

# Derivative free data-driven stabilization of continuous-time linear systems from input-output data

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**Abstract**—This letter presents a data-driven framework for the design of stabilizing controllers from input-output data in the continuous-time, linear, and time-invariant domain. Rather than relying on measurements or reliable estimates of input and output time derivatives, the proposed approach uses filters to derive a parameterization of the system dynamics. This parameterization is amenable to the application of linear matrix inequalities enabling the design of stabilizing output feedback controllers from input-output data and the knowledge of the order of the system.

**Index Terms**—Data-driven control, linear systems, LMI.

## I. INTRODUCTION

This letter aims to extend the results given in [2] to the case where only input-output measurements are available. To pursue this objective, some filters, borrowed from the adaptive control literature, are first used to obtain a state-space system representation of the dynamics of the plant so that stabilization of such a system ensures stabilization of the closed loop. Although the obtained system is linear and time-invariant (LTI), it is affected by a disturbance whose dynamics depend just on those of the filters. Hence, the framework proposed in [2] is adapted to handle this disturbance.

As for the framework proposed in [2], and unlike most existing techniques [3]–[15], the approach proposed in this letter does not assume that measurements or reliable estimates of the successive time derivatives of the output are available. Other approaches to designing stabilizing controllers from input-output data that do not use the time derivatives of input and output have been proposed in [16], [17] for the single-input and single-output (SISO) and for the multi-input and multi-output (MIMO) cases, respectively. The main differences from [16], [17] are the following: (i) the data-driven parameterization proposed in this letter is independent of the time derivatives of the filter state; (ii) in the SISO case, the data matrix in this letter has  $2n + 1$  rows rather than  $3n + 1$  as in [16], [17]; (iii) differently from [16], [17], where a dynamic extension is used to deal with unmeasurable states, in this letter a filtering strategy is proposed to remove disturbances; (iv) sufficient conditions ensuring that the data matrix has full rank are provided.

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This paper is an extended version of reference [1].

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**Notation:** Let  $I_n$  and  $0_{n,m}$  denote the  $n$ -dimensional *identity* and  $(n \times m)$ -dimensional *zero* matrices, respectively. Let  $\mathbf{e}_i^n$  denote the  $i$ th column of  $I_n$ . Let

$$S_n = [\mathbf{e}_n^n \ \cdots \ \mathbf{e}_1^n]$$

denote the  $n$ -dimensional *reversal* matrix. Let  $\text{col}(A, B) = [\mathbf{A}^\top \ \mathbf{B}^\top]^\top$ . Let  $\lambda(A)$  denote the *spectrum* of  $A$ . The symbol  $\otimes$  denotes the Kronecker product. A matrix is *Toeplitz* if each descending diagonal from left to right is constant. A polynomial is *Hurwitz* if all its roots have negative real parts. A matrix  $A \in \mathbb{R}^{n \times n}$  is *Hurwitz* if its characteristic polynomial is Hurwitz. Two polynomials in a single variable are *coprime* if they do not have a common root. Let  $A^\dagger$  be the *Moore-Penrose pseudoinverse* of the matrix  $A$ .  $\mathbb{R}[c_1, \dots, c_n]$  denotes the *ring* of all the polynomials in  $c_1, \dots, c_n$  with coefficients in  $\mathbb{R}$ . Let  $\imath = \sqrt{-1}$  denote the *imaginary unit* in  $\mathbb{C}$ . Let  $\text{He}(A) = A + A^\top$  and  $\text{Sk}(A) = A - A^\top$  be the *symmetric* and *skew-symmetric* parts of  $A$ , respectively. The matrix  $A \in \mathbb{R}^{n \times n}$  is *positive definite*, denoted  $A \succ 0$ , if  $x^\top A x > 0$ ,  $\forall x \in \mathbb{R}^n \setminus \{0\}$ . Let  $\mathfrak{H}_n(\{d_k\}_{k=1}^{N-1})$  denote the *Hankel matrix of depth*  $n$  of the discrete-time sequence  $\{d_k\}_{k=1}^{N-1}$ . The sequence  $\{d_k\}_{k=1}^{N-1}$ ,  $d_k \in \mathbb{R}^m$ , is *persistently exciting of order*  $n$  if  $\text{rank}(\mathfrak{H}_n(\{d_k\}_{k=1}^{N-1})) = nm$ .

## II. THE PROPOSED DATA-DRIVEN APPROACH FOR THE SYNTHESIS OF AN OUTPUT FEEDBACK CONTROLLER

Consider a continuous-time, LTI, MIMO system described in differential operator representation [18, Sec. 2.1]

$$\begin{aligned} y^{(n)}(t) + A_1 y^{(n-1)}(t) + \cdots + A_{n-1} y^{(1)}(t) + A_n y(t) \\ = B_1 u^{(n-1)}(t) + \cdots + B_{n-1} u^{(1)}(t) + B_n u(t), \end{aligned} \quad (1)$$

where  $y(t) \in \mathbb{R}^p$ ,  $u(t) \in \mathbb{R}^m$ ,  $A_i \in \mathbb{R}^{p \times p}$ , and  $B_i \in \mathbb{R}^{p \times m}$ ,  $i = 1, \dots, n$ . Assume that a single input-output trajectory of system (1) in the interval  $[0, T]$  is available:

$$\mathcal{D} = \{(u(t), y(t)) \in \mathbb{R}^m \times \mathbb{R}^p, t \in [0, T], \text{ such that (1) holds}\}.$$

The main goal of this letter is to design a stabilizing output feedback controller for system (1) just using the dataset  $\mathcal{D}$  and the knowledge of the order  $n$  of system (1). Note that the knowledge of  $n$  is usually assumed when applying adaptive approaches for the synthesis of an output feedback controller; see, e.g., [19, Chap. 7]. Algorithm 1 illustrates the proposed approach to design a stabilizing output feedback controller by using the dataset  $\mathcal{D}$  and the knowledge of  $n$ .

The effectiveness of Algorithm 1 is proven in Section III in the SISO case and in Section IV for a class of MIMO systems.

**Algorithm 1** The proposed data-driven approach

**Input:** dataset  $\mathcal{D}$ , parameters  $c_1, \dots, c_n, \beta \in \mathbb{R}$ , order  $n$  of (1) such that any of its minimal realizations has dimension  $pn$

**Output:** a stabilizing output feedback controller or a **failure**

1: define the matrices

$$A_r = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ -c_n & \dots & -c_1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0_{n-1,1} \\ 1 \end{bmatrix}, \quad (2)$$

2: let  $A_{r,m} = A_r \otimes I_m$ ,  $B_{r,m} = B_r \otimes I_m$ ,  $A_{r,p} = A_r \otimes I_p$ , and  $B_{r,p} = B_r \otimes I_p$

3: filter the data in  $\mathcal{D}$  using the following filters

$$\dot{\zeta}(t) = A_{r,m}\zeta(t) + B_{r,m}u(t), \quad \zeta(0) = 0_{mn,1}, \quad (3a)$$

$$\dot{\mu}(t) = A_{r,p}\mu(t) + B_{r,p}y(t), \quad \mu(0) = 0_{pn,1}, \quad (3b)$$

$$\dot{\phi}(t) = -\beta\phi(t) + \chi(t), \quad \phi(0) = 0_{2n,1}, \quad (3c)$$

$$\dot{v}(t) = -\beta v(t) + u(t), \quad v(0) = 0, \quad (3d)$$

$$\delta(t) = \chi(t) - \beta\phi(t), \quad (3e)$$

where  $\chi(t) = \text{col}(\zeta(t), \mu(t))$

4: fix sampling times  $0 < t_1 < t_2 < \dots < t_N$  and define

$$\Delta_N = \begin{bmatrix} \delta(t_1) & \dots & \delta(t_N) \end{bmatrix} \in \mathbb{R}^{(m+p)n \times N}, \quad (4a)$$

$$\Phi_N = \begin{bmatrix} \phi(t_1) & \dots & \phi(t_N) \end{bmatrix} \in \mathbb{R}^{(m+p)n \times N}, \quad (4b)$$

$$\Upsilon_N = \begin{bmatrix} v(t_1) & \dots & v(t_N) \end{bmatrix} \in \mathbb{R}^{m \times N} \quad (4c)$$

5: find a matrix  $W_N$  such that, for all  $x_0 \in \mathbb{R}^{pn}$ ,  $[(e^{A_r^\top t_1} - e^{-\beta t_1} I_{pn})x_0 \dots (e^{A_r^\top t_N} - e^{-\beta t_N} I_{pn})x_0]W_N = 0$

6: define the filtered data

$$\bar{\Delta}_N = \Delta_N W_N, \quad \bar{\Phi}_N = \Phi_N W_N, \quad \bar{\Upsilon}_N = \Upsilon_N W_N \quad (5)$$

7: **if**  $\text{rank}(\text{col}(\bar{\Phi}_N, \bar{\Upsilon}_N)) = (m+p)n + m$  **then**

8: solve the LMI

$$\text{He}(\bar{\Delta}_N Z^\top) \prec 0, \quad \text{He}(Z \bar{\Phi}_N^\top) \succ 0, \quad (6a)$$

$$\text{Sk}(Z \bar{\Phi}_N^\top) = 0_{2n,2n} \quad (6b)$$

9: compute the gain

$$K_f = \bar{\Upsilon}_N Z^\top (Z \bar{\Phi}_N^\top)^{-1} \quad (7)$$

10: **return** the controller obtained by interconnecting (3a), (3b) and letting  $u(t) = K_f \text{col}(\zeta(t), \mu(t))$

11: **else**

12: **failure** the proposed approach cannot be applied

### III. THE SINGLE-INPUT SINGLE-OUTPUT CASE

The main goal of this section is to prove the effectiveness of Algorithm 1 to solve the problem of designing a stabilizing output feedback controller in the SISO case. First, in Section III-A, it is shown how to obtain a model whose state is measurable from input-output data. Then, in Section III-B, it is shown how the gathered measurements can be used to design a stabilizing output feedback controller.

#### A. Data-driven modeling with input-output data

If  $p = m = 1$  (i.e., in the SISO case), then (1) becomes

$$\begin{aligned} y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_{n-1} y^{(1)}(t) + a_n y(t) \\ = b_1 u^{(n-1)}(t) + \dots + b_{n-1} u^{(1)}(t) + b_n u(t), \end{aligned} \quad (8)$$

where  $y(t) \in \mathbb{R}$ ,  $u(t) \in \mathbb{R}$ ,  $y^{(i)}(t) = \frac{d^i y(t)}{dt^i}$ ,  $u^{(i)}(t) = \frac{d^i u(t)}{dt^i}$ ,  $a_i \in \mathbb{R}$ , and  $b_i \in \mathbb{R}$ ,  $i = 0, \dots, n$ . The following assumption is made hereafter in this section.

**Assumption 1.** The polynomials  $s^n + a_1 s^{n-1} + \dots + a_n$  and  $b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n$  are coprime.

Assumption 1 entails the fact that the transfer function

$$\frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

completely describes system (8).

The description (8) is equivalent to the state-space form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad (9)$$

where  $x(t) \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^n$ , and  $C \in \mathbb{R}^{1 \times n}$  are in *observability canonical form*, i.e.,  $C = [0_{1,n-1} \quad 1]$ ,

$$A = \begin{bmatrix} 0_{1,n-1} & -a_n \\ I_{n-1} & \vdots \\ & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_n \\ \vdots \\ b_1 \end{bmatrix},$$

by [18, Sec. 2.3]. In particular, by letting

$$O = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}B & \dots & CB & 0 \end{bmatrix},$$

be the observability matrix and the Toeplitz matrix of the Markov parameters of system (9), the relation between the initial conditions of the descriptions (8) and (9) is

$$\begin{bmatrix} y^{(0)}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = Ox(0) + M \begin{bmatrix} u^{(0)}(0) \\ \vdots \\ u^{(n-1)}(0) \end{bmatrix}. \quad (10)$$

By [18, Sec. 2.3 and Sec. 5.1], under Assumption 1, the pair  $(A, B)$  is reachable and the pair  $(A, C)$  is observable.

Since the time derivatives of the output  $y(\cdot)$  and of the input  $u(\cdot)$  are usually not measurable in practical settings, consider the filters (3a) and (3b) of the input and output of (9).

**Remark 1.** The filters (3a) and (3b) are usually employed in the adaptive control literature [19], [20] to estimate the parameter vector  $\theta = [b_1 \dots b_n \ a_1 \dots a_n]^\top$ . In fact, letting  $\chi(t) = \text{col}(\zeta(t), \mu(t))$  and  $\psi(t) = y(t) + [-c_n \ -c_{n-1} \ \dots \ -c_1]\mu(t)$ , the vector  $\theta$  can be estimated using the *gradient algorithm*

$$\dot{\hat{\theta}}(t) = \frac{\psi(t) - \hat{\theta}^\top(t)\chi(t)}{1 + \chi^\top(t)\chi(t)}\chi(t).$$

If  $\chi(t)$  is *persistently exciting* according to [19, Def. 4.3.1], then  $\hat{\theta}(t)$  converges exponentially to  $\theta$ ; see [19, Thm. 4.3.2].

Unlike adaptive approaches, which estimate the parameters of system (8) to design a control law, the main goal of this letter is to design a stabilizing control law using only input-output data, bypassing the identification step. To this end, consider the next theorem, proved in Appendix A.

**Theorem 1.** Let Assumption 1 hold and consider the interconnection of systems (9), (3a), and (3b), whose dynamics are

$$\dot{\eta}(t) = A_i \eta(t) + B_i u(t), \quad \chi(t) = C_i \eta(t), \quad (11)$$

where  $\eta(t) = \text{col}(\zeta(t), \mu(t), x(t))$ ,  $\eta(t) \in \mathbb{R}^{3n}$ ,  $\chi(t) = \text{col}(\zeta(t), \mu(t))$ ,  $\chi(t) \in \mathbb{R}^{2n}$ ,  $C_i = \begin{bmatrix} I_{2n} & 0_{2n,n} \end{bmatrix}$ , and

$$A_i = \begin{bmatrix} A_r & 0_{n,n} & 0_{n,n} \\ 0_{n,n} & A_r & B_r C \\ 0_{n,n} & 0_{n,n} & A \end{bmatrix}, \quad B_i = \begin{bmatrix} B_r \\ 0_{n,1} \\ B \end{bmatrix}. \quad (12a)$$

By letting  $x(0) = x_0$ , the dynamics of  $\chi(t)$  are given by

$$\dot{\chi}(t) = A_f \chi(t) + B_f u(t) + G_f e^{A_r^\top t} x_0, \quad (13)$$

the pair  $(A_f, B_f)$  is reachable, and

$$A_f = \left[ \begin{array}{c|c} \frac{0_{n-1,1} \mid I_{n-1}}{-c_n \cdots -c_1} & 0_{n,n} \\ \hline \frac{0_{n-1,n}}{b_n \cdots b_1} & \frac{0_{n-1,1} \mid I_{n-1}}{-a_n \cdots -a_1} \end{array} \right],$$

$$B_f = \left[ \begin{array}{c} \frac{0_{n-1,1}}{1} \\ 0_{n,1} \end{array} \right], \quad G_f = \left[ \begin{array}{c} 0_{2n-1,n} \\ 0_{1,n-1} \mid 1 \end{array} \right].$$

By Theorem 1, the dynamics of the state  $\chi(t) \in \mathbb{R}^{2n}$  of the filters (3a) and (3b) are LTI and are affected by a disturbance that depends on  $c_1, \dots, c_n$  and on the initial condition  $x_0$  of system (9). Consider the next corollary.

**Corollary 1.** Let Assumption 1 hold and let  $K_f$  be such that  $A_f + B_f K_f$  is Hurwitz. If the polynomial  $s^n + c_1 s^{n-1} + \dots + c_{n-1} s + c_n$  is Hurwitz, then the origin is globally asymptotically stable for the feedback interconnections of system (9), the filters (3a) and (3b), and  $u(t) = K_f \chi(t)$ .

*Proof.* The proof follows directly from Appendix A and the fact that if  $s^n + c_1 s^{n-1} + \dots + c_{n-1} s + c_n$  is Hurwitz and  $K_f$  is such that  $A_f + B_f K_f$  is Hurwitz, then the matrix

$$\begin{bmatrix} A_f + B_f K_f & G_f \\ 0_{n,2n} & A_r^\top \end{bmatrix}$$

is Hurwitz.  $\square$

By Corollary 1, if  $s^n + c_1 s^{n-1} + \dots + c_{n-1} s + c_n$  is Hurwitz, then it suffices to design a feedback gain  $K_f$  such that  $A_f + B_f K_f$  is Hurwitz yielding a stabilizing controller consisting of the filters (3a), (3b) and the control law  $u(t) = K_f \chi(t)$ . Since the dynamics of  $\chi$  are LTI with a disturbance generated by an LTI known exosystem, this objective can be pursued by suitably adapting the framework proposed in [2]. Namely, consider the additional filters (3c), (3d), and (3e), where  $\beta \in \mathbb{R}$  is a parameter assumed to satisfy the following assumption.

**Assumption 2.** The parameter  $\beta$  satisfies  $-\beta \notin \lambda(A_r)$ .

Assumption 2 holds for all  $\beta \in \mathbb{R}$  such that  $(-\beta)^n + c_1(-\beta)^{n-1} + \dots + c_n \neq 0$ . Consider the following lemma, whose proof is given in Appendix B.

**Lemma 1.** Let Assumptions 1 and 2 hold. By letting  $\Gamma^* \in \mathbb{R}^{2n \times n}$  be the unique solution to  $\Gamma A_r^\top + \beta \Gamma = G_f$ , define  $\epsilon(t) = \Gamma^*(e^{A_r^\top t} - e^{-\beta t} I_n) x_0$ . For all  $t \geq 0$ , one has

$$\delta(t) = A_f \phi(t) + B_f v(t) + \epsilon(t). \quad (14)$$

Building on Lemma 1 and using a construction similar to that employed in [2], [21], consider the following assumption.

**Assumption 3.** Let  $E_N = [\epsilon(t_1) \cdots \epsilon(t_N)] \in \mathbb{R}^{2n \times N}$ . There is a matrix  $W_N \in \mathbb{R}^{N \times \bar{N}}$ ,  $\bar{N} \in \mathbb{N}$ , such that  $E_N W_N = 0_{2n, \bar{N}}$ , and, by defining the filtered data matrices as in (5), the following rank condition holds

$$\text{rank}(\text{col}(\bar{\Phi}_N, \bar{\Upsilon}_N)) = 2n + 1. \quad (15)$$

By Lemma 1, if Assumptions 1–3 hold, then the matrices  $A_f$  and  $B_f$  can be obtained as

$$\begin{bmatrix} A_f & B_f \end{bmatrix} = \bar{\Delta}_N (\text{col}(\bar{\Phi}_N, \bar{\Upsilon}_N))^\dagger. \quad (16)$$

In fact, by Lemma 1, one has  $\Delta_N = A_f \Phi_N + B_f \Upsilon_N + E_N$ . Hence, since  $W_N$  is such that  $E_N W_N = 0$ , one has  $\bar{\Delta}_N = A_f \bar{\Phi}_N + B_f \bar{\Upsilon}_N$ . Hence, in principle, a gain  $K_f$  such that  $A_f + B_f K_f$  is Hurwitz can be designed by estimating  $A_f$  and  $B_f$  using (16), provided that Assumptions 1–3 hold. In Section III-B, it is shown how to directly design  $K_f$  using the matrices in (5) bypassing the intermediate estimation step.

The main goal of the remainder of this section is to provide conditions ensuring that Assumption 3 is satisfied. Consider the following lemma, whose proof is given in Appendix C.

**Lemma 2.** Let  $N \geq n + 1$  and  $t_1 < \dots < t_N$  be fixed. There exist  $N - n$  linearly independent  $\alpha \in \mathbb{R}^N$  such that

$$\sum_{i=1}^N \alpha_i (e^{A_r^\top t_i} - e^{-\beta t_i} I_n) = 0_{n,n}.$$

By Lemma 2, if  $N \geq n + 1$ , then there exists  $W_N \in \mathbb{R}^{N \times (N-n)}$ ,  $\text{rank}(W_N) = N - n$ , depending only on  $t_1, \dots, t_N$ ,  $c_1, \dots, c_n$ , and  $\beta$ , such that  $E_N W_N = 0_{2n, N-n}$ ,  $\forall x_0 \in \mathbb{R}^n$ . In particular, the matrix  $W_N$  can be obtained by stacking the  $N - n$  linearly independent  $\alpha \in \mathbb{R}^N$  characterized in Lemma 2. The next remark illustrates how to obtain  $\bar{\Delta}_N$ ,  $\bar{\Phi}_N$ , and  $\bar{\Upsilon}_N$  using a discrete-time filter in the case that  $t_1 < \dots < t_N$  are uniformly spaced.

**Remark 2.** Let  $N \geq n + 2$ . If  $t_1, \dots, t_N$  are uniformly spaced,  $t_i = iT_S$ ,  $i = 1, \dots, N$ , for some sampling time  $T_S > 0$ , then by the Cayley-Hamilton theorem [22, Ex. 7.2.2], by letting  $W_N \in \mathbb{R}^{N \times (N-n-1)}$  be the lower triangular Toeplitz matrix whose first column is  $[w_{n+1} \cdots w_1 \ w_0 \ 0_{1, N-n-2}]^\top$ , where  $w_0 z^{n+1} + \dots + w_{n+1} = (z - e^{-\beta T_S}) \det(zI - e^{A_r^\top T_S})$ , one has  $\text{rank}(W_N) = N - n - 1$  and  $E_N W_N = 0$ ,  $\forall x_0 \in \mathbb{R}^n$ . Note that  $\bar{\Delta}_N$ ,  $\bar{\Phi}_N$ , and  $\bar{\Upsilon}_N$  can be obtained by feeding the discrete-time signals  $\{\delta_i(t_k)\}_{k=1}^N$ ,  $\{\phi_i(t_k)\}_{k=1}^N$ ,  $i = 1, \dots, 2n$ , and  $\{v(t_k)\}_{k=1}^N$ , to the finite impulse response (FIR) filter

$$\frac{z^{n+1} + w_1 z^n + \dots + w_n z + w_{n+1}}{z^{n+1}},$$

and discarding its first  $n$  outputs.

Consider the following assumption.

**Assumption 4.** The polynomials  $s^n + a_1 s^{n-1} + \dots + a_n$  and  $s^n + c_1 s^{n-1} + \dots + c_n$  are coprime.

By letting  $a_1, \dots, a_n \in \mathbb{R}$  and  $\beta \in \mathbb{R}$  be fixed, let  $q \in \mathbb{R}[c_1, \dots, c_n]$  be the resultant [23, §6, Chap. 3] of  $(s^n +$

$a_1 s^{n-1} + \dots + a_n)(s + \beta)$  and  $s^n + c_1 s^{n-1} + \dots + c_n$ . Assumptions 2 and 4 hold if and only if  $\begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^\top \notin \mathbf{V}(q)$ , where  $\mathbf{V}(q) = \{ \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^\top \in \mathbb{R}^n : q(c_1, \dots, c_n) = 0 \}$ . Thus, Assumptions 2 and 4 hold generically. These assumptions are made in Proposition 1, proved in Appendix D, to simplify the analysis of the interconnection of (9) and (3), ruling out the presence of resonances, to establish sufficient conditions ensuring that Assumption 3 holds. Since the interconnection of (8) and (3) is not reachable (see Appendix D), such a proposition does not follow directly from the Willems et al. fundamental lemma [24, Cor. 2] or [25, Lem. 1].

**Proposition 1.** *Let Assumptions 1, 2 and 4 hold. Let  $T_S > 0$  be a given sampling time such that*

$$\nexists \ell \in \{-\beta\} \cup \lambda(A) \cup \lambda(A_r), h \in \mathbb{Z} \setminus \{0\} \text{ such that } \ell + i2h\pi T_S^{-1} \in \{-\beta\} \cup \lambda(A) \cup \lambda(A_r), \quad (17)$$

*By letting  $N \geq 8n + 4$ ,  $t_k = kT_S$ ,  $k \in \{0, \dots, N\}$ , let  $W_N \in \mathbb{R}^{N \times (N-n-1)}$  be the lower triangular Toeplitz matrix whose first column is  $\begin{bmatrix} w_{n+1} & \dots & w_1 & w_0 & 0_{1, N-n-2} \end{bmatrix}^\top$ , where  $w_0 z^{n+1} + w_1 z^n + \dots + w_{n+1} = (z - e^{-\beta T_S}) \det(zI - e^{A_r^\top T_S})$ . By applying to the interconnection of system (9) and the filters (3), the input  $u(t) = d_k$ ,  $\forall t \in [kT_S, (k+1)T_S)$ ,  $k \in \{0, \dots, N-1\}$ , if  $\{d_k\}_{k=1}^{N-1}$  is persistently exciting of order  $4n+2$ , then Assumption 3 holds.*

Note that (17) holds for almost all  $T_S > 0$  (see the discussion in [2]). Therefore, since Assumptions 2 and 4 are also generically satisfied, Proposition 1 provides an easily implementable approach to gather informative data.

By letting  $L_f \in \mathbb{R}^{1 \times 2n}$ , the result of Proposition 1 can be extended to include a feedback term in the control input:  $u(t) = L_f \chi(t) + d_k$ ,  $\forall t \in [kT_S, (k+1)T_S)$ . If Assumptions 1 and 2 hold,  $T_S > 0$  is such that (17) holds with  $\lambda(A) \cup \lambda(A_r)$  substituted by  $\lambda(A_f + B_f L_f)$ ,  $\lambda(A + B_f L_f) \cap \lambda(A_r) = \emptyset$ , and  $\{d_k\}_{k=1}^{N-1}$  is persistently exciting of order  $4n+2$ , then a reasoning wholly similar to that given in Appendix D can be used to conclude that Assumption 3 holds.

#### B. Data-driven synthesis of a stabilizing gain

By Corollary 1, if  $K_f$  is such that  $A_f + B_f K_f$  is Hurwitz, then the feedback interconnection of system (9), the filters (3a), (3b), and  $u(t) = K_f \chi(t)$  is globally asymptotically stable. Thus, consider the following theorem, which follows directly from (16) and [2, Thm. 2].

**Theorem 2.** *Let Assumptions 1–3 hold. Then, any matrix  $Z \in \mathbb{R}^{2n \times h}$  such that the LMI (6) holds is such that the gain  $K_f$  given in (7) is stabilizing. Conversely, if  $K_f$  is stabilizing, then it can be written as in (7), with  $Z$  satisfying (6).*

It is worth pointing out that an approach wholly similar to that given in [2, Thm. 3] can be used to guarantee robustness of such a scheme with respect to measurement and input noises.

#### IV. EXTENSION TO A CLASS OF MULTI-INPUT MULTI-OUTPUT SYSTEMS

The main objective of this section is to extend the results given in Section III to a class of MIMO systems. Toward this

goal, consider a generic MIMO system in state space form

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \quad y(t) = \bar{C}\bar{x}(t), \quad (18)$$

with  $\bar{x}(t) \in \mathbb{R}^h$ ,  $u(t) \in \mathbb{R}^m$ , and  $y(t) \in \mathbb{R}^p$ . The next assumption is made throughout this section to ensure that system (18) is a realization of the differential form (1).

**Assumption 1'.** *System (18) is reachable and observable,  $pn = h$ , and  $\text{rank}(\bar{O}) = h$ ,  $\bar{O} = \text{col}(\bar{C}, \bar{C}\bar{A}, \dots, \bar{C}\bar{A}^{n-1})$ .*

**Remark 3.** Under Assumption 1', by letting  $\bar{M}$  be the matrix obtained from  $M$  by substituting  $A$ ,  $B$ , and  $C$  with  $\bar{A}$ ,  $\bar{B}$ , and  $\bar{C}$ , respectively, one has that

$$y^{(n)}(t) = \bar{C}\bar{A}^n \bar{x}(t) + \sum_{i=0}^{n-1} \bar{C}\bar{A}^{n-i-1} \bar{B}u^{(i)}(t)$$

and  $\bar{x}(t)$  can be uniquely expressed in terms of  $y^{(0)}(t), \dots, y^{(n-1)}(t), u^{(0)}(t), \dots, u^{(n-1)}(t)$  as

$$\bar{x}(t) = O^{-1}(\text{col}(y^{(0)}(t), \dots, y^{(n-1)}(t)) - \bar{M}\text{col}(u^{(0)}(t), \dots, u^{(n-1)}(t))).$$

Thus, under Assumption 1', Eq. (1) completely describes the dynamics of system (18).

Define the matrices  $C_M = \begin{bmatrix} 0_{p,p(n-1)} & I_p \end{bmatrix}$  and

$$A_M = \left[ \begin{array}{c|c} \frac{0_{p,p(n-1)}}{I_{p(n-1)}} & \begin{bmatrix} -A_n \\ \vdots \\ -A_1 \end{bmatrix} \end{array} \right], \quad B_M = \begin{bmatrix} B_n \\ \vdots \\ B_1 \end{bmatrix}.$$

Under Assumption 1', the pair  $(A_M, B_M)$  is reachable since the triplet  $(A_M, B_M, C_M)$  is a realization of (1), and all minimal (i.e., reachable and observable) realizations have the same order [26, Sec. 17.1]. In particular, by letting

$$O_M = \begin{bmatrix} C_M \\ \vdots \\ C_M A_M^{n-1} \end{bmatrix}, \quad M_M = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_M A_M^{n-2} B & \dots & C_M B_M & 0 \end{bmatrix},$$

the relation between the initial conditions of the description in differential operator form (1) and the one in state space form

$$\dot{x}(t) = A_M x(t) + B_M u(t), \quad y(t) = C_M x(t),$$

is, by Remark 3, given by the following relation

$$\begin{bmatrix} y^{(0)}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = O_M x(0) + M_M \begin{bmatrix} u^{(0)}(0) \\ \vdots \\ u^{(n-1)}(0) \end{bmatrix}. \quad (19)$$

Hence, consider the following theorem, whose proof is given in Appendix E, that is the equivalent of Theorem 1 in the MIMO case considered in this section.

**Theorem 1'.** *Let Assumption 1' hold and consider the interconnection of systems (1), (3a), (3b), whose dynamics are*

$$\dot{\eta}(t) = A_{M,i} \eta(t) + B_{M,i} u(t), \quad \chi(t) = C_{M,i} \eta(t), \quad (20)$$

where  $\eta(t) = \text{col}(\zeta(t), \mu(t), x(t))$ ,  $\eta(t) \in \mathbb{R}^{(m+2p)n}$ ,  $\chi(t) = \text{col}(\zeta(t), \mu(t))$ ,  $\chi(t) \in \mathbb{R}^{(m+p)n}$ ,  $C_i = \begin{bmatrix} I_{(m+p)n} & 0_{(m+p)n, pn} \end{bmatrix}$ , and

$$A_{M,i} = \begin{bmatrix} A_{r,m} & 0_{mn,pn} & 0_{mn,pn} \\ 0_{pn,mn} & A_{r,p} & B_{r,p}C_M \\ 0_{pn,mn} & 0_{pn,pn} & A_M \end{bmatrix}, \quad (21a)$$

$$B_{M,i} = \begin{bmatrix} B_{r,m} \\ 0_{pn,m} \\ B_M \end{bmatrix}. \quad (21b)$$

By letting  $x(0) = x_0$ , where  $x(0)$  satisfies (19), the dynamics of  $\chi(t)$  are given by

$$\dot{\chi}(t) = A_{M,f}\chi(t) + B_{M,f}u(t) + G_{M,f}e^{A_{r,p}^\top t}x_0, \quad (22)$$

the pair  $(A_{M,f}, B_{M,f})$  is reachable, and

$$A_{M,f} = \left[ \begin{array}{c|c} \frac{A_{r,m}}{B_n \cdots B_1} & \frac{0_{mn,pn}}{0_{(n-1)p,p} \mid I_{(n-1)p}} \\ \hline & \frac{0_{(n-1)p,p} \mid I_{(n-1)p}}{-A_n \cdots -A_1} \end{array} \right],$$

$$B_{M,f} = \begin{bmatrix} B_{r,m} \\ 0_{pn,m} \end{bmatrix}, \quad G_{M,f} = \begin{bmatrix} 0_{mn,pn} \\ B_{r,p}C_M \end{bmatrix}.$$

By leveraging Theorem 1', the following lemma, which is the equivalent of Lemma 1 in the MIMO case considered in this section and whose proof is given in Appendix F, characterizes the response of the filters (3c), (3d), and (3e).

**Lemma 1'.** *Let Assumptions 1' and 2 hold. Consider the response of the filters (3c), (3d), and (3e). One has that*

$$\delta(t) = A_{M,f}\phi(t) + B_{M,f}v(t) + \epsilon_e(t), \quad (23)$$

for all  $t \geq 0$ , where

$$\epsilon_e(t) = G_{M,f}(A_{r,p}^\top + \beta I_{pn})^{-1}(e^{A_{r,p}^\top t} - e^{-\beta t}I_{pn})x_0.$$

Since the minimal polynomial of  $A_{r,p}$  is  $s^n + c_1s^{n-1} + \dots + c_n$ , if the sampling times are uniformly spaced, i.e.  $t_i = iT_S$ ,  $i = 1, \dots, N$ , then the matrix  $W_N$  given in Remark 2 is such that  $[\epsilon_e(t_1) \cdots \epsilon_e(t_N)]W_N = 0$ , for all  $x_0 \in \mathbb{R}^{pn}$ . Thus, consider the following proposition, whose proof is given in Appendix G, that extends Proposition 1 to the MIMO case considered in this section.

**Proposition 1'.** *Let Assumptions 1' and 2 hold. Additionally, suppose that  $\sigma(A_M) \cap \sigma(A_r) = \emptyset$ . Let  $T_S > 0$  be a given sampling time such that*

$$\nexists \ell \in \{-\beta\} \cup \lambda(A_M) \cup \lambda(A_r), h \in \mathbb{Z} \setminus \{0\} \text{ such that } \ell + i2h\pi T_S^{-1} \in \{-\beta\} \cup \lambda(A_M) \cup \lambda(A_r), \quad (24)$$

Let  $W_N \in \mathbb{R}^{N \times (N-n-1)}$  be the lower triangular Toeplitz matrix whose first column is  $[w_{n+1} \cdots w_1 \ w_0 \ 0_{1, N-n-2}]^\top$ ,  $w_0 z^{n+1} + \dots + w_{n+1} = (z - e^{-\beta T_S}) \det(zI - e^{A_r^\top T_S})$ . By applying to the interconnection of system (1) and the filters (3), the input  $u(t) = d_k$ ,  $\forall t \in [kT_S, (k+1)T_S)$ ,  $k \in \{0, \dots, N-1\}$ , if  $\{d_k\}_{k=1}^{N-1}$  is persistently exciting of order  $2((m+p)n+m)$ , then the matrices  $\bar{\Delta}_N$ ,  $\bar{\Phi}_N$ , and  $\bar{\Upsilon}_N$  defined in (5) satisfy

$$\text{rank}(\text{col}(\bar{\Phi}_N, \bar{\Upsilon}_N)) = (m+p)n + m.$$

By the same reasoning given in Section III-B, under the hypotheses of Proposition 1', by solving the LMI (6), a gain  $K_f$  such that  $A_{M,f} + B_{M,f}K_f$  is stabilizing can be obtained as in (7). Then, by the same reasoning used to prove Corollary 1, a stabilizing controller can be obtained using the filters (3a), (3b) and by letting  $u(t) = K_f\chi(t)$ .

## V. CONCLUSIONS

A data-driven framework for the design of output feedback controllers for LTI continuous-time systems has been presented. By combining input-output filtering with an LMI-based synthesis procedure, the proposed method avoids reliance on time derivatives and bypasses explicit model identification. Constructive conditions ensuring the applicability of the framework have been provided. Future work will focus on estimating the order  $n$  of system (1) from data and on weakening Assumption 1' in the MIMO case.

## APPENDIX

### A. Proof of Theorem 1

It is first proved that,  $\forall k \in \mathbb{N}$ ,  $k \geq 1$ , one has

$$A_i^k B_i = \begin{bmatrix} A_r^k B_r \\ \sum_{i=0}^{k-1} A_r^i B_r C A^{k-i-1} B \\ A^k B \end{bmatrix}. \quad (25)$$

By definition of  $A_i$  and  $B_i$ , Eq. (25) holds for  $k = 1$ . Assuming that it holds for some  $k \in \mathbb{N}$ ,  $k \geq 1$ , by computing  $A_i \cdot A_i^k B_i$ , one has that (25) holds with  $k$  substituted by  $k+1$ . Therefore, Eq. (25) holds by induction.

It is then proved that

$$A_r^k B_r + \sum_{j=1}^k c_j A_r^{k-j} B_r = e_{n-k}^n, k = 1, \dots, n-1, \quad (26a)$$

$$A_r^{n+i} B_r + \sum_{j=1}^n c_j A_r^{n+i-j} B_r = 0_{n,1}, \forall i \in \mathbb{N}. \quad (26b)$$

Note that  $B_r = e_n^n$  and  $A_r B_r + c_1 B_r = e_{n-1}^n$ . Hence, Eq. (26a) holds for  $k = 1$ . Assume that (26) holds for some  $k \in \mathbb{N}$ ,  $1 \leq k < n$ . Then, one has  $A_r e_{n-k}^n = e_{n-k-1}^n - c_{k+1} e_n^n = A_r^{k+1} B_r + c_1 A_r^k B_r + \dots + c_k A_r B_r$ . Hence, Eq. (26a) holds by induction for  $k = 1, \dots, n-1$ . On the other hand, by the Cayley-Hamilton theorem, one has that (26b) holds.

Equations (25) and (26) are now used to find a basis for the image of the reachability matrix  $R_i = [B_i \cdots A_i^{3n-1} B_i]$  of the pair  $(A_i, B_i)$ . By letting  $c_0 = 1$ , by (25), one has

$$\sum_{j=0}^k c_j A_i^{k-j} B_i = \sum_{j=0}^k c_j \text{col} \left( A_r^{k-j} B_r, \sum_{i=0}^{k-j-1} A_r^{k-j-i-1} B_r C A^i B, A^{k-j} B \right).$$

Note that

$$\begin{aligned} \sum_{j=0}^k c_j \sum_{i=0}^{k-j-1} A_r^{k-j-i-1} B_r C A^i B \\ = \sum_{i=0}^{k-1} \left( \sum_{j=0}^{k-i-1} c_j A_r^{k-j-i-1} B_r \right) C A^i B. \end{aligned}$$

By (26a), one has  $\sum_{j=0}^{k-i-1} c_j A_r^{k-j-i-1} B_r = \mathbf{e}_{n-k+i+1}^n$ , which implies, for  $k = 1, \dots, n-1$ ,

$$\begin{aligned} \sum_{j=0}^k c_j A_i^{k-j} B_i \\ = \text{col} \left( \mathbf{e}_{n-k}^n, \sum_{i=0}^{k-1} \mathbf{e}_{n-k+i+1}^n C A^i B, \sum_{j=0}^k c_j A^{k-j} B \right). \end{aligned}$$

Similarly, by (26b), one has, for  $k = n, \dots, 3n-1$ ,

$$\begin{aligned} \sum_{j=0}^n c_j A_i^{k-j} B_i \\ = \text{col} \left( 0_{n,1}, \sum_{i=0}^{n-1} \mathbf{e}_{n-i}^n C A^{k-i-1} B, \sum_{j=0}^n c_j A^{k-j} B \right). \end{aligned}$$

Therefore, by letting  $J_f \in \mathbb{R}^{3n \times 3n}$  be the upper triangular Toeplitz matrix whose first row is  $[1 \ c_1 \ \dots \ c_n \ 0_{1,2n-1}]$ , one has

$$R_i J_f = \begin{bmatrix} S_n & 0_{n,2n} \\ M S_n & O R_{2n} \\ F_n S_n & q_r(A) R_{2n} \end{bmatrix},$$

where  $R_{2n} = [B \ AB \ \dots \ A^{2n-1}B]$ ,  $F_n = [(\sum_{i=0}^{n-1} c_i A^{n-i-1})B \ \dots \ B]$ , and  $q_r(s) = s^n + c_1 s^{n-1} + \dots + c_n$ . Under Assumption 1, one has  $\text{rank}(R_{2n}) = n$ . Since  $O$  is invertible and  $S_n^{-1} = S_n$ , by taking linear combinations of the columns of  $R_i J_f$ , one has

$$\text{span}(R_i) = \text{span} \left( \begin{bmatrix} I_n & 0_{n,n} \\ 0_{n,n} & I_n \\ \bar{U}_1 & \bar{U}_2 \end{bmatrix} \right),$$

where  $\bar{U}_1 = F_n - q_r(A)O^{-1}M$  and  $\bar{U}_2 = q_r(A)O^{-1}$ . Since  $(A, B, C)$  is a realization of (8), one has  $CA^n O^{-1} = [-a_n \ \dots \ -a_1]$  and  $[CA^{n-1}B \ \dots \ CB] - CA^n O^{-1}M = [b_n \ \dots \ b_1]$ ; see Remark 3. Hence, it results that

$$\begin{aligned} C\bar{U}_2 &= \sum_{i=0}^{n-1} (c_{n-i} - a_{n-i}) C A^i O^{-1} \\ &= [c_n - a_n \ \dots \ c_1 - a_1]. \end{aligned}$$

Further, by [18, Eq. (28) at p. 324], one has

$$\begin{aligned} C\bar{U}_1 &= CF_n - C\bar{U}_2 M \\ &= [CA^{n-1}B + \sum_{i=1}^{n-1} a_i CA^{n-1-i}B \ \dots \ CB] \\ &= B^\top. \end{aligned}$$

Finally, since  $O^{-1}B_r = \mathbf{e}_1^n$ , by [18, Eq. (11) at p. 199], and  $A^{i-1}\mathbf{e}_1^n = \mathbf{e}_i^n$ ,  $i = 1, \dots, n$ , one has

$$\bar{U}_2 B_r = [c_n - a_n \ \dots \ c_1 - a_1]^\top.$$

Thus, let

$$P = \begin{bmatrix} I_n & 0_{n,n} & 0_{n,n} \\ 0_{n,n} & I_n & 0_{n,n} \\ \bar{U}_1 & \bar{U}_2 & I_n \end{bmatrix}.$$

By classical results on the Kalman decomposition for reachability [26, Sec. 16.2], one has

$$P^{-1}A_i P = \begin{bmatrix} A_{i,r,r} & A_{i,r,u} \\ 0_{n,2n} & A_{i,u,u} \end{bmatrix}, \quad P^{-1}B_i = \begin{bmatrix} B_{i,r} \\ 0_{n,1} \end{bmatrix},$$

and the pair  $(A_{i,r,r}, B_{i,r})$  is reachable. Note that  $C_i P = C_i$ ,  $A_r + B_r C \bar{U}_2 = A^\top$ ,  $P^{-1} \text{col}(\chi(t), x(t)) = \text{col}(\chi(t), x(t) - \bar{U}_1 \zeta(t) - \bar{U}_2 \mu(t))$ ,  $A_{i,r,r} = \begin{bmatrix} A_r & 0_{n,n} \\ B_r C \bar{U}_1 & A_r + B_r C \bar{U}_2 \end{bmatrix}$ ,  $B_{i,r} = \begin{bmatrix} B_r \\ 0_{n,1} \end{bmatrix}$ ,  $A_{i,r,u} = \begin{bmatrix} 0_{n,n} \\ B_r C \end{bmatrix}$ , and  $A_{i,u,u} = A - \bar{U}_2 B_r C = A_r^\top$ . Hence, the statement follows by the fact that  $A_{i,r,r} = A_f$ ,  $B_{i,r} = B_f$ ,  $A_{i,r,u} = G_f$ , and  $\kappa(t) = x(t) - \bar{U}_1 \zeta(t) - \bar{U}_2 \mu(t)$  satisfies  $\dot{\kappa}(t) = A_{i,u,u} \kappa(t)$  with  $\kappa(0) = x_0$ .

### B. Proof of Lemma 1

Following a reasoning similar to that used in the proof of [2, Lem. 1], the solution to system (3c), (3d), (3e) is

$$\phi(t) = \int_0^t e^{-\beta(t-\tau)} \chi(\tau) d\tau, \quad (27)$$

and  $v(t) = \int_0^t e^{-\beta(t-\tau)} u(\tau) d\tau$ . Integrating the right-hand side of (27) by parts and using (13), one obtains that  $\beta \phi(t) = \chi(t) - \int_0^t e^{-\beta(t-\tau)} \dot{\chi}(\tau) d\tau = \chi(t) - A_f \phi(t) - B_f v(t) - (\int_0^t e^{-\beta(t-\tau)} G_f e^{A_r^\top \tau} d\tau) x_0$ . Under Assumption 2, since  $-\beta \notin \lambda(A_r)$ , the equation  $\Gamma A_r^\top + \beta \Gamma = \Gamma(A_r^\top + \beta I_n) = G_f$  admits the unique solution  $\Gamma^* = G_f(A_r^\top + \beta I_n)^{-1}$ . Hence, letting  $\epsilon(t) = \Gamma^* e^{A_r^\top t} x_0 - e^{-\beta t} \Gamma^* x_0$ , one has

$$\epsilon(t) = \left( \int_0^t e^{-\beta(t-\tau)} G_f e^{A_r^\top \tau} d\tau \right) x_0.$$

Thus, Eq. (14) holds for  $t \geq 0$ .

### C. Proof of Lemma 2

By [22, Ex. 5.13.16] and the Cayley-Hamilton theorem, there exist  $\vartheta_1^i, \dots, \vartheta_n^i \in \mathbb{R}$  such that  $e^{A_r^\top t_i} = \sum_{j=1}^n \vartheta_j^i (A_r^\top)^{j-1}$ , for each  $i \in \{1, \dots, N\}$ . Therefore, by letting

$$\Theta = \begin{bmatrix} \vartheta_1^1 - e^{-\beta t_1} & \dots & \vartheta_1^N - e^{-\beta t_N} \\ \vdots & \ddots & \vdots \\ \vartheta_n^1 & \dots & \vartheta_n^N \end{bmatrix} \in \mathbb{R}^{n \times N},$$

$$\begin{aligned} [e^{A_r^\top t_1} - e^{-\beta t_1} I_n \ \dots \ e^{A_r^\top t_N} - e^{-\beta t_N} I_n] \\ = [I_n \ A_r^\top \ \dots \ (A_r^\top)^{n-1}] (\Theta \otimes I_n). \end{aligned}$$

Since  $\text{rank}(\Theta) \leq n$ , by the rank plus nullity theorem [22, Eq. (4.4.15)],  $\dim(\ker(\Theta)) \geq N - n$ . By [22, Ex. 5.8.15], for each  $\alpha \in \ker(\Theta)$ , one has  $(\Theta \otimes I_n)(\alpha \otimes I_n) = (\Theta \alpha) \otimes I_n = 0_{n^2, n}$ . Therefore, it holds that  $[e^{A_r^\top t_1} - e^{-\beta t_1} I_n \ \dots \ e^{A_r^\top t_N} - e^{-\beta t_N} I_n](\alpha \otimes I_n) = \sum_{i=1}^N \alpha_i (e^{A_r^\top t_i} - e^{-\beta t_i} I_n) = 0_{n,n}$ , for all  $\alpha \in \ker(\Theta)$ .

#### D. Proof of Proposition 1

Following [2, App. A], consider the interconnection of system (9) and the filters (3), whose dynamics are

$$\dot{\xi}(t) = A_e \xi(t) + B_e u(t) + G_e \varphi(t), \quad \dot{\varphi}(t) = A_r^\top \varphi(t) \quad (28)$$

where  $\xi(t) = [v^\top(t) \quad \phi^\top(t) \quad \chi^\top(t)]^\top$  and

$$A_e = \begin{bmatrix} -\beta & 0_{1,2n} & 0_{1,2n} \\ 0_{2n,1} & -\beta I_{2n} & I_{2n} \\ 0_{2n,1} & 0_{2n,2n} & A_f \end{bmatrix}, \quad B_e = \begin{bmatrix} 1 \\ 0_{2n,1} \\ B_f \end{bmatrix},$$

$$G_e = \begin{bmatrix} 0_{1,n} \\ 0_{2n,n} \\ G_f \end{bmatrix}, \quad C_e = \begin{bmatrix} 0_{2n,1} & I_{2n} & 0_{2n,2n} \\ 1 & 0_{1,2n} & 0_{1,2n} \end{bmatrix},$$

whose output  $\varkappa(t) = C_e \xi(t)$  from the initial condition  $\xi(0) = 0_{4n+1,1}$ ,  $\varphi(0) = x_0$ , is  $\varkappa(t) = [\phi^\top(t) \quad v^\top(t)]^\top$ . Under Assumption 4, the Sylvester equation  $\Pi A_r^\top - A^\top \Pi = B_r C$  admits a unique solution  $\Pi^* \in \mathbb{R}^{n \times n}$ . Therefore, the matrix  $\Pi_f^* = \text{col}(0_{n,n}, \Pi^*)$  solves  $\Pi_f A_r^\top - A_f \Pi_f = G_f$  and hence, under Assumption 2, the matrix  $\Pi_e^* = \text{col}(0_{1,n}, \Pi_f^*(A_r^\top + \beta I_n)^{-1}, \Pi_f^*)$  solves

$$\Pi_e A_r^\top - A_e \Pi_e = G_e. \quad (29)$$

By linearity, the solution to system (28) from the initial condition  $\xi(0) = 0_{4n+1,1}$  satisfies  $\xi(t) = \xi^a(t) + \xi^b(t)$ , with

$$\xi^a(t) = A_e \xi^a(t) + B_e u(t), \quad \xi^a(0) = -\Pi_e^* x_0, \quad (30a)$$

$$\xi^b(t) = A_e \xi^b(t) + G_e \varphi(t), \quad \xi^b(0) = \Pi_e^* x_0. \quad (30b)$$

Since  $\varphi(t) = e^{A_r^\top t} x_0$  and  $\Pi_e^*$  solves (29), the solution to system (30b) is  $\xi^b(t) = \Pi_e^* e^{A_r^\top t} x_0$ . Therefore, one has

$$\varkappa(t) = C_e \xi^a(t) + C_e \xi^b(t) = \varkappa^a(t) + C_e \Pi_e^* e^{A_r^\top t} x_0.$$

By [2, App. A], the dynamics of  $\varkappa^a(t)$  are given by

$$\dot{\varkappa}^a(t) = A_v \varkappa^a(t) + B_v u(t) + G_v \vartheta(t), \quad \dot{\vartheta}(t) = -\beta \vartheta(t), \quad (31)$$

with initial conditions  $\varkappa^a(0) = \varkappa_0^a$ ,  $\varkappa_0^a = -C_e \Pi_e^* x_0$ ,  $\vartheta(0) = \vartheta_0$ ,  $\vartheta_0 = F \Pi_e^* x_0$ ,  $F = [B_f \quad A_f + \beta I_{2n} \quad -I_{2n}]$ ,  $A_v = \begin{bmatrix} A_f & B_f \\ 0_{1,2n} & -\beta \end{bmatrix}$ ,  $B_v = \begin{bmatrix} 0_{2n,1} \\ 1 \end{bmatrix}$ ,  $G_v = \begin{bmatrix} I_{2n} \\ 0_{1,2n} \end{bmatrix}$ , and  $(A_v, B_v)$  is reachable. Since  $(A_f, B_f)$  is reachable, by the PBH test for reachability [18, Thm. 6.2.6],  $\text{rank}([A_f + \beta I_{2n} \quad B_f]) = 2n$ ,  $\forall \beta \in \mathbb{R}$ . Therefore, one has  $[A_f + \beta I_{2n} \quad B_f] [A_f + \beta I_{2n} \quad B_f]^\dagger = I_{2n}$ . Hence, since  $A_v = \begin{bmatrix} A_f + \beta I_{2n} & B_f \\ 0_{1,2n} & 0 \end{bmatrix} - \beta I_{2n+1}$ , the matrix  $\Pi_v^* = [A_f + \beta I_{2n} \quad B_f]^\dagger$  solves

$$A_v \Pi_v^* + \beta \Pi_v^* = G_v. \quad (32)$$

By linearity, the solution to system (31) from the initial condition  $\varkappa^a(0) = \varkappa_0^a$  satisfies  $\varkappa^a(t) = \varkappa_1^a(t) + \varkappa_{11}^a(t)$ ,

$$\varkappa_1^a(t) = A_v \varkappa_1^a(t) + B_v u(t), \quad \varkappa_1^a(0) = \varkappa_0^a - \Pi_v^* \vartheta_0, \quad (33a)$$

$$\varkappa_{11}^a(t) = A_v \varkappa_{11}^a(t) + G_v \vartheta(t), \quad \varkappa_{11}^a(0) = \Pi_v^* \vartheta_0. \quad (33b)$$

Since  $\Pi_v^*$  solves (32), the solution to (33b) is  $\varkappa_{11}^a(t) = \Pi_v^* e^{-\beta t} \vartheta_0$ . Therefore, the vector  $\varkappa(t)$  satisfies

$$\varkappa(t) = \varkappa_1^a(t) + C_e \Pi_e^* e^{A_r^\top t} x_0 + e^{-\beta t} \Pi_v^* F \Pi_e^* x_0,$$

where  $\varkappa_1^a(t)$  solves (33a).

Note that  $\lambda(A_v) = \{-\beta\} \cup \lambda(A_f) = \{-\beta\} \cup \lambda(A) \cup \lambda(A_r)$ . Since  $(A_v, B_v)$  is reachable, by [25, Lem. 1], if  $\{d_k\}_{k=1}^{N-1}$  is persistently exciting of order  $4n+2$ ,  $u(t) = d_k$  for all  $t \in [kT_S, (k+1)T_S)$ , and (17) holds, then  $\text{rank}(H) = 4n+2$ ,

$$H = \begin{bmatrix} \varkappa_{1,1}^a & \cdots & \varkappa_{1,N-2n-1}^a \\ \mathfrak{H}_{2n+1}(\{d_k\}_{k=1}^{N-1}) \end{bmatrix},$$

where  $\varkappa_{1,k}^a = \varkappa_1^a(kT_S)$ ,  $k \in \mathbb{N}$ ,  $k \geq 1$ . In particular, one has  $\varkappa_{1,k+1}^a = A_v^d \varkappa_{1,k}^a + B_v^d d_k$ , where  $A_v^d = e^{A_v T_S}$  and  $B_v^d = (\int_0^{T_S} e^{A_v^\top \tau} d\tau) B_v$ . Note that, letting  $w_0, w_1, \dots, w_{n+1}$  be defined as in Remark 2, one has

$$\sum_{i=0}^{n+1} w_{n+1-i} (C_e \Pi_e^* e^{A_r^\top t_{i+j}} x_0 + e^{-\beta t_{i+j}} \Pi_v^* F \Pi_e^* x_0) = 0,$$

$j = 1, \dots, N - n - 1$ . Thus, one has

$$\begin{bmatrix} \bar{\Phi}_N \\ \bar{\Upsilon}_N \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n+1} w_i \varkappa_{1,n+2-i}^a & \cdots & \sum_{i=0}^{n+1} w_i \varkappa_{1,N-i}^a \end{bmatrix}.$$

Hence, by letting  $w_i = 0$  for  $i \in \mathbb{Z}$ ,  $i \geq n+1$ , define  $\Lambda = [\sum_{i=0}^{2n+1} w_i (A_v^d)^{2n+1-i} \sum_{i=0}^{2n} w_i (A_v^d)^{2n-i} B_v^d \cdots w_0 B_v^d]$ . By [27, Thm. 3.2.1], if (17) holds, then  $(A_v^d, B_v^d)$  is reachable. Therefore, if (17) holds, then  $\text{rank}(\Lambda) = 2n+1$ . Hence, since

$$[\sum_{i=0}^{n+1} w_i \varkappa_{1,2n+2-i}^a \cdots \sum_{i=0}^{n+1} w_i \varkappa_{1,N-i}^a] = \Lambda H,$$

by the Frobenius inequality [22, Ex. 4.5.17], one has  $\text{rank}(\Lambda H) \geq \text{rank}(\Lambda) + \text{rank}(H) - (4n+2) = 2n+1$ . Thus, since  $\Lambda H \in \mathbb{R}^{(2n+1) \times (N-2n-1)}$ ,  $\text{rank}(\Lambda H) = 2n+1$ .

#### E. Proof of Theorem 1

The proof follows the same lines of that of Theorem 1 given in Section A with minor adaptations to deal with the MIMO case. It is first proved that,  $\forall k \in \mathbb{N}$ ,  $k \geq 1$ , one has

$$A_{M,i}^k B_{M,i} = \begin{bmatrix} A_{r,m}^k B_{r,m} \\ \sum_{i=0}^{k-1} A_{r,p}^i B_{r,p} C_M A_M^{k-i-1} B_M \\ A_{M,M}^k B_M \end{bmatrix}. \quad (34)$$

By definition of  $A_{M,i}$  and  $B_{M,i}$ , Eq. (34) holds for  $k = 1$ . Assuming that it holds for some  $k \in \mathbb{N}$ ,  $k \geq 1$ , by computing  $A_{M,i} \cdot A_{M,i}^k B_{M,i}$ , one has that (34) holds with  $k$  substituted by  $k+1$ . Therefore, Eq. (34) holds by induction.

Equations (34) and (26) are now used to find a basis for the image of the reachability matrix  $R_{M,i} = [B_{M,i} \cdots A_{M,i}^{(m+2p)n-1} B_{M,i}]$  of the pair  $(A_{M,i}, B_{M,i})$ . Since  $A_{r,m} = A_r \otimes I_m$ ,  $B_{r,m} = B_r \otimes I_m$ ,  $A_{r,p} = A_r \otimes I_p$ , and  $B_{r,p} = A_r \otimes I_p$ , by letting  $c_0 = 1$ , by (34) and (26), one has  $\sum_{j=0}^k c_j A_{M,i}^{k-j} B_{M,i} = \text{col}(\mathbf{e}_{n-k}^n \otimes I_m, \sum_{i=0}^{k-1} (\mathbf{e}_{n-k+i+1}^n \otimes I_p) C_M A_M^{k-i-1} B_M, \sum_{j=0}^k c_j A_M^{k-j} B_M)$ ,  $k = 1, \dots, n-1$ , and  $\sum_{j=0}^n c_j A_{M,i}^{k-j} B_{M,i} = \text{col}(0_{n,1} \otimes I_m, \sum_{i=0}^{n-1} (\mathbf{e}_{n-i}^n \otimes I_p) C_M A_M^{k-i-1} B_M, \sum_{j=0}^n c_j A_M^{k-j} B_M)$ ,  $k = n, \dots, 3n-1$ . Therefore, by letting  $J_f \in \mathbb{R}^{(m+2p)n \times (m+2p)n}$  be the upper triangular Toeplitz matrix whose first row is  $[1 \quad c_1 \quad \cdots \quad c_n \quad 0_{1,(m+2p-1)n-1}]$ , one has

$$R_i(J_f \otimes I_m) = \begin{bmatrix} S_n \otimes I_m & 0_{nm,(m+2p-1)nm} \\ M_M(S_n \otimes I_m) & O_M R_M \\ F_M(S_n \otimes I_m) & q_r(A_M) R_M \end{bmatrix},$$

where  $q_r(s) = s^n + c_1 s^{n-1} + \dots + c_n$ , and

$$R_M = [ B_M \quad \dots \quad A_M^{(m+2p-1)n} B_M ],$$

$$F_M = [ (\sum_{i=0}^{n-1} c_i A_M^{n-i-1}) B_M \quad \dots \quad B_M ].$$

Under Assumption 1', the pair  $(A_M, B_M)$  is reachable and hence  $\text{rank}(R_M) = pn$ . Since  $O_M$  is invertible and  $S_n^{-1} = S_n$ , by taking linear combinations of the columns of  $R_i J_f$ , one has

$$\text{span}(R_{M,i}) = \text{span} \left( \begin{bmatrix} I_{mn} & 0_{mn,pn} \\ 0_{pn,mn} & I_{pn} \\ \tilde{U}_1 & \tilde{U}_2 \end{bmatrix} \right),$$

where  $\tilde{U}_1 = F_M - q_r(A_M)O_M^{-1}M_M$  and  $\tilde{U}_2 = q_r(A_M)O_M^{-1}$ . Since  $(A_M, B_M, C_M)$  is a realization of (1), one has  $C_M A_M^{n-1} O_M^{-1} = [-A_n \quad \dots \quad -A_1]$  and  $[C_M A_M^{n-1} B_M \quad \dots \quad C_M B_M] - C_M A_M^{n-1} O_M^{-1} M_M = [B_n \quad \dots \quad B_1]$ ; see Remark 3. Hence, it results that  $C_M \tilde{U}_2 = [c_n I_p - A_n \quad \dots \quad c_1 I_p - A_1]$  and  $C_M \tilde{U}_1 = [B_n \quad \dots \quad B_1]$ . Finally, since  $O_M^{-1} B_{r,p} = e_1^n \otimes I_p$ , one has  $\tilde{U}_2 B_{r,p} = \text{col}(c_n I_p - A_n, \dots, c_1 I_p - A_1)$ . Thus, let

$$P = \begin{bmatrix} I_{mn} & 0_{mn,pn} & 0_{mn,pn} \\ 0_{pn,mn} & I_{pn} & 0_{pn,pn} \\ \tilde{U}_1 & \tilde{U}_2 & I_{pn} \end{bmatrix}.$$

By classical results on the Kalman decomposition for reachability [26, Sec. 16.2], one has

$$P^{-1} A_{M,i} P = \begin{bmatrix} A_{M,i,r} & A_{M,i,u} \\ 0_{pn,(m+p)n} & A_{M,i,u} \end{bmatrix},$$

$$P^{-1} B_{M,i} = \begin{bmatrix} B_{M,i,r} \\ 0_{pn,m} \end{bmatrix},$$

and the pair  $(A_{M,i,r}, B_{M,i,r})$  is reachable. Note that

$$A_{M,i,r} = \begin{bmatrix} A_{r,m} & 0_{mn,pn} \\ B_{r,p} C_M \tilde{U}_1 & A_{r,p} + B_{r,p} C_M \tilde{U}_2 \end{bmatrix},$$

$$B_{M,i,r} = \begin{bmatrix} B_{r,m} \\ 0_{pn,m} \end{bmatrix}, \quad A_{M,i,r,u} = \begin{bmatrix} 0_{mn,pn} \\ B_{r,p} C_M \end{bmatrix},$$

$$A_{M,i,u,u} = A_M - \tilde{U}_2 B_{r,p} C_M = A_{r,p}^T.$$

Hence, the statement follows by the fact that  $A_{M,i,r} = A_{M,f}$ ,  $B_{M,i,r} = B_{M,f}$ ,  $A_{i,r,u} = G_{M,f}$ , and  $\kappa(t) = x(t) - \tilde{U}_1 \zeta(t) - \tilde{U}_2 \mu(t)$  satisfies  $\dot{\kappa}(t) = A_{M,i,u,u} \kappa(t)$  with  $\kappa(0) = x_0$ .

### F. Proof of Lemma 1'

Following the same construction employed in Appendix B, one has that the following relation holds for all  $t \geq 0$ :

$$\beta \phi(t) = \chi(t) - \int_0^t e^{-\beta(t-\tau)} \dot{\chi}(\tau) d\tau$$

$$= \chi(t) - A_{M,f} \phi(t) - B_{M,f} v(t)$$

$$- \left( \int_0^t e^{-\beta(t-\tau)} G_{M,f} e^{A_{r,p}^T \tau} d\tau \right) x_0.$$

Under Assumption 2, since  $-\beta \notin \lambda(A_{r,p})$ , the equation  $\Gamma A_{r,p}^T + \beta \Gamma = \Gamma(A_{r,p}^T + \beta I_{pn}) = G_{M,f}$  admits the unique solution  $\Gamma^* = G_{M,f}(A_{r,p}^T + \beta I_{pn})^{-1}$ . Thus, the statement follows by the fact that  $(\int_0^t e^{-\beta(t-\tau)} G_{M,f} e^{A_{r,p}^T \tau} d\tau) x_0 = \epsilon_e(t)$ .

### G. Proof of Proposition 1'

By adapting the construction given in Appendix D for the SISO case, consider the interconnection of system (1) and the filters (3), whose dynamics are

$$\dot{\xi}(t) = A_{M,e} \xi(t) + B_{M,e} u(t) + G_{M,e} \varphi(t), \quad (35a)$$

$$\dot{\varphi}(t) = A_{r,p}^T \varphi(t) \quad (35b)$$

where  $\xi(t) = [v^\top(t) \quad \phi^\top(t) \quad \chi^\top(t)]^\top$  and

$$A_{M,e} = \begin{bmatrix} -\beta I_m & 0_{m,(m+p)n} & 0_{m,(m+p)n} \\ 0_{(m+p)n,m} & -\beta I_{(m+p)n} & I_{(m+p)n} \\ 0_{(m+p)n,m} & 0_{(m+p)n,(m+p)n} & A_{M,f} \end{bmatrix},$$

$$B_{M,e} = \begin{bmatrix} I_m \\ 0_{(m+p)n,m} \\ B_{M,f} \end{bmatrix},$$

$$G_{M,e} = \begin{bmatrix} 0_{pn} \\ 0_{(m+p)n,pn} \\ G_{M,f} \end{bmatrix},$$

$$C_{M,e} = \begin{bmatrix} 0_{(m+p)n,m} & I_{(m+p)n} & 0_{(m+p)n,(m+p)n} \\ I_m & 0_{m,(m+p)n} & 0_{m,(m+p)n} \end{bmatrix},$$

whose output  $\varkappa(t) = C_{M,e} \xi(t)$  from the initial condition  $\xi(0) = 0_{2(m+p)n+m,1}$ ,  $\varphi(0) = x_0$ , is  $\varkappa(t) = [\phi^\top(t) \quad v^\top(t)]^\top$ . If  $\sigma(A_M) \cap \sigma(A_r) = \emptyset$ , then the Sylvester equation  $\Pi_M A_{r,p}^T - A_M^T \Pi_M = B_{r,p} C_M$  admits an unique solution  $\Pi_M^* \in \mathbb{R}^{pn \times pn}$ . Therefore, the matrix  $\Pi_{M,f}^* = \text{col}(0_{mn,pn}, \Pi_M^*)$  solves  $\Pi_{M,f} A_{r,p}^T - A_{M,f}^T \Pi_{M,f} = G_{M,f}$  and hence, under Assumption 2, the matrix  $\Pi_{M,e}^* = \text{col}(0_{m,pn}, \Pi_{M,f}^* (A_{r,p}^T + \beta I_{pn})^{-1}, \Pi_{M,f}^*)$  solves

$$\Pi_{M,e} A_{r,p}^T - A_{M,e} \Pi_{M,e} = G_{M,e}. \quad (36)$$

By linearity, the solution to (35) from the initial condition  $\xi(0) = 0_{2(m+p)n+m,1}$  satisfies  $\xi(t) = \xi^a(t) + \xi^b(t)$ , with

$$\dot{\xi}^a(t) = A_{M,e} \xi^a(t) + B_{M,e} u(t), \quad \xi^a(0) = -\Pi_{M,e}^* x_0, \quad (37a)$$

$$\dot{\xi}^b(t) = A_{M,e} \xi^b(t) + G_{M,e} \varphi(t), \quad \xi^b(0) = \Pi_{M,e}^* x_0. \quad (37b)$$

Since  $\varphi(t) = e^{A_{r,p}^T t} x_0$  and  $\Pi_{M,e}^*$  solves (36), the solution to system (37b) is  $\xi^b(t) = \Pi_{M,e}^* e^{A_{r,p}^T t} x_0$ . Therefore, one has

$$\varkappa(t) = C_{M,e} \xi^a(t) + C_{M,e} \xi^b(t) = \varkappa^a(t) + C_{M,e} \Pi_{M,e}^* e^{A_{r,p}^T t} x_0.$$

By [2, App. A], the dynamics of  $\varkappa^a(t)$  are given by

$$\dot{\varkappa}^a(t) = A_{M,v} \varkappa^a(t) + B_{M,v} u(t) + G_{M,v} \tilde{\vartheta}(t), \quad (38a)$$

$$\dot{\tilde{\vartheta}}(t) = -\beta \tilde{\vartheta}(t), \quad (38b)$$

with initial conditions  $\varkappa^a(0) = \varkappa_0^a$ ,  $\varkappa_0^a = -C_{M,e} \Pi_{M,e}^* x_0$ ,  $\tilde{\vartheta}(0) = \tilde{\vartheta}_0$ ,  $\tilde{\vartheta}_0 = F_M \Pi_{M,e}^* x_0$ ,  $F_M = [B_{M,f} \quad A_{M,f} + \beta I_{(m+p)n} \quad -I_{(m+p)n}]$ ,  $A_{M,v} = [A_{M,f} \quad B_{M,f}]$ ,  $B_{M,v} = [0_{m,(m+p)n} \quad I_m]$ ,  $G_{M,v} = [I_{(m+p)n}]$ , and  $(A_{M,v}, B_{M,v})$  is reachable. Since  $(A_{M,f}, B_{M,f})$  is reachable, by the PBH test for reachability [18, Thm. 6.2.6],  $\text{rank}([A_{M,f} + \beta I_{(m+p)n} \quad B_{M,f}]) = (m+p)n$ ,  $\forall \beta \in \mathbb{R}$ . Therefore, one has  $[A_{M,f} + \beta I_{(m+p)n} \quad B_{M,f}] [A_{M,f} + \beta I_{(m+p)n} \quad B_{M,f}]^\dagger =$



$I_{(m+p)n}$ . Hence, since  $A_{M,v} = \begin{bmatrix} A_{M,f} + \beta I_{(m+p)n} & B_{M,f} \\ 0_{m,(m+p)n} & 0_{m,m} \end{bmatrix} - \beta I_{(m+p)n+m}$ ,  $\Pi_{M,v}^* = [A_{M,f} + \beta I_{(m+p)n} \quad B_{M,f}]^\top$  solves

$$A_{M,v}\Pi_{M,v} + \beta\Pi_{M,v} = G_{M,v}. \quad (39)$$

By linearity, the solution to system (38) from the initial condition  $\mathcal{x}^a(0) = \mathcal{x}_0^a$  satisfies  $\mathcal{x}^a(t) = \mathcal{x}_I^a(t) + \mathcal{x}_{II}^a(t)$ ,

$$\dot{\mathcal{x}}_I^a(t) = A_{M,v}\mathcal{x}_I^a(t) + B_{M,v}u(t), \quad (40a)$$

$$\dot{\mathcal{x}}_{II}^a(t) = A_{M,v}\mathcal{x}_{II}^a(t) + G_{M,v}\tilde{\mathcal{O}}(t), \quad (40b)$$

where  $\mathcal{x}_I^a(0) = \mathcal{x}_0^a - \Pi_{M,v}^*\tilde{\mathcal{O}}_0$  and  $\mathcal{x}_{II}^a(0) = \Pi_{M,v}^*\tilde{\mathcal{O}}_0$ . Since  $\Pi_{M,v}^*$  solves (39), the solution to (40b) is  $\mathcal{x}_{II}^a(t) = \Pi_{M,v}^*e^{-\beta t}\tilde{\mathcal{O}}_0$ . Therefore, the vector  $\mathcal{x}(t)$  satisfies

$$\mathcal{x}(t) = \mathcal{x}_I^a(t) + C_{M,e}\Pi_{M,e}^*e^{A_{r,p}^\top t}x_0 + e^{-\beta t}\Pi_{M,v}^*F_M\Pi_{M,e}^*x_0,$$

where  $\mathcal{x}_I^a(t)$  solves (40a).

Note that  $\lambda(A_{M,v}) = \{-\beta\} \cup \lambda(A_{M,f}) = \{-\beta\} \cup \lambda(A_M) \cup \lambda(A_r)$ . Since  $(A_{M,v}, B_{M,v})$  is reachable, by [25, Lem. 1], if  $\{d_k\}_{k=1}^{N-1}$  is persistently exciting of order  $2((m+p)n+m)$ ,  $u(t) = d_k$  for all  $t \in [kT_S, (k+1)T_S)$ , and (24) holds, then  $\text{rank}(H) = (m+p)n+m+m((m+p)n+m)$ ,

$$H = \begin{bmatrix} \mathcal{x}_{I,1}^a & \cdots & \mathcal{x}_{I,N-(m+p)n+m}^a \\ \mathfrak{H}_{(m+p)n+m}(\{d_k\}_{k=1}^{N-1}) \end{bmatrix},$$

where  $\mathcal{x}_{I,k}^a = \mathcal{x}_I^a(kT_S)$ ,  $k \in \mathbb{N}$ ,  $k \geq 1$ . In particular, one has  $\mathcal{x}_{I,k+1}^a = A_{M,v}^d\mathcal{x}_{I,k}^a + B_{M,v}^d d_k$ , where  $A_{M,v}^d = e^{A_{M,v}T_S}$  and  $B_{M,v}^d = (\int_0^{T_S} e^{A_{M,v}\tau} d\tau)B_{M,v}$ . Note that, letting  $w_0, w_1, \dots, w_{n+1}$  be defined as in Remark 2, one has  $\sum_{i=0}^{n+1} w_{n+1-i}(C_{M,e}\Pi_{M,e}^*e^{A_{r,p}^\top t}x_0 + e^{-\beta t}F_M\Pi_{M,v}^*x_0) = 0$ ,  $j = 1, \dots, N-n-1$ . Thus, one has that

$$\begin{bmatrix} \bar{\Phi}_N \\ \bar{\Upsilon}_N \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n+1} w_i \mathcal{x}_{I,n+2-i}^a & \cdots & \sum_{i=0}^{n+1} w_i \mathcal{x}_{I,N-i}^a \end{bmatrix}.$$

Hence, by letting  $w_i = 0$  for  $i \in \mathbb{Z}$ ,  $i \geq n+1$ , define  $\Lambda$  as in Eq. (41). By [27, Thm. 3.2.1], if (24) holds, then  $(A_{M,v}^d, B_{M,v}^d)$  is reachable. Therefore, if (24) holds, then  $\text{rank}(\Lambda) = (m+p)n+m$ . Hence, since

$$\begin{bmatrix} \sum_{i=0}^{n+1} w_i \mathcal{x}_{I,(m+p)n+m+1-i}^a & \cdots & \sum_{i=0}^{n+1} w_i \mathcal{x}_{I,N-i}^a \end{bmatrix} = \Lambda H,$$

by the Frobenius inequality [22, Ex. 4.5.17], one has  $\text{rank}(\Lambda H) \geq \text{rank}(\Lambda) + \text{rank}(H) - ((m+p)n+m+m((m+p)n+m)) = (m+p)n+m$ . Thus, one has that  $\text{rank}(\Lambda H) = (m+p)n+m$ .

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$$\Lambda = \begin{bmatrix} \sum_{i=0}^{(m+p)n+m} w_i (A_{M,v}^d)^{(m+p)n+m-i} & \sum_{i=0}^{(m+p)n+m-1} w_i (A_{M,v}^d)^{(m+p)n+m-1-i} B_{M,v}^d & \cdots & w_0 B_{M,v}^d \end{bmatrix}. \quad (41)$$