Some Notes on Discrete Derivatives

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1 Basic concepts

Let k be a field containing $\mathbb{Q}$ (e.g. $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}, ...$), so that $\mathbb{N} \subseteq k$. Then let $S = k^{\mathbb{N}}$ be the ring of sequences $(f(0), f(1), f(2), \cdots)$ with entries in k (i.e. function $f : \mathbb{N} \to k$), where addition and multiplication are defined entry-wise. A polynomial $f \in k[x]$ defines a function $k \to k$ by restriction $f|_{\mathbb{N}} : \mathbb{N} \to k$. Since $\mathbb{N}$ is infinite, the map $f \mapsto f|_{\mathbb{N}}$ is injective (**Exercise**: Prove this!), and we may therefore view $k[x]$ as a subring of S. For $f \in S$, we define four k-linear operators $S \to S$, Δ (the discrete derivative), Σ (the partial sum operator), σ_L (left shift), and σ_R (right shift) as follows:

$$\begin{aligned}
(\Delta f)(x) &= f(x+1) - f(x) \\
(Sf)(x) &= f(0) + f(1) + \cdots + f(x-1) \\
(\sigma_L f)(x) &= f(x+1) \\
(\sigma_R f)(x) &= f(x-1)
\end{aligned}$$

with zero entries filled in as appropriate. For example, if $f = (6, 5, 10, 3, 4, \cdots)$, then

$$\begin{aligned}
\Delta f &= (-1,\ 5,\ -7,\ 1,\ \cdots) \\
Sf &= (0,\ 6,\ 11,\ 21,\ 24,\ 28,\ \cdots) \\
\sigma_L f &= (5,\ 10,\ 3,\ 4,\ \cdots) \\
\sigma_R f &= (0,\ 6,\ 5,\ 10,\ 3,\ 4,\ \cdots)
\end{aligned}$$

Proposition 1. Let $f \in S$ be a sequence.

(i) $\Delta f = 0$ if and only if f is a constant sequence. Thus, $\Delta f = \Delta g$ if and only if $g = f + c$, where c is a constant sequence.

(ii) If f is polynomial[1] of degree $d > 0$, then Δf is polynomial of degree $d - 1$.

(iii) Let c be a constant sequence. For any $n > 0$, there is a polynomial g of degree n such that $c = \Delta^n g$.

(iv) $\Delta^n f = 0$ if and only if f is a polynomial of degree $< n$.

Proof. (i) "If" is obvious. For the other implication, say $\Delta f = 0$. Then for any $x \in \mathbb{N}$, $f(x+1) = f(x)$ and by induction we see that f is a constant sequence equal to $f(0)$. Since Δ is k-linear, $\Delta f = \Delta g$ means $\Delta(f - g) = 0$ if and only if $f - g = c$ for some constant c, as desired.

(ii) Let f be polynomial of degree $d > 0$, say obtained as the restriction of $ax^d + bx^{d-1} + \cdots$ for $a \neq 0$. We can compute

$$\begin{aligned}
(\Delta f)(x) = f(x+1) - f(x) &= a(x+1)^d + b(x+1)^{d-1} - ax^d - bx^{d-1} + [\text{lower degree terms}] \\
&= a(x^d + dx^{d-1}) + bx^{d-1} - ax^d - bx^{d-1} + [\text{lower degree terms}] \\
&= adx^{d-1} + [\text{lower degree terms}]
\end{aligned}$$

[1] Call a sequence $f \in S$ *polynomial of degree d* if there is a (necessarily unique) polynomial $\widetilde{f} \in k[x]$ such that $f = \widetilde{f}|_{\mathbb{N}}$.

and we see that Δf has degree precisely $d-1$.

(iii) Let g_0 be the sequence described by $g_0(x) = x^n$. By a repeated application of (ii), $\Delta^n g_0$ is a polynomial of degree 0, i.e., a nonzero constant sequence a (in fact, $a = n!$). Let $g(x) = (c/a)g_0(x)$, and we can see by k-linearity that $\Delta^n g = c$.

(iv) The "if" direction follows from (ii). For the "only if" direction, we showed in (i) that if $\Delta^1 f = 0$, then f has degree < 1 (i.e., f is constant). Fix $n > 1$, let $\Delta^n f = 0$, and suppose for the sake of induction that we've shown that for all $g \in S$ and all $m < n$, $\Delta^m g = 0$ implies $\deg g < m$. Note that $\Delta(\Delta^{n-1} f) = 0$, so by (i), $\Delta^{n-1} f$ is equal to some constant c. By (iii), there is a polynomial g of degree $n-1$ such that $c = \Delta^{n-1} g$. Since the operator Δ^{n-1} is k-linear and $\Delta^{n-1} f = \Delta^{n-1} g$, we have $\Delta^{n-1}(f-g) = 0$, and by the inductive hypothesis, $f - g$ is a polynomial p of degree strictly smaller than $n-1$. Thus, $f = g + p$ is a polynomial of degree $\deg f \leq \deg g < n$, as desired. □

2 The Fundamental Theorem of Difference Calculus

Call a sequence $g : \mathbb{N} \to k$ an *antiderivative* for $f : \mathbb{N} \to k$ if $f = \Delta g$. In Proposition 1, we showed that any two antiderivatives of f differ by a constant. Below, we will show that for any sequence $f \in S$, the partial sum sequence Σf is always an anti-derivative.

Proposition 2. Let $f \in S$ be a sequence. Then for all $x \in \mathbb{N}$,

$$(\Delta \Sigma f)(x) = f(x)$$
$$(\Sigma \Delta f)(x) = f(x) - f(0)$$

Proof. Both calculations are straightforward.

$$\begin{aligned}
(\Delta(\Sigma f))(x) &= \Sigma f(x+1) - \Sigma f(x) \\
&= (f(0) + f(1) + \cdots + f(x-1) + f(x)) - (f(0) + f(1) + \cdots + f(x-1)) \\
&= f(x).
\end{aligned}$$

When we compute $\Sigma(\Delta f)$, we obtain the following telescoping sum:

$$\begin{aligned}
(\Sigma(\Delta f))(x) &= (\Delta f)(0) + (\Delta f)(1) + (\Delta f)(2) + \cdots + (\Delta f)(x-1) \\
&= [f(1) - f(0)] + [f(2) - f(1)] + [f(3) - f(2)] + \cdots + [f(x) - f(x-1)] \\
&= f(x) - f(0).
\end{aligned}$$

□

From this, we can deduce that every polynomial sequence has an antiderivative. Is that antiderivative itself a polynomial sequence? If so, what is its degree? The general case will follow (by k-linearity) from the case of monomials.

Proposition 3. For $d \geq 0$, the monomial sequence f described by $f(x) = x^d$ has a polynomial antiderivative of degree $d+1$.

Proof. The case $d = 0$ is the case $n = 1$ of Proposition 1, part (iii). So, suppose for the sake of induction that the result holds for all degrees $e < d$. Take $f(x) = x^d$, let $h(x) = x^{d+1}$, and compute

$$\begin{aligned}
(\Delta h)(x) &= (x+1)^{d+1} - x^{d+1} \\
&= \left(x^{d+1} + dx^d\right) - x^{d+1} + [\text{lower degree terms}]
\end{aligned}$$

By the inductive hypothesis, each term cx^e in $(\Delta h)(x)$ of degree strictly smaller than d has an polynomial antiderivative of degree $e + 1 \leq d$. Factoring out Δ by k-linearity, we obtain

$$(\Delta h)(x) = dx^d + (\Delta p)(x)$$

for p some polynomial of degree at most d. Thus,

$$f = \Delta \left(\frac{h - p}{d} \right)$$

with $h - p$ a polynomial of degree exactly $d + 1$, as desired. $\square$

Theorem 1. Every polynomial function $f : \mathbb{N} \to k$ of degree d has a polynomial antiderivative of degree $d + 1$.

Proof. Write $f(x) = \sum_{i=0}^{d} a_i x^i$, and for each i, let $g_i(x)$ be the antiderivative of x^i guaranteed to exist by Proposition 3. Then

$$f(x) = \sum_{i=0}^{d} a_i (\Delta g_i)(x) = \left[\Delta \left(\sum_{i=0}^{d} a_i g_i \right) \right] (x).$$

The g_i with the largest degree is g_d, of degree $d + 1$. $\square$

Corollary. If f is a polynomial of degree $d \geq 0$, then Σf is a polynomial of degree $d + 1$.

3 The Binomial Sequences

For each $d \geq 0$, define the dth *binomial sequence* B_d by

$$B_d(x) = \frac{x(x-1)\cdots(x-(d-1))}{d!} \in k[x] \subseteq S.$$

By convention, B_0 is the constant sequence equal to 1, and note that $B_1(x) = x$. We may write

$$B_d = \left(\underbrace{0, \cdots, 0}_{d \text{ times}}, \binom{d}{d}, \binom{d+1}{d}, \binom{d+2}{d}, \binom{d+3}{d}, \cdots \right)$$

e.g.,

$$B_2 = \left(0, 0, \binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \cdots \right)$$

Proposition 4. The binomial sequences $B_0, \cdots, B_d$ are a k-basis for the subspace of S consisting of all polynomial sequences of degree $\leq d$.

Proof. Each B_d for $d \geq 1$ is a product of d linear polynomials, and hence has degree d. A set of polynomials, each of different degree, is automatically linearly independent. The $d + 1$ linearly independent polynomial sequences $B_0, B_1, \cdots, B_d$ inside the $d + 1$ dimensional space of all polynomial sequences of degree at most d must be a basis. $\square$

Why would we use the basis $B_0(x), \cdots, B_d(x)$ instead of the more obvious basis $x^0, \cdots, x^d$? One advantage of using the binomial sequences is that they behave better with respect to the property of being *integer valued.* We will call a polynomial $f \in k[x]$ *integer valued* if $f(\mathbb{Z}) \subseteq \mathbb{Z}$. It is possible for a polynomial to be integer valued even though its expansion in the $x^0, \cdots, x^d$ basis uses rational numbers. For example, $B_2(x) = x(x-1)/2 = -\frac{1}{2}x + \frac{1}{2}x^2$ is an integer valued polynomial whose coefficients in the x^0, x^1, x^2 basis (for the space of degree ≤ 2 polynomial sequences) are non-integer rationals. It turns out that this doesn't happen with the B_0, B_1, B_2 basis for the same space.

Theorem 2. Let the binomial sequences B_d be defined as above.

(i) The binomial sequences are integer valued: $B_d(\mathbb{Z}) \subseteq \mathbb{Z}$.

(ii) If $f = \sum_d c_d B_d$ is a polynomial, then $f(\mathbb{Z}) \subseteq \mathbb{Z}$ if and only if $c_d \in \mathbb{Z}$ for all d.

Proof. For (i), there are three relevant ranges of integers to consider.

- If $a = 0, 1, \cdots, (d-1)$, then $B_d(a) = 0 \in \mathbb{Z}$.
- If $a \geq d$, then $B_d(a) = \binom{a}{d} \in \mathbb{Z}$.
- If $a = -b$ for $b > 0$, then

$$B_d(a) = (-1)^d \frac{b(b+1)\cdots(b+(d-1))}{d!} = (-1)^d \binom{b+(d-1)}{d} \in \mathbb{Z}.$$

(ii) Since $B_d(\mathbb{Z}) \subseteq \mathbb{Z}$ for all d, if we know that all coefficients c_d are integers, then clearly $f(x) = \sum_d c_d B_d(x) \in \mathbb{Z}$ for any $x \in \mathbb{Z}$. So, suppose that at least one coefficient of f is *not* in $\mathbb{Z}$, and take c_{d_0} to be the one with the largest index $d_0 \in \mathbb{N}$. We briefly take note of the property that $B_d(x) = 0$ for any natural number $x < d$, with $B_d(d) = 1$. Thus,

$$f(d_0) = \sum_d c_d B_d(d_0) = c_{d_0} + \sum_{d > d_0} c_d B_d(d_0).$$

Since all coefficients of f indexed higher than d_0 must take integer values, $\sum_{d>d_0} c_d B_d(d_0)$ is an integer. If, in addition, $f(d_0)$ were an integer, then we would have

$$c_{d_0} = f(d_0) - \sum_{d>d_0} c_d B_d(d_0) \in \mathbb{Z}$$

which is a contradiction. Thus, for some integer d_0, $f(d_0) \notin \mathbb{Z}$, as desired. $\square$

Proposition 5. $\Delta B_d = B_{d-1}$ and $\Sigma B_d = B_{d+1}$.

Proof. We can show this directly.

$$\begin{aligned}
(\Delta B_d)(x) &= \frac{(x+1)(x)(x-1)\cdots(x-d+2)}{d!} - \frac{x(x-1)(x-2)\cdots(x-d+1)}{d!} \\
&= \frac{x(x-1)\cdots(x-d+2)}{d!}\left((x+1)-(x-d+1)\right) \\
&= \frac{x(x-1)\cdots(x-((d-1)-1))}{(d-1)!} = B_{d-1}(x).
\end{aligned}$$

For the corresponding result with the Σ operator, we use the Fundamental Theorem to see that $B_{d+1}(x) = B_{d+1}(x) - B_{d+1}(0) = (\Sigma\Delta B_{d+1})(x) = (\Sigma B_d)(x)$. $\square$

4 Examples

We show how x^n can be written as a linear combination of the B_d for $n = 0, 1, 2, 3$.

- $(n = 0)$ $x^0 = 1 = B_0(x)$.
- $(n = 1)$ $x^1 = x/1! = B_1(x)$.
- $(n = 2)$ We have

  $$B_2(x) = \frac{x(x-1)}{2} = \frac{1}{2}(x^2 - x)$$

  hence $x^2 = 2B_2(x) + B_1(x)$.
- $(n = 3)$ Expand

  $$B_3(x) = \frac{x(x-1)(x-2)}{6} = \frac{1}{6}(x^3 - 3x^2 + 2x)$$

  giving

  $$\begin{aligned}
  x^3 &= 6B_3(x) + 3x^2 - 2x \\
  &= 6B_3(x) + 6B_2(x) + B_1(x).
  \end{aligned}$$

We can use this to obtain a nice formula for the sequence $S_n = 1^n + 2^n + \cdots + (x-1)^n$ for $n = 1, 2, 3$. We remark that $1^d + 2^d + \cdots + (x-1)^d = (Sf_d)(x)$ where $f_d(x) = x^d$. Given an expansion $x^d = \sum_i a_i B_i(x)$, Proposition 5 tells us that $(Sf_d)(x) = \sum_i a_i B_{i+1}(x)$.

- $(n = 1)$ Recall our expressions for the various x^n in terms of B_d

$$1^1 + 2^1 + \cdots + (x-1)^1 = \Sigma x = \Sigma B_1(x) = B_2(x) = \frac{x(x-1)}{2}.$$

- $(n = 2)$ We have

$$\begin{aligned} 1^2 + 2^2 + \cdots + (x-1)^2 = \Sigma x^2 = \Sigma(2B_2(x) - B_1(x)) &= 2B_3(x) + B_2(x) \\ &= \frac{x(x-1)(x-2)}{3} + \frac{x(x-1)}{2} \\ &= \frac{x(x-1)(2x-1)}{6} \end{aligned}$$

- $(n = 3)$ Finally,

$$\begin{aligned} 1^3 + 2^3 + \cdots + (x-1)^3 &= 6B_4(x) + 6B_3(x) + B_2(x) \\ &= \frac{x(x-1)(x-2)(x-3)}{4} + \frac{4x(x-1)(x-2)}{4} + \frac{2x(x-1)}{4} \\ &= \frac{x(x-1)}{4}\left((x-2)(x-3) + 4(x-2) + 2\right) \\ &= \frac{x^2(x-1)^2}{4}. \end{aligned}$$