Python/MATLAB for Physics
USPAS February 2022
Homework 2 Assignment

February 9, 2022

Due: 10am cst, Fri Feb. 11th

Problem 2.1: Numerical Integration of a Synchrotron Oscillation

The synchrotron equations of motion for a particle in a stationary RF bucket are given by:

$$\frac{d\delta}{dt} = \left(\frac{eV}{\beta^2 E}\right) f_0 \sin\phi$$
$$\frac{d\phi}{dt} = (2\pi h\eta) f_0 \delta$$

The synchrotron tune (for small oscillations) is given by:

$$\nu_s = \sqrt{2\pi h|\eta| \left(\frac{eV}{\beta^2 E}\right)}$$

The Hamiltonian for the synchroton equations of motion is given by:

$$H = \frac{1}{2}(2\pi h|\eta|) f_0 \delta^2 + \left(\frac{eV}{\beta^2 E}\right) f_0 \left(1 - \cos\phi\right)$$

Program

Write a MATLAB program to calculate the trajectory of a particle governed by the synchrotron equations of motion. Implement a symplectic numerical integration method (such as Euler-Cromer), with a timestep of f_0^{-1}. Use $\frac{eV}{\beta^2 E} = 0.05$ and $2\pi h\eta = -0.05$.

Plot $\delta(t)$ of a particle with several initial coordinates, with $\delta_0 = 0$ and with ϕ_0 varying from 0 to π. Use linecolors that fall along a colormap (or a continuous gradient) and a legend that identifies each line. It should be clear from the plot that the higher oscillation amplitudes have lower frequencies of oscillation.

For one of the (nonzero) particle trajectories, calculate the value of H at each timestep and plot $(H(t) - H_0)/H_0$. Is this Hamiltonian precisely conserved? Does the trajectory appear to be symplectic?

Problem 2.2: Stability of a FODO Cell

Linear beamline elements (dipole, drift spaces, and quadrupoles) change the trajectory of a particle in a manner that can be represented by a transfer matrices:

$$M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

In the thin lense approximation, the transfer matrices associated with a dipole (D) of bending radius ρ and bending angle θ, a drift space (O) of length L, a focusing quadrupole (Q_F) of focal length f, and a defocusing quadrupole (Q_D) of focal length f are given by:

$$D = \begin{pmatrix} \cos(\theta) & \rho\sin(\theta) \\ -\frac{\sin(\theta)}{\rho} & \cos(\theta) \end{pmatrix},\ O = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix},\ Q_F = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix},\ Q_D = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

A dipole in the horizontal plane is a drift space of length $\rho\theta$ in the vertical plane. A horizontally focusing quadrupole is vertically defocusing quadrupole and a horizontally defocusing quadrupole is vertically focusing quadrupole.

The transfer matrix for a series of beamline elements can be obtained by the product of the transfer matrices for each beamline element. A series of beamline elements with transfer matrix M is stable if the following condition is met

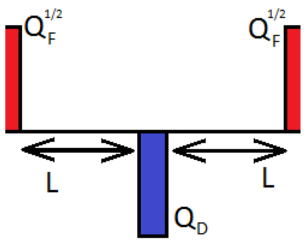
$$\frac{1}{2}\left|\text{Trace}(M)\right| < 1$$

in both the horizontal and the vertical plane.

A Focusing-defocusing (FODO) lattice is a series equally spaced quadrupoles that alternate between focusing and defocusing. A FODO cell is composed of the following beamline elements acting on the horizontal plane:

$Q_{F_1}^{1/2}$: A focusing quad with focal length $2f_1$,
O a drift space of length L,
Q_{D_2}: a defocusing quad with focal length f_2,
O a drift space of length L, and then
$Q_{F_1}^{1/2}$: a focusing quad with focal length $2f_1$.

The FODO cell is shown schematically below:



Program

Write a Python program to multiply the transfer matrices associated with each beamline element of a FODO cell, to obtain the transfer matrix for the overall FODO cell. Calculate the transfer matrix for the FODO cell for both the horizontal and vertical plane. Output the stability condition for both the horizontal and vertical plane. Make a plot of the stable values of $\frac{L}{2f_1}$ and $\frac{L}{2f_2}$ from 0 to 1.

Problem 2.3: Damped Driven Harmonic Oscillator

The second-order differential equation for a damped harmonic oscillator is

$$\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = 0$$

where ω is the characteristic frequency and where β is a damping constant.

If we add an external sinesoidal driving force to the system, the second-order differential equation becomes

$$\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = f_0\sin(\omega_0 t) \tag{1}$$

where f_0 is the amplitude of the sinesoidal driving force and ω_0 is the freqquency of the sinesoidal driving force.

Program

Write a program to analytically solve the second-order differential equation for the damped harmonic oscillator with an external sinesoidal driving force (given in Eq. 1 above). Output the right-hand-side of the solution.

To obtain what is known as the particular solution, define `C1` and `C2` as symbolic variables. Take the right-hand-side of the solution, and make the substitutions `C1=0` and `C1=0`.

Obtain the coefficient of the $\cos(\omega_0 t)$ term by making the substitution `t=0`. Multiply this expression by ω^2/f_0. This is the on-phase amplitude of the particular solution, in units of $f_0\omega^{-2}$.

Plot this expression for $\beta = 0.1\omega$ and for ω_0 varying from 0 to 2ω. On the same figure, plot the expression for $\beta = 0.05\omega$ and for ω_0 varying from 0 to 2ω.