

# Why is timestepping so hard?

Jemma Shipton

January 2023

# Solving time dependent problems

## Ordinary differential equation

We wish to approximate the solution of

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad (1)$$

where  $f$  is a 'nice' function that we know.

We know, or can evaluate,

1. the value of  $y$  at  $t_0$ ,
2. the *slope*, given by  $\frac{dy}{dt}$  for any value of  $y$ .

## Integrating equation 1

$$y(t) = y(t_0) + \int_{t_0}^t f(\tau, y(\tau)) d\tau \quad (2)$$

# Euler's method

Approximate  $f(t, y) \approx f(t_0, y_0)$  for  $t_0 < t < t_0 + \Delta t$ :

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^t f(\tau, y(\tau)) d\tau, \\ &\approx y_0 + (t - t_0)f(t_0, y_0). \end{aligned}$$

Given a sequence  $t_0, t_0 + \Delta t, t_0 + 2\Delta t, \dots$  we can generate a sequence of approximate solutions

$$y_{n+1} = y_n + \Delta t f(t_n, y_n),$$

where  $y_n$  is our approximate solution for  $y$  at time  $t_n = t_0 + n\Delta t$ .

# Euler's method

Approximate  $f(t, y) \approx f(t_0, y_0)$  for  $t_0 < t < t_0 + \Delta t$ :

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^t f(\tau, y(\tau)) d\tau, \\ &\approx y_0 + (t - t_0)f(t_0, y_0). \end{aligned}$$

Given a sequence  $t_0, t_0 + \Delta t, t_0 + 2\Delta t, \dots$  we can generate a sequence of approximate solutions

$$y_{n+1} = y_n + \Delta t f(t_n, y_n),$$

where  $y_n$  is our approximate solution for  $y$  at time  $t_n = t_0 + n\Delta t$ .

But how do we know if it is any good?

## Other methods

### Trapezoidal rule

Instead of approximating the derivative by a constant value at  $t_0$ , use the average of the endpoint values:

$$\begin{aligned}y(t) &= y(t_0) + \int_{t_0}^t f(\tau, y(\tau)) d\tau, \\ &\approx y_0 + \frac{1}{2}(t - t_0)(f(t_0, y_0) + f(t, y)).\end{aligned}$$

which gives the trapezoidal rule:

$$y_{n+1} = y_n + \frac{1}{2}\Delta t(f(t_n, y_n) + f(t_{n+1}, y_{n+1})).$$

### Theta method

$$y_{n+1} = y_n + \Delta t(\theta f(t_n, y_n) + (1 - \theta)f(t_{n+1}, y_{n+1})).$$

# Other methods

## Implicit midpoint rule

Instead of approximating the derivative by the average of the endpoint values, take the value at the midpoint:

$$y'(t) \approx f\left(t_n + \frac{1}{2}\Delta t, \frac{1}{2}(y_n + y_{n+1})\right).$$

which gives the implicit midpoint rule:

$$y_{n+1} = y_n + \Delta t f\left(t_n + \frac{1}{2}\Delta t, \frac{1}{2}(y_n + y_{n+1})\right).$$

# But how do we know if it is any good?

## Local truncation error

This is the error that the method makes in one step:

$$\tau = ||y(\Delta t) - y_1||$$

## Global truncation error

This is the error that the method makes over many steps:

$$E = ||y(T) - y_N||, \quad T = N\Delta t$$

For numerical methods for solving ordinary differential equations, the global truncation error is one order less than the local truncation error.

# But how do we know if it is any good?

For the forward Euler method we have:

## Local truncation error

$$\begin{aligned}\tau &= \|y(\Delta t) - y_1\|, \\ &= \|y_0 + \Delta t y'(t_0) + \mathcal{O}(\Delta t^2) - (y_0 + \Delta t f(t_0, y_0))\|, \\ &= \mathcal{O}(\Delta t^2).\end{aligned}$$

where we have used the Taylor series expansion for  $y(\Delta t)$ , substituted for  $y_1$  from the forward Euler formula, and used the original ODE to cancel terms.

## Global truncation error

This is one order less than the local truncation error, meaning that forward Euler has global truncation error of  $\mathcal{O}(\Delta t)$ .



## Consistency

A consistent method is one for which the local truncation error approaches zero as  $\Delta t \rightarrow 0$ .

## Convergence

A convergent method produces a numerical solution that tends to the true solution, i.e. the global truncation error goes to zero, as the timestep is decreased.

## Order

If the local truncation error is of order  $\mathcal{O}(h^{p+1})$  then the method is said to have *order of accuracy*  $p$ .

# Accuracy

- ▶ Formal definitions of accuracy rely on Taylor series and therefore assume smoothness of solutions that may not be realistic.
- ▶ The accuracy of the solution is considered as  $\Delta t \rightarrow 0$ , which is not what happens in practice.
- ▶ Durrant: 'If the function is sufficiently rough and  $t$  is sufficiently coarse, neither formula is likely to produce a good approximation, and the superiority of one over the other will be largely a matter of chance.'

# Stability

Consistency of order  $p$  and *stability* imply convergence of order  $p$ . Define the *amplification factor*  $A$  as the ratio of the numerical solution at consecutive timesteps:

$$A = \frac{\phi_{n+1}}{\phi_n}.$$

## Dahlquist equation

Simple, but important, test problem:

$$\phi_t = \lambda\phi, \quad \phi(0) = \phi_0, \quad \lambda \in \mathbb{C}.$$

This has solution

$$\phi(t) = \phi_0 e^{\lambda t}.$$

Solutions grow if  $\Re(\lambda) > 0$  and decay if  $\Re(\lambda) < 0$ .

# Timestepping notebook

Exercises to implement different timestepping schemes and investigate the stability.

# Partial differential equations

# Diffusion equation

This partial differential equation governs diffusion of a substance  $\phi$ :

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2},$$

where  $\nu$  is the diffusivity.

This equation is also known as the heat equation as it can be used to model the conduction of heat.

# Transport equation

This partial differential equation governs the transport, or advection, of a substance  $\phi$ :

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0,$$

where  $c$  is the transport speed.

The solution of this equation is of the form

$$\phi(x, t) = F(x - ct).$$

[You can see this by substituting this expression into the transport equation and using the chain rule.]

## More complexity...

We can build up the complexity of the problems we consider by:

Combining advection and diffusion:

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2}.$$

Introducing nonlinearity:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

Considering higher dimensions:

$$\frac{\partial \phi}{\partial t} + \mathbf{c} \cdot \nabla \phi = \nu \nabla^2 \phi.$$

Introducing some forcing:

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2} + f(t, x).$$