Part I

Predicate Logic Symbolisation

Chapter 1
The Language of PL

1.1 Decomposing Statements

Consider the following argument:

Phoenix is a logician.
All logicians are kind.
$\therefore$ Phoenix is kind.

This argument is valid. To see this, take any situation in which Phoenix is a logician and all logicians are kind. We do not know whether there are other logicians in this case, but we know that Phoenix is one of them. We further know from the second premise that anyone who is a logician is someone who is kind. So it must be that Phoenix is kind. Therefore, the argument is valid, because every case in which the premises are true is a case in which the conclusion is also true.

Logic Corner

Logicians sometimes like to work with counterexamples to demonstrate that arguments are valid. Here's how this might go with the Phoenix argument:

> Imagine there were a counterexample, i.e., a case in which the premises are true but the conclusion is false. From the truth of the first premise and the falsity of the conclusion, this would be a case in which Phoenix is a logician that isn't kind. If that were so, we would have at least one logician who isn't kind, which would make the second premise false. So our case would fail to be a counterexample. Hence, there cannot be a counterexample to the argument. The argument is therefore valid.

Because the argument is quite simple, appealing to counterexamples might seem to you like an overkill. But remember this strategy when you face arguments in the wild and wonder if they are valid: assume there's a counterexample, i.e., a case in which the premises are true and the conclusion is false, and try to get a contradiction.

Mathematics Corner

Working with counterexamples can be very efficient in mathematics. For those of you who have to sit in mathematics exams and have to provide proofs, try the counterexample methodology: assume that there is a case in which conditions set out in the problem hold, but the result you've been asked to proved doesn't. Use this information to derive a contradiction. If you succeed, you know that the result follows *validly* from the conditions of the problem, and so you have an efficient proof.

Let's try now to symbolise it in TFL, and see if we get the same diagnosis. We might use this symbolisation key:

L: Phoenix is a logician.
A: All logicians are kind.
F: Phoenix is kind.

And the argument itself becomes:

	Phoenix is a logician.		L
	All logicians are kind.		A
$\therefore$	Phoenix is kind.	$\therefore$	F

This is *invalid* in TFL, as can be see with a valuation that assigns 1 to L and A, and 0 to F. Something has gone wrong. We have a valid argument whose symbolisation tell us that it's invalid. What's going on?

The problem is not that we have made a mistake while symbolising the argument. This is the best symbolisation we can give *in* TFL. The problem lies with TFL itself. 'All logicians are kind' is about two kinds of things: logicians and things that are kind. Our symbolisation in TFL fails to make this distinction, so we lose the connection between Phoenix being a logician and Phoenix being kind. In other words, connectives do not allow us to decompose statements like 'All logicians are kind' into simpler components. This is why we need a more powerful formal system.

To symbolise arguments like the preceding one, we will have to develop a new logical language which will allow us to *split the atom*. We will call this language Predicate Logic (PL). The details of PL will be explained throughout this chapter, but here is the basic idea. First, we have *names*. Names allow us to refer directly to things, like Phoenix. In PL, we indicate names with lowercase italic letters. For instance, we might let 'p' be a name for Phoenix, or let 'b' be a name for SpongeBob.

Second, we have predicates, which we use to symbolise properties of things, like *being a logician* or *wearing square pants*. In PL, we indicate predicates with uppercase italic letters. For instance, we might use the letter 'L' to symbolise the property of *being a logician*, or the letter 'P' to symbolise the property of *wearing squarepants*. We can then combine names and predicates to symbolise statements: $L(p)$ symbolises 'Phoenix is a logician' and $P(b)$ symbolises 'SpongeBob wears squarepants'.

Finally, we have symbols that allow us to symbolise the part of statements that talk about *all* or *some* things. We call those quantifiers. In this course, we will work with two quantifiers: the existential quantifier '$\exists$' and the universal quantifier '$\forall$'. We will use quantifiers to symbolise statements like 'All logicians are kind' as $\forall x\,[Lx \to Kx]$'. Combining all ingredients together, we symbolise the Phoenix argument in PL like this:

$$\begin{array}{ll} & L(p) \\ & \forall x\,[Lx \to Kx] \\ \therefore & K(p) \end{array}$$

That is the general idea, but PL is significantly more subtle than TFL, so we will come at it slowly.

1.2 Names

The tallest building in Auckland is 328 metres high. This is a true fact. Here's another true fact: the Sky Tower is 328 metres high. The two sentences express the same thing, because 'the tallest building in Auckland' and 'Sky Tower' are two ways to refer to the same thing. The Sky Tower *is* the tallest building in Auckland. Although they refer to the same thing, however, 'the tallest building in Auckland' and 'Sky Tower' perform different linguistic roles. 'The tallest building in Auckland' is a description, whereas 'Sky Tower' is a name. A description refers to a thing by listing some of its properties. A name refers to a thing without describing it. A thing is given a name, and the name refers to that thing, by stipulation. As people, we all have a name, which is given to us at birth, but we can also refer to people by describing them. Thus, 'Emily Parke' and 'the philosopher of science at the University of Auckland' refer to the same person.

In PL, names are by default lower-case letters 'a' through to 't'. On some occasions, we might also use letters u to z if the context makes it nicer, for instance if we wanted to use w for Willi or z for Zeus. You can think of these names along the lines of the proper names that all of us have, but with one difference. More than one person may be given the name 'Emily Parke', and usually context makes it clear which person the name refers to. We have to live with this type of ambiguity, allowing context to individuate the fact that 'Emily Parke' refers to a philosopher of science at the University of Auckland, and not some other Emily. In PL, we do not tolerate any such ambiguity. Each name must pick out *exactly* one thing. However, two different names may pick out the same thing.

As with TFL, we can provide symbolisation keys. These indicate, temporarily, what a name will pick out, for example:

e: Emily
g: Gregor
s: the Sky Tower

1.3 Predicates

The simplest predicates are about properties of individuals, such as 'being a dog', 'being a square' or 'being a member of the Avengers'.

In general, you can think of predicates as things which combine with names to to make statements. For example you can combine 'Hulk' and 'is a member of The Avengers' to make the statement 'Hulk is a member of the Avengers'.

In PL, PREDICATES are capital letters A through Z. When we include predicates in symbolisation keys, we will use variables to make it clear how many things a predicate applies to. In PL, variables are italic lowercase letters 'u' through 'z'. Until we get to §4, the predicates we will use refer to individual properties, such as 'being a dog' or 'being a logician'. Later, we will use more complex predicates that refer to relations between things, such as 'being siblings' or 'being taller than'. We call a predicate that refers to an individual property a *one-place predicate*, a predicate that refers to a relation between two things a *two-place predicate*, a predicate that refers to a relation between three things a *three-place predicate*, and so on. The use of variables makes it clear what type of predicates we are using, so Hx is a one-place predicate, Rxy is a two-place predicate, and $S(xyz)$ is a three-place predicate.

Notation Corner

There are various conventions about the use of brackets in writing predicates. Some write $H(x)$ instead of Hx for one-place predicates, and some write $R(x, y)$ or xRy for two-place predicates. We write Hx and Rxy in this textbook.

For now, let's focus on one-place predicates. Combining names and predicates, a symbolisation key looks like this:

e: Emily Parke
h: Hulk
Ax: x is angry.
Px: x is a philosopher of science.

With this symbolisation key, we can symbolise statements that use these names and predicates in combination. For example, consider these statements:

1. Hulk is angry.
2. Emily isn't angry.
3. Hulk is angry, but Emily isn't.

Statement 1 is straightforward. We symbolise it thus:

$$Ah$$

Statement 2 expresses that Emily doesn't have the property of being angry, so is the negation of the statement $A(e)$. We symbolise it thus:

$$\neg A(e)$$

Statement 3 is the conjunction of Statements 1 and 2. This illustrates an important point: PL has all of the truth-functional connectives of TFL. We thus symbolise Statement 3, a conjunction, like this:

$$(A(h) \wedge \neg A(e))$$

Practice Exercises

Exercise A:

Identify all the names in the following story, and list them in a symbolisation key:

> Once upon a time a cat showed up at a house in Ponsonby. She was fluffy and lovely, but seemed to be lost. At the house was a family of four, Raz and Aki, and their two children, a boy and a girl. The kids asked if they could keep the cat, because she looked lost and hungry. Their parents said that they would first reach out to their community on Facebook and other social media, but that they could take care of her while they tried to find her home. The cat came to be known as 'The Queen of Ponsonby', because of the way she seemed to demand a lot of attention and get upset when no one would feed her. Eventually someone called Umut responded to the family's post on Facebook and claimed to be the cat's owner. When Umut came to pick her up, he told them her name was Shiro, and was very thankful that Raz and Aki's family help in reuniting them. The children were sad when Shiro had to leave, but they were also happy that she would go back to her house. When they talked about the lovely times they had with her, they always refer to her as 'The Queen of Ponsonby'.

Refer to this symbolisation key for the next two problem sets:

e: Emily Parke
h: Hulk
Ax: x is an Avenger.
Mx: x is a Marvel superhero.
Px: x is a philosopher of science.

Exercise B:

Symbolise the following statements in PL:

1. Hulk is a Marvel superhero.
2. Emily Parke is not an Avenger or a Marvel superhero.
3. Hulk isn't a philosopher of science, but he is an Avenger and a Marvel superhero.
4. If Hulk is a philosopher of science, then he isn't an Avenger.
5. Neither Hulk nor Emily Parke are Avengers, Marvel superheroes, or philosophers of science.

Exercise C:

Express the following formulas of PL in English:

1. $\neg Ap$
2. $Pe \wedge (Mh \wedge Ah)$
3. $Me \rightarrow \neg Pe$

Exercise D:

For each statement, provide a symbolisation key and suggest a symbolisation. Pay special attention to how many places your predicates have.

1. The Lion King is a great movie.
2. Titanic is a better movie than Avatar.
3. Hamilton is south of Auckland and north of Wellington.
4. Hamilton is in between Wellington and Auckland.
5. Wellington is in the middle of New Zealand.
6. Wellington is in the middle of Auckland, Gisborne, Nelson and Christchurch.

Chapter 2

Quantifiers

Consider again the argument we started with:

Phoenix is a logician.
All logicians are kind.
∴ Phoenix is kind.

With the work we've done so far, we are able to symbolise that Phoenix is a logician with something like $L(p)$ and that Phoenix is kind with $K(p)$, but we are not able yet to symbolise the second premise 'all logicians are kind'. If we had a list of all people (dead or alive), with full information telling us who is a logician and who is kind, we might be able to express 'all logicians are kind' with a very long conjunction like this:

> 'Jeremy is a logician and is kind' and 'Emily is a logician and is kind' and 'Phoenix is a logician and is kind' and ...

This strategy would only work if we were talking about finitely many things, which isn't always the case. We know for instance that 'all prime numbers are only divisible by 1 and themselves', but we wouldn't be able to symbolise this as a conjunction by listing all the cases. The first reason is that the conjunction would be infinite, because there are infinitely many prime numbers, and the second reason is that we do not know which numbers are prime numbers. Yet we can express true things about infinite collections of things (like prime numbers), even thought we don't know all the members of the collections). In PL, we symbolise this with QUANTIFIERS. allow us to talk about (finite or infinite) collections of things in one statement.

2.1 Introducing Quantifiers

In this book, we will focus on two types of quantifiers, the existential quantifier and the universal quantifier. We use the symbol $\exists$ for the existential quantifier and the symbol $\forall$ for the universal quantifier. The existential quantifier expresses that *at least one thing* of a collection has a certain property, for example that at least one planet has life, or that at least one philosopher at the University of Auckland specialises in philosophy of science. On the opposite end, the universal quantifiers refers to *everything* in a collection, for instance that all logicians are kind, or that all planets orbit a star.

Linguistics Corner

There are further quantifiers between these two, for instance a quantifier that expresses that *most things* in a collection have a property, such as most birds can fly, or most prime numbers are odd. We will not include those in our formal language, because they require more complexity than we want to go into in this book. As you will see, dealing with the existential and universal quantifiers is challenging enough.

When we write formulas with quantifiers, we always use variables. There's no exception to this rule: a quantifier must always be followed by a variable. Formulas with quantifiers will look like this in their simplest form, when they apply only to predicates:

$$\forall x\,[Ax] \qquad \exists y\,[Hy]$$

We always use brackets to indicate the *scope* of the quantifier. In this book we are using [square brackets], just so that you can more easily keep track of quantifier scope. However, they function just like normal (parentheses), as we can see below:

$$\forall x\,[Ax \wedge \neg Hx] \qquad \exists y\,[Hy \rightarrow \neg(Ay \vee \neg By)]$$

We will come back to the notion of the *scope* of a quantifier shortly. There is no special reason to use 'x' rather than some other variable. The statements

'$\forall x\,[Hx]$', '$\forall y\,[Hy]$', '$\forall z\,[Hz]$' use different variables, but are all logically equivalent.

Notation Corner

There are various conventions on how to use brackets in formulas with quantifiers. Some put brackets around the quantifiers and write $(\forall x\,)Hx$, others put brackets around the predicate, as in $\forall x\,(Hx)$. You could be extra-thorough and write $(\forall x\,)(H(x))$. In this textbook, we don't put brackets around the quantifier, and we always put square brackets around the scope of the quantifier, as in $\forall x\,[Ax \wedge Bx]$. You can drop the scope brackets if it contains a single predicate, as in $\forall x\,Ax$.

Let's look at some examples, based on this symbolisation key:

p: Phoenix
Lx: x is a logician.
Kx: x is kind.

How do we express the following statements?

1. Some logicians are kind.
2. Phoenix is kind, but some logician isn't.
3. All logicians are kind.

Statement 1 expresses that there is at least one logician who is kind. We can express this with this formula:

$$\exists x\,[Lx \wedge Kx]$$

This formula says that there is something that is both a logician and is kind. We can express statement 2 as a conjunction of two statements, one expressing that Phoenix is kind, the other that at least one logician isn't kind:

$$K(p) \wedge \exists x\,[Lx \wedge \neg Kx]$$

For the last statement, let's first see how you *don't* express that all logicians are kind:

$$\forall x\,[Lx \wedge Kx]$$

This formula not only says that all logicians are kind, but that *everything is a kind logician*. Obviously, there are things that are neither logicians nor kind, like chairs or rocks. What we want is to express not that everything is a kind logician, but that *everything that happens to be a logician is also kind*. We need to restrict the reach of the quantifier to talk about logicians only. We achieve this by using a conditional:

$$\forall x\,[Lx \to Kx]$$

This says of each thing that *if* it is a logician (some things aren't, but if it is...) then it is kind. The universal quantifier talks about everything, and we use the conditional to restrict it to talk about logicians only. With the restriction as the antecedent of the conditional, we can then express that everything in that restriction is kind.

> **Logic Corner**
>
> Let us use the opportunity to introduce two important rules of thumb for the use of quantifiers in PL. The first is that the existential quantifier $\exists$ is usually accompanied by a conjunction $\wedge$, the second that the universal quantifier $\forall$ is usually accompanied by a conditional $\to$. The existential quantifier typically expresses a conjunction of properties that hold of at least one thing. The universal quantifier typically expresses that things which meet some conditions also have other properties. This will become second nature for you before too long, but for now remember this: $\exists$ goes with $\wedge$, and $\forall$ goes with $\to$.

Putting everything together, we can now express the Phoenix argument we saw at the start of this section in PL:

	Phoenix is a logician.		$L(p)$
	All logicians are kind.		$\forall x\,[Lx \to Kx]$
$\therefore$	Phoenix is kind.	$\therefore$	$K(p)$

2.2 Quantifiers and Scope

Let's expand our symbolisation key:

p: Phoenix
Lx: x is a logician.
Kx: x is kind.
Px: x is a person.

How would you symbolise this statement?

4. If someone is a logician, then someone is kind.

There are two different quantifiers in the statement, so we expect to use two different variables, one for each quantifier. The first thing to notice is that the statement is a conditional:

someone is a logician $\quad\rightarrow\quad$ someone is kind

Now that we've identified the form of the statement and isolated the quantified bits, we can complete the symbolisation:

5. $\exists x\,[Px \wedge Lx] \rightarrow \exists y\,[Py \wedge Ky]$

It would be a mistake to symbolise 4 as:

$$\exists x\,[(Px \wedge Lx) \rightarrow Kx]$$

This says that if someone is a logician, then *they* are kind. Statement 4, however, didn't specify of the *same* person that they are kind if they are a logician. All 4 says is that if *someone* is a logician, then *someone* (maybe the same person, but not necessarily the same) is kind. There are two quantifiers in 5, and each has a different *scope*. The scope of $\exists x$ is $[Px \wedge Lx]$, and the scope of $\exists y$ is $[Py \wedge Ky]$.

It would also be a mistake to symbolise 4 as:

$$\exists x\,[Px \wedge Lx] \rightarrow Kx$$

This is an incomplete statement, that would read, literally, as: 'If someone is a logician, then x is kind.' The variable x in the Kx is left dangling; it is outside the scope of the quantifier $\forall x$.

The SCOPE OF A QUANTIFIER, then, is the formula that is included in the brackets attached to the quantifier: $\forall x\,[\ldots]$ or $\exists x\,[\ldots]$. The variable that is attached to a quantifier may only be used within the scope of the quantifier. Once the bracket indicating the scope of the quantifier closes, you can no longer use it. So you can write:

$$\exists x\,[(Ax \wedge \neg Bx) \rightarrow \neg(Cx \vee \neg Dx)]$$

or

$$\exists x\,[Ax \wedge \exists y\,[By] \rightarrow Cx]$$

but you cannot write:

$$\exists x\,[Ax] \rightarrow (Bx)$$

or

$$\exists x\,[Ax \wedge By]$$

There you have it. Those are the basic ingredients of the language of PL.

2.3 The Power of Paraphrase

When symbolising English statements in PL, it is important to understand the structure of the statements you want to symbolise. What matters is the final symbolisation in PL, and sometimes you will be able to move from an English language statement directly to a statement of PL. Other times, it helps to paraphrase the statement one or more times. Each successive paraphrase should move from the original statement to something that you can more easily symbolise directly in PL. We will use this symbolisation key:

s: Sam Smith
Fx: x is famous.
Px: x is a person.
Sx: x is a pop star.

Consider these statements:

6. If Sam Smith is a pop star, then they are famous.
7. If someone is a pop star, then they are famous.

It may look as though the two statements are quite similar, but we will see that their symbolisation reveals quite a different logical structure. First, the same pronouns appear in the consequent of statements 6 and 7 (‘they’), although they mean different things. In 6, we use the pronoun ‘they’ as a pronoun to refer specifically to Sam Smith. In 7, we use the pronoun ‘they’ as a pronoun to refer to an arbitrary pop star. To make this clear, it helps to paraphrase the original statements, removing pronouns altogether:

8. If Sam Smith is a pop star, then *Sam Smith* is famous.
9. If someone is a pop star, then *that person* is famous.

The paraphrases 8 and 9 express the same things as 6 and 7, albeit in less elegant ways. With these paraphrases, we can proceed to symbolisation in PL, starting with statement 8:

10. $Ss \to Fs$

For statement 9, we need to make a decision as to whether the statement is universal or existential. You might be tempted to think that 9 is an existential statement because it uses 'someone', but this would be a mistake. Don't worry, that's a common mistake. In fact, 9 talks about arbitrary people that are pop stars, not specific ones, so it is a universal statement. It says, of anyone, that if they are a pop star, then they are famous, which suggests this further paraphrase:

11. Any person who is a pop star is famous.

This looks more like the universal statements we've symbolised previously:

12. $\forall x\,[(Px \wedge Sx) \to Fx]$

The symbolisation makes it clear that 6 and 7, even though they look alike, have a different logical structure in PL. The structure of sentences in everyday language isn't always the same as the structure of their symbolisation in PL. Should you use an existential or a universal quantifier? Paraphrasing with different words can help you reveal the logical structure of sentences. When you paraphrase statements, don't worry about expressing things nicely. What is important is to reveal the logical structure, and so make your symbolisation easier.

Practice Exercises

Refer to this symbolisation key for the next two problem sets:

h: Hulk
p: Phoenix
Ax: x is an Avenger.
Lx: x is a logician.
Mx: x angry.
Px: x is a person.

Exercise A: Symbolise the following statements in PL:

1. Some Avengers are angry.
2. Phoenix is a logician, not an Avenger, whereas Hulk is an Avenger, but not a logician.
3. No Avenger is a logician.
4. All logicians are angry, but Phoenix isn't.
5. Everything is an Avenger, a logician, and is angry.

Exercise B:
Express the following formulas of PL in English:

1. $\exists x\,[Px \wedge Lx]$
2. $\forall x\,[Lx \to Px]$
3. $\neg\forall x\,[Ax \to Mx]$
4. $\neg\exists x\,[Ax \wedge Lx]$

⋆ **Exercise C:**
Identify the scope of each quantifier in the following formulas:

1. $\exists x\,[Fx] \wedge \exists x\,[Gx]$
2. $\exists x\,[Fx] \wedge \exists y\,[Fy]$
3. $\exists x\,[Fx \wedge \exists y\,[Gy]]$
4. $\exists x\,[Fx \to \exists y\,[(Gy \vee Hy) \wedge \neg\forall z\,[Fx \vee Gz]]]$
5. $\neg(Fa \wedge \exists x\,[Ga \wedge Hx])$

Exercise D:

Paraphrase the following statements to make their logical structure explicit (where possible):

1. Every pop star is famous
2. If everyone is a pop star, then Sam Smith is.
3. If everyone is a pop star, then they are famous.
4. If they are famous, then Sam Smith is a pop star.
5. If everyone is a pop star, then everyone is famous.
6. If Sam Smith is a pop star, then anyone is.
7. If someone is a pop star, then Sam smith is.
8. If someone is a pop star, then they are famous.
9. If Sam Smith is a pop star, then someone is famous.
10. If Sam Smith is a pop star, then anyone is famous.
11. If nobody is a pop star, then Sam Smith isn't.
12. No one is a pop star that isn't famous.
13. If Sam Smith isn't famous, then nobody is.
14. If Sam Smith isn't famous, then someone is.
15. If anyone is, Sam Smith is a pop star.
16. If they are a pop star, then they are famous.

Exercise E: Use the following symbolisation key to symbolise the above statements in PL, based on your paraphrases.

s: Sam Smith
Fx: x is famous.
Px: x is a person.
Sx: x is a pop star.

Chapter 3
Single Quantifier Statements

We now have all of the pieces of PL. Symbolising more complicated statements will only be a matter of knowing the right way to combine predicates, names, quantifiers, and connectives. There is a skill to this, and there is no substitute for practice.

3.1 Common Quantifier Phrases

Consider these statements:

1. Every animal in the paddock is a sheep.
2. Some sheep are rams.
3. Not all sheep in the paddock are ewes.
4. None of the sheep in the paddock are wethers.

We will work with this symbolisation key:

Ax: x is an animal
Ex: x is a ewe
Px: x is in the paddock
Rx: x is a ram
Sx: x is a sheep
Wx: x is a wether

Try to symbolise the four statements above before reading on.

The first step when symbolising in PL is to figure out if a statement is universal or existential. A statement is universal if it talks about all things of the same type, and it is existential if it talks about some of them. Universal statements often use words like 'every' or 'all', like 'all mice are afraid of cats', whereas existential statements use words like 'some' or 'there is', like 'Some cats like dogs'. Let's start with universal statements:

Statement 1 *Every animal in the paddock is a sheep.*

Statement 1 is about *every* animal in the paddock, so is a universal statement. It talks about everything that is an animal in the paddock, and says of those things that they are sheep. As per our rule of thumb from the previous section, we expect our symbolisation to use a combination of a universal quantifier $\forall$ and a conditional $\rightarrow$. In the antecedent of the conditional, we list the conditions that must be met to qualify, in this case to be an animal and to be in the paddock. In the consequent, we express the relevant properties ascribed to the things that meet the conditions set out in the antecedent, namely that they are sheep. So the symbolisation looks like this:

$$\forall x\,[(Ax \land Px) \rightarrow Sx]$$

This formulas says something like this, if you were to read it aloud: 'for all x, if x is an animal and x is in the paddock, then x is a sheep''. This is how the language of PL expresses that every animal in the paddock is a sheep.

Statement 2 *Some sheep in the paddock are rams.*

Statement 2 is about *some* of the sheep in the paddock, and says that at least one of them is a ram. According to our rule of thumb, we expect to use the existential quantifier $\exists$ and conjunction $\land$ in our symbolisation. What we need to express is that there's at least one thing that is a sheep and is in the paddock and is *also* a ram:

$$\exists x\,[(Sx \land Px) \land Rx]$$

Remember our convention about the use of brackets in long conjunctions. The convention is useful with existential statements when they have several conjuncts, in which case we can drop brackets:

$$\exists x\,[Sx \land Px \land Rx]$$

But never drop the bracket that comes after the existential quantifier. Never write something like this:

$$\exists x\, Sx \land Px \land Rx$$

The next two common quantifier phrases are negative statements, which are either negated universal or negated existential statements. Each can be expressed in two different ways, depending on whether we take the main connective to be a negation or a quantifier. These examples will illustrate the difference:

Statement 3 *Not all sheep in the paddock are ewes.*

We have two ways of symbolising this statement. Statement 3 starts with a negation 'not', and denies that all sheep in the paddock are ewes. This is one way of approaching the symbolisation. First symbolise that all sheep in the paddock are ewes with $\forall x\,[(Sx \wedge Px) \rightarrow Ex]$, then negate it to obtain:

$$\neg\forall x\,[(Sx \wedge Px) \rightarrow Ex]$$

The second way of approaching the symbolisation is to ask what would be sufficient to make the statement true. All it takes is for one sheep in the paddock to be a ram, in which case it is fair to say that not all sheep in the paddock are ewes. This is the second way we can think of the symbolisation:

$$\exists x\,[Sx \wedge Px \wedge \neg Ex]$$

Logic Corner

Logicians are used to the equivalence between these two symbolisations, because of two logical facts. The first fact is that the pattern of quantifier and negation $\neg\forall$ is logically equivalent to $\exists\neg$. The second is that the negation of a conditional $\neg(A \rightarrow B)$ is equivalent to the conjunction of the antecedent and the negation of the consequent $(A \wedge \neg B)$. This is why logicians think there's no effective difference between these symbolisations:

$$\begin{gathered}\neg\forall x\,[(Sx \wedge Px) \rightarrow Ex]\\ \exists x\,[\neg((Sx \wedge Px) \rightarrow Ex)]\\ \exists x\,[Sx \wedge Px \wedge \neg Ex]\end{gathered}$$

Statement 4 *None of the sheep in the paddock are wethers.*

As with the previous example, there are two ways to symbolise this statement. The first way is to take it at face value with the main connective being a negation. The question to ask, then, is what statement is being negated. Are we denying that all sheep in the paddock are wethers, or that at least one them is? The second option is correct. What we are saying is that we can't find at least one sheep in the paddock that is a wether. So we start by symbolising the statement to be negated as $\exists x\,[Sx \wedge Px \wedge Wx]$. We then put a negation in front of it to obtain our first symbolisation:

$$\neg\exists x\,[Sx \wedge Px \wedge Wx]$$

The second way to approach the symbolisation is to ask this question: does the statement talk about all the sheep in the paddock? Yes! And it says of them that they are not wethers. So here's a symbolisation:

$$\forall x\,[(Sx \wedge Px) \rightarrow \neg Wx]$$

Logic Corner

Logicians are also used to the equivalence between these two symbolisations, because the pattern $\neg\exists$ is logically equivalent to $\forall\neg$. So they think that there's no difference between these symbolisations:

$$\neg\exists x\,[Sx \wedge Px \wedge Wx]$$
$$\forall x\,[\neg(Sx \wedge Px \wedge Wx]$$
$$\forall x\,[(Sx \wedge Px) \rightarrow \neg Wx]$$

3.2 Ambiguous Predicates

Suppose we want to symbolise this statement:

5. Adina is a skilled surgeon.

We could do it with this symbolisation key:

a: Adina.
Kx: x is a skilled surgeon.

The symbolisation of 5 is then:

$$Ka$$

Suppose instead that we want to symbolise this argument:

> The hospital will only hire a skilled surgeon. All surgeons are greedy. Billy is a surgeon, but is not skilled. Therefore, Billy is greedy, but the hospital will not hire him.

We need to distinguish being a *skilled surgeon* from merely being a *surgeon*. So we introduce a predicate for each in our symbolisation key:

b: Billy
Gx: x is greedy.
Hx: The hospital will hire x.
Kx: x is skilled.
Rx: x is a surgeon.

The argument can then be symbolised in this way:

$$\begin{array}{rl} & \forall x\,[\neg(Rx \wedge Kx) \rightarrow \neg Hx] \\ & \forall x\,[Rx \rightarrow Gx] \\ & Rb \wedge \neg Kb \\ \therefore & Gb \wedge \neg Hb \end{array}$$

Next suppose that we want to symbolise this argument:

> Carol is a skilled surgeon and a tennis player. Therefore, Carol is a skilled tennis player.

This argument is invalid. The premise tells us that Carol is a skilled surgeon, and that she is a tennis player, but being a good surgeon doesn't make you a skilled tennis player. With this in mind, suppose we try to expand the previous symbolisation key thus:

b: Billy
c: Carol
Gx: x is greedy.
Hx: The hospital will hire x.
Kx: x is skilled.
Rx: x is a surgeon.
Tx: x is a tennis player.

A symbolisation of the argument might then look like this:

$$\begin{array}{rl} & Rc \wedge Kc \wedge Tc \\ \therefore & Tc \wedge Kc \end{array}$$

This symbolisation is a disaster. Can you see why? It takes an invalid argument and symbolises it as a valid argument in PL. The problem is that there is a difference between being *skilled as a surgeon* and *skilled as a tennis player*. Symbolising this argument correctly requires two separate predicates, one for each type of skill. We thus use a different symbolisation key:

c: Carol
Gx: x is a skilled surgeon.
Px: x is a skilled tennis player.
Rx: x is a surgeon.
Tx: x is a tennis player.

Now we can properly symbolise the argument in this way:

$$
\begin{array}{rl}
 & (Rc \wedge Gc) \wedge Tc \\
\therefore & Tc \wedge Pc
\end{array}
$$

The moral of these examples is that you need to be careful of symbolising predicates in an ambiguous way. Similar problems can arise with predicates like *good*, *bad*, *big*, and *small*. Just as skilled surgeons and skilled tennis players have different skills, big dogs, big mice, and big problems are big in different ways.

Is it enough to have a predicate that means 'x is a skilled surgeon', rather than two predicates 'x is skilled' and 'x is a surgeon'? Sometimes. As statement 5 shows, sometimes we do not need to distinguish between skilled surgeons and other surgeons.

Must we always distinguish between different ways of being skilled, good, bad, or big? No. As the argument about Billy shows, sometimes we only need to talk about one kind of skill. If you are symbolising an argument that is just about dogs, it is fine to define a predicate that means 'x is big.' If you are symbolising an argument that is about dogs and mice, however, it is probably best to use a predicate for 'x is big for a dog' and another one for 'x is big for a mouse.'

3.3 Aristotelian Syllogisms

Aristotle of Stagira (384-322 BCE) is often regarded as the founder of formal logic. Some of his observations are still part of logic today, even if we have become slightly more advanced in the subsequent 2300 years. Three of the distinctions he made when classifying simple statements are still relevant:

1. *Subject / Predicate*: Every simple statement has a subject (what we are talking textit), and a predicate (what we say about the subject).
2. *Universal / Particular*: Perhaps the simplest ways of having a subject subject are to talk about everything in that subject (the universal), or one thing in that subject (the particular).
3. *Affirmative / Negative*: A simple statement either affirms that a subject has a predicate, or denies (negates) that affirmation.

We can use these three distinctions to build up a taxonomy of simple statements, exactly as Aristotle did; and as we did at the start of this chapter:

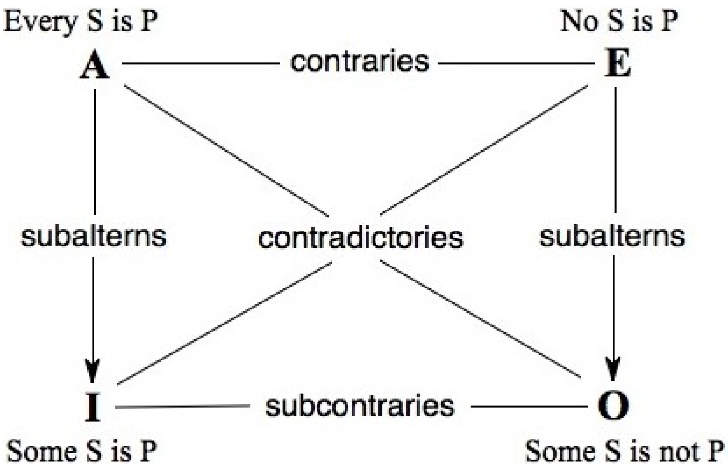
A All Stamps are Paper (Universal Subject + Affirmative Predicate)
E All Stamps aren't Paper (Universal Subject + Negative Predicate)
I Some Stamps are Paper (Particular Subject + Affirmative Predicate)
O Some Stamps aren't Paper (Particular Subject + Negative Predicate)

There is a mechanical way to convert Aristotle's four simple statement types into modern PL. If a statement has a Universal Subject, it uses a '$\forall$' quantifier, and the Subject and Predicate are connected with an '$\rightarrow$'. If a statement has a Particular Subject, it uses a '$\exists$' quantifier, and the Subject (first predicate) and Predicate (second predicate) are connected with an '$\wedge$'. The Subject (first predicate) is never negated. If a statement is Negative, we negate the Predicate (second predicate). For example:

A All Stamps are Paper – $\forall x\,[Sx \rightarrow Px]$
E All Stamps aren't Paper – $\forall x\,[Sx \rightarrow \neg Px]$
I Some Stamps are Paper – $\exists x\,[Sx \wedge Px]$
O Some Stamps aren't Paper – $\exists x\,[Sx \wedge \neg Px]$

These four statement types are all that are allowed for Aristotle. We have already seen PLallows a lot more complexity, but it took until the late 1800s before we figured out how to make that work. Having only four statement types also allowed them to describe logical relations between all the statement types, via the Square of Opposition.

The Square of Opposition



In this book, we aren't interested in the logical relations of subalternation, contrariety, or subcontrariety, just contradiction. This means that the Aristotelian Square of Opposition, the jewel of logic carefully passed down on scrolls for 2,000 years, reduces to a straight-forward observation about contradiction.

Aristotle's A-type statement 'All Stamps are Paper' can be symbolised as $\forall x\,[Sx \to Px]$, which is the same as $\neg\exists x\,[Sx \wedge \neg Px]$, which symbolises the negation of the O-type statement 'Some Stamps aren't Paper'. That is, the A and O-type statements are negations of each other, and so are contradictory. Similarly, the E and I-type statements are contradictory.

Note that Aristotelian Syllogisms are **NOT** a simple version of PL; have a different notion validity, and so are a different logic system. But for our current purposes, they do provide a ring-fenced set of highly structured statements for exploration and practice. The Universal Affirmative, Universal Negative, Particular Affirmative, and Particular Negative statement types are still the key building blocks of PL, 2,300 years later.

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Over the next 2,000 years, the number of valid syllogisms varied between 15 and 24, as logicians subtly changed their ideas about validity, primarily by changing their minds about whether every Frog being Green meant that there were Green Frogs. This is the infamous *Existential Import.* Actually, they didn't talk about Frogs, but properties of God; this had major theological significance, and sometimes logic changed with the election of a new Pope.

Practice Exercises

Refer to this symbolisation key for the next two problem sets:

Ax: x is airborne.
Ix: x is infectious.
Hx: x is infectious to humans.
Vx: x is a virus.

Exercise A: Symbolise the following statements in PL:

1. All viruses are infectious.
2. Not all viruses are infectious.
3. All viruses are not infectious.
4. Some viruses are infectious.
5. No virus is infectious.
6. Some viruses are not infectious.
7. Some viruses are infectious, but not to humans.
8. All viruses are infectious, but none is infectious to humans unless it's airborne.
9. If all viruses are airborne, then some are infectious to humans.
10. If no virus is airborne, then none is infectious to humans.
11. Not all viruses that are infectious to humans are airborne.
12. Some viruses that aren't infectious to humans are airborne.

Exercise B: Using the logical equivalence between negated quantifiers ($\neg\exists$ vs. $\forall\neg$; and $\neg\forall$ vs. $\exists\neg$), as well as equivalences known from TFL, link the statements from the first column to those in the second column that are logically equivalent:

$\neg\forall x\,[Fx]$	$\exists x\,[\neg\neg Fx]$
$\forall x\,[\neg Fx]$	$\exists x\,[\neg Fx]$
$\neg\forall x\,[\neg Fx]$	$\neg\exists x\,[\neg Fx]$
$\neg\neg\forall x\,[Fx]$	$\neg\exists x\,[Fx]$
$\exists x\,[\neg(Px \wedge Qx)]$	$\neg\exists x\,[Px] \vee \neg\exists x\,[Qx]$
$\neg\forall x\,[Px \wedge Qx]$	$\neg\forall x\,[Px \wedge \neg Qx]$
$\neg\neg\exists x\,[Px \to Qx]$	$\neg\forall x\,[Px \wedge Qx]$
$\exists x\,[Px] \to \forall x\,[\neg Qx]$	$\exists x\,[\neg Px \vee \neg Qx]$

Exercise C: Provide two symbolisation keys for the following argument, and discuss how the treatment of ambiguous predicates affect the validity of the argument:

> Willi is a fluffy cat, but a lousy mouser. Therefore, Willi is a lousy cat.

Exercise D: Using the following symbolisation key:

Kx: x knows the combination to the safe
Px: x is a person
Sx: x is a spy
Vx: x is a vegetarian
h: Hofthor
i: Ingmar

Symbolise the following statements in PL:

1. Neither Hofthor nor Ingmar is a vegetarian.
2. No spy knows the combination to the safe.
3. No one knows the combination to the safe unless Ingmar does.
4. Hofthor is a spy, but no vegetarian is a spy.

⋆ **Exercise E:** Using this symbolisation key:

Ax: x is an alligator.
Mx: x is a monkey.
Rx: x is a reptile.
Zx: x lives at the zoo.
a: Amos
b: Bouncer
c: Cleo

Symbolise each of the following statements in PL:

1. Amos, Bouncer, and Cleo all live at the zoo.
2. Bouncer is a reptile, but not an alligator.
3. Some reptile lives at the zoo.
4. Every alligator is a reptile.
5. Anything that lives at the zoo is either a monkey or an alligator.
6. There are reptiles which are not alligators.
7. If any animal is an reptile, then Amos is.
8. If any animal is an alligator, then it is a reptile.

Exercise F: For each argument, write a symbolisation key and symbolise the argument in PL.

1. Phoenix is a logician. All logicians wear funny hats. So Phoenix wears a funny hat.
2. Nothing on my desk escapes my attention. There is a computer on my desk. Thus there is a computer that does not escape my attention.
3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.
4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.
5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

Exercise G: These are some of the syllogistic figures identified by Aristotle and his successors, along with their medieval names.

Symbolise each argument in PL.

Barbara : All Bs are Cs. All As are Bs. $\therefore$ All As are Cs.
Baroco : All Cs are Bs. Some A is not B. $\therefore$ Some A is not C.
Baralipton : All Bs are Cs. All As are Bs. $\therefore$ Some C is A.
Bocardo : Some B is not C. All As are Bs. $\therefore$ Some A is not C.
Celantes : No Bs are Cs. All As are Bs. $\therefore$ No Cs are As.
Celarent : No Bs are Cs. All As are Bs. $\therefore$ No As are Cs.
Cemestres : No Cs are Bs. No As are Bs. $\therefore$ No As are Cs.
Cesare : No Cs are Bs. All As are Bs. $\therefore$ No As are Cs.
Dabitis : All Bs are Cs. Some A is B. $\therefore$ Some C is A.
Darii : All Bs are Cs. Some A is B. $\therefore$ Some A is C.
Datisi : All Bs are Cs. Some A is B. $\therefore$ Some A is C.
Disamis : Some B is C. All As are Bs. $\therefore$ Some A is C.
Ferison : No Bs are Cs. Some A is B. $\therefore$ Some A is not C.
Ferio: No Bs are Cs. Some A is B. $\therefore$ Some A is not C.
Festino : No Cs are Bs. Some A is B. $\therefore$ Some A is not C.
Frisesomorum : Some B is C. No As are Bs. $\therefore$ Some C is not A.

Exercise H: Bonus question. One or two of the syllogisms listed above aren't valid in PL. See if you can figure out which ones, before we introduce the formal methods for testing validity in PL.

Chapter 4

Multiple generality

So far, we have only considered statements that require one-place predicates and one quantifier. The full power of PL really becomes apparent when we start to use many-place predicates and multiple quantifiers.

4.1 Many-place Predicates

All of the predicates that we have considered so far concern properties that things might have. They are one-place predicates. Other predicates concern the *relation* between things, such as:

1. Patrick and Sylvie are siblings.
2. Auckland is to the north of Dunedin.
3. Hamilton is in between Auckland and Tauranga.

Statement 1 is about two things, Patrick and Sylvie, and relates them by saying that they are siblings. Statement 2 also relates two things, Auckland and Dunedin, by saying that one is to the north of the other. Statement 3, however, relates three things, Hamilton, Auckland and Tauranga with a relation of *in-betweenness*. We call the relations in statements 1 and 2 two-place predicates, and the relation in statement 3 a three-place predicate. In principle, there is no upper limit on the number of places that our predicates may have. In practice, more than three is rare.

To keep track of the number of places in relations, as well as their order, we use variables. This is best explained by an example. Suppose we want to symbolise the following statements:

4. Karl loves Jesse.
5. Jesse is loved by Karl.
6. Jesse loves themselves.
7. Karl loves Jesse, but Jesse doesn't love Karl.

We will start with the following symbolisation key:

j: Jesse
k: Karl
Lxy: x loves y

The symbolisation key has two names, j and k, and a two-place predicate Lxy. The use of the variables x and y show that L is a two-place predicate. Furthermore, the order in which they appear in Lxy and in the English key shows that it is x that loves y, and not the other way around. Statements 4 and 5 both express that Karl loves Jesse, but in different voices: 'Karl loves Jesse' is in the active voice, 'Jesse is loved by Karl' in the passive voice. This difference in voices is lost in PL; we symbolise statements 4 and 5 in the same way:

$$Lk, j$$

Statement 6 expresses that Jesse loves Jesse, which illustrates that the same name may occur more than once in a relation:

$$Ljj$$

Statement 7 is a conjunction. The first conjunct expresses that Karl loves Jesse, which we symbolise as

$$Lk, j$$

The second conjunct says that Jesses doesn't love Karl back:

$$\neg Lj, k$$

This again illustrates the importance of the order of the places in predicates: Karl loves Jesse, but Jesse doesn't love Karl. Another way of saying this is that the loving relation holds from Karl to Jesse, but not from Jesse to Karl. Putting things together, we symbolise statement 7 like this:

$$Lkj \wedge \neg Ljk$$

When we are dealing with predicates with more than one place, we need to pay careful attention to the order of the places.

4.2 Quantifier Order

Consider the ambiguous statement ‘everyone loves someone’. Why is it ambiguous? Because it could mean either of the following:

8. Everyone loves a person, though not necessarily the same person.
9. There is some particular person whom everyone loves.

The language of PL may not be the most beautiful language you’ve encountered, but it is very good at avoiding ambiguity. It avoids ambiguity not only by using the order of variables in predicates, but also the order of quantifiers in formulas. The difference between 8 and 9 is that the first expresses a $\forall\exists$ pattern of quantification whereas the second expresses a $\exists\forall$ pattern. More precisely, take this symbolisation key:

Px: x is a person
Lxy: x loves y

We can then symbolise statements 8 and 9 like this:

$$\forall x\, \exists y\, [(Px \wedge Py) \rightarrow Lxy]$$
$$\exists x\, \forall y\, [(Px \wedge Py) \rightarrow Lyx]$$

The point of the example is to illustrate that the order of the quantifiers matters a great deal. Indeed, to switch them around is called a *quantifier shift fallacy.* Here is an example, which comes up in various forms throughout the philosophical literature:

For every person, there is some truth they cannot know. ($\forall\exists$)
$\therefore$ There is some truth that no person can know. ($\exists\forall$)

This argument form is invalid. It’s just as bad as:

Everyday someone is struck by lightning. ($\forall\exists$)
$\therefore$ Some poor person is struck by lightning everyday. ($\exists\forall$)

This is why we ask you to take great care with the order of quantification.

4.3 Stepping-stones to Symbolisation

Once we have the possibility of multiple quantifiers and many-place predicates, symbolisation in PL can become quite tricky. However, there are several stepping stones you can make use of. These steps are best illustrated by example. Consider this symbolisation key:

g: Geraldo
Dx: x is a dog
Fxy: x is a friend of y
Oxy: x owns y

Now let's try to symbolise these statements:

10. Geraldo is a dog owner.
11. There are dog owners.
12. All of Geraldo's friends are dog owners.
13. Every dog owner is a friend of a dog owner.
14. Every dog owner's friend owns a dog of a friend.

Paraphrasing can help with identifying the quantifiers that are appropriate for symbolising a statement. Statement 10 can first be paraphrased as saying that 'Geraldo owns a dog'. This is what it means to be a dog owner. Next, is this a universal or an existential statement? The statement doesn't say that Geraldo owns every dog, but that Geraldo owns at least one dog, so we can paraphrase it as 'There is a dog that Geraldo owns'. This is easier to symbolise:

$$\exists x\,[Dx \wedge Ogx]$$

Statement 11 says that at least one person is the owner of a dog, so it refers to two objects (the person and the dog) that don't have names, so will have two quantifiers, and their associated variables. It may help to take them one at the time. Let's first paraphrase 11 as, 'There is some x such that x owns a dog'. This takes care of the first quantifier, so we can write, as an intermediary step, '$\exists x\,[x$ owns a dog$]$'. Now the fragment we have left as 'x

owns a dog' is much like statement 10, except that it is not specifically about Geraldo. So we can symbolise statement 11 by:

$$\exists x\, \exists y\, [Dy \wedge Oxy]$$

We should pause to clarify something here. In working out how to symbolise the last statement, we wrote down '$\exists x\, [x$ owns a dog]'. To be very clear: this is *neither* an PL statement *nor* an English statement, because it uses bits of PL ('$\exists$', 'x') and bits of English ('owns a dog'). It is really is *just a stepping-stone* on the way to symbolising the entire English statement with a PL statement.

Statement 12 talks about all of Geraldo's friends, so it is a universal statement. That's why we first paraphrase it as, 'Every friend of Geraldo also owns a dog'. Using our stepping-stone tactic, we might write

$$\forall x\, [Fxg \to x \text{ is a dog owner}]$$

This leaves us to deal with the fragment 'x owns a dog', which we've already encountered in statement 10. Because the variable x is already used to talk about Geraldo's friends, we need to introduce a new variable, y:

$$\forall x\, [Fxg \to \exists y\, [Dy \wedge Oxy]]$$

Speeding up the intermediate steps, statement 13 can be paraphrased as 'For any x that owns a dog, there is someone who owns a dog and whom x is a friend of'. Using our stepping-stone tactic, this becomes

$$\forall x\, [x \text{ owns a dog} \to \exists y\, [y \text{ owns a dog} \wedge Fxy]]$$

Completing the symbolisation, we end up with

$$\forall x\, [\exists z\, [Dz \wedge Oxz] \to \exists y\, [\exists w\, [Dw \wedge Oyw] \wedge Fxy]]$$

Statement 14 is the trickiest yet. First we paraphrase it as 'For any x that is a friend of someone who owns a dog, x owns a dog, and that dog is also owned by a friend of x'. Using our stepping-stone tactic, this becomes:

$$\forall x\, [x \text{ is a friend of someone who owns a dog} \to \\ x \text{ owns a dog which is also owned by a friend of } x]$$

Breaking this down a bit more:

$$\forall x\, [\exists y\, [Fxy \wedge y \text{ owns a dog}] \to \\ \exists z\, [Dz \wedge Oxz \wedge z \text{ is owned by a friend of } x]]$$

And a bit more:

$$\forall x\,[\exists y\,[Fxy \wedge \exists w\,[Dw \wedge Oyw]] \;\rightarrow\; \exists z\,[Dz \wedge Oxz \wedge \exists v\,[Fvx \wedge Ovz]]]$$

And we are done. We told you, things get tricky quickly!

Hopefully, using these stepping-stones between English and PL will help you simplifying the problem of symbolising complex statements into formulas by breaking down the tasks into simpler ones.

4.4 De-Symbolising

So far we've looked at symbolising statements, which goes from a natural language (English) to a formal language (PL). When you start working with formal languages like PL, however, you will be working directly with the symbolisation, but will still have to communicate about it in natural language. You will then need to be able to read formulas, or as we like to put it, to *de-symbolise*. Using a stepping-stone approach is very useful for that. Let's see some examples, working with this symbolisation key:

Ix: x is an instrument.
Mx: x is a musician.
Pxy: x plays y.

We will desymbolise these two formulas back to English:

15. $\forall x\,[Mx \rightarrow \exists y\,[Iy \wedge Pxy]]$
16. $\exists x\,[Mx \wedge \forall y\,[Iy \rightarrow Pxy]]$

Start with formula 15. One step in desymbolising a formula is to simply read every symbol out loud, as in:

> For every x, if x is a musician, then there is a y such that y is an instrument and x plays y.

This is (nearly) a sentence of English, as you have transformed most parts of the formula into English, but it's only a stepping stone towards a naturally flowing English sentence. Our stepping-stone approach to desymbolising has as an end-goal to produce a nice sentence of English.

Let's try again, taking each part of the formula in turn. One useful way to divide up a complex formula is quantifier by quantifier. Formula 15 has two quantifiers. We will start by transforming the outermost quantifier:

> For every x, if x is a musician, then $\exists y\,[Iy \land Pxy]$

Now that we know what we are talking about, namely everything that is a musician, we can express this part of the formula in English with:

> Every musician . . . $\exists y\,[Iy \land Pxy]$

Next we transform the existential quantifier:

> Every musician . . . there is a y such that y is an instrument and Pxy.

This tells us that every musician does something to an instrument. What is it?

> Every musician . . . there is a y such that y is an instrument and x plays y.

Now we have all the information we need: for every musician there is an instrument that the musician plays, or written elegantly:

> Every musician plays an instrument.

And this is how we desymbolise formula 15 into English. For formula 16, we follow a similar procedure. We transform the outermost quantifier first:

> There is a x such that x is a musician and $\forall y\,[Iy \to Pxy]$

So we know that something is going on with a musician:

> Some musician . . . $\forall y\,[Iy \to Pxy]$

Next we transform the universal quantifier:

> Some musician . . . for all y, if y is an instrument, then Pxy.

What does the musician do to every instruments? They play it:

> Some musician . . . for all y, if y is an instrument, then x plays y.

Now we have all the information needed to express it in English:

> Some musician plays all instruments.

4.5 Suppressed Quantifiers

Logic can often help to clarify statements, especially where the quantifiers are left implicit or their order is ambiguous or unclear. The clarity of expression and thinking afforded by PL can give you a significant advantage in argument, as can be seen in the following takedown by British political philosopher Mary Astell (1666–1731) of her contemporary, the theologian William Nicholls. In Discourse IV: The Duty of Wives to their Husbands of his *The Duty of Inferiors towards their Superiors, in Five Practical Discourses* (London 1701), Nicholls argued that women are naturally inferior to men. In the preface to the 3rd edition of her treatise *Some Reflections upon Marriage, Occasion'd by the Duke and Duchess of Mazarine's Case; which is also considered,* Astell responded as follows:

> 'Tis true, thro' Want of Learning, and of that Superior Genius which Men as Men lay claim to, she [Astell] was ignorant of the *Natural Inferiority* of our Sex, which our Masters lay down as a Self-Evident and Fundamental Truth. She saw nothing in the Reason of Things, to make this either a Principle or a Conclusion, but much to the contrary; it being Sedition at least, if not Treason to assert it in this Reign.
>
> For if by the Natural Superiority of their Sex, they mean that *every* Man is by Nature superior to *every* Woman, which is the obvious meaning, and that which must be stuck to if they would speak Sense, it wou'd be a Sin in *any* Woman to have Dominion over *any* Man, and the greatest Queen ought not to command but to obey her Footman, because no Municipal Laws can supersede or change the Law of Nature; so that if the Dominion of the Men be such, the *Salique Law,*[1] as unjust as *English Men* have ever thought it, ought to take place over all the Earth, and the most glorious Reigns in the *English, Danish, Castilian*, and other Annals, were wicked Violations of the Law of Nature!
>
> If they mean that *some* Men are superior to *some* Women this is no great Discovery; had they turn'd the Tables they might have seen that *some* Women are Superior to *some* Men. Or had they been pleased to remember their Oaths of Allegiance and Supremacy, they might have known that *One* Woman is superior to *All* the Men in these Nations, or else they have sworn to very

[1]The Salique law was the common law of France which prohibited the crown be passed on to female heirs.

> little purpose.[2] And it must not be suppos'd, that their Reason and Religion wou'd suffer them to take Oaths, contrary to the Laws of Nature and Reason of things.[3]

We can symbolise the different interpretations Astell offers of Nicholls' claim that men are superior to women: He either meant that every man is superior to every woman, i.e.,

$$\forall x\,[Mx \to \forall y\,[Wy \to Sxy]]$$

or that some men are superior to some women,

$$\exists x\,[Mx \land \exists y\,[Wy \land Sxy]].$$

The latter is true, but then so is

$$\exists y\,[Wy \land \exists x\,[Mx \land Syx]].$$

(some women are superior to some men), so that, Astell says, would be "no great discovery". In fact, since the Queen is superior to all her subjects, it's even true that some woman is superior to every man, i.e.,

$$\exists y\,[Wy \land \forall x\,[Mx \to Syx]].$$

But this is incompatible with the "obvious meaning" of Nicholls' claim, i.e., the first reading. So what Nicholls claims amounts to treason against the Queen!

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[2] In 1706, England was ruled by Queen Anne.

[3] Mary Astell, *Reflections upon Marriage*, 1706 Preface, iii–iv, and Mary Astell, *Political Writings*, ed. Patricia Springborg, Cambridge University Press, 1996, 9–10.

Practice Exercises

Exercise A: Using the following symbolisation key:

i: me
Px: x is a person.
Wx: x is a winner.
Tx: x is a time.
Lxy: x loves y.
$Hxyz$: x loves y at z.

symbolise the following statements in PL:

1. Everybody loves a winner.
2. Nobody loves me.
3. Everybody loves somebody sometime.

Exercise B: Using the following symbolisation key:

i: me
Px: x is a person.
Wx: x is a winner.
Tx: x is a time.
Lxy: x loves y.
$Hxyz$: x loves y at z.

Desymbolise the following formulas of PL into English:

1. $\exists x\,[Px \wedge \forall y\,[Wy \rightarrow Lxy]]$
2. $\neg\forall x\,[Lxi]$
3. $\exists x\,[Px \wedge \forall y\,[Py \rightarrow \exists z\,[Tz \wedge Hxyz]]]$

$\star$ **Exercise C:** Using this symbolisation key:

a: Amos
b: Bouncer
c: Cleo
Ax: x is an alligator
Mx: x is a monkey
Rx: x is a reptile
Zx: x lives at the zoo
Lxy: x loves y

symbolise each of the following statements in PL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.
2. If both Bouncer and Cleo are alligators, then Amos loves them both.
3. Cleo loves a reptile.
4. Bouncer loves all the monkeys that live at the zoo.
5. All the monkeys that Amos loves love him back.
6. Every monkey that Cleo loves is also loved by Amos.
7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.

Exercise D: Using this symbolisation key:

r: Rave
h: Shane
d: Daisy
Ax: x is an animal
Dx: x is a dog
Sx: x likes samurai movies
Lxy: x is larger than y

symbolise the following statements in PL:

1. Rave is a dog who likes samurai movies.
2. Rave, Shane, and Daisy are all dogs.
3. Shane is larger than Rave, and Daisy is larger than Shane.
4. All dogs like samurai movies.
5. Only dogs like samurai movies.
6. There is a dog that is larger than Shane.
7. If there is a dog larger than Daisy, then there is a dog larger than Shane.
8. No animal that likes samurai movies is larger than Shane.
9. No dog is larger than Daisy.
10. Any animal that dislikes samurai movies is larger than Rave.
11. There is an animal that is between Rave and Shane in size.
12. There is no dog that is between Rave and Shane in size.
13. No dog is larger than itself.
14. Every dog is larger than some dog.
15. There is an animal that is smaller than every dog.
16. Any animal that is larger than any dog does not like samurai movies.

Exercise E: Using this symbolisation key:

Cx: x is a candy.
Mx: x has marzipan in it.

Sx: x has sugar in it.
Tx: Boris has tried x.
Bxy: x is better than y.

symbolise the following statements in PL:

1. Boris has never tried any candy.
2. Marzipan is always made with sugar.
3. Some candy is sugar-free.
4. No candy is better than itself.
5. Boris has never tried sugar-free candies.
6. Any candy with sugar is better than any candy without it.

Exercise F: Using this symbolisation key:

e: Eli
f: Francesca
g: the guacamole
Dx: x is a dish.
Fx: x is food.
Px: x is a person.
Rx: x has run out.
Tx: x is on the table.
Lxy: x likes y.

symbolise the following English statements in PL:

1. All the food is on the table.
2. If the guacamole has not run out, then it is on the table.
3. Everyone likes the guacamole.
4. If anyone likes the guacamole, then Eli does.
5. Francesca only likes the dishes that have run out.
6. Francesca likes no one, and no one likes Francesca.
7. Eli likes anyone who likes the guacamole.
8. Eli likes anyone who likes the people that he likes.
9. If there is a person on the table already, then all of the food must have run out.

⋆ **Exercise G:** Using this symbolisation key:

e: Elmer
j: Jane
p: Patrick
Dx: x dances ballet.

Fx: x is female.
Mx: x is male.
Px: x is a person.
Cxy: x is a child of y.
Sxy: x is a sibling of y.

symbolise the following statements in PL:

1. All of Patrick's children are ballet dancers.
2. Jane is Patrick's daughter.
3. Patrick has a daughter.
4. Jane is an only child.
5. All of Patrick's sons dance ballet.
6. Patrick has no sons.
7. Jane is Elmer's niece.
8. Patrick is Elmer's brother.
9. Patrick's brothers have no children.
10. Jane is an aunt.
11. Everyone who dances ballet has a brother who also dances ballet.
12. Every woman who dances ballet is the child of someone who dances ballet.

Chapter 5
Identity

Consider this statement:

1. Aristotle is more philosophical than anyone on YouTube.

Using the symbolisation key:

a: Aristotle
Yx: x is a person on Youtube
Pxy: x is more philosophical than y

we could attempt to symbolise statement 1 with:

$$\forall x\,[Yx \to Pax]$$

This symbolisation isn't accurate, however. Can you see why? The problem is that Aristotle is also on YouTube, and if Aristotle is more philosophical than anyone on YouTube, then Aristotle is more philosophical than themself, which is absurd. What statement 1 is meant to say is rather something like this:

2. Aristotle is more philosophical than *anyone else* on YouTube.

but we do not have the resources in PL to express *anyone else*. What we have in PL are two quantifiers, the universal quantifier $\forall$ that talks about everything, and the existential quantifier $\exists$ that talks about at least one thing. Neither quantifier allows us to talk about everything *except* Aristotle. You might be tempted by this symbolisation:

$$\forall x\,[Yx \to Pax] \land \neg Paa$$

This says that Aristotle is more philosophical than anyone on YouTube (including Aristotle) and that Aristotle isn't more philosophical than Aristotle. But this formula is a contradiction, and that won't do. The solution is to add identity to PL. Identity is the subject of this chapter.

5.1 Adding Identity

An IDENTITY between terms expresses that we have two ways of talking about one thing. For instance, identity holds between Diana Prince and Wonder Woman, because they are the same person. In PL, we treat identity as a special predicate and we add a special symbol for it, the symbol '='. We *always* adopt the following symbolisation key for identity:

$x = y$: x is identical to y

To say that x and y are identical does not mean *merely* that the objects they refer to are indistinguishable, or that all of the same things are true of them. Rather, it means that x and y refer to *the very same* object.

For the next examples, we will use this symbolisation key:

d: Diana Prince
p: Peter Parker
w: Wonder Woman
$x = y$: x is identical to y

We use identity to symbolise statements like this:

3. Diana Prince is Wonder Woman.

We can then symbolise statement 3 in this way:

$$d = w$$

Simple! This means that the names 'd' and 'w' both refer to (name) the same thing. One notational difference about identity is that we write $d = w$ instead of $=(d, w)$ like other predicates. This makes it easier to read.

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This way of writing identity is what is an *infix* notation, because the identity symbol is written in between its two arguments x and y. In contrast, we have been using a *prefix* notation for our predicates. In terms of expressivity, the difference in notation doesn't make a difference. For some coding purposes, however, prefix notation is taken to be easier to work with, especially with complex codes.

Like other predicates, we can express that identity doesn't hold between two things with a negation. For instance, we could symbolise that Peter Parker isn't Wonder Woman like this:

$$\neg(p = w)$$

but this is not very pretty. We prefer to use the following notation:

$$p \neq w$$

which expresses that Peter Parker isn't (identical to) Wonder Woman.

With this new symbol added to PL, we can now come back to our example about Aristotle. We now use this symbolisation key:

a: Aristotle
Yx: x is a person on Youtube
Pxy: x is more philosophical than y
$x = y$: x is identical to y

To express that Aristotle is more philosophical than *anyone else* on YouTube, we use identity to exclude Aristotle:

$$\forall x\,[(Yx \land x \neq a) \to Pax]$$

This formula says that Aristotle is more philosophical than anyone on Youtube, except themself. We can also use identity to symbolise something more complex, like:

4. Aristotle is the most philosophical person on YouTube

How might we express this in PL? Here's one reasonable symbolisation:

$$Ya \land \forall x\,[Yx \to (\forall z\,[(Yx \land z \neq x) \to Pxz] \leftrightarrow x = a)]$$

This says that Aristotle is on Youtube, that if anyone is more philosophical than anyone else on Youtube then that person is Aristotle (to avoid ties), and that he is more philosophical than anyone else on Youtube.
Hint To read this formula: break down the $\leftrightarrow$ into an $\to$ and a $\to$, and pay attention to the bracketing, and the quantifier scope for z.

Having identity allows us to express more nuanced concepts. But we are also going to have to be more careful, and to accustom ourselves to reading and creating longer formulas.

5.2 Using Identity

We have already seen a couple of ways to use identity. It's time to go through some common applications a little more systematically.

We can state that two English names are identical, as we did when declaring Diana Prince to be Wonder Woman. We symbolised this in PL by $d = w$.

We can also do this with name-like definite descriptions. With this key:

k: The cutest kitten in Auckland.
b: Belinda.
a: Auckland.
$c(x)$: The cutest kitten in x.

'Belinda is the cutest kitten in Auckland' can be symbolised as $b = k$. Or if we allow functions in our language, we could symbolise this statement as $b = c(a)$. It would also be true that $k = c(a)$.

We can also state that a name satisfies a quantifier. With this key:

j: Jeremy.
Lx: x is your lecturer.

'One of your lecturers is Jeremy' can be symbolised by $\exists x\,[Lx \wedge x = j]$. You could also symbolise this simply as Lj, although that might be more suited to symbolising 'Jeremy is your lecturer', a statement with a similar meaning but that we would use in different contexts. The opportunities for subtlety continue to grow!

We can also claim that names are non-identical, such as when we symbolised that Peter Parker was not Wonder Woman, by $p \neq w$. There are a number of common words whose meaning is a kind of non-identity.

Other To be 'other' is to not be the same as. With this key:

i: Me.
u: You.
Px: x is a person.
Txy: x is thinking of y.

'I was thinking of someone other than you' can be symbolised by the formula $\exists x\,[Px \wedge Tix \wedge x \neq u]$. We might want to add $(u \neq i)$ if we think that grammatical truth is important to the argument.

Another / Else To be 'another' is something like being 'other', but usually in addition to. 'I was thinking of another person, not just of you' can be symbolised using the above key by the formula $\exists x\,[Px \wedge Tix \wedge x \neq u] \wedge Tiu$. We could also symbolise 'I was not just thinking of you' similarly, although we don't know I'm thinking of another *person*: $\exists x\,[Tix \wedge x \neq u] \wedge Tiu$. And 'I wasn't thinking of you at all' could be symbolised as $\neg Tiu$, but if we interpret this ambiguous statement as 'I was thinking of someone else' it could be symbolised as $\exists x\,[Px \wedge Tix \wedge x \neq u] \wedge \neg Tiu$.

No one else / No one but The phrase 'no one else' is a negation of 'someone else'. Suppose we stated 'I was thinking of you and no one else', or 'I was thinking of no one but you', or 'All I thought of was you'. Then using the above key, we can symbolise these sweet whispers as $Tiu \wedge \forall x\,[Tix \to x = u]$ or more compactly $\forall x\,[Tix \leftrightarrow x = u]$. Isn't logic romantic!

Also 'Also' is something like 'and', but the two conjuncts must refer to distinct (non-identical) objects. With this key:

i: Me.
u: You.
Fxy: x is a friend of y.
Hxy: x is hanging with y.

We can symbolise 'I hang out with all my friends, and also with you' as $\forall x\,[Fix \to (Hix \wedge x \neq u)] \wedge Hiu$, which is pretty brutal. Contrast that with 'I hang out with all my friends, including you' which we would symbolise as $\forall x\,[Fix \to Hix] \wedge Fiu$.

Except / Unless 'Except' works roughly like 'unless', except that it can be used for excluding individuals. We can express most 'except' statements using an 'unless', replacing individuals in the exception with a statement. For example, 'I hang out with all my friends, except you' means roughly the same as 'I hang out with all my friends, unless they are you', or 'I hang out with all my friends, if they are not you'.

We can symbolise any of these statements using the above key as $\forall x\,[(Fix \wedge x \neq u) \to Hix] \wedge Fiu$. Compare this with the similar 'also' sentence above. Here you are a friend, and we leave it open whether I hang out with you or not. If you think that the English sentence logically excludes that I hang out with you, we can change the formula from an '$\to$' to a '$\leftrightarrow$': $\forall x\,[(Fix \wedge x \neq u) \leftrightarrow Hix] \wedge Fiu$.

Only As promised in Chapter **??**, we are finally returning to discuss 'only' in more detail. It's worth have another read of §**??** first. With this key:

j: Janice.
Hx: x can talk.

We can symbolise 'Only humans talk' as $\forall x\,[\neg Hx \to \neg Tx]$, or $\forall x\,[Tx \to Hx]$. But it doesn't mean that all humans talk $\forall x\,[Hx \to Tx]$, and so it can't be symbolised as $\forall x\,[Hx \leftrightarrow Tx]$. This is just like what we said when symbolising 'only' in TFL. But with identity, we can also consider individuals. And our rule of thumb of negating both the antecedent and consequent of our conditional still applies when we are making statements about individuals. The statement 'Only Janice can talk' would be symbolised as $\forall x\,[x \neq j \to \neg Tx]$; we do not include Tj, as it does not automatically also mean that in all contexts.

The One and Only / The Only The expression 'The one and only' is the name-level analogue to the propositional 'if and only if'. It is often shortened to 'the only'. This expression does include both halves of the logical content. The expression 'Janice is the (one and) only person who can talk' is roughly 'Janice is one person who can talk AND only Janice can talk', which can be would be symbolised as $Tj \wedge \forall x\,[x \neq j \to \neg Tx]$, or as the equivalent but more compact $\forall x\,[Tx \leftrightarrow x = j]$.

The and One This now gives us another approach to understanding and symbolising definite descriptions. To say 'the A' is to say that there is at least one thing that is A, and that only that thing is A. With this key:

a: The accountant.
Ax: x is an accountant.
Sx: x smirked.

We can symbolise 'The accountant smirked' as either: Sa by treating 'the accountant' as standing for a name, or as $\exists x\,[Ax \wedge \forall y\,[Ay \to x = y] \wedge Sx]$. This latter approach would naturally lead us to consider statements where a definite description does not refer to exactly one object as being false. Considerations from our previous discussion on definite descriptions then help us decide whether we think this is an adequate symbolisation or not.

5.3 Counting on Identity

At least. . .

We can also use identity to say how many things there are of a particular kind. For example, consider these statements:

5. There is at least one apple
6. There are at least two apples
7. There are at least three apples

We will use the symbolisation key:

Ax: x is an apple

Statement 5 does not require identity. It can be adequately symbolised by '$\exists x\,[Ax]$': There is an apple; perhaps many, but at least one.

It might be tempting to also symbolise statement 6 without identity. Yet consider the statement '$\exists x\,\exists y\,[Ax \wedge Ay]$'. Roughly, this says that there is something x that is an apple and something y that is an apple. But since nothing prevents these from being the same apple, this would be true even if there were only one apple. In order to make sure that we are dealing with *different* apples, we need an identity predicate. Statement 6 needs to say that the two apples that exist are not identical, and so we symbolise it by:

$$\exists x\,\exists y\,[(Ax \wedge Ay) \wedge x \neq y].$$

Statement 7 requires talking about three different apples. Now we need three existential quantifiers, and we need to make sure that each will pick out something different:

$$\exists x\,\exists y\,\exists z\,[(Ax \wedge Ay \wedge Az) \wedge (x \neq y \wedge y \neq z \wedge x \neq z)].$$

Note that it is *not* enough to use '$x \neq y \wedge y \neq z$' to symbolise 'x, y, and z are all different.' For that would be true if x and y were different but $x = z$. Nor can we use the mathematical short-hand '$x \neq y \neq z$' As logicians, we aspire to avoid the casual notational abuse and general sloppiness of mathematicians.

At most... Now consider these statements:

8. There is at most one apple
9. There are at most two apples

Statement 8 can be paraphrased as 'It is not the case that there are at least *two* apples'. This is just the negation of statement 6:

$$\neg\exists x\,\exists y\,[(Ax \wedge Ay) \wedge x \neq y]$$

But statement 8 can also be approached in another way. It means that if you pick out an object and it's an apple, and then you pick out an object and it's also an apple, you must have picked out the same object both times. With this in mind, it can be symbolised by

$$\forall x\,\forall y\,[(Ax \wedge Ay) \rightarrow x = y]$$

The two statements will turn out to be logically equivalent.

In a similar way, statement 9 can be approached in two equivalent ways. It can be paraphrased as, 'It is not the case that there are *three* or more distinct apples', so we can write:

$$\neg\exists x\,\exists y\,\exists z\,[Ax \wedge Ay \wedge Az \wedge x \neq y \wedge y \neq z \wedge x \neq z]$$

Alternatively we can read it as saying that if you pick out an apple, and an apple, and an apple, then you will have picked out (at least) one of these objects more than once. Thus:

$$\forall x\,\forall y\,\forall z\,[(Ax \wedge Ay \wedge Az) \rightarrow (x = y \vee x = z \vee y = z)]$$

Another way of saying that is 'if you are showing me three apples, then you must have shown me at least one apple twice'.

This method of counting can get tiresome if we are dealing with complex predicates. For example, consider the statement 'at least 2 cats chased a mouse or two into someone's room'. We might symbolise this as:

$$\begin{aligned}\exists x\,\exists y\,\exists z\,\exists t\,\exists u\,[(&Cat(x) \wedge Cat(y) \wedge Person(z) \wedge Room(t) \wedge Owns(zt)\\ &\wedge Mouse(u) \wedge Chased(xut) \wedge Chased(yut) \wedge (x \neq y)\\ &\wedge \neg\exists v\,\exists w\,[Mouse(v) \wedge Mouse(w) \wedge Chased(xvt) \wedge Chased(yvt)\\ &\qquad \wedge Chased(xwt) \wedge Chased(ywt) \wedge (u \neq v \wedge u \neq w \wedge v \neq w)]].\end{aligned}$$

Exactly... We can now consider precise statements, like:

10. There is exactly one apple.
11. There are exactly two apples.
12. There are exactly three apples.

Statement 10 can be paraphrased as, 'There is *at least* one apple and there is *at most* one apple'. This is just the conjunction of statement 5 and statement 8. So we can write:

$$\exists x\,[Ax] \land \forall x\,\forall y\,[(Ax \land Ay) \to x = y]$$

But it is perhaps more straightforward to paraphrase statement 10 as, 'There is a thing x which is an apple, and everything which is an apple is just x itself'. Thought of in this way, we write:

$$\exists x\,[Ax \land \forall y\,[Ay \to x = y]]$$

Similarly, statement 11 may be paraphrased as, 'There are *at least* two apples, and there are *at most* two apples'. Thus we could write

$$\exists x\,\exists y\,[(Ax \land Ay) \land x \neq y] \land \\ \forall x\,\forall y\,\forall z\,[(Ax \land Ay \land Az) \to (x = y \lor x = z \lor y = z)]$$

More efficiently, though, we can paraphrase it as 'There are at least two different apples, and every apple is one of those two apples'. Then we write:

$$\exists x\,\exists y\,[x \neq y \land \forall z\,[Az \leftrightarrow (x = z \lor y = z)]]$$

Finally, consider these statements:

13. There are exactly two things
14. There are exactly two objects

It might be tempting to add a predicate to our symbolisation key, to symbolise the English predicate '_____ is a thing' or '_____ is an object', but this is unnecessary. Words like 'thing' and 'object' do not sort wheat from chaff: they apply trivially to everything, which is to say, they apply trivially to every thing. So we can symbolise Statement 13 or 14 with either of:

$$\exists x\,\exists y\,[x \neq y] \land \neg\exists x\,\exists y\,\exists z\,[x \neq y \land y \neq z \land x \neq z]$$
$$\exists x\,\exists y\,[x \neq y \land \forall z\,[x = z \lor y = z]]$$

Practice Exercises

⋆ **Exercise A:** Using this symbolisation key:

a: Andrew
k: Kim
Px: x is a person
Lx: x loves y
$x = y$: x is identical to y

symbolise each of the following statements in PL:

1. Everyone loves someone else.
2. Kim loves no one but herself.
3. Andrew is the only one who doesn't love Kim.
4. Only Kim loves Andrew.

Exercise B: Explain why:

- '$\exists x \forall y [Ay \leftrightarrow x = y]$' is a good symbolisation of 'there is exactly one apple'.
- '$\exists x \exists y [x \neq y \wedge \forall z [Az \leftrightarrow (x = z \vee y = z)]]$' is a good symbolisation of 'there are exactly two apples'.

Exercise C: Using the following symbolisation key:

h: Hofthor
i: Ingmar
Kx: x knows the combination to the safe.
Px: x is a person.
Sx: x is a spy.
Vx: x is a vegetarian.
Txy: x trusts y.

symbolise the following statements in PL:

1. Hofthor trusts a vegetarian.
2. Everyone who trusts Ingmar trusts a vegetarian.
3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.
4. Only Ingmar knows the combination to the safe.
5. Ingmar trusts Hofthor, but no one else.
6. The person who knows the combination to the safe is a vegetarian.
7. The person who knows the combination to the safe is not a spy.

⋆ **Exercise D:** Using the following symbolisation key:

Bx: x is black.
Cx: x is a club.
Dx: x is a deuce.
Jx: x is a jack.
Mx: x is a man with an axe.
Ox: x is one-eyed.
Sx: x is a card.
Wx: x is wild.

symbolise each statement in PL:

1. All clubs are black cards.
2. There are no wild cards.
3. There are at least two clubs.
4. There is more than one one-eyed jack.
5. There are at most two one-eyed jacks.
6. There are two black jacks.
7. There are four deuces.
8. The deuce of clubs is a black card.
9. One-eyed jacks and the man with the axe are wild.
10. If the deuce of clubs is wild, then there is exactly one wild card.
11. The man with the axe is not a jack.
12. The deuce of clubs is not the man with the axe.

Exercise E: Using the following symbolisation key:

Ax: x is an animal.
Bx: x is in Farmer Brown's field.
Hx: x is a horse.
Px: x is a Pegasus.
Wx: x has wings.

symbolise the following statements in PL:

1. There are at least three horses in the world.
2. There are at least three animals in the world.
3. There is more than one horse in Farmer Brown's field.
4. There are three horses in Farmer Brown's field.
5. There is a single winged creature in Farmer Brown's field; any other creatures in the field must be wingless.

6. The Pegasus is a winged horse.
7. The animal in Farmer Brown's field is not a horse.
8. The horse in Farmer Brown's field does not have wings.

Exercise F: In this chapter, we symbolised 'Nick is the traitor' by '$\exists x\,[Tx \wedge \forall y\,[Ty \rightarrow x = y] \wedge x = n]$'. Two equally good symbolisations would be:

- $Tn \wedge \forall y\,[Ty \rightarrow n = y]$
- $\forall y\,[Ty \leftrightarrow y = n]$

Explain why these would be equally good symbolisations.