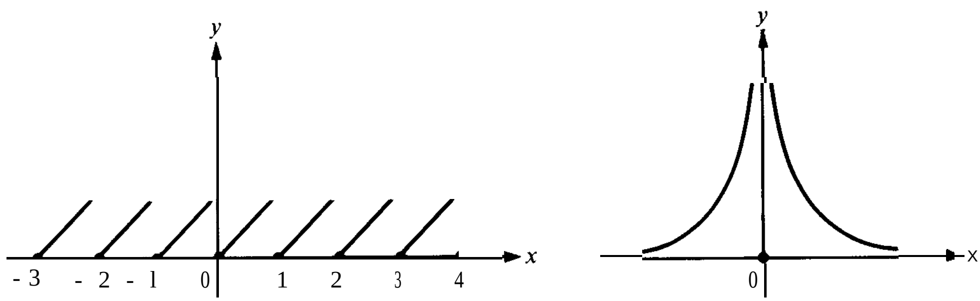
3

CONTINUOUS FUNCTIONS

3.1 Informal description of continuity

This chapter deals with the concept of continuity, one of the most important and also one of the most fascinating ideas in all of mathematics. Before we give a precise technical definition of continuity, we shall briefly discuss the concept in an informal and intuitive way to give the reader a feeling for its meaning.

Roughly speaking, the situation is this: Suppose a function f has the value $f(p)$ at a certain point p. Then f is said to be continuous at p if at every nearby point x the function



(a) A jump discontinuity at each integer. (b) An infinite discontinuity at 0.

FIGURE 3.1 Illustrating two kinds of discontinuities.

value $f(x)$ is close to $f(p)$. Another way of putting it is as follows: If we let x move toward p, we want the corresponding function values $f(x)$ to become arbitrarily close to $f(p)$, regardless of the manner in which x approaches p. We do *not* want sudden jumps in the values of a continuous function, as in the examples in Figure 3.1.

Figure 3.1(a) shows the graph of the function f defined by the equation $f(x) = x - [x]$, where $[x]$ denotes the greatest integer $\leq x$. At each integer we have what is known as a *jump discontinuity*. For example, $f(2) = 0$, but as x approaches 2 from the left, $f(x)$ approaches the value 1, which is not equal to $f(2)$. Therefore we have a discontinuity at 2. Note that $f(x)$ *does* approach $f(2)$ if we let x approach 2 *from the right,* but this by itself is not enough to establish continuity at 2. In a case like this, the function is called *continuous from the right* at 2 and *discontinuous from the left* at 2. Continuity at a point requires both continuity from the left and from the right.

In the early development of calculus almost all functions that were dealt with were continuous and there was no real need at that time for a penetrating look into the exact meaning of continuity. It was not until late in the 18th Century that discontinuous functions began appearing in connection with various kinds of physical problems. In particular, the work of J. B. J. Fourier (1758-1830) on the theory of heat forced mathematicians of the early 19th Century to examine more carefully the exact meaning of such concepts as ***function*** and ***continuity.*** Although the meaning of the word "continuous" seems intuitively clear to most people, it is not obvious how a good definition of this idea should be formulated. One popular dictionary explains continuity as follows :

Continuity: Quality or state of being continuous.

Continuous: Having continuity of parts.

Trying to learn the meaning of continuity from these two statements alone is like trying to learn Chinese with only a Chinese dictionary. A satisfactory mathematical definition of continuity, expressed entirely in terms of properties of the real-number system, was first formulated in 1821 by the French mathematician, Augustin-Louis Cauchy (1789-1857). His definition, which is still used today, is most easily explained in terms of the limit concept to which we turn now.

3.2 The defmition of the limit of a function

Let f be a function defined in some open interval containing a point p, although we do not insist that f be defined at the point p itself. Let A be a real number. The equation

$$\lim_{x \to p} f(x) = A$$

is read: "The limit of $f(x)$, as x approaches p, is equal to A," or "$f(x)$ approaches A as x approaches p." It is also written without the limit symbol, as follows:

$$f(x) \to A \qquad \text{as} \quad x \to p .$$

This symbolism is intended to convey the idea that we can make $f(x)$ as close to A as we please, provided we choose x sufficiently close to p.

Our first task is to explain the meaning of these symbols entirely in terms of real numbers. We shall do this in two stages. First we introduce the concept of a ***neighborhood*** of a point, then we define limits in terms of neighborhoods.

DEFINITION OF NEIGHBORHOOD OF A POINT. *Any open interval containing a point p as its midpoint is called a neighborhood of p.*

Notation. We denote neighborhoods by $N(p)$, $N_1(p)$, $N_2(p)$, etc. Since a neighborhood $N(p)$ is an open interval symmetric about p, it consists of all real x satisfying $p - r < x < p + r$ for some $r > 0$. The positive number r is called the ***radius*** of the neighborhood. We designate $N(p)$ by $N(p; r)$ if we wish to specify its radius. The inequalities $p - r < x < p + r$ are equivalent to $-r < x - p < r$, and to $|x - p| < r$. Thus, $N(p; r)$ consists of all points x whose distance from p is less than r.

In the next definition, we assume that A is a real number and thatfis a function defined on some neighborhood of a point p (except possibly at p). The function f may also be defined at p but this is irrelevant in the definition.

DEFINITION OF LIMIT OF A FUNCTION. *The symbolism*

$$\lim_{x \to p} f(x) = A \qquad [or\ f(x) \to A \qquad \text{a s} \quad x - p]$$

means that for every neighborhood $N_1(A)$ there is some neighborhood N,(p) such that

(3.1) $$f(x) \in N_1(A) \qquad whenever\ x \in N_2(p) \qquad and\ \ x \neq p\,.$$

The first thing to note about this definition is that it involves *two* neighborhoods, *N,(A)* and *N,(p).* The neighborhood *N,(A)* is specified *first;* it tells us how close we wish $f(x)$ to

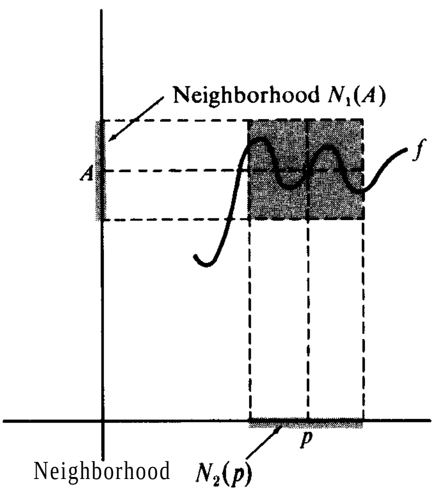


FIGURE 3.2 Here $\lim_{x \to p} f(x) = A$, but there is no assertion about f at p.

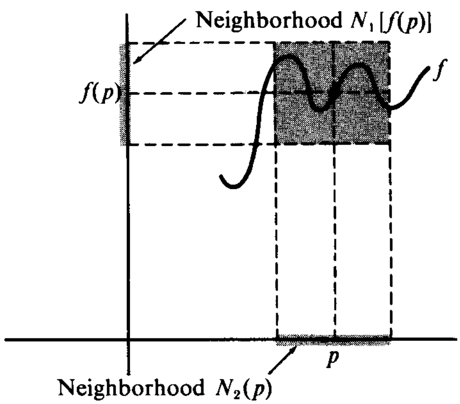


FIGURE 3.3 Here f is defined at p and $\lim_{x \to p} f(x) = f(p)$, hence f is continuous at p.

be to the limit A. The second neighborhood, *N,(p),* tells us how close x should be to p so that $f(x)$ will be within the first neighborhood *N,(A).* The essential part of the definition is that, for *every $N_1(A)$, no matter how small,* there is *some* neighborhood *N,(p)* to satisfy (3.1). In general, the neighborhood *N,(p)* will depend on the choice of *N,(A).* A neighborhood *N,(p)* that works for one particular $N_1(A)$ will also work, of course, for every larger *N,(A),* but it may not be suitable for any smaller *N,(A).*

The definition of limit may be illustrated geometrically as in Figure 3.2. A neighborhood $N_1(A)$ is shown on the y-axis. A neighborhood *N,(p)* corresponding to $N_1(A)$ is shown on the x-axis. The shaded rectangle consists of all points (x, y) for which x $\in$ *N,(p)* and y $\in N_1(A)$. The definition of limit asserts that the entire graph offabove the interval *N,(p)* lies within this rectangle, except possibly for the point on the graph above p itself.

The definition of limit can also be formulated in terms of the *radii* of the neighborhoods $N_1(A)$ and $N_2(p)$. It is customary to denote the radius of $N_1(A)$ by ϵ (the Greek letter *epsilon)* and the radius of $N_2(p)$ by δ (the Greek letter *delta).* The statement $f(x) \in N_1(A)$ is equivalent to the inequality $|f(x) - A| < \epsilon$, and the statement $x \in N_2(p)$, $x \neq p$, is equivalent to the inequalities $0 < |x - p| < \delta$. Therefore, the definition of limit can also be expressed as follows :

The symbol $\lim_{x\to p} f(x) = A$ *means that for every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that*

$$|f(x) - A| < \epsilon \qquad \text{whenever} \qquad 0 < |x - p| < \delta . \tag{3.2}$$

We note that the three statements,

$$\lim_{x\to p} f(x) = A , \qquad \lim_{x\to p} (f(x) - A) = 0 , \qquad \lim_{x\to p} |f(x) - A| = 0 ,$$

are all equivalent. This equivalence becomes apparent as soon as we write each of these statements in the ϵ, δ-terminology (3.2).

In dealing with limits as $x \to p$, we sometimes find it convenient to denote the difference $x - p$ by a new symbol, say h, and to let $h \to 0$. This simply amounts to a change in notation, because, as can be easily verified, the following two statements are equivalent:

$$\lim_{x\to p} f(x) = A , \qquad \lim_{h\to 0} f(p + h) = A .$$

EXAMPLE 1. *Limit of a constant function.* Let $f(x) = c$ for all x. It is easy to prove that for every p, we have $\lim_{x\to p} f(x) = c$. In fact, given any neighborhood $N_1(c)$, relation (3.1) is trivially satisfied for any choice of $N_2(p)$ because $f(x) = c$ for all x and $c \in N_1(c)$ for all neighborhoods $N_1(c)$. In limit notation, we write

$$\lim_{x\to p} c = c .$$

EXAMPLE 2. *Limit of the identity function.* Here $f(x) = x$ for all x. We can easily prove that $\lim_{x\to p} f(x) = p$. Choose any neighborhood $N_1(p)$ and take $N_2(p) = N_1(p)$. Then relation (3.1) is trivially satisfied. In limit notation, we write

$$\lim_{x\to p} x = p .$$

"One-sided" limits may be defined in a similar way. For example, if $f(x) \to A$ as $x \to p$ through values greater than p, we say that A is the *right-hand limit* of f at p, and we indicate this by writing

$$\lim_{x\to p+} f(x) = A .$$

In neighborhood terminology this means that for every neighborhood $N_1(A)$, there is some neighborhood $N_2(p)$ such that

$$f(x) \in N_1(A) \qquad \text{whenever } x \in N_2(p) \quad \text{and } x > p. \tag{3.3}$$

Left-hand limits, denoted by writing $x \to p-$, are similarly defined by restricting x to values less than p.

If f has a limit A at p, then it also has a right-hand limit and a left-hand limit at p, both of these being equal to A. But a function can have a right-hand limit at p different from the left-hand limit, as indicated in the next example.

EXAMPLE 3. Let $f(x) = [x]$ for all x, and let p be any integer. For x near p, $x < p$, we have $f(x) = p - 1$, and for x near p, $x > p$, we have $f(x) = p$. Therefore we see that

$$\lim_{x\to p-} f(x) = p - 1 \qquad \text{and} \qquad \lim_{x\to p+} f(x) = p\,.$$

In an example like this one, where the right- and left-hand limits are unequal, the limit of f at p *does not exist.*

EXAMPLE 4. Let $f(x) = 1/x^2$ if $x \neq 0$, and let $f(0) = 0$. The graph of f near zero is shown in Figure 3.1(b). In this example, f takes arbitrarily large values near 0 so it has no right-hand limit and no left-hand limit at 0. To prove rigorously that there is no real number A such that $\lim_{x\to 0+} f(x) = A$, we may argue as follows: Suppose there were such an A, say $A \geq 0$. Choose a neighborhood $N_1(A)$ of length 1. In the interval $0 < x < 1/(A + 2)$, we have $f(x) = 1/x^2 > (A + 2)^2 > A + 2$, so $f(x)$ cannot lie in the neighborhood $N_1(A)$. Thus, every neighborhood $N(0)$ contains points $x > 0$ for which $f(x)$ is outside $N_1(A)$, so (3.3) is violated for this choice of $N_1(A)$. Hence f has no right-hand limit at 0.

EXAMPLE 5. Let $f(x) = 1$ if $x \neq 0$, and let $f(0) = 0$. This function takes the constant value 1 everywhere except at 0, where it has the value 0. Both the right- and left-hand limits are 1 at every point p, so the limit of $f(x)$, as x approaches p, exists and equals 1. Note that the limit of f is 1 at the point 0, even though $f(0) = 0$.

3.3 The definition of continuity of a function

In the definition of limit we made no assertion about the behavior of f at the point p itself. Statement (3.1) refers to those $x \neq p$ which lie in $N_2(p)$, so it is not necessary that f be defined at p. Moreover, even if f is defined at p, its value there need not be equal to the limit A. However, if it happens that f is defined at p and if it also happens that $f(p) = A$, then we say the function f is continuous at p. In other words, we have the following definition.

DEFINITION OF CONTINUITY OF A FUNCTION AT A POINT. *A function f is said to be continuous at a point p if*

(a) *f is defined at p, and*

(b) $\lim_{x\to p} f(x) = f(p)$.

This definition can also be formulated in terms of neighborhoods. A function f is continuous at p if for every neighborhood $N_1[f(p)]$ there is a neighborhood $N_2(p)$ such that

$$f(x) \in N_1[f(p)] \qquad \text{whenever } x \in N_2(p)\,. \tag{3.4}$$

Since $f(p)$ always belongs to $N_1[f(p)]$, we do not need the condition $x \neq p$ in (3.4). In the ϵ, δ-terminology, where we specify the radii of the neighborhoods, the definition of continuity can be restated as follows:

A function f is continuous at p if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \qquad \text{whenever } |x - p| < \delta.$$

The definition of continuity is illustrated geometrically in Figure 3.3. This is like Figure 3.2 except that the limiting value, A, is equal to the function value $f(p)$ so the entire graph of f above $N_2(p)$ lies in the shaded rectangle.

EXAMPLE 1. *Constant functions are continuous everywhere.* If $f(x) = c$ for all x, then

$$\lim_{x \to p} f(x) = \lim_{x \to p} c = c = f(p)$$

for every p, so f is continuous everywhere.

EXAMPLE 2. *The identity function is continuous everywhere.* If $f(x) = x$ for all x, we have

$$\lim_{x \to p} f(x) = \lim_{x \to p} x = p = f(p)$$

for every p, so the identity function is continuous everywhere.

EXAMPLE 3. Let $f(x) = [x]$ for all x. This function is continuous at every point p which is not an integer. At the integers it is discontinuous, since the limit of f does not exist, the right- and left-hand limits being unequal. A discontinuity of this type, where the right- and left-hand limits exist but are unequal, is called a *jump discontinuity*. However, since the right-hand limit equals $f(p)$ at each integer p, we say that f is *continuous from the right* at p.

EXAMPLE 4. The function f for which $f(x) = 1/x^2$ for $x \neq 0$, $f(0) = 0$, is discontinuous at 0. [See Figure 3.1(b).] We say there is an *infinite discontinuity* at 0 because the function takes arbitrarily large values near 0.

EXAMPLE 5. Let $f(x) = 1$ for $x \neq 0$, $f(0) = 0$. This function is continuous everywhere except at 0. It is discontinuous at 0 because $f(0)$ is not equal to the limit of $f(x)$ as $x \to 0$. In this example, the discontinuity could be removed by redefining the function at 0 to have the value 1 instead of 0. For this reason, a discontinuity of this type is called a *removable discontinuity*. Note that jump discontinuities, such as those possessed by the greatest-integer function, cannot be removed by simply changing the value of f at one point.

3.4 The basic limit theorems. More examples of continuous functions

Calculations with limits may often be simplified by the use of the following theorem which provides basic rules for operating with limits.

TMEOREM 3.1. *Let f and g be functions such that*

$$\lim_{x\to p} f(x) = A, \qquad \lim_{x\to p} g(x) = B.$$

Then we have

(i) $\lim_{x\to p} [f(x) + g(x)] = A + B$,

(ii) $\lim_{x\to p} [f(x) - g(x)] = A - B$,

(iii) $\lim_{x\to p} f(x) \cdot g(x) = A \cdot B$,

(iv) $\lim_{x\to p} f(x)/g(x) = A/B$ if $B \neq 0$.

Note: An important special case of (iii) occurs when f is constant, say $f(x) = A$ for all x. In this case, (iii) is written as $\lim_{x\to p} A \cdot g(x) = A B$.

The proof of Theorem 3.1 is not difficult but it is somewhat lengthy so we have placed it in a separate section (Section 3.5). We discuss here some simple consequences of the theorem.

First we note that the statements in the theorem may be written in a slightly different form. For example, (i) can be written as follows:

$$\lim_{x\to p} [f(x) + g(x)] = \lim_{x\to p} f(x) + \lim_{x\to p} g(x).$$

It tells us that the limit of a sum is the sum of the limits.

It is customary to denote by $f + g$, $f - g$, $f \cdot g$, and f/g the functions whose values at each x under consideration are

$$f(x) + g(x), \qquad f(x) - g(x), \qquad f(x) \cdot g(x), \qquad \text{and} \qquad f(x)/g(x),$$

respectively. These functions are called the *sum, difference, product,* and *quotient* off and g. Of course, the quotient f/g is defined only at those points for which $g(x) \neq 0$. The following corollary to Theorem 3.1 is stated in this terminology and notation and is concerned with continuous functions.

THEOREM 3.2. *Let f and g be continuous at a point p. Then the sum $f + g$, the difference $f - g$, and the product $f \cdot g$ are also continuous ut p. The same is true of the quotient f/g if $g(p) \neq 0$.*

Proof. Since f and g are continuous at p, we have $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to p} g(x) = g(p)$. Therefore we may apply the limit formulas in Theorem 3.1 with $A = f(p)$ and $B = g(p)$ to deduce Theorem 3.2.

We have already seen that the identity function and constant functions are continuous everywhere. Using these examples and Theorem 3.2, we may construct many more examples of continuous functions.

EXAMPLE 1. *Continuity of polynomials.* If we take $f(x) = g(x) = x$, the result on continuity of products proves the continuity at each point for the function whose value at each x is x^2. By mathematical induction, it follows that for every real c and every positive integer n, the function f for which $f(x) = cx^n$ is continuous for all x. Since the sum of two continuous functions is itself continuous, by induction it follows that the same is true for the sum of any finite number of continuous functions. Therefore every polynomial $p(x) = \sum_{k=0}^{n} c_k x^k$ is continuous at all points.

EXAMPLE 2. *Continuity of rational functions.* The quotient of two polynomials is called a *rationalfunction.* If r is a rational function, then we have

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials. The function r is defined for all real x for which $q(x) \neq 0$. Since quotients of continuous functions are continuous, we see that every rational function is continuous wherever it is defined. A simple example is $r(x) = 1/x$ if $x \neq 0$. This function is continuous everywhere except at $x = 0$, where it fails to be defined.

The next theorem shows that if a function g is squeezed between two other functions which have equal limits as $x \to p$, then g also has this limit as $x \to p$.

THEOREM 3.3. **SQUEEZING PRINCIPLE.** *Suppose that $f(x) \le g(x) \le h(x)$ for all $x \neq p$ in some neighborhood $N(p)$. Suppose also that*

$$\lim_{x\to p} f(x) = \lim_{x\to p} h(x) = a\ .$$

Then we also haue $\lim_{x\to p} g(x) = a$.

Proof. Let $G(x) = g(x) - f(x)$, and $H(x) = h(x) - f(x)$. The inequalities $f \le g \le h$ imply $0 \le g - f \le h - f$, or

$$0 \le G(x) \le H(x)$$

for all $x \neq p$ in $N(p)$. To prove the theorem, it suffices to show that $G(x) \to 0$ as $x \to p$, given that $H(x) \to 0$ as $x \to p$.

Let $N_1(0)$ be any neighborhood of 0. Since $H(x) \to 0$ as $x \to p$, there is a neighborhood $N_1(p)$ such that

$$H(x) \in N_1(0) \qquad \text{whenever } x \in N_1(p) \qquad \text{and } x \neq p\ .$$

We can assume that $N_1(p) \subseteq N(p)$. Then the inequality $0 \le G \le H$ states that $G(x)$ is no further from 0 than $H(x)$ if x is in $N_2(p)$, $x \neq p$. Therefore G(x) $\in$ N,(O) for such x, and hence G(x) $\to$ 0 as x $\to p$. This proves the theorem. The same proof is valid if all the limits are one-sided limits.

The squeezing principle is useful in practice because it is often possible to find squeezing functions f and h which are easier to deal with than g. We shall use the result now to prove that every indefinite integral is a continuous function.

THEOREM 3.4. CONTINUITY OF INDEFINITE INTEGRALS. *Assume f is integrable on [a, x] for every x in [a, b], and let*

$$A(x) = \int_a^x f(t)\,dt\ .$$

Then the indefinite integral A is continuous at each point of [a, b]. (At each endpoint we have one-sided continuity.)

Proof. Choose p in [a, b]. We are to prove that $A(x) \to A(p)$ as x $\to p$. We have

$$A(x) - A(p) = \int_p^x f(t)\,dt\ . \tag{3.5}$$

Now we estimate the size of this integral. Sincefis bounded on $[a, b]$, there is a constant $M > 0$ such that $-M$ <f(t) $\leq$ A4 for all t in $[a, b]$. If x > p, we integrate these inequalities over the interval $[p$, x] to obtain

$$-M(x - p) \leq A(x) - A(p) \leq M(x - p)\ .$$

If x < p, we obtain the same inequalities with x $-$ p replaced by p $-$ x. Therefore, in either case we can let x $\to p$ and apply the squeezing principle to find that $A(x) \to A(p)$. This proves the theorem. If p is an endpoint of $[a, b]$, we must let x $\to p$ from inside the interval, so the limits are one-sided.

EXAMPLE 3. *Continuity of the sine and cosine.* Since the sine function is an indefinite integral, sin x $= \int_0^x \cos t\,dt$, the foregoing theorem tells us that the sine is continuous everywhere. Similarly, the cosine is everywhere continuous since $\cos x = 1 - \int_0^x \sin t\,dt$. The continuity of these functions can also be deduced without making use of the fact that they are indefinite integrals. An alternate proof is outlined in Exercise 26 of Section 3.6.

EXAMPLE 4. In this example we prove an important limit formula,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\ , \tag{3.6}$$

that is needed later in our discussion of differential calculus. Since the denominator of the quotient $(\sin x)/x$ approaches 0 as x $\to$ 0, we cannot apply the quotient theorem on limits to deduce (3.6). Instead, we use the squeezing principle. From Section 2.5 we have the fundamental inequalities

$$0 < \cos x < \frac{\sin x}{x} < \frac{1}{\cos x},$$

valid for $0 < x < \frac{1}{2}\pi$. They are also valid for $-\frac{1}{2}\pi < x < 0$ since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, and hence they hold for all $x \neq 0$ in the neighborhood $N(0; \frac{1}{2}\pi)$. When $x \to 0$, we find $\cos x \to 1$ since the cosine is continuous at 0, and hence $1/(\cos x) \to 1$. Therefore, by the squeezing principle, we deduce (3.6). If we define $f(x) = (\sin x)/x$ for $x \neq 0$, $f(0) = 1$, then f is continuous everywhere. Its graph is shown in Figure 3.4.

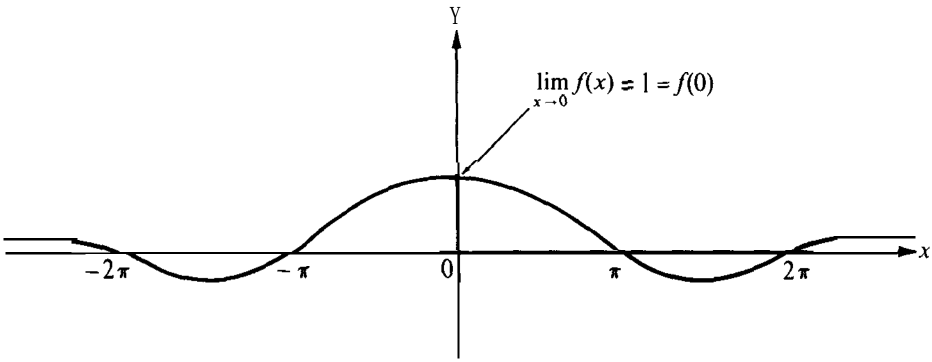


FIGURE 3.4 $f(x) = (\sin x)/x$ if $x \neq 0$, $f(0) = 1$. This function is continuous everywhere.

EXAMPLE **5.** ***Continuity of f when $f(x) = x^r$ for $x > 0$, where r is a positive rational number.*** From Theorem 2.2 we have the integration formula

$$\int_0^x t^{1/n}\, dt = \frac{x^{1+1/n}}{1 + 1/n},$$

valid for all $x > 0$ and every integer $n \geq 1$. Using Theorems 3.4 and 3.1, we find that the function A given by $A(x) = x^{1+1/n}$ is continuous at all points $p > 0$. Now let $g(x) = x^{1/n} = A(x)/x$ for $x > 0$. Since g is a quotient of two continuous functions it, too, is continuous at all points $p > 0$. More generally, if $f(x) = x^{m/n}$, where m is a positive integer, then f is a product of continuous functions and hence is continuous at all points $p > 0$. This establishes the continuity of the rth-power function, $f(x) = x^r$, when r is any positive rational number, at all points $p > 0$. At $p = 0$ we have right-hand continuity.

The continuity of the rth-power function for rational r can also be deduced without using integrals. An alternate proof is given in Section 3.13.

3.5 Proofs of the basic limit tbeorems

In this section we prove Theorem 3.1 which describes the basic rules for dealing with limits of sums, products, and quotients. The principal algebraic tools used in the proof are two properties of absolute values that were mentioned earlier in Sections 14.8 and 14.9. They are (1) the triangle inequality, which states that $|a + b| \le |a| + |b|$ for all real a and b, and (2) the equation $|ab| = |a|\,|b|$, which states that the absolute value of a product is the product of absolute values.

***Proofs* of (i) *and* (ii).** Since the two statements

$$\lim_{x\to p} f(x) = A \qquad \text{and} \qquad \lim_{x\to p}\,[f(x) - A] = 0$$

are equivalent, and since we have

$$f(x) + g(x) - (A + B) = [f(x) - A] + [g(x) - B]\,,$$

it suffices to prove part (i) of the theorem when the limits A and B are both zero.

Suppose, then, that $f(x) \to 0$ and $g(x) \to 0$ as $x \to p$. We shall prove that $f(x) + g(x) \to 0$ as $x \to p$. This means we must show that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) + g(x)| < \epsilon \qquad \text{whenever } 0 < |x - p| < \delta\,. \tag{3.7}$$

Let ϵ be given. Since $f(x) \to 0$ as $x \to p$, there is a $\delta_1 > 0$ such that

$$|f(x)| < \frac{\epsilon}{2} \qquad \text{whenever } 0 < |x - p| < \delta_1\,. \tag{3.8}$$

Similarly, since $g(x) \to 0$ as $x \to p$, there is a $\delta_2 > 0$ such that

$$|g(x)| < \frac{\epsilon}{2} \qquad \text{whenever } 0 < |x - p| < \delta_2 \tag{3.9}$$

If we let δ denote the smaller of the two numbers δ_1 and δ_2, then both inequalities (3.8) and (3.9) are valid if $0 < |x - p| < \delta$ and hence, by the triangle inequality, we find that

$$|f(x) + g(x)| \le |f(x)| + |g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This proves (3.7) which, in turn, proves (i). The proof of (ii) is entirely similar, except that in the last step we use the inequality $|f(x) - g(x)| \le |f(x)| + |g(x)|$.

Proof of (iii). Suppose that we have proved part (iii) for the special case in which one of the limits is 0. Then the general case follows easily from this special case. In fact, all we need to do is write

$$f(x)g(x) - AB = f(x)[g(x) - B] + B[f(x) - A]\,.$$

The special case implies that each term on the right approaches 0 as $x \to p$ and, by property (i), the sum of the two terms also approaches 0. Therefore, it remains to prove (iii) in the special case where one of the limits, say B, is 0.

Suppose, then, that $f(x) \to A$ and $g(x) \to 0$ as $x \to p$. We wish to prove thatf(x)g(x) $\to 0$ as $x \to p$. To do this we must show that if a positive ϵ is given, there is a $\delta > 0$ such that

(3.10) $$|f(x)g(x)| < \epsilon \qquad \text{whenever } 0 < |x - p| < \delta .$$

Since f(x) $\to$ A as $x \to p$, there is a 8, such that

(3.11) $$|f(x) - A| < 1 \qquad \text{whenever } 0 < \text{Ix} - p| < \delta_1 .$$

For such x, we have $|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A| < 1 + |A|$, and hence

(3.12) $$|f(x)g(x)| = |f(x)|\,|g(x)| < (1 + |A|)\,|g(x)|.$$

Since g(x) $\to 0$ as $x \to p$, for every $\epsilon > 0$ there is a δ_2 such that

(3.13) $$|g(x)| < \frac{\epsilon}{1 + |A|} \qquad \text{whenever } 0 < |x - p| < \delta_2 .$$

Therefore, if we let δ be the smaller of the two numbers δ_1 and δ_2 , then both inequalities (3.12) and (3.13) are valid whenever $0 < |x - p| < \delta$, and for such x we deduce (3.10). This completes the proof of (iii).

Proof of (iv). Since the quotient $f(x)/g(x)$ is the product of $f(x)/B$ with $B/g(x)$, it suffices to prove that $B/g(x) \to 1$ as $x \to p$ and then appeal to (iii). Let $h(x) = g(x)/B$. Then $h(x) \to 1$ as x $\to p$, and we wish to prove that $1/h(x) \to 1$ as $x \to p$.

Let $\epsilon > 0$ be given. We must show that there is a $\delta > 0$ such that

(3.14) $$\left|\frac{1}{h(x)} - 1\right| < \epsilon \qquad \text{whenever } 0 < |x - p| < \delta .$$

The difference to be estimated may be written as follows.

(3.15) $$\left|\frac{1}{h(x)} - 1\right| = \frac{|h(x) - 1|}{|h(x)|}$$

Since $h(x) \to 1$ as x $\to p$, we can choose a $\delta > 0$ such that both inequalities

(3.16) $$|h(x) - 1| < \frac{\epsilon}{2} \qquad \text{and} \qquad |h(x) - 1| < \frac{1}{2}$$

are satisfied whenever $0 < |x - p| < \delta$. The second of these inequalities implies $h(x) > \frac{1}{2}$ so $1/|h(x)| = 1/h(x) < 2$ for such x. Using this in (3.15) along with the first inequality in (3.16), we obtain (3.14). This completes the proof of (iv).

3.6 Exercises

In Exercises 1 through 10, compute the limits and explain which limit theorems you are using in each case.

1. $\lim_{x\to 2} \frac{1}{x^2}$.
2. $\lim_{x\to 0} \frac{25x3 + 2}{75x7 - 2}$.
3. $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.
4. $\lim_{x\to 1} \frac{2x^2 - 3x + 1}{x-l}$,
5. $\lim_{h\to 0} \frac{(t + h)^2 - t^2}{h}$.
6. $\lim_{x\to 0} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $\quad a \neq 0$.
7. $\lim_{a\to 0} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $\quad x \neq 0$.
8. $\lim_{x\to a} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $\quad a \neq 0$.
9. $\lim_{t\to 0} \tan t$.
10. $\lim_{t\to 0} (\sin 2t + t^2 \cos 5t)$.
11. $\lim_{x\to 0+} \frac{|x|}{x}$.
12. $\lim_{x\to 0-} \frac{|x|}{x}$.
13. $\lim_{z\text{-}o+} \frac{\sqrt{x^2}}{x}$.
14. $\lim_{x\to 0-} \frac{\sqrt{x^2}}{x}$.

Use the relation $\lim_{x\to 0} (\sin x)/x = 1$ to establish the limit formulas in Exercises 15 through 20.

15. $\lim_{2\text{-}o} \frac{\sin 2x}{x} = 2$.
16. $\lim_{x\to 0} \frac{\tan 2x}{\sin x} = 2$.
17. $\lim_{x\to 0} \frac{\sin 5x}{\sin x} = 5$.
18. $\lim_{x\text{-}o} \frac{\sin 5x - \sin 3x}{x} = 2$.
19. $\lim_{x\to 0} \frac{\sin x - \sin a}{x - a} = \cos a$.
20. $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.
21. Show that $\lim_{x\text{-}o} \frac{1 - \sqrt{1 - x^2}}{x^2} = \frac{1}{2}$. [*Hint:* $(1 - \sqrt{u})(1 + \sqrt{u}) = 1 - u$.]
22. A function f is defined as follows:

$$f(x) = \begin{cases} \sin x & \text{if } x \le c, \\ ax + b & \text{if } x > c, \end{cases}$$

where a, b, c are constants. If b and c are given, find all values of a (if any exist) for which f is continuous at the point $x = c$.

23. Solve Exercise 22 if f is defined as follows:

$$f(x) = \begin{cases} 2 \cos x & \text{if } x \le c, \\ ax^2 + b & \text{if } x > c. \end{cases}$$

24. At what points are the tangent and cotangent functions continuous?
25. Let $f(x) = (\tan x)/x$ if $x \neq 0$. Sketch the graph of f over the half-open intervals $[-\frac{1}{4}\pi, 0)$ and $(0, \frac{1}{4}\pi]$. What happens to $f(x)$ as $x \to 0$? Can you define $f(0)$ so that f becomes continuous at 0?

26. This exercise outlines an alternate proof of the continuity of the sine and cosine functions.
(a) The inequality $|\sin x| < |x|$, valid for $0 < |x| < \frac{1}{2}\pi$, was proved in Exercise 34 of Section 2.8. Use this inequality to prove that the sine function is continuous at 0.
(b) Use part (a) and the identity $\cos 2x = 1 - 2\sin^2 x$ to prove that the cosine is continuous at 0.
(c) Use the addition formulas for $\sin (x + h)$ and $\cos (x + h)$ to prove that the sine and cosine are continuous at any real x.
27. Figure 3.5 shows a portion of the graph of the functionfdefined as follows:

$$f(x) = \sin \frac{1}{x} \quad \text{if } x \neq 0.$$

For $x = 1/(n\pi)$, where n is an integer, we have $\sin (1/x) = \sin (na) = 0$. Between two such points, the function values rise to + 1 and drop back to 0 or else drop to -1 and rise back to 0.

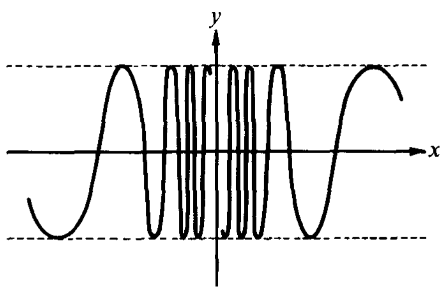


FIGURE 3.5 $f(x) = \sin (1/x)$ if $x \neq 0$. This function is discontinuous at 0 no matter how f(0) is defined.

Therefore, between any such point and the origin, the curve has an infinite number of oscillations. This suggests that the function values do not approach any fixed value as $x \to 0$. Prove that there is no real number A such thatf(x) $\to$ A as $x \to 0$. This shows that it is not possible to define $f(0)$ in such a way that f becomes continuous at 0.

[*Hint:* Assume such an A exists and obtain a contradiction.]

28. For $x \neq 0$, let $f(x) = [1/x]$, where $[t]$ denotes the greatest integer $\leq t$. Sketch the graph of f over the intervals $[-2, -\frac{1}{5}]$ and $[\frac{1}{5}, 2]$. What happens to $f(x)$ as $x \to 0$ through positive values? through negative values? Can you define $f(0)$ so that f becomes continuous at O?
29. Same as Exercise 28, when $f(x) = (-1)^{[1/x]}$ for $x \neq 0$.
30. Same as Exercise 28, whenf(x) = $x(-1)^{[1/x]}$ for $x \neq 0$.
31. Give an example of a function that is continuous at one point of an interval and discontinuous at all other points of the interval, or prove that there is no such function.
32. Letf(x) = $x \sin (1/x)$ if $x \neq 0$. Define $f(0)$ so thatfwill be continuous at 0.
33. Letf be a function such that $|f(u) - f(v)| \leq |u - v|$ for all u and v in an interval [a, b].
(a) Prove that f is continuous at each point of [a, b].
(b) Assume that f is integrable on $[a, b]$. Prove that

$$\left| \int_a^b f(x)\,dx - (b-a)f(a) \right| \leq \frac{(b-a)^2}{2}.$$