Math 53 Final Formula Sheet Jeffrey Shen

Trig Identities

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin(2x) = 2 \sin x \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$\tan(2x) = \dfrac{2 \tan x}{1 - \tan^2 x}$

Chapter 10 | Parametric Eq, Polar

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$, if $\dfrac{dx}{dt} \neq 0$

Area $= \int_\alpha^\beta g(t) f'(t)\, dt$ if $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$

Arc Length $= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta |v(t)|\, dt$

Surface Area (around x-axis) $= 2\pi \int_\alpha^\beta y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = r\cos\theta$ $y = r \sin\theta$

$r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Tangent of Polar: $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

Cardioid/Limacon: $r = a \pm b\cos\theta$, $r = a \pm b \sin\theta$

Roses: $r = a\cos(n\theta)$ $r = a\sin(n\theta)$
odd: petals $= n$ even: petals $= 2n$

Area Polar $= \frac{1}{2}\int_a^b r^2\, d\theta$

Arc length $= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Chapter 12 | Vectors, Space

$a \cdot b = |a||b|\cos\theta$ $0 \le \theta \le \pi$

$\text{comp}_a \vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ $\text{proj}_a \vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ $0 \le \theta \le \pi$

Vol of parallelpiped $= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

Dist from point to plane $= \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$

Dist from point to line: $h = \dfrac{|\vec{AB} \times \vec{AP}|}{|\vec{AB}|}$

Intersections
① Check if direction vector parallel
② set $x_1 = x_2, y_1 = y_2, z_1 = z_2$ in terms of t and s

Find distance between lines using cross product and $\text{comp}_a \vec{b}$

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperbolic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperboloid One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Hyperboloid Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Parallel a, b
$a \times b = 0$

Prop Vectors
1. $a \times b = -b \times a$
2. $(ca) \times b = c(a \times b) = a \times (cb)$
3. $a \times (b + c) = a \times b + a \times c$
4. $(a+b) \times c = a \times c + b \times c$
5. $a \cdot (b \times c) = (a \times b) \cdot c$
6. $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

Chapter 13 | Vector Func

Arc length vector $= \int_a^b |\vec{r}'(t)|\, dt$

Parameterize in terms of s $r(t) \Rightarrow r(s)$
① Solve for s
② Use eq to solve for t in terms of s
③ Plug t equation into $\vec{r}(t)$

Unit Tangent Vector: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ $\vec{B}(t) = \vec{T} \times \vec{N}$

Curvature (How quickly changes direction): $K = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

radius of osculating plane: $r = \frac{1}{K}$

$a = v'\vec{T} + Kv^2\vec{N}$ where $v = |\vec{v}|$

$a_T = \dfrac{r'(t)\cdot r''(t)}{|r'(t)|}$ $a_N = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$

Chapter 14 | Partials

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ ① Approach from dif lines ② convert to polar coor

Tangent Plane: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Linear approx: $L(x,y) = f_x(x-x_0) + f_y(y-y_0) + z_0$

Case 1: $z = f(x,y)$ $x = g(t)$ $y = h(t)$

$\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

Case 2: $z = f(x,y)$ $x = g(s,t)$ $y = h(s,t)$

$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $\dfrac{\partial z}{\partial t} = \ldots$

$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$

$D_u f(x,y) = \nabla f \cdot \vec{u}$ in direction unit vector $\vec{u}$

$\nabla f(x,y) = \langle f_x, f_y \rangle$ $\max D_u f = |\nabla f|$

∇ is $\vec{n}$ of tangent plane

$F_x(x-x_0) + F_y(y-y_0) = 0$

or $\text{orth} = \vec{AP} - \text{proj}_{\vec{AB}} \vec{AP}$

Max/Min
Critical point $f_x = f_y = 0$ or one DNE
$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$
① $D > 0$, $f_{xx} > 0 \Rightarrow$ local min
② $D > 0$, $f_{xx} < 0 \Rightarrow$ local max
③ $D < 0 \Rightarrow$ not min or max (saddle point)
④ $D = 0 \Rightarrow$ no info, any

Steps
① Find vals of f at critical points
② Find extreme values on Boundary D
③ Largest max smallest min

Lagrange max/min: f constraint: $g = k$
$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

Chapter 15 | Mult Integrals

$V = \iint_R f(x,y)\, dA$

Avg value $= \dfrac{\iint_R f(x,y)\, dA}{\text{area of } R} = \dfrac{\text{Volume}}{\text{Area}}$

Polar
$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Applications
$m = \iint_D \rho(x,y)\, dA$ ρ: lamina
$= \iiint_V \rho(x,y,z)\, dV$

Moment
$M_x = \iint_D y\, \rho(x,y)\, dA$
$M_y = \iint_D x\, \rho(x,y)\, dA$

Center of mass
$\bar{x} = \dfrac{M_y}{m} = \dfrac{1}{m}\iint_D x\, \rho(x,y)\, dA$
$\bar{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\iint_D y\, \rho(x,y)\, dA$

Moment of Inertia:
$I_x = \iint_D y^2\, \rho(x,y)\, dA$
$I_y = \iint_D x^2\, \rho(x,y)\, dA$
$I_0\ (\text{origin}) = \iint_D (x^2+y^2)\, \rho(x,y)\, dA$

Probability:
$f(x,y) \ge 0$ $P = \iint_R f(x,y)\, dA = 1$

Triple Integrals
Cylindrical
$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1}^{u_2} f(x,y,z)\, r\, dz\, dr\, d\theta$

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Spherical
$\int_\alpha^\beta \int_{h_1}^{h_2} \int_{u_1}^{u_2} f(x,y,z)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$ $x^2 + y^2 + z^2 = \rho^2$

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian: $x = g(u,v)$ $\quad y = h(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

1. Find $u=$, $v=$ suitable
2. Find $x=$, $y=$
3. Find Jacobian plug in abs val
4. Plug into equation, solve

Chapter 16 Vector Calc

Conservative if some f that $F = \nabla f$

16.2 Line Integrals

$$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

16.3 Fundamental Theorem for Line Integrals

Cond	Then
1. C smooth curve 2. f differentiable 3. ∇f continuous on C	$\int_C \nabla f\cdot d\vec{r} = f(r(b)) - f(r(a))$

Thm

Cond	Then
1. $\oint F\cdot dr = 0$ for every closed path C in D	$\int_C F\cdot dr$ is independent of path in D
1. open (no boundary points) 2. connected (2 pts joined by path) 3. Independent of path	There is f where $\nabla f = F$
1. Open, simply connected (no holes) 2. P and Q have continuous first partials 3. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D	$\vec{F}$ is Conservative

16.4 Green's Theorem

1. C is positively oriented 2. piecewise-smooth 3. simple closed curve 4. P and Q have continuous partial derivatives on open region	let D be region bound by C $\oint P\,dx + Q\,dy =$ $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

16.5 Curl and Divergence

curl $F = \nabla \times F$

div $F = \nabla \cdot F$

Func 3 vars continuous partials	$\Rightarrow$	curl$(\nabla f) = 0$
$\vec{F}$ conservative	$\Rightarrow$	curl$(F) = 0$
curl$(F) \neq 0$	$\Rightarrow$	$\vec{F}$ not conservative
curl$(F) = 0$ & simply connected & continuous Partials	$\Rightarrow$	$\vec{F}$ conservative
has second partials	$\Rightarrow$	div(curl(F)) $= 0$
div$(F) \neq 0$	$\Rightarrow$	F not a curl

16.6 Parametric Surfaces and Areas

Parametric Surface: $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Surface of Revolution (about x-axis): $x = x$ $\quad y = f(x)\cos\theta$ $\quad z = f(x)\sin\theta$

Normal Vector to tangent plane is $r_u \times r_v$

Surface Area: $A(s) = \iint_D |r_u \times r_v|\,dA = \iint_D \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dA$

16.7 Surface Integrals

Surface Integral / Flux:

$$\iint_S F\cdot dS = \iint_S F\cdot n\,dS = \iint_D F\cdot(r_u\times r_v)\,dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$$

$$= \iint_D f(r(u,v))\,|r_u\times r_v|\,dA = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2+1}$$

16.8 Stokes' Theorem

1. S is piecewise smooth surface 2. Bounded by simple, closed piecewise-smooth boundary curve C w/ positive Orientation 3. Continuous partial derivatives	$\int_C F\cdot dr =$ $\iint_S \text{curl}\,\vec{F}\cdot dS$

16.9 Divergence Theorem

1. E is simple solid region 2. S is boundary surface of E w/ positive outward orientation 3. continuous partials on open region contains E	$\iint_S F\cdot dS =$ $\iiint_E \text{div}\,F\,dV$

Vol(Sphere) $= \frac{4}{3}\pi r^3$