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Master's Thesis

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Finding Steiner Minimal Trees in Euclidean d-Space

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Abstract

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Abbreviations

BnB Branch and Bound.

ESTP Euclidean Steiner Tree Problem.

FST Full Steiner Tree.

MST Minimum Spanning Tree.

SMT Steiner Minimal Tree.

1 Introduction

1.1 Objectives

1.2 Related work

1.3 Structural outline

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2 Preliminaries

This chapter contains all the preliminaries, basic concepts and definitions, which are needed to understand the work done in the thesis, and which are not necessarily known to the reader in full.

Note: This can be a bit more verbose.

2.1 Topology

The combinatorial structure of of a tree, or other form of geometric network, is called a topology. I.e. in the topology we only think about the adjacencies of the points and not their coordinates.

2.2 Steiner tree

There are, at least, two different definitions of a Steiner tree both of which a equivalent. The first, used by Gilbert and Pollak [1] defines a Steiner tree, as a tree that cannot be shortened by a small perturbation or by "splitting" a terminal by inserting a Steiner point.

Here a Steiner point is an extra vertex which one is allowed to insert into the graph and use as a "connection point" for edges, as can be seen in Figure 2.1.

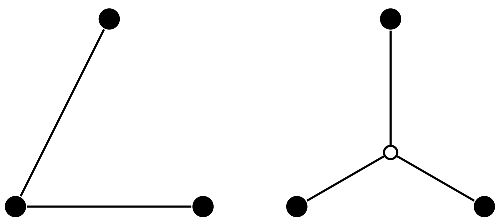

Figure 2.1: The length of the tree on the left can be shortened by inserting an extra point as has been done on the right.

Another definition, used by Smith [2], defines the Steiner tree a tree with the following properties:

1. It contains N "regular points" $\vec{x}_1 \cdots \vec{x}_N$, and possibly K additional "Steiner points" $\vec{x}_{N+1} \cdots \vec{x}_{N+K}$.
2. Each Steiner point has valence 3, and the edges emanating from it lie in a plane and have mutual angle 120°
3. Each regular point has valence between 1 and 3 (and generally $\leqslant 2$).
4. $0 \leqslant K \leqslant N - 2$.

These two definitions are equivalent, however the second one gives some important information about the structure of the Steiner tree. This thesis will therefore in general use the second definition.

That the two definitions are indeed equivalent can seen by considering each of the property of the second definition.

The first property is by definition, and to name the regular points and the Steiner points.

The second property can be shown in the following way. First of consider that any Steiner point must have at least valence three. This should be obvious as every point with valence one can simply be removed to make the length the same or shorter as they do not lead to any regular point Every Steiner point with valence two can be removed, and its edges be replaced with one edge which is of the same length or shorter as per the triangle inequality theorem.

We now show that no pair of edges emanating from a Steiner point can have a mutual angle less than 120°. Suppose the contrary is the case, and the lines PR and RQ meet with angle PRQ $< 120°$. If we use the mechanical interpretation given by Gilbert and Pollak [1] the two edges "pull" on point R with a resultant force of magnitude

Note: I don't quite get why this is the case?

$$F = 2\cos\theta/2 > 1$$

ToDo: Finish the part about the angles being no less than 120

Using that the mutual angles can be no less than 120° it follows that every Steiner point can have a valence of at most 3, as $120° = 360°$. We therefore have, using the first part about the valence that a Steiner point always has exactly valence 3, and as the mutual angles can be no less than 120°, they must be exactly 120°.

ToDo: Write proof of third property

The fourth property follows in the following way from property 3 and 4. Any tree has one edge fewer than vertices. Thus a Steiner tree has $N + K - 1$ edges.

Every edge has two endpoints meaning there are $2N + 2K - 2$ edges. As every Steiner point has valence three they account for $3K$ of the endpoints. For the regular points we must split them in three groups – N_1, those that have valence 1. N_2, those that have valence 2 and, N_3, those that have valence 3. Thus the regular points account for the rest of the endpoints as $N_1 + 2N_2 + 3N_3 \geqslant N$. We can then write the equation as

$$3K + N_1 + 2N_2 + 3N_3 = 2N + 2K - 2 \tag{2.1}$$

$$K = 2N - 2 - (N_1 + 2N_2 + 3N_3) \tag{2.2}$$

$$0 \leqslant K \leqslant N - 2 \tag{2.3}$$

Full Steiner tree

A Full Steiner Tree (FST) is a Steiner tree where all regular points have valence 1, which is equivalent to the tree having exactly $N-2$ Steiner points, the maximal number of possible Steiner points. This can easily be seen by considering the definition of Steiner trees in Section 2.2. This corresponds to the proof of the fourth property, but here $N_1 = N$ and $N_2 = N_3 = 0$, meaning that the number of Steiner points will be exactly $N - 2$.

It can furthermore be shown that any Steiner tree is a union of edge-disjoint FSTs, meaning that any edge of a Steiner Minimal Tree (SMT) joining to regular points is also an edge of Minimum Spanning Tree (MST) of the regular points. This is relatively easily seen, by realizing that every three points connected to a Steiner point is a FST for those three points. These are then either connected to the rest of the tree with a Steiner point, in which case they are part of a bigger FST, or they are connected to a regular point, in which case we can we can remove that edge to have one FST and the rest of the tree.

The topology of a FST is called a full Steiner topology, also abbreviated FST. Here we have the fixed coordinates of the regular points, but not the coordinates of the Steiner points, and as such we only think about the adjacencies of the points. Whether we are talking about the topology or tree will be clear from the context.

Steiner minimal tree

A minimal tree on N points $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ is the tree having these points as its vertices and having the smallest possible sum of the length of all its edges. When we are allowed to insert any number of extra points into the tree and connect the edges using these to further shorten the tree this is known as a SMT [1].

Note: This could probably be fleshed out a bit more

2.3 Euclidean Steiner tree problem

The Euclidean Steiner Tree Problem (ESTP) is the problem of finding the SMT for the set of points N in Euclidean d-space. I.e each point $x_i, 1 \leqslant i \leqslant n$ is a d-vector, and the metric used to calculate the length of the edges connecting the points is the L^2-norm, the Euclidean norm.

The work of this thesis is only concerned with the Euclidean version of the Steiner tree problem in d-space.

* * *

3 Smiths algorithm

The following chapter will introduce the main focus of this thesis, namely the algorithm for finding SMTs in d-space proposed by Smith [2].

The chapter will mainly be concerned with the algorithmic design of the algorithm, and the proofs and theorems surrounding it. It will however also delve into more some parts of the implementation done by Smith when this is relevant.

3.1 Overview

In general the algorithm follows the following form, proposed by Gilbert and Pollak [1].

1. Enumerate all Steiner topologies on N regular points and K, $0 \leqslant K \leqslant N-2$ Steiner points.
2. Optimize the coordinates of the Steiner points for each topology, to find the shortest possible Euclidean embedding (tree) of that topology.
3. Select, and output, the shortest tree found.

In other words this approach might be seen as an exhaustive search of the solution space, and the approach described is simply to "try all possibilities".

Smith however does some things a little differently to avoid having to optimize every topology. Firstly he only looks at FSTs as it turns out we can regard any topology which is non-full, as a FST where some points overlap.

Secondly the algorithm not only finds the topologies for N regular points, but for $3, 4, \ldots, N$ regular points, all as FSTs having $1, 2, \ldots, N-2$ Steiner points. The reason for this is to allow a Branch and Bound (BnB) approach where we wish to prune all descendants of a topology if we have a better upper bound than the length of the current topology.

The optimization of the algorithm is done by a iterative process which updates all Steiner points of a topology every iteration, by Smith described as an iteration analogous to a Gauss-Seidel iteration [2, p. 145]. The equations of each iteration are solved using Gauss-elimination.

Note: Am I missing something in the overview?

3.2 Topologies

The first step of the algorithm is, as described in Section 3.1, to generate topologies. It is therefore natural that we need some way of representing and generating these topologies.

The algorithm only considers FSTs where $K = N - 2$. This simplification is allowed, as we can simply regard any Steiner tree with $K \leqslant N - 2$ as a FST where some edges have length zero and thus some points have "merged".

Note however that even with this simplification the number of FSTs is still exponential in N, which is clear from Corollary 3.1.1.

Representation

It turns out that every FST can be represented using a vector, in particular we utilize the following theorem

Theorem 3.1. *There is an 1–1 correspondence between full Steiner topologies with $N \geqslant 3$ regular points, and $(N-3)$-vectors $\vec{a}$, whose ith entry a_i is an integer between $1 \leqslant a_i \leqslant 2i + 1$.*

The proof of this theorem is done constructively by induction on N. It is clear that the smallest FST, or any Steiner topology for that matter, we can construct must have $N = 3$, as the number of Steiner points is $N - 2 = 1$. Thus we start with the initial null vector $\vec{a} = ()$ corresponding to the unique FST for the points 1, 2 and 3 connected through the respective edges 1, 2 and 3 and one Steiner point $N + 1$ as seen in Figure 3.1a. After this first step, each entry of the topology vector is considered, one at a time, where the ith entry of the topology vector describes the insertion of the $(N + 1 + i)$th Steiner point on the edge a_i and its connection to the $(i + 3)$th regular point. Thus for the ith insertion we will have $2i+1$[1] different edges on which we can insert the Steiner point $N + 1 + i$ and connect it to the regular point $i + 3$.

Furthermore we get a corollary saying that the number of FSTs is exponential in N

[1] At the first iteration we clearly have 3 edges to insert on, and as each insertion generates two new edges we have $3 + \underbrace{2 + \cdots + 2}_{2(i-1)} = 2i + 1$.

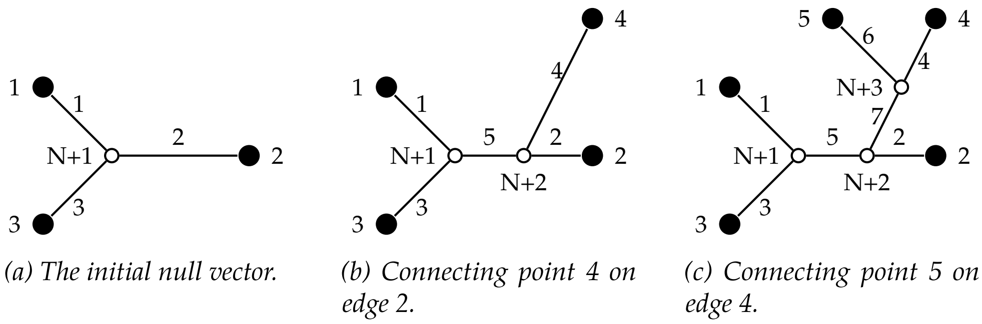

Figure 3.1: Construction of the FSTs corresponding to the vector $\vec{a} = (2, 4)$.

Corollary 3.1.1. *The number of FSTs on* N *regular points is*

$$\prod_{i=0}^{N-3} 2i + 1 = 1 \cdot 3 \cdot 5 \cdots (2N - 5)$$

I.e the number of FSTs is exponential in N.

Which is clear as we must insert $N - 2$ Steiner points, where the null vector is the 0th iteration. Thus the last iteration must be $N - 3$, and for each iteration we have $2i + 1$ different insertions.

The way Smith chooses to enumerate the edges is not explained outright, but only in the form of a visual example [2]. However one must take care to keep the enumeration consistent to avoid generating the same topologies more than once. Thus we do the enumeration in the same way as Smith, and as in Figure 3.1. That is when we insert Steiner point $N + 1 + i$ on the edge $a_i = (a_i, j), N < j < N + i + 1$, we split it such that we get the following three edges

- edge $a_i = (a_i, N + 1 + i)$
- edge $2i + 2 = (i + 3, N + 1 + i)$
- edge $2i + 3 = (j, N + 1 + i)$

Generation

Using the representation described in Section 3.2 the problem of generating all topologies can now be done as a backtracking problem generating all $(N - 3)$-topology vectors.

ToDo: Describe the backtracking here

To further speed up the generation of topologies, or rather to avoid generating unnecessary topologies, Smith also utilizes the following theorem

Theorem 3.2. *For any set of* N *distinct regular points in any Euclidean space, the length of the shortest tree, interconnecting* $N-1$ *points, with topology vector* $a_1 \cdots a_{N-4}$ *is no greater than the length of the shortest tree, interconnecting* N *points, with topology vector* $a_1 \cdots a_{N-3}$.

ToDo: Proof of above theorem

The algorithm utilizes this to prune in the following way. Imagine we have found some upper bound for the SMT. If we then optimize any generated topology vector which does not yet include all the regular points, and it turns out to have length greater than the upper bound, we can prune any topologies that we would have generated from this vector, as the length of the larger topologies cannot become any smaller than the length of the current, and thus cannot become smaller than the length of the upper bound.

Thus the implementation of the algorithm generates and optimizes topologies depth-first to ensure we get an upper bound as quickly as possible. If it did breadth-first we would not be able to prune anything, as we would get all the full topologies as the last to optimize.

3.3 Optimization of a prespecified topology

Error function

Iteration

FixMe: This section could probably be better named

ToDo: TODO

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4 Implementation

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5 Benchmarks and Tests

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6 Discussion

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7 Conclusion

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Bibliography

[1] E. N. Gilbert and H. O. Pollak, "Steiner Minimal Trees," *Society for Industrial and Applied Mathematics (SIAM) Journal*, vol. 16, no. 1, pp. 1–29, 1968.

[2] W. D. Smith, *Algorithmica*. 1992.