

Module 6 Exercises | Joshua Shapiro | 24 May 2017

5. Suppose $\Pr[\text{heads}] = p$ in the example above. Write down $\Pr[X = i]$, $i \in \{0, 1, 2, 3\}$, in terms of p .

- **Answer:**

- $\Pr[X = 0] = (1 - p)^3$
- $\Pr[X = 1] = p * (1 - p)^2$
- $\Pr[X = 2] = p^2 * (1 - p)$
- $\Pr[X = 3] = p^3$

8. CODE: If $p = 0.6$, what is the probability that the first head appears on the 3rd flip? Verify your answer using `Coin.java` and `CoinExample.java`.

- **Answer:**

- $\Pr[X = 3] = (1 - p)^2 p = (0.4)^2 * 0.6 = 0.096$. **This has been verified in code.**

9. Suppose I compare two parameter values for the Geometric distribution: $p = 0.6$ and $p = 0.8$. For which of the two values of p is $\Pr[X = 3]$ higher?

- **Answer:**

- **When $p = 0.6$:** $\Pr[X = 3] = (1 - p)^2 p = (0.4)^2 * 0.6 = 0.096$
- **When $p = 0.8$:** $\Pr[X = 3] = (1 - p)^2 p = (0.2)^2 * 0.8 = 0.032$
- **$\Pr[X = 3]$ is higher when the probability of getting heads is lower.**

10. Compute (by hand) $\Pr[X > k]$ when $X \sim \text{Geometric}(p)$.

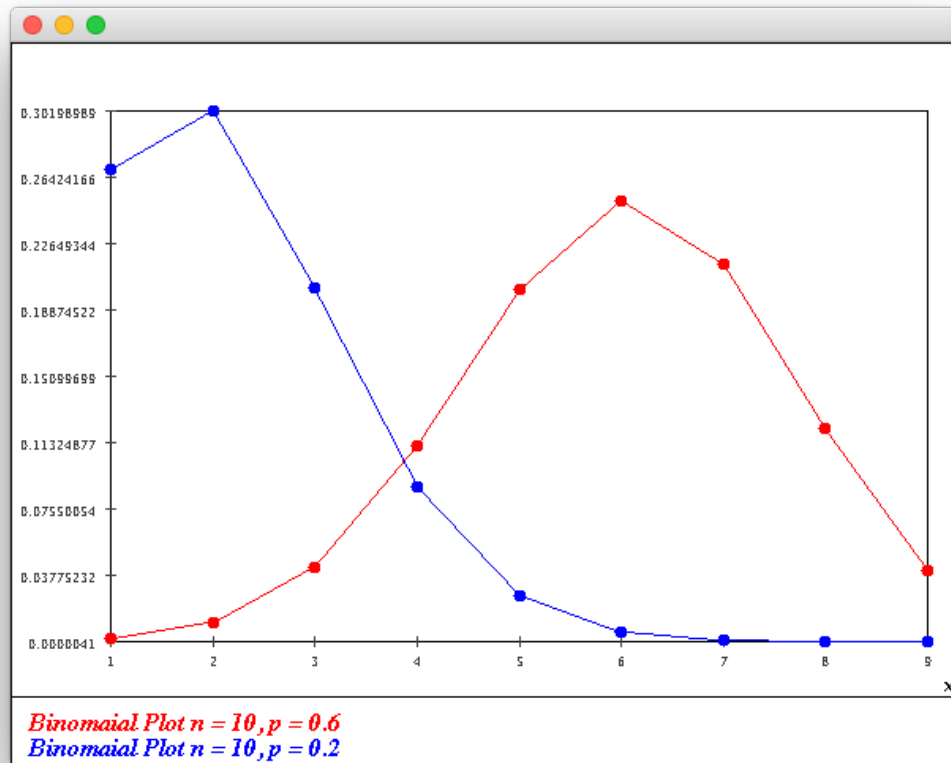
- **Answer:**

- $\Pr[X > k] = 1 - (\sum_{i=1}^k \Pr[X = i]) = 1 - (\sum_{i=1}^k (1 - p)^{i-1} p)$

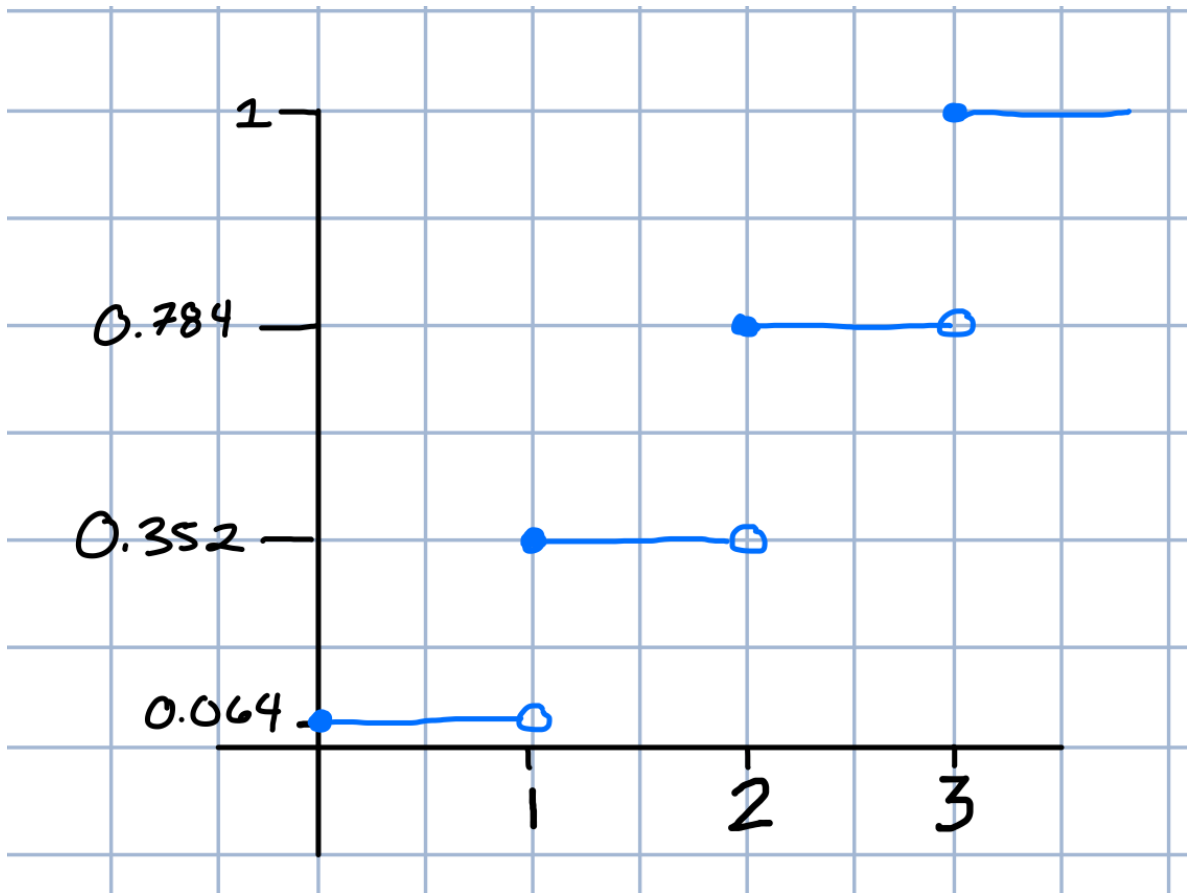
11. CODE: Suppose we flip a coin n times and count the number of heads using a coin for which $\Pr[H] = p$.

- Write code to compute $\Pr[X = k]$ using the formula $\Pr[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$. Write your code in `Binomial.java`.
- Plot a graph of $\Pr[X = k]$ vs. k for the case $n = 10$, $p = 0.6$ and for the case $n = 10$, $p = 0.2$.
- Write a simulation to estimate $\Pr[X = 3]$ when $n = 10$, $p = 0.6$. You can use `Coin.java` and `CoinExample2.java` for this purpose. Verify the estimate using the earlier formula.

- **Answer:**



- See BinomialPlot.java for the code.
- The simulation shows $\Pr[3 \text{ H in } 10 \text{ flips}] = 0.042469$. This matches the code in Bino-mail.java which produced $\Pr[3 \text{ H in } 10 \text{ flips}] = 0.042467328000000006$
13. CODE: Add code to Poisson.java to compute $\Pr[X = k]$ and plot a graph of $\Pr[X = k]$ vs. k when $\gamma = 2$. Use the Taylor series for e^x to prove that $\sum_k \Pr[X = k]$ adds up to 1.
- **Answer:**
 - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} = \sum_{n=0}^k \frac{x^n}{n!}$
 - $\sum_{i=0}^k \Pr[X = k] = \sum_{i=0}^k e^{-\gamma} * \frac{\gamma^i}{i!} = e^{-\gamma} \sum_{i=0}^k \frac{\gamma^i}{i!}$
 - **As shown on the first bullet,** $\sum_{i=0}^k \frac{\gamma^i}{i!} = e^{\gamma}$
 - $\Rightarrow \sum_{i=0}^k \Pr[X = k] = e^{-\gamma} * e^{\gamma} = e^0 = 1 \blacksquare$
14. CODE: Download BusStop.java and BusStopExample3.java, and modify the latter to estimate the probability that exactly three buses arrive during the interval $[0, 2]$. Compare this with $\Pr[X = 3]$ when $X \sim \text{Poisson}(2)$.
- **The BusStop yields a result of 0.1796. The Poisson code from 13 yields a result of 0.1804470443154836.**
19. Consider the distribution for the 3-coin-flip example:
- $\Pr[X = 0] = 0.064$
 - $\Pr[X = 1] = 0.288$
 - $\Pr[X = 2] = 0.432$
 - $\Pr[X = 3] = 0.216$
 - Sketch the CDF on paper.



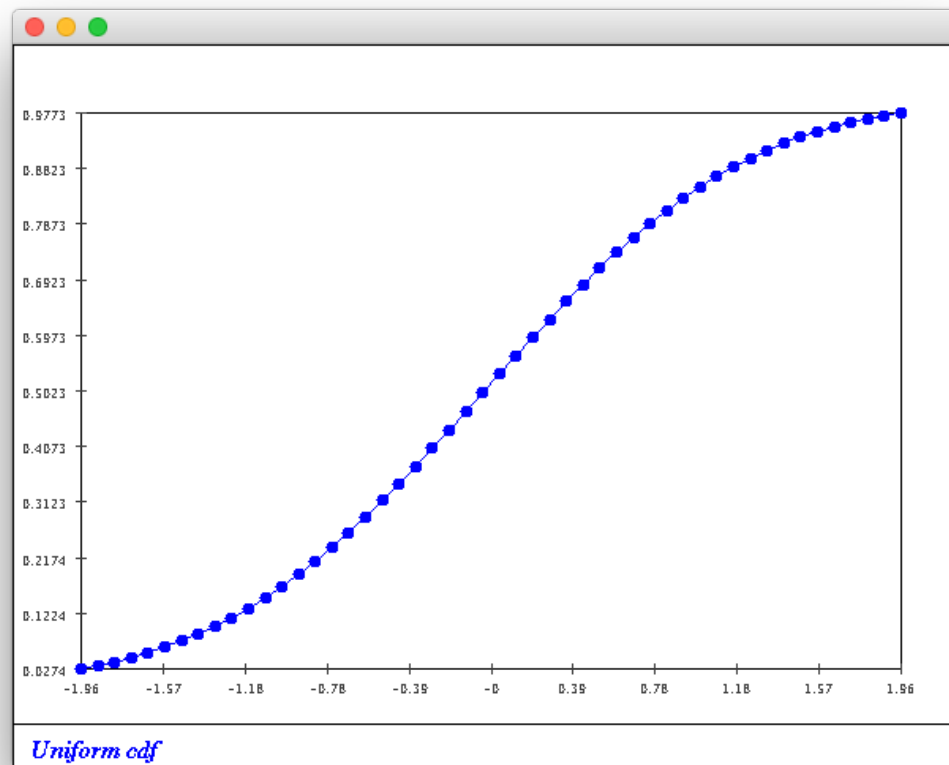
23. What is an example of a continuous rv associate with the QueueControl.java application?

- **avgWaitTime**

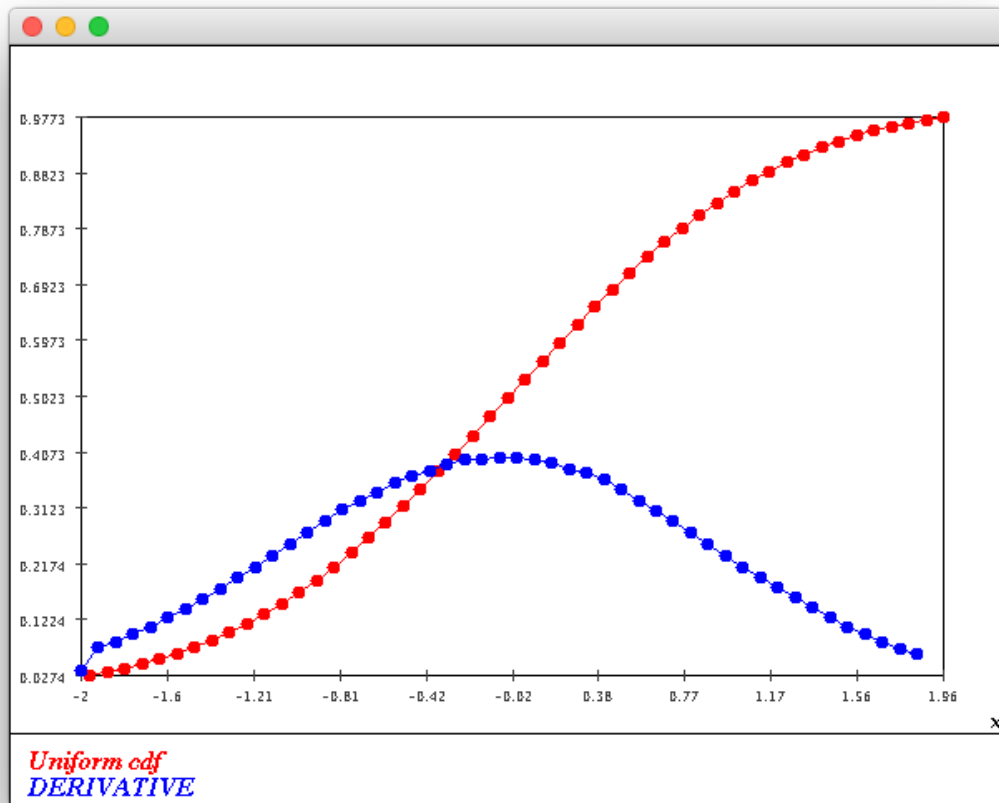
25. CODE: The program GaussianCDF.java estimates the CDF of a Gaussian rv. Execute the program to plot the CDF. Then, use this CDF to compute the following probabilities:

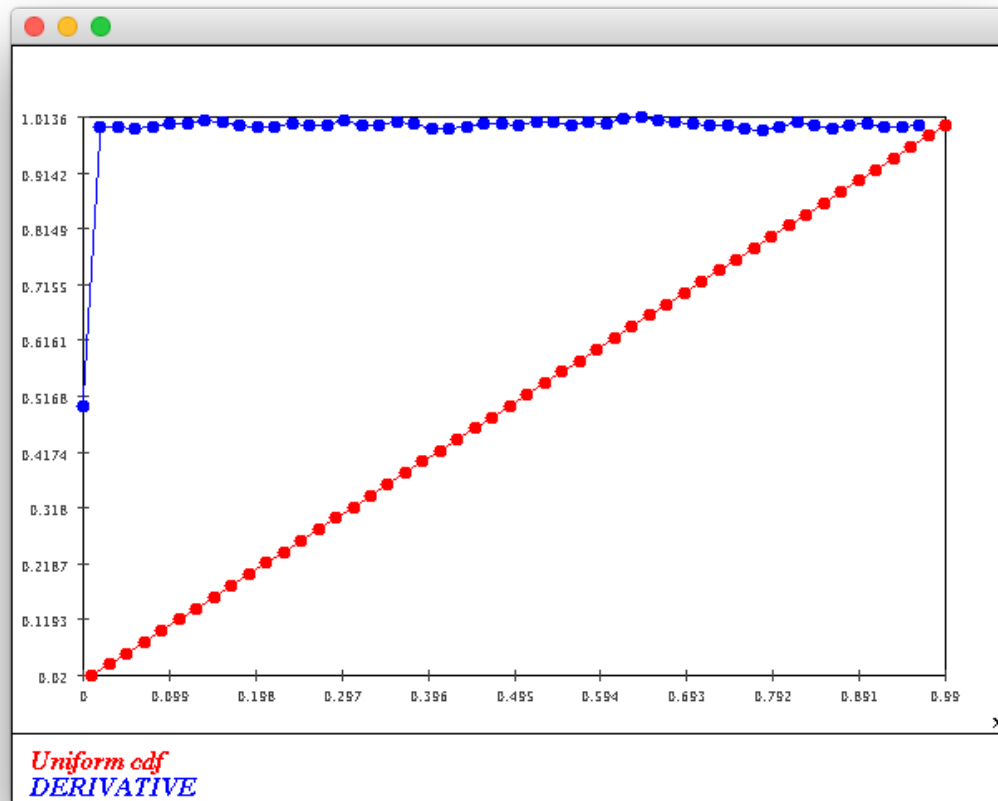
- $\Pr[0 < X \leq 2]$
- $\Pr[X > 0]$

- **Answers:**

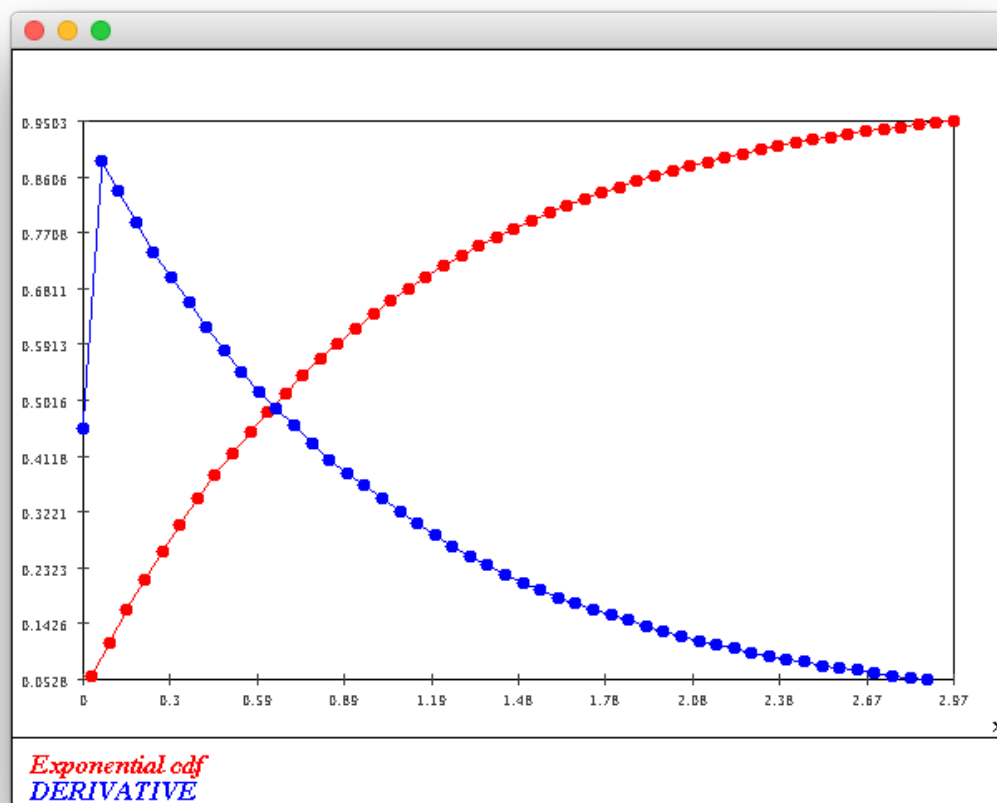


- - $Pr[0 < X \leq 2] = 0.462294$. In reality this should be 0.5. The error can be explained by the step size of the function.
 - $Pr[X > 0] = 0.512404$. In reality this should be 0.5. The error can be explained by the step size of the function.
26. CODE: Modify UniformCDF.java and GaussianCDF.java to compute the derivative of each. What is the shape of $F'(y)$ in each case?





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27. If X denotes the first interarrival time in the bus-stop problem, estimate the CDF of X as follows:
- Assume that values fall in the range $[0,3]$ (i.e., disregard values outside this range).
 - Use `ExponentialCDF.java` as a template, and add modified code from `UniformCDF.java`
 - Next, compute the derivative of this function and display it.



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28. Complete the calculation above. What would you get if $\Pr[H] = 0.5$?
- **Answers:**
 - **For $\Pr[H] = 0.6$:** $E[X] = \sum_{k \in \{0,1,2,3\}} k \Pr[X = k] = 0 \Pr[X = 0] + 1 \Pr[X = 1] + 2 \Pr[X = 2] + 3 \Pr[X = 3] = 0 * 0.064 + 1 * 0.288 + 2 * 0.432 + 3 * 0.216 = 1.8$
 - **For $\Pr[H] = 0.5$:** $E[X] = \sum_{k \in \{0,1,2,3\}} k \Pr[X = k] = 0 \Pr[X = 0] + 1 \Pr[X = 1] + 2 \Pr[X = 2] + 3 \Pr[X = 3] = 0 * 0.125 + 1 * 0.375 + 2 * 0.375 + 3 * 0.125 = 1.5$
29. How does this relate to the 3-coin-flip example?
- **Since the 3-coin-flip example can be modeled as a Binomial distribution, the formula $E[X] = np$ can be used to find the expectation. When plugging in 0.6 or 0.5, we get the same values we did above (1.8 and 1.5 respectively).**
31. What does $\frac{n_k}{n}$ become in the limit? Unfold the sum for the 3-coin-flip example to see why this is true.
- **It becomes $\Pr[X = k]$.**
 - **In the three-coin-flip example, the sum becomes $\sum_k k \frac{n_k}{n} = 0 * \frac{n_0}{n} + 1 * \frac{n_1}{n} + 2 * \frac{n_2}{n} + 3 * \frac{n_3}{n} = 0 * \Pr[X = 0] + 1 * \Pr[X = 1] + 2 * \Pr[X = 2] + 3 * \Pr[X = 3]$**
32. CODE: Download Coin.java and CoinExample3.java and let X = the number of heads in 3 coin flips.
- Compute the average value of X using $\frac{1}{n} S_n$
 - Estimate $\Pr[X = k]$ using $\frac{n_k}{n}$
 - Compute $\sum_k k \frac{n_k}{n}$ using the estimate of $\frac{n_k}{n}$
 - Compare with the $E[X]$ calculation you made earlier.

- Average value of X using $S_n/n = 1.800082$
 - Estimated probability of 0 heads: 0.064165
 - Estimated probability of 1 heads: 0.287286
 - Estimated probability of 2 heads: 0.432851
 - Estimated probability of 3 heads: 0.215698
 - Summation of $k * n_k/n = 1.800082$
 - These values align closely with the actual probabilities and expected value.
33. CODE: Use `Coin.java` and `CoinExample4.java` and let X = the number of flips needed to go get the first heads when $\Pr[\text{Heads}] = 0.1$. Compute the average value of X using $\frac{1}{n}S_n$ as you did in the previous exercise. Compare with the $E[X]$ calculation from earlier.
- The code produces the average value of $X = 10.008696$. Using the $E[X]$ formula for geometric distributions, we get $E[X] = 10$. These answers are within a reasonable distance apart.
34. CODE: Try this computation with the uniform, Gaussian, and exponential distributions using `UniformCDF2.java`, `GaussianCDF2.java`, and `ExponentialCDF2.java`. Explore what happens when more intervals are used in the expectation computation than in the CDF estimation.
35. CODE: Estimate the density of the time spent in the system by a random customer in the `QueueControl` example. To do this, you need to build a density histogram of values of the variable `timeInSystem` in `QueueControl.java`.
36. Suppose $X \sim \text{Exponential}(\gamma)$ with CDF $F(x)$. Write down an expression for $F^{-1}(y)$, the inverse of F .
37. CODE: Add code to `DiscreteGenExample.java` to implement the above generator, and to test it by building a histogram.
38. CODE: Add code to `ExponentialGenerator.java` to implement the above idea. Use the inverse-CDF you computed earlier. The test code is written to produce a histogram. Use your modified version of `PropHistogram.java` to make a density histogram. Compare the result with the actual density (using $\gamma = 4$). How do you know your code worked?