Homework 4 (Computational)

Due on Tuesday, October 30, at 5:20 pm

Solve the following 3 problems to get full credit for this homework assignment. Please follow the instructions for homework assignments. I reserve the right to deduct points if you do not follow these rules. The computational part of the homework can be submitted in groups of up to 3 people. When you submit your homework to homework.am.math@gmail.com make sure that you state who was in your group.

1. In the theoretical part of the homework we considered the quadratically constrained quadratic program

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & \frac{1}{2} x^\mathsf{T} A_0 x + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & \frac{1}{2} x^\mathsf{T} A_i x + q_i^\mathsf{T} x + c_i \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

   with $A_0 \in \mathbf{S}_{++}^n$, and $A_i \in \mathbf{S}_+^n$, $i = 1, \ldots, m$. Set $n = 12$ and $m = 3$. Use random vectors $q_i \in \mathbf{R}^n$, random scalars c_i, and matrices A_i constructed from random matrices $Q_i \in \mathbf{R}^{n,n}$. Use the `randn` command in combination with `rng(1,'v5normal')` to ensure reproducibility. More precisely, the matrices A_i are constructed as follows: $A_i = Q_i^\mathsf{T} Q_i$, $i = 0, 1, 2, 3$ with Q_i random. To compute A_0 add a small perturbation by the identity. That is, $A_0 = Q_0^\mathsf{T} Q_0 + \varepsilon \, \text{diag}(1, \ldots, 1)$, where $\epsilon > 0$ is the floating-point relative accuracy. Solve the primal problem using CVX. Use the CVX command `quad_form(x,B)` to implement the quadratic form $x^\mathsf{T} B x$. Compute the optimal dual variables (how this can be done during the solve of the primal problem can be seen in `ex09_dual.m` in the OPTIK repository). Compute the duality gap by evaluating the dual function for the computed (optimal) dual variables. Notice that `ex09_dual.m` shows you how to access the optimal value for the primal problem. Submit your script.

2. We consider a game with two players K(laus) and L(isa). The player K makes choices $k \in \{1, \ldots, n\}$ and player L makes choices $l \in \{1, \ldots, m\}$. Then, say, player K makes a payment of $a_{kl} \geq 0$ to player L, where

$$A = \left[a_{kl}\right]_{k=1, l=1}^{m,n} \in \mathbf{R}^{m,n}$$

   is the payoff matrix for the game. The goal of player K is to make the payment as small as possible. The goal of player L is to maximize the payment. Each player makes his or her choice randomly and independently of the other player's choice. The associated probability distributions are

$$\text{prob}(k = i) = u_i, \;\; i = 1, \ldots, n, \quad \text{and} \quad \text{prob}(l = i) = v_i, \;\; i = 1, \ldots, m.$$

   Thus, $u \in \mathbf{R}^n$ (player K) and $v \in \mathbf{R}^m$ (player L) correspond to the associated probability distributions for each player's strategy. The expected payoff from player L to player K is

$$\sum_{k=1}^{n} \sum_{l=1}^{m} u_k v_l a_{kl} = u^\mathsf{T} A v.$$

   Suppose player L knows the strategy u of player K. Player L will choose v to maximize the expected payoff $u^\mathsf{T} A v$, i.e.,

$$\sup_{v} \, \{u^\mathsf{T} A v : v \succeq 0, \;\; e_m^\mathsf{T} v = 1\} = \max_{i=1,\ldots,m} \, (A^\mathsf{T} u)_i$$

The best player K can do is to choose u to minimize the worst-case payoff to player L, i.e.,

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,m} (A^\mathsf{T} u)_i \\ \text{subject to} & u \succeq 0, \quad e_n^\mathsf{T} u = 1. \end{array} \tag{1}$$

In a similar way, player K will choose u to minimize $u^\mathsf{T} A v$, which results in an expected payoff of

$$\inf_u \{u^\mathsf{T} A v : u \succeq 0, e_n^\mathsf{T} u = 1\} = \min_{i=1,\ldots,n} (Av)_i.$$

So, player L will choose a strategy v that will maximize the expected payoff, i.e.,

$$\begin{array}{ll} \text{maximize} & \min_{i=1,\ldots,n} (Av)_i \\ \text{subject to} & v \succeq 0, \quad e_m^\mathsf{T} v = 1. \end{array} \tag{2}$$

Initialize A as a random 10×10 matrix (i.e., $m = n$). Use the `randn(n)` command in combination with `rng(1,'v5normal')` to ensure reproducibility. Formulate the equivalent linear program of (1) and (2) and solve the resulting optimization problems using CVX. Compare the optimal values (i.e., the expected payoff) of both problems. Also derive the dual of the linear program equivalent to (1), compute the optimal dual variables, and evaluate the dual function. Compute the duality gap. Does strong duality hold? Compare the dual problem to (2) and explain your observations (as comments in your script). Submit your script.

3. Write a CVX program to solve the following unconstrained and constrained optimization problems.
   (a) Solve the unconstrained (regularized) optimization problem

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \|x - y^\delta\|_2 + \beta \|Bx\|_2, \tag{3}$$

where $y^\delta \in \mathbf{R}^n$ is an observed quantity of interest perturbed by noise and B is the bidiagonal matrix

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \in \mathbf{R}^{n-1,n}.$$

The first term in (3) is represents the mismatch between the data and the unknown x. The second term is a regularization model (in particular, a smoother; why?). Suppose $y \in \mathbf{R}^n$ is a vector with entries $y_i = 0.5 \sin((2\pi/n)(i-1)) \sin(0.01(i-1)), \quad i = 1, \ldots, n$. Use $n = 4096$. (You can reduce the memory requirements by storing B as a sparse matrix.) Construct the vector y^δ via $y^\delta = y + 0.05\delta y$, where $\delta y \in \mathbf{R}^n$ is a random perturbation. Use the `randn` command in combination with `rng(1,'v5normal')` to compute δy.

Solve the optimization problem 100 times for different choices of the regularization parameter β. Pick $\beta > 0$ from a logarithmically spaced vector ranging from 1e−10 to 1e10 with 100 entries. Plot the trade-off curve $\|x_{\text{sol}} - y^\delta\|_2$ (residual norm; x-axis) versus $\|Bx_{\text{sol}}\|_2$ (solution norm; y-axis) for different choices of β. Here, $x_{\text{sol}} \in \mathbf{R}^n$ represents the solution for a particular choice of β (i.e., x_{sol} is a function of / parametrized by β). To be able to generate this curve you need to evaluate the two building blocks (mismatch/data term and regularization model) for each x_{sol} and store the resulting values.

Identify the regularization parameter $\beta^\star$ that corresponds to the value $d^\star = \|x - y^\delta\|_2$ located at the ''corner'' of the curve (point with largest curvature (approximately)). Solve (3) using the identified $\beta^\star$ (solution 1). Subsequently solve the constrained optimization problem

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & \|Bx\|_2 \\ \text{subject to} \quad & \|x - y^\delta\|_2 \leq \alpha d^\star \end{aligned}$$

for $\alpha \in \{\frac{1}{3}, 1, 3\}$ (solutions 2, 3, and 4). Plot the four solutions x_{sol}, y, as well as y^δ. Also report the relative error

$$\epsilon = \frac{\|x_{\text{sol}} - y\|_2^2}{\|y\|_2^2}$$

for the found solutions. Submit your script.

(b) Repeat the experiment in (a) for non-smooth data y. In particular, $y \in \mathbf{R}^n$ is a vector with entries $y_i = \phi_i + 0.5\sin((2\pi/n)(i-1))$, $i = 1, \ldots, n$. Choose $n = 4096$. The vector $\phi \in \mathbf{R}^n$ represents a step function of the form

$$\phi_i = \begin{cases} 1 & i \in \Phi \\ -1 & \text{otherwise} \end{cases}$$

with $\Phi = \{1, 2, \ldots, 1024\} \cup \{2049, 2050 \ldots, 3072\} \subset \mathbf{N}_+$. Construct the vector y^δ via $y^\delta = y + 0.1\delta y$, where $\delta y \in \mathbf{R}^n$ is a random perturbation. Use the `randn` command in combination with `rng(1,'v5normal')` to ensure reproducibility. Use $\alpha \in \{\frac{1}{2}, 1, 2\}$ for the constrained optimization problem. Do we achieve the same reconstruction error as in the former experiment? Submit your script.

(c) Repeat the experiment in (b) with a (actually not so) small twist: Use the regularization operator $\|Bx\|_1$. How is the reconstruction different from the former experiment? Comment on your findings in your script. Submit your script.