Homework 6

ME 7751 Fall 2021

Christopher Bogaev

Mechanical Engineering
Georgia Institute of Technology

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1 Program

Included Files

ME7751_HW6_Bogaev_Christopher.pdf
ME7751_HW6_Problem1_2.m
ME7751_HW6_Problem1_3.m
ME7751_HW6_Problem1_4.m
ME7751_HW6_Problem1_5.m
ME7751_HW6_Problem1_6.m

Directions

Run ME7751_HW6_Problem1_2.m to compute and generate the graphics for Problem 1.2. Adjustments to the spacial domain, time discretization, and spacial discretization can be made in lines 4 - 19.

Run ME7751_HW6_Problem1_3.m to compute and generate the graphics for Problem 1.3. Adjustments to the spacial domain, time discretization, and spacial discretization can be made in lines 4 - 19.

Run ME7751_HW6_Problem1_4.m to compute and generate the graphics for Problem 1.4. Adjustments to the spacial domain, time discretization, and spacial discretization can be made in lines 4 - 19.

Run ME7751_HW6_Problem1_5.m to compute and generate the graphics for Problem 1.5. Adjustments to the spacial domain, time discretization, and spacial discretization can be made in lines 4 - 19.

Run ME7751_HW6_Problem1_6.m to compute and generate the graphics for Problem 1.5. Adjustments to the spacial domain, time discretization, and spacial discretization can be made in lines 4 - 34.

2 Results and Discussion

Problem 1

The two-dimensional cavity of Figure 1 is filled with an incompressible Newtonian fluid. The fluid is driven by the lid moving with a constant velocity U. This problem is widely used as a benchmark to validate CFD models due to its simple geometry but nontrivial flow solution.

Numerically solve the cavity flow of Figure 1 using the Lattice Boltzmann Method (LBM) with the D2Q9 lattice and BGK collision operator.

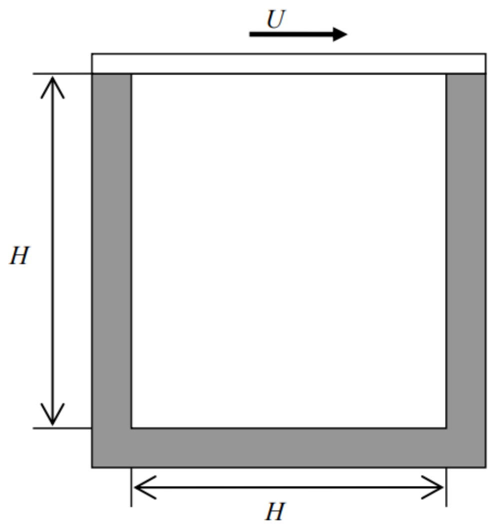

Figure 1: Two-Dimensional Cavity

Problem 1.1

Traditionally, a quantitative study of fluids relies on either a continuum (macroscopic) or atomistic (microscopic) description. However, an intermediate (mesoscopic) level of description is possible wherein fluids are represented in terms of the probability (density) $f(r, v, t)$ of finding a given particle at a given position in space, r and time t, with a given velocity v [5]. This intermediate description of fluids is possible with kinetic theory.

In the Boltzmann transport equation (1), the left hand side $\partial f/\partial t + c \cdot \nabla f$ describes particle streaming and the right hand side Ω describes inter-particle collisions.

$$\frac{\partial f}{\partial t} + c \cdot \nabla f = \Omega \tag{1}$$

The Bhatnagar, Gross and Krook (BGK) collision model is used to model the inter-particle collisions Ω where f^{eq} is the Maxwell-Boltzmann equilibrium distribution function.

$$\Omega = \omega(f^{eq} - f) = \frac{1}{\tau}(f^{eq} - f) \tag{2}$$

The streaming and collision steps are performed on the D2Q9 lattice model of Figure 2 wherein the collision step is described in (3) and the streaming step is described in (4) for $k = 1, 2, ..., 8$ streaming directions.

$$f_k(x, y, t + \Delta t) = f_k(x, y, t)[1 - \omega] + \omega f_k^{eq}(x, y, t) \tag{3}$$

$$f_k(x + \Delta x, y + \Delta y, t + \Delta t) = f_k(x, y, t + \Delta t) \tag{4}$$

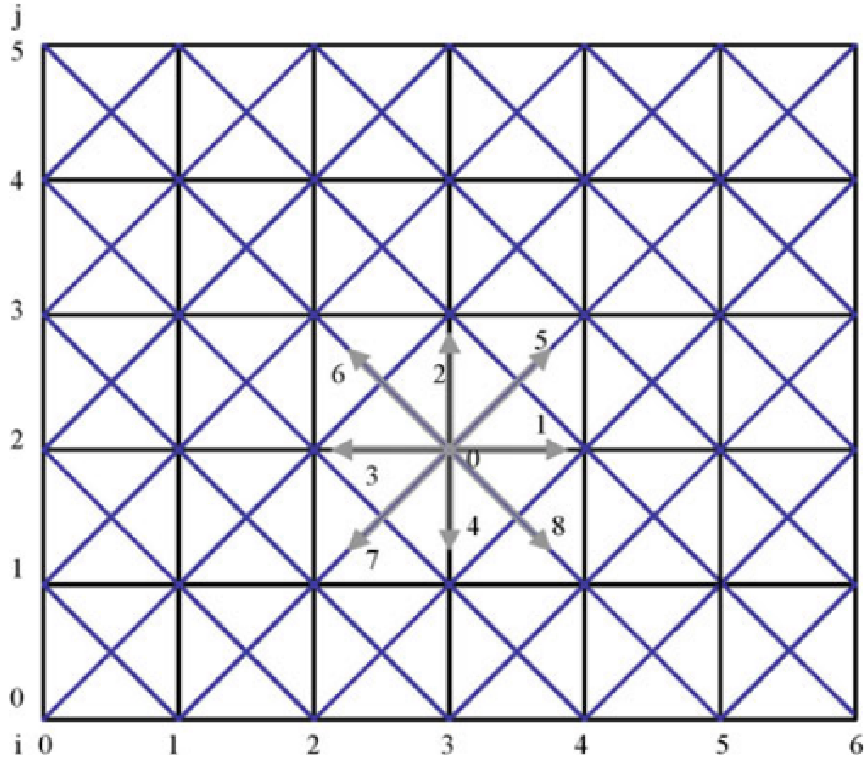

Figure 2: D2Q9 Lattice Model [5]

The bounce back method of Figure 3 is used to model solid stationary or moving boundary condition, nonslip condition, or flow-over obstacles [5]. This method ensures conservation of mass and momentum at the boundaries by extending the streaming process into the wall. In the configuration of Figure 3, $f_5 = f_7, f_2 = f_4$ and $f_6 = f_8$, where f_7, f_4 and f_8 are know from the prior streaming process (3).

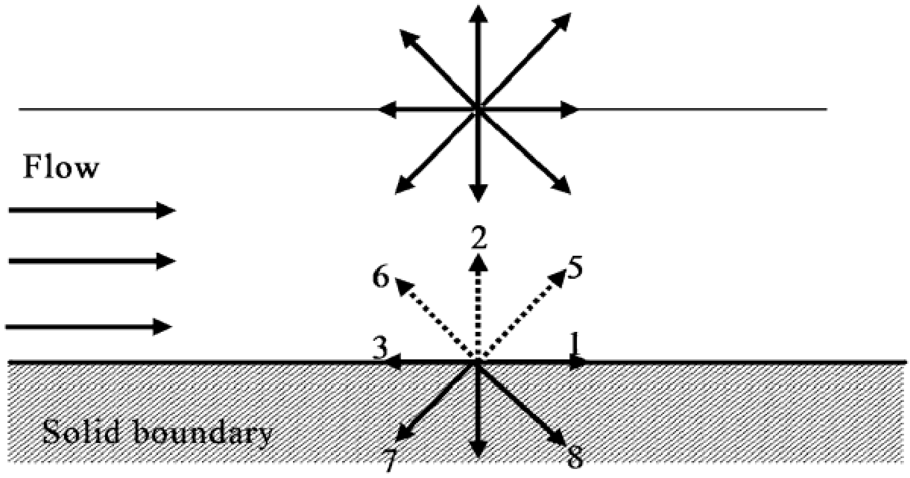

Figure 3: Bounce Back Scheme [5]

The relations between the mesoscopic and the macroscopic quantities for fluid density ρ and fluid velocity u are given by (5) and (6) respectively.

$$\rho(r,t) = \int mf(r,c,t)dc \tag{5}$$

$$\rho(r,t)u(r,t) = \int mcf(r,c,t)dc \tag{6}$$

Problem 1.2

In the square cavity of Figure 1, the Reynolds number, defined by $\mathrm{Re} = UH/\nu$ characterizes the flow patterns. The steady state solutions were computed for both $\mathrm{Re} = 100$, $\mathrm{Re} = 400$, and $\mathrm{Re} = 1000$. The residual between successive velocity field computations was used as the halting criteria for steady state.

Streamlines were plotted for both $\mathrm{Re} = 100$, $\mathrm{Re} = 400$, and $\mathrm{Re} = 1000$ in Figures 4, 5, and 6 respectively. Additionally, for both $\mathrm{Re} = 100$ and $\mathrm{Re} = 400$, the x component of velocity was plotted along the vertical centerline in Figure 7 and compared against those found in literature [4]. Similarly, the y component of velocity was plotted along the horizontal centerline in Figure 8 and compared against those found in literature [4]. Excellent agreement was found between the results from the Lattice Boltzmann method and those found in literature [4].

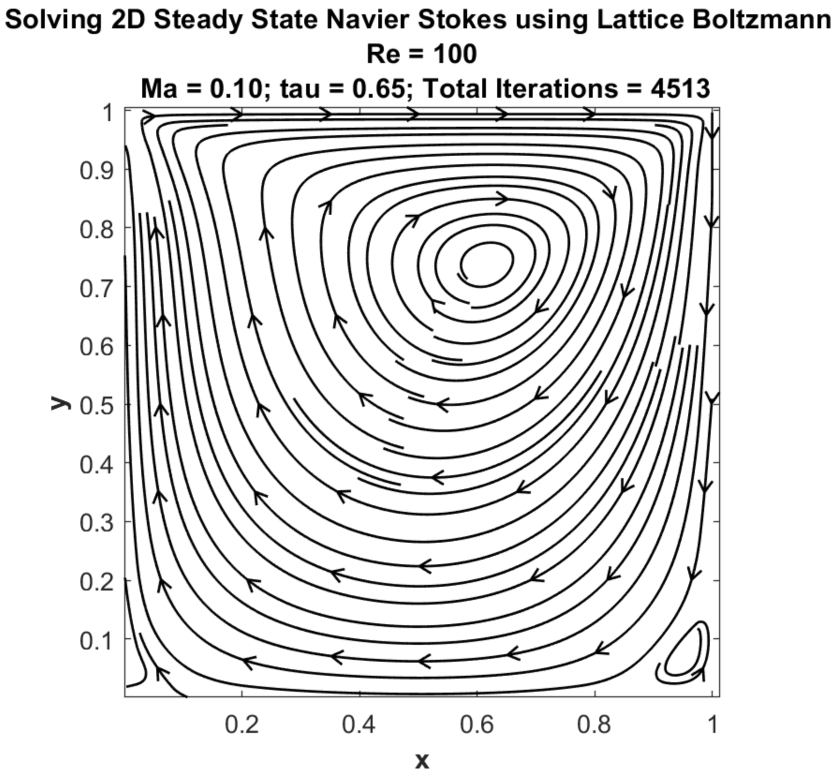

Figure 4: 2D Steady State Solution for Re = 100

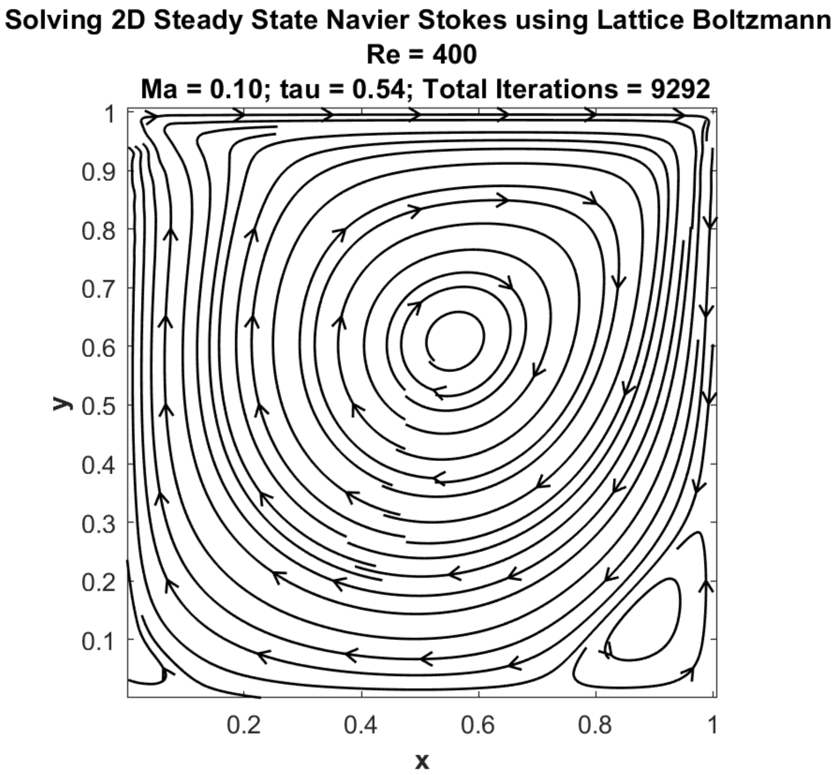

Figure 5: 2D Steady State Solution for Re = 400

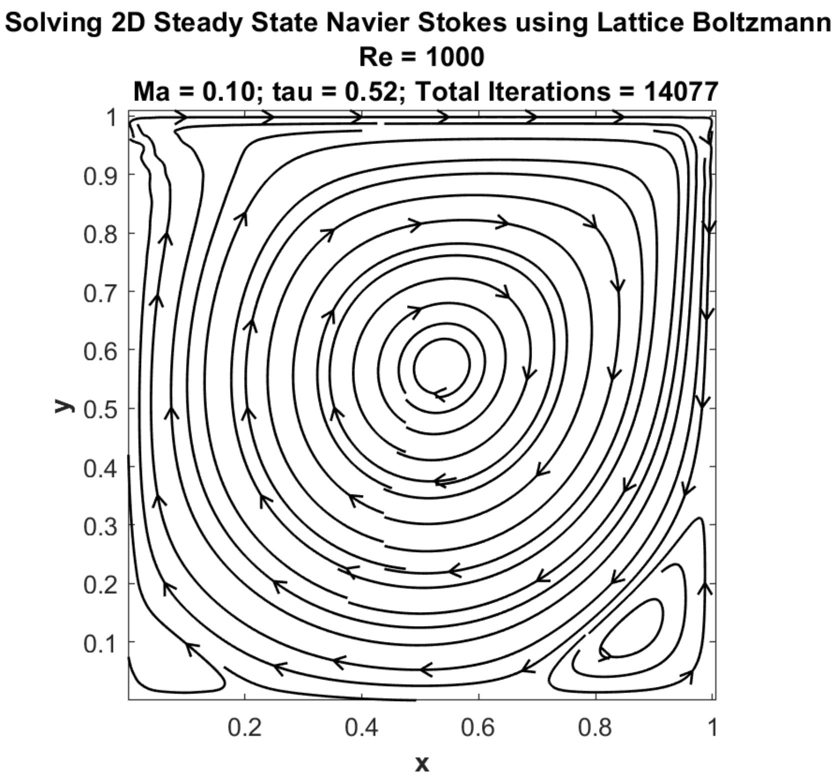

Figure 6: 2D Steady State Solution for Re = 1000

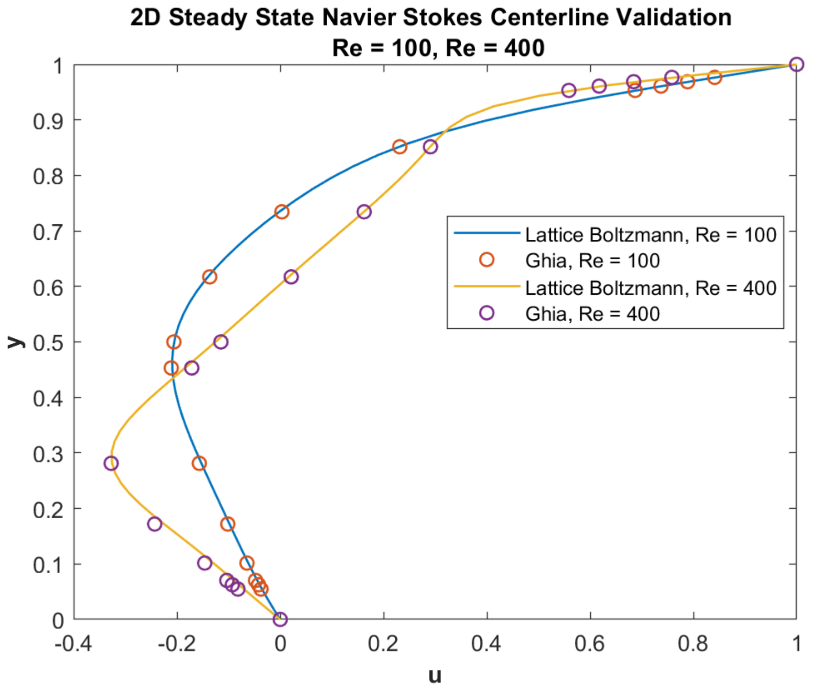

Figure 7: 2D Steady State U-Velocity Centerline Validation

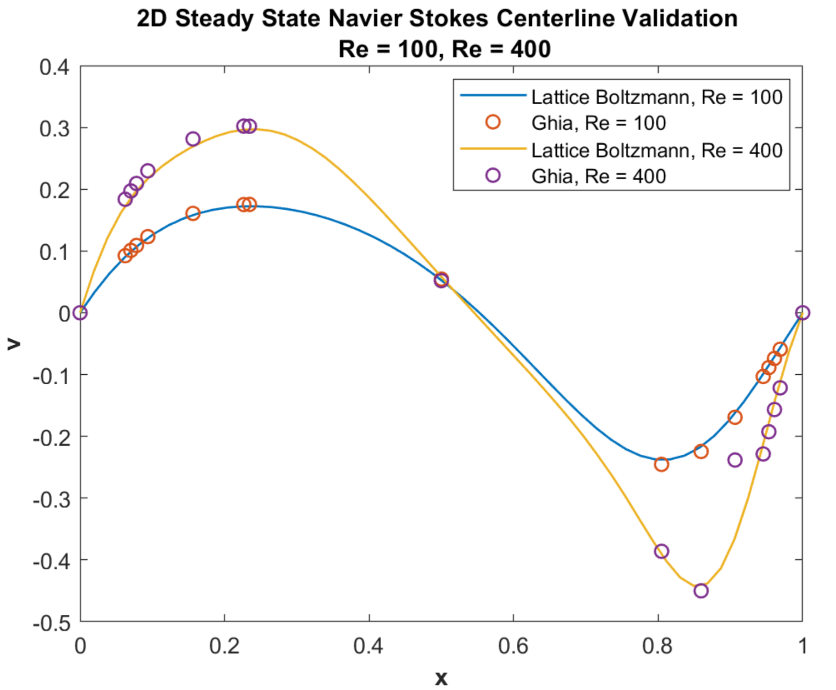

Figure 8: 2D Steady State V-Velocity Centerline Validation

Problem 1.3

The two centerline u and v velocity profiles, for Re $=$ 100 and Re $=$ 400, are qualitatively shown to converge to the literature values [4] in Figures 9 and 10 for $\mathrm{Ma} = 0.2, 0.1, 0.05$. Excellent agreement was found between the results from the projection method and those found in literature [4] for $\mathrm{Ma} \leq 0.05$ and $\tau \geq 0.55$.

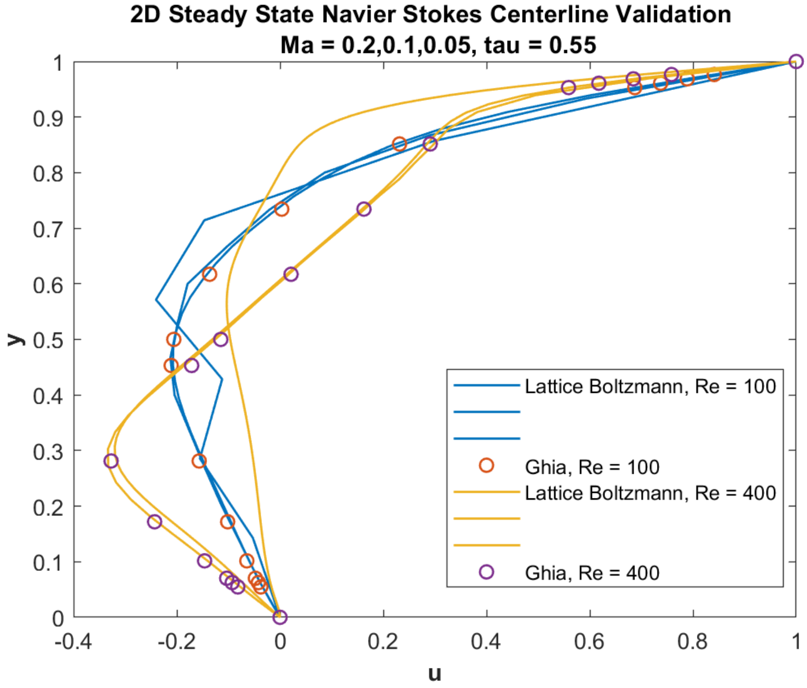

Figure 9: 2D Steady State U-Velocity Centerline Validation

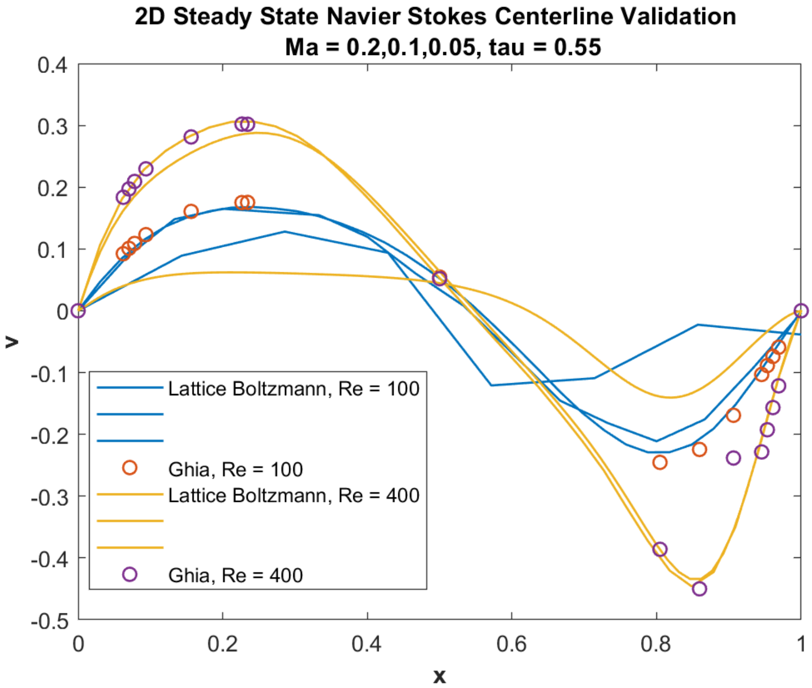

Figure 10: 2D Steady State V-Velocity Centerline Validation

Problem 1.4

The spacial convergence rate of the Lattice Boltzmann method was numerically computed for Ma $= 0.2, 0.1, 0.05$, $\tau = 0.55$ and Re $= 100$ and Re $= 400$ in Figure 11. The spacial convergence rate was computed by taking the L2 norm of the u and v centerline velocities from the method with respect to those found in literature [4]. The theoretical spacial convergence rate was determined to be $\mathcal{O}(\mathrm{Ma}^2)$ matching the numerical spacial convergence rate in Figure 9.

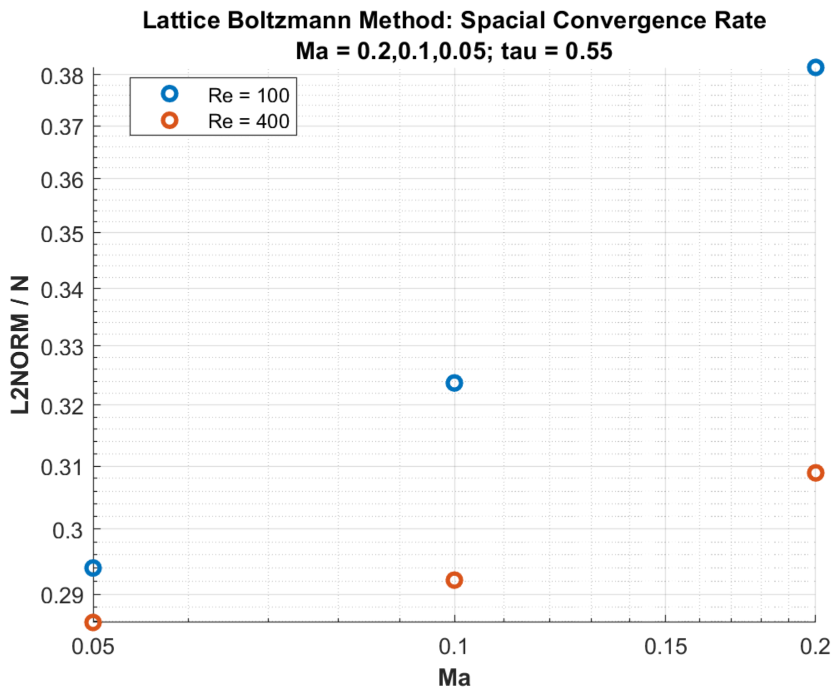

Figure 11: 2D Steady State Spacial Convergence Rate

Problem 1.5

The spacial convergence rate of the Lattice Boltzmann method was used as an indicator for method stability and was numerically computed for $\mathrm{Ma} = 0.2, 0.1$, $\tau = 0.51, 0.55, 0.60, 0.65$ and $\mathrm{Re} = 100$ in Figure 12. Comparing the results to the stability criteria, the method is stable for $\mathrm{Ma} \leq 0.2$ and $\tau > 0.5$ thus matching the theoretical stability criteria.

The physical Reynolds number must match the lattice Reynolds number, as shown in (7), for accurate results. However, the lattice Reynolds number is influenced by the lattice Mach number Ma, the lattice speed of sound c_s, the grid size N, the grid step Δx, the lattice time step Δt, and the relaxation time τ. Additionally, the lattice kinematic viscosity α, expressed in (8), must be sufficiently large for method stability, limiting the stable values of τ. To match the lattice Reynolds number at high physical Reynolds number, the required grid size becomes large, increasing computational complexity and impacting the feasible range of Reynolds numbers.

$$\mathrm{Re} = \frac{UH}{\nu} = \frac{\mathrm{Ma}\, c_s \Delta x N_x}{\alpha} \tag{7}$$

$$\alpha = \frac{\Delta x^2}{3\Delta t}(\tau - 0.5) \tag{8}$$

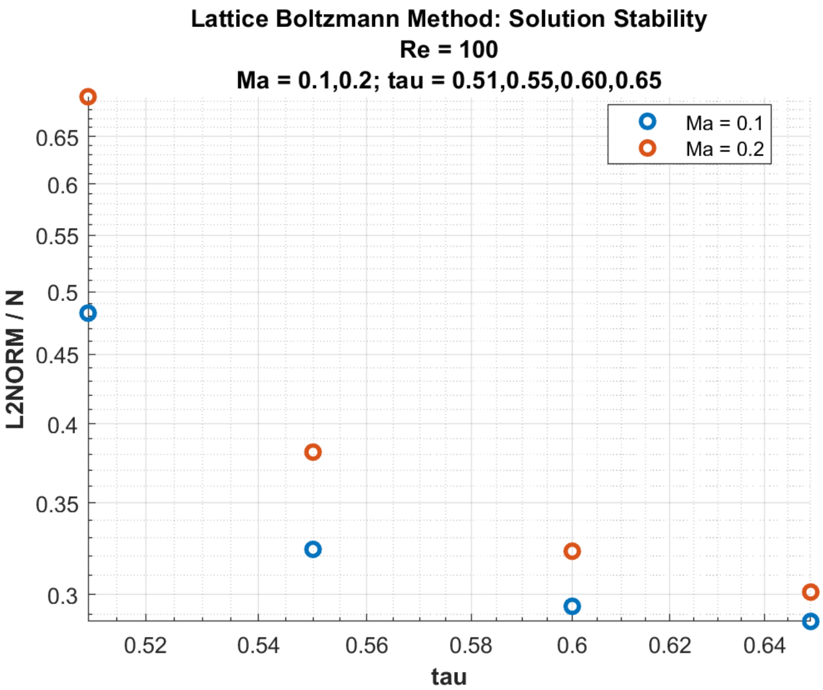

Figure 12: 2D Steady State Solution Stability

Problem 1.6

For both Re = 100 and Re = 400, the x and y components of velocity were plotted along the vertical centerline in Figures 13 and 14 and compared against those obtained from the artificial compressibility method, projection method and those found in literature [4]. Excellent agreement was found between the results from the projection method, the artificial compressibility method, the Lattice Boltzmann method and those found in literature [4].

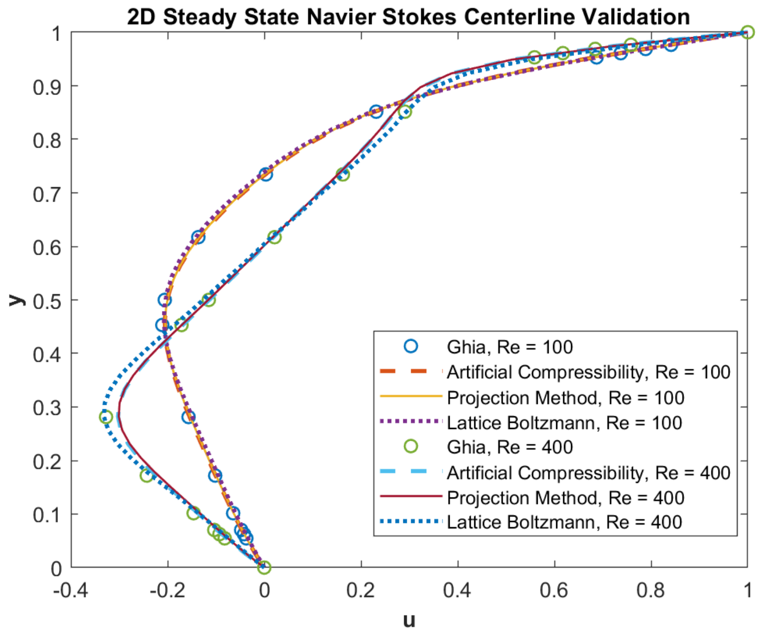

Figure 13: 2D Steady State U-Velocity Centerline Comparison

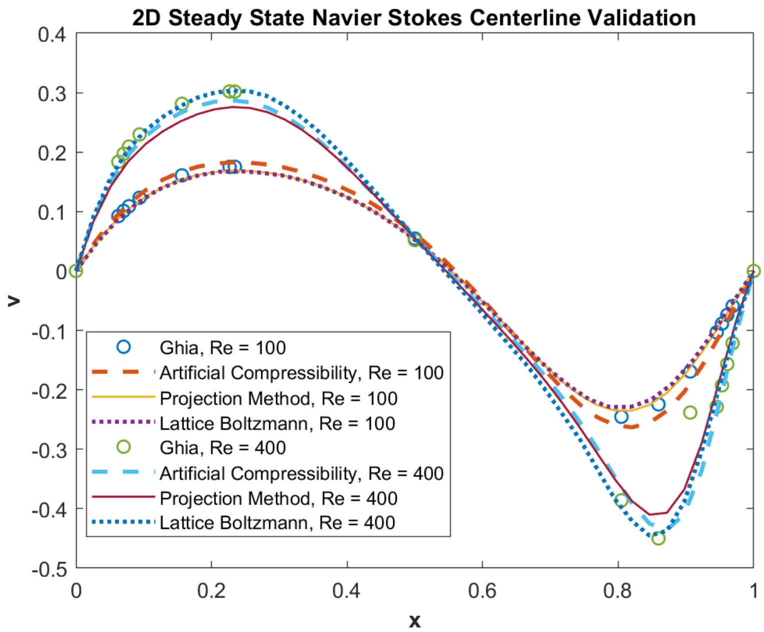

Figure 14: 2D Steady State V-Velocity Centerline Comparison

References

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[2] D. Anderson, J. Tannehill, R. Pletcher *Computational fluid mechanics and heat transfer*. McGraw-Hill, New York, (1984)

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[4] Ghia, U. K. N. G., Kirti N. Ghia, and C. T. Shin. *High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method*. Journal of computational physics 48.3 (1982): 387-411.

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