Supplement to Proposition 5.2 of Rizzo's book

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1 Why is there a $1/k$ factor in the definition of $\hat{\theta}^S$?

Consider estimating

$$\theta = \int_0^1 g(x)\,\mathrm{d}x$$

with stratified sampling method. Divide the integral interval $[0, 1]$ into k sub-intervals

$$[0, 1/k], [1/k, 2/k], \ldots, [(k-1)/k, 1].$$

Then

$$\theta = \left(\int_0^{\frac{1}{k}} + \cdots + \int_{\frac{k-1}{k}}^1\right) g(x)\,\mathrm{d}x = \sum_{j=1}^k \theta_j,$$

with

$$\theta_j = \int_{\frac{j-1}{k}}^{\frac{j}{k}} g(x)\,\mathrm{d}x, \quad j = 1, \ldots, k.$$

Within the j-th stratum, we apply simple Monte Carlo method to estimate θ_j with m replicates. That is,

1. generate m random numbers

$$x_1^{(j)}, \ldots, x_m^{(j)} \sim \mathrm{Unif}\left(\frac{j-1}{k}, \frac{j}{k}\right)$$

2. Estimate θ_j with the simple Monte Carlo estimator

$$\tilde{\theta}_j = \frac{1}{km}\sum_{i=1}^m g(x_i^{(j)}).$$

Note that Rizzo's book defines $\hat{\theta}_j = \frac{1}{m}\sum_{i=1}^m g(x_i^{(j)})$, so we have the relationship that

$$\tilde{\theta}_j = \frac{1}{k}\hat{\theta}_j.$$

Similar strategies apply to all k sub-intervals. Hence

$$\hat{\theta}^S = \sum_{j=1}^k \tilde{\theta}_j = \frac{1}{km} \sum_{j=1}^k \sum_{i=1}^m g(x_i^{(j)}) = \frac{1}{k} \sum_{j=1}^k \hat{\theta}_j.$$

2 One more comment on the meanings of (θ_j, σ_j^2)

In Rizzo's book, they are *the mean and variance of* $g(U)$ *on stratum* j. To be explicit, let $U \sim \text{Unif}(0,1)$, when we truncate U within the j-th sub-interval $[(j-1)/k, j/k]$, such truncated uniform distribution is $U^{(j)} \sim \text{Unif}(\frac{j-1}{k}, \frac{j}{k})$. Then $\theta_j = \text{E}\{g(U^{(j)})\}$ and $\sigma_j^2 = \text{var}\{g(U^{(j)})\}$.