Math 4242
Summer 2018
Midterm 1
6/22/2018
Time Limit: 50 minutes

Name (Print): ____________________

Solutions

Instructor ____________________

Midterm 1 contains 7 pages (including this cover page) and 5 problems. Please check to see if any pages are missing.

Work individually without reference to a textbook, notes, the internet, or a calculator.

Show your work on each problem. Specifically:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Unsupported answers will not receive credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will likely receive partial credit.
- **Circle your final answer** for problems involving a series of computations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	15	
3	15	
4	10	
5	15	
Total:	65	

Please do not write in the table to the right.

1. (10 points) Below are six statements about $n \times n$ invertible matrices A and B.
Circle T for true and F for false.
Provide a counterexample for each false statement.
(You do not need to justify the true statements.)

(2 pts) (a) $(AB)^{-1} = B^{-1}A^{-1}$ T F

(2 pts) (b) $\det\left((AB)^T\right) = \det A \det B$ T F

Fall 2020 ppl det NOT on exam

(2 pts) (c) $(A + B)$ must be nonsingular T F

$A = I$ $B = -I$
$A^{-1} = A$ $B^{-1} = B$
so invertible

But $A + B =$ 0 matrix which is singular

(2 pts) (d) $\operatorname{rank}(AB) = n$ T F

(2 pts) (e) $\ker(A) = \mathbf{0}$ T F

2. (15 points) The augmented matrix $[A|\mathbf{b}]$ for a linear system is shown below.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & a & 1 \\ -1 & -1 & 2 & 0 & -4 \\ 0 & 2 & -4 & 1 & a \end{array}\right]$$

What value(s) of a (if any) would imply that the system has

(i) no solution?
(ii) exactly one solution?
(iii) infinitely many solutions?

First, we have 3 eqn's and 4 variables so there is never a unique solution.

Now row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 1 \\ -1 & -1 & 2 & 0 & -4 \\ 0 & 2 & 4 & 1 & a \end{array}\right] \xrightarrow{r_1+r_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 1 \\ 0 & -1 & 2 & a & -3 \\ 0 & 2 & -4 & 1 & a \end{array}\right]$$

$$\xrightarrow{r_2+r_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 1 \\ 0 & -1 & 2 & a & -3 \\ 0 & 0 & 0 & 2a+1 & a-6 \end{array}\right]$$

We see that we have an infinitely many solutions as long as $2a+1 \neq 0$ since then we would have 3 pivots.
Else when $a = \frac{-1}{2}$, the system is inconsistent.
To summarize i) $a = \frac{-1}{2}$ iii) else.
ii) never

3. (15 points) Compute the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 3 & 17 & 1 \\ 0 & -4 & 2 \end{bmatrix}.$$

$$\xrightarrow{-3r_1+r_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{2r_2+r_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & -2 & 1 \end{bmatrix}$$

So the LU decomp of A is

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 3 & 17 & 1 \\ 0 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

4. (10 points) Let $\mathcal{P}^{(n)}$ be the vector space of all polynomial functions of degree less than or equal to n, with addition and scalar multiplication defined in the standard way.

(5 pts) (a) Let W be the set of polynomials $p(x)$ in $\mathcal{P}^{(n)}$ such that $p(2) = 0$. Is W a vector subspace of $\mathcal{P}^{(n)}$? Justify your answer.

We show that W is a subspace.

① If $p(x) = 0$, then $p(2) = 0$ so $0 \in W$.

② Let $p, q \in W$. Then $(p+q)(2) = p(2) + q(2)$
$= 0 + 0 = 0$. So $p+q \in W$.

③ Let $c \in \mathbb{R}$, $p \in W$. Then $(cp)(2) = cp(2) =$
$c \cdot 0 = 0$. So $cp(x) \in W$.

Since W satisfies the 3 properties, W is a subspace of $\mathcal{P}^{(n)}$.

(5 pts) (b) Let V be the subset of $\mathcal{P}^{(n)}$ consisting of polynomials of even degree and the zero polynomial. Is V a vector subspace of $\mathcal{P}^{(n)}$? Justify your answer.

Claim: V is not a subspace.

Pf: Let $p(x) = x^2 - x + 1 \in V$ and $q(x) = -x^2 \in V$.

But $p(x) + q(x) = \cancel{x^2} - x + 1 - \cancel{x^2}$
$= -x + 1 \notin V$.

This has degree 1 so V is not closed under addition (property 2). So V is not a subspace.

5. (15 points) The vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^4$ form the columns of the 4×4 matrix A:

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} \\ | & | & | & | \end{bmatrix}$$

The row echelon form of A is

$$\begin{bmatrix} \boxed{3} & 1 & 7 & -1 \\ 0 & \boxed{-4} & 8 & 2 \\ 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 3 pivots

(7 pts) (a) Do $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, and $\mathbf{v_4}$ form a basis for $\mathbb{R}^4$? Justify your answer.

They do not. Since $rk(A) = \#$ of pivots which is 3, the A is not full rank ($n=4$). By FTOLA, the columns $v_1 v_2 v_3 v_4$ do not form a basis.

(8 pts) (b) Is $\mathbf{v_3}$ in the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$? If not, explain why. If yes, write $\mathbf{v_3}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, and $\mathbf{v_4}$.

Yes, $v_3 \in \mathrm{span}(v_1, v_2)$ in fact.

We know that the dependencies of the RREF is the same for A.

$\frac{2}{3} + \frac{7}{3} = 3$

Row reducing further

$$\begin{bmatrix} 3 & 1 & 7 \\ 0 & -4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{3}r_2 + r_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

Only part that matters

Dependency is $3\begin{pmatrix}1\\0\end{pmatrix} + -2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$

Thus $3v_1 + (-2)v_2 = v_3$. So $v_3 \in \mathrm{span}(v_1 v_2)$.

This page is for your reference only. There are no questions to answer.

$\boxed{\text{Definition}}$ A *vector space* is a set V equipped with two operations:

(i) *Addition*: adding any pair of vectors $\mathbf{v}$, $\mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$.

(ii) *Scalar Multiplication*: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

These operations obey the following axioms, valid for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$

(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.

(e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$ and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.

(f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.

(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

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