}{2\Delta + \mathrm{i}\Gamma} a_+ - \frac{\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma} a_-$$

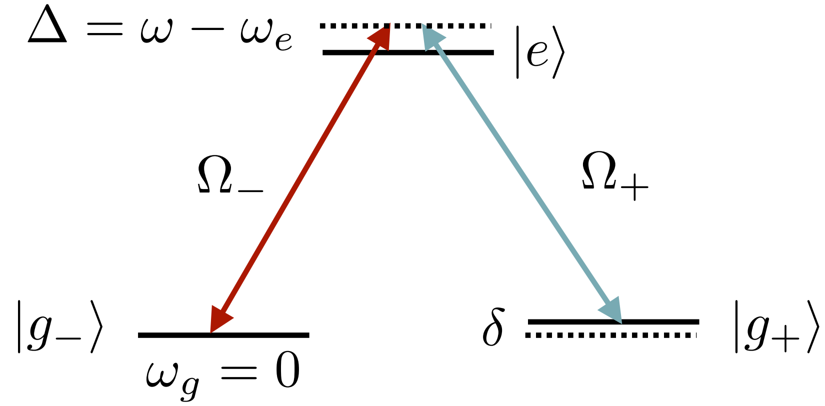
$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_-(t) & 0 \\ -\frac{1}{2}\Omega_-(t) & -\Delta - \mathrm{i}\frac{1}{2}\Gamma & -\frac{1}{2}\Omega_+(t) \\ 0 & -\frac{1}{2}\Omega_+(t) & \delta \end{pmatrix} \begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix}$$

- We can now insert this value into the equations of motion for the other amplitudes

$$\mathrm{i}\frac{d}{dt}a_+ = \delta a_+ - \frac{1}{2}\Omega_+(t)a_0 = \delta a_+ + \frac{\Omega_+(t)^2}{4\Delta + 2\mathrm{i}\Gamma}a_+ + \frac{\Omega_+(t)\Omega_-(t)}{4\Delta + 2\mathrm{i}\Gamma}a_-$$

$$\mathrm{i}\frac{d}{dt}a_- = -\frac{1}{2}\Omega_-(t)a_0 = \frac{\Omega_+(t)\Omega_-(t)t}{4\Delta + 2\mathrm{i}\Gamma}a_+ + \frac{\Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma}a_-$$

3.1 - Lambda system



Λ

- After the adiabatic elimination of the excited state, we have described the lambda system as an effective two-level system.

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_-(t) & 0 \\ -\frac{1}{2}\Omega_-(t) & -\Delta - \mathrm{i}\frac{1}{2}\Gamma & -\frac{1}{2}\Omega_+(t) \\ 0 & -\frac{1}{2}\Omega_+(t) & \delta \end{pmatrix}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix}$$

$$\dot{a}_0 \approx 0$$

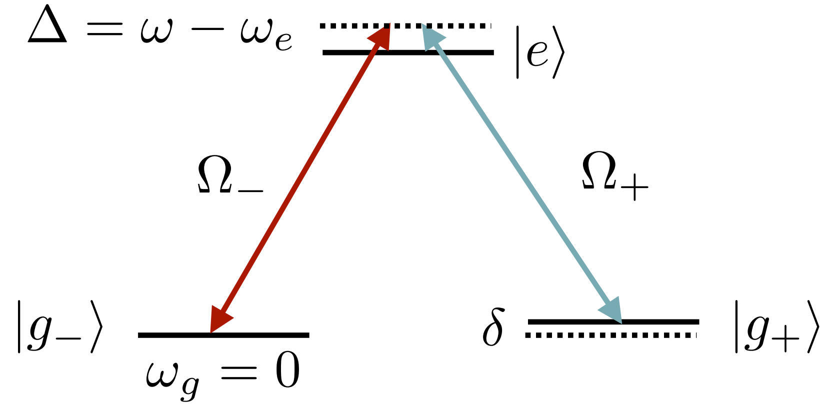
$$\mathrm{i}\frac{d}{dt}a_+ = \delta a_+ + \frac{\Omega_+(t)^2}{4\Delta + 2\mathrm{i}\Gamma}a_+ + \frac{\Omega_+(t)\Omega_-(t)}{4\Delta + 2\mathrm{i}\Gamma}a_-$$

$$\mathrm{i}\frac{d}{dt}a_- = \frac{\Omega_+(t)\Omega_-(t)t}{4\Delta + 2\mathrm{i}\Gamma}a_+ + \frac{\Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma}a_-$$

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \delta + \frac{\Omega_+(t)^2}{4\Delta+2\mathrm{i}\Gamma} & \frac{\Omega_+(t)\Omega_-(t)}{4\Delta+2\mathrm{i}\Gamma} \\ \frac{\Omega_+(t)\Omega_-(t)}{4\Delta+2\mathrm{i}\Gamma} & \frac{\Omega_-(t)^2}{4\Delta+2\mathrm{i}\Gamma} \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

- We can now use all our results from solving the Rabi problem earlier!

3.1 - Lambda system



Λ

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \delta + \frac{\Omega_+(t)^2}{4\Delta+2\mathrm{i}\Gamma} & \frac{\Omega_+(t)\Omega_-(t)}{4\Delta+2\mathrm{i}\Gamma} \\ \frac{\Omega_+(t)\Omega_-(t)}{4\Delta+2\mathrm{i}\Gamma} & \frac{\Omega_-(t)^2}{4\Delta+2\mathrm{i}\Gamma} \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

- We can define an effective de-tuning and an effective Rabi-frequency between the two ground-states in terms of the real Rabi-Frequencies

$$-\Delta_{\text{eff}} \equiv \delta + \frac{\Omega_+(t)^2 - \Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma} \qquad -\Omega_{\text{eff}} = \frac{\Omega_+(t)\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}$$

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} -\Delta_{\text{eff}}(t) & -\frac{1}{2}\Omega_{\text{eff}}(t) \\ -\frac{1}{2}\Omega_{\text{eff}}(t) & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

- **Remark:** The diagonal term contains the standard AC Stark shifts to the ground-states, which are absorbed into the de-tuning. Interestingly, the effective de-tuning can therefore not only be changed by the actual de-tuning, but also by the Rabi-frequency/Intensity of the laser.
- **Remark:** The processes in this effective dynamics involve two photons (the Rabi-Frequencies always shows up in pairs). Those processes are also called Raman processes.

Recap

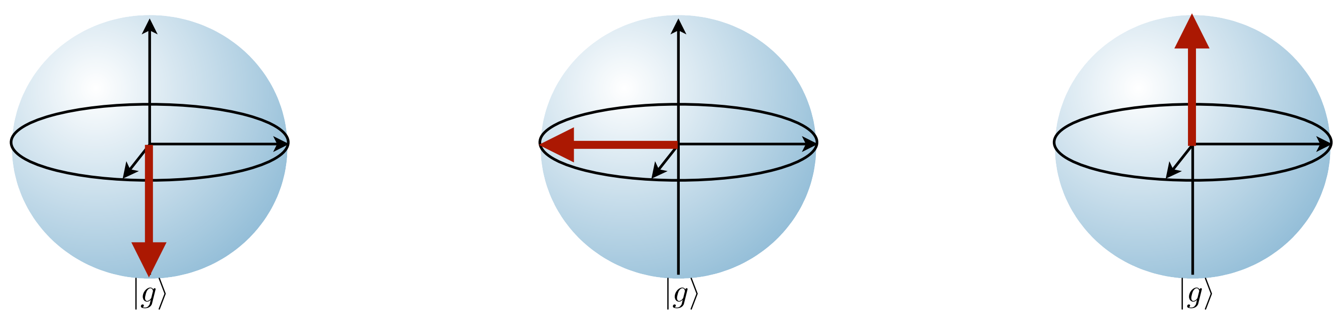
- We looked at the full dynamics of a two-level atom (computed the time-evolution operator)

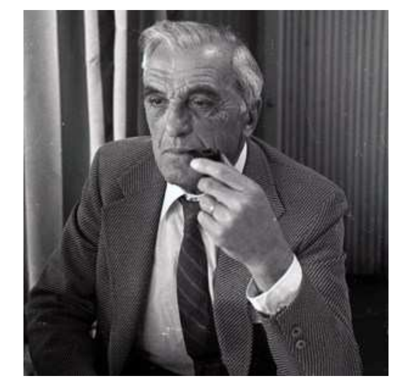
$$\hat{U} = \mathrm{e}^{-\mathrm{i}t\Delta/2} \begin{pmatrix} \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) + \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \\ \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) - \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \end{pmatrix}$$

- Any quantum state of a two-level atom can be described by a point on the **Bloch sphere**, consequently, any laser pulse can be described by a rotation of a **Bloch vector.**

$$c_e = \cos(\theta/2)$$

$$c_g = \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi}$$





- We derived the Schrödinger equation of a laser-coupled N-level atom initially in the ground-state in perturbation theory. The perturbation theory is fully valid if the laser is far-detuned from any internal transition. Then each level gives rise to an AC stark shift of the ground-state.

$$\hat{\hat{H}}_{\text{eff}} = \delta\omega_g(t)\,|g\rangle\,\langle g|$$

$$\delta\omega_g = \sum_{k\neq g} \frac{1}{2}\left(\frac{(\omega_k-\omega_g)|\Omega_{gk}(t)|^2}{(\omega_k-\omega_g)^2-\omega^2}\right) \equiv \frac{1}{2}\alpha(\omega)|\mathcal{E}|^2$$

- The condition for the validity of the RWA becomes obvious in the perturbation theory: $|\Omega_{ng}(t)|/2 \ll \omega_n - \omega_g + \omega$

- We started to look at an application of our tools for a **three-level** Lambda system.