Math 6366 :: Optimization I :: Fall 2018

Exam 1

Time: 32 hours

9AM on 09/29/18 through 5PM on 09/30/18

This exam has 12 questions, for a total of 150 points.

Name: Jonathan Schuba

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You must sign the Honor Code Statement and submit it together with your work online by uploading your PDF file to Blackboard by 5pm on Sunday, September 30.

• Although there are 32 hours after its opening on Saturday 9AM, the exam is not supposed to take anything close to 32 hours. You are given 32 hours so you can take a number of long breaks (to eat, sleep, study for other classes, take other exams, etc.).

• This is an open book exam. You may use any books or notes, but you may not discuss the exam with anyone until September 30, after everyone has taken the exam.

• The problems are all equally weighted. This is to make things simple, and not because the problems are all of equal difficulty.

• You will be graded on clarity and conciseness as well as accuracy and correctness. Please take the time and make the effort to make your solutions clear. Please try to use standard and simple notation, introducing only as many new symbols as is absolutely necessary. The solutions are not long, so if you find that your solution to a problem goes on and on for pages, you should try to figure out a simpler one. I expect neat, legible exams from everyone.

• If you cannot find the solution to a problem because you think the problem statement is wrong, state this in your answer.

• If you run out of space, use one of the blank pages at the end of the exam and indicate this clearly.

• Please respect the honor code. Although I allow you to work on homework assignments in small groups, you cannot discuss the exam with anyone, until everyone has taken it.

Good luck!

Honor Code Statement

I am familiar with the University of Houston Academic Honesty policy that is published in the University of Houston Handbook. I am aware that violations of the code include, but are not limited to:

- Giving or receiving unauthorized aid during any exam.
- Discussing the contents of the exam with students who have not taken the exam.
- Plagiarism.

Furthermore, I realize that any compromise of these standards, or failure to report an infringement by another student, will result in sanctions up to and including course failure.

With my signature, I affirm that: "On my honor, I commit to adherence with the University of Houston policies for academic honesty".

Please sign your full name here: [signature]

Print your name here: Jonathan Schuba

Name: J. Schuba PeopleSoft ID: 1670581

1. Determine which of the following sets are convex.

(a) (10 points) The set of points closer to a given point than a given set, i.e.,

$$K = \{x \in \mathbf{R}^n : \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\},$$

where $S \subseteq \mathbf{R}^n$.

$\boxed{\text{Convex}}$

Consider a point u in K, and any point $y \in S$

$$\|u - x_0\|_2^2 \le \|u - y\|_2^2$$

$$\cancel{u^Tu} - 2u^Tx_0 + x_0x_0 \le \cancel{u^Tu} - 2u^Ty + y^Ty$$

$$2u^T(y - x_0) \le y^Ty - x_0^Tx_0 \qquad \text{Let } a = 2(y - x_0)$$

$$a^Tu \le b \qquad b = y^Ty - x_0^Tx_0$$

So, each $y \in S$ determines a halfspace, and u must lie in the intersection of these halfspaces, which is a convex set.

(b) (10 points) The set of points closer to one set S_1 than another set S_2, i.e.,

$$K = \{x \in \mathbf{R}^n : \mathrm{dist}(x, S_1) \le \mathrm{dist}(x, S_2)\},$$

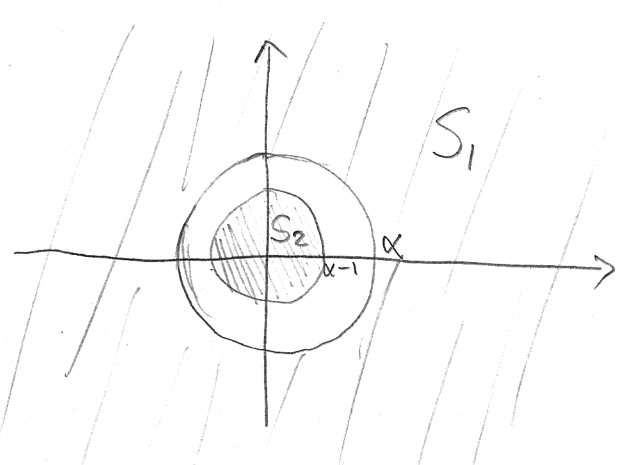
where $S_i \subseteq \mathbf{R}^n$, $i = 1, 2$, and $\mathrm{dist}(x, S) = \inf\{\|x - y\|_2 : y \in S\}$.

$\boxed{\text{Not Convex.}}$ Can construct a counter-example.

Consider $\alpha > 1$

$S_1 = \{x : \|x\|_2 \ge \alpha\}$ which is not convex

$S_2 = \{x : \|x\|_2 \le \alpha - 1\}$ which is convex



Points in K will be

$$\{x : \|x\|_2 \ge \alpha - \tfrac{1}{2}\}$$

and is not convex.

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2.19d

2. (10 points) Let f be the linear fractional function $f(x) = (Ax+b)/(c^T x+d)$, $\mathbf{dom}\, f = \{x : c^T x + d > 0\}$. Give a simple description of the inverse image $f^{-1}(C) = \{x \in \mathbf{dom}\, f : f(x) \in C\}$ of the convex set

$$C = \left\{ y \in \mathbf{R}^n : \sum_{i=1}^{n} y_i A_i \preceq B \right\}$$

under f, where $A_i \in \mathbf{S}^p$, $i = 1, \ldots, n$, $B \in \mathbf{S}^p$, and C represents the solution set of a linear matrix inequality.

$$f^{-1}(C) = \left\{ x \in \text{dom} f : \sum_{i=1}^{n} f(x)_i A_i \preceq B \right\}$$

$$= \left\{ x \in \text{dom} f : \sum_{i=1}^{n} \left(\frac{Ax+b}{c^T x + d} \right)_i A_i \preceq B \right\}$$

$$= \left\{ x \in \text{dom} f : \sum_{i=1}^{n} (Ax+b)_i A_i \preceq (c^T x + d) B \right\}$$

$$= \left\{ x \in \text{dom} f : \sum_{i=1}^{n} (a_i^T x + b_i) A_i \preceq (c^T x + d) B \right\}$$

$$= \left\{ x \in \text{dom} f : \left(\sum_{i=1}^{n} x^T a_i A_i \right) - x^T c B \preceq dB - \sum_{i=1}^{n} b_i A_i \right\}$$

$$= \left\{ x \in \text{dom} f : \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} (a_{ij} A_i) + c_j B \right) \preceq dB - \sum_{i=1}^{n} b_i A_i \right\}$$

So, $f^{-1}(C)$ is the x in dom f which are solutions to a linear matrix inequality.

2.32

3. (10 points) Find the dual cone of $\{Ax : x \succeq 0\} \subseteq \mathbf{R}^m$, where $A \in \mathbf{R}^{m,n}$.

dual cone $K^* = \{ y : x^T y \geq 0 \ \forall x \in K \}$

$$K^* = \{ y : (Ax)^T y \geq 0 ;\ x \succeq 0 \}$$

$$= \{ y : x^T (A^T y) \geq 0 ;\ x \succeq 0 \}$$

if $x \succeq 0$, then for $x^T (A^T y)$ to be non-negative, we must have $A^T y \succeq 0$, so,

$$\boxed{K^* = \{ y : A^T y \succeq 0 \}}$$

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4. (10 points) Show that the function $f : \mathbf{R}^n \to \mathbf{R}$, $f(x) = \max_{i=1,\dots,n} x_i$, is convex on $\mathbf{R}^n$.

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2) \quad \text{with } 0 \le \theta \le 1$$

$$\max_j (\theta x_{1j} + (1-\theta)x_{2j}) \le \theta \max_i x_{1i} + (1-\theta) \max_i x_{2i}$$

This holds because, for each j in $\{1, \dots, n\}$; $x_{1j} \le \max_i x_{1i}$ and $x_{2j} \le \max_i x_{2i}$

5. (10 points) Answer the following true or false questions. (You do not need to justify your answer.)

(a) A function is convex if and only if its epigraph $\mathbf{epi}\, f$ is a convex set.

True

(b) If for each $y \in Y$, where Y is an arbitrary set, $f(x,y)$ is convex in x, then the function g, defined as $g(x) = \sup_{y \in Y} f(x,y)$ is convex in x (with appropriate domain $\mathbf{dom}\, g$).

True

(c) If for each $y \in Y$, where Y is an arbitrary set, $f(x,y)$ is convex in x, then the function g, defined as $g(x) = \inf_{y \in Y} f(x,y)$ is convex in x (with appropriate domain $\mathbf{dom}\, g$).

False

(d) Let $f : \mathbf{R}^n \to \mathbf{R}$ be defined as the composition $f(x) = (h \circ g)(x)$. This function f is *concave* if $h : \mathbf{R} \to \mathbf{R}$ is convex, the extended-value extension $\tilde{h} : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ is nondecreasing, and $g : \mathbf{R}^n \to \mathbf{R}$ is convex.

False

(e) The conjugate of the conjugate f^* of any function $f : \mathbf{R}^n \to \mathbf{R}$ is f, i.e., $f^{**} = f$.

False

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6. (10 points) Consider the quadratic functions $f_1 : \mathbf{R}^n \to \mathbf{R}$, $\mathrm{dom}\, f_1 = \mathbf{R}^n$, and $f_2 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $\mathrm{dom}\, f_2 = \mathbf{R}^n \times \mathbf{R}^n$, given by

$$f_1(x) = \frac{1}{2}x^\mathsf{T}Ax + q^\mathsf{T}x + c \quad \text{and} \quad f_2(x,y) = x^\mathsf{T}Ax + 2x^\mathsf{T}By + y^\mathsf{T}Cy,$$

where $A, C \in \mathbf{S}^n$, $B \in \mathbf{R}^{n,n}$, $q \in \mathbf{R}^n$, and $c \in \mathbf{R}$. Under what condition on the matrix A is f_1 convex in x? Under what condition on the matrix A is f_1 concave in x? Under what condition on a block matrix consisting of A, C, and B is f_2 convex in (x,y)? Justify your answer.

$\nabla f_1(x) = Ax + q^T$

$\nabla^2 f_1(x) = A$

So f_1 is convex when $A \succcurlyeq 0$ (A pos semidef.)

$\nabla_x f_2(x,y) = 2Ax + 2By$

$\nabla_y f_2(x,y) = 2x^TB^T + 2Cy$

$\nabla^2_{xx} = 2A \qquad \nabla^2_{xy} = 2B$

$\nabla^2_{yx} = 2B^T \qquad \nabla^2_{yy} = 2C$

$\nabla^2 f_2(x,y) = 2\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ must be pos. semi-def.

7. (10 points) A function $g : \mathbf{R}^n \to \mathbf{R}^n$ is called *monotone* if $(g(x) - g(y))^\mathsf{T}(x-y) \geq 0$ for all $x, y \in \mathrm{dom}\, g$. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone.

~~This was a homework question.~~

Take x, y in dom f. Since f is convex, the following first-order conditions hold:

$f(y) \geq f(x) + \nabla f(x)^T(y-x)$

$f(x) \geq f(y) + \nabla f(y)^T(x-y)$

add them ...

$f(y) + f(x) \geq f(x) + f(y) - \nabla f(x)^T(x-y) + \nabla f(y)^T(x-y)$

$0 \geq (\nabla f(y)^T - \nabla f(x)^T)(x-y)$

$0 \leq (\nabla f(x)^T - \nabla f(y)^T)(x-y)$

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8. For each of the following functions $f : \mathbf{R}^2 \to \mathbf{R}$ determine whether it is convex, concave, quasiconvex, or quasiconcave. **Hint:** One way to answer this is through the Hessian.

(a) (10 points) Let $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

$\nabla_{x_1} f(x_1, x_2) = x_2$

$\nabla_{x_2} f(x_1, x_2) = x_1$

$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is neither neg or pos semidef.

$\boxed{\text{quasiconcave}}$

$\{ x_1, x_2 : f(x_1, x_2) \geq \alpha \}$ are all convex sets. superlevel

(b) (10 points) Let $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ on $\mathbf{R}^2_{++}$, where $0 \leq \alpha \leq 1$.

$\nabla_{x_1} f(x_1, x_2) = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$

$\nabla_{x_2} f(x_1, x_2) = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix}$$

$$= \alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{bmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{bmatrix}$$

$$= -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \begin{bmatrix} \frac{1}{x_1} & -\frac{1}{x_2} \end{bmatrix} \preceq 0$$

$\boxed{\text{Concave}}$

All principal minors are non-positive $\Leftrightarrow$ negative semi def.

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9. (10 points) Show that the function $f : \mathbf{R}^n \to \mathbf{R}$,

$$f(x) = -(\det(A_0 + \sum_{i=1}^{n} x_i A_i))^{1/m}$$

on $\{x \in \mathbf{R}^n : A_0 + \sum_{i=1}^{n} x_i A_i \succ 0\}$, where $A_i \in \mathbf{S}^m$, is convex. **Hint:** You can use the argument that the composition of a convex function with an affine function/mapping preserves convexity. You can additionally use the fact that the geometric mean is a concave function on $\mathbf{R}^n_{++}$ (see question 10).

Let $X = A_0 + \sum_{i=1}^{n} x_i A_i \succ 0$

This is an affine transformation, so it is convex.

Rewrite $f(X) = -\det(X)^{1/m}$

restrict f to the line $Z + tV$

$$g(t) = -\det(Z + tV)^{1/m}$$

$$= -\det(Z^{1/2}(I + tZ^{-1/2} V Z^{-1/2}) Z^{1/2})^{1/m}$$

$$= -\det(Z)^{1/m} \det(I + tZ^{-1/2} V Z^{-1/2})^{1/m}$$

$$= \underbrace{-}_{\text{negative}} \underbrace{\det(Z)^{1/m}}_{\text{constant}} \underbrace{\prod_{i=1}^{m} (1 + t\lambda_i)^{1/m}}_{\text{geometric mean (concave)}}$$

where λ_i are eigenvalues of $Z^{-1/2} V Z^{-1/2}$

Since g is additive inverse of concave function, $f(x)$ is convex.

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10. (10 points) Verify that the geometric mean $f(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ is concave on $\operatorname{dom} f = \mathbf{R}^n_{++}$ based on the fact that

$$\nabla^2 f(x) = -\frac{\prod_{i=1}^{n} x_i^{1/n}}{n^2}\left(n \operatorname{diag}\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right) - qq^\mathsf{T}\right), \quad \text{where } q_i = \frac{1}{x_i}.$$

Hint: We used a similar argument to show that $f(x) = \log(\sum_{i=1}^{n} \exp x_i)$ (log-sum-exp) is convex.

Need to show that $\nabla^2 f(x)$ is negative semidef.

Let $v \in \mathbb{R}^n$

$$v^T \nabla^2 f(x) v = -\frac{\prod_{i=1}^{n} x_i^{1/n}}{n^2}\left(n \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} - \left(\sum_{i=1}^{n} \frac{v_i}{x_i}\right)^2\right)$$

Let $a = \vec{1}$ vector of ones

$b_i = \dfrac{v_i}{x_i}$

$$= -\underbrace{\frac{\prod_{i=1}^{n} x_i^{1/n}}{n^2}}_{\substack{\text{positive} \\ \text{since } x_i > 0 \\ n > 0}}\underbrace{\left((a^T a)(b^T b) - (a^T b)^2\right)}_{\substack{\text{Positive by} \\ \text{Cauchy-Schwartz inequality}}}$$

Negative means

$v^T \nabla^2 f(x) v \leq 0$ for any $v \in \mathbb{R}^n$

and $f(x)$ is concave

(Just copied this from the book. page 74)

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11. Show that the following sets are convex.

(a) (10 points) The hyperbolic set $S = \{x \in \mathbf{R}^2_+ : x_1x_2 \geq 1\}$.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in S. and $0 \leq \theta \leq 1$

Consider

$$(\theta x_1 + (1-\theta)y_1)(\theta x_2 + (1-\theta)y_2)$$

$$= \theta^2 x_1x_2 + (1-\theta)^2 y_1y_2 + \theta(1-\theta)x_1y_2 + \theta(1-\theta)x_2y_1$$

We know that $x_1x_2 \geq 1$, $y_1y_2 \geq 1$ and $y_2 \geq \frac{1}{y_1}$, $x_2 \geq \frac{1}{x_1}$

$$\geq \theta^2(1) + (1-\theta)^2(1) + \theta(1-\theta)\frac{x_1}{y_1} + \theta(1-\theta)\frac{y_1}{x_1}$$

$$= 1 \underbrace{+2\theta^2 + 2\theta + \theta(1-\theta)\left(\frac{x_1}{y_1} + \frac{y_1}{x_1}\right)}$$

$$\geq 1$$

all terms are positive or zero

Therefore, $\theta x + (1-\theta)y$ is in S, and S is convex.

(b) (10 points) The set $S = \{x \in \mathbf{R}^n_+ : \prod_{i=1}^n x_i \geq 1\}$. **Hint:** If $\alpha, \beta \geq 0$ and $\lambda \in [0,1]$ then $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$.

$$\prod_{i=1}^n (\theta x_i + (1-\theta)y_i) \geq \prod_{i=1}^n x_i^\theta y_i^{(1-\theta)}$$

for $x, y \in S$

$0 \leq \theta \leq 1$

$$= \left(\prod_{i=1}^n x_i\right)^\theta \left(\prod_{i=1}^n y_i\right)^{(1-\theta)}$$

$$\geq 1$$

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12. (10 points) Show that $f : \mathbf{R}^{n,n} \to \mathbf{R}$, $f(X) = \dfrac{\det X}{\operatorname{tr} X}$ with $\operatorname{dom} f = \mathbf{S}^n_{++}$ is log-concave, where $\operatorname{tr} : \mathbf{R}^{n,n} \to \mathbf{R}$ represents the trace.

3.49d

$$\log(f(X)) = \log \det(X) - \log \operatorname{tr}(X)$$
$$= \log \det(X) + \log\left(\frac{1}{\operatorname{tr} X}\right)$$

We know from the book, and also question 9, that $\log(\det(X))$ is concave, So we need to show that

$$\log\left(\frac{1}{\operatorname{tr}(X)}\right)$$

is concave. Since log is concave on R_{++}, we need only show that $\frac{1}{\operatorname{tr}(X)}$ is convex. (Clearly $\frac{1}{\operatorname{tr}(X)} > 0$ since $X \in S^n_{++}$)

Take A, B in S^n_{++}; $0 \le \theta \le 1$

$$\frac{1}{\operatorname{tr}(\theta A + (1-\theta)B)} \le \frac{\theta}{\operatorname{tr} A} + \frac{1-\theta}{\operatorname{tr} B}$$

$$\frac{1}{\theta \operatorname{tr} A + (1-\theta)\operatorname{tr} B} \le \frac{\theta \operatorname{tr} B + (1-\theta)\operatorname{tr} A}{\operatorname{tr} A \operatorname{tr} B}$$

$$1 \le \frac{\theta^2 \operatorname{tr} A \operatorname{tr} B + \theta(1-\theta)\operatorname{tr}^2 A + \theta(1-\theta)\operatorname{tr}^2 B + (1-\theta)^2 \operatorname{tr} A \operatorname{tr} B}{\operatorname{tr} A \operatorname{tr} B}$$

$$1 \le \theta^2 + (1-\theta)^2 + \theta(1-\theta)\frac{\operatorname{tr}^2 A + \operatorname{tr}^2 B}{\operatorname{tr} A \operatorname{tr} B}$$

$$1 \le 1 + \underbrace{2\theta + 2\theta^2 + \theta(1-\theta)\frac{\operatorname{tr}^2 A + \operatorname{tr}^2 B}{\operatorname{tr} A \operatorname{tr} B}}$$

All terms are positive (or zero) since trace of pos def matrix is positive

Therefore, $\frac{1}{\operatorname{tr}(X)}$ is convex, and $f(X)$ is log concave.