Applicative Functors in Isabelle/HOL

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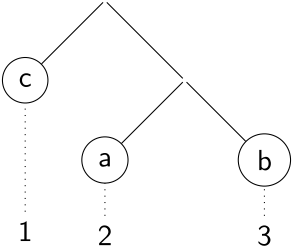
Outline

Example: Tree Labels

```
data Tree a = Leaf a | Node (Tree a) (Tree a)
```



Inspired by G. Hutton and D. Fulger, "Reasoning About Effects: Seeing the Wood Through the Trees." in *Proceedings of the Symposium on Trends in Functional Programming*, (Nijmegen, The Netherlands, 2008).

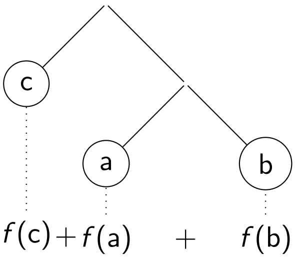
Composing Stateful Computations

Standard solution: state monad

```
fresh = do
  x <- get
  put (x + 1)
  return x

numberTree (Leaf _) = do
  x <- fresh
  return (Leaf x)
numberTree (Node l r) = do
  l' <- numberTree l
  r' <- numberTree r
  return (Node l' r')
```

Short Circuit Evaluation



f is partial! `f :: a -> Maybe Int`

```
evalTree (Leaf x) = f x
evalTree (Node l r) = do
  l' <- evalTree l
  r' <- evalTree r
  return (l' + r')
```

Cue Applicative Functors

```
class Functor f => Applicative f where
  pure  :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

numberTree (Leaf _) = pure Leaf <*> fresh
numberTree (Node l r) =
  pure Node <*> numberTree l <*> numberTree r

evalTree (Leaf x) = f x
evalTree (Node l r) =
  pure (+) <*> evalTree l <*> evalTree r
```

The Laws

To do: Applicative functor laws
To do: Notation pure, $\diamond$

C. McBride and R. Paterson, “Applicative Programming with Effects.” *Journal of Functional Programming, 18* (1). 2008, 1–13.

Lifting Terms and Equations

Lift the term $\mathsf{f}\, a + b$:

$$\begin{array}{ccccccccc} & (+) & & (& \mathsf{f} & & a) & & b \\ \mathsf{pure} & (+) & \diamond & (& \mathsf{pure}\ \mathsf{f} & \diamond & a) & \diamond & b \end{array}$$

To do: Idiomatic term

Lift an equation: Addition is commutative

$$x + y = y + x$$

$$\mathsf{pure}\,(+) \diamond x \diamond y = \mathsf{pure}\,(+) \diamond y \diamond x$$

Hinze's Lemmas (1)

Lemma (Normal Form)

Let e be an idiomatic term with variables $x_1, \ldots, x_n$. There exists f such that

$$e = \mathsf{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n.$$

Can lift equations

1. where both sides have the same list of variables, and
2. no variable is repeated.

R. Hinze, "Lifting Operators and Laws." 2010. Retrieved June 6, 2015, `http://www.cs.ox.ac.uk/ralf.hinze/Lifting.pdf`

Hinze's Lemmas (2)

Lemma (Lifting Lemma)

To do.

A Proof Method for Isabelle

Project Goal

Implement a proof method for Isabelle/HOL which lifts equations to applicative functors.

$$\text{base equation} \Longrightarrow \text{lifted equation}$$

Proof method: User interface for goal state transformation

$$\text{base equation} \longleftarrow \text{lifted equation}$$

Overview of Operation

$$e_1 = e_2$$

1. Transform into *canonical forms*

$$\Longleftarrow \quad \text{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n = \text{pure}\, g \diamond x_1 \diamond \ldots x_n$$

2.

$$\Longleftarrow \quad f = g$$

3.

$$\Longleftarrow \quad \forall y_1 \ldots y_n.\ f y_1 \ldots y_n = g y_1 \ldots y_n$$

Combinatory Logic

To do.

Instances of Applicative

To do.

To do