Textbook: 4.4.12abd, 4.4.21, 4.4.29abc

DON'T TURN THESE IN, THIS IS NOT HOMEWORK! JUST FOR PRACTICE!
See Lecture Notes from 11-11 for 4.4.12b and 4.4.29b.

Solution (4.4.21). Let V be an inner product space. (a) To show that $V^\perp = \{0\}$, we must prove that (i) $\{0\} \subseteq V^\perp$ and (ii) $V^\perp \subseteq \{0\}$. We'll do them in order.

(i) First, by bilinearity

$$\langle \vec{0}, v\rangle = \langle \vec{0} - \vec{0}, v\rangle = \langle \vec{0}, v\rangle - \langle \vec{0}, v\rangle = 0$$

which means $\langle \vec{0}, v\rangle = 0$ no matter what v is. Therefore $0 \in V^\perp$ and $\{0\} \subseteq V^\perp$.

(ii) Second, $V^\perp = \{x \in V \mid \langle v, x\rangle = 0 \ \ \forall v \in V\}$, so x is orthogonal to every vector in V! If $x \in V^\perp$, then $x \in V$ in particular. So it must be that $x \in V^\perp \cap V$. But we showed that this was already $\{0\}$ in class! Therefore all $x \in V^\perp$ were $x = 0$ all along, which means $V^\perp \subseteq \{0\}$.

We can conclude that $V^\perp = \{0\}$.

(b) We can make a similar argument to show that $\{0\}^\perp = V$. By definition $\{0\}^\perp = \{v \in V \mid \langle v, 0\rangle = 0\}$. But for all $v \in V$, no matter what vector we pick, we already proved that

$$\langle v, \vec{0}\rangle = 0.$$

All vectors $v \in V$ are orthogonal to $\vec{0}$! So $\{0\}^\perp = V$.