Atomic physics : Basics of quantum optics/light matter interactions

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The goal of this class (and the next ones) is to introduce basic concepts of **light - matter interactions** on the atomic level = **Quantum optics**. Basically, we will see how to use lasers to manipulate quantum states of atoms.

- Book suggestions:
  - "*Photons and Atoms: Introduction to Quantum Electrodynamics*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Atom-Photon Interactions: Basic Process and Appilcations*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Quantum Optics* " D. F. Walls, G. J. Milburn (accessible and complete)
  - "*Quantum Computation and Quantum Information*", *M. A. Nielsen, I. Chuang* (standard text for quantum information theory, has also a good part on master equations, available online for free)
  - "*The Quantum World of Ultra-Cold Atoms and Light Book I*", P. Zoller and C. Gardiner (also "Quantum Noise") (very advanced and concise)

- Outline (may vary)

1. Atom-field interaction Hamiltonian (RWA)/Rabi Problem and AC Stark shift.

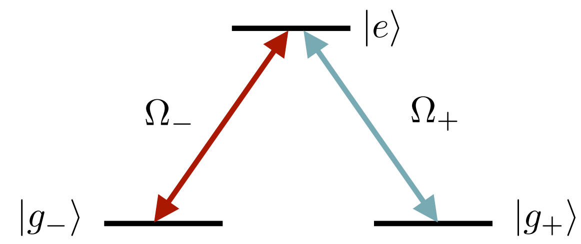
2. The Bloch sphere for two-level atoms and perturbation theory for multi-level atoms.

3. Three level systems/STIRAP and field quantization.

4. Atom decay, density matrix and master equations. **This time**

Last time

- We looked at a “Lambda” three level system:



- We have introduced the approximation of an adiabatic **elimination** of one of the states, reducing the problem to an effective two-level system.

- We introduced the concept of **adiabatic population transfer**, using e.g. a delayed laser-pulse sequence. This scheme is called **STIRAP** and is a very popular scheme in atomic physics and beyond. It works very well, since the system adiabatically remains in a **dark state**, which is immune to decay.

- We have shown that generally a system prepared in an **eigenstate of an Hamiltonian**, **remains in the instantaneous eigenstate** if the **rate of change** in the Hamiltonian is **smaller than the energy-gap** to other eigenstates. This is called the **adiabatic approximation.**

$$E_n(t) - E_k(t) \gg \langle \phi_k(t)| \left[\frac{d}{dt} \hat{H}(t) \right] |\phi_n(t)\rangle$$

- We have started to look at field quantization. We have seen that the solutions of the classical Maxwell equations can be seen viewed as classical harmonic oscillators. Treating these oscillators quantum-mechanically is called **canonical quantization**. We arrived at an operator description of the EM field

$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}} \left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} - \mathrm{h.c.} \right)$$

$$[\hat{a}_{\boldsymbol{k}\lambda}, \hat{a}^\dagger_{\boldsymbol{k}'\lambda'}] = \delta_{k_x,k'_x}\delta_{k_y,k'_y}\delta_{k_z,k'_z}\delta_{\lambda,\lambda'}$$

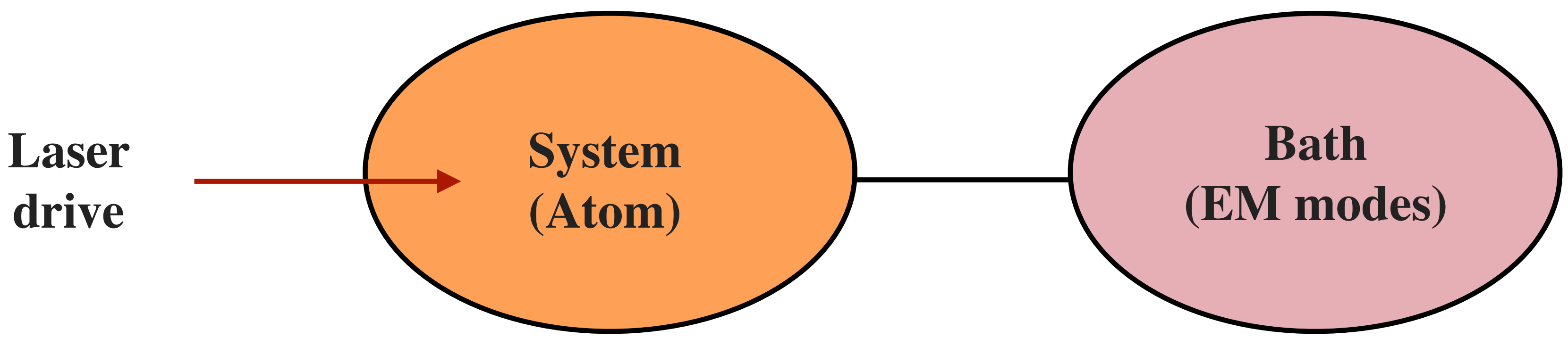
- With this full quantum-mechanical description of both the atom and the EM environment, we can now treat atomic decay properly. We have as an example looked at the decay according to Fermi’s Golden rule

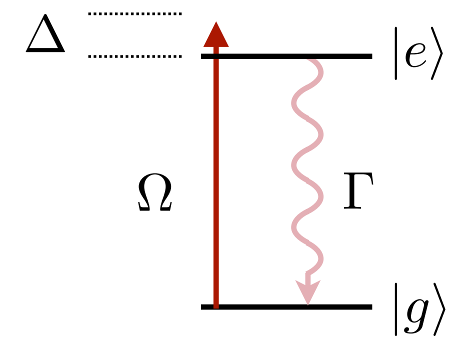
$$\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3 |\boldsymbol{\mu}_{ge}|^2}{3\pi c^3 \hbar \epsilon_0}$$

This time - Outline

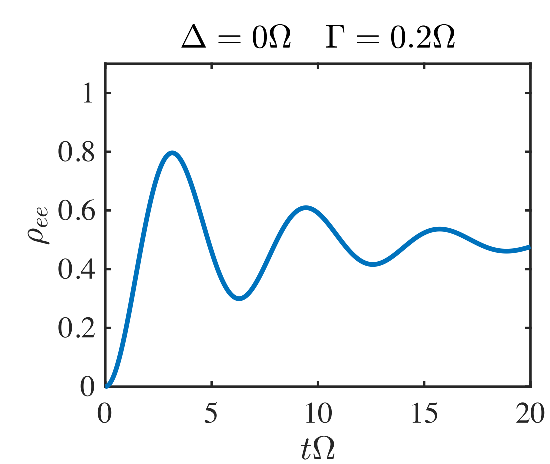
- We will start with a few remarks on field quantization: Schrödinger vs. Heisenberg.

- As a main part of this class we will derive the equations of motion for an atom that is coupled to the electromagnetic vacuum. In particular, we consider a two-level atom that can decay, while being excited by a laser. This is an important example for a **master equation**, called **“Optical Bloch equations”**.





- Lastly, we will analyze how the general dynamics under the Optical Bloch equations looks like.



4.1 - Schrödinger vs. Heisenberg picture

- We start with a small extra remark on the quantization of the EM field … we had:

$$\boxed{\hat{H}_{0F} = \sum_{\boldsymbol{k},\lambda} \hbar\omega_k \left(\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2} \right)} \qquad \boxed{\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}} \left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \mathrm{h.c.} \right)}$$

- The operator representing the electric field is time-dependent, and it is actually Heisenberg-picture operator:
- Let's show this: $\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} \equiv \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} \equiv \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} + \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda}$

with the definition $\quad \boldsymbol{C}_{\boldsymbol{k},\lambda} = \sqrt{\dfrac{\hbar\omega_k}{2(V\epsilon_0)}}\boldsymbol{\epsilon}_\lambda$

Then: $\quad [\hat{H}_{0F}, \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}] = \hbar\omega_k \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}[\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}] = -\hbar\omega_k \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\hat{a}_{\boldsymbol{k},\lambda}$

$$[\hat{a}_{\boldsymbol{k}\lambda}, \hat{a}^\dagger_{\boldsymbol{k}'\lambda'}] = \delta_{k_x,k'_x}\delta_{k_y,k'_y}\delta_{k_z,k'_z}\delta_{\lambda,\lambda'}$$

$$[\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}] = \hat{a}^\dagger_{\boldsymbol{k},\lambda}\cancel{[\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}]} + [\hat{a}^\dagger_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}]\hat{a}_{\boldsymbol{k},\lambda} = -\hat{a}_{\boldsymbol{k},\lambda}$$

On the other hand: $\quad \boxed{\dfrac{d}{dt}\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}} = (-\mathrm{i}\omega_k)\mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} = \omega_k\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} = \boxed{\dfrac{\mathrm{i}}{\hbar}\left[\hat{H}_{0F}, \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}\right]}$

The operator follows the ***Heisenberg equation*** *of motion for the free field Hamiltonian*

… and trivially also: $\quad \dfrac{d}{dt}\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = \left(\dfrac{d}{dt}\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}\right)^\dagger = \left(\dfrac{\mathrm{i}}{\hbar}\left[\hat{H}_{0F}, \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}\right]\right)^\dagger = -\dfrac{\mathrm{i}}{\hbar}\left[\left(\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}\right)^\dagger, \hat{H}_{0F}\right] = \dfrac{\mathrm{i}}{\hbar}\left[\hat{H}_{0F}, \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda}\right]$

4.1 - Schrödinger vs. Heisenberg picture

Heisenberg picture

$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega}{2(V\epsilon_0)}}\left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \mathrm{h.c.}\right)$$

- In the **Schrödinger picture** the field operator becomes time-independent:

Schrödinger picture

$$\hat{H}_{0F} = \sum_{\boldsymbol{k},\lambda} \hbar\omega_k \left(\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2}\right)$$

$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega}{2(V\epsilon_0)}}\left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - \mathrm{h.c.}\right)$$

- And formally the transformation can be done with the basis change

Schrödinger picture: $\mathrm{i}\dfrac{d}{dt}|\psi\rangle = \hat{H}_{0F}|\psi\rangle$ $\qquad \hat{U} = \mathrm{e}^{\mathrm{i}\hat{H}_{0F}t}$ $\qquad |\tilde{\psi}\rangle = \hat{U}|\psi\rangle$ $\qquad \hbar \equiv 1$

Heisenberg picture

$$\mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle = \hat{U}\hat{H}_{0F}\hat{U}^\dagger|\tilde{\psi}\rangle - \hat{H}_{0F}|\tilde{\psi}\rangle = 0 \quad \longrightarrow \quad |\tilde{\psi}(t+\Delta t)\rangle = |\tilde{\psi}(t)\rangle$$

The state does not evolve.

- The transformation gives the time-dependence to the ladder operators:

$$\hat{U}\hat{a}_{\boldsymbol{k},\lambda}\hat{U}^\dagger = \mathrm{e}^{(\mathrm{i}\omega_k t)\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}}\,\hat{a}_{\boldsymbol{k},\lambda}\,\mathrm{e}^{(-\mathrm{i}\omega_k t)\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}} = \cdots = \hat{a}_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\omega_k t}$$

Proof with one of the Baker-Campbell-Hausdorff identities (exercise)

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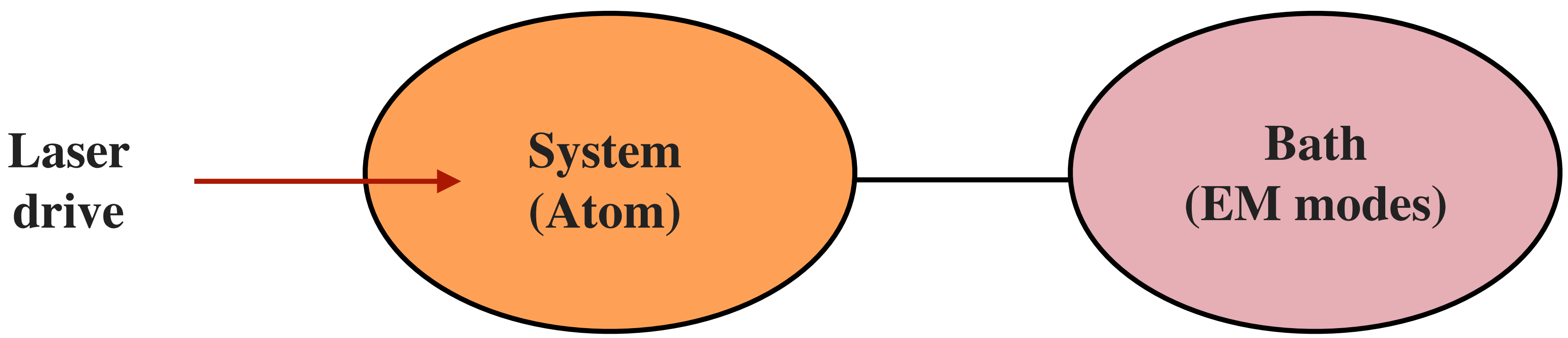
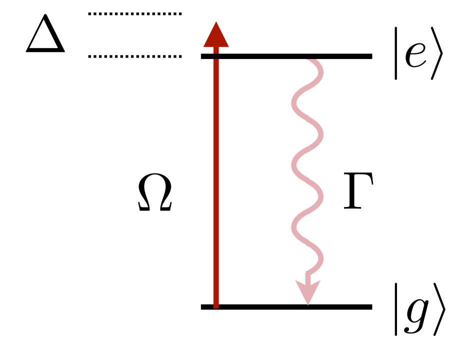
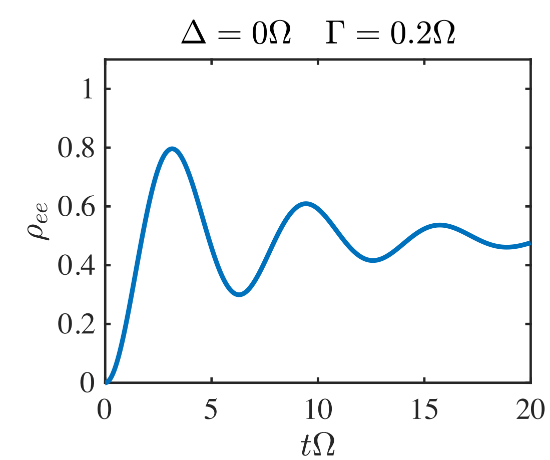
- We will start with a few remarks on field quantization: Schrödinger vs. Heisenberg.

- As a main part of this class we will derive the equations of motion for an atom that is coupled to the electromagnetic vacuum. In particular, we consider a two-level atom that can decay, while being excited by a laser. This is an important example for a **master equation**, called **“Optical Bloch equations”**.





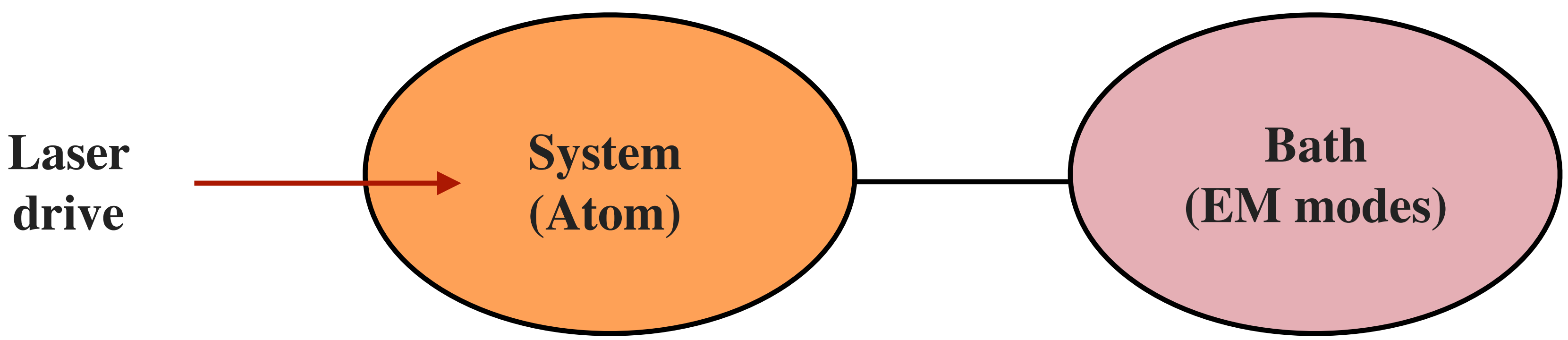
- Lastly, we will analyze how the general dynamics under the Optical Bloch equations looks like.

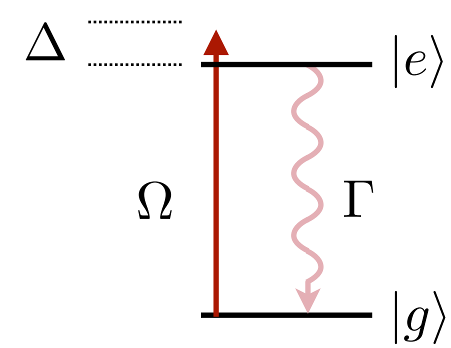


4.3 - Derivation of a master equation

Our **goal** is to study the **dynamics of a single atom**, **driven by a laser** in a vacuum. Our description should include the possibility for the **atom to decay**.

- Problem: $$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$





- **System:** $$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$ $$\hat{\sigma}^+ = |e\rangle\langle g| \qquad \hat{\sigma}^- = |g\rangle\langle e|$$

- **Bath:** $$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

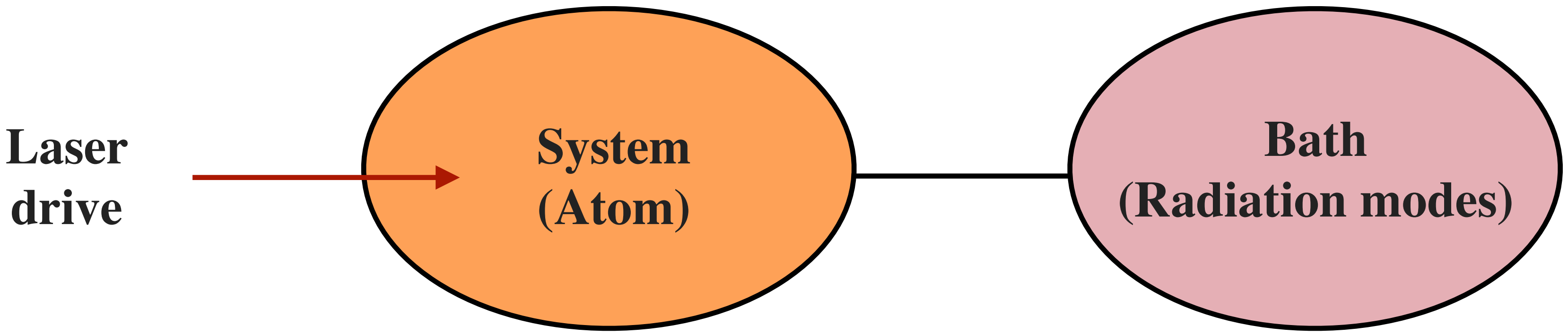
ω *Laser frequency*

ω_k *Mode frequencies*

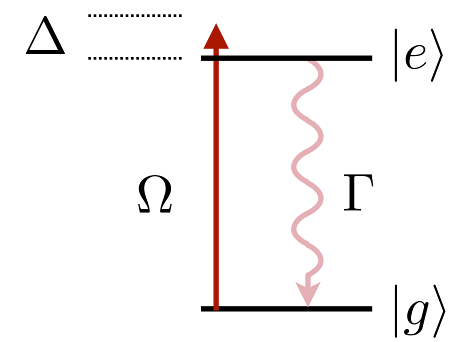
- **S-B coupling:** $$\hat{H}_{\mathrm{int}} = -\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \mathrm{h.c.}\right)$$

… with $$\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(2\pi^3)\epsilon_0}}\,\boldsymbol{\epsilon}^*_\lambda \hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

4.3 - Derivation of a master equation



$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$



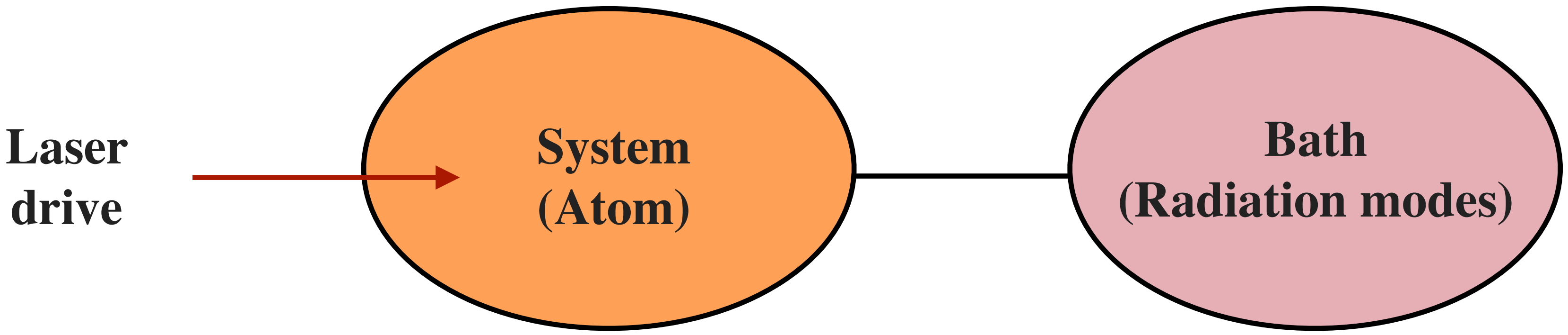
- **System:** $$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$
- **Bath:** $$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$
- **S-B coupling:** $$\hat{H}_{\mathrm{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \mathrm{h.c.}\right) \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(2\pi^3)\epsilon_0}}\boldsymbol{\epsilon}^*_\lambda \hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Remark 1:** We subtracted a constant energy, which is the energy of the vacuum field plus the ground-state energy.
- **Remark 2:** We have already **applied the rotating wave approximation**, both for the system and the S-E coupling. But we have still written the problem in the lab frame.
- **Remark 3:** We have positioned the atom at: $\boldsymbol{r} = (0, 0, 0)$

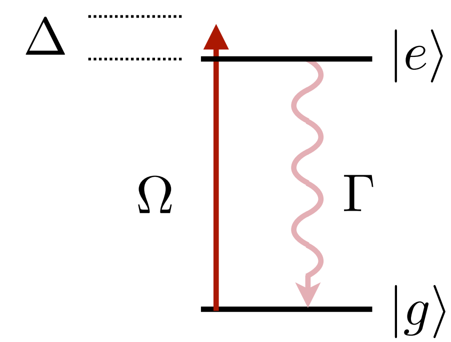
Let's define again: $$\boldsymbol{C}_{\boldsymbol{k},\lambda} = \sqrt{\frac{\hbar\omega_k}{2(2\pi)^3\epsilon_0}}\boldsymbol{\epsilon}_\lambda$$ … to write the field more compact: $$\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Remark 4:** Note that we already replaced the sum over the k variable by an integration, and thus got rid of the quantization volume.

4.3 - Derivation of a master equation



$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$



- **System:** $\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+ \hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$
- **Bath:** $\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$
- **S-B coupling:** $\hat{H}_{\text{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \text{h.c.}\right)$ $\qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$

- **Our problem:** We have an initial state, which we assume to be $|\Phi(0)\rangle = |\psi(t)\rangle \otimes |\text{vac}\rangle$

  … then we want to solve the Schrödinger equation $\mathrm{i}\frac{d}{dt}|\Phi\rangle = \left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\text{int}}\right]|\Phi\rangle$

  BUT: We're only interested in equations for the dynamics of the **system**, not the environment/bath.

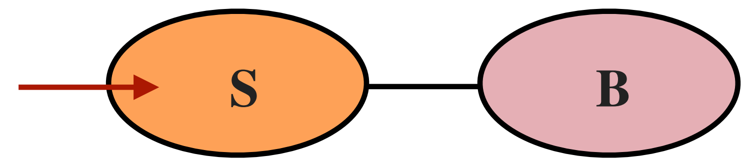
  Therefore, our goal is to derive equations for the **density matrix:** $\hat{\rho}_S(t) = \mathrm{tr}_B\left(|\Phi\rangle\langle\Phi|\right) \qquad \frac{d}{dt}\hat{\rho}_S(t) = \ldots?$

  (… the trace is taken over the whole bath)

- **Note:** It is impossible to only use a pure-state description for the atom, as we will see and analyze later.

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+ \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 1:** Let's look at the evolution of the field … in the **Heisenberg picture:**

$\hbar \equiv 1$

- The mode operators will evolve like

$$\frac{d}{dt}\hat{a}_{\boldsymbol{k},\lambda} = \mathrm{i}\left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\mathrm{int}}, \hat{a}_{\boldsymbol{k},\lambda}\right] = \mathrm{i}\left[\hat{H}_{0A}, \hat{a}_{\boldsymbol{k},\lambda}\right] + \mathrm{i}\left[\hat{H}_{0F}, \hat{a}_{\boldsymbol{k},\lambda}\right] + \mathrm{i}\left[\hat{H}_{\mathrm{int}}, \hat{a}_{\boldsymbol{k},\lambda}\right]$$

$$= 0 + \mathrm{i}\omega_k[\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda},\, \hat{a}_{\boldsymbol{k},\lambda}] + \mathrm{i}(-1)(-\mathrm{i})(\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda})[\hat{a}^\dagger_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}]\hat{\sigma}^-(t)$$

$$= -\mathrm{i}\omega_k\hat{a}_{\boldsymbol{k},\lambda} + \left(\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}\right)\hat{\sigma}^-(t)$$

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

$$[\hat{a}^\dagger_{\boldsymbol{k}\lambda}, \hat{a}_{\boldsymbol{k}\lambda}] = -1$$

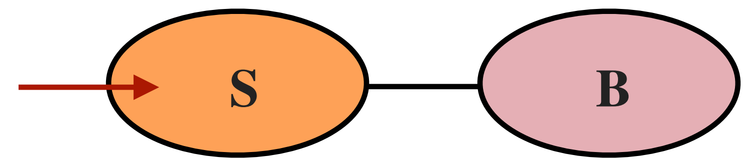
- **Note:** The atom decay operator acts like a source term for the mode, which makes sense.

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$



- **Step 1:** Let's look at the evolution of the field … in the **Heisenberg picture:**

$$\frac{d}{dt}\hat{a}_{\boldsymbol{k},\lambda} = -\mathrm{i}\omega_k\hat{a}_{\boldsymbol{k},\lambda} + \left(\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\right)\hat{\sigma}^-(t)$$

- A formal integration of the equation gives the time-evolution:

$$\hat{a}_{\boldsymbol{k},\lambda}(t) = \mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\int_0^t ds\,\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

- Proof:

$$\frac{d}{dt}\hat{a}_{\boldsymbol{k},\lambda}(t) = (-\mathrm{i}\omega_k)\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\left\{(-\mathrm{i}\omega_k)\mathrm{e}^{-\mathrm{i}\omega_k t}\int_0^t ds\,\mathrm{e}^{\mathrm{i}\omega_k s}\hat{\sigma}^-(s) + \cancel{\mathrm{e}^{-\mathrm{i}\omega_k t}}\left[\cancel{\mathrm{e}^{\mathrm{i}\omega_k t}}\hat{\sigma}^-(t)\right]\right\}$$

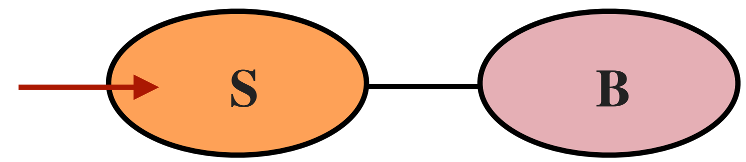
$$= (-\mathrm{i}\omega_k)\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\left\{(-\mathrm{i}\omega_k)\int_0^t ds\,\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s) + \hat{\sigma}^-(t)\right\}$$

$$= (-\mathrm{i}\omega_k)\left\{\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\int_0^t ds\,\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)\right\} + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\hat{\sigma}^-(t)$$

$$= (-\mathrm{i}\omega_k)\hat{a}_{\boldsymbol{k},\lambda}(t) + (\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda})\hat{\sigma}^-(t)$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+ \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 1:** Let's look at the evolution of the field … in the **Heisenberg picture:**

$$\hat{a}_{\boldsymbol{k},\lambda}(t) = \mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + (\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}) \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k (t-s)}\hat{\sigma}^-(s)$$

- We can now sum this over all field modes, to arrive at the evolution of the total field operator (at the atom position)

$$\hat{\boldsymbol{E}}^{(-)}(t) = \sum_\lambda \int d^3k\, \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}(t) = \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}(t)$$

$$= \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0) + \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}(\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}) \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k (t-s)}\hat{\sigma}^-(s)$$

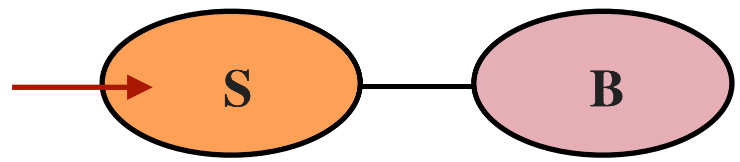
$$\equiv \hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) + \sum_\lambda \int d^3k\, \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}(\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}) \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k (t-s)}\hat{\sigma}^-(s)$$

… where we defined the total **"input field"** $\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) \equiv \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0)$

It is basically the Heisenberg operator for the full freely evolving field.

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+ \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 1:** Let's look at the evolution of the field … in the **Heisenberg picture:**

$$\hat{\boldsymbol{E}}^{(-)}(t) = \hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) + \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}(\boldsymbol{\mu}_{ge}\cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}) \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

$$\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) \equiv \mathrm{i}\sum_\lambda \int d^3k\, \boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0)$$

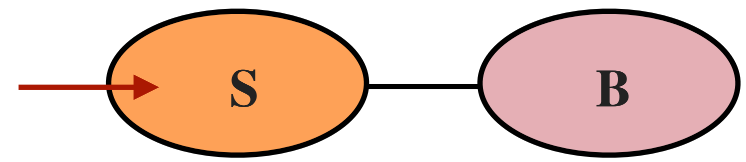
- In the interaction Hamiltonian part we have the term $\hat{H}_{\mathrm{int}} = -\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\hat{\sigma}^+ + \mathrm{h.c.}$

… so we can also look at the evolution of this term

$$-\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}(t) = -\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda \int d^3k\, |\boldsymbol{\mu}_{ge}\cdot \boldsymbol{C}^*_{\boldsymbol{k},\lambda}|^2 \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

$$-\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}(t) = -\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda\int d^3k\,|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\boldsymbol{k},\lambda}|^2\int_0^t ds\,\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s) \qquad \boldsymbol{C}_{\boldsymbol{k},\lambda} = \sqrt{\frac{\omega_k}{2(2\pi)^3\epsilon_0}}\boldsymbol{\epsilon}_\lambda$$

- Again, we can turn the integration over k into an integration over frequencies for spherical coordinates

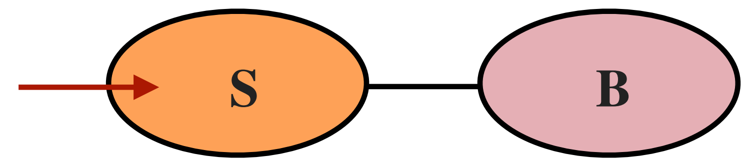
$$-\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}(t) = -\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda\int d\omega_k\int d\Omega\frac{\omega_k^2}{c^3}|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{C}^*_{\omega_k,\lambda}|^2\int_0^t ds\,\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

This looks like the Fermi's Golden Rule integral, but unfortunately we don't have a delta-function replacing the mode frequency with the atomic frequency.

- Everything up to now is exact (except for the RWA). We have written down a formal solution of an operator representing the coupling-strength of the overall EM field to the atomic transition.
- … one could keep going and further iteratively (formally) integrate the equations of motion. Instead, here we will make an important approximation:
- **Step 2:** **Markov approximation**

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

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- **Step 2:** **Markov approximation**

$$\boldsymbol{C}_{\boldsymbol{k},\lambda} = \sqrt{\frac{\omega_k}{2(2\pi)^3\epsilon_0}}\boldsymbol{\epsilon}_\lambda$$

$$-\boldsymbol{\mu}_{ge} \cdot \hat{\boldsymbol{E}}^{(-)}(t) = -\boldsymbol{\mu}_{ge} \cdot \hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\sum_\lambda \int d\omega_k \int d\Omega \frac{\omega_k^2}{c^3}|\boldsymbol{\mu}_{ge} \cdot \boldsymbol{C}^*_{\omega_k,\lambda}|^2 \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

- The coupling of the field to the atom currently depends on the **whole frequency range** of the bath.
  … but we can assume that only a frequency in a range around the atomic frequency matters. A typical range should be given by $\sim \omega_e \pm \Gamma$ … with the Fermi's Golden Rule decay rate Γ … typically: $\Gamma/\omega_e \sim 10^{-6}$

- Furthermore the coupling term $\sim \omega_k^3$

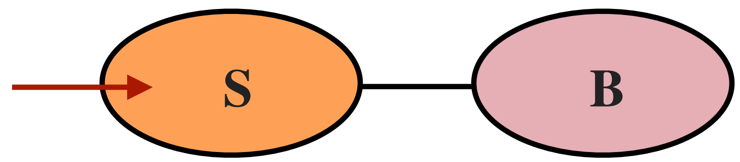
  … so if we consider a term that couples the atom to a photon frequency $\omega_e + \delta\omega$

  … we assume: $\omega_k^3 = (\omega_e + \delta\omega)^3 = \omega_e^3 \left(1 + \frac{\delta\omega}{\omega_e}\right)^3 \approx \omega_e^3$

- **Remark:** There are other interpretations of the **Markov approximation** that we will come to later.

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

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- **Step 2:** **Markov approximation**

- In the coupling term $\quad \omega_k \to \omega_e \qquad \boldsymbol{C}_{\omega_k,\lambda} = \sqrt{\frac{\omega_k}{2(2\pi)^3\epsilon_0}}\boldsymbol{\epsilon}_\lambda \to \sqrt{\frac{\omega_e}{2(2\pi)^3\epsilon_0}}\boldsymbol{\epsilon}_\lambda$

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) = -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda \int d\omega_k \int d\Omega \frac{\omega_k^2}{c^3}|\boldsymbol{\mu}_{ge}\cdot \boldsymbol{C}^*_{\omega_k,\lambda}|^2 \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda \int d\omega_k \int d\Omega \frac{\omega_e^3}{2(2\pi)^3 c^3\epsilon_0}|\boldsymbol{\mu}_{ge}\cdot \boldsymbol{\epsilon}^*_\lambda|^2 \int_0^t ds\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\hat{\sigma}^-(s)$$

- Now the bath integral is a delta-function: $\displaystyle \int d\omega_k\, \mathrm{e}^{-\mathrm{i}\omega_k(t-s)} = 2\pi\delta(t-s)$

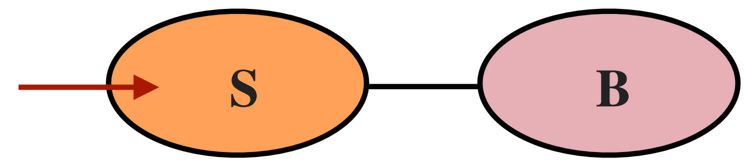
$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\mathrm{in}}(t) - \mathrm{i}\sum_\lambda \int d\Omega \frac{2\pi\omega_e^3}{2(2\pi)^3 c^3\epsilon_0}|\boldsymbol{\mu}_{ge}\cdot \boldsymbol{\epsilon}^*_\lambda|^2 \int_0^t ds\, \delta(t-s)\ \hat{\sigma}^-(s)$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge} \cdot \left(\hat{\sigma}^- \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+ \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$



- **Step 2:** **Markov approximation**

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\sum_\lambda \int d\Omega \frac{2\pi\omega_e^3}{2(2\pi)^3 c^3 \epsilon_0} |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}^*_\lambda|^2 \int_0^t ds\, \delta(t-s)\hat{\sigma}^-(s)$$

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\frac{1}{2}\sum_\lambda \int d\Omega \frac{\omega_e^3}{8\pi^2 c^3 \epsilon_0} |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}^*_\lambda|^2 \hat{\sigma}^-(t)$$

there is a very sneaky factor of 1/2, because we integrate the delta-function not over an infinite range

$$\delta(x) = \lim_{a\to 0} \frac{1}{|a|\sqrt{\pi}} \mathrm{e}^{-\left(\frac{x}{a}\right)^2}$$

- The last integral looks oddly familiar … remember Fermi's Golden rule result:

$$\Gamma = \frac{\omega_e^3}{8\pi^2 c^3 \epsilon_0} \sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2 = \frac{\omega_e^3 |\boldsymbol{\mu}_{ge}|^2}{3\pi c^3 \epsilon_0} \qquad \hbar = 1 \quad \omega_g = 0$$

- Therefore we arrive at (Markov property):

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-(t)$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

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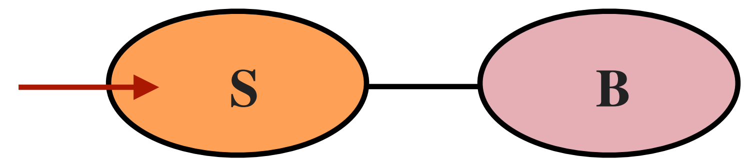

- **Step 2:** **Markov approximation**

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-(t)$$

Markov property

- Let's have a look at the effect of the Markov property:
- We specialize on the case where initially the state of the total system has the form: $|\Phi(0)\rangle = |\psi(0)\rangle \otimes |\text{vac}\rangle$

System *Bath*

Schrödinger picture *Heisenberg picture*

- Then:

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}(0)\right)|\Phi(t)\rangle = -\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}(t)\right)|\Phi(0)\rangle$$

$$\approx \left\{-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-(t)\right\}|\Phi(0)\rangle$$

$$= \left\{-\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-(t)\right\}|\Phi(0)\rangle \qquad \hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) \equiv \mathrm{i}\sum_{\lambda}\int d^3k\,\boldsymbol{C}_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\omega_k t}\hat{a}_{\boldsymbol{k},\lambda}(0)$$

$$= \left\{-\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-\right\}|\Phi(t)\rangle$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e\hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda}\hbar\omega_k\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

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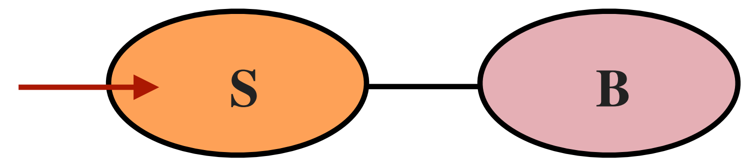


- **Step 2:** **Markov approximation**

- Let's have a look at the effect of the Markov property:

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)|\Phi(t)\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-|\Phi(t)\rangle$$

Markov property for vacuum bath state in the Schrödinger picture.

- Furthermore, for the term in the interaction Hamiltonian, this means:

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\hat{\sigma}^+|\Phi(t)\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-|\Phi(t)\rangle$$

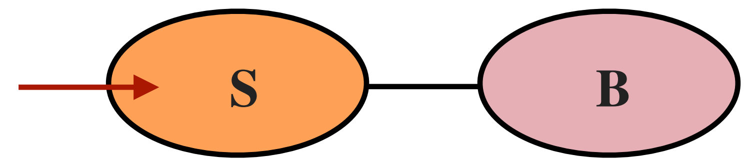
- **Interpretation:** At any time, the effect of applying the interaction Hamiltonian to a state of the system is equivalent to only modify a property in the **System!** This basically means, that all the dynamics coming from the system-bath interaction only changes a property of the system and the bath remains **unchanged.** This is the essence of the Markov approximation, i.e. that the bath has **no memory of the system.**

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$



- **Step 2:** **Markov approximation**

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\hat{\sigma}^+\left|\Phi(t)\right\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-\left|\Phi(t)\right\rangle$$

- **Interpretation:** At any time, the effect of applying the interaction Hamiltonian to a state of the system is equivalent to only modify a property in the **System!** This basically means, that all the dynamics coming from the system-bath interaction only changes a property of the system and the bath remains **unchanged.** This is the essence of the Markov approximation, i.e. that the bath has **no memory of the system.**

- **Interpretation "part 2":** In hindsight, the same conclusion can be obtained from the previous term

$$-\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}(t) \approx -\boldsymbol{\mu}_{ge}\hat{\boldsymbol{E}}^{(-)}_{\text{in}}(t) - \mathrm{i}\sum_{\lambda}\int d\Omega \frac{2\pi\omega_e^3}{2(2\pi)^3c^3\epsilon_0}\left|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}^*_\lambda\right|^2\int_0^t ds\, \delta(t-s)\, \hat{\sigma}^-(s)$$

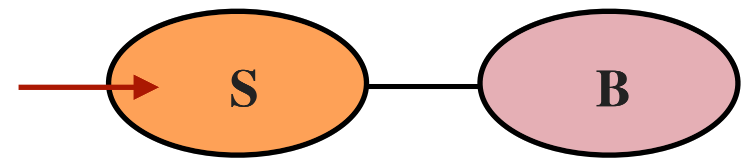
… by assuming that we have the delta-function here, we see that the state of the field at time t does only depend on the system state at time t. It does not depend on the past of the system … **no memory.**

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$



- **Step 3:** Derivation of the master equation

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\hat{\sigma}^+\left|\Phi(t)\right\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-\left|\Phi(t)\right\rangle$$

- Looking at the evolution of the full Schrödinger equation $\mathrm{i}\frac{d}{dt}\left|\Phi\right\rangle = \left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\text{int}}\right]\left|\Phi\right\rangle$

… we can now write it with a new effective Hamiltonian:

$$\mathrm{i}\frac{d}{dt}\left|\Phi\right\rangle = \left[\hat{H}_{\text{eff}} + \hat{H}_{0F} - \left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\hat{\sigma}^-\right)\right]\left|\Phi\right\rangle$$

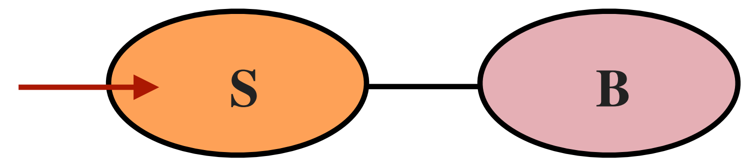
… with $\hat{H}_{\text{eff}} = \hat{H}_{0A} - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-$ *(this looks like the naive way to introduce dissipation we used in STIRAP)*

- **Note** that the Hamiltonian is *not* hermitian! $\hat{H}^\dagger_{\text{eff}} \neq \hat{H}_{\text{eff}}$

- This means that the evolution does not preserve the norm of the state … this is simply because dynamics cannot be described by Hamiltonian dynamics and state vectors, instead we need a density matrix description.

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 3:** Derivation of the master equation

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\hat{\sigma}^+\left|\Phi(t)\right\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-\left|\Phi(t)\right\rangle$$

- For the (reduced) density matrix we have $\hat{\rho}_S(t) = \mathrm{tr}_B\left(\left|\Phi(t)\right\rangle\left\langle\Phi(t)\right|\right)$ … von Neumann equation

$$\frac{d}{dt}\left(\left|\Phi\right\rangle\left\langle\Phi\right|\right) = \left(\frac{d}{dt}\left|\Phi\right\rangle\right)\left\langle\Phi\right| + \left|\Phi\right\rangle\left(\frac{d}{dt}\left\langle\Phi\right|\right)$$

$$\mathrm{i}\frac{d}{dt}\left|\Phi\right\rangle = \left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\text{int}}\right]\left|\Phi\right\rangle$$

$$= -\mathrm{i}\left\{\left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\text{int}}\right]\left|\Phi\right\rangle\left\langle\Phi\right| - \left|\Phi\right\rangle\left\langle\Phi\right|\left[\hat{H}_{0A} + \hat{H}_{0F} + \hat{H}_{\text{int}}\right]\right\}$$

$$= -\mathrm{i}\left\{\left[\hat{H}_{\text{eff}} + \hat{H}_{0F} - \left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\hat{\sigma}^-\right)\right]\left|\Phi\right\rangle\left\langle\Phi\right| - \left|\Phi\right\rangle\left\langle\Phi\right|\left[\hat{H}^\dagger_{\text{eff}} + \hat{H}_{0F} - \left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\hat{\sigma}^+\right)\right]\right\}$$

(we used the adjoint relation) $\quad -\left\langle\Phi(t)\right|\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\right)\hat{\sigma}^- = \left\langle\Phi(t)\right|\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-$

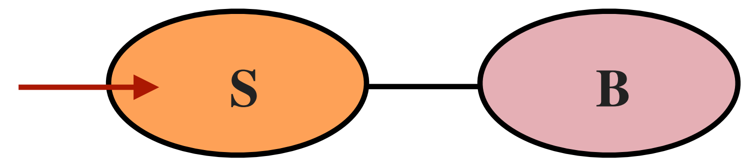
$$\hat{H}_{\text{eff}} = \hat{H}_{0A} - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-$$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e\hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$

$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda}\hbar\omega_k\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\mathrm{int}} = -\sum_\lambda\int d^3k\,\boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$



- **Step 3:** Derivation of the master equation

$$\frac{d}{dt}\left(|\Phi\rangle\langle\Phi|\right) = -\mathrm{i}\left\{\left[\hat{H}_{\mathrm{eff}} + \hat{H}_{0F} - \left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\hat{\sigma}^-\right)\right]|\Phi\rangle\langle\Phi| - |\Phi\rangle\langle\Phi|\left[\hat{H}^\dagger_{\mathrm{eff}} + \hat{H}_{0F} - \left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\hat{\sigma}^+\right)\right]\right\}$$

- For the (reduced) density matrix we have $\hat{\rho}_S(t) = \mathrm{tr}_B\left(|\Phi(t)\rangle\langle\Phi(t)|\right)$

$$\frac{d}{dt}\hat{\rho}_S(t) = \mathrm{tr}_B\left[\frac{d}{dt}|\Phi\rangle\langle\Phi|\right] = -\mathrm{i}\left(\hat{H}_{\mathrm{eff}}\overbrace{\mathrm{tr}_B[|\Phi\rangle\langle\Phi|]}^{\hat{\rho}_S} - \mathrm{tr}_B[|\Phi\rangle\langle\Phi|]\hat{H}^\dagger_{\mathrm{eff}}\right)$$

The system Hamiltonian is technically: $\underset{S}{\hat{H}_{\mathrm{eff}}}\otimes\underset{B}{\mathbb{1}}$

$$-\mathrm{i}\,\mathrm{tr}_B\left[\hat{H}_{0F}|\Phi\rangle\langle\Phi|\right]\cancel{+\mathrm{i}\,\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\hat{H}_{0F}\right]}$$

$\mathbb{1}\otimes\hat{H}_{0\mathrm{F}}$ $\rightarrow$ $\mathrm{tr}_B\ldots = \mathrm{tr}\ldots$ $\rightarrow$ $\mathrm{tr}(\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{A})$

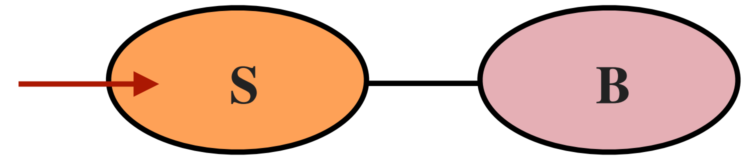
Cyclic invariance of trace

$$+\mathrm{i}\hat{\sigma}^-\mathrm{tr}_B\left[\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\right)|\Phi\rangle\langle\Phi|\right] - \mathrm{i}\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\right]\hat{\sigma}^+$$

Careful: Ordering is important here!

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_\lambda \int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 3:** Derivation of the master equation

$$-\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)|\Phi(t)\rangle = -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^-|\Phi(t)\rangle$$

$$\frac{d}{dt}\hat{\rho}_S = \mathrm{tr}_B\left[\frac{d}{dt}|\Phi\rangle\langle\Phi|\right] = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\rho}_S - \hat{\rho}_S\hat{H}^\dagger_{\text{eff}}\right)$$

$$+\mathrm{i}\hat{\sigma}^-\mathrm{tr}_B\left[\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\right)|\Phi\rangle\langle\Phi|\right] - \mathrm{i}\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}\right)\right]\hat{\sigma}^+$$

- Cyclic invariance of trace $\quad \mathrm{tr}(\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{A})$

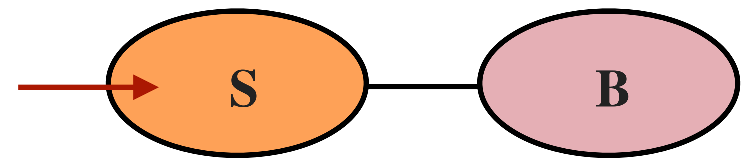
$$-\mathrm{i}\mathrm{tr}_B\left[\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(-)}|\Phi\rangle\langle\Phi|\right)\right]\hat{\sigma}^+ = \mathrm{i}(-\mathrm{i})\frac{\Gamma}{2}\hat{\sigma}^-\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\right]\hat{\sigma}^+ = \frac{\Gamma}{2}\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+$$

$$+\mathrm{i}\hat{\sigma}^-\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\right)\right] = \mathrm{i}(-\mathrm{i})\frac{\Gamma}{2}\hat{\sigma}^-\mathrm{tr}_B\left[|\Phi\rangle\langle\Phi|\right]\hat{\sigma}^+ = \frac{\Gamma}{2}\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+$$

(we used the adjoint relation) $\qquad -\langle\Phi(t)|\left(\boldsymbol{\mu}_{ge}\cdot\hat{\boldsymbol{E}}^{(+)}\right) = \langle\Phi(t)|\,\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+$

4.3 - Derivation of a master equation

$$\hat{H}_{0A} = \hbar\omega_e \hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\hbar\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\hbar\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^-$$



$$\hat{H}_{0F} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_k \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}$$

$$\hat{H}_{\text{int}} = -\sum_{\lambda}\int d^3k\, \boldsymbol{\mu}_{ge}\cdot\left(\hat{\sigma}^-\hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} + \hat{\sigma}^+\hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda}s\right) \qquad \hat{\boldsymbol{E}}^{(-)}_{\boldsymbol{k},\lambda} = \mathrm{i}\boldsymbol{C}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \qquad \hat{\boldsymbol{E}}^{(+)}_{\boldsymbol{k},\lambda} = -\mathrm{i}\boldsymbol{C}^*_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}$$

- **Step 3:** Derivation of the master equation
- **In total:** $$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\rho}_S - \hat{\rho}_S\hat{H}^\dagger_{\text{eff}}\right) + \frac{\Gamma}{2}\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+ + \frac{\Gamma}{2}\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+$$

$$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\rho}_S - \hat{\rho}_S\hat{H}^\dagger_{\text{eff}}\right) + \Gamma\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+ \qquad \hat{H}_{\text{eff}} = \hat{H}_{0A} - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-$$

- We arrived at the **master equation for the system-dynamics** (by using of the Markov property)
- The dynamics has **two contributions**

1. The **first term** looks like an **evolution with a non-hermitian Hamiltonian** $-\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\rho}_S - \hat{\rho}_S\hat{H}^\dagger_{\text{eff}}\right)$
2. The **second term** fixes the proper normalization of the state, it is therefore also called **recycling term** $\Gamma\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+$

4.3 - Summary: Derivation of a master equation

- **Master equation:** $$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left(\hat{H}_{\mathrm{eff}}\hat{\rho}_S - \hat{\rho}_S\hat{H}_{\mathrm{eff}}^\dagger\right) + \Gamma\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+$$ $$\hat{H}_{\mathrm{eff}} = \hat{H}_{0A} - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+\sigma^-$$

- **Alternative form:** $$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left[\hat{H}_{0A},\hat{\rho}_S\right] + \frac{\Gamma}{2}\left(-\hat{\sigma}^+\hat{\sigma}^-\hat{\rho}_S - \hat{\rho}_S\hat{\sigma}^+\hat{\sigma}^- + 2\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+\right)$$

- This last form is called **Lindblad master equation**, and it is a very general form of a master equation. Here the **Lindblad (jump) operator** for excited state decay is:

$$\hat{L} = \sqrt{\frac{\Gamma}{2}}\hat{\sigma}^- \longrightarrow \text{“Dissipator”} \quad \mathcal{L}(\hat{\rho}_S) = -\hat{L}^\dagger\hat{L}\hat{\rho}_S - \hat{\rho}_S\hat{L}^\dagger\hat{L} + 2\hat{L}\hat{\rho}_S\hat{L}^\dagger$$

- Evolution under a Lindblad master equation is generally a proper evolution for a density matrix since it preserves the trace of the density matrix, which is simply seen by the fact that

$$\mathrm{tr}\left[\frac{d}{dt}\hat{\rho}_S\right] = \mathrm{tr}\left\{-\mathrm{i}\left[\hat{H},\hat{\rho}_S\right] + \mathcal{L}(\hat{\rho}_S)\right\} = 0$$

Cyclic invariance of trace!

- **Remark:** Here we have only derived one master equation in a particular setup (atom decay with a vacuum bath).

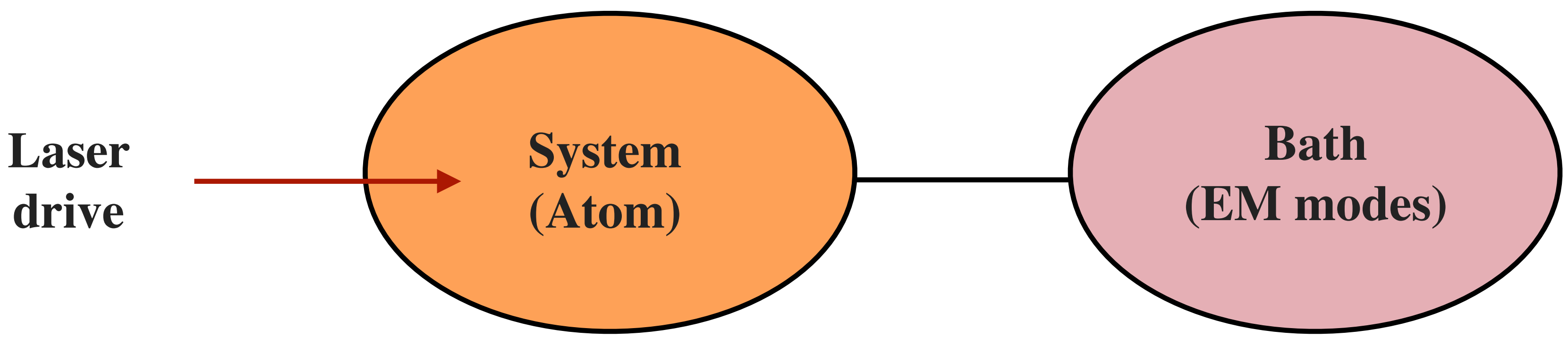
- Generally master equations can be derived for many different types of “oscillator baths”, or also for Hamiltonians with randomly fluctuating parameters. In almost all cases master equations can be written in Lindblad form, only with different (possibly multiple) Lindblad operators.

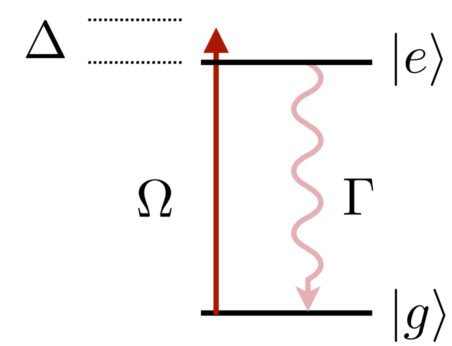
- **Example:** For atoms, generally another important dissipative channel is **“de-phasing”**. Fluctuating magnetic fields surrounding an atom, give rise to a Lindblad term with: $$\hat{L}_{\mathrm{deph}} = \sqrt{\frac{\gamma}{2}}\hat{\sigma}^z$$

This time - Outline

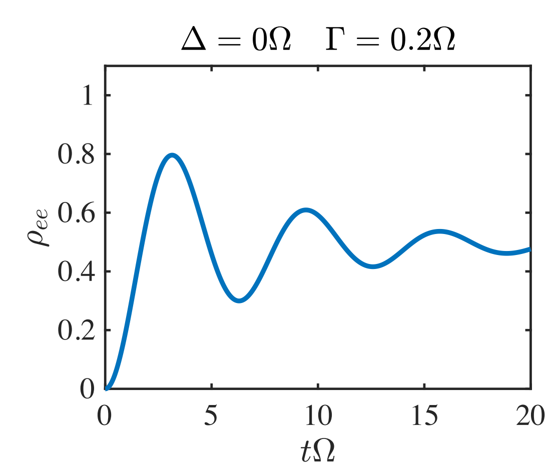
- We will start with a few remarks on field quantization: Schrödinger vs. Heisenberg.

- As a main part of this class we will derive the equations of motion for an atom that is coupled to the electromagnetic vacuum. In particular, we consider a two-level atom that can decay, while being excited by a laser. This is an important example for a **master equation**, called **"Optical Bloch equations"**.

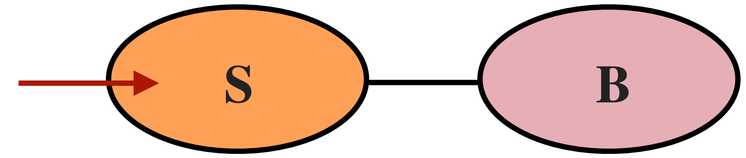




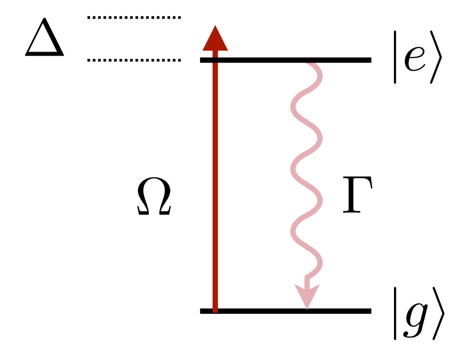
- Lastly, we will analyze how the general dynamics under the Optical Bloch equations looks like.



4.4 - Optical Bloch Equations



$$\hat{H}_{\text{eff}} = \omega_e \hat{\sigma}^+ \hat{\sigma}^- - \frac{1}{2}\Omega(t)\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}^+ - \frac{1}{2}\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega t}\hat{\sigma}^- - \mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^+ \sigma^-$$



$$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\rho}_S - \hat{\rho}_S \hat{H}_{\text{eff}}^\dagger\right) + \Gamma \hat{\sigma}^- \hat{\rho}_S \hat{\sigma}^+$$

- Let's write down the equations explicitly for the density matrix elements $\hat{\rho}_S = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$

- The excited state population: $\rho_{ee} = \mathrm{tr}(\hat{\sigma}^+ \hat{\sigma}^- \hat{\rho}_S) = \mathrm{tr}(|e\rangle\langle e|\,\hat{\rho}_S)$

- The ground state population: $\rho_{gg} = \mathrm{tr}(\hat{\sigma}^- \hat{\sigma}^+ \hat{\rho}_S) = \mathrm{tr}(|g\rangle\langle g|\,\hat{\rho}_S)$

- The "coherences": $\rho_{eg} = \mathrm{tr}(\hat{\sigma}^+ \hat{\rho}_S) = \rho_{ge}^*$

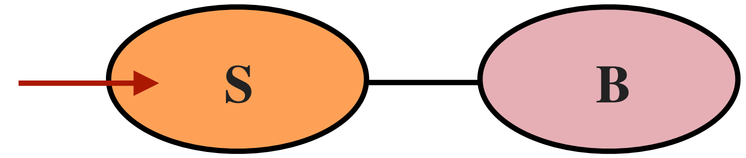
- As in the Rabi-Problem we go into a rotating frame (the RWA was already applied) $\hat{U} = \mathrm{e}^{\mathrm{i}\omega t}|e\rangle\langle e| + |g\rangle\langle g|$

  … which transforms away the optical frequencies

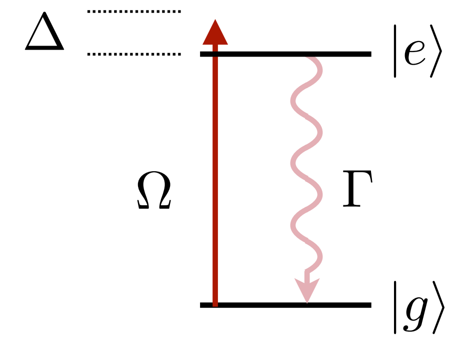
$$\hat{\tilde{H}}_{\text{eff}} = \hat{U}\hat{H}_{\text{eff}}\hat{U}^\dagger - \omega\hat{\sigma}^+\hat{\sigma}^- = \left(-\Delta - \mathrm{i}\frac{\Gamma}{2}\right)\hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\Omega(t)\hat{\sigma}^+ - \frac{1}{2}\Omega^*(t)\hat{\sigma}^-$$

$$\hat{\tilde{\rho}}_S = \hat{U}\hat{\rho}_S\hat{U}^\dagger = \begin{pmatrix} \rho_{ee} & \rho_{eg}\mathrm{e}^{-\mathrm{i}\omega t} \\ \rho_{ge}\mathrm{e}^{\mathrm{i}\omega t} & \rho_{gg} \end{pmatrix} \qquad \Gamma\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+ \rightarrow \Gamma\hat{\sigma}^-\hat{\tilde{\rho}}_S\hat{\sigma}^+$$

4.4 - Optical Bloch Equations



$$\hat{\tilde{H}}_{\text{eff}} = \left(-\Delta - \mathrm{i}\frac{\Gamma}{2}\right)\hat{\sigma}^+\hat{\sigma}^- - \frac{1}{2}\Omega(t)\hat{\sigma}^+ - \frac{1}{2}\Omega^*(t)\hat{\sigma}^-$$



$$\frac{d}{dt}\hat{\tilde{\rho}}_S = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\tilde{\rho}}_S - \hat{\tilde{\rho}}_S\hat{H}^\dagger_{\text{eff}}\right) + \Gamma\hat{\sigma}^-\hat{\tilde{\rho}}_S\hat{\sigma}^+$$

$$\hat{\tilde{\rho}}_S = \begin{pmatrix} \rho_{ee} & \tilde{\rho}_{eg} \\ \tilde{\rho}_{ge} & \rho_{gg} \end{pmatrix} \qquad \hat{\tilde{H}}_{\text{eff}} = \begin{pmatrix} -\Delta - \mathrm{i}\frac{\Gamma}{2} & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

- **Note:** Without decay, we simply recover the Rabi problem.
- Explicitly … *(exercise in matrix multiplications)* …

$$\frac{d}{dt}\rho_{ee} = \mathrm{i}\frac{1}{2}\Omega\tilde{\rho}_{ge} - \mathrm{i}\frac{1}{2}\Omega^*\tilde{\rho}_{eg} - \Gamma\rho_{ee}$$

$$\frac{d}{dt}\rho_{gg} = -\mathrm{i}\frac{1}{2}\Omega\tilde{\rho}_{ge} + \mathrm{i}\frac{1}{2}\Omega^*\tilde{\rho}_{eg} + \Gamma\rho_{ee}$$

$$\frac{d}{dt}\tilde{\rho}_{eg} = -\mathrm{i}\frac{1}{2}\Omega(\rho_{ee} - \rho_{gg}) + \left(\mathrm{i}\Delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{eg}$$

As expected:

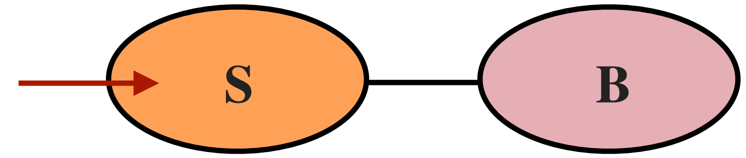
$$\frac{d}{dt}\rho_{ee} + \frac{d}{dt}\rho_{gg} = 0 \qquad \textit{trace preserved}$$

Recycling term:

$$\Gamma\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \rho_{ee} & \tilde{\rho}_{ge} \\ \tilde{\rho}_{eg} & \rho_{gg} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \Gamma\begin{pmatrix} 0 & 1 \\ 0 & \rho_{ee} \end{pmatrix}$$

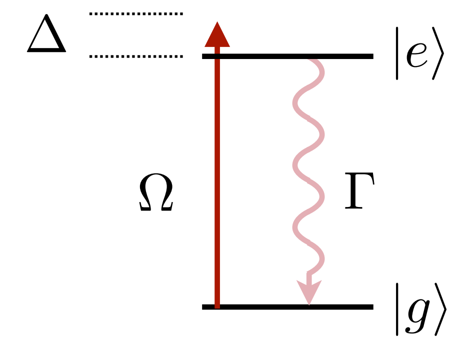
Population decays from the upper state. The recycling term puts this population into the ground-state and leads to norm preservation … without this term population is simply lost!

4.4 - Optical Bloch Equations



$$\frac{d}{dt}\rho_{ee} = \mathrm{i}\frac{1}{2}\Omega\tilde{\rho}_{ge} - \mathrm{i}\frac{1}{2}\Omega^*\tilde{\rho}_{eg} - \Gamma\rho_{ee}$$
$$\frac{d}{dt}\tilde{\rho}_{eg} = -\mathrm{i}\frac{1}{2}\Omega(2\rho_{ee} - 1) + \left(\mathrm{i}\Delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{eg}$$

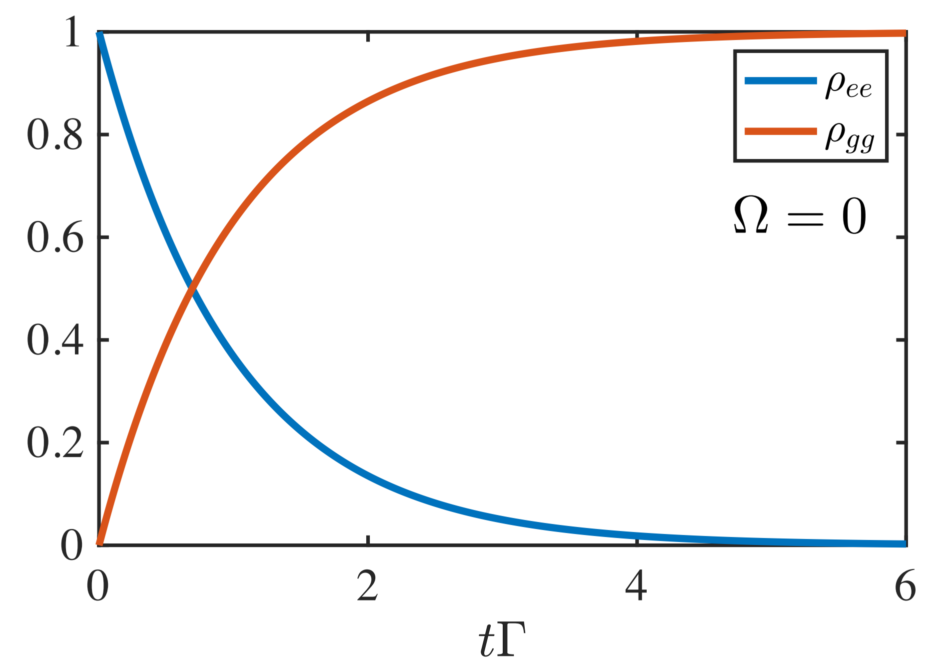
$$\tilde{\rho}_{eg} = \rho_{eg}\mathrm{e}^{-\mathrm{i}\omega t}$$
$$\rho_{gg} = 1 - \rho_{ee}$$



- **Without driving field:** No coupling between equations $\frac{d}{dt}\rho_{ee} = -\Gamma\rho_{ee}$

$$\rho_{ee}(t) = \rho_{ee}(0)\mathrm{e}^{-\Gamma t} \qquad \rho_{gg}(t) = 1 - \rho_{gg}(0)\mathrm{e}^{-\Gamma t}$$

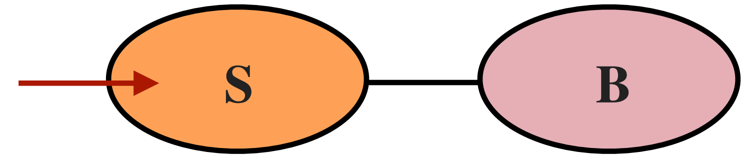
- We recover exponential decay (Fermi's Golden Rule result)



- But we now also have a decay of "coherence" $\frac{d}{dt}\tilde{\rho}_{eg} = \left(\mathrm{i}\Delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{eg}$ $\qquad \rho_{eg} = \rho_{eg}(0)\mathrm{e}^{-\mathrm{i}\omega_e t - \frac{\Gamma}{2}t}$

… the "coherence" dampens half as fast as the population.

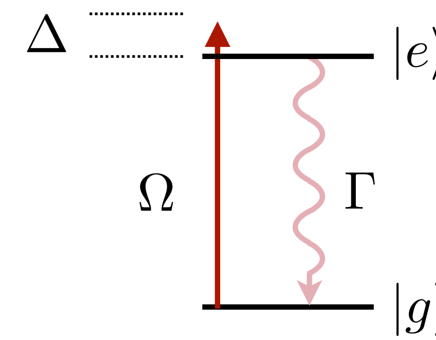
4.4 - Optical Bloch Equations



$$\boxed{\begin{aligned} \frac{d}{dt}\rho_{ee} &= \mathrm{i}\frac{1}{2}\Omega\tilde{\rho}_{ge} - \mathrm{i}\frac{1}{2}\Omega^{*}\tilde{\rho}_{eg} - \Gamma\rho_{ee} \\ \frac{d}{dt}\tilde{\rho}_{eg} &= -\mathrm{i}\frac{1}{2}\Omega(2\rho_{ee}-1) + \left(\mathrm{i}\Delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{eg} \end{aligned}}$$

$$\tilde{\rho}_{eg} = \rho_{eg}\mathrm{e}^{-\mathrm{i}\omega t}$$

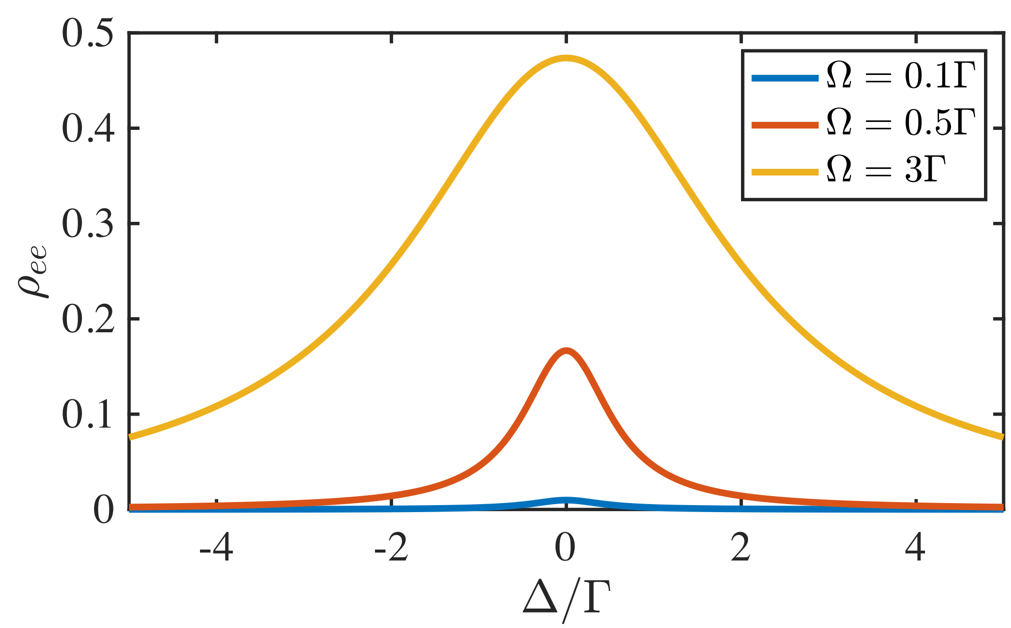
$$\rho_{gg} = 1 - \rho_{ee}$$



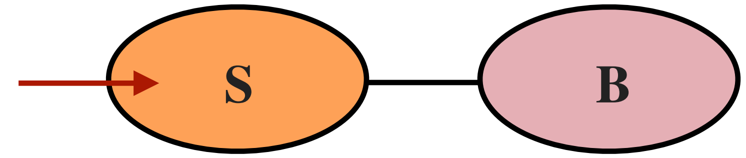
- **Steady-state:** In contrast to the Rabi-Problem, for long times the solution will now reach a stationary state as a consequence of a dynamical equilibrium between atom decay and laser drive. The steady-state is found by

$$\frac{d}{dt}\rho_{ee} = 0 \qquad \frac{d}{dt}\tilde{\rho}_{eg} = 0 \qquad \textit{(exercise)} \qquad \boxed{\rho_{ee} = \frac{|\Omega^2|}{4\Delta^2 + \Gamma^2 + 2|\Omega|^2}}$$

- This is a very similar result as for the power-broadening in the Rabi-Problem. The maximum population in the steady-state is 1/2 … but this can never really be reached, because without decay there never is a true stationary state, instead there are persisting Rabi oscillations.



4.4 - Optical Bloch Equations

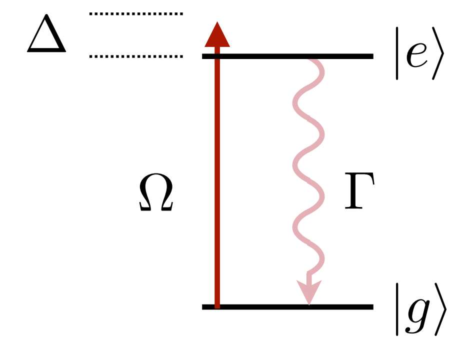


$$\frac{d}{dt}\rho_{ee} = \mathrm{i}\frac{1}{2}\Omega\tilde{\rho}_{ge} - \mathrm{i}\frac{1}{2}\Omega^*\tilde{\rho}_{eg} - \Gamma\rho_{ee}$$

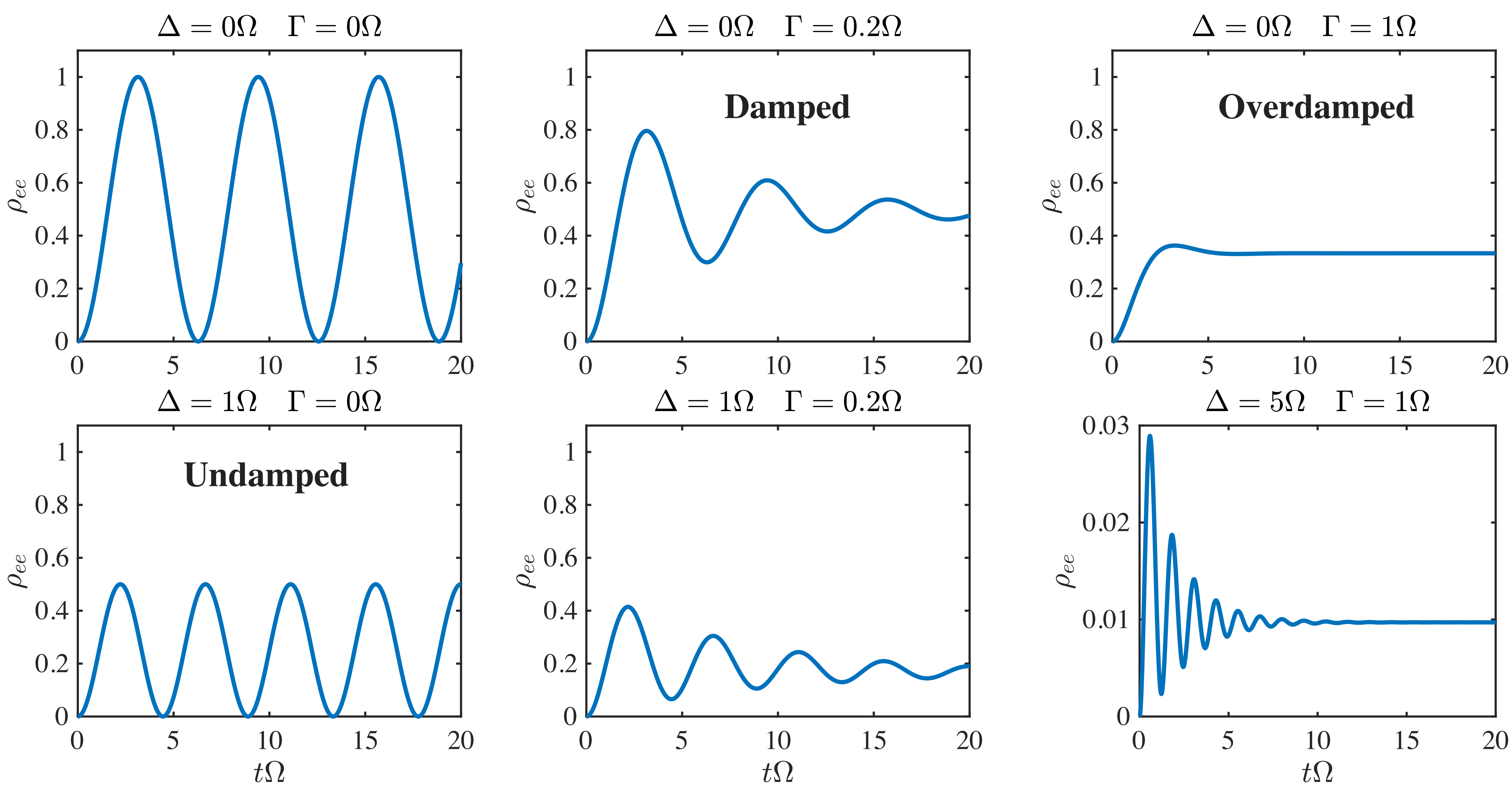
$$\frac{d}{dt}\tilde{\rho}_{eg} = -\mathrm{i}\frac{1}{2}\Omega(2\rho_{ee} - 1) + \left(\mathrm{i}\Delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{eg}$$

$$\tilde{\rho}_{eg} = \rho_{eg}\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\rho_{gg} = 1 - \rho_{ee}$$



- **Transient behavior:** Generally we will find damped oscillations, e.g. when starting in the ground-state:

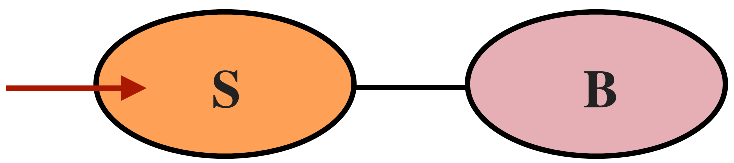


Recap

- According to Fermi's Golden Rule an atom in the excited state decays with a rate $$\Gamma = \frac{\omega_{eg}^3 |\boldsymbol{\mu}_{ge}|^2}{3\pi c^3 \hbar \epsilon_0}$$

- The more general question is: How can we (properly) describe the full dynamics of an atom coupled to an EM vacuum. We derived a general master equation for the **reduced system density matrix**, and such an equation is called **master equation**:

$$\frac{d}{dt}\hat{\rho}_S = -\mathrm{i}\left[\hat{H}_{0A}, \hat{\rho}_S\right] + \frac{\Gamma}{2}\left(-\hat{\sigma}^+\hat{\sigma}^-\hat{\rho}_S - \hat{\rho}_S\hat{\sigma}^+\hat{\sigma}^- + 2\hat{\sigma}^-\hat{\rho}_S\hat{\sigma}^+\right)$$



- A key approximation in deriving this equation is the **Markov approximation**. Essentially we used the fact that the bath has no memory of the system dynamics, and always remains in the same vacuum state. Explicitly, in a tedious calculation we showed that we can approximate:

$$-\left(\boldsymbol{\mu}_{ge} \cdot \hat{\boldsymbol{E}}^{(-)}\right)|\Phi(t)\rangle \approx -\mathrm{i}\frac{\Gamma}{2}\hat{\sigma}^- |\Phi(t)\rangle$$

… and used this to derive the above master equation.

- The master equation we derived has a **Lindblad** form, which preserves the trace (norm) of the quantum state. Also many other dissipative processes can be written down in such a master equation form.

- The master equation with atom decay and coherent laser drive is called **Optical Bloch equations**. In contrast to the Rabi problem it now features a stationary **steady-state**. In general, laser excitation of an atom now takes the form of a damped oscillation: