# DATA130004: Solution to Quiz 2

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1. Suppose $x_1, \dots, x_n$ are independent and identically distributed with variance $\sigma^2$. It is known that the plug-in estimator of variance $\hat{\theta} = \sum_{i=1}^n (x_i - \bar{x})^2/n$ has bias equal to $-\sigma^2/n$. Then show that the Jackknife estimate of the bias of $\hat{\theta}$, $\widehat{\text{bias}}_{jack} = -s^2/n$, where $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/(n-1)$.

   *Proof:* By definition, one has $\hat{\theta} = \sum_{i=1}^n x_i^2/n - \bar{x}^2$, so

   $$\hat{\theta}_{(j)} = \frac{1}{n-1}\left(\sum_{i=1}^n x_i^2 - x_j^2\right) - \left(\frac{n\bar{x} - x_j}{n-1}\right)^2$$

   and

   $$\hat{\theta}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^n \hat{\theta}_{(j)} = \frac{1}{n}\sum_{i=1}^n x_i^2 - \frac{1}{n}\sum_{j=1}^n \left(\frac{n\bar{x} - x_j}{n-1}\right)^2$$

   For the last term in the above equation, some algebra shows that

   $$\sum_{j=1}^n \left(\frac{n\bar{x} - x_j}{n-1}\right)^2 = \frac{n^2(n-2)}{(n-1)^2}\bar{x}^2 + \frac{1}{(n-1)^2}\sum_{i=1}^n x_i^2$$

   and hence

   $$\hat{\theta}_{(\cdot)} = \frac{n-2}{(n-1)^2}\sum_{i=1}^n (x_i - \bar{x})^2.$$

   Consequently,

   $$\widehat{\text{bias}}_{jack} = (n-1)(\hat{\theta}_{(\cdot)} - \hat{\theta}) = -\frac{1}{n(n-1)}\sum_{i=1}^n (x_i - \bar{x})^2 = -\frac{s^2}{n}.$$

2. Consider a $p$-dimensional normal distribution $X = (Y, Z)^\top$ with two partitions $Y \in \mathbb{R}^q, Z \in \mathbb{R}^{p-q}, 0 < q < p$. Correspondingly, the mean of $X$ is $\mu = (\mu_Y, \mu_Z)^\top$ and the covariance of $X$ is

   $$\Sigma = \begin{pmatrix} \Sigma_{YY} & \Sigma_{YZ} \\ \Sigma_{ZY} & \Sigma_{ZZ} \end{pmatrix}.$$

(a) Now derive the conditional distribution of $Z$ given $Y$.
Hint: make a non-singular transformation $AX$ where

$$A = \begin{pmatrix} I_q & 0 \\ -\Sigma_{ZY}\Sigma_{YY}^{-1} & I_{p-q} \end{pmatrix}.$$

(b) When $p = 3$, $\mu = (0, 0, 0)^\top$ and

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}$$

where $\rho \in [-1, 1]$, describe the Gibbs sampler algorithm to generate random samples from the trivariate normal distribution.

(c) From the above updating rules, what is your finding about the relationship among $X_1$, $X_2$ and $X_3$?
Hint: check the inverse of $\Sigma$.

*Proof:*

(a) With the transformation $A$, $AX$ has two independent multivariate normal partitions. Hence,

$$Z|Y \sim N(\mu_Z - \Sigma_{ZY}\Sigma_{YY}^{-1}\mu_Y, \Sigma_{ZZ} - \Sigma_{ZY}\Sigma_{YY}^{-1}\Sigma_{YZ}).$$

Similarly,

$$Y|Z \sim N(\mu_Y - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\mu_Z, \Sigma_{YY} - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Sigma_{ZY}).$$

(b) Apply the results in (a) and derive all three conditionals.
Set a starting value $x^{(0)} \in \mathbb{R}^3$, and iterate for $t = 1, 2, \ldots$

(1) Generate $x_1^{(t+1)} \sim N(\rho x_2^{(t)}, 1 - \rho^2)$.
(2) Generate $x_2^{(t+1)} \sim N(\frac{\rho}{1+\rho^2}(x_1^{(t+1)} + x_3^{(t)}), \frac{1-\rho^2}{1+\rho^2})$
(3) Generate $x_3^{(t+1)} \sim N(\rho x_2^{(t+1)}, 1 - \rho^2)$.

(c) The updating rules indicate that generating $X_1$ involves only $X_2$ and so is generating $X_3$.
Statistically, under the multivariate normal distribution assumption, $\Sigma_{13}^{-1} = 0$ means that conditioning on $X_2$, $X_1$ and $X_3$ are independent.