Problem Set #1

MACS 30150, Dr. Evans
Nan Ge

Question 5.1 If the individual lives for one period T = 1, what is the condition that characterizes the optimal amount of cake to eat in period 1? Write the problem in the equivalent way of showing what the condition is for the optimal amount of cake to save for the next period W_{T+1} or W_2.

$$\begin{aligned} &\max_{c_1 \in [0, W_1]} u\left(c_1\right) \\ &\text{s.t} \quad W_2 = W_1 - c_1 \end{aligned} \tag{1}$$

Optimal decision is characterized such as $W_2 = 0$ or $c_1 = W_1$

Question 5.2 T = 2

$$\begin{aligned} &\max_{c_t \in [0, W_t]} u(c_1) + \beta \cdot u(c_2) \\ &\text{s.t} \quad W_{t+1} = W_t - c_t \\ &\qquad W_{T+1} \geq 0 \end{aligned} \tag{2}$$

The first order condition with regard to W_2 implies that

$$-u'(W_1 - W_2) + \beta \cdot u'(W_2) = 0$$

Therefore, the optimal decision rule is characterized by

$$\begin{aligned} u'(W_1 - W_2) &= \beta \cdot u'(W_2 - W_3) \\ W_3 &= 0 \end{aligned} \tag{3}$$

Question 5.3 T=3

$$\begin{aligned} &\max_{c_t \in [0, W_t]} u(c_1) + \beta \cdot u(c_2) \\ &\text{s.t} \quad W_{t+1} = W_t - c_t \\ &\qquad W_{T+1} \geq 0 \end{aligned} \tag{4}$$

Same as 5.2, the optimal decision rule for this question is characterized by

$$\begin{aligned} u'(W_1 - W_2) &= \beta \cdot u'(W_2 - W_3) \\ u'(W_2 - W_3) &= \beta \cdot u'(W_3 - W_4) \\ W_4 &= 0 \end{aligned} \tag{5}$$

Question 5.4
The value function V_{T-1} is

$$V_{T-1}\left(W_{T-1}\right) = \max_{W_T} u\left(W_{T-1} - \psi_{T-1}(W_{T-1})\right) + \beta V_T\left(\psi_{T-1}(W_{T-1})\right) \tag{6}$$

The condition that characterizes the optimal choice in period T-1 is

$$u'\left(W_{T-1} - \psi_{T-1}(W_{T-1})\right) = \beta V_T'\left(\psi_{T-1}(W_{T-1})\right) \tag{7}$$

Question 5.5
At time $t = T - 1$, the recursive problem is expressed by (6). The optimal decision rule is characterized by (7). When $u(c) = ln(c)$, the optimal decision rule is such that

$$\frac{1}{W_{T-1} - \psi_{T-1}(W_{T-1})} = \beta \frac{1}{W_T}$$

Notice that $V_T(W_T) = u(W_T)$.
The solution is

$$\psi_{T-1}(\overline{W}) = \frac{\beta}{1+\beta}\overline{W}$$
$$V_{T-1}(\overline{W}) = ln(\frac{1}{1+\beta}\overline{W}) + \beta ln(\frac{\beta}{1+\beta}\overline{W})$$

Furthermore, at t = T, we have

$$\psi_T(W_T) = 0$$
$$V_T(\overline{W}) = ln(\overline{W})$$

To conclude, when $T < \infty$, $\psi_T(\overline{W})$ and $V_T(\overline{W})$ depend on time T.

Question 5.6
The Bellman Equation at T-2 is

$$V_{T-2}(W_{T-2}) = ln(W_{T-2} - WT - 1) + \beta ln(W_{T-1} - W_T) + \beta^2 ln(W_T) \tag{8}$$

The Envelope Theorem gives

$$\begin{aligned} W_{T-1} - W_T &= \beta(W_{T-2} - WT - 1) \\ W_T &= \beta(W_{T-1} - W_T) \end{aligned} \tag{9}$$

The solutions are

$$\begin{aligned} \psi(W_{T-2}) = W_{T-1} &= \frac{\beta + \beta^2}{1+\beta+\beta^2} \\ \psi(W_{T-1}) = W_T &= \frac{\beta^2}{1+\beta+\beta^2} \end{aligned} \tag{10}$$

Let $W_{T-2} = W_0$, then the value function is

$$V_{T-2}(W_0) = ln(\frac{W_0}{1+\beta+\beta^2}) + \beta \cdot ln(\frac{\beta W_0}{1+\beta+\beta^2}) + \beta^2 \cdot ln(\frac{\beta^2 W_0}{1+\beta+\beta^2})$$

Question 5.7
Claim:

$$\begin{aligned}\psi_{T-s}(W_{T-s}) &= \frac{\sum_{i=1}^{s}\beta^i}{\sum_{i=0}^{s}\beta^i}W_{T-s} \\ V_{T-s}(W_{T-s}) &= \sum_{i=0}^{s}\beta^i \cdot ln\left(\frac{\beta^i}{\sum_{j=0}^{s}\beta^j}W_{T-s}\right)\end{aligned} \tag{11}$$

We will prove using induction.
When s = 1,

$$\psi_{T-1}(W_{T-1}) = W_T = 0$$
$$V_{T-1}(W_{T-1}) = ln(W_{T-1})$$

Suppose (11) holds when t = T. Then at t = T+1, we could prove that

$$u'(\psi_{T-1}(W_{T-1}) - \psi_T(W_T)) = \beta V'(\psi_T(W_T))$$
$$V(\psi_{T-1}(W_{T-1})) = u(\psi_{T-1}(W_{T-1}) - \psi_T(W_T)) + \beta V(\psi_T(W_T))$$

When $s \to \infty$,

$$\lim_{s\to\infty}\psi_{T-s}\left(W_{T-s}\right) = \lim_{s\to\infty}\frac{\frac{\beta(1-\beta^s)}{1-\beta}}{\frac{1-\beta^s}{1-\beta}}W_{T-s} = \beta W_{T-s}$$
$$\lim_{s\to\infty}V_{T-s}\left(W_{T-s}\right) = \left(\sum_{i=0}^{\infty}\beta^i ln(1-\beta) + \sum_{i=1}^{\infty}\beta^i ln(\beta^i)\right)W_{T-s} = \left(\frac{ln(1-\beta)}{1-\beta} + \frac{ln\beta}{(1-\beta)^2}\right)W_{T-s}$$

$\square$

Question 5.8
When time horizon is finite, the Bellman equation is

$$V\left(W\right) \equiv \max_{W'\in[0,W]} u\left(W - W'\right) + \beta V\left(W'\right)$$

Question 5.9-5.22
Please see to the attached jupyter notebook.

References