McGill

MATH 578 - NUMERIC ANALYSIS 1

Problem Set III - Convergence Methods

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Numerical Methods

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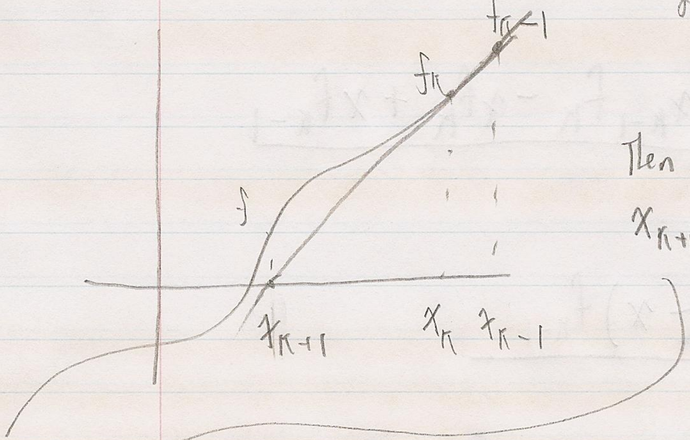


Let $f(x)$ be differential once wrt to x.

Then

$$x_{k+1} = x_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) f_k$$

To find the rate of convergence we assume $f(x)$ has simple roots of multiplicity 1 that exists, points x_{k-1} and x_k are taken so that the secant method converges and $f(x)$ is twice differentiable wrt x.

Let $(0,0)$ be the location of the root in question; $f(x=0)=0$
$e_{k+1} = x_{k+1} - x$ where e_{k+1} is the error

Since $f(x)$ is twice differentiable we write the Taylor series about point x (the root) to x_{k+1}

$$f(x+e_k) = f(x) + e_k \cdot f'(x) + \frac{e_k^2 f''(x)}{2} + O[e_k^3]$$

We do not write more of $O[e_k^3]$ terms as they will not be important when we get down further on the proof for the rate of convergence.

if $e_{k+1} \stackrel{\Delta}{=} x_{k+1} - x$ $\quad$ subing in the definition of x

$$= \left\{x_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) f_k\right\} - x$$

$$= \frac{x_k(f_k - f_{k-1}) - (x_k - x_{k-1}) f_k - x(f_k - f_{k-1})}{f_k - f_{k-1}}$$

$$= \frac{\cancel{x_k f_k} - x_k f_{k-1} - \cancel{x_k f_k} + x_{k-1} f_k - x f_k + x f_{k-1}}{f_k - f_{k-1}}$$

$$= \frac{(x_{k-1} - x) f_k - (x_k - x) f_{k-1}}{f_k - f_{k-1}}$$

We know $e_{k+1} = x_{k+1} - x$ so $e_{k-1} = x_{k-1} - x$
$e_k = x_k - x$

$$= \frac{e_{k-1} f_k - e_k f_{k-1}}{f_k - f_{k-1}}$$

$$f_k = f(x + e_k) = \overset{0}{\cancel{f(x)}} + e_k f'(x) + \frac{e_k^2}{2} f''(x) + O[e_k^3]$$

$$f_{k-1} = f(x + e_{k-1}) = \overset{0}{\cancel{f(x)}} + e_{k-1} f'(x) + \frac{e_{k-1}^3}{2} f''(x) + O[e_{k-1}^3]$$

$$= \frac{e_{k-1}\{e_k f'(x) + \frac{e_k^2}{2} f''(x) + O[e_k^3]\} - e_k\{e_{k-1} f'(x) + \frac{e_{k-1}^2}{2} f''(x) + O[e_{k-1}^3]\}}{e_k f'(x) + \frac{e_k^2}{2} + f''(x) + O[e_k^3] - \{e_{k-1} f'(x) + \frac{e_{k-1}^3}{2} f''(x) + O[e_{k-1}^3]\}}$$

$$= \frac{\cancel{e_{k-1} e_k f'(x)} + \frac{e_{k-1} \cdot e_k^2}{2} f''(x) + e_{k-1} O[e_k^3] - \cancel{e_k \cdot e_{k-1} f'(x)} - \frac{e_k \cdot e_{k-1}^2}{2} f''(x) + e_k \cdot O[e_{k-1}^3]}{f'(x)[e_k - e_{k-1}] + f''(x)\left[\frac{e_k^2}{2} - \frac{e_{k-1}^2}{2}\right] + (e_k - e_{k-1})\left[O[e_k^3 - e_{k-1}^3]\right]}$$

Since $O[e_{k-1}^3] > O[e_k^3]$ if the secant method converges we will rewrite $O[e_{k-1}^3] - O[e_k^3]$ as $O[e_{k-1}^3]$

$$= \frac{\frac{e_k e_{k-1}}{2} f''(x)\left\{ e_k - e_{k-1} \right\} + O[e_{k-1}^4]}{\cancel{[e_k - e_{k-1}]}\left\{ f'(x) + f''(x)\left[\frac{e_k + e_{k-1}}{2}\right] + O[e_{k-1}^3]\right\}}$$

$$e_{k+1} = e_k e_{k-1} \frac{f''(x) \quad + 2O[e_{k-1}^4]}{2f'(x) + 2f''(x)\left[\frac{e_k + e_{k-1}}{2}\right] + O[e_{k-1}^3]}$$

as $k \to \infty$ $e_k + e_{k-1} << f'(x)$ and $O[e_k^4] << O[e_{k-1}^3] << f'(x)$

$$\therefore e_{k+1} = \frac{e_k e_{k-1} f''(x)}{2f'(x)}$$

Let us rewrite as $e_{k+1} = Ae_k^r$ where r is the rate of convergence

$$e_k = Ae_{k-1}^r$$

or $\frac{1}{A}(e_k)^{1/r} = e_{k-1}$

$$Ae_k^r = Ae_k^r \cdot \frac{1}{A}(e_k)^{1/r}$$

$$e_k^r \approx Ce_k^{r+1/r}$$

Only if $r = r + \frac{1}{r}$ does the above equation have the same kind of convergence

$\therefore r^2 = r+1$ or $r^2 - r - 1 = 0$

with quadratic theory $r = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

A negative level of convergence makes no sense as the would get larger and not converge. We take the positive value

$$\boxed{\therefore\ r = \frac{1+\sqrt{5}}{2} \quad \text{for the secant method if it converges} \quad r - \text{rate of convergence}}$$

Problem #2 - Estimate Convergence Rate of Various Methods

For the "real" answer to $x^2 - 4\sin(x) = 0$ we use a binomial search with a tolerance of the min and max values to be 3×10^{-16}. Any number lower than this gives an error as we run into the errors associated with machine zero.

Rate of Convergence		Starting Conditions
Bisection	8.6	x_min = 0.5, x_max 4
Secant	21.8	$x = 5.9$, $x_{i-1} = 6$
Newton's	22.7	$x = 4$
FPI	12	$x = 2$
Aitken's Method	29.3	$x = 2$

All these methods converge much faster than expected. The comparison between them are from fastest to converge to slowest

1. Aitken
2. Newton
3. Secant
4. FPI
5. Bisection

Keep in mind I did not standardize the above equations with respect to each other and their rate of convergence may depend on what their initial values are.

Also, from question 1 the rate of convergence was calculated to be roughly 1.6 while numerically it is 21.8... this is a huge difference.

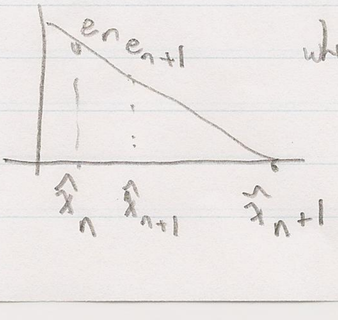
Aitken Method was calculated using the following formula

Knowing

$x_{n+1} = g(x)$

$e_n = g(x_n) - x_n = x_{n+1}$

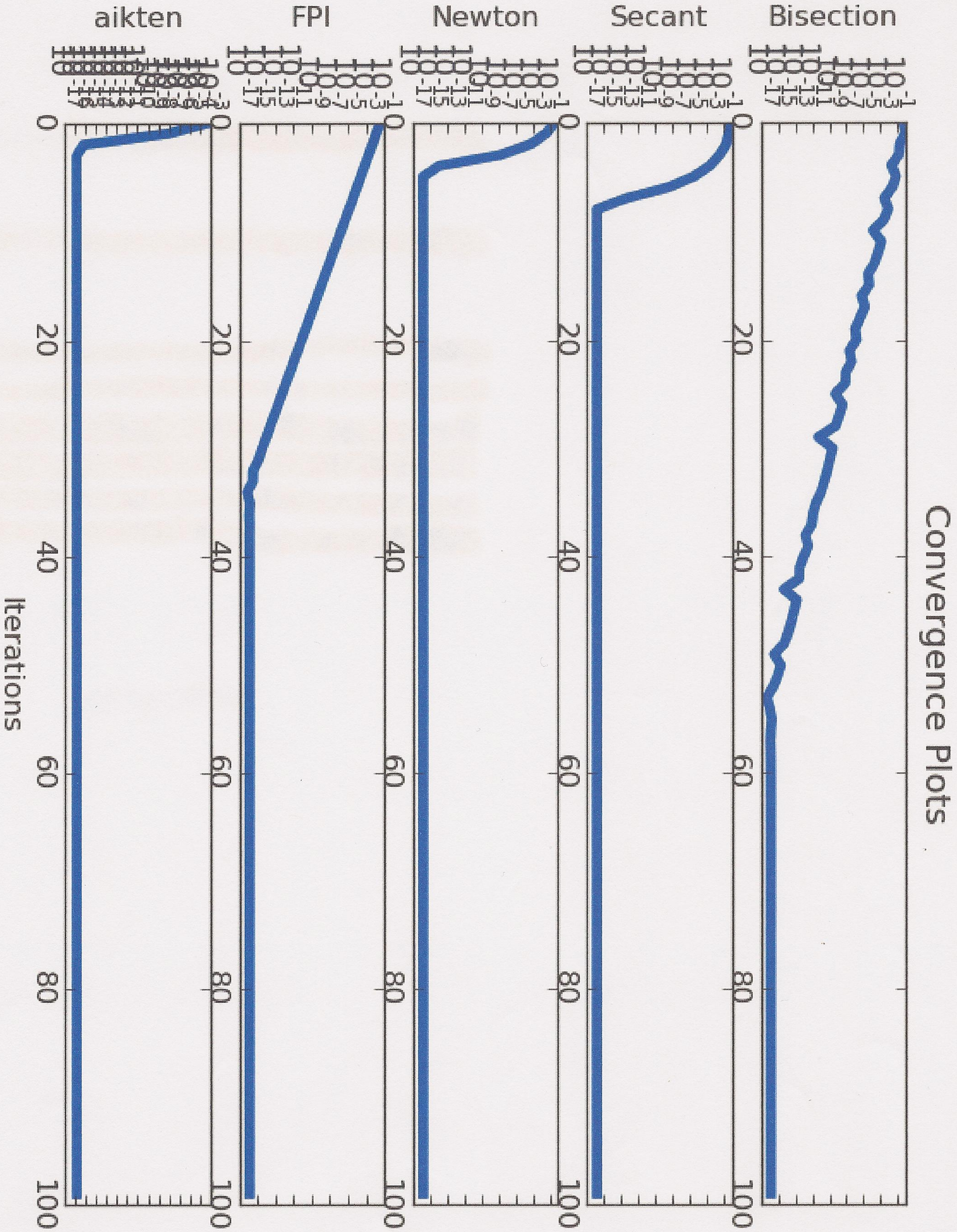
$e_{n+1} = g(x_{n+1}) - x_n$



where $x_{n+1} = x_n - \frac{e_n^2}{e_{n+1} - e_n}$

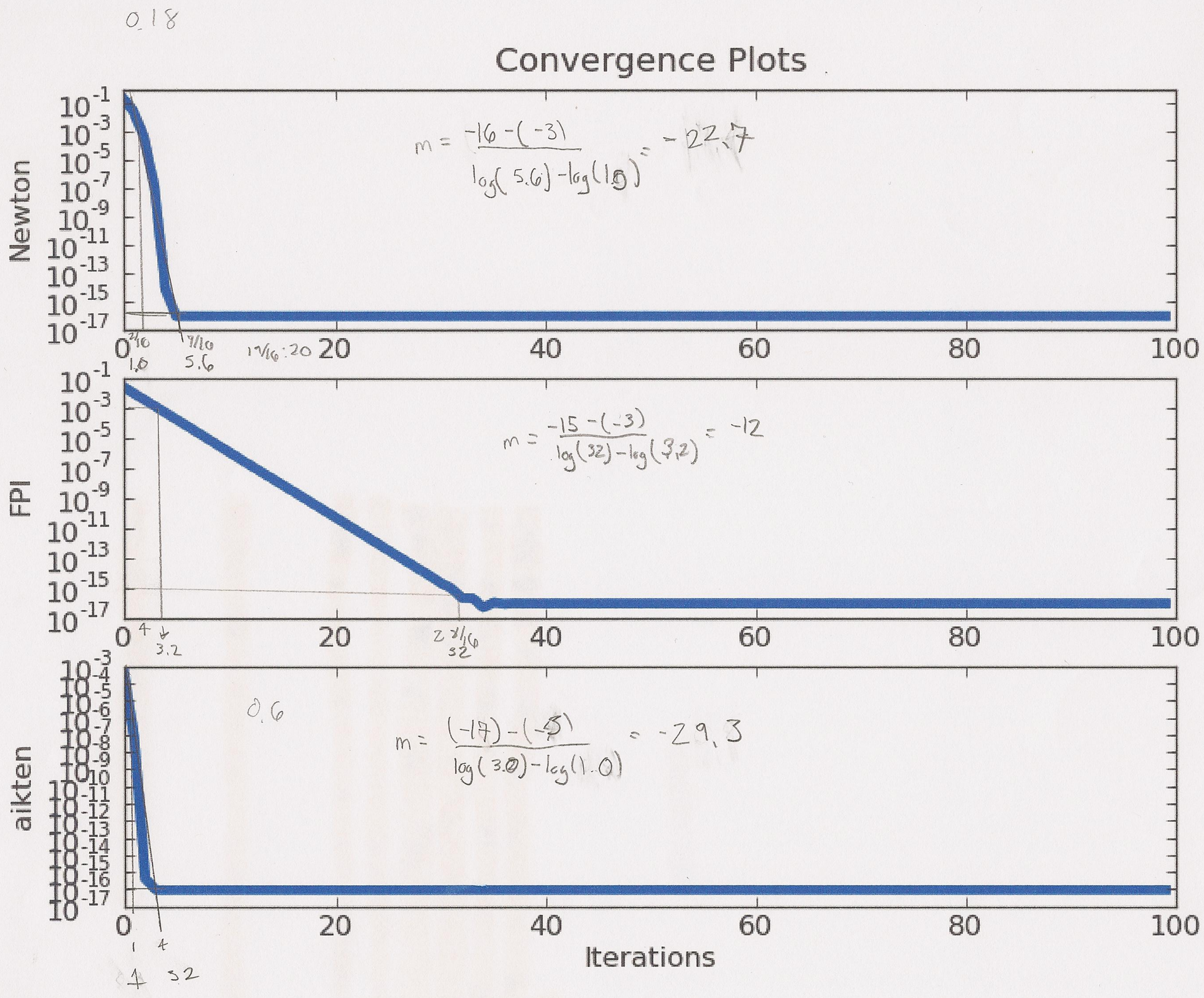
where e_n and e_{n+1} are calculated using the FPI scheme

Problem 2

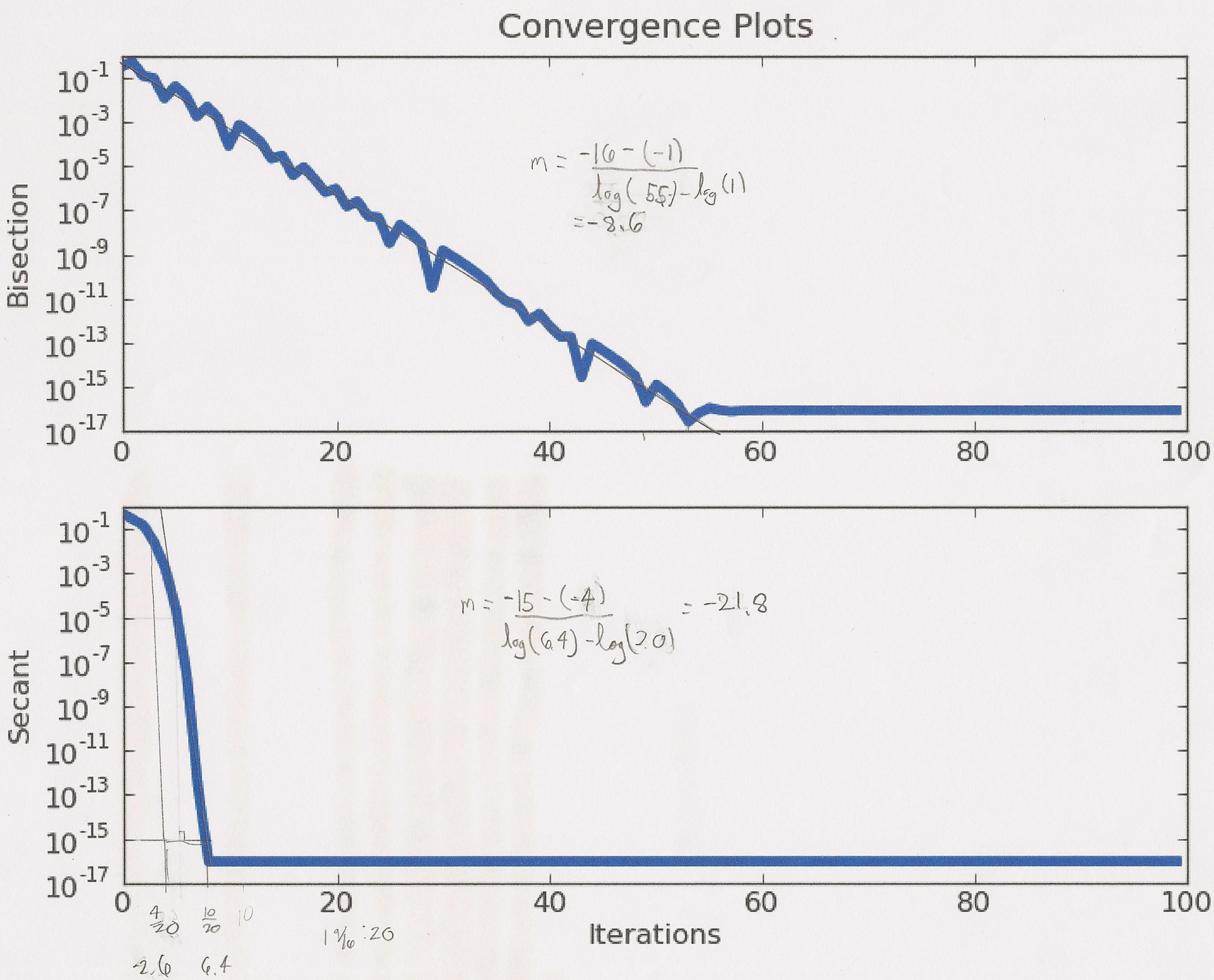


Convergence Plots
Bisection
Secant
Newton
FPI
aikten
Iterations

Problem 2


0.18
Convergence Plots
Newton
FPI
aikten
Iterations
m = (-16 - (-3)) / (log(5.6) - log(1.0)) = -22.7
m = (-15 - (-3)) / (log(32) - log(3.2)) = -12
0.6
m = ((-17) - (-5)) / (log(3.0) - log(1.0)) = -29.3

Problem 2



Convergence Plots
Bisection
Secant
Iterations
10^-1
10^-3
10^-5
10^-7
10^-9
10^-11
10^-13
10^-15
10^-17
0
20
40
60
80
100
m = (-16 - (-1)) / (log(55) - log(1))
= -8.6
m = (-15 - (-4)) / (log(6.4) - log(2.0)) = -21.8
4/20
10/20
10
1 9/10 : 20
-2.6
6.4

Problem #3

We get the correct answer by assuming FPI with 10,000 iterations which when graphed converges to machine 0. We could also solve this analytically by seeing $x_1 = x_2$ as the equations are equal to each other if the x_1 is replaced with x_2

Overall Newton's scheme converges faster than the FPI scheme. The Newton's scheme takes about 5 iterations while FPI takes 35. We may be able to speed up the FPI scheme using Aitken's Method

$$(x_1, x_2) = (0.56714\ldots, 0.56714\ldots)$$

Newton's Method

$$x_{k+1} = x_k + J(x_k)^{-1} \cdot f(x^k)$$

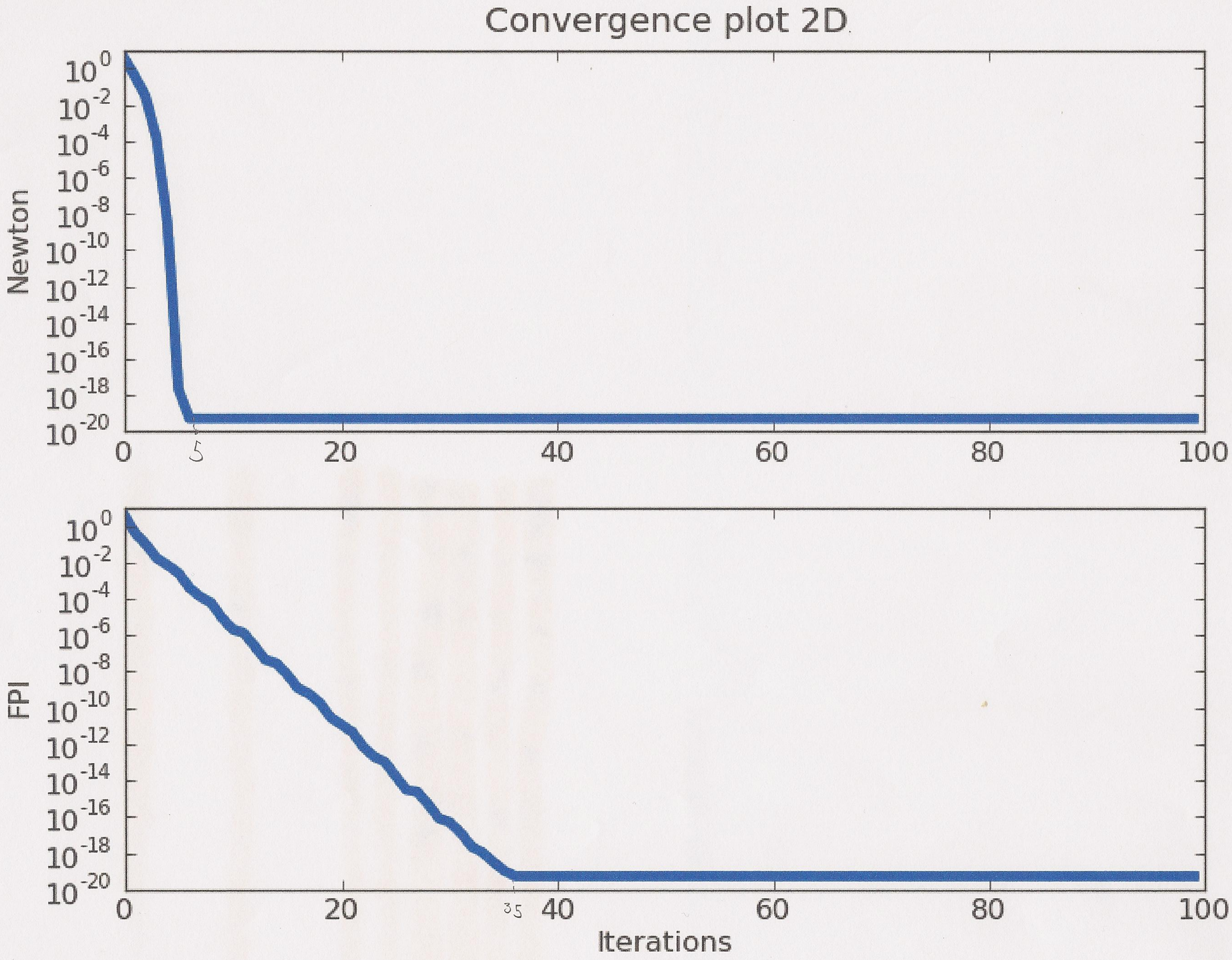
where $J(x_k) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2-e^{x_1} & -1 \\ -1 & 2-e^{-x_2} \end{pmatrix}$

$$x_1^{k+1} = x_1^k + \frac{(2x_1 - x_2 - e^{-x_1})}{(2-e^{x_1})x_1^k - x_2^k}$$

$$x_2^{k+1} = x_2^k + \frac{(-x_1 + 2x_2 - e^{-x_2})}{-x_1^k + (2-e^{-x_2})x_2^k}$$

FPI

$$x_1^{k+1} = F_1(x_1^k, x_2^k)$$
$$x_2^{k+1} = F_2(x_1^k, x_2^k)$$



Convergence plot 2D
Newton
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
0
5
20
40
60
80
100
FPI
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
0
20
35
40
60
80
100
Iterations

Problem #4 - Numerical Methods

On $[0,1]$ Let $(u^2 u_x)_x = 1$ where $u(0) = 0$
$u(1) = 0$

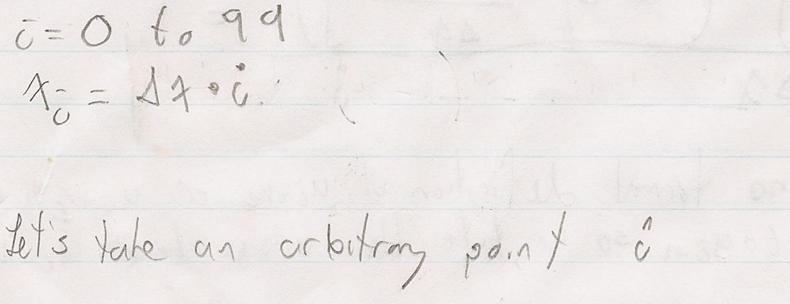
with $N = 100$ pts

Δx is evenly spaced with $\Delta x = \frac{1}{100-1} = \frac{1}{99}$

$i = 0$ to 99

$x_i = \Delta x \cdot i$

Let's take an arbitrary point i



if $\left(u^2 \frac{\partial u}{\partial x}\right)_x = 1$ at $i = 1$ we get $\left(u_i^2 u_{ix}\right)_x = 1$

if we take the derivative at u_i to be the equation below

$$\frac{\partial u_i}{\partial x} \approx \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} + O[\Delta x^2]$$

where we ignore the Δx^2 terms as they will be smaller in comparison to the Δx term

we get

$$\left(u_i^2\left[\frac{u_{i+1/2} - u_{i-1/2}}{\Delta x}\right]\right)_x = 1$$

To get the next derivative we assume the same definition of the equation above and set i to $i+1/2$ and $i-1/2$ which gives

$$\frac{\left(u_{i+1/2}^2\left[\frac{u_{i+1} - u_i}{\Delta x}\right]\right) - \left(u_{i-1/2}^2\left[\frac{u_i - u_{i-1}}{\Delta x}\right]\right)}{\Delta x} = 1$$

Since we have no formal definition of $u_{i+1/2}$ or $u_{i-1/2}$ we assume that from u_i to u_{i+1} so we take the average to get $u_{i+1/2}$... likewise for $u_{i-1/2}$

Let $u_{i+1/2} = \frac{u_i + u_{i+1}}{2}$ and $u_{i-1/2} = \frac{u_i + u_{i-1}}{2}$

we get

$$\left(\frac{u_i + u_{i+1}}{2}\right)^2\left[\frac{u_{i+1} - u_i}{\Delta x^2}\right] - \left(\frac{u_i + u_{i-1}}{2}\right)^2\left[\frac{u_i - u_{i-1}}{\Delta x^2}\right] = 1$$

Simplyfying gives

i) $(u_i + u_{i+1})^2[u_{i+1} - u_i] - (u_i + u_{i-1})^2[u_i - u_{i-1}] = 4(\Delta x)^2$

The equation above is non-linear (nasty) and will try to be solved using fixed-point iteration. I am going to try trial and error and hope the g_i's functions I choose will converge.

Here is the outline

1. ReWrite Equation i) so that $g_i(x_1, x_2, x_3 \dots x_{100}) = x_i$ where $x_i = i \cdot \Delta x$
2. Pick an arbitrary $\vec{x}_{guess} = [0, 0, \dots, 0]$ as it is seems to fit into the B.C
3. Iterate and see if it converges

2.5 Use ghost points so x_{-1} and x_{101} are used with the BC

$$(u_i + u_{i+1})^2 [u_{i+1} - u_i] - 4(\Delta x)^2 = [u_i - u_{i-1}]^2 (u_i - u_{i-1})$$

$$(u_i - u_{i-1}) = \frac{(u_i + u_{i+1})^2 [u_{i+1} - u_i] - 4(\Delta x)^2}{[u_i + u_{i-1}]^2}$$

$$u_i = u_{i-1} + \frac{(u_i + u_{i+1})^2 [u_{i+1} - u_i] - 4(\Delta x)^2}{[u_i + u_{i-1}]^2} = g(x_i)$$

$u(x_0) = 0$

$$\text{when } g(x_0) = \frac{0 + (0 + u_{i+1})^2 [u_{i+1} - 0] - 4(\Delta x)^2}{[0 + 0]^2}$$

$$= \infty$$

so we will let $g(x_0) = 0$
$g(x_{99}) = 0$

This equation does not converge

A few other ways were tried as it took trial and error to get an answer to converge.

Basically I tried variations of solving for u_i using the equation.

The problems I encountered

: if $u_i = \sqrt{\gamma}$ where $\gamma < 0$ there was an error

$u_i = \frac{A}{B}$ if B was small or 0 there was an error

Below are a few $g(x)$ I tried

$$u_i = u_{i-1} + \frac{(u_i + u_{i+1})^2 (u_{i+1} - u_i) - 4(\Delta x)^2}{(u_i + u_{i-1})^2}$$

$$u_i = u_{i+1} - \frac{(u_i + u_{i-1})^2 (u_i - u_{i-1}) + 4(\Delta x)^2}{(u_i + u_{i+1})^2}$$

$$u_i = \sqrt[3]{(u_i + u_{i+1})^2 (u_{i+1} - u_i) - 4(\Delta x)^2 - u_i^2 + u_{i-1} (u_{i-1})^2 + u_{i-1}^3}$$

$$u_i = \sqrt{\frac{(u_i + u_{i+1})^2 (u_{i+1} - u_i) - 4(\Delta x)^2}{(u_i - u_{i-1})^2}} + u_{i-1}$$

$$u_i = \frac{\Delta x}{(u_{i+1} - u_{i-1})^2} \left(1 - \frac{u_i (u_{i+1} - 2u_i + u_{i-1})}{\Delta x^2} \right)$$

$$u_i = \left(\frac{2\Delta x}{u_{i-1} - u_i} \right)^2 \left(1 - \frac{u_i (u_{i+1} - 2u_i + u_{i-1})}{\Delta x^2} \right)$$

$$u_i = \frac{1}{(u_{i+1})^2}\left(2u_i^{2+u_{i+1}} - u_i(u_{i-1})^2 + \frac{2}{\Delta x^2}\left(1 - u_i\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right)\right)\right)$$

$$u_i = \frac{1}{(u_{i-1})^2}\left(2u_i^2 u_{i+1} - u_i u_{i+1}^2 + \frac{2}{\Delta x}\left(1 - u_i\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right)\right)\right)$$

$$u_i = \frac{0.5}{(u_{i+1} - u_i)^2}\left[\Delta x^2 - u_i^2\left(u_{i+1} - 2u_i + 2u_{i-1}\right)\right]$$

$$u_i = -0.25\left(\frac{2\Delta x^2}{u_i} - (u_{i+1} - u_{i-1})^2 - 2u_{i+1} - 2u_{i-1}\right)$$

Until Finally I found a solution that converged.
the end solution is

$$u_i = \sqrt{\underbrace{-0.25\left(2\Delta x^2 - u_i(u_{i+1} - u_{i-1})^2 - u_i \cdot 2 \cdot u_{i+1} - u_i \cdot 2 \cdot u_{i-1}\right)}}$$

which only works if is greater than zero which means $u_i > 0$

I can attach some very rough notes on how I managed to solve the equation to get a converging form if need be.

I had to re-arrange the equation so

$$\frac{\partial}{\partial x}\left(u^2 \frac{\partial u}{\partial x}\right) = u^2 \frac{\partial^2 u}{\partial x^2} + 2u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) = 1$$

$$u_i^2\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right] + 2u_i\left[\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right]^2 = 1$$

expanding the main equation gives

$$2u_i\left(u_{i+1}^2 - 2u_i \cdot u_{i+1} + u_i^2\right) + u_i^2\left(u_{i+1} - 2u_i + u_{i-1}\right) = (\Delta x)^2$$

$$u_i = \frac{1}{2(u_{i+1}-u_i)^2}\left[(\Delta x)^2 - u_i^2\left(u_{i+1} - 2u_i + u_{i-1}\right)\right]$$

$$\frac{u_i}{2\Delta x^2}\left(u_{i+1} - u_{i-1}\right)^2 + u_i\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) = 1$$

$$\frac{u_i\left[(u_{i+1} - u_{i-1})^2 + 2(u_{i+1} - 2u_i + u_{i-1})\right]}{2(\Delta x^2)} = 1$$

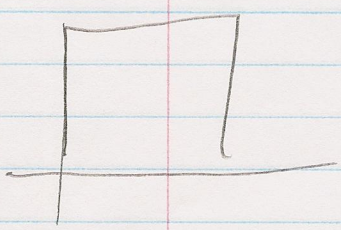
$$u_i\left[(u_{i+1}^2 - 2u_i u_{i-1} + u_{i-1}^2 + 2u_{i+1} - 4u_i + 2u_{i-1}\right] = 2(\Delta x^2)$$

$$u_i\left[(u_{i+1} - u_{i-1})^2 + 2u_{i+1} - 4u_i + 2u_{i-1}\right] = 2(\Delta x)^2$$

$$-4u_i^2 = 2(\Delta x)^2 - u_i(u_{i+1} - u_{i-1})^2 - 2u_i \cdot u_{i+1} - 2u_i \cdot u_{i-1}$$

$$u_i = \frac{1}{-4}\left(\frac{2(\Delta x)^2}{u_i} - (u_{i+1} - u_{i-1})^2 - 2u_{i+1} - 2u_{i-1}\right)$$

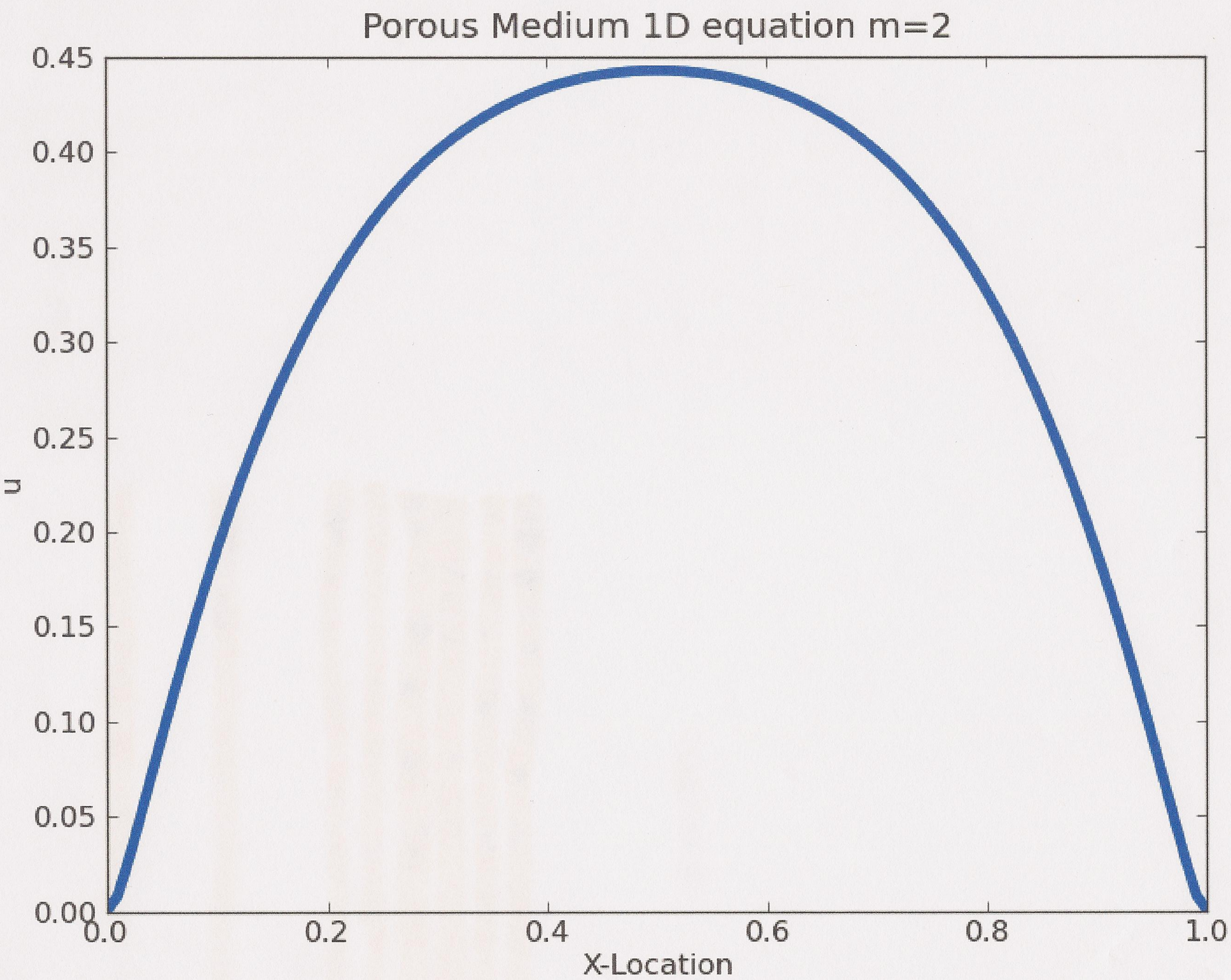
$$u_i \neq 0$$

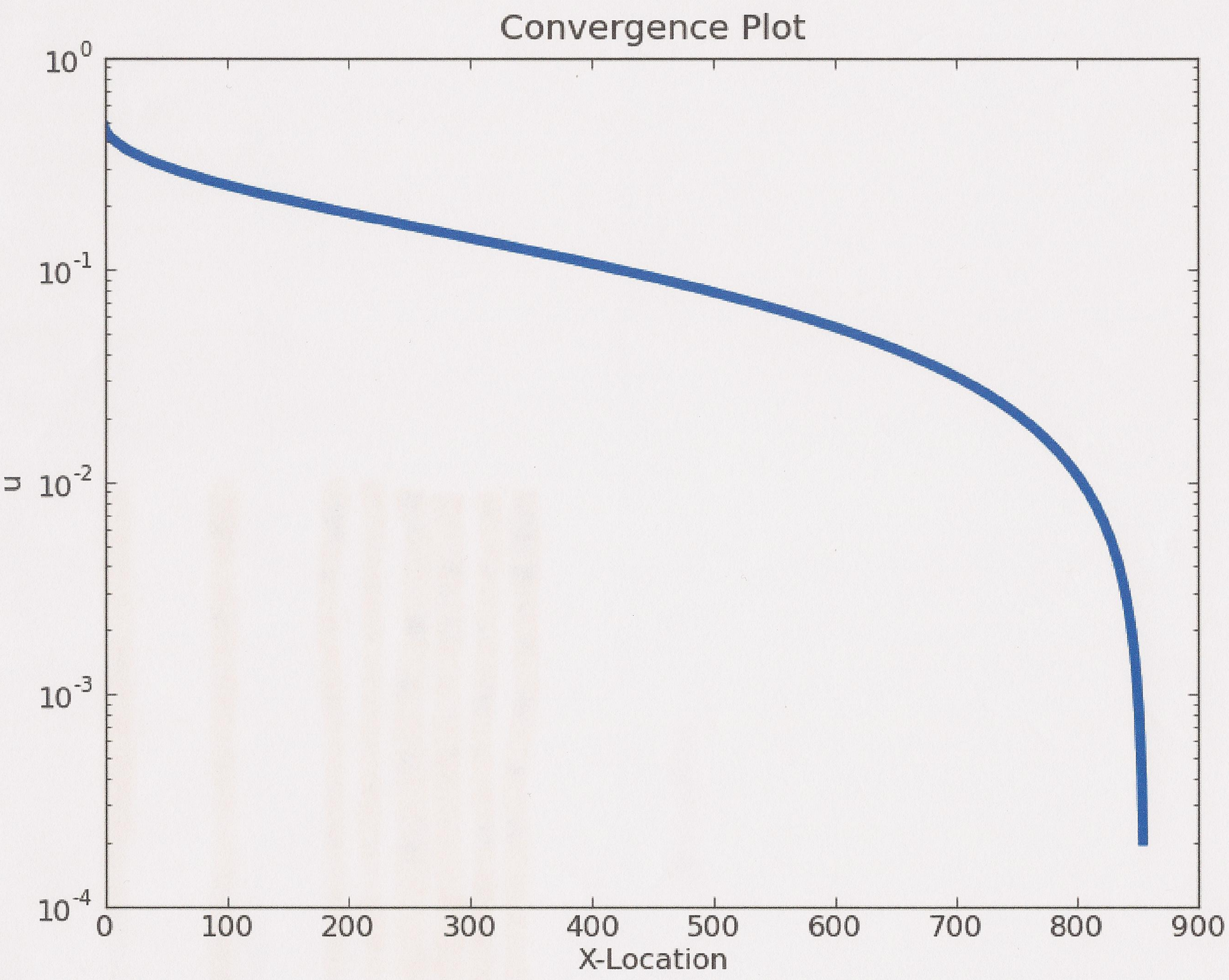


$u_i =$

Final solution

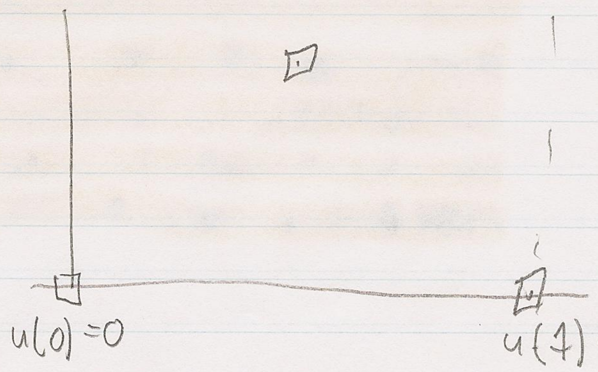
$$u_i = \sqrt{-0.25\left(2\Delta x^2 - (u_{i+1} - u_{i-1})^2 - 2u_{i+1} - 2u_{i-1}\right)}$$


Porous Medium 1D equation m=2
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
u
0.0
0.2
0.4
0.6
0.8
1.0
X-Location


Convergence Plot
10^0
10^-1
u
10^-2
10^-3
10^-4
0
100
200
300
400
500
600
700
800
900
X-Location

Near the boundaries the equation has a discontinuity and this problem does not converge to machine zero. The reason why is that the BC and main equation is incompatible near the edges

Imagine there are only 3 points on the domain (the minimum needed to make a 2nd order derivative.
we get



we know $\Delta x > 0$ and $u_i > 0$ from the $g(A)$ function
we have to solve the following equation

$$(u_i + u_{i+1})^2 (u_{i+1} - u_i) - (u_i + u_{i-1})^2 (u_i - u_{i-1}) = 4(\Delta x^2)$$

where $u_{i+1} = 0$; $u_{i-1} = 0$ we get

$$u_i^2 \cdot -u_i \quad - u_i^2 \cdot u_i = 4(\Delta x)^2$$
$$-2u_i^3 = 4(\Delta x)^2 \quad \text{and since } \Delta x > 0, \; u_i > 0$$

we get a negative equal to a positive which is inconsistent
This is why the BC do not work with this equation