Chapter 4 Optimal Control Problems in Infinite Dimensional Function Space

4.1 Introduction

In this chapter, we will consider optimal control problems in function space where we will restrict ourselves to state equations that are both linear in the state and linear in the control:
We assume that E is a reflexive, separable Banach space with norm $\|\cdot\|_E$ and dual space E^* and X is another separable Banach space with norm $\|\cdot\|_X$ and separable dual space X^*. We further suppose $A : D(A) \subseteq E \to E^*$ to be the infinitesimal generator of a compact, analytic semigroup $S(\cdot)$ in E (cf. section 4.2 below) and $B : X^* \to E$ to be a bounded linear operator. Finally, let $g : C([0,T];E) \to C([0,T])$ and $h : L^\infty([0,T];X^*) \to L^\infty([0,T])$ be strictly convex, coercive and continuously Fréchet-differentiable mappings with Fréchet-derivatives $g'(\cdot)$ and $h'(\cdot)$, respectively.

We consider the control constrained optimal control problem:

$$\text{(4.1a)} \qquad \text{minimize } J(y,u) \ := \ \int_0^T \Big(g(y(t)) + h(u(t)) \Big) \, dt \ ,$$

$$\text{(4.1b)} \qquad \text{subject to } \dot{y}(t) \ = \ Ay(t) \ + \ Bu(t) \quad , \quad t \in [0,T] \ ,$$

$$\text{(4.1c)} \qquad y(0) \ = \ y_0 \ ,$$

$$\text{(4.1d)} \qquad u \in L^\infty([0,T];K) \ , \ K \subseteq X^*,$$

where $\dot{y} := dy/dt, y_0 \in E$, and K is assumed to be weakly* closed and convex.
We note that $L^\infty([0,T];X^*)$ denotes the linear space of functions v with $v(t) \in X^*$ for almost all $t \in [0,T]$ such that $\|v(t)\|_{X^*}$ is essentially bounded in $t \in [0,T]$. It is a normed space with respect to the norm

$$\|v\|_{L^\infty([0,T];X^*)} \ := \ \operatorname*{ess\ sup}_{t\in[0,T]} \|v(t)\|_{X^*} \ .$$

Consequently, the control constraint (4.1d) implies

$$u(t) \in K \text{ for almost all } t \in [0,T] \ .$$

In addition to the above assumptions, we require the control constraint set $L^\infty([0,T];K)$ to be $L^1([0,T];X)$-weakly compact.

In the sequel, for Banach spaces E and F with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ we refer to $\mathcal{L}(E,F)$ as the normed space of bounded linear operators

$L : E \to F$ with norm

$$\|L\| \ := \ \sup_{y \in E \setminus \{0\}} \frac{\|Ly\|_F}{\|y\|_E} \ .$$

4.2 Evolution equations in function space

Let E be a reflexive, separable Banach space with norm $\| \cdot \|_E$ and dual E^* and $A : D(A) \subseteq E \to E^*$ be a densely defined, closed linear operator. We consider the following initial-value problem (**Cauchy problem**) for an **abstract evolution equation**

$$\text{(4.2a)} \qquad \dot{y}(t) = \ Ay(t) \quad , \quad t \geq 0 \ ,$$

$$\text{(4.2b)} \qquad y(0) = \ y_0 \in E \ .$$

The Cauchy problem is said to be **well posed**, if

- for every $y_0 \in D(A)$ there exists a solution y of (4.2a) satisfying (4.2b),
- there exists a positive function $C = C(t), t \geq 0$, that is bounded on bounded subsets of the nonnegative real line such that

$$\text{(4.3)} \qquad \|y(t)\|_E \ \leq \ C(t) \ \|y_0\|_E \ , \ t \geq 0 \ .$$

If the Cauchy problem is well posed, the **solution operator** $S(\cdot)$ is given by

$$S(t)y_0 \ = \ y(t) \quad , \quad t \geq 0 \ .$$

It can be easily extended to a mapping $S(\cdot) : [0, \infty) \to \mathcal{L}(E, E)$ according to

$$S(t)z \ = \ \lim_{n \to \infty} S(t)z_n \ , \ t \geq 0 \ ,$$

where $z \in E$ and $z_n \in D(A), n \in \mathbb{N}$, such that $\lim\limits_{n \to \infty} z_n = z$.
The solution operator S is an operator valued function that is **strongly continuous** in $t \in \mathbb{R}_+$, i.e., the mapping $t \longmapsto S(t)y$ is continuous in the norm of E for every $y \in E$. Moreover, it has the properties

$$\text{(4.4a)} \qquad S(0) = \ I \ ,$$

$$\text{(4.4b)} \qquad S(t_1 + t_2) = \ S(t_1)S(t_2) \quad , \quad t_i \in \mathbb{R}_+ \ , \ 1 \leq i \leq 2 \ .$$

Definition 4.1 (Strongly continuous and analytic semigroup)
An operator valued function $S : \mathbb{R}_+ \to \mathcal{L}(E, E)$ that satisfies (4.4a) and (4.4b) is called a **semigroup**. If it is strongly continuous, it is said to be a **strongly continuous semigroup**. Moreover, if it is analytic in $t > 0$, it is referred to as an **analytic semigroup**.

Given a strongly continuous semigroup S, we define an operator $A : D(A) \subset E \to E^*$ according to

$$(4.5) \qquad Ay \; := \; \lim_{h \to 0_+} h^{-1}\Big(S(h) - I\Big)y \quad , \quad y \in D(A) \; ,$$

where $D(A) \subset E$ consists of all $y \in E$ for which the limit exists.

Definition 4.2 (Infinitesimal generator)
The operator $A : D(A) \subset E \to E^*$ given by (4.5) is called the **infinitesimal generator** of the strongly continuous semigroup S.

Theorem 4.1 (Properties of the infinitesimal generator)
The infinitesimal generator $A : D(A) \subset E \to E^*$ of a strongly continuous semigroup S is a **closed densely defined operator** such that the Cauchy problem (4.2a), (4.2b) is well posed.

Proof: In order to show that A is densely defined, let $y \in E$ and for $\varepsilon > 0$ define

$$(4.6) \qquad y_\varepsilon \; := \; \varepsilon^{-1} \int_0^\varepsilon S(t)y \; dt \; .$$

Using (4.4b), for $h \leq \varepsilon$ we find

$$\begin{aligned} h^{-1}\Big(S(h) - I\Big)y_\varepsilon \; &= \; \varepsilon^{-1}\Big(h^{-1} \int_\varepsilon^{\varepsilon+h} S(t)y \; dt - h^{-1} \int_0^h S(t)y \; dt\Big) \\ &\to \varepsilon^{-1}\Big(S(\varepsilon)y - y\Big) \quad \text{as } h \to 0_+ \; , \end{aligned}$$

where we have used the continuity of $S(\cdot)y$. Hence, $y_\varepsilon \in D(A)$ and

$$Ay_\varepsilon \; = \; \varepsilon^{-1}\Big(S(\varepsilon)y - y\Big) \; .$$

Similarly, one can show $y_\varepsilon \to y$ as $\varepsilon \to 0_+$ which proves the denseness of A.
For the proof of the closedness of A, let $y, z \in E$ and $\{y_n\} \subset D(A)$ such that $y_n \to y$ and $Ay_n \to z$ as $n \to \infty$. Then

$$h^{-1}\Big(S(h) - I\Big)y \; = \; \lim_{n \to \infty} h^{-1}\Big(S(h) - I\Big)y_n \; = \; \lim_{n \to \infty} (Ay_n)_h \; = \; z_h \; ,$$

where $(Ay_n)_h$ and z_h as in (4.6). The limit process $h \to 0_+$ reveals $y \in D(A)$ and $Ay = z$.

Theorem 4.2 (Hille-Yosida theorem)
An operator $A : D(A) \subset E \to E^*$ is the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ with

$$(4.7) \qquad \|S(t)\| \; \leq \; C \exp(\omega t) \quad , \quad t \in \mathbb{R}_+$$

for some $C \in R_+, \omega > 0$, if and only if A is a closed densely defined operator whose resolvent $R(\lambda; A) := (\lambda I - A)^{-1}$ exists for all $\lambda > \omega$ and satisfies

$$\|R(\lambda;A)^n\| \ \leq \ C\ (\lambda - \omega)^{-n} \quad , \quad n \in \mathbb{N} \ . \tag{4.8}$$

Proof: We refer to [1].

For the optimality conditions associated with (4.1a)-(4.1d) we need the following result:

Theorem 4.3 (Adjoint semigroup)
Let $A : D(A) \subset E \to E^*$ be the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ in a reflexive Banach space E. Then, its adjoint $A^* : D(A^*) \subset E^* \to E$ is the infinitesimal generator of the adjoint semigroup $S^*(\cdot) = S(\cdot)^*$ in E^*.

Proof: We refer to Theorem 5.2.6 in [2].

Instead of the Cauchy problem (4.2a),(4.2b) let us consider the slightly more general problem

$$\dot{y}(t) = \ Ay(t) + f(t) \quad , \quad t \geq 0 \ , \tag{4.9a}$$

$$y(0) = \ y_0 \in E \ , \tag{4.9b}$$

where A is assumed to be the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ in E and $f \in C([0,T]; E)$.

Theorem 4.3 (Representation of a solution)
The solution $y = y(t), t \in \mathbb{R}_+$, of (4.9a),(4.9b) has the representation

$$y(t) \ = \ S(t)y_0 \ + \ \int\limits_0^t S(t-\tau)f(\tau)\ d\tau \ . \tag{4.10}$$

Proof: Let $g(\tau) := S(t-\tau)y(\tau)$. Then, there holds

$$\begin{aligned} \dot{g}(\tau) \ &= \ -AS(t-\tau)y(\tau) + S(t-\tau)\dot{y}(\tau) \ = \\ &= \ -AS(t-\tau)y(\tau) + S(t-\tau)\Big(Ay(\tau) + f(\tau)\Big) \ = \\ &= \ S(t-\tau)f(\tau) \ , \end{aligned}$$

whence

$$g(t) - g(0) \ = \ y(t) - S(t)y_0 \ = \ \int\limits_0^t S(t-\tau)f(\tau)\ d\tau \ .$$

4.3 Existence and uniqueness of an optimal solution

We consider the control constrained optimal control problem (4.1a)-(4.1d) under the assumptions made in section 4.1.
Since X is a separable reflexive Banach space, it follows that functions $u \in L^\infty([0,T];X^*)$ have a measurable norm. Thus, given $y_0 \in E$ and $u \in L^\infty([0,T];X^*)$, the solution $y = y(t), t \in \mathbb{R}_+$, of (4.1a) is given by

$$(4.11) \qquad y(t) \ = \ S(t)y_0 \ + \ \int\limits_0^t S(t-\tau)Bu(\tau) \ d\tau \ .$$

Indeed, for $z \in E^*$, we have $\langle z, Bu(\tau)\rangle = \langle B^* z, u(\tau)\rangle$ so that $Bu(\cdot)$ is an E-valued, E^*-weakly measurable function. Consequently, the integrand in (4.11) is a strongly measurable E-valued function so that the integral is a Lebesgue-Bochner integral. Moreover, the integral is continuous so that the solution y is a continuous E-valued function.
It follows that (4.11) defines a mapping

$$(4.12) \qquad \begin{aligned} G \quad &: \quad L^\infty([0,T];X^*) \to C([0,T];E) \\ Gu(t) \ &:= \ S(t)y_0 \ + \ \int\limits_0^t S(t-\tau)Bu(\tau) \ d\tau \ , \end{aligned}$$

which we refer to as the **control-to-state map**. We note that due to the boundedness of the operator B and the compactness of the semigroup $S(\cdot)$, the control-to-state map G is a compact linear operator.
Substituting $y = Gu$ in (4.1a), we get the **reduced optimal control problem**

$$(4.13) \qquad \inf_{u \in L^\infty([0,T];K)} J_{red}(u) := \int\limits_0^T \Big(g(Gu(t)) + h(u(t)) \Big) \ dt \ .$$

Theorem 4.4 (Existence and uniqueness)
Under the assumptions on the data, the control constrained optimal control problem (4.1a)-(4.1d) admits a unique solution (y^*, u^*) with $y^* \in C([0,T];E)$ and $u^* \in L^\infty([0,T];K)$.

Proof: Let $\{u_n\}_\mathbb{N}, u_n \in L^\infty([0,T];K), n \in \mathbb{N}$, be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} J_{red}(u_n) \ = \ \inf_{u \in L^\infty([0,T];K)} J_{red}(u) \ .$$

Due to the coerciveness assumption on the functionals g and h, the sequence $\{u_n\}_\mathbb{N}$ is bounded in $L^\infty([0,T];K)$. Since $L^\infty([0,T];K)$ is

$L^1([0,T];X)$-weakly compact, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and an element $u^* \in L^\infty([0,T];K)$ such that for all $v \in L^1([0,T];X)$

$$\int_0^T \langle u_n(t), v(t)\rangle_{X^*,X} \ dt \to \int_0^T \langle u^*(t), v(t)\rangle_{X^*,X} \ dt \quad (n \in \mathbb{N}', n \to \infty) \ .$$

Due to the compactness of the state-to-control map G, we have

$$Gu_n(t) \to Gu^*(t) \quad (n \in N', n \to \infty) \quad \text{uniformly} \ \text{ in } t \in [0,T] \ .$$

Due to its convexity, the reduced functional J_{red} is lower semicontinuous with respect to the $L^1([0,T];X)$-weak topology of $L^\infty([0,T];X^*)$, and hence

$$\lim\inf_{n\to\infty} J_{red}(u_n) \ \leq \ J_{red}(u^*) \ ,$$

which allows to conclude with $y^* = Gu^*$.

4.4 Optimality conditions

The necessary optimality conditions for the reduced optimal control problem (4.13) are given by the variational inequality

$$\langle J_{red}'(u^*), u^* - u\rangle \ \leq \ 0 \quad , \quad u \in L^\infty([0,T];K) \ . \tag{4.14}$$

Evaluating the Gâteaux derivative $J_{red}'(u^*)$ allows to characterize the optimal solution (y^*,u^*) by means of an **adjoint state** $p^* = p^*(t)$ satisfying a **terminal value problem** for an associated **adjoint equation**. We note that in this case the optimality conditions are also sufficient due to the convexity of the objective functional.

Theorem 4.5 (Optimality conditions)
Let $(y^*,u^*) \in C([0,T];E) \times L^\infty([0,T];K)$ be the optimal solution of the control constrained optimal control problem (4.1a)-(4.1d). Then, there exists an adjoint state $p^* \in C([0,T];E^*)$ such that the following optimality conditions are satisfied

$$\dot y^*(t) - Ay^*(t) - Bu^*(t) = 0 \ , \ t \in [0,T] \quad , \quad y^*(0) = y_0 \ , \tag{4.15a}$$

$$\dot p^*(t) + A^*p^*(t) - g'(y^*(t)) = 0 \ , \ t \in [0,T] \quad , \quad p^*(T) = 0 \ , \tag{4.15b}$$

$$u^* \in L^\infty([0,T];K) \ , \tag{4.15c}$$

$$\int_0^T \Big(\langle u^*(t) - u(t), B^*p^*(t)\rangle_{X^*,X} \ + \ + \ h'(u^*(t))(u^*(t) - u(t))\Big) \ dt \ \leq \ 0 \quad , \quad u \in L^\infty([0,T];K) \ . \tag{4.15d}$$

Proof: Observing the definition (4.13) of the reduced objective functional J_{red}, the variational inequality (4.14) reads as follows

$$
\begin{aligned}
(4.16)\quad & \int_0^T \Big(\langle g'(Gu^*(t)), GB(u^*(t) - u(t))\rangle \ + \\
& \quad + h'(u^*(t))(u^*(t) - u(t))\Big)\ dt \ \le \ 0 \quad , \quad u \in L^\infty([0,T];K) \ .
\end{aligned}
$$

For the first part in (4.16) we obtain

$$
\begin{aligned}
(4.17)\qquad & \int_0^T \Big(\langle g'(Gu^*(t)), GB(u^*(t) - u(t))\rangle \ dt \ = \\
& = \int_0^T \langle u^*(t) - u(t), B^*G^*g'(Gu^*(t))\rangle \ dt \ .
\end{aligned}
$$

We know that $Gu^*(t) = y(t) = S(t)y_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau$. For the evaluation of G^*, assume $v \in C([0,T];E)$. Then, there holds

$$
\begin{aligned}
& \int_0^T \langle \int_0^t S(t-\tau)Bu(\tau)\ d\tau, v(t)\rangle \ dt \ = \\
= \ & \int_0^T \int_0^\tau \langle S(\tau - t)Bu(t), v(\tau)\rangle \ dtd\tau \ = \\
= \ & \int_0^T \langle u(\tau), \int_0^\tau B^*S^*(\tau - t)v(t)\ dt\rangle \ d\tau \ = \\
= \ & \int_0^T \langle u(t), B^* \int_0^t S^*(t-\tau)v(\tau)\ d\tau\rangle \ dt \ .
\end{aligned}
$$

Since $S^*(\cdot)$ is an analytic semigroup with infinitesimal generator A^*, we may interpret $z(t) := \int_0^t S^*(t-\tau)v(\tau)\ d\tau$ as the solution of the initial value problem

$$
\dot z(t) \ = \ A^*z(t) + v(t) \ , \ t \in [0,T] \quad , \quad z(0) = 0 \ .
$$

We set

$$p^*(t) \ := \ \int\limits_{T}^{T-t} S^*(\tau - (T-t)) g'(y^*(T-\tau)) \ d\tau$$

and employ the transformation $\hat{\tau} = T - \tau$ which gives

$$p^*(t) \ = \ - \int\limits_{0}^{t} S^*(t-\hat{\tau}) g'(y^*(\hat{\tau})) \ d\hat{\tau} \ . \tag{4.18}$$

It follows that p^* satisfies the terminal value problem (4.15b). Substituting $G^* g'(y^*(t))$ in (4.17) by $p^*(t)$ readily gives (4.15d).

References

[1] N. Dunford and T. Schwartz; Linear Operators. Part I. Interscience, New York, 1958

[2] H.O. Fattorini; Infinite Dimensional Optimization and Control Theory. Cambridge University Press, Cambridge, 1999