Math 6366 Optimization: Homework 04

Due on Oct 30, 2018

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Problem 1

Consider the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

The auxiliary function has the form:

$$\phi(x) = f_0(x) + \beta p(x) = f_0(x) + \beta \|Ax - b\|_2^2$$

Supposing $\tilde{x}$ is a minimizer of ϕ, show how to find the dual feasible point from $\tilde{x}$. Find the corresponding lower bound on the optimal value.

Solution:

$$\begin{aligned} \phi(x) &= f_0(x) + \beta(Ax - b)^\top (Ax - b) \\ \phi(x) &= f_0(x) + \beta[x^\top A^\top Ax - 2(Ax)^\top b + b^\top b] \\ \phi'(x) &= f_0'(x) + 2\beta A^\top (Ax - b) \end{aligned}$$

if $\tilde{x}$ minimizes ϕ then it also minimizes:

$$L(x, \nu) = f_0(x) + \nu^\top (Ax - b)$$

with $\nu^\top = 2\beta(A\tilde{x} - b)$, and then:

$$\begin{aligned} g(\nu) &= \inf_x (f_0(x) + \nu^\top (Ax - b) \\ &= f_0(\tilde{x}) + 2\beta \|A\tilde{x} - b\|_2^2 \end{aligned}$$

This provides a lower bound on f_{opt}, since

$$f_0(x) \geq g(nu) = f_0(\tilde{x}) + 2\beta \|A\tilde{x} - b\|_2^2$$

Problem 2

The weak duality inequality, $gopt \leq fopt$, holds when $gopt = \infty$ and $fopt = \infty$. Show that it holds in the two cases below as well.

a If $fopt = \infty$ we must have $gopt = \infty$.

b If $gopt = \infty$ we must have $fopt = \infty$.

Solution:

a If $fopt = -\infty$, the primal is unbounded below. Therefore $L(x, \lambda) = f_0(x) + \sum_i \lambda_i f_i(x)$ is unbounded below, and $gopt = -\infty$.

b If $gopt = \infty$, the dual is unbounded above, and the primal is infeasible. To see this, consider if the primal were feasible, so that $f_i(x) \leq 0$ for all i. Then for $\lambda \geq 0$.

$$\begin{aligned} g(\lambda) &= inf(f_0(x) + \sum_i \lambda_i f_i(x)) \\ &= f_0(\tilde{x}) + \sum_i \lambda_i f_i(x)) \end{aligned}$$

Which implies the dual is bounded from above.

Problem 3

Derive a dual problem for the unconstrained problem

$$\underset{x}{\text{minimize}} \quad \sum_{i=1}^{k} \|A_i x + b_i\|_2 + 1/2\|x - x_{\text{ref}}\|_2^2$$

Solution:
Introduce variables y_i and rewrite:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{i=1}^{k} \|y_i\|_2 + 1/2\|x - x_{\text{ref}}\|_2^2 \\ \text{subject to} \quad & A_i x + b_i - y_i = 0 \quad \forall i \end{aligned}$$

The Lagrangian is:

$$L(x, y, \nu) = \sum_{i=1}^{k} \|y_i\|_2 + 1/2\|x - x_{\text{ref}}\|_2^2 + \sum_{i=1}^{k} \nu_i^\top (A_i x + b_i - y_i)$$

to find the infimum wrt x, take the derivative

$$\begin{aligned} \nabla_x L(x, y, \nu) &= 0 + x - x_{\text{ref}} + \sum_{i=1}^{k} \nu_i^\top A_i \\ 0 &= x - x_{\text{ref}} + \sum_{i=1}^{k} \nu_i^\top A_i \end{aligned}$$

So the min of L is $-\infty$ unless $\sum_{i=1}^{k} \nu_i^\top A_i = 0$, whereupon $x = x_{\text{ref}}$. Then we have:

$$\begin{aligned}
g(\nu) &= \inf_y \Big(\sum_{i=1}^k \|y_i\|_2 + \sum_{i=1}^k \nu_i^\top (A_i x_{\text{ref}} + b_i - y_i)\Big) \\
&= \sum_{i=1}^k \nu_i^\top b_i + \sum_{i=1}^k \nu_i^\top A_i x_{\text{ref}} + \sum_{i=1}^k \inf_y (\|y_i\|_2 - \nu_i^\top y_i) \\
&= \sum_{i=1}^k [\nu_i^\top b_i - \sup_y (\nu_i^\top y_i - \|y_i\|_2)] \qquad \text{since } \nu_i^\top A_i = 0 \\
&= \sum_{i=1}^k [\nu_i^\top b_i - \|\nu_i\|_{2*}]
\end{aligned}$$

So the dual problem is:

$$\begin{aligned}
\underset{\nu}{\text{maximize}} \quad & g(\nu) \\
\text{subject to} \quad & \sum_{i=1}^k \nu_i A_i = 0
\end{aligned}$$

Problem 4

Consider the problem

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0 \quad \forall i
\end{aligned}$$

where f_i are differentiable and convex. Suppose $x and \lambda$ satisfy the KKT conditions. Show that this implies that $\nabla f_0(x)^\top (x x^*) \geq 0$ for all feasible x.

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Solution:

The KKT conditions are:

$$f_i(x^*) \leq 0 \tag{1}$$

$$\lambda_i^* \geq 0 \tag{2}$$

$$\lambda_i^* f_i(x^*) = 0 \tag{3}$$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) = 0 \tag{4}$$

If x is feasible, we have:

$$f_i(x) \leq 0$$

and we have:

$$\begin{aligned}
0 \geq f_i(x) &\geq f_i(x^*) + \nabla f_i(x^*)^\top (x - x^*) \\
0 &\geq \sum \lambda_i^* [f_i(x^*) + \nabla f_i(x^*)^\top (x - x^*)] \quad \text{from (2)} \\
&= \sum \lambda_i^* f_i(x^*) + \sum \lambda_i^* \nabla f_i(x^*)^\top (x - x^*) \\
&= \sum \lambda_i^* \nabla f_i(x^*)^\top (x - x^*) \quad \text{from (3)} \\
&= -\nabla f_0(x^*)^\top (x - x^*) \quad \text{from (4)}
\end{aligned}$$

Which establishes the result.

Problem 5

Find the dual function of the linear program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Provide the dual problem, and make the implicit equality constraints explicit.
Solution:

$$\begin{aligned} g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\ &= \inf_x [c^\top x + \lambda^\top (Gx - h) + \nu^\top (Ax - b)] \\ &= -\lambda^\top h - \nu^\top b + \inf_x (c^\top + \lambda^\top G + \nu^\top A)x \end{aligned}$$

the latter term is a linear function, so:

$$g(\lambda, \nu) = \begin{cases} -\lambda^\top h - \nu^\top b & \text{if} \quad c^\top + \lambda^\top G + \nu^\top A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is

$$\begin{aligned} \underset{x}{\text{maximise}} \quad & -\lambda^\top h - \nu^\top b \\ \text{subject to} \quad & \lambda \succeq 0 \\ & c^\top + \lambda^\top G + \nu^\top A = 0 \end{aligned}$$

Problem 7

Consider the optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & tr(Y(x)) \\ \text{subject to} \quad & x \succeq 0 \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

where $Y(x) := (\sum(x_i y_i y_i^\top))^{-1}$, the vectors y_i are given, and the domain is given by... Derive the dual problem. Simplify the dual problem as much as you can.

Solution:
Rewrite:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & tr(X^{-1}) \\ \text{subject to} \quad & -x \preceq 0 \\ & \mathbf{1}^\top x = 1 \\ & X - \sum(x_i y_i y_i^\top) = 0 \end{aligned}$$

The Lagrangian is

$$\begin{aligned} L(X, x, \lambda, \nu, N) &= tr(X^{-1}) - \lambda^\top x + \nu(\mathbf{1}^\top x - 1) + \langle N, X - \sum(x_i y_i y_i^\top)\rangle \\ &= tr(X^{-1}) + \rangle N, X\rangle - \sum \lambda_i x_i + \sum \nu x_i - \langle N, \sum(x_i y_i y_i^\top)\rangle - \nu \\ &= tr(X^{-1}) + tr(NX) + \sum x_i(-\lambda_i + \nu - y_i N y_i^\top) - \nu \end{aligned}$$

The minimum over x is $-\infty$ unless $-\lambda_i + \nu - y_i N y_i^\top = 0$.
Taking the derivative wrt X yields:

$$\begin{aligned} \nabla_X L = 0 &= -X^{-2} + N \\ N &= -X^{-2} \\ N^{-1/2} &= X \end{aligned}$$

The dual function is:

$$g(\lambda, \nu, N) = \begin{cases} 2tr(N^{1/2}) - \nu & -\lambda_i + \nu - y_i N y_i^\top = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is:

$$\begin{aligned} \text{maximise} \quad & 2tr(N^{1/2}) - \nu \\ \text{subject to} \quad & -\lambda_i + \nu - y_i N y_i^\top = 0 \end{aligned}$$

Problem 8

Consider the equality constrained least-squares problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - b\|_2^2 \\ \text{subject to} \quad & Gx = h \end{aligned}$$

Provide the KKT conditions and derive expressions for the primal solution x^* and the dual solution ν^*.

Solution:

$$\begin{aligned} L(x, \nu) &= \|Ax - b\|_2^2 + \nu^\top (Gx - h) \\ &= x^\top A^\top Ax + (G^\top \nu - 2A^\top b)^\top x - b^\top b - \nu^\top h \\ \text{taking the derivative} \qquad \nabla_x L = 0 &= 2A^\top Ax + G^\top \nu - 2A^\top b \\ x &= 1/2(A^\top A)^{-1}(2A^\top b - G^\top \nu) \end{aligned}$$

The dual is then:

$$g(\nu) = -(1/4)(G^\top \nu - 2A^\top b)^\top (A^\top A)^{-1} (G^\top \nu - 2A^\top b) - \nu^\top h$$

The KKT optimality conditions provide the following equations:

$$Gx^* = h$$

$$2A^\top (Ax^* - b) + G^\top \nu^* = 0$$

Solving the equations for x^* and ν^* yields some very long equations. (Please don't make me type them up.)