Graph Theory

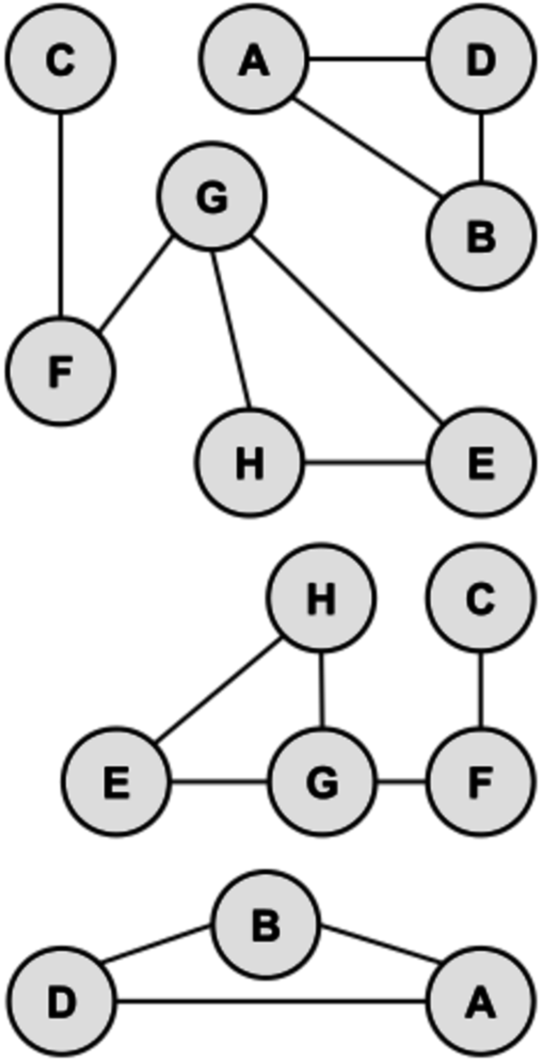
ACSL Contest #4

What is graph theory?

- In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects.
- "Many problems are naturally formulated in terms of points and connections between them. For example, a computer network has PCs connected by cables, an airline map has cities connected by routes, and a school has rooms connected by hallways. A graph is a mathematical object which models such situations."
  - Another cool example: graph theory has applications in helping robotic swarms coordinate and reach a consensus based on each individual's sensor data.

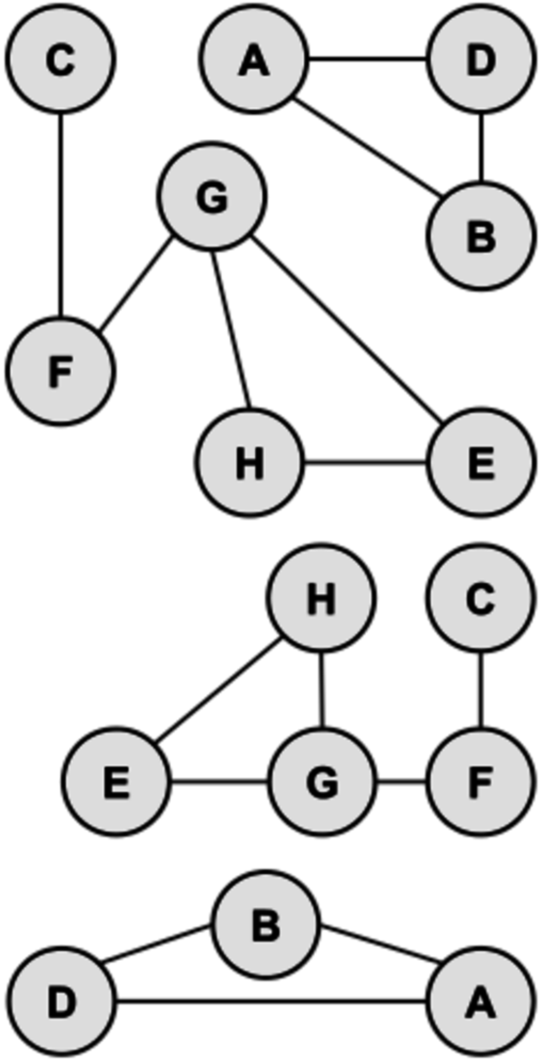
Overview

- A **graph** is a collection of vertices and edges.
- An **edge** is a connection between two **vertices** (sometimes referred to as **nodes**).
- One can draw a graph by marking points for the vertices and drawing lines connecting them for the edges, but the graph is defined independently of the visual representation.
- The precise way to represent this graph is to identify its set of vertices {A, B, C, D, E, F, G}, and its set of edges between these vertices {AB, AD, BD, CF, FG, GH, GE, HE}.



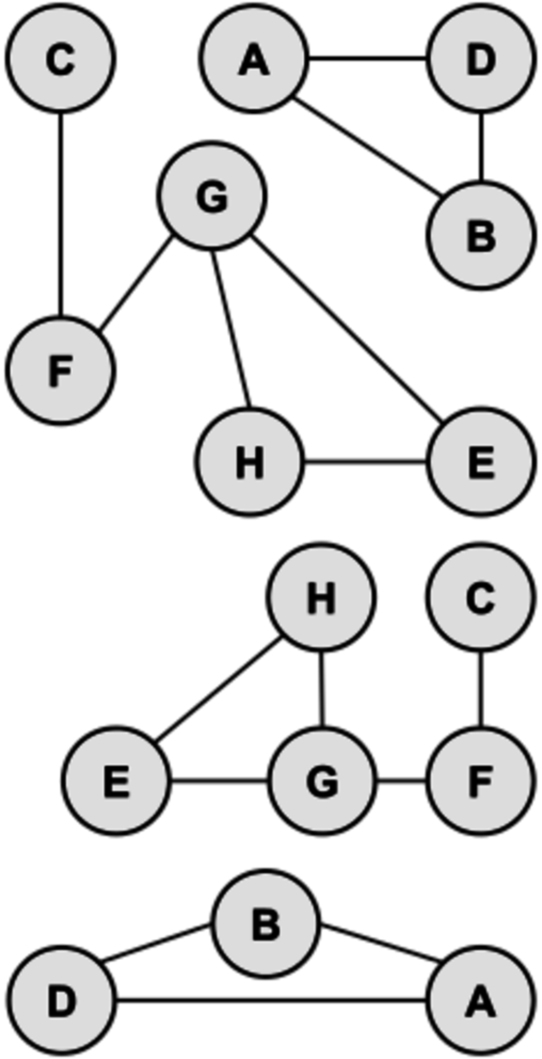
Terminology

- The edges of these graphs have no directions meaning that the edge from one vertex A to another vertex B is the same as from vertex B to vertex A. Such a graph is called an **undirected graph**.
- Similarly, a graph having a direction associated with each edge is known as a **directed graph**.
- A **path** from vertex x to y in a graph is a list of vertices, in which successive vertices are connected by edges in the graph.
  - For example, FGHE is path from F to E in the graph above.
- A **simple path** is a path with no vertex repeated.
  - For example, FGHEG is not a simple path.



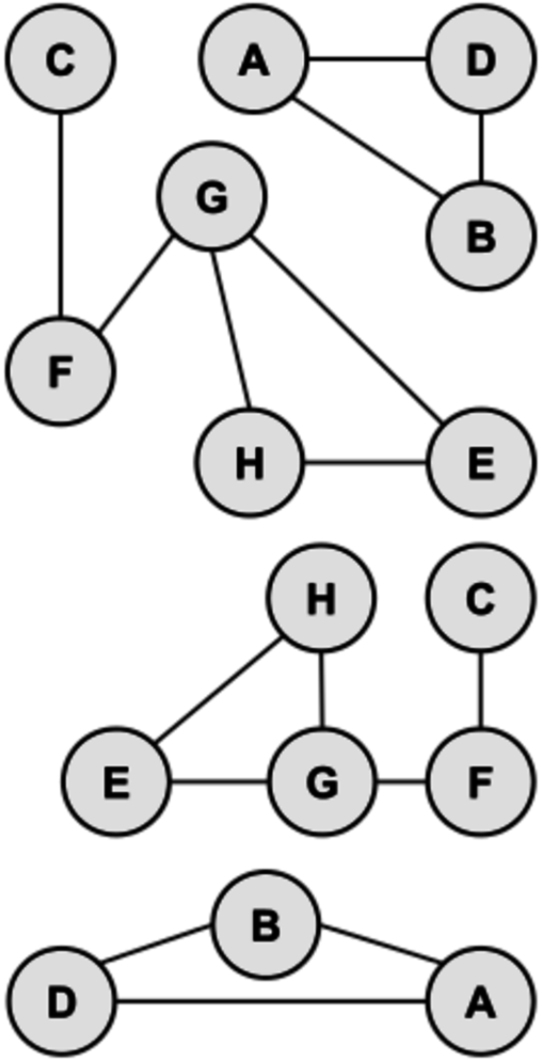
Terminology

- A **cycle** is a simple path, except that the first and last vertex are the same (a path from a point back to itself).
  - For example, the path HEGH is a cycle in our example.
- Vertices must be listed in the order that they are traveled to make the path; any of the vertices may be listed first. Thus, HEGH and EHGE are different ways to identify the same cycle. For clarity, we list the start / end vertex twice: once at the start of the cycle and once at the end.



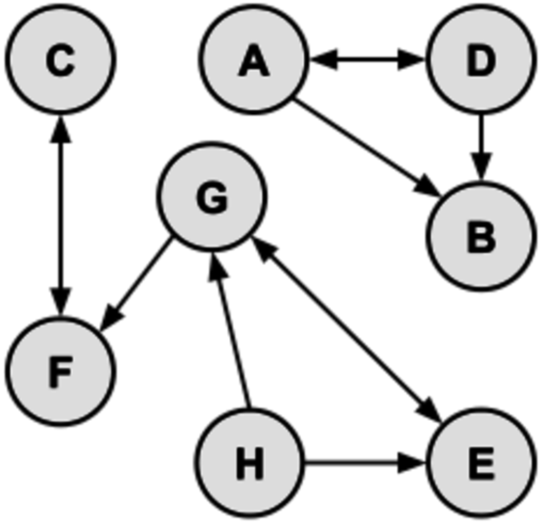
Terminology

- ACSL will denote the number of vertices in a given graph by V and the number of edges by E.
  - For example, when given a problem you'll be provided the sets V = {A, B, C, D} and E = {AB, AD, BA, BD, CA, DB, DC}
- Note that E can range anywhere from V to V^2 (or V(V-1)/2 in an undirected graph).
- Graphs with all edges present are called **complete graphs**.
- Graphs with relatively few edges present (say less than V*log(V)) are called **sparse**.
- Graphs with relatively few dges missing are called **dense**.



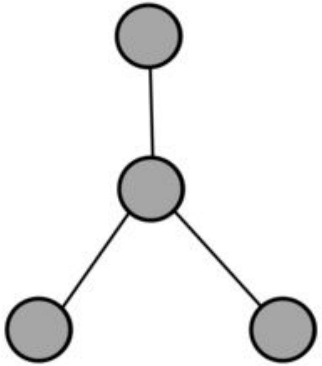
Directed Graphs

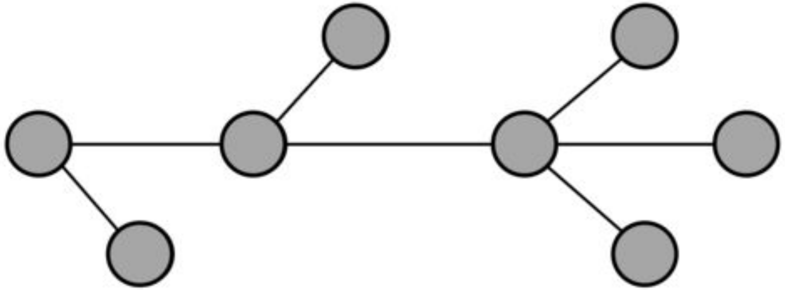
- **Directed graphs** are graphs which have a direction associated with each edge. An edge xy in a directed graph can be used in a path that goes from x to y but not necessarily from y to x.
    - This graph is defined as the set of vertices V = {A,B,C,D,E,F,G,H} and the set of edges {AB,AD,DA,DB,EG,GE,HG,HE,GF,CF,FC}.
    - There is one directed path from G to C (namely, GFC); however, there are no directed paths from C to G. Note that a few of the edges have arrows on both ends, such as the edge between A and D.
    - These dual arrows indicate that there is an edge in each direction, which essentially makes an undirected edge.
- An **undirected graph** can be thought of as a directed graph with all edges occurring in pairs in this way.
- A directed graph with no cycles is called a **dag** (directed acyclic graph).



Trees and Forests

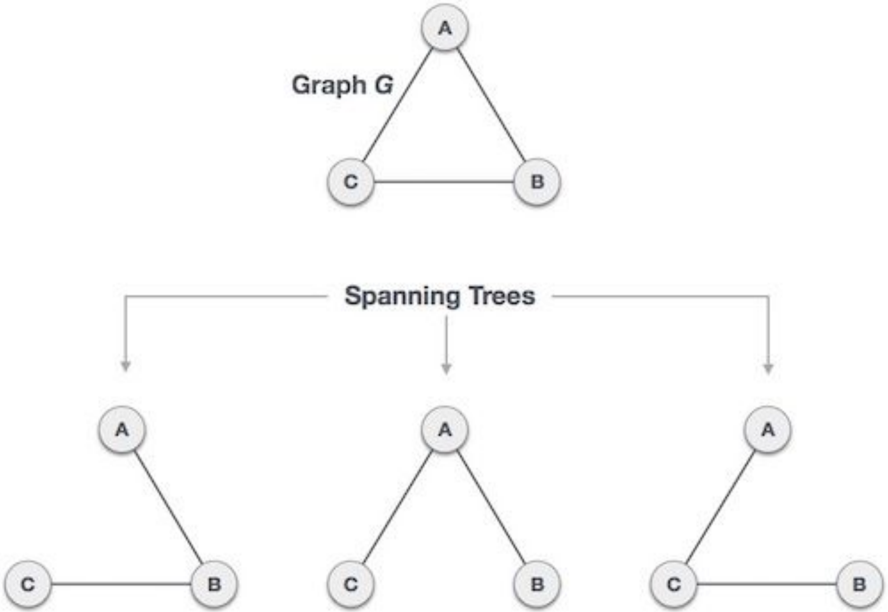
- A graph with no cycles is called a **tree**. There is only one path between any two nodes in a tree. A tree with N vertices contains exactly N-1 edges.
- The two graphs shown below are trees because neither has any cycles and all vertices are connected. The graph on the left has 4 vertices and 3 edges; the graph on the right has 8 vertices and 7 edges. Note that in both cases, because they are trees, the number of edges is one less than the number of vertices.





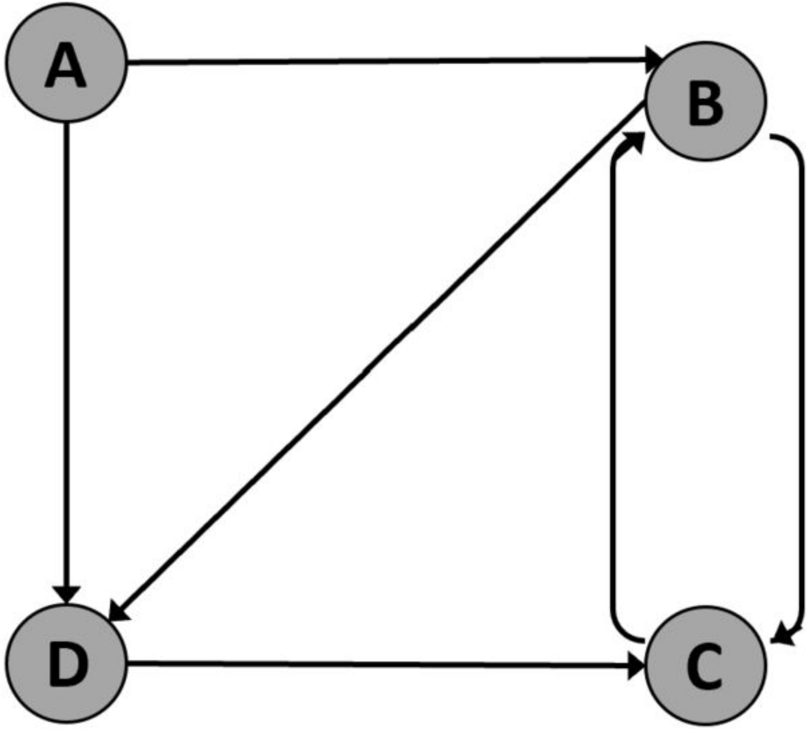
Trees and Forests

- A group of disconnected trees is called a **forest**.
- A **weighted graph** is a graph that has a weight (also referred to as a cost) associated with each edge.
  - For example, in a graph used by airlines where cities are vertices and edges are cities with direct flights connecting them, the weight for each edge might be the distance between the cities.
- A **spanning tree** of a graph is a subgraph that contains all the vertices and forms a tree. A **minimal spanning tree** can be found for weighted graphs in order to minimize the cost across an entire network.

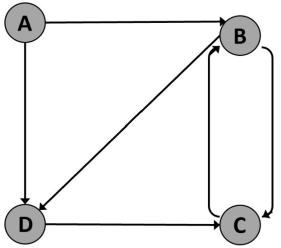


Adjacency Matrices

- It is frequently convenient to represent a graph by a matrix known as an adjacency matrix.
- Consider the following directed graph:



Adjacency Matrices



- To draw the adjacency matrix we create an N by N grid and label the rows and columns for each vertex.
- Then, place a 1 for each edge in the cell whose row and column correspond to the starting and ending vertices of the edge.
- Finally, place a 0 in all other cells.

M =

	A	B	C	D
A				
B				
C				
D				

=

	A	B	C	D
A		1		1
B			1	1
C		1		
D			1	

=

	A	B	C	D
A	0	1	0	1
B	0	0	1	1
C	0	1	0	0
D	0	0	1	0

Adjacency Matrices

- By construction, cell (i, j) in the matrix with a value of 1 indicates a direct path from vertex i to vertex j.
- If we square the matrix, the value in cell (i, j) indicates the number of paths of length 2 from vertex i to vertex j.
- For example, the M^2 says that there are two paths of length 2 from A to C (A —> B —> C and A —> D —> C)
- This also says that there is exactly one path of length 2 from A to D (A —> B —> D), exactly 1 path of length 2 from B to B (B —> C —> B) and so on.

$$M^2 = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 2 & 1 \\ B & 0 & 1 & 1 & 0 \\ C & 0 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

Adjacency Matrices

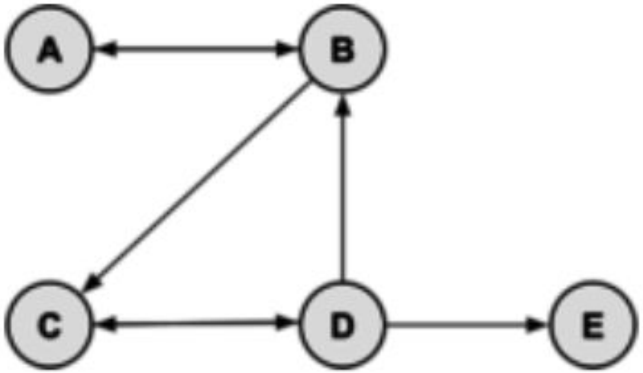
- In general, if we raise M to the pth power, the resulting matrix indicates which paths of length p exist in the graph.
- The value in M^p(i, j) is the number of paths from vertex i to vertex j.

$$M^2 = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 2 & 1 \\ B & 0 & 1 & 1 & 0 \\ C & 0 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

Sample Problem #1

Find the number of different cycles contained in the directed graph with vertices {A, B, C, D, E} and edges {AB, BA, BC, CD, DC, DB, DE}.

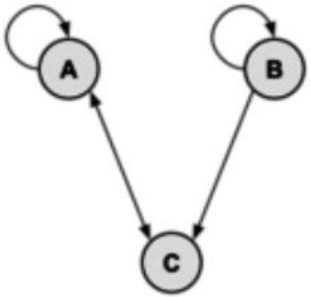
Solution: The graph is as follows:



By inspection, the cycles are: ABA, BCDB, and CDC. Thus, there are 3 cycles in the graph.

Sample Problem #2

In the following directed graph, find the total number of different paths from vertex A to vertex C of length 2 or 4.



Solution:

Let matrix M represent the graph. Recall that the number of paths from vertex i to vertex j of length p equals $M^p[i,j]$. The values of M, M^2 and M^4 are:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, M^4 = \begin{bmatrix} 5 & 0 & 3 \\ 4 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

There is 1 path of length 2 from A to C (cell [1,3] in M^2). By inspection, the only path of length 2 is A → A → C.

There are 3 paths of length 4 (cell [1,3] in M^4) and they are A → A → A → A → C, A → A → C → A → C, A → C → A → A → C.

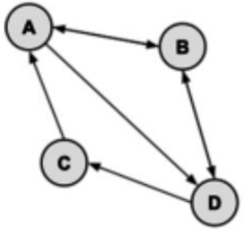
Sample Problem #3

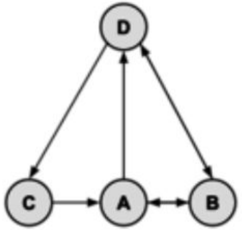
Given the adjacency matrix, draw the directed graph.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution:

There must be exactly 4 vertices: V = {A, B, C, D}. There must be be exactly 7 edges: E = {AB, AD, BA, BD, CA, DB, DC}. Here are two valid drawings of the graph:

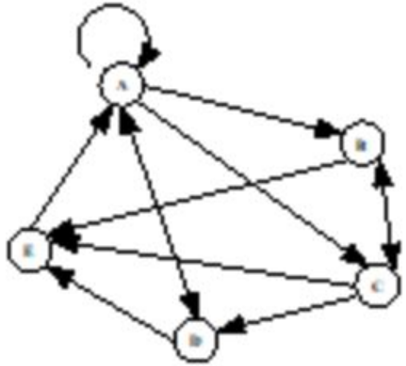




Senior: Past Contests' Problems

1. Graph Theory

How many path of length 2 exist in the directed graph at the right?



1. Graph Theory

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}^2 = \begin{vmatrix} 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

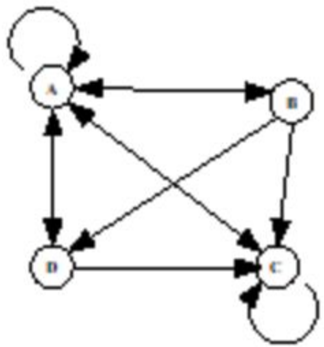
Adding all the entries in the second matrix gives the number of paths of length 2. There are 29 of them.

1. 29

Senior: Past Contests' Problems

2. Graph Theory

How many cycles starting at vertex A are there in the directed graph at the



2. Graph Theory There are 8 cycles: AA, ABA, ABCA, ABDA, ABDCA, ACA, ADA, and ADCA

2. 8

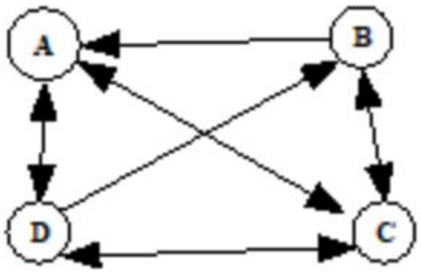
Intermediate: Past Contests' Problems

1. Graph Theory
Given the adjacency matrix, draw the directed graph.

0	0	1	1
1	0	1	0
1	1	0	1
1	1	1	0

1. Graph Theory

The directed graph is shown at the right.

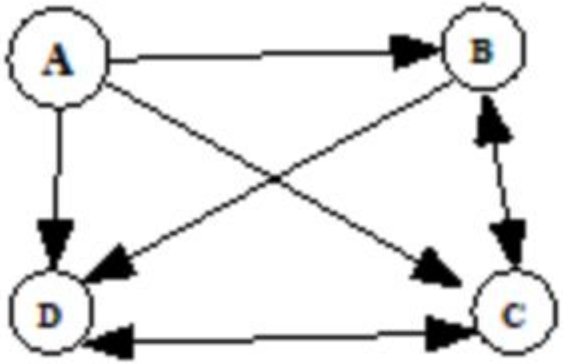


1. As shown

Intermediate: Past Contests' Problems

2. Graph Theory

How many cycles are there from vertex B?



2. Graph Theory
There are 2 cycles from B: BCB and BDCB.

2. 2