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Sets ①

Cardinality [|A|]: size of set

Empty set [∅]: {}

Subset [A ⊆ B]: elem in A in B

Proper subset [A ⊂ B]: A ⊆ B, but A ≠ B

Intersection [A ∩ B]: both A and B

Disjoint: A ∩ B = ∅

Union [A ∪ B]: either A or B

Set Difference [B − A (B \ A)]:

Natural [N] Rational [Q]

Integer [Z] Complex [C]

Cross Product [A × B]: (u, k), u ∈ A, k ∈ B

Power set [P(S)]: set of subsets

Logic ②

1) Conjunction [P ∧ Q]: and
2) Disjunction [P ∨ Q]: or
3) Negation [¬P]: not
4) Implication [P ⇒ Q]: implies
   a) Contrapositive [¬Q ⇒ ¬P]
   b) Converse [Q ⇒ P]

DeMorgan's Laws:

$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Proofs ③

a | b 2 | 10 = 5 b = aq

1) Direct Proof [P ⇒ Q]
2) Proof By Contraposition [¬Q ⇒ ¬P] = [P ⇒ Q]
3) Proof By Contradiction [P]
   ¬P, show R, ¬R, P holds
4) Proof By Cases [P]
   proof result in all cases
5) Induction

Pigeonhole Principle: *
n pigeons, k holes
if n > k, at least 1 hole > 1 pigeon

Induction ④

Prove for all natural #s

1) Base Case: Eq holds for initial value
2) Inductive Hypothesis: for n = k suppose P(k) holds
3) Inductive Step: Assuming Inductive hypothesis, show P(k+1)

Strong Induction
Assume holds 0 ≤ n ≤ k for k ≥ 15

Hyper cube
$V: 2^n$ V has n degrees
$E: \frac{n}{2} 2^n$
Edge is one bit different

Stable Matching ⑤

Propose-and-Reject Algo *
Loop each day until no offers rejected
Morning: job proposes to most preferable candidate who hasn't rejected
Afternoon: candidate collects offers and put most liked on a string, reject others
Evening: Rejected job crosses candidate who rejected
- Always halts since one job must eliminate candidate terminate at n^2
- No rogue couples
- Lemma: every subsequent day C has job offer she likes as much as J
- Well Ordering Principle *
  any non-empty set of natural nums has smallest num
- Propose-and-reject is job optimal, candidate pessimal

Graph Theory ⑥

Graph [G] = (V, E) set of vertices and edges

Edge: {u, v} pair of vertices, line segments

Vertices: points in a graph

Directed Graph: G = (V, E) but set of Edges are ordered arrow (u, v)

Edge e is incident on vertices u, v and u, v are neighbors, adjacent

degree(u) = |{v ∈ V: {u, v} ∈ E}|

Path: sequence of edges {v, v₁}{v₂ ...} distinct

Cycle (circuit): simple path starts and ends at same place, distinct

Walk: sequence of edges w/ repeated vertices

Tour: walk starts and ends same vertex

Connected: If has path to reach distinct vertices

Eulerian walk/tour: uses each edge exactly once

even degree graph: all vertices have even degree

Planar: drawn in plane w/o crossing

Faces: regions that subdivide plane

Euler's formula: v + f = e + 2 for every planar

Planar graphs e ≤ 3v − 6

Non-planar graphs can pass test

Complete graphs have max num of edges

Trees removing edge disconnects
① connected, no cycles
② connected, n−1 edges
③ connected, removal of edge disconnects
④ no cycles, addition creates cycles

Mod Arithmetic ⑦

range {0, 1, ..., N−1} x mod m remainder r

Bijections: b ∈ B unique preimage a ∈ A f(a) = b
1) onto (surjective): every b ∈ B has a a ∈ A preimage
2) 1-to-1 (injective): B can't have many A

Inverse: xy ≡ 1 gcd(m, x) = 1 → x

d = gcd(m, x) = am + bx b is multiplicative inverse of x mod m

CRT

unique x that satisfies $x \equiv a_1 \pmod{n_1} \ldots x \equiv a_k \pmod{n_k}$

$x = \sum_{i=1}^{k} a_i b_i \bmod N$ where $b_i = \frac{N}{n_i}\left(\frac{N}{n_i}\right)^{-1}_{n_i}$

$N = \prod_{i=1}^{k} n_i$ $\left(\frac{N}{n_i}\right)^{-1}_{n_i}$ inverse (mod n_i) of $\frac{N}{n_i}$

RSA ⑧

p & q large primes, N = pq mod N

e is relatively prime to (p−1)(q−1)

public key: (N, e)

private key: d = inverse e mod (p−1)(q−1)

Encryption: message x compute $E(x) = x^e \bmod N$

Decryption: y = E(x) $D(y) = y^d \bmod N = x$

Fermat's Little Theorem *

for prime p and any a ∈ {1, 2, ..., p−1} we have $a^{p-1} \equiv 1 \bmod p$

Polynomials ⑨

1) Non-zero polynomial degree d has at most d roots
2) Given d+1 pairs with x_i distinct, unique polynomial p(x) degree at most d st $p(x_i) = y_i$ for $1 \le i \le d+1$

Lagrange Interpolation *

$$\Delta_i(x) = \frac{\prod_{j \ne i}(x - x_j)}{\prod_{j \ne i}(x_i - x_j)}$$

$$p(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x)$$

$p(x) = q'(x)\, q(x) + r(x)$

working in GF(m)

polynomial degree 2 in GF(m) total?
m^3 bc each coefficient can take m values

Secret Sharing
1) Any group k can figure out
2) No group ≤ k−1 have any info

code = s q is prime larger than n and s
n officials

P(x) degree k−1 where P(0) = S and P(1) to first official, P(2) to second ...

1) Any k officials use lagrange interpolation to find P
2) Group k−1 cannot reconstruct

Graph Theory

max edges for vertex $\frac{n(n+1)}{2}$

Bipartite planar graph: e = 2v − 4 not enough to prove

bipartite: two disjoint sets, no 2 vertices of same set are adjacent

We assume 3f ≤ 2e for face at least 3 sides
can change 5f = 2e

Removing edge for cycle, still connected

Not Planar

$K_{3,3}$

K_5

4D cube

Stable Matching

$(n-1)^2+1$ at most rejections

$n(n-1)+1$ at most proposals

companies get worse candidates over time

candidates get better job offers

Proofs (Examples)

Direct

1) Sum of digits of n divisible by $9 \Rightarrow 9 | n$

let n be written as $n = abc$ $n = 100a + 10b + c$

$a+b+c = 9k$ $\Rightarrow 100a + 10b + c = 9(k + 11a + b)$

2) $\sqrt{2}$ is irrational

Contradiction

Use if a^2 is even $\Rightarrow a$ is even

Must be $\sqrt{2} = a/b$ $\Rightarrow 2 = \frac{a^2}{b^2}$ must be some $a = 2c$

since $a^2 = 2b^2$ state a,b share no common factors

prove a,b even

3) Every $n \in \mathbb{N}$ $n \geq 12$, $n = 4x + 5y$ $x,y \in \mathbb{N}$

Induction

Base Case: $n = 12, 13, 14, 15$

Induction Hypo: Assume holds for all $12 \leq n \leq k$ $k \geq 15$

Prove for $n = k+1 \geq 16$ $k+1-4 \geq 4x' + 5y'$ $x = x'+1$ $y = y'$

4) Improvement Lemma

Induction

If job J makes offer to candidate C on kth day every subsequent Day C has job she likes as much as J

Proof: induction on i $i \geq k$

Base ($i = k$) receives offer, have J or better

Induction step: prove $i+1$ had offer from job J' on a string she likes as much as J. J' proposes again $i+1$, will either have J' or another better

5) Matching is always stable

Direct

No job can be in a rogue couple. Consider couple (J,C) Suppose J prefers C^* to C, C^* prefers current job to J, (J, C^*) not rogue. made offer to C^* but C^* likes current more. No job J in a rogue couple.

6) Matching is job/employer optimal

Contradiction

Exists day job got rejected from optimal candidate

J rejected by C^* for J^* in $T: \{(J, C^*), \ldots (J^*, C')\}$,

(J^*, C^*) is rogue C^* prefers J^* J^* made offer to C^*

7) Euler's formula: For every connected planar graph,

Induction

$v + f = e + 2$

Induction on e Base: $e = 0$ $v = f = 1$

If tree) $f = 1$ $e = v - 1$

Not tree) has cycle, take cycle delete edge reduce e and f by 1 not changing v

8) Let m, x be $+\mathbb{Z}$ $\gcd(m,x) = 1$, x has multiplicative inverse mod m and unique

Direct

$0, x, 2x, (m-1)x$ distinct mod m so $ax \equiv 1$ mod m for one a

Suppose $ax \equiv bx \pmod m$ Then $(a-b)x \equiv 0$ $(a-b)x \equiv km$ but x, m relatively prime

$a-b$ must be integer multiple of m $a-b$ range 1 to $(m-1)$

* Need $\gcd(m,x) = 1$ for inverse

9) CRT

Direct

$\left(\frac{N}{n_i}\right)^{-1}_{n_i}$ exists $\frac{N}{n_i} = \Pi_{j \neq i}$ integer coprime n_i, inverse exists

$x \bmod n_i = \left(\sum_{i=1}^{k} a_i b_i\right) \bmod N \bmod n_i$

$= a_i b_i \bmod n_i$

$= a_i \bmod n_i$

10) FLT p any $a \in \{1, 2, \ldots p-1\}$ $a^{p-1} \equiv 1 \bmod p$

$S = \{1, 2, \ldots p-1\}$ $a, 2a, 3a, \ldots (p-1)a$ if $\gcd(p,a) = 1$ are distinct

none of them zero $p-1$ of them $S' = \{a \bmod p, 2a \bmod p, \ldots (p-1)a \bmod p\}$

$(p-1)! \bmod p$ $a^{p-1}(p-1)! \bmod p$ product of nums

$(p-1)! \bmod p \equiv a^{p-1}(p-1)! \bmod p$ $a^{p-1} \equiv 1$

p is prime every int has inverse

11) RSA $(x^e)^d = x \bmod N$ $x \in \{0, 1, \ldots N-1\}$

Direct Cases

$ed \equiv 1 \bmod (p-1)(q-1)$ $ed = 1 + k(p-1)(q-1)$

$x^{ed} - x = x^{1+k(p-1)(q-1)} - 1 = x(x^{k(p-1)(q-1)} - 1)$ show $= 0 \bmod N$

1) Not multiple of p 2) multiple of p

$x \neq 0 \bmod p$ FLT: $x^{p-1} = 1 \bmod p$ $x(x^{k(p-1)\ldots})$ divisible by p

$x^{k(p-1)(q-1)} - 1 = 0 \bmod p$

must also be divisible by q so divisible by product N

12) Prime Number Theorem $\pi(n)$: primes $\leq n$ for $n \geq 17$ $\pi(n) \geq \frac{n}{\ln n}$

13) $d+1$ pairs has unique polynomial

Contradiction

Suppose another $q(x)$ $q(x_i) = y_i$ then $r(x) = p(x) - q(x)$

r is at most degree d $r(x_i) = p(x_i) - q(x_i) = 0$ at $d+1$ points so $r(x)$ at least $d+1$ roots

14) Euler's Totient Theorem

n and a are coprime $a^{\phi(n)} \equiv 1 \pmod n$ $\phi(n)$: # $\mathbb{Z}^+ \leq n$ coprime to n

FLT model $\{m_1, m_2, \ldots m_{\phi(n)}\} = \phi(n)$ set $\{am_1, am_2, \ldots am_{\phi(n)}\}$

m_i and a coprime to n. Suppose shared factor p $p|a$ or $p|m_i$

Injective: $f(x) = f(y)$ $ax \equiv ay$ a has inverse

Surjective: take y prime n $f^{-1}(a^{-1}y) \equiv y \pmod n$ $f(x) = y$ $a^{-1}y$ prime n

General

Use variable to suppose things

Sets: De Morgans $\forall \exists$

$\neg(\forall(x) P(x)) \equiv \exists x (\neg P(x))$

10) Error Correcting Codes

Erasure Errors: n packets, k packets lost
Need $n+k$ to retrieve

1. Polynomial $P(x)$ degree $n-1$,
2. mod q, send $P(i)=m$, $n+k \le q$
3. Recover w/ any n points
   Use Lagrange interpolation: points → Polynomial

General Errors: n packets, k packets corrupted
Need $n+2k$ to retrieve, Berlekamp-Welch Alg

1. Polynomial $P(x)$ degree $n-1$, $\in F(q)$
2. Error Locator Polynomial $E(x)=(x-e_1)(x-e_2)\dots(x-e_k)$
   Plug in $Q(i)=r_i E(i)$ $1 \le i \le n+2k$ where $Q(i)=P(i)E(i)$
3. Solve for error correcting $E(x)$ errors $e_1, e_2 \dots$
   and $P(x)=\frac{Q(x)}{E(x)}$ use long division

Distance Properties Reed-Solomon Codes
Hamming distance: positions where strings differ
$$d(\vec{s},\vec{r})=\sum_{i=1}^{L} 1(r_i \ne s_i) \quad 1 \text{ if true}$$
Min distance of two codes d
At $d/2$, can impersonate two codes equally

11) Counting

First Rule Counting: succession of k choices where n_1 ways first choice, then for every first choice n_2 second choice
Total choices: $n_1 \times n_2 \times \dots \times n_k$

Second Rule Counting: succession choices order does not matter
Ways of choosing k elements from n total elements
Total: $\binom{n}{k}=\frac{n!}{(n-k)!\,k!}$

Ex) Select multiset size k with set size n, use binary strings w/ 1 for bin-edge can model $\binom{n+k-1}{k}$ k fruits

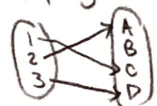
Zeroth Rule of Counting; if set A bijection set B, $|A|=|B|$

Combinatorial Proofs: Proofs by stories told from multiple points of view

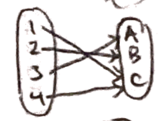
Permutations: rearrangement, $n!$ distinct ways
Derangement: Permutation w/ no fixed points
Inclusion Exclusion:
Disjoint: $|A_1 \cup A_2| = |A_1|+|A_2|$, since $|A_1 \cap A_2|=0$
$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\dots,n\}: |S|=k} \left|\bigcap_{i \in S} A_i\right|$$
Stirling's Approx: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
Simple: $\left(\frac{n}{e}\right)^n$

12) Countability and Computability

Bijections: $f: A \to B$ every $a \in A$ unique image $b=f(a)$ $b \in B$ preimage $f(a)=b$
injection (1-to-1): distinct input to distinct output
): $x \ne y \Rightarrow f(x) \ne f(y)$



surjective (onto): every element in range has preimage
$(\forall y \exists x)(f(x)=y)$



Bijection: injection and surjective

Countable set S if bijection between S and N or $\subseteq N$
N, Z, Q have same cardinality
R is not using Cantor's diagonalization
Ex) Enumerate real nums in infinite list diagonal $= r$, we find r and 1 to every digit, must be real but is 1 num off from nth digit

No program can test if in infinite loop, self reference, cannot separate programs from data

13) Discrete Probability

Probability Space: Sample space Ω, probability $P[\omega]$
1) Non-negative: $0 \le P[\omega] \le 1$ for $\omega \in \Omega$
2) Total 1: $\sum_{\omega \in \Omega} P[\omega]=1$

Event A is subset sample space $A \subseteq \Omega$
$P[A]=\sum_{\omega \in A} P[\omega]$
Event $\bar{A}$ is complement of A $\bar{A}=1-A$ $A^c \cup A=\Omega$
$A \cap A^c=\emptyset$
Throw m balls into n bins n^m sample space
1) What is sample space? (experiment, possible outcomes)
2) What is probability of each outcome? (sample point)
3) Event we are interested in? (What subset of sample space)
4) Add up probabilities of sample points in it.

14) Conditional Probability, Independence, Combination

Conditional Probability of A given B, events $A,B \subseteq \Omega$
$$P[A|B]=\frac{P[A \cap B]}{P[B]}$$

Bayes Rule: Flip $P[A|B]$ $P[B|A]$ General
$$P[A|B]=\frac{P[A \cap B]}{P[B]}=\frac{P[B|A]\,P[A]}{P[B]}=\frac{P[B|A_i]P[A_i]}{\sum_{j=1}^{n} P[B|A_j]P[A_j]}$$

Total Probability Rule
$$P[B]=P[A \cap B]+P[\bar{A} \cap B]=P[B|A]P[A]+P[B|\bar{A}]P[\bar{A}]$$
$$P[A|B]=\frac{P[B|A]P[A]}{P[B|A]P[A]+P[B|\bar{A}](1-P[A])}$$

When A is partitioned $A_1 \cup A_2 \cup \dots \cup A_n$
Total Prob Rule: $P[B]=\sum_{i=1}^{n} P[B|A_i]P[A_i]$

Independent: $P[A \cap B]=P[A] \cdot P[B]$
$$P[A|B]=\frac{P[A \cap B]}{P[B]}=P[A]$$

Mutual Independence: Events $A_1 \ldots A_n$ $B_i \in \{A_i, \bar{A}_i\}, i = 1 \ldots n$

$$P[B_1 \cap \ldots \cap B_n] = \prod_{i=1}^{n} P[B_i]$$

Pairwise Independence: each pair is independent

Mutual Independence $\Rightarrow$ Pairwise Independence
$\not\Leftarrow$

Product Rule (not Mutually independent)

$$P[\cap_{i=1}^{n} A_i] = P[A_1] \times P[A_2 | A_1] \times \ldots \times P[A_n | \cap_{i=1}^{n-1} A_i]$$

Principle Inclusion-Exclusion: $A_1 \ldots A_n$ probability space

$$P[A_1 \cup \ldots \cup A_n] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1, \ldots, n\}: |S| = k} P[\cap_{i \in S} A_i]$$

$$P[\cup_{i=1}^{n} A_i] = \sum_{i=1}^{n} P[A_i] - \sum_{i<j} P[A_i \cap A_j] + \sum_{i<j<k} P[A_i \cap A_j \cap A_k] + (-1)^{n-1} P[\cap A_i]$$

Mutually Exclusive: $A_1 \ldots A_n$ ($A_i \cap A_j = \emptyset$ all i, j)

$$P[\bigcup_{i=1}^{n} A_i] = \sum_{i=1}^{n} P[A_i]$$

Union Bound: $A_1 \ldots A_n$ all $n \in \mathbb{Z}^+$

$$P[\bigcup_{i=1}^{n} A_i] \le \sum_{i=1}^{n} P[A_i]$$

15 Random Variables

Random variable: depends on outcome of probabilistic experiment

X on Ω func $X: \Omega \to \mathbb{R}$ $X(\omega)$ for all $\omega \in \Omega$

Distribution of X is collection of values $\{(a, P[X=a]) : a \in A\}$

The collection of events form partition

Bernoulli Distribution: takes in $\{0,1\}$

$$P[X=i] = \begin{cases} p & \text{if } i=1 \\ 1-p & \text{if } i=0 \end{cases} \qquad X \sim \text{Bernoulli}(p)$$

Binomial Distribution: values of X, prob of $X=i$ sum of sample pts $\binom{n}{i}$ and $p^i(1-p)^{n-i}$

$$P[X=i] = \binom{n}{i} p^i (1-p)^{n-i} \qquad X \sim \text{Bin}(n, p)$$

Hypergeometric Distribution: sample w/o replacement, not independent

$$P[Y=k] = \binom{n}{k} \frac{\frac{B!}{(B-k)!} \frac{(N-B)!}{(N-B-(n-k))!}}{\frac{N!}{(N-n)!}} = \frac{\binom{B}{k}\binom{N-B}{n-k}}{\binom{N}{n}}$$

$N = B + W$ balls sample $n \le N$ $\qquad Y \sim \text{Hypergeometric}(N, B, n)$

Joint Distribution X and Y $\{((a,b), P[X=a, Y=b]) : a \in A, b \in B\}$

marginal distribution $P[X=a] = \sum_{b \in B} P[X=a, Y=b]$

RV X and Y independent: $P[X=a, Y=b] = P[X=a] P[Y=b]$

Indicator RV $I_1 \ldots I_n$ mutually independent

Summarize distributions w/ Expectation

Expectation discrete rv X sum over all possible values "typical" value

$$E[X] = \sum_{a \in A} a \times P[X=a]$$

Linearity of Expectation: $E[X+Y] = E[X] + E[Y]$

$E[cX] = cE[X]$

Geometric Distribution: how long before event happens

$$P[X=i] = (1-p)^{i-1} p \qquad X \sim \text{Geometric}(p)$$

General

- Binary String to Solve balls bins
- Inclusion Exclusion to flip and find $P[A \cap B \cap \bar{C}]$
- $d+1$ points determine d degree polynomial
- $E(x) = x - e_1$ where e_1 is x val of corrupted packet
- Not all polynomials have d roots
- Use symmetry when you can

16 Variance and Covariance

Variance: For RV X w/ $E[X]=\mu$,

$$Var(X)=E[(X-\mu)^2]=E[X^2]-E[X]^2$$

Standard Deviation

$$\sigma(X) := \sqrt{Var(X)}$$

Independent RV $X=X_1+X_2+\dots+X_n$ if $X_i=X_j$

$$Var(X)=\sum_{i=1}^{n} Var(X_i)= n Var(X_1)$$

$$\sigma(X)=\sqrt{n}\cdot\sigma(X_1)$$

$$E[X]=nE[X_1]$$

For RV X,Y

$E[XY]=E[X]E[Y]$ $Var(X+Y)=Var(X)+Var(Y)$

Covariance

$$Cov(X,Y)=E[XY]-E[X]\cdot E[Y]$$

$$Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)$$

19 Geometric, Poisson Distributions

Geometric: tossing tails for i-1 before heads with

$$P[X=i]=(1-p)^{i-1}p \qquad X\sim Geometric(p)$$

$$E[X]=\frac{1}{p} \qquad Var(X)=\frac{1-p}{p^2}$$

Poisson: avg num λ per time or space determines prob func

$$P[X=i]=\frac{\lambda^i}{i!}e^{-\lambda} \qquad X\sim Poisson(\lambda)$$

$$E[X]=\lambda \qquad Var(X)=\lambda$$

Independent Poisson RV

Let $X\sim Poisson(\lambda)$ and $Y\sim Poisson(\mu)$

$$X+Y\sim Poisson(\lambda+\mu)$$

Similar to Binomial$(n,\frac{\lambda}{n})$

17 Concentration Inequalities & Law of Large Nums

Markov's Inequality

For nonnegative RV X $X(\omega)\geq 0$ $\forall\ \omega\in\Omega$, c constant

$$P[X\geq c]\leq\frac{E[X]}{c}$$

Chebyshev's Inequality iid

RV X w/ finite expectation $E[X]=\mu$ constant c

$$P[|X-\mu|\geq c]\leq\frac{Var(X)}{c^2}$$

$$P[|X-\mu|\geq k\sigma]\leq\frac{1}{k^2} \quad \text{where } \sigma=\sqrt{Var(X)}$$

Law of Large Numbers

for iid $X_1,X_2,X\dots$ $S_n=X_1+X_2+\dots+X_n$ $E[X_i]=\mu$

$$P[|\tfrac{1}{n}S_n-\mu|<\varepsilon]\to 1 \quad n\to\infty$$

20 Continuous Probability Distributions

No longer probability points, but intervals

Probability Density Function (pdf) "probability per unit length"

For RV X is func $f:\mathbb{R}\to\mathbb{R}$

1. f is nonnegative. $f(x)\geq 0$ for all $x\in\mathbb{R}$
2. The total integral of f is equal to 1: $\int_{-\infty}^{\infty}f(x)\,dx=1$

$$P[a\leq X\leq b]=\int_a^b f(x)\,dx \quad \text{for all } a<b$$

Cumulative Distribution Function (cdf)

$$F(x)=P[X\leq x]=\int_{-\infty}^{x}f(z)\,dz$$

pdf: $f(x)=\frac{dF(x)}{dx}$

$$E[X]=\int_{-\infty}^{\infty}xf(x)\,dx$$

$$Var(X)=E[X^2]-E[X]^2=\int_{-\infty}^{\infty}x^2f(x)\,dx-\left(\int_{-\infty}^{\infty}xf(x)\,dx\right)^2$$

Joint Distribution "probability per unit area"

2 RV X,Y is func $f:\mathbb{R}^2\to\mathbb{R}$

1. f is nonnegative: $f(x,y)\geq 0$
2. total integral equals 1: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=1$

$$P[a\leq X\leq b, c\leq Y\leq d]=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

Independence

$$P[a\leq X\leq b, c\leq Y\leq d]=P[a\leq X\leq b]\,P[c\leq Y\leq d]$$

Marginal Dist

$$f_X(x)=\int_c^{c} f(x,y)\,dy$$

Conditional

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$$

Exponential Distribution

Continous version of geometric

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{ow} \end{cases} \qquad X \sim Exp(\lambda)$$

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

Normal Distribution

for any $\mu \in \mathbb{R}$ and $\sigma > 0$, cont RV X

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \qquad X \sim N(\mu, \sigma^2)$$

Standard normal distribution $\mu = 0$, $\sigma^2 = 1$

If $X \sim N(\mu, \sigma^2)$, then $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$ $\Leftrightarrow$

If $Y \sim N(0,1)$ then $X = \sigma Y + \mu \sim N(\mu, \sigma^2)$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$

Can relate to $N(0,1)$ by $P[X \leq a] = P[Y \leq \frac{a-\mu}{\sigma}]$

Independent Normal RV

Let $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$

RV $Z = aX + bY$ normally distributed

$$\mu = a\mu_x + b\mu_y \qquad \sigma^2 = a^2\sigma_x^2 + b^2\sigma_y^2$$

(21) Central Limit Theorem

Distribution of sample avg $\frac{S_n}{n}$ for large enough n looks like normal distribution w/ mean μ var $\frac{\sigma^2}{n}$

50% of mass in width 0.67σ of either side

99.7% in interval width 3σ either side

CLT

Let X_1, X_2 be sequence of iid w/ $E[X_i] = \mu$ $Var = \sigma^2$

Let $S_n = \sum_{i=1}^{n} X_i$ $\quad \frac{S_n - n\mu}{\sigma\sqrt{n}}$ conv to $N(0,1)$ as $n \to \infty$

$$P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq c\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx \qquad n \to \infty$$

Don't use for probabilites smaller than $O(1/\sqrt{n})$

Approx finite n

Expectation

$$E[X] = \sum_{i=1}^{n} i \, P[X=i]$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[cX] = cE[X]$$

Variance

$$Var(X) = E[X^2] - E[X]^2$$

$$Var(cX) = c^2 Var(X)$$

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Independent

$$P[X=a, Y=b] = P[X=a]P[Y=b]$$

$$E[XY] = E[X]E[Y]$$

$$Var(X+Y) = Var(X) + Var(Y)$$

$$P[A \cap B] = P[A]P[B] \qquad P[A \mid B] = P[A]$$

Distribution Formulas

Binomial	$X \sim Bin(n,p)$	$\binom{n}{i} p^i (1-p)^{n-i}$	$E[X] = np$	$Var(X) = np(1-p)$
Geometric	$X \sim Geometric(p)$	$p(1-p)^{i-1}$	$E[X] = \frac{1}{p}$	$Var(X) = \frac{1-p}{p^2}$
Poisson	$X \sim Poisson(\lambda)$	$\frac{\lambda^i}{i!} e^{-\lambda}$	$E[X] = \lambda$	$Var(X) = \lambda$
Bernouilli	$X \sim Bernouilli(p)$	$\begin{cases} p & \text{if } i=1 \\ 1-p & \text{if } i=0 \end{cases}$	$E[X] = p$	$Var(X) = p(1-p)$

Cont

Exponential	$X \sim Exp(\lambda)$	$f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$	$E[X] = \frac{1}{\lambda}$	$Var(X) = \frac{1}{\lambda^2}$
Normal	$X \sim N(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$	$E[X] = \mu$	$Var(X) = \sigma^2$
Uniform	interval $[0, \ell]$	$f(x) = \frac{1}{\ell}$	$E[X] = \frac{\ell}{2}$	$Var(X) = \frac{\ell^2}{12}$