Homework 3
Due on Thursday, October 11, at 5:20 pm

Answer the following 10 questions and solve the problems on the computational part to get full credit for this homework assignment. Please follow the instructions for homework assignments. I reserve the right to deduct points if you do not follow these rules.

1. Consider the optimization problem

$$
\begin{aligned}
\underset{x \in \mathbf{R}^2}{\text{minimize}} \quad & f_0(x) \\
\text{subject to} \quad & 2x_1 + x_2 \geq 1 \\
& x_1 + 3x_2 \geq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0.
\end{aligned}
$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x) = x_1 + x_2$.

(b) $f_0(x) = \max\{x_1, x_2\}$.

(c) $f_0(x) = x_1^2 + 9x_2^2$.

2. Verify that $x^\star = \begin{bmatrix} 1 \\ 0.5 \\ -1 \end{bmatrix}$ is optimal for the optimization problem

$$
\begin{aligned}
\underset{x \in \mathbf{R}^3}{\text{minimize}} \quad & \frac{1}{2} x^\mathsf{T} A x + q^\mathsf{T} x + r \\
\text{subject to} \quad & -1 \leq x_i \leq 1, \quad i = 1, 2, 3,
\end{aligned}
$$

where $A = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}$, $q = \begin{bmatrix} -22 \\ -14.5 \\ 30 \end{bmatrix}$ and $r = 1$.

3. Show that the following three convex problems are equivalent (compare (a) and (b) and (a) and (c)). Carefully explain how the solution of each problem is obtained from the solution of the other problems. The problem data are the matrix $A \in \mathbf{R}^{m,n}$ (with rows a_i^T), the vector $b \in \mathbf{R}^m$, and the constant $\alpha > 0$.

(a) The robust least-squares problem

$$
\underset{x \in \mathbf{R}^n}{\text{minimize}} \sum_{i=1}^{m} \phi(a_i^\mathsf{T} x - b_i),
$$

where $\phi : \mathbf{R} \to \mathbf{R}$ (which is known as *Huber penalty function*) is defined as

$$
\phi(u) := \begin{cases} u^2 & \text{if } |u| \leq \alpha, \\ \alpha(2|u| - \alpha) & \text{if } |u| > \alpha. \end{cases}
$$

(b) The least squares problem with variable weights

$$\begin{aligned} \underset{x\in\mathbf{R}^n,\, w\in\mathbf{R}^m}{\text{minimize}} \quad & \sum_{i=1}^{m} \frac{(a_i^\mathsf{T} x - b_i)^2}{w_i + 1} + \alpha^2 e_m^\mathsf{T} w \\ \text{subject to} \quad & w \succeq 0, \end{aligned}$$

where $e_m = (1, \dots, 1) \in \mathbf{R}^m$ and $\{(x, w) \in \mathbf{R}^n \times \mathbf{R}^m : w \succ -e_m\}$ is the domain. **Hint:** Optimize over w for a fixed x to establish a relation with the problem in part (a).

(This problem can be interpreted as a weighted least-squares problem in which we are allowed to adjust the weight of the ith residual. The weight is one if $w_i = 0$, and decreases if we increase w_i. The second term in the objective penalizes large values of w, i.e., large adjustments of the weights.)

(c) The quadratic program

$$\begin{aligned} \underset{\Phi}{\text{minimize}} \quad & \sum_{i=1}^{m} (u_i^2 + 2\alpha v_i) \\ \text{subject to} \quad & -u - v \preceq Ax - b \preceq u + v \\ & 0 \preceq u \preceq \alpha e_m \\ & v \succeq 0, \end{aligned}$$

where $\Phi := (u, v, x) \in \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^n$ and $e_m = (1, \dots, 1) \in \mathbf{R}^m$.

4. Consider the unconstrained optimization problem

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \, \|A(x)\|_2$$

where $\|A\|_2$ is the spectral norm and $A(x) = A_0 + \sum_{i=1}^n A_i$. Derive the equivalent semidefinite program. Use the fact that $A^\mathsf{T} A \succeq s^2 I$, where $I = \operatorname{diag}(1, \dots, 1) \in \mathbf{R}^{n,n}$.

5. Consider the vector optimization problem with respect to the positive semidefinite cone

$$\begin{aligned} \underset{X\in\mathbf{S}_+^n}{\text{minimize}} \quad & X \\ \text{subject to} \quad & X \succeq A_i, \quad i = 1, \dots, m, \end{aligned}$$

where $A_i \in \mathbf{S}^n$, $i = 1, \dots, m$, are given. State the minimization problem if you were to apply scalarization to solve this problem.

6. Let $A \in \mathbf{R}^{m,n}$, $x \in \mathbf{R}^n$, and $b \in \mathbf{R}^m$. Formulate the following problems as linear programs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent linear program.

(a) $\text{minimize}\, \|Ax - b\|_\infty$ (unconstrained ℓ^∞-norm approximation).

(b) $\text{minimize}\, \|Ax - b\|_1$ (unconstrained ℓ^1-norm approximation).

(c) $\text{minimize}\, \|Ax - b\|_1 \quad \text{subject to} \quad \|x\|_\infty \le 1$.

7. Let the induced/operator norm of the ℓ^∞ vector norm, denoted by $\|\cdot\|_\infty : \mathbf{R}^{m,n} \to \mathbf{R}$, be given by

$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$

This norm is sometimes called the max-row-sum norm. Consider the problem of approximating a matrix, in the max-row-sum norm, by a linear combination of other matrices. That is, we are given $k+1$ matrices $A_i \in \mathbf{R}^{m,n}$, $i = 0, \dots, k$, and need to find $x \in \mathbf{R}^k$ that minimizes

$$\|A_0 + \sum_{i=1}^{k} x_i A_i\|_\infty$$

Express this problem as a linear program. Explain the significance of any extra variables in your linear program. Carefully explain how your program formulation solves this problem, e.g., what is the relation between the feasible set for your linear program and this problem?

8. We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n$, $t = 0, ..., n_t$, and actuator or input signal $u(t) \in \mathbf{R}$ for $t = 0, ..., n_t - 1$. The dynamics of the system are given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \;\; t = 0, ..., n_t - 1,$$

where $A \in \mathbf{R}^{n,n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$. The minimum fuel optimal control problem is to choose the inputs $u(0), ..., u(n_t - 1)$ so as to minimize the total fuel consumed, which is given by $f = \sum_{t=0}^{n_t-1} f(u(t))$ subject to the constraint that $x(n_t) = x_{\text{des}}$, where n_t is the (given) time horizon, and $x_{\text{des}} \in \mathbf{R}^n$ is the (given) desired final or target state. The function $f : \mathbf{R} \to \mathbf{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half. Formulate the minimum fuel optimal control problem as an linear program.

9. Consider the quadratically constrained quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} x^\mathsf{T} A x + q^\mathsf{T} x + r \\ \text{subject to} & x^\mathsf{T} x \leq 1, \end{array}$$

with $A \in \mathbf{S}^n_{++}$. Show that $x^\star = -(A + \lambda I)^{-1} q$ where $\lambda = \max\{0, \tilde{\lambda}\}$ and $\tilde{\lambda}$ is the largest solution of the nonlinear equation $q^\mathsf{T}(A + \lambda I)^{-2} q = 1$.

10. Express the geometric program

$$\underset{x}{\text{minimize}} \max\{p(x), q(x)\},$$

where p and q are posynomials, as a convex optimization problem.