Math 53 Final

15.9 Change of Variables in Multiple Integrals

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian: $x = g(u,v)$ $y = h(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

1. Find $u =$, $v =$ suitable
2. Find $x =$, $y =$
3. Find Jacobian and plug in abs value
4. Plug in to equation

Chapter 16: Vector Calculus

16.1 Vector Fields

Vector Field assigns to each point (x,y) in D vector $F(x,y)$

Gradient Field $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$

Conservative if some f that $F = \nabla f$

f is potential function of F

16.2 Line Integrals

Line Integral of F along C arc length

$$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt = \int_C F\cdot T\,ds$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

16.3 Fundamental Theorem for Line Integrals

Thm: Let C be smooth curve $\vec{r}(t)$ $a \le t \le b$
f be differentiable func whose ∇f is continous on C

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Independent of path if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for $\int_C F \cdot dr$
conservative vector field depends only on initial and terminal point

Thm: F is vector field continous on open (doesn't contain boundary points) connected (any 2 points can be joined by path) region D. If $\int_C F \cdot dr$ is independent of path,
there is f that $\nabla f = F$

Thm:

Condition	Then
① Open simply-connected (no holes) region D	$\vec{F}$ is conservative
② P and Q have continous first partials	
③ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D	

Thm: $\int_C F \cdot dr$ is independent of path in D if and only if $\int_C F \cdot dr = 0$ for every closed path C in D

16.4 Green's Theorem

Positive orientation is counter clockwise

Thm:

Conditions

1. C is positively oriented
2. piecewise-smooth
3. simple closed curve
4. P and Q have continous partial derivatives on open region

Then!

let D be region bound by curve C

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

16.5 Curl and Divergence

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times F$$

Thm: If fn of 3 vars has continous second-partials,
curl $(\nabla f) = 0$

Thms:

$\vec{F}$ conservative	=>	curl $(F) = 0$
curl$(F) \neq 0$	=>	$\vec{F}$ not conservative
curl $(F) = 0$ & simply connected & continous partial derivatives	=>	$\vec{F}$ conservative
$F = \langle P, Q, R \rangle$ have second partials	=>	div(curl(F)) $= 0$
div $(F) \neq 0$	=>	$\vec{F}$ is not a curl

div $F = \nabla \cdot F$

16.6 Parametric Surfaces and their Areas

Parametric Surface : $r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Surface of Revolution : $x = x$ $\quad y = f(x)\cos\theta \quad z = f(x)\sin\theta$
(about x-axis)

Tangent plane contains r_u and r_v vectors and
Normal vector to tangent plane is $r_u \times r_v$

Surface Area : $$A(S) = \iint_D \| r_u \times r_v \| \, dA$$

or

$$A(S) = \iint_D \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} \, dA$$

16.7 Surface Integrals

Surface Integral of f over surface S

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u \times r_v|\,dA$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}\,dA$$

Surface Integral of F over S
(Flux of F across S)

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS$$

$$\iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v)\,dA$$

$$\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

16.8 Stokes' Theorem

Thm:

Conditions:

1. S is piecewise smooth surface
2. Bounded by simple, closed piecewise-smooth boundary curve C w/ positive orientation
3. Continuous partial derivatives

Then:

$$\int_C F \cdot dr = \iint_S \text{curl } F \cdot dS$$

16.9 Divergence Theorem

Thm

Conditions:

1. E is a simple solid region
2. S be the boundary surface of E positive outward orientation
3. continuous partial derivaties on an open region contains E

Then:

$$\iint_S F \cdot dS = \iiint_E \text{div } F \; dV$$