Yesterday...

Any square matrix A is a change of basis to a matrix J, where

$$J = \begin{bmatrix} J_{\lambda_1, k_1} & & \\ & \ddots & \\ & & J_{\lambda_r, k_r} \end{bmatrix}$$

eigenvector

generalized eigenvectors

$$J_{\lambda, k} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} \quad k \times k$$

Every time we lack an eigenvector we put a 1 above the eigenvalue.

let's say

$$A = SJS^{-1}$$

let $J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$,

A has 1 eigenvector, w.

$$S = \begin{pmatrix} w & ? & ? & ? \end{pmatrix} = \begin{pmatrix} w & v_2 & v_3 & 4 \end{pmatrix}$$

try to solve for $v_2 v_3 v_4$ using the $A = SJS^{-1}$ eq'n.

maybe $v_2 v_3 v_4$ can be found in terms of A, w, λ.

$$AS = SJS^{-1}S \rightsquigarrow AS = SJ$$

$$A\begin{pmatrix} w & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} w & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

$$A\begin{pmatrix} w & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} w & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

$$\begin{pmatrix} Aw & Av_2 & Av_3 & Av_4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda w & w + \lambda v_2 & v_2 + \lambda v_3 & v_3 + \lambda v_4 \end{pmatrix}$$

$Aw = \lambda w$ ✓ w is an eigenvector for λ

$$\left.\begin{aligned} Av_2 &= \lambda v_2 + w \\ Av_3 &= \lambda v_3 + v_2 \\ Av_4 &= \lambda v_4 + v_3 \end{aligned}\right\}$$

You can solve for $v_2 v_3 v_4$ by multiplying by $A-\lambda I$ repeatedly

$$Av_2 - \lambda v_2 = w$$

$$(A-\lambda I)\boxed{v_2} = w$$

$$(A-\lambda I)\boxed{v_3} = v_2$$

$$(A-\lambda I)\boxed{v_4} = v_3$$

could solve for v_2 using row reduction!

$\underline{w\ v_2\ v_3\ v_4}$

Jordan chain

Def : A Jordan chain for a square matrix A is a sequence of vectors $\underbrace{w_1, w_2, \ldots, w_j}_{\text{generalized eigenvectors}} \in \mathbb{C}^n$ such that

$$Aw_1 = \lambda w_1$$

$$Aw_2 = \lambda w_2 + w_1$$

$$Aw_3 = \lambda w_3 + w_2$$

$$\vdots$$

$$Aw_j = \lambda w_j + w_{j-1}.$$

Def : A nonzero vector w is called a generalized eigenvector if for some k and λ,

$$(A - \lambda I)^k w = 0.$$

How?

Claim: Given a Jordan chain

$w_1 \ldots w_j$, w_1 is an eigenvector,

$w_2 \ldots w_j$ are generalized eigenvectors.

Pf:

$$Aw_1 = \lambda w_1 \qquad \underline{(A - \lambda I)w_1 = 0}$$

$$Aw_2 = \lambda w_2 + w_1 \qquad \underline{(A - \lambda I)w_2 = w_1}$$

$$Aw_3 = \lambda w_3 + w_2 \iff (A - \lambda I)w_3 = w_2$$

$$\vdots \qquad \vdots$$

$$Aw_j = \lambda w_j + w_{j-1} \qquad (A - \lambda I)w_j = w_{j-1}$$

$$(A - \lambda I)^2 w_2 = (A - \lambda I)\overbrace{(A - \lambda I) w_2}$$

$$= \overbrace{(A - \lambda I) w_1} = 0$$

w_2 is a generalized eigenvector!

$$\boxed{(A-\lambda I)^k w_k}$$

k^{th} element of the chain

$$= \underbrace{(A-\lambda I)(A-\lambda I)\ \ldots\ldots\ (A-\lambda I)}_{k \text{ times}} w_k$$

$$= \underbrace{(A-\lambda I)\ \ldots\ldots\ (A-\lambda I)}_{k-1 \text{ times}} w_{k-1}$$

$\vdots$

$$= (A-\lambda I)w_1 = \boxed{0}$$

So w_k is a generalized eigenvector.

Thm Every square matrix A can be written in the form

$$A = SJS^{-1}$$

where
$$J = \begin{bmatrix} J_{\lambda_1, k_1} & & \\ & \ddots & \\ & & J_{\lambda_r, k_r} \end{bmatrix}$$

The columns of S are family of Jordan chains of A.

Pf: Every time we don't have enough eigenvectors for λ, compute Jordan chain starting at an eigenvector you do have.

$(A - \lambda I) w_2 = w_1$ always solvable.

$(A - \lambda I) w_3 = w_2$ always solvable ...

$\vdots$

$(A - \lambda I) w_j = w_{j-1}$ solvable $\square$

<u>Ex</u>: let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$.

Not diagonalizable.

$\det(A - \lambda I) = 0$

$-(\lambda - 1)^3 = 0$

$\lambda = 1, \underline{\lambda = 1}, \underline{\lambda = 1}$

3 eigenvectors? No...

We lack 2 eigenvectors!

$\ker(A - 1 I) = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$.

We need 2 generalized eigenvectors!

$A = S J S^{-1}$

Jordan chain we need to compute

$S = \begin{pmatrix} 0 & & \\ 1 & \boxed{v_2 \quad v_3} \\ 1 & & \end{pmatrix}$

finish by finding these!

$J = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

By def.

$(A - 1I)v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Solve $\quad v_2 = c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$(A - \lambda I)v_3 = v_2$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

A solution is (solve using row reduction)

$$v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

done!

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$$

Ex

$$A = \begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix}$$

$$P_\lambda(A) = 2\lambda^2 - \lambda^3 = 0$$

$$\lambda = 0, \lambda = 0, \lambda = 2$$

$$V_0 = \ker(A - 0I) = \ker(A)$$

$$= \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(compute by row reduction)

$$V_2 = \ker(A - 2I) = \operatorname{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda = 2$

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad ??}_{\text{Jordan chain for } \lambda = 0} \qquad \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\text{Jordan chain for } \lambda = 2}$$

need 1 generalized eigenvector

let $w_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ w_2 generalized

then $(A - 0I) w_2 = w_1$

$$A w_2 = w_1$$

$$\begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \underset{\text{free}}{-\frac{1}{2}} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is free

$$x = \frac{1}{2}z - \frac{1}{2}$$

$$y = 1$$

$$z = t$$

eventually to A

aka $\frac{1}{2}w_1$

$$w_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t - \frac{1}{2} \\ 1 \\ t \end{pmatrix} = \cancel{t\begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}} + \boxed{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}}$$

$w_2 =$

$$\underbrace{\overset{\lambda = 0}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} \qquad \overset{\lambda = 0}{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}}}_{\text{Jordan chain of length 2}} \qquad \underbrace{\overset{\lambda = 2}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}_{\text{Jordan chain of length 1}}$$

$$\begin{cases} A = SJS^{-1} \\ J = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 2 \end{bmatrix} \\ S = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \end{cases}$$

ans!