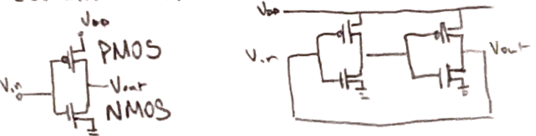
EECS 16B Midterm 1 Notes — Jeffrey Shen

① RC Circuits

Inverter: logical block output inverse, 0 <=> 1

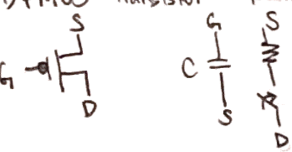
Oscillator: device oscillates between 0 and 1

V_{in}	V_{out}	NMOS	PMOS
V_{DD}	0	on	off
0	V_{DD}	off	on



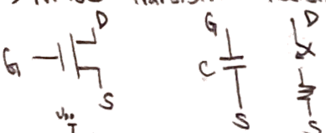
CMOS Inverter   Ring Oscillator

1) PMOS Transistor   Resistor-Switch Model

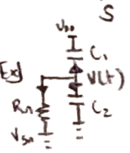


$V_{SG} \geq V_{tp}: V_{SG} = V_S - V_G$

2) NMOS Transistor   Resistor-Switch Model



$V_{GS} \geq V_{tn}: V_{GS} = V_G - V_S$



$d(V(t)) = \frac{V(t)}{R_n(C_1+C_2)}$

$V(t) = C_0 e^{-t/RC}$: Charging

$V(t) = V_{dd}(-e^{-t/RC}+1)$: Discharging

$P = \frac{V_{DD}^2}{R}$ when $V_{in} = V_{DD}$, or 0

② Input

$\frac{d}{dt} V(t) = \lambda v(t) + \lambda u(t)$ $V(t) = V_0 e^{\lambda t} + \lambda \int_0^t u(\theta) e^{\lambda(t-\theta)} d\theta$

③ Vector Differential Equations

$A\vec{v}_\lambda = \lambda \vec{v}_\lambda$ $\vec{v}_\lambda \neq 0$ eigenvector $\det(A - \lambda I_n) = 0$

$$\begin{array}{ccc} \vec{x}(t) & \xrightarrow{A_x = VAV^{-1}} & \frac{d}{dt}\vec{x}(t) \\ V \uparrow \downarrow V^{-1} & & V \uparrow \downarrow V^{-1} \\ \vec{y}(t) & \xrightarrow{\Lambda = VA_yV^{-1}} & \frac{d}{dt}\vec{y}(t) \end{array}$$

} Different Basis

$A_x \vec{x}(t) = V\Lambda V^{-1} \vec{x}(t)$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $V = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$

Solving:

1) Find Diff Eq $\frac{d}{dt}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$

2) Find Eigenvalue/vector $\det(\lambda I - A) \Rightarrow \lambda_1, \lambda_2, \vec{v}_1, \vec{v}_2$

$V = [\vec{v}_1, \vec{v}_2]$ $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

3) Change of Basis $\tilde{\vec{x}} = V^{-1}\vec{x}$ $\frac{d}{dt}\vec{x}(t) = V\Lambda V^{-1}\vec{x}$

$\frac{d}{dt} V^{-1}\vec{x}(t) = V^{-1} V \Lambda V^{-1} \vec{x} \Rightarrow \frac{d}{dt}\tilde{\vec{x}}(t) = \Lambda \tilde{\vec{x}}(t)$

4) Solve Eq & Initial $\tilde{\vec{x}}[1] = K_1 e^{\lambda t}$, $\tilde{\vec{x}}[2] = K_2 e^{\lambda_2 t}$ $\begin{bmatrix} \tilde{\vec{x}}[1](0) \\ \tilde{\vec{x}}[2](0) \end{bmatrix} = V^{-1} \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix}$

5) Find eq (original basis) $\vec{x} = V\tilde{\vec{x}}$

If Charging

1) Make substitution $\frac{d}{dt}\vec{x}(t) = A\vec{x} + \vec{b} \Rightarrow A\hat{x} = A\vec{c} + \vec{b} \Rightarrow \frac{d}{dt}\hat{x} = A\hat{x}$

2) Find initial $\hat{\vec{v}} = \hat{A}^{-1}(\hat{A}\vec{v} + \vec{b}) = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$

3) Find eq (original basis) $\vec{v} = \hat{A}^{-1}(\hat{A}\hat{\vec{v}} - \vec{b}) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Inductors: stores energy in magnetic field [H]

$V(t) = L\frac{dI_L(t)}{dt}$ Series: $L_1 + L_2$ Parallel: $\frac{L_1L_2}{L_1+L_2}$ $\frac{\text{prod}}{\text{sum}}$

③ Complex Nums

$j = \sqrt{-1}$ $z = a + jb$ Conjugate: $\bar{z} = a - jb$ $|z| = \sqrt{a^2+b^2}$

$z = |z|e^{j\theta} = |z|(\cos\theta + j\sin\theta)$ $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

rotation matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

④ Phasors

$V(t) = V_0\cos(\omega t + \phi_v)$
$i(t) = I_0\cos(\omega t + \phi_i)$

V_0, I_0: amplitude
ω: angular frequency
ϕ: phase shift

$\omega = 2\pi f$ $f = \frac{1}{T}$ cycles/sec, period

$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$

$v(t) = V_0\cos(\omega t + \phi) = Re(V_0 e^{j(\omega t + \phi)}) = Re(V_0 e^{j\phi} e^{j\omega t})$

Phasor: not time dependent $\tilde{V} = V_0 e^{j\phi}$ $\sin(t + \frac{\pi}{2}) = \cos(t)$

$V_0\cos(\omega t + \phi) = \frac{1}{2}(\tilde{V}e^{j\omega t} + \bar{\tilde{V}}e^{-j\omega t})$

Impedance: depend on $j\omega$ $Z = \frac{\tilde{V}}{\tilde{I}}$

Resistor: $Z_R = R$ Capacitor: $Z_C = \frac{1}{j\omega C}$ Inductor: $Z_L = j\omega L$

Transfer function:

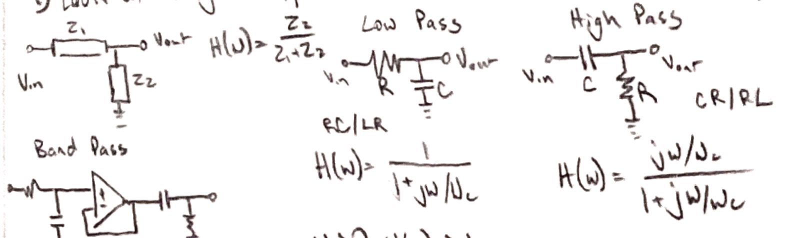
$H(j\omega) = \frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)}$

KCL works for phasors
ω is constant throughout

⑤ Circuit Filters

$H(\omega) = M(\omega)e^{j\theta}$ magnitude of transfer func

1) Express sinusoidal input as phasors (coefficient of $e^{j\omega t}$)
2) Solve output V/I phasors as func of input phasor
3) Determine transfer func
4) Look at magnitude/phase shift of transfer func at input freq



$H(\omega) = \frac{1}{1 + j\omega/\omega_c}$ $H(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$

$\omega \to 0, H(\omega) \to 1$ $\omega \to \infty, H(\omega) \to 1$

Unity Gain Buffer to access output voltage of filter w/o drawing current, prevent second loading first

Make current draw low by increasing impedance

$f_{cutoff} = \frac{1}{2\pi RC}$

Bode Plots:

log-log plot (Magnitude $H(\omega)$)   Phase of $H(\omega)$ angle-log plot

$\frac{d}{dt} x(t) = \lambda x(t) \qquad x(t) = x(0)e^{\lambda t}$

$\frac{d}{dt} x(t) = \lambda x(t) + \alpha \qquad x(t) = x(0)e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$

$\frac{d}{dt} x(t) = \lambda x(t) + u(t) \qquad x(t) = x(0)e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau$

Change of variables $\tilde{V}_{out} = V_{out} - V_{in}$

ω_c : cutoff freq

ω_c st $|H(\omega_c)| = \frac{1}{\sqrt{2}} |H(\omega_c)_{max}|$ | $\frac{1}{RC}$ or $\frac{R}{L}$

If transfer functions in series w/ unity gain buffers, we can multiply transfer funcs

Resonance

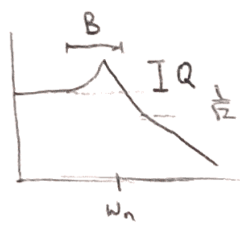
λ → real = decay "envelop"
→ imaginary = oscillation

Impedance - generalized resistance $[\Omega]$ $Z = R + jX$

Resistance Reactance

$\zeta = \frac{1}{2} \frac{R}{\sqrt{\frac{L}{C}}}$ "damping ratio"

$Q = \frac{1}{2\zeta}$



Bandwidth $B = \frac{\omega_n}{Q} = 2\omega_n \zeta$

Resonance frequency: $\frac{1}{\sqrt{LC}}$ when impedances of inductor and capacitor are equal

For differential equation $\frac{d^2}{dt^2} x(t) + a \frac{d}{dt} x(t) + b x(t) = 0$

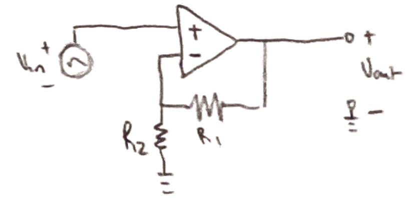
$\lambda^2 + a\lambda + b = 0 \qquad \lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$

1) Critically damped: $a^2 - 4b = 0$ repeated eigenvalue, real
2) Undamped: a is 0 eigenvalues purely imaginary
3) Underdamped: $a^2 - 4b < 0$ eigenvalues complex, real and imag
4) Overdamped $a^2 - 4b > 0$ eigenvalues purely real

Circuits

Unity Gain Buffer: $\frac{V_{out}}{V_{in}} = 1$

Non-Inverting Amplifier: $V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right)$



$i(0) = \omega = 0 = j\omega L = 0$ Inductors = short circuit
Capacitors = open circuit

NMOS

High = 1	1
Low = 0	0

PMOS

High = 1	0
Low = 0	1

$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$

$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

Phasors

1) Adopt Cosine reference (time domain)
2) Transfer to phasor domain
   $i \to \tilde{I} \qquad L \to Z_L = j\omega L$
   $v \to \tilde{V} \qquad C \to Z_C = \frac{1}{j\omega C}$
   $R \to Z_R = R$
3) Cast Equations in phasor form
4) Solve for unknown
5) Transform back to Time domain
   $i = Re[\tilde{I} e^{j\omega t}]$
   $= 6\cos(\omega t - 105°)$

Units

$m = 10^{-3}$	$k = 10^3$
$\mu = 10^{-6}$	$M = 10^6$
$n = 10^{-9}$	$G = 10^9$
$p = 10^{-12}$	$T = 10^{12}$

EECS 16B MT 2 Notes Jeffrey Shen

6A) State Space Models

State variables: internal variable representing state of a dynamic system

State vector: vector of state variables

State model: vector differential equation of vector

General form State Equations: n states and m inputs

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t)) \quad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Linear: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \quad A \in \mathbb{R}^{n \times n} \quad B \in \mathbb{R}^{n \times m}$

Equilibrium State: for all $t \geq t_0$, remains at $\vec{x}^*$

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}^*) = 0 \qquad \vec{x}(t) = \vec{x}^* \quad t \geq t_0$$

6B) Equilibrium & Linearization

Linear if:
1) Scaling $f(ax) = af(x)$
2) Additivity $f(x+y) = f(x) + f(y)$

Linear systems w/ inputs @ equilibrium: constant $\vec{u}$

$$A\vec{x} + B\vec{u} = 0$$

Linearization: approx nonlinear system

Taylor: $f(x) \approx f(x^*) + \nabla f(x)|_{x=x^*}(x - x^*)$

1) Find $\vec{x}^*$ set $\frac{d}{dt}\vec{x} = 0$, find $\vec{x}^*$
2) Find $\nabla f(x)$
3) $\nabla f(x)|_{\vec{x}=\vec{x}^*}$
4) $\frac{d}{dt}(\vec{x}(t)) = \nabla f(\vec{x})|_{\vec{x}^*}\tilde{x}(t)$

Jacobian: $$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1(x_1, x_2 \ldots x_n)}{\partial x_1} & \frac{\partial f_1(\vec{x}, x_2 \ldots x_n)}{\partial x_2} & \frac{\partial f_1(x_1, x_2 \ldots)}{\partial x_n} \\ \frac{\partial f_2(x_1, x_2, x_n)}{\partial x_1} & \frac{\partial f_2(\vec{x}, x_2 \ldots x_n)}{\partial x_2} & \frac{\partial f_2(x_1 x_2)}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n(x_1, x_2 \ldots x_n)}{\partial x_1} & \frac{\partial f_n(\vec{x}, x_2 \ldots x_n)}{\partial x_2} & \frac{\partial f_n(x_1 x_2 \ldots)}{\partial x_n} \end{bmatrix}$$

$f_i \downarrow$ $\quad x_i \rightarrow$

$\vec{x}^*$ is equilibrium

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)) \qquad f(\vec{x}^*) = 0 \qquad \tilde{x}(t) := \vec{x}(t) - \vec{x}^*$$

$$A \triangleq \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*} \qquad \frac{d}{dt}\tilde{x}(t) \approx A\tilde{x}(t)$$

7A) Linearization & Discrete Time Systems

Linearizing systems w/ inputs: equilibrium $\vec{x}^*$ and $\vec{u}^*$ $\quad f(\vec{x}^*, \vec{u}^*) = 0$

$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$ $\quad$ 1) $\tilde{x}(t) := \vec{x}(t) - \vec{x}^*$ $\quad \tilde{u}(t) := \vec{u}(t) - \vec{u}^*$

2) Find $\nabla_x f(\vec{x}, \vec{u})$ and $\nabla_u f(\vec{x}, \vec{u})$ set

$A := \nabla_x f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*} \qquad B := \nabla_u f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}$

3) Linearization: $\frac{d}{dt}\tilde{x}(t) \approx A\tilde{x}(t) + B\tilde{u}(t)$

Discrete Time System: evolves w/ difference equation

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

7B) Discretization

Changing State Variables: transform to $\vec{z}$

1) $\vec{z} := T\vec{x}$

2) $A_{new} = TAT^{-1} \qquad B_{new} = TB$

3) $\frac{d}{dt}\vec{z}(t) = A_{new}\vec{z}(t) + B_{new}\vec{u}(t)$

$$x(kT+T) - x(kT) = \int_{kT}^{kT+T} RHS \, d\tau$$

Digital Control: continuous vs. discrete

input is sampled every T units time $\quad \vec{x}(0), \vec{x}(T), \vec{x}(2T)$

$\vec{x}_d(k) := \vec{x}(kT) \qquad \vec{u}(t) = \vec{u}_d(k) \quad t \in [kT, (k+1)T]$

Cont → Discrete

Discrete → Continuous (zero order hold)

Discretization w/ single state single input

$$x_d(k+1) = A_d\vec{x}_d(k) + B_d\vec{u}_d(k)$$

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$$

$$x_d(k+1) = \lambda_d x_d(k) + b_d u_d(k)$$

where $\lambda_d = e^{\lambda T} \qquad b_d = b\int_0^T e^{\lambda s} ds = \begin{cases} bT & \text{if } \lambda = 0 \\ b\frac{e^{\lambda T} - 1}{\lambda} & \text{if } \lambda \neq 0 \end{cases}$

8A) Controllability

let $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ $\quad$ let $A_d = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix}$ $\quad B_d = \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix}$

Using $\vec{z} = V^{-1}\vec{x}$ where V is eigenvectors

$$\vec{z}_d[k+1] = A_d\vec{z}_d[k] + B_d V^{-1} B u_d[k] \qquad \vec{x}_d[k] = V\vec{z}_d[k]$$

$$\vec{x}_d[k+1] = VA_dV^{-1}\vec{x}_d[k] + VB_dV^{-1}Bu_d[k]$$

Controllability

$$\vec{x}(t) = A^t\vec{x}(0) + \begin{bmatrix} B & AB & \ldots & A^{t-2}B & A^{t-1}B \end{bmatrix} \begin{bmatrix} \vec{u}(t-1) \\ \vec{u}(t-2) \\ \vdots \\ \vec{u}(1) \\ \vec{u}(0) \end{bmatrix}$$

If there exists t and input sequence $\vec{u}(0), \vec{u}(1), \ldots, \vec{u}(t-1)$ such that $\vec{x}(t) = \vec{x}_{target}$ then it is controllable

Controllability $\Leftrightarrow$ $\text{span}\{\vec{b}, A\vec{b}, \ldots, A^{n-2}\vec{b}, A^{n-1}\vec{b}\} = \mathbb{R}^n$

8B) System Identification

$$\vec{y} = D\vec{p} + \vec{e}$$

$y \in \mathbb{R}^l$ measurements; $l \times k$ matrix; unknown parameters; measurement error

$m \times n$

$m > n$

Given

scalar: $x(t+1) = \lambda x(t) + bu(t) + e(t)$

$$\underbrace{\begin{bmatrix} x(0) & u(0) \\ x(1) & u(1) \\ \vdots & \vdots \\ x(l-1) & u(l-1) \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(l-1) \end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(l) \end{bmatrix}}_{\vec{y}}$$

vector: $\vec{x}(l)^T = \vec{x}(l-1)^T A^T + \vec{u}(l-1)^T B^T + \vec{e}(l-1)^T$

$$\underbrace{\begin{bmatrix} \vec{x}(0)^T & \vec{u}(0)^T \\ \vec{x}(1)^T & \vec{u}(1)^T \\ \vdots & \vdots \\ \vec{x}(l-1)^T & \vec{u}(l-1)^T \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}_{[\vec{p}_1 \ldots \vec{p}_n]} + \underbrace{\begin{bmatrix} \vec{e}(0)^T \\ \vec{e}(1)^T \\ \vdots \\ \vec{e}(l-1)^T \end{bmatrix}}_{[\vec{e}_1 \ldots \vec{e}_n]} = \underbrace{\begin{bmatrix} \vec{x}(1)^T \\ \vec{x}(2)^T \\ \vdots \\ \vec{x}(l)^T \end{bmatrix}}_{[\vec{y}_1 \ldots \vec{y}_n]}$$

$$\hat{\vec{p}} = (D^TD)^{-1}D^T\vec{y}_i$$

9A) Singular Value Decomposition (SVD)

SVD separates rank r matrix $A \in \mathbb{R}^{m \times n}$ into sum of r rank 1

Find 1) orthonormal vectors $\vec{u}_1, \ldots \vec{u}_r \in \mathbb{R}^m$

2) orthonormal vectors $\vec{v}_1, \ldots \vec{v}_r \in \mathbb{R}^n$

3) real positive numbers $\sigma_1, \ldots \sigma_r$ $\qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$

$$A = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \ldots \sigma_r\vec{u}_r\vec{v}_r^T$$

Orthonormal: columns a_i are

1) Orthogonal $\langle a_i, a_j \rangle = 0 \quad i \neq j$

9B) Finding SVD

Using A^TA

1) Find eigenvalues of A^TA , $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$, $(A-\lambda I)=0$
2) Find orthonormal eigenvectors $\vec{v}_i$, (Plug in $(A-\lambda I)$ matrix and find where $=\vec{0}$)

$$A^TA\vec{v}_i = \lambda_i \vec{v}_i$$

3) Let $\sigma_i = \sqrt{\lambda_i}$ get $\vec{u}_i$ $i=1,\dots,r$

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$$

Using AA^T

1) Find eigen values of AA^T
2) Find orthonormal eigenvectors $\vec{u}_i$

$$AA^T\vec{u}_i = \lambda_i \vec{u}_i$$

3) Let $\sigma_i = \sqrt{\lambda_i}$ get $\vec{v}_i$

$$\vec{v}_i = \frac{1}{\sigma_i} A^T\vec{u}_i$$

$U^TU = I$
$V^TV = I$

11A) SVD Cont.

$m \times n$: $A = U \quad \Sigma \quad V^T$

$U = [\vec{u}_1 \dots \vec{u}_r \dots \vec{u}_m]$ $m \times m$

$$S = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \sigma_r & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right]$$
$m \times n$

$[\vec{v}_1 \dots \vec{v}_r \dots \vec{v}_n]$ $n \times n$

! U and V must be orthonormal so append orthonormal basis if needed

Geometric Interpretation:

$A\vec{x}$ composition of

1) $V^T\vec{x}$: reorients $\vec{x}$ w/o changing length
2) $\Sigma V^T \vec{x}$: stretches along axis w/ singular value
3) $U\Sigma V^T \vec{x}$: reorients vector w/o changing length

if $\|\vec{x}\|=1$ then $\|A\vec{x}\| \leq \sigma_1$, for $\vec{x}=\vec{v}_1$ $\|A\vec{x}\| = \sigma_1$

Symmetric Matrices: $Q = Q^T$ (Spectral Theorem)

Symmetric Matrix has

- real eigen values
- real orthonormal eigenvectors
- diagonalizable $(A = V\Lambda V^T)$

Rank-Nullity Theorem:

For $A = n \times m$

$\text{rank}(A) + \text{Null}(A) = m$

$(AB)^T = B^TA^T$

If Q can be written as $Q = R^TR$ for some R, eigenvalues are nonnegative

General

- Stable if eigenvalue <1
- Plugging into x with θ value $\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$ gives eigen vectors
- inverse of orthogonal matrix is same as transpose

Controllability

To reach $x(n)$:

$$\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = \begin{bmatrix} \vec{b} & A\vec{b} & \dots & A^{n-1}\vec{b} \end{bmatrix}^{-1} \vec{x}(n)$$

EECS 16B Final Notes Jeffrey Shen

11B Applications of SVD

- Least Squares with SVD tall matrix $m \times n$, $m > n$

$$A = U \begin{bmatrix} S \\ 0_{(m-n) \times n} \end{bmatrix} V^T \qquad \vec{y} = A\vec{x} + \vec{e}$$
$$= \Sigma$$
$$\boxed{\vec{x} = V S^{-1} U_1^T \vec{y}}$$

Minimum Norm Solution wide matrix $m \times n$, $m < n$

- Many choices for $\vec{x}$ so we want to choose shortest length

$$A = U \begin{bmatrix} S & 0_{m \times (n-m)} \end{bmatrix} V^T \qquad \vec{y} = A\vec{x} = USV_1^T\vec{x}$$
$$= \Sigma$$
$$\boxed{\vec{x} = V_1 S^{-1} U^T \vec{y}} \qquad V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \quad V^T = \begin{bmatrix} -v_1- \\ -v_2- \end{bmatrix}$$

Principle Component Analysis (PCA)

- finds most informative directions in a data set

1) From each measurement subtract avg $A \in \mathbb{R}^{m \times n}$
2) "covariance matrix": $\frac{1}{m-1} A^T A$: $n \times n$ matrix

- Eigenvalues of matrix are singular values of A except for scaling factor $m-1$
- Orthonormal eigenvectors correspond to $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ of SVD
- Eigen vectors corresponding to largest singular values are principle components and identify dominant directions

12A Min Energy / Stability

Use min norm least Energy: have

$$\begin{bmatrix} P_{target} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{l-1}\vec{b} \end{bmatrix}}_{C_l} \begin{bmatrix} u_d(l-1) \\ u_d(l-2) \\ \vdots \\ u_d(0) \end{bmatrix}$$

to get desired state in l time steps

$$\begin{bmatrix} u(l-1) \\ u(l-2) \\ \vdots \\ u(0) \end{bmatrix} = C_l^T (C_l C_l^T)^{-1} \begin{bmatrix} P_{target} \\ 0 \end{bmatrix}$$

Controllability matrix target state vector

Stability of Linear State Models

System: $x(t+1) = \lambda x(t) + bu(t)$

$$x(t) = \lambda^t x(0) + \lambda^{t-1} bu(0) + \lambda^{t-2} bu(1) + \dots + bu(t-1) \qquad x(t) = \lambda^t x(0) + \sum_{0}^{t-1} \lambda^{t-1-k} bu(k)$$

Stability: Stable: $x(t)$ is bounded for any initial and bounded input

Unstable: can find initial and bounded input st $|x(t)| \to \infty$ as $t \to \infty$

$|\lambda| > 1$: unstable - $|\lambda^t|$ grows unbounded $|\lambda|^t |x(0)| \to \infty$

$|\lambda| < 1$: Stable - $\left|\sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k)\right| \le |b| M \sum_{s=0}^{t-1} |\lambda|^s \to \frac{1}{1-|\lambda|}$

$|\lambda| = 1$: "marginal stability" - if $b=0$ $|\lambda^t x(0)| = |x(0)|$ else $b \neq 0$ unbounded

Vector Case

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) \qquad \vec{x}(t) = A^t x(0) + \sum_{k=0}^{t-1} A^{t-1-k} Bu(k)$$

$|\lambda_i| < 1$ for all eigenvalues : stable

$|\lambda_i| > 1$ for at least one eigenvalue : unstable

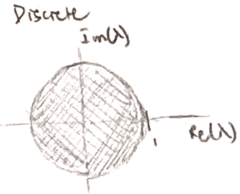
controllable $\Rightarrow$ stabilizeable

stabilizeable $\not\Rightarrow$ controllable

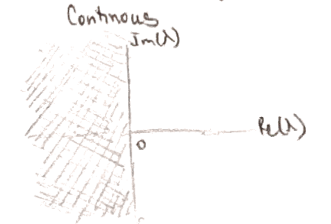
12B Stability

$$x(t) = e^{\lambda t} x(0) + b\int_0^t e^{\lambda(t-s)} u(s)\, ds$$

Discrete



Continous



$\boxed{\text{Stable: } |\lambda| < 1}$ $\boxed{\text{Stable: } Re\{\lambda\} < 0}$

Transient Behavior

- Oscillates if has imaginary
- grows unbounded if outside stability zone
- converges if inside stability zone

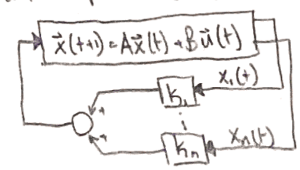
13A State Feedback Control

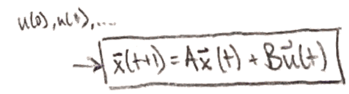
$\vec{x}(t+1) = A\vec{x} + B\vec{u}$ choose $\vec{u} = K\vec{x}$

then $\vec{x}(t+1) = (A + BK)\vec{x}(t)$

Choose K so all eigenvalues inside unit circle

Closed Loop Control	Open Loop Control
Can shape transients (well damped conv)	Sensitive to disturbances





1) Find A+BK
$$\begin{bmatrix} a_{11} + b_1 k_1 & a_{12} + b_1 k_2 \\ a_{21} + b_2 k_1 & a_{22} + b_2 k_2 \end{bmatrix} \qquad BK = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} [k_1 \ k_2]$$

2) Find eigenvalues/equation
$$\lambda_1 = 1 + k_1 \qquad \lambda_2 = 2 \qquad \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

3) Solve for values of k or declare unstable

$|\lambda_2| > 1$ unstable

13B Feedback, Controller Canonical Form

Controller Canonical Form

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & & a_n \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} \dots - a_1$$

$$\det(\lambda I - (A_c + B_c K)) = \lambda^n - (a_n + k_n)\lambda^{n-1} \dots - (a_2 + k_2)\lambda - (a_1 + k_1)$$

can tune each k value by the coefficient of polynomial to reach any set of eigenvalues

If system is controllable, can assign eigenvalues of A+BK with choices of K

- CCF has eigenvalues on bottom row

(14A) Upper Triangularization

- NxN matrix is diagonalizable (values only in diagonal) if it has n linearly independent eigenvectors, distinct eigenvalues
- Any square matrix can be upper triangular
- Uppertriangular matrix has eigenvalues along diagonal

Gram-Schmidt Process

Algo takes linearly independent vectors $\{s_1, \dots s_n\}$ and generates an orthonormal set of vectors $\{q_1 \dots q_n\}$ that span same space

1) Find unit vector $\vec{q}_1$ $\text{span}(\{q_1\}) = \text{span}(\{s_1\})$

$$\vec{q}_1 = \frac{\vec{S}_1}{\|\vec{S}_1\|}$$

2) Find $\vec{q}_2$

1. span same
2. orthogonal
3. normal

$$Z_2 = \vec{S}_2 - \alpha\vec{q}_1 \quad ①$$

get orthogonal vector with OMP

$$\text{proj}_{\vec{q}_1} \vec{S}_2 = \vec{q}_1^T \vec{s}_2 \vec{q}_1 \qquad ② \quad \vec{Z}_2 = \vec{S}_2 - \vec{q}_1^T \vec{S}_2 \vec{q}_1$$

$$③ \quad \vec{q}_2 = \frac{\vec{Z}_2}{\|\vec{Z}_2\|}$$

Algo

$$q_1 = \frac{\vec{S}_1}{\|\vec{S}_1\|}$$

for $i = 2 \to n$:

$$\vec{Z}_i = \vec{S}_i - \sum_{j=1}^{i-1} (\vec{s}_i^T \vec{q}_j)\vec{q}_j$$

$$\vec{q}_i = \frac{\vec{Z}_i}{\|\vec{Z}_i\|}$$

Conceptual

PCA: How to approx higher dimension data into lower dimension essence. First Principle Component is which 1D line best approx, by projecting data points onto it

SVD: Decompose A into sum of rank 1 matrices ith rank 1 matrix formed from taking outer product of normalized column vectors $\vec{u}_i$ and normalized row vectors $\vec{v}_i^T$ scaled by σ_i. Adding each matrix for singular value on top of each other

Phasors: By converting to phasor domain, you take snapshot of time domain, solve it relative to snapshot and phasor extends to all phases so you can convert back to time domain, and result will describe solution