Representation Theory in Braid Groups

by

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None

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Abstract

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Write an abstract here.

Acknowledgments

Any acknowledgements?

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Chapter 1

SAMPLE CHAPTER 1

1.1 First Section of Introduction

This is an equation:

$$c^2 = a^2 + b^2. \tag{1.1}$$

Chapter 2

SAMPLE CHAPTER 2

2.1 MY FIRST SECTION

There are lots of great resources on the internet to help you learn LaTeX. Perhaps start with examples like the ones at

`http://en.wikibooks.org/wiki/LaTeX/Sample_LaTeX_documents.`

It is important to cite references. [1, 2, 5]

Organize the paper into sections and subsections.

2.1.1 Interesting subsection title

You get the idea. Hey, this one has some displayed math,

$$\frac{2}{x} = \sin(\epsilon),$$

not that it makes any sense whatsoever. And here is how you do a numbered equation,

$$\int_0^{y^2} f(x)\,dx = \sqrt{z+y}. \tag{2.1}$$

Don't forget to punctuate your equations as part of the sentence. You can do inline math, too, as in $f(x) = \lim_{n\to\infty} nf(x)/n$, which is trivial.

2.1.2 Another interesting subsection title

Okay, not really.

Figure 2.1: This is a sample figure. It is not interesting. Note that figures will "float" to wherever LaTeX wants to put them.

Chapter 3

REPRESENTATION THEORY

3.1 Introduction to Representations

Over the course of this chapter, we will develop the theory and utility of representations. At a glance, representations give us the ability to dial back the complexity of a mysterious group by viewing its elements as matrices. Thanks to the rigorous development and study of linear algebra, groups of matrices are well-understood structures. Representations allow us to unravel the mystery of an unknown group structure and reveal a group's fundamental properties as results of linear algebra techniques.

Definition 3.1. *(First Definition) A* ***representation*** *of degree n is a group homomorphism that maps a group into* $GL_n(\mathbb{C})$

$$\phi : G \to GL_n(\mathbb{C})$$

We say that ϕ *is a representation of* G*. If* ϕ *is an injective homomorphism, we say that the representation is* ***faithful****. Otherwise, the representation is called* ***degenerate****.*

To illustrate to concept of representations, we will consider the group of all roots of unity, G, for the following examples. We can construct multiple homomorphisms from G to showcase different kinds of representations.

Note: Since $GL_1(\mathbb{C})$ as $\mathbb{C}$ are isomorphic, we identify each $1x1$ matrix with its corresponding entry with its element in $\mathbb{C}$.

Example 3.2. *(Trivial Representation)*

$$\text{Let} \quad \begin{aligned} \phi : G &\to GL_1(\mathbb{C}) \\ g &\mapsto 1 \end{aligned}$$

This map is the trivial homomorphism from G *to* $GL_1(\mathbb{C})$ *and therefore it easily satisfies the requirement of a degree* 1 *representation of* G*. We say that* ϕ *is the* ***trivial representation*** *of* G*.*

Example 3.3. *(Nontrivial Degree 1 Representation)*

By construction of G*, if* $g \in G$*, then* $g = e^{\frac{2\pi i m}{n}}$ *where* $m, n \in \mathbb{Z}$

$$\text{Let} \quad \begin{aligned} \phi : G &\to GL_1(\mathbb{C}) \\ g &\mapsto g \end{aligned}$$

where we view G as a multiplicative subgroup of $\mathbb{C}$. This observation trivializes the argument that ϕ is a homomorphism. Therefore, ϕ is a degree 1 representation of G.

Example 3.4. *(Degree 2 Representation)*

$$\text{Let} \quad \begin{aligned} \phi : G &\to GL_2(\mathbb{C}) \\ e^{2\pi i \frac{m}{n}} &\mapsto \begin{bmatrix} \cos(\frac{2\pi m}{n}) & \sin(\frac{2\pi m}{n}) \\ -\sin(\frac{2\pi m}{n}) & \cos(\frac{2\pi m}{n}) \end{bmatrix} \end{aligned}$$

To show this map is a homomorphism, we will take two elements of G, say $e^{2\pi i \frac{x}{y}}$ and $e^{2\pi i \frac{a}{b}}$ and track the image of their product under ϕ.

$$\begin{aligned} \phi(e^{2\pi i \frac{x}{y}} * e^{2\pi i \frac{a}{b}}) &= \phi(e^{2\pi i (\frac{x}{y} + \frac{a}{b})}) \\ &= \begin{bmatrix} \cos(2\pi(\frac{x}{y} + \frac{a}{b})) & \sin(2\pi(\frac{x}{y} + \frac{a}{b})) \\ -\sin(2\pi(\frac{x}{y} + \frac{a}{b})) & \cos(2\pi(\frac{x}{y} + \frac{a}{b})) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\pi\frac{x}{y})\cos(2\pi\frac{a}{b}) - \sin(2\pi\frac{x}{y})\sin(2\pi\frac{a}{b}) & \sin(2\pi\frac{x}{y})\cos(2\pi\frac{a}{b}) + \cos(2\pi\frac{x}{y})\sin(2\pi\frac{a}{b}) \\ -\sin(2\pi\frac{x}{y})\cos(2\pi\frac{a}{b}) - \cos(2\pi\frac{x}{y})\sin(2\pi\frac{a}{b}) & \cos(2\pi\frac{x}{y})\cos(2\pi\frac{a}{b}) - \sin(2\pi\frac{x}{y})\sin(2\pi\frac{a}{b}) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\pi\frac{x}{y}) & \sin(2\pi\frac{x}{y}) \\ -\sin(2\pi\frac{x}{y}) & \cos(2\pi\frac{x}{y}) \end{bmatrix} \begin{bmatrix} \cos(2\pi\frac{a}{b}) & \sin(2\pi\frac{a}{b}) \\ -\sin(2\pi\frac{a}{b}) & \cos(2\pi\frac{a}{b}) \end{bmatrix} \\ &= \phi(e^{2\pi i \frac{x}{y}}) * \phi(e^{2\pi i \frac{a}{b}}) \end{aligned} \tag{3.1}$$

Since, ϕ has been shown to be a homomorphism, we can conclude that ϕ is also a degree 2 representation of G.

Is ϕ faithful or degenerate?A faithful representation would have a trivial kernel. Suppose $\phi(e^{2\pi i \frac{x}{y}}) = I_2$ (I_n is the Identity Matrix of dimension $n \times n$).

$$\begin{bmatrix} \cos(2\pi\frac{x}{y}) & \sin(2\pi\frac{x}{y}) \\ -\sin(2\pi\frac{x}{y}) & \cos(2\pi\frac{x}{y}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.2}$$

Comparing entrywise, we see that $\cos(2\pi \frac{x}{y}) = 1$ *and* $\pm\sin(2\pi \frac{x}{y}) = 0$. *Using any of these equations, we see that* $\frac{x}{y} = n$ *for some* $n \in \mathbb{Z}$. *Therefore,* $ker(\phi) = \mathbb{Z}$ *and this representation is degenerate.*

Alternatively, we can formulate the definition of a representation in a different context, illuminating a useful interpretation that will be used extensively throughout this paper.

Definition 3.5. *(Second Definition) Let* G *be a group, let* V *be a linear vector space, and let* $\mathcal{L}(V)$ *be the group of linear operators on* V *together with the operation of composition. A* ***representation*** *of* G *is a group homomorphism that maps* G *into* $\mathcal{L}(V)$.

$$\phi : G \to \mathcal{L}(V)$$

The degree of the representation is the dimension of V.

Remark 3.6. *In the case where we have a finite dimensional vector space, we can make an interesting observation. Suppose* V *is finite dimensional and* G *is a group. It is easy to identify both definitions of representations with one another. Let* $\{e_i\}_{i=1}^n$ *be a basis for* V. *Let* $\phi : G \to \mathcal{L}(V)$ *be a representation of* G. *Then* $\forall g \in G$, $U_g \coloneqq \phi(g)$ *is a linear operator on* V. U_g *has a corresponding matrix,* $M(U_g)$, *with coefficients defined by the image of our basis vectors of* V.

$$M(U_g) = \begin{array}{cc} \begin{array}{cccc} U_g(e_1) & U_g(e_2) & \dots & U_g(e_n) \end{array} & \\ \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} & \begin{array}{c} e_1 \\ e_2 \\ \vdots \\ e_n \end{array} \end{array}$$

$$U_g(e_j) = \sum_{i=1}^{n} m_{ij} e_i$$

Does the map $\begin{array}{c} \psi : G \to GL_n(\mathbb{C}) \\ g \mapsto M(U_g) \end{array}$ *satisfy the criteria to be considered a representation (by the first definition)? If* $g, h \in G$, *then*

$$\psi(gh) = M(U_{gh}) = M(U_g \circ U_h) = M(U_g) * M(U_h) = \psi(g)\psi(h)$$

Where the homomorphism property of ϕ is used in succession with link between composition and multiplication of operators and their corresponding matrices. Therefore, we have established a very special link between th

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APPENDICES

Appendix A

APPENDIX A TITLE

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