Atomic physics: Basics of quantum optics/light matter interactions

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The goal of this class (and the next ones) is to introduce basic concepts of **light - matter interactions** on the atomic level = **Quantum optics**. Basically, we will see how to use lasers to manipulate quantum states of atoms.

- Outline (may vary)

1. Atom-field interaction Hamiltonian (RWA)/Rabi Problem and AC Stark shift. **This time**

2. The Bloch sphere for two-level atoms and perturbation theory for multi-level atoms.

3. Decay of a two-level atom.

4. General dissipation, density matrix and master equations.

5. Atom in a cavity (the simplest possible QED system).

Background - Reminder

Problem: We consider a single electron atom (fixed in space) that interacts with an oscillating electric field in the optical domain. We are interested in the internal electronic dynamics induced by the field (e.g. by laser light)

- Atom-Field Interaction Hamiltonian (in the "dipole approximation"):

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

Electric field operator

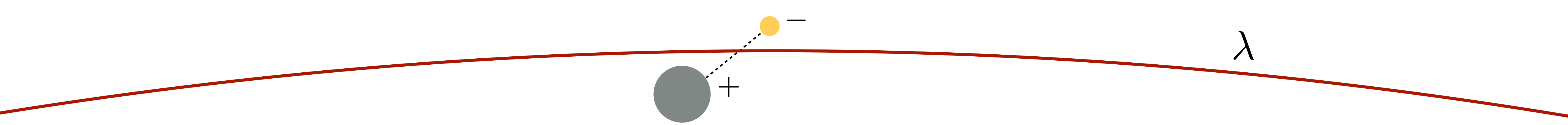
Free atom Hamiltonian

(bold: vectors)

Dipole operator

Free field Hamiltonian

$\hat{\boldsymbol{\mu}} = -e\,\hat{\boldsymbol{r}}$



- The dipole approximation assumes that the distance of the electron from the nucleus is much smaller than the wavelength of the photon it couples to, then the field is roughly constant over the size of the atomic wave function.

- In practice this is a very safe assumption $\quad \lambda \sim 500\,\mathrm{nm} \gg a_0 \sim 0.05\,\mathrm{nm}$

- **NB:** In making the dipole approximation we neglect magnetic dipole (and electric quadrupole) interactions by a factor of $(\alpha = 1/137)^2$

Atom-field interaction Hamiltonian

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

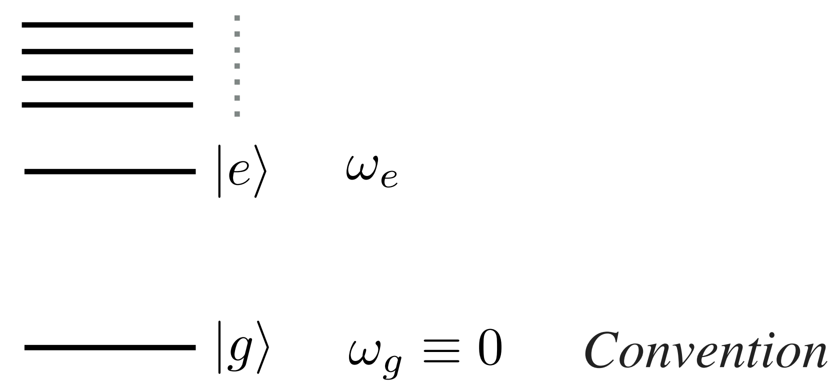
Problem: We consider a single electron atom (fixed in space) that interacts with an oscillating electric field in the optical domain. We are interested in the internal electronic dynamics induced by the field (e.g. by laser light)

- Since we are interested in the dynamics of the atom in a time-dependent laser field, we have to solve the Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{AF} |\psi(t)\rangle \quad \ldots \text{ with } \ldots \quad \hat{E} = \hat{E}(t)$$

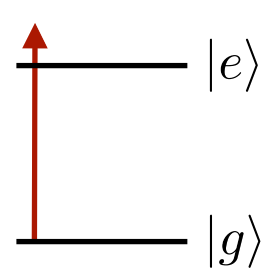
- We solved the free atom problem earlier and can expand the atom Hamiltonian into the atomic levels:

$$\hat{H}_{0A} = \sum_n \hbar\omega_n |n\rangle \langle n|$$



- Importantly, the energy of the photon in the optical domain is comparable to the level transitions

$$\omega = 2\pi \frac{c}{\lambda}$$



$$\omega_e \sim \omega$$

Atom-field interaction Hamiltonian

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

- What about the field? $\hat{H}_{0F}$ $\hat{\boldsymbol{E}}$

- Let's keep in mind that the EM field is also a quantum object and should thus be described by a Hamiltonian. Here, however, we do not consider any of those Quantum Electrodynamic (QED) effects and just *assume a classical field* for now. This means the Hamiltonian for the field is just a number (i.e. a meaningless constant) and

$$\hat{H}_{AF} \rightarrow \hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}_{\text{cl.}}$$

- **FYI, Phenomenological motivations for the classical picture:**

  - Quantum mechanically a laser field can be described by a photon state with a very large photon occupation number. Then a single interaction with an atom changes this number by ~ one, which means that the field is not significantly changed by the interaction.

  - More mathematically for the photon mode "a": $\langle \hat{a}^\dagger \hat{a} \rangle \gg [\hat{a}, \hat{a}^\dagger]$ … and the operator character of photons is not important.

  - Fully mathematically: A laser can be described by a so-called "coherent state". For such a state the Hamiltonian can be transformed (Mollow transformation) to the classical form, and thus for coherent states the classical form is in fact exact.

Atom in a laser pulse

$$\hat{H}_{AF} = \hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}_{\text{cl.}} \qquad i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{AF} |\psi(t)\rangle$$

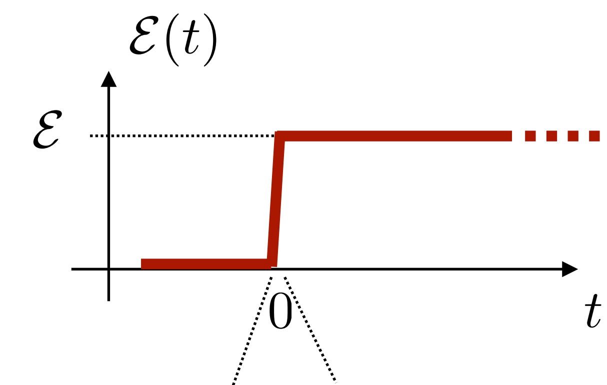
- We parametrize the oscillating electric field $\boldsymbol{E}_{\text{cl.}} = \mathcal{E}(t)\boldsymbol{\epsilon} e^{-i\omega t} + \text{c.c.} \equiv \boldsymbol{E}(t)^{-} + \boldsymbol{E}(t)^{+}$

Polarization vector

- with $\mathcal{E}(t)$ a "slowly" varying function on the optical time-scale: $\frac{d}{dt}|\mathcal{E}|/\mathcal{E} \ll \omega$
- This is usually very good, as $\omega \sim 10^{15}$ Hz
- A special case of a pulse shape one usually considers is a rectangular pulse

$$\mathcal{E}(t) = \mathcal{E} \ \ldots \text{ for } \quad 0 \leq t \leq T$$
$$\mathcal{E}(t) = 0 \ \ldots \text{ otherwise}$$

… where the step function must be still considered slow on the optical time-scale



"very fast, but still slow compared to optical frequency"

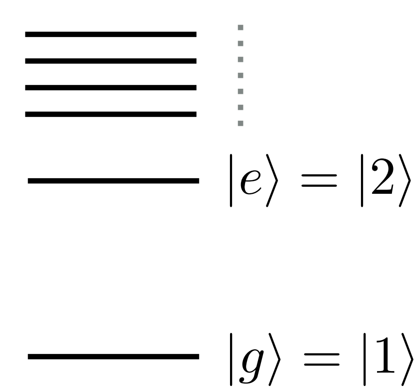
- **Schrödinger equation** to solve: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \left[\hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \left(\boldsymbol{E}(t)^{+} + \boldsymbol{E}(t)^{-}\right)\right]|\psi(t)\rangle$

… with $-\hat{\boldsymbol{\mu}} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] = -\sum_{n,k} \boldsymbol{\mu}_{nk} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] |n\rangle\langle k|$

the dipole operator expansion into the atomic states:

$$\boldsymbol{\mu}_{nk} = \langle n| \hat{\boldsymbol{\mu}} |k\rangle$$

Can be computed with integrations with spherical harmonics in real space … or numerically for complicated atoms…. or measured.



Atom in a laser pulse

- **Schrödinger equation** to solve: $$\boxed{\mathrm{i}\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \left[\hat{H}_{0A} - \hat{\boldsymbol{\mu}}\cdot\left(\boldsymbol{E}(t)^{+} + \boldsymbol{E}(t)^{-}\right)\right]\left|\psi(t)\right\rangle}$$

$$\boldsymbol{E}(t)^{-} = \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega t} \qquad \boldsymbol{E}(t)^{+} = \mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*}\mathrm{e}^{\mathrm{i}\omega t}$$

… with $$-\hat{\boldsymbol{\mu}}\cdot\left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] = -\sum_{n,k}\boldsymbol{\mu}_{nk}\cdot\left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right]\left|n\right\rangle\left\langle k\right| \qquad \boldsymbol{\mu}_{nk} = \left\langle n\right|\hat{\boldsymbol{\mu}}\left|k\right\rangle$$

- The term: $\boldsymbol{E}^{+}(t)\left|n\right\rangle\left\langle k\right|$

… can be thought of as a transition in the atom from k to n, and a corresponding creation of a photon with energy $\hbar\omega$

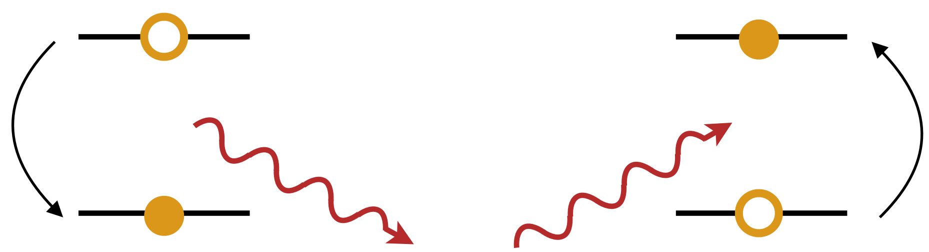
(This is more clear in a full QED picture, where the fields become "photon creation operators")

- The term: $\boldsymbol{E}^{-}(t)\left|n\right\rangle\left\langle k\right|$

… can be thought of as a transition in the atom from k to n, and a corresponding absorption of a photon with energy $\hbar\omega$

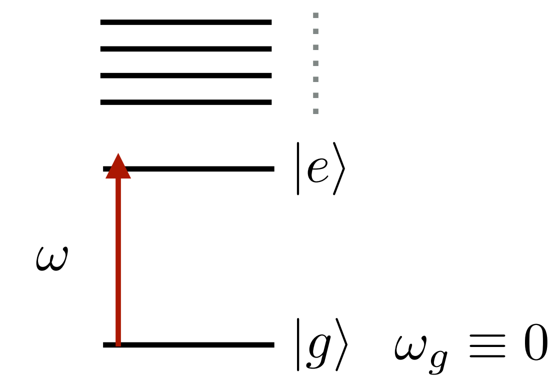
- Up to now, we thus are including terms that can be non-resonant, i.e. terms that include for example the excitation of the atom with a simultaneous emission of a photon

"resonant"



"non-resonant"

Strategy for solving the problem



- In the following we consider a situation where the atom is excited from the **ground-state** by a laser which is tuned close to resonant with the **first excited state**, i.e. the laser frequency is assumed to be close to the transition frequency.

$$\omega \approx \omega_e$$

- … we also assume the transition dipole moment $\boldsymbol{\mu}_{eg}$ is significant between the two states.

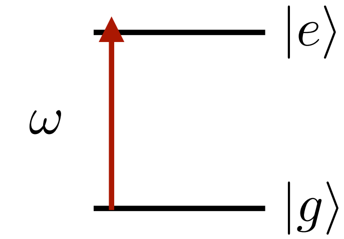
- **Strategy**

1. Two-level System (TLS) + Rotating Wave approximation (RWA): In a first step we solve the Schrödinger equation for the resonantly coupled TLS (consisting of "*g*" and "e"). We keep only the terms in the Hamiltonian which correspond to resonant couplings (RWA).

$$\hat{H} = \hbar\omega_e \left|e\right\rangle \left\langle e\right| - \boldsymbol{\mu}_{eg} \cdot \boldsymbol{E}^{-}(t) \left|e\right\rangle \left\langle g\right| - \boldsymbol{\mu}_{ge} \cdot \boldsymbol{E}^{+}(t) \left|g\right\rangle \left\langle e\right|$$

2. Non-resonant couplings: We treat the remaining "non-resonant" interactions in perturbation theory. This will both justify the RWA and the fact that other levels can be ignored. Furthermore, it gives rise to "AC Stark shifts" of atomic levels.

Rotating wave approximation for a TLS



- We start by including non-resonant terms

$$\hat{H} = \omega_e \left|e\right\rangle \left\langle e\right| - \boldsymbol{\mu}_{eg} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] \left|e\right\rangle \left\langle g\right| - \boldsymbol{\mu}_{ge} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] \left|g\right\rangle \left\langle e\right|$$

- **Ansatz:** $\left|\psi(t)\right\rangle = a_g(t)\left|g\right\rangle + a_e(t)\left|e\right\rangle$ $\quad \mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle$ $\quad$ *Note:* $\hbar \equiv 1$

$$\mathrm{i}\dot{a}_e = \omega_e a_e - \boldsymbol{\mu}_{eg} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge} \cdot \left[\boldsymbol{E}^{+}(t) + \boldsymbol{E}^{-}(t)\right] a_e$$

$$\boldsymbol{E}(t)^{-} = \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\boldsymbol{E}(t)^{+} = \mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*}\mathrm{e}^{\mathrm{i}\omega t}$$

$$\mathrm{i}\dot{a}_e = \omega_e a_e - \boldsymbol{\mu}_{eg} \cdot \left[\mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*}\mathrm{e}^{\mathrm{i}\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega t}\right] a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge} \cdot \left[\mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*}\mathrm{e}^{\mathrm{i}\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}\omega t}\right] a_e$$

- **Transformation to the rotating frame at the optical frequency** $\quad a_e(t) = \tilde{a}_e(t)\mathrm{e}^{-\mathrm{i}\omega t}$ $\quad \tilde{a}_e(t) = a_e(t)\mathrm{e}^{\mathrm{i}\omega t}$

“new frame”

$$\mathrm{i}\dot{\tilde{a}}_e = \mathrm{i}\dot{a}_e\mathrm{e}^{\mathrm{i}\omega t} - \omega\tilde{a}_e$$

$$\mathrm{i}\dot{\tilde{a}}_e\mathrm{e}^{-\mathrm{i}\omega t} + \omega\tilde{a}_e\mathrm{e}^{-\mathrm{i}\omega t} = \mathrm{i}\dot{a}_e$$

$$\mathrm{i}\dot{\tilde{a}}_e + \omega\tilde{a}_e = \omega_e\tilde{a}_e - \boldsymbol{\mu}_{eg} \cdot \left[\mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*}\mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right] a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge} \cdot \left[\mathcal{E}(t)^{*}\boldsymbol{\epsilon}^{*} + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}2\omega t}\right] \tilde{a}_e$$

Rotating wave approximation for a TLS

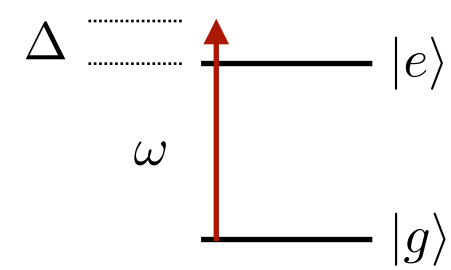
- **Transformation to the rotating frame at the optical frequency**

$$\mathrm{i}\dot{\tilde{a}}_e + \omega \tilde{a}_e = \omega_e \tilde{a}_e - \boldsymbol{\mu}_{eg} \cdot \left[\mathcal{E}(t)^* \boldsymbol{\epsilon}^* \mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right] a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge} \cdot \left[\mathcal{E}(t)^* \boldsymbol{\epsilon}^* + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}2\omega t}\right] \tilde{a}_e$$

- Define “de-tuning”

$$\boxed{\Delta = \omega - \omega_e}$$



$$\mathrm{i}\dot{\tilde{a}}_e = -\Delta \tilde{a}_e - \boldsymbol{\mu}_{eg} \cdot \left[\mathcal{E}(t)^* \boldsymbol{\epsilon}^* \mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right] a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge} \cdot \left[\mathcal{E}(t)^* \boldsymbol{\epsilon}^* + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}2\omega t}\right] \tilde{a}_e$$

- We see that the equations of motion have terms rotating at twice the optical frequency and terms that evolve on the “slow” time scale

- **Remember:** $\frac{d}{dt}|\mathcal{E}|/\mathcal{E} \ll \omega$ … and we consider always: $(\Delta, |\boldsymbol{\mu}_{eg} \cdot \mathcal{E}\boldsymbol{\epsilon}|) \ll \omega$

- **The goal now is to show that the fast oscillating terms can be neglected (= RWA).** To show this we use perturbation theory, and a similar perturbation theory calculation will later (more formally) also be useful to compute effects of other “far-detuned” levels.

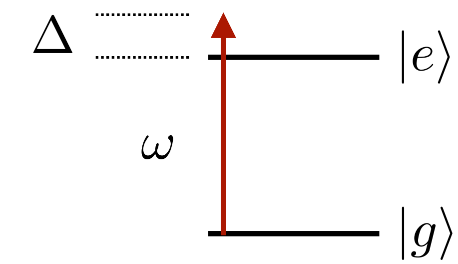
Rotating wave approximation for a TLS

$$\mathrm{i}\dot{\tilde{a}}_e = -\Delta\tilde{a}_e - \boldsymbol{\mu}_{eg}\cdot\left[\mathcal{E}(t)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right]a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge}\cdot\left[\mathcal{E}(t)^*\boldsymbol{\epsilon}^* + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}2\omega t}\right]\tilde{a}_e$$

- Let's integrate the equations of motion (first one)

$$\mathrm{i}\int_0^\tau dt\,\dot{\tilde{a}}_e = \int_0^\tau dt\,\left\{-\Delta\tilde{a}_e - \boldsymbol{\mu}_{eg}\cdot\left[\mathcal{E}(t)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right]a_g\right\}$$



- We do the integration on a particular time-step τ

$$\frac{1}{\omega} \quad \ll \quad \tau \quad \ll \quad \frac{1}{|\boldsymbol{\mu}_{eg}\cdot\mathcal{E}\boldsymbol{\epsilon}|} \quad \frac{1}{\Delta}$$

optical time-scales ("instantaneous") *"time-step for integration"* *system dynamics*

- For one time-step, the **system amplitudes remain constant** (because the time is too short to see any system-scale dynamics). On the other hand **many optical oscillations have taken place** in the time-step.

- Then: $$\mathrm{i}\tilde{a}_e(\tau) - \mathrm{i}\tilde{a}_e(0) \approx -\Delta\tilde{a}_e(0)\tau - \boldsymbol{\mu}_{eg}\cdot\left[\frac{\mathcal{E}(0)^*}{2\mathrm{i}\omega}\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}2\omega\tau} + \mathcal{E}(0)\boldsymbol{\epsilon}\tau\right]a_g(0)$$

  … so that the counter-rotating terms are proportional to a factor inverse proportional to the optical frequency.

- **NB:** that this somewhat reflects energy-time uncertainty: There is a rapid change of energy in the system, but this is ok, as long as $\Delta E \Delta t > h$

Rotating wave approximation for a TLS

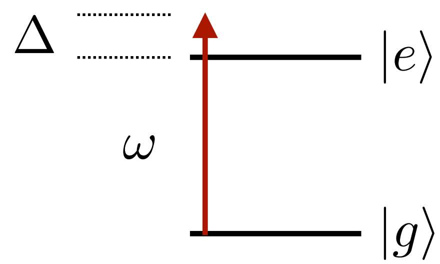
- Evolution in a single time-step is:

$$\mathrm{i}\dot{\tilde{a}}_e = -\Delta\tilde{a}_e - \boldsymbol{\mu}_{eg}\cdot\left[\mathcal{E}(t)^*\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}2\omega t} + \mathcal{E}(t)\boldsymbol{\epsilon}\right]a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge}\cdot\left[\mathcal{E}(t)^*\boldsymbol{\epsilon}^* + \mathcal{E}(t)\boldsymbol{\epsilon}\mathrm{e}^{-\mathrm{i}2\omega t}\right]\tilde{a}_e$$

$$\mathrm{i}\tilde{a}_e(\tau) - \mathrm{i}\tilde{a}_e(0) \approx -\Delta\tilde{a}_e(0)\tau - \boldsymbol{\mu}_{eg}\cdot\left[\frac{\mathcal{E}(0)^*}{2\mathrm{i}\omega}\boldsymbol{\epsilon}^*\mathrm{e}^{\mathrm{i}2\omega\tau} + \mathcal{E}(0)\boldsymbol{\epsilon}\tau\right]a_g(0)$$

$$\mathrm{i}\tilde{a}_g(\tau) - \mathrm{i}\tilde{a}_g(0) \approx -\boldsymbol{\mu}_{ge}\cdot\left[\mathcal{E}(0)^*\boldsymbol{\epsilon}^*\tau + \frac{\mathcal{E}(0)\boldsymbol{\epsilon}}{2\mathrm{i}\omega}\mathrm{e}^{-\mathrm{i}2\omega\tau}\right]\tilde{a}_e(0)$$



- On a coarse-grained integration time-scale we can drop terms: $\propto \cancel{\frac{|\boldsymbol{\mu}_{eg}\cdot\mathcal{E}\boldsymbol{\epsilon}|}{\omega}}$
  (later we will be more formal with the perturbation theory)

- Then the effective equations of motion read:

$$\mathrm{i}\dot{\tilde{a}}_e = -\Delta\tilde{a}_e - \boldsymbol{\mu}_{eg}\cdot\mathcal{E}(t)\boldsymbol{\epsilon}a_g$$

$$\mathrm{i}\dot{a}_g = -\boldsymbol{\mu}_{ge}\cdot\mathcal{E}(t)^*\boldsymbol{\epsilon}^*\tilde{a}_e$$

- Or nicer matrix form:

$$\mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle \equiv \mathrm{i}\frac{d}{dt}\begin{pmatrix}\tilde{a}_e\\ a_g\end{pmatrix} = \begin{pmatrix}-\Delta & -\boldsymbol{\mu}_{eg}\cdot\mathcal{E}(t)\boldsymbol{\epsilon}\\ -\boldsymbol{\mu}_{ge}\cdot\mathcal{E}(t)^*\boldsymbol{\epsilon}^* & 0\end{pmatrix}\begin{pmatrix}\tilde{a}_e\\ a_g\end{pmatrix} \equiv \hat{\tilde{H}}\,|\tilde{\psi}\rangle$$

Rabi Problem

$$\mathrm{i}\frac{d}{dt}|\tilde{\psi}\rangle \equiv \mathrm{i}\frac{d}{dt}\begin{pmatrix}\tilde{a}_e \\ a_g\end{pmatrix} = \begin{pmatrix}-\Delta & -\boldsymbol{\mu}_{eg}\cdot\mathcal{E}(t)\boldsymbol{\epsilon} \\ -\boldsymbol{\mu}_{ge}\cdot\mathcal{E}(t)^*\boldsymbol{\epsilon}^* & 0\end{pmatrix}\begin{pmatrix}\tilde{a}_e \\ a_g\end{pmatrix} \equiv \hat{\tilde{H}}|\tilde{\psi}\rangle$$

- We define the (complex) Rabi frequency $\Omega(t) = 2(\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon})\mathcal{E}(t)$

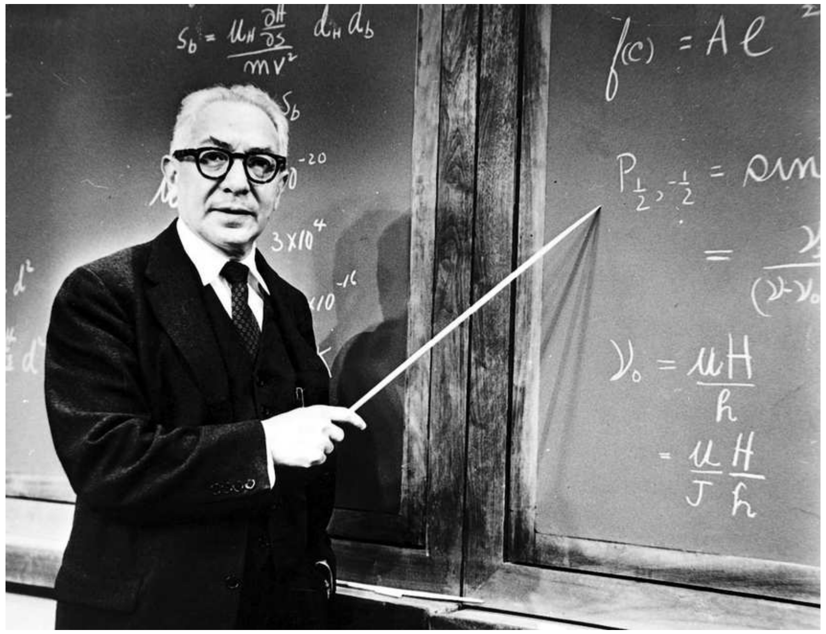
Note: $\hbar \equiv 1$

- Then the Hamiltonian for the TLS (in the RWA) becomes

$$\hat{\tilde{H}} = \begin{pmatrix}-\Delta & -\frac{1}{2}\Omega(t) \\ -\frac{1}{2}\Omega(t)^* & 0\end{pmatrix}$$

$$\hat{\tilde{H}} = -\Delta|e\rangle\langle e| - \frac{1}{2}\Omega(t)|e\rangle\langle g| - \frac{1}{2}\Omega(t)^*|g\rangle\langle e|$$

- This is known as Rabi-Problem (and is equivalent to for example oscillations of a spin-1/2 system in a magnetic field)



Isidor Isaac Rabi (1898-1988)

Rabi Problem

$$\hat{\tilde{H}} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega(t) \\ -\frac{1}{2}\Omega(t)^* & 0 \end{pmatrix}$$

$$\hat{\tilde{H}} = -\Delta \left|e\right\rangle \left\langle e\right| - \frac{1}{2}\Omega(t) \left|e\right\rangle \left\langle g\right| - \frac{1}{2}\Omega(t)^* \left|g\right\rangle \left\langle e\right|$$

- **NB:** Another common way to write the Hamiltonian is by using Pauli "spin" matrices

$$\hat{\tilde{H}} = -\frac{\Delta}{2}\hat{\sigma}^z - \frac{1}{2}\Omega(t)\hat{\sigma}^+ - \frac{1}{2}\Omega(t)^*\hat{\sigma}^-$$

- **With:** $\hat{\sigma}^- = \left|g\right\rangle \left\langle e\right| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

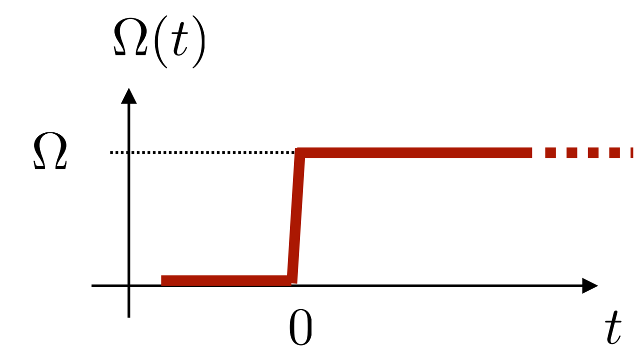
$$\hat{\sigma}^+ = (\hat{\sigma}^-)^\dagger = \left|e\right\rangle \left\langle g\right| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}^z = \left|e\right\rangle \left\langle e\right| - \left|g\right\rangle \left\langle g\right| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Here, the definition of the ground-state energy has changed.

Rabi Problem - Diagonalization

- We will now solve the case of a square pulse $\Omega(t) \equiv \Omega \qquad 0 \leq t \leq T$



- Then, while the light is on, the Hamiltonian is $$\boxed{\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}}$$

- To generally solve the problem we have to diagonalize the hermitian 2 x 2 matrix (easy), then

- **Eigenvalues:** $E_+ \qquad E_-$

$$\hat{H} = E_+ \left|+\right\rangle \left\langle+\right| + E_- \left|-\right\rangle \left\langle-\right|$$

- **Eigenvectors:** $\left|+\right\rangle \qquad \left|-\right\rangle$

- After the switch-on, the solution of the Schrödinger equation becomes $\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle$

Time-evolution operator

$$\left|\psi(t)\right\rangle = \mathrm{e}^{-\mathrm{i}t\hat{H}}\left|\psi(0)\right\rangle \equiv \hat{U}\left|\psi(0)\right\rangle$$

… and assuming the initial state being the ground-state $\left|\psi(0)\right\rangle = \left|g\right\rangle$

$$\boxed{\left|\psi(t)\right\rangle = \mathrm{e}^{-\mathrm{i}tE_+}\left|+\right\rangle\left\langle+|g\right\rangle + \mathrm{e}^{-\mathrm{i}tE_-}\left|-\right\rangle\left\langle-|g\right\rangle}$$

Rabi Problem - Diagonalization

$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

- To generally solve the problem we have to diagonalize the hermitian 2 x 2 matrix (easy)

- **Mathematical complement** (see e.g. Cohen-Tannoudji)

$$\boldsymbol{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

- Parametrization into 2 angles:

$$H_{12} = |H_{12}|\mathrm{e}^{\mathrm{i}\phi} \qquad 0 \leq \phi \leq 2\pi$$

$$\tan(\theta) = \frac{2|H_{12}|}{H_{11} - H_{22}} \qquad 0 \leq \theta \leq \pi$$

… one can show:

- **Eigenvalues:** $$E_\pm = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}$$

- **Eigenvectors:** $$|+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$

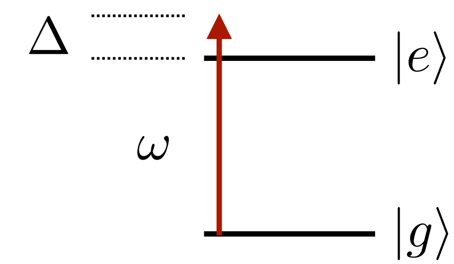
- Exercise: Proof (straightforward)

Rabi Problem - Diagonalization

$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

- **Eigenvalues:** $$E_\pm = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}$$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2}$$

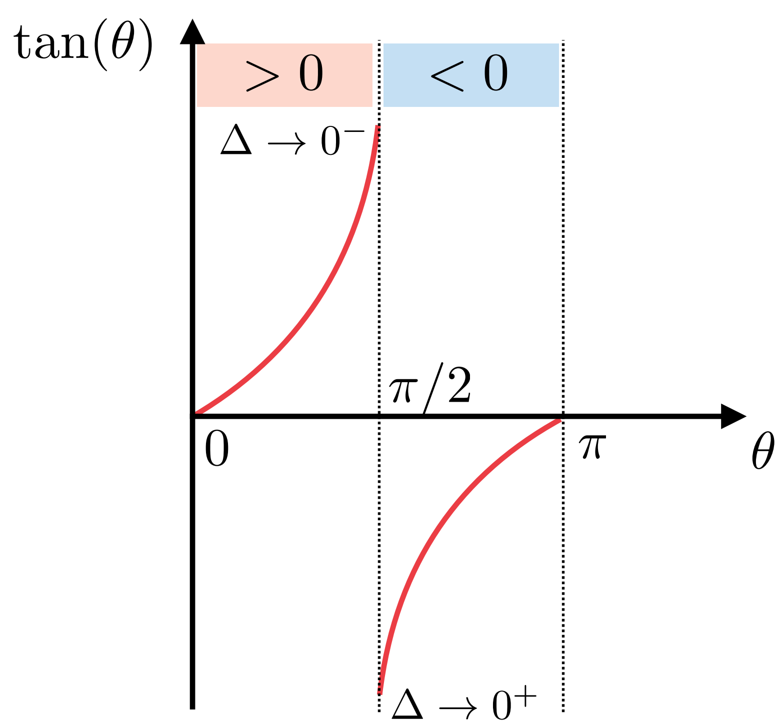


- **Eigenvectors:** $$|+\rangle = \begin{pmatrix} +\cos(\theta/2)e^{-i\phi/2} \\ +\sin(\theta/2)e^{+i\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix}$$

$\phi = \arg(\Omega)$ *… we don't really care about this. The electric field can have an arbitrary phase, and the relative phase between "g" and "e" is the same in both eigenstates.*

- **Parameters:**

$$\tan(\theta) = \frac{2|H_{12}|}{H_{11} - H_{22}} = -\frac{|\Omega|}{\Delta}$$ *… important parameter!* $0 \leq \theta \leq \pi$

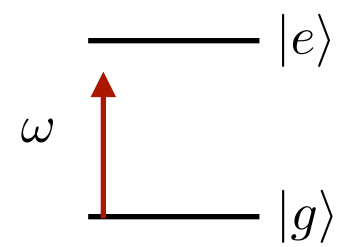


$|\Omega| \equiv 1$

$\Delta \to 0^-$	$\theta \to \pi/2^-$	*"red detuned"* $\Delta < 0$
$\Delta \to -\infty$	$\theta \to 0^+$	
$\Delta \to 0^+$	$\theta \to \pi/2^+$	*"blue detuned"* $\Delta > 0$
$\Delta \to +\infty$	$\theta \to \pi^-$	

Far red detuned dressed system

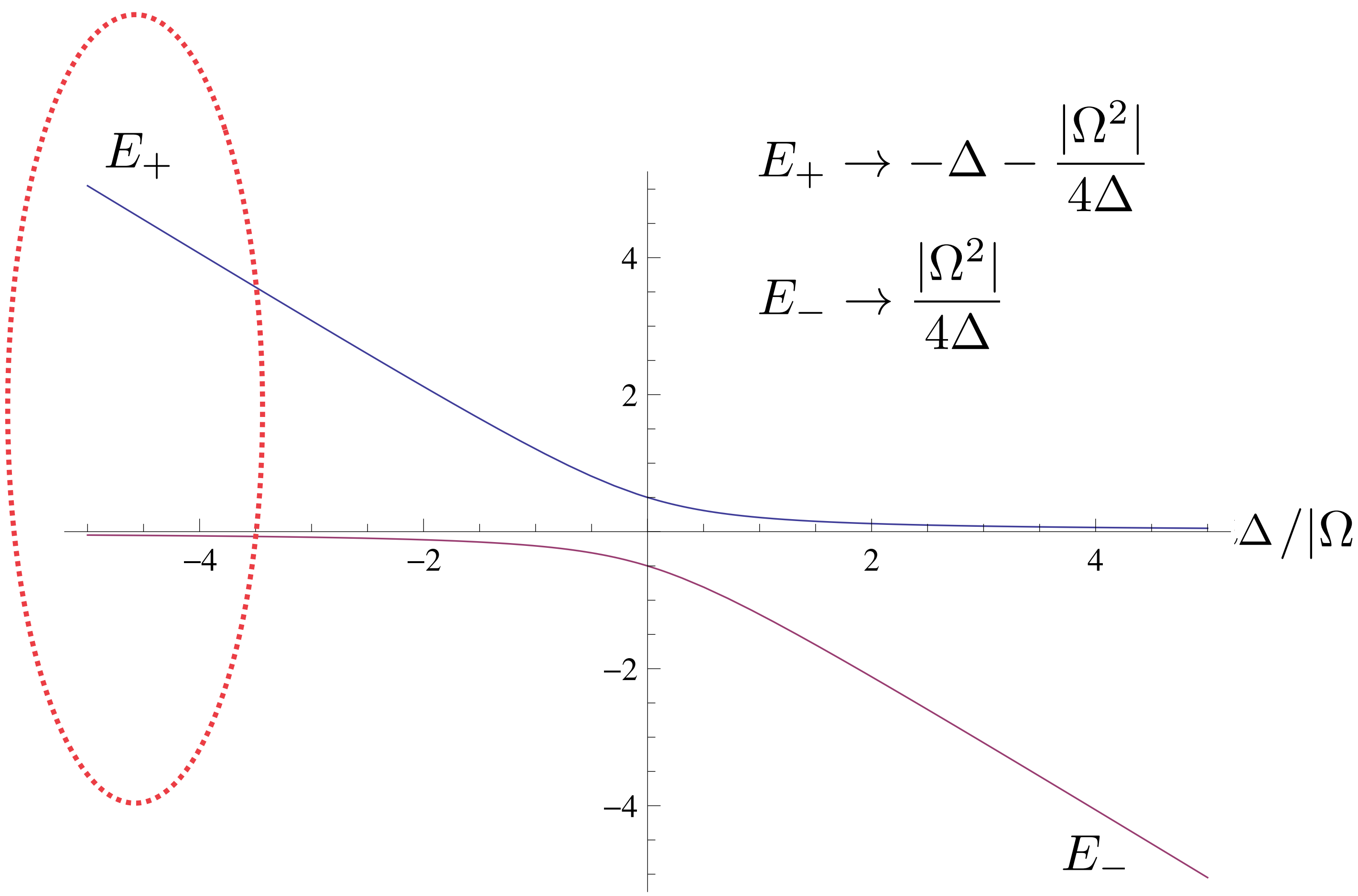
$$\Delta \ll 0$$



$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2}$$

$$|+\rangle = \begin{pmatrix} +\cos(\theta/2)e^{-i\phi/2} \\ +\sin(\theta/2)e^{+i\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix}$$



$$E_+ \rightarrow -\Delta - \frac{|\Omega^2|}{4\Delta}$$

$$E_- \rightarrow \frac{|\Omega^2|}{4\Delta}$$

$$\Delta \rightarrow -\infty \qquad \theta \rightarrow 0^+$$

$$\sin(\theta/2) \rightarrow 0 - \frac{|\Omega|}{2\Delta}$$

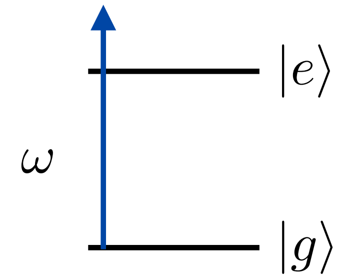
$$\cos(\theta/2) \rightarrow 1$$

$$|+\rangle \rightarrow \left(|e\rangle - e^{i\phi}\frac{|\Omega|}{2\Delta}|g\rangle\right)$$

$$|-\rangle \rightarrow \left(|g\rangle + e^{-i\phi}\frac{|\Omega|}{2\Delta}|e\rangle\right)$$

- In this limit, basically "+" becomes "e" and "-" becomes "g"
- The "g" states thus obtains an energy shift **down** in energy due to mixing in the excited state … this is called an **"AC Stark shift"**.

Far blue detuned dressed system

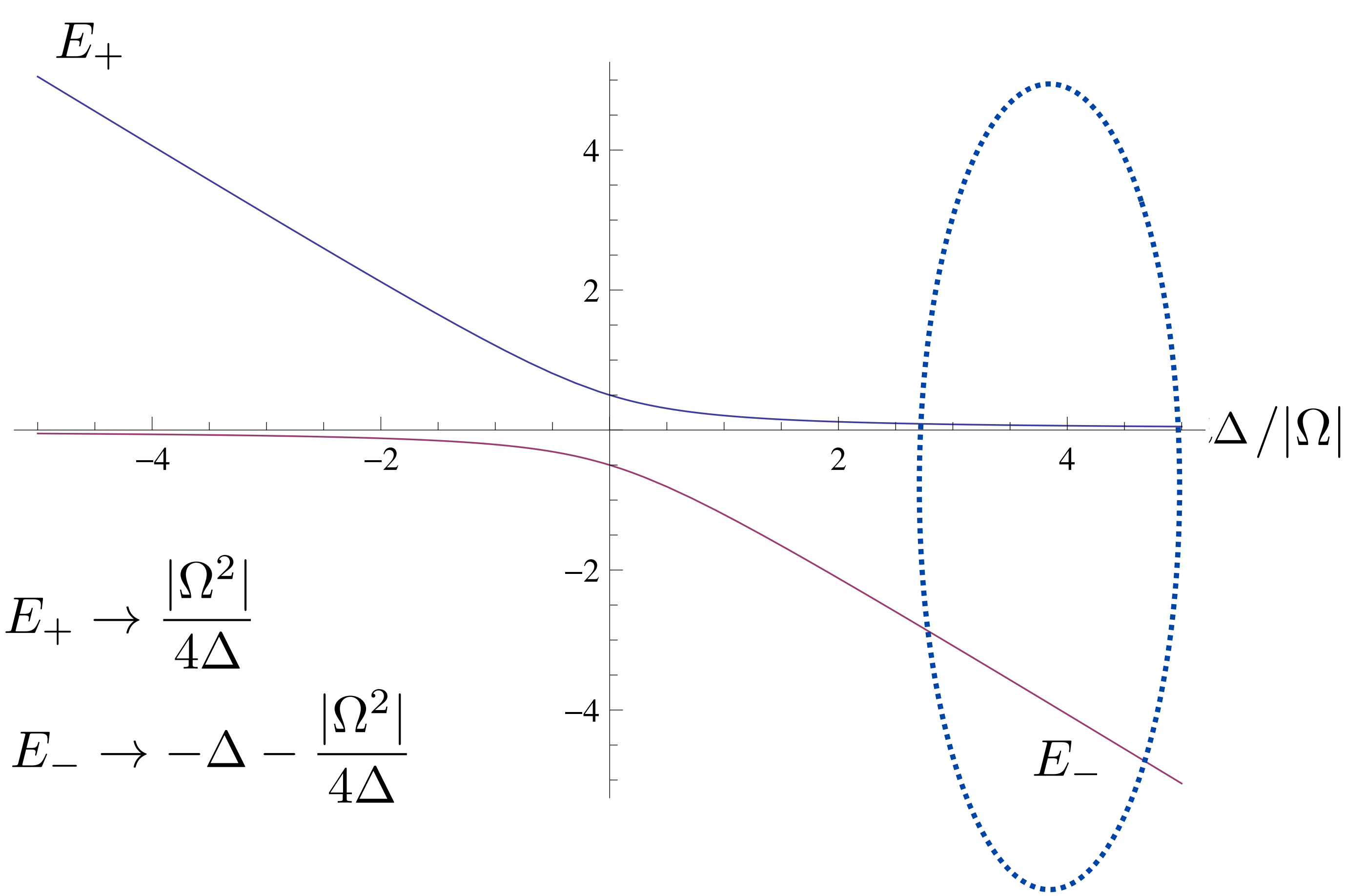


$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

$$\Delta \gg 0$$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2}$$

$$|+\rangle = \begin{pmatrix} +\cos(\theta/2)e^{-i\phi/2} \\ +\sin(\theta/2)e^{+i\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix}$$



$$E_+ \to \frac{|\Omega^2|}{4\Delta}$$

$$E_- \to -\Delta - \frac{|\Omega^2|}{4\Delta}$$

$$\Delta \to +\infty \qquad \theta \to \pi^-$$

$$\sin(\theta/2) \to 1$$

$$\cos(\theta/2) \to \frac{|\Omega|}{2\Delta}$$

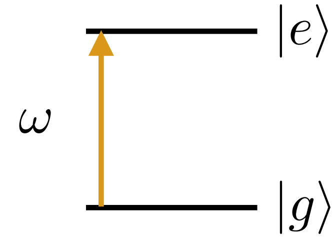
$$|+\rangle \to \left(|g\rangle + e^{-i\phi}\frac{|\Omega|}{2\Delta}|e\rangle\right)$$

$$|-\rangle \to \left(|e\rangle - e^{i\phi}\frac{|\Omega|}{2\Delta}|g\rangle\right)$$

- In this limit, basically "+" becomes "g" and "-" becomes "e"
- The "g" states thus obtains an energy shift **up** in energy due to mixing in the excited state … this is called an **"AC Stark shift"**.

On resonance dressed system

$$\Delta = 0$$

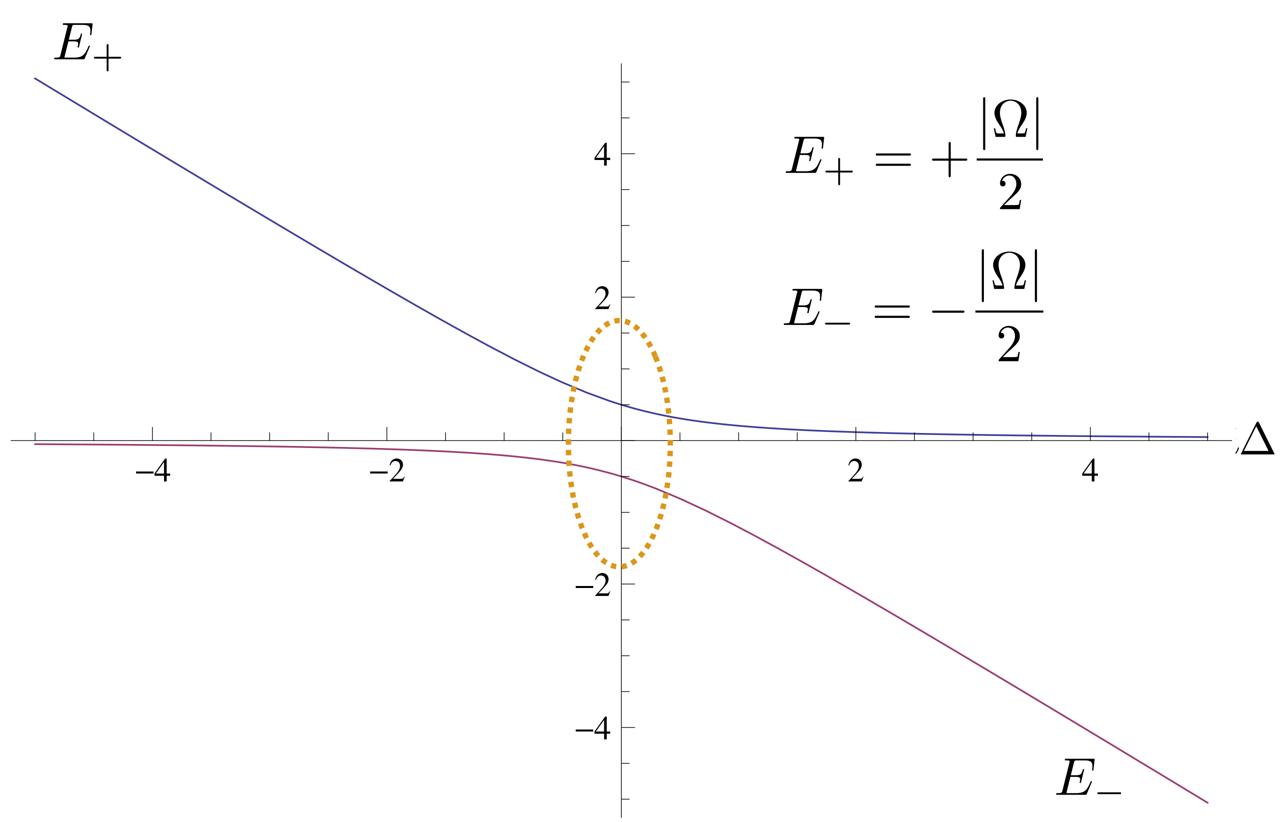


$$\hat{H} = \begin{pmatrix} -\Delta & -\frac{1}{2}\Omega \\ -\frac{1}{2}\Omega^* & 0 \end{pmatrix}$$

$$|\Omega| \equiv 1$$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2}$$

$$|+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \begin{matrix} |e\rangle \\ |g\rangle \end{matrix} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$



$$\Delta \to 0^- \qquad \theta \to \pi/2^-$$

$$\Delta \to 0^+ \qquad \theta \to \pi/2^+$$

$$\cos(\theta/2) \to \frac{1}{\sqrt{2}} \qquad \sin(\theta/2) \to \frac{1}{\sqrt{2}}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + \mathrm{e}^{-\mathrm{i}\phi}\,|e\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle - \mathrm{e}^{-\mathrm{i}\phi}\,|e\rangle\right)$$

- In this limit, both "+" and "-" are ideal equal super-positions of "e" and "g" (with a relative phase set by the laser).
- The two dressed states are separated by the **Rabi-Splitting** $|\Omega|$

General time-evolution in the Rabi-Problem

- **Remember:**

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}t\hat{H}}\,|\psi(0)\rangle \equiv \hat{U}\,|\psi(0)\rangle \qquad |\psi(0)\rangle = |g\rangle \qquad |\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}tE_+}\,|+\rangle\langle+|g\rangle + \mathrm{e}^{-\mathrm{i}tE_-}\,|-\rangle\langle-|g\rangle$$

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \qquad |+\rangle = \begin{pmatrix} +\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\sin(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix} \begin{array}{c} |e\rangle \\ |g\rangle \end{array} \qquad |-\rangle = \begin{pmatrix} -\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ +\cos(\theta/2)\mathrm{e}^{+\mathrm{i}\phi/2} \end{pmatrix}$$

- … to get an idea, evolution of excited state population: $|\langle e|\psi(t)\rangle|^2$

(note that the population is independent of the frame)

$$\langle e|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}tE_+}\,\langle e|+\rangle\langle+|g\rangle + \mathrm{e}^{-\mathrm{i}tE_-}\,\langle e|-\rangle\langle-|g\rangle$$

$$= \mathrm{e}^{-\mathrm{i}tE_+}\cos(\theta/2)\sin(\theta/2) - \mathrm{e}^{-\mathrm{i}tE_-}\cos(\theta/2)\sin(\theta/2)$$

$$\sin(\alpha) = 2\sin(\alpha/2)\cos(\alpha/2)$$

$$= \frac{1}{2}\sin(\theta)\left(\mathrm{e}^{-\mathrm{i}tE_+} - \mathrm{e}^{-\mathrm{i}tE_-}\right)$$

$$= \frac{1}{2}\sin(\theta)\mathrm{e}^{-\mathrm{i}t\Delta/2}\left(\mathrm{e}^{-\mathrm{i}t\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2}} - \mathrm{e}^{\mathrm{i}t\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2}}\right)$$

$$\sin(\alpha) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{+\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha}\right)$$

$$= \mathrm{i}\sin(\theta)\mathrm{e}^{-\mathrm{i}t\Delta/2}\sin\left(t\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2}\right)$$

$$\sin^2(\alpha) = \frac{1}{2}\left[1-\cos(2\alpha)\right]$$

$$|\langle e|\psi(t)\rangle|^2 = \sin^2(\theta)\sin^2\left(t\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2}\right) = \sin^2(\theta)\frac{1}{2}\left[1-\cos\left(t\sqrt{\Delta^2+|\Omega|^2}\right)\right]$$

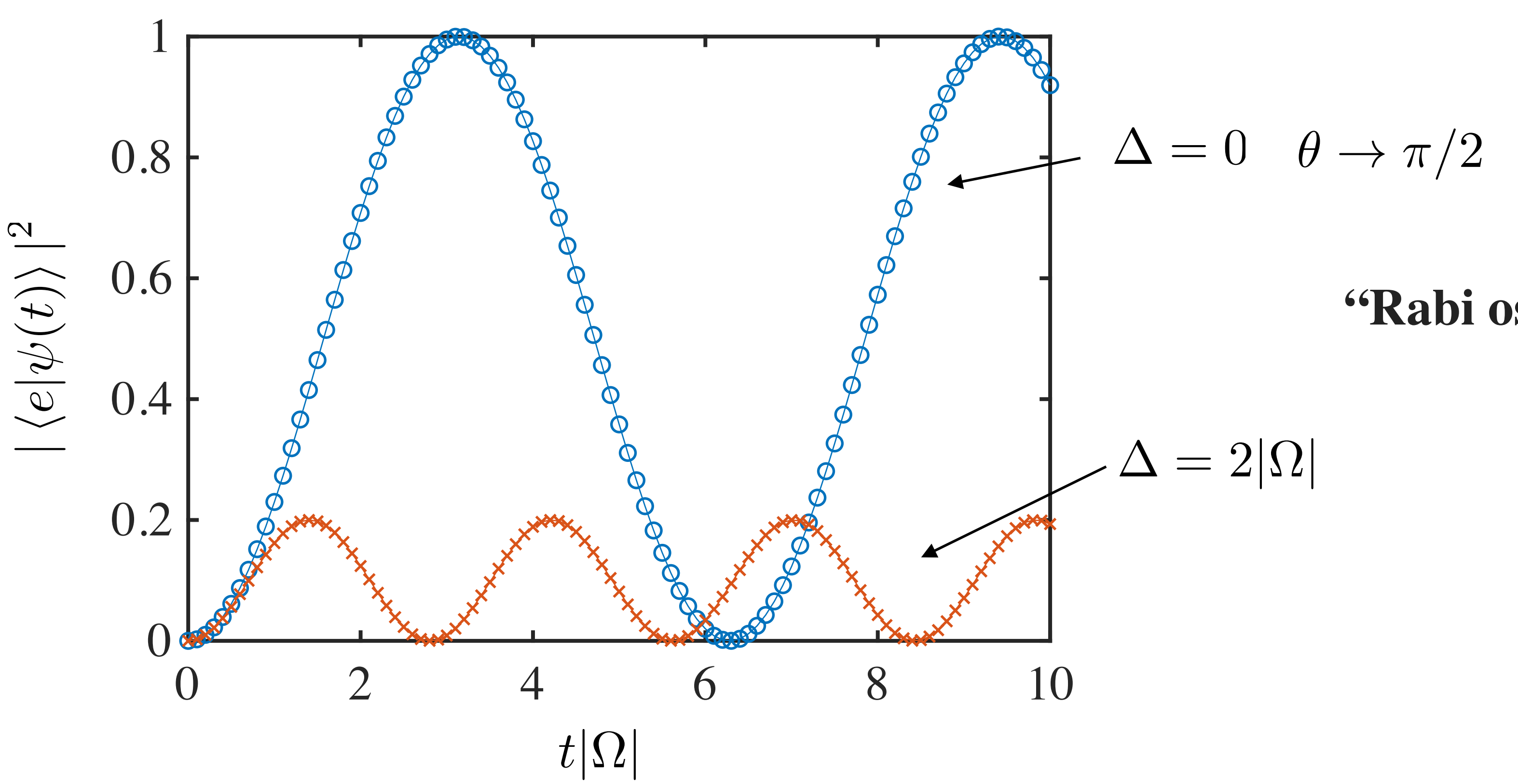
Rabi oscillations

- Evolution of excited state population:

$$|\langle e|\psi(t)\rangle|^2 = \sin^2(\theta)\frac{1}{2}\left[1 - \cos\left(t\sqrt{\Delta^2 + |\Omega|^2}\right)\right]$$

- We see that the population evolution takes place at the effective frequency: $\Omega_{\text{eff}} \equiv \sqrt{\Delta^2 + |\Omega|^2}$

- We find oscillations of the excited state population with frequency Ω_{eff}



"Rabi oscillations"

Power broadening

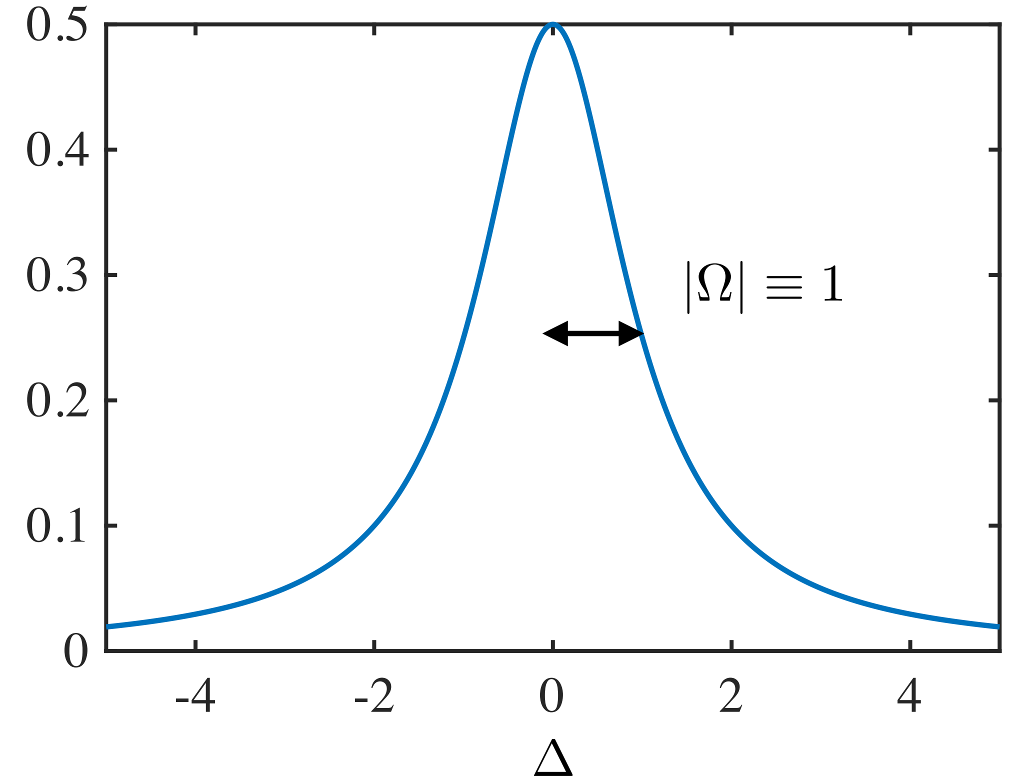
- Furthermore: $$|\langle e|\Psi(t)\rangle|^2 = \sin^2(\theta)\frac{1}{2}\left[1-\cos\left(t\Omega_{\text{eff}}\right)\right]$$

$$\tan(\theta) = -\frac{|\Omega|}{\Delta} \qquad \theta = \arctan\left(-\frac{|\Omega|}{\Delta}\right) \qquad \sin^2(\theta) = \frac{(|\Omega|/\Delta)^2}{1+(|\Omega|/\Delta)^2} = \frac{|\Omega|^2}{\Delta^2+|\Omega|^2}$$

$$\sin(\arctan(\alpha)) = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

$$|\langle e|\Psi(t)\rangle|^2 = \frac{1}{2}\frac{|\Omega|^2}{\Delta^2+|\Omega|^2}\left[1-\cos\left(t\Omega_{\text{eff}}\right)\right]$$

$$\frac{1}{2}\frac{|\Omega|^2}{\Delta^2+|\Omega|^2}$$



- **"Power broadening"** … at each point in time, as a function of the de-tuning the excited state population has a Lorentzian form of a width given by the Rabi frequency.

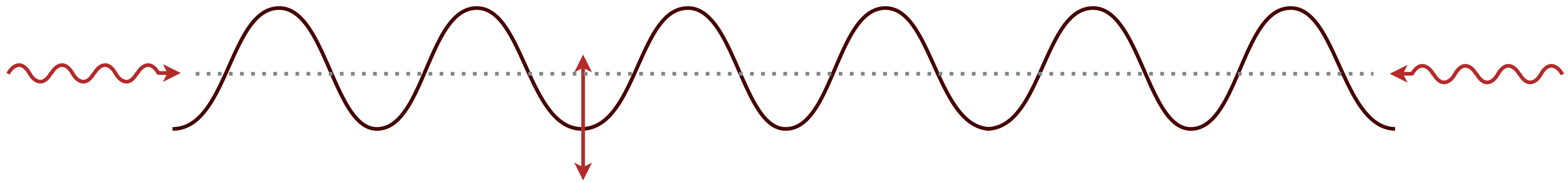
- The highest excited state population is reached at

$$t_{\max} = \frac{\pi}{\Omega_{\text{eff}}} \qquad |\langle e|\Psi(t_{\max})\rangle|^2 = \frac{|\Omega|^2}{\Delta^2+|\Omega|^2}$$

AC Stark shift - Application: Optical lattice

- Counter-propagating lasers: Standing wave lattice laser, wave-length λ
- Field: $$E(x) = E_0 \sin(k_L x) e^{-i\omega_L t} + \text{c.c.} \qquad k_L = \frac{2\pi}{\lambda}$$

(For simplicity written in 1D and without polarization)



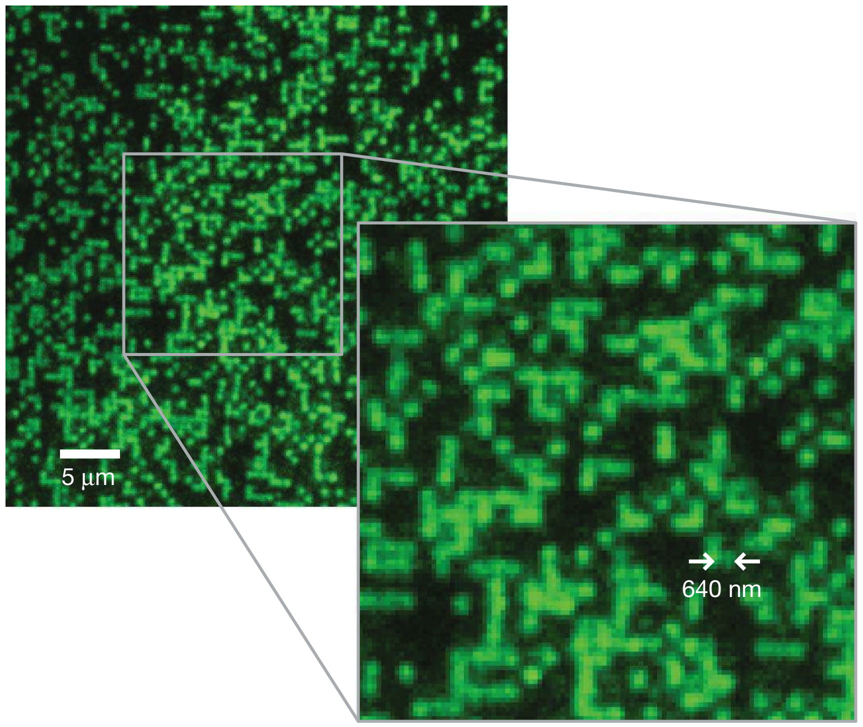
- The laser is a spatially stationary, time-dependent electric field: There are nodes where the field is alway zero, and there are points where it oscillates between a maximum/minimum with the optical laser frequency.

- Ground-state energy due to **AC Stark shift:** $$E_g = 0 + \frac{|\tilde{E}(x)\mu_{eg}|^2}{4\Delta}$$

- **One can now trap atoms in lattice structures!**

- **Platform for quantum computing/quantum simulation applications.**

Experiments: *Munich, Harvard/MIT, Zürich, Innsbruck, Glasgow, Paris, JILA, Innsbruck, Hamburg, Okazaki, Pisa, Florence, Oxford, Cambridge, Austin, Chicago, Penn State, Kyoto, Toronto, Stony Brook, Illinois, ...*



Recap

- We looked at the problem of a classical oscillating (optical frequency) electric field coupled to an atom in the dipole approximation:

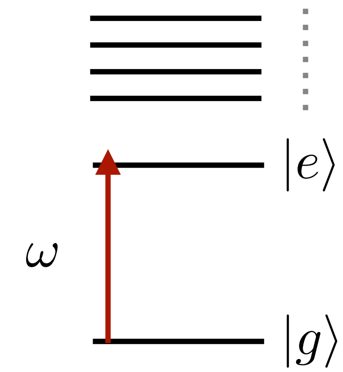
$$\hat{H}_{AF} = \hat{H}_{0A} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}_{\text{cl.}} \qquad i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{AF} |\psi(t)\rangle$$

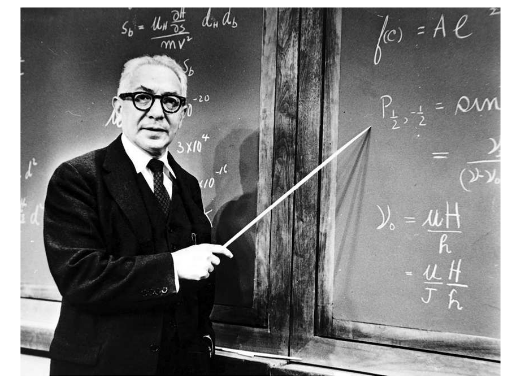
- The problem can be transformed to a rotating frame (at an optical frequency). Then, "counter-rotating" terms can be eliminated in perturbation theory: **Rotating wave approximation (RWA)**.

- In the case where a laser is tuned close to an internal transition of the atom, also other atomic level can be eliminated in perturbation theory (more on the details on this theory later).

- Then we arrive at the "Rabi-problem" for a two-level system

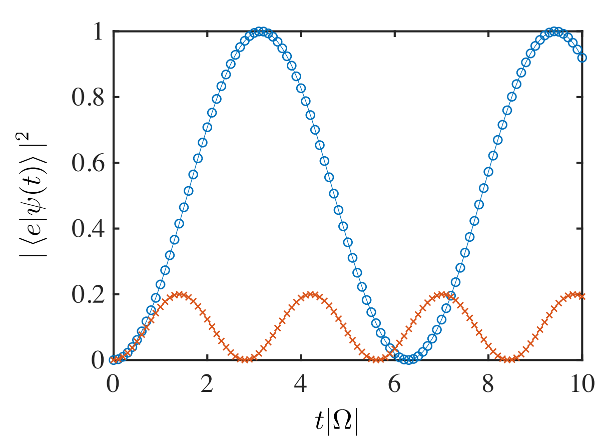
$$\hat{H} = -\Delta |e\rangle \langle e| - \boldsymbol{\mu}_{eg} \cdot \boldsymbol{E}^{-}(t) |e\rangle \langle g| - \boldsymbol{\mu}_{ge} \cdot \boldsymbol{E}^{+}(t) |g\rangle \langle e|$$





- We solved the Rabi-Problem for the case were initially the atom is in the ground state. Then, the key result are **Rabi oscillations** at the frequency:

$$\Omega_{\text{eff}} \equiv \sqrt{\Delta^2 + |\Omega|^2}$$



- In the far-detuned case we find level (AC Stark) shifts which can be used for atom trapping (optical lattices).