CONDITONS ON THE EXISTENCE OF λ FOR CONSTANT AND LINEAR HILBERT POLYNOMIALS

FAUSTAS UDRENAS

Theorem 0.1 (Zhang, Donato, Udrenas; (2020)). *Let $N > 0$ and $H_I(d)$ be a Hilbert Polynomial in $N + 1$ variables.*

- *For $H_I(d) = K$ for some $K \in \mathbb{R}$, then λ exists if and only if $K \in \mathbb{N}$. λ takes the form*
$$\lambda = (1^{[K]})$$
- *For $H_I(d) = Md - r$ for some $M, r \in \mathbb{R}, M \neq 0$, λ exists if and only if $M \in \mathbb{N}$, $r \in \mathbb{Z}$ and*
$$r \leq \frac{M^2 - 3M}{2}$$
λ takes the form
$$\lambda = (2^{[M]}, 1^{[\frac{M^2-3M}{2} - r]})$$

Proof for constant Linear Polynomials $H_I(d) = K$:
" $\implies$ "
Assume that λ exists.
First we show that for all $\lambda_i \in \lambda, \lambda_i = 1$. Consider λ_1. By Macaulays theorem, this is the largest lambda value in the lambda partition. Assume $\lambda_1 = r > 1\ rin\mathbb{N}$. Thus by Macaulays theorem, $H_I(d)$ contains the term

$$\binom{d+r-1}{r-1} = \frac{(d+r-1)!}{(r-1)!(d)!} = \frac{(d+r-1)\cdot(d+r-2)\cdot\ldots\cdot(d+1)}{(r-1)!}$$

But notice that this would imply that $H_I(d)$ contains a d^{r-1} term, which is a contradiciton since $r-1 \geq 1$ and $H_I(d) = K$ is a constant polynomial. Thus we conclude $r \leq 1$ but by Macaulys theorem, we have that $r = 1$. Thus we have that $\lambda_i = 1\ \ \forall \lambda_i \in \lambda$ but notice that

$$\binom{d+\lambda_i-i}{\lambda_i-1} = \binom{d+1-i}{0} = 1$$

Thus if λ exists and $\lambda_i = 1$ for all $\lambda_i \in \lambda$ then

$$\sum_{i=1}^{|\lambda|} \binom{d+\lambda_i-1}{\lambda_i-1} = |\lambda| \cdot 1 = |\lambda|$$

where $|\lambda|$ is the number of $\lambda_i \in \lambda$ Thus this implies

$$K = |\lambda|$$

and since $|\lambda| \in \mathbb{N}$ we have that $K \in \mathbb{N}$. From this we conclude

$$\lambda = (1^{[K]})$$

"$\Longleftarrow$"
Assume $K \in \mathbb{N}$.
Then we can write $H_I(d)$ as

$$H_I(d) = 1 + 1 + \ldots + 1 = K$$

and note

$$\binom{d+1-i}{1-1} = \binom{d+1-i}{0} = 1, \forall i$$

so

$$H_I(d) = \binom{d+1-1}{1-1} + \binom{d+1-2}{1-1} + \ldots + \binom{d+1-K}{1-1} = K \quad \implies \lambda = (1^{[K]})$$

$\square$

Proof for Linear Hilbert Polynomials $H_I(d) = Md - r$:
" $\implies$ "
Assume λ exists. Now first we show that for all $\lambda_i \in \lambda, \lambda_i = 1$ or $\lambda_i = 2$. Consider λ_1. By Macaulays theorem, this is the largest lambda value in the lambda partition. Assume $\lambda_1 = F > 2$, $F \in \mathbb{N}$. Thus by Macaulays theorem, $H_I(d)$ contains the term

$$\binom{d+F-1}{F-1} = \frac{(d+F-1)!}{(F-1)!(d)!} = \frac{(d+F-1)\cdot(d+F-2)\cdot\ldots\cdot(d+1)}{(F-1)!}$$

But this would imply that $H_I(d) = Md - r$ contains a d^{F-1} term and $F - 1 \geq 2$ which clearly is a contradiction. Thus $F \leq 2$. Now notice that for $\lambda_i = 2$ tells us that the following term is in the sume that makes up $H_I(d)$

$$\binom{d+2-i}{2-1} = \binom{d+2-i}{1} = d+2-i$$

Recall that the term correpsonding to when λ_i has no linear term in d, thus, the number of 2's in the λ partition determine the coefficient of the linear term of the hilbert polynomial $H_I(d)$. Thus, $M \in \mathbb{N}$.
However notice that for $\lambda = (2^M)$ we have that

$$H_I(d) = \sum_{i=1}^{M}\binom{d+2-i}{1} = \sum_{i=1}^{M} d+2-i = \sum_{i=1}^{M} d+\sum_{i=1}^{M} 2-\sum_{i=1}^{M} i = Md+2M-\frac{(M)(M+1)}{2}$$

$$\implies H_I(d) = Md - \frac{M^2-3M}{2}$$

Thus notice that for any 1's in the λ-partition will only add 1's to this polynomial. Thus if $r > \frac{M^2-3M}{2}$ then λ cannot exist, thus

$$r \leq \frac{M^2-3M}{2}$$

Additionally notice that $\frac{M^2-3M}{2}$ is always an integer for any $M \in \mathbb{N}$. Thus adding ones to $\frac{M^2-3M}{2}$ still results in an integer value. Thus r is an integer.

" $\Longleftarrow$ " Let $H_I(d) = Md - r$, $M \in \mathbb{N}$, $r \in \mathbb{Z}$ and

$$r \leq \frac{M^2 - 3M}{2}$$

Then choose $\lambda = (2^{[M]}, 1^{[\frac{M^2-3M}{2} - r]})$ Thus we get that

$$\lambda \implies \sum_{i=1}^{\frac{M^2-3M}{2}-r+M} \binom{d+\lambda_i - i}{\lambda_i - 1} = \sum_{i=1}^{M}(d+2-i) + \sum_{i=M+1}^{\frac{M^2-3M}{2}-r+M} 1$$

$$= Md + \frac{M^2-3M}{2} + \frac{M^2-3M}{2} - r = Md - r$$

so for this Hilbert polynomial with these conditions, the λ exists.

$\square$

Conjecture 0.2. *Let $N = 3$ and $H_I(d)$ be a Hilbert Polynomial in $N+1$ variables.*
λ exists if and only if all conditions below are satisfied

- $H_I(d) = ax^2 + x + c$ *for some* $a, b \in \mathbb{Q}, c \in \mathbb{Z}$
- a *is a multiple of* $\frac{1}{2}$
- $b - (2a[2-a]) \in \mathbb{N}$
- $c - \frac{1}{3}\left(4a^3 - 12a^2 + 11a\right) \geq \left[(2a^2-4a-b)(2-2a) - \left(\frac{(2a^2-4a-b)(2a^2-4a-b+1)}{2}\right)\right]$