Chapter 1

Truth Trees With Identity

We introduced Identity in Chapter ??, and Identity tables in Chapter ??. It should come as no surprise that we can also add Identity to Truth Trees.

The rules for Identity are a little different from the other Truth Tree rules.

The first rule is that we can always introduce the logical truth $(\alpha = \alpha)$ into any branch, for any name α.

$$\alpha = \alpha \qquad \texttt{id}$$

The only use for this rule is to close branches that already contain $(\alpha \neq \alpha)$, which is shorthand for $\neg(\alpha = \alpha)$.

The second rule is more complicated. If one formula in the current branch is an identity statement $\alpha = \beta$, then we can substitute some or all instances of α with β, or β with α, in a copy of another formula in the current branch.

$$\begin{array}{cl} \alpha = \beta & \\ A & \\ A[\alpha/\beta] & \qquad \texttt{sub} \end{array}$$

This subrule uses 2 formulas (the identity, and the formula being copied), so both need to be listed. But we aren't decomposing these formulas, so they aren't ticked (✓) – they can be used as many times as needed. We represent this rule by $[\alpha/\beta]$, where we replace α with β in the formula.

An example will help to make these rules clearer. Let's test this argument: $(a = b), (a = a) \to Fa \therefore Fb$.

$$
\begin{array}{lccl}
1. & \multicolumn{2}{c}{a = b \checkmark} & \text{Premise} \\
2. & \multicolumn{2}{c}{(a = a) \to Fa \checkmark} & \text{Premise} \\
3. & \multicolumn{2}{c}{\neg Fb} & \text{Neg Conc} \\
4. & \neg(a = a) & Fa & 2 \to \\
5. & (a = a) & | & \texttt{id} \\
6. & \times & Fb & 1, 4\ [a/b] \\
 & & \times &
\end{array}
$$

Adding $(a = a)$ on line #5 lets us close the branch. Otherwise we'd need a special closing rule for $\neg(a = a)$. On line #6 we copy the formula Fa on line #4 and replace an a with a b, because line #1 says that a and b are identical.

Practice Exercises

$\star$ **Exercise A:** Let A be this formula:

$$a = b \to (Rca \vee Pb)$$

Write all formulas that can be obtained from the following substitutional instances:

1. $A[a/b]$
2. $A[b/a]$
3. $A[a/c]$
4. $A[c/a]$
5. $A[b/c]$
6. $A[c/b]$

1.1 Logical Truths of Identity

In Chapter **??** we learned several rules to help us to complete Identity tables. The first rule was 'Reflexivity', which says that everything is identical with itself: $\forall x\,(x = x)$. The second rule was 'Symmetry', which says that if a is identical to b, then b is identical to a: $\forall x\,\forall y\,(x = y \to y = x)$. The third rule was 'Transitivity', which says that if anything is identical to two other things, they are identical with each other: $\forall x\,\forall y\,\forall z\,((x = y \wedge y = z) \to x = z)$. Let's check these rules are logical truths, using Truth Trees.

1.	$\neg\forall x\,(x = x)$ ✓	Root
2.	$\exists x\,(x \neq x)$ ✓a	1 ¬∀
3.	$a \neq a$	2 ∃ a
4.	$a = a$	`id`
	×	

The last line is one in which we use the rule of identity to close the branch.

1.	$\neg\forall x\,\forall y\,(x = y \to y = x)$ ✓	Root
2.	$\exists x\,(\neg\forall y\,(x = y \to y = x))$ ✓a	1 ¬∀
3.	$\neg\forall y\,(a = y \to y = a)$ ✓	2 ∃ a
4.	$\exists y\,(\neg(a = y \to y = a))$ ✓b	3 ¬∀
5.	$\neg(a = b \to b = a)$ ✓	4 ∃ b
6.	$a = b$	5 ¬ →
7.	$\neg(b = a)$	5 ¬ →
8.	$a = a$	`id`
9.	$b = a$	$8, 6$ $[a/b]$
	×	

1.	$\neg\forall x\,\forall y\,\forall z\,((x = y \wedge y = z) \to x = z)$ ✓	Root
2.	$\exists x\,(\neg\forall y\,\forall z\,((x = y \wedge y = z) \to x = z))$ ✓a	1 ¬∀
3.	$\neg\forall y\,\forall z\,((a = y \wedge y = z) \to a = z)$ ✓	2 ∃ a
4.	$\exists y\,(\neg\forall z\,((a = y \wedge y = z) \to a = z))$ ✓b	3 ¬∀
5.	$\neg\forall z\,((a = b \wedge b = z) \to a = z)$ ✓	4 ∃ b
6.	$\exists z\,(\neg((a = b \wedge b = z) \to a = z))$ ✓c	5 ¬∀
7.	$\neg((a = b \wedge b = c) \to a = c)$ ✓	6 ∃ c
8.	$a = b \wedge b = c$	7 ¬ →
9.	$\neg(a = c)$	7 ¬ →
10.	$a = b$	8 ∧
11.	$b = c$	8 ∧
12.	$a = c$	$10, 11$ $[b/a]$
	×	

We also learned how to complete predicate tables from information in identity tables, and identity tables from predicate tables. If two names are identical, then they have exactly the same truth values for each predicate: $\forall x\,\forall y\,((x = y) \to (Px \leftrightarrow Py))$. And if two names had different values for at least one predicate, then they were not identical: $\forall x\,\forall y\,(\neg(Px \leftrightarrow Py) \to (x \neq y))$. These formulas are equivalent, so we'll only test the first one.

1.	$\neg\forall x\,\forall y\,((x = y) \to (Px \leftrightarrow Py))$ ✓		Root
2.	$\exists x\,(\neg\forall y\,((x = y) \to (Px \leftrightarrow Py)))$ ✓a		1 ¬∀
3.	$\neg\forall y\,((a = y) \to (Pa \leftrightarrow Py))$ ✓		2 ∃ a
4.	$\exists y\,(\neg((a = y) \to (Pa \leftrightarrow Py)))$ ✓b		3 ¬∀
5.	$\neg((a = b) \to (Pa \leftrightarrow Pb))$ ✓		4 ∃ b
6.	$a = b$		5 →
7.	$\neg(Pa \leftrightarrow Pb)$ ✓		5 →
8.	Pa	$\neg Pa$	7 ¬ ↔
9.	$\neg Pb$	Pb	7 ¬ ↔
10.	Pb	$\neg Pb$	$6, 8\ [a/b]$
	×	×	

One final logical truth about Identity: $\forall x\,(\forall y\,((x = y) \to Fy) \leftrightarrow Fx)$.

1.	$\neg\forall x\,(\forall y\,((x = y) \to Fy) \leftrightarrow Fx)$ ✓			Root
2.	$\exists x\,(\neg(\forall y\,((x = y) \to Fy) \leftrightarrow Fx))$ ✓a			1 ¬∀
3.	$\neg(\forall y\,((a = y) \to Fy) \leftrightarrow Fa)$ ✓			2 ∃ a
4.	$\forall y\,((a = y) \to Fy)\ \backslash a$		$\neg\forall y\,((a = y) \to Fy)$ ✓	3 ¬ ↔ a
5.	$\neg Fa$		Fa	3 ¬ ↔ a
6.	$(a = a) \to Fa$ ✓			4 ∀ a
7.	$\neg(a = a)$	Fa		6 →
8.	$a = a$	×		`id`
9.	×		$\exists y\,(\neg((a = y) \to Fy))$ ✓b	4 ¬∀
10.			$\neg((a = b) \to Fb)$ ✓	4 ∃ b
11.			$a = b$	10 ¬ →
12.			$\neg Fb$	10 ¬ →
13.			$\neg Fa$	$11, 12\ [b/a]$
			×	

1.2 Arguments with Identity

Checking arguments that use Identity for validity is the same as any other argument. Consider $Rab, a = c \mathrel{.\dot{.}.} Rbc$:

1.	Rab	Premise
2.	$a = c$	Premise
3.	$\neg Rbc$	Neg Conc
4.	Rcb	$1, 2$ $[a/c]$
5.	$\neg Rba$	$2, 3$ $[c/a]$
	$\uparrow$	

R	a	b	c
a	?	1	?
b	0	?	0
c	?	1	?

=	a	b	c
a	1	0	1
b	0	1	0
c	1	0	1

Counter-examples are read off an open branch of the tree, just like any other counter-example. Once you've read off the counter-example, you'll need to do a little more work to complete the identity and other predicate tables, exactly as we've shown you in Chapter **??**.

Is this argument valid?

$$\exists x\,(Fx), \exists x\,(Gx), \forall x\,(\neg(Fx \wedge Gx)) \therefore \exists x\,\exists y\,(x \neq y)$$

1.	$\exists x\,(Fx)$		Premise
2.	$\exists x\,(Gx)$		Premise
3.	$\forall x\,(\neg(Fx \wedge Gx))$		Premise
4.	$\neg\exists x\,\exists y\,(x \neq y)$		Neg Conc
5.	Fa		1 ∃ a
6.	Gb		2 ∃ b
7.	$\forall x\,\neg\exists y\,(x \neq y)$		4 ¬∃
8.	$\neg\exists y\,(a \neq y)$		7 ∀ a
9.	$\forall y\,(\neg(a \neq y))$		8 ¬∃
10.	$\neg\neg(a = b)$		9 ∀ a
11.	$a = b$		10 ¬¬ a
12.	$\neg(Fa \wedge Ga)$		2 ∀ a
13.	$\neg Fa$	$\neg Ga$	12 ∨
14.	$\times$	Ga	$5, 11$ $[b/a]$
		$\times$	

Is this argument valid?

$$Rab \to Sa, a = c, Rcb \therefore Sa$$

1.	$Rab \to Sa$		Premise
2.	$a = c$		Premise
3.	Rcb		Premise
4.	$\neg Sa$		Neg Conc
5.	$\neg Rab$	Sa	$1 \to$
6.	$\neg Rcb$	$\times$	$3, 5\ [a/c]$
	$\times$		

Is this argument valid?

$$\forall x\, \forall y\, ((Fx \wedge Gy) \to x \neq y)\ \therefore\ \neg \exists x\, (Fx \wedge Gx)$$

1.	$\forall x\, \forall y\, ((Fx \wedge Gy) \to x \neq y)\ \backslash a$		Premise
2.	$\neg\neg\exists x\, (Fx \wedge Gx)$		Neg Conc
3.	$\exists x\, (Fx \wedge Gx)\ \checkmark a$		2 $\neg\neg$
4.	$Fa \wedge Ga$		3 $\exists$ a
5.	$\forall y\, ((Fa \wedge Gy) \to a \neq y)\ \backslash a$		1 $\forall$ a
6.	$(Fa \wedge Ga) \to a \neq a\ \checkmark$		5 $\forall$ a
7.	$\neg(Fa \wedge Ga)$	$\neg(a = a)$	$6 \to$
8.	$\times$	$a = a$	`id`
		$\times$	

Is this argument valid?

$$\forall x\, (Fx \to x = a), \forall x\, (Fx \vee Gx), \exists x\, (\neg Gx)\ \therefore\ Fa$$

1.	$\forall x\, (Fx \to x = a)\ \backslash b$			Premise
2.	$\forall x\, (Fx \vee Gx)\ \backslash b$			Premise
3.	$\exists x\, (\neg Gx)\ \checkmark b$			Premise
4.	$\neg Fa$			Neg Conc
5.	$\neg Gb$			3 $\exists$ b
6.	$Fb \vee Gb\ \checkmark$			2 $\forall$ b
7.	Fb		Gb	6 $\vee$
8.	$Fb \to b = a\ \checkmark$		$\times$	1 $\forall$ b
9.	$\neg Fb$	$b = a$		$8 \to$
10.	$\times$	Fa		$7, 9\ [b/a]$
		$\times$		

Identity can be used for counting. Here's an argument involving counting:

> Suppose there are at most two things, and Angela and Belinda are both frogs, and Angela is not Belinda. Then obviously, everything is a frog.

And here's a symbolisation of this argument, using initials as an implicit symbolisation key:

$$\exists x\, \exists y\, (\forall z\, (z = x \lor z = y)), Fa, Fb, a \neq b \therefore \forall x\, (Fx)$$

1.	$\exists x\, \exists y\, (\forall z\, (z = x \lor z = y))$ ✓d	Premise
2.	Fa	Premise
3.	Fb	Premise
4.	$(a \neq b)$	Premise
5.	$\neg \forall x\, (Fx)$	Neg Conc
6.	$\exists x\, (\neg Fx)$ ✓c	5 ¬∀
7.	$\neg Fc$	6 ∃ c
8.	$\exists y\, (\forall z\, (z = d \lor z = y))$ ✓e	1 ∃ d
9.	$\forall z\, (z = d \lor z = e)$ ✓	8 ∃ e
10.	$a = d \lor a = e$ ✓	9 ∀ a
11.	$b = d \lor b = e$ ✓	9 ∀ b
12.	$c = d \lor c = e$ ✓	9 ∀ c

13.	$a = d$		$a = e$		10 ∨
14.	$b = d$	$b = e$	$b = e$	$b = d$	11 ∨
15.	$a = b$	|	$a = b$	|	$13, 14\ [d/b]$; $13, 14\ [e/b]$
16.	×	Fd	×	Fd	$2, 13\ [a/d]$; $2, 13\ [b/d]$
17.		Fe		Fe	$3, 14\ [b/e]$; $3, 14\ [a/e]$
18.	$c = d$	$c = e$	$c = d$	$c = e$	12 ∨
19.	Fc	Fc	Fc	Fc	$14, 18\ [d/c]$; $14, 18\ [e/c]$
	×	×	×	×	

You can see the slightly different substitutions in the parallel branches on lines #15-#19. Identity trees with many parallel branches – like this one – can get messy. When the different substitutions become too messy to keep track of, we can abuse notation, and just write:

19.	Fc	Fc	Fc	Fc	$14, 18[/]$

Practice Exercises

⋆ **Exercise A:** Use a tree to show that the following arguments are valid.

1. $\forall x(x = a \vee x = b)$, $\neg(Fa \wedge Ga)$, $Gb \to Hb \therefore \forall x((Fx \wedge \neg Hx) \to \neg Gx)$
2. $\exists x \forall y(Fy \leftrightarrow (x = y)) \therefore \exists x Fx$
3. $\exists x Fx$, $\forall x \forall y((Fx \wedge Fy) \to x = y) \therefore \exists x \forall y(Fy \leftrightarrow (x = y))$
4. $\forall x \forall y(Fx \to x = y)$, $\forall x(Fx \vee Gx) \therefore (\forall x Fx \vee \forall x Gx)$
5. $\exists x \forall y(Fy \leftrightarrow (x = y)) \therefore \forall x \forall y((Fx \wedge Fy) \to (x = y))$
6. $\exists x \forall y(Fy \leftrightarrow (x = y))$, $\forall x(Fx \vee Gx) \therefore \forall x \forall y((Gx \vee Gy) \vee x = y)$

Exercise B: Use a tree to test whether the following arguments are valid.

1. $(\forall x \forall y \forall z((Rxy \wedge Rxz) \to y = z) \vee \forall x \forall y \forall z((Rxy \wedge Rzy) \to x = z))$, $\forall x Rxx \therefore \forall x \forall y(Rxy \leftrightarrow (x = y))$
2. $\forall x \neg Rxx$, $\forall x \forall y((Rxy \wedge Ryx) \to x = y)$, $\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$ $\therefore \forall x \forall y \forall z(\neg Rxy \vee \neg Ryz \vee \neg Rzx)$
3. $\exists x \forall y(Fy \leftrightarrow (x = y))$, $\exists x \forall y(Gy \leftrightarrow (x = y))$, $\neg \exists x(Fx \wedge Gx)$ $\therefore \exists x \exists y(x \neq y \wedge \forall z((Fz \vee Gz) \leftrightarrow (z = x \vee z = y)))$
4. $\forall x \forall y((Rxy \wedge Ryx) \to x = y)$, $\forall x(Fx \to \exists y(Rxy \wedge \neg Fy))$ $\therefore \forall x \exists y \neg(Ryx \wedge Fx)$
5. $\forall x \exists y \forall z(Rxz \leftrightarrow (z = y)$, $\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$ $\therefore \forall x \exists y(Rxy \wedge Ryx)$