Homework 4
Due on Tuesday, October 30, at 5:20 pm

Answer the following 10 questions to get full credit for this homework assignment. Please follow the instructions for homework assignments. I reserve the right to deduct points if you do not follow these rules.

1. Let $f_0 : \mathbf{R}^n \to \mathbf{R}$ be a convex and differentiable function. We consider the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{1}$$

where $A \in \mathbf{R}^{m,n}$ with $\operatorname{rank} A = m$. Based on the problem given above we can form an auxiliary function of the form

$$\phi(x) = f_0(x) + \beta p(x) = f_0(x) + \beta \|Ax - b\|_2^2$$

This auxiliary function consists of the objective plus the penalty term $p(x)$, the contribution of which is controlled by the parameter $\beta > 0$. The idea is that a minimizer of the auxiliary function, $\tilde{x}$, should be an approximate solution of the original problem. Intuition suggests that a larger parameter β will provide a better approximation $\tilde{x}$ to a solution of the original problem. Suppose $\tilde{x}$ is a minimizer of ϕ. Show how to find, from $\tilde{x}$, a dual feasible point of (1). Find the corresponding lower bound on the optimal value of (1).

2. The weak duality inequality, $g_{\text{opt}} \leq f_{\text{opt}}$, holds when $g_{\text{opt}} = -\infty$ and $f_{\text{opt}} = \infty$. Show that it holds in the two cases below as well.
   (a) If $f_{\text{opt}} = -\infty$ we must have $g_{\text{opt}} = -\infty$.
   (b) If $g_{\text{opt}} = \infty$ we must have $f_{\text{opt}} = \infty$.

3. Derive a dual problem for the unconstrained problem

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \sum_{i=1}^{k} \|A_i x + b_i\|_2 + \frac{1}{2}\|x - x_{\text{ref}}\|_2^2.$$

The problem data are $A_i \in \mathbf{R}^{m_i,n}$, $b_i \in \mathbf{R}^{m_i}$, and $x_{\text{ref}} \in \mathbf{R}^n$. First, introduce new variables $y_i \in \mathbf{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

4. Consider the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

where f_i, $i = 0, \dots, m$ are differentiable and convex. Suppose $x^\star \in \mathbf{R}^n$ and $\lambda^\star \in \mathbf{R}^m$ satisfy the KKT conditions. Show that this implies that $\nabla f_0(x^\star)^\mathsf{T}(x - x^\star) \geq 0$ for all feasible x.

5. Find the dual function of the linear program

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & c^\mathsf{T} x \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b. \end{aligned}$$

Provide the dual problem, and make the implicit equality constraints explicit.

6. Consider the quadratically constrained quadratic program

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & \frac{1}{2} x^\mathsf{T} A_0 x + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & \frac{1}{2} x^\mathsf{T} A_i x + q_i^\mathsf{T} x + c_i \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

with $A_0 \in \mathbf{S}^n_{++}$, and $A_i \in \mathbf{S}^n_{+}$, $i = 1, \ldots, m$; i.e., the objective and constraints are convex quadratic. According to Slater's condition, when does strong duality hold for this problem?

7. Consider the optimization problem

$$\begin{aligned} \underset{x \in \mathbf{R}^k}{\text{minimize}} \quad & \operatorname{tr}(Y(x)) \\ \text{subject to} \quad & x \succeq 0 \\ & e_k^\mathsf{T} x = 1 \end{aligned}$$

where $Y(x) := (\sum_{i=1}^k x_i y_i y_i^\mathsf{T})^{-1}$, the vectors $y_i \in \mathbf{R}^n$ are given, and the domain is given by $\{x : \sum_{i=1}^k x_i y_i y_i^\mathsf{T} \succ 0\}$. Derive the dual problem. To do so, introduce a new variable $X \in \mathbf{S}^n$ and an equality constraint $X = \sum_{i=1}^k x_i y_i y_i^\mathsf{T}$, and then apply Lagrange duality. Simplify the dual problem as much as you can.

8. Consider the equality constrained least-squares problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - b\|_2^2 \\ \text{subject to} \quad & Gx = h, \end{aligned}$$

where $A \in \mathbf{R}^{m,n}$ with $\operatorname{rank} A = n$, and $G \in \mathbf{R}^{p,n}$, with $\operatorname{rank} G = p$. Provide the KKT conditions and derive expressions for the primal solution $x^\star$ and the dual solution $\nu^\star$.

9. Consider the problem

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m, \end{aligned} \tag{2}$$

where the functions $f_i : \mathbf{R}^n \to \mathbf{R}$ are differentiable and convex. In an exact penalty method, we solve the auxiliary problem

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \phi(x) = f_0(x) + \beta p(x), \tag{3}$$

where $p(x) = \max_{i=1,\ldots m} \max\{0, f_i(x)\}$ and $\beta > 0$ is a parameter. The term p penalizes deviations of x from feasibility. The method is called an exact penalty method if for sufficiently large β, solutions of the auxiliary problem (3) also solves the original, constrained problem (2).

(a) Show that ϕ is convex.

(b) The auxiliary problem can be expressed as

$$\begin{aligned} \underset{x \in \mathbf{R}^n,\ y \in \mathbf{R}}{\text{minimize}} \quad & f_0(x) + \beta y \\ \text{subject to} \quad & f_i(x) \leq y, \quad i = 1, \ldots, m \\ & 0 \leq y. \end{aligned}$$

Find the Lagrange dual of this problem and express it in terms of the Lagrange dual function g of the original problem (2).

(c) Use the result in (b) to prove the following property: Suppose $\lambda^\star$ is an optimal solution of the Lagrange dual of (2) and that strong duality holds. If $\beta > e_m^\mathsf{T}\lambda^\star$ then any solution of the auxiliary problem in (3) is also an optimal solution of (2).

10. Express the dual problem of

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & c^\mathsf{T} x \\ \text{subject to} \quad & f(x) \leq 0 \end{aligned}$$

with $c \neq 0$ in terms of the conjugate f^*. Explain why the resulting problem is convex. It is not assumed that f is convex.