Analytical Solution for the Generalized Fermat-Torricelli Problem

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We present explicit analytical solution for the problem of minimization of the function $F(x,y)=\sum_{j=1}^{3} m_j\sqrt{(x-x_j)^2+(y-y_j)^2}$, i.e. we find the coordinates of stationary point and the corresponding critical value as functions of $\{m_j,x_j,y_j\}_{j=1}^3$. In addition, we also discuss inverse problem of finding such values for m_1,m_2,m_3 for which the corresponding function F possesses a prescribed position of stationary point.

1 Introduction

Given the coordinates of three noncollinear points $P_1=(x_1,y_1), P_2=(x_2,y_2), P_3=(x_3,y_3)$ in the plane, find the coordinates of the point $P_*=(x_*,y_*)$ which gives a solution for the optimization problem

$$\min_{(x,y)} F(x,y) \quad \text{for} \quad F(x,y)=\sum_{j=1}^{3} m_j\sqrt{(x-x_j)^2+(y-y_j)^2}\ . \tag{1.1}$$

Here m_1,m_2,m_3 are assumed to be real positive numbers and will be subsequently referred to as *weights*.

The stated problem in its particular case of equal weights $m_1=m_2=m_3=1$ is known since 1643 as (classical) Fermat-Torricelli problem. It has a unique solution which coincides either with one of the point P_1,P_2,P_3 or with the so-called *Fermat* or *Fermat-Torricelli point* [2, 4] of the triangle $P_1P_2P_3$.

Generalization of the problem to the case of unequal weights was investigated since XIX century. This generalization is known under different names: Steiner problem, Weber problem, problem of railway junction[2] [3, 8], the three factory problem [6]. The two last names were inspired by the optimal transportation problem like a following one. Let the cities P_1,P_2 and P_3 be the sources of iron ore, coal and water respectively. In order to produce one tonne of steel one the steel works need m_1 tonnes of iron, m_2 tonnes of coal and m_3 tonnes of water. Assuming that the freight charge for tonne-kilometer is independent of the nature of the cargo, find the optimal position for steel works connected with P_1,P_2,P_3 via straight roads so as to minimize the transport costs.

In the rest of the paper this problem will be referred to as the *generalized Fermat-Torricelli problem.* Existence and uniqueness of its solution is guaranteed by the following result [4]:

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[2] (*Germ.*) Problem des Knotenpunktes

Theorem 1.1 *Denote by $\alpha_1, \alpha_2, \alpha_3$ the corner angles of the triangle $P_1P_2P_3$. If the conditions*

$$\left\{ \begin{array}{lll} m_1^2 & < & m_2^2 + m_3^2 + 2\, m_2 m_3 \cos \alpha_1 , \\ m_2^2 & < & m_1^2 + m_3^2 + 2\, m_1 m_3 \cos \alpha_2 , \\ m_3^2 & < & m_1^2 + m_2^2 + 2\, m_1 m_2 \cos \alpha_3 \end{array} \right. \tag{1.2}$$

are fulfilled then there exists a unique solution $P_* = (x_*, y_*) \in \mathbb{R}^2$ for the generalized Fermat-Torricelli problem lying inside the triangle $P_1P_2P_3$. This point is a stationary point for the function $F(x,y)$, i.e. a real solution of the system

$$\sum_{j=1}^{3} \frac{m_j(x-x_j)}{\sqrt{(x-x_j)^2+(y-y_j)^2}} = 0, \quad \sum_{j=1}^{3} \frac{m_j(y-y_j)}{\sqrt{(x-x_j)^2+(y-y_j)^2}} = 0 \ . \tag{1.3}$$

If any of the conditions (1.2) is violated then $F(x,y)$ attains its minimum value at the corresponding vertex of the triangle.

Let us overview some approaches for finding the point P_*. The first one is geometrical: the point is found as intersection point of special construction of lines or circles. For the equal weighted case, Torricelli proved that the circles circumscribing the equilateral triangles constructed on the sides of and outside the triangle $P_1P_2P_3$ intersect in the point P_*; for an alternative Simpson construction of P_* see [5]. For the general (unequal weighted) case see [3, 8].

The second approach is based on mechanical model[3]: a horizontal board is drilled with the holes at the points P_1, P_2, P_3 (or at the vertices of a triangle similar to $P_1P_2P_3$) ; three strings are tied together in a knot at one end, the loose ends are passed through the holes and are attached to physical weights proportional to m_1, m_2, m_3 respectively below the board. The equilibrium position of the knot yields the solution [3].

The third approach, based on gradient descent method, was originated in the paper [14]; further developments and comments can be found in [7, 10].

The present paper is devoted to the fourth approach — analytical one. We look for explicit expressions for the coordinates of stationary point P_* as functions of $\{m_j, x_j, y_j\}_{j=1}^3$. Although the existence of such a solution by radicals, i.e. in a finite number of operations like standard arithmetic ones and extraction of (positive integer) roots, is not questioned in any review article on the problem, we failed to find in publications the constructive and universal version of an algorithm even for the classical (i.e. equal weighted) case.

[3]Sometimes incorrectly called Pólya's mechanical model.

2 Algebra

Theorem 2.1 *Under the conditions (1.2), the coordinates of stationary point* (x_*, y_*) *of the function* $F(x, y)$ *are as follows:*

$$x_* = \frac{K_1K_2K_3}{4S\sigma d}\left(\frac{x_1}{K_1} + \frac{x_2}{K_2} + \frac{x_3}{K_3}\right),\ y_* = \frac{K_1K_2K_3}{4S\sigma d}\left(\frac{y_1}{K_1} + \frac{y_2}{K_2} + \frac{y_3}{K_3}\right) \tag{2.1}$$

with

$$F(x_*, x_*) = \min_{(x,y)} F(x, y) = \sqrt{d}\ .$$

Here

$$d = \frac{1}{2\sigma}(m_1^2K_1 + m_2^2K_2 + m_3^2K_3) \tag{2.2}$$

$$= 2\,S\sigma + \frac{1}{2}\left[m_1^2(r_{12}^2 + r_{13}^2 - r_{23}^2) + m_2^2(r_{23}^2 + r_{12}^2 - r_{13}^2) + m_3^2(r_{13}^2 + r_{23}^2 - r_{12}^2)\right]\ . \tag{2.3}$$

and

$$r_{j\ell} = \sqrt{(x_j - x_\ell)^2 + (y_j - y_\ell)^2} = |P_jP_\ell| \quad \mathit{for}\ \{j, \ell\} \subset \{1, 2, 3\}\ ;$$

$$S = |x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1|\ ; \tag{2.4}$$

$$\sigma = \frac{1}{2}\sqrt{-m_1^4 - m_2^4 - m_3^4 + 2\,m_1^2m_2^2 + 2\,m_1^2m_3^2 + 2\,m_2^2m_3^2}\ ; \tag{2.5}$$

and

$$\left\{\begin{array}{lcl} K_1 &=& (r_{12}^2 + r_{13}^2 - r_{23}^2)\sigma + (m_2^2 + m_3^2 - m_1^2)S, \\ K_2 &=& (r_{23}^2 + r_{12}^2 - r_{13}^2)\sigma + (m_1^2 + m_3^2 - m_2^2)S, \\ K_3 &=& (r_{13}^2 + r_{23}^2 - r_{12}^2)\sigma + (m_1^2 + m_2^2 - m_3^2)S. \end{array}\right. \tag{2.6}$$

Proof. Firstly, we established via direct computations the validity of the following equalities:

$$K_1K_2 + K_1K_3 + K_2K_3 = 4\sigma Sd\ , \tag{2.7}$$

and a dual one for (2.2):

$$r_{23}^2K_1 + r_{13}^2K_2 + r_{12}^2K_3 = 2\,Sd\ . \tag{2.8}$$

Secondly, let us deduce the following relationships:

$$\sqrt{(x_* - x_j)^2 + (y_* - y_j)^2} = \frac{m_jK_j}{2\sigma\sqrt{d}} \quad \mathit{for}\ j \in \{1, 2, 3\}\ . \tag{2.9}$$

To prove (2.9) for $j = 1$ we first represent x_* and y_* given by (2.1) with the aid of (2.7):

$$x_* = \frac{1}{\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}}\left(\frac{x_1}{K_1} + \frac{x_2}{K_2} + \frac{x_3}{K_3}\right),\ y_* = \frac{1}{\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}}\left(\frac{y_1}{K_1} + \frac{y_2}{K_2} + \frac{y_3}{K_3}\right). \tag{2.10}$$

Thus,

$$
\begin{aligned}
(x_* - x_1)^2 + (y_* - y_1)^2 &= \left(\frac{K_1K_2K_3}{4\,\sigma S d}\right)^2 \left[\left(\frac{x_2}{K_2}+\frac{x_3}{K_3}-\frac{x_1}{K_2}-\frac{x_1}{K_3}\right)^2 + \left(\frac{y_2}{K_2}+\frac{y_3}{K_3}-\frac{y_1}{K_2}-\frac{y_1}{K_3}\right)^2\right] \\
&= \left(\frac{K_1K_2K_3}{4\,\sigma S d}\right)^2 \Big[\frac{(x_2-x_1)^2+(y_2-y_1)^2}{K_2^2} + \frac{(x_3-x_1)^2+(y_3-y_1)^2}{K_3^2} \\
&\qquad + 2\,\frac{(x_2-x_1)(x_3-x_1)+(y_2-y_1)(y_3-y_1)}{K_2K_3}\Big] \\
&= \left(\frac{K_1K_2K_3}{4\,\sigma S d}\right)^2 \left[\frac{r_{12}^2}{K_2^2}+\frac{r_{13}^2}{K_3^2}+2\frac{1/2(r_{12}^2+r_{13}^2-r_{23}^2)}{K_2K_3}\right] \\
&= \frac{K_1^2}{(4\,\sigma S d)^2}\left[r_{12}^2K_3^2 + r_{13}^2K_2^2 + (r_{12}^2+r_{13}^2-r_{23}^2)K_2K_3\right] \\
&= \frac{K_1^2}{(4\,\sigma S d)^2}\left[(r_{12}^2K_3 + r_{13}^2K_2)(K_2+K_3) - r_{23}^2K_2K_3\right] \\
&\overset{(2.8)}{=} \frac{K_1^2}{(4\,\sigma S d)^2}\left[(2\,Sd - r_{23}^2K_1)(K_2+K_3) - r_{23}^2K_2K_3\right] \\
&= \frac{K_1^2}{(4\,\sigma S d)^2}\left[2\,Sd(K_2+K_3) - r_{23}^2(K_1K_2+K_1K_3+K_2K_3)\right] \\
&\overset{(2.7)}{=} \frac{K_1^2}{(4\,\sigma S d)^2}\left[2\,Sd(K_2+K_3) - 4\,r_{23}^2\sigma S d\right] = \frac{2\,SdK_1^2}{(4\,\sigma S d)^2}\left[K_2+K_3-2\,r_{23}^2\sigma\right] \\
&\overset{(2.6)}{=} \frac{K_1^2}{8\,Sd\sigma^2}\left[2\,m_1^2 S\right] = \frac{m_1^2K_1^2}{4\sigma^2 d}\ .
\end{aligned}
$$

Similar arguments hold for $j \in \{2,3\}$ in (2.9). To complete the proof of those equalities it should be additionally verified that the values K_1, K_2, K_3 are nonnegative. This will be done in the next section.

Now let us prove the first statement of the theorem. Substitute (2.1) into the left-hand side of the first equation of (1.3). The resulting expression can be represented with the aid of formulae (2.7) and (2.9) as

$$
\frac{x_*-x_1}{K_1}+\frac{x_*-x_2}{K_2}+\frac{x_*-x_3}{K_3} = x_*\left(\frac{1}{K_1}+\frac{1}{K_2}+\frac{1}{K_3}\right) - \left(\frac{x_1}{K_1}+\frac{x_2}{K_2}+\frac{x_3}{K_3}\right) \overset{(2.10)}{=} 0.
$$

Similar arguments are valid for the second equation from (1.3). Finally compute $F(x_*, y_*)$:

$$
F(x_*, y_*) = \sum_{j=1}^{3} m_j\sqrt{(x_*-x_j)^2+(y_*-y_j)^2} \overset{(2.9)}{=} \sum_{j=1}^{3}\frac{m_j^2K_j}{2\sigma\sqrt{d}} \overset{(2.2)}{=} \frac{2\,\sigma d}{2\sigma\sqrt{d}} = \sqrt{d}\ .
$$

□

Tests 1

	P_1 m_1	P_2 m_2	P_3 m_3	P_* $\sqrt{d}$
1.	$(2,6)$ 2	$(1,1)$ 3	$(5,1)$ 4	$\left(\frac{1}{2866}\left(4103+1833\sqrt{15}\right)\ ,\ \frac{1}{8598}\left(29523-4481\sqrt{15}\right)\right)$ $\approx(3.9086, 1.4152)$ $\sqrt{d}=2\sqrt{79+15\sqrt{15}}\approx 23.4174$
2.	$(2,6)$ 3	$(1,1)$ 5	$(5,1)$ 4	$\left(\frac{751}{485},\frac{647}{485}\right)$ $\approx(1.5484, 1.3340)$ $\sqrt{d}=\sqrt{970}\approx 31.1448$
3.	$(0,0)$ $3/2$	$(2,0)$ 2	$(-\sqrt{2},\sqrt{2})$ 2	$\left(1-\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{110}}\ ,\ \frac{1}{\sqrt{2}}-\frac{3}{\sqrt{55}}-\frac{3}{\sqrt{110}}\right)$ $\approx(0.0068, 0.0165)$ $\sqrt{d}=\sqrt{32+\frac{23}{\sqrt{2}}+3\sqrt{\frac{55}{2}}}\approx 7.9997$
4.	$(39,57)$ 18	$(22,42)$ 41	$(42,75)$ 52	$\approx(37.0432, 61.4053)$ $\sqrt{d}=\sqrt{3068047+3915\sqrt{7511}}\approx 1845.8994$

The exact coordinates of P_* in Test 1.4 are as follows:

$$\begin{aligned} x_* &= \tfrac{296577529815837}{9297789607234}+\tfrac{357441196078431}{6020318770684015}\sqrt{7511}\ , \\ y_* &= \tfrac{271001243105952}{4648894803617}+\tfrac{432306390086253}{12040637541368030}\sqrt{7511}\ . \end{aligned}$$

3 Geometry

Let us give an interpretation for some constants appeared in Theorem 2.1. Being rewritten in an alternative form, the constant (2.4)

$$S=\left|\det\begin{pmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{pmatrix}\right|$$

is recognized as the doubled area of the triangle $P_1P_2P_3$. As for the constant (2.5), factorization of the radicand in its right-hand side leads one to the form

$$\sigma=2\sqrt{\frac{m_1+m_2+m_3}{2}\left(\frac{m_1+m_2+m_3}{2}-m_1\right)\left(\frac{m_1+m_2+m_3}{2}-m_2\right)\left(\frac{m_1+m_2+m_3}{2}-m_3\right)}$$

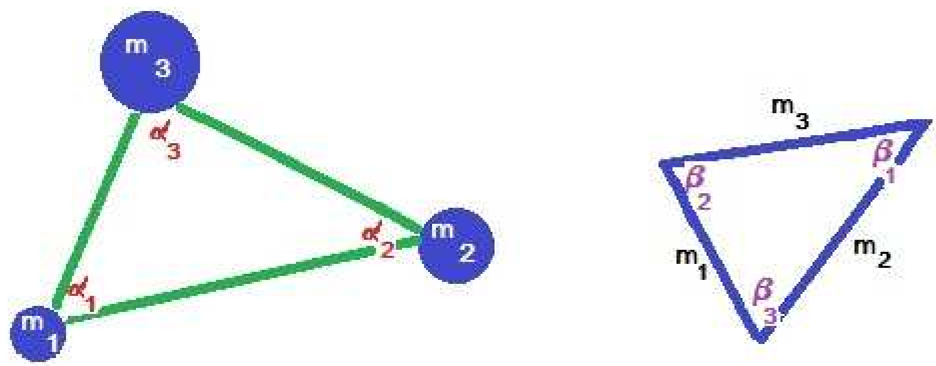

Figure 1:

which can be treated as the Heron's formula for the doubled area of a triangle formed by the triple of weights m_1, m_2, m_3. Under the restrictions (1.2), such a triangle exists. Construct now this triangle and denote its angles as shown in Figure 1.

Then the first formula from (2.6) can be represented with the aid of the law of cosines as

$$K_1 = \sigma S \left(\frac{r_{12}^2 + r_{13}^2 - r_{23}^2}{S} + \frac{m_2^2 + m_3^2 - m_1^2}{\sigma} \right) = \sigma S \left(\frac{2\, r_{12} r_{13} \cos \alpha_1}{S} + \frac{2\, m_2 m_3 \cos \beta_1}{\sigma} \right)$$

$$= 2\, \sigma S (\cot \alpha_1 + \cot \beta_1) \ .$$

On rewriting the first condition from (1.2) in the form $\cos \alpha_1 + \cos \beta_1 > 0$, one can conclude that $\cot \alpha_1 + \cot \beta_1 > 0$ and, thus, $K_1 > 0$. In a similar way the expressions for K_2 and K_3 can be deduced, and established that under the restrictions (1.2) they both are positive. This completes the proof of Theorem 2.1.

Remark. Set the *dual* generalized Fermat-Torricelli problem: let the triangle be composed of the sides with the lengths equal to m_1, m_2 and m_3; let the weights r_{12}, r_{23}, r_{13} be placed in its vertices as shown in Figure 2.

The minimum value for the objective function will be the same as in the direct problem since (2.3) is equivalent to

$$2\, S\sigma + \frac{1}{2} \left[r_{12}^2 (m_1^2 + m_2^2 - m_3^2) + r_{13}^2 (m_1^2 + m_3^2 - m_2^2) + r_{23}^2 (m_2^2 + m_3^2 - m_1^2) \right] \ .$$

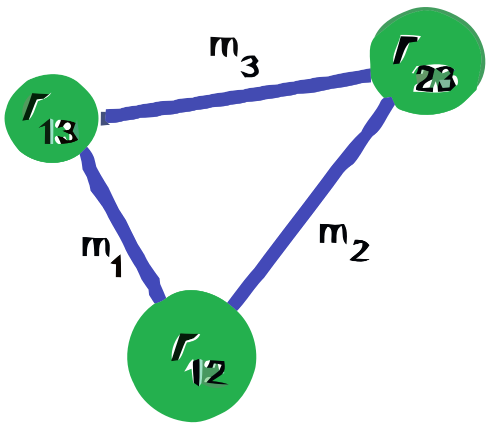

Figure 2:

4 Classical Fermat-Torricelli problem

Consider now the case $m_1 = m_2 = m_3 = 1$.

Theorem 4.1 *Let all the angles of the triangle $P_1P_2P_3$ be less than $2\pi/3$, or, equivalently:*

$$r_{12}^2 + r_{13}^2 + r_{12}r_{13} - r_{23}^2 > 0, \ r_{23}^2 + r_{12}^2 + r_{12}r_{23} - r_{13}^2 > 0, \ r_{13}^2 + r_{23}^2 + r_{13}r_{23} - r_{12}^2 > 0 \ .$$

The coordinates of Fermat-Torricelli point for this triangle are as follows:

$$x_* = \frac{k_1k_2k_3}{2\sqrt{3}Sd}\left(\frac{x_1}{k_1} + \frac{x_2}{k_2} + \frac{x_3}{k_3}\right), \ y_* = \frac{k_1k_2k_3}{2\sqrt{3}Sd}\left(\frac{y_1}{k_1} + \frac{y_2}{k_2} + \frac{y_3}{k_3}\right) \tag{4.1}$$

with the corresponding minimum value of the objective function:

$$F(x_*, x_*) = \min_{(x,y)} \sum_{j=1}^{3} \sqrt{(x - x_j)^2 + (y - y_j)^2} = \sqrt{d} \ .$$

Here

$$d = \frac{1}{\sqrt{3}}(k_1 + k_2 + k_3) = \frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{2} + \sqrt{3}\,S \tag{4.2}$$

and

$$k_1 = \frac{\sqrt{3}}{2}(r_{12}^2 + r_{13}^2 - r_{23}^2) + S, \ k_2 = \frac{\sqrt{3}}{2}(r_{23}^2 + r_{12}^2 - r_{13}^2) + S, \ k_3 = \frac{\sqrt{3}}{2}(r_{13}^2 + r_{23}^2 - r_{12}^2) + S.$$

with the rest of the parameters coinciding with those from Theorem 2.1.

It turns out that expressions (4.1), being represented as rational fractions with respect to $\{x_j, y_j\}_{j=1}^3$, can be reduced further to the form where denominators become "area free" [12]:

Corollary. Under the conditions of Theorem 4.1, the coordinates of Fermat-Torricelli point are as follows:

$$\begin{aligned} x_* &= \frac{1}{2\sqrt{3}\,d}[(x_1+x_2+x_3)|\tilde{S}| + \sqrt{3}\left(x_1 r_{23}^2 + x_2 r_{13}^2 + x_3 r_{12}^2\right) \\ &\quad +3\,\mathbf{sgn}(\tilde{S}) \left| \begin{array}{ccc} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_2x_3+y_2y_3 & x_1x_3+y_1y_3 & x_1x_2+y_1y_2 \end{array} \right|], \end{aligned} \tag{4.3}$$

$$\begin{aligned} y_* &= \frac{1}{2\sqrt{3}\,d}[(y_1+y_2+y_3)|\tilde{S}| + \sqrt{3}\left(y_1 r_{23}^2 + y_2 r_{13}^2 + y_3 r_{12}^2\right) \\ &\quad -3\,\mathbf{sgn}(\tilde{S}) \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_2x_3+y_2y_3 & x_1x_3+y_1y_3 & x_1x_2+y_1y_2 \end{array} \right|] . \end{aligned} \tag{4.4}$$

Here

$$\tilde{S} = \det \left(\begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \right) .$$

Remark. Result of the last corollary can be extended to the generalized Fermat-Torricelli problem: numerators and denominators of the formulae (2.1) can be reduced by the common factor S. We do not present here the resulting expressions since they are inelegantly ponderous [12].

Tests 2

	P_1	P_2	P_3	x_*	y_*	$\sqrt{d}$
1.	$(1,1)$	$(3,5)$	$(7,2)$	$\frac{2}{687}\left(1029+79\sqrt{3}\right)$ ≈ 3.3939	$\frac{1}{687}\left(1053+647\sqrt{3}\right)$ ≈ 3.1639	$\sqrt{41+22\sqrt{3}}$ ≈ 8.8941
2.	$(1,2)$	$(3,3)$	$(4,1)$	$\frac{1}{6}(15+\sqrt{3})$ ≈ 2.7886	$\frac{1}{2}(3+\sqrt{3})$ ≈ 2.3660	$\sqrt{10+5\sqrt{3}}$ ≈ 4.3197
3.	$(0,0)$	$(399,0)$	$(\frac{5005}{38}, \frac{9555\sqrt{3}}{38})$	$\frac{21255}{133}$ ≈ 159.8120	$\frac{8580\sqrt{3}}{133}$ ≈ 111.7368	784
4.	$(0,0)$	$(2,0)$	$(0,1)$	$\frac{1}{13}+\frac{4}{39}\sqrt{3}$ ≈ 0.2545	$\frac{8}{13}-\frac{7}{39}\sqrt{3}$ ≈ 0.3045	$\sqrt{5+2\sqrt{3}}$ ≈ 2.9093

Test 2.3 is generated from [11], test 2.4 is taken from [9].

5 Inverse problem

Given the coordinates of the point $P_* = (x_*, y_*)$ lying inside the triangle $P_1P_2P_3$, find the values for the weights m_1, m_2, m_3 with the aim for the corresponding function $F(x, y)$ to posses a minimum point precisely at P_*.

Theorem 5.1 *Let the vertices of the triangle $P_1P_2P_3$ be counted counterclockwise. Then for the choice*

$$m_1^* = |P_*P_1| \cdot \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_* & x_2 & x_3 \\ y_* & y_2 & y_3 \end{array} \right| , \ m_2^* = |P_*P_2| \cdot \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_* & x_3 \\ y_1 & y_* & y_3 \end{array} \right| , \ m_3^* = |P_*P_3| \cdot \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_* \\ y_1 & y_2 & y_* \end{array} \right| \tag{5.1}$$

the function

$$F(x, y) = \sum_{j=1}^{3} m_j^* \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

has its stationary point at P_*. Provided that the latter is chosen inside the triangle $P_1P_2P_3$ the values (5.1) are all positive and

$$F(x_*, y_*) = \min_{(x,y)} F(x, y) = \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 + y_*^2 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{array} \right| . \tag{5.2}$$

Proof. Substitute $x = x_*, y = y_*$ and the values (5.1) into the left-hand side of the first equation from (1.3):

$$(x_* - x_1) \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_* & x_2 & x_3 \\ y_* & y_2 & y_3 \end{array} \right| + (x_* - x_2) \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_* & x_3 \\ y_1 & y_* & y_3 \end{array} \right| + (x_* - x_3) \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_* \\ y_1 & y_2 & y_* \end{array} \right| . \tag{5.3}$$

Represent this combination of the third order determinants in the form of the fourth order determinant, namely

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ 0 & x_* - x_1 & x_* - x_2 & x_* - x_3 \end{array} \right|$$

(expansion by its last row coincides with (5.3)). Now add the second row to the last one:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_* & x_* & x_* & x_* \end{array} \right| .$$

In this determinant the first row is proportional to the last one; therefore the determinant equals just zero. The second equality from (1.3) can be verified in a similar manner.

Let us evaluate $F(x_*, y_*)$:

$$
\begin{aligned}
F(x_*, y_*) &= \left[(x_* - x_1)^2 + (y_* - y_1)^2\right] \begin{vmatrix} 1 & 1 & 1 \\ x_* & x_2 & x_3 \\ y_* & y_2 & y_3 \end{vmatrix} \\
&+ \left[(x_* - x_2)^2 + (y_* - y_2)^2\right] \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_* & x_3 \\ y_1 & y_* & y_3 \end{vmatrix} \\
&+ \left[(x_* - x_3)^2 + (y_* - y_3)^2\right] \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_* \\ y_1 & y_2 & y_* \end{vmatrix} .
\end{aligned}
$$

To prove the equality (5.2) let us split it into the x-part and the y-part. First keep the x-terms in brackets of the previous formula:

$$
(x_* - x_1)^2 \begin{vmatrix} 1 & 1 & 1 \\ x_* & x_2 & x_3 \\ y_* & y_2 & y_3 \end{vmatrix} + (x_* - x_2)^2 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_* & x_3 \\ y_1 & y_* & y_3 \end{vmatrix} + (x_* - x_3)^2 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_* \\ y_1 & y_2 & y_* \end{vmatrix} .
$$

Similar to the proof of the first part of the theorem, represent this linear combination as the determinant of the fourth order

$$
= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ 0 & (x_* - x_1)^2 & (x_* - x_2)^2 & (x_* - x_3)^2 \end{vmatrix} .
$$

Multiply the first row by $(-x_*^2)$, the second one by $2\,x_*$ and add the obtained rows to the last one:

$$
= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 & x_1^2 & x_2^2 & x_3^2 \end{vmatrix} . \tag{5.4}
$$

The y-part of the equality (5.2) can be proved in exactly the same manner with the resulting determinant differing from (5.4) only in its last row. The linear property of the determinant with respect to its rows completes the proof of (5.2). $\square$

Remark. Solution of the inverse problem is determined up to a common positive multiplier, i.e. the solution triple (m_1, m_2, m_3) is defined by the value of the ratio $m_1 : m_2 : m_3$. Up to this remark[4], solution of the inverse problem is unique: we have proved this statement via direct computations starting from the formulae (2.1).

Example 5.1 *Let* $P_1 = (2, 6), P_2 = (1, 1), P_3 = (5, 1)$ *and*

$$P_* = \left(\frac{1}{2866} \left(4103 + 1833\sqrt{15} \right), \ \frac{1}{8598} \left(29523 - 4481\sqrt{15} \right) \right) \ .$$

Find the values for the weights m_1^*, m_2^*, m_3^* *from Theorem 5.1.*

Solution. Formulae (5.1) give:

$$\begin{aligned} m_1^* &= \frac{2(20925 - 4481\sqrt{15})}{18481401} \sqrt{316380606 + 35999826\sqrt{15}}, \\ m_2^* &= \frac{2(15105 - 2342\sqrt{15})}{6160467} \sqrt{75400161 - 9169767\sqrt{15}}, \\ m_3^* &= \frac{8(-1185 + 15988\sqrt{15})}{18481401} \sqrt{8335761 - 2050623\sqrt{15}}, \end{aligned}$$

with

$$F(x_*, y_*) = \frac{1}{4299}(-333980 + 193436\sqrt{15}) \ .$$

Now compare the obtained result with the one represented in Test 1.1: according with the last remark one might expect that

$$m_1^* : m_2^* : m_3^* = 2 : 3 : 4 \ .$$

Let us verify this fact:

$$\frac{m_1^*}{m_2^*} = \frac{1}{3} \cdot \frac{20925 - 4481\sqrt{15}}{15105 - 2342\sqrt{15}} \cdot \sqrt{\frac{316380606 + 35999826\sqrt{15}}{75400161 - 9169767\sqrt{15}}}$$

$$= \frac{1}{3} \cdot \frac{(20925 - 4481\sqrt{15})(15105 + 2342\sqrt{15})}{145886565}$$

$$\times \sqrt{\frac{(316380606 + 35999826\sqrt{15})(75400161 + 9169767\sqrt{15})}{4423914876311586}}$$

[4] In the language of railway transportation, it is equivalent to the fact that the optimal position of junction is independent of the currency of the state: it does not matter whether in roubles or in dollars the freight charge is nominated.

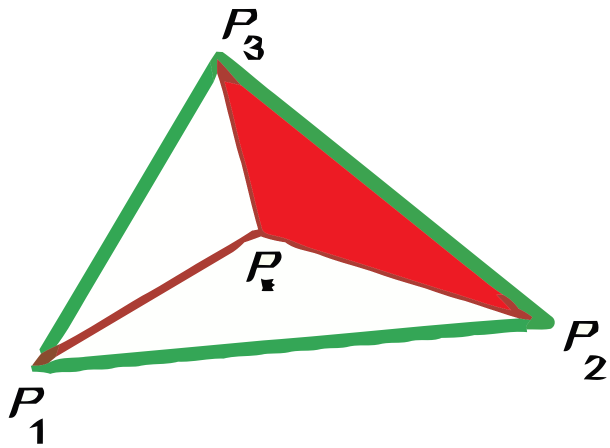

Figure 3:

$$= \frac{1}{3} \cdot \frac{671 - 79\sqrt{15}}{617} \cdot \sqrt{\frac{543856 + 106018\sqrt{15}}{83521}} = \frac{1}{3} \cdot \frac{671 - 79\sqrt{15}}{617} \cdot \frac{671 + 79\sqrt{15}}{289} = \frac{2}{3} \, .$$

□

Let us discuss now geometrical meaning of the constants from Theorem 5.1. The value m_1^* equals the doubled product of the distance from P_1 to P_* by the area of the triangle $P_*P_2P_3$.

The first statement of the theorem is equivalent to the fact that

$$\overrightarrow{P_*P_1} \cdot S_{\triangle P_* P_2 P_3} + \overrightarrow{P_*P_2} \cdot S_{\triangle P_* P_3 P_1} + \overrightarrow{P_*P_3} \cdot S_{\triangle P_* P_1 P_2} = \overrightarrow{\mathbb{O}} \, .$$

Finally, the constant (5.2) is connected with the following one

$$h = -\frac{1}{S} \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 + y_*^2 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{array} \right| ,$$

which is known [13] as *the power of the point P_* with respect to the circle* through the points P_1, P_2 and P_3 (circumscribed circle of the triangle). If one denotes by C the circumcenter of the triangle $P_1P_2P_3$ then

$$h = |CP_*|^2 - |CP_j|^2 \qquad (5.5)$$

and provided that P_* lies inside this triangle, this value is negative.

Results of the present section can evidently be extended to the case of three (and more) dimensions:

Theorem 5.2 *Let the points* $\{P_j = (x_j, y_j, z_j)\}_{j=1}^4$ *be noncoplanar, and, in addition, be counted in such a manner that the value of the determinant*

$$V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{5.6}$$

is positive. Then for the choice

$$\left\{ m_j^* = |P_* P_j| \cdot V_j \right\}_{j=1}^4 \tag{5.7}$$

where V_j *equals the determinant obtained on replacing the* j*-th column of (5.6) by the column*[5] $[1, x_*, y_*, z_*]^\top$*, the function*

$$F(x, y, z) = \sum_{j=1}^4 m_j^* \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}$$

has its stationary point at $P_* = (x_*, y_*, z_*)$*. If* P_* *lies inside the tetrahedron* $P_1 P_2 P_3 P_4$ *then the values (5.7) are all positive and*

$$F(x_*, y_*, z_*) = \min_{(x,y,z)} F(x, y, z)$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 & x_4 \\ y_* & y_1 & y_2 & y_3 & y_4 \\ z_* & z_1 & z_2 & z_3 & z_4 \\ x_*^2 + y_*^2 + z_*^2 & x_1^2 + y_1^2 + z_1^2 & x_2^2 + y_2^2 + z_2^2 & x_3^2 + y_3^2 + z_3^2 & x_4^2 + y_4^2 + z_4^2 \end{vmatrix} . \tag{5.8}$$

Geometrical meanings of the values appeared in the last theorem are similar to their counterparts from Theorem 5.1. For instance, the value (5.6) equals six times the volume of tetrahedron $P_1 P_2 P_3 P_4$, while the value (5.8) divided by V is known [13] as *the power of the point* P_* *with respect to a sphere* circumscribed to that tetrahedron; it is equivalent to (5.5) where C this time stands for the circumcenter of the tetrahedron.

[5] Here $^\top$ denotes transposition.

6 Conclusions

Analytical solution for the generalized Fermat-Torricelli problem is presented. The three point case is completely solved in "extended radicals": in addition to elementary and extraction of roots operations the sign function is utilized in the formulae. For further investigations remains the treatment of the multidimensional n point case, although some theoretical results like [1] do not give reason to hope for the nice (e.g. extended radicals) solution.

References

[1] Bajaj C. "The algebraic degree of geometric optimization problems." *Discr. Comput. Geom.*, 1988. V. 3, pp. 177191

[2] Courant R., Robbins H. *What is Mathematics?* London. Oxford University Press. 1941

[3] Dingeldey F. *Sammlung von Aufgaben zur Anwendung der Differenzial- und Integralrechnung. Erster Teil. Aufgaben zur Anwendung der Differenzialrechnung.* Teubner. Leipzig. 1910, pp. 134–136

[4] "Fermat-Torricelli problem." *Encyclopedia of Mathematics.* http://www.encyclopediaofmath.org/index.php?title=Fermat-Torricelli_problem&oldid=22419

[5] Gonzalez Martinez D. "The Fermat point." http://jwilson.coe.uga.edu/EMAT6680Fa10/Gonzalez/Assignment 6/THE FERMAT POINT.htm

[6] Greenberg I., Robertello R.A. "The three factory problem". *Math. Magazine.* 1965. V. 38, pp. 67–72

[7] Kuhn H.W. "A note on Fermat's problem." *Math. Programming.* 1973. V. 4, No. 1, pp. 98–107

[8] Launhardt W. "Kommercielle Tracirung der Verkehrswege." *Zeitschrift f. Architekten u.Ingenieur-Vereinis im Konigreich Hannover.* 1872. V.18, pp. 516–534.

[9] Llambay A.B., Llambay P.B., Pilotta E.A. "On characterizing the solution for the Fermat-Weber location problem." Universidad Nacional de Córdoba. Argentina. Facultad de Matemática, Astronomıa y Fısica. Serie "A". Trabajos de Matemática. N 94/09, July 29, 2009. http://www.famaf.unc.edu.ar/publicaciones/documents/serie_a/AMat94.pdf

[10] Ostresh L.M. "On the convergence of a class of iterative methods for solving the Weber location problem". *Operations Research.* 1978. V.26, No. 4, pp.597–609

[11] Project Euler. Problem 143. http://projecteuler.net/problem=143

[12] Ulanov E.A., Uteshev A.Yu. "Analytical solution for the generalized Fermat-Torricelli-Steiner problem." *Control processes and stability: Proceedings of the 42th international scientific conference of graduates and postgraduates.* A.S.Eriomin, N.V.Smirnov eds. St.Petersburg. St.Petersburg State University, 2011. pp. 201206 (*in Russian*).

[13] Uspensky J.V. *Theory of Equations.* New York. McGraw-Hill. 1948. pp. 251–255

[14] Weiszfeld E. "Sur le point pour lequel la somme des distances de n points donnés est minimum". *Tohoku Math. Journal.* 1937. V. 43, pp. 355386.