Math 53 Midterm 2

Chapter 14: Partial Derivatives

14.1 Functions of Several Variables

- Function f of two vars $f(x,y)$
- Domain is region in xy-plane, range is z, $f(x,y)$
- Level curves are curves with equations $f(x,y)=k$ where k is constant

14.2 Limits and Continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

- If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along path C_1 and
  $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ on path C_2 where $L_1 \neq L_2$ lim DNE

Approach from different lines

Ex) $x=0$ $\lim_{(0,y)} f(x,y) = L$ | $y=x$ $\lim_{(x,x)} f(x,y) = L$ | $y=mx$ $\lim_{(x,mx)} f(x,y)=L$

- function is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
  continuous on D if f is continuous at every point (a,b) in D

14.3 Partial Derivatives

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) \qquad f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y)$$

Clairaut's Theorem: $f_{xy}(a,b) = f_{yx}(a,b)$

Laplace Equation: $u_{xx} + u_{yy} = 0$

14.4 Tangent Planes and Linear Approximations

Eq of tangent plane: of surface $z=f(x,y)$ at point $P(x_0,y_0,z_0)$

$$z-z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

- If partials exist near (a,b) and are continuous at (a,b)
  then f is differentiable at (a,b)

$$L(x,y) = f_x(x-x_0) + f_y(y-y_0) + z_0$$

- $dw = w_x\, dx + w_y\, dy + w_z\, dz$

14.5 Chain Rule

① $z = f(x,y)$ where $x = g(t)$ $y = h(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Case 2: $z = f(x,y)$ $x = g(s,t)$ $y = h(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

14.6 Directional Derivatives and the Gradient Vector

If f is differential function of x and y f has directional derivative in direction of unit vector $\vec{u} = \langle a, b\rangle$

$$D_u f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b = \nabla f \cdot \vec{u}$$

Gradient: fastest increase

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$$

- Max of Directional Derivative $D_u f(\vec{x})$ is $|\nabla f(\vec{x})|$ in when $\vec{u}$ has same direction as gradient $\nabla f(x)$

Tangent Plane

Gradient is $\vec{n}$ of tangent plane

Plane: $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$

14.7 Max and Min Values

- If f has a local max or min at (a,b) the first order partials then $f_x(a,b) = 0$ and $f_y(a,b) = 0$
- A point (a,b) is a critical point if $f_x(a,b) = 0$ and $f_y(a,b)$ or one of partials are DNE

If (a,b) is a critical point

$D = f_{xx}(a,b) f_{xy}(a,b) - [f_{xy}(a,b)]^2$

① $D > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$ is local min

② $D > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$ is local max

③ $D < 0$, $f(a,b)$ not local min or max (saddle point)

④ $D = 0$, no information, could be any

If f is continuous on closed, bounded set D in $\mathbb{R}^2$ there is max $f(x_1,y_1)$ and min $f(x_2,y_2)$ in D

Steps to find max, min

① Find vals of f at critical points of f in D

② Find extreme values of f on boundary of D

③ largest is max, smallest is min

14.8 Lagrange Multipliers

To find min max of $f(x,y,z)$ constraint $g(x,y,z)=k$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ $\quad g(x,y,z)=k$

$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \qquad g(x,y,z)=k$

Two Constraints

$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$

Chapter 15: Multiple Integrals

15.1 Double Integrals over Rectangles

$V = \iint_R f(x,y)\, dA$

Avg value $= \dfrac{\iint_R f(x,y)\, dA}{\text{area of } R} = \dfrac{\text{Volume total}}{\text{Area of base}}$

15.2 Double Integrals over General Regions

- Make sure each (partial or single) integral is doable
- If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M A(D)$$

15.3 Double Integrals in Polar

$r^2 = x^2 + y^2$ $\quad x = r\cos\theta \quad y = r\sin\theta$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

15.4 Applications of Double Integrals

Mass: $m = \iint_D \rho(x,y)\,dA$ $\qquad \rho(x,y)$ is lamina func

Moment: $M_x = \iint_D y\,\rho(x,y)\,dA$ $\qquad M_y = \iint_D x\,\rho(x,y)\,dA$

Center of mass: $\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint x\,\rho(x,y)\,dA$ $\qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint y\,\rho(x,y)\,dA$

Moment of Inertia

$I_x = \iint_D y^2 \rho(x,y)\,dA$ $\qquad I_y = \iint_D x^2 \rho(x,y)\,dA$

I_0 (around origin) $= \iint_D (x^2+y^2)\,\rho(x,y)\,dA = \iint_D r^2\,\rho(x,y)\,dA$

Probability

- Probability isn't negative scale 0-1

$f(x,y) \ge 0 \qquad \iint_R f(x,y)\,dA = 1$

15.6 Triple Integrals

$\iiint_E f(x,y,z)\,dV$ $\qquad V(E) = \iiint_E dV$

15.7 Cylindrical Coordinates Triple Integrals

$x = r\cos\theta \quad y = r\sin\theta \quad z = z \quad \Big| \quad r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$

$$\iiint_E f(x,y,z)\,dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta)\; r\,dz\,dr\,d\theta$$

15.8 Spherical Coordinates Triple Integrals

$\rho \geq 0 \quad , \quad 0 \leq \phi \leq \pi$

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi \quad \rho^2 = x^2 + y^2 + z^2$

$$\iiint_E f(x,y,z)\,dV = \int_c^d\int_{\alpha}^{\beta}\int_a^b f\;\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$