Foundational Problems and a Program for a Schrödinger–Style Formulation of Equity Prices

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1 Purpose and Scope

This note identifies and expands three foundational problems that must be addressed to convert the "quantum–tunneling equities" program from a set of applications to a rigorous framework:

P1. *Non-uniqueness of prices under identical fundamental information.* Formalize and prove the claim that, even when two time points have identical information about fundamentals, the equilibrium stock price need not be identical.

P2. *Deriving (rather than positing) a Schrödinger-type evolution for prices.* Construct a principled derivation from axioms to an operator/PDE representation, clarifying mappings (diffusion $\leftrightarrow$ imaginary time, open-system effects, domains/self-adjointness).

P3. *Disciplining applications.* Separate application papers from foundational proofs; specify a roadmap where each dependency is resolved before empirical or computational claims.

2 Problem P1: Non-uniqueness of Prices Under Identical Information

2.1 Set-up: Information Equivalence and the Object to be Proved

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $(D_t)_{t\geq 0}$ denote a (discounted) dividend process adapted to $\mathbb{F}$. Let S_t denote the (ex-dividend) price process for a single risky asset.

Definition 2.1 (Information equivalence)**.** *Two stopping times τ_1, τ_2 are* fundamentally information-equivalent *if the σ-algebras about primitives coincide up to $\mathbb{P}$-null sets, e.g.*

$$\mathcal{F}_{\tau_1}^{fund} = \mathcal{F}_{\tau_2}^{fund} \quad a.s.,$$

where $\mathcal{F}_t^{fund}$ is generated by the fundamental state (cash flows, states impacting preferences/technologies) but excludes *extrinsic public randomization and population-level higher-order beliefs.*

Definition 2.2 (Price non-uniqueness at identical fundamentals)**.** *We say* price non-uniqueness at identical fundamentals *holds if there exist τ_1, τ_2 that are fundamentally information-equivalent such that*

$$\mathbb{P}(S_{\tau_1} \neq S_{\tau_2}) > 0.$$

The goal is to *prove* that non-uniqueness can arise in rational equilibrium without changing fundamentals.

2.2 Mechanisms Yielding Non-uniqueness

We isolate three mechanisms. Each delivers a formal route to the desired statement.

(M1) Sunspot/Extrinsic Randomization Equilibria. Let agents be risk-averse with common priors. Suppose a public *sunspot* process Z_t, independent of fundamentals, is available and can coordinate beliefs about future market depth or liquidity demand. In incomplete markets, multiple equilibria can exist where equilibrium pricing kernels depend on Z_t.

Assumption 2.1 (Incomplete markets and strategic complementarities)**.** *There is a continuum of risk-averse agents with CARA utility over wealth, constraints induce endogenous price impact, and demand schedules are strategic complements (depth depends on expected depth).*

Proposition 2.1 (Sunspot price multiplicity)**.** *Under the above, there exist equilibria* (S_t) *and* $(\tilde{S}_t)$ *with identical fundamentals and identical* $\mathcal{F}_t^{fund}$ *such that* S_t *is* Z_t*-measurable while* $\tilde{S}_t$ *is* $\sigma(Z_t)$*-independent, yielding* $\mathbb{P}(S_t \neq \tilde{S}_t) > 0$ *for some* t*.*

Proof sketch. Equilibrium price is pinned down by market-clearing with endogenous depth $k(Z_t)$. Fixed-point arguments on demand elasticity show multiple solutions for k; selection can be coordinated by the extrinsic Z_t. Since fundamentals are held fixed, price differences are attributable to equilibrium selection, not informational content. Formalization uses a Kakutani fixed-point on best-response correspondences with a sunspot index. $\square$

(M2) Heterogeneous Beliefs with Common Information. Even if $\mathcal{F}_{\tau_1}^{\text{fund}} = \mathcal{F}_{\tau_2}^{\text{fund}}$, agents can disagree on *models* (likelihoods) consistent with that σ-algebra. With short-sale constraints or convex portfolio constraints, equilibrium prices can be strictly increasing in the cross-sectional dispersion of beliefs.

Assumption 2.2 (Model heterogeneity and constraints)**.** *Agents share* $\mathcal{F}_t^{fund}$ *but have beliefs* $\{\mathbb{P}^{(i)}\}$ *equivalent to* $\mathbb{P}$ *with distinct Radon–Nikodým derivatives* $L^{(i)} = \frac{\mathrm{d}\mathbb{P}^{(i)}}{\mathrm{d}\mathbb{P}}$ *on* $\mathcal{F}_t^{fund}$*. Short selling is limited.*

Proposition 2.2 (Dispersion-driven price variation)**.** *Fix fundamentals. For two dates* τ_1, τ_2 *with identical* $\mathcal{F}^{fund}$*, if dispersion of* $L^{(i)}$ *across agents differs (e.g. due to model reweighting or selection of active traders), then competitive equilibrium prices satisfy* $S_{\tau_1} \neq S_{\tau_2}$ *with positive probability.*

Proof sketch. In mean-variance or CARA-normal settings with constraints, the market-clearing price equals a weighted average of subjective risk-neutral valuations, with weights depending nonlinearly on active agents and constraints. Varying dispersion or the active set changes those weights without altering $\mathcal{F}^{\text{fund}}$. $\square$

(M3) Microstructure/Inventory and Path Dependence. Let the efficient price be $P_t = \mathbb{E}^{\mathbb{Q}}\left[\sum_{u \geq t} \beta^{u-t} D_u \mid \mathcal{F}_t^{\text{fund}}\right]$. Observed S_t incorporates a microstructure distortion Δ_t driven by dealer inventories and order-flow imbalance Y_t:

$$S_t = P_t + \Delta_t, \qquad \Delta_t = \phi Y_t + (\text{inventory term}).$$

If $\mathcal{F}_{\tau_1}^{\text{fund}} = \mathcal{F}_{\tau_2}^{\text{fund}}$ but the *order-flow state* differs (yet conveys no new fundamental information), then $S_{\tau_1} \neq S_{\tau_2}$.

Proposition 2.3 (Non-ergodicity via order flow)**.** *If* Y_t *evolves with persistence independent of fundamentals, then price differences at equal fundamentals occur with positive probability.*

Remark 2.1 (Minimality)*.* Each mechanism (M1)–(M3) is sufficient; (M1) gives clean existence via equilibrium selection; (M2) highlights belief-model heterogeneity; (M3) connects to empirics (transient impact, liquidity).

2.3 Target Theorem and Proof Strategy

Theorem 2.1 (Price non-uniqueness at identical fundamentals)**.** *There exists a competitive equilibrium in an economy with identical fundamental information at* τ_1, τ_2 *such that* $\mathbb{P}(S_{\tau_1} \neq S_{\tau_2}) > 0$*.*

Proof plan. (1) Construct an incomplete-market economy with depth externalities; (2) introduce an extrinsic i.i.d. public signal Z; (3) show the best-response correspondence has multiple fixed points indexed by Z; (4) verify welfare/feasibility and selection measurability; (5) conclude S differs across time indices with identical $\mathcal{F}^{\text{fund}}$.

2.4 Empirical Tests (Falsifiability)

- **Sunspot test:** regress price innovations on instruments measurable with respect to Z_t that are orthogonal to fundamentals; test for incremental explanatory power.
- **Dispersion test:** use options-implied densities across broker cohorts to proxy cross-sectional $L^{(i)}$ dispersion; test price level sensitivity holding $\mathcal{F}^{\text{fund}}$ signals fixed.
- **Microstructure test:** control for fundamental news; estimate $S_t - P_t$ on lagged order flow and inventory; persistence at identical fundamentals supports (M3).

3 Problem P2: Deriving a Schrödinger-Type Equation for Finance

3.1 Axioms and the Required Map

We do *not* posit a Schrödinger equation; we derive a PDE/operator representation from standard no-arbitrage and Markovian state dynamics, then identify when (and only when) it can be represented in Schrödinger form.

Assumption 3.1 (State and measure)**.** *Let $X_t \in \mathbb{R}$ denote log-price (or a low-dimensional Markov feature vector). Under $\mathbb{Q}$ (risk-neutral measure),*

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t, \qquad S_t = \mathrm{e}^{X_t}. \tag{1}$$

Let $f(x,t) = \mathbb{E}^{\mathbb{Q}}[\,\Phi(X_T) \mid X_t = x\,]$ be a priced claim.

Proposition 3.1 (Generator and backward PDE)**.** *Let $\mathcal{L}_t$ be the generator*

$$(\mathcal{L}_t g)(x) = \mu(x,t)g'(x) + \tfrac{1}{2}\sigma^2(x,t)g''(x).$$

Then f solves the backward *Kolmogorov equation*

$$\partial_t f + \mathcal{L}_t f - r(x,t)f = 0, \qquad f(\cdot, T) = \Phi(\cdot), \tag{2}$$

where r is the short rate (possibly state dependent) embedded by discounting.

3.2 Heat Equation, Wick Rotation, and Schrödinger Form

Constant-coefficient case. Suppose $\mu \equiv 0$ under $\mathbb{Q}$ (martingale for discounted asset) and $\sigma \equiv \sigma > 0$, $r \equiv 0$ for clarity. Then (2) reduces to the heat equation

$$\partial_t f = -\tfrac{1}{2}\sigma^2 \partial_{xx} f. \tag{3}$$

Introduce *imaginary time* $\tau := \mathrm{i}t$ and define $\psi(x,\tau) := f(x,t(\tau))$. Formally,

$$\partial_\tau \psi = \mathrm{i}\,\partial_t f = -\mathrm{i}\,\tfrac{1}{2}\sigma^2 \partial_{xx} f \quad \Longleftrightarrow \quad \mathrm{i}\,\partial_\tau \psi = -\tfrac{1}{2}\sigma^2\,\partial_{xx}\psi.$$

Thus we obtain a free-particle Schrödinger equation with effective constants

$$\frac{\hbar_{\text{eff}}^2}{2m_{\text{eff}}} = \frac{\sigma^2}{2}. \tag{4}$$

Potential term. When $r(x,t) \neq 0$ or drift-removal requires a Girsanov transform introducing a killing term, (2) becomes

$$\partial_t f = -\left[-\tfrac{1}{2}\sigma^2 \partial_{xx} + V(x,t)\right] f,$$

with $V(x,t) := r(x,t) - \mu(x,t)\,\partial_x(\cdot)$ after completing squares or transforming to eliminate first derivatives (see below). Under Wick rotation,

$$\mathrm{i}\,\partial_\tau \psi = \left[-\frac{\hbar_{\text{eff}}^2}{2m_{\text{eff}}}\partial_{xx} + U(x,\tau)\right]\psi, \quad U(x,\tau) := V(x,t(\tau)). \tag{5}$$

3.3 Eliminating First Derivatives: Doob h-Transform

If $\mu \neq 0$, the operator contains a first-derivative term. To reach a symmetric form, set

$$f(x,t) = \mathrm{e}^{-\phi(x,t)}\,u(x,t),$$

and choose ϕ to cancel the first derivative in the transformed operator. For time-homogeneous $\mu = \mu(x)$, $\sigma = \sigma(x)$, take

$$\phi'(x) = \frac{\mu(x)}{\sigma^2(x)} \quad \Rightarrow \quad \mathcal{L}f = \mathrm{e}^{-\phi}\left[\tfrac{1}{2}\sigma^2 u'' - \underbrace{\left(\tfrac{1}{2}\sigma^2\phi'^2 - \tfrac{1}{2}(\sigma^2\phi')'\right)}_{=:V_{\text{eff}}(x)} u\right].$$

Then u solves a heat-type equation with potential V_{eff}, and the Schrödinger form (5) follows by Wick rotation. This is a Doob h-transform ensuring formal self-adjointness.

3.4 Operator-Theoretic Requirements (No Hand-Waving)

Define the formal Hamiltonian

$$H := -\frac{\hbar_{\text{eff}}^2}{2m_{\text{eff}}}\,\partial_{xx} + U(x),$$

on $L^2(\Omega_x)$ with domain $\mathrm{Dom}(H) \subseteq H^2(\Omega_x)$ and appropriate boundary conditions.

Assumption 3.2 (Domain and boundary conditions)**.** *Ω_x is either $\mathbb{R}$ or a bounded interval $[a,b]$; boundary conditions are reflecting (Neumann), absorbing (Dirichlet), or Robin; $U \in L^1_{loc}$ and bounded below.*

Proposition 3.2 (Self-adjointness and spectral decomposition)**.** *Under the assumption above, H is essentially self-adjoint on $C_c^\infty(\Omega_x)$ (on $\mathbb{R}$ with Kato-bounded U) or self-adjoint with standard* *boundary conditions on $[a,b]$. Therefore a complete spectral resolution exists, enabling expansion*

$$\psi(x,\tau)=\sum_n c_n\,\mathrm{e}^{-\,\mathrm{i}E_n\tau/\hbar_{\mathrm{eff}}}\,\varphi_n(x)$$

(or a continuous integral on $\mathbb{R}$), with $\{E_n,\varphi_n\}$ the spectrum/eigenfunctions of H.

Remark 3.1 (Finance vs. Physics). *Unitary* Schrödinger evolution presumes closed dynamics. Financial pricing under $\mathbb{Q}$ is *diffusive/dissipative*. The exact correspondence is **imaginary time** (heat kernel) $\leftrightarrow$ Euclidean quantum mechanics. If one insists on real-time unitary evolution for prices, the physically correct mathematical object is an *open quantum system* with Lindblad generator $\mathcal{L}_{\mathrm{GKSL}}$, not a bare Hamiltonian. That is a separate paper (see §4).

3.5 Path Integral (Feynman–Kac) Derivation

For bounded U, the Euclidean propagator equals the risk-neutral pricing kernel:

$$f(x,t)=\mathbb{E}_x^{\mathbb{Q}}\left[\exp\left(-\int_t^T U(X_s,s)\mathrm{d}s\right)\Phi(X_T)\right]=\int\mathcal{D}x\,\exp\big(-\mathcal{S}_E[x]\big)\,\Phi(x_T),$$

with Euclidean action

$$\mathcal{S}_E[x]=\int_t^T\left\{\frac{m_{\mathrm{eff}}}{2\hbar_{\mathrm{eff}}^2}\,\dot{x}^2+U(x,s)\right\}\mathrm{d}s.$$

This is a rigorous statement of Feynman–Kac; it *is* the bridge, avoiding ad-hoc postulates.

3.6 Units, Calibrations, and Identifications

From (4):

$$\hbar_{\mathrm{eff}}=\sigma\sqrt{m_{\mathrm{eff}}}.$$

We may set $m_{\mathrm{eff}}=1$ by convention, identifying $\hbar_{\mathrm{eff}}=\sigma$; more generally, place macro/sector/liquidity effects into $m_{\mathrm{eff}}(x,t)$, retaining $\hbar_{\mathrm{eff}}$ as a universal scale or vice versa. This choice must be fixed *before* empirical work.

3.7 Boundary Conditions as Market Microstructure

- **Dirichlet** at $x=a$ models absorbing default/kill barriers.
- **Robin** implements state-tax or transaction-friction penalties at boundaries.
- **Moving boundaries** $a(t)$ encode regime switches/limits-up-down; use time-dependent domains + Trotter–Kato product formulas.

3.8 What Must Be Proved in the "Derivation Paper"

(D1) A precise Doob transform from (1) to a symmetric generator; conditions for H to be self-adjoint.

(D2) Existence/uniqueness of solutions to (5) (Euclidean and real time) with the chosen domain/boundaries.

(D3) Validity of the Wick rotation map and its limitations for state/time-dependent coefficients.

(D4) Identification and calibration of $U(x,t)$ from observable primitives (rates, liquidity premia, drawdown asymmetry).

4 Problem P3: Discipline the Applications and Paper Roadmap

Applications that *use* tunneling before P1–P2 are settled risk begging the question. The program should be split as follows.

Paper A (Foundations): Price Non-uniqueness at Identical Fundamentals

- **Main Theorem:** Existence of equilibria with $\mathcal{F}^{\text{fund}}$-identical dates but different prices (via M1).
- **Extensions:** Heterogeneous beliefs (M2), microstructure (M3).
- **Empirics:** Instruments and identification for each mechanism.

Paper B (Derivation): From Risk-Neutral Diffusion to Schrödinger Form

- **Main Result:** Doob transform ⇒ symmetric generator ⇒ heat kernel ⇒ Euclidean Schrödinger; proper operator domains and spectral theory; mapping to $U(x)$.
- **What is *not*** claimed: real-time unitary evolution for financial prices (unless via Lindblad).

Paper C (Applications): Tunneling, Barriers, and Asymmetry

- Only after A–B are complete.
- Empirical calibration of $U(x)$ (beta layers, liquidity, drawdown asymmetry).
- **Results:** Spectral gap vs. mean-reversion, tunneling probabilities across calibrated barriers, path-dependent knock-outs as moving boundaries.

5 Technical Appendices (Skeletons to be Filled)

5.1 A. Doob Transform Details

Let

$$(\mathcal{L}g)(x) = \mu(x)g'(x) + \tfrac{1}{2}\sigma^2(x)g''(x).$$

Set $h(x) = \exp\Big(\int^x \frac{\mu(y)}{\sigma^2(y)}\,\mathrm{d}y\Big)$ and define

$$\tilde{\mathcal{L}} := h\,\mathcal{L}\,h^{-1} = \tfrac{1}{2}\sigma^2(x)\,\partial_{xx} - V_{\text{eff}}(x), \quad V_{\text{eff}}(x) = \tfrac{1}{2}\sigma^2\left(\frac{\mu}{\sigma^2}\right)^2 - \tfrac{1}{2}\left(\sigma^2\,\frac{\mu}{\sigma^2}\right)'.$$

State conditions for $V_{\text{eff}} \in L^1_{\text{loc}}$ and Kato-Rellich bounds to ensure $-\frac{1}{2}\sigma^2\partial_{xx} + V_{\text{eff}}$ is self-adjoint and bounded below.

5.2 B. From Heat Kernel to Spectral Representation

If H is self-adjoint and bounded below, then

$$\mathrm{e}^{-tH} = \int_0^\infty \mathrm{e}^{-t\lambda}\, \mathrm{d}E_H(\lambda),$$

with E_H the spectral measure. For bounded domains, eigen-expansion yields

$$f(x,t) = \sum_n \mathrm{e}^{-tE_n} \langle \Phi, \varphi_n \rangle\, \varphi_n(x).$$

This directly maps to "imaginary-time" decay rates E_n and barrier-tunneling estimates.

5.3 C. Open-System (Lindblad) Extension (For Future Work)

If one insists on real-time dynamics capturing frictions/flows, postulate a density matrix ρ on L^2 and

$$\partial_t \rho = -\frac{\mathrm{i}}{\hbar_{\mathrm{eff}}}[H,\rho] + \sum_k \Big(L_k \rho L_k^\dagger - \tfrac{1}{2}\{L_k^\dagger L_k, \rho\} \Big),$$

with L_k encoding liquidity dissipation, order-flow shocks, or inventory leakage. This is *not* needed for pricing via Feynman–Kac, but may be useful for microstructure-consistent real-time evolution.

6 Checklist of Claims to Prove Before Applications

C1. (From Paper A) Constructive equilibrium with sunspot multiplicity; verify integrability and measurability.

C2. (From Paper B) Doob transform validity; self-adjointness of H on the chosen domain/boundaries.

C3. (Calibration) Map $(\alpha, \beta, \gamma, \delta)$-style potential parameters to observable primitives; prove identifiability or provide conditions for partial identification.

C4. (Empirics) Out-of-sample tests distinguishing (M1)–(M3).

7 Minimal Working Examples (for the Overleaf Companion)

Example 1 (Gaussian Diffusion, Fixed Barrier)

Let $\mathrm{d}X_t = \sigma\, \mathrm{d}W_t$, $U(x) = \kappa\, \mathbb{1}_{\{x \in I\}}$ for some interval I. Then $H = -\frac{1}{2}\sigma^2 \partial_{xx} + \kappa \mathbb{1}_I$ on $L^2(\mathbb{R})$. Compute heat kernel via Trotter product and bound tunneling probability across I by the Agmon distance:

$$\mathbb{P}_{x_0}[X_T \in J \text{ without entering } I] \lesssim \exp\Big(- \int_{\text{class. path}} \sqrt{2\,U(x)}/\sigma \, \mathrm{d}x \Big).$$

This provides a non-heuristic tunneling proxy consistent with Euclidean action.

Example 2 (Order-Flow Distortion)

Let $S_t = P_t + \phi Y_t$ with $Y_t = \rho Y_{t-1} + \varepsilon_t$, $\rho \in (0,1)$, $\varepsilon_t \perp \mathcal{F}_t^{\text{fund}}$. Take $\tau_1 \neq \tau_2$ with $\mathcal{F}_{\tau_1}^{\text{fund}} = \mathcal{F}_{\tau_2}^{\text{fund}}$ but $Y_{\tau_1} \neq Y_{\tau_2}$; then $\mathbb{P}(S_{\tau_1} \neq S_{\tau_2}) > 0$ is immediate, illustrating (M3).

Deliverable structure for the project repository (not part of the paper):

- `paper-A-nonuniqueness.tex`: full proofs for P1.
- `paper-B-derivation.tex`: full operator/PDE derivation for P2.
- `paper-C-applications.tex`: empirical/simulation work, only after A–B.

Summary

The applications are *not* the foundation. Prove P1 via equilibrium multiplicity (with heterogeneous beliefs and microstructure as corroborating channels). Derive P2 via Doob transform $\to$ heat kernel $\to$ Euclidean Schrödinger with rigorous operator theory. Only then pursue tunneling and barrier asymmetries in applications.