Chapter 3 Continuous-Time Optimal Control

3.1 Resource allocation as a bilinear control problem

We consider a producer who produces with production rate $y(t)$ at time $t \in [0,T], T > 0,$. He allocates a certain fraction $0 \le u(t) \le 1$ of the production to reinvestment and the rest $1 - u(t)$ to the production of a storable good. The producer wants to choose $u = u(t)$ such that the total amount of the stored product is maximized

$$\text{maximize } J(y,u) \ := \ \int_0^T (1 - u(t))y(t) \ dt \ . \tag{3.1}$$

We call $y = y(t)$ the state and $u = u(t)$ the control. According to our assumptions above, the state y evolves in time according to the following initial-value problem for a first order ordinary differential equation

$$\dot{y}(t) = \ \gamma u(t) y(t) \quad , \quad t \in [0,T] \ , \tag{3.2a}$$

$$y(0) = \ y_0 \ , \tag{3.2b}$$

where $\dot{y} := dy/dt$ and $\gamma > 0$ and y_0 are given constants.
The control u is subject to the constraints

$$u(t) \in U := \{w \in \mathbb{R} \mid 0 \le w \le 1\} \quad , \quad t \in [0,T] \ . \tag{3.3}$$

As we shall see below, the optimization problem (3.1),(3.2),(3.3) is a particular example of a continuous-time optimal control problem. Since the right-hand side in (3.2a) is a bilinear function with respect to y and u, it is called a bilinear control problem. Bilinear control problems are the simplest examples of nonlinear control problems.

3.2 Continuous-time optimal control problems

We assume that

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \ , \\ g &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \ , \\ h &: \mathbb{R}^n \to \mathbb{R} \end{aligned}$$

are functions such that

- f and g are continuously differentiable in the first argument and continuous with respect to the second argument,
- h is continuously differentiable.

We further suppose that $U \subset \mathbb{R}^m$ is a given set and $y_0 \in \mathbb{R}^n, T > 0$ are given as well.

We consider the optimization problem:
Find $(y, u) \in C^1([0,T]) \times L^\infty([0,T])$ such that

$$\text{(3.4a)} \qquad \text{minimize} \quad J(y,u) \;:=\; h(y(T)) \;+\; \int_0^T g(y(t),u(t))\; dt \;,$$

$$\text{(3.4b)} \qquad \text{subject to} \quad \dot{y}(t) \;=\; f(y(t),u(t)) \quad , \quad t \in [0,T] \;,$$

$$\text{(3.4c)} \qquad y(0) \;=\; y_0 \;,$$

$$\text{(3.4d)} \qquad u(t) \in U \quad \text{f.a.a. } t \in [0,T] \;.$$

The optimization problem (3.2) represents a control-constrained **continuous-time optimal control problem**. The function $y \in C^1([0,T])$ is said to be the **state** and the function $u \in L^\infty([0,T])$ is referred to as the **control**. The set U is called the **control constraint set**. If $(y^*, u^*) \in C^1([0,T]) \times L^\infty([0,T])$ satisfies (3.4), y^* is called the **optimal state** and u^* the **optimal control**. The value $J^*(t,y)$ of the objective functional J for $t \in [0,T]$ and $y \in \mathbb{R}^n$ is said to be the **optimal value function**.

Example (Linear-quadratic optimal control problems):
An important class of continuous-time optimal control problems are the so-called **linear-quadratic optimal control problems** where the objective functional J in (3.4a) is quadratic in y and u, and the system of ordinary differential equations (3.4b) is linear:
Let $Q_T \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times n}$ be symmetric, positive semidefinite matrices, $R \in \mathbb{R}^{m\times m}$ be symmetric positive definite and suppose that $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. Then, the optimization problem

$$\text{(3.5a)} \qquad \text{minimize} \quad J(y,u) \;:=\; \langle y(T), Q_T y(T)\rangle \;+$$
$$+ \int_0^T \Big(\langle y(t), Qy(t)\rangle + \langle u(t), Ru(t)\rangle \Big)\; dt \;,$$

$$\text{(3.5b)} \qquad \text{subject to} \quad \dot{y}(t) \;=\; Ay(t) \;+\; Bu(t) \quad , \quad t \in [0,T] \;,$$

$$\text{(3.5c)} \qquad y(0) \;=\; y_0 \;,$$

$$\text{(3.5d)} \qquad u(t) \in U \quad \text{f.a.a. } t \in [0,T] \;,$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, is called a **linear-quadratic optimal control problem**.

3.3 Hamilton-Jacobi-Bellman equation

In this section, we show that the optimal value function of a continuous-time optimal control problem satisfies a first order partial differential equation, called the **Hamilton-Jacobi-Bellman (HJB-) equation**. We will first derive the HJB-equation by a heuristic argument based on a discretization of the objective functional in (3.4a) and the state equation (3.4b) by finite differences and afterwards provide a slightly more rigorous sufficiency proof.

We start from an equidistant partition

$$\text{(3.6)} \qquad \Delta_h \ := \ \{0 =: t_0 < t_1 < \cdots < t_{N+1} := T \ , \quad h := t_{k+1} - t_k = T/(N+1) \ , \ 0 \le k \le N\}$$

of the time interval $[0,T]$ into N subintervals $[t_k, t_{k+1}], 0 \le k \le N$, of length h. We denote by $y_h(t_k)$ and $u_h(t_k)$ approximations of the state y and the control u at $t_k, 0 \le k \le N+1$, and we approximate the integral in (3.4a) by

$$\int_0^T g(y(t), u(t)) \ dt \ \approx \ h \sum_{k=0}^{N} g(y(t_k), u(t_k) \ .$$

We further approximate the time derivative d/dt in (3.4b) by the forward difference quotient according to

$$\dot{y}(t_k) \ \approx \ h^{-1}(y(t_{k+1}) - y(t_k)) \quad , \quad 0 \le k \le N \ .$$

We are thus led to the discrete-time optimal control problem

$$\text{(3.7a)} \quad \text{minimize} \ \ J_h(y_h, u_h) \ := \ h(y_h(T)) \ + \ h \sum_{k=0}^{N} g(y_h(t_k), u_h(t_k) \ ,$$

subject to

$$\text{(3.7b)} \quad y_h(t_{k+1}) \ = \ y_h(t_k) \ + \ hf(y_h(t_k), u_h(t_k)) \quad , \quad 0 \le k \le N \ ,$$

$$\text{(3.7c)} \qquad y_h(0) \ = \ y_0 \ ,$$

$$\text{(3.7d)} \qquad u_h(t_k) \in U \quad , \quad 0 \le k \le N+1 \ .$$

Denoting by $J_h^*(t, y_h)$ the optimal value function of the discrete-time optimization problem (3.7) and applying the dynamic programming principle, we obtain the equations

$$(3.8a) \qquad J_h^*(T, y_h(T)) \;=\; h(y_h(T)) \;,$$

$$(3.8b) \qquad J_h^*(t_k, y_h(t_k)) \;=\; \min_{u_h(t_k)\in U} \Big(hg(y_h(t_k), u_h(t_k)) \; +$$
$$+ \; J_h^*(t_{k+1}, y_h(t_k) + hf(y_h(t_k), u_h(t_k))\Big) \;,\; 0 \le k \le N \;.$$

Assuming sufficient smoothness of the optimal value, Taylor expansion around $(t_k, y_h(t_k))$ yields

$$(3.9) \qquad J_h^*(t_{k+1}, y_h(t_k) + hf(y_h(t_k), u_h(t_k)) \;=\; J_h^*(t_k, y_h(t_k)) \; +$$
$$+ \; h\nabla_t J_h^*(t_k, y_h(t_k)) + h\nabla_y J_h^*(t_k, y_h(t_k))^T f(y_h(t_k), u_h(t_k)) + o(h) \;,$$

where the upper index ^{T} denotes - as usual - the transpose.
Substituting (3.9) in (3.8b) gives rise to

$$J_h^*(t_k, y_h(t_k)) = \min_{u_h(t_k)\in U} \Big(hg(y_h(t_k), u_h(t_k) + J_h^*(t_k, y_h(t_k)) \; +$$
$$+ \; h\nabla_t J_h^*(t_k, y_h(t_k)) + h\nabla_y J_h^*(t_k, y_h(t_k))^T f(y_h(t_k), u_h(t_k)) + o(h)\Big) \;.$$

Assuming that for $t = t_k = kh$ we have that $y_h(t_k) \to y(t)$, $u_h(t_k) \to u(t)$ and $J_h^*(t_k, y_h(t_k)) \to J^*(t, y(t))$ as $k \to \infty, h \to 0$, it follows that the optimal value function $J^*(t, y(t))$ satisfies a final-value problem for a first-order nonlinear partial differential equation

$$(3.10a) \qquad \min_{u\in U} \Big(g(y,u) + \nabla_t J^*(t,y) + \nabla_y J^*(t,y)^T f(y,u)\Big) = 0 \;,$$

$$(3.10b) \qquad J^*(T, y) = h(y) \;.$$

On the other hand, the following result shows that under the assumption of a smooth solution of the HJB-equation (3.10) related to a feasible pair $(\hat{y}(t), \hat{u}(t))$ associated with (3.4) (i.e., satisfying (3.4b)(3.4c) and (3.4d)) such that the minimum in (3.10a) is attained for that $\hat{u}$, this solution corresponds to the optimal value function $J^*(t, y)$ of (3.4).

Theorem 3.1 (HJB-equation and optimal value)
We assume that $V(t, y), t \in [0, T], y \in \mathbb{R}^n$, is a continuously differentiable solution of the HJB-equation

$$(3.11a) \qquad \min_{u\in U} \Big(g(y,u) + \nabla_t V(t,y) + \nabla_y V(t,y)^T f(y,u)\Big) = 0 \;,$$

$$(3.11b) \qquad V(T, y) = h(y) \;.$$

We further suppose that $(\hat{y}(t), \hat{u}(t)), t \in [0, T]$, is an admissible pair of states and controls for (3.4) in the sense that $\hat{u}$ is a piecewise continuous function in t satisfying the constraints (3.4d) and $\hat{y}$ is the unique solution of (3.4b),(3.4c) with respect to the control $\hat{u}$. Finally, we assume that the minimum in (3.11a) is attained for $\hat{u}$. Then, V corresponds to the optimal value function in (3.4), i.e.,

$$V(t,y) \;=\; J^*(t,y) \quad , \quad t \in [0,T] \; , \; y \in \mathbb{R}^n \; . \tag{3.12}$$

and the control $\hat{u}$ corresponds to the optimal control, i.e.,

$$\hat{u}(t) \;=\; u^*(t) \quad , \quad t \in [0,T] \; . \tag{3.13}$$

Proof: Let $\tilde{u}(t), t \in [0,T]$, be a piecewise continuous admissible control and let $\tilde{y}(t), t \in [0,T]$, be the associated state. Then, (3.11a) implies

$0 \le g(\tilde{y}(t),\tilde{u}(t)) + \nabla_t V(t,\tilde{y}(t)) + \nabla_y V(t,\tilde{y}(t))^T f(\tilde{y}(t),\tilde{u}(t)) \; , \; t \in [0,T]$.

Since $\tilde{y}$ satisfies $\dot{\tilde{y}}(t) = f(\tilde{y}(t),\tilde{u}(t))$, this readily gives

$$0 \;\le\; g(\tilde{y}(t),\tilde{u}(t)) \;+\; \frac{d}{dt}\Big(V(t,\tilde{y}(t))\Big) \; , \; t \in [0,T],$$

whence by integration over $[0,T]$

$$0 \;\le\; \int_0^T g(\tilde{y}(t),\tilde{u}(t)) \; dt \;+\; V(T,\tilde{y}(T)) \;-\; V(0,\tilde{y}(0)).$$

Observing the initial condition $\tilde{y}(0) = y_0$ and the terminal condition $V(T,y) = h(y)$ results in

$$V(0,\tilde{y}(0)) \;\le\; h(\tilde{y}(T)) \;+\; \int_0^T g(\tilde{y}(t),\tilde{u}(t)) \; dt.$$

Now, if we consider $(\tilde{y},\tilde{u}) = (\hat{y},\hat{u})$, the preceding inequalities become equalities, and therefore, we obtain

$$V(0,\hat{y}(0)) \;=\; h(\hat{y}(T)) \;+\; \int_0^T g(\hat{y}(t),\hat{u}(t)) \; dt.$$

Since $V(0,\hat{y}(0)) \le V(0,\tilde{y}(0))$ for any admissible pair $(\tilde{y},\tilde{u})$, the pair $(\hat{y},\hat{u})$ must be optimal and hence,

$$V(0,y_0) \;=\; J^*(0,y_0).$$

Repeating the same argument for an arbitrary initial time $0 < t < T$, gives the assertion.

Example: We consider the following continuous-time optimal control problem for a scalar ordinary differential equation:

$$\text{(3.14a)} \qquad \text{minimize } J(y,u) := \frac{1}{2} y(T)^2 ,$$

$$\text{(3.14b)} \qquad \text{subject to } \dot{y}(t) = u(t) \quad , \quad t \in [0,T] ,$$

$$\text{(3.14c)} \qquad y(0) = y_0 ,$$

$$\text{(3.14d)} \qquad -1 \le u(t) \le +1 , \ t \in [0,T] .$$

By inspection we see that a natural control policy is to move the state y as fast as possible to zero and to keep it at zero, once it achieves this value, i.e.,

$$\text{(3.15)} \qquad \hat{u}(t) := \hat{\mu}(t,y) = -\operatorname{sgn}(y) = \begin{cases} 1 , \text{ if } y < 0 \\ 0 , \text{ if } y = 0 \\ -1 , \text{ if } y > 0 \end{cases} .$$

Consequently, for an initial time t and initial state y the associated value function $\hat{J}(t,y)$ is given by

$$\text{(3.16)} \qquad \hat{J}(t,y) = \frac{1}{2} \Big(\max(0, |y| - (T-t))\Big)^2 .$$

The value function is depicted in Fig. 3.1.

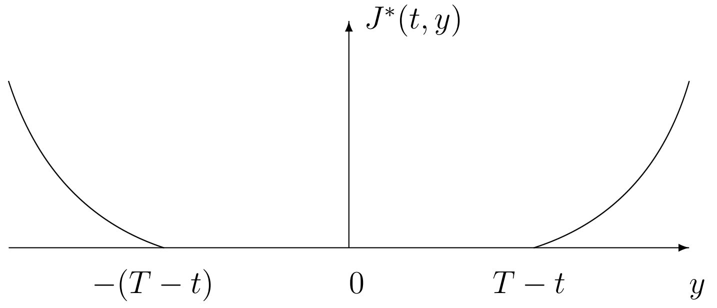

Fig. 3.1 Optimal value function

The value function $\hat{J}(t,y)$ satisfies the terminal condition $\hat{J}(T,y) = y^2/2$. Moreover, the derivatives of $\hat{J}(t,y)$ can be easily computed:

$$\text{(3.17)} \qquad \nabla_t \hat{J}(t,y) = \max(0, |y| - (T-t)) ,$$

$$\text{(3.18)} \qquad \nabla_y \hat{J}(t,y) = \operatorname{sgn}(y) \max(0, |y| - (T-t)).$$

Substituting these expressions into (3.10a), the HJB equation takes the form

$$\min_{|u|\le 1} \ (1+\mathrm{sgn}(y)) \max(0,|y|-(T-t)) \ = \ 0 \ . \tag{3.19}$$

Obviously, (3.19) holds true for all $t \in [0,T]$ and all $y \in \mathbb{R}$. Additionally, the minimum is attained for $\hat{u}$. Consequently, Theorem 3.1 tells us that $\hat{u}$ is the optimal control and $\hat{J}(t,y)$ is the optimal value function for (3.14a)-(3.14d). However, the optimal control is not unique: For $|y(t)| \le T-t$, any control from $[-1,+1]$ does the job.

Example (Linear-quadratic optimal control problems):
For the linear-quadratic optimal control problem (3.5a)-(3.5d), the HJB equation is of the form

$$\begin{aligned} &\min_{u\in\mathbb{R}^m} \Big(\langle y,Qy\rangle \ + \ \langle u,Ru\rangle \ + \ \nabla_t V(t,y) \ + \\ &\qquad\qquad + \ \langle \nabla_y V(t,y), Ay+Bu\rangle \Big) \ = \ 0, \end{aligned} \tag{3.20a}$$

$$V(T,y) \ = \ \langle y, Q_T y\rangle \ . \tag{3.20b}$$

We are looking for a solution of the form

$$V(t,y) \ = \ y^T K(t) y \ , \tag{3.21}$$

where $K(t) \in \mathbb{R}^{n\times n}$ is assumed to be symmetric.
Differentiation with respect to t and x yields

$$\nabla_t V(t,y) \ = \ y^T \dot{K}(t) y \quad , \quad \nabla_y V(t,y) \ = \ 2K(t)y \ . \tag{3.22}$$

Substituting (3.22) into (3.20a) results in

$$0 \ = \ \min_u \mathcal{L}(y,u) \ , \tag{3.23}$$

$$\mathcal{L}(y,u) := y^T Q y + u^T R u + y^T \dot{K}(t) y + 2y^T K(t) A y + 2 y^T K(t) B u \ .$$

The optimality condition is

$$\mathcal{L}_u(y,u) \ = \ 2B^T K(t) y + 2Ru = 0 \ ,$$

which gives rise to

$$u \ = \ -R^{-1} B^T K(t) y \ . \tag{3.24}$$

Substituting (3.24) into (3.23) gives

$$0 \ = \ y^T \Big(\dot{K}(t) + K(t)A + A^T K(t) - K(t) B R^{-1} B^T K(t) + Q \Big) y \quad \text{for all } (t,y) \ .$$

Hence, the matrix $K(t)$ must satisfy the following **continuous-time Riccati equation**

$$\dot{K}(t) \ = \ -K(t)A - A^T K(t) + K(t) B R^{-1} B^T K(t) - Q \tag{3.25}$$

with the terminal condition

$$K(T) \; = \; Q_T \; . \tag{3.26}$$

Conversely, if a symmetric $K(t) \in \mathbb{R}^{n \times n}$ satisfies the continuous-time Riccati equation (3.25) with the terminal condition (3.26), then $V(t,x) = x^T K(t) x$ is a solution of the HJB-equation. Consequently, by Theorem 3.1 the optimal value function is

$$J^*(t,y) \; = \; y^T K(t) y \; , \tag{3.27}$$

and the optimal policy turns out to be

$$\mu^*(t,y) \; = \; -R^{-1} B^T K(t) y \; . \tag{3.28}$$

3.4 Pontrjagin's minimum principle

Pontrjagin's minimum principle is a necessary optimality condition for continuous-time optimal control problems (cf. (3.4a)-(3.4d))

$$\text{(3.29a)} \qquad \text{minimize} \;\; J(y,u) \;:=\; h(y(T)) \;+\; \int_0^T g(y(t),u(t))\,dt \;,$$

$$\text{(3.29b)} \qquad \text{subject to} \;\; \dot{y}(t) \;=\; f(y(t),u(t)) \quad , \quad t \in [0,T] \;,$$

$$\text{(3.29c)} \qquad y(0) \;=\; y_0 \;,$$

$$\text{(3.29d)} \qquad u(t) \in U \quad \text{f.a.a. } t \in [0,T] \;.$$

It states the existence of an **optimal adjoint state** $p^*(t)$ satisfying an **adjoint state equation** such that the optimal state $y^*(t)$, the optimal control $u^*(t)$, and the optimal adjoint state $p^*(t)$ are related by an algebraic equation. This algebraic equation is given in terms of the **Hamiltonian function** $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ which is defined according to

$$\text{(3.30)} \qquad H(y,u,p) \;:=\; g(y,u) \;+\; p^T f(y,u) \;.$$

We impose the following two assumptions:

(A1) Convexity assumption
For every state y the set

$$D \;:=\; \{f(y,u) \mid u \in U\}$$

is convex.

Remark: The convexity assumption is, for instance, satisfied, if the control set U is convex and f,g are linear in u.

(A2) Regularity assumption
Let $u_k(t) \in U, 1 \le k \le 2$, be two admissible controls and let $y_1(t), t \in [0,T]$, be the state associated with u_1. We assume that for any $\varepsilon \in [0,1]$ the solution $y_\varepsilon(t), t \in [0,T]$, of the system

(3.31a)
$$\dot{y}_\varepsilon(t) = \;(1-\varepsilon)f(y_\varepsilon(t),u_1(t)) \;+\; \varepsilon f(y_\varepsilon(t),u_2(t)) \;,\; 0 \le t \le T \;,$$

(3.31b)
$$y_\varepsilon(0) = \; y_1(0) \;,$$

satisfies

$$\text{(3.32)} \qquad y_\varepsilon(t) \;=\; y_1(t) \;+\; \varepsilon\xi(t) \;+\; o(\varepsilon) \;,$$

where $\xi(t), t \in [0,T]$, is the solution of the following initial-value problem for a linear system of ordinary differential equations

(3.33a) $\dot{\xi}(t) = \nabla_y f(y_1(t), u_1(t))\xi(t) + f(y_1(t), u_2(t)) - f(y_1(t), u_1(t))$

(3.33b) $\xi(0) = 0.$

Remark: It is easy to see that in case of a linear system

$$\dot{y}(t) \ = \ Ay(t) \ + \ Bu(t)$$

the regularity assumption **(A2)** is satisfied. In fact, (3.31a) and (3.33a) take the form

$$\begin{aligned} \dot{y}_\varepsilon(t) &= Ay_\varepsilon(t) + Bu_1(t) + \varepsilon B\Big(u_2(t) - u_1(t)\Big) \ , \\ \dot{\xi}(t) &= A\xi(t) + B\Big(u_2(t) - u_1(t)\Big) \ , \end{aligned}$$

so that

$$\varepsilon(t) \ = \ y_1(t) \ + \ \varepsilon\xi(t) \quad , \quad t \in [0,T] \ .$$

Theorem 3.2 (Pontrjagin's minimum principle)
Assume that the convexity assumption **(A1)** and the regularity assumption **(A2)** are satisfied and let $y^*(t), u^*(t), t \in [0,T]$, be the optimal state and the optimal control for (3.29a)-(3.29d). Then, there exists an **optimal adjoint state** $p^*(t), t \in [0,T]$, which satisfies the **terminal value problem** for the **adjoint state equation**

(3.34a) $$\dot{p}^*(t) = \ -\nabla_y H(y^*(t), u^*(t), p^*(t)) \ , \ t \in [0,T] \ ,$$

(3.34b) $$p^*(T) = \ \nabla h(y^*(T)) \ .$$

Moreover, we have

(3.35) $$u^*(t) \ = \ \arg\min_{u \in U} H(y^*(t), u, p^*(t)) \ ,$$

and there exists a constant $C \in \mathbb{R}$ such that

(3.36) $$H(y^*(t), u^*(t), p^*(t)) \ = \ C \quad , \quad t \in [0,T] \ .$$

Proof: We give the proof first in case $J(y,u) = h(y(T))$. The convexity assumption **(A1)** ensures that for any admissible control $u(t) \in U, t \in [0,T]$, and any $\varepsilon \in [0,1]$ there exists $\bar{u}(t), t \in [0,T]$, such that

$$f(y_\varepsilon(t), \bar{u}(t)) \ = \ (1-\varepsilon) f(y_\varepsilon(t), u^*(t)) + \varepsilon f(y_\varepsilon(t), u(t)) \ .$$

Hence, the state $y_\varepsilon(t)$ of (3.31a) corresponds to the control $\bar{u}(t)$. The optimality of $y^*(t)$ and the regularity assumption **(A2)** imply

$$\begin{aligned} h(y^*(T)) \ \le \ h(y_\varepsilon(T)) \ &= \ h(y^*(T) + \varepsilon\xi(T) + o(\varepsilon)) \ = \\ &= \ h(y^*(T)) + \varepsilon \nabla h(y^*(T))^T \xi(T) + o(\varepsilon) \ , \end{aligned}$$

whence

$$\nabla h(y^*(T))^T \xi(T) \ \geq \ 0 \ . \tag{3.37}$$

Now, let $W(t,\tau)$ be the **Wronski matrix** associated with the linear system (3.33) of ordinary differential equations, i.e.,

$$\frac{\partial W(t,\tau)}{\partial \tau} = \ -W(t,\tau)\nabla_y f(y^*(\tau),u^*(\tau))^T \ , \tag{3.38a}$$

$$W(t,t) = \ I \ . \tag{3.38b}$$

Then, the solution of (3.33) can be written in closed form according to

$$\begin{aligned} \xi(t) \ &= \ W(t,\tau)\xi(\tau) \ + \\ &+ \int\limits_\tau^t W(t,s)\Big(f(y^*(s),u(s)) - f(y^*(s),u^*(s))\Big)ds \ . \end{aligned} \tag{3.39}$$

Since $\xi(0) = 0$, we deduce

$$\xi(T) \ = \ \int\limits_0^T W(T,t)\Big(f(y^*(t),u(t)) - f(y^*(t),u^*(t))\Big)dt \ . \tag{3.40}$$

Now, we define
(3.41)

$$p^*(t) \ := \ W(T,t)^T p^*(T) \ , \ t \in [0,T] \quad , \quad p^*(T) \ := \ \nabla h(y^*(T)) \ .$$

Differentiation with respect to t yields

$$\dot{p}^*(t) \ = \ \frac{\partial W(T,t)^T}{\partial t} p^*(T) \ .$$

Using this with (3.38a) and (3.41), we find that $p^*(t)$ satisfies

$$\dot{p}^*(t) = \ -\nabla_y f(y^*(t),u^*(t))^T p^*(t) \quad , \quad t \in [0,T] \ , \tag{3.42a}$$

$$p^*(T) = \ \nabla h(y^*(T)) \ , \tag{3.42b}$$

which is the adjoint state equation for $H(y,u,p) = p^T f(y,u)$.
It remains to prove the minimum principle (3.35). The relations (3.37), (3.40), and (3.41) readily imply

$$\begin{aligned} 0 \ &\leq \ p^*(T)^T \xi(T) \ = \\ &= \ p^*(T)^T \int\limits_0^T W(T,t)\Big(f(y^*(t),u(t)) - f(y^*(t),u^*(t))\Big)dt \ = \\ &= \ \int\limits_0^T p^*(t)^T \Big(f(y^*(t),u(t)) - f(y^*(t),u^*(t))\Big)dt \ . \end{aligned} \tag{3.43}$$

We claim that (3.43) implies that for all $t \in [0,T]$ where $u^*(t)$ is continuous there holds

$$p^*(t)^T f(y^*(t), u^*(t)) \;\le\; p^*(t)^T f(y^*(t), u(t)) \quad \text{for all } u \in U \; . \tag{3.44}$$

We prove (3.44) by contradiction. In fact, if (3.44) does not hold true, then for some $\hat{u} \in U$ and $t_0 \in [0,T]$, where u^* is continuous, we have

$$p^*(t_0)^T f(y^*(t_0), u^*(t_0)) \;>\; p^*(t_0)^T f(y^*(t_0), \hat{u}) \; .$$

The continuity of u^* also implies the existence of an interval $I(t_0) \subset [0,T]$ such that

$$p^*(t)^T f(y^*(t), u^*(t)) \;>\; p^*(t)^T f(y^*(t), \hat{u}) \quad , \quad t \in I(t_0) \; .$$

Choosing

$$u(t) \;=\; \begin{cases} \hat{u} \; , \; t \in I(t_0) \\ u^*(t) \; , \; t \notin I(t_0) \end{cases} \quad ,$$

then leads to a contradiction of (3.43).
The general case

$$J(y,u) \;=\; h(y(T)) \;+\; \int_0^T g(y(t), u(t)) \; dt$$

can be reformulated as the terminal cost problem

$$\bar{J}(y,u) \;=\; h(y(T)) \;+\; y(T)$$

by introducing the additional differential equation

$$\dot{y}(t) \;=\; g(y(t), u(t)) \quad , \quad t \in [0,T] \; .$$

Using analogous arguments as before allows to conclude.

Example (Resource allocation (revisited)): In the resource allocation problem considered before, a producer wants to allocate a portion $u(t)$ of his production rate $y(t)$ to reinvestment and $1-u(t)$ to production of a storable good. He wants to maximize the total amount of product stored

$$\int_0^T (1-u(t)) y(t) \; dt$$

subject to

$$\begin{aligned} \dot{y}(t) \;&=\; \gamma u(t) y(t) \; , \; t \in [0,T] \quad , \quad y(0) = y_0 \; , \\ 0 \;&\le\; u(t) \;\le\; 1 \; , \; t \in [0,T] \; . \end{aligned}$$

The Hamiltonian is given by

$$H(y,u,p) \;=\; (1-u)y \;+\; \gamma p u y \; ,$$

and the adjoint equation reads as follows

$$\dot{p}^*(t) \;=\; -\gamma u^*(t)p^*(t) - 1 + u^*(t)\ ,\ t \in [0,T] \quad , \quad p^*(T) = 0\ .$$

If we maximize the Hamiltonian over $u \in [0,1]$, we obtain the optimal control

$$u^*(t) \;=\; \begin{cases} 0\ ,\ p^*(t) < 1/\gamma \\ 1\ ,\ p^*(t) \geq 1/\gamma \end{cases} .$$

We discuss the optimal control and the adjoint state for the following cases:

Case 1 ($t > 1 - T/\gamma$): Since $p^*(T) = 0$, for t close to T we have $p^*(t) < 1/\gamma$ and hence,

$$u^*(t) \;=\; 0 \quad , \quad t \in [\max(0, T - 1/\gamma), T]\ .$$

The adjoint equation takes the form $\dot{p}^*(t) = -1$, whence

$$p^*(t) \;=\; T - t \quad , \quad t \in [\max(0, T - 1/\gamma), T]\ .$$

If $T < 1/\gamma$, the optimal control is $u^*(t) = 0\ ,\ t \in [0,T]$. This means that for a small time horizon, the best strategy is not to reinvest at any time.

Case 2 ($t \leq T - 1/\gamma$ and $T > 1/\gamma$): For $t = T - 1/\gamma$ we have $p^*(t) = 1/\gamma$ and thus

$$u^*(t) \;=\; 1 \quad , \quad t \in [0, T - 1/\gamma]\ .$$

Consequently, for $t < T - 1/\gamma$, the adjoint equation is of the form $\dot{p}^* = -\gamma p^*(t)$ with the solution

$$p^*(t) \;=\; \gamma^{-1}\exp(\gamma(T - 1/\gamma - t)) \quad , \quad t \in [0, T - 1/\gamma]\ .$$

Example (Brachistochrone): The brachistochrone problem is due to Johann Bernoulli: Given two points A and B, find a curve $\mathcal{C}_{AB}$ from A to B such that a body with mass m that moves along the curve under the force of gravity reaches B in minimum time T (cf. Fig. 3.2).

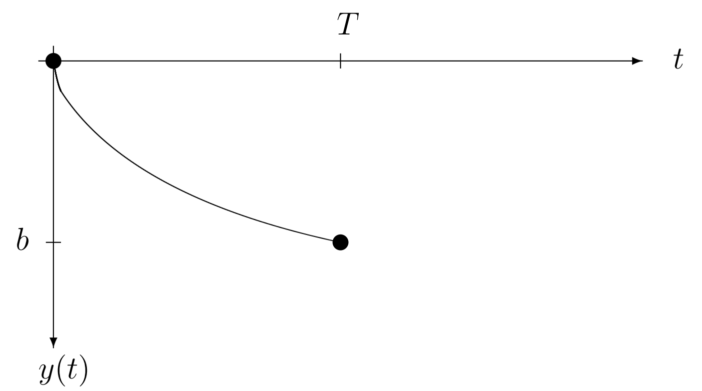

Fig. 3.2. The brachistochrone problem

In terms of the function $y = y(t), t \in [0,T]$, the length of the curve from $y(t)$ to $y(t+dt)$ is given by

$$\sqrt{1 \ + \ (\dot{y}(t))^2} \ dt \ . \tag{3.45}$$

We denote by $v(t)$ the velocity of the body and by γ the gravitational acceleration. Since the kinetic energy $E_{kin}(t) := mv(t)^2/2$ and the potential energy $E_{pot}(t) := m\gamma y(t)$ are in equilibrium at each point of the curve, it follows that

$$v(t) \ = \ \sqrt{2\gamma y(t)} \ . \tag{3.46}$$

Hence, the brachistochrone problem can be formulated as the **minimum-time problem**:

$$\text{minimize} \quad \int_0^T \frac{\sqrt{1+(\dot{y}(t))^2}}{\sqrt{2\gamma y(t)}} \ dt \tag{3.47}$$

Setting $u(t) := \dot{y}(t), t \in [0,T]$, and substituting $v(t)$ in (3.47) by (3.46), we obtain the continuous-time optimal control problem:

$$J(y^*,u^*) \ = \ \min_{(y,u)} J(y,u) \quad , \quad J(y,u) \ = \ \int_0^T \frac{\sqrt{1+(u(t))^2}}{\sqrt{2\gamma y(t)}} \ dt \tag{3.48}$$

subject to the boundary value problem

$$\dot{y}(t) = \ u(t) \quad , \quad t \in [0,T] \ , \tag{3.49a}$$

$$y(0) = \ 0 \quad , \quad y(T) \ = \ b \ . \tag{3.49b}$$

The Hamiltonian associated with (3.48),(3.49) is

$$H(y,u,p) \ = \ g(y,u) + pu \quad , \quad g(y,u) \ := \ \frac{\sqrt{1+(u(t))^2}}{\sqrt{2\gamma y(t)}} \ . \tag{3.50}$$

The optimality condition

$$H_u(y^*,u^*,p^*) \ = \ 0$$

gives rise to

$$p^*(t) \ = \ -\nabla_u g(y^*(t),u^*(t)) \ . \tag{3.51}$$

Pontrjagin's minimum principle tells us that the Hamiltonian is constant along an optimal trajectory, i.e.,

$$g(y^*(t),u^*(t)) \ - \ \nabla_u g(y^*(t),u^*(t))u^*(t) \ = \ const. \quad , \quad t \in [0,T] \ ,$$

whence

$$\frac{\sqrt{1+(u^*(t))^2}}{\sqrt{2\gamma y^*(t)}} \ - \ \frac{(u^*(t))^2}{\sqrt{1+(u^*(t))^2}\sqrt{2\gamma y^*(t)}} \ =$$

$$= \frac{1}{\sqrt{1+(u^*(t))^2}\sqrt{2\gamma y^*(t)}} = const. \quad , \quad t \in [0,T] \ .$$

Using $\dot{y}^*(t) = u^*(t)$, for some constant $C \in \mathbb{R}$ we thus obtain

$$y^*(t)\Big(1 + (\dot{y}^*(t))^2\Big) = C \quad , \quad t \in [0,T] \ .$$

It follows that the optimal trajectory y^* satisfies the differential equation

$$\dot{y}^*(t) = \sqrt{\frac{C - y^*(t)}{y^*(t)}} \quad , \quad t \in [0,T] \ , \tag{3.52}$$

whose solution is known as a **cycloid**. The unknown parameters C and T of the cycloid can be determined by the boundary conditions $y^*(0) = 0$ and $y^*(T) = b$.

References

[1] D.P. Bertsekas; Dynamic Programming and Optimal Control. Vol. 1. 3rd Edition. Athena Scientific, Belmont, MA, 2005