Homework 1

Due on Tuesday, September 11, at 05:20 pm

Answer 10 of the following 13 questions to get full credit for this homework assignment. Please follow the instructions for homework assignments. I reserve the right to deduct points if you do not follow these rules.

1. According to the Cauchy–Schwarz inequality we have $|\langle x, y\rangle| \leq \|x\|_2\|y\|_2$ for any $x, y \in \mathbf{R}^n$, where $\langle\cdot,\cdot\rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is the inner product. Proof the inequality and show that the equality only holds if the vectors x and y are linearly dependent.

2. Let $Q \in \mathbf{R}^{n,n}$ be a positive definite matrix, i.e., $\forall x \in \mathbf{R}^n \backslash \{0\} : x^\mathsf{T} Q x > 0$. Show that

$$\|x\|_Q := \sqrt{x^\mathsf{T} Q x} = \langle x, Qx\rangle^{1/2}$$

   is a norm.

3. Sketch the following sets in $\mathbf{R}^2$ and determine from you figure which sets are convex and which sets are not convex.
   (a) $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$
   (b) $\{(x_1, x_2) : 0 < x_1^2 + x_2^2 \leq 1\}$
   (c) $\{(x_1, x_2) : |x_1| + |x_2| \leq 1\}$

4. Let $C = \{x \in \mathbf{R}^n : Ax \leq b\}$. Show that C is a convex set.

5. Show that the convex hull $H := \operatorname{conv} S$ of a set $S \subseteq \mathbf{R}^n$ is the intersection of all convex sets C that contain S, i.e., H is equal to $\bigcap_{C \supseteq S,\, C \text{ convex}} C =: M$.

6. What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n : a^\mathsf{T} x = b_1\}$ and $\{x \in \mathbf{R}^n : a^\mathsf{T} x = b_2\}$?

7. Let $\{x_i\}_{i=1}^m$, $x_i \in \mathbf{R}^n$. Consider a set of points V that are closer (in Euclidean norm) to x_0 than the other x_i, i.e., $V := \{x \in \mathbf{R}^n : \|x - x_0\|_2 \leq \|x - x_i\|_2,\ \ i = 1, \ldots, m\}$. V is called a Voronoi region around x_0 with respect to $\{x_i\}_{i=1}^m$. Show that V is a polyhedron. Express V in the form $V = \{x : Ax \preceq b\}$.

8. Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality $C = \left\{x \in \mathbf{R}^n : x^\mathsf{T} A x + b^\mathsf{T} x + c \leq 0\right\}$ with $A \in \mathbf{S}^n$, $\mathbf{S}^n := \{M \in \mathbf{R}^{n,n} : M = M^\mathsf{T}\}$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}$. Show that C is convex if $A \succeq 0$. **Hint:** One possible way of showing this is to use the fact that a set is convex if its intersection with all lines of the form $\{u + \lambda v : \lambda \in \mathbf{R}\}$, where u and v are vectors, is a convex set.

9. Determine which of the following sets are convex.
   (a) The set $S = \{x \in \mathbf{R}^n : \alpha \leq a^\mathsf{T} x \leq \beta,\ \ \alpha, \beta \in \mathbf{R}, a \in \mathbf{R}^n\}$.
   (b) The set of points closer to one set S_1 than another set S_2, i.e.,

$$K = \{x \in \mathbf{R}^n : \operatorname{dist}(x, S_1) \leq \operatorname{dist}(x, S_2)\},$$

   where $S_i \subseteq \mathbf{R}^n$ and $\operatorname{dist}(x, S) = \inf\{\|x - y\|_2 : y \in S\}$.
   (c) The set of points whose distance to a point $x_0 \in \mathbf{R}^n$ does not exceed a fixed fraction $\alpha \in [0, 1]$ of the distance to $y \in \mathbf{R}^n$, $y \neq x_0$, i.e., the set $\{x \in \mathbf{R}^n : \|x - x_0\|_2 \leq \alpha\|x - y\|_2\}$.

10. Let $S \subseteq \mathbf{R}^n$ and let $\|\cdot\|$ be some norm in $\mathbf{R}^n$.
    (a) Let $S_\alpha := \{x \in \mathbf{R}^n : \mathrm{dist}(x, S) \le \alpha\}$, where $\mathrm{dist}(x, S) = \inf_{y \in S} \|x - y\|$ with $\alpha \ge 0$. Show that if S is convex, so is S_α.
    (b) Let $S_{-\alpha} := \{x \in \mathbf{R}^n : \mathcal{B}(x, \alpha) \subseteq S\}$, where $\mathcal{B}$ is the ball (in the norm $\|\cdot\|$) centered around x with radius α. Show that if S is convex, then $S_{-\alpha}$ is convex.

11. Let $A \in \mathbf{R}^{m,n}$, $b \in \mathbf{R}^m$, and b is in the range of A. Show that there exists an x satisfying $x \succ 0$, $Ax = b$, if and only if there exists no λ with $A^\mathsf{T}\lambda \succeq 0$, $A^\mathsf{T}\lambda \ne 0$, $b^\mathsf{T}\lambda \le 0$. **Hint:** First prove the following fact from linear algebra: $c^\mathsf{T}x = d$ for all x satisfying $Ax = b$ if and only if there is a vector λ such that $c = A^\mathsf{T}\lambda$, $d = b^\mathsf{T}\lambda$. Then use a separating hyperplane argument.

12. Let $C_i \subseteq \mathbf{R}^n$, $i = 1, 2$, be nonempty convex sets. Show that $S = \theta_1 C_1 + \theta_2 C_2$, where

$$C_1 + C_2 := \{z \in \mathbf{R}^n : z = z_1 + z_2, z_1 \in C_1, z_2 \in C_2\}$$

is convex for any $\theta \in \mathbf{R}, i = 1, 2$.

13. Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be the linear fractional function

$$f(x) = (Ax + b)/(c^\mathsf{T}x + d), \quad \mathrm{dom}\, f = \{x : c^\mathsf{T}x + d > 0\}.$$

In this problem we study the inverse image of a convex set C under f, i.e.,

$$f^{-1}(C) = \{x \in \mathrm{dom}\, f : f(x) \in C\}.$$

For each of the following sets $C \subseteq \mathbf{R}^n$, give a simple description of $f^{-1}(C)$.
    (a) The halfspace $C = \{y : g^\mathsf{T}y \le h, \ \ g \ne 0\}$.
    (b) The ellipsoid $C = \{y : y^\mathsf{T}P^{-1}y \le 1, \ \ P \in \mathbf{S}^n_{++}\}$.