Math 442: Applications of Geometry

March 13, 2024

Presented by
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CAL POLY

Outline:

1. Celestial Navigation
2. Mathematical Billiards (my research area)
3. More Geometry, More Billiards! (also my research area)

1 Celestial Navigation

Question

On a clear night, how can you tell which direction is north by looking at the stars?

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Answer

By finding the north star (also known as Polaris)! From where you're standing, the star will be roughly pointing in the northern direction. Or, geometrically speaking, minimize the distance from the north star to the horizon, and that point on the horizon will be north from wherever you happen to be.

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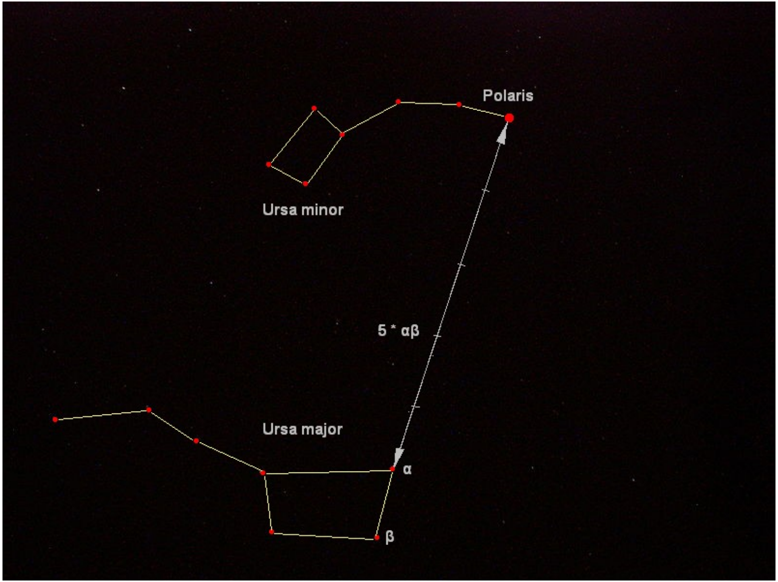
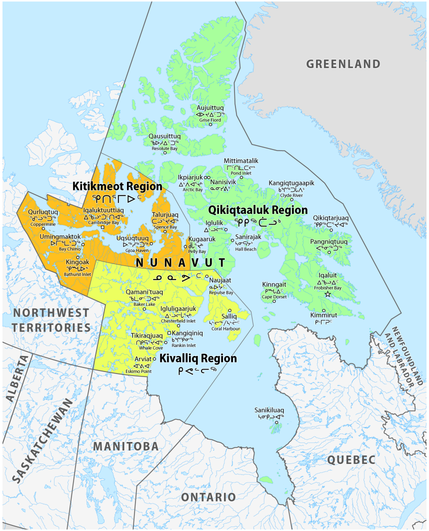
Follow-up Question

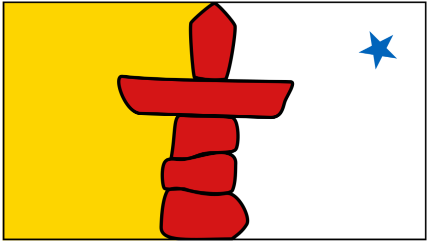
But how do we find the north star?



Source: `https://en.wikipedia.org/wiki/Polaris`

The north star also appears in a variety of contexts, from literature, history, religion, and exploration. It also plays an important role in Indigenous and First Nations culture.





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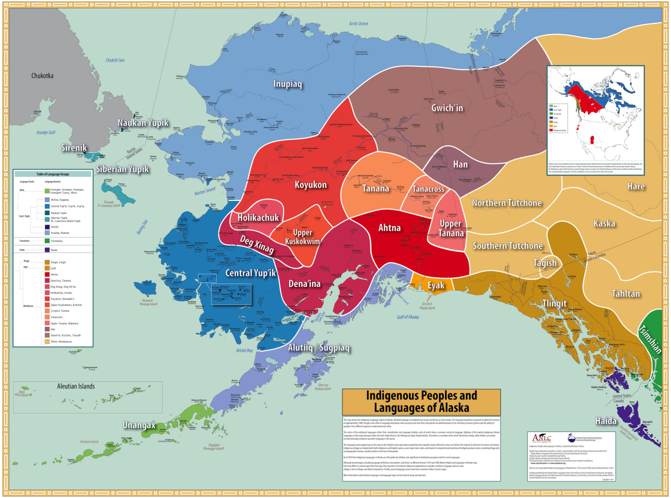


https://www.uaf.edu/anla/collections/map/

Problem

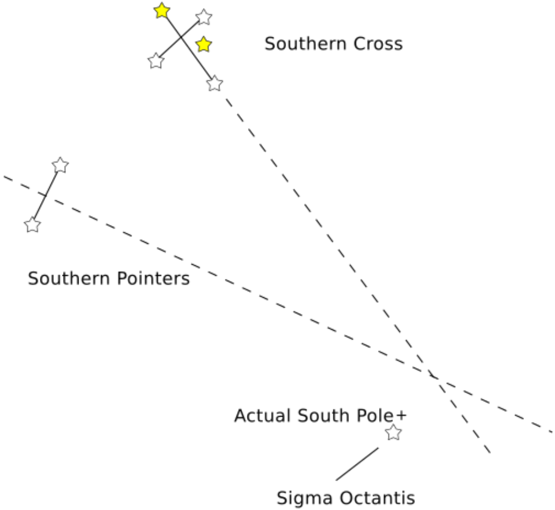
Not everyone lives in the northern hemisphere or close enough to the equator where Polaris is visible. For those in the southern hemisphere, there's a literal planet in the way between you and the north star!

Flags of the southern hemisphere

All of these flags share the same constellation, called the Southern Cross (or Crux).

Application of Geometry #1

Use the Southern Cross (and the two pointer stars) to help us find which direction is south:

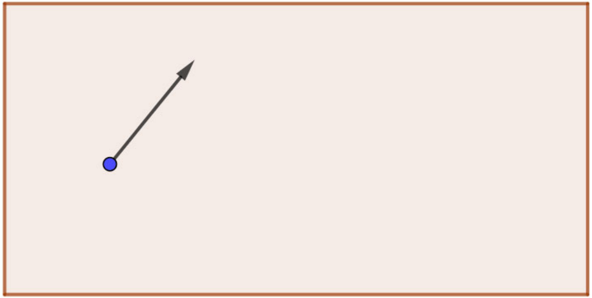


"The constellations that figure in the National Flag match the aspect of the sky, in the city of Rio de Janeiro, at 8 hours and 30 minutes of November 15th of 1889 (12 sidereal hours) and must be considered as viewed by an observer outside the celestial sphere." – Brazilian law on the flag. This is why the Southern Cross is reversed!

2 Mathematical Billiards

Mathematical billiards is a dynamical system consisting of a planar connected set Ω (the *billiard table*) and a point-mass in its interior (the *ball*) which moves in a straight line at constant speed.

When the ball hits the boundary, the ball reflects *elastically*, meaning it follows the rule "angle of incidence = angle of reflection."



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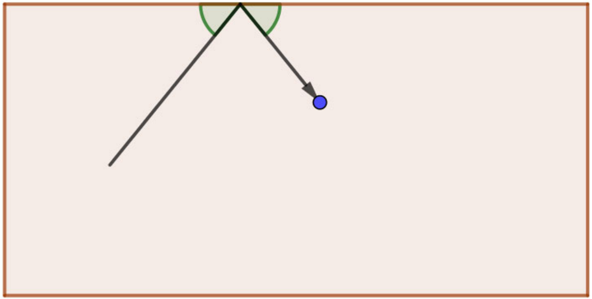

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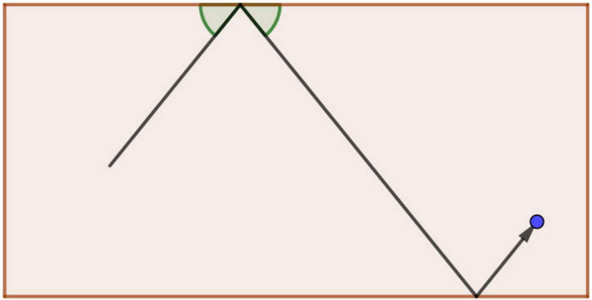

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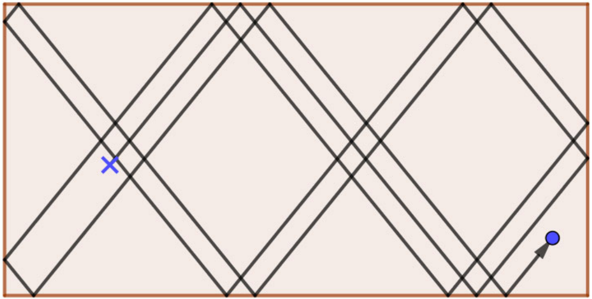
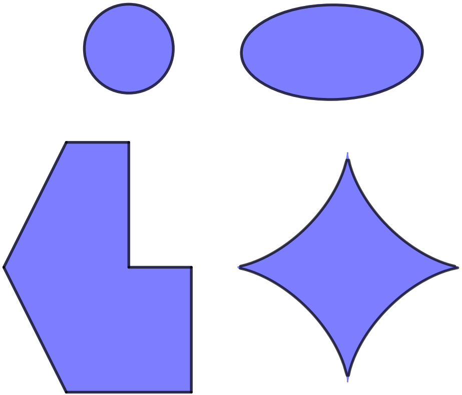
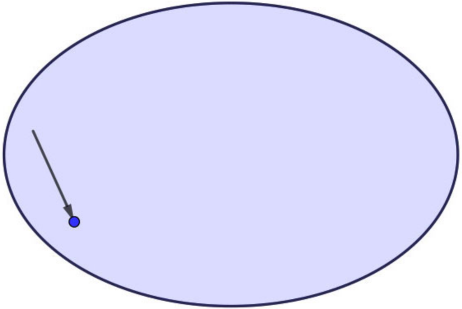


The dynamics of billiards is completely determined by the geometry of the table (e.g. its shape)!

One can choose tables of different shapes



If the boundary of the table is not a straight line, we need to more accurately describe the reflection law:



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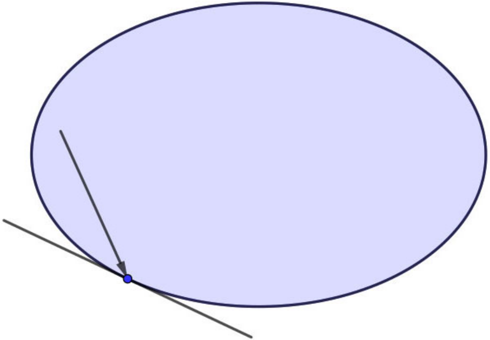

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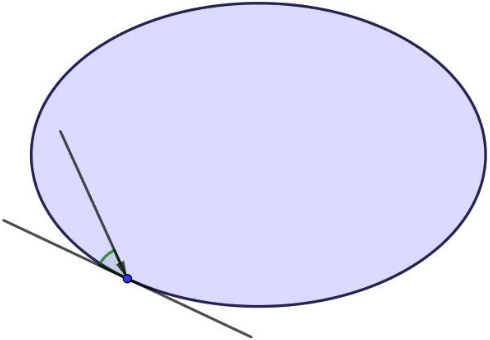

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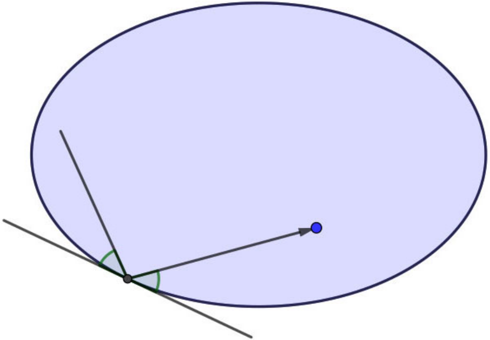
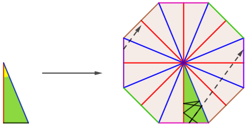
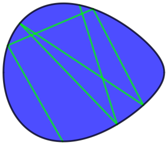


New reflection law: use the angle made with the tangent to the boundary at the point of impact. Can also describe this reflection law as $v^- = v_t + v_n \mapsto v_t - v_n = v^+$

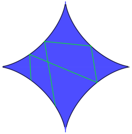
These are just several types of planar billiards that are studied:



Polygonal billiards:
-Related to the study of geodesic flow on a translation surface (with singular points)
-Teichmüller theory



(Strictly) Convex billiards:
-Birkhoff billiards (G. Birkhoff, 1927 - pragmatic example of Hamiltonian systems)
-The billiard map is a *twist map*
-Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics



Concave (or dispersive) billiards:
-Nearby orbits tend to move apart (exponentially)
-Hyperbolicity and chaotic behaviour (Sinai, 1970)
-Study of statistical properties of orbits

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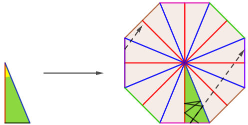

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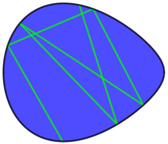

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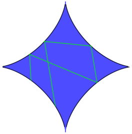

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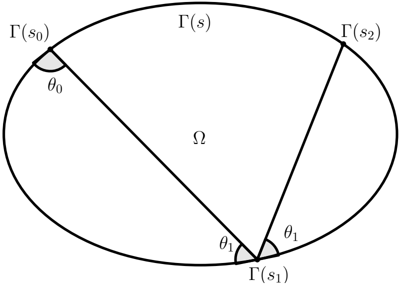
Introduction to Birkhoff Billiards

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with smooth boundary $\partial\Omega$. Fix an orientation of the boundary and measure the angle θ_i with respect to the positive tangent at the point p_i. Parametrize $\partial\Omega = \Gamma$ by arc length s so that the billiard map is

$$T : \partial\Omega \times (0, \pi) \longrightarrow \partial\Omega \times (0, \pi)$$

$$(s_i, \theta_i) \longrightarrow (s_{i+1}, \theta_{i+1})$$



Example 1 (Circular Billiard)

CIRCULAR BILLIARDS,

FOR TWO PLAYERS.

INVENTED, IN 1889, BY

LEWIS CARROLL.

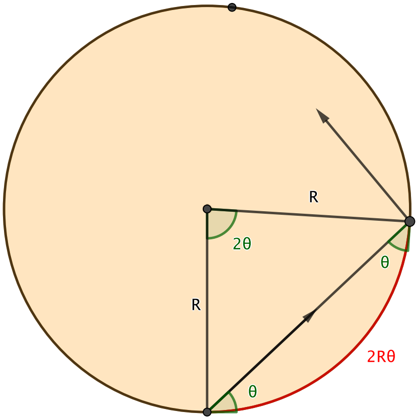
The Table is circular, with a cushion all round it, no pockets, and three white spots arranged in an equilateral triangle.

Rules.

1.

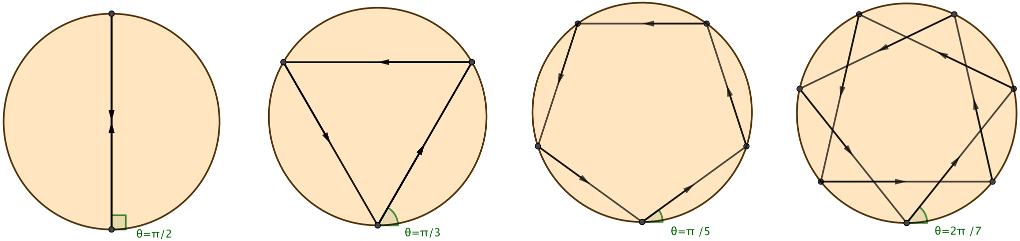
String for lead. Then the player, who is not to begin, places the 3 balls (red, white, and spot-white) on the spots provided for them. The first stroke must be played at the *red* ball.

2.



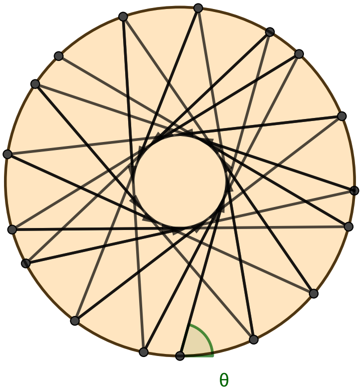
The map becomes $T(s, \theta) = (s + 2R\theta, \theta)$ and the angle θ is an integral of motion.

If θ is a rational multiple of π, the orbit is periodic:



For every $p/q \in (0, 1/2]$ there exist *infinitely many* periodic orbits which bounce q times (*period*) and turn p times around the circle (*winding number*) before closing. We call p/q the *rotation number*.

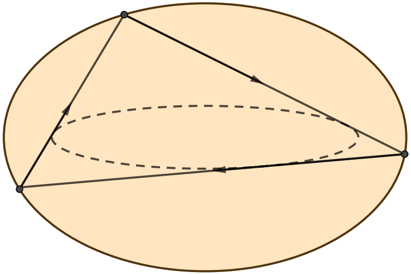
If θ is not a rational multiple of π then the trajectory hits the boundary at a dense set of points.



The trajectory is always tangent to a circle (an example of a *caustic*).

Definition 2

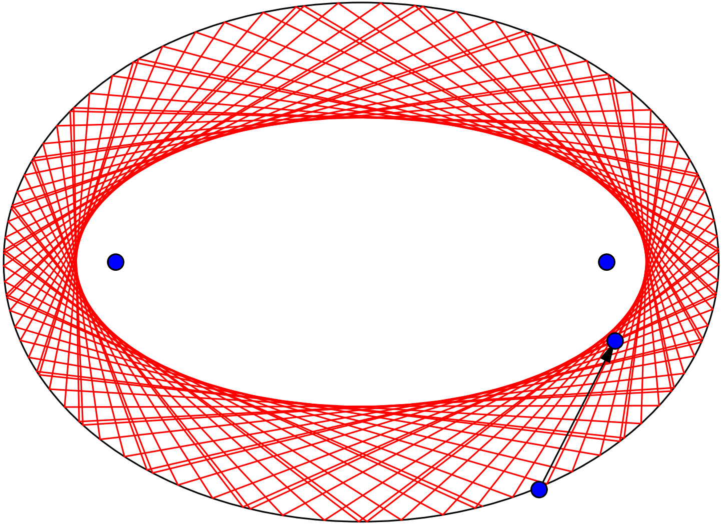
A *caustic* is a curve with the property that each trajectory (or its extension) that is tangent to it stays tangent after each reflection.

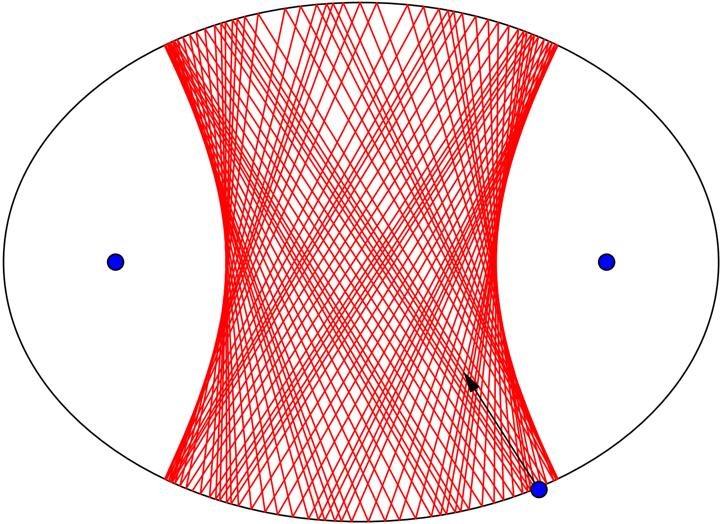


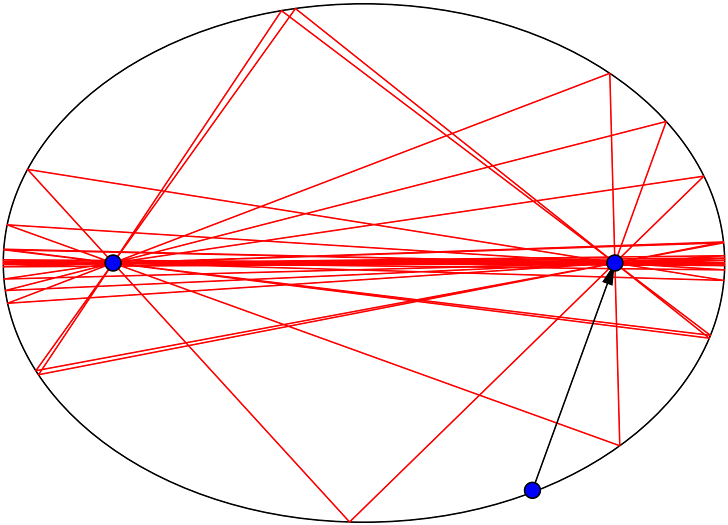
Example 3 (Elliptical Billiard)

The New York Times (1st July 1964) ran a full-page ad for *Elliptipool*, played on an elliptical table with a single pocket at one focus. The ad said that on the following day the game would be demonstrated at Stern's department store by movie stars Paul Newman and Joanne Woodward.

Example 4 (Elliptical Billiard)







Applications of Geometry #2

Let's prove some of these properties of elliptical billiards.

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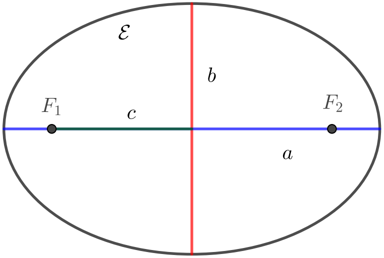
Consider a billiard inside the ellipse

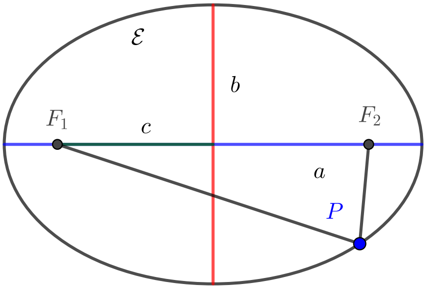
$$\mathcal{E}: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2}, \qquad a > b > 0.$$

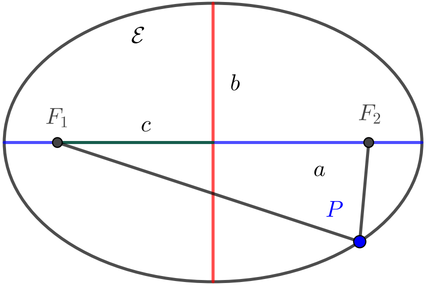


The focal distance $c = \sqrt{a^2 - b^2}$ and foci are at $F_1 = (-c, 0)$ and $F_2 = (c, 0)$

Definition 5

An ellipse is the closed plane curve such that for all point P on the curve, the sum of the two distances to the focal points is constant. Specifically, $|PF_1| + |PF_2| = 2a$.





Key Observation

Points inside $\mathcal{E}$ satisfy $|PF_1| + |PF_2| < 2a$ while points outside of $\mathcal{E}$ satisfy $|PF_1| + |PF_2| > 2a$.

Proposition

Let P be a point on the ellipse $\mathcal{E}$. The angle bisectors of the lines $\overleftrightarrow{PF_1}$ and $\overleftrightarrow{PF_2}$ are the tangent and normal lines to $\mathcal{E}$ at P.

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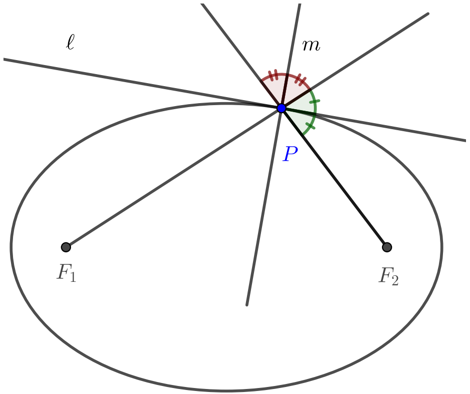
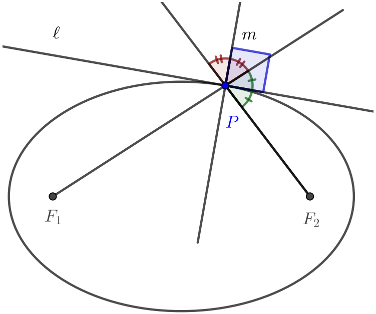


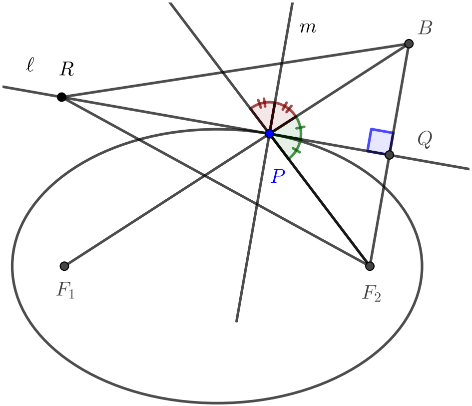
Proof

- Let ℓ and m be the angle bisectors of $\overleftrightarrow{PF_1}$ and $\overleftrightarrow{PF_2}$.
- By Axiom 7 and Theorem 86, $2(\textit{red angle}) + 2(\textit{green angle}) = 180^\circ \implies (\textit{red angle}) + (\textit{green angle}) = 90^\circ$. So ℓ is perpendicular to m.
- This means that if we can show ℓ is the tangent line, then m will automatically be the normal line.



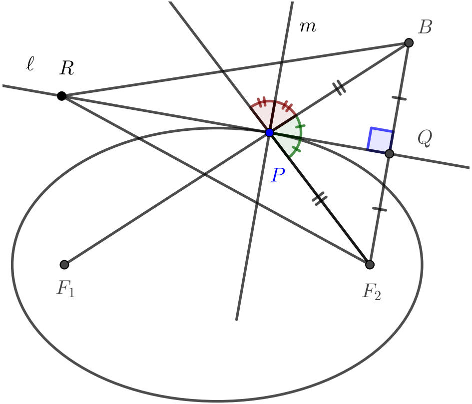
Proof (cont.)

- Let R be any point on ℓ other than P.
- Construct the line perpendicular to ℓ through F_2. This line intersects ℓ at Q and intersects $\overleftrightarrow{PF_1}$ at B. Connect R to B and R to F_2.



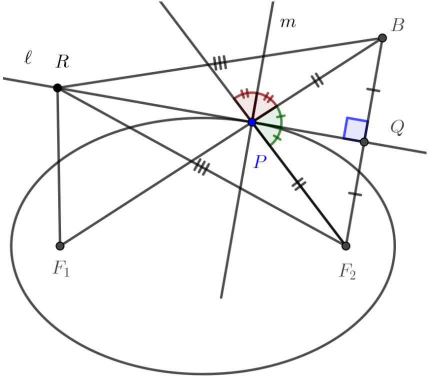
Proof (cont.)

- Since $PQ \cong PQ$, $\Delta PQF_2 \cong \Delta PQB$ by ASA.
- By CPCFC, $QF_2 \cong QB$ and $PB \cong PF_2$. This makes ℓ the perpendicular bisector of F_2B.



Proof (cont.)

- Since $RQ \cong RQ$, $\Delta RQF_2 \cong \Delta RQB$ by SAS.
- By CPCFC, $RF_2 \cong RB$.
- Construct RF_1.

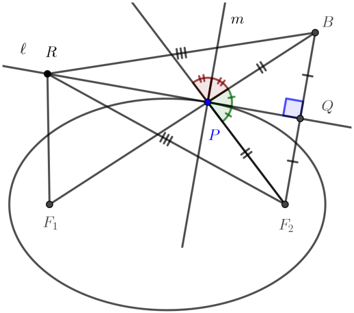


Proof (cont.)

- Moreover,

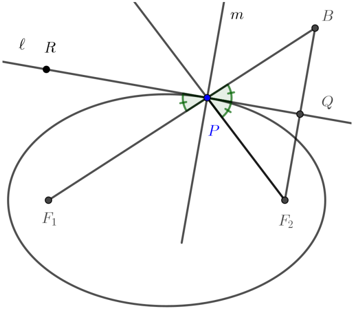
$$|RF_1| + |RF_2| = |RF_1| + |RB| \geq |F_1B| = |PF_1| + |PB| = |PF_1| + |PF_2| = 2a.$$

- Therefore R is outside $\mathcal{E}$, and so all points on ℓ other than P are outside $\mathcal{E}$.
- The inequality is strict unless $R = P$. Therefore P is the only point that ℓ shares with $\mathcal{E}$, making ℓ the tangent line to $\mathcal{E}$ and m the normal line. $\square$



Corollary

The polygonal line F_1PF_2 is a billiard trajectory.

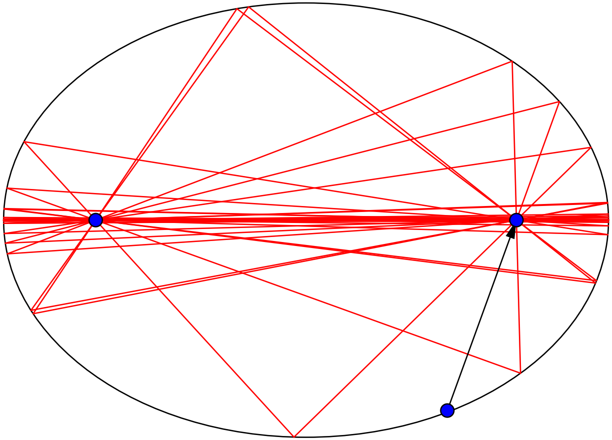


Proof

By Corollary 38 (vertical angles), $\angle RPF_1 \cong \angle BPQ$, and so $\angle RPF_1 \cong \angle QPF_2$. □

Corollary Restatement

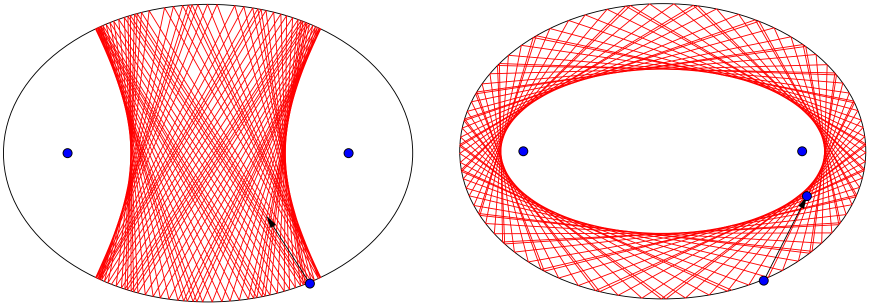
Every billiard trajectory in $\mathcal{E}$ which passes through one focus will pass through the other focus after each reflection with the boundary.



Theorem

Consider the billiard in an ellipse $\mathcal{E}$ with foci F_1 and F_2.

1. If a segment of a billiard trajectory crosses F_1F_2, then every segment of the trajectory will cross F_1F_2.
2. If a segment of a billiard trajectory does not cross F_1F_2, then all segments of this trajectory do not cross F_1F_2.

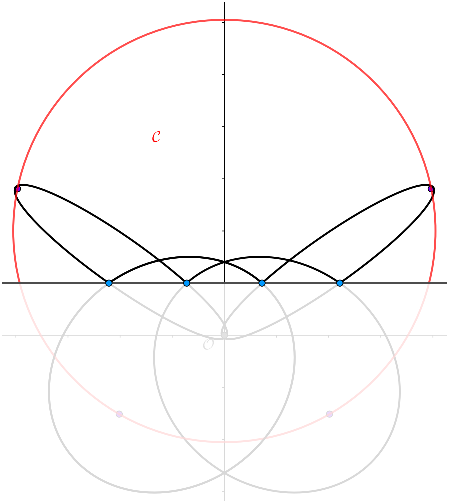


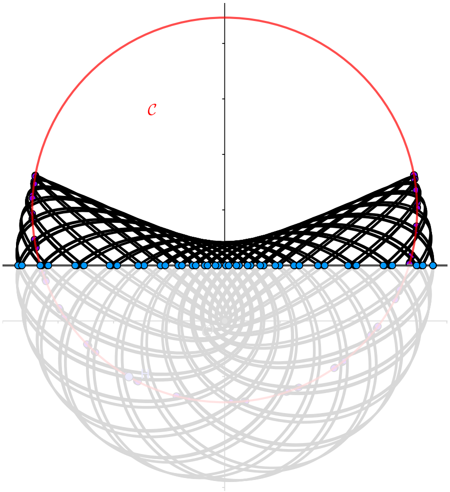
Applications of Geometry #3

SG playing elliptipool at the University of Jyväskylä, Finland, Feb. 2019

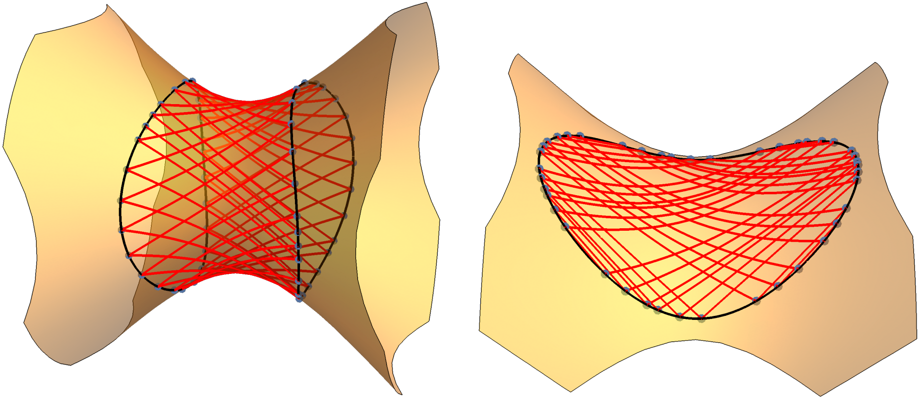
3 More Geometry, More Billiards!

Current Research #1





Current Research #2



Thank you!

`Problems worthy of attack`
`Prove their worth by hitting back.`
–Piet Hein, “Problems”, Grooks (1966)