Survival Analysis

Session 2: From Data to Distribution

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Recap: Survival Identities

In survival analysis we consider the random variable **“*X*: Time until event”**
x = 0.1 weeks, 2.3 weeks…

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f(*x*): Density function
The relative likelihood of experiencing the event around time *x*.

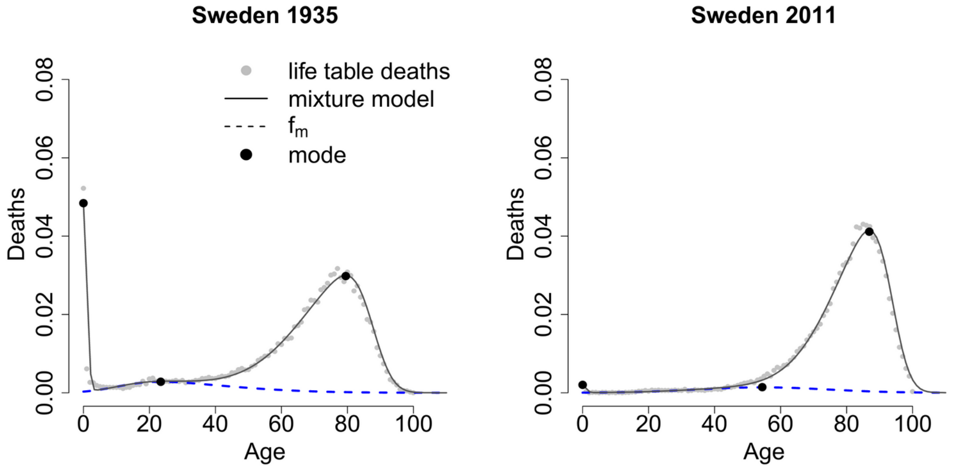

Fig. 2 Model fit on life table deaths for Sweden in 1935 and 2011. The solid line shows the overall mixture model. The dotted line highlights the fit of the Skew Normal employed to estimate accidental and premature mortality. The big dots point out the three modal ages of the distribution

Zanotto etal. (2021). A Mixture-Function Mortality Model.

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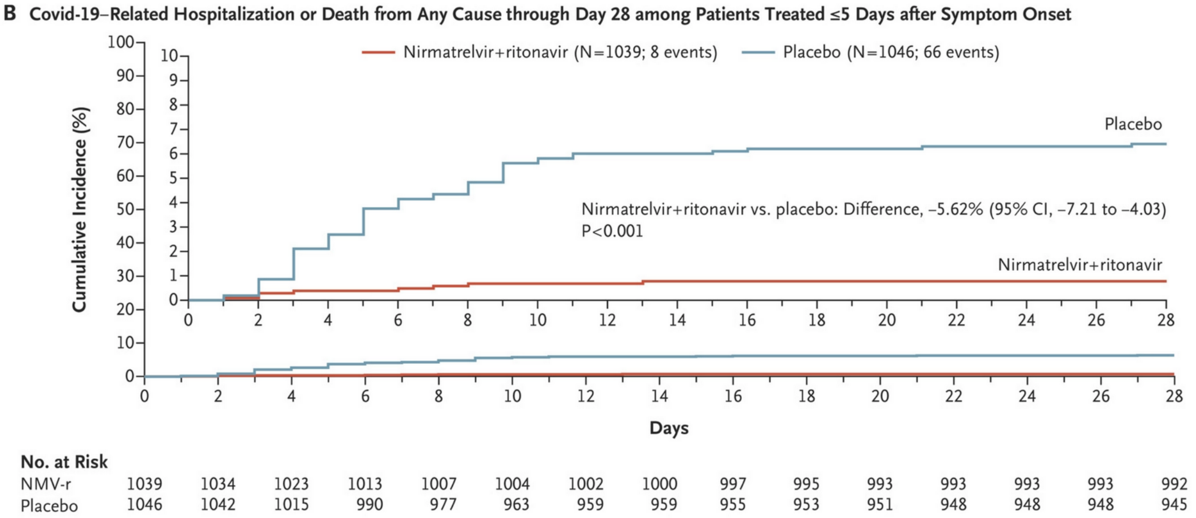
The relative likelihood of experiencing the event around time *x*.

$F(x)=\int_0^x f(x)\,dx$

F(*x*): Distribution function
aka Cumulative function
The probability of experiencing the event until time *x*.
F(*x*) = P(*X*≤*x*)



Hammond etal. (2022).
Oral Nirmatrelvir for High-Risk, Nonhospitalized Adults with Covid-19.

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f(*x*): Density function
The relative likelihood of experiencing the event around time *x*.

$F(x) = \int_0^x f(x)\,dx$

$S(x) = \int_x^{Inf} f(x)\,dx$

F(*x*): Distribution function
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The probability of experiencing the event until time *x*.
F(*x*) = P(*X*≤*x*)

$S(x)=1-F(x)$

S(*x*): Survival function

The probability of *not* experiencing the event until time *x*.
S(*x*) = P(*X*>*x*)

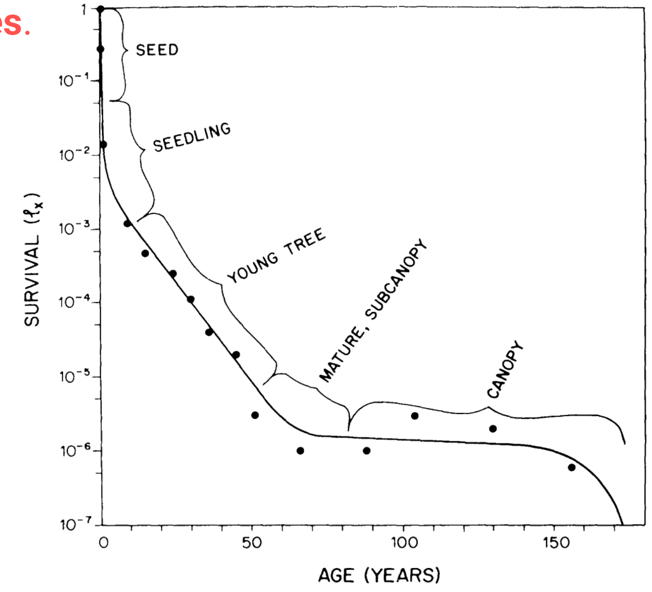

FIGURE 1. Survivorship curve for *Euterpe globosa*, over 7 orders of magnitude. All values of l_x except that at 9 years are completely independent of each other, thus providing an internal check on accuracy.

Valen (1975). Life, Death, and Energy of a Tree.

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h(*x*): Hazard function
The instantaneous rate of new events at time *x* among those who did not experience the event yet.

$$h(x) = \lim_{h \to 0} P(x \le X < x+h \mid X \ge x)/h$$

f(*x*): Density function
The relative likelihood of experiencing the event around time *x*.

$F(x)={}_0\int^x f(x)\,dx$

$S(x)={}_x\int^{\text{Inf}} f(x)\,dx$

$h(x)=-S'(x)/S(x)$

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The probability of experiencing the event until time *x*.

$$F(x) = P(X \le x)$$

$S(x)=1-F(x)$

S(*x*): Survival function

The probability of *not* experiencing the event until time *x*.

$$S(x) = P(X > x)$$

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Klüver (2022). The survival of interest groups.

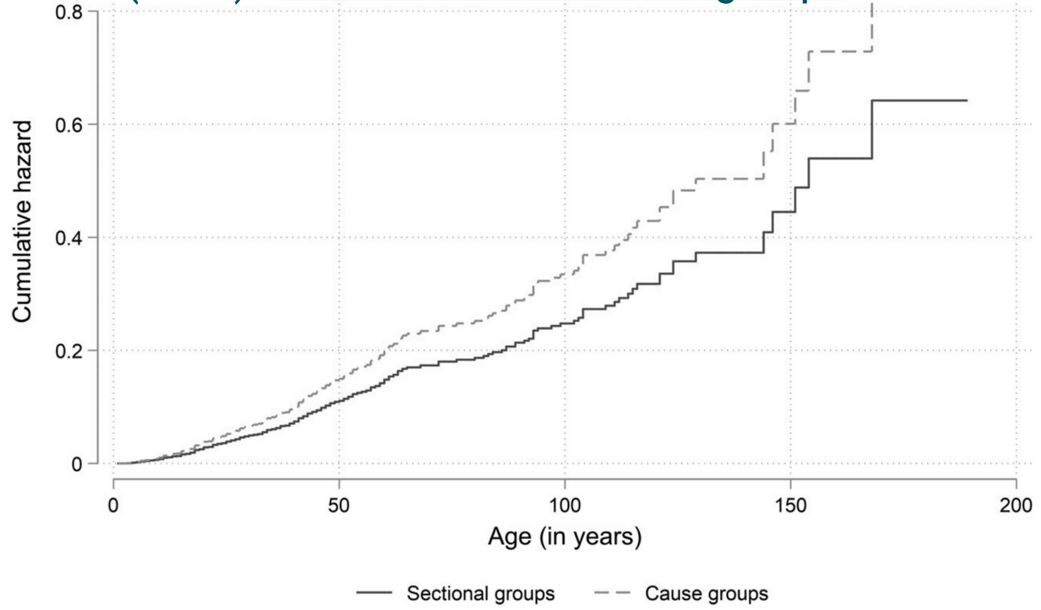

Figure 3. The effect of interest group type.

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$F(x) = {}_0\int^x$

$H(x) = {}_0\int^x h(x)\,dx$

$= -\log S(x)$

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aka Cumulative
The probability
experiencing the event until
time *x*.
$F(x) = P(X \le x)$

time *x*.
$S(x) = P(X > x)$

H(*x*): Cumulative Hazard
The integral of h(x).

$S(x) = \exp(-H(x))$

Last Week’s Homework

Choose a time-to-event setting that interests you and look up a constant rate related to that setting. What is the time scale for your setting? When does the time-to-event start? When have have half of the population experienced the event given the chosen rate?

Example: Today we looked at the time until I catch COVID. I choose the rate 2,428 infections per 100,000 persons per 7 days from the local COVID incidences and assumed this rate to be constant. The timescale was “weeks into the semester” and it starts at the first week of the semester. I used the survival function of the exponential distribution to calculate the time until the probability of catching COVID reached 50%.
$S(x) = \exp(-\int_0^x \lambda \mathrm{d}x) = \exp(-\lambda x)$

What Does Survival Data Look Like?

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x = 0.1 month, 2.3 months…

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Time unit? Months

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End of observation? 10 years follow up

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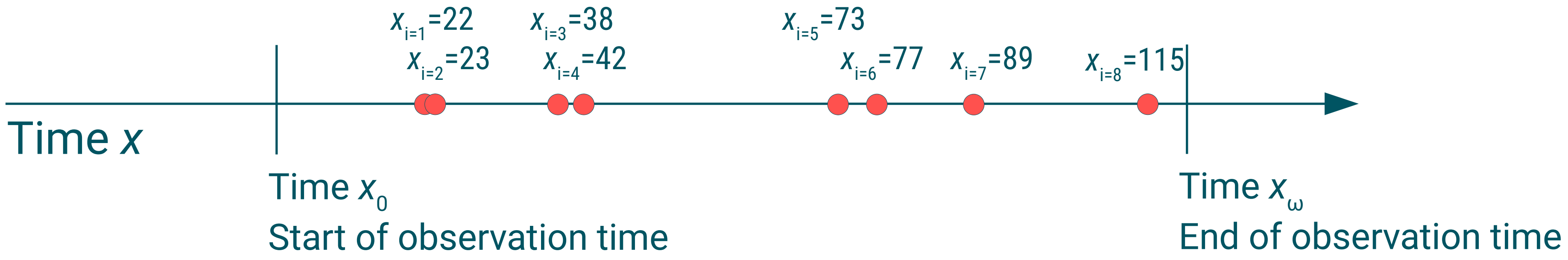
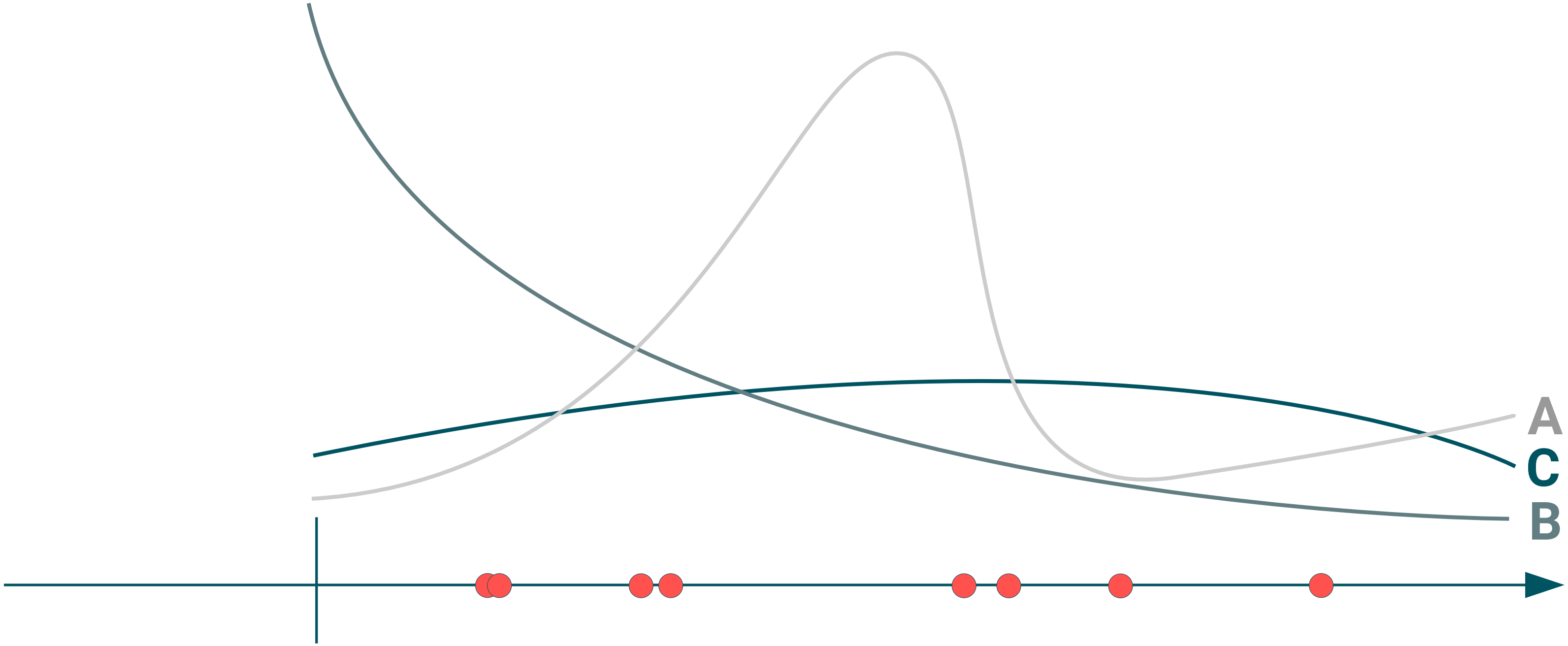
Who? 8 breast cancer patients
Start of observation? Cancer diagnosis
Event of interest? Death
Time unit? Months
End of observation? 10 years follow up

Observation index i	Observed event time x
1	22
2	23
3	38
4	42
5	73
6	77
7	89
8	115

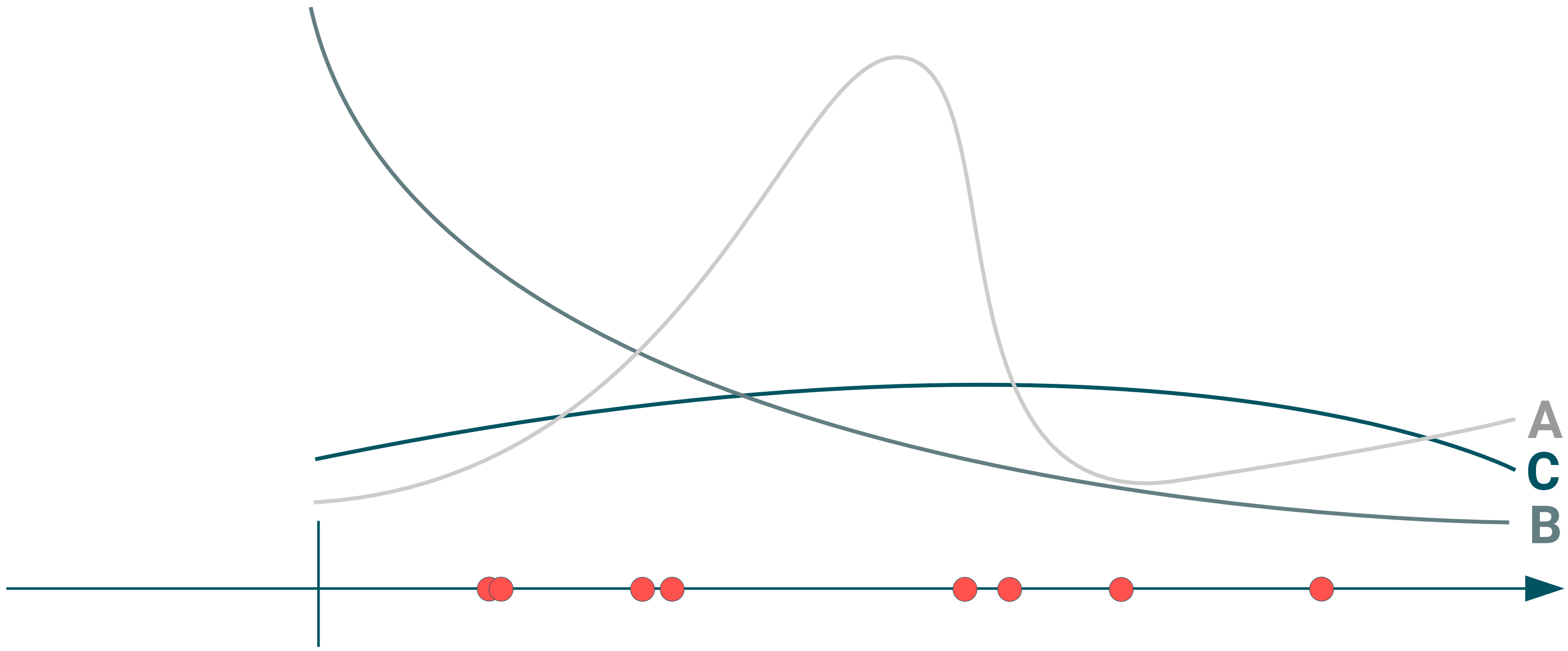


Inferring the Survival Density from Data



Which distribution most likely corresponds to the data?

Inferring the Survival Density from Data



Which distribution most likely corresponds to the data?
→ Maximum Likelihood Estimation

Censoring

But what if some people did not experience the event during the observation time?

Next week → Censored observations

Homework

Using the breastcancer data choose a distribution from the table to the right and fit it to the data via Maximum Likelihood. Use R. You can adapt the script we wrote today.

TABLE 2.2
Hazard Rates, Survival Functions, Probability Density Functions, and Expected Lifetimes for Some Common Parametric Distributions

Distribution	*Hazard Rate* $h(x)$	*Survival Function* $S(x)$	*Probability Density Function* $f(x)$	*Mean* $E(X)$
Exponential $\lambda > 0,\ x \geq 0$	λ	$\exp[-\lambda x]$	$\lambda \exp(-\lambda x)$	$\frac{1}{\lambda}$
Weibull $\alpha, \lambda > 0,$ $x \geq 0$	$\alpha\lambda x^{\alpha-1}$	$\exp[-\lambda x^{\alpha}]$	$\alpha\lambda x^{\alpha-1}\exp(-\lambda x^{\alpha})$	$\frac{\Gamma(1+1/\alpha)}{\lambda^{1/\alpha}}$
Gamma $\beta, \lambda > 0,$ $x \geq 0$	$\frac{f(x)}{S(x)}$	$1 - I(\lambda x, \beta)^{*}$	$\frac{\lambda^{\beta} x^{\beta-1}\exp(-\lambda x)}{\Gamma(\beta)}$	$\frac{\beta}{\lambda}$
Log normal $\sigma > 0, x \geq 0$	$\frac{f(x)}{S(x)}$	$1 - \Phi\left[\frac{1\mathrm{n}x - \mu}{\sigma}\right]$	$\frac{\exp\left[-\frac{1}{2}\left(\frac{1\mathrm{n}x-\mu}{\sigma}\right)^2\right]}{x(2\pi)^{1/2}\sigma}$	$\exp(\mu + 0.5\sigma^2)$
Log logistic $\alpha, \lambda > 0, x \geq 0$	$\frac{\alpha x^{\alpha-1}\lambda}{1+\lambda x^{\alpha}}$	$\frac{1}{1+\lambda x^{\alpha}}$	$\frac{\alpha x^{\alpha-1}\lambda}{[1+\lambda x^{\alpha}]^2}$	$\frac{\pi \mathrm{Csc}(\pi/\alpha)}{\alpha\lambda^{1/\alpha}}$ if $\alpha > 1$
Normal $\sigma > 0,$ $-\infty < x < \infty$	$\frac{f(x)}{S(x)}$	$1 - \Phi\left[\frac{x-\mu}{\sigma}\right]$	$\frac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]}{(2\pi)^{1/2}\sigma}$	μ
Exponential power $\alpha, \lambda > 0, x \geq 0$	$\alpha\lambda^{\alpha} x^{\alpha-1}\exp\{[\lambda x]^{\alpha}\}$	$\exp\{1 - \exp[(\lambda x)^{\alpha}]\}$	$\alpha e \lambda^{\alpha} x^{\alpha-1}\exp[(\lambda x)^{\alpha}] - \exp\{\exp[(\lambda x)^{\alpha}]\}$	$\int_0^{\infty} S(x)dx$
Gompertz $\theta, \alpha > 0, x \geq 0$	$\theta e^{\alpha x}$	$\exp\left[\frac{\theta}{\alpha}(1 - e^{\alpha x})\right]$	$\theta e^{\alpha x}\exp\left[\frac{\theta}{\alpha}(1 - e^{\alpha x})\right]$	$\int_0^{\infty} S(x)dx$
Inverse Gaussian $\lambda \geq 0, x \geq 0$	$\frac{f(x)}{S(x)}$	$\Phi\left[\left(\frac{\lambda}{x}\right)^{1/2}\left(1 - \frac{x}{\mu}\right)\right] - e^{2\lambda/\mu}\Phi\left\{-\left[\frac{\lambda}{x}\right]^{1/2}\left(1 + \frac{x}{\mu}\right)\right\}$	$\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left[\frac{\lambda(x-\mu^2)}{2\mu^2 x}\right]$	μ
Pareto $\theta > 0, \lambda > 0$ $x \geq \lambda$	$\frac{\theta}{x}$	$\frac{\lambda^{\theta}}{x^{\theta}}$	$\frac{\theta\lambda^{\theta}}{x^{\theta+1}}$	$\frac{\theta\lambda}{\theta - 1}$ if $\theta > 1$
Generalized gamma $\lambda > 0, \alpha > 0,$ $\beta > 0, x \geq 0$	$\frac{f(x)}{S(x)}$	$1 - I[\lambda x^{\alpha}, \beta]$	$\frac{\alpha\lambda^{\beta} x^{\alpha\beta-1}\exp(-\lambda x^{\alpha})}{\Gamma(\beta)}$	$\int_0^{\infty} S(x)dx$

$^{*}\ I(t, \beta) = \int_0^t u^{\beta-1}\exp(-u)du/\Gamma(\beta).$

Materials for this lecture

github.com/jschoeley/survival_analysis-ur-ss22

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