Intro to Tensors Notes

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Contents

Chapter 0

Tensor Definitions

Based on the video series by eigenchris

Definition 0.0.1: Tensors as multidimensional arrays

Examples:

1. Rank 0 Tensor: **Scalar**
$$[4] \text{ or } 4$$
2. Rank 1 Tensor: **Vector**
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
3. Rank 2 Tensor: **2D-Matrix**
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
4. Rank 3 Tensor: **3D-Matrix** "Cube" of values

Although tensors can be *represented* as matrices/arrays, this definition doesn't describe what tensors *actually* are since this definition doesn't explain the geometric meaning of tensors.

Definition 0.0.2: Tensors as objects invariant under a change in coordinates

1. Tensors have **components** that change in a **predictable** way when the coordinates are changed
2. Vectors are **invariant**, but vector components are not
3. Example of something invariant under coordinate transformation: length
4. Converting tensor components from one coordinate system to another is called a **Forward Transformation**, while doing the reverse if **Backwards Transformation**

Definition 0.0.3: Tensors as a combination of vectors and convectors combined using the tensor product

Best definition, but a bit abstract.

Definition 0.0.4: Tensors as partial derivatives and gradients that transform with the Jacobean matrix

Chapter 1

Forward and Backwards Transformations

Old Basis:

$$\{\vec{e_1}, \vec{e_2}\}$$

New Basis:

$$\left\{\tilde{\vec{e_1}}, \tilde{\vec{e_2}}\right\}$$

1.1 Forward Transformation

Convert from the old basis to the new basis:

$$\tilde{\vec{e_1}} = c_1\vec{e_1} + c_2\vec{e_2}$$
$$\tilde{\vec{e_2}} = c_3\vec{e_1} + c_4\vec{e_2}$$

where $c_1, c_2, c_3,$ and c_4 are scalar constants.

We can rewrite the above linear equations in matrix form by defining a **forward transformation matrix**, F

$$F = \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\vec{e_1}} & \tilde{\vec{e_2}} \end{bmatrix} = \begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} F = \begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

Note that the F matrix is flipped along the diagonal (transposed) from what we'd get if we multiplied the forward transformation matrix before the old basis.

$$\tilde{F} = F^T = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\vec{e_1}} \\ \tilde{\vec{e_2}} \end{bmatrix} = \tilde{F} \begin{bmatrix} \vec{e_1} \\ \vec{e_2} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} \vec{e_1} \\ \vec{e_2} \end{bmatrix}$$

1.2 Backwards Transformation

Similarly, we can convert from the new basis to the old basis with a **backwards transformation**

$$\vec{e_1} = a_1\tilde{\vec{e_1}} + a_2\tilde{\vec{e_2}}$$
$$\vec{e_2} = a_3\tilde{\vec{e_1}} + a_4\tilde{\vec{e_2}}$$

where $a_1, a_2, a_3,$ and a_4 are scalar constants.

Rewrite in terms of the backwards transformation matrix, B

$$B = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

$$\begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e_1}} & \tilde{\vec{e_2}} \end{bmatrix} B = \begin{bmatrix} \tilde{\vec{e_1}} & \tilde{\vec{e_2}} \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

If we multiply F and B we get the identity matrix:

$$BF = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus one is the inverse of the other:

$$B = F^{-1}$$

1.3 Generalizing to N-Dimensions

We can generalize to n dimensions

$$F = \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1n} \\ F_{21} & F_{22} & \dots & F_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n1} & F_{n2} & \dots & F_{nn} \end{bmatrix}$$

$$\tilde{\vec{e_1}} = F_{11}\vec{e_1} + F_{21}\vec{e_2} + \ldots + F_{n1}\vec{e_n}$$

$$\tilde{\vec{e_2}} = F_{12}\vec{e_1} + F_{22}\vec{e_2} + \ldots + F_{n2}\vec{e_n}$$

$$\vdots$$

$$\tilde{\vec{e_n}} = F_{1n}\vec{e_1} + F_{2n}\vec{e_2} + \ldots + F_{nn}\vec{e_n}$$

This can be written more simply as:

$$\tilde{\vec{e_i}} = \sum_{j=1}^{n} F_{ji}\vec{e_j}$$

Similarly we have

$$\vec{e_i} = \sum_{j=1}^{n} B_{ji}\tilde{\vec{e_j}}$$

We still have that F and B are inverses

$$FB = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Proof:

$$\vec{e_i} = \sum_{j=1}^{n} B_{ji}\tilde{\vec{e_j}}$$

$$\vec{e_i} = \sum_{j=1}^{n} B_{ji} \left(\sum_{k=1}^{n} F_{kj}\vec{e_k} \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} F_{kj}B_{ji}\vec{e_k} \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} F_{kj}B_{ji}\vec{e_k} \right)$$

Note that we want $\vec{e_1} = \vec{e_1}$, $\vec{e_2} = \vec{e_2}$, etc. so that implies that

$$\sum_j F_{kj} B_{ji} = \delta_{ki} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

which means that the matrix FB has 1's on the diagonals and 0's elsewhere (identity matrix).

$\square$

Chapter 2

Vector Definition

A vector is an example of a tensor.

Definition 2.0.1: Vector Definition

1. Naive definition: Array of numbers
   - The list of numbers are the *components* of the vector, not the vector itself. Remember that vectors are invariant under a coordinate transformation while the components are not.
2. Arrow in space
   - A bit better than the above definition
   - Not all vectors can be visualized as arrows (like functions!)
   - An arrow is just a special type of vector: A *Euclidean vector*
3. Member of a Vector Space

Definition 2.0.2: Vector Space

$$\left(\underbrace{V}_{\text{Set of vectors}}, \underbrace{S}_{\text{Set of Scalars}}, \underbrace{+}_{\text{Vector addition}}, \underbrace{\cdot}_{Vector scaling operator} \right)$$

Vectors can be

- added together $+$
- Multiplied by a scalar $\cdot$

For these notes, when we say *vector*, we mean a euclidean vector.

Chapter 3

Vector Transformation Rules

To convert the vector components from the old basis $\left\{\vec{e_1}, \vec{e_2}\right\}$ to components in the new basis, $\left\{\tilde{\vec{e_1}}, \tilde{\vec{e_2}}\right\}$, we apply the *backwards* transformation on the old basis components:

Vector, v, in the old basis:

$$\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{\vec{e_i}}$$

In the new basis

$$\vec{v} = \tilde{v_1}\tilde{\vec{e_1}} + \tilde{v_2}\tilde{\vec{e_2}} = \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \end{bmatrix}_{\tilde{\vec{e_i}}}$$

where b_1, b_2, b_3 and $, b_4$ are scalars.

Converting vector components in the old basis to new

$$B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{\vec{e_i}} = \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \end{bmatrix}_{\tilde{\vec{e_i}}}$$

When the basis vectors get larger, the "measuring stick" gets bigger, which means the components of the vector get smaller so that the length remains the same.

Similarly, when the basis vectors are rotated clockwise, from the perspective of the vector, the components have to be rotated counterclockwise to maintain the same orientation in space.

Since the vector components transform *contrary* to the basis vectors, we say that the vector components are *contravariant.*

3.1 Generalization to n-dimensions

Proof:

$$\vec{v} = \sum_{i=1}^{n} v_i\vec{e_i} = \sum_{j=1}^{n} \tilde{v_j}\tilde{\vec{e_j}}$$

$$\widetilde{\vec{e_j}} = \sum_{i=1}^{n} F_{ij}\vec{e_i}$$

$$\vec{e_j} = \sum_{i=1}^{n} B_{ij}\widetilde{\vec{e_i}}$$

$$\vec{v} = \sum_{j=1}^{n} \tilde{v_j}\tilde{\vec{e_j}} = \sum_{j=1}^{n} \tilde{v_j}\left(\sum_{i=1}^{n} F_{ij}\vec{e_i}\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} F_{ij}\widetilde{v_j}\vec{e_i}\right)$$

$$\vec{v_i} = \sum_{j=1}^{n} F_{ij}\widetilde{\vec{v_j}}$$

This proves that converting the vector components in the new basis to the old basis requires the forward transformation $\square$

3.2 Notation

Since the vector components are contravariant (do the opposite of the basis vectors), we write the index for the vector components as a superscript:

$$\vec{v} = \sum_{i=1}^{n} v^i \vec{e}_i = \sum_{i=1}^{n} \widetilde{v^i} \widetilde{\vec{e}_i}$$

Chapter 4

What are covectors?

Covectors are our second example of a tensor.

You can think of covectors as row vector. Row vectors are not simply just column vectors flipped on its side (only true in *orthonormal basis*).

Covectors/Row vectors can both be thought of as functions that act on a vector:

$$\alpha : V \longrightarrow \mathbb{R}$$

Covectors exhibit linearity:

$$\alpha\left(\vec{v} + \vec{w}\right) = \alpha(\vec{v}) + \alpha\left(\vec{w}\right)$$
$$\alpha(n\vec{v}) = n\alpha(\vec{v})$$

Covectors are also elements of a vector space, called the dual vector space, V^*

$$(n\alpha)(\vec{v}) = n\alpha(\vec{v})$$

$$(\beta + \alpha)(\vec{v}) = \beta(\vec{v}) + \alpha(\vec{v})$$

where α is the covector

4.1 Visualizing Covectors

Take the 2-D case:

$$\alpha = \begin{bmatrix} 2 & 1 \end{bmatrix}$$
$$\alpha(\vec{v}) = \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = 2x + 1y$$

We can visualize the covector by looking at the isolines (i.e. where $\alpha = k$, with k as a constant)

$$2x + 1y = 0$$
$$2x + 1y = 1$$
$$2x + 1y = 2$$
$$\vdots$$

2D Covectors form a "stack" of lines. Graphically, the value of α acting on $\vec{v}$ is the number of isolines that $\vec{v}$ pierces

Chapter 5

Covector Components

Like vectors, covectors are invariant, but their components are not.

Covectors form a vector space, V^*, and are not part of V, so we need new basis vectors, ϵ^1 and ϵ^2: $V \to \mathbb{R}$. They are defined as follows:

$$\epsilon^1\left(\vec{e_1}\right) = 1$$
$$\epsilon^2\left(\vec{e_1}\right) = 0$$
$$\epsilon^1\left(\vec{e_2}\right) = 0$$
$$\epsilon^2\left(\vec{e_2}\right) = 1$$

or simply

$$\epsilon^i\left(\vec{e_j}\right) = \delta_{ij}$$

Let's see what the two covector bases looks like when acting on a vector, $\vec{v}$

$$\epsilon^1\left(\vec{v}\right) = \epsilon^1\left(v^1\vec{e_1} + v^2\vec{e_2}\right)$$

Since covectors are linear:

$$= \epsilon^1\left(v^1\vec{e_1} + v^2\vec{e_2}\right) = v^1\epsilon^1\left(\vec{e_1}\right) + v^2\epsilon^1\left(\vec{e_2}\right) = v^1 \cdot 1 + v^2 \cdot 0 = v^1$$

Similarly,

$$\epsilon^2\left(\vec{v}\right) = v^2$$

Now consider a general covector, α acting on $\vec{v}$

$$\alpha\left(\vec{v}\right) = \alpha\left(v^1\vec{e_1} + v^2\vec{e_2}\right) = v^1\alpha(e_1) + v^2\alpha(e_2)$$

Using the definitions of $\epsilon^1(\vec{v}) = v^1$ and $\epsilon^2(\vec{v}) = v^2$ gives:

$$\alpha(\vec{v}) = \epsilon^1\left(\vec{v}\right) \cdot \alpha\left(e_1\right) + \epsilon^2\left(\vec{v}\right) \cdot \alpha\left(e_2\right)$$

Let $\alpha_1 = \alpha(\vec{e_1})$ and $\alpha_2 = \alpha(\vec{e_2})$

$$\alpha\left(\vec{v}\right) = \epsilon^1\left(\vec{v}\right)a_1 + \epsilon^2\left(\vec{v}\right)a_2 = \left(a_1\epsilon^1 + a_2\epsilon^2\right)\left(\vec{v}\right)$$
$$\implies \alpha = \alpha_1\epsilon^1 + \alpha_2\epsilon^2$$

We see that we can rewrite any generic covector, α, as a linear combination of ϵ^1 and ϵ^2. Thus the covectors ϵ^1 and ϵ^2 form a *dual basis*.

To summarize, with any set of basis vectors ($\vec{e_1}$ and $\vec{e_2}$), we can define basis convectors (ϵ^1 and ϵ^2). Then we can represent any convector as a linear combination of these basis convectors.

To convert between components of a convector from an old dual basis to a new dual basis, you use the *Forward matrix*

$$\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}_{\epsilon^i} F = \begin{bmatrix}\widetilde{\alpha_1} & \widetilde{\alpha_2}\end{bmatrix}_{\widetilde{\epsilon^i}}$$

Converting from the new basis to the old basis requires the *Backwards matrix*:

$$\begin{bmatrix} \widetilde{\alpha_1} & \widetilde{\alpha_2} \end{bmatrix}_{\widetilde{\epsilon^i}} B = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}_{\epsilon^i}$$

Note this is the opposite compared to what happens for a change of basis for normal vector components.

This shows that the vector components are not always the same as convector components. See the eigenchris' video for examples.

Chapter 6

Convector Transformation Rules

We now know how dual vector components transform. This chapter will show how to go from an old dual vector basis to a new dual basis.

We want to find the matrix, Q, where

$$\widetilde{\epsilon^1} = Q_{11}\epsilon^1 + Q_{12}\epsilon^2$$
$$\widetilde{\epsilon^2} = Q_{21}\epsilon^1 + Q_{22}\epsilon^2$$