4D Euclidean Space Visualization

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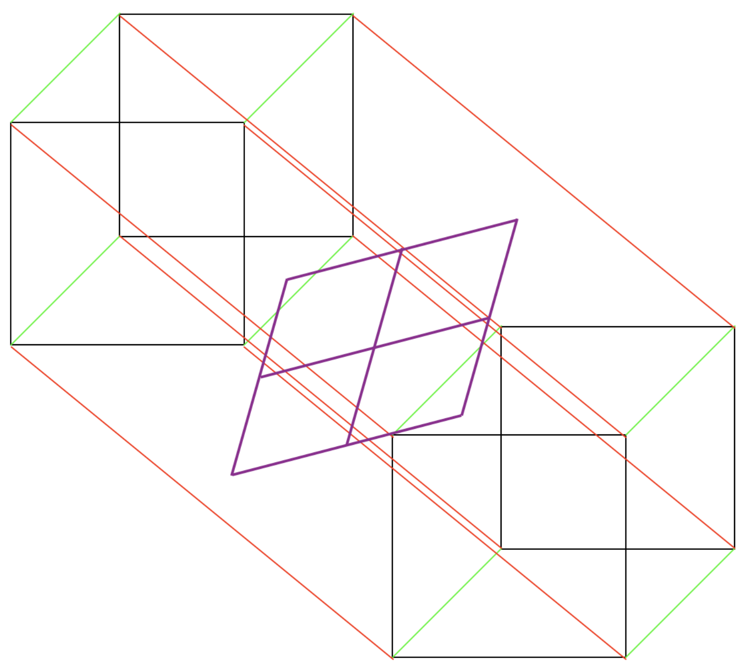
1 Introduction

Our goal is to visualize 4D Euclidean space on a 2D screen.

We will attempt a projection from 4D space to 2D space.

User can translate and rotate the 2D frame around in 4D space.



In this diagram, the black squares represent the x and y axis, the green lines represent the z axis, and the red lines represent the w axis.
The purple frame represents the view screen somewhere in 4d space.

2 Transformations and Projections

2.1 The goal

At a high level this transformation from 4d space to 2d space can be thought of as a function:

$$T : \mathbb{R}^4 \to \mathbb{R}^2$$

which can be represented as a 2x4 matrix A in the equation

$$A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = T \left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right) = \begin{pmatrix} X \\ Y \end{pmatrix}$$

2.2 Finding a good T

If $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, then $T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so T is a projection onto the x-y plane.

Generally, If $T = \begin{pmatrix} 1_i \\ 1_j \end{pmatrix}$, then $Tv = \begin{pmatrix} v_i \\ v_j \end{pmatrix}$, so T is a projection onto the v_i-v_j plane.

We want a way to rotate from a projection onto one plane to another.

Let $R_{ij} \in \mathbb{R}^{2\times 4} =$

$$\begin{pmatrix} \cos(\theta)_i & -\sin(\theta)_j \\ \sin(\theta)_i & \cos(\theta)_j \end{pmatrix}$$

So $R_{yw} \in \mathbb{R}^{2\times 4} =$

$$\begin{pmatrix} 0 & \cos(\theta)_i & 0 & -\sin(\theta)_j \\ 0 & \sin(\theta)_i & 0 & \cos(\theta)_j \end{pmatrix}$$

Notice that 2×4 matrices cannot be chained together, we could have a situation like:

$$\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} & & & \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} & & & \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} & & & \end{pmatrix}$$

with these matrices, $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^4 \to \mathbb{R}^4$
There are now 6 planes to rotate between
$\theta, \alpha, \beta, \gamma, \phi, \psi$

Imagine

$$T(\frac{1}{6}(\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0 & 0\\ \sin(\theta) & \cos(\theta) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}+\begin{pmatrix}\cos(\alpha) & 0 & -\sin(\alpha) & 0\\ 0 & 1 & 0 & 0\\ \sin(\alpha) & 0 & \cos(\alpha) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}+\begin{pmatrix}\cos(\beta) & 0 & 0 & -\sin(\beta)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sin(\beta) & 0 & 0 & \cos(\beta)\end{pmatrix}+$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\gamma) & -\sin(\gamma) & 0\\ 0 & \sin(\gamma) & \cos(\gamma) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}+\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\phi) & 0 & -\sin(\phi)\\ 0 & 0 & 1 & 0\\ 0 & \sin(\phi) & 0 & \cos(\phi)\end{pmatrix}+\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos(\psi) & -\sin(\psi)\\ 0 & 0 & \sin(\psi) & \cos(\psi)\end{pmatrix}))$$

with T

$$=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}$$