Homework 2
Due on Tuesday, September 25, at 05:20 pm

Answer the following 9 questions and solve the problems on the computational part to get full credit for this homework assignment. Please follow the instructions for homework assignments. I reserve the right to deduct points if you do not follow these rules.

1. Answer the following questions that relate functions f to their epigraph $\mathrm{epi}\, f$.
   (a) When is the epigraph of a function a halfspace?
   (b) When is the epigraph of a function a convex cone?
   (c) When is the epigraph of a function a polyhedron?

2. Here we explore the second-order conditions for convexity on an affine set. Let $F \in \mathbf{R}^{n,m}$, $\tilde{x} \in \mathbf{R}^n$. The restriction of $f : \mathbf{R}^n \to \mathbf{R}$ to the affine set $\{Fx + \tilde{x} : x \in \mathbf{R}^m\}$ is defined as the function $\tilde{f} : \mathbf{R}^m \to \mathbf{R}$ with $\tilde{f}(x) = f(Fx + \tilde{x})$, $\mathrm{dom}\, \tilde{f} = \{x : Fx + \tilde{x} \in \mathrm{dom}\, f\}$. Suppose f is twice differentiable with a convex domain.
   (a) Show that $\tilde{f}$ is convex if and only if for all $x \in \mathrm{dom}\, \tilde{f}$ we have that $F^\mathsf{T} \nabla^2 f(Fx + \tilde{x}) F \succeq 0$.
   (b) Suppose that $A \in \mathbf{R}^{p,n}$ is a matrix whose nullspace $\mathrm{N}(A)$ is equal to the range of F and $\mathrm{rank}\, A = n - \mathrm{rank}\, F$. Show that $\tilde{f}$ is convex if and only if for all $x \in \mathrm{dom}\, \tilde{f}$ there exists a $\lambda \in \mathbf{R}$ such that
   $$\nabla^2 f(Fx + \tilde{x}) + \lambda A^\mathsf{T} A \succeq 0.$$
   Hint: You can use the following result: If $B \in \mathbf{S}^n$ and $A \in \mathbf{R}^{p,n}$, then $x^\mathsf{T} B x \geq 0$ for all $x \in \mathrm{N}(A)$ if and only if there exists a λ such that $B + \lambda A^\mathsf{T} A \succeq 0$.

3. A function $g : \mathbf{R}^n \to \mathbf{R}^n$ is called *monotone* if for all $x, y \in \mathrm{dom}\, g$,
   $$(g(x) - g(y))^\mathsf{T}(x - y) \geq 0.$$
   Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone.

4. We say the function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is convex-concave if $f(x, y)$ is a concave function of y, for each fixed x, and a convex function of x, for each fixed y. We also require its domain to have the product form $\mathrm{dom}\, f = A \times B$, where $A \subseteq \mathbf{R}^n$ and $B \subseteq \mathbf{R}^m$ are convex.
   (a) Give a second-order condition for a twice differentiable function $f : \mathbf{R}^n \times \mathbf{R}^m \to R$ to be convex-concave, in terms of its Hessian $\nabla^2 f(x, y)$.
   (b) Suppose that $f : \mathbf{R}^n \times \mathbf{R}^m \to R$ is convex-concave and differentiable, with $\nabla f(\tilde{x}, \tilde{y}) = 0$. Show that the saddle-point property holds: For all x, y, we have $f(\tilde{x}, y) \leq f(\tilde{x}, \tilde{y}) \leq f(x, \tilde{y})$. Show that this implies that f satisfies the strong max-min property:
   $$\sup_{y \in \mathbf{R}^m} \inf_{x \in \mathbf{R}^n} f(x, y) = \inf_{x \in \mathbf{R}^n} \sup_{y \in \mathbf{R}^m} f(x, y)$$
   (and their common value is $f(\tilde{x}, \tilde{y})$).

5. For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.
   (a) $f(x) = \exp(x) - 1$ on $\mathbf{R}$.
   (b) $f(x_1, x_2) = 1/x_1x_2$ on $\mathbf{R}^2_{++}$.
   (c) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

6. Show that $f(X) = \operatorname{tr}(X^{-1})$ is convex on $\operatorname{dom} f = \mathbf{S}^n_{++}$, where $\operatorname{tr} : \mathbf{R}^{n,n} \to \mathbf{R}$ is the trace. **Hint:** The arguments are similar to the example of the log-determinant function $f : \mathbf{S}^n \to \mathbf{R}$ with $f(X) = \log\det X$ and $\operatorname{dom} f = \mathbf{S}^n_{++}$ we considered in class.

7. Show that the following functions are convex. **Hint:** Use arguments based on composition rules to make your point (i.e., use arguments concerning the characteristics of h and g to establish that $f(x) = (h \circ g)(x) = h(g(x))$ is convex).
   (a) $f(x) = -\log(-\log(\sum_{i=1}^m \exp(a_i^\mathsf{T} x + b_i)))$ on $\operatorname{dom} f = \{x : \sum_{i=1}^m \exp(a_i^\mathsf{T} x + b_i) < 1\}$. You can use the fact that the log-sum-exp function $\log(\sum_{i=1}^m \exp(y_i))$ is convex.
   (b) $f(x, u, v) = -\log(uv - x^\mathsf{T} x)$ on $\operatorname{dom} f = \{(x, u, v) : uv > x^\mathsf{T} x,\ u, v > 0\}$. You can use the fact that $x^\mathsf{T} x/u$ is convex.
   (c) $f(x, t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\operatorname{dom} f = \{(x, t) : t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex on $\{(x, u) : u > 0\}$.

8. Show that the following statements are valid. **Hint:** Use arguments based on the fact that the perspective of a convex function is convex to make your point.
   (a) For $p > 1$ the function
   $$f(x, t) = \frac{\sum_{i=1}^n |x_i|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$
   is convex on $\{(x, t) : t > 0\}$.
   (b) The function
   $$f(x) = \frac{\|Ax + b\|_2^2}{c^\mathsf{T} x + d}$$
   is convex on $\{x : c^\mathsf{T} x + d > 0\}$, where $A \in \mathbf{R}^{m,n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$.

9. Show that the logistic function $f(x) = \exp(x)/(1 + \exp(x))$ with $\operatorname{dom} f = \mathbf{R}$ is log-concave.