# Intro to Tensors Notes

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# Chapter 1

# Tensor Definitions

**Definition 1.0.1: Tensors as multidimensional arrays**

Examples:

1. Rank 0 Tensor: **Scalar**
$$[4] \text{ or } 4$$
2. Rank 1 Tensor: **Vector**
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
3. Rank 2 Tensor: **2D-Matrix**
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
4. Rank 3 Tensor: **3D-Matrix** "Cube" of values

Although tensors can be *represented* as matrices/arrays, this definition doesn't describe what tensors *actually* are since this definition doesn't explain the geometric meaning of tensors.

**Definition 1.0.2: Tensors as objects invariant under a change in coordinates**

1. Tensors have **components** that change in a **predictable** way when the coordinates are changed
2. Vectors are **invariant**, but vector components are not
3. Example of something invariant under coordinate transformation: length
4. Converting tensor components from one coordinate system to another is called a **Forward Transformation**, while doing the reverse if **Backwards Transformation**

**Definition 1.0.3: Tensors as a combination of vectors and convectors combined using the tensor product**

Best definition, but a bit abstract.

**Definition 1.0.4: Tensors as partial derivatives and gradients that transform with the Jacobean matrix**

# Chapter 2

# Forward and Backwards Transformations

Old Basis:

$$\{\vec{e_1}, \vec{e_2}\}$$

New Basis:

$$\{\tilde{\vec{e_1}}, \tilde{\vec{e_2}}\}$$

## 2.1 Forward Transformation

Convert from the old basis to the new basis:

$$\tilde{\vec{e_1}} = c_1\vec{e_1} + c_2\vec{e_2}$$
$$\tilde{\vec{e_2}} = c_3\vec{e_1} + c_4\vec{e_2}$$

where $c_1, c_2, c_3,$ and $c_4$ are scalar constants.

We can rewrite the above linear equations in matrix form by defining a **forward transformation matrix**, $F$

$$F = \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\vec{e_1}} & \tilde{\vec{e_2}} \end{bmatrix} = \begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} F = \begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

Note that the $F$ matrix is flipped along the diagonal (transposed) from what we'd get if we multiplied the forward transformation matrix before the old basis.

$$\tilde{F} = F^T = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\vec{e_1}} \\ \tilde{\vec{e_2}} \end{bmatrix} = \tilde{F} \begin{bmatrix} \vec{e_1} \\ \vec{e_2} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} \vec{e_1} \\ \vec{e_2} \end{bmatrix}$$

## 2.2 Backwards Transformation

Similarly, we can convert from the new basis to the old basis with a **backwards transformation**

$$\vec{e_1} = a_1\tilde{\vec{e_1}} + a_2\tilde{\vec{e_2}}$$
$$\vec{e_2} = a_3\tilde{\vec{e_1}} + a_4\tilde{\vec{e_2}}$$

where $a_1, a_2, a_3,$ and $a_4$ are scalar constants.
Rewrite in terms of the backwards transformation matrix, $B$

$$B = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} B = \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

If we multiply $F$ and $B$ we get the identity matrix:

$$BF = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus one is the inverse of the other:

$$B = F^{-1}$$

## 2.3 Generalizing to N-Dimensions

We can generalize to $n$ dimensions

$$F = \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1n} \\ F_{21} & F_{22} & \dots & F_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n1} & F_{n2} & \dots & F_{nn} \end{bmatrix}$$

$$\tilde{\vec{e}}_1 = F_{11}\vec{e}_1 + F_{21}\vec{e}_2 + \dots + F_{n1}\vec{e}_n$$

$$\tilde{\vec{e}}_2 = F_{12}\vec{e}_1 + F_{22}\vec{e}_2 + \dots + F_{n2}\vec{e}_n$$

$$\vdots$$

$$\tilde{\vec{e}}_n = F_{1n}\vec{e}_1 + F_{2n}\vec{e}_2 + \dots + F_{nn}\vec{e}_n$$

This can be written more simply as:

$$\tilde{\vec{e}}_i = \sum_{j=1}^{n} F_{ji}\vec{e}_j$$

Similarly we have

$$\vec{e}_i = \sum_{j=1}^{n} B_{ji}\tilde{\vec{e}}_j$$

We still have that $F$ and $B$ are inverses

$$FB = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

***Proof:***

$$\vec{e}_i = \sum_{j=1}^{n} B_{ji}\tilde{\vec{e}}_j$$

$$\vec{e}_i = \sum_{j=1}^{n} B_{ji}\left(\sum_{k=1}^{n} F_{kj}\vec{e}_k\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} F_{kj}B_{ji}\vec{e}_k\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} F_{kj}B_{ji}\vec{e}_k\right)$$

Note that we want $\vec{e_1} = \vec{e_1}$, $\vec{e_2} = \vec{e_2}$, etc. so that implies that

$$\sum_j F_{kj} B_{ji} = \delta_{ki} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

which means that the matrix $FB$ has 1's on the diagonals and 0's elsewhere (identity matrix).

$\square$

# Chapter 3

# Vector Definition

A vector is an example of a tensor.