Applications of Representation Theory to Braids and Anyonic Systems

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Outline

1 Introduction to Representation Theory

Question

How does one study a mathematical structure? (Sets, Groups, Vector Spaces, etc.)

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Answer

By studying its maps!

Maps help us to identify unknown structures with understood ones.

Example 1

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In this thesis, we look to understand the behavior of groups through the lens of a special kind of map.

Definition 3

Let G be a group. A **representation** of degree n is a group homomorphism that maps a G into $GL_n(\mathbb{C})$.

$$\phi : G \to GL_n(\mathbb{C})$$

We say that ϕ is a representation of G.

If ϕ is an injective homomorphism, we say that the representation is **faithful**. Otherwise, the representation is called **degenerate**.

Question

What is so special about representations?

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Answer

Matrices are some of the most well-understood strucutres in all of mathematics. We can use linear algebra to gain insight into group properties.

Example 4

Let G be the group of all roots of unity (under multiplication). Explicitly, $G = \{e^{\frac{2\pi im}{n}} \mid m, n \in \mathbb{Z}\}$. One representation can be created by

$$\phi : G \to GL_2(\mathbb{C})$$

$$e^{2\pi i \frac{m}{n}} \mapsto \begin{bmatrix} \cos(\frac{2\pi m}{n}) & \sin(\frac{2\pi m}{n}) \\ -\sin(\frac{2\pi m}{n}) & \cos(\frac{2\pi m}{n}) \end{bmatrix}$$

This representation is degree 2 and is degenerate.

There is another equivalent definition for representations:

Definition 5

Let G be a group, V be an n-dimensional vector space, and let $\mathcal{L}(V)$ be the group of linear operators on V under composition. A **representation** of G of degree n is a group homomorphism that maps the G into $\mathcal{L}(V)$

$$\phi : G \to \mathcal{L}(V)$$

We identify matrices in $GL_n(\mathbb{C})$ with matrices of operators in $\mathcal{L}(V)$.

We will switch between definitions when useful

Example 6

Let G be the complex unit circle ($G = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$) and the operation of multiplication and let $V = \mathbb{C}$. Then,

$$\phi : G \to \mathcal{L}(V)$$

$$e^{i\theta} \mapsto U_{e^{i\theta}}$$

where $U_{e^{i\theta}} : \mathbb{C} \to \mathbb{C}$ such that $U_{e^{i\theta}}(re^{i\psi}) = e^{i\theta} re^{i\psi} \; \forall re^{i\psi} \in G$

Depending on V, this is either a one or two degree representation. It is faithful.

Question

When are two representations representing the same information?

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Definition 7

Two representations, ϕ and ψ, are said to be **equivalent representations** if there exists some invertible operator/matrix (depending on definition of representation), M, such that

$$\phi = M\psi M^{-1}$$

Representation equivalence (conjugation) is an equivalence relation.

This is directly related to similarity transformations and matrix canonical forms.

We can detect matrix similarity with the following operation:

Definition 8

The **character** of a representation, ϕ, on $g \in G$, denoted $\chi^{\phi}(g)$, is defined by

$$\chi^{\phi}(g) = \mathit{trace}(\phi(g))$$

The trace operation satisfies $\mathit{trace}(AB) = \mathit{trace}(BA)$ for any two matrices.

Theorem 9

If two representations are equivalent, then character of both representations are the same.

Much like in the study of linear algebra, we will explore the decomposition of representations into its invariant components

Definition 10

Let ϕ be a representation of the group G and $\phi(G) := \{\phi(g) = \phi_g : V \to V \mid g \in G\}$. Let $W \subset V$. W is said to be an **invariant subspace** with respect to $\phi(G)$ if $\forall v \in W$ and $\forall g \in G$, $\phi_g(v) \in W$.

Definition 11

A representation is said to be **irreducible** if there is no nontrivial, invariant subspace with respect to it. Otherwise, we say that the representation is reducible.

We look to the the eigendecomposition of matrices of a representation to find such subspaces.

In order to be more explicit in our discussion, we introduce new notation:

Definition 12

If M and N are square matrices, then let

$$M \oplus N := \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

we call this new matrix the **direct sum** of M and N.

Example 13

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Following our natrual inclination, we can use our notion of block-diagonal matrix decompositions to explicitly characterize representations.

Theorem 14

Let ϕ be a representation of a group G. Then there exists a set of irreducible representations of G, $\{\psi_i\}_{i=1}^{j}$, such that

$$\phi = T\left(\bigoplus_{i=1}^{j} \alpha_i * \psi_i\right) T^{-1}$$

where $\alpha_i * \psi_i := \underbrace{\psi_i \oplus \psi_i \oplus \ldots \oplus \psi_i}_{\alpha_i \text{times}}$ and T is some invertible matrix/operator.

We prove this theorem using induction on the degree of representation.

The results of this theorem show us that we can completely characterize any representation in terms of its irreducible components. With this in mind, we can characterize important properties of irreducible representations

Theorem 15

Schur's Theorem: *Let ϕ and ψ be two irreducible representations of the group G. Let M be a matrix defined such that $M\phi(g) = \psi(g)M \ \forall g \in G$. Then M is invertible or 0.*

Schur's Theorem tells us that irreducible representations are either completely unrelated or equivalent, much like eigenspaces.

Theorem 16

Let ϕ be an irreducible representation and M be a matrix/operator such that $M\phi(g) = \phi(g)M \ \forall g \in G$. Then M is a mulptiple of the identity matrix/map.

This identity will be incredibly useful for finding representations when we know that certain matrices will commute.

Theorem 17

If G is an abelian group, then any irreducible representation of G can be viewed as a degree one representation.

This identity trivializes the classification of abelian groups, giving their representations complete sets of eigenvalues.

Irreducible representations are not the only special kind of representation. When V is an inner-product space, we have more structure.

Definition 18

Let V be an inner-product space. Let $U \in \mathcal{L}(V)$. U is said to be **unitary** if U is surjective and $\forall x, y \in V$, $\langle x, y \rangle = \langle U(x), U(y) \rangle$.

Definition 19

A representation, ϕ, of a group, G, is said to be a **unitary representation** if $\forall g \in G$, $\phi(g)$ is unitary.

Theorem 20

Every representation of a finite group on an inner-product space is equivalent to a unitary representation

The above theorem gives us the utility of being able to tranform any representation into its unitary form. We can make our irreducible representations unitary as well.

When we consider the unitary irreducible representations of a group, we can make striking conclusions.

Theorem 21

Let G be a finite group, let Φ be the set of distinct (inequivalent to others in set) irreducible, unitary representation of G. Let $\phi, \psi \in \Phi$ with degrees n_ϕ and n_ψ respectively. Let $\phi_g := \phi(g)$ and $\psi_g := \psi(g)$. Then the following equality holds:

$$\frac{n_\phi}{|G|} \sum_{g \in G} [\phi_g]_{ij} \left[\psi_g^\dagger\right]_{kl} = \begin{cases} 1 & \text{if } \psi = \phi,\ j = k,\ \text{and } i = l \\ 0 & \text{else} \end{cases}$$

where for any matrix $M = [m]_{ij}$, $\left[M^\dagger\right]_{ij} = \overline{m}_{ji}$. We refer to this as the ***orthonormality condition*** *of unitary irreducible representations.*

There is no doubt complexity here, but this theorem can be made more digestible.

We create a $|G|$-dimensional vector space in the following way: For fixed ϕ, i, j,

$$\sqrt{\frac{n_\phi}{|G|}}\left([\phi_{g_1}]_{ij},\ [\phi_{g_2}]_{ij},\ \ldots\ ,\ [\phi_{g_{|G|}}]_{ij}\right)$$

Then, $\frac{n_\phi}{|G|}\sum_{g\in G}[\phi_g]_{ij}\left[\psi_g^\dagger\right]_{kl}$ is just an inner product of vectors in this space.

Further,

$$\begin{cases} 1 & \text{if } \psi = \phi,\ j = k,\ \text{and } i = l \\ 0 & \textit{else} \end{cases}$$

is the conclusion that these vectors are orthonormal to one another in choice of ϕ, i,and j.

We have a another very similar property that unitary irreducible representations meet:

Theorem 22

Let G be a finite group, let $\Phi = \{\phi \mid \phi$ is a unitary, irreducible representation of $G\}$, and let n_ϕ denote the degree of the representation ϕ. Then for any pair $g, h \in G$,

$$\sum_{\phi \in \Phi} \sum_{i=1}^{n_\phi} \sum_{j=1}^{n_\phi} \frac{n_\phi}{|G|} \left[\phi_g\right]_{ij} \left[\phi_h^\dagger\right]_{ji} = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{else} \end{cases}$$

This is referred to as the ***completeness condition*** *of unitary, irreducible representations.*

Here, we mean complete in the sense that our irreducible unitary decomposition is a complete characterization of a generic repesentation.

This is a consequence of the follwoing relationship we derive from the previous theorem:

Theorem 23

For any finite group, G, $\sum_{\phi \in \Phi} n_\phi^2 = |G|$ where Φ is the set of a distinct, irreducible representations of G.

Here, we see that for any finite group, there is a finite set of irreducible representations that we can decompose generic representations into.

Generalizing the properties of unitary representations to groups with infinite order requires more careful construction (Haar Measure).

Generalizing representation theory to infinite-dimensional vector spaces also requires care, but we can explicitly approach this

Definition 24

A **matrix element** of a representation ϕ of a group G on a inner-product space V is a function defined in the following way:

$$f : G \to \mathbb{C}$$

$$g \mapsto \langle \phi(g)v, w \rangle$$

for some fixed $v, w \in V$.

When V is finite-dimensional, then the choice of standard basis vectors gets us matrix entry.

When we define representations on infinite-dimensional vector spaces, we will identify them by their matrix elements on orthonormal bases.

We further generalize our notion of representations:

Definition 25

A **multi-valued representation** is a multi-valued mapping of a group into $GL_n(\mathbb{C})$ which is a group homomorphism in the sense that at least one of the outputs (from each input) may be used to satisfy the homomorphism property.

2-valued representations will be of interest to us.

Strategy 1

Look for eigenvalues of generators (one-dimensional representations)

Definition 26

A **Lie Algebra** is a vector space, $\mathfrak{g}$, together with a bilinear operation, [. , .] that satisfies the the following identities:

- $[x, x] = 0 \ \forall x \in \mathfrak{g}$
- $[x, [y, z]] + [y, [z, x]] + [z[y, x]] = 0 \ \forall x, y, z \in \mathfrak{g}$

Definition 27

The **universal enveloping algebra** of a Lie algebra is the largest embedding of Lie algebra into an algebra.

Strategy 2

- Embed into Lie Algebra
- Embed to Universal Enveloping Algebra
- Find a set of mutually commuting generators of above
- Create representation on those generators

Definition 28

Let G be a group, N be a normal subgroup, ϕ be a representation of G, and V be the underlying vector space of the representation. Then for any $v \in V$, the **little group** of v is the subgroup of G/N who's image under ϕ leave v invariant.

Strategy 3

- Find a nice normal subgroup, quotient out
- Pick a nonzero starting vector
- Find representations of the little group of said vector
- Transform vector to recover remaining vectors, repeat process for general representation
- We are left with the smallest invariant subspace (generated by one vector)

2 Braid Group and Anyons

Definition 29

The braid group, B_n, is generated by $n-1$ generators (denoted $\sigma_1, \ldots, \sigma_{n-1}$) that have the following "braid relations":

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \forall i, j \in \{1, \ldots, n-1\} \text{ where } |i-j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall i \in \{1, \ldots, n-2\}$$

Definition 30

A **geometric braid** on $n \in \mathbb{N}$ strands is a set $b \subset \mathbb{R}^2 \times [0,1]$ formed by n disjoint, intervals topologically equivalent to $[0,1]$ such that we can define a projection mapping from $\mathbb{R}^2 \times [0,1]$ to $[0,1]$ that maps each strand homeomorphically to $[0,1]$ and the following conditions hold:

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$$

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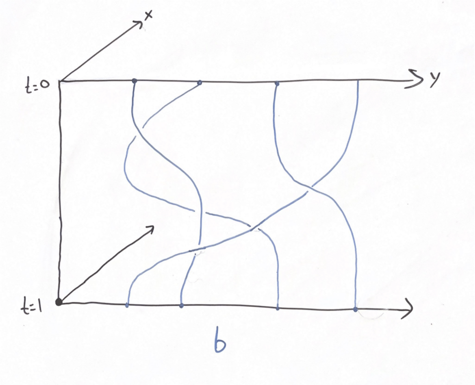


Figure: Generic geometric braid

Definition 31

Two braids, b_1 and b_2, are said to be **isotopic** to each other if one can be continuously deformed into the other. This is an equivalence relation.

$$b_1 b_2 := \{(x,y,t) \mid (x,y,2t) \in b_1 \text{ when } 0 \le t \le \frac{1}{2} \text{ and } (x,y,2t-1) \in b_2 \text{ when } \frac{1}{2} \le t \le 1\} \tag{1}$$

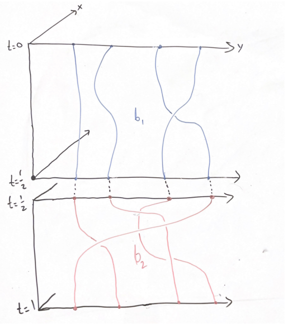


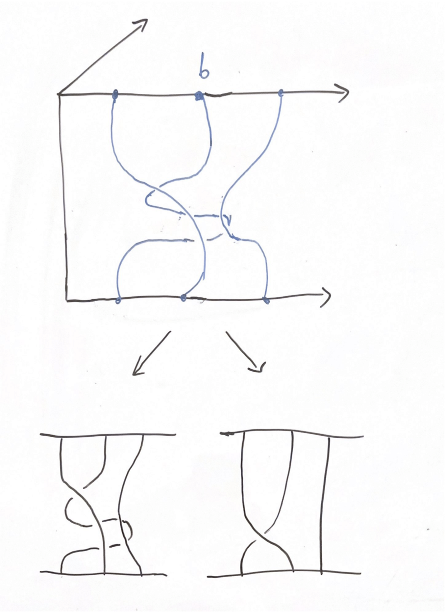
Figure: The product of two geometric braids.

Definition 32

A **braid diagram** on n strands is set, $\mathcal{D} \subset \mathbb{R} \times [0,1]$ made up as a union of n intervals topologically equivalent to $[0,1]$ (called strands) such that the following conditions are met:

- There exists a projection map from $\mathbb{R} \times [0,1]$ to $[0,1]$ that maps each strand homeomorphically to $[0,1]$.
- Every element of $\{1, 2, \ldots, n\} \times \{0, 1\}$ is a starting or endpoint of a unique strand.
- Every element in $\mathcal{D}$ belongs to either one or two strands. When an elment belongs to two, one strand must be designated as overgoing and the other undergoing (referred to as a crossing of $\mathcal{D}$

While it is definitely true that every geometric braid has a braid diagram, the identification is not unique.


b

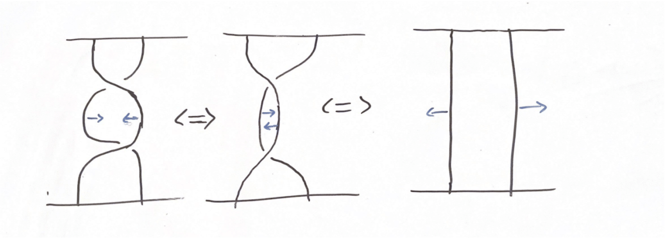


Figure: Ω_1^{-1}

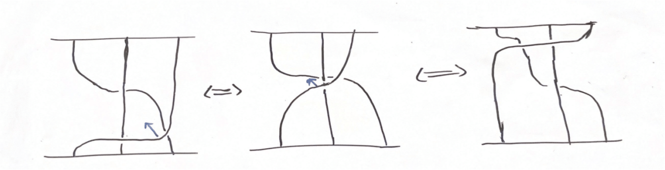


Figure: Θ

Definition 33

Two braid diagrams are said to be **R-equivalent** if one can be transformed into the other by means of a finite sequence of isotopies and Reidemeister moves. This is an equivalence relation.

Braids can be decomposed in the following way:

Theorem 34

Let $\mathcal{B}_n$ be the group of geometric braids on n strands. Then for any $\beta \in \mathcal{B}_n$, β has a natural decomposition:

$$\beta = \sigma_{i_1} \dots \sigma_{i_k}$$

where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n-1\}$.

Now that we have characterized the group, we can discuss representations. We consider the ring of matrices over the ring $\mathbb{Z}[t, t^{-1}]$

For $i \in \{1, \dots, n-1\}, n > 1$, define

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} \tag{2}$$

Definition 35

The **Burau Representation** of B_n is the map

$$\phi_n : B_n \to M_n(\mathbb{Z}[t, t^{-1}])$$

$$\sigma_i \mapsto U_i$$

If we compose this map with a ring homomorphism on $\mathbb{Z}[t, t^{-1}]$ that evaluates elements at some complex number with magnitude one, the result is a unitary representation.

Let us shift our discussion towards anyonic systems. In quantum mechanics, we study wave functions.

$$\int_\Omega |\psi|^2 d\mu = 1 \tag{3}$$

Wave function encodes information depending on inputs. For our purposes, inputs are positions.

Suppose we have a two particle system where particles are indistinguishable.

After one particle exchange:

$$\psi(r_1, r_2) = e^{i\theta}\psi(r_2, r_1) \quad \text{for some } \theta \in [0, 2\pi) \tag{4}$$

After two particle exchanges:

$$\psi(r_1, r_2) = e^{2i\theta}\psi(r_1, r_2) \quad \text{for some } \theta \in [0, 2\pi) \ \textit{a.e.} \tag{5}$$

This means $\theta = 0, \pi$: Bosons or Fermions.

However, in two dimensions, it is not always the case that returning back to starting position returns our value. Rather, the anyons begin to braid. This is more clearly seen with n anyons.

We say that the particles obey θ-statistics if the wave function picks up a factor of $e^{i\theta}$ upon exchange. Anyons obey θ-statistics where $\theta \neq 0, \pi$.

Using this concept, we can build a degree one representations of B_n.

Representation

$$\phi_\theta : B_n \to \mathbb{C}$$
$$\beta \mapsto e^{i\theta}$$

If $\beta \in B_n$ with decomposition $\beta = \sigma_{i_1}^{m_1} \dots \sigma_{i_k}^{m_k}$ for some $i_1, \dots, i_k \in \{1, \dots, n-1\}$, $m_j \in \mathbb{N}$, and if σ is the underlying permutation of β, then

$$\psi(\sigma(r_1), \sigma(r_2), \dots, \sigma(r_n)) = e^{i\theta(m_1 + \dots + m_k)} \psi(r_1, r_2, \dots, r_m) \tag{6}$$

Since the image of this homomorphism is an abelian group, we say these anyons are abelian.

Another representation can be defined in the following way:

Theorem 36

Let $V := \bigotimes_{i=1}^{m} \psi$ *where* $m > 1$ *and let* $R : V \otimes V \to V \otimes V$ *be an invertible, linear operator. Then the following map is a representation of* B_n

$$\phi_R : B_n \to GL_m(\mathbb{C})$$

$$\sigma_i \mapsto M(U_i)$$

where $M(U_i)$ *is the matrix of the operator* U_i *defined by*

$$U_i : V \to V$$

$$(v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_m) \mapsto (v_1 \otimes \ldots \otimes R(v_i \otimes v_{i+1}) \otimes \ldots \otimes v_m)$$

$$M(U_i) = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & M(R) & 0 \\ 0 & 0 & I_{m-i-1} \end{bmatrix} \tag{7}$$

Question

What happens if anyons obey different statistics?

We continue our work in the context of monoidal category theory

Definition 37

A monoidal category is a category, C, equipped with the following structure:

- a bifunctor $\otimes : C \times C \to C$
- an object e which acts as the identity object
- three natural isomorphisms defined in the following way:
  - The associator, α, whose components are $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$
  - The left unitor, λ, whose components are $\lambda_a : e \otimes a \cong a$
  - The right unitor, ρ, whose components are $\rho_a : a \otimes e \cong a$

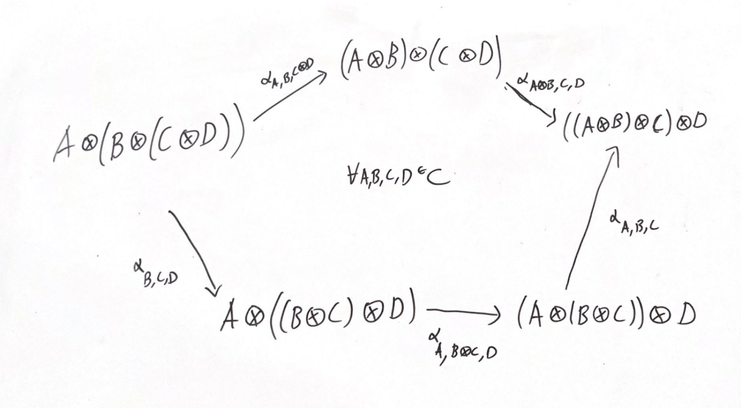


Figure: Commutative Pentagon Diagram for Monoidal Categories

Here, the tensor product corresponds to the fusion of two anyons. We will be characterizing the Fibonacci anyonic system.

If *e* is the identity anyon type (vacuum), and τ is a nontrivial anyon type, then our system is characterized by this rule:

$$\tau \otimes \tau = 1 \oplus \tau \tag{8}$$

We say that τ has multi-fusion channels. We represent all possible fushion paths in tree diagrams in the following way:

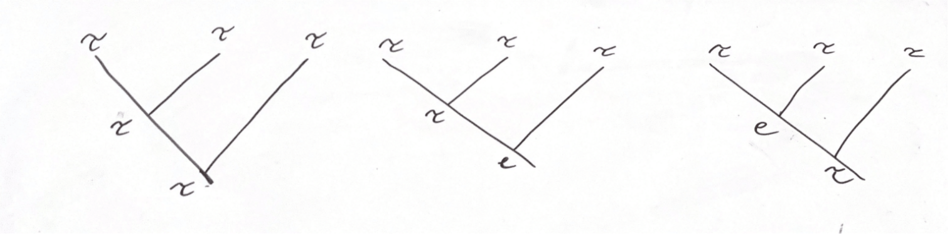


Figure: These tree diagrams are labelled according to the fusions rules of Fibonacci anyons.

The Fibonacci sequence counts the number of fushion paths that can be taken. These fusion paths represent an orthonormal basis of some underlying vector space.

There is nothing to stop fushion from occuring in any order.

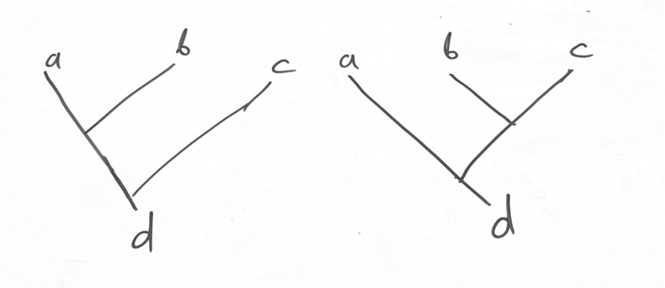


Figure: Both tree diagrams represent a different fusion order on the same anyon types.

Transforming from one basis to another should be a linear, unitary operation, and as such can be depicted in a matrix. We call these transformations F-moves. Denote F_d^{abc} as the F-move that moves $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$ over total charge d.

We use the commutative diagram to construct systems of equations to solve for each F explicitly

F_d^{abc} is the identity matrix if any the fusion associates any identity particles. Therefore, we must solve for $F_\tau^{\tau\tau\tau}$.

$$[F_\tau^{\tau\tau\tau}]_{11} = [F_\tau^{\tau\tau\tau}]_{12}[F_\tau^{\tau\tau\tau}]_{21} \tag{9}$$

$$F_\tau^{\tau\tau\tau} = \begin{bmatrix} \frac{1}{\Phi} & \frac{1}{\sqrt{\Phi}} \\ \frac{1}{\sqrt{\Phi}} & -\frac{1}{\Phi} \end{bmatrix} \tag{10}$$

where Φ is the golden ratio.

Choosing to reintroduce braiding to the set up gives us the ability to embed ourselves in another kind of category: a Braiding Monoidal Category

Here, braiding and fusion have to behave well with one another

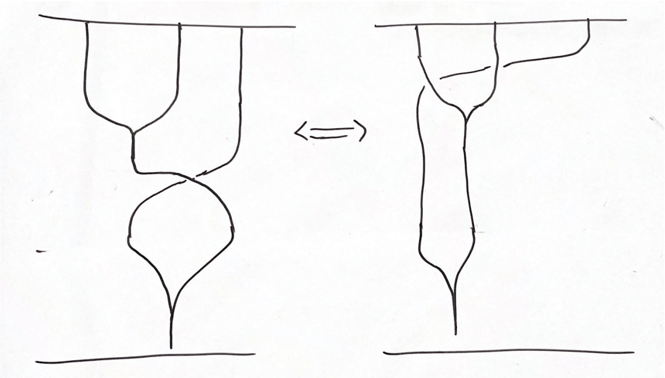


Figure: Compatibility of braiding and fusion.

Here, we refer to individual anyon exchanges as R-moves and these can be encoded into matrices due to their basis transforming abilities on our fusion paths. We take the convention that $R_c^{a,b}$ swaps a and b with total charge c fused. We solve a similar system of equations to find explicit R matrices

$$(R_e^{\tau,\tau})^2 \frac{1}{\Phi} = R_\tau^{\tau,\tau} \frac{1}{\Phi} + \frac{1}{\Phi^2} \tag{11}$$

$$R_e^{\tau,\tau} R_\tau^{\tau,\tau} \frac{1}{\sqrt{\Phi}} = (1 - R_\tau^{\tau,\tau}) \frac{1}{\Phi^{\frac{3}{2}}} \tag{12}$$

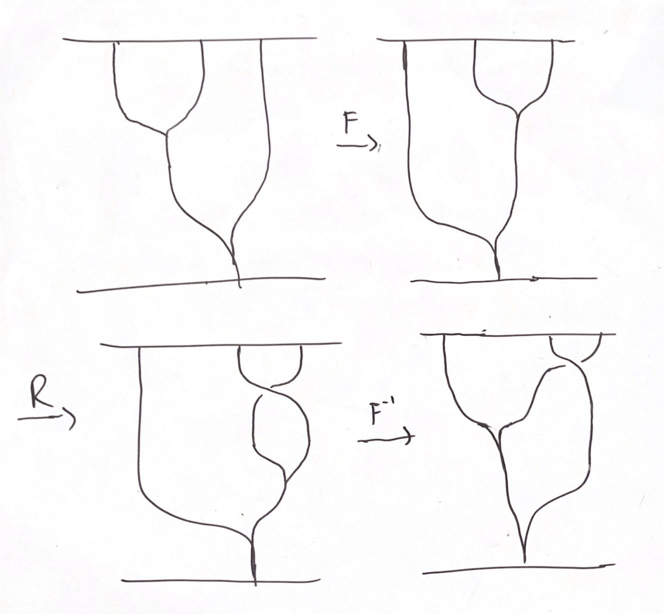
$$-(R_e^{\tau,\tau})^2 \frac{1}{\Phi} = R_\tau^{\tau,\tau} \frac{1}{\Phi^2} + \frac{1}{\Phi} \tag{13}$$

It is difficult to find solutions to these equations in general. If we make the assumption that our state space is one dimensional, these matrices should be one dimensional with values $R_e^{\tau\tau} = e^{i\frac{4\pi}{5}}$ and $R_\tau^{\tau\tau} = e^{i\frac{-3\pi}{5}}$.

In order to encode a true braiding effect in this set up, we need to ensure that the braid takes into account fusions. Braiding matrices take the following form:

$$B = F^{-1}RF \tag{14}$$

We need more specificity to the scenario to calculate a specific solution for this set up.


F
R
F⁻¹

Thank You