Atomic physics : Basics of quantum optics/light matter interactions

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The goal of this class (and the next ones) is to introduce basic concepts of **light - matter interactions** on the atomic level = **Quantum optics**. Basically, we will see how to use lasers to manipulate quantum states of atoms.

- Book suggestions:
  - "*Photons and Atoms: Introduction to Quantum Electrodynamics*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Atom-Photon Interactions: Basic Process and Appilcations*" C. Cohen−Tannoudji, J. Dupont−Roc, G. Grynberg (very complete and detailed)
  - "*Quantum Optics* " D. F. Walls, G. J. Milburn (accessible and complete)
  - "*Quantum Computation and Quantum Information*", *M. A. Nielsen, I. Chuang* (standard text for quantum information theory, has also a good part on master equations, available online for free)
  - "*The Quantum World of Ultra-Cold Atoms and Light Book I*", P. Zoller and C. Gardiner (also "Quantum Noise") (very advanced and concise)

- Outline (may vary)

1. Atom-field interaction Hamiltonian (RWA)/Rabi Problem and AC Stark shift.

2. The Bloch sphere for two-level atoms and perturbation theory for multi-level atoms.

3. Three level systems/STIRAP and field quantization. **This time**

4. Atom decay, density matrix and master equations.

Last time

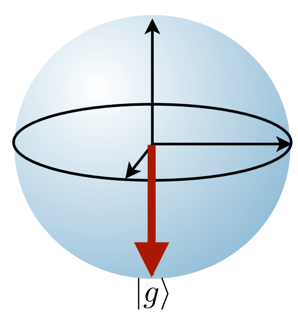
- We looked at the full dynamics of a two-level atom (computed the time-evolution operator)

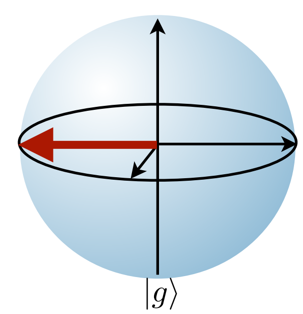
$$\hat{U} = \mathrm{e}^{-\mathrm{i}t\Delta/2} \begin{pmatrix} \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) + \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \\ \mathrm{i}\sin(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) & \cos\left(t\frac{1}{2}\Omega_{\text{eff}}\right) - \mathrm{i}\cos(\theta)\sin\left(t\frac{1}{2}\Omega_{\text{eff}}\right) \end{pmatrix}$$

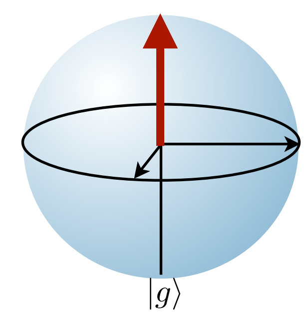
- Any quantum state of a two-level atom can be described by a point on the **Bloch sphere**, consequently, any laser pulse can be described by a rotation of a **Bloch vector.**

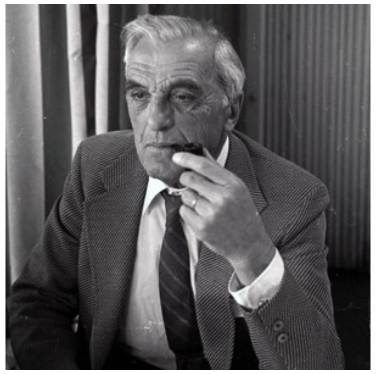
$$c_e = \cos(\theta/2)$$

$$c_g = \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi}$$









- We derived the Schrödinger equation of a laser-coupled N-level atom initially in the ground-state in perturbation theory. The perturbation theory is fully valid if the laser is far-detuned from any internal transition. Then each level gives rise to an AC stark shift of the ground-state.

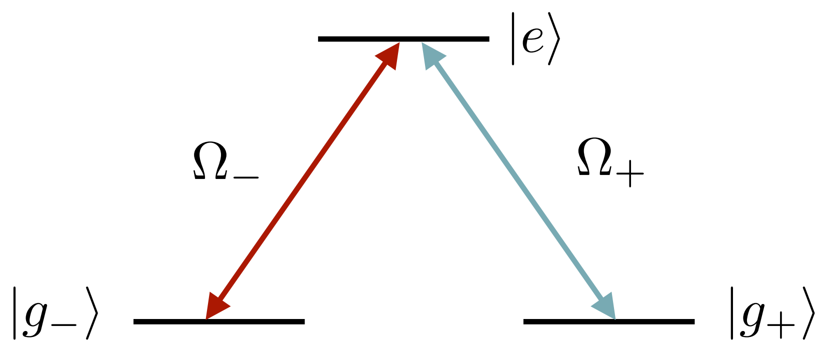
$$\hat{\tilde{H}}_{\text{eff}} = \delta\omega_g(t)\,|g\rangle\langle g|$$

$$\delta\omega_g = \sum_{k\neq g} \frac{1}{2}\left(\frac{(\omega_k-\omega_g)|\Omega_{gk}(t)|^2}{(\omega_k-\omega_g)^2-\omega^2}\right) \equiv \frac{1}{2}\alpha(\omega)|\mathcal{E}|^2$$

- The condition for the validity of the RWA becomes obvious in the perturbation theory: $|\Omega_{ng}(t)|/2 \ll \omega_n - \omega_g + \omega$
- We started to look at an application of our tools for a **three-level** Lambda system.

This time - Outline

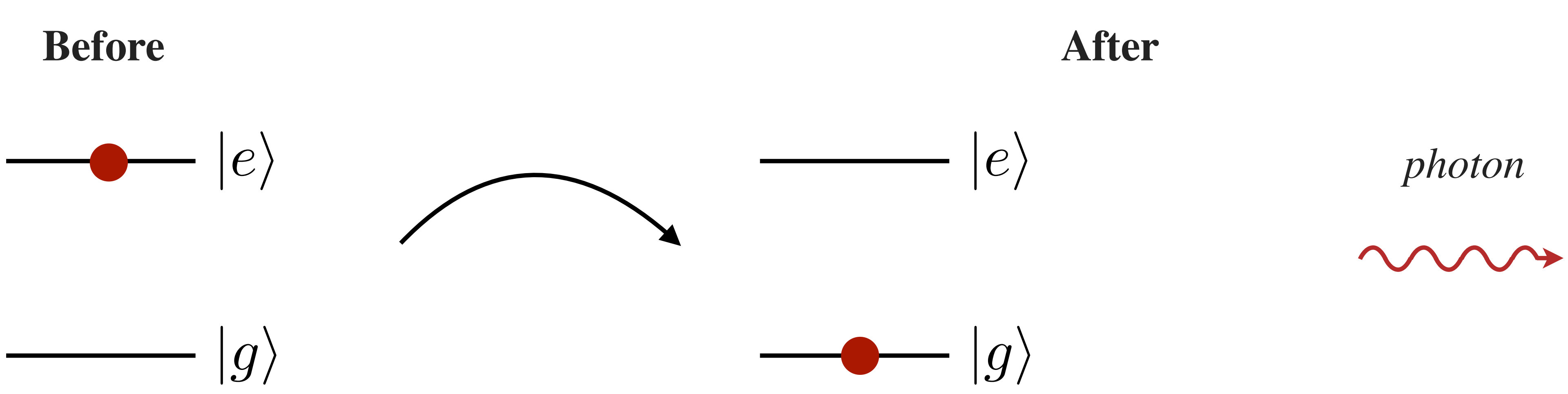
- First we will review the calculations for the three level system that we started last time. We will discuss the **STIRAP protocol**, a scheme to transfer population from one ground-state to another one.



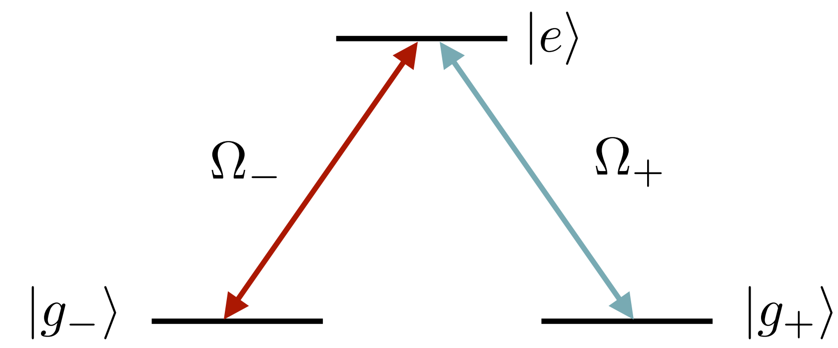
- In order to treat the decay of an atom in an excited state properly, we will need to treat the environment, i.e. the electromagnetic (EM) field fully quantum-mechanically. We will introduce the **quantization of the EM field**.

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

- Lastly we will quickly look at decay and compute the decay-rate of an excited state with **Fermi's Golden Rule**



3.1 - Three level systems



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- Last time we introduced the “Lambda” configuration: We have **two lasers** tuned to **two transitions** to the **same excited state.**

- In the RWA:

$$\boxed{\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_-(t) & 0 \\ -\frac{1}{2}\Omega_-(t) & -\Delta - \mathrm{i}\frac{1}{2}\Gamma & -\frac{1}{2}\Omega_+(t) \\ 0 & -\frac{1}{2}\Omega_+(t) & \delta \end{pmatrix}\begin{pmatrix} a_- \\ a_0 \\ a_+ \end{pmatrix}}$$

- Then: Adiabatic elimination of “e”

$$\boxed{\dot{a}_0 \approx 0} \qquad a_0 \approx -\frac{\Omega_+(t)}{2\Delta + \mathrm{i}\Gamma}a_+ - \frac{\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}a_-$$

- Reduction to effective two-level system with (tunable) parameters:

$$-\Delta_{\mathrm{eff}} \equiv \delta + \frac{\Omega_+(t)^2 - \Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma} \qquad -\Omega_{\mathrm{eff}} = \frac{\Omega_+(t)\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}$$

$$\boxed{\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} -\Delta_{\mathrm{eff}}(t) & -\frac{1}{2}\Omega_{\mathrm{eff}}(t) \\ -\frac{1}{2}\Omega_{\mathrm{eff}}(t) & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}}$$

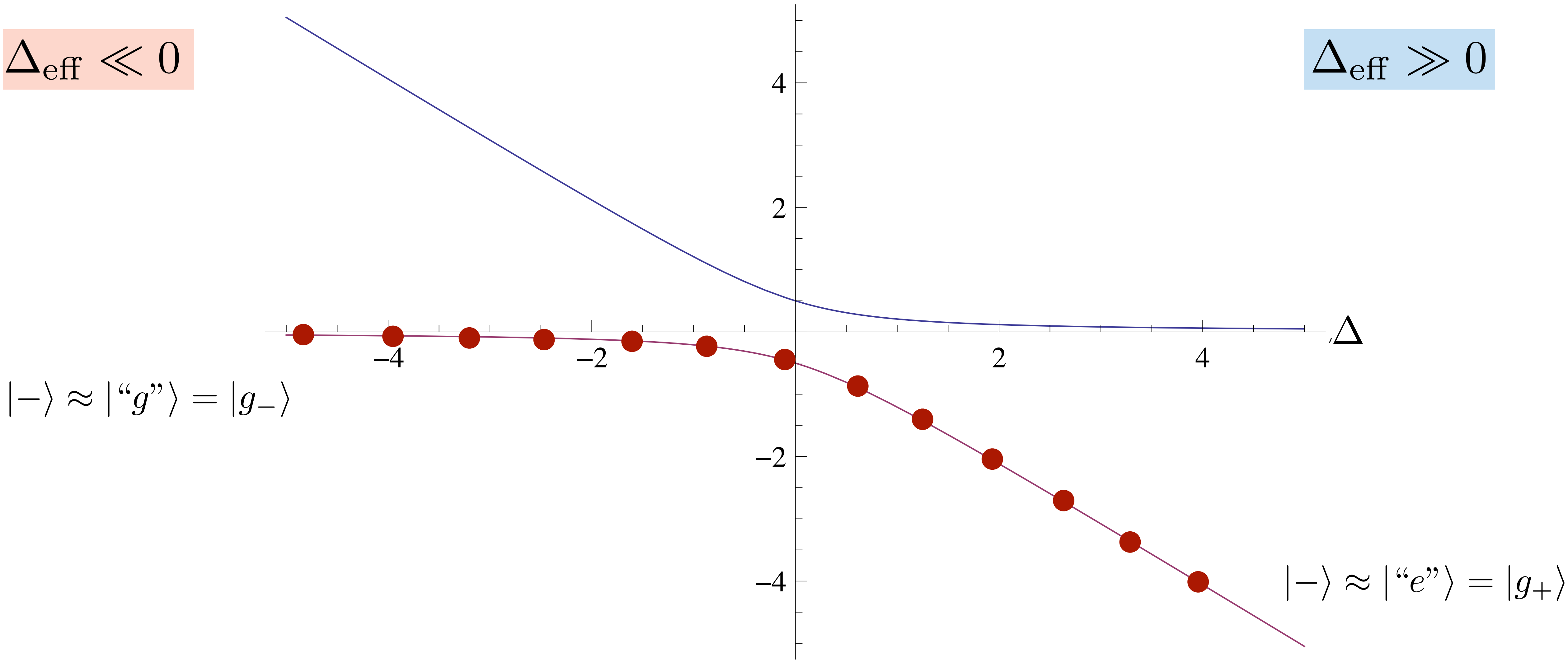
- We can re-use our two-level results!

3.1 - STIRAP

$$-\Delta_{\text{eff}} \equiv \delta + \frac{\Omega_+(t)^2 - \Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma} \qquad -\Omega_{\text{eff}} = \frac{\Omega_+(t)\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}$$

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} -\Delta_{\text{eff}}(t) & -\frac{1}{2}\Omega_{\text{eff}}(t) \\ -\frac{1}{2}\Omega_{\text{eff}}(t) & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

- **Reminder:** Eigenstates of the Rabi-Problem:



- If we are **red de-tuned** in the effective model then we will be in an **eigenstate** which is the **left ground-state**.
  If we are **blue de-tuned** we will be in an **eigenstate** which is the **right ground-state.**
  The general idea of the STIRAP scheme is to adiabatically (very slowly) change the effective detuning, and to bring population from **left** to **right**.

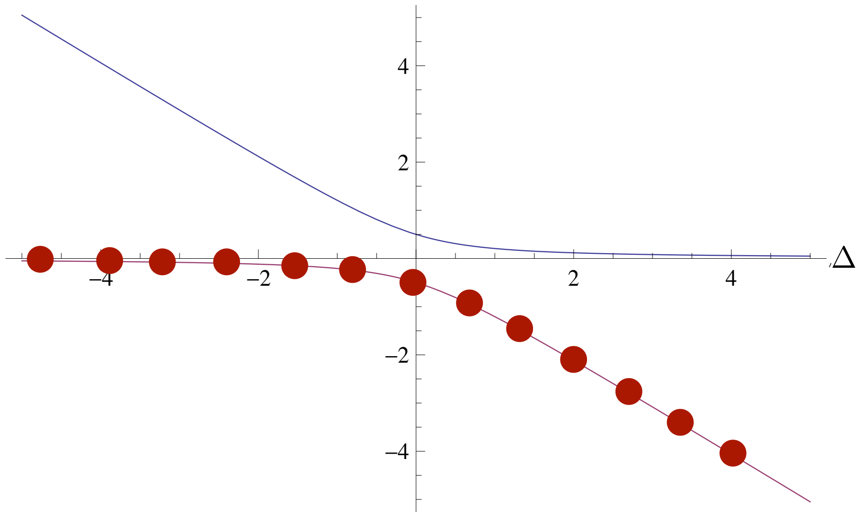
3.1 - STIRAP

$$-\Delta_{\text{eff}} \equiv \delta + \frac{\Omega_+(t)^2 - \Omega_-(t)^2}{4\Delta + 2\mathrm{i}\Gamma} \qquad -\Omega_{\text{eff}} = \frac{\Omega_+(t)\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}$$

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} -\Delta_{\text{eff}}(t) & -\frac{1}{2}\Omega_{\text{eff}}(t) \\ -\frac{1}{2}\Omega_{\text{eff}}(t) & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$\Delta_{\text{eff}} \ll 0$

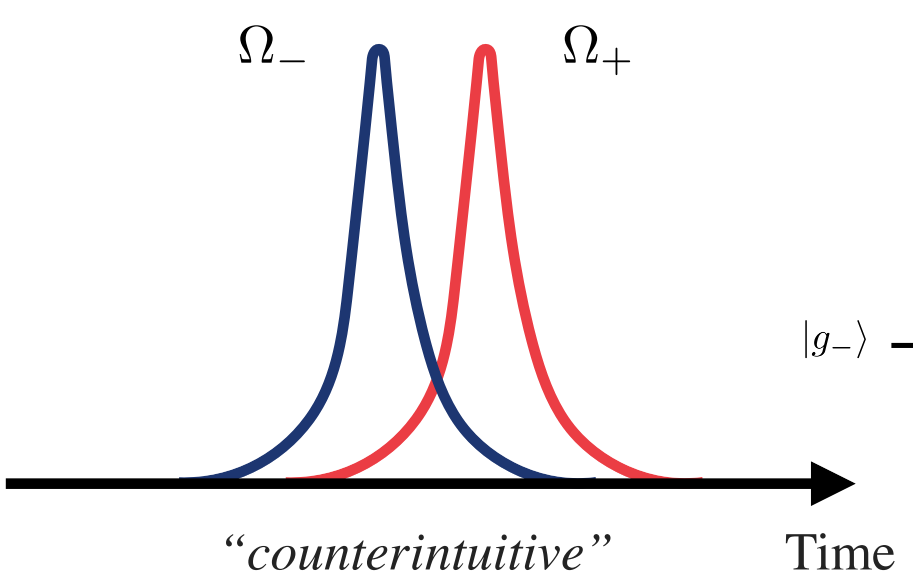
$|-\rangle \approx |g_-\rangle$

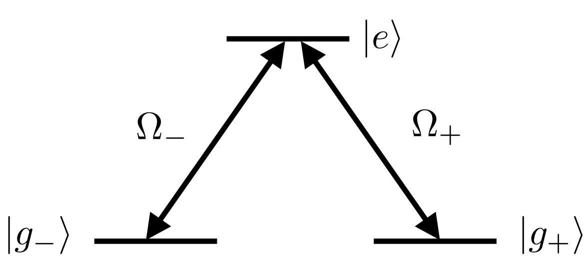


$\Delta_{\text{eff}} \gg 0$

$|-\rangle \approx |g_+\rangle$

- **STIRAP:** The general idea is to simply use two time-delayed laser pulses:





- Light off: $|\psi\rangle = |g_-\rangle$
- "+" is turned on:

  $\Delta_{\text{eff}} \ll 0 \quad |\psi\rangle \approx |-\rangle \approx |g_-\rangle$
- "-" is turned on while "+" is switched off:

  $\Delta_{\text{eff}} \gg 0 \quad |\psi\rangle \approx |-\rangle \approx |g_+\rangle$
- Light off: $|\psi\rangle \approx |g_+\rangle$

3.1 - Adiabatic approximation of the time-dependent Schrödinger equation

- **Why and when does this work?**
- **Generally:** Take a time-dependent Hamiltonian $\hat{H}[\alpha(t)]$ … and the Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}[\alpha(t)]\left|\psi(t)\right\rangle$$

- *In our example:* $\alpha(t) = \Delta_{\text{eff}}(t)$
- The idea of the adiabatic approximation is to expand the system into "instantaneous" eigenstates

$$\hat{H}(t)\left|\phi_n(t)\right\rangle = E_n(t)\left|\phi_n(t)\right\rangle$$

"diagonalize the problem for each point in time"

- *In our example:* These are just the dressed states $\left|\pm\right\rangle$
- **Ansatz:** $\left|\psi(t)\right\rangle = \sum_n c_n(t)\left|\phi_n(t)\right\rangle$

$$\mathrm{i}\frac{d}{dt}\left[\sum_n c_n(t)\left|\phi_n(t)\right\rangle\right] = \hat{H}(t)\sum_n c_n(t)\left|\phi_n(t)\right\rangle$$

$$\mathrm{i}\sum_n\left[\dot{c}_n(t)\left|\phi_n(t)\right\rangle + c_n(t)\frac{d}{dt}\left|\phi_n(t)\right\rangle\right] = \hat{H}(t)\sum_n c_n(t)\left|\phi_n(t)\right\rangle$$

$\left\langle\phi_k(t)\right|$

$$\mathrm{i}\sum_n\left[\dot{c}_n(t)\left\langle\phi_k(t)|\phi_n(t)\right\rangle + c_n(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle\right] = \left\langle\phi_k(t)\right|\hat{H}(t)\sum_n c_n(t)\left|\phi_n(t)\right\rangle$$

$$\boxed{\mathrm{i}\dot{c}_k(t) = E_k(t)c_k(t) - \mathrm{i}\sum_n c_n(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle}$$

3.1 - Adiabatic approximation of the time-dependent Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}[\alpha(t)]\left|\psi(t)\right\rangle \qquad \hat{H}(t)\left|\phi_n\right\rangle = E_n(t)\left|\phi(t)\right\rangle \qquad \left|\psi(t)\right\rangle = \sum_n c_n(t)\left|\phi_n(t)\right\rangle$$

$$\mathrm{i}\dot{c}_k(t) = E_k(t)c_k(t) - \mathrm{i}\sum_n c_n(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle$$

- Let's look at the second term on the RHS … take

from $\hat{H}(t)\left|\phi_n(t)\right\rangle = E_n(t)\left|\phi_n(t)\right\rangle$

$$\frac{d}{dt}\left[\hat{H}(t)\left|\phi_n(t)\right\rangle\right] = \left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle + \hat{H}(t)\left[\frac{d}{dt}\left|\phi_n(t)\right\rangle\right] = \left[\frac{d}{dt}E_n(t)\right]\left|\phi_n(t)\right\rangle + E_n(t)\left[\frac{d}{dt}\left|\phi_n(t)\right\rangle\right]$$

$\left\langle\phi_k(t)\right|$

$$\left\langle\phi_k(t)\right|\left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle + E_k(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle = \left[\frac{d}{dt}E_n(t)\right]\delta_{kn} + E_n(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle$$

- Thus for $k \neq n$

$$\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle = \frac{\left\langle\phi_k(t)\right|\left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle}{E_n(t) - E_k(t)}$$

- Insert this into:

3.1 - Adiabatic approximation of the time-dependent Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}[\alpha(t)]\left|\psi(t)\right\rangle \qquad \hat{H}(t)\left|\phi_n\right\rangle = E_n(t)\left|\phi(t)\right\rangle \qquad \left|\psi(t)\right\rangle = \sum_n c_n(t)\left|\phi_n(t)\right\rangle$$

$$\mathrm{i}\dot{c}_k(t) = E_k(t)c_k(t) - \mathrm{i}\sum_n c_n(t)\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle$$

$$\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_n(t)\right\rangle = \frac{\left\langle\phi_k(t)\right|\left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle}{E_n(t)-E_k(t)}$$

- Insert this into:

$$\begin{aligned}\mathrm{i}\dot{c}_k(t) = c_k(t)&\left[E_k(t) - \mathrm{i}\left\langle\phi_k(t)\right|\frac{d}{dt}\left|\phi_k(t)\right\rangle\right]\\ &- \mathrm{i}\sum_{n\neq k} c_n(t)\cancel{\frac{\left\langle\phi_k(t)\right|\left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle}{E_n(t)-E_k(t)}}\end{aligned}$$

- We have derived an equation of motion for the instantaneous eigenstate probability amplitudes. The term in the **first line does not change population between eigenstates**.
- The term in the second line does mix the eigenstates. Neglecting the term in the second line is the so-called **adiabatic approximation**. In this approximation, once an eigenstate is prepared, the system remains in that eigenstate and only acquires a phase.
- This is a good approximation whenever: $E_n(t) - E_k(t) \gg \left\langle\phi_k(t)\right|\left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle$

Adiabatic approximation: If the rate of change in a time-dependent Hamiltonian is slower than the gaps between the eigenstate, a system prepared in an eigenstate remains in that eigenstate.

3.1 - STIRAP

$$
\mathrm{i}\dot{c}_k(t) = c_k(t)\left[E_k(t) - \mathrm{i}\left\langle \phi_k(t)\right| \frac{d}{dt}\left|\phi_k(t)\right\rangle\right]
\cancel{- \mathrm{i}\sum_{n\neq k} c_n(t)\frac{\left\langle \phi_k(t)\right| \left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle}{E_n(t) - E_k(t)}}
$$

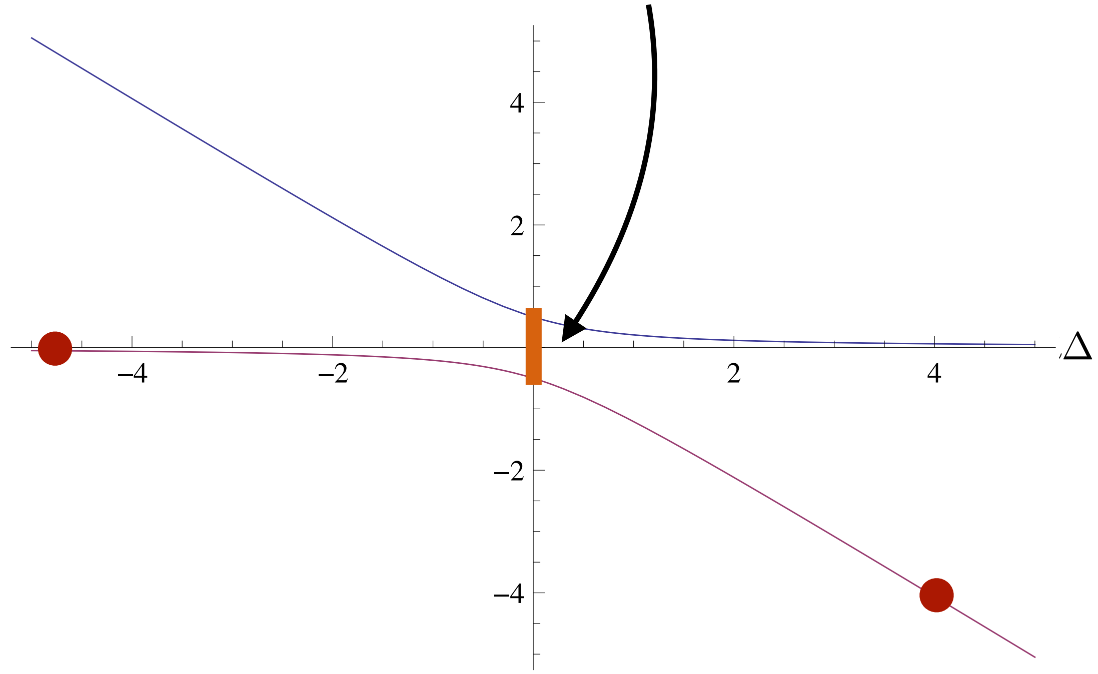
$$
E_n(t) - E_k(t) \gg \left\langle \phi_k(t)\right| \left[\frac{d}{dt}\hat{H}(t)\right]\left|\phi_n(t)\right\rangle
$$

Adiabatic approximation: If the rate of change in a time-dependent Hamiltonian is slower than the gaps between the eigenstate, a system prepared in an eigenstate remains in that eigenstate.

- This means that the STIRAP scheme works, as long as $\frac{d}{dt}\Delta_{\text{eff}} \ll \hbar|\Omega_{\text{eff}}|$ $\quad -\Omega_{\text{eff}} = \frac{\Omega_+(t)\Omega_-(t)}{2\Delta + \mathrm{i}\Gamma}$

$\Delta_{\text{eff}} \ll 0$

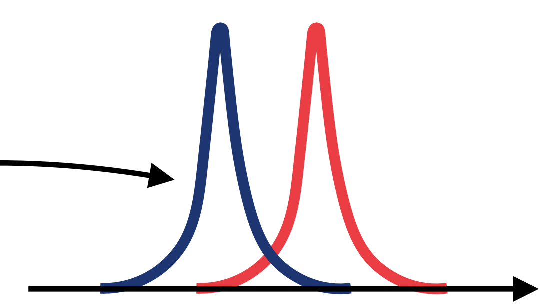
$|-\rangle \approx |g_-\rangle$



$\Delta_{\text{eff}} \gg 0$

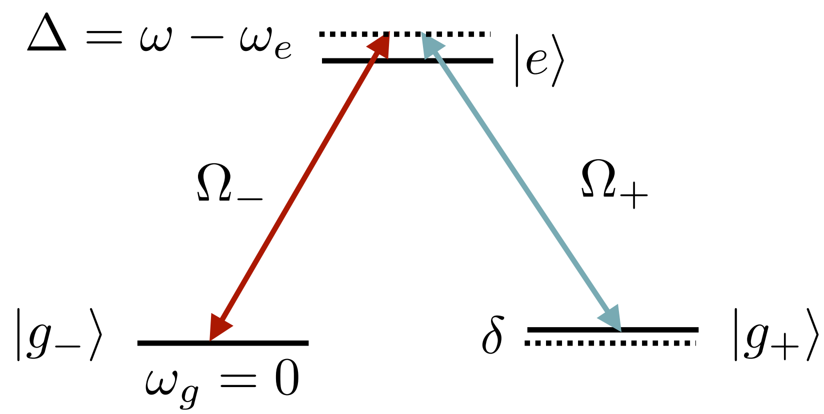
$|-\rangle \approx |g_+\rangle$

- This means that the shape of the delayed pulses cannot be too steep.



- **Note:** For fast adiabatic change the tunneling between eigenstates is called “Landau-Zener tunneling”

3.1 - STIRAP



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- Why the scheme works **<u>really</u> well:**

- We have solved the problem in the large de-tunings, where we could eliminate the excited state to reduce the problem to a two-level system… let's now go quickly back to the full three level system.

- Hamiltonian in laser frame for $\delta = 0$

$$\hat{H}_\Lambda = -\Delta |e\rangle \langle e| - \frac{1}{2}\left(\Omega_- |e\rangle \langle g_-| + \text{h.c.}\right) - \frac{1}{2}\left(\Omega_+ |e\rangle \langle g_+| + \text{h.c.}\right)$$

$$= \begin{pmatrix} 0 & -\frac{1}{2}\Omega_- & 0 \\ -\frac{1}{2}\Omega_- & -\Delta & -\frac{1}{2}\Omega_+ \\ 0 & -\frac{1}{2}\Omega_+ & 0 \end{pmatrix}$$

- In this case it's also not too hard to find the eigenvalues of the 3x3 matrix:

$$|\pm\rangle = \frac{1}{\sqrt{E_\pm^2 + \Omega_r^2}}\left[\Omega_+ |g_-\rangle + E_\pm |e\rangle + \Omega_- |g_+\rangle\right] \qquad E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + \Omega_r^2}$$

$$|0\rangle = \frac{1}{\Omega_r}\left[\Omega_+ |g_-\rangle - \Omega_- |g_+\rangle\right] \qquad E_0 = 0$$

$$\Omega_r = \sqrt{\Omega_+^2 + \Omega_-^2}$$

3.1 - STIRAP

$$\hat{H}_\Lambda = -\Delta \left|e\right\rangle \left\langle e\right| - \frac{1}{2}\left(\Omega_- \left|e\right\rangle \left\langle g_-\right| + \text{h.c.}\right) - \frac{1}{2}\left(\Omega_+ \left|e\right\rangle \left\langle g_+\right| + \text{h.c.}\right) \quad = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_- & 0 \\ -\frac{1}{2}\Omega_- & -\Delta & -\frac{1}{2}\Omega_+ \\ 0 & -\frac{1}{2}\Omega_+ & 0 \end{pmatrix}$$

$$\left|\pm\right\rangle = \frac{1}{\sqrt{E_\pm^2 + \Omega_r^2}}\left[\Omega_+ \left|g_-\right\rangle + E_\pm \left|e\right\rangle + \Omega_- \left|g_+\right\rangle\right] \qquad E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + \Omega_r^2}$$

$$\left|0\right\rangle = \frac{1}{\Omega_r}\left[\Omega_+ \left|g_-\right\rangle - \Omega_- \left|g_+\right\rangle\right] \qquad E_0 = 0$$

$$\Omega_r = \sqrt{\Omega_+^2 + \Omega_-^2}$$

- Why … test it! Especially interesting is the (nontrivial) **eigenstate with eigenvalue zero!**

$$\hat{H}_\Lambda \left|0\right\rangle = \frac{1}{2\Omega_r}\left(\Omega_-\Omega_+ \left|e\right\rangle - \Omega_-\Omega_+ \left|e\right\rangle\right) = 0$$

- **Discussion:**

  It is a superposition of the two ground-states. A state prepared in this state is not coupled by the light (laser terms in the Hamiltonian), it is thus "dark" and called a **dark state**.

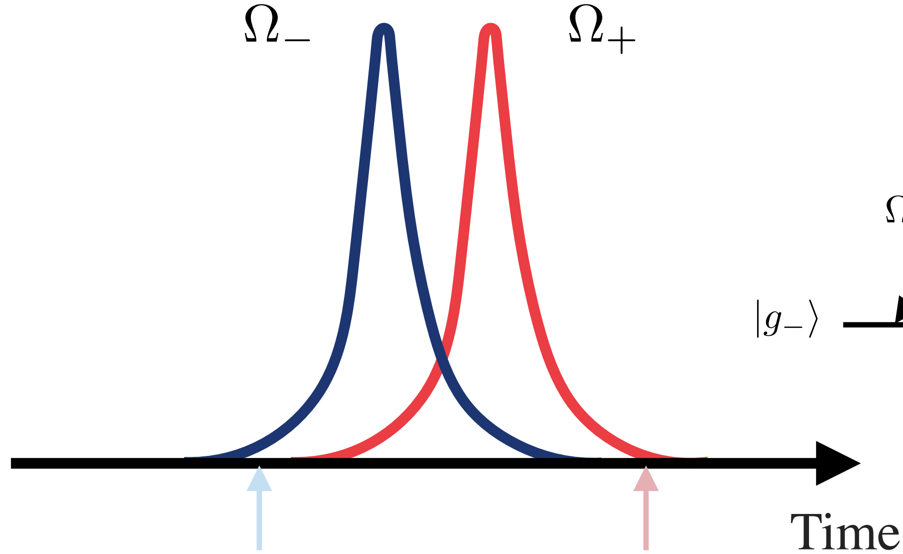
  An atom prepared in this state does not see the light and will never be excited to the *e* state.

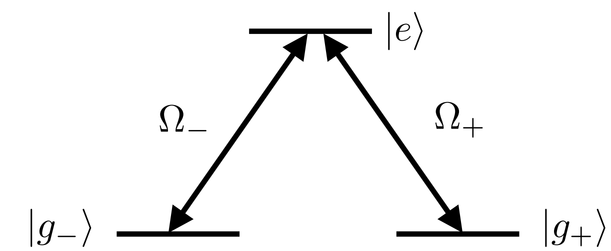
  Since the state has no admixture of the excited state, it is completely *immune* to decay of the excited state!

3.1 - STIRAP

$$\hat{H}_\Lambda = -\Delta \left|e\right\rangle\left\langle e\right| - \frac{1}{2}\left(\Omega_- \left|e\right\rangle\left\langle g_-\right| + \text{h.c.}\right) - \frac{1}{2}\left(\Omega_+ \left|e\right\rangle\left\langle g_+\right| + \text{h.c.}\right) = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_- & 0 \\ -\frac{1}{2}\Omega_- & -\Delta & -\frac{1}{2}\Omega_+ \\ 0 & -\frac{1}{2}\Omega_+ & 0 \end{pmatrix}$$

$$\left|0\right\rangle = \frac{1}{\Omega_r}\left[\Omega_+ \left|g_-\right\rangle - \Omega_- \left|g_+\right\rangle\right] \qquad E_0 = 0$$





- If we review the pulse sequence scheme:

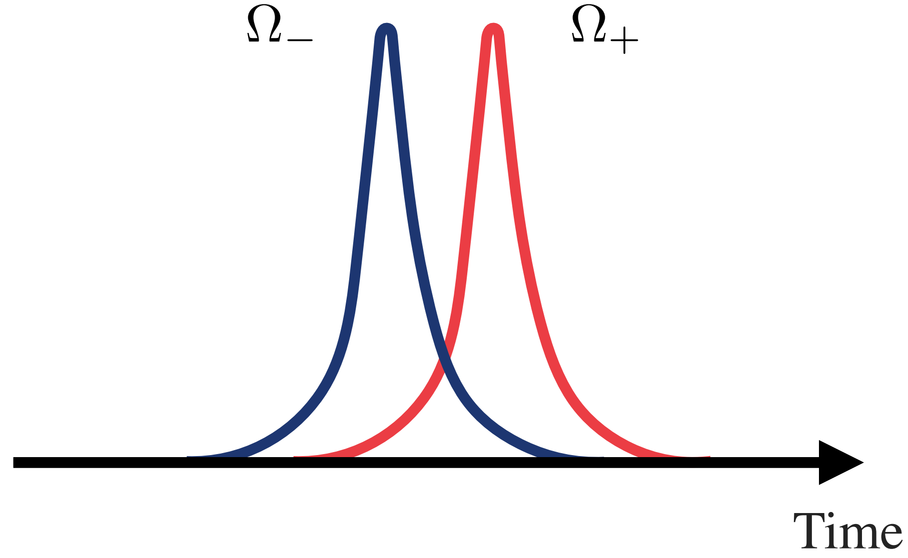
- In the beginning:

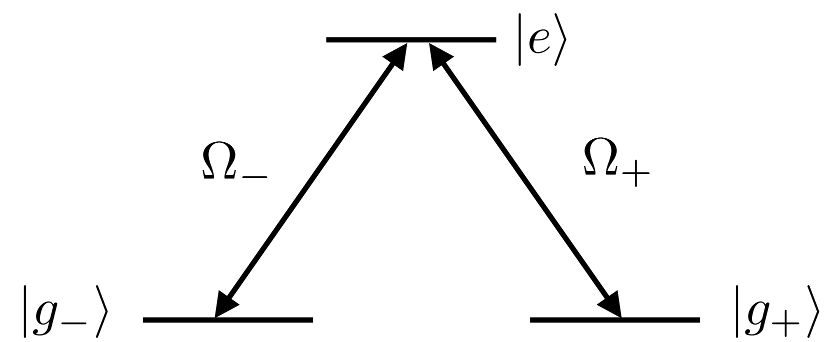
$$\Omega_- = 0 \qquad \Omega_+ > 0 \qquad \Omega_-/\Omega_+ = 0 \quad \longrightarrow \quad \left|0\right\rangle = \left|g_-\right\rangle = \left|\psi_0\right\rangle$$

$$\Omega_- > 0 \qquad \Omega_+ = 0 \qquad \Omega_+/\Omega_- = 0 \quad \longrightarrow \quad \left|0\right\rangle = \left|g_+\right\rangle = \left|\psi_0\right\rangle$$

- The **dark state** is adiabatically connected to the two ground states! The adiabatic sweep goes through the **dark state**. Since this state is immune to decay, the scheme is very robust.

3.1 - STIRAP



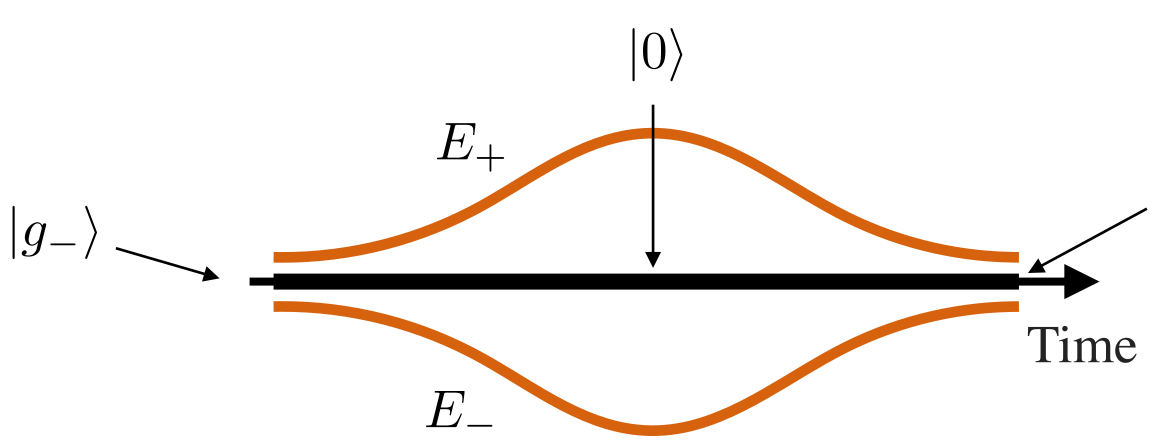


- The **dark state** is adiabatically connected to the two ground states! The adiabatic sweep goes through the **dark state**. Since this state is immune to decay, the scheme is very robust.

$$|\pm\rangle = \frac{1}{\sqrt{E_\pm^2 + \Omega_r^2}} \left[\Omega_+ |g_-\rangle + E_\pm |e\rangle + \Omega_- |g_+\rangle\right] \qquad E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + \Omega_r^2}$$

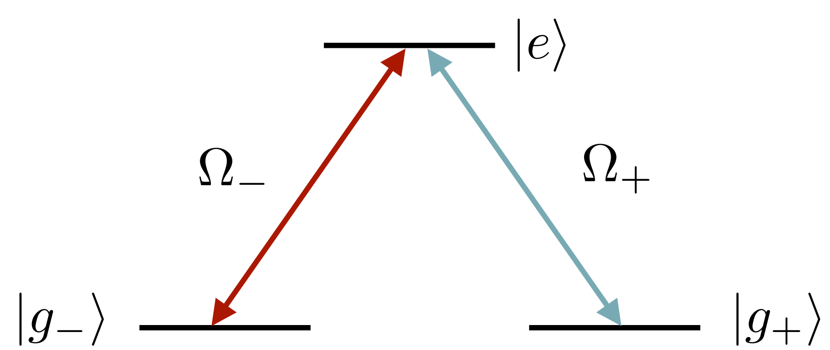
$$|0\rangle = \frac{1}{\Omega_r} \left[\Omega_+ |g_-\rangle - \Omega_- |g_+\rangle\right] \qquad E_0 = 0$$

- Adiabatic energy levels:



This time - Outline

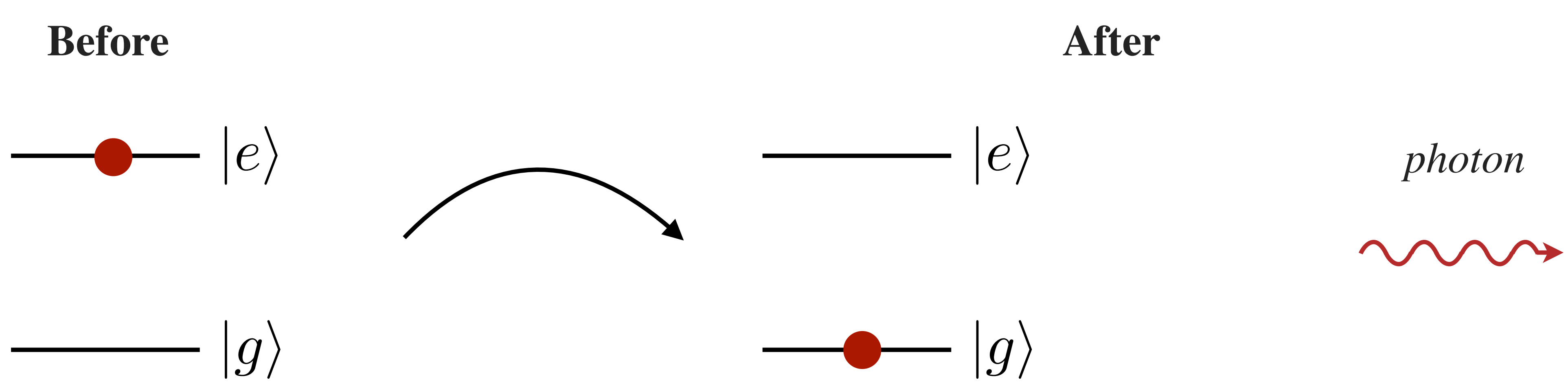
- First we will review the calculations for the three level system that we started last time. We will discuss the **STIRAP protocol**, a scheme to transfer population from one ground-state to another one.



- In order to treat the decay of an atom in an excited state properly, we will need to treat the environment, i.e. the electromagnetic (EM) field fully quantum-mechanically. We will introduce the **quantization of the EM field**.

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

- Lastly we will (maybe) look at decay and compute the decay-rate of an excited state with **Fermi's Golden Rule**



4.1 - Basics of atomic decay

- **Reminder:** Let's go back to the very beginning

- Atom-Field Interaction Hamiltonian (in the "dipole approximation"):

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

Electric field **operator**

Free atom Hamiltonian

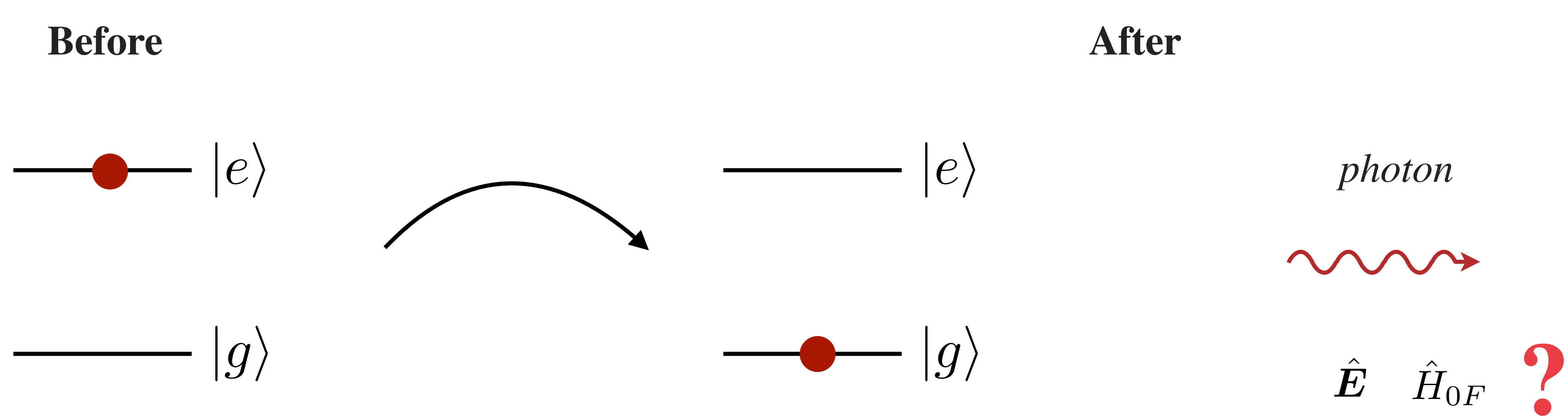
Free field **Hamiltonian**

Dipole operator

$\hat{\boldsymbol{\mu}} = -e\,\hat{\boldsymbol{r}}$

(bold: vectors)

In the full quantum problem, the surrounding free "field" is described by an **Hamiltonian** and the electric field couples to the atom via an **operator**. For a decaying atom, energy will be transferred from the atom to the field. Therefore, to treat decay properly, we **first have to understand how to quantum-mechanically describe the electro-magnetic environment.**



4.1 - Reminder slide: Classical electrodynamics

- **Reminder:** In the absence of sources, the electromagnetic fields are described by Maxwell's equations

$$\boxed{\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 \qquad & \nabla \times \boldsymbol{E} &= -\frac{\partial}{\partial t}\boldsymbol{B} \\ \nabla \cdot \boldsymbol{B} &= 0 \qquad & \nabla \times \boldsymbol{B} &= \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E} \end{aligned}}$$

(SI units) $\qquad \frac{1}{c^2} = \epsilon_0 \mu_0$

… and both can be written in terms of a scalar and a vector potential: $\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A}$

… and we can choose a "Coulomb gauge" for the fields are only determined by the vector potential.

$$\nabla \cdot \boldsymbol{A} \equiv 0 \qquad \phi = 0$$

- Then, the vector potential follows a wave equation:

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E} \quad \longrightarrow \quad \nabla \times \nabla \times \boldsymbol{A} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\boldsymbol{A}\right) \quad \longrightarrow \quad \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}$$

$$\longrightarrow \quad \boxed{\nabla^2 \boldsymbol{A} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}}$$

$$\boxed{\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}}$$

- Energy of the field:

$$H = \frac{1}{2}\int d^3r \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{\epsilon_0}{2}\int d^3r \,\left(E^2 + c^2 B^2\right)$$

4.1 - Reminder slide: Classical electrodynamics — EM waves

$$\boxed{\nabla^2 \boldsymbol{A} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}}$$

$$\boxed{\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}}$$

- A solution of this equation is given by plane waves: $\boldsymbol{A}_{\boldsymbol{k}} = \left(\boldsymbol{\epsilon}\alpha \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \text{c.c.}\right)$

… since

$$\nabla^2 \boldsymbol{A} = -k^2 \boldsymbol{A}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\frac{\omega^2}{c^2}\boldsymbol{A}$$

$$\boxed{\omega = \omega(k) = kc}$$

$$\boldsymbol{E}_{\boldsymbol{k}} = \mathrm{i}\omega\left(\boldsymbol{\epsilon}\alpha \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \text{c.c.}\right)$$

$$\boldsymbol{B}_{\boldsymbol{k}} = \mathrm{i}k\left(\boldsymbol{\epsilon}_\perp\alpha \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \text{c.c.}\right)$$

- What is the mean energy if we assume our field to be in this solution? $H = \frac{\epsilon_0}{2}\int d^3r\ \left(E^2 + c^2B^2\right)$

$$E^2 = \boldsymbol{E}_{\boldsymbol{k}}\cdot\boldsymbol{E}_{\boldsymbol{k}} = -\omega^2\left(\alpha \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \alpha^*\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right)\left(\alpha \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \alpha^*\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right)$$

$$= -\omega^2\left(\cancel{\alpha^2\mathrm{e}^{2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}} + \cancel{(\alpha^*)^2\mathrm{e}^{-2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}} - \alpha^*\alpha - \alpha\alpha^*\right) = \omega^2\left(\alpha^*\alpha + \alpha\alpha^*\right)$$

When integrating over all space and time these terms average away!
(same calculation for B-field)

- In a large volume V:

$$\boxed{\overline{H} =} \lim_{T\to\infty}\frac{1}{T}\left[\int_0^T dt\,\frac{\epsilon_0}{2}\int d^3r\ \left(E^2 + c^2B^2\right)\right] = V\frac{\epsilon_0}{2}\left[\left(\omega^2 + k^2c^2\right)\left(\alpha^*\alpha + \alpha\alpha^*\right)\right] = \boxed{V\epsilon_0\omega^2(\alpha^*\alpha + \alpha\alpha^*)}$$

4.1 - Classical electrodynamics — EM wave as harmonic oscillator

$$\boxed{\overline{H} = V\epsilon_0\omega^2(\alpha^*\alpha + \alpha\alpha^*)}$$

- Now we define some variables corresponding to the real and imaginary parts of the *alphas*.

$$q \equiv \sqrt{V\epsilon_0}\sqrt{\frac{1}{m}}(\alpha + \alpha^*) \qquad p \equiv -\mathrm{i}\omega\sqrt{V\epsilon_0}\sqrt{m}(\alpha - \alpha^*)$$

- Then, we find that

$$H_{\mathrm{HO}} = \boxed{\frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2} = -(V\epsilon_0)\frac{\omega^2}{2}(\alpha - \alpha^*)^2 + (V\epsilon_0)\frac{\omega^2}{2}(\alpha + \alpha^*)^2 = V\epsilon_0\omega^2(\alpha^*\alpha + \alpha\alpha^*) \boxed{= \overline{H}}$$

This single solution (mode) of the Maxwell equations is a **harmonic oscillator**!

- Therefore, we can now immediately understand the quantum version of this harmonic oscillator by imposing commutation relations (turning the variables into operators):

$$\boxed{q \to \hat{q} \qquad p \to \hat{p} \qquad [\hat{q}, \hat{p}] = \mathrm{i}\hbar}$$

This is called **canonical quantization** of the field.

4.1 - Canonical quantization of the EM field

$$\boxed{\overline{H} = V\epsilon_0\omega^2(\alpha^*\alpha + \alpha\alpha^*) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2} \qquad q \equiv \sqrt{V\epsilon_0}\sqrt{\frac{1}{m}}(\alpha + \alpha^*) \qquad p \equiv -\mathrm{i}\omega\sqrt{V\epsilon_0}\sqrt{m}(\alpha - \alpha^*)$$

$$q \to \hat{q} \qquad p \to \hat{p} \qquad [\hat{q}, \hat{p}] = \mathrm{i}\hbar$$

- **Remember** the problem of a simple quantum harmonic oscillator … it is solved with the transformation

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \qquad \hat{p} = -\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger) \qquad \ldots \textit{with} \qquad [\hat{a}, \hat{a}^\dagger] = 1$$

- We see that the *alphas* are the variables that now correspond to the harmonic oscillator raising and lowering operators.

- To quantize the harmonic oscillator we therefore can replace $\alpha \to \hat{a}\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}} \qquad \alpha^* \to \hat{a}^\dagger\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}}$

- Then

$$\boxed{q \to} \sqrt{V\epsilon_0}\sqrt{\frac{1}{m}}\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}}(\hat{a} + \hat{a}^\dagger) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\boxed{= \hat{q}}$$

$$\boxed{p \to} -\mathrm{i}\omega\sqrt{V\epsilon_0}\sqrt{m}\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}}(\hat{a} - \hat{a}^\dagger) = -\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)\boxed{= \hat{p}}$$

4.1 - Vacuum Hamiltonian and electric field operator

- **Canonical quantization:** $$\boxed{\alpha \to \hat{a}\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}} \qquad \alpha^* \to \hat{a}^\dagger\sqrt{\frac{\hbar}{2\omega(V\epsilon_0)}}}$$

- With this we can now immediately write down the Hamiltonian for the mode

$$\overline{H} = (V\epsilon_0)\omega^2(\alpha^*\alpha + \alpha\alpha^*) \to \boxed{\hat{H}_{0F}^{(k)}} = (V\epsilon_0)\omega^2 \frac{\hbar}{2\omega(V\epsilon_0)} \left(\hat{a}^\dagger a + \hat{a}\hat{a}^\dagger\right) \boxed{= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}$$

$$[\hat{a}, \hat{a}^\dagger] = 1 \qquad \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

- Now, the harmonic oscillator ladder operators create and destroy “photons” in the mode. The energy of a single “quantum” = photon is $\hbar\omega$.
- Also the electric field is now written as operator:

$$\boldsymbol{E}_{\boldsymbol{k}} = \mathrm{i}\omega\left(\boldsymbol{\epsilon}\alpha\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \mathrm{c.c.}\right) \longrightarrow \boxed{\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}}\left(\boldsymbol{\epsilon}_\lambda\hat{a}_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \mathrm{h.c.}\right)}$$

- We now see that the electric field operator for the mode is an operator is an operator that destroys and creates a photon of a mode.
- In the classical field limit (large photon occupation), the ladder operator becomes a number: $\hat{a} \to \mathcal{E}$

4.1 - Summary slide: Quantized electric field

The solution of the Maxwell equations are plane waves (modes), and that the dynamics of each plane wave is that of a classical harmonic oscillator. We have quantized this harmonic oscillator, and can now write our operators for the EM field in terms of photon-creation (and destruction) operators. This works for each mode of free space.

- **In total we therefore have:**

$$\hat{H}_{0F} = \sum_{k,\lambda} \hbar\omega_k \left(\hat{a}^\dagger_{\boldsymbol{k},\lambda} \hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2} \right)$$

$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}} \left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \text{h.c.} \right)$$

… which follows from the canonical quantization:

$$\alpha_{\boldsymbol{k},\lambda} \to \hat{a}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar}{2\omega_k (V\epsilon_0)}} \qquad \alpha^*_{\boldsymbol{k},\lambda} \to \hat{a}^\dagger_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar}{2\omega_k (V\epsilon_0)}} \qquad [\hat{a}_{\boldsymbol{k}\lambda}, \hat{a}^\dagger_{\boldsymbol{k}'\lambda'}] = \delta_{k_x,k'_x}\delta_{k_y,k'_y}\delta_{k_z,k'_z}\delta_{\lambda,\lambda'}$$

- **Remark:** There is an infinite number of modes in free space. Note that we have defined our problem on a finite Volume V. This volume is called the *"quantization volume"*. In the end we can take the limit of infinite volume.
- **Remark:** The mode index is 3D, and furthermore we introduced a discrete index "lambda" for 3 orthogonal polarizations.
- **Remark:** The fact that photon operators for different modes commute with each other follows trivially from the fact that the different oscillators for different modes are not coupled with each other. This is a general property of the linear wave equations. In particular it's easy to show that the energy of two modes:

$$H = \frac{\epsilon_0}{2} \int d^3r \left[(\boldsymbol{E_1} + \boldsymbol{E_2})^2 + c^2 (\boldsymbol{B}_1 + \boldsymbol{B}_2)^2 \right] = \cdots = H_1 + H_2$$

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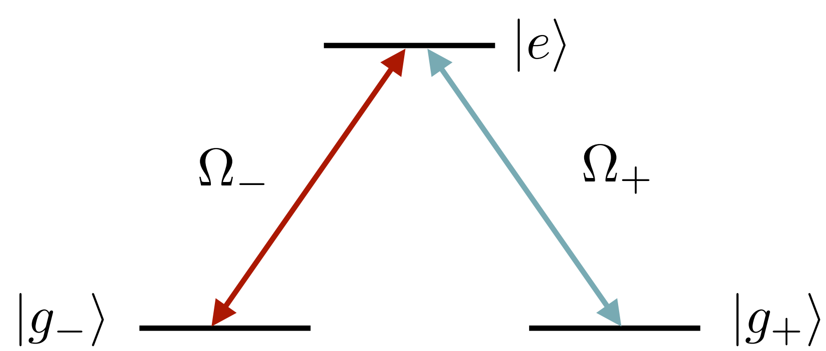

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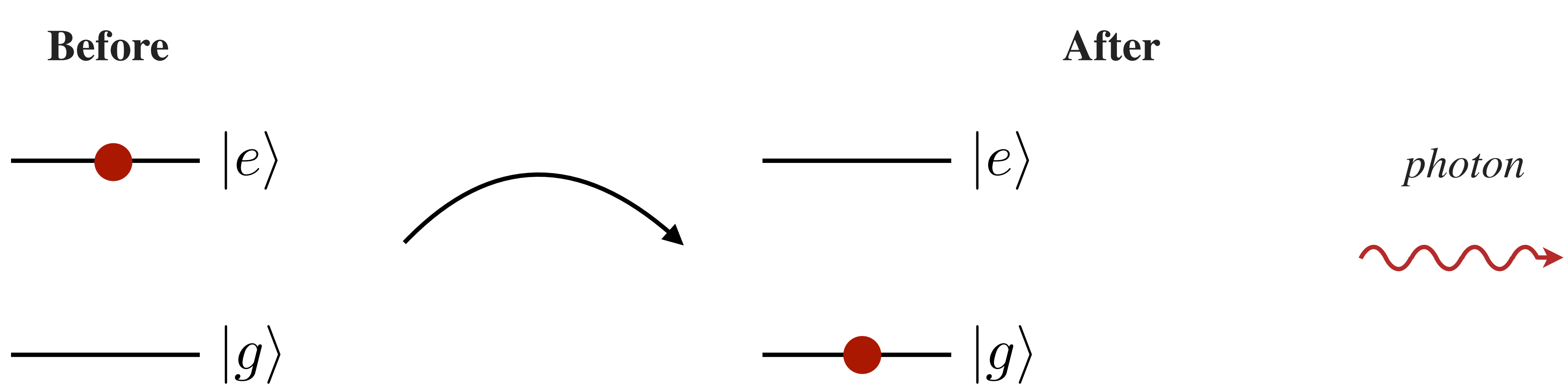


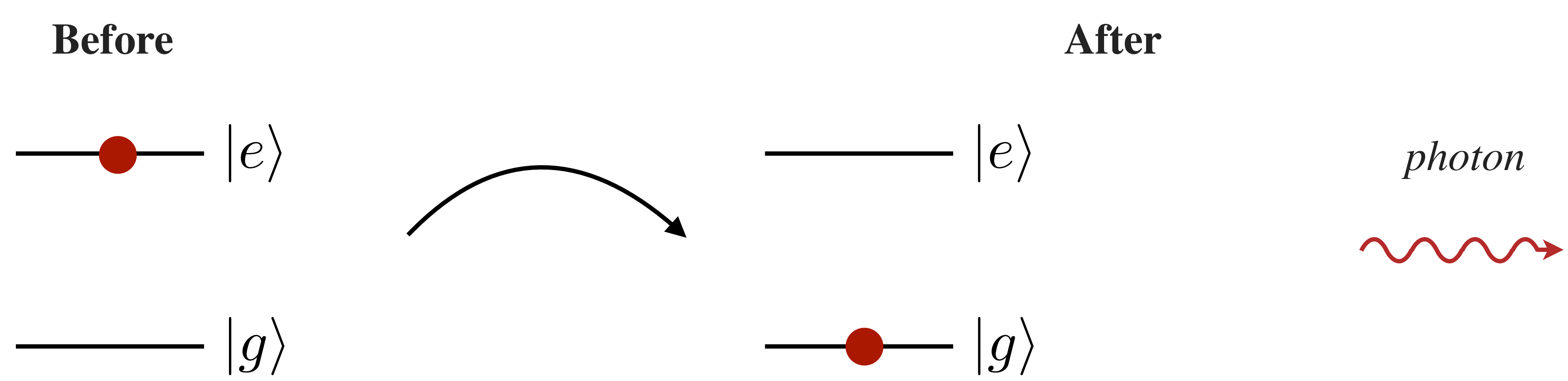
4.2 - Atomic decay with Fermi's Golden Rule

- We have the full Hamiltonian problem:

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

… as a warm-up we will use it to compute an atom decay rate using Fermi's Golden Rule.

- **The problem:** An atom is prepared in an excited atomic state. Our goal is to describe a spontaneous transition to the ground-state



- **Fermi's Golden Rule:**

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

Free Hamiltonian

Perturbation, coupling between a system and a bath

$|i\rangle$ *initial state*

$|f\rangle$ *final states*

$$\Gamma_{f \leftarrow i} = \frac{2\pi}{\hbar} \sum_f \delta(E_i - E_f) |\langle f | \hat{H}_1 | i \rangle|^2$$

Fermi's golden rule transition rates

4.2 - Atomic decay with Fermi's Golden Rule

$$\hat{H}_{AF} = \hat{H}_{0A} + \hat{H}_{0F} - \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\Gamma_{f \leftarrow i} = \frac{2\pi}{\hbar} \sum_f \delta(E_i - E_f) |\langle f| \hat{H}_1 |i\rangle|^2$$

Fermi's golden rule transition rates

- We can now identify

$$\hat{H}_0 = \hat{H}_{0A} + \hat{H}_{0F} \qquad \hat{H}_1 = -\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}$$

Before

$$|i\rangle = |e\rangle \otimes |\mathrm{vac}\rangle$$

No photon

After

$$|f\rangle = |g\rangle \otimes |1_{\boldsymbol{k},\lambda}\rangle = |g\rangle \otimes \hat{a}^\dagger_{\boldsymbol{k},\lambda} |\mathrm{vac}\rangle$$

one photon with momentum k and polarization lambda

- So we need to compute $\left| \sum_{\boldsymbol{k}',\lambda} \langle f| \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda} |i\rangle \right|^2 = \ldots$

… using

$$\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}} \left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k}',\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}} - \mathrm{h.c.}\right)$$

(Schrödinger picture)

$$\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_{ge}(|e\rangle\langle g| + |g\rangle\langle e|)$$

… and then we will have to sum over a continuum of *f* modes with different energies and polarizations.

4.2 - Atomic decay with Fermi's Golden Rule

$$|i\rangle = |e\rangle \otimes |\text{vac}\rangle \qquad |f\rangle = |g\rangle \otimes \hat{a}^\dagger_{\boldsymbol{k},\lambda} |\text{vac}\rangle$$

$$\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_{ge}(|e\rangle\langle g| + |g\rangle\langle e|)$$

$$\Gamma_{f\leftarrow i} = \frac{2\pi}{\hbar}\sum_f \delta(E_i - E_f)|\langle f|\hat{H}_1|i\rangle|^2$$

Fermi's golden rule transition rates

- We see: $\langle f|\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda}|i\rangle = \langle g|\otimes\langle\text{vac}|\hat{a}_{\boldsymbol{k},\lambda}\left(\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda}\right)|e\rangle\otimes|\text{vac}\rangle$

$$= \boldsymbol{\mu}_{ge}\cdot\langle e|\otimes\langle\text{vac}|\hat{a}_{\boldsymbol{k},\lambda}\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda}|e\rangle\otimes|\text{vac}\rangle = \boldsymbol{\mu}_{ge}\cdot\langle\text{vac}|\hat{a}_{\boldsymbol{k},\lambda}\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda}|\text{vac}\rangle$$

$$\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}}\left(\boldsymbol{\epsilon}_\lambda\hat{a}_{\boldsymbol{k}',\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}} - \text{h.c.}\right)$$

- Using: $\hat{a}_{\boldsymbol{k}'\lambda}|\text{vac}\rangle = 0$

$$\langle f|\hat{H}_1|i\rangle = \sum_{\boldsymbol{k}',\lambda}\langle f|\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{E}}_{\boldsymbol{k}',\lambda}|i\rangle = \boldsymbol{\mu}_{ge}\cdot\langle\text{vac}|\hat{a}_{\boldsymbol{k},\lambda}\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda}|\text{vac}\rangle$$

$$\langle f|\hat{H}_1|i\rangle = -\mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}}(\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}^*_\lambda)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\langle\text{vac}|\hat{a}_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}|\text{vac}\rangle = -\mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}}(\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}^*_\lambda)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$|\langle f|\hat{H}_1|i\rangle|^2 = \frac{\hbar\omega_k}{2(V\epsilon_0)}|(\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda)|^2$$

Geometric part, depending on dipole moment direction and light polarization

Photon energy dependent part

(note: we assume a real dipole moment)

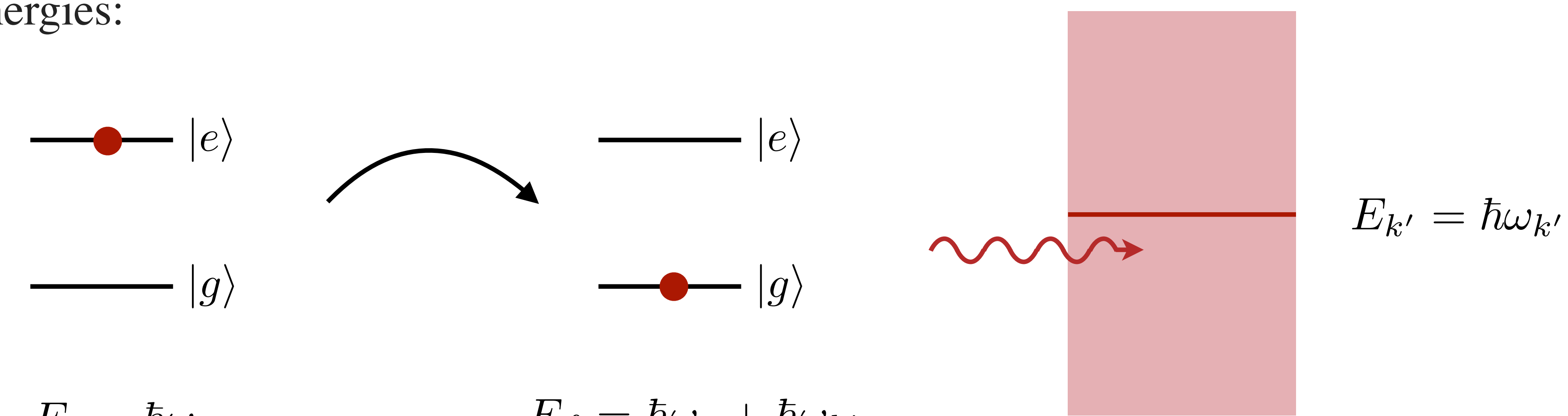
4.2 - Atomic decay with Fermi's Golden Rule

$$|\langle f|\,\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda}\,|i\rangle|^2 = \frac{\hbar\omega_k}{2(V\epsilon_0)}|(\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda)|^2$$

$$\Gamma_{f\leftarrow i} = \frac{2\pi}{\hbar}\sum_f \delta(E_i - E_f)|\langle f|\,\hat{H}_1\,|i\rangle|^2$$

Fermi's golden rule transition rates

- Now we have to consider the whole continuum of f modes with energies:



$E_i = \hbar\omega_e$ $\qquad$ $E_f = \hbar\omega_g + \hbar\omega_{k'}$

- **Remark:** Just as for the standard quantum mechanical oscillator, technically there is a "ground-state energy", this means that also for the vacuum of the EM field there is an (infinite) ground-state energy (which we subtracted here).

- The delta function becomes $\qquad \delta(E_i - E_f) \to \frac{1}{\hbar}\delta(\omega_{eg} - \omega_{k'}) \qquad \omega_{eg} = \omega_e - \omega_g$

- The decay-rate is thus

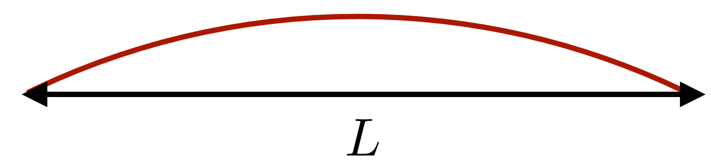
$$\Gamma_{g\leftarrow e} = \frac{2\pi}{\hbar}\sum_{\boldsymbol{k},\lambda}\delta(\omega_{eg} - \omega_k)\frac{\omega_k}{2(V\epsilon_0)}|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2$$

4.2 - Atomic decay with Fermi's Golden Rule

$$\Gamma_{g\leftarrow e} = \frac{2\pi}{\hbar}\sum_{\boldsymbol{k},\lambda}\delta(\omega_{eg}-\omega_k)\frac{\omega_k}{2(V\epsilon_0)}|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2$$

- To do the summation we can turn the sum into an integral:

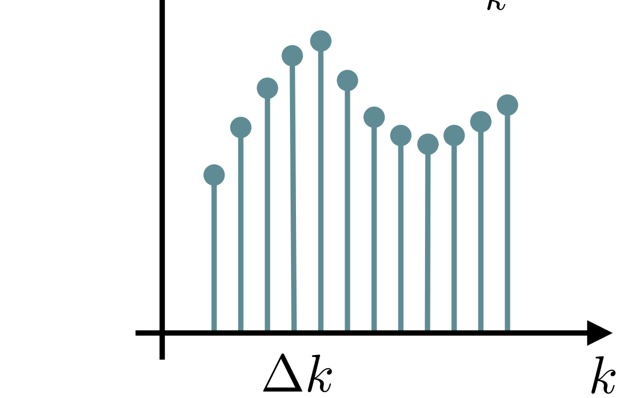
… assume modes in a (periodic) box of Volume $V = L^3$



$$\boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

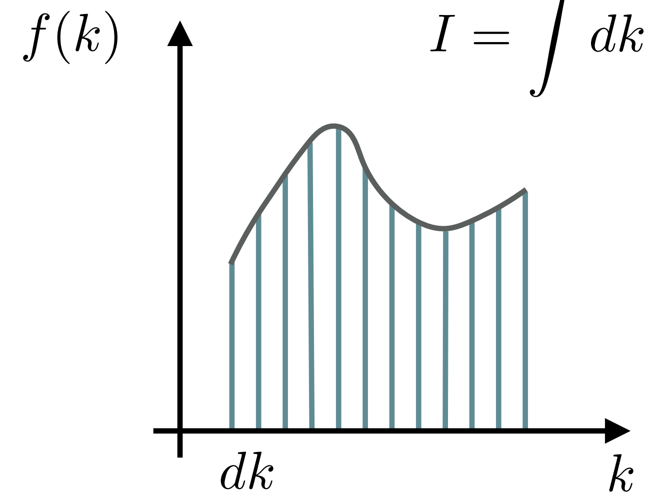
$$n_x = 0, 1, 2, \ldots$$

$$S = \sum_k f(k)$$



$$S = \lim_{\Delta k\to 0}\frac{1}{\Delta k} I$$

$$I = \int dk\, f(k)$$



- In our case $\Delta k = \dfrac{2\pi}{L}$

$$\sum_{\boldsymbol{k}} \ldots = \lim_{L\to\infty}\frac{L^3}{(2\pi)^3}\int d^3k \ldots$$

The quantization volume drops out

$$\Gamma_{g\leftarrow e} = \frac{2\pi}{\hbar}\frac{1}{2(2\pi)^3\epsilon_0}\sum_{\lambda}\int d^3k\, \delta(\omega_{eg}-\omega_k)\omega_k|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2$$

4.2 - Atomic decay with Fermi's Golden Rule

$$\boxed{\Gamma_{g\leftarrow e} = \frac{2\pi}{\hbar}\frac{1}{2(2\pi)^3\epsilon_0}\sum_{\lambda}\int d^3k\,\delta(\omega_{eg}-\omega_k)\omega_k|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2}$$

- Using spherical coordinates $\int d^3k \cdots = \int_0^\infty dk\,k^2 \int d\Omega \cdots = \int_0^\infty dk\,k^2 \int_0^\pi d\theta\,\sin(\theta) \int_0^{2\pi} d\phi \ldots$

- And the dispersion relation: $k = \frac{\omega_k}{c} \quad dk = \frac{1}{c}d\omega_k$

$$\Gamma_{g\leftarrow e} = \frac{2\pi}{\hbar}\frac{1}{2(2\pi c)^3\epsilon_0}\int d\omega_k\,\delta(\omega_{eg}-\omega_k)\omega_k^3\sum_{\lambda}\int d\Omega|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2 = \frac{2\pi}{\hbar}\frac{\omega_{eg}^3}{2(2\pi c)^3\epsilon_0}\sum_{\lambda}\int d\Omega|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2$$

- **In total:** Decay rate with Fermi's Golden Rule:

$$\boxed{\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3}{8\pi^2c^3\hbar\epsilon_0}\sum_{\lambda}\int d\Omega|\boldsymbol{\mu}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2}$$

4.2 - Atomic decay with Fermi's Golden Rule

$$\boxed{\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3}{8\pi^2 c^3 \hbar \epsilon_0} \sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2}$$

- **Discussion:** Importantly, the decay rate scales with $\boxed{\sim \omega_{eg}^3 \quad \sim |\boldsymbol{\mu}_{eg}|^2}$
- The geometric term $\sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2$

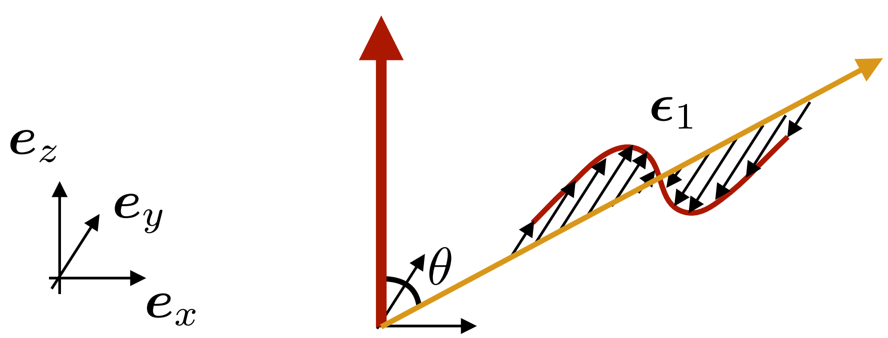
  … contains an integration over possible emissions of the photons in all directions and over all possible polarizations

- If the dipole moment is aligned with the polarization $\quad \boldsymbol{\mu}_{ge} \parallel \boldsymbol{\epsilon} \rightarrow \boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon} = 1$

  … if not $\quad \boldsymbol{\mu}_{ge} \perp \boldsymbol{\epsilon} \rightarrow \boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon} = 0$

  Emission only with polarization along the dipole

- **Choose:** $\boldsymbol{\mu}_{ge} \parallel \boldsymbol{e}_z$



$\hat{\boldsymbol{n}}$ *"Mode propagation direction"*

… and choose orthogonal polarization axes $\quad \boldsymbol{\epsilon}_1 = \boldsymbol{e}_y$, $\quad \boldsymbol{\epsilon}_2 = \hat{\boldsymbol{n}} \times \boldsymbol{\epsilon}_1$

$$\boldsymbol{\mu}_{eg} \cdot \boldsymbol{\epsilon}_1 = |\boldsymbol{\mu}_{eg}| \boldsymbol{e}_z \cdot \boldsymbol{e}_y = 0 \qquad \boldsymbol{\mu}_{eg} \cdot \boldsymbol{\epsilon}_2 = |\boldsymbol{\mu}_{eg}| \boldsymbol{e}_z \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{e}_y) = |\boldsymbol{\mu}_{eg}| \hat{\boldsymbol{n}} \cdot (\boldsymbol{e}_y \times \boldsymbol{e}_z) = |\boldsymbol{\mu}_{eg}| \hat{\boldsymbol{n}} \cdot \boldsymbol{e}_x = |\boldsymbol{\mu}_{eg}| \sin(\theta)$$

$$\boxed{\sum_\lambda |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2 = |\boldsymbol{\mu}_{eg}|^2 \sin^2(\theta)}$$

… note that this is a usual dipole pattern

4.2 - Atomic decay with Fermi's Golden Rule

$$\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3}{8\pi^2 c^3 \hbar \epsilon_0} \sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2$$

- **Discussion:** Importantly, the decay rate scales with $\sim \omega_{eg}^3 \quad \sim |\boldsymbol{\mu}_{eg}|^2$

- The geometric term $\sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2$

$$\sum_\lambda |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2 = |\boldsymbol{\mu}_{eg}|^2 \sin^2(\theta)$$

… can now be integrated over the solid angle:

$$\int d\Omega \cdots = \int_0^\pi d\theta \, \sin(\theta) \int_0^{2\pi} d\phi \, \ldots$$

$$\sum_\lambda \int d\Omega |\boldsymbol{\mu}_{ge} \cdot \boldsymbol{\epsilon}_\lambda|^2 = |\boldsymbol{\mu}_{ge}|^2 \int d\Omega \, \sin^2(\theta) = |\boldsymbol{\mu}_{ge}|^2 2\pi \int_0^\pi d\theta \, \sin^3(\theta) = |\boldsymbol{\mu}_{ge}|^2 2\pi \frac{4}{3}$$

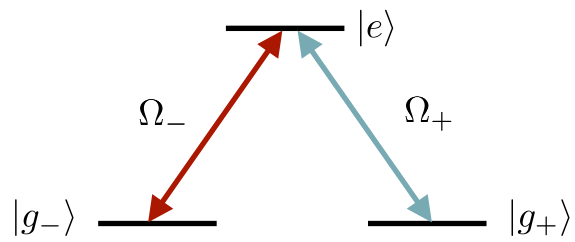
- **And overall:** For an atom with two levels *e* and *g* the decay rate from is given by

$$\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3 |\boldsymbol{\mu}_{ge}|^2}{3\pi c^3 \hbar \epsilon_0}$$

- **Remark:** This is just simple decay. Next, we will look at how we can described the full dynamics of a decaying atom.

Recap

- We looked at a "Lambda" three level system:



- We have introduced the approximation of an adiabatic **elimination** of one of the states, reducing the problem to an effective two-level system.

- We introduced the concept of **adiabatic population transfer**, using e.g. a delayed laser-pulse sequence. This scheme is called **STIRAP** and is a very popular scheme in atomic physics and beyond. It works very well, since the system adiabatically remains in a **dark state**, which is immune to decay.

- We have shown that generally a system prepared in an **eigenstate of an Hamiltonian**, **remains in the instantaneous eigenstate** if the **rate of change** in the Hamiltonian is **smaller than the energy-gap** to other eigenstates. This is called the **adiabatic approximation.**

$$E_n(t) - E_k(t) \gg \langle \phi_k(t) | \left[\frac{d}{dt} \hat{H}(t) \right] | \phi_n(t) \rangle$$

- We have started to look at field quantization. We have seen that the solutions of the classical Maxwell equations can be seen viewed as classical harmonic oscillators. Treating these oscillators quantum-mechanically is called **canonical quantization**. We arrived at an operator description of the EM field

$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda} = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2(V\epsilon_0)}} \left(\boldsymbol{\epsilon}_\lambda \hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} - \mathrm{h.c.} \right)$$

$$[\hat{a}_{\boldsymbol{k}\lambda}, \hat{a}^\dagger_{\boldsymbol{k}'\lambda'}] = \delta_{k_x,k'_x}\delta_{k_y,k'_y}\delta_{k_z,k'_z}\delta_{\lambda,\lambda'}$$

- With this full quantum-mechanical description of both the atom and the EM environment, we can now treat atomic decay properly. We have as an example looked at the decay according to Fermi's Golden rule

$$\Gamma_{g\leftarrow e} = \frac{\omega_{eg}^3 |\boldsymbol{\mu}_{ge}|^2}{3\pi c^3 \hbar \epsilon_0}$$