Applications of Representation Theory to Braids and Anyonic Systems

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Outline

1 Introduction to Representation Theory

Question

How does one study a mathematical structure? (Sets, Groups, Vector Spaces, etc.)

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Answer

By studying its maps!

Maps help us to identify unknown structures with understood ones.

Example 1

Cayley's Theorem: $\phi : G \to S_G$

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In this thesis, we look to understand the behavior of groups through the lens of a special kind of map.

Definition 3

Let G be a group. A **representation** of degree n is a group homomorphism that maps a G into $GL_n(\mathbb{C})$.

$$\phi : G \to GL_n(\mathbb{C})$$

We say that ϕ is a representation of G.

If ϕ is an injective homomorphism, we say that the representation is **faithful**. Otherwise, the representation is called **degenerate**.

Question

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Answer

Matrices are some of the most well-understood strucutres in all of mathematics. We can use linear algebra to gain insight into group properties.

Example 4

Let G be the group of all roots of unity (under multiplication). Explicitly, $G = \{e^{\frac{2\pi i m}{n}} \mid m, n \in \mathbb{Z}\}$. One representation can be created by

$$\phi : G \to GL_2(\mathbb{C})$$

$$e^{2\pi i \frac{m}{n}} \mapsto \begin{bmatrix} \cos(\frac{2\pi m}{n}) & \sin(\frac{2\pi m}{n}) \\ -\sin(\frac{2\pi m}{n}) & \cos(\frac{2\pi m}{n}) \end{bmatrix}$$

This representation is degree 2 and is degenerate.

There is another equivalent definition for representations:

Definition 5

Let G be a group, V be an n-dimensional vector space, and let $\mathcal{L}(V)$ be the group of linear operators on V under composition. A **representation** of G of degree n is a group homomorphism that maps the G into $\mathcal{L}(V)$

$$\phi : G \to \mathcal{L}(V)$$

We identify matrices in $GL_n(\mathbb{C})$ with matrices of operators in $\mathcal{L}(V)$.

We will switch between definitions when useful

Example 6

Let G be the complex unit circle ($G = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$) and the operation of multiplication and let $V = \mathbb{C}$. Then,

$$\phi : G \to \mathcal{L}(V)$$

$$e^{i\theta} \mapsto U_{e^{i\theta}}$$

where $U_{e^{i\theta}} : \mathbb{C} \to \mathbb{C}$ such that $U_{e^{i\theta}}(re^{i\psi}) = e^{i\theta} re^{i\psi} \; \forall re^{i\psi} \in G$

Depending on V, this is either a one or two degree representation. It is faithful.

Question

When are two representations representing the same information?

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Definition 7

Two representations, ϕ and ψ, are said to be **equivalent representations** if there exists some invertible operator/matrix (depending on definition of representation), M, such that

$$\phi = M\psi M^{-1}$$

Representation equivalence (conjugation) is an equivalence relation.

This is directly related to similarity transformations and matrix canonical forms.

We can detect matrix similarity with the following operation:

Definition 8

The **character** of a representation, ϕ, on $g \in G$, denoted $\chi^\phi(g)$, is defined by

$$\chi^\phi(g) = trace(\phi(g))$$

The trace operation satisfies $trace(AB) = trace(BA)$ for any two matrices.

Theorem 9

If two representations are equivalent, then character of both representations are the same.

Much like in the study of linear algebra, we will explore the decomposition of representations into its invariant components

Definition 10

Let ϕ be a representation of the group G and $\phi(G) := \{\phi(g) = \phi_g : V \to V \mid g \in G\}$. Let $W \subset V$. W is said to be an **invariant subspace** with respect to $\phi(G)$ if $\forall v \in W$ and $\forall g \in G$, $\phi_g(v) \in W$.

Definition 11

A representation is said to be **irreducible** if there is no nontrivial, invariant subspace with respect to it. Otherwise, we say that the representation is reducible.

We look to the the eigendecomposition of matrices of a representation to find such subspaces.

In order to be more explicit in our discussion, we introduce new notation:

Definition 12

If M and N are square matrices, then let

$$M \oplus N := \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

we call this new matrix the **direct sum** of M and N.

Example 13

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Following our natrual inclination, we can use our notion of block-diagonal matrix decompositions to explicitly characterize representations.

Theorem 14

Let ϕ be a representation of a group G. Then there exists a set of irreducible representations of G, $\{\psi_i\}_{i=1}^{j}$, such that

$$\phi = T\left(\bigoplus_{i=1}^{j} \alpha_i * \psi_i\right) T^{-1}$$

where $\alpha_i * \psi_i := \underbrace{\psi_i \oplus \psi_i \oplus \ldots \oplus \psi_i}_{\alpha_i \text{times}}$ and T is some invertible matrix/operator.

We prove this theorem using induction on the degree of representation.

The results of this theorem show us that we can completely characterize any representation in terms of its irreducible components. With this in mind, we can characterize important properties of irreducible representations

Theorem 15

Schur's Theorem: *Let ϕ and ψ be two irreducible representations of the group G. Let M be a matrix defined such that $M\phi(g) = \psi(g)M \ \forall g \in G$. Then M is invertible or 0.*

Schur's Theorem tells us that irreducible representations are either completely unrelated or equivalent, much like eigenspaces.

Theorem 16

Let ϕ be an irreducible representation and M be a matrix/operator such that $M\phi(g) = \phi(g)M \ \forall g \in G$. Then M is a mulptiple of the identity matrix/map.

This identity will be incredibly useful for finding representations when we know that certain matrices will commute.

Theorem 17

If G is an abelian group, then any irreducible representation of G can be viewed as a degree one representation.

This identity trivializes the classification of abelian groups, giving their representations complete sets of eigenvalues.

Irreducible representations are not the only special kind of representation. When V is an inner-product space, we have more structure.

Definition 18

Let V be an inner-product space. Let $U \in \mathcal{L}(V)$. U is said to be **unitary** if U is surjective and $\forall x, y \in V$, $\langle x, y \rangle = \langle U(x), U(y) \rangle$.

Definition 19

A representation, ϕ, of a group, G, is said to be a **unitary representation** if $\forall g \in G$, $\phi(g)$ is unitary.

Theorem 20

Every representation of a finite group on an inner-product space is equivalent to a unitary representation

The above theorem gives us the utility of being able to tranform any representation into its unitary form. We can make our irreducible representations unitary as well.

When we consider the unitary irreducible representations of a group, we can make striking conclusions.

Theorem 21

Let G be a finite group, let Φ be the set of distinct (inequivalent to others in set) irreducible, unitary representation of G. Let $\phi, \psi \in \Phi$ with degrees n_ϕ and n_ψ respectively. Let $\phi_g := \phi(g)$ and $\psi_g := \psi(g)$. Then the following equality holds:

$$\frac{n_\phi}{|G|}\sum_{g\in G}[\phi_g]_{ij}\left[\psi_g^\dagger\right]_{kl} = \begin{cases} 1 & \text{if } \psi = \phi,\ j = k,\ \text{and } i = l \\ 0 & \text{else} \end{cases}$$

*where for any matrix $M = [m]_{ij}$, $\left[M^\dagger\right]_{ij} = \overline{m}_{ji}$. We refer to this as the **orthonormality condition** of unitary irreducible representations.*

There is no doubt complexity here, but this theorem can be made more digestible.

We create a $|G|$-dimensional vector space in the following way: For fixed ϕ, i, j,

$$\sqrt{\frac{n_\phi}{|G|}}\left([\phi_{g_1}]_{ij},\ [\phi_{g_2}]_{ij},\ \ldots\ ,\ [\phi_{g_{|G|}}]_{ij}\right)$$

Then, $\frac{n_\phi}{|G|}\sum_{g\in G}[\phi_g]_{ij}\left[\psi_g^\dagger\right]_{kl}$ is just an inner product of vectors in this space.

Further,

$$\begin{cases} 1 & \text{if } \psi=\phi,\ j=k,\ \text{and } i=l \\ 0 & \textit{else} \end{cases}$$

is the conclusion that these vectors are orthonormal to one another in choice of ϕ, i,and j.

We have a another very similar property that unitary irreducible representations meet:

Theorem 22

Let G be a finite group, let $\Phi = \{\phi \mid \phi$ is a unitary, irreducible representation of $G\}$, and let n_ϕ denote the degree of the representation ϕ. Then for any pair $g, h \in G$,

$$\sum_{\phi\in\Phi}\sum_{i=1}^{n_\phi}\sum_{j=1}^{n_\phi}\frac{n_\phi}{|G|}\left[\phi_g\right]_{ij}\left[\phi_h^\dagger\right]_{ji} = \begin{cases}1 & \text{if } g=h\\ 0 & \text{else}\end{cases}$$

This is referred to as the ***completeness condition*** *of unitary, irreducible representations.*

Here, we mean complete in the sense that our irreducible unitary decomposition is a complete characterization of a generic repesentation.

This is a consequence of the follwoing relationship we derive from the previous theorem:

Theorem 23

For any finite group, G, $\sum_{\phi\in\Phi} n_\phi^2 = |G|$ *where* Φ *is the set of a distinct, irreducible representations of G.*

Here, we see that for any finite group, there is a finite set of irreducible representations that we can decompose generic representations into.

Generalizing the properties of unitary representations to groups with infinite order requires more careful construction (Haar Measure).

Generalizing representation theory to infinite-dimensional vector spaces also requires care, but we can explictly approach this

Definition 24

A **matrix element** of a representation ϕ of a group G on a inner-product space V is a function defined in the following way:

$$f : G \to \mathbb{C}$$

$$g \mapsto \langle \phi(g)v, w \rangle$$

for some fixed $v, w \in V$.

When V is finite-dimensional, then the choice of standard basis vectors gets us matrix entry.

When we define representations on infinite-dimensional vector spaces, we will identify them by their matrix elements on orthonormal bases.

We further generalize our notion of representations:

Definition 25

A **multi-valued representation** is a multi-valued mapping of a group into $GL_n(\mathbb{C})$ which is a group homomorphism in the sense that at least one of the outputs (from each input) may be used to satisfy the homomorphism property.

2-valued representations will be of interest to us.

2 $SO(2)$: The Rotation Group in Two Dimensions

$SO(2)$ is a group whose elements correspond to the action of rotating vectors in two-dimensional space about some central point.

For our discussion, we will take vectors to be in $\mathbb{R}^2$, rotations occuring counterclockwise about the origin.

Angle measurement will be real-valued.

We will take $\{e_1, e_2\}$ to be the standard basis of $\mathbb{R}^2$.

If we let $(x, y) \in \mathbb{R}^2$ be rotated by θ into (x', y'), we can visualize it in the following way:

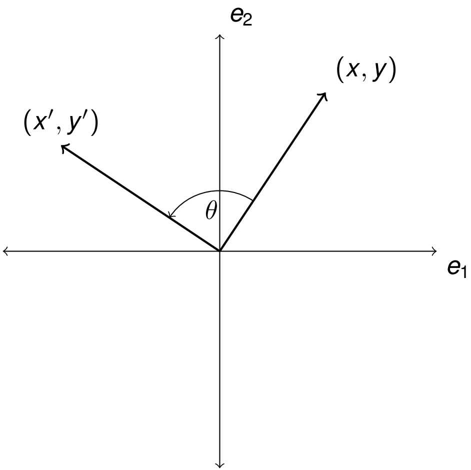


Figure: Rotation of a generic vector in $\mathbb{R}^2$ by angle θ

If we identify each point with its polar coordinates, we can make a direct calculation for (x', y'):

$$x' = x\cos(\theta) - y\sin(\theta) \tag{1}$$

$$y' = x\sin(\theta) + y\cos(\theta) \tag{2}$$

Or alternatively

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

If we take all matrices of this form, we can verify that they form a group under matrix multiplication.

We exercise care to ensure that we identify each rotation (modulo 2π) with its corresponding matrix to show equivalence. The resulting group is $SO(2)$.

We can discuss some properties of $SO(2)$ matrices. Generally, we reference these matrices as $R(\theta)$

Theorem 26

For any $R(\theta) \in SO(2)$, $R(\theta)$ is an orthogonal matrix.

$$R(\theta)R(\theta)^{\intercal} = I_n$$

Theorem 27

For any θ, $R(\theta)$ is a special matrix.

$$\det(R(\theta)) = 1$$

Hence, the group is entitled the **special orthogonal group (in two dimensions)**

$SO(2)$ is also a special type of group called a Lie group.

Definition 28

A **Lie Group** is a group that is a finite-dimensional smooth manifold in which the operations of multiplication and inversion are differentiable.

Intuitively, varying the parameter of θ varies group elements smoothly. We can use this principle to construct a "generator" of sorts.

If we consider what the rotation matrix would be for an arbitrarily small angle, $d\theta$, we can write

$$R(d\theta) := I_2 + (-i)d\theta J \tag{4}$$

where the factor of $-i$ is conventionally use.

$R(d\theta)$ differes from the identity matrix by a factor of $d\theta$.

This realization gives us two ways to calculate the instantaneous change after rotating by angle θ.

$$R(\theta + d\theta) = R(\theta) - i d\theta R(\theta) J \tag{5}$$

$$R(\theta + d\theta) = R(\theta) + \frac{d}{d\theta} R(\theta) d\theta \tag{6}$$

Putting both equations gives us a differential equation solved by

$$R(\theta) = e^{-i\theta J} = \sum_{n=0}^{\infty} \frac{(-i\theta J)^n}{n!} \tag{7}$$

We can explicitly solve our equation for J by taking (4) explictly

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + (-i)d\theta * \begin{bmatrix}j_{11} & j_{12}\\j_{21} & j_{22}\end{bmatrix} = R(d\theta) = \begin{bmatrix}\cos(d\theta) & -\sin(d\theta)\\\sin(d\theta) & \cos(d\theta)\end{bmatrix} = \begin{bmatrix}1 & -d\theta\\d\theta & 1\end{bmatrix} \tag{8}$$

To get

$$J = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \tag{9}$$

While J is not in the group, it still is referred to as the generator of $SO(2)$ due to (7).

Now that we have characterized the group $SO(2)$ we can find its irreducible representations.

$SO(2)$ is abelian, so all irreducible representations are degree one.

We look to the invariant subspaces of our generator for motivation.

If λ is an eigenvalue of J, then $e^{-i\theta\lambda}$ is an eigenvalue of $R(\theta)$.

$$R(\theta)(v) = \left(\sum_{i=0}^{\infty} \frac{(-i\theta J)^n}{n!}\right)(v) = \left(\sum_{i=0}^{\infty} \frac{(-i\theta\lambda)^n}{n!}\right) v = \left(e^{-i\theta\lambda}\right) v \tag{10}$$

We place a restriction on our eigenvalue possibilities in order to appeal to the physical constraint:

$$e^{-i2\pi n\lambda} = 1 \Rightarrow \lambda \in \mathbb{Z} \tag{11}$$

Making $\lambda \in \mathbb{Z}$.

Therefore, we can define irreducible representations for $SO(2)$ in the following way:

$$\begin{aligned} \phi_m : SO(2) &\to \mathbb{C} \\ R(\theta) &\mapsto e^{-i\theta m} \end{aligned}$$

These representations are also unitary:

$$\begin{aligned} \langle \phi_m(x), \phi_m(y) \rangle &= \langle e^{-i\theta m} x, e^{-i\theta m} y \rangle \\ &= e^{-i\theta m} \overline{e^{-i\theta m}} \langle x, y \rangle \\ &= \langle x, y \rangle \end{aligned} \tag{12}$$

3 $SO(3)$: The Rotation Group in Three Dimensions

Elements of the group $SO(3)$ represents all possible ways to take the physical action or rotating a vector in three-dimensional space about some central point.

For our discussion, we will take vectors to be in $\mathbb{R}^3$.

Angle measurement will be real-valued.

We will take $\{e_1, e_2, e_3\}$ to be the standard basis of $\mathbb{R}^3$.

There are two main ways to construct rotation matrices in $SO(3)$.

Method 1: Axis-Angle

Let $n \in \mathbb{R}^3$. We will treat n as an axis about which we rotate by angle ψ. n is completely characterized by spherical coordinate angles, θ and ϕ. We call denote rotation $R_n(\psi)$.

H

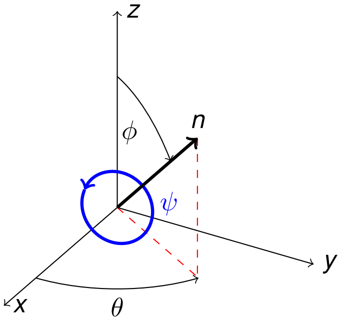


Figure: Generic rotation in $SO(3)$ characterized by $R_n(\psi)$

There is a redundancy in this construction illustrated below

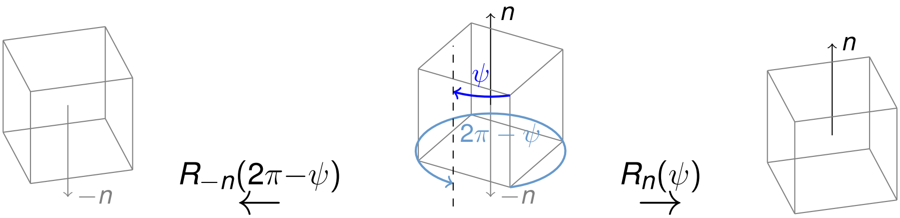


Figure: Identifying equivalent rotations about vectors pointing in opposite directions.

As a result, we restrict our parameters θ, ϕ, ψ to

$$0 \leq \psi \leq \pi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

Using this convention we can evaluate a geometric identity through the lens of $SO(3)$ matrices.

Theorem 29

Let $n, n' \in \mathbb{R}^3$ and $\psi \in [0, \pi]$. Let $R \in SO(3)$ be a rotation that takes n and points it in the direction of n'. Then, following identity holds:

$$R_n(\psi) = R^{-1} R_{n'}(\psi) R$$

We can assess the validity of this idea from the following successive images

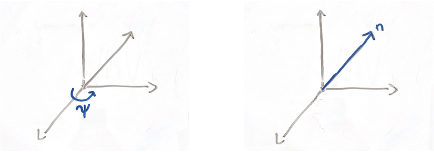


Figure: Theorem 29: $R_n(\psi)$

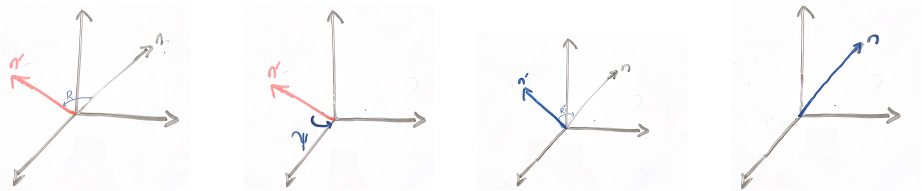


Figure: Theorem 29: $R^{-1}R_{n'}(\psi)R$

A direct consecuence of this is

Theorem 30

All rotations (about any axis) by a fixed angle ψ share a conjugacy class (denoted C_ψ).

We can now use our first method of classifying rotations in $SO(3)$ to help build a new way to classify them

Method 2: Euler Angles

Definition 31

Let (x, y, z) be the fixed frame and let (x', y', z') be any rotated frame. Then this rotation's **Euler Angles** decompose the rotation in the following way:

$$R(\alpha, \beta, \gamma) := R_{z'}(\gamma) R_n(\beta) R_z(\alpha)$$

where n points in the direction defined by the intersection of the xy and $x'y'$ planes (or equivalently, the $z \times z'$ direction). Note: we use the subscript z and z' as a shorthand to mean the z-axis and z'-axis.

We can see this idetification in the following successive images:

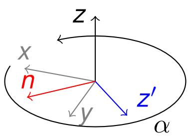


Figure: Euler Angles: Rotating Fixed Frame about z-axis by α

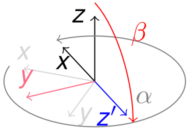


Figure: Euler Angles: Rotating Intermediate Frame about new y-axis by β

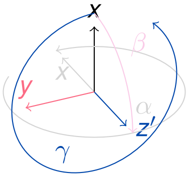


Figure: Euler Angles: Rotating Intermediate Frame about new z-axis by γ

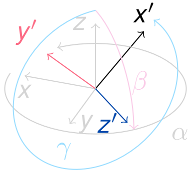


Figure: Euler Angles: Rotated Frame

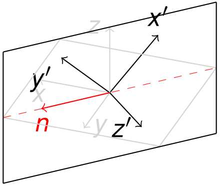


Figure: Euler Angles: Identifying n

These rotations illustrate the following identities

$$R_{z'}(\gamma) = R_n(\beta)R_z(\gamma)R_n(\beta)^{-1} \tag{13}$$

$$R_n(\beta) = R_z(\alpha)R_y(\beta)R_z(\alpha)^{-1} \tag{14}$$

Leading to the conclusion that every rotation in $SO(3)$ is characterized by:

$$R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) \tag{15}$$

We can use our knowledge of $\mathbb{R}^3$ to immediately build rotation matrices about the coordinate axes

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Putting it all together, a general rotation matrix, $R(\alpha, \beta, \gamma)$, takes in the following form:

$$\begin{bmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{bmatrix}$$

Now that we have characterized the matrices in this group, we want to search for generators.

We can find generators of rotations about a fixed axis in the same way as we did in $SO(2)$.

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad J_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 32

For any arbitrary direction, $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, the generator, J_n, can be written in the following way:

$$J_n = n_1 J_x + n_2 J_y + n_3 J_z$$

As a result, we can write any rotation in the following equivalent ways depending on our convention:

$$R_n(\psi) = e^{-i\psi n_1 J_x} e^{-i\psi n_2 J_y} e^{-i\psi n_3 J_z} \tag{16}$$

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{17}$$

In order to find invariant subspaces, we seek eigenvectors of all our generators.

General Strategy

Find a set of generators that commute, and therefore, have matching eigenvalues and matching eigenvectors.

Unfortunately, none of our generators commute.

Definition 33

The **commutator** of two matrices A and B is defined by to be

$$[A, B] = AB - BA$$

Theorem 34

Let $J_k, J_l \in \{J_x, J_y, J_z\}$. Consider the word xyz, and all 6 rearrangements of it (identifying each rearrangement with a permutation in S_3). Then

$$[J_k, J_l] = \begin{cases} 0 & \textit{if } k = l \\ i\,\mathrm{sign}(klm) J_m & \textit{else} \end{cases}$$

where m is the label of the remaining generator in $\{J_x, J_y, J_z\}$.

Definition 35

A **Lie Algebra** is a vector space, $\mathfrak{g}$, together with a bilinear operation, [. , .] that satisfies the the following identities:

- $[x, x] = 0 \; \forall x \in \mathfrak{g}$
- $[x, [y, z]] + [y, [z, x]] + [z[y, x]] = 0 \; \forall x, y, z \in \mathfrak{g}$

$$\mathfrak{so}(3) := \{a_1 J_x + a_2 J_y + a_3 Jz \mid a_1, a_2, a_3 \in \mathbb{C}\}$$

Does not solve our problem . . . yet.

Definition 36

The **universal enveloping algebra** of a Lie algebra is the largest embedding of Lie algebra into an algebra.

This space is constructed as a tensor space with the commutator relations of the Lie algebra quotiented out.

Definition 37

A **Casimir element** of a Lie algebra, A, is any element in the center of the universal enveloping algebra of A.

Our Casimir element is one of physical significane (angular-momentum operator)

$$\mathfrak{J}^2 := (\mathfrak{J}_x \otimes \mathfrak{J}_x) \oplus (\mathfrak{J}_y \otimes \mathfrak{J}_y) \oplus (\mathfrak{J}_z \otimes \mathfrak{J}_z) = \mathfrak{J}_x^2 + \mathfrak{J}_y^2 + \mathfrak{J}_z^2$$

Since $\mathfrak{J}^2$ commutes with the generators, so does its image through an irreducible representation.

Schur's Theorem:

$$\phi_{\mathfrak{J}^2} = \lambda I_n \tag{18}$$

Choose generators (by convention) $\{\mathfrak{J}^2, \mathfrak{J}_z\}$

Define raising and lowering generators:

$$\mathfrak{J}_\pm = \mathfrak{J}_x \pm i\mathfrak{J}_y \tag{19}$$

New commutator relationships:

$$[\mathfrak{J}_z, \mathfrak{J}_+] = \mathfrak{J}_+ \tag{20}$$

$$[\mathfrak{J}_z, \mathfrak{J}_-] = -\mathfrak{J}_- \tag{21}$$

$$[\mathfrak{J}_+, \mathfrak{J}_-] = 2\mathfrak{J}_z \tag{22}$$

$$\mathfrak{J}^2 = (\mathfrak{J}_z)^2 - \mathfrak{J}_z + \mathfrak{J}_+\mathfrak{J}_- = (\mathfrak{J}_z)^2 + \mathfrak{J}_z + \mathfrak{J}_-\mathfrak{J}_+ \tag{23}$$

For any eigenvector v_m of $\phi_{\mathfrak{J}_z}$, corresponging to eigenvalue m.

$$\phi_{\mathfrak{J}_z \mathfrak{J}_+}(v_m) = (m+1)\phi_{\mathfrak{J}_+}(v_m) \tag{24}$$

$$\phi_{\mathfrak{J}_z \mathfrak{J}_-}(v_m) = (m-1)\phi_{\mathfrak{J}_-}(v_m) \tag{25}$$

We can construct an eigenbasis of our vectorspace by repeated applications of the raising and lowering generators.

This process will eventually terminate for finite degree representation.

If k is the last nonzero power of $\phi_{\mathfrak{J}_+}$ to v_0, and s is the eigenvalue of $\phi_{\mathfrak{J}_+} v_{k-1}$ to the map $\phi_{\mathfrak{J}_z}$, then

$$\phi_{\mathfrak{J}^2} \phi_{\mathfrak{J}_+}^k (v_{k-1}) = s(s+1)\phi_{J_+}^k v_{k-1} \tag{26}$$

Theorem 38

The irreducible representations of the $A_{\mathfrak{so}(3)}$ are characterized by eigenvalues that take on positive integer and positive half-integers. If λ is one such eigenvalue, then we can construct our eigenvectors, $\{v_m\}_{m=-\lambda}^{\lambda}$, corresponding to $\phi_{\mathfrak{J}_z}$ with eigenvalue m, using the raising or lowering operators. This gives us a degree $2s+1$ representation characterized by the following relationships:

$$\phi_{\mathfrak{J}^2}(v_m) = s(s+1)v_m$$

$$\phi_{\mathfrak{J}_z}(v_m) = mv_m$$

$$v_{m+1} = \frac{1}{\sqrt{\lambda(\lambda+1) - m(m \pm 1)}}\phi_{\mathfrak{J}_\pm}(v_m)$$

$$\phi_j(R(\alpha,\beta,\gamma)) = e^{-i\alpha\phi_{\mathfrak{J}_z}} e^{-i\beta\phi_{\mathfrak{J}_y}} e^{-i\gamma\phi_{\mathfrak{J}_z}} \tag{27}$$

4 Euclidean Groups

Definition 39

The n-dimensional **Euclidean Group, E_n,** is the group of all continuous, isometric, linear transformations on $\mathbb{R}^n$.

For any transformation of this kind, all vectors, $x \in \mathbb{R}^n$, get mapped to $x' \in \mathbb{R}^n$ in the following way

$$x' = Rx + b$$

for some fixed $R \in SO(n)$ and $b \in \mathbb{R}^n$.

If we focus on E_2, we can construct an invertible matrix to represent each transformation

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & b_1 \\ \sin(\theta) & \cos(\theta) & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \tag{28}$$

We can derive the generators of E_2 in the same way as we did for $SO(2)$ and $SO(3)$.

Our rotational generator is the same as in $SO(2)$

$$J = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

Further, we can use the same process to derive the generators for translations in two directions.

If we denote T_n to be the group of translations in n-dimensions.

Starting with T_1, we let $T_b \in T_1$ where $b \in \mathbb{R}$

$$T_{dx} = I - idxP \tag{30}$$

where P is our generator.

We construct our system of equations:

$$T_{x+dx} = T_x + dx\frac{d}{dx}T_x \tag{31}$$

$$T_{x+dx} = T_{dx}T_x \tag{32}$$

We conclude with

$$T(x) = e^{-iPx} \tag{33}$$

As we generalize to E_2,

$$P_x = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

and therefore for any $b \in \mathbb{R}^2$,

$$T_b = e^{-ib_1 P_x} e^{-ib_2 P_y} \tag{35}$$

Some useful properties of E_2 include

Theorem 40

For any θ,b

$$R(\theta)T_bR(\theta)^{-1} = T_{R(\theta)b}$$

Theorem 41

For any $g(\theta, b) \in E_2$, there is a natural decomposition of the transformation into a pure rotation times a pure translation.

$$g(\theta, b) = g(0, b)g(\theta, 0)$$

When discussing the search for irreducible representations, we follow the same path.

Our commutator relationships:

$$[P_x, P_y] = 0 \tag{36}$$

$$[J, P_x] = iP_y \tag{37}$$

$$[J, P_y] = -iP_x \tag{38}$$

Our raising and lowering generators:

$$P_\pm := P_x \pm iP_y \tag{39}$$

And journeying into the universal enveloping algebra, the Casimir element:

$$\mathfrak{P}^2 = (\mathfrak{P}_x)^2 + (\mathfrak{P}_y)^2 = \mathfrak{P}_+\mathfrak{P}_- = \mathfrak{P}_-\mathfrak{P}_+ \tag{40}$$

If we let ψ be our irreducible representation of our universal enveloping algebra,

$$\psi(\mathfrak{P}^2)v_m = pv_m, \quad p > 0 \tag{41}$$

$$\psi(\mathfrak{J})v_m = mv_m, \quad m \in \mathbb{Z} \tag{42}$$

Unfortunately, there is no termination in the construction of our basis, so we must resort to characterizing our representations by discussing their matrix elements.

Theorem 42

The faithful, unitary, irreducible representations of E_2 are characterized by the following equation defined in terms of matrix elements. If ψ_p is a representation corresponding to a $p > 0$, then $[\psi_p(g)]_{mm'}$ corresponds to the matrix element specified by the choice of v_m, $v_{m'}$, and g. The representations, ψ_p have the following form:

$$[\psi_p(g(\theta, b))]_{mm'} = e^{i(m-m')\phi} J_{m-m'}(pr) e^{-im\theta}$$

where r and ϕ are the polar coordinates for the vector b.

Regarding E_3, we can decompose any transformation as

$$x' = \begin{bmatrix} \underset{3\times 3}{R(\alpha,\beta,\gamma)} & \underset{3\times 1}{b} \\ \underset{1\times 3}{0} & 1 \end{bmatrix} x \tag{43}$$

Our generators $\{J_x, J_y, J_z, P_x, P_y, P_z\}$ as defined in $SO(3)$ and implicitly T_3 satisfy

$$T_b = e^{-ib_1 P_x} e^{-ib_2 P_y} e^{-ib_3 P_z} \tag{44}$$

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{45}$$

Our commutator relations:

$$[P_k, P_l] = 0 \quad \forall k, l \in \{x, y, z\} \tag{46}$$

$$[J_k, J_l] = \begin{cases} 0 & \text{if } k = l \\ isign(klm) J_m & else \end{cases} \tag{47}$$

where m is the remaining label for the unused generator in the commutator.

$$[P_k, J_l] = \begin{cases} 0 & \text{if } k = l \\ isign(klm) P_m & else \end{cases} \tag{48}$$

Useful Identities:

Theorem 43

For any $R \in SO(3)$, $T_b \in T_3$,

$$RT_bR^{-1} = T_{b'}$$

Theorem 44

For any $g \in E_3$,

$$g = R(\alpha, \beta, \gamma)T_b \text{ for some } b \in \mathbb{R}^3, R \in SO(3)$$

In the universal enveloping algebra, we actually have two Casimir operators:

$$\mathfrak{P}^2 := (\mathfrak{P}_x)^2 + (\mathfrak{P}_y)^2 + (\mathfrak{P}_z)^2 \tag{49}$$

$$\mathfrak{J} * \mathfrak{P} := \mathfrak{J}_x\mathfrak{P}_x + \mathfrak{J}_y\mathfrak{P}_y + \mathfrak{J}_z\mathfrak{P}_z \tag{50}$$

Constructing our irreducible representations as we have done previously will result in a trivial case. Eigenvalues are no longer going to help us.

Definition 45

Let G be a group, N be a normal subgroup, ϕ be a representation of G, and V be the underlying vector space of the representation. Then for any $v \in V$, the **little group** of v is the subgroup of G/N who's image under ϕ leave v invariant.

Strategy

- Find a nice normal subgroup, quotient out (T_3)
- Pick a nonzero starting vector
- Find representations of the little group of said vector
- Transform vector to recover remaining vectors, repeat process for general representation
- We are left with the smallest invariant subspace (generated by one vector)

Let our starting choice in this case be $p_0 = \mu_z e_z$.

The only rotations that leave p_0 unaffected are about the $z - axis$. The little group of p_0 is isomorphic to $SO(2)$.

The irreducible representations of $SO(2)$ are indexed by $\lambda \in \mathbb{Z}$.

$$\psi_{\mathfrak{J}_z} p_0 = \lambda p_0 \tag{51}$$

Further, we have the following relationships between our other generators:

$$\psi_{\mathfrak{P}_x} p_0 = \psi_{\mathfrak{P}_y} p_0 = 0 \tag{52}$$

$$\psi_{\mathfrak{P}_z} p_0 = \mu_z p_0 \tag{53}$$

$$\psi_{\mathfrak{J}\mathfrak{P}} p_0 = \mu_z \lambda p_0 \tag{54}$$

$$\psi_{\mathfrak{P}^2} p_0 = \mu_z^2 p_0 \tag{55}$$

We then conclude that

$$\psi(R_z(\theta))p_0 = e^{-i\lambda\theta}p_0 \tag{56}$$

$$\psi(T_b)p_0 = e^{-i\mu b_3}p_0 \tag{57}$$

Now that we know the value our representation can take on this vector, let us rotate this vector p_0 by the rotation $R(\alpha, \beta, 0)$ to p.

We can repeat this process, finding the eigenvalues corresponding to our generators.

Theorem 46

The irreducible unitary representations are characterized by $\lambda \in \mathbb{Z}$ and the following equations:

$$\psi(T_b)p = e^{-ib_1\mu_x} e^{-ib_2\mu_y} e^{-ib_3\mu_z} p \tag{58}$$

$$\psi(R(\alpha, \beta, \gamma))p = e^{-i\lambda\xi} p' \tag{59}$$

where $p' = R(\alpha, \beta, \gamma)p$, the cylindrical angles of p' are θ' and ϕ', and ξ is the angle ascertained from the equation $R(0, 0, \xi) = R(\phi', \theta', 0)^{-1} R(\alpha, \beta, \gamma) R(\phi, \theta, 0)$.

5 Lorentz and Poincaré Groups

The coming spaces we will study are non-Euclidean

Definition 47

An **event** is an ordered triple together with an additional parameter, meant to represent time. An event is conventionally indexed by the integers $\mu = 0, 1, 2, 3$. Referring to an event as x references the event a four (component) vector, and specifying $x^{\mu=0}$ gives us ct where c is the speed of light and t is a time.

Definition 48

The **length** of an event is defined by the following equation:

$$\mid x \mid^2 := (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 - c^2t^2$$

$$\underbrace{||x||^2}_{\text{Euclidean Norm}} = x^\mathsf{T} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \qquad \underbrace{|x|^2}_{\text{Length}} = x^\mathsf{T} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

Definition 49

A **Homogeneous Lorentz Transformation** is a continuous, length-preserving, linear transformation defined on our space-time vector space. By convention, we refer to such maps with the variable Λ. A **Proper Homogeneous Lorentz Transformation** has a positive entry in the 00^{th} component of its matrix.

Some Proper Homogeneous Lorentz Transformation examples:

Let $R \in SO(3)$, then one example could be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & [R]_{11} & [R]_{12} & [R]_{13} \\ 0 & [R]_{21} & [R]_{22} & [R]_{23} \\ 0 & [R]_{31} & [R]_{32} & [R]_{33} \end{bmatrix} \tag{60}$$

Another kind is referred to as a Lorentz Boost

$$\Lambda_x = \begin{bmatrix} \cosh(\xi) & \sinh(\xi) & 0 & 0 \\ \sinh(\xi) & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{61}$$

Definition 50

The set of all proper, homogeneous, Lorentz transformations forms the **Lorentz Group** (under matrix multiplication). This group is denoted $\tilde{L}_+$.

Definition 51

The four dimensional vector space on which length of events (elements of the space) is defined as in Definition (5.2) is referred to as **Minkowski Space**. Elements of the Minkowski Space are defined to transform under Lorentz transformations in the way we expect vectors in $\mathbb{R}^4$ to be transformed under matrices in $GL_4(\mathbb{R})$. We call the elements of the Minkowski Space four-vectors or Lorentz-vectors. We will denote the Minkowski Space by M.

We can define a linear functional on this space that acts like an inner product on this space

$$x\dot{y} := -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 \tag{62}$$

The equation $|x|^2 = ||(x^1, x^2, x^3)||^2 - c^2t^2 = 0$ decomposes Minkowski Space in the following way:

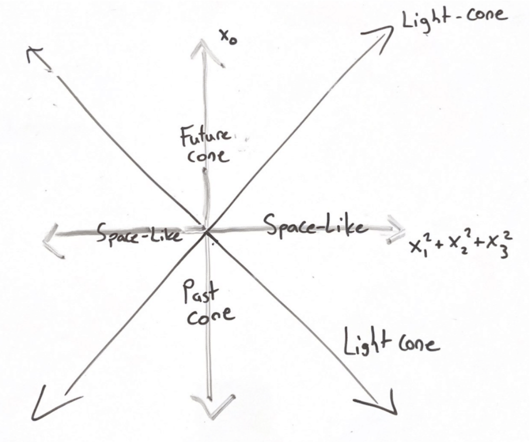


Figure: Decomposition of Minkowski Space based on length

Useful Identity in $\widetilde{L}_+$:

Theorem 52

Any $\Lambda \in \widetilde{L}_+$ can be uniquely written in the following way:

$$\Lambda = R(\theta, \phi, 0)L_z(\xi)R(\alpha, \beta, \gamma)^{-1}$$

where $L_z(\xi)$ is a Lorentz boost along the z-axis by angle ξ and the parameters for the Euler angles of the rotations from $SO(3)$ are defined as expected.

Putting the Lorentz group together with T_4 gives us a new structure to study:

Definition 53

The set of all compositions of translations proper homogeneous Lorentz transformations is called the **Poincaré Group**, denoted $\widetilde{P}$

All transformations in the Poincaré group can be written as

$$\begin{bmatrix} \underset{4\times 4}{\Lambda} & \underset{4\times 1}{b} \\ \underset{1\times 4}{0} & \underset{1\times 1}{1} \end{bmatrix} \tag{63}$$

Some Useful Identities:

Theorem 54

For any $g(\Lambda, b) \in \widetilde{P}$

$$g(\Lambda, b) = T_b \Lambda \tag{64}$$

Theorem 55

For any $\Lambda \in \widetilde{L}_+$ and $T_b \in T_4$,

$$\Lambda T_b \Lambda^{-1} = T_{\Lambda b} \tag{65}$$

In the pursuit of generators, we find the natural choices

$$P_t = \begin{bmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad J_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad J_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And where K_μ is the generator for the Lorentz Boost along the μ-axis.

$$K_x = \begin{bmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad K_y = \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad K_z = \begin{bmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Lambda(\omega) = e^{-i\omega_1 J_x} e^{-i\omega_2 J_y} e^{-i\omega_3 J_x} e^{-i\omega_4 K_x} e^{-i\omega_5 K_y} e^{-i\omega_6 K_z} \tag{66}$$

Too many commutator relationships to be concise.

We define one Casimir operator to be as excpected

$$\mathfrak{C}_1 := \mathfrak{P}_t^2 - \mathfrak{P}_x^2 - \mathfrak{P}_y^2 - \mathfrak{P}_z^2 \tag{67}$$

If ψ is an irreducible representation, then we want to treat the eigenvalue of $\psi_{\mathfrak{C}_1}$ by cases in order to calculate every unitary irreducible representaion. If the eigenvalue is λ, we will calculate the representations in the case when $\lambda > 0$.

In this case, vectors must be time-like.

Choosing starting vector $v_0 : (\sqrt(\lambda), 0, 0, 0)$. Our quotient group will be $\widetilde{P}/T_4$. Little group: $SO(3)$.

Representations are indicated by s, a half or whole integer (from $SO(3)$). Explicitly,

$$\psi_{\mathfrak{J}^2}(v_0)_m = s(s+1)(v_0)_m \tag{68}$$

$$\psi_{\mathfrak{J}_z}(v_0)_m = m(v_0)_m \tag{69}$$

$$\psi_{\mathfrak{P}_\mu}(v_0)_m = \begin{cases} \sqrt{\lambda_1} & \text{if } \mu = t \\ 0 & \text{else} \end{cases} \tag{70}$$

where α, β are the cylindrical angles of the

We transform this starting vector with Lorentz transformations:

$$v_p := H_p((v_0)_m) = R(\alpha, \beta, 0)L_z(\xi)(v_0)_m \tag{71}$$

where α, β are the cylindrical angles of the vector v_p.

Theorem 56

Time-like Unitary Irreducible Representations of $\widetilde{P}$ The subspace generated by the vectors $\{(v_p)_m\}_{p,m}$ is invariant under $\widetilde{P}$. The irreducible, unitary representations of elements of this group are characterized in the following way:

$$\psi_{T_b}(v_p)_m = e^{-ib_0p_0}e^{-ib_1p_1}e^{-ib_2p_2}e^{-ib_3p_3}(v_p)_m$$

$$\psi_{\Lambda}(v_p)_m = [\phi_s(H(p')\Lambda H(p))]_{m'm}(v_{p'})_{m'}$$

where $p' = \Lambda p$ and ϕ_s is the irreducible representation of $SO(3)$ defined by s.

6 Braid Group and Anyons

Definition 57

The braid group, B_n, is generated by $n-1$ generators (denoted $\sigma_1, \ldots, \sigma_{n-1}$) that have the following "braid relations":

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \forall i, j \in \{1, \ldots, n-1\} \text{ where } |i-j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall i \in \{1, \ldots, n-2\}$$

Definition 58

A **geometric braid** on $n \in \mathbb{N}$ strands is a set $b \subset \mathbb{R}^2 \times [0,1]$ formed by n disjoint, intervals topologically equivalent to $[0,1]$ such that we can define a projection mapping from $\mathbb{R}^2 \times [0,1]$ to $[0,1]$ that maps each strand homeomorphically to $[0,1]$ and the following conditions hold:

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$$

$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), \ldots, (n,0,1)\}$$

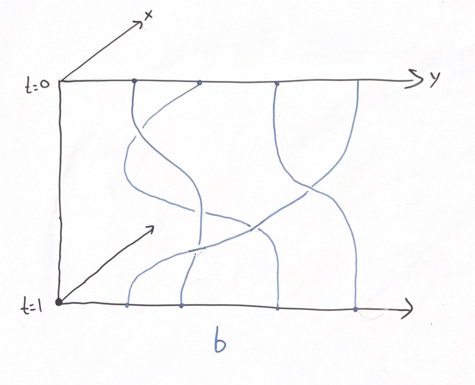


Figure: Generic geometric braid

Definition 59

Two braids, b_1 and b_2, are said to be **isotopic** to each other if one can be continuously deformed into the other. This is an equivalence relation.

$$b_1 b_2 := \{(x, y, t) \mid (x, y, 2t) \in b_1 \text{ when } 0 \leq t \leq \frac{1}{2} \text{ and } (x, y, 2t - 1) \in b_2 \text{ when } \frac{1}{2} \leq t \leq 1\} \tag{72}$$

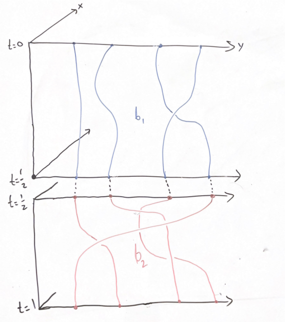


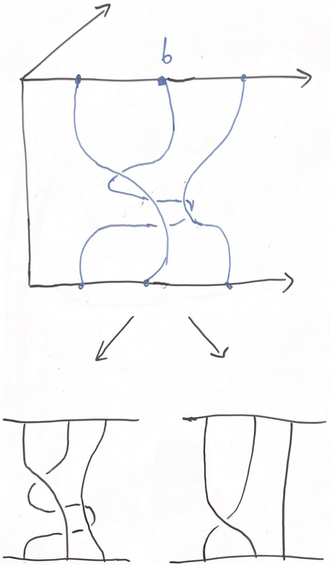
Figure: The product of two geometric braids.

Definition 60

A **braid diagram** on n strands is set, $\mathcal{D} \subset \mathbb{R} \times [0, 1]$ made up as a union of n intervals topologically equivalent to $[0, 1]$ (called strands) such that the following conditions are met:

- There exists a projection map from $\mathbb{R} \times [0, 1]$ to $[0, 1]$ that maps each strand homeomorphically to $[0, 1]$.
- Every element of $\{1, 2, \ldots, n\} \times \{0, 1\}$ is a starting or endpoint of a unique strand.
- Every element in $\mathcal{D}$ belongs to either one or two strands. When an elment belongs to two, one strand must be designated as overgoing and the other undergoing (referred to as a crossing of $\mathcal{D}$

While it is definitely true that every geometric braid has a braid diagram, the identification is not unique.


b

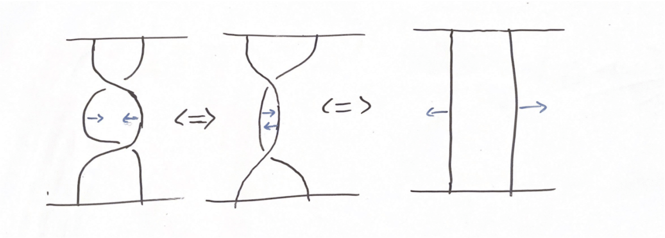


Figure: Ω_1^{-1}

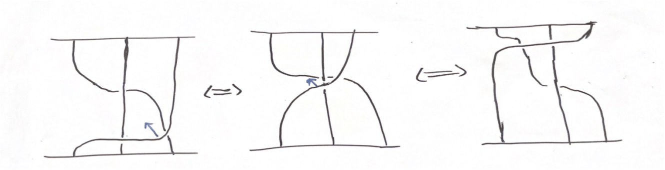


Figure: Θ

Definition 61

Two braid diagrams are said to be **R-equivalent** if one can be transformed into the other by means of a finite sequence of isotopies and Reidemeister moves. This is an equivalence relation.

Braids can be decomposed in the following way:

Theorem 62

Let $\mathcal{B}_n$ be the group of geometric braids on n strands. Then for any $\beta \in \mathcal{B}_n$, β has a natural decomposition:

$$\beta = \sigma_{i_1} \dots \sigma_{i_k}$$

where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n-1\}$.

Now that we have characterized the group, we can discuss representations. We consider the ring of matrices over the ring $\mathbb{Z}[t, t^{-1}]$

For $i \in \{1, \ldots, n-1\}, n > 1$, define

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} \tag{73}$$

Definition 63

The **Burau Representation** of B_n is the map

$$\phi_n : B_n \to M_n(\mathbb{Z}[t, t^{-1}])$$

$$\sigma_i \mapsto U_i$$

If we compose this map with a ring homomorphism on $\mathbb{Z}[t, t^{-1}]$ that evaluates elements at some complex number with magnitude one, the result is a unitary representation.

Let us shift our discussion towards anyonic systems. In quantum mechanics, we study wave functions.

$$\int_\Omega |\psi|^2 d\mu = 1 \tag{74}$$

Wave function encodes information depending on inputs. For our purposes, inputs are positions.

Suppose we have a two particle system where particles are indistinguishable.

After one particle exchange:

$$\psi(r_1, r_2) = e^{i\theta}\psi(r_2, r_1) \quad \text{for some } \theta \in [0, 2\pi) \tag{75}$$

After two particle exchanges:

$$\psi(r_1, r_2) = e^{2i\theta}\psi(r_1, r_2) \quad \text{for some } \theta \in [0, 2\pi) \textit{ a.e.} \tag{76}$$

This means $\theta = 0, \pi$: Bosons or Fermions.

However, in two dimensions, it is not always the case that returning back to starting position returns our value. Rather, the anyons begin to braid. This is more clearly seen with n anyons.

We say that the particles obey θ-statistics if the wave function picks up a factor of $e^{i\theta}$ upon exchange. Anyons obey θ-statistics where $\theta \neq 0, \pi$.

Using this concept, we can build a degree one representations of B_n.

Representation

$$\phi_\theta : B_n \to \mathbb{C}$$
$$\beta \mapsto e^{i\theta}$$

If $\beta \in B_n$ with decomposition $\beta = \sigma_{i_1}^{m_1} \dots \sigma_{i_k}^{m_k}$ for some $i_1, \dots, i_k \in \{1, \dots, n-1\}$, $m_j \in \mathbb{N}$, and if σ is the underlying permutation of β, then

$$\psi(\sigma(r_1), \sigma(r_2), \dots, \sigma(r_n)) = e^{i\theta(m_1+\dots+m_k)}\psi(r_1, r_2, \dots, r_m) \tag{77}$$

Since the image of this homomorphism is an abelian group, we say these anyons are abelian.

Another representation can be defined in the following way:

Theorem 64

Let $V := \bigotimes_{i=1}^{m} \psi$ where $m > 1$ and let $R : V \otimes V \to V \otimes V$ be an invertible, linear operator. Then the following map is a representation of B_n

$$\phi_R : B_n \to GL_m(\mathbb{C})$$

$$\sigma_i \mapsto M(U_i)$$

where $M(U_i)$ is the matrix of the operator U_i defined by

$$U_i : V \to V$$

$$(v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_m) \mapsto (v_1 \otimes \ldots \otimes R(v_i \otimes v_{i+1}) \otimes \ldots \otimes v_m)$$

$$M(U_i) = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & M(R) & 0 \\ 0 & 0 & I_{m-i-1} \end{bmatrix} \tag{78}$$

Question

What happens if anyons obey different statistics?

We continue our work in the context of monoidal category theory

Definition 65

A monoidal category is a category, C, equipped with the following structure:

- a bifunctor $\otimes : C \times C \to C$
- an object e which acts as the identity object
- three natural isomorphisms defined in the following way:
  - The associator, α, whose components are $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$
  - The left unitor, λ, whose components are $\lambda_a : e \otimes a \cong a$
  - The right unitor, ρ, whose components are $\rho_a : a \otimes e \cong a$

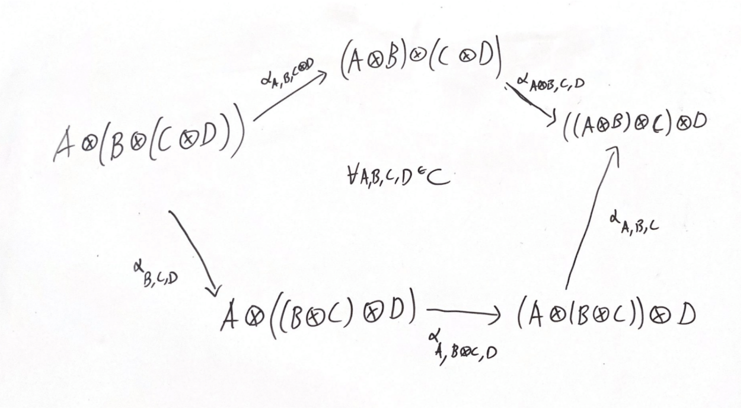


Figure: Commutative Pentagon Diagram for Monoidal Categories

Here, the tensor product corresponds to the fusion of two anyons. We will be characterizing the Fibonacci anyonic system.

If e is the identity anyon type (vacuum), and τ is a nontrivial anyon type, then our system is characterized by this rule:

$$\tau \otimes \tau = 1 \oplus \tau \tag{79}$$

We say that τ has multi-fusion channels. We represent all possible fushion paths in tree diagrams in the following way:

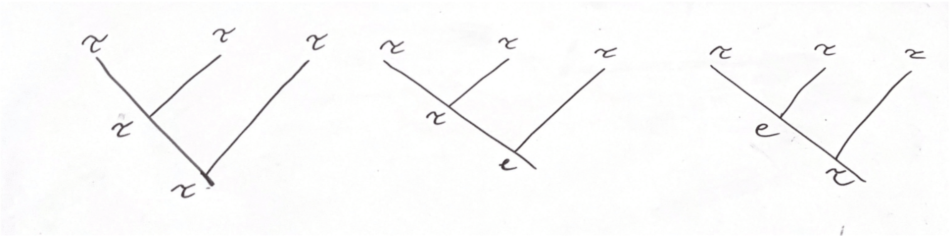


Figure: These tree diagrams are labelled according to the fusions rules of Fibonacci anyons.

The Fibonacci sequence counts the number of fushion paths that can be taken. These fusion paths represent an orthonormal basis of some underlying vector space.

There is nothing to stop fushion from occuring in any order.

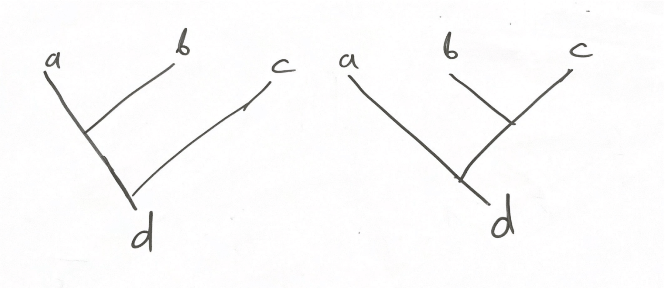


Figure: Both tree diagrams represent a different fusion order on the same anyon types.

Transforming from one basis to another should be a linear, unitary operation, and as such can be depicted in a matrix. We call these transformations F-moves. Denote F_d^{abc} as the F-move that moves $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$ over total charge d.

We use the commutative diagram to construct systems of equations to solve for each F explicitly

F_d^{abc} is the identity matrix if any the fusion associates any identity particles. Therefore, we must solve for $F_\tau^{\tau\tau\tau}$.

$$[F_\tau^{\tau\tau\tau}]_{11} = [F_\tau^{\tau\tau\tau}]_{12}[F_\tau^{\tau\tau\tau}]_{21} \tag{80}$$

$$F_\tau^{\tau\tau\tau} = \begin{bmatrix} \frac{1}{\Phi} & \frac{1}{\sqrt{\Phi}} \\ \frac{1}{\sqrt{\Phi}} & -\frac{1}{\Phi} \end{bmatrix} \tag{81}$$

where Φ is the golden ratio.

Choosing to reintroduce braiding to the set up gives us the ability to embed ourselves in another kind of category: a Braiding Monoidal Category

Here, braiding and fusion have to behave well with one another

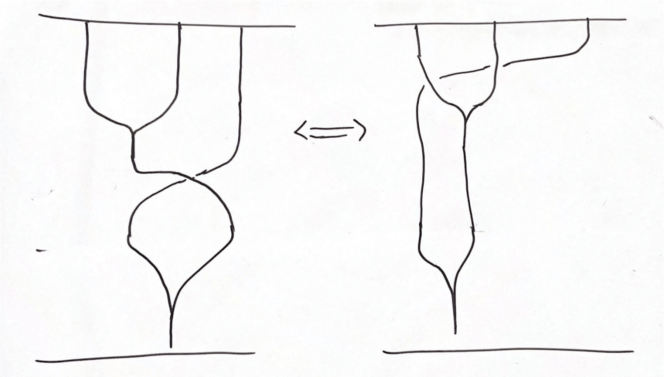


Figure: Compatibility of braiding and fusion.

Here, we refer to individual anyon exchanges as R-moves and these can be encoded into matrices due to their basis transforming abilities on our fusion paths. We take the convention that $R_c^{a,b}$ swaps a and b with total charge c fused. We solve a similar system of equations to find explicit R matrices

$$(R_e^{\tau,\tau})^2 \frac{1}{\Phi} = R_\tau^{\tau,\tau} \frac{1}{\Phi} + \frac{1}{\Phi^2} \tag{82}$$

$$R_e^{\tau,\tau} R_\tau^{\tau,\tau} \frac{1}{\sqrt{\Phi}} = (1 - R_\tau^{\tau,\tau}) \frac{1}{\Phi^{\frac{3}{2}}} \tag{83}$$

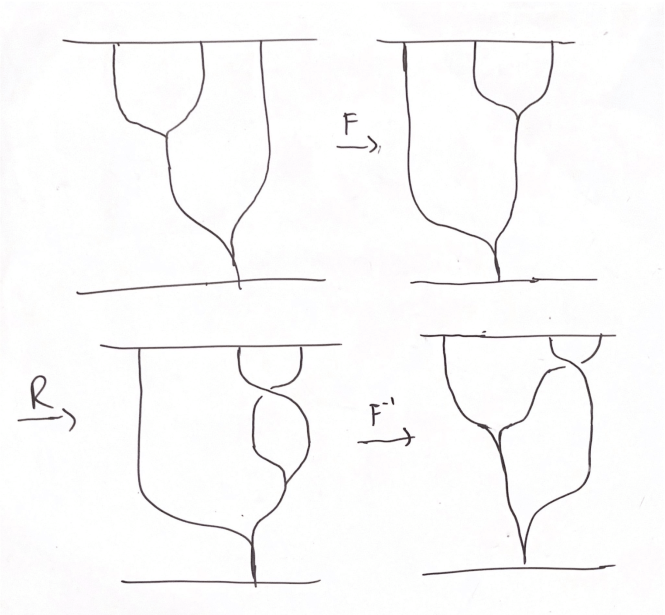
$$-(R_e^{\tau,\tau})^2 \frac{1}{\Phi} = R_\tau^{\tau,\tau} \frac{1}{\Phi^2} + \frac{1}{\Phi} \tag{84}$$

It is difficult to find solutions to these equations in general. If we make the assumption that our state space is one dimensional, these matrices should be one dimensional with values $R_e^{\tau\tau} = e^{i\frac{4\pi}{5}}$ and $R_\tau^{\tau\tau} = e^{i\frac{-3\pi}{5}}$.

In order to encode a true braiding effect in this set up, we need to ensure that the braid takes into account fusions. Braiding matrices take the following form:

$$B = F^{-1}RF \tag{85}$$

We need more specificity to the scenario to calculate a specific solution for this set up.


F
R
F⁻¹

Thank You