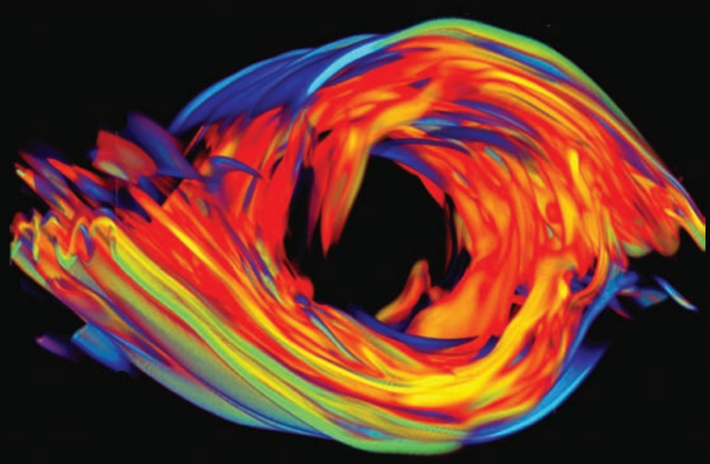
INTRODUCTION TO PLASMA DYNAMICS



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Foreword

The enormous role played by the plasma in nature is now generally recognised, and along with it we have clearer understanding of its value now and in the future, especially for the daily practice of mankind. There is no doubt that the XXI century will be the century of not only computer science, biology and space exploration, but also the century of plasma technology. This will provide new sources of energy (thermonuclear fusion), methods of direct conversion (MHD generators and plasma thermocouples), effective means of ion-plasma processing of materials (in both microelectronics and heavy machinery), wave electromagnetic radiation sources (including 'eternal' household lamps with adjustable range and plasma TVs with the unlimited size of thin screens), space plasma thrusters, etc.

Parallel to the development of plasma technologies there have been intensive experimental and theoretical studies of the plasma state of matter and its hidden features. Such research not only accelerate technological development, but also can dramatically improve the understanding of natural phenomena, in particular the amazing processes in the magnetospheres of planets, stars and nebulae. The need for progress is evidenced by at least the fact that we still do not have a convincing model of the solar prominences and flares, which are so strongly influenced by the magnetosphere and many processes on the Earth.

Naturally, in view of this, it is necessary to write different books on plasma, both specialized and broadly covering this area of science. However, the familiarity with published books says that if narrowly themed books still appear on store shelves, the books on 'common plasma dynamics' are rare[1]. There is a demand for books written for graduate students, postgraduates and engineers. One of the first bricks to close this loophole should be, in my opinion, this 'Introduction to plasma dynamics'.

[1]Four introductory volumes of the Encyclopedia of Low-Temperature Plasma were published in Russia at the beginning of 2001. At the moment, this publication contains the largest amount of information on plasma but it is limited to the particle energy in the range 1–10 eV.

I would like to specify the features of this book.[2]

First of all, it is really an introduction. It should introduce the reader to the terms, the basic experimental facts and the main ‘basic’ models – the systems of equations that describe the broad areas of plasma dynamics processes. ‘Introduction ...’ uses specific examples to show how to use these models and, finally, to familiarize the reader with specific examples of modern plasma technology. Thus, we would like to, having mastered this "Introducing ...", to ensure that the reader has enough open-mindedness to start studying the literature.

What has been said is explained by the widespread lack of books on the subject of plasma and ignorance of the actual diversity of plasma technology and completely inadequate attention to two fundamental points: the creation of plasma and its loss on the walls of the working volume.

Indeed, in the laboratory conditions plasma is the ephemeral substance, limited in space and time. It is artificially created, passing the stage of a neutral substance (gas, liquid, solid) to a final state of the plasma with the required parameters. On the other hand, the plasma is surrounded by ‘plasma consuming’ walls. Naturally, the book on plasma dynamics should adequately reflect this specificity of plasma.

Although the above described general formulation by the author is unlikely to cause objections, however, there are clearly visible difficulties of the program in general. Here there are two main problems.

The first of them is the fact that the world of plasma processes has no natural physical boundaries. Once heating of the material and its ionisation start, it is hard to stop. It is natural to want to further increase the energy of the particles, increasing their density, lifetime, etc. As a result, from low-temperature plasma and cold walls we move to a system in which there is a plasma of medium to higher and higher energies, etc. to the black holes, which until recently (2000) were dealt with by astronomers only, and now their ‘germs’ are beginning to be established on the Earth in accelerators.

Thus, plasma physics is, in fact, unique. So, for example, solid state physics or electrodynamics can in many cases be studied without knowing anything about the plasma. But this does not work with the

[2] I believe the term ‘plasma dynamics’ is more suitable for the content of the book than the very general term ‘plasma physics’. This is explained by the fact that the book deals extensively with plasma flows. I also believe that the term ‘plasma electrodynamics’ is less suitable, although the book of H. Alfvén, the founder of modern plasma dynamics, had the title ‘Cosmic electrodynamics’. At the same time, H. Alfvén formally referred to the field (not corpuscular) plasma component and I believe this to be less accurate.

plasma did not work. In any experiment we have to deal with the interaction of the plasma with solid (liquid) bodies (electrodes, walls, probes) and the processes of radiation in a wide wavelength range, etc. Therefore, the general plasma dynamics should include more or less all sections of physics.

Thus, plasma is actually a link between different forms of matter from a dilute gas to extremely dense substances. It is not surprising that the analogues of plasma processes can be seen in other environments. Therefore, the terms like ‘plasma-like environment’ (semiconductors, metals, electrolytes) or ‘quark–gluon plasma’ have appeared, though in the latter case it is a qualitatively different substance of black holes.

Of course, some boundaries can be made between plasma processes and facilities, but they are conditional and temporary. This ‘Introduction’ will focus on the relatively rarefied, non-degenerate and non-relativistic plasma, with the emphasis in the applied sense placed on the ‘laboratory’ plasma dynamics, i.e. man-made plasma systems. These are discussed in detail in the final chapter of the book. As for ‘cosmic" plasma dynamics’, a few fragments associated with the Earth and the Sun, are discussed in Chapter 9.

As can be seen, the present book covers quite a wide range of issues on a fundamental level, but not all. To characterise the most pressing issues, it is necessary to prepare at least three more major reviews. Here is a brief description of them.

– Overview of plasma dynamics (PD) of essentially three-dimensional non-stationary processes. This must include questions about the stability of the equilibrium configurations and flows, turbulence, the dynamics of solar activity, etc.

– Overview of PD of dense (‘quantum’) plasma. It should include the dynamics of strongly coupled plasma, formation of the superdense states of laser targets, processes in white dwarfs, etc. Thus, this volume should be linked to plasma physics and the physics of solids.

– Review of relativistic PD. Relativism has long become a laboratory of PD. Of course, in ‘classic’ laboratories systems with relativistic electrons are studied. These are generators of high-current (mega-ampere) electron beams, and also a number of plasma systems (tokamaks, Z-pinches of different types) in which ‘unwelcome’ fluxes of relativistic electrons appear. However, the laboratories in which the processes with elementary particles with energies greater than 10^9 eV are studied, have been using for many years special storage devices for such particles (colliders) in which the particles collide with each other. Although the functions of the distribution of the relativistic ions in the colliders are almost δ-shaped, nevertheless, here we are already concerned with ion relativistic plasma.

So the basis for the three volumes is already ready.

The second difficulty is the following. Confining ourselves to the ideal (in the kinetics sense) rarefied, non-relativistic plasma, we have not solved the problem of selecting the principles of systematization of the material. The usual courses of plasma physics as an analysis of the specific processes in arbitrarily chosen plasma configurations. But because the 'simple' plasma systems do not exist, then the attention of the authors is focused on a few systems, most often on homogeneous plasma (waves) or thermonuclear fusion systems (tokamaks, etc.). Clearly, for any general review of plasma phenomena (and technology) this way is unacceptable. The basis of the Introduction is the consideration of the principles of modelling or, in other words, the principles of the hierarchy of the 'basic' models and their relationships with one another, as well as a demonstration of their work.

So, we have a section dedicated to structures (morphology) of magnetic fields, one- and zero-dimensional (block) plasma models. Then we sequentially consider single-, two- and multi-component simulation models, and then the 'different kinetics' and ionisation processes, radiation transport, plasma interaction with the walls. Thus, we are able to show how to approach the analysis of rather complex plasma systems with a large number of degrees of freedom, and also take into account the great diversity of the environment in which it is located. Of course, each specific model, being a part of the hierarchy, has its own field of activity and its own characteristics. We illustrate these features, mainly on the example of the three types of plasma processes: static configurations, linear oscillations and stationary flows, such as shock waves, i.e. consider zero-dimensional, one-dimensional, two-dimensional ((x_1, x_2) or (x, t)) and three-dimensional (x_1, x_2, t) models. Where there are particularly interesting situations that are outside of this triad, we describe them as well.

At the end of the book, in chapter 8, we examine several general problems, associated with oscillations in PD systems. As already mentioned, in chapter 9 we discuss a number of cosmic PD systems, and chapter 10 is concerned with the examples of advanced plasma technologies.

The wide range of the subjects investigated in the book may result in a number of inaccuracies. Unfortunately, because of subjective and objective circumstances I could not have avoided them. Therefore, I would be grateful to the readers for their understanding.

In the book, I present the portraits of the scientists who contributed greatly to the discussed areas of science and technology. Of course, I could not have mentioned all of them. Therefore, the selection of the portraits is partly subjective. I hope that others will do this better.

Finally, the number of publications on plasma physics and related disciplines is very large. I have tried to mention easily available sources that are easy to read (for example encyclopedias) and early, fundamental publications.

In a sense, the prototype structure of the Introduction is my book, ‘The physical basis of cosmic electroreactive thrusters’, volume 1, ‘Elements of flow dynamics in the ERT’ (1978). Further improvement of this structure is based on lectures on plasma dynamics that I presented in a special course for about 20 years at the Physics Department of the Moscow State University. The then head of the Department of Mathematics, Prof. A.G. Sveshnikov, who invited me to read this special course, advised me to prepare and publish an expanded version of the lectures in the book form. I am very grateful to him for that. The appearance of this book owes much also to my old friend, from my school days, Nikolai Ivanovich Dolbin, one of the founders of a new section of mechanics – magnetoelasticity. Nikolai wrote and edited many of my lectures. His untimely death in 1995 deprived me not only of a friend but also of an assistant at the final stage of the Introduction. But the memory of his enthusiasm and his work were very important for me at the end of this book.

Finally, I am grateful to my colleagues from the Kurchatov Institute, M.V. Keldysh MIREA Research Centre, and also colleagues and friends from abroad who carried out many of the experiments described in this book.

Introduction

I.1. What is plasma?

Plasma is an ionised gas. However, not every cloud of ionised gas is plasma. The main special feature of the plasma state can be explained by the following hypothetical experiment (Fig. I.1.1).

It is assumed that a photon with the energy $\hbar\omega$ falls on the single stationary atom with ionisation energy I and ionises the atom (Fig. I.1a). As a result, an electron with the energy

$$\varepsilon_1^{(e)} = h\omega - I \tag{I.1.1}$$

separates from the resultant almost stationary ion. We now consider two identical atoms, situated at a certain distance a from each other and irradiated consecutively with a small delay one atom after the other (Fig. I.1b). From the first atom, the electron is emitted as previously with the energy $\varepsilon_1^{(e)}$ but the second atom leaves with a lower energy

$$\varepsilon_2^{(e)} = \varepsilon_1^{(e)} - (\delta\varepsilon)_1 \tag{I.1.2}$$

because this atom must overcome attraction to the first ion.

Evidently

$$(\delta\varepsilon)_1 \sim \frac{e^2}{a} \tag{I.1.3}$$

If we now consider three atoms at distances $\sim a$, then in consecutive ionisation, the last (third) electron leaves with the energy

$$\varepsilon_3^{(e)} = \varepsilon_1^{(e)} - (\delta\varepsilon)_2$$

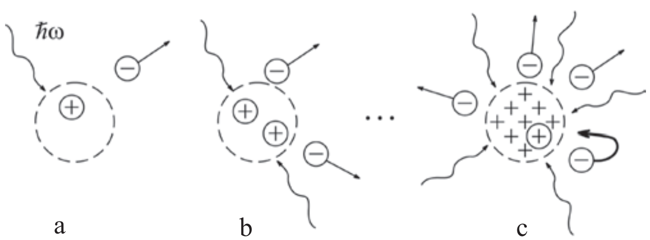

Fig. I.1.1. Transition from the set of ions and electrons to plasma.

and

$$(\delta\varepsilon^{(e)})_2 \sim 2(\delta\varepsilon^{(e)})_1 \text{ and so on}$$

Evidently, at some number of atoms N_* (we assume that the atoms occupy some spherical volume with radius a), attraction to this volume of the ions will be such that the last electron can no longer leave to infinity (Fig. I.1.1c), since the kinetic energy of this electron at the moment of ionisation is not sufficient for overcoming attraction to the already formed ions, i.e.

$$\varepsilon_{N_*}^{(e)} < 0 \tag{I.1.4}$$

Increasing the number of atoms further and repeating the procedure of consecutive ionisation, it can be seen that at the given ionisation energy I and photon energy $\hbar\omega$ up to $N = N_*$, it is not possible to increase the charge of the cloud, and the increase of the total number of the charged ions N_i in the cloud will result in a decrease of the fraction of non-compensated charges

$$\delta \equiv \frac{N_i - N_e}{N_i} \sim \frac{N_*}{N_i} \to 0 \tag{I.1.5}$$

Thus, the cloud of charged particles becomes 'quasi-neutral', i.e. N_e becomes almost equal to N_i. This quasi-neutral gas is also referred to as plasma (Langmuir, Tonks, 1923).

The strength of bonding of the electrons and ions in plasma can be evaluated on the basis of the following example. Let the volume of air in 1 cm^3 in the form of a sphere under normal conditions be fully ionised. Consequently, the volume will contain N~$5{\cdot}10^{19}$ ions and the same number of electrons. It is assumed that somebody (or something) is capable of taking all the ions in one 'hand' and all of the electrons in the other hand and we shall try to separate them from each other. Evidently, the maximum force of interaction of positive and negative 'spheres' is found at the distance between the spheres of $r \approx a$, where $a \sim 0.5$ cm is the radius of the spheres. This interaction force is determined by the Coulomb law:

$$F \approx \frac{N^2 e^2}{a^2} \tag{I.1.6}$$

Substituting into this equation $N = 5{\cdot}10^{19}$ cm^{-3}, $e = 4.8{\cdot}10^{-10}$ CGS (ESU unit), we obtain the gigantic force

$$F \approx 2.5 \times 10^{12} \text{ tonnes}$$

i.e. 2.5 trillion tonnes! However, the ionisation of this amount of gas required only ~100 J, and its mass is ~1 mg.

This unity of giant and tiny and does not contain any information on the huge possibilities offered by plasma.

Debye radius

The criterion of quasi-neutrality may be given a more constructive form. For this purpose, we estimate critical density n_*, assuming that N_* heavy particles are situated inside *a* sphere with radius *a*. In this case, the potential on the surface of the sphere will be:

$$\phi_* = \frac{N_* e}{a} \tag{I.1.7}$$

The last electron, which can travel to infinity, has the initial energy ε_1, and is determined by the condition

$$e\phi_* = \varepsilon_1 \tag{I.1.8a}$$

Or

$$\frac{N_* e^2}{a} = \varepsilon_1 \tag{I.1.8b}$$

Introducing the critical density of electrons n_*

$$N_* = \frac{4\pi}{3} a^3 n_*$$

equation (I.1.7) can be written in the form

$$\frac{\varepsilon_1}{4\pi e^2 n_*} = \theta a^2, \;\; \theta = \frac{1}{3} \tag{I.1.9}$$

The term on the left side of the equation has the dimension of the square of length. At any n and ε, this length is equal to

$$r_D \equiv \sqrt{\frac{\varepsilon}{4\pi e^2 n}} \tag{I.1.10}$$

and is referred to as the Debye radius.

Therefore, it may be assumed that the critical density at which the transition of the charged cloud to a quasi-neutral formation, i.e. to plasma, starts is the density at which the Debye radius becomes smaller than the radius of the cloud of the charged particles. Correspondingly, the cloud of the ionised gas becomes the 'actual plasma' is

$$r_D \equiv \sqrt{\frac{\varepsilon}{4\pi e^2 n}} \ll a \tag{I.1.11}$$

where n and a are the characteristic values of the density and size of the cloud.

We estimate the value of n_* for a sphere with radius a = 1 cm at the electron energy ε = 1 eV (1 eV = $1.6 \cdot 10^{-12}$ erg). Taking into account equation (I.1.8) we obtain

$$n_* = \frac{3\varepsilon}{4\pi e^2 a^2} \approx 1.7 \times 10^6 \, \mathrm{cm}^{-3}$$

Two comments should be made here.

a. As indicated by the above considerations, the criteria of the plasma state (I.1.5) or (I.1.11) are also valid in the case in which we examine a cloud in which there are also neutral atoms and molecules, in addition to the charged particles. They simply do not have any role. However, the degree of ionisation is indicated by the terms: 'fully ionised plasma' and 'incompletely ionised plasma'.

b. The quasi-neutral medium is referred normally to as plasma, if it is gaseous. However, there are quasi-neutral media which are not gaseous. They include solid (metals, semiconductors) or liquid (melts, electrolytes) conductors with moving electrically charged particles. They are now referred to as plasma-like media.

Plasma is often referred to as the fourth state of matter. This is justified, but the equation (I.1.11) shows that the transition to the plasma state is determined not only by the properties of the substance (n, ε) but also by external factors – the size of the system a. By this plasma differs at first sight from the classic states of matter: solid, liquid, gaseous, and the transitions between the states do not appear to depend on the macrodimensions of the specimen. However, in the case of small specimens, the phase transition temperature is also not constant. For example, the melting point of tin is T_m = 505 K but if the size of a piece of tin is reduced 10 nm, T_m decreases to 480 K.

This can also be said of other phase transitions of the first kind. It is evident that the important role of the macrodimensions of the plasma is explained in the first instance by the long-range character of Coulomb interactions.

In some cases it is suggested that the plasma is not the fourth state of matter because of the large size of the temperature range within which the transition from the 'neutral' to 'completely ionised gas' takes place. However, this objection cannot be regarded as important

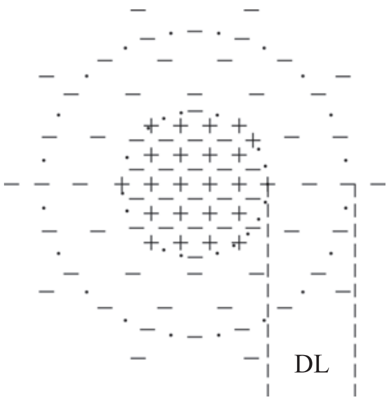

Fig. I.1.2. Distribution of charges in a plasma cloud, dispersed in vacuum, DL is the Debye layer.

because the criterion of the plasma state does not include the degree of ionisation.

Debye shells of 'free' plasma formations in the absence of the magnetic field[1] (**H** = 0)

Plasma formations – plasma blobs in vacuum, in a dense atmosphere, in vessels, usually have a 'shell' in which the quasi neutrality is violated. These layers will be referred as 'Debye layers'. They are often referred to as 'double' layers or 'Langmuir layers'.

To understand the nature of formation of these layers, it is assumed that a plasma cloud is located in vacuum. The cloud can be produced by, for example, evaporation of a piece of solid hydrogen with its subsequent ionisation as a result of radiation from all sides by laser radiation. What takes place subsequently? Assuming that the random energy of the ions is of the order or lower than the electron energy, it can be seen that the electrons, as the particles with higher mobility, start to leave the cloud and rapidly produce a deficit of electrons of the order N_* in the surface layer of the plasma volume. This deficit is determined by the equations (I.1.7) and (I.1.8) and further departure of the electrons is interrupted (Fig. I.1.2). Finally, the electrons will continue to be emitted but they will be pulled into the cloud by its positive charge, and only the faster electrons which form in a considerably smaller number of cases (as a result of collisions) will leave the volume.

Consequently, an almost vacuum field will form at a large distance from the cloud ($r \gg r_D$):

$$\phi \sim \frac{eN_*}{r}$$

[1]In the presence of magnetic fields, the situation is greatly complicated and this will be discussed in section 3.5.

In the area around the surface, within the limits of the layer with the thickness

$$r_D = \sqrt{\frac{\varepsilon}{4\pi e^2 n}}$$

there is a layer of non-compensated ions – this layer may also be referred to as the Debye layer, and in the depth of the cloud the plasma will be simply neutral.

Further evolution of the cloud, as shown in chapter 3, is identical with the expansion of a gas sphere with temperature $T_{ef} = T_i / \overline{z}_i + T_e$, where $\overline{z}_i$ is the mean ion charge.

The Debye layer at dielectric walls

It is well known that a discharge burns in the so-called daylight lamps and its main volume is occupied by the slightly ionised plasma formed in this case (for more details see section 10.2).

Its electrons have a temperature of ~*2* eV and rapidly move in all directions, including the direction to the wall. At the same time, because of collisions with neutral atoms, the ion temperature is close to room temperature and the velocity of the ions is more than four orders of magnitude lower than that of electrons. Consequently, the internal surface of the tube is rapidly negatively charged. At the same time, the velocity of travel of the electrons to the wall rapidly decreases down to the velocity of travel of the ions (Fig. I.1.3).

Thus, a regime is established in which:

$$j_n^{(i)} = j_n^{(e)}(1 - \sigma) \tag{I.1.12}$$

Here j_n is the density of the flux reaching the wall, and σ is the coefficient of secondary electronic emission of the wall, i.e. the number of electrons emitted by the wall under the effect of a single incident

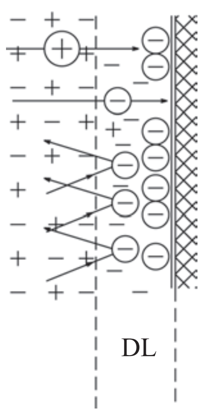

Fig.I.1.3. The Debye layer on a dielectric wall.

electron. The ions and electrons arriving at the wall recombine with each other. In (I.1.12) it is assumed that $\sigma < 1$. The case in which $\sigma > 1$ is investigated in chapter 7.

The field of the electrons 'sitting' on the wall penetrates into the plasma volume again to the thickness $\sim r_D$. This can be explained approximately as follows. The Maxwell equation for the electrical field around a flat wall has the form:

$$\frac{d^2\phi}{dx^2} = -4\pi e(n_i - n_e) \tag{I.1.13}$$

It is assumed that the electrons are described by the Boltzmann distribution

$$n_e = n_0 \exp\left\{\frac{e\phi}{kT_e}\right\}$$

and we restrict ourselves to calculations of only the long-range zone where it may be assumed that $\phi \ll kT_e$, and the ion density is constant. Consequently

$$n_i = n_0,\; n_e \approx n_0\left(1 + \frac{e\phi}{kT_e}\right)$$

and equation (I.1.13) is simplified to the maximum extent

$$\frac{d^2\phi}{dx^2} = \frac{4\pi e^2 n_0}{kT_e}\phi \tag{I.1.14a}$$

Consequently, the electrical field, generated by the wall, rapidly decreases in intensity with increase of the distance from the wall

$$\phi = \phi_0 \exp\left\{-\frac{x}{r_D}\right\},\quad r_D = \sqrt{\frac{kT_e}{4\pi e^2 n_0}} \tag{I.1.14b}$$

Therefore, in many cases, the Debye radius is regarded as the screening radius. In the above considerations, the daylight lamp acted as a suitable example. The resultant equation (I.1.14b) is valid always if $\sigma < 1$, and the wall is insulated.

Langmuir frequency

In addition to the Debye radius, another characteristic parameter of plasma is the so-called Langmuir or plasma frequency[2]. The simplest model which makes it possible to derive an equation for this frequency is constructed using the following procedure (Fig. I.1.4). We take a

[2]In fact, there may be large differences between these two terms, but they are often regarded as synonymous.

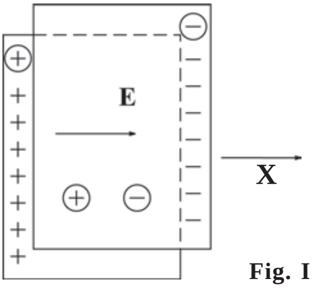

Fig. I.1.4. Block model of Langmuir oscillations.

plasma layer and treat it as a set of two non-deformed layers–blocks, electronic and ionic. If the electronic block is moved by x_e, and the ionic block by x_i, charges with the density $\pm q$ appear on the sides of the plasma sheet, where

$$q = en_0(x_i - x_e)$$

These charges generate an electrical field with the strength

$$E = -4\pi en_0(x_i - x_e) \tag{I.1.15a}$$

in the volume.

It is assumed that the thicknesses of the projecting part of the electronic and ionic layers are negligibly small. The appearance of the field (I.1.15a) in the quasi-neutral volume of the layer results in the oscillation of the electronic and ionic blocks in relation to each other. Evidently, they are described by the equations (related to 1 cm^2 of the surface area of the layer)

$$(m_e n_0 l)\frac{d^2 x_e}{dt^2} = (en_0 l)4\pi en_0(x_i - x_e) \tag{I.1.15b}$$

$$(Mn_0 l)\frac{d^2 x_i}{dt^2} = -(en_0 l)4\pi en_0(x_i - x_e). \tag{I.1.15c}$$

Here l is the thickness of the layer, m is the electron mass, M is the ion mass. Shortening in these equations $n_0 l$ and deducting the second equation from the first one gives

$$\ddot{\xi} = -\omega_0^2\xi,$$

where $\xi = x_i - x_e$, and

$$\omega_0^2 = 4\pi e^2 n_0\left(\frac{1}{m} + \frac{1}{M}\right). \tag{I.1.16}$$

The frequencies ω_{0e} and ω_{0i} determined by the equations

$$\omega_{0e}^2 = \frac{4\pi e^2 n_0}{m}, \ \omega_{0i}^2 = \frac{4\pi e^2 n_0}{M}, \tag{I.1.17}$$

are referred to as the electronic and ionic Langmuir frequencies, respectively, and the frequency

$$\omega_0 = \sqrt{\omega_{0e}^2 + \omega_{0i}^2} \tag{I.1.18}$$

is the plasma frequency. Evidently, with a high accuracy $\omega_0 \approx \omega_{0e}$.

We estimate the scale of ω_{0e} using the density $n = 10^{14}$ cm^{-3} characteristic of future thermonuclear reactors. Substituting into (I.1.17) we obtain

$$\omega_{0e} \approx 5 \cdot 10^4 \sqrt{n_0} \approx 5 \cdot 10^{11} \mathrm{s}^{-1}.$$

Evidently, the electronic block oscillates around the almost stationary ionic block. The amplitudes of these oscillations relate to each other as M/m.

If the amplitude of the velocity of the electronic block is V_{em}, than because of the harmonic form of the oscillations its maximum displacement is

$$D \approx \frac{V_{em}}{\omega_{0e}} = \sqrt{\frac{mV_{em}^2}{4\pi e^2 n_0}}. \tag{I.1.19}$$

This value, with the accuracy to the multiplier ~1, coincides with the equation for the Debye radius (I.1.10).

I.2. Region of rarefied non-relativistic plasma in the coordinates *n*, *T*

We now examine how large the ‘ocean’ of plasmas in which we are interested is and the ‘shores’ which it sweeps. This area is defined by three conditions.

1. As already mentioned in the preface, in this book we examine only rarefied plasmas[3] in the sense that the kinetic energy of their particles $(3/2)kT$ is considerably higher than the potential energy of the pair of the particles at the mean distance $r_0 \sim n^{-1/3}$

$$\frac{e^2}{r_o} \ll \frac{3}{2} kT. \tag{I.2.1a}$$

[3]We shall use this term instead of the widely used term ‘ideal’ gas (‘ideal plasma’), and the word ‘ideal’ will be used as the synonym of dissipation-free synonym.

If we introduce the Debye radius in the form (I.1.15b), the condition (I.2.1a) has the form

$$N_D \gg \frac{1}{20}, N_D = \frac{4\pi}{3} r_D^3 n_0;\ r_D = \sqrt{\frac{kT}{4\pi e^2 n_0}} \tag{I. 2.1b}$$

where N_D is the number of particles in the Debye sphere. For example, at $n_0 = 10^{13}$ cm^{-3} and $kT = 10^4$ eV, we have $N_D \sim 3 \cdot 10^8$. It should be stressed that the number of particles in the Debye sphere N_D is proportional to $n_0^{-1/2}$, i.e. increases with the decrease of plasma density n_0.

2. Both the electronic and ionic components are assumed to be non-relativistic (c is the speed of light)

$$kT_e \ll mc^2,\ kT_i \ll M_i c^2 \tag{I.2.2}$$

For the electrons this restriction means that the temperature in the case of, for example, Maxwell's distribution, in a tokamak should be considerably lower than 500 keV and, in particular, as shown by the calculations of radiation, should be lower than 50 keV.

3. Finally, it is assumed that the plasma is of the non-quantum nature, i.e. the mean distance between the particles is

$$r_0 \approx n_0^{-1/3} \gg \lambda_\mathrm{B} \tag{I.2.3}$$

where $\lambda_\mathrm{B} = h/mv$ is the de Broglie wavelength.

Regardless of these restrictions, the parameters of the investigated plasma change in a very wide range. Information on this is provided by Fig. I.2.1, in which the graph in the coordinates T (characteristic substance temperature) and n (concentration of the substance particles) indicates the regions related to the rarefied classic plasma (it is crosshatched) and the adjacent media[4].

We characterise briefly the regions shown in Fig.I.2.1, moving from bottom to top. It can be seen immediately that the boundaries between the regions are conditional and depend on the specific properties of the matter and external conditions. The region I ($T \lesssim 0.2$ eV) is the region of neutral (non-ionised) gases and solids. For every specific substance the region is 'spotted', because the density of the saturated vapours usually differs by an order of magnitude from the density of the condensed liquid or solid. It should be mentioned that the entire set of the technologies of mankind from stone hammer to space technology originated from this region.

[4]After all, any plasma formation is characterised by a large number of parameters. However, these parameters cannot be listed in a convenient manner and, therefore, we must restrict ourselves to two coordinates.

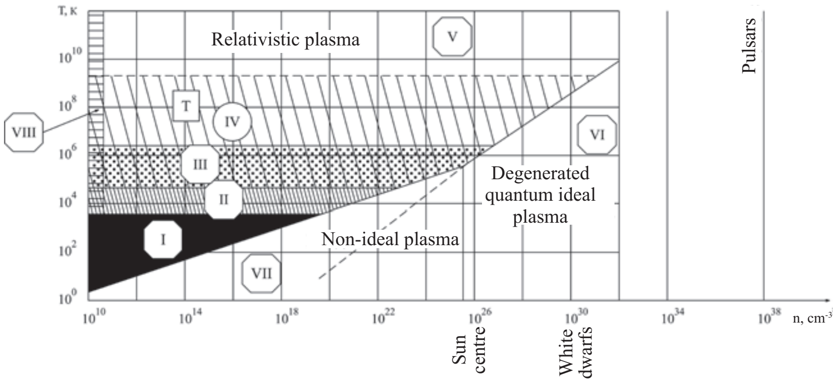

Fig. I.2.1. Diagram of the regions of rarefied classic plasma and of surrounding media in the temperature–density coordinates.

In this region, the energy of the particles is limited by the level of the energy of chemical bonds, and density by the density of the solid (n_{max} ~10^{23} cm^{-3}).

Above this region, there is region II – ‘low-temperature plasma’ ($T_i \sim T_e \lesssim 5$ eV) with the highest population in the temperature range ~0.3–0.6 eV and the degree of ionisation is not high (from fractions to units of percent). This region includes the plasma of the classic discharges (arc, glow, etc) and also of the surface layers of the Sun. When the density increases to 10^{18}–10^{19} cm^{-3}, the plasma becomes ‘dense’, i.e. the condition (I.2.1a) is violated and we transfer to the region VII.

This is followed by the region III ($5 \lesssim T \lesssim 1000$ eV). This is the region of completely ionised plasma with ‘mean’ energy. In specific cases, it is necessary to separate the two sub-regions. They will be referred to as III$_z$ and III$_0$. Region III$_0$ is the region in which all the ions have lost their electronic shells so that they represent Coulomb centres. The simplest example are hydrogen ions. However, if the charge of the nucleus is $Z \gg 1$, inner shells are retained in most cases in the ions. They form the region III$_z$. Therefore, the Coulomb interaction between these ions takes place here only during their interactions at relatively large distances. At short distances, the nature of interaction becomes more complicated (chapter 6). The boundary between the regions III$_z$ and III$_0$ depends on the type of substance. With the increase of density, especially in the case of relatively high values of z, we also arrive in the region of ‘dense’ plasma.

The next region is region IV ($10^2 \lesssim T \lesssim 5\cdot10^4$ eV). This is the region of ‘hot’ plasma. In the laboratory conditions, this plasma can be produced in the stationary form only on hydrogen and light elements

(He, Li) as a result of powerful bremsstrahlung. The plasma of the light elements in this energy range is of considerable interest for the problem of controlled thermonuclear fusion (see section 10.5). It should be mentioned that the temperature in the centre of the Sun is of the order of 1000 eV, but the density (mostly hydrogen) is of the order of 100 g/cm^3, i.e., n_0~$7 \cdot 10^{25}$ cm^{-3}.

Boundary regions will now be briefly discussed. Region V is the region of relativistic plasma. Initially, the electrons become relativistic here. This plasma in the presence of the magnetic field generates synchrotron radiation due to centripetal acceleration.

In the cosmic conditions, the synchrotron mechanism is responsible for radio emission of the stars, including pulsars. If the electron-relativistic plasma formation is large, and gamma radiation is suppressed to a large degree, then at T_e ~1 MeV we observe the birth of electron–positron pairs. Evidently, this atmosphere exists in the pulsars (neutron stars).

At the present time, the stationary formations, which can be regarded as ion-relativistic plasma, are produced in laboratories. In fact, as already mentioned in the preface, unique 'traps' are used in the technology of accelerators of charged particles, i.e. storage rings into which the particles, accelerated to high energies, are introduced. If particles and antiparticles are injected into such a ring with a transverse magnetic field, they will move against each other. Consequently, it is possible to examine the collisions of particles at ultra-relativistic energies. These storage rings are referred to as colliders. Evidently, a quasi-neutral ion–relativistic medium is produced in such a single-ring collider at equal concentrations of the particles and antiparticles. In particular, this assumption is valid if we use a collider consisting of two storage rings with ions of the same sign. Consequently, the density of the protons in the zone of intersection of the fluxes of the protons moving in the opposite directions (with the energy of up to ~10^{12} eV) reaches 10^8 cm^{-3}.

This zone is a unique volume of the non-isothermal (i.e. $T_e \ll \varepsilon_i$) plasma with a highly anisotropic function of the distribution of the ions.

The region VI is the plasma with the quantum degeneration of the electrons. In other words, here in contrast to (I.2.3) the distance between the particles of the order of the de Broglie wavelength is:

$$r_0 \sim \frac{1}{n_0^{1/3}} \sim \lambda_{\mathrm{B}}. \tag{I.2.4}$$

Therefore, the electron energy is not characterised by temperature. but, because of the Pauli principle, it is characterised by the Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}. \tag{I.2.5}$$

In other words, if $kT < \varepsilon_F$, then in the rarefaction criterion in (I.1.18) we should use ε_F and not kT. Consequently, in the range of high n, the electronic component of the ‘rarefied’ plasma is ideal if:

$$\frac{e^2}{r_0} \ll \varepsilon_F. \tag{I.2.6}$$

Region VII is the region of the classic dense plasma. It is characterised by high density and low temperature. In the laboratory conditions, in most cases the density $n_0 \gtrsim 10^{19}$ cm^{-3}, $T \lesssim 1$ eV. A suitable example of the systems with the non-ideal plasma are discharges at high pressures. As regards the properties (structure), the non-ideal plasma is closer to liquids than gases. Taking into account the non-ideal state criterion

$$e^2 \sqrt[3]{n} \sim \frac{3}{2} kT,$$

it may be seen that the boundary line between the classic rarefied and dense plasma in the graph I.2.1 is described by the straight line

$$\ln n = 3 \ln T + \text{const}.$$

The lower boundary of quantum plasma as regards density is determined by the condition ($r_0 = n^{-1/3}$):

$$\frac{e^2}{r} = e^2 \sqrt[3]{n} = \varepsilon_F$$

and depends only on density. It is the critical density $n^* \approx 5 \cdot 10^{25}$ cm^{-3} and corresponds to $\varepsilon_F \sim 30$ eV. It should be mentioned that the density $n^* \approx 5 \cdot 10^{25}$ cm^{-3} is at present produced in systems with laser compression of the targets. It may be seen that the transition to the quantum range takes place at a density considerably higher than the rarefied–dense plasma transition in the classic situation. Therefore, at $kT \lesssim 30$ eV, the boundary of rarefied plasma is determined by the classic criterion (I.2.3).

Finally, region VIII in this graph, close to the vertical axis, corresponds to the set of charged particles. In reality, there are no restrictions on energy (protons with the energy of $>10^{12}$ eV have already been produced in accelerators), but these sets are restricted in density because of the volume charge.

Outside the investigated regions there is the white dwarf plasma (n_0 >10^{30} cm^{-3}), the neutron matter of pulsars and quarks-gluon plasma of black holes. However, this is far away from the region of 'rarefied plasma' in which we are interested.

Figure I.2.1 shows the region T – thermonuclear plasma in stationary magnetic traps.

The figure shows how wide is the range of the parameters which relate to the classic rarefied plasmas. It is clear that the prospects for the practical application of plasma in this range are enormously encouraging.

I. 3. History of plasma investigations [50, 52]

I.3.1. Investigations up to the 30th of the 20th century

The origin of plasma investigations belongs in the middle of the 18th century. At that time, precursors of electrophoretic machines and Leyden flasks were produced and were used to entertain the public on squares or palaces[5] with different electrical phenomena and, in particular, spark discharges.

At the same time (1750–1752), B. Franklin 'tamed' the lightning in 1769 the Grand Duke of Tuscany ordered to put lightning rods at all about gunpowder stores of the Duchy. The attitude to electricity became serious.

The development of galvanic elements by Volta resulted in the discovery of the electrical arc at the very beginning of the 19th century (Petrov, Davy). In the second half of the 19th century, the electric arc was used in completely new practical applications: light source (Yablochkov, 1870), welding tool (Benardos, Slavyanov, 1880).

The development of technological applications of arc charges was accompanied, in the middle of the 19th century, by laying the foundations of plasma physics. This was associated with the examination of the electrical properties of materials, initially in the form of Faraday electrolysis laws (1830) and subsequently by influencing fluxes of particles (on the cathode and channel beams) of magnetic and electrical fields in electric discharge tubes with a low gas pressure. These are discharges were detected for the first time by Faraday and subsequently by Geisler in many variants (1850s), but detailed investigations of the detected 'cathode rays' started in the 1870s. In particular, W. Crookes proved that they represent particle fluxes. This

[5]In a series of portraits of students of the Smol'nyi Institute, produced by Levitskii in 1776, there is a portrait of E.I. Molchanova next to an electrical machine.

lecture, presented in 1879 at a meeting of the Royal Institute, had the name: 'On radiant matter or the fourth aggregate state'. Presenting convincing experimental data, indicating the corpuscular nature of cathode rays, Crookes concluded: 'In examination of this fourth state of matter it becomes apparent that we have finally at our disposal the 'final' particles which can be justifiably regarded as forming the base of the physics of the universe.... We have entered here the region in which the matter and energy appear to be merged into a single unit... I would like to assume that the main problems of the future will be to find 'solutions' in this region'.

Indeed, this was a truly brilliant foresight!

However, approximately 16 years were required for the final acceptance of the electrons and this took place in 1895 due to Thompson who determine the ratio e/m for cathode particles. In physics, the word 'electron' appeared in 1891, and after 1900 was given todays meaning. This resulted in the birth of electronics and the physics of plasma or, more accurately, gas discharges.

Subsequently, the events progressed very rapidly: the Planck equation was derived for radiation (1900); Rutherford experiments, describing the atomic nucleus[6], were formulated (1911); the Bohr atom model was constructed (1930), quantum mechanics by Heisenberg–De Broglie–Schrödinger–Dirac (the first half of the 1920s). Overall, the foundations of plasma physics were laid.

However, intensive work was also carried out on the development of the technology of charged particles: mass-spectroscopy (Thompson, 1912, Aston, 1920); the first oscilloscope ('Brown tube'), electronic and radio valves (De Forest), electrostatic accelerators of charged particles (van de Graaf, 1931), etc.

Work also started on the development of plasma technology. In 1908, 'neon tubes' – Geissler discharges, were used for advertising purposes. Work also started on plasma switches: mercury rectifiers, tyratrons, arc extinguishing chambers for breakers of powerful electrical circuits, development of lamps with high illumination capacity based on air discharge, etc.

All this development was important but, on the whole, occurred at the periphery of advances in physics. There is no adequate term for this new region, i.e. gas discharge physics: arc, glow, spark, etc. Up to the middle of the 1920s, only studies by Townsend (1910s) were regarded as fundamental and concerned with the mobility of charged particles in slightly ionised gases. Investigations carried out

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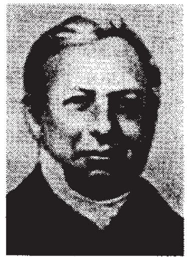
[6]By the way, the spinthariscope with which Rutherford made his observations, was invented by the same Crookes.


D. Bernoulli


L. Euler


G. Liouville

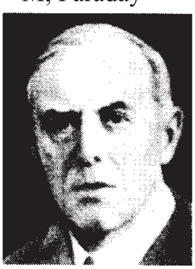

L. Boltzmann

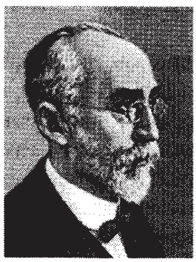

M, Faraday

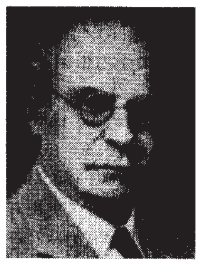

D. Maxwell


W. Crookes


G. Tomson


G. Townsend


H. Lorenz

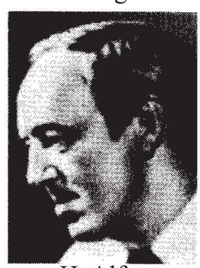

E. Hall


I. Langmuir

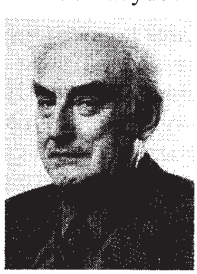

M. Saha


B.I. Davydov


A.A. Vlasov


H. Alfven


Ya.B. Fainberg


V.L. Ginzburg


Ya.B. Zel'dovich


A.L. Chizhevskii

Founders of hydro- plasma dynamics

by Townsend into the effect of the transverse magnetic field on the mobility of electrons in the electrical field proved to be very important for future investigations. Townsend showed, both theoretically and by experiments (in the case weak ionisation!) that the mobility of the electrons is inversely proportional to the square of the strength of the transverse magnetic field

$$\mu_{\perp}^{(e)} \sim \frac{1}{H_{\perp}^{2}} \tag{I.3.1}$$

This dependence is referred to as the 'classic' dependence. However, only in the 60s (see below) it was possible to apply this dependence in the experiments in the volume of the efficiently ionised and relatively dense plasma.

The separation of plasma physics from the physics of discharges is usually attributed to I. Langmuir. His fundamental studies in this area were carried out in the 1920s and were characterised by three fundamental achievements.

Firstly, he developed a completely new vacuum technology - diffusion pumps which made it possible to produce efficiently high and pure vacuum. Secondly, Langmuir perfected the theory and techniques of measuring the plasma parameters using electrostatic probes. These devices made it possible to determine, on a completely new level, the spatial distribution of electronic temperature (T_e), electrical potential (ϕ) and the density in plasma configurations. Therefore, it was not surprising that the electrostatic probes were referred to as 'Langmuir' probes, although the first probing experiments were carried out in 1887 by Lecher.

Thirdly, Langmuir and Tonks were the authors of a number of theoretical studies, in particular, the discovery of specific plasma oscillations with 'Langmuir' frequency

$$\omega_0 = \sqrt{\frac{4\pi e^2 n}{m}}$$

Finally, Langmuir introduced the concept of 'plasma' and this was used to define this state of matter which was defined previously.

1920s are also characterised by the development of the theory of thermodynamically equilibrium plasma based on Saha's equation (see section 6.5.7). Important achievements of this period were the discovery of the ionosphere of the Earth (Heaviside) and the initial attempts for simulation of the Northern polar lights (chapter 9).

I.3.2. Investigations and developments in 1930s and 1940s

The 30s of the previous century were regarded as the years of evolution development of experimental investigations and rapid development of theory. In astronomy, spectral methods were improved and the characteristics of cosmic objects were determined more accurately. Radio physicists described the structure of the ionosphere of the Earth and constructed models of propagation of radio waves in the ionosphere. In the laboratories, using mainly the Langmuir probe method (improved by D. Bohm and M.J. Druyvestein) and, to a lesser extent, spectroscopy, detailed investigations were carried out on different varieties of discharges. This was the basis for the development of a number of gas discharge devices. Of the completely new devices, developed in that period, it is necessary to mention the 'Penning' cell – one of the first gas discharge devices with the magnetic field which is still used at present, with the field magnetising the electrons (more details on this class of devices are presented in section 10.6). However, most significant advances took place in the theory. This included the construction of the effective models of fragments of classical discharges which are in good agreement with experiments (Schottky, Engel, Mecker, etc). Work also started on the general kinetics of plasma. In 1936, L.D. Landau, modifying the collisional term of the Boltzmann kinetic equation for Coulomb interactions, derived the collisional Landau term. These equations are discussed in chapter 5. At the same time, B.I. Davydov derived, on the basis of the kinetic Boltzmann equation, for slightly ionised plasma, the adequate Boltzmann–Davydov collisional term and derived the function of distribution of the electrons in the presence of a uniform electrical field. Finally, in 1938, A.A. Vlasov, ignoring the Liouville equation, formulated his system of equations with a self-consistent field which reflects most adequately the kinetics of completely ionised plasma. The Vlasov (or, more accurately, Vlasov-Maxwell) equations are the subject of chapter 4.

The comparatively smooth although quite rapid development of plasma physics in the 1930s was completely interrupted in the 1940s, mostly because of the Second world war. Its effect on the development of plasma physics and technology was huge. This was associated with the development of microwave technology for radio location and its basis – powerful generators of the waves in the centimetre and decimetre range, waveguides and super-sensitive (at that time) receivers. These achievements resulted in the development of radio-diagnostics of plasma, generation of microwave discharges and radio astronomy which provided invaluable information on the plasma processes taking place in the universe.

Of even greater importance in this area was the work on the development of the atomic bomb. It is important to mention three moments:

a. One of the stages of solution of this problem was the development of equipment for 'electromagnetic' separation of uranium isotopes U^{238} and U^{235}. For this purpose, it was necessary to develop powerful sources of ions and high-productivity separators capable of working with high ionic currents. These beams had to be quasi-neutral, i.e. represent plasma formations.

b. Investigations of the behaviour of plasma in the gas discharge source of ions (carried out using the Penning cell) and in the separator resulted in a conclusion that the Townsend law (I.2.6) does not work and, consequently, D. Bohm proposed – on the basis of processing experimental data – equations (the so-called scalings) for the transfer coefficient which differed from the classic equations by substituting the duration of the free path of the electron τ_e by the quantity

$$\tau_{\mathrm{B}} = \left(\frac{2\pi}{\omega_{\mathrm{H}}}\right) \tag{I.3.2}$$

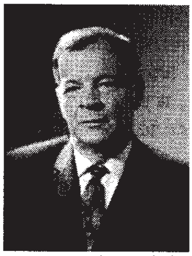
Thus, in particular, the diffusion coefficients – classic (Townsend) and Bohm's, have the form ($\omega_e\tau_e \gg 1$):

$$D^{(\mathrm{C})} = \frac{mc^2 k(T_i + T_e)}{e^2 H^2 \tau_e}, \quad D^{(\mathrm{B})} = \frac{ckT}{16eH} \tag{I.3.3}$$

It may be seen immediately that $D^{(\mathrm{B})}$, at relatively high T and H, is many orders of magnitude higher than $D^{(\mathrm{C})}$. If the 'Bohn diffusion' were fatal, it would not be possible to develop magnetic traps for controlled thermonuclear fusion (CTF) and the majority of plasma accelerators.

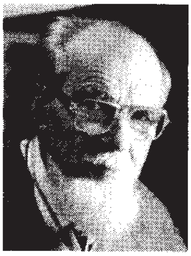
Fortunately, in the 60s, it was established that Bohm diffusion is determined by anomalous processes in the plasma, i.e. this type of diffusion is not universal, and the transfers can be reduced to the level determined by the classic equations, i.e. Townsend equations. However, to determine and confirm this fact, more than 50 years of work of scientists was required because in the initial stage when the simplest plasma systems were examined, the Bohm equation reasonably corresponded to the experimental data obtained in greatly differing systems. In any case, this equation represented a unique 'reference point' and


O.A. Lavrent'ev

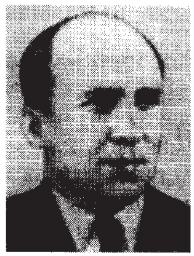

L.A. Artsimovich


M.A. Leontovich


L. Spitzer


N.V. Fillippov


S.I. Braginskii


I.M. Gel'fand


A.B. Mikhailovskii

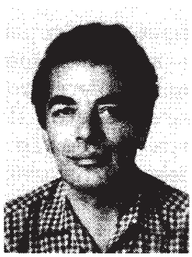

G.I. Budker


M. Rosenbluth

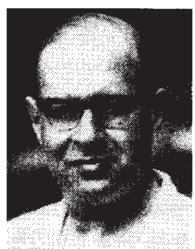

M.S. Ioffe

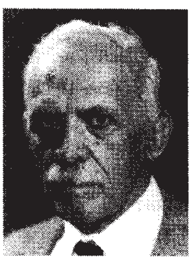

N.G. Basov


V.D. Shafranov


L.S. Solov'ev


R.Z. Sagdeev


M.S. Rabinovich


I.N. Golovin


N.A. Yavlinskii


M.I. Guseva


S.V. Mirnov

Pioneers of controlled thermonuclear synthesis this obviously contributed to the systematisation of experimental facts.

c. Finally, the development and investigations of the electromagnetic method of separation of the isotopes proved to be an excellent school, and its principals in Russia later headed experimental investigations of controlled thermonuclear fusion (L.A. Artsimovich, etc).

Explosions of the atomic bombs resulted in the formation of a plasma with very high parameters. Outstanding scientists Ya.B. Zel'dovich, I.E. Tamm, A.D. Sakharov, and others) were invited to investigate the processes of explosion of the atomic bombs and the resultant shock waves and these investigations became a stimulus for the investigations of advanced low-temperature plasma systems with multi-electron atoms.

I.3.3. Investigations in the 1950s and 1960s. Problem of controlled thermonuclear fusion

The problem of the hydrogen bomb provided an impetus for the concept of controlled thermonuclear fusion[7] and, at the same time, the development of a large new area of plasma investigations – physics of ‘hot plasma’, i.e. plasma with a temperature of ~10 keV = 10^8 K, in ‘peaceful’ laboratory conditions.

These temperatures are essential to ensure that in a reactor two nuclei of light elements – in the simplest case the nuclei of the hydrogen isotopes deuterium ($D \equiv {}^2H$) and tritium ($T \equiv {}^3H$) can overcome the coulomb repulsion, coming together to distances of $\sim 10^{-13}$ cm over which the effect of nuclear forces operates, and merge, forming a helium nucleus (α-particle) and a neutron:

$$D + T \rightarrow {}_2He^4 + n + 17\ \text{MeV}$$

The reaction is accompanied by the generation of a large amount of energy. It is very important to note that in any water from a puddle to an ocean every ~5000 hydrogen atoms are associated with 1 deuterium atom. In other words, in contrast to gas, oil, coal, the reserves of thermonuclear fuel are almost inexhaustible because synthesis can also take place by the D + D scheme (for more details see 10.5). At the same time (the end of the 1950s) a road to ‘energy Eldorado’ was also shown: it is necessary to avoid using energy-absorbing walls (dense gases, liquids, glass tubes, etc.) and transfer to sustaining the plasma by electromagnetic fields.

[7]Explosion – uncontrolled synthesis – takes place in the hydrogen bomb.

The problem of controlled thermonuclear fusion in Russia is described by Academician Yu.B. Khariton, the scientific supervisor of the Centre for the Development of Nuclear Weapons, who knew well events of that period: 'Igor' Evengen'evich (Tamm)...did not ignore a letter he received in the summer of 1950 ... from the unknown Oleg Lavrent'ev, who served as a sergeant in the Far East Sakhalin military area. The author, a self-taught scientist, proposed to use the system of electrostatic thermal isolation for producing high-temperature deuterium plasma. Tamm asked young Sakharov to examine Lavrent'ev's idea. Later, Sakharov wrote that this 'proactive and creative person raised the issue of colossal importance'[8]. Very soon, it became clear to Sakharov that there are real possibilities offered by the application of magnetic thermal isolation. He and Tamm started detailed calculations[9]. This shows clearly the role of O.A. Lavrent'ev in the births of the problem of controlled thermonuclear fusion [22].

In addition to the concept of controlled thermonuclear fusion, the end of the 1940s and the beginning of the 1950s were marked by another important event. This period was characterised by the publication by the Swedish astrophysicist H. Alfvén of 'Cosmic Electrodynamics', with the author completely unknown to a large number of plasma scientists [21]. In his book, the author investigated the self-consistent dynamics of ideally conducting plasma and magnetic field. This book showed the significant role of electromagnetic processes in the space, and the proposed model of plasma dynamics, referred to as magnetic hydrodynamics (MHD), was immediately regarded as highly attractive because of its novelty and 'beauty'. Consequently, the astrophysical processes on huge scales, taking place with the velocities measured in hundreds and thousands of km/s, including solar corona, protuberances, magnetosphere of the Earth, etc, became accessible for serious theoretical analysis. Therefore, it is not surprising that giants such as Fermi and Chandrasekhar also provided a contribution to the development of MHD models very soon after publication of Alfvén's book. MHD rapidly found its application also on the Earth because of investigations into controlled thermonuclear fusion and similar tasks. Magnetic hydrodynamics is the subject of the main volume of chapter 2 and part of chapter 9 of this book. In his book, Alfvén also proposed an extremely effective 'drift' approach (section 1.2) for describing the dynamics of single particles in slightly non-uniform fields.

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[8]A. Sakharov, Memories. New York, Chekhov Publishing House, 1990, p. 186.

[9]Yu.B. Khariton, et al., I.E. Tamm through the eyes of physicists of Arzamas-16, Memories of I.E. Tamm, ed. E.L. Feinberg, Moscow, 1995.

Thus, at the very beginning of the 1950s, plasma physicists had at their disposal a basis for the kinetic and hydrodynamic description of classic discharges and plasma rapidly heated by ‘nuclear’ explosions, and also the ideology of the dynamics of strongly heated plasma of cosmic objects, with sustainment of this plasma in the laboratory conditions using electromagnetic fields. Consequently, the 1950s and 1960s became the years of inventions of the circuits of greatly differing devices and a large number of experiments for realisation of these phenomena were conducted.

This activity was especially strong as a result of the problem of controlled thermonuclear fusion, whose physical model of the end grandiose prospects[10] excited leading physicists and, most importantly, administrators on higher levels, dealing with financing. As a result, on May 5, 1951, I.V. Stalin signed the Decree of the Government on the Organization of a magnetic fusion reactor (MFR). This day is now regarded as the start of work on controlled thermonuclear fusion in the former USSR.

The leader in the area of investigations in the area of controlled thermonuclear fusion in the former Soviet Union was The Institute of Atomic Energy (IAE), directed by I.V. Kurchatov[11]. The direct supervisor of work on controlled thermonuclear fusion at the IAE at the beginning of the 1950s was L.A. Artsimovich. In his department, the theoretical sector, headed by M.A. Leontovich, was soon formed. The initial scheme of MFR – magnetic thermonuclear reactor (the name used by Sakharov) – was not capable of operation. Therefore, Artsimovich proposed to start investigations of high-current (J_p ~100 kA) electrode pulsed (τ_p ~ 1 μs) discharges in straight dielectric tubes, filled with deuterium at a pressure of ~1 torr. Later, these devices were referred to as *Z*-pinches because it was found that a rapid increase of current is accompanied by the formation of a skin layer with high conductivity and the magnetic fields surrounding it constrict this plasma ‘tube’ which collects the ionising gas (for more details see section 1.6).

As a result, when the pinching sheath reaches the axis, there is a short time (~ 0.1 μs) when a dense (n ~ 10^{19} cm^{-3}) and hot (T ~ 100 eV) plasma forms. Moreover, in the summer of 1952 it was found in experiments in the system constructed by N.V. Filippov that neutrons form at a certain point in the pinch. It seemed that the problem of UTC was solved in principle. But it soon became clear that the generation

[10]Processing in a fusion reactor of the 0.02% deuterium contained in any natural water supplies generates 300 times more energy than an equal volume of gasoline.

[11]I.V. Kurchatov was the supervisor of the entire nuclear power program of the country since 1945. Under his leadership the former USSR was the first country in Europe to construct a nuclear reactor, develop the first atomic bomb, and launch the first nuclear power plant.

of neutrons is not due to randomly moving (heated) particles but to particles accelerated in electric fields arising in the process of discharge. And, more than that, all attempts to raise the density and temperature of the plasma at the time of maximum pinch, and thus increase the neutron yield, were unsuccessful[12]. Soon became clear reason for the failure. It turned out that in the compressed cord develops instability, limiting the compression ratio (see § 1.6).

For the first time there was a terrible scourge of all the schemes of magnetic confinement of the hot plasma – instability. Hopes to solve the problem by using *Z*-pinches of different variations were finally extinguished in 1957. At this time there was a literally whooping search for new schemes of plasma 'traps'. Most exotic schemes were proposed, but the clear favorite in the late 50's were the so-called 'mirror trap' ('mirror cells', 'open traps', 'mirror traps'), proposed in the USSR by G.I. Budker (1954) and in the U.S. by R. Post (1952–53). More details are described in section 1.7.

Although shortly theorists (Rosenbluth, Longmayer, Kadomtsev) predicted the inevitability of a strong convective instability in mirror traps, they also proposed methods to combat them (section 1.7), which was confirmed in the brilliant experiments by M.S. Joffe (IAE, 1961). But the still remaining 'weak' instability and 'openness' of these traps led to large losses. Expectations related to the mirror cells also started to melt.

After these failures the late 50's saw the beginning of serious work on toroidal discharges in a strong magnetic field. Thus, tokamaks[13] were born. Their differences from the early toroidal discharges in England, the USA and the USSR were associated with a strong longitudinal magnetic field (see section 10.5). The need for such a strong field to stabilise the helical instability was predicted by theorists (V.D. Shafranov, M. Kruskal).

Soon modelling experiments by N.A. Yavlinsky confirmed the Kruskal–Shafranov criterion and Yavlinsky also constructed the first tokamak together with L.A. Artsimovich and I.N. Golovin (1960). Great misfortune caused the death of N.A. Yavlinsky in 1962 in a plane crash, and then the direct control of studies of these systems was taken over by L.A. Artsimovich. After extremely hard and painstaking work, at the IAEA International Conference in Novosibirsk in 1968, he presented the plasma parameters achieved in the tokamak T-3: electron density $n_e = 5 \cdot 10^{13}$ cm^{-3}, the temperature of ions and electrons, respectively

[12]Later, in one of the versions of Z-pinch – the so-called 'plasma focus' - *n* and *T* were increased by an order of magnitude, but it was still not enough to ensure a positive energy output.
[13]Tokamak – Toroidal Camber with Magnetic Coils (first letters of the Russian name).

$T_i \approx 200$ eV, $T_i \approx 500$ eV and the energy confinement time of the plasma $\tau_E \approx 0.01$ s[14].

The impression of the report was stunning and this started the expansion of tokamaks – of course with both qualitative and quantitative changes – at virtually every laboratory in the world. This leading position of tokamaks still lasts.

Investigations of controlled thermonuclear fusion abroad

Until now, we have been discussing the main stages of the development of investigations of controlled thermonuclear fusion in the former USSR and then Russia. We shall now briefly mention the special features of the investigations carried out in the 50s and 60s abroad. At that time, the leading laboratories were those of Great Britain and the USA.

In Great Britain, investigations of discharge in toroidal chambers have been carried out since 1947–1948. Z-pinches were also studied but the results obtained did not match those at the Institute of Atomic Energy. Therefore, the document presented by I.V. Kurchatov in 1956 in the Harwell Atomic Centre describing the investigations of the Z-pinches in the Institute of Atomic Energy surprised English scientists, not only because of the physical achievements but also the fact that the results were presented which had been secret up to them.

In England, special attention was given to toroidal discharges with a weak longitudinal magnetic field. The largest equipment used for this purpose at the time was ZETA equipment, and the results of investigations obtained in this equipment were published in 1958. They were quite modest, nevertheless, this type of discharge has a number of interesting properties and under the name ‘the pinch with the inverted field‘ (RFP) is being studied even now.

In the U.S., research on CTF was inspired in the second half of the 40’s by Fermi, Teller, and other founders of nuclear technology. But the first systems appeared in 1950–51. Direct discharges were also studied, but most promising were stellarators (see section 10.5) – toroidal systems without current inside the plasma, in contrast to the tokamak.

Stellarators were proposed by the outstanding astrophysicist L. Spitzer Jr. He also became the head of experimental studies of these systems. The stellarators produced sufficiently high plasma parameters, but they conceded at the end of the 60s to the results obtained in tokamaks, so in America, researchers modified their stellarators to tokamaks.

However, the revival of the stellarator program started soon in the Soviet Union where the theory of perturbations of toroidal magnetic

[14]This time was almost 30 times longer than that calculated by Bohm.

K.E. Tsiolkovskii W. von Braun S.P. Korolev M.V. Keldysh

Founders of classical space exploration

H. Kaufman A.M. Andrianov G.Ya. Shchepkin A.I. Morozov

Founders of space electroreactive thrusters

fields was developed (A.I. Morozov, L.S. Solov'ev, I.M. Gel'fand and their colleagues) and this stimulated the development of methods for the control of magnetic configurations directly in the systems[15]. Because of this, stellarators with the required structure of the magnetic field lines were built in the USSR. Here it is important to mention studieds at the FIAN (now the P.N. Lebedev Physical Institute, Russian Academy of Science) by M.S. Rabinovich, I.S. Shpigel', L.M. Kovriznykh, and others. This is currently the most advanced area of CTF research, after tokamak (see section 10.5).

We have described the formation and early evolution of the most popular magnetic plasma confinement systems, although there are many more. In addition to systems with magnetic confinement of a lot of effort has been invested in the idea of ‘inertial fusion’ enunciated in the late 60’s by N.G. Basov and O.N. Krokhin. The gist of it is that in a very short time (~10^{-9}) a small target (diameter ~1 mm) is hit with a powerful stream of laser or other light (particle fluxes can also be used); the target material is heated to thermonuclear temperatures and

[15]Until then the structure of the magnetic field in traps in US stellarators was not measured, and the magnetic field was evaluated only on the basis of calculations which naturally did not take into account the defects of manufacture of the magnetic circuit.

kept at these temperature for just the right time due to inertia (see Appendix B).

Ending this brief review of the early stages of CTF studies and taking into follow-up studies, we can sum up.

1. The foundations of the science of the properties of the hot plasma in a wide range of parameters and conditions have been laid. The progress made in the field of fusion would not be possible without the hard work of talented theorists. In the Soviet Union in the first place we must note Academician M.A. Leontovich, who created a powerful theoretical division at the IAE. A large role in the development of theoretical questions of hot plasma and also in original experimental studies was played by the team of Academician G.I. Budker (Novosibirsk Institute of Nuclear Physics), as well as scientists at the Khar'kov Physico-Technical Institute, The Physical Institute of the Russian Academy of Sciences, etc. A major contribution to the theory of fusion was made by mathematicians from the M.V. Keldysh Institute of Applied Mathematics, primarily the teams of Academicians I.M. Gel'fand, A.N. Tikhonov, and A.A. Samarskii. Fundamental contributions to the theory and numerical simulations have also been made by many foreign scientists (Rosenbluth, Kruskal, Bernstein, Furth).
2. An entirely new engineering of large magnetic systems, large vacuum chambers, and powerful electrical pulse has been developed. The methods for the production of hot plasma using pulsed high-current discharges, powerful ion sources and powerful generators of microwaves (gyrotrons) have been proposed.
3. Contactless methods of local diagnostics of the plasma density distribution functions of electrons and ions, magnetic and electric fields, and other variables have been developed.
4. Effective international cooperation has been organised (section 10.5) due to the efforts of I.V. Kurchatov, L.A. Artsimonovich, E.P. Velikhov, and B.B. Kadomtsev. In the framework of international collaboration, realisation of the ITER tokamak project was launched in 2007.

Now, despite all these impressive, but ultimately partial achievements, the CTF problem can not be considered as solved, although tokamaks JET (European Community) and JT-60 (Japan) produced over short periods of time the fusion power, compatible with the power going to heat the plasma (section 10.5).

It is important to emphasize that today this situation occurs in all areas of CTF research, related to both magnetic confinement and inertial confinement systems. This is despite the fifty years of focused and intense work, which cost more than 30 billion dollars and with the participation in these works in some periods to about 10 000 people. So now an intensive search for the 'alternative' schemes for plasma confinement has been launched.

Specifically, the approaches and CTF systems will be discussed below, but for now we emphasize that the research on fusion is not the only source of modern plasma dynamics.

I.3.4. Studies in the 50's and 60's. The problem of electroreactive thrusters

An important and, in many areas, controlling role in the development of plasma dynamics, especially middle energies ($\varepsilon_i \sim 30$–1000 eV), has been played by the problem of electric reactive (rocket) thrusters (ERT), which began to flourish in the years following 1958–59 when the Soviet Union launched into an orbit the Sputnik in 1957, although at a theoretical level, the idea of electric propulsion had been studied before this period.

The need to change to higher exhaust velocities is quite evident, since the thrust developed by the rocket engine is

$$\mathbf{F} = \dot{m}\mathbf{w},$$

where $\dot{m}$ is the mass flow rate, and $\mathbf{w}$ is the gas exhaust velocity. Therefore, for small $\mathbf{w}$ the supply of the working material for missiles becomes very large.

The exhaust velocity of the gases from the modern thermal thrusters is ~(3–4) km/s. At the same time, for most flights near the solar system (Mercury – asteroid belt) with the maneuvers we require exhaust velocities of ~20–40 km/s. However, this range of velocities with acceptable efficiency could not be produced with satisfactory efficiency by the available plasmatrons or sources of ion fluxes. It was natural to try to solve the problem using plasma accelerators, but for this it was necessary to learn to create epithermal electrostatic fields in the plasma volume. In the mid-60s, this problem was solved in principle at the IAE by A.I. Morozov and G.Ya. Shchepkin who developed in stationary plasma thrusters (SPT), in which the actual conductivity $\sigma_\perp$ was almost 1000 lower than Bohm' estimtate. Other types of electric reactive engines, ionic (Kaufman) and plasma (Andrianov, et al.) were also developed. However, the SPT are favourites even at the present time. This is discussed in section 10.4.

Thus, by joint efforts on the problem of controlled thermonuclear fusion, where the emphasis was on creating traps to confine the plasma, (i.e. ‘magnetised’ diffusion) and ERT where the focus was on generating flows (i.e. create epithermal electrostatic fields in plasma) made it possible to completely overcome Bohm’s limit.

Of course, these fundamental achievements have opened a whole new world of plasma systems and a number of concrete are discussed below.

I.3.5. Other backgrounds of plasma dynamics

But not only the plasma systems of medium- and high-energy have been developed over the years. The physics and applications of low-temperature plasma were quite different and seemingly exhausted. The most important ‘starter’ achievement here was the research of powerful shock waves (SW) and flows resulting from the explosion of nuclear bombs. In these shock waves the temperature reaches many thousands and millions of degrees and there are intensive molecular reactions of excitation and ionisation of the air particles.

This cycle of research was summarized in a book by Ya.B. Zel’dovich and Yu.P. Raizer ‘Physics of shock waves and high-temperature gas dynamic phenomena’, two editions of which appeared in the first half of the 60s. The processes taking place at high temperatures are accompanied by powerful radiation, which largely determines the dynamics of the plasma. Specific features of the dynamics of dense plasma in the presence of intense radiation subsequently led to a scientific field known as ‘radiation plasma dynamics’ (N.P. Kozlov, Yu.S. Protasov, and others).

Besides this direction, an important role for the development of low-temperature plasma physics in the new phase was played by the invention of gas and plasma lasers (1960). Another great – but, unfortunately, not completed – series of papers in the late 50’s – early 60’s dealt with MHD generators, with which they tried to raise the efficiency of power plants. At that time (early 60s) the work on the creation of plasma chemical reactors sharply intensified. There is a new large area: plasma chemistry science (L.S. Polok, V.M. Smirnov, et al.).

It is also important to mention the development of plasma technology for surface treatment (V.M. Gusev and M.I. Guseva, V.G. Padalka and V.T. Tolok, and others).

These directions of the completely earth ‘laboratory plasma dynamics’ are continuing to develop rapidly, branch and find more and more applications. A number of studies to create instruments for plasma technology will be discussed in later chapters and are summarized in chapter 10.

The achievements of laboratory plasma dynamics are finding more and more applications both in the practice of contemporary space exploration (including plasma thrusters) and the analysis of plasma dynamic processes in the magnetosphere of the Earth and planets, as well as the dynamics of stellar and galactic masses (chapter 9).

I.4. Features of plasma research

Let us look at the features of the theoretical analysis of plasma systems. Some features of plasma experimental studies have been described in detail in many books, for example, in [8], and we will not discuss this here. Only the following should be mentioned. Since the majority of the methods of plasma diagnostics are contactless, the plasma parameters can be determined also remotely, not in the immediate vicinity of experimental equipment. This can be carried out through the Internet using advanced network technologies[16]. See also the book by N.P. Norenkov and A.M. Zimin: Information technologies in education, N.E. Bauman Moscow State Technical University, Moscow, 2004.

It would seem that the behaviour of plasma dynamic systems should be close to that of conventional gas-dynamic systems. Indeed, in some cases it is so, but such cases are few, but even they have very non-trivial features. In general, however, there are fundamental differences between the two types of systems.

Somewhat arbitrary, these differences in relation to laboratory conditions can be combined into four groups.

I. First of all, in contrast to the classical three states of matter, plasma does not exist by itself on the Earth (we are now considering the magnetosphere). It should especially be created and is annihilated by contact with dense gas or walls. Therefore, analysis of processes in any real system must take into account its formation from a neutral gas (or evaporation products of condensed matter), and its destruction by contact with 'cold environment'. In other words, it is imperative to take into account the presence of the ionized and neutral components, at least in the initial stage of formation of the plasma configuration and around of the bounding walls, even if the bulk of the volume plasma is completely ionised. This explains the principal role of the 'low-temperature' plasma in any plasma system. It will be mainly discussed in chapters 6 and 7 of

[16]Such technologies have been applied in, for example, large tokamaks, and also at the N.E. Bauman Moscow State Technical University by A.M. Zimin in lectures on plasma spectroscopy (http://lud.bmstu.ru).

this book. Further, in the ordinary gas dynamics, the wall, for example, the aircraft wing, is a surface on which the boundary condition

$$\mathbf{v}\big|_{\Gamma} = \left(v_n, v_t\right)\big|_{\Gamma} = 0$$

Here v_n is the normal and v_t the tangential velocity component. This boundary condition is sufficient to calculate, using the Navier–Stokes equations, not only the flow, such as the flow around an airplane wing, but also the viscous boundary layer. If the speed of the wing is close to the speed of sound, it is still necessary to consider the heat exchange between the wing and the oncoming flow. Another situation is the one in which the plasma particles with energies measured in many units, tens, hundreds of electron volts interact with the wall. Here the recombination of ions and electrons takes place, as well as sputtering of the surface by ions, electron emission from the wall, charging of the wall, etc. These processes often play a large role and their consideration in the analysis of specific systems is essential. This will be discussed in Chapter 7.

II. The second difficulty of modeling is that the plasma, especially in the laboratory, is fundamentally multi-component. Even the simplest hydrogen plasma in fusion reactors, apart from a small number of impurities and neutral atoms, contains substantially different components: hydrogen ions, electrons, magnetic and electric fields, as well as radiation, which are all in the self-consistent dynamics. The presence of multi-charged ions drastically increases the number of possible situations.

The magnetic field is a powerful factor, which regulates the behavior of particles in the plasma volume. Therefore, it is the consideration of the features of its structure in chapter I where we start description of the foundations of plasma dynamics.

III. Conventional gas dynamics deals with neutral particles (atoms, molecules). These particles interact with each other only by direct contact. Figuratively speaking, they are 'deaf and blind'. Quite a different picture arises in the plasma. The Coulomb fields of the ions and electrons relatively slowly ($1/r^2$) decrease with distance. Therefore, being even at a large distance, they disrupt each other's path. Such a perturbation will be stronger, the smaller the particle velocity. The effective cross-section of paired collisions can be estimated by the formula[17]

$$\sigma^{(\text{coul})} \sim \pi p^2, \tag{I.4.1}$$

where p is the characteristic 'impact parameter' defined by the condition

[17]Coulomb collisions are examined in detail in chapter 5.

$$\frac{|e_1 e_2|}{p} = \varepsilon_{\mathrm{rel}}. \tag{I.4.2}$$

Here $\varepsilon_{\mathrm{rel}}$ is the relative kinetic energy in the reference system connected with the centre of masses

$$\varepsilon_{\mathrm{rel}} = \frac{\mu u^2}{2}, \; \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2},$$

where μ is the reduced mass of the two particles, and $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity. In the case where the particle distribution is close to Maxwellian and $T_i \sim T_e = T$, we can write

$$\sigma^{(\mathrm{coul})} \sim \frac{10^{-13}}{T^2}, \tag{I.4.3}$$

where the cross section is measured in cm^2, and T is in eV.

This shows that at $T_e \sim 1$ eV,[18] and even more so when $T_e \sim 0.3$ eV the Coulomb cross section is very large compared to the gas-kinetic cross sections of neutral atoms or molecules ($\sim 10^{-16}$–10^{-15} cm^2). However, the Coulomb cross sections decrease rapidly with increasing temperature, and if we have a hydrogen plasma where the ions do not have electron shells, already at $T_e \sim 100$ eV the cross section is an order of magnitude smaller than the gas kinetic one, and at thermonuclear temperatures ($T_e \sim 10^4$ eV) the collision cross section decreases to about 10^{-21} cm^2 in a plasma with a density of air at atmospheric pressure ($n \sim 3 \cdot 10^{19}$ cm^{-3}) the mean free path λ of the Coulomb particles is ~30 cm. Even this simple example shows that by warming plasma we can 'almost imperceptibly' move from the environment well described by hydrodynamics (at $\lambda \ll L$) to the environment that require a kinetic description ($\lambda \gtrsim L$), where L is the scale of the system.

IV. Theoretical hydrodynamics is usually based on the macrostructures described quite efficiently by a system of hydrodynamics equations – laminar flows in pipes, the flow around thin profiles, in single vortices and shock waves, etc. In the real world, we have to deal with a spontaneously occurring mesostructure in the form of turbulence, which often rises and the macro level. In fact, only the emergence of powerful computers has made it possible to calculate the mesostructure of gas-dynamic flows. And we are talking about the media where the particles interact by the 'push method' with almost constant collision cross sections.

In plasma, the situation with the mesostructures is greatly complicated. Now the particles can interact over long distances, to

[18] Temperature 1 eV corresponds to ≈11 400 K.

create volumes with a predominance of particles of the same sign, break into groups with different velocities and, accordingly, different collision sections. We can say that plasma is charged particles immersed in an oscillating electric field. And not just oscillating but also leading to self-organisation of long-lived mesostructures. Today, even the powerful computers are not able to fully reproduce the real situation with the mesolevel taken into account.

We will draw conclusions from the above. As can be seen, there is no relatively simple and, at the same time, universal mathematical (theoretical) model. Only one thing remains: to consider the hierarchy of models consistently and more accurately describing the specific system under consideration. The most general models (single particle dynamics, the equations of hydrodynamic type, different kinetics), which will be called 'base' models, will be the focus of attention in chapters 1–6.

Mesostructures are dealt with in chapter 8. Cosmic systems are dsicussed in chapter 9. Chapter 10 (Plasma Technology) describes plasma devices used in practice in everyday life and technology. In addition to purely cognitive goals, the aim of chapter 10 is to give the reader to feel the pace and forn of the penetration of plasma technology in our lives and once again demonstrate how 'slender' plasma is connected with a rigid environment. Fragments of physics of the systems that are in chapter 10 are discussed in previous chapters, and the review is performed by step-by-step, namely in the transition to the next chapter the features of the model become more and more complicated. As a result, the reader, learning the basic material, becomes acquainted with the physics of many important plasma systems (traps, accelerators, classical discharges, etc.).

In view of this, it may be advisable after reading this introduction to skip to chapter 10, and only then begin the study of sequential chapters.

1

Fields, particles, blocks (point models)

In this chapter, we consider the basis of the plasma systems (**E**- and **H**-fields and the dynamics of single particles in these fields) which are described by the Maxwell and Newton equations or, more accurately, Newton–Lorenz equations.

The chapter starts with the Maxwell equations which should be known to the reader, and they are presented here mainly for demonstrating the selection of notations. Further, in contrast to conventional textbooks, special attention in this chapter is given to the morphology of the magnetic field, i.e. the structure of magnetic lines of force in a number of systems. This morphology plays a significant role in many advanced plasma dynamic systems (PDS). The PDS is a complex of plasma configuration, characterised by the required properties, a shell isolating to a certain degree the plasma from the destructive influence of the surrounding medium, and devices supplying power to the system (for the supply of working substances and electric power). In a general case, the PDS includes control and diagnostics systems.

The chapter is divided into two parts. The first part contains three paragraphs concerned with the general properties of electromagnetic fields, important for plasma dynamics (section 1.1), followed by the dynamics of single particles in electromagnetic fields of different structure (section 1.2) and finally, plasma systems in which the processes – in the first approximation – permit simulation by blocks and description using conventional (zero-dimensional) differential equations (section 1.3). The model of this type has already been described in the Introduction where the equation was derived for the frequency of Langmuir oscillations. The plasma volume was approximated by two rigid blocks: ionic and electronic. The dynamics of these blocks was described by two differential equations for the centre of masses.

The second part of the chapter is concerned with the investigation of a number of models of plasma systems using methods described in the first part. Special attention is given to electromagnetic waves in cold plasma, systems of classic corpuscular optics (electromagnetic lenses and separators), plasma traps, used in investigations of controlled thermonuclear synthesis (CTS), etc.

1.1. Electromagnetic fields

1.1.1. Maxwell equations

In all PDS, electromagnetic fields are present and operate actively in the zones of ionisation, sustainment and acceleration of plasma. It is important to stress that in this case the electromagnetic fields are not simply some energy carriers (as is the case of, for example, heat), and they are a factor determining the entire plasma dynamics. A suitable example of these are ionic sources in which a system of electrons is used to produce the geometry of the equipotentials of the electrical field in the accelerating gap which ensures acceleration and focusing of the flux into a relatively narrow beam without affecting the electrodes (section 1.4).

Another example are magnetic traps for plasma sustainment (sections 1.6–1.7). Finally, the Debye fields have already been discussed.

It can be said without exaggeration that in these PDS the fields are ‘constructional material’. It is therefore natural to start explanation of the fundamentals of the theory of processes in the PDS by mentioning the main properties of the electromagnetic field described by Maxwell equations [53].

In the Gaussian unit system, these equations have the form:

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}; \tag{1.1.1a}$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \tag{1.1.1b}$$

$$\operatorname{div}\mathbf{H} = 0; \tag{1.1.1c}$$

$$\operatorname{div}\mathbf{E} = 4\pi q_e \tag{1.1.1d}$$

Here **E** and **H** are the strength of the electrical and magnetic fields in vacuum, respectively, **j** is the volume density of flows of all particles,

q_e is the volume density of all charges.

The system (1.1.1) can have different forms. We shall mention some of these forms which will be required in further considerations.

1. *Static fields*. In this case, the system (1.1.1) has the following form:

$$\operatorname{rot}\mathbf{E} = 0; \tag{1.1.2a}$$

$$\operatorname{div}\mathbf{E} = 4\pi q_e; \tag{1.1.2b}$$

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}; \tag{1.1.3a}$$

$$\operatorname{div}\mathbf{H} = 0.$$

(1.1.3b)

This shows that in the statics the fields **E** and **H** are independent of each other, and the electrostatic field is potential

$$\mathbf{E} = -\nabla\phi, \tag{1.1.4a}$$

and potential ϕ satisfies at $q_e \neq 0$ the Poisson equation

$$\Delta\phi = -4\pi q_e. \tag{1.1.4b}$$

However, if $q_e = 0$, ϕ is governed by the Laplace equation

$$\Delta\phi = 0. \tag{1.1.4c}$$

The situation in the case of the magnetic field is slightly more complicated.

If we consider the field in the volume where $\mathbf{j} = 0$, then here, as in the case with the **E**-field, the magnetic field is potential (rot $\mathbf{H} = 0$):

$$\mathbf{H} = \nabla\phi_m, \tag{1.1.5a}$$

and potential ϕ_m satisfies the Laplace equation

$$\Delta\phi_m = 0. \tag{1.1.5b}$$

However, if $\mathbf{j} \neq 0$, we should use the continuity equation (1.1.3b) which can also be satisfied by introducing the vector potential **A** and assuming that

$$\mathbf{H} = \text{rot } \mathbf{A}. \tag{1.1.5c}$$

The vector potential is ambiguous because without changing **H,** which also has a meaning, we can add the gradient of an arbitrary function f to the vector potential:

$$\mathbf{A} \sim \mathbf{A} + \nabla f. \tag{1.1.5d}$$

This circumstance often enables us to simplify the formal side in specific calculations.

Substituting (1.1.5c) into (1.1.3a) we obtain

$$\Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}. \tag{1.1.5e}$$

Here, we have taken into account the vector identity

$$\text{rot rot } \mathbf{a} = \nabla\text{div } \mathbf{a} - \Delta\mathbf{a},$$

and utilising the arbitratiness in the definition of **A**, we set div **A** = 0.

2. *Introduction of dielectric permittivity*. System (1.1.1) describes the field in vacuum generated by the currents and charges present there. In fact, this is a microscopic Lorenz model. **j** and q_e include both 'molecular' currents jmol and charges q_{mol} and also secondary 'macrocurrents' $\mathbf{j}_{sec}$ and 'macrocharges' $\mathbf{q}_{sec}$. In classic electrodynamics, the molecular current and charges are usually separated and their influence is described by means of the vectors of electrostatic induction **D** and magnetic induction **B**, retaining in the Maxwell equations only external (secondary) currents and charges in relation to the medium. There are a number of cases in which the plasma should be treated as a medium in which the displacement of the particles can be taken into account efficiently in the same manner. In most cases, it is sufficient to introduce single induction **D**. Consequently, the system of the Maxwell equations assumes the following form:

$$\text{rot}\,\mathbf{H} + \frac{4\pi}{c}\mathbf{j} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}; \tag{1.1.6a}$$

$$\text{div}\,\mathbf{H} = 0; \tag{1.1.6b}$$

$$\text{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \tag{1.1.6c}$$

$$\text{div } \mathbf{D} = 4\pi q_{sec} \tag{1.1.6d}$$

The vector of electrostatic induction **D** is connected with the strength of the field **E** by the relationship

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad j_{mol} = \frac{\partial \mathbf{P}}{\partial t}. \tag{1.1.6e}$$

Here **P** is the polarisation vector of the medium. If polarisation is directly proportional to the strength of the electrical field

$$\mathbf{P} = \overleftrightarrow{\varkappa}\mathbf{E}, \tag{1.1.6f}$$

we can introduce dielectric permittivity of the medium $\overleftrightarrow{\varepsilon} = 1 + 4\pi\overleftrightarrow{\varkappa}$ and write

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E}, \tag{1.1.6g}$$

The linear relationship between **P** and **E** is observed in particular in propagation of waves with a low amplitude in plasma. The letters with the two sided arrows above them indicate tensors. A specific example of calculation of $\overleftrightarrow{\varepsilon}$ and of the system (1.1.6) will be investigated in the section 1.5.

1.1.2. Conservation laws

Energy conservation law

In analysis of the operation of the PDS it is convenient to have general information on the dynamics of electromagnetic fields which can be extracted from the law of conservation of the energy of electromagnetic fields. If the equations (1.1.1a) and (1.1.1b) are multiplied in a scalar manner by **E** and **H**, respectively, and the second equation is deducted from the first one, we obtain

$$\mathbf{jE} + \frac{\partial}{\partial t}\left(\frac{E^2}{8\pi} + \frac{H^2}{8\pi}\right) = \frac{c}{4\pi}(\mathbf{E}\,\mathrm{rot}\,\mathbf{H} - \mathbf{H}\,\mathrm{rot}\,\mathbf{E}). \tag{1.1.7a}$$

Taking into account the identity **a** rot **b** – **b** rot **a** = div [**b**, **a**], equation (1.1.7a) can be presented in a new form

$$\mathbf{jE} + \frac{\partial}{\partial t}\left(w_E + w_H\right) = -\mathrm{div}\,\mathbf{\Pi}. \tag{1.1.7b}$$

Here w_E and w_H are the volume densities of the energy of the magnetic and electrical fields:

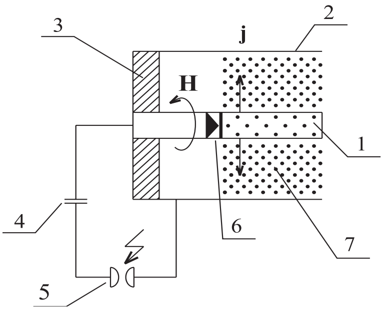


Fig. 1.1.1. Diagram of a pulsed plasma gun with an intrinsic magnetic field. 1) cathode, 2) anode, 3) insulator, 4) condenser batteries, 5) switch, 6) pulsed valve for letting in gas placed inside the cathode, 7) plasma bunch with the current flowing through it.

$$w_H = \frac{\mathbf{H}^2}{8\pi}; \quad w_E = \frac{\mathbf{E}^2}{8\pi} \tag{1.1.7c}$$

Π is the Umov–Pointing vector:

$$\Pi = \frac{c}{4\pi}[\mathbf{E},\mathbf{H}], \tag{1.1.7d}$$

which characterises the strength and the direction of the energy flow of the electromagnetic field.

It should be mentioned that in practice it is often convenient to use the integral form of the equation (1.1.7b). Denoting the volume element by dV, we obtain

$$\iiint \mathbf{j}\cdot\mathbf{E}dV + \frac{\partial}{\partial t}\iiint\left(\frac{\mathbf{H}^2}{8\pi}+\frac{\mathbf{E}^2}{8\pi}\right)dV = -\oint\!\!\oint \mathbf{\Pi}\, d\mathbf{S}. \quad \text{‘(1.1.7e)}$$

To illustrate the law (1.1.7e) we can use a coaxial high-current pulsed plasma accelerator – a ‘plasma gun’ – with an intrinsic magnetic fields (Fig. 1.1.1).

In this accelerator, there are azimuthal magnetic and radial electrical fields which ensure, in accordance with (1.1.7), the required supply of energy to the accelerated plasma.

Stress tensor

We now transfer to the law of conservation of the momentum for the electromagnetic field. We start with the ampere force, acting on the investigated volume:

$$\mathbf{F} = \frac{1}{c}\iiint[\mathbf{j},\mathbf{H}]dV\,. \tag{1.1.8}$$

Maxwell showed that integral (1.1.8) can be transformed in the general form to a surface integral. This means that the volume force (1.1.8) can be regarded as the result of the effect of some surface force, namely:

$$\mathbf{F} = -\oiint \overleftrightarrow{\mathbf{T}}^{(H)} d\mathbf{S}. \tag{1.1.9}$$

The minus sign here is associated with the selection of the vector of the surface element $d\mathbf{S}$ and the selection of the sign of $\overleftrightarrow{T}^{(H)}$ – the Maxwell strength tensor for the magnetic field.

In the Cartesian coordinates (see for example [53])

$$\overleftrightarrow{T}^{(H)} = \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix} = \begin{vmatrix} \dfrac{H^2}{8\pi} - \dfrac{H_x^2}{4\pi}; & -\dfrac{H_x H_y}{4\pi}; & -\dfrac{H_x H_z}{4\pi}; \\ -\dfrac{H_x H_y}{4\pi}; & \dfrac{H^2}{8\pi} - \dfrac{H_y^2}{4\pi}; & -\dfrac{H_y H_z}{4\pi}; \\ -\dfrac{H_x H_z}{4\pi}; & -\dfrac{H_y H_z}{4\pi}; & \dfrac{H^2}{8\pi} - \dfrac{H_z^2}{4\pi}; \end{vmatrix} \tag{1.1.10}$$

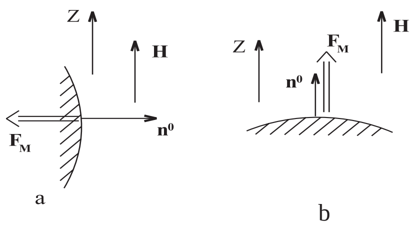
If the axis of the coordinates in the given point are selected in such a manner that the magnetic field has a single component of the field $H_Z = H$, then

$$\overleftrightarrow{T}^{(H)} = \begin{vmatrix} \dfrac{H^2}{8\pi} & 0 & 0 \\ 0 & \dfrac{H^2}{8\pi} & 0 \\ 0 & & -\dfrac{H^2}{8\pi} \end{vmatrix}. \tag{1.1.11}$$

This shows that if the normal to the surface element is normal to the z axis (Fig. 1.1.2a), the force is directed in the direction opposite to the normal and is equal to:

$$d\mathbf{F} = -\frac{H^2}{8\pi} d\mathbf{S} \equiv -p_M d\mathbf{S}, \tag{1.1.12}$$

i.e. the magnetic field, tangential to the boundary, exerts a pressure on this boundary. $p_H = H^2/8\pi$ is usually referred to as the magnetic



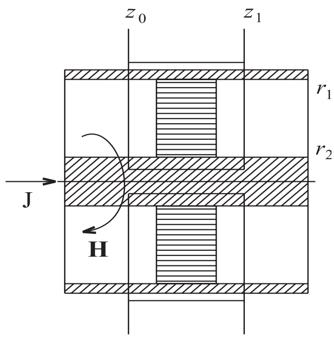


Fig. 1.1.2. (left) Effect of the magnetic field on the surface of the plasma volume; a) field parallel to the surface; b) field normal to the surface.
Fig. 1.1.3 (right). Scheme for calculating the force acting on the plasma bunch in a pulsed gun.

pressure[1]. However, if the normal to the surface element is directed along the *z* axis, then it is subjected to the effect of the tension force in the same direction (Fig. 1.1.2b), equal to:

$$d\,\mathbf{F} = \frac{H^2}{8\pi} d\,\mathbf{S}. \tag{1.1.13}$$

Equations (1.1.12) and (1.1.13) can be 'felt' by holding two magnets.If the magnets make contact through the same poles they will be repulse each other and this corresponds to (1.1.2) if the surface is represented by the median plane *S*. On the other hand, if the magnets are brought together by the different poles, they will attract each other (1.1.13).

The effect of Maxwell tensions can be conventionally described if the magnetic lines of force are regarded as identical with the stretched and at the same time swollen rubber cords. These cords will, on the one hand, try to shorten (tension) and on the other hand pull away from each other (pressure).

If the boundary intersects the lines of force under an angle, then the force acting on the boundary will be determined by the composition of the ‘shortening’ and ‘repulsive’ capacities of the field.

As an example of application of the stress tensor, we consider acceleration of a plasma bunch in a coaxial pulsed gun (Fig. 1.1.3)

[1]It should be mentioned that the magnetic field H = 5000 Oe = 0.5 T generates a pressure of $p_M \approx 1$ kg/cm^2. However, the magnetic field of 100 kOe = 10 T – and such fields are required for a number of the PDS – generates a pressure of 400 kg/cm^2 acting on the coils of the PDS. The fields in the megaoersted range can be produced only in the pulsed regime.

which takes place as a result of the interaction of the azimuthal magnetic field H_θ with the radial component of the density of current j_r, flowing between the electrodes. Acceleration can be interpreted on the basis of (1.1.3a) also as the result of a decrease of magnetic pressure. The value of F_z will be calculated using a toroidal volume embracing the entire plasma bunch. In this case, only the azimuthal component of the fields differs from zero and, consequently

$$F_z = \oiint n_z^0 \frac{H_\theta^2}{8\pi} dS = 2\pi \int_{r_i}^{r_2} \left(H_\theta^2 \Big|_{z_0} - H_\theta^2 \Big|_{z_1} \right) \frac{rdr}{8\pi}, \tag{1.1.14}$$

i.e. this includes the magnetic pressure $p_H = H^2/8\pi$ at the edges of the volume. If the entire current is closed through the plasma, then at $z = z_1$ the magnetic field converts to zero and therefore

$$F_z = \frac{1}{4} \int_{r_1}^{r_2} H_\theta^2 \Big|_{z_0} rdr. \tag{1.1.15}$$

Section z_0 was selected in front of the bunch, i.e. in the area where the current flows only through the electrodes. Consequently, in this section:

$$H_\theta = \frac{2J}{cr}, \tag{1.1.16}$$

where J is the total discharge current. Substituting (1.1.16) into (1.1.15) we obtain, irrespective of the detailed pattern of the flows in the plasma)

$$F_z = \frac{J^2}{c^2} \ln \frac{r_2}{r_1}. \tag{1.1.17}$$

These considerations also hold in the case of a stationary discharge in a coaxial system with cylindrical electrodes.

We now consider the effect of the electrical field on the volume charge

$$\mathbf{F} = \int q_e \mathbf{E} dV . \tag{1.1.18a}$$

The integral is taken over the entire space occupied by the charge. It appears that the Maxwell equations can also be used in this case to reduce the force **F** to surface force in accordance with (1.1.9)

$$\mathbf{F} = \oiint \overleftrightarrow{\mathbf{T}}^{(E)} d\mathbf{S}. \tag{1.1.18b}$$

and the structure $\overleftrightarrow{T}^{(E)}$ is completely identical with (1.1.10) if **H** is substituted by **E**.

The electromagnetic forces, applied to the volume, make the volume can rotate. The momentum of the electrondynamic force in relation to the z axis is:

$$M_z = \iiint f_\theta r dV = \frac{1}{c}\iiint r(j_z H_r - j_r H_z)dV. \tag{1.1.19a}$$

Here f_0 is the azimuthal component of the ampere force.

If the field axially symmetric, then from the Maxwell equations (1.1.1a) in the stationary case we have:

$$j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial rH_\theta}{\partial r}; \quad j_r = -\frac{c}{4\pi}\frac{\partial H_\theta}{\partial z}. \tag{1.1.19b}$$

Using these relationships and the continuity equation div **H** = 0, equation (1.1.19a) can be transformed to the following form

$$M_z = \frac{1}{4\pi}\oiint (n_r^0 r H_r H_\theta + n_z^0 r H_z H_\theta) dS. \tag{1.1.19c}$$

Here, as in (Int.1.9), we use the external normal to the external surface. Equation (Int.2.1b) shows that the momentum of the forces forms only when the azimuthal H_θ and poloidal (H_r, H_z) fields exist simultaneously.

1.1.3. Morphology of magnetic fields [54]

The processes in plasma systems depend very strongly on the morphology (geometry, structure) of the magnetic lines of force. From the physical viewpoint, this is associated with the high mobility of the particles, in particular, electrons, along the lines of force and the comparatively low mobility of these particles across this lines. Therefore, calculation of the morphology of the magnetic field is an essential stage of development of any plasma dynamic system with a magnetic field.

According to the definition, *magnetic lines of force* $\mathbf{r}(s)$ have at every point the tangent $d\mathbf{r}$, parallel to the strength of the field **H**

$$d\mathbf{r} \| \mathbf{H}. \tag{1.1.20a}$$

In the Cartesian coordinate system this can be written in the following form

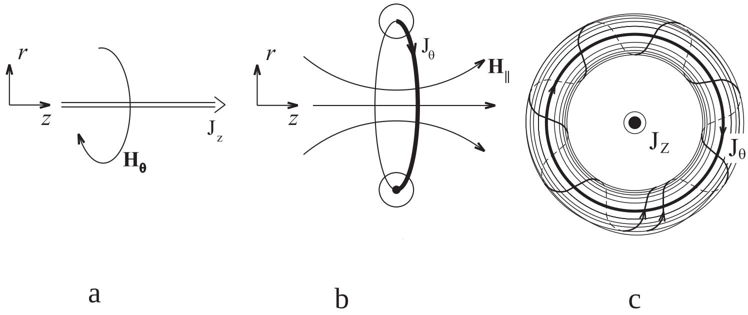


Fig. 1.1.4. The simplest axisymmetric field: a) azimuthal magnetic field of a straight filament with current; b) the poloidal magnetic field of a ring with current; c) the axisymmetric three-component magnetic field whose lines of force do not usually close with each other and do not extend to infinity.

$$\frac{dx}{H_x}=\frac{dy}{H_y}=\frac{dz}{H_z}. \tag{1.1.20b}$$

Correspondingly, in the cylindrical coordinate system

$$\frac{dr}{H_r}=\frac{rd\theta}{H_\theta}=\frac{dz}{H_z}. \tag{1.1.20c}$$

In physics courses, attention is usually given to two very partial cases of the magnetic field: the magnetic field of a straight filament with current (Fig. 1.1.4a) and the field of a ring with current (Fig. 1.1.4b). These fields are symmetric. The field with a single azimuthal component H_0 is referred to as the azimuthal field, and the field of the components H_r, H_z, at $H_\theta = 0$ is termed the poloidal field.

In a general case, the magnetic lines of force may be of different types. However, it should be mentioned here that the 'school' assumption according to which the magnetic force fields are either closed or extend to infinity is completely inaccurate. This holds only for the fields shown in Fig. 1.1.4a, b. This however does not apply to the symmetric field formed by combining the filament with current and a ring (Fig. 1.1.4c). Here, the main part of the lines of force does not close on itself, and only the line positioned along the axis of the system extends to infinity (I.E. Tamm, 1928, [53]).

Magnetic surfaces

In addition to the concept of the magnetic line of force, another important concept in morphology is the concept of the 'magnetic surface'. This is the surface formed by magnetic lines of force. Here

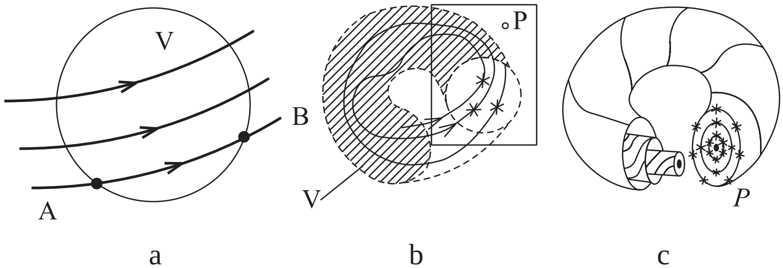


Fig. 1.1.5. The concept of the 'magnetic surface': a) working volume V, filled with 'segments' of the lines of force; b) the toroidal volume V, whose boundary is not intersected by the magnetic line; P is the 'imaging' plane; c) the toroidal magnetic field with magnetic surfaces.

we have two completely different situations, depending on whether the working volume V_0 contains only 'segments' of the lines of force (Fig. 1.1.5a) or whether the lines of force are completely situated in such a volume (Fig. 1.1.5b).

In the first case, we have no *apriori* criterion for construction (composition) of the magnetic surfaces from lines of force. They can be completely arbitrary. The situation in the second case is completely different. The structures of the magnetic field in the toroidal volume can differ greatly (see Appendix A at the end of this book). However, in most cases, fields consisting of well-defined magnetic surfaces are of special interest. This means (Fig. 1.1.5c) that on the 'imaging' plane P, intersecting the torus, there are (closed) lines μ which are characterised by the fact that the lines of force passing through them intersect the imaging plane P again at points of this line μ although, generally speaking, not at the points from which they originated.

Therefore, it appears that in many cases it is not possible to construct fields in which we could define the system of inserted well-defined (in the sense of the image on the plane P) surfaces on which we can place all the lines of force of the given toroidal volume (this problem is discussed in greater detail in Appendix A). As shown later, only the toroidal fields, consisting of such inserted surfaces[2], may act as efficient 'vessels' ('traps') for sustaining plasma (see section 1.5 of this chapter, and also section 2.4).

Two components of symmetric fields
An important group is formed by the axisymmetric fields whose components do not depend on the azimuth θ

[2]Or slightly 'eroded' surfaces, i.e. thin layers

$$\mathbf{H} = (H_r(r, z), H_\theta(r, z), H_z(r, z)). \tag{1.1.21a}$$

It may easily be seen that any such field should be treated as a superposition of two independent fields: poloidal $\mathbf{H}_{\text{pol}} = (H_r(r, z), 0, H_z(r, z))$ and the azimuthal field $\mathbf{H}_\theta(r, z)$, i.e. in a general case

$$\mathbf{H} = \mathbf{H}_{\text{pol}} + \mathbf{H}_\theta. \tag{1.1.21b}$$

In fact, every one of these fields satisfies separately the system of the Maxwell equations:

$$\operatorname{rot}\mathbf{H}_{\text{pol}} = \frac{4\pi}{c}\mathbf{j}_\theta; \quad \operatorname{div}\mathbf{H}_{\text{pol}} = 0 \tag{1.1.22a}$$

$$\operatorname{rot}\mathbf{H}_\theta = \frac{4\pi}{c}\mathbf{j}_{\text{pol}}; \quad \operatorname{div}\mathbf{H}_\theta = 0. \tag{1.1.22b}$$

Here $\mathbf{j}_{\text{pol}} = (j_r(r, z), 0, j_z(r, z))$. Attention should be given to the 'crossed' relationship $(\mathbf{H}_{\text{pol}}, \mathbf{H}_\theta) \leftrightarrow (\mathbf{j}_\theta, \mathbf{j}_{\text{pol}})$.

The 'function of the flux' can be introduced $\psi(r, z)$ taking into account the fact that the poloidal components of the field satisfy the equation $\operatorname{div}\mathbf{H}_{\text{pol}} = 0$. This function is connected with the components of the field by the relationship:

$$H_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}; \; H_z = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{1.1.23a}$$

Comparing (1.1.5c) and (1.1.23a), we obtain that at $A_r = A_z = 0$,

$$\psi = \mathbf{r}A_\theta. \tag{1.1.23b}$$

It may easily be seen that the magnetic flux Φ, passing through the ring restricted by the radii r_1, r_2 and located in the plane z = const, is connected with ψ by the relationship:

$$\Phi(r_1, r_2, z) = 2\pi\int_{r_1}^{r_2} H_z(r, z)\,dr = 2\pi\big(\psi(r_2, z) - \psi(r_1, z)\big). \tag{1.1.24}$$

It should be mentioned that from the equations (1.1.1) and (1.1.23a) we obtain the equation for the function of the magnetic flux (at $\partial\mathbf{E}/\partial t = 0$):

$$\Delta^*\psi = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2} = -\frac{4\pi}{c} rj_\theta. \tag{1.1.25}$$

Here Δ^* is the modified Laplace operator[3].

At the same time, the azimuthal field, characterised by the component $H_\theta(r, z)$, as indicated by the Maxwell equation (1.1.1a), is reduced to the equation

$$\frac{\partial}{\partial r} rH_\theta(r,z) = \frac{4\pi}{c} rj_z(r,z). \tag{1.1.26a}$$

Introducing the total current $J_z(r, t)$, passing inside a circle with radius r at the given value of z

$$J(r,z) = 2\pi\int_0^r j_z(r,z)rdr, \tag{1.1.26b}$$

we can integrate the equation (1.1.26a) and obtain the equation, generalising the well-known equation for the field of the straight filament:

$$H_\theta = \frac{2}{cr} J(r,z). \tag{1.1.26c}$$

It is easy to verify that the superposition of the two fields may be used to describe the general field also at other types of symmetry: flat, when the strength of the field does not depend on the Cartesian coordinate z, and the helical symmetry in the case in which the components of the strength of the field depend only on r and $\omega \equiv \theta - \alpha z$, where $\alpha = 2\pi/L$, L is the pitch of the helix (see below).

Lines of force of axisymmetric fields

The previously mentioned relationship (1.1.24) between ψ and $\mathbf{\Phi}$ shows that the equation ψ = const describes the magnetic surfaces or the lines of force of the poloidal field in the plane (r, z). This claim can easily be verified also formally.

For this purpose, we write the system of equations (1.1.20c), which determine the lines of force, in the form

---

[3] The conventional Laplace operator in the case of axial symmetry in the cylindrical coordinate has the form:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}.$$

$$H_z dr - H_r dz = 0; \qquad \frac{dz}{d\theta} = \frac{rH_z(r,z)}{H_\theta(r,z)}. \tag{1.1.27}$$

Substituting into the first equation the expression (1.1.23a) for H_r and H_z, we immediately obtain the first integral $\frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial z}dz = 0$, i.e.

$$\psi(r, z) = \text{const}. \tag{1.1.28}$$

This equation defines the toroidal surfaces on which the lines of force are situated. In order to determine completely the form of the force of lines, it is necessary to solve the second equation (1.1.27), substituting here the dependence $r = r\ (z, \psi)$, determined from the equations (1.1.28). The resultant equation can be solved easily by the numerical procedure, and for the analytical solution it is usually necessary to introduce the toroidal coordinates of some.

Previously, we have discussed and showed (Fig. 1.1.5c) that the toroidal magnetic field can often be regarded as a system of inserted surfaces, i.e. after passing around the torus, the line of force intersects the imaging plane P at the points situated on the same curve. In the axisymmetric case, the equations $\psi(r, z)$ describes strictly the magnetic surfaces which transfer into themselves after passing around the torus, if in addition to the poloidal field there is also azimuthal field H_θ. The azimuthal field does not disrupt the symmetry of these magnetic surfaces ψ = const., but the lines of force now transform to helices wound onto the toroidal surface $\psi(r, z)$ = const. Naturally, it is assumed that the line determined by the equation ψ = const in the plane (r, z) is closed.

The helical lines of force, situated on the magnetic surfaces ψ = const., are usually not closed and are continuously wound and cover densely 'their' surface. The density of the coating indicates that the line of force passes as close as possible to any surface point. However, this infinitely thin line of force does not cover the entire surface. This is due to the fact that the set of bypasses around the torus is countable, and to coat the entire surface we should have a continuum of the bypasses. In fact, this comment is of the formal nature because the 'real' thickness of the line of force is of the order of the electronic Larmor radius, i.e. it is finite. Such a 'thick line' completely covers the magnetic surface.

It should be stressed that in the investigated doubly connected axisymmetric case the system of the inserted magnetic surfaces exist *only in the presence of the azimuthal electrical current in the volume.* This situation is realised in tokamaks (section 1.7).

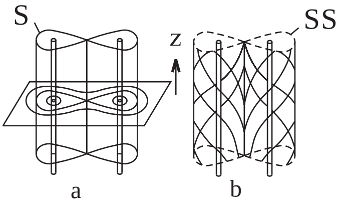


Fig. 1.1.6. Concept of the separatrix: a) the structure of the magnetic field of direct bars with the currents with the same direction, $H_z = 0$. S is the separatrix; b) the same for the superposition of the field H_z, SS is the separatrix surface.

Separatrix surfaces

Another fundamental concept of the morphology of the magnetic fields is the concept of the 'separatrix surface', i.e. the surface separating the surfaces (or the lines of force) which cannot be transferred into each other without disrupting the structure of the surfaces. This can be conveniently illustrated on the example of flat (i.e. independent of z) fields whose magnetic surfaces are formed by two straight parallel currents of the same direction (Fig. 1.1.6). The lines of force of this field are divided into three groups:

a) those which close around the left conductor,

b) those which close around the right conductor,

c) the lines covering both conductors.

The boundary between these groups is the self-intersecting line of force in the form of the number eight (lemniscate of Bernoulli). This line is referred to as the separatrix. The self-intersection of the separatrix is due to the fact that in the centre of the system both components of the field H_z, H_y convert to zero, and the equation of the line of force (1.1.20b) does not give an unambiguous solution, i.e. in this case

$$\left.\frac{dy}{dx}\right|_0 = \frac{H_x}{H_y} = \frac{0}{0}. \tag{1.1.29}$$

Since the magnetic field outside the conductor is vortex-free, the field satisfies the Laplace equation and it can easily be verified that, expanding the scalar potential ϕ_M in the vicinity of $x = 0$ with respect to the degrees x and y, the branches of the separatrix at zero are mutually perpendicular.

Now, a direct magnetic field $\mathbf{H}_z$ = const., oriented along the axis (this field is the analogue of the azimuthal field $\mathbf{H}_\theta$ in the axisymmetric system) is superposed on the field of two conductors. Consequently, we obtain smooth cylindrical magnetic surfaces everywhere, with the exception of the one which is formed by the separatrix. It is important to stress that the separatrix surfaces do not necessarily surround the regions containing conductors with current. A suitable example here are the helical fields, shown in Fig. 1.1.8 which will be discussed later.

The field of the system of parallel direct bars with current

The separatrices may have greatly differing forms, even in the symmetric field[4]. Here we note only the configuration of the magnetic lines of force of the field of the system of N straight bars with equal currents, distributed around a circle at equal distances from each other. This field can be described quite easily by a complex potential. We explain this procedure.

It was mentioned previously that in the region where $\mathbf{j} = 0$ the magnetic fiels can be described by both scalar potential ϕ_m and by vector potential $\mathbf{A}$. For a flat field, at $H_z = 0$ it is sufficient to assume that only component A_z, which corresponds to the direction of current which forms the field, differs from zero. The equalities $A_z(x, y) = \text{const.}$ are the equations of the lines of force. From (1.1.5c) we obtain

$$H_x = \frac{\partial A_z}{\partial y}, \quad H_y = -\frac{\partial A_z}{\partial x}.$$

At the same time, $\mathbf{H} = \nabla\phi_m$. Therefore,

$$\frac{\partial\phi_m}{\partial x} = \frac{\partial A_z}{\partial y}; \quad \frac{\partial\phi_m}{\partial y} = -\frac{\partial A_z}{\partial x}. \tag{1.1.30a}$$

This is nothing else but the Cauchy–Rieman conditions for the complex function, known in the theory of the functions of the complex variable

$$w(z) = A_z + i\varphi_m. \tag{1.1.30b}$$

Finally, taking into account the equation for the field of the straight filament (1.1.26c)

$$H = \frac{2J}{c\left|\mathbf{r}-\mathbf{a}\right|},$$

it is quite easy to obtain the explicit expression for w for a single conductor

$$w = A_z + i\phi_m = \frac{2J}{c}\ln(z-a), \quad z = x+iy. \tag{1.1.30c}$$

Here a is the complex coordinate of the filament. The coordinates of the system of N bars symmetrically distributed around the circle have the form

[4] If there is no symmetry, the form of the separatrices is very complicated (see below).

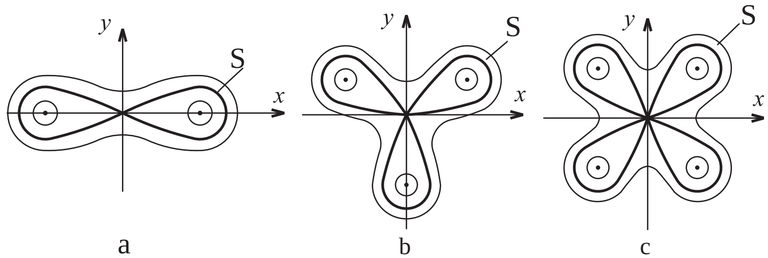


Fig. 1.1.7. Lines of force of the magnetic field of two parallel straight currents (a); lines of force of the field of three bars (b); the lines of force of four bars with a current flowing in the same direction (c), *S* are separatrices.

$$a_k = \mathrm{R}\ \exp\left\{\frac{2\pi}{N}k\right\}, \quad k = 0, \ldots, N-1. \tag{1.1.31a}$$

It may easily be seen that the total field is described by the equation:

$$w_N = A_{z,N} + i\phi_{m,N} = \frac{2J}{c}\ln(\mathrm{z}^n - R^n). \tag{1.1.31b}$$

Consequently, transferring to the polar coordinates ($z = re^{i\theta}$),

$$A_{z,N} = \frac{J}{c}\ln\left(r^{2N} - 2r^N R^N \cos N\theta + R^{2N}\right), \tag{1.1.31c}$$

and, consequently, the equations of the lines of force of the field of *N* bars have the form

$$r^{2N} - 2r^N R^N \cos N\theta = \mathrm{const}. \tag{1.1.31d}$$

The lines of force at $N = 2, 3, 4$ are shown in Fig. 1.1.7. The zero point at $N = 2$ is often referred to as the *x*-point.

Helical fields

Here we mention only one important group of fields having the characteristic feature that in this case the formation of the inserted magnetic surfaces does not require any current inside the volume of the field.

For this purpose, we examine symmetric ‘straight’ fields, formed by current-carrying helices wound on a straight circular cylinder (Fig. 1.1.8).

It is assumed that all the helices have the same pitch *L*. To simplify considerations, it is assumed that they are distributed at equal

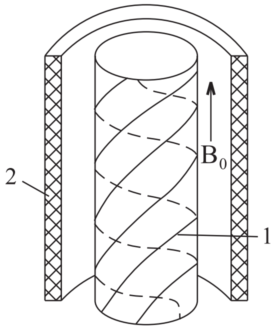


Fig. 1.1.8. A system of formation of the magnetic field of helical current-carrying conductors (1) and a homogeneous magnetic field (2).

distances around the azimuth. If the field is formed by N helices with the current with the same direction, these are 'N-pass' helical fields. This field is characterised by helical symmetry because it depends only on two coordinates (r, $\omega \equiv \theta - \alpha$, $\alpha = 2\pi/L$)

$$\mathbf{H} = (H_r(r, \theta - \alpha z), H_\theta(r, \theta - \alpha z), H_z(r, \theta - \alpha z)). \tag{1.1.32a}$$

In a general case, the field can be described by two components of the vector-potential

$$\mathbf{A} = (0, A_\theta(r, \theta - \alpha z), A_z(r, \theta - \alpha z)). \tag{1.1.32b}$$

The components of the magnetic field in this case are equal to:

$$\begin{aligned} H_r &= \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} = \frac{1}{r}\frac{\partial A_z}{\partial \omega} + \alpha\frac{\partial A_\theta}{\partial \omega} = \frac{1}{r}\frac{\partial}{\partial \omega}(A_z + \alpha r A_\theta) \\ H_\theta &= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = -\frac{\partial A_z}{\partial r} \\ H_z &= \frac{1}{r}\frac{\partial r A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} = \frac{1}{r}\frac{\partial r A_\theta}{\partial r}. \end{aligned} \tag{1.1.33}$$

Substituting these expressions into (1.1.20c) we immediately determine the first integral

$$\Psi_{(r,\omega)} \equiv A_z + \alpha r A_\theta = \text{const}. \tag{1.1.34}$$

This integral describes the magnetic surfaces of the helical field. The resultant morphology of the fields will be discussed for the case in which the field can be regarded as a sum of the homogeneous field H_0, oriented along the axis z, and the field of the n-th helical harmonics.

Such a vacuum field, satisfying the Laplace equation, is described by the scalar potential

$$\phi_m = H_0 z + \frac{h}{\alpha} I_n(n\rho)\sin n\omega, \quad \rho \equiv \alpha r, \tag{1.1.35a}$$

and the equation of the magnetic surfaces, corresponding to this field, has the form

$$\Psi = \frac{H_0}{2\alpha}\left(\rho^2 - \frac{2h}{H_0}\rho I_n'(n\rho)\cos n\omega\right) = \text{const.} \tag{1.1.35b}$$

Here I_n is the Bessell function of the imaginary argument.

Figure 1.1.9 shows the sections through the magnetic surfaces by the plane $z = 0$, i.e. the lines ψ = const for different n, obtained by the numerical solution of the equations (1.1.35b). Attention should be given to the fact that at all n there is a separatrix which separates the magnetic surfaces closed in the plane (r, ω), from the surfaces extending to infinity [54].

If these straight fields are now closed in a torus, we obtain magnetic configurations of stellarators proposed by L. Splitzer, as mentioned in the Introduction (section Int.3.3).

1.1.4. Metric characteristics of magnetic fields

Up to now, when discussing the morphology of the magnetic lines of force and surfaces, we have not mentioned the quantitative characteristics.

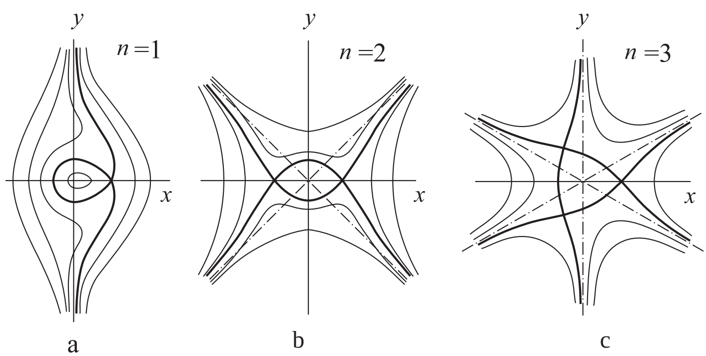


Fig. 1.1.9. Section of the magnetic helical surfaces at n = 1 (a); the same at n = 2 (b); the same at n = 3 (*c*). The ratio of the amplitude of the helical field to the homogeneous field is $h/H_0 < 1$.

1. *Angle of twist (rotational transform angle)*. The angle of 'twist' or the 'rotational transform angle' is an important characteristic of toroidal magnetic surfaces. This angle is determined as the limit of the ratio of the sum of the angles of rotation $\Delta\theta_k$ of the imaging point in plane P (see Fig. 1.1.5) to N – the number of passes around the torus

$$\bar{\theta} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \Delta\theta_k . \tag{1.1.36}$$

The toroidal magnetic surface on which $\bar{\theta}/2\pi$ is a rational number, is referred to as the rational magnetic surface, and on this surface either all or some of the lines of force are closed. If not all lines are closed, and only some of them are, the latter are used as asymptotes for the non-closed lines. Correspondingly, the surface is referred to as irrational, if $\bar{\theta}/2\pi$ is irrational. If the angle $\bar{\theta}$ changes in transition from one magnetic surface to another, it is then concluded that the toroidal magnetic field has a 'shear'.

2. *The specific volume of the magnetic force tube*. It is important to mention another metric parameter which is often used in examining systems of either finite segments or closed magnetic lines of force. This is the so-called specific volume of the magnetic force tube (or, in other words, the 'line of force').

$$U = \lim_{S\to 0} \frac{V}{\Phi}. \tag{1.1.37a}$$

Here S is the cross-section of the magnetic tube, V is its volume, Φ is the magnetic flux inside the tube. Taking into account that the flux inside the tube is constant and equal to $\Phi = HS$, we can write

$$U = \lim_{S\to 0} \frac{1}{\Phi} \int_a^b S dl = \lim_{S\to 0} \int_a^b \frac{S dl}{\Phi} = \lim_{S\to 0} \int_a^b \frac{S dl}{SH} = \int_a^b \frac{dl}{H}, \tag{1.1.37b}$$

where a and b are the start and end of the tube. However, if the magnetic line of force is closed, then

$$U = \oint \frac{dl}{H}. \tag{1.1.37c}$$

Quantity U has a significant role in the criteria of stability of the static plasma configurations.

3. *The beta parameter*. In plasma systems with a magnetic field, the plasma generates its pressure p and the magnetic force its pressure

p_M. Consequently, we can introduce an important dimensionless 'beta parameter'

$$\beta = \frac{p}{p_M} = \frac{8\pi p}{H^2}. \tag{1.1.38}$$

1.1.5. Perturbation of the field morphology

We now examine the effect of perturbations on the morphology of the magnetic lines of force, in particular, the superposition of small random fields on the main field which is assumed to be symmetric[5].

Two completed different situations may form in this case.

(a) The working volume contains segments of the lines of force. In this case, if the strength of the perturbation field **h** is a small in comparison with the strength of the main field $\mathbf{H}_0$, the displacement of the line of force will be on the scale

$$\delta r \sim \int_A^B \frac{h}{H_0} ds \sim \frac{h}{H_0} L, \tag{1.1.39}$$

and at $h/H_0 \to 0$ $|\delta r| \to 0$. Here L is the length of the line of force.

(b) The situation greatly changes if we examine toroidal fields. This is due to the fact that in this case the lengths of the lines of force are usually infinite. Consequently, complete rearrangement of the morphology of the magnetic field can take place. If the amplitude of the pertubing field is small (or, more accurately, as small as possible), the rearrangement of the morphology is 'visually' regular, as shown clearly on the imaging plane. In particular, a peculiar resonance forms here between the harmonics of the perturbing field

$$\mathbf{A}_1 = \sum \mathbf{a}_n(r, z)\cos(n\theta + \alpha_n) \tag{1.1.40}$$

and the lines of force on the rational toroidal surfaces which are closed after n passages of the torus. As a result, the initially infinitely thin resonant surface is split and a chain of 'magnetic islands' forms on the imaged surface (Fig. 1.1.10) or a chain of magnetic fibres, if we are discussing the three-dimensional pattern. The width of the islands increases with the increase of the amplitude of perturbation

[5]This range of phenomena with special refernce fo the problems of controlled thermonuclear synthesis was studied initially by A.I. Morozov, L.S. Solov'ev and I.M. Gel'fand [54].

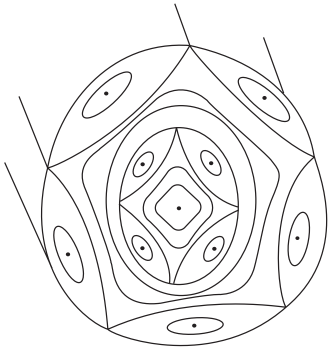


Fig. 1.1.10. Typical scheme of splitting under the effect of weak perturbations of magnetic surfaces, as seen on the representative surface.

and zones with stochastic deposition of the imaging points (‘dynamic chaos’) appear. Separatrix magnetic surfaces are especially sensitive to perturbations. The non-linear stage of the rearrangement can be plotted only by means of numerical calculations. An example of the evolution of the pattern on the imaging plane for a helical field, perturbed by a resonance ‘corrugated’ field

$$\phi_m = z + 3I_3(3r)\sin 3(\phi - z) + \varepsilon I_0(3r)\sin 3z \qquad (1.1.41)$$

at different perturbations ε is shown in Fig. 1.1.11.

It may be seen that a three-petal rosette initially appears in the centre and subsequently the morphology starts to be disrupted as a whole. As indicated by the images, the separatrix is characterised by the highest sensitivity to perturbations. It is shown that in the zone of disruption of the separatrix, the coordinates of the intersection of the lines of force with the imaging plane appear to be random. This phenomenon is referred to as ‘dynamic chaos’. This will be discussed

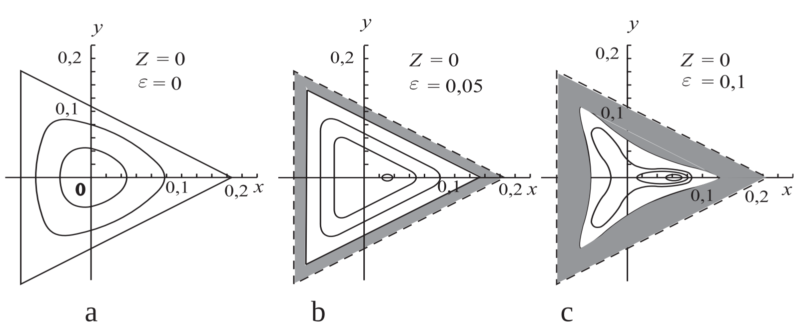


Fig. 1.1.11. Splitting of the surface of the field (1.1.41) at different perturbation amplitudes at ε = 0 (a), ε = 0.05 (b), ε = 0.1 (c); grey colour indicates the region of chaos.

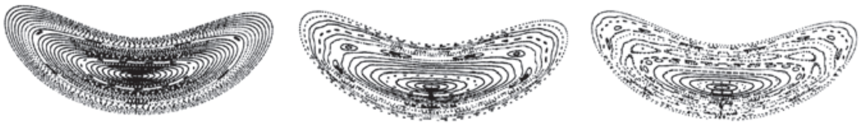


Fig. 1.1.12. Splitting of the magnetic surfaces of the stellarator type field with the increase of plasma pressure: a) $\beta = 0$; b) $\beta = 1.6\%$, c) $\beta = 21\%$.

in greater detail in section 8.4. The appearance of perturbations of the symmetric fields may be associated with various circumstances, often unavoidable. A suitable example is the bending of a straight symmetric helical field into a torus or the perturbation of the magnetic field by the pressure of the plasma situated in it (Fig. 1.1.12). The splitting of the surfaces in a tokamak was detected for the first time in experiments by Mirnov [56].

1.2. Movement of particles in electromagnetic fields

Taking into account the fact that particles in many plasma dynamic systems are subjected to relatively rare collisions. The theory of movement of an individual particle in external electromagnetic fields is of fundamental importance for these systems. This movement is described by the Newton–Lorenz equation:

$$\frac{md\mathbf{v}}{dt} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}]\right); \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}. \tag{1.2.1a}$$

The following Lagrange function corresponds to the above equation:

$$L = \frac{mv^2}{2} + \frac{e}{c}\mathbf{v}\mathbf{A} - e\phi, \tag{1.2.1b}$$

where **A** is the vector potential of the magnetic field and ϕ is the scalar potential of the electrical field. The theory of the solutions of the equation (1.2.1a) has been described in many reviews (see, for example [55]). Here, we shall mention only several factors important for further considerations. The numerical solution of the equations (1.2.1a) for any **E**- and **H**-fields is in most cases quite easy but the analytical approach, although it is efficient only for special $\mathbf{E}(x, t)$, $\mathbf{H}(x, t)$, describes much better many features of the investigated movements.

1.2.1. Conservation laws

If the potentials of the fields (ϕ and **A**) not depend on one of the four coordinates x, y, z, t, then the conservation laws (Neter theorem) corresponds to this coordinate. In particular, if the fields are characterised by azimuthal symmetry, i.e. are independent of θ, then the generalised momentum of the amount of motion is conserved in movement of the particle:

$$mr^2\dot{\theta} + \frac{e}{c}\psi \equiv D \equiv \text{const.} \tag{1.2.2}$$

Here r, θ, and z are the polar coordinates of the particle; ψ is the function of the magnetic flux. It should be mentioned that the conservation of the momentum does not depend on the variability of the fields with time nor on the presence of the axisymmetric electrical field.

The application of the conservation law (1.2.2) reduces the problem of movement of the particle in the space of three measurements (r, θ, z) to the problem of movement of the particle in the two-dimensional space with the coordinates (r, z). These equations have the form

$$m\ddot{r} = -\frac{\partial U(r,z)}{\partial r}; \quad m\ddot{z} = -\frac{\partial U(r,z)}{\partial z}, \tag{1.2.3a}$$

where $U(r, z)$ is the so-called generalised potential:

$$U \equiv e\phi + \frac{e^2}{2mc^2r^2}\left(D - \frac{e}{c}\psi\right)^2. \tag{1.2.3b}$$

The law of conservation of energy is fulfilled if the fields are constant with time

$$\frac{mv^2}{2} + e\varphi = \text{const.} \tag{1.2.4}$$

It should be stressed that the magnetic field is not included in the expression for the law of energy conservation.

1.2.2. Movement of the particle in uniform constant electrical and magnetic fields

If **E** and **H** are constant with spect to magnitude and direction, equation (1.2.1a) is linear in relation to the sought function $\mathbf{v}(t)$. Its properties will be investigated.

Let **H** be parallel to the axis z, and **E** is situated, for example, in plane (y, z). It is natural to expand the movement of the particle into two movements – parallel and normal to the magnetic field

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}.$$

Similarly, it is also convenient to expand **E** (Fig. 1.2.1):

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}.$$

Only $\mathbf{E}_{\parallel}$ acts along the axis z on the particle, and uniform accelerated motion is observed along this coordinate:

$$z = z_0 + V_{\parallel 0} t + \left(\frac{\mathbf{e} E_{\parallel}}{m} \right) \frac{t^2}{2}. \tag{1.2.5}$$

The movement across the magnetic field at $\mathbf{E}_{\perp} = 0$ is reduced to the rotation around the circle with a frequency

$$\omega_H = \frac{eH}{mc} \tag{1.2.6}$$

and the radius

$$\rho_H = \frac{v}{\omega_H}. \tag{1.2.7}$$

This rotation is referred to as Larmor or cyclotron rotation.

Thus, at $\mathbf{E}_{\perp} = 0$ and $\mathbf{E}_{\parallel} \neq 0$ movement takes place along a helix with the pitch increasing along z.

If $\mathbf{E}_{\parallel} = 0$, and $\mathbf{E}_{\perp} \neq 0$, then in addition to Larmor rotation, there is also the 'electrical' drift of the particles along the x axis with the velocity

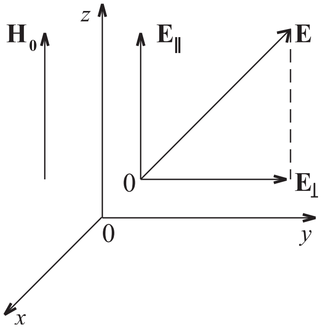


Fig. 1.2.1. Selection of the coordinate system for calculations in section 1.2.2.

$$\mathbf{u}_E = \frac{c[\mathbf{E},\mathbf{H}]}{H^2}. \tag{1.2.8}$$

Consequently, as a result of the linearity of the equation (1.2.1) with the given assumptions, we can write $\mathbf{v}_\perp = \mathbf{u}_E + \mathbf{v}_1$, where $\mathbf{v}_1$ is the velocity of 'natural rotation' in the absence of the electrical field.

Attention should be given to the fact that the velocity of the drift does not depend on the particle charge nor on particle mass.

Equation (1.2.8) shows that the concept of the drift is rational (i.e. $\mathbf{u} < \mathbf{c}$) if $E_\perp < H$.[6] This always takes place if the magnetic field is strong. Otherwise, the presence of the magnetic field can be generally ignored. Evidently, if $H > E_\perp$, then only the rotation around the Larmor circle remains in the reference system moving with the drift velocity. The electrical field disappears in this reference system. This claim is general and directly follows from the Lorenz transform equations for the electromagnetic field. If $E > H$, we can transfer to a system in which the **H**-field disappears, and the particle moves only under the effect of the **E**-field.

Depending on the ratio of the drift velocity u_E and the velocity of movement around a circle with a radius $\rho_H = v_1/\omega_H$, the form of the trajectories of the particles differs (Fig. 1.2.2):

a) $u = 0$, $v_1 \neq 0$ – circle;
b) $u < v_1$ – trochoid;
c) $u = v_1$ – cycloid;
d) $u > v_1$ – hypocycloid, similar to a sinusoid.

However, if $\mathbf{v}_1 \to 0$, the trajectory degenerates into a straight line.

A particle which at some "initial moment" ceases to move (rest state) moves along the cycloid (Fig. 1.2.2c). The height of the cycloid is

$$h = \frac{2u}{\omega_H} = \frac{2c^2Em}{eH^2}. \tag{1.2.9}$$

In conclusion, it should be mentioned that if the electrical force is replaced by some other force, for example, the gravitational force, the movement of the charge particle will be identical to movement under the effect of the electrical field. In particular, the velocity of the drift is

[6] In order to transfer to the case in which $E_\perp > H$, it is necessary to consider the relativistic mechanics and, consequently, we obtain the general drift equation $\mathbf{v} = c\alpha\mathbf{K}$, where $\mathbf{K} = \frac{[\mathbf{E},\mathbf{H}]}{E^2 + H^2}$, $\alpha = \frac{2}{1+\sqrt{1-4K^2}}$.

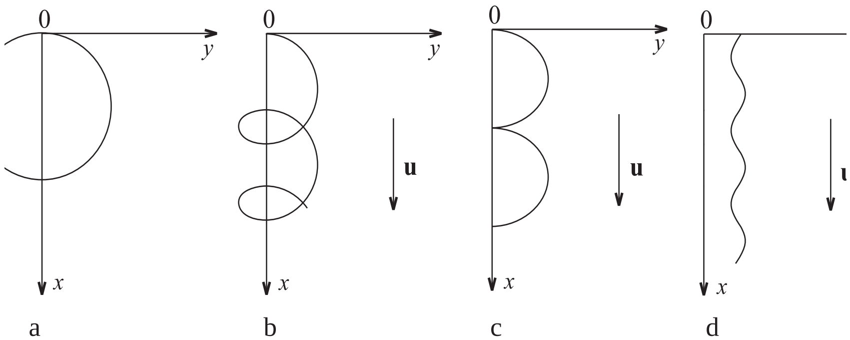


Fig. 1.2.2. Trajectory of the particles in relation to the ratio of the speed of Larmor rotation and the drift speed: a) circle, b) trochoid, c) cycloid; d) hypocycloid.

$$\mathbf{u}_F = \mathbf{c}\frac{[\mathbf{F},\mathbf{H}]}{\mathbf{e}H^2}. \qquad (1.2.10)$$

but now the direction of the drift depends on the sign of the charge.

1.2.3. Dynamics of particles in constant magnetic and alternating electrical field

A simple but important case is the movement of particles in uniform fields in which the magnetic field is constant and the electrical field depends on time t only

$$m\frac{d\mathbf{v}}{dt} - \frac{e}{c}[\mathbf{v},\mathbf{H}_0] = e\mathbf{E}(t). \qquad (1.2.11)$$

These equations are used for the analysis of linear waves in uniform cold plasma in a magnetic field (section 1.5). For this reason, we confine ourselves to the case in which the dependence of $\mathbf{E}$ on t is harmonic[7]

$$\mathbf{E}(t) = \mathbf{E}_1 e^{-i\omega t}, \qquad (1.2.12a)$$

and the process may be regarded as steady, i.e.

$$\mathbf{v} = \mathbf{v}_1 e^{-i\omega t}. \qquad (1.2.12b)$$

Here $\mathbf{E}_1$ and $\mathbf{v}_1$ are the amplitudes of oscillations.

[7]The real field $\mathbf{E}(t)_{real} = Re\mathbf{E}_1 e^{-i\omega t}$ is the real part of the field (1.2.12a).

Equation (1.2.11) with these assumptions can be solved efficiently in the Cartesian coordinates, using the direction of the magnetic field $\mathbf{H}_0$ as the z axis. The remaining axis are selected arbitrarily. Consequently

$$m\frac{dv_x}{dt}-\frac{e}{c}v_y H = eE_x;$$

$$m\frac{dv_y}{dt}-\frac{e}{c}v_x H = eE_y; \qquad (1.2.13)$$

$$m\frac{dv_z}{dt}=eE_z.$$

Substituting (1.2.12) into (1.2.13) we obtain a system of linear algebraic equations

$$-i\omega v_x-\omega_H v_y=\left(\frac{e}{m}\right)E_x,$$

$$-i\omega v_y+\omega_H v_x=\left(\frac{e}{m}\right)E_y, \qquad (1.2.14)$$

$$-i\omega v_z=\left(\frac{e}{m}\right)E_z,$$

From this:

$$v_x=\frac{e}{m}\frac{(-i\omega E_x+\omega_H E_y)}{\omega_H^2-\omega^2};\; v_y=\frac{e}{m}\frac{(-\omega_H E_x-i\omega E_y)}{\omega_H^2-\omega^2};\; v_z=-\frac{ie}{m\omega}E_z. \qquad (1.2.15)$$

or in the tensor form

$$\mathbf{v}=(v_x,v_y,v_z)=\frac{e}{m}\begin{pmatrix}-\frac{i\omega}{\Omega^2} & \frac{\omega_H}{\Omega^2} & 0\\ -\frac{\omega_H}{\Omega^2} & -\frac{i\omega}{\Omega^2} & 0\\ 0 & 0 & \frac{i}{\omega}\end{pmatrix}\begin{pmatrix}Ex\\ Ey\\ Ez\end{pmatrix}, \qquad (1.2.16a)$$

where $\Omega^2\equiv\omega_H^2-\omega^2$.

If we know the velocity, the displacement of the particle can be determined automatically from

$$\xi = \frac{i}{\omega}\mathbf{v} = \overleftrightarrow{S}\mathbf{E}. \tag{1.116b}$$

Here $\overleftrightarrow{S}$ is the displacement tensor:

$$\overleftrightarrow{S} = \frac{ie}{\omega m}\begin{pmatrix} -\dfrac{i\omega}{\Omega^2} & \dfrac{\omega_H}{\Omega^2} & 0 \\ -\dfrac{\omega_H}{\Omega^2} & -\dfrac{i\omega}{\Omega^2} & 0 \\ 0 & 0 & \dfrac{i}{\omega} \end{pmatrix}. \tag{1.2.16c}$$

This equation will be used in the section 1.5.

1.2.4. Movement of a particle in a nonuniform high-frequency field

The plasma systems often contain two characteristic time or space scales. For example, in the Earth magnetosphere the electron rotates around a Larmor circle with a diameter of several centimetres and at the same time along a line of force, reflecting from some ‘plugs’ situated at a distance of ~10^4 km (see section 9.2). Evidently, in this case we are not interested in every Larmor circle; the displacement of the particles averaged out with respect to the rotation over a relatively long time t is more important.

The method of transformation of the exact equation of motion in the equation which contains only parameters of the averaged motion is referred to as the ‘averaging method’ [57].

In this section, we apply this method to the simplest example: to one-dimensional movement in an electrical field rapidly oscillating in time and non-uniform in space.

$$m\frac{d^2x}{dt^2} = eE_0(x)\sin\omega t. \tag{1.2.17}$$

This equation describes, for example, the movement of the electron in a flat high-frequency resonator (Fig. 1.2.3).[8] We are interested in the case in which the field carries out a large number of rotations during

[8] It should be mentioned that the modulation of density in the plasma volume can often play the role of a resonator Langmuir oscillations (chapter 8).

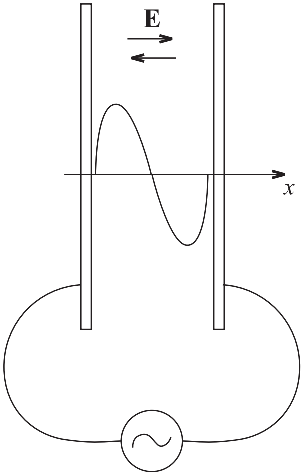


Fig. 1.2.3. Electron in a high-frequency resonator.

time τ, i.e. the time required for the electron to pass through the scale of the non-uniformity scale of the field:

$$\omega\tau \gg 2\pi. \tag{1.2.18}$$

To solve the equation (1.2.17) under the condition (1.2.18), the coordinate of the particle x is regarded as the sum of the coordinate of the average position $\overline{x}(t)$ and the high-frequency displacement $\xi(t)$:

$$x = \overline{x}(t) + \xi(t). \tag{1.2.19a}$$

ξ is selected accurately on the basis of the condition:

$$\int_0^T \xi(t)dt = 0, \quad T \equiv \frac{2\pi}{\omega}. \tag{1.2.19b}$$

Substituting (1.2.19a) into equation (1.2.17) and confining ourselves to 'linear' terms $\sim\xi$:

$$\frac{d^2\overline{x}}{dt^2} + \frac{d^2\xi}{dt^2} = \frac{e}{m}\left[E(\overline{x}) + \frac{\partial E(\overline{x})\xi}{\partial(\overline{x})}\right]\sin\omega t. \tag{1.2.20a}$$

The formal integration of this equation with respect to t in the period T and subsequent division by T gives

$$\frac{d^2\overline{x}}{\partial t^2} = \frac{e}{m}\frac{\partial E_0}{\partial \overline{x}}\left\langle \xi \sin\omega t\right\rangle \tag{1.2.20b}$$

where