Math 53 Midterm 1 Study Guide

Chapter 10: Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Parametric equations: $x = f(t)$ $y = g(t)$

- Know how curves are drawn and direction, how much

10.2 Calculus with Parametric Curves

Tangents: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ if $\dfrac{dx}{dt} \neq 0$

2nd Derivative: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$
(Concavity)

Area: $A = \int_\alpha^\beta g(t) f'(t)\, dt$ if $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$

Arc Length: $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Surface Area: $S = 2\pi \int_\alpha^\beta y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ if rotated around x-axis

10.3 Polar Coordinates

Polar coordinate system: (r, θ)

$x = r\cos\theta$	$y = r\sin\theta$	rect → polar
$r^2 = x^2 + y^2$	$\tan\theta = \frac{y}{x}$	polar → rect

Polar graphs

$r = c$: circle $\theta = c$: line

Cardioid/Limacon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$

Inner loop: $b > a$ Cardiod (heart): $a = b$

Roses: $r = a\cos(n\theta)$ $r = a\sin(n\theta)$

if n is odd: petals $= n$ even: petals $= 2n$

Tangents: $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ since $x = r\cos\theta$, $y = r\sin\theta$

10.4 Areas and Lengths in Polar Coordinates

Area: $A = \frac{1}{2}\int_a^b r^2\, d\theta$

Arc Length: $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Chapter 12: Vectors and the Geometry of Space

12.1 3D Systems

Distance: $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Eq of sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

12.2 Vectors

vector has both magnitude and direction

$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Magnitude: $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$a+b = \langle a_1+b_1, a_2+b_2 \rangle$ $a-b = \langle a_1-b_1, a_2-b_2 \rangle$ $ca = \langle ca_1, cb_1 \rangle$

$$\vec{a} = |a| \langle \cos\theta, \sin\theta \rangle$$

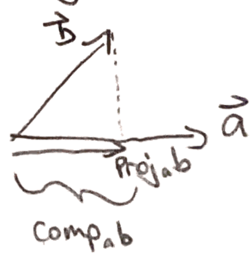
12.3 Dot Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$ where θ is the angle between vectors $\vec{a}$ and $\vec{b}$

if $\vec{a} \cdot \vec{b} = 0$, vectors are orthogonal

Projections



Scalar proj of b onto a

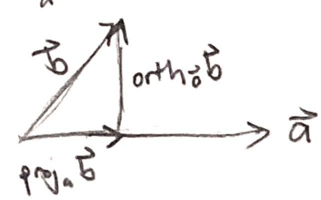
$$\mathrm{comp}_a b = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Vector proj of b onto a

$$\mathrm{proj}_a b = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \, \vec{a}$$

Orthogonal Vector

$$\mathrm{orth}_{\vec{a}} \vec{b} = \vec{b} - \mathrm{proj}_{\vec{a}} \vec{b}$$



$$\text{Work} = F \cdot D \cos\theta = \vec{F} \cdot \vec{D}$$

12.4 Cross Product

$$\text{Det} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$a \times b = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

if $\vec{a} \times \vec{b} = 0$, vectors $\vec{a}$ and $\vec{b}$ are parallel

$|\vec{a} \times \vec{b}|$ is equal to parallelogram made by $\vec{a}$ and $\vec{b}$

Scalar triple product : $\vec{a} \cdot (\vec{b} \times \vec{c})$

(Volume of parallelpiped) = $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

Torque : $\vec{T} = \vec{r} \times \vec{F}$ $\quad |T| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$

12.5 Equations of Lines and Planes

vector eq of Line: $\vec{r} = \vec{r}_0 + t\vec{v}$ where $\vec{v}$ is a direction vec and $\vec{r}_0$ is a vector on the line

parametric eqs : $x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$

Symmetric eqs : $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

Planes

Scalar eq of plane : $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where $\langle a, b, c \rangle$ is a vector normal to plane

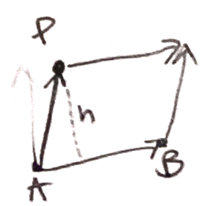
linear eq of plane : $ax + by + cz + d = 0$

- Use normal vectors to find angle between planes

Distance from point to plane : $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ where (x_1, y_1, z_1) is point and plane $ax + by + cz + d = 0$

$D = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$

Distance from a point to line : $h = \frac{|\vec{AB} \times \vec{AP}|}{|\vec{AB}|}$ $\quad \frac{(\text{area of parallelogram})}{(\text{base})}$



Intersections

Check is direction vectors are parallel (scalar multiples)

Set $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ in terms of t and s

12.6 Cylinders and Quadric Surfaces

Cylinder is a surface consisting of all lines through plane

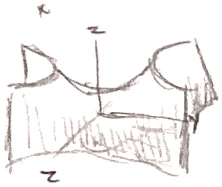
Quadric Surfaces

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Use traces, setting $z, y, x = c$ and graph

Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — traces are ellipses; $a=b=c$ sphere

Elliptic Paraboloid : $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ — H traces are ellipses; V traces are parabolas



Hyperbolic Paraboloid : $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ — H traces are hyperbolas; V traces are parabolas

Cone : $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ — H traces are ellipses; V traces $x=k$ hyperbolas; $k \neq 0$ pairs of lines

Hyperboloid of One Sheet : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ — H traces ellipses; V traces Hyperbolas

Hyperboloid of Two Sheets : $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — H traces in $z=k$ ellipses; V traces are hyperbolas

Chapter 13: Vector Functions

13.1 Vectors Functions and Space Curves

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

13.2 Derivatives and Integrals of Vector Func

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Unit Tangent Vector: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

13.3 Arc Length and Curvature

Arc Length: $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |\vec{r}'(t)|\,dt$

$$\frac{ds}{dt} = |r'(t)|$$

Reparametrize in terms of s:

$r(t) \Rightarrow r(s)$

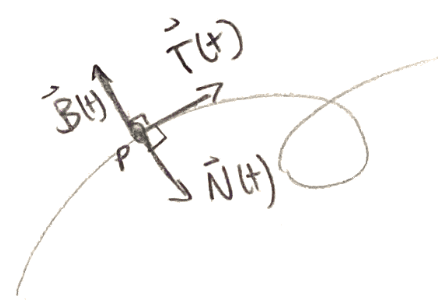
1. Solve for s
2. use eq to solve for t in terms of s
3. Plug t equation into $\vec{r}(t)$

Curvature: how quickly curve changes direction

$r = \frac{1}{K}$ of osculating circle

$$K = \frac{|\vec{T}'(t)|}{|r'(t)|} \quad \text{or} \quad \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Normal and Binormal vectors



$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The vectors $\vec{N}(t)$ and $\vec{T}(t)$ are on the osculating plane

13.4 Motion in Space: Velocity and Acceleration

position: $\vec{r}$

velocity: $\vec{v} = r'(t)$

acceleration: $\vec{a} = r''(t) = v'(t)$

Force: $\vec{F} = m\vec{a}(t)$

Projectile Motion

$a = \langle 0, -9.8 \rangle$ $\quad v(0) = v_0 \quad$ α = angle

$$r(t) = \langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \rangle$$

$$d = \frac{v_0^2 \sin 2\alpha}{g} = \frac{2v_0^2 \sin\alpha \cos\alpha}{g}$$

Acceleration

$$\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}$$ where $v = |\vec{v}|$

v'

$a_T = v'$ $\qquad$ $a_N = \kappa v^2$

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|} \qquad a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$$