Skewnormal

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1 Parametrization

Let $y \in \mathbb{R}^d$, we say that $y \sim \mathcal{SN}_d\left(\xi, \bar{\Sigma}, \delta\right)$ if

$$p(y \mid \xi, \bar{\Sigma}, \delta) = \int_0^\infty 2\phi_{d+1}\left(\left[y^\top, z\right]^\top \mid \mu, \Omega\right) dz \tag{1}$$

with $\mu = \left[\xi^\top, 0\right]^\top$ and $\Omega = \begin{pmatrix} \omega\Sigma\omega & \omega\delta \\ \delta^\top\omega & 1 \end{pmatrix}$ and $\bar{\Sigma} = \omega\Sigma\omega$ is the decomposition of the covariance matrix in correlation matrix and the diagonal matrix with variances.
By defining $\theta = \omega\delta$ and $\Psi = \bar{\Sigma} - \theta\theta^\top$, we can rewrite the previous density as

$$p(y \mid \xi, \bar{\Sigma}, \delta) \propto \int_0^\infty 2\phi_1(z)\phi_d(y \mid \xi + \theta z, \Psi)\, dz. \tag{2}$$

Note that $|\Omega| = \left|\omega\Sigma\omega - \omega\delta\delta^\top\omega\right| = \left|\omega\left(\Sigma - \delta\delta^\top\right)\omega\right| = |\Psi|$ and

$$\Omega^{-1} = \begin{pmatrix} \Psi^{-1} & -\Psi^{-1}\theta \\ -\theta^\top\Psi^{-1} & 1 + \theta^\top\Psi^{-1}\theta \end{pmatrix}$$

Moreover, by rearranging the terms from equation (2) we can write:

$$\begin{aligned}
p(y \mid \xi, \bar{\Sigma}, \delta) &\propto \int_0^\infty 2\phi_1(z \mid \bar{\mu}, \bar{\sigma}^2)\phi_d(y \mid \xi, \Psi) \exp\left[\frac{\bar{\mu}^2}{2\bar{\sigma}^2}\right] dz \\
&= \phi_d(y \mid \xi, \Psi) \exp\left[\frac{\bar{\mu}^2}{2\bar{\sigma}^2}\right] 2\int_{-\bar{\mu}/\bar{\sigma}}^\infty \phi_1(z) dz \\
&= 2\phi_d(y \mid \xi, \Psi) \exp\left[\frac{\bar{\mu}^2}{2\bar{\sigma}^2}\right] \Phi_1\left(\frac{\bar{\mu}}{\bar{\sigma}}\right) \\
&= 2\phi_d(y \mid \xi, \Psi) \exp\left[\frac{1}{2}(y - \xi)^\top \alpha\alpha^\top (y - \xi)\right] \Phi_1\left(\alpha^\top (y - \xi)\right) \\
&= 2\phi_d\left(y \mid \xi, \Psi - \alpha\alpha^\top\right) \Phi_1\left(\alpha^\top (y - \xi)\right)
\end{aligned}$$

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where we have used

$$\bar{\mu} = \frac{(y-\xi)^\top \Psi^{-1}\theta}{1+\theta^\top\Psi^{-1}\theta} \qquad \bar{\sigma}^2 = \left(1+\theta^\top\Psi^{-1}\theta\right)^{-1}$$

and we defined

$$\alpha = \frac{\Psi^{-1}\theta}{\sqrt{1+\theta^\top\Psi^{-1}\theta}}$$

II Constraints

In order for Ψ (and thus Ω) to be positive definite, a constrain should be put on δ and $\bar{\Sigma}$. First of all, recall that the matrix $\theta\theta^\top$ has only one strictly positive eigenvalue, equal to $\|\theta\|^2$, while all the others are 0. As it can be proven that Ψ is SPD if and only if the smallest eigenvalue of $\bar{\Sigma}$ is larger than $\|\theta\|^2$, we can require that

$$\|\theta\|^2 = \delta^\top\omega\omega\delta \leqslant \min_i \lambda_i(\bar{\Sigma})$$

III Data generation mechanism

To generate samples from a skewnormal we exploit equation (1) and we proceed in the following way:

- We compute μ and Ω from the parameters ξ, $\bar{\Sigma}$ and δ
- We generate a sample from a $(d+1)$-variate normal distribution: $Z \sim \mathcal{N}_{d+1}(\mu, \Omega)$
- if $Z[d+1] \geqslant 0$ then $y = Z[:d]$, else $y = -Z[:d]$.

References

[1] Rajendra Bhatia
Matrix analysis. Springer-Verlag New York, 1997.