Boolean Algebra

ACSL Contest #3

What is Boolean Algebra and why does it matter?

- Boolean algebra is the branch of algebra in which the values of variables and constants have exactly two values: true and false, usually denoted 1 and 0 respectively.

```
s = 0
x = 1
while (s < 100):
    if (x % 2 == 0) and (x % 3 != 0)
        then s = s + x
    x = x + 1
```

- It's the basis for digital circuits that make up a computer's hardware.

Operators Overview

- Basic operators: AND, OR, and NOT
- Secondary operators: XOR (eXclusive OR) and XNOR (eXclusive NOR).
    - Secondary operators are secondary in the sense that they can be composed of basic operators.

AND Operator

- The AND of two values is true only whenever both values are true. It's notated as xy or x * y.

x	y	xy
0	0	0
0	1	0
1	0	0
1	1	1

OR Operator

- The OR of two values is true whenever either or both values are true. It is written as x + y.

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT Operator

- The NOT of a value is its opposite; that is , the not of a true value is false whereas the not of a false value is true.
- It is written as $\bar{x}$ or $\neg x$

x	$\bar{x}$
0	1
1	0

XOR Operator

- The XOR of two values is true whenever the values are different.
- It uses the $\oplus$ operator and can be built from the basic operators:

$$x \oplus y = x\bar{y} + \bar{x}y$$

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

XNOR Operator

- The XNOR of two values is true whenever the values are the same.
  - It is the NOT of the XOR function.
- It uses the $\odot$ operator:

$$x \odot y = \overline{x \oplus y}$$

- Can be built from:

$$x \odot y = xy + \overline{xy}$$

x	y	$x \odot y$
0	0	1
0	1	0
1	0	0
1	1	1

Laws

- A law of boolean algebra is an identity such as x+ (y+z) = (x+y)+z between two boolean terms.
- A boolean term is an expression built up from variables, the constants 0 and 1, and operations and, or, not, xor, and xnor.
- As in ordinary algebra, parentheses are used to group terms.
- NOT AND is OR and NOT OR is AND
- When a not is represented with an overhead horizontal line, there is an implicit grouping of the terms under the line:

$x \cdot \overline{y + z}$ is evaluated as if it were written $x \cdot \overline{(y + z)}$

Order of Precedence

1. NOT
2. AND
3. XOR and XNOR
4. OR

Operators with the same level of precedence are evaluated from left to right.

Fundamental Identities

Commutative Law – The order of application of two separate terms is not important.	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative Law – Regrouping of the terms in an expression doesn't change the value of the expression.	$(x + y) + z = x + (y + z)$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Idempotent Law – A term that is *or′* ed or *and′* ed with itself is equal to that term.	$x + x = x$	$x \cdot x = x$
Annihilator Law – A term that is *or′* ed with 1 is 1; a term *and′* ed with 0 is 0.	$x + 1 = 1$	$x \cdot 0 = 0$
Identity Law – A term *or′* ed 0 or *and′* ed with a 1 will always equal that term.	$x + 0 = x$	$x \cdot 1 = x$
Complement Law – A term *or′* ed with its complement equals 1 and a term *and′* ed with its complement equals 0.	$x + \overline{x} = 1$	$x \cdot \overline{x} = 0$
Absorptive Law – Complex expressions can be reduced to a simpler ones by absorbing like terms.	$x + xy = x$ $x + \overline{x}y = x + y$ $x(x + y) = x$	
Distributive Law – It's OK to multiply or factor-out an expression.	$x \cdot (y + z) = xy + xz$ $(x + y) \cdot (p + q) = xp + xq + yp + yq$ $(x + y)(x + z) = x + yz$	
DeMorgan's Law – An *or* (*and*) expression that is negated is equal to the *and* (*or*) of the negation of each term.	$\overline{x + y} = \overline{x} \cdot \overline{y}$	$\overline{x \cdot y} = \overline{x} + \overline{y}$
Double Negation – A term that is inverted twice is equal to the original term.	$\overline{\overline{x}} = x$	
Relationship between XOR and XNOR	$x \odot y = \overline{x \oplus y} = x \oplus \overline{y} = \overline{x} \oplus y$	

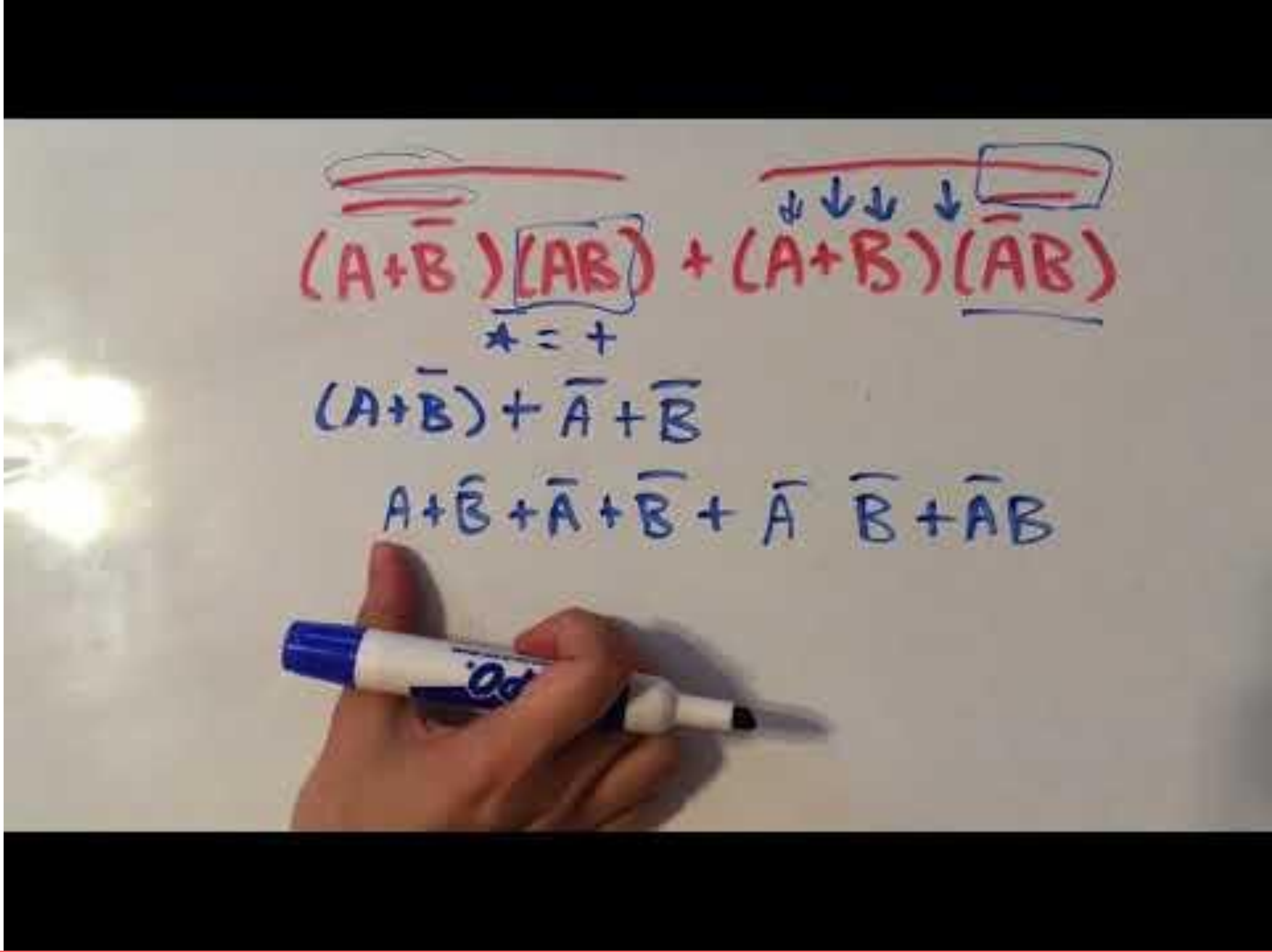
Absorptive Law Proof via Truth Table

Prove using truth table: $A + \bar{A}.B = A + B$

Proof

A	B	$A + B$	$\bar{A}$	$\bar{A}.B$	$A + \bar{A}.B$
0	0	0	1		
0	1	1	1		
1	0	1	0		
1	1	1	0		

Solved Example



Example Problem 1

Simplify the following expression as much as possible: $\overline{\overline{A(A+B)} + B\overline{A}}$

Solution:

The simplification proceeds as follows:

$$\overline{\overline{A(A+B)} + B\overline{A}}$$

$$\begin{aligned}
&= \left(\overline{\overline{A(A+B)}}\right) \cdot \left(\overline{B\overline{A}}\right) && \text{(DeMorgan's Law)} \\
&= (A(A+B)) \cdot \left(\overline{B} + \overline{\overline{A}}\right) && \text{(Double Negation; DeMorgan's Law)} \\
&= A \cdot \left(\overline{B} + A\right) && \text{(Absorption; Double Negation)} \\
&= A && \text{(Absorption)}
\end{aligned}$$

Example Problem 2: Question

Find all orderd pairs (A, B) that make the following expression *true*: $\overline{\overline{(A + B)} + \overline{A}B}$

- There are typically two approaches to solving this type of problem.
  - One approach is to simplify the expression as much as possible, until it's obvious what the solutions are.
  - The other approach is to create a truth table of all possible inputs, with columns for each subexpression.

Problem 2: Solutions

The simplification approach is as following:

$$
\begin{aligned}
&\overline{\overline{(A+B)}+\overline{A}B} \\
&\quad = \overline{\overline{A+B}} \cdot \overline{\overline{A}B} \\
&\quad = (A+B)\cdot(\overline{\overline{A}}+\overline{B}) \\
&\quad = (A+B)\cdot(A+\overline{B}) \\
&\quad = AA + A\overline{B} + BA + B\overline{B} \\
&\quad = A + A(\overline{B}+B) + 0 \\
&\quad = A + A(1) \\
&\quad = A + A \\
&\quad = A
\end{aligned}
$$

This means that all inputs are valid whenever A is *true*: $(1,0)$ and $(1,1)$

The truth table approach is as following. Each column is the result of a basic operation on two other columns.

#1	#2	#3	#4	#5	#6	#7	#8
		OR of Col#1, Col#2	NOT of Col#3	NOT of Col#1	ADD of Col#1, Col#2	OR of Col#4, Col#6	NOT of Col#7
A	B	$A+B$	$\overline{A+B}$	$\overline{A}$	$\overline{A}B$	$\overline{A+B}+\overline{A}B$	$\overline{\overline{A+B}+\overline{A}B}$
0	**0**	0	1	1	0	1	**0**
0	**1**	1	0	1	1	1	**0**
1	**0**	1	0	0	0	0	**1**
1	**1**	1	0	0	0	0	**1**

The rightmost column is the expression we are solving; it is *true* for the 3rd and 4th rows, where the inputs are $(1,0)$ and $(1,1)$.

Past Contest Senior

1. Boolean Algebra

Simplify the following Boolean expression:

$$\overline{A}(B+\overline{A})+A\overline{B}+B(A+\overline{A}B)$$

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1. Boolean Algebra $\overline{A}(B+\overline{A})+A\overline{B}+B(A+\overline{A}B) = \overline{A}B+\overline{A}\,\overline{A}+A\overline{B}+BA+B\overline{A}B$ $= \overline{A}B+\overline{A}+A\overline{B}+AB+\overline{A}B = \overline{A}(B+1)+A(\overline{B}+B)$ $= \overline{A}+A=1$	1. 1

Past Contest Senior

2. Boolean Algebra

How many ordered triples make the following Boolean expression TRUE?

$$\overline{A(\overline{B+\overline{C}})}$$

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2. Boolean Algebra $\overline{A(\overline{B+\overline{C}})} \;=\; \overline{A}+\overline{\overline{B+\overline{C}}} \;=\; \overline{A}+B+\overline{C}$ $\overline{A}+B+\overline{C} = 0$ only when each term is 0. This only happens for (1, 0, 1). Therefore 7 ordered triples make the expression TRUE.	2. 7

Past Contest Intermediate

1. Boolean Algebra

Simplify the following Boolean algebra expression:

$$A(\overline{A}+B)+\overline{B}(A+B)$$

1. Boolean Algebra $A(\overline{A}+B)+\overline{B}(A+B) = A\overline{A}+AB+\overline{B}A+\overline{B}B$ $= 0+AB+\overline{B}A+0$ $= A(B+\overline{B}) = A$	1. A

Past Contest Intermediate

2. Boolean Algebra

How many ordered triples make the following Boolean expression FALSE?

$$A\overline{B} + \overline{B\,C} + \overline{A}\,\overline{C} + \overline{B}\,\overline{C}$$

2. Boolean Algebra $A\overline{B} + \overline{B\,C} + \overline{A}\,\overline{C} + \overline{B}\,\overline{C} = A\overline{B} + \overline{B} + \overline{C} + \overline{A}\,\overline{C} + \overline{B}\,\overline{C}$ $= \overline{B}(A+1) + \overline{C}(1+\overline{A}+\overline{B})$ $= \overline{B} + \overline{C}$ If $\overline{B}+\overline{C} = 0$, then $\overline{B} = 0 \wedge \overline{C} = 0$. Therefore $\overline{B} = 0 \rightarrow B = 1 \wedge \overline{C} = 0 \rightarrow C = 1.$ (*, 1, 1)	2. 2