

Affine LPV systems: realization theory, input-output equations and relationship with linear switched systems

Mihály Petreczky[†], Guillaume Mercère^{*} and Roland Tóth[‡]

[†]Ecole des Mines de Douai, F-59500 Douai, France,

mihaly.petreczky@mines-douai.fr

^{*} University of Poitiers, Laboratoire d’Informatique et d’Automatique pour les Systèmes
B.P. 633 86022 Poitiers Cedex, France

guillaume.mercere@univ-poitiers.fr

[‡] Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
R.Toth@tue.nl.

Abstract—We formulate a Kalman-style realization theory for discrete-time affine LPV systems. By an affine LPV system we mean an LPV system whose matrices are affine functions of the scheduling parameter. In this paper we characterize those input-output behaviors which exactly correspond to affine LPV systems. In addition, we characterize minimal affine LPV systems which realize a given input-output behavior. Furthermore, we explain the relationship between Markov-parameters, Hankel-matrices, existence of an affine LPV realization and minimality. The results are derived by reducing the problem to the realization problem for linear switched systems. In this way, as a secondary contribution, we formally demonstrate the close relationship between LPV systems and linear switched systems. In addition we show that an input-output map has a realization by an affine LPV system if and only if it satisfies certain types of input-output equations.

I. INTRODUCTION

The paper presents a Kalman-style realization theory for discrete-time affine LPV systems. An affine LPV system (*abbreviated by ALPV*) is linear parameter-varying systems whose matrices are affine functions of the scheduling parameters. By the input-output behavior of an ALPV we will mean the input-output map induced by the zero initial state. The paper aims at answering the following questions.

- How can we characterize those input-output maps which can be described ALPVs ? What is the role of Hankel-matrices in this characterization ?
- What can be said about minimal ALPVs realizing the given input-output map ? What is the relationship between minimal ALPVs, and reachability and observability of such systems ? Are all minimal ALPV realizations of the same input-output map isomorphic ?
- How can we characterize the input-output equations solutions of which correspond to input-output maps of ALPVs ?

In this paper we will show the following.

- We prove that reachability and observability of ALPVs is equivalent to minimality and that minimal realizations of the same input-output map are isomorphic. Note that isomorphism in this setting means a linear state-space

transformation which does not depend on the scheduling parameter.

We also show that any ALPV can be transformed into a minimal one while preserving its input-output map. In addition, we characterize reachability and observability in terms of rank conditions for extended reachability and observability matrices.

- We define the Markov-parameters as functions of the input-output map. We then show that the Hankel-matrix constructed from the Markov-parameters has a finite rank if and only if the corresponding input-output map has a realization by an ALPV. We show that the Kalman-Ho algorithm of [31] can be used to compute an ALPV realization from the Hankel-matrix, and we provide a bound on the size of the Hankel sub-matrices which guarantees correctness of the algorithm.
- We also present a class of input-output equations which characterize ALPVs precisely: an input-output map is a solution of such an input-output equation if and only if it admits a realization by an ALPV .
- Finally, as a secondary result, we establish a formal equivalence between the realization problems for ALPVs and for linear switched systems. The solution of the latter problem is known [23], [22], [21] and it is equivalent to that of recognizable formal power series and state-affine systems [7], [28], [12]. We then use realization theory of linear switched systems to derive a Kalman-style realization theory for ALPVs.

Note that in this paper we consider ALPVs with a fixed initial state. Just as in the linear switched case [21], [22] it is possible to extend these results to the case of an arbitrary set of initial states.

Motivation and novelty To the best of our knowledge, the paper is new. Many of the concepts (Hankel-matrix, Markov-parameters, extended reachability/observability matrix, etc.) used in the paper have already appeared before. However, what is truly novel in this paper is that it formulates a Kalman-style realization theory for ALPVs, while using

the existing concepts from the literature. In addition, the equivalence between ALPV realizations and input-output equations is also new, to the best of our knowledge.

A Kalman-like realization theory offers several benefits for system identification. It allows the characterization of identifiability and equivalence of state-space representations. The latter is important for model validation. Kalman-like realization theory also provides a tool for finding identifiable canonical parameterizations and characterizing the manifold structure of systems, including hybrid and nonlinear systems, [26], [29], [13], [20], [14], [15], [19]. In turn, this knowledge could be used for deriving new parametric identification algorithms, see [20], [14] for the linear case. Realization theory also leads to model reduction techniques, such as balanced truncation and moment matching [1]. This is also true for linear switched systems [25], [24] and ALPVs [31].

Finally, the paper formulates the precise relationship between the realization problems for ALPVs and linear switched systems. While this relationship is part of the folklore, it has not been stated formally yet.

Relationship with existing work The field of identification of LPV systems is a mature one with a vast literature and several applications, without claiming completeness, we mention [36], [38], [35], [37], [18], [34], [30], [32], [16], [9], [8], [6], [3], [11]. As it was mentioned before, many of the concepts used in this paper were published before. In particular, the idea of Hankel-matrix appeared in [31], [36], [38], [35], [37]. However, [31], [36], [38], [35], [37] focuses on the identification problem, which is related to, but different from the realization problem studied in this paper. The Markov-parameters were already described in [31], [35]. In contrast to the existing work, in this paper the Markov-parameters and Hankel-matrix are defined directly for input-output maps, without assuming the existence of a finite dimensional ALPV realization. In fact, the finite rank of the Hankel-matrix represents the necessary and sufficient condition for the existence of an ALPV realization. The Kalman-Ho realization algorithm was discussed in [31], but it was formulated with the assumption that an ALPV realization exists. Moreover, the conditions under which the algorithm yields a true realization of the input-output map were not discussed in detail in [31]. Extended observability and reachability matrices were presented in [35], [31]. However, their system-theoretic interpretation and relationship with minimality were not explored.

Realization theory of more general linear parameter-varying systems was already developed in [30]. In [30] the system matrices are allowed to depend on the scheduling parameter in a non-linear way. Moreover, in [30] no conditions involving the rank of the Hankel-matrix were formulated for the existence of a state-space realization. Hence, the results of [30] do not always imply the ones presented in this paper. The minimality conditions of [30] imply those of this paper. However, an ALPV may be minimal in the sense of this paper, and may fail to be minimal in the sense of [30]. Intuitively this is not at all surprising, since it is conceivable that by allowing more complicated dependence

on the scheduling parameter we can get rid of some states.

In particular, minimal ALPVs in the sense of this paper are related by constant state-space isomorphism. This is in contrast to [30], where the isomorphism relating state-space representations may depend on the scheduling parameter. Note that a minimal ALPVs in the sense of this paper need not be minimal in the sense of [30]. Hence, there might exist several state-space isomorphisms between ALPVs which are minimal in the sense of this paper. Some of these isomorphisms might depend on the scheduling parameters. However, the results of this paper imply that there will be a constant state-space isomorphism. This is also consistent with [17].

Although realization theory of ALPVs is quite similar to that of linear switched systems, there are important differences. In particular, there exist no parallel for linear switched systems of the equivalence between realizability and existence of input-output equations. In fact, ALPVs seem to behave more like state-affine systems [27], [28] for which an analogous result exists.

It is well known that there is a correspondence between LPVs and LFT representations [36], [33]. In [2], [5], [4] the theory of recognizable formal power series was used to develop realization theory for LFT representations. In this paper we reduce the realization problem of ALPVs to that of linear switched systems. The latter problem can also be solved by using recognizable formal power series [21], [22], [23]. Hence, there is an analogy between our approach and that of [2], [5], [4]. Note that the transformations between ALPVs and LFT representations involve non-trivial transformations of the system matrices. Moreover, the resulting class of LFT representations seem to differ from the one in [2], [5], [4]. For this reason, it is unclear how the results of this paper could be derived directly from [2], [5], [4] and whether such an approach would be simpler than the current one.

Outline In §II we review the definition of ALPVs and the related system-theoretic concepts. In §III we establish the formal relationship between ALPVs and linear switched systems. In §IV we present a Kalman-style realization theory for ALPVs. Finally, in §V we present the input-output equations describing the behavior of ALPVs.

Notation Denote by \mathbb{N} the set of natural numbers including 0. The notation described below is standard in automata theory, see [10]. Consider a (possibly infinite) set X . Denote by X^+ the set of finite non-empty sequences of elements of X , i.e. each $w \in X^+$ is of the form $w = a_1 a_2 \cdots a_k$, $a_1, a_2, \dots, a_k \in X$, $k > 0$. The length of the sequence w above is denoted by $|w|$. We denote by wv the concatenation of the sequences $w, v \in X^+$, i.e. if $w = a_1 \cdots a_k$ and $v = v_1 \cdots v_l$, $a_1, \dots, a_k, v_1, \dots, v_l \in X$, then $wv = a_1 \cdots a_k v_1 \cdots v_l$. We denote by ϵ the *empty sequence*. We define $X^* = X^+ \cup \{\epsilon\}$ as the set of all finite sequences of elements of X , including the empty sequence. By convention, $|\epsilon| = 0$, and the concatenation is extended to X^* as follows: for all $w \in X^*$, $w\epsilon = \epsilon w = w$. For each $j = 1, \dots, m$, e_j is the j th unit vector of \mathbb{R}^m , i.e. $e_j = (\delta_{1,j}, \dots, \delta_{n,j})$, $\delta_{i,j}$ is the Kronecker symbol. If Z is a subset of a vector

space, then $\text{Span}Z$ denotes the vector space spanned by the elements of Z .

II. DISCRETE-TIME LPV SYSTEMS

In this section we present the formal definition of ALPVs along with a number of relevant system-theoretic concepts for ALPVs.

Definition 1: A discrete-time affine linear parameter-varying system (abbreviated by ALPV) is of the form

$$\Sigma \left\{ \begin{array}{lcl} x(t+1) & = & \sum_{q=1}^D (A_q x(t) + B_q u(t)) p_q(t) \\ y(t) & = & \sum_{q=1}^D (C_q x(t)) p_q(t). \end{array} \right. \quad (1)$$

Here $\mathcal{P} \subseteq \mathbb{R}^D$ is the space of scheduling parameters, D is a positive integer, $p(t) = (p_1(t), \dots, p_D(t)) \in \mathcal{P}$ is the scheduling signal, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^r$ is the output and $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$, $C_q \in \mathbb{R}^{r \times n}$, $q \in Q = \{1, \dots, D\}$ are the system matrices. We will use the following short notation.

$$(r, m, n, \mathcal{P}, \{(A_q, B_q, C_q)\}_{q=1}^D)$$

Notation 1: In the sequel, $Q = \{1, \dots, D\}$.

The definition above also allows for affine dependence on the scheduling parameters. To this end, choose \mathcal{P} to be of the form $\mathcal{P} = \{(p_1, \dots, p_D) \mid p_1 = 1, (p_2, \dots, p_D) \in \hat{\mathcal{P}}\}$ for some set $\hat{\mathcal{P}} \subseteq \mathbb{R}^{D-1}$. Moreover, if the affine hull of \mathcal{P} equals \mathbb{R}^{D-1} , then the linear span of \mathcal{P} will be equal to \mathbb{R}^D . The latter property is important, because in the sequel we often use the technical assumption that \mathcal{P} contains a basis of \mathbb{R}^D .

Note that in our definition the output y_t at time t does not depend on the input at time t . This restriction is made in order to simplify notation and most of the results can be easily extended to include direct dependence of y_t on u_t .

Throughout the section, Σ denotes an ALPV of the form (1). The dynamics of Σ is driven by the inputs $\{u(t)\}_{t=0}^\infty$ and the scheduling parameters $\{p(t)\}_{t=0}^\infty$. The state of the system at time t is $x(t)$. If $\mathcal{P} = \{e_1, \dots, e_D\}$, where e_i denotes the i th standard basis vector, $i = 1, \dots, D$, then the ALPV Σ can be viewed as a *linear switched system* with the set of discrete modes being equal to $Q = \{1, \dots, D\}$.

In order to enable formal discussion, we define a number of standard concepts such as input-output maps, reachability, etc. for ALPVs.

Notation 2 (Generalized inputs): Denote $\mathcal{U} = \mathcal{P} \times \mathbb{R}^m$. We denote by \mathcal{U}^* (resp. \mathcal{U}^+) the set of all finite (resp. non-empty and finite) sequences of elements of \mathcal{U} . A sequence

$$w = (p(0), u(0)) \cdots (p(t), u(t)) \in \mathcal{U}^+, t \geq 0 \quad (2)$$

describes the scenario, when the scheduling parameter $p(i)$ and the input $u(i)$ are fed to Σ at time i , for $i = 0, \dots, t$.

Definition 2 (State and output): Let $x \in \mathbb{R}^n$ be a state of Σ . Define the *input-to-state map* $x_{\Sigma,x} : \mathcal{U}^+ \rightarrow \mathbb{R}^n$ and *input-output map* $y_{\Sigma,x} : \mathcal{U}^+ \rightarrow \mathbb{R}^r$ of Σ as follows. For any $w \in \mathcal{U}^+$ of the form (2), define $x_{\Sigma,x}(w)$ as the state $x(t)$ of Σ at time t , and define $y_{\Sigma,x}(w)$ as the output $y(t)$ of Σ at time t , if the initial state $x(0)$ of Σ equals x , and the inputs

$\{u(i)\}_{i=0}^t$ and the scheduling signal $\{p(i)\}_{i=0}^t$ are fed to Σ . Note that for $t = 0$, $x_{\Sigma,x}(w) = x$.

The definition above implies that the potential input-output behavior of an ALPV can be formalized as a map

$$f : \mathcal{U}^+ \rightarrow \mathbb{R}^r. \quad (3)$$

The value $f(w)$ for w of the form (2) represents the output of the underlying black-box system at time t , if the inputs $\{u(i)\}_{i=0}^t$ and the scheduling parameters $\{p(i)\}_{i=0}^t$ are fed to the system. This black-box system may or may not admit a description by a ALPV. Next, we define when an ALPV describes (realizes) f .

Definition 3 (Realization): The ALPV Σ of the form (1) is a *realization* of an input-output map f of the form (3), if f equals the input-output map of Σ which corresponds to the zero initial state, i.e. $f = y_{\Sigma,0}$. The map $y_{\Sigma,0}$ will be referred to as the *input-output map of Σ* and it will be denoted by y_Σ .

Similarly to [22], [21], the results of this paper could be extended to families of input-output maps and multiple initial states. However, in order to keep the notation simple, we deal only with the case when the initial state is zero.

Definition 4 (Input-output equivalence): Two ALPVs Σ_1 and Σ_2 are said to be *input-output equivalent*, if $y_{\Sigma_1} = y_{\Sigma_2}$.

Definition 5 (Reachability): Let Σ be an ALPV of the form (1). We say that Σ is reachable, if the linear span of all the states of Σ which are reachable from the zero initial state yields the whole space \mathbb{R}^n .

Definition 6 (Observability): The ALPV Σ is called *observable* if for any two states $x_1, x_2 \in \mathbb{R}^n$, $y_{\Sigma,x_1} = y_{\Sigma,x_2}$ implies $x_1 = x_2$.

That is, observability means that if we pick any two distinct states of the system, then for **some** input and scheduling signal, the resulting outputs will be different.

Note that the concepts of reachability and observability presented above are strongly related to extended controllability and observability matrices from subspace identification of ALPVs [35]. Later on, we will show that the ALPV is reachable if and only if the extended controllability matrix is full rank, and the ALPV is observable if and only if the extended observability matrix is full rank.

Finally, we recall the notion of isomorphism for ALPVs.

Definition 7 (ALPV isomorphism): Consider a ALPV Σ_1 of the form (1) and a ALPV Σ_2 of the form

$$\Sigma_2 = (r, m, n^a, \mathcal{P}, \{(A_q^a, B_q^a, C_q^a)\}_{q=1}^D)$$

with $n_a = n$. A nonsingular matrix $\mathcal{S} \in \mathbb{R}^{n \times n}$ is said to be an *ALPV isomorphism* from Σ_1 to Σ_2 , if

$$\forall q = 1, \dots, D : A_q^a \mathcal{S} = \mathcal{S} A_q, \quad B_q^a = \mathcal{S} B_q, \quad C_q^a \mathcal{S} = C_q.$$

Note that in the definition of an ALPV isomorphism, the state-space transformation \mathcal{S} does not depend on the scheduling parameter. Finally, below we define what we mean by the dimension minimality of a ALPV.

Definition 8 (Dimension): The dimension of Σ , denoted by $\dim \Sigma$, is the dimension n of its state-space.

Definition 9 (Minimality): Let f be an input-output map. An ALPV Σ is a *minimal realization* of f , if Σ is a realization of f , and for any ALPV $\hat{\Sigma}$ which is a realization of f , $\dim \Sigma \leq \dim \hat{\Sigma}$. We say that Σ is *minimal*, if Σ is a minimal realization of its own input-output map y_Σ .

III. RELATIONSHIP BETWEEN LINEAR SWITCHED SYSTEMS AND ALPVs

In this section we establish a formal relationship between ALPVs and linear switched systems. We start by stating the following assumption.

Assumption 1: In the rest of the paper, unless stated otherwise, we will assume that the linear span of elements of \mathcal{P} equals \mathbb{R}^D , i.e. \mathcal{P} does not belong to any of the proper linear subspaces of \mathbb{R}^D .

Note that the assumption above is not restrictive. Indeed, if \mathcal{P} belongs to a \hat{D} dimensional proper linear subspace \mathcal{X} of \mathbb{R}^D , then we can define a linear map $\mathbf{S} : \mathbb{R}^D \rightarrow \mathbb{R}^{\hat{D}}$ such that \mathbf{S} is injective on \mathcal{X} and replace the set of scheduling parameters by $\hat{\mathcal{P}} = \mathbf{S}(\mathcal{P})$. Since \mathbf{S} is linear, the parameters of the resulting new LPV system will depend on the parameters in an affine way.

Next, we introduce the concept of *generalized convolution representation* for input-output maps. This concept will allow us to concentrate on input-output maps for which there is a hope that they can be realized by ALPVs.

Notation 3: Let $\underline{p} = p(0) \cdots p(t)$ be a sequence of scheduling parameters and let $v = q_0 \cdots q_t \in Q^+$, $q_0 \cdots q_t \in Q$. Then $\underline{p}^v = p_{q_0}(0)p_{q_1}(1) \cdots p_{q_t}(t)$.

Definition 10 (Convolution representation): Let f be an input-output map of the form (3). The map f has a *generalized convolution representation* (abbreviated as **GCR**), if there exists a map $S^f : \{v \in Q^+ \mid |v| > 1\} \rightarrow \mathbb{R}^{r \times m}$ such that for each $w \in \mathcal{U}^+$ of the form (2),

$$f(w) = \sum_{k=0}^{t-1} \left\{ \sum_{v \in Q^+, |v|=t-k+1} S^f(v) \underline{p}_{k:t}^v \right\} u(k), \quad (4)$$

where $\underline{p}_{k:t} = p(k)p(k+1) \cdots p(t)$.

The convolution representation states that $f(w)$ is linear in control input and that it is a homogeneous polynomial of degree one in the scheduling parameters. The values of the map S^f play the role of the coefficients of this polynomial. Note that the concept of **GCR** above is a special case of *impulse response representation (IRR)* in [30]. Note that since in the ALPVs of interest the output at time t does not depend on the input at time t , the summation in (4) goes only up to $t-1$. Below we show that S^f is uniquely determined by f and that the existence of a **GCR** implies that without loss of generality we can assume that $\mathcal{P} = \mathbb{R}^D$.

Lemma 1: If f has a **GCR**, then the map S^f is uniquely determined by f . Moreover, there exists a unique extension f_{ext} of f to \mathcal{U}_{ext}^+ , where $\mathcal{U}_{ext} = (\mathbb{R}^D \times \mathbb{R}^m)$, such that f_{ext} also admits a **GCR** and $S^f = S^{f_{ext}}$.

Proof: [Proof of Lemma 1] The fact that f_{ext} exists relies on the fact that (4) is defined for any values of $p(0), \dots, p(t) \in \mathbb{R}^D$, and by noticing that the right-hand

side of (4) is a sum of terms, each of which multilinear in $p(i), \dots, p(t)$, $i = 0, \dots, t$. Recall that function $g(z_1, \dots, z_k)$ is multi-linear, if for each $i = 1, \dots, k$, if we fix $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_k$ and we vary only z_i , then g is a linear function of z_i . Then set f_{ext} as the value of the right-hand side of (4). If the value of $f(w)$ is known for $p(0), \dots, p(t)$ where $p(0), \dots, p(t)$ run through a bases of \mathbb{R}^D , then these values uniquely determine the value of the right-hand side of (4), and thus f_{ext} exists and it is unique and $S^{f_{ext}} = S^f$. Finally, the uniqueness of S^f follow by noticing that $S^f(v)u = f_{ext}((e_{q_0}, u)(e_{q_1}, 0) \cdots (e_{q_t}, 0))$ for $v = q_0 \cdots q_t$, $q_0, \dots, q_t \in Q$, $u \in \mathbb{R}^m$. ■

In the sequel, we will restrict attention to input-output maps which admit a **GCR**. This is not a strong restriction, since the input-output maps of ALPVs always admits a **GCR**.

Lemma 2: The ALPV Σ of the form (1) is a realization of an input-output map f if and only if f has a **GCR** and for all $v = q_0 \cdots q_t \in Q^+$, $q_0, \dots, q_t \in Q$, $t > 0$

$$S^f(v) = C_{q_t} A_{q_{t-1}} A_{q_{t-2}} \cdots A_{q_1} B_{q_0}. \quad (5)$$

If $t = 1$, then $A_{q_{t-1}} A_{q_{t-2}} \cdots A_{q_1}$ is interpreted as the identity matrix.

Now we are ready to state the relationship between ALPVs and linear switched systems. To this end, we introduce the following notation.

Notation 4 (Switched generalized inputs): Denote $\mathcal{P}_{sw} = \{e_1, \dots, e_D\}$ and $\mathcal{U}_{sw} = (\mathcal{P}_{sw} \times \mathbb{R}^m)$.

Recall that we can view linear switched systems as a subclass of ALPVs, such that the space of scheduling parameters equals \mathcal{P}_{sw} . Potential input-output maps of linear switched systems are maps of the form $f : \mathcal{U}_{sw}^+ \rightarrow \mathbb{R}^r$ such that f admits a **GCR**. Linear switched systems and their input-output maps in the sense of [23] correspond to linear switched systems and their input-output maps in the above sense, if one identifies the scheduling parameter e_q with the discrete mode $q \in Q$. We refer the reader to [23] for the notion of realization, minimality, observability, span-reachability, isomorphism. Alternatively, all these notions are special cases of the corresponding concepts for ALPVs, if one identifies linear switched systems as a subclass of ALPVs. Note that the concept of span-reachability from [23] corresponds to the concept of reachability as defined in Definition 5.

Definition 11: For each $f : \mathcal{U}^+ \rightarrow \mathbb{R}^r$ admitting a **GCR**, define the *associated switched input-output map* $\mathfrak{I}(f) : \mathcal{U}_{sw}^+ \rightarrow \mathbb{R}^r$ as follows. Let f_{ext} be the extension of f to \mathcal{U}_{ext}^+ as described in Lemma 1 and define $\mathfrak{I}(f)$ as the restriction of f_{ext} to $\mathcal{U}_{sw}^+ \subseteq \mathcal{U}_{ext}^+$.

By noticing that $S^f = S^{f_{ext}} = S^{\mathfrak{I}(f)}$ we can in fact conclude that the correspondence between f and $\mathfrak{I}(f)$ is one-to-one. Next we will establish a correspondence between ALPVs and linear switched systems.

Definition 12: Let Σ be a ALPV of the form (1). Define the *linear switched systems* $\mathfrak{S}(\Sigma)$ associated with Σ as the linear switched system $\mathfrak{S}(\Sigma) = (r, m, n, \mathcal{P}_{sw}, \{(A_q, B_q, C_q)\}_{q=1}^D)$.

The following theorem collects the properties of the correspondence between linear switched systems and ALPVs.

- Theorem 1:* 1) An ALPV Σ is a realization of the input-output map f , if and only if $\mathfrak{S}(\Sigma)$ is a realization of $\mathfrak{I}(f)$.
 2) For any ALPV Σ , $\dim \mathfrak{S}(\Sigma) = \dim \Sigma$.
 3) Two ALPVs Σ_1 and Σ_2 are isomorphic if and only if $\mathfrak{S}(\Sigma_1)$ is isomorphic to $\mathfrak{S}(\Sigma_2)$.
 4) The ALPV Σ is reachable, observable, minimal if and only if $\mathfrak{S}(\Sigma)$ is respectively reachable, observable, or minimal.

Proof: [Sketch of the proof of Theorem 1] The only non-trivial statement is that \mathfrak{S} preserves reachability and observability. Let Σ be an ALPV of the form (1). First we show that Σ is reachable if and only if $\mathfrak{S}(\Sigma)$ is reachable. To this end, consider the map input-to-state map $x_{\Sigma,0} : \mathcal{U}^+ \rightarrow \mathbb{R}^n$ of Σ . Notice that $x_{\Sigma,0}$ can be extended to act on \mathcal{U}_{ext}^+ and that for any input $w \in \mathcal{U}_{ext}^+$ of the form (2), $x_{\Sigma,0}$ is a sum of terms, each of which is multilinear in $p(0), \dots, p(t)$. Hence, the linear span of the values of $x_{\Sigma,0}(w)$, $w \in \mathcal{U}_{ext}^+$ equals the linear span of values of $x_{\Sigma,0}(w)$, $w \in (\mathbf{Z} \times \mathbb{R}^m)^+$, where \mathbf{Z} is a basis of \mathbb{R}^D . Since by Assumption 1 \mathcal{P} contains such a basis of \mathbb{R}^D and \mathcal{P}_{sw} is a basis of \mathbb{R}^D , it follows that the linear span of $x_{\Sigma,0}(w)$, $w \in \mathcal{U}^+$ equals the linear span of $x_{\Sigma,0}(w)$, $w \in \mathcal{U}_{sw}^+$. Finally, notice that $x_{\Sigma,0}(w) = x_{\mathfrak{S}(\Sigma),0}(w)$ for all $w \in \mathcal{U}_{sw}^+$. Hence, Σ is reachable if and only if $\mathfrak{S}(\Sigma)$ is reachable.

Next, we show that Σ is observable if and only if $\mathfrak{S}(\Sigma)$ is observable. To this end, notice that $y_{\Sigma,x}$ can be extended to \mathcal{U}_{ext}^+ and that for any $w \in \mathcal{U}_{ext}^+$ of the form (2), $y_{\Sigma,x}(w)$ is a sum of terms, each of which is multilinear in $p(0), \dots, p(t)$. Hence, y_{Σ,x_1} and y_{Σ,x_2} agree on \mathcal{U}^+ , if they agree on any set $(\mathbf{Z} \times \mathbb{R}^m)^+$, where \mathbf{Z} is a basis of \mathbb{R}^D . Since \mathcal{P}_{sw} is a basis of \mathbb{R}^D and by Assumption 1 \mathcal{P} contains a basis of \mathbb{R}^D , it then follows that y_{Σ,x_1} and y_{Σ,x_2} are equal on \mathcal{U}^+ if and only if they are equal on \mathcal{U}_{sw}^+ . Notice that for $i = 1, 2$, $y_{\mathfrak{S}(\Sigma),x_i}$ coincides with the restriction of y_{Σ,x_i} to the set \mathcal{U}_{sw}^+ . This then implies that Σ is observable if and only if $\mathfrak{S}(\Sigma)$ is observable. ■

IV. KALMAN-STYLE REALIZATION THEORY

In this section we exploit Section III and realization theory of linear switched systems [21], [22], [24], [23] to formulate a Kalman-style realization theory for ALPVs.

We start with presenting a characterization of minimality.

Theorem 2 (Minimality): An ALPV is minimal, if and only if it is reachable and observable. If two minimal ALPVs are equivalent, then they are isomorphic.

The theorem above is a direct consequence of Theorem 1 and [23, Theorem 3].

Similarly to linear switched systems [23], one can construct example of an ALPV Σ which is minimal (reachable, observable), while none of the linear subsystems (A_q, B_q, C_q) , $q \in Q$ is minimal (resp. reachable, observable).

Next, we present rank conditions for observability and reachability. To this end, recall from [31], [35] the defini-

tion of extended reachability and observability matrices for ALPVs. That is, let Σ be of the form (1). We define the extended reachability matrices \mathcal{R}_i , $i \in \mathbb{N}$ for Σ as follows: $\mathcal{R}_0 = [B_1, B_2, \dots, B_D]$ and for all $i \in \mathbb{N}$, let

$$\mathcal{R}_{i+1} = [R_i, A_1 \mathcal{R}_i, A_2 \mathcal{R}_i \dots, A_D \mathcal{R}_i]$$

Similarly, we define the extended observability matrices \mathcal{O}_i for Σ recursively as follows: $\mathcal{O}_0 = [C_1^T, C_2^T, \dots, C_D^T]^T$ and for all $i \in \mathbb{N}$,

$$\mathcal{O}_{i+1} = [O_i^T, A_1^T \mathcal{O}_i^T, A_2^T \mathcal{O}_i^T, \dots, A_D^T \mathcal{O}_i^T]^T.$$

Notice that \mathcal{R}_{n-1} equals the reachability matrix of the switched system $\mathfrak{S}(\Sigma)$ and \mathcal{O}_{n-1} equals the observability matrix of $\mathfrak{S}(\Sigma)$. For the definition of reachability and observability matrices for linear switched systems see [23]. Hence, Theorem 1 and [23, Theorem 4] yield the following rank conditions.

Theorem 3: The ALPV Σ is reachable if and only if $\text{rank } \mathcal{R}_{n-1} = n$, and Σ is observable if and only if $\text{rank } \mathcal{O}_{n-1} = n$.

Theorem 3 yields algorithms for reachability, observability and minimality reduction of ALPVs. These algorithms are the same as those for linear switched systems [23].

Next, we present the necessary and sufficient conditions for the existence of a ALPV realization for an input-output map. To this end, we need the notion of the Hankel-matrix and Markov-parameters of an input-output map. In the sequel, f denotes a map of the form (3), and we assume that f has a GCR.

Definition 13 (Markov-parameters): The Markov-parameter $M^f(v)$ of f indexed by the sequence $v \in Q^*$ is the following $rD \times Dm$ matrix

$$M^f(v) = \begin{bmatrix} S^f(1v1), & \dots, & S^f(Dv1) \\ S^f(1v2), & \dots, & S^f(Dv2) \\ \vdots & \vdots & \vdots \\ S^f(1vD), & \dots, & S^f(DvD) \end{bmatrix}. \quad (6)$$

That is, $M^f(v)$ can be viewed as a $D \times D$ block matrix, such that the (i, j) th entry of $M^f(v)$ equals $S^f(jvi)$, $j, i \in Q$.

If f has an ALPV realization Σ , then from Lemma 2 it follows that $M^f(v)$ can be expressed as product of matrices of Σ : if Σ is as in (1), then $M^f(\epsilon) = \tilde{C} \tilde{B}$ and for all $v = q_1, \dots, q_k \in Q$, $k > 0$,

$$M^f(v) = \tilde{C} A_{q_k} A_{q_{k-1}} \cdots A_{q_1} \tilde{B}, \quad (7)$$

where $\tilde{C} = [C_1^T, \dots, C_D^T]^T$, $\tilde{B} = [B_1, \dots, B_D]$.

Note that the values of the map S^f , and hence the Markov-parameters $\{M^f(v)\}_{v \in Q^*}$ can be obtained from the values of f . A naive way to compute S^f is to compute the derivatives of f with respect to the scheduling parameter. It is easy to see that the Markov-parameters f and $\mathfrak{I}(f)$ coincide, i.e. $M^f(v) = M^{\mathfrak{I}(f)}(v)$, $v \in Q^*$. Moreover, when applied to linear switched systems, the Markov-parameters from Definition 13 coincide with the ones in [23, Definition 12].

Note that the definition of Markov-parameters does not assume the existence of an ALPV realization of f . In fact,

even if f does not admit a finite dimensional state realization, its Markov-parameters remain well-defined. The reason for this choice is that we want to use the Markov-parameters to characterize the existence of a finite dimensional ALPV realization of f . This will be achieved by constructing a Hankel-matrix from the Markov-parameters and by proving that f has an ALPV realization if and only if the rank of that Hankel-matrix is finite. Of course, for this to make sense, we have to define the Markov-parameters and the Hankel-matrix as objects which are well-defined even in the absence of a finite dimensional state-space representation.

In order to define the Hankel-matrix of f , we will introduce a lexicographic ordering on the set Q^* .

Definition 14 (Lexicographic ordering): Recall that $Q = \{1, \dots, D\}$. We define a lexicographic ordering \prec on Q^* as follows. For any $v, s \in Q^*$, $v \prec s$ holds if either **(a)** $|v| < |s|$, or **(b)** $0 < |v| = |s| = k$, $v \neq s$ and the following holds: $v = q_1 \cdots q_k$, $s = s_1 \cdots s_k$, $q_1, \dots, q_k, s_1, \dots, s_k \in Q$, and for some $l \in \{1, \dots, k\}$, $q_l < s_l$ with the usual ordering of integers and $q_i = s_i$ for $i = 1, \dots, l - 1$. Note that \prec is a complete ordering and

$$Q^* = \{v_1, v_2, \dots\} \quad (8)$$

with $v_1 \prec v_2 \prec \dots$. Note that $v_1 = \epsilon$ and for all $i \in \mathbb{N}$, $q \in Q$, $v_i \prec v_i q$.

Definition 15 (Hankel-matrix): Define the Hankel-matrix H_f of f as the following infinite matrix

$$H_f = \begin{bmatrix} M^f(v_1 v_1), & M^f(v_2 v_1), & \dots, & M^f(v_k v_1), & \dots \\ M^f(v_1 v_2), & M^f(v_2 v_2), & \dots, & M^f(v_k v_2), & \dots \\ M^f(v_1 v_3), & M^f(v_2 v_3), & \dots, & M^f(v_k v_3), & \dots \\ \vdots & \vdots & \dots & \vdots & \dots \end{bmatrix},$$

i.e. the $rD \times mD$ block of H_f in the block row i and block column j equals the Markov-parameter $M^f(s)$, where the word $s = v_j v_i \in Q^*$ is the concatenation of the words v_j and v_i from (8).

Note that $H^f = H^{\mathcal{I}(f)}$ and the definition of the Hankel-matrix coincides with the one for linear switched systems [23, Definition 13].

Theorem 4 (Main result on existence): The map f has a realization by an ALPV if and only if f has a **GCR** and $\text{rank}H_f < +\infty$. Any minimal ALPV realization of f has dimension equal to $\text{rank}H_f$.

The theorem above is a direct consequence of Theorem 1 and [23, Theorem 5].

Finally, we prove the correctness of the Kalman-Ho-like realization algorithm for ALPVs from [31]. A similar algorithm was formulated for linear switched systems in [24], [23]. To this end, we need the following definition. For every $L \in \mathbb{N}$, denote by $\mathbf{N}(L) = \sum_{j=0}^L \mathcal{D}^j$ the number such all the sequences $v \in Q^*$ of length at most L . Due to the properties of lexicographic ordering, it follows that $\{v_1, \dots, v_{\mathbf{N}(L)}\} = \{v \in Q^* \mid |v| \leq L\}$.

Definition 16: Denote by $H_{f,L,M}$ the $\mathbf{N}(L)rD \times \mathbf{N}(M)mD$ upper-left sub-matrix of H_f .

If f is realized by an ALPV Σ , then $H_{f,L,M} = \mathcal{O}_L \mathcal{R}_M$, where \mathcal{O}_L is the K th extended observability matrix and \mathcal{R}_M

is the M th extended reachability matrix of Σ . In this case $H_{f,L,M}$ coincides with the Hankel-matrix defined in [31]. The Kalman-Ho algorithm goes as follows. Compute the factorization

$$H_{f,L,L+1} = \mathbf{O} \mathbf{R}$$

such $\mathbf{O} \in \mathbb{R}^{rDN(L) \times n}$, $\mathbf{R} \in \mathbb{R}^{n \times mDN(L+1)}$ and $\text{rank} \mathbf{O} = \text{rank} \mathbf{R} = n$ for $n = \text{rank} H_{f,L,L+1}$. One way to compute this factorization is by SVD decomposition as in [31], i.e. if $H_{f,L,L+1} = USV^T$ is the SVD decomposition of $H_{f,L,L+1}$ where S is the diagonal part, then set $\mathbf{O} = US^{1/2}$ and $\mathbf{R} = S^{1/2}V^T$. Let $\bar{\mathbf{R}}$ be the matrix formed by the first $\mathbf{N}(L)mD$ columns of \mathbf{R} . For each $q \in Q$, let \mathbf{R}_q be the $n \times \mathbf{N}(L)mD$ matrix, such that the j th $n \times mD$ block column of \mathbf{R}_q equals to the k th $n \times mD$ block column of \mathbf{R} , where k is such that $v_j q = v_k$. Here v_k and v_j are the j th and k th elements of the lexicographic ordering (8). Construct Σ of the form (1) such that $[B_1, \dots, B_D]$ equals the first mD columns of \mathbf{R} , $[C_1^T, C_2^T, \dots, C_D^T]^T$ equals the first rD rows of \mathbf{O} and $A_q = \mathbf{R}_q \bar{\mathbf{R}}^+$, where $\bar{\mathbf{R}}^+$ is the Moore-Penrose pseudoinverse of $\bar{\mathbf{R}}$.

Theorem 5: If $\text{rank} H_{f,L,L} = \text{rank} H_f$, then Σ computed by the algorithm above is a minimal realization of f . The condition $\text{rank} H_{f,L,L} = \text{rank} H_f$ holds, if there exists an ALPV realization Σ of f such that $\dim \Sigma \leq L + 1$.

The theorem above is a direct consequence of Theorem 1 and [23, Theorem 6].

V. INPUT-OUTPUT EQUATIONS FOR ALPVs

In this section we use the results of realization theory to establish a relationship between ALPVs and input-output equations. In the sequel, f is assumed to be an input-output map $f : \mathcal{U}^+ \rightarrow \mathbb{R}^r$ and it is assumed that f admits a **GCR**. In order to avoid excessive notation, in this section we assume that $r = 1$. However, all the results can easily be extended to several outputs.

Definition 17 (Input-output equations): An affine polynomial equation $E(\mathbf{P}, \mathbf{Y}, \mathbf{U})$ of order n is a polynomial in variables $\mathbf{P} = \{P_{i,j}\}_{i=0, \dots, n, j \in Q}$, $\mathbf{Y} = \{Y_i\}_{i=0}^n$, $\mathbf{U} = \{U_{i,j}\}_{i=1, \dots, n, j=1, \dots, m}$ such that

$$E(\mathbf{P}, \mathbf{Y}, \mathbf{U}) = \sum_{j=0}^n Q_j(\mathbf{P}) Y_j + \sum_{i=1}^n \sum_{j=1}^m L_{i,j}(\mathbf{P}) U_{i,j} \quad (9)$$

where $Q_0(\mathbf{P})$, $Q_i(\mathbf{P})$, $L_{i,j}(\mathbf{P})$ are polynomials, $i = 1, \dots, n$, $j = 1, \dots, m$ and $Q_0(\mathbf{P}) \neq 0$.

Definition 18: Assume that E is an affine polynomial equation of the form (9). Then the input-output map f is said to satisfy the equation E , if for each w of the form (2) with $t > n$, $E(f, w) = 0$, where $E(f, w)$ denotes the value of $E(\mathbf{P}, \mathbf{Y}, \mathbf{U})$ with the following substitution $P_{i,j} = p_j(t-i)$, $U_{i,l} = u_l(t-i)$, $Y_i = f((p(0), u(0)) \cdots (p(t-i), u(t-i)))$ for $j \in Q$, $l = 1, \dots, m$, $i = 0, \dots, n$.

Theorem 6: Assume that the set of scheduling parameters \mathcal{P} is an open subset of \mathbb{R}^D . The input-output map f has a realization by an ALPV if and only if f satisfies an affine polynomial equation of the form (9).

In [30] it was shown that input-output maps of LPV systems with a meromorphic dependence on parameters correspond to input-output maps which satisfy linear autoregressive equations with respect to outputs and inputs. The coefficients of these autoregressive equations were meromorphic functions of the time-shifted scheduling parameters. Affine polynomial input-output equations represent a special case of the autoregressive equations of [30]. Theorem 6 says that input-output maps described by these type of equations (and which, in addition, admit a **GCR**) correspond precisely to input-output maps realizable by ALPVs.

The proof of Theorem 6 is an adaptation of the proof of the analogous statement for state-affine systems [27], [28]. The proof is divided into several lemmas, proofs of which are presented in the appendix.

Lemma 3: If the interior of \mathcal{P} not empty, then f satisfies the input-output equation (9) if and only if its extension f_{ext} from Lemma 1 satisfies (9).

From Lemma 3 it follows that without loss of generality, we can assume $\mathcal{P} = \mathbb{R}^D$.

Assumption 2: In the sequel, we assume that $\mathcal{P} = \mathbb{R}^D$.

For any sequence $\underline{p} = p_1 p_2 \cdots p_k \in \mathcal{P}^+$, $p_1, \dots, p_k \in \mathcal{P}$, $k > 0$ define the map $f^{\underline{p}} : \mathcal{U}^+ \rightarrow \mathbb{R}$ as follows:

$$\forall w \in \mathcal{U}^+ : f^{\underline{p}}(w) = f(w(p_1, 0)(p_2, 0) \cdots (p_k, 0))$$

Recall that $w(p_1, 0) \cdots (p_k, 0)$ denotes the concatenation of the sequence w with the sequence $(p_1, 0) \cdots (p_k, 0)$. Intuitively, $f^{\underline{p}}(w)$ equals the response of f , if first we feed in the inputs and scheduling parameters prescribed by w and then for the last k time steps we feed in the zero input and the scheduling parameters p_1, \dots, p_k .

Lemma 4: There exists an affine polynomial input-output equation E of the form (9) such that f satisfies E , if and only if there exists polynomials $Q_i(\mathbf{P})$, $i = 0, \dots, n$ such that $Q_0 \neq 0$, and for any $p_1, \dots, p_{n+1} \in \mathcal{P}$,

$$\sum_{j=0}^n Q_j(p_1, \dots, p_{n+1}) f^{p_1 p_2 \cdots p_{n+1-j}} \quad (10)$$

Before formulating the next statement, recall the set of all maps $g : \mathcal{U}^+ \rightarrow \mathbb{R}^r$ forms a vector space with respect to point-wise addition and multiplication by scalar.

Lemma 5: The map f satisfies (10) for some Q_j , $j = 0, \dots, n$ if and only if $\mathcal{W}_f = \text{Span}\{f^{\underline{p}} \mid \underline{p} \in (\mathbb{R}^D)^+\}$ is finite dimensional.

Lemma 6: The input-output map f has a realization by a ALPV if and only if $\mathcal{W}_f = \text{Span}\{f^{\underline{p}} \mid \underline{p} \in (\mathbb{R}^D)^+\}$ is finite dimensional.

The proof of Lemma 6 boils down to showing that there is a linear isomorphism between \mathcal{W}_f and the linear space spanned by the rows of the Hankel-matrix H_f of f . Hence, \mathcal{W}_f is finite dimensional if and only if $\text{rank } H_f < +\infty$. By Theorem 4, the latter is equivalent to the existence of an ALPV realization of f . Theorem 6 follows from the lemmas above as follows. From Lemma 6, f has a realization by a ALPV if and only if \mathcal{W}_f is finite dimensional. By Lemma 5 and Lemma 4, the latter is equivalent to existence of an affine polynomial equation of the form (9) such that f satisfies E .

VI. CONCLUSION

We have presented realization theory for the class of affine LPV systems. In addition, we have shown that realization theory of this class of LPV systems is equivalent to that of for linear switched systems. We have also presented an equivalent input-output representation for affine LPV systems.

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APPENDIX

Proof: [Proof of Lemma 3] Note that $E(f_{ext}, w)$ is a polynomial expression in $p(0), u(0)), \dots, (p(t), u(t)) \in \mathcal{U}_{ext}$ for each w of the form (2). Since $E(f_{ext}, w) = 0$ for all $w \in \mathcal{U}^+$ and \mathcal{U} is an open subset of \mathcal{U}_{ext} , it then follows that $E(f_{ext}, w) = 0$ for all $w \in \mathcal{U}_{ext}$. ■

Proof: [Proof of Lemma 4] By substitution it is clear that if (9) holds, then (10) holds. Conversely assume that (10) holds. With the notation of (9), notice that for $Y_i = f((p(0), u(0)) \cdots (p(t-i), u(t-i))), i = 0, \dots, n$,

$$Y_i = f^{p(t-n)p(t-n+1)\cdots p(t-i)}(v) + \\ + \sum_{j=i+1}^n R_{j,i}(p(t-j), \dots, p(t-i))u(t-j),$$

where $v = (p(0), u(0)) \cdots (p(t-n-1), u(t-n-1))$, and $R_{j,i}, j = n, \dots, i+1$ are suitable polynomials. Consider the expression $\sum_{i=0}^n Q_i(p(t-n), \dots, p(t))Y_i$. By grouping together the terms $(Q_i R_{j,i})(p(t-n), \dots, p(t))u(t-j)$ in a suitable way, we can obtain polynomials $L_{j,l}, j = 1, \dots, n$ and $l = 1, \dots, m$ such that then (9) holds. ■

Proof: [Proof of Lemma 5] Assume that f satisfies (10). It then follows from $f^{p_1, \dots, p_{r+i}}(w) = f^{p_{k+1}, \dots, p_{r+i}}(w(p_1, 0) \cdots (p_r, 0)), i = 1, \dots, n+1, w \in \mathcal{U}^+$ that for any $p_1, \dots, p_{r+n+1} \in \mathcal{P}, r \geq 0$,

$$\sum_{j=0}^n f^{p_1, \dots, p_{n-j+r+1}} Q_j(p_{r+1}, \dots, p_{r+n+1}) = 0. \quad (11)$$

Define $W_{f,k} = \text{span}\{f^{p_1 \cdots p_k} \mid p_1, \dots, p_k \in \mathcal{P}\}$. Note that as \mathcal{P} contains an open set, there exists a finite subset E of \mathcal{P} such that the elements of E span \mathbb{R}^D . Since f^{p_1, \dots, p_k} is multi-linear in p_1, \dots, p_k , f^{p_1, \dots, p_k} is a linear combination of $f^{v_1, \dots, v_k}, (v_1, \dots, v_k) \in E^k$. Since E^k is a finite set, it then follows that $W_{f,k}$ is finite dimensional.

Since $Q_0 \neq 0$, there exists an open and dense subset $Z \subseteq \mathcal{P}^{r+n+1}$ such that for any $(p_1, \dots, p_{n+r+1}) \in Z$, $Q_0(p_r, \dots, p_{n+r+1}) \neq 0, r = 1, \dots, k$. By dividing (11) with $Q_0(p_r, \dots, p_{n+r+1}) \neq 0$, and using induction on r , it then follows that $f^{p_1, \dots, p_{n+r+1}} \in W_{f,n}$. Since Z is dense, it then follows that for any $\underline{p} = p_1 \cdots p_{r+n+1} \in \mathcal{P}^+$, there exist $\underline{p}_j = p_1^j \cdots p_{r+n+1}^j, p_1^j, \dots, p_{r+n+1}^j \in Z, j \in \mathbb{N}$, such that $\lim_{j \rightarrow \infty} p_i^j = p_i, i = 1, \dots, r+n+1$. As $f^{\underline{p}}$ is polynomial (multi-linear) in the entries of p_1, \dots, p_{n+r+1} , it then follows $\lim_{j \rightarrow \infty} f^{\underline{p}_j}(w) = f^{\underline{p}}(w)$ for all $w \in \mathcal{U}^+$, i.e. $f^{\underline{p}_j} \in W_{f,n}$ converges to $f^{\underline{p}}$ point-wise. Since $W_{f,n}$ is finite dimensional vector space, it then follows that $f^{\underline{p}} \in W_{f,n}$.

Indeed, let g_1, \dots, g_K be a basis of $W_{f,n}$. Then there exist $w_1, \dots, w_K \in \mathcal{U}^+$ and such that the matrix $S = (S_{r,l})_{r,l=1, \dots, K}, S_{r,l} = g_l(w_r) \in \mathbb{R}$, is invertible. Define $v_j = (f^{\underline{p}_j}(w_1), \dots, f^{\underline{p}_j}(w_K))^T$, and $v = (f^{\underline{p}}(w_1), \dots, f^{\underline{p}}(w_K))^T$. It then follows that $f^{\underline{p}_j} = \sum_{k=1}^K \alpha_k^j g_k$, where $\alpha^j = (\alpha_1^j, \dots, \alpha_K^j)^T$ satisfies $\alpha_j = S^{-1}v_j$. We claim that if $(\alpha_1, \dots, \alpha_K)^T = \alpha = S^{-1}v$, then $f^{\underline{p}} = \sum_{k=1}^K \alpha_k g_k$. Assume the contrary. Then for some $w \in \mathcal{U}^+, f^{\underline{p}}(w) \neq \sum_{k=1}^K \alpha_k g_k(w)$. Notice that $\lim_{j \rightarrow \infty} v_j = v$ and hence $\alpha = S^{-1}v = \lim_{j \rightarrow \infty} S^{-1}v_j = \lim_{j \rightarrow \infty} \alpha^j$. Hence, $\sum_{k=1}^K \alpha_k g_k(w) = \lim_{j \rightarrow \infty} \sum_{k=1}^K \alpha_k^j g_k(w) = \lim_{j \rightarrow \infty} f^{\underline{p}_j}(w) = f^{\underline{p}}(w)$, which is a contradiction.

Hence, $f^{\underline{p}} \in W_{f,n}$, for all $\underline{p} \in \mathcal{P}^+, |\underline{p}| = n+r+1$, and thus $W_{f,n} = W_f$, i.e. W_f is finite dimensional.

Conversely, assume that W_f is finite dimensional. For each $v = q_1 \cdots q_k \in Q^+, q_1, \dots, q_k \in Q$, denote by f^v the map $f^{e_{q_1} \cdots, e_{q_k}}$. As it was noted above, $f^{p_1 \cdots p_k}$ is multilinear in p_1, \dots, p_k and hence W_f equals the linear span of f^{z_1}, \dots, f^{z_d} for some $z_1, \dots, z_d \in Q^+$. Notice that for any $\underline{p} \in (\mathbb{R}^D)^+, |\underline{p}| = k, f^{\underline{p}} = \sum_{v \in Q^+, |v|=k} f^v \underline{p}^v$. Since for every $v \in Q^+$, f^v is a linear combination of $f^{z_i}, i = 1, \dots, d$, there exist polynomials $P_{i,k}$ is kD variables, such that $f^{p_1 \cdots p_k} = \sum_{j=1}^d P_{j,k}(p_1, \dots, p_k) f^{z_j}$ for any $p_1, \dots, p_k \in \mathbb{R}^D$.

Consider now the $d \times (d+1)$ polynomial matrix \mathbf{D}_{d+1} in variables $X_i = (X_{i,1}, \dots, X_{i,D}), i = 1, 2, \dots, d+1$ such that (i, j) the entry of \mathbf{D}_{d+1} equals $P_{i,j}(X_1, \dots, X_j)$. Let's view \mathbf{D}_{d+1} as a matrix with elements in $\mathbb{R}(X_1, \dots, X_{d+1})$. Here, $\mathbb{R}(X_1, \dots, X_{d+1})$ is the quotient field of the polynomial ring $\mathbb{R}[X_1, \dots, X_{d+1}]$. Since \mathbf{D}_{d+1} has only d rows

and $d + 1$ columns, the columns of \mathbf{D}_{d+1} must be linearly dependent. It then follows that there exist polynomials $D_j, N_j \in \mathbb{R}[X_1, \dots, X_{d+1}]$, $N_j \neq 0$, $j = 1, \dots, k^*$, such that $D_{k^*} \neq 0$ and $\sum_{j=1}^{k^*} P_{i,j} \frac{D_j}{N_j} = 0$. By multiplying the equation above by the product of $N_1 \cdots N_{k^*}$ we get that

$$\forall i = 1, \dots, d : \sum_{j=1}^{k^*} P_{i,j} R_j = 0 \quad (12)$$

for some polynomial R_1, \dots, R_{k^*} , $R_{k^*} \neq 0$. Notice that the polynomial $P_{i,j}$ depend only on the variables X_1, \dots, X_j , hence R_1, \dots, R_{k^*} can be chosen to be polynomials only in X_1, \dots, X_{k^*} . If $k^* = 1$, then $P_{i,1} = 0$ and hence $f^p = 0$ for all $p \in \mathcal{P}$. Hence, $f^{p_1, \dots, p_k}(w) = f^{p_k}(w(p_1, 0) \cdots (p_{k-1}, 0)) = 0$ for all $w \in \mathcal{U}^+$, p_1, \dots, p_k , $k > 0$. Then (10) holds for $n = 1$ with any choice of Q_1 and Q_0 . If $k^* > 1$, then set $n = k^*$, $Q_i = R_{k^*-i}$, $i = 1, \dots, k^* - 1$. Using the fact that $f^{p_1, \dots, p_i} = \sum_{j=1}^d P_{j,i}(p_1, \dots, p_i) f^{z_j}$ and (12), it then follows that (10) holds for all $p_1, \dots, p_k \in \mathbb{R}^D$. ■

Proof: [Proof of Lemma 6] Denote by \mathcal{H} the linear span of the rows of the Hankel-matrix H_f . Notice that each element of \mathcal{H} can be viewed as a sequence of $1 \times Dm$ matrices. We define the linear map $\Phi : \mathcal{W}_f \rightarrow \mathcal{H}$ as follows: $\Phi(f^{\underline{p}}) = (H_{v_1}, H_{v_2}, \dots)$, such that for each $v \in Q^*$,

$$H_v = \sum_{s \in Q^+, |s|=|\underline{p}|-1} [\underline{p}^{s1}, \dots, \underline{p}^{sD}] M^f(vs).$$

In other words, $H_v = [H_{v,1}, \dots, H_{v,D}]$, where $H_{v,q} = \sum_{s \in Q^+, |s|=|\underline{p}|} S^f(qvs) \underline{p}^s$. Moreover, for any w of form (2),

$$f^{\underline{p}}(w) = \sum_{k=0}^{t-1} H_{q_{k+1} \cdots q_t, q_k} u(k) p_{q_1}(k) \cdots p_{q_t}(t).$$

Hence, it is clear that Φ is an injective linear map. Moreover, the row of H_f indexed by the integer $l = (i-1)D+q$, $q \in Q$, $i = 1, \dots$ equals $\Phi(f^{e_q})$ if $i = 1$, or $\Phi(f^{e_{q_1} \cdots e_{q_k}, e_q})$, if $i > 1$ and $q_1, \dots, q_k \in Q$ are such that $v_i = q_1 \cdots q_k$, where v_i is i th sequence of the lexicographic ordering (8). Hence, Φ is a linear isomorphism from \mathcal{W}_f to the space spanned by the rows of H_f . The rest of follows from Theorem 4. ■