Applicative Functors in Isabelle/HOL: Notes

Joshua Schneider

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1   Project Overview

1.1   Introduction

Our primary goal is to implement an Isabelle/HOL proof method which reduces lifted equations to their base form. Here, lifting refers to a transition from operations on base types to related operations on some structure. Hinze [1] studied the conditions under which lifting preserves the validity of equations. He noticed that lifting can be defined in an intuitive fashion if the target structure is an applicative functor [2]: a unary type constructor f with associated constants[1]

$$pure_f :: \alpha \Rightarrow \alpha f,$$
$$(\diamond_f) :: (\alpha \Rightarrow \beta)f \Rightarrow \alpha f \Rightarrow \beta f.$$

The operator $\diamond_f$ is left-associative. We omit the subscripts if the functor is clear from the context. Moreover, the following laws must be satisfied:

$$pure\, id \diamond u = u \qquad\qquad\qquad\text{(identity)}$$
$$pure\,(\cdot) \diamond u \diamond v \diamond w = u \diamond (v \diamond w) \qquad\qquad\text{(composition)}$$
$$pure\, f \diamond pure\, x = pure\,(f x) \qquad\qquad\text{(homomorphism)}$$
$$u \diamond pure\, x = pure\,(\lambda f.\, f x) \diamond u \qquad\qquad\text{(interchange)}$$

The identity type constructor defined by $\alpha\, id = \alpha$ is a trivial applicative functor for $pure\, x = x$, $f \diamond x = f x$. We can take any abstraction-free term t and replace each constant c by $pure\, c$, and each instance of function application $f x$ by $f \diamond x$. The rewritten term is equivalent to t under the identity functor interpretation, or identity "idiom" as coined in [2]. By choosing a different applicative functor, we obtain a different interpretation of the same term structure. In fact, this is how we define the lifting of t to an idiom. We also permit variables, which remain as such in the lifted term, but range over the structure instead. A term consisting only of $pure$ and $\diamond$ applications and free variables is called an idiomatic expression.

Example 1. Another applicative functor can be constructed from sets. For each type α there is a corresponding type $\alpha\, set$ of sets with elements in α; $pure$ denotes the singleton set constructor $x \mapsto \{x\}$; $F \diamond X$ takes a set of functions F

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[1]Types are given in Isabelle notation.

and a set of arguments X with compatible type, applying each function to each argument:

$$F \diamond X = \{fx \mid f \in F,\, x \in X\}.$$

We can lift addition on natural numbers to the set idiom by defining the operator

$$(\oplus) :: nat\ set \Rightarrow nat\ set \Rightarrow nat\ set,$$
$$X \oplus Y = pure\,(+) \diamond X \diamond Y = \{x + y \mid x \in X,\, y \in Y\}.$$

The associative property of addition

$$\forall xyz.\ (x + y) + z = x + (y + z)$$

can be translated to sets of natural numbers

$$\forall XYZ.\ (X \oplus Y) \oplus Z = X \oplus (Y \oplus Z),$$

where it holds as well, as one can check with a slightly laborious proof. Note that the two sides of the latter equation are the lifted counterparts of the former, respectively. ▲

As we have seen, lifting can be generalized to equations. There is actually a more fundamental relationship between the two equations from above example—the lifted form can be proven for all applicative functors, not just *set*, using only the base property and the applicative functor laws. We want to automate this step with a proof method.

Not all equations can be lifted in all idioms, though. Stronger conditions are required if the list of quantified variables is different for each side of the equation. (The left-to-right order is relevant, but not the nesting within the terms.) These conditions must basically ensure that the functor does not add "too many effects" which go beyond the simple embedding of a base type. Such effects may be evoked if a variable takes an impure value, i.e., a value which is not equal to $pure\,x$ for any x.

Example 2. We try to lift the fact that zero is a left absorbing element for multiplication of integers, $\forall x :: int.\ 0 \cdot x = 0$, to sets. Note that the variable x occurs only on the left. But the lifted equation does not hold: If x, now generalized to *int set*, is instantiated with the empty set, then

$$pure\,(\cdot) \diamond pure\,0 \diamond \{\} = \{\} \neq pure\,0.$$

Here the effect of $\{\}$ is that it cancels out everything else if it occurs somewhere in an idiomatic expression. This makes it impossible to lift any equation with a variable occuring only on one side to *set*. ▲

2 Background

2.1 Proving with Isabelle

Isabelle was originally designed as a framework for interactive theorem proving, without being restricted to a specific logical system [3]. However, one chooses a particular *object-logic* in order to construct a theory and prove theorems.

In this paper, we focus on the Isabelle/HOL object-logic [4]. It implements the higher-order logic which was used in the HOL system [5], another proving environment. Isabelle/HOL (or HOL from here on) is arguably the most popular object-logic of Isabelle, as it comes with an extensive library of readily formalized mathematics. It also supports modelling of functional programs by means of datatypes and recursive functions, making it suitable for verification tasks. We will further discuss the choice of HOL for the extension presented here in Section ⟨To do.⟩.

The basis of HOL is a slightly extended variant of simply-typed lambda calculus. Therefore, every object (and every term representing such an object) has a certain type attached to it. We use lower-case greek letters α, β, γ as meta-variables for types. The language of types consists of base types, type variables, and compound types. Base types are represented by their name and include fundamental types like the booleans *bool* and the natural numbers *nat*. Type variables stand for an arbitrary types. In Isabelle syntax, they are distinguished by a prefixed "'", e.g. $'a$, $'b$, $'c$. The polymorphism in HOL is quite restricted, though, because higher-ranked types cannot be expressed: There is no explicit quantifier on the type level. This rules out functions which take a polymorphic function as an argument and apply it to values of different types. Compound types are built up of a type constructor and a list of types. The type constructor determines the number of argument types. For example, the unary type constructor *set* denotes sets with elements of a certain type. The argument is written on the left as in *nat set*, the type of sets of natural numbers. Multiple types are written in parentheses: (α, β)*fun*. *fun* is the special type constructor for (total) functions from α to β. More commonly, the infix operator $\Rightarrow$ is used. It is right-associative, i.e. $\alpha \Rightarrow \beta \Rightarrow \gamma$ is notation for $(\alpha, (\beta, \gamma)fun)fun$. Note that type constructors are different from types and must always be concrete. In particular, it is not possible to use a variable in place of a type constructor!

Terms follow the standard rules of lambda calculus. Atomic terms are constants and variables. Application of a function f to an argument x is written $f\,x$. Functions with multiple arguments are commonly curried in HOL; we can drop parentheses accordingly : $f\,x\,y$ is the same as $(f\,x)\,y$. Abstraction of a term t over the variable x is written $\lambda x.\,t$. Terms must be well-typed, of course. The types of variables and polymorphic constants can usually be omitted, since they are inferred automatically. Explicit type constraints are denoted by $t :: \alpha$ and may occur anywhere in a term. While all terms are represented internally roughly as shown above, Isabelle comes with extensible notation support.

Example 3. We already introduced the type *nat*. Number literals can be used directly. Common arithmetic operators are available, like[2] *plus* :: $nat \Rightarrow nat \Rightarrow nat$. We can also use infix operators:

$$\lambda(x :: nat)\,y.\,1 + x * y$$

is a function which multiplies two natural numbers and adds one to the result. Another important type family are sets. They can be specified as finite collections $\{\}$, $\{a, b, c\}$ etc., and by using set comprehension: Let P be a predicate $\alpha \Rightarrow bool$. Then $\{x.\,P\,x\}$ is the set of those values $x :: \alpha$ such that $P\,x$ is true, and $\{f\,x \mid x.\,P\,x\}$ is the image of that set under f.

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[2]These functions actually have a more generic type (they are overloaded). We will look at this later on.

3

Logical formulas are centered around truth values. Thus, the usual connectives like conjunction $\wedge$ and implication $\longrightarrow$ operate on type *bool*. Quantifiers work just as expected: The term

$$\forall (x :: nat)\, y.\ x + y = y + x$$

states that addition of natural numbers is commutative. Note that $=$ is just another operator of polymorphic type $'a \Rightarrow' a \Rightarrow bool$. Internally, quantifiers are represented as constants applied to lambda abstractions, which handle the variable binding. ▲

In order to achieve the goal of supporting different object-logics, Isabelle contains an immediate layer, the meta-logic Pure. It is an "intuitionistic fragment of higher-order logic" [6, p. 27]. In Pure itself there is only the type *prop* of propositions. A term of this type combined with a proof relative to some context forms a *theorem*. For now, the context is an abstract entity which may contain local hypotheses. Assumptions can also be recorded explicitly in a proposition, using meta-implication $\Longrightarrow$. Note that this is technically different from HOL's implication $\longrightarrow$, though there are theorems which allow conversion between the two. The meta-quantifier $\bigwedge$ denotes universal quantification (with a similar relationship to $\forall$); it is used to restrict the scope of variables in assumptions. Meta-equality $\equiv$ is the third main operator of Pure. The generic rewriting and simplification tools work with such equations. Again, conversion with $=$ is possible. Object-logics embed their own notion of a proposition into Pure via a truth judgement. In HOL, this is the constant *Trueprop* $:: bool \Rightarrow prop$, which turns an object-level formula into a proposition stating that said formula is indeed true. It is usually left implicit. See [7, Chapter 2] for further details about Pure.

Pure has two main uses within the Isabelle framework: representation and manipulation of deduction rules, and goal states. For the former, consider the traditional introduction rule for conjunction,

$$\frac{\Gamma_1 \vdash P \qquad \Gamma_2 \vdash Q}{\Gamma_1 \cup \Gamma_2 \vdash P \wedge Q},$$

which we want to turn into a theorem. Deduction in Isabelle handles the contexts Γ_i automatically, and therefore do not have to be stated in the corresponding proposition. The dependency of the conclusion $P \wedge Q$ on the hypotheses P and Q translates to repeated meta-implication. This gives us the proposition

$$\bigwedge PQ.\ P \Longrightarrow Q \Longrightarrow P \wedge Q$$

(the *Trueprop* markers have been omitted, and P, Q range over *bool*). There is one quirk, however. The outermost meta-quantified variables (and all type variables) are turned into schematic variables, which are free variables distinguished by the prefix *?*. Thus, the introduction rule as it is supplied by HOL appears as

$$\texttt{conjI:} \quad ?P \Longrightarrow ?Q \Longrightarrow ?P \wedge ?Q.$$

Schematic (type) variables are eligible for instantiation during *resolution*, which is Isabelle's primary tool for proof construction. Resolution combines two rules (theorems), identifying an assumption of the second with the conclusion of the

first by higher-order unification. Additionally, the first rule is brought into the context of that assumption.

A goal state is a theorem that represents a partially completed proof of some proposition, the goal. While the proof is incomplete, the remaining subgoals are tracked as assumptions. Proof steps transform the goal state. The proof is finished once only the goal remains.

In contemporary use of Isabelle, user input to the system is expressed in the Isar language [8, 9, 6]. It aims to encode proofs in a way that is formal, i.e. has precise semantics, but still resembles informal patterns of reasoning. The basic organization unit in Isar is a *theory*. The body of a theory consists of a sequence of commands, which consecutively augment the logical context by declarations of various kinds. Other theories may be imported in the beginning, leading to a acyclic graph of theory dependencies. Commands constitute the so-called outer syntax of Isar. Terms and types occuring within them are parsed separately, according to the inner syntax. They are usually embedded in quotes '"..."' to disambiguate them. In certain cases, a command may put the theory state into proof mode. After a proof is finished, the associated goal becomes a *fact*.

Some commonly used specifications are:

- The **definition** command introduces new constants by means of defining equations.

- To do. **datatype**, **primrec**

- Facts can be given names for further use in proofs. The canonical command for this is **lemma** and its variants **theorem** and **corollary**.

- To do. **locale**?

Finally, there two syntactical categories which are repeatedly used in commands: *Proof methods* denote (possibly parameterized) operations on the goal state. *Attributes* invoke further processing steps on facts, either transforming them or causing additional declarations.

3 Requirements and Basic Design

3.1 Requirements

Our primary goal is to implement an Isabelle/HOL proof method which reduces lifted equations to their base form. This proof method should be generic and work with arbitrary idioms. The following is the minimal set of user actions we shall support:

a) Declare applicative functors to the theory context. Given a type constructor f, the functions $pure_f$ and $\diamond_f$, and proofs for the relevant functor laws, the functor is registered with the package such that it can be used in subsequent invocations of the proof method.

b) Prove lifted equations $a'[\vec{x}] = b'[\vec{x}]$, where a' and b' are idiomatic expressions with free variables $\vec{x}$, using the base equation $\forall \vec{y}.\, a[\vec{y}] = b[\vec{y}]$. More precisely, if there is a subgoal stating the former, applying the proof methods transforms the goal state to the latter. The functor should be either detected automatically, or specified by the user.

The first requirement ensures that our package is reusable, while the second is the core functionality. However, usability is also a concern: In the realm of interactive theorem proving, it is not sufficient to just verify formal objects—we are not extending the logic, after all, but providing a shortcut for a certain principle. We must balance the clarity of the resulting proof document and the amount of work that the user has to put into developing a proof. The following features are possible extensions which may help in this regard:

c) Declare lifted constants and other terms to the theory context. Generally speaking, if a term t with free variables v_i can be expressed as $pure\, t' \diamond v_1 \diamond \cdots \diamond v_n$ for some t', then the corresponding equation can be registered. The lifting proof methods rewrites with these equations at the beginning, such that the base equation will refer to t'. This way, the user does not have to transform everything into idiomatic format first.

d) More flexibility regarding the logical structure of the input proposition. This includes bound variables (quantified by $\forall$ or $\bigwedge$), complex subgoals with premises, and cases where the conceptual variables of the lifted equation have been subsituted by some terms.

e) Related proof methods and attributes, for example for forward lifting of proven base equations.

f) Inspection and tracing output. This is particular useful if something does not work as expected.

g) Extend the notion of lifting beyond equations. It is possible to define lifting for other logical operators. For example, the cancellation law $a + b = a + c \longrightarrow b = c$ consists of two equations, joined by implication. We can interpret it in different domains, e.g., for integers and for streams of integers with lifted $+$. In this example, the law is true for both interpretations. We are not able to handle such propositions with just a method for equations, though.

The current version at the time writing supports c) and d), as well as e) and f) to some extent. $\boxed{\text{To do.}}$ g) not discussed in detail, but see final section. In the remainder of this section, we will explain the design decisions we have made in order to fulfill the requirements. Furthermore, the registration infrastructure and the basic proof approach are explained.

3.2 Choice of Embedding

In Isabelle, it is not possible to construct an abstract framework for applicative functors in such a way that it is inhabited by all instances. We already referred to the fact that type constructors are fixed. Another issue is the lack of polymorphism in the inner logic: We cannot have, say, a schematic variable *?pure* and use it with different types within the same proposition or proof. One solution is to define a custom logic, including a term language, axioms and meta theorems, and formalize it using the available specification tools. This is a *deep embedding* of the logic [10]. Then it would be possible to derive the Normal Form Lemma as a regular inference rule, for example. However, we want to prove propositions about arbitrary HOL objects, not just their encodings in the

embedded logic. Some machinery, which performs the encoding and transfers results, is necessary.

- finite number of types involved per term $\implies$ could use sum types

- number of types in sum is linear in size of terms

- would introduce a large number of projections/abstractions

To do.
A different approach, which is the one we will take, is a *shallow embedding*. The "formulæ" (here, idiomatic terms) are expressed directly in HOL. Due to aforementioned restrictions, meta-theorectical results must be provided in specialized form for each case. We make use of the powerful ML interface of Isabelle to program the proof construction. The correctness of the proofs is still verified by the system, of course. On the other hand, the correctness of the construction algorithm has to be asserted externally.

3.3   User Interface

Since Isabelle's core logic does not allow parameterization of type constructors, we need a custom mechanism for registering applicative functors with the system. In order to apply the proof method, the user must provide beforehand

a) corresponding *pure* and $\diamond$ instances, and

b) a proof of the applicative functor laws, optionally with extended properties.

Lifted constants may be registered with an attribute, which can be applied to facts *lhs = rhs*, where *rhs* is an idiomatic expression. These must be suitable for rewriting.

The complete set of subgoal forms to support has not been determined yet. To do. As a minimal requirement, after unfolding lifted constants, HOL equations of idiomatic expressions shall be handled. Only the outermost functor f is considered per invocation. The conceptual variables of the lifted expressions may be instantiated with arbitrary terms. However, the method actually proves the fully universally quantified form—for every subterm not matching $pure_f\,\_$ or $\_\diamond_f\,\_$, a new, locally quantified variable is introduced. We call these subterms *opaque*. The method attempts to transform the first subgoal to the base form of the equation, in other words, its identity functor interpretation. Variable names shall be preserved, if possible. Finally, it is desirable to have some kind of debugging facility for tracing intermediate steps. This may be especially useful if the proof fails because additional laws are required.

Example 4. Continuing with the set idiom from Example 1, assume that the user wants to prove an instantiation of the associativity law for $\oplus$ as part of a larger proof. *set* and $\oplus$ have been declared to the enclosing theory. The current Isabelle goal state is

$$1. \quad (X \oplus Y) \oplus fZ = X \oplus (Y \oplus fZ).$$

The variables X, Y and Z have been fixed in the proof context, and f is some constant, all of appropriate type. This illustrates how we allow a larger variety of propositions, making it easier to apply the method without too much preparation. After applying the proof method, the new proof obligation reads

$$1. \quad \bigwedge xyu.\ (x+y)+u = x+(y+u),$$

which is easily discharged. The corresponding Isabelle/Isar fragment could be

> **fix** $X\ Y\ Z$
>
> $\ldots$
>
> **have** "$(X \oplus Y) \oplus fZ = X \oplus (Y \oplus fZ)$" **by** *af_lifting algebra*

af_lifting is our new proof method, and the standard *algebra* method completes the subproof. ▲

3.4 Proof Strategy

The proof method starts with testing the first subgoal for the expected structure. If the test succeeds, the applicative functor f is known, such that the relevant theorems can be accessed subsequently. We then rewrite the subgoal using the declared rules for lifted constants. Only those related to f are used, the reason being that overeager, unwanted unfolding may be difficult to reverse. The result of this preparation is a subgoal which is simple equation of two idiomatic expressions.

The following step depends on which additional properties of f have been provided. All approaches have in common that both expressions are replaced by others in *canonical form*:

$$pure\ g \diamond s_1 \diamond \cdots \diamond s_m = pure\ h \diamond t_1 \diamond \cdots \diamond t_n,$$

If either $m \neq n$ or $t_i \neq s_i$ for some i (as terms modulo $\alpha\beta\eta$-conversion), the proof method fails. Otherwise, we apply appropriate congruence rules until the subgoal is reduced to $g = h$. Since g and h are at least n-ary functions, we can further apply extensionality, reaching the subgoal

$$\bigwedge x_1 \ldots x_n.\ gx_1 \cdots x_n = hx_1 \cdots x_n.$$

This is the transformed proof state presented to the user.

Hinze's Normal Form Lemma [1, p. 7] asserts the existence of a certain normal form for idiomatic expressions where each variable occurs only once. This normal form has the desired structure. As it turns out, we can compute it for arbitrary terms. This is convenient because opaque parts are handled implicitly. However, the result might be too general and unprovable, so we will use the normal form only if no other properties are available. The details of the normalization algorithm are described in Section 4. There we will also show that the transformed equation is exactly the generalized base form of the original equation.

Hinze then explored under what circumstances a larger variety of equations can be lifted. He found that sufficient conditions can be expressed in terms of combinators as known from combinatory logic. We take this idea and adapt it

our framework in Section 5. In contrast to the normal form approach, we gain flexibility regarding the opaque terms in the canonical form. This means that the algorithm must determine the sequence $\vec{t} = \vec{s}$ of opaque terms prior to the transformations. The set of available combinators further limits the admissible sequences. Because this relationship cannot be generalized easily, we restrict ourselves to certain combinator sets.

4 Normal Form Conversion

McBride and Paterson [2] noted that idiomatic expressions can be transformed into an application of a pure function to a sequence of impure arguments. They called this the *canonical form* of the expression. Hinze [1, Section 3.3] gave an explicit construction based on the monoidal variant of applicative functors. Transforming the terms in this way is useful for our purpose, because the arguments of the remaining *pure* terms reflect the equation that was lifted. The soundness of the algorithm depends only on the applicative laws, making it the most general approach regarding functors (but not regarding lifted equalities). We will later show that all transformations based on the applicative laws yield a unique canonical form, modulo $\alpha\beta\eta$-conversion. Therefore, we follow Hinze and denote by *normal form* this particular canonical form. The distinction is necessary, as we will consider other canonical forms in Section 5, which are justified by additional laws.

In the following, we define lifting and normalization formally, based on a syntactic representation of idiomatic terms. While this presentation is more abstract than what is actually happening in Isabelle, it makes it easier to demonstrate correctness and some other properties. Unlike Hinze, we use the *pure*/$\diamond$ formalism. Then we describe the implementation of the normalization procedure in Isabelle/ML.

4.1 The Idiomatic Calculus

In Section 1.1, we introduced idiomatic expressions built from *pure* and $\diamond$ constants of an applicative functor. This structure maps straightforward to a recursive datatype, given that there is a representation for arguments of *pure*. These must have some structure as well such that the applicative laws can be expressed. It should also be possible to have "opaque" idiomatic subterms, which cannot (or should not) be written as a combination of *pure* and $\diamond$. This is primarily useful for variables ranging over lifted types, but as demonstrated in Example 4, more complex terms may occur too. Therefore it makes sense to refer to general lambda terms in both cases; then we can define semantics consistently. Types are ignored here for simplicity. However, all results are compatible with the restrictions of simply typed lambda calculus.

Definition 1 (Untyped lambda terms)**.** Let $\mathcal{V}$ be an infinite set of variable symbols. We assume that f, g, x, y are disjoint variables in the following formulas. The set of untyped lambda terms is defined as

$$\mathcal{T} ::= \mathcal{V} \mid (\mathcal{T}\,\mathcal{T}) \mid \boldsymbol{\lambda}\mathcal{V}.\,\mathcal{T} \tag{4.1}$$

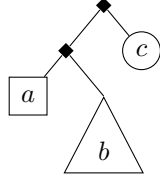An equivalence relation on $\mathcal{T}$ is a $\mathcal{T}$-congruence iff it is closed under application

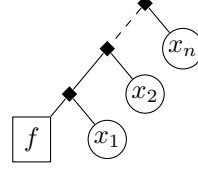Figure 1: $(\text{pure}\, a\, \text{`ap`}\, b)\, \text{`ap`}\, \text{term}\, c$ as a tree.

Figure 2: A term in canonical form.

and abstraction. Let $=_{\alpha\beta\eta}$ be the smallest $\mathcal{T}$-congruence containing α-, β-, and η-conversion. ▲

Definition 2 (Idiomatic terms)**.** The set of idiomatic terms is defined as

$$\mathcal{I} ::= \text{term}\, \mathcal{T} \mid \text{pure}\, \mathcal{T} \mid \mathcal{I}\, \text{`ap`}\, \mathcal{I}. \tag{4.2}$$

By convention, the binary operator `ap` associates to the left. An $\mathcal{I}$-congruence is an equivalence relation closed under `ap`. We overload notation and reuse $=_{\alpha\beta\eta}$ for idiomatic terms, where it stands for structural equality modulo substitution of $=_{\alpha\beta\eta}$-equivalent lambda terms. The $\mathcal{I}$-congruence $\simeq$ is induced by the rules

$$x \simeq \text{pure}\,(\boldsymbol{\lambda}x.\, x)\, \text{`ap`}\, x \tag{4.3}$$

$$g\, \text{`ap`}\, (f\, \text{`ap`}\, x) \simeq \text{pure}\, \mathbf{B}\, \text{`ap`}\, g\, \text{`ap`}\, f\, \text{`ap`}\, x \tag{4.4}$$

$$\text{pure}\, f\, \text{`ap`}\, \text{pure}\, x \simeq \text{pure}\,(f\, x) \tag{4.5}$$

$$f\, \text{`ap`}\, \text{pure}\, x \simeq \text{pure}\,((\boldsymbol{\lambda}x.\, \boldsymbol{\lambda}f.\, f\, x)\, x)\, \text{`ap`}\, f \tag{4.6}$$

$$s =_{\alpha\beta\eta} t \implies s \simeq t \tag{4.7}$$

where $\mathbf{B}$ abbreviates $\mathbf{B} \equiv \boldsymbol{\lambda}g.\, \boldsymbol{\lambda}f.\, \boldsymbol{\lambda}x.\, g\,(f\, x)$. ▲

term represents arbitrary values in the lifted domain, whereas pure lifts a value. The introduction rules for the relation $\simeq$ are obviously the syntactical counterparts of the applicative laws. Together with symmetry, substitution, etc., they give rise to a simple calculus of equivalence judgements. The intuitive meaning of $s \simeq t$ is that the terms can be used interchangeably. For example, there is a derivation for

$$\text{pure}\, g\, \text{`ap`}\, (f\, \text{`ap`}\, x) \simeq \text{pure}\,(\mathbf{B}\, g)\, \text{`ap`}\, f\, \text{`ap`}\, x \tag{4.8}$$

from (4.4), where g is instantied with pure g, and a substitution along (4.5) on the right-hand side.

Idiomatic terms are visualized naturally as trees. This will be helpful in explaining term transformations. Figure 1 shows the conventions: Inner nodes correspond to `ap`, leaves are either pure terms (boxes) or opaque terms (circles). Whole subterms may be abbreviated by a triangle. A term has canonical form if it consists of a single pure node to which a number of opaque terms (or none) are applied in sequence. Figure 2 gives a general example. A formal construction follows:
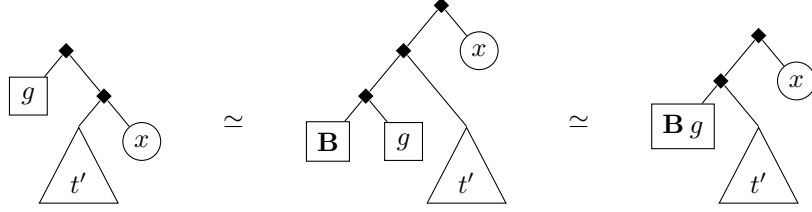
Figure 3: The "pure-rotate" step.

Definition 3 (Canonical form). The set $\mathcal{C} \subset \mathcal{I}$ of idiomatic terms in canonical form is defined inductively as

$$\textsf{pure}\, x \in \mathcal{C}, \tag{4.9}$$

$$t \in \mathcal{C} \implies t \text{ `ap` } \textsf{term}\, s \in \mathcal{C}. \tag{4.10}$$

▲

It is not entirely obvious how a canonical form can be derived from equations (4.3)–(4.6). Rewriting blindly with these is prone to infinite recursion. Therefore we need a more controlled algorithm. As we have said before, we use *normal form* to refer to the particular canonical form derived from those equations. Consider an idiomatic term t. If t is a single pure term, then it is already in normal form. The case $t = \textsf{term}\, x$ is also easy: Due to (4.3), we have $t \simeq \textsf{pure}\, (\boldsymbol{\lambda} x.\, x) \text{ `ap` } t$, which is in normal form. But in the case of $t = u \text{ `ap` } v$, various steps could be performed, depending on the subterms. We simplify the situation by normalizing each subterm recursively, so we get an equivalent term $u' \text{ `ap` } v'$ where $u', v' \in \mathcal{C}$.

Now let us assume that u' is just $\textsf{pure}\, g$. If v' is also a pure term, they can be combined along (4.5). Otherwise, the term looks like the one on the left of Figure 3. As is shown there, the term tree can be rotated such that one opaque term moves to the outer-most level. This is the same equivalence as stated in (4.8). Because the remaining part again has the shape "pure term applied to normal form", we proceed recursively. In pattern-matching style, the transformation 'pure-nf' reads

$$\text{pure-nf}(\textsf{pure}\, g \text{ `ap` } (f \text{ `ap` } x)) = \text{pure-nf}\, (\textsf{pure}\, (\mathbf{B}g) \text{ `ap` } f) \text{ `ap` } x \tag{4.11}$$

$$\text{pure-nf}(\textsf{pure}\, f \text{ `ap` } \textsf{pure}\, x) = \textsf{pure}\, (fx) \tag{4.12}$$

Lemma 1. *For all $g \in \mathcal{T}$ and $t \in \mathcal{C}$, $\text{pure-nf}(\textsf{pure}\, g \text{ `ap` } t)$ is well-defined, and[3] $\text{pure-nf}(\textsf{pure}\, g \text{ `ap` } t) \in \mathcal{C} \simeq \textsf{pure}\, g \text{ `ap` } t$.*

Proof. We prove all claims simultaneously by induction on $t \in \mathcal{C}$, where g is arbitrary.

Case 1. Assume $t = \textsf{pure}\, x$ for some $x \in \mathcal{T}$. Only the second equation applies, so we have

$$\text{pure-nf}(\textsf{pure}\, g \text{ `ap` } t) = \textsf{pure}\, (g\, x).$$

All pure terms are in $\mathcal{C}$, and equivalence follows from (4.5).

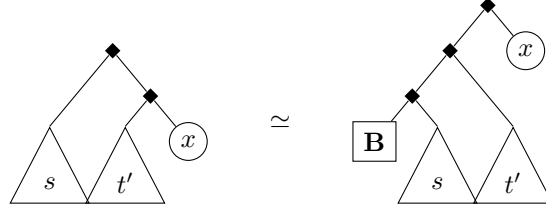
[3] $a \in S \simeq b$ abbreviates "$a \in S$ and $a \simeq b$".

Figure 4: The "rotate" step.

Case 2. Assume $t = t'$ 'ap' term x for some $t' \in \mathcal{C}$, $x \in \mathcal{T}$, and that the hypothesis holds for t' and all g. Only the first equation applies, so

$$\text{pure-nf}(\text{pure } g \text{ 'ap' } t) = \text{pure-nf}(\text{pure } (\mathbf{B}g) \text{ 'ap' } t') \text{ 'ap' term } x.$$

Instantiating the induction hypothesis, we find that

$$\text{pure-nf}(\text{pure } (\mathbf{B} \, g) \text{ 'ap' } t') \in \mathcal{C} \simeq \text{pure } (\mathbf{B} \, g) \text{ 'ap' } t'$$

is well-defined. $\mathcal{C}$ is closed under application to opaque terms (4.10), hence pure-nf(pure g 'ap' $t) \in \mathcal{C}$. Finally, we have

$$\text{pure-nf}(\text{pure } g \text{ 'ap' } t) \simeq \text{pure } (\mathbf{B} \, g) \text{ 'ap' } t' \text{ 'ap' term } x$$

$$\overset{(4.8)}{\simeq} \text{pure } g \text{ 'ap' } (\text{pure } t' \text{ 'ap' term } x) = \text{pure } g \text{ 'ap' } t. \quad \square$$

Going back to u' 'ap' v', we assumed that u' is a pure term. The case where v' is pure instead can be translated to the former by

$$\text{nf-pure}(f \text{ 'ap' pure } x) = \text{pure-nf}(\text{pure } ((\boldsymbol{\lambda}x. \, \boldsymbol{\lambda}f. \, fx) \, x) \text{ 'ap' } f) \tag{4.13}$$

Lemma 2. *For all $t \in \mathcal{C}$ and $x \in \mathcal{T}$,* nf-pure(t 'ap' pure x) *is well-defined, and* nf-pure(t 'ap' pure x) $\in \mathcal{C} \simeq t$ 'ap' pure x.

Proof. Follows from Lemma 1 and (4.6). $\square$

Finally, we look at general u', v'. A term rotation is useful again, see Figure 4. Before recursion, we must normalize the subterm pure $\mathbf{B}$ 'ap' s. But we already know how to do this: by 'pure-nf'. The base case is reached when v' is a single pure term, which is the domain of 'nf-pure'. The corresponding transformation is therefore

$$\text{nf-nf}(g \text{ 'ap' } (f \text{ 'ap' } x)) = \text{nf-nf}(\text{pure-nf}(\text{pure } \mathbf{B} \text{ 'ap' } g) \text{ 'ap' } f) \text{ 'ap' } x \tag{4.14}$$

$$\text{nf-nf}(t) = \text{nf-pure}(t) \quad \text{(otherwise)} \tag{4.15}$$

Lemma 3. *For all $s, t \in \mathcal{C}$,* nf-nf(s 'ap' t) *is well-defined, and* nf-nf(s 'ap' t) $\in \mathcal{C} \simeq s$ 'ap' t.

Proof. The proof is similar to the one of Lemma 1, by induction on $t \in \mathcal{C}$ and arbitrary $s \in \mathcal{C}$.

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Algorithm 1 Normalization of idiomatic terms.

$$\text{normalize}(\mathsf{pure}\,x) = \mathsf{pure}\,x$$
$$\text{normalize}(\mathsf{term}\,x) = \mathsf{pure}\,(\boldsymbol{\lambda}x.\,x)\,\text{`ap`}\,\mathsf{term}\,x$$
$$\text{normalize}(x\,\text{`ap`}\,y) = \text{nf-nf}(\text{normalize}\,x\,\text{`ap`}\,\text{normalize}\,y)$$

$$\text{nf-nf}(g\,\text{`ap`}\,(f\,\text{`ap`}\,x)) = \text{nf-nf}\,(\text{pure-nf}\,(\mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,g)\,\text{`ap`}\,f)\,\text{`ap`}\,x$$
$$\text{nf-nf}(t) = \text{nf-pure}(t) \quad \text{(otherwise)}$$

$$\text{pure-nf}(\mathsf{pure}\,g\,\text{`ap`}\,(f\,\text{`ap`}\,x)) = \text{pure-nf}\,(\mathsf{pure}\,(\mathbf{B}g)\,\text{`ap`}\,f)\,\text{`ap`}\,x$$
$$\text{pure-nf}(\mathsf{pure}\,f\,\text{`ap`}\,\mathsf{pure}\,x) = \mathsf{pure}\,(fx)$$

$$\text{nf-pure}(f\,\text{`ap`}\,\mathsf{pure}\,x) = \text{pure-nf}\,(\mathsf{pure}\,((\boldsymbol{\lambda}x.\,\boldsymbol{\lambda}f.\,fx)\,x)\,\text{`ap`}\,f)$$

Case 1. Assume $t = \mathsf{pure}\,x$ for some $x \in \mathcal{T}$. The second equation applies, so we have
$$\text{nf-nf}(s\,\text{`ap`}\,t) = \text{nf-pure}(s\,\text{`ap`}\,\mathsf{pure}\,x).$$

Since $s \in \mathcal{C}$, the claim follows directly from Lemma 2.

Case 2. Assume $t = t'\,\text{`ap`}\,\mathsf{term}\,x$ for some $t' \in \mathcal{C}$, $x \in \mathcal{T}$, and that the hypothesis holds for t' and all $s \in \mathcal{C}$. Only the first equation applies,
$$\text{nf-nf}(s\,\text{`ap`}\,t) = \text{nf-nf}(\text{pure-nf}(\mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,s)\,\text{`ap`}\,t')\,\text{`ap`}\,\mathsf{term}\,x.$$

We have $\text{pure-nf}(\mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,s) \in \mathcal{C} \simeq \mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,s$ from Lemma 1. Thus we can instantiate the induction hypothesis, and the transformed term is indeed in normal form. Furthermore,

$$\text{nf-nf}(s\,\text{`ap`}\,t) \overset{\text{(IH)}}{\simeq} \text{pure-nf}(\mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,s)\,\text{`ap`}\,t'\,\text{`ap`}\,\mathsf{term}\,x$$
$$\simeq \mathsf{pure}\,\mathbf{B}\,\text{`ap`}\,s\,\text{`ap`}\,t'\,\text{`ap`}\,\mathsf{term}\,x$$
$$\overset{(4.4)}{\simeq} s\,\text{`ap`}\,(t'\,\text{`ap`}\,\mathsf{term}\,x) = s\,\text{`ap`}\,t. \qquad \square$$

Algorithm 1 summarizes all pieces of the normal form transformation. 'normalize' is the entry point and performs the main recursion mentioned in the beginning. We haven't proved the desired property for 'normalize' yet, but this is just a straightforward induction.

Lemma 4. *For all $t \in \mathcal{I}$, normalize t is well-defined, and normalize $t \in \mathcal{C} \simeq t$.*

Proof. By induction on t, Lemma 3, and equation (4.3). $\qquad \square$

At this point, we know that it is always possible to obtain a certain canonical form. This is not sufficient to setup the complete proving process, though. We need to learn a bit more about the structure of idiomatic terms and how it relates to lifting.

Definition 4 (Opaque subterms)**.** The sequence of opaque subterms of an idiomatic term is defined by the recursive function

$$\mathrm{opaq}(\mathsf{pure}\,x) = [], \quad \mathrm{opaq}(\mathsf{term}\,x) = [x], \quad \mathrm{opaq}(s\,\text{'}\mathsf{ap}\text{'}\,t) = \mathrm{opaq}\,s\,@\,\mathrm{opaq}\,t.$$

@ denotes concatenation of lists. ▲

Definition 5 (Unlifting)**.** Let t be some idiomatic term, and $n = |\mathrm{opaq}\,t|$. Let $v_{i \in \{1..n\}}$ be new variable symbols that do not occur anywhere in t. The "unlifted" lambda term corresponding to t is defined as

$$\downarrow t = \boldsymbol{\lambda} v_1.\,\cdots \boldsymbol{\lambda} v_n.\,\mathrm{vary}_1\,t,$$

where

$$\mathrm{vary}_i(\mathsf{pure}\,x) = x, \tag{4.16}$$

$$\mathrm{vary}_i(\mathsf{term}\,x) = v_i, \tag{4.17}$$

$$\mathrm{vary}_i(s\,\text{'}\mathsf{ap}\text{'}\,t) = (\mathrm{vary}_i\,s)\,(\mathrm{vary}_{i+|\mathrm{opaq}\,s|}\,t). \tag{4.18}$$

▲

Example 5. The definition of $\downarrow$ may need some explanation. Consider the idiomatic term

$$t \equiv \mathsf{pure}\,f\,\text{'}\mathsf{ap}\text{'}\,x\,\text{'}\mathsf{ap}\text{'}\,(\mathsf{pure}\,g\,\text{'}\mathsf{ap}\text{'}\,y\,\text{'}\mathsf{ap}\text{'}\,z).$$

Its unlifted term is

$$\downarrow t = \boldsymbol{\lambda} a.\,\boldsymbol{\lambda} b.\,\boldsymbol{\lambda} c.\,f\,a\,(g\,b\,c).$$

The applicative structure is the same, but all opaque terms have been substituted for new bound variables, which are assigned from left to right. We do not define lifting formally here, but it should be clear that this is some sort of inverse operation, given that variables appear only once. ▲

The interesting properties about these two concepts is that they are preserved by the equivalence relation $\simeq$, and can be directly read from the canonical form. Furthermore, we can leverage them to show the uniqueness of the normal form.

Lemma 5. *For equivalent terms* $s \simeq t$*, the sequences of opaque terms are equivalent w.r.t* $=_{\alpha\beta\eta}$*, and* $\downarrow s =_{\alpha\beta\eta} \downarrow t$*.*

Proof. (Sketch.) By induction on the relation $s \simeq t$, where we show $\mathrm{vary}_i\,s =_{\alpha\beta\eta} \mathrm{vary}_i\,t$ instead. The index i is arbitrary. The part regarding opaque terms is shown easily for each case: We note that in (4.3)–(4.6), the opaque terms are identical for both sides. By the induction hypothesis, this is also true for the necessary closure rules for symmetry, transitivity, and substitution. It is obvious that (4.7) also preserves opaque terms. Regarding unlifted terms, we have

Case 1. (4.3)

$$\mathrm{vary}_i(\mathsf{pure}\,(\boldsymbol{\lambda} x.\,x)\,\text{'}\mathsf{ap}\text{'}\,x) = (\boldsymbol{\lambda} x.\,x)\,(\mathrm{vary}_i\,x) =_{\alpha\beta\eta} \mathrm{vary}_i\,x.$$

Case 2. (4.4) Let $j = i + |\mathrm{opaq}\,g|$ and $k = j + |\mathrm{opaq}\,f|$.

$$\mathrm{vary}_i(\mathsf{pure}\,\mathbf{B}\,\text{'}\mathsf{ap}\text{'}\,g\,\text{'}\mathsf{ap}\text{'}\,f\,\text{'}\mathsf{ap}\text{'}\,x) = \mathbf{B}\,(\mathrm{vary}_i\,g)\,(\mathrm{vary}_j\,f)\,(\mathrm{vary}_k\,g)$$
$$=_{\alpha\beta\eta}\,(\mathrm{vary}_i\,g)\,((\mathrm{vary}_j\,f)\,(\mathrm{vary}_k\,g))$$
$$=\,\mathrm{vary}_i(g\,\text{'}\mathsf{ap}\text{'}\,(f\,\text{'}\mathsf{ap}\text{'}\,x)).$$

Case 3. (4.5) and (4.6) are similar.

Case 4. (4.7) By induction.

Case 5. Symmetry, transitivity, and substitution: These follow from the induction hypothesis and the corresponding properties of $=_{\alpha\beta\eta}$.

$\square$

Lemma 6. *Let* pure f *be the single pure term in* $t \in \mathcal{C}$*. Then* $f =_{\alpha\beta\eta} \downarrow t$*.*

Proof. By induction on $\mathcal{C}$. The base case is trivial. For the step case, we need to prove that

$$f' =_{\alpha\beta\eta} \downarrow (g \text{ `ap` term } x) = \boldsymbol{\lambda} v_1. \ldots \boldsymbol{\lambda} v_n. \boldsymbol{\lambda} v_{n+1}. (\text{vary}_1 \, g) \, v_{n+1},$$

where pure f' is the single pure term in g, $n = |\text{opaq} \, g|$, and v_i are new variables. The right-hand side can be eta-reduced to

$$\boldsymbol{\lambda} v_1. \ldots \boldsymbol{\lambda} v_n. (\text{vary}_1 \, g) = \downarrow g \,.$$

From the induction hypothesis we get $f' =_{\alpha\beta\eta} \downarrow g$, which concludes the proof.
$\square$

This lemma is why we need eta-equivalence and not just use $=_{\alpha\beta}$. In fact, we can find a counterexample of equivalent canonical forms that do not agree in the pure function if $\rightarrow_\eta$ is not available:

$$\text{pure} \, (\boldsymbol{\lambda} x. \, x) \text{ `ap` } (\text{pure} \, f \text{ `ap` } x) \simeq \text{pure} \, f \text{ `ap` } x \simeq \text{pure} \, (\boldsymbol{\lambda} x. \, f \, x) \text{ `ap` } x.$$

The middle term is derived from (4.3), the latter from (4.4).

Corollary 1. *The normal form is structurally unique. Formally, if $s, t \in \mathcal{C}$ and $s \simeq t$, then $s =_{\alpha\beta\eta} t$.*

Now we have all tools ready to complete the picture. The following theorem shows that (limited) lifting is possible with just the applicative laws. Its proof hints towards the implementation in Isabelle, which is of course based on the normal form: Under the condition that the opaque terms are equivalent, normalizing two idiomatic terms reduces the problem to the "unlifted" terms.

Theorem 1. *Let $s, t \in \mathcal{I}$ with $\text{opaq} \, s =_{\alpha\beta\eta} \text{opaq} \, t$. If $\downarrow s =_{\alpha\beta\eta} \downarrow t$ (the base equation), then $s \simeq t$.*

Proof. From Lemma 4 we obtain normal forms $s' \simeq s$ and $t' \simeq t$. With the base equation and Lemma 5 we get $\downarrow s' =_{\alpha\beta\eta} \downarrow t'$. By Lemma 6 and the condition on the opaque subterms, it follows that $s' =_{\alpha\beta\eta} t'$ and further $s \simeq t$. $\square$

4.2 Implementation

We introduced an algorithm for computing the normal form of an idiomatic term, and argued for its central role in lifting. In order to be useful in the context of theorem proving, just providing the normal form is not sufficient. We need to establish a formal proof of the equivalence. In Isabelle, this means constructing a theorem $t = t'$, where t' is the normal form of t. We observe that

Function	Pattern		Substitution	Name
normalize	term x	$\simeq$	pure $(\boldsymbol{\lambda}x.\ x)$ 'ap' term x	
	$?x$	$=$	$pure\,(\lambda x.\ x) \diamond ?x$	`I_intro`
nf-nf	g 'ap' $(f$ 'ap' $x)$	$\simeq$	pure $\mathbf{B}$ 'ap' g 'ap' f 'ap' x	
	$?g \diamond (?f \diamond ?x)$	$=$	$pure\,(\lambda g f x.\ g(f x)) \diamond ?g \diamond ?f \diamond ?x$	`B_intro`
pure-nf	pure g 'ap' $(f$ 'ap' $x)$	$\simeq$	pure $(\mathbf{B}\ g)$ 'ap' f 'ap' x	
	$pure\ ?g \diamond (?f \diamond ?x)$	$=$	$pure\,(\lambda f x.\ ?g(f x)) \diamond ?f \diamond ?x$	`B_pure`
pure-nf	pure f 'ap' pure x	$\simeq$	pure $(f\ x)$	
	$pure\ ?f \diamond pure\ ?x$	$=$	$pure\,(?f\,?x)$	`merge`
nf-pure	f 'ap' pure x	$\simeq$	pure $((\boldsymbol{\lambda}x.\ \boldsymbol{\lambda}f.\ f\ x)\ x)$ 'ap' f	
	$?f \diamond pure\ ?x$	$=$	$pure\,(\lambda f.\ f\,?x) \diamond ?f$	`swap`

Table 1: Fixed transformations of Algorithm 1, with corresponding rewrite rules. Identity cases are omitted.

in Algorithm 1 each equation can be recast in the following way: The input term is first transformed in a fixed way that changes the outermost constitution. Then, the whole term or subterms thereof are substituted by other functions— or recursively—, possibly multiple times. For instance, in the first equation (4.14) for nf-nf, the term g 'ap' $(f$ 'ap' $x)$ (where g, f, x should be understood as placeholders for concrete terms) is rearranged to pure $\mathbf{B}$ 'ap' g 'ap' f 'ap' x. The subterm pure $\mathbf{B}$ 'ap' g is passed to pure-nf and replaced by the result, say g'. nf-nf acts on g' 'ap' f, yielding f', such that the final term is f' 'ap' x.

Table 1 shows all initial transformations. It is no coincidence that these are the well-known equivalences (4.3)–(4.6) and (4.8), which were used in the correctness proofs. Each transformation has been augmented with the corresponding HOL equation. The mapping from syntactic idiomatic terms to HOL terms is obvious; *pure* and ($\diamond$) are the concrete constants of the idiom under consideration. The variables standing for arbitrary opaque terms become new schematic variables, representing unknowns which may be instantiated. Rule `B_pure` is easily proven by rewriting with `B_intro` and `merge`. The other rules are the applicative laws and thus are available from the registration infrastructure, see also ⌐To do.⌐. Applying a transformation to a term means instantiating the rule such that the left hand side is equal to that term. Isabelle provides a low-level reasoning framework for combining *conversions* [11], which we use here. A conversion is a function which takes a term t and returns a theorem $t = u$ for some u.[4] The ML function `Conv.rewr_conv` turns a proven equation into such a conversion function. When the resulting conversion is applied, it attempts to instantiate the schematic variables in the equation. Note that conversions can fail; here if there is no match. Basic conversion combinators are the `then_conv` operator, which chains two conversions sequentially (and fails if any of these does), and the `else_conv` operator, which attempts to apply the first conversion, but uses the second if the first fails. Because rewrite conversions act like pattern matching guards, we combine them with `else_conv` to recreate the case switching of the algorithm. Combinators like `Conv.arg1_conv` facilitate the rewriting of subterms—this one takes a conversions and modifies it such that it applies to the left operand of a binary operator. With these tools

---

[4]Isabelle's conversions use the Pure equality $\equiv$.

```
1  fun normalform_conv ctxt af =
2    let
3      val rules = facts_of_afun af;
4
5      val leaf_conv = rename_rewr_conv (fn t => [("x", term_to_vname t)])
6        (#I_intro rules);
7      val merge_conv = Conv.rewr_conv (#merge rules);
8      val swap_conv = Conv.rewr_conv (#swap rules);
9      val rotate_conv = rename_rr_conv "x" (#B_intro rules);
10     val pure_rotate_conv = rename_rr_conv "x" (#B_pure rules);
11
12     fun normalize_pure_nf ct = ((pure_rotate_conv then_conv
13       Conv.arg1_conv normalize_pure_nf) else_conv merge_conv) ct;
14     val normalize_nf_pure = swap_conv then_conv normalize_pure_nf;
15     fun normalize_nf_nf ct = ((rotate_conv then_conv
16         Conv.arg1_conv (Conv.arg1_conv normalize_pure_nf then_conv
17           normalize_nf_nf)) else_conv
18       normalize_nf_pure) ct;
19
20     fun normalize ct =
21       let val t = Thm.term_of ct
22       in if can (dest_comb ctxt af) t
23         then (Conv.arg1_conv normalize then_conv
24             Conv.arg_conv normalize then_conv normalize_nf_nf) ct
25         else if can (dest_pure ctxt af) t
26           then Conv.all_conv ct
27           else leaf_conv ct
28       end;
29   in normalize end;
```

Figure 5: ML implementation of normalization.

at hand, we can construct the normal form conversion. The recursive pieces can be expressed just as recursive auxiliary functions. However, we lose a bit of generality by using `Conv.arg1_conv` to access the left operand of ($\diamond$), because it requires that ($\diamond$) is not a beta-redex. This could be solved by creating a specialized combinator, which does pattern matching instead of simple term deconstruction.

Figure 5 shows the ML code. It is parameterized by the applicative functor `af`, which contains all the necessary rules. In the helper function `normalize`, the different cases are selected by predicates like `can (dest_comb ctxt af) t`, which is true only if term `t` fits the pattern _ $\diamond_{\mathtt{af}}$ _. We do this here because it is the only recursive conversion which is not guarded by a rewrite rule before recurring, so we want to protect against rampant rewriting. Finally, `rename_rewr_conv` and `rename_rr_conv` (not shown) adjust the names of bound variables in the argument rule to the term to convert. While these names are only hints in Isabelle and obviously subject to alpha conversion, they influence the display of the proof state. To elaborate, consider a HOL term $p \diamond (q \diamond r)$. Without the adjustment, the normal form would be $pure\,(\lambda x x_a x_b.\, x\,(x_a x_b)) \diamond p \diamond q \diamond r$. The name component x stems from the lambda abstraction in rule `B_intro`. If the term is part of an equation, the variables of the reduced base equation

will also be called x, x_a, and x_b. This creates unnecessary confusion, and to help the user, we try to preserve the variable names of the input term.

5 Lifting with Combinators

5.1 Motivation

The normalization approach to solving lifted equations works only if the opaque terms on both sides coincide. This is not true for all equations of interest. Let's revisit the set version of addition of natural numbers, $\oplus$ from Example 1. This operator is also commutative, so it should be possible to prove

$$X \oplus Y = Y \oplus X.$$

After unfolding and normalization, we get

$$pure\,(\lambda xy.\,x + y) \diamond X \diamond Y = pure\,(\lambda yx.\,y + x) \diamond Y \diamond X. \tag{5.1}$$

Clearly, this can't be solved with a standard congruence rule, because we would have to to prove that X is equal to Y. Since we are concerned with transferring properties from a base domain, we don't want to assume anything about those opaque subterms.

Hinze showed that such equations can be solved if certain *combinators* can be lifted. Informally, combinators are functions which rearrange their arguments in a specific manner. We have already used two combinators, **I** and **B**. Lifting their defining equations (see Table 2) gives us the identity and composition laws, respectively. If the lifted combinator performs the same rearrangement with arbitrary functorial values, one can translate that particular term structure between the two layers. In this case, we simply say that the combinator *exists*. To continue with (5.1), we could attempt to change the order of Y and X on the right-hand side. Note that these appear as arguments to a pure function. The **C** combinator, also known as 'flip' in functional programming, does what we want: $\mathbf{C}fxy = fyx$. The lifted equation is

$$pure\,\mathbf{C} \diamond f \diamond x \diamond y = f \diamond y \diamond x, \tag{5.2}$$

and it is indeed true for set idiom! From this we get

$$pure\,(+) \diamond X \diamond Y = pure\,(\mathbf{C}(+)) \diamond X \diamond Y. \tag{5.3}$$

The right-hand side is no longer the normal form of $Y \oplus X$, but still a canonical form (which is why we distinguish these two). But now the argument lists on both sides coincide. We reduce to

$$\lambda xy.\,x + y = \lambda xy.\,y + x,$$

which is extensionally equivalent to the base equation $x + y = y + x$. The availability of equation (5.2) is a quite powerful condition, because it will allow us to permute opaque terms freely. If permutations exist such that both sides of an equation in canonical form align regarding their opaque terms, reduction by congruence is possible again. This will again lead to the expected base equation.

Symbol	Reduction
B	$\mathbf{B}xyz = x(yz)$
I	$\mathbf{I}x = x$
C	$\mathbf{C}xyz = xzy$
K	$\mathbf{K}xy = x$
W	$\mathbf{W}xy = xyy$
S	$\mathbf{S}xyz = xz(yz)$
H	$\mathbf{H}xyz = xy(zy)$

Table 2: Useful combinators.

However, the combinator **C** does not exist for all applicative functors. For example, the order of values in a state monad may be significant.

Combinators appeared originally in the context of logic [12]. They were studied because it is possible to write logical formulas without variables using only applications of suitable combinators, as opposed to the usual lambda calculus. Table 2 lists all combinators which are used throughout this text, together with their defining equations. There are certain sets of combinators which are sufficient to express all lambda terms, $\{\mathbf{S}, \mathbf{K}\}$ being one of them. In other sets, only a limited part of terms is representable. Hinze's Lifting Lemma shows that all terms and thus all equations can be lifted while preserving the variable structure if **S** and **K** exist. He also notes that other combinator set are useful, because there are idioms where more than $\{\mathbf{B}, \mathbf{I}\}$, but not all combinators exist. Generally speaking, additional combinators enlarge the set of equations which can be lifted.

The original proof of the Lifting Lemma [1, pp. 11–14] uses induction on the structure of idiomatic terms; it is not entirely obvious how it can be generalized to other combinators sets, as it depends on the availability of **K** to lift tuple projections. In this section we present an implementation of this generalized lifting, whose underlying concept works with arbitrary combinators. However, it depends on an abstraction algorithm and the structure of representable terms, which are difficult to derive automatically. Therefore we will restrict ourselves to certain sets ("bases") with fixed algorithms, while understanding that the scope can be extended if needed.

5.2 Generic Lifting

We start with the relationship of combinators and lambda terms. The equations in Table 2 can be expressed as abstractions $\mathbf{I} = \lambda x.\, x$ etc. If we substitute occurrences of combinators in a term (signified by $=_\delta$), new abstractions are introduced, which may be beta-reduced afterwards:

$$\mathbf{WB} =_\delta (\lambda fx.\, fxx)(\lambda gfx.\, g(fx)) =_\beta \lambda xy.\, x(xy).$$

The question arises when and how this process can be reversed, meaning that all abstractions are replaced by suitable combinators. In Curry et. al. [12, Section 6A], terms with variables, but no abstractions are considered. A syntactical operation is defined, denoted $[x]t$, where t is such a term and x is a variable. The desired property is that x does not occur in $[x]t$, and $([x]t)x =_{\delta\beta} t$. Due to its notation, the operation is known as *bracket abstraction*. There is an obvious

correspondence with lambda abstractions $\lambda x.\,t$. Bracket abstraction however stands for a concrete applicative term, whereas a lambda is an object of the syntax itself. Replacing lambdas $\lambda x.\,t$ by brackets $[x]t$ performs the shift to a combinator representation. Curry et. al. give several possible definitions for bracket abstraction. They note that these follow a scheme they refer to as an algorithm—a sequence of rules, where each rule is a partial definition. The rules may invoke abstraction recursively. In particular, the following rules are used:

$$
\begin{aligned}
&[x]x = \mathbf{I}, && &&(i)\\
&[x]t = \mathbf{K}t && \text{if } x \text{ not free in } t, &&(k)\\
&[x]tx = t && \text{if } x \text{ not free in } t, &&(\eta)\\
&[x]st = \mathbf{B}s([x]t) && \text{if } x \text{ not free in } s, &&(b)\\
&[x]st = \mathbf{C}([x]s)t && \text{if } x \text{ not free in } t, &&(c)\\
&[x]st = \mathbf{S}([x]s)([x]t). && &&(s)
\end{aligned}
$$

The algorithm which consists of rules (i), (k) and (s), in that order, is written succinctly as (iks). The algorithm attempts to use the rules in their left-to-right order, applying the first one whose restrictions are satisfied by the term at hand. Each abstraction algorithm A introduces a certain set of basic combinators, which we refer to as $C(A)$. It is sound only if certain postulates about those combinators, which are again the equations in Table 2, are assumed.

Example 6. Using the (iks) algorithm, one gets

$$[x]xxy \overset{(s)}{=} \mathbf{S}([x]xx)([x]y) \overset{(s),(k)}{=} \mathbf{S}(\mathbf{S}([x]x)([x]x))(\mathbf{K}y) \overset{(i)}{=} \mathbf{S}(\mathbf{SII})(\mathbf{K}y).$$

Attempting to use the $(ik\eta bc)$ algorithm with the same abstraction quickly comes to a stop:

$$[x]xxy \overset{(c)}{=} \mathbf{C}([x]xx)y,$$

which is undefined. ▲

As we can see, not all algorithms are total. Therefore, there is a trade-off between the combinators required and the terms for which abstraction is possible. Bunder [13] presents an analysis of the situation for certain algorithms and combinator sets, based on rigorous definitions for term translation and definability. We will come back to this later, when we discuss how to order the variables in an idiomatic term such that abstraction is defined. For now, the concept of bracket abstraction with the example of rules (i)–(s) is sufficient.

Next, we attempt to transfer these concepts to idiomatic terms. On the one hand, this is quite intuitive since the latter are also formed by an application operator, and pure terms can be identified with constants. But we do not have any "idiomatic abstractions". Hinze actually defines these in terms of abstract combinators and an extensionality property of the idiom. For our purpose it is sufficient to work directly with bracket abstraction, and we assume that all combinators are lifted, i.e. expressible as a pure term. To clarify the following discussion, we adjust our $\mathcal{I}$ formalism and replace opaque terms $\mathsf{term}\,x$ with variables.

Definition 6. The set of generic idiomatic terms $\mathcal{I}'$ is defined by

$$\mathcal{I}' ::= \mathsf{var}\,\mathcal{V} \mid \mathsf{pure}\,\mathcal{T} \mid \mathcal{I}' \text{ `ap` } \mathcal{I}'. \qquad (5.4)$$

We reuse the congruence $\simeq$ from Definition 2 for generic terms. The set of variables $\mathrm{var}(t)$ of t is defined as the set of all arguments to var occuring in t. The sequence of variables $\overrightarrow{\mathrm{var}}(t)$ is defined similarly to opaq. Unlifting (see Definition 5) is also transferred, but uses the variable x in subterms $\mathsf{var}\,x$ instead of inventing new ones. ▲

Using this definition, it is clear what the rules for idiomatic abstraction are:

$$[x]'(\mathsf{var}\,x) = \mathsf{pure}\,\mathbf{I}, \qquad\qquad\qquad\qquad\qquad (i')$$

$$[x]'t = \mathsf{pure}\,\mathbf{K} \text{ `ap` } t \qquad\qquad \text{if } x \notin \mathrm{var}(t), \qquad (k')$$

$$[x]'(t \text{ `ap` } \mathsf{var}\,x) = t \qquad\qquad \text{if } x \notin \mathrm{var}(t), \qquad (\eta')$$

$$[x]'(s \text{ `ap` } t) = \mathsf{pure}\,\mathbf{B} \text{ `ap` } s \text{ `ap` } [x]'t \qquad \text{if } x \notin \mathrm{var}(s), \qquad (b')$$

$$[x]'(s \text{ `ap` } t) = \mathsf{pure}\,\mathbf{C} \text{ `ap` } [x]'s \text{ `ap` } t \qquad \text{if } x \notin \mathrm{var}(t), \qquad (c')$$

$$[x]'(s \text{ `ap` } t) = \mathsf{pure}\,\mathbf{S} \text{ `ap` } [x]'s \text{ `ap` } [x]'t. \qquad\qquad\qquad (s')$$

In general, the algorithm A' on idiomatic terms is obtained from algorithm A on regular terms by lifting its rules in this fashion, preserving order.

Before we show the connection to the canonical form, there is one thing which remains to be considered. The interchange law allows us to move a variable out of the left subterm of an application, given that the right subterm is pure. This is not captured by rules (b') and (i'), which are the only ones from above which are valid in all idioms. We define a combinator $\mathbf{T}xy = yx$ and the rules

$$[x]st = \mathbf{T}t([x]s) \qquad\qquad \text{if } t \text{ contains no variables}, \qquad (t)$$

$$[x]'(s \text{ `ap` } t) = \mathsf{pure}\,\mathbf{T} \text{ `ap` } t \text{ `ap` } [x]'s \qquad \text{if } \mathrm{var}(t) = \emptyset. \qquad (t')$$

Soundness of rule (t') can be shown to be equivalent to the interchange law. It is important to understand that $\mathbf{T}$ does not have to exist in the idiom; these rules do not fit exactly in the pattern of the other rules. The $\mathbf{T}$ combinator is also necessary to formulate the most generic rule for the $\mathbf{W}$ combinator. Without the interchange law, it could only be used for terms $t \text{ `ap` } \mathsf{var}\,x \text{ `ap` } \mathsf{var}\,x$, i.e. those where the same variable is applied twice in direct succession. In an idiom, there may be arbitrary pure terms "inbetween" the variables. We use the (w) rule when the variable occurs in both operands of an application, just like the (s) rule.

$$[x]st = \mathbf{W}(\mathbf{B}(\mathbf{T}[x]t)(\mathbf{B}\mathbf{B}[x]s)) \qquad \text{if } [x]t \text{ contains no variables}. \qquad (w)$$

(w') is derived similarly to the other rules.

As with ordinary terms, we demand a soundness property for idiomatic bracket abstraction, namely that $[x]'t' \text{ `ap` } \mathsf{var}\,x \simeq_C t'$ holds true. The definitions for the additional combinators C get lifted to $\mathsf{pure}\,\mathbf{I} \text{ `ap` } x \simeq_C x$ and so on, consistently extending our congruence relation $\simeq$ to $\simeq_C$.

Lemma 7. *Let $t' \in \mathcal{I}'$ be a generic idiomatic term, and $x \in \mathcal{V}$ a variable. For an abstraction algorithm A' consisting of a subset of rules (i')–(t'), we have $\downarrow[x]'t' = [x]\downarrow t'$ and $[x]'t' \text{ `ap` } \mathsf{var}\,x \simeq_{C(A')} t'$, assuming that $[x]'t'$ is defined. Also, bracket abstraction does not add variables: $\mathrm{var}([x]'t') \subseteq \mathrm{var}(t')$.*

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Proof. This statement uses that fact that the rules in A' are very similar to those of A. In particular, rule (r') is applied first when evaluating $[x]'t'$ iff rule (r) is applied first to $[x]\!\downarrow\! t'$. The remainder of the proof is a simple induction. We show the case involving rule (c') as an example, the other cases are similar. Thus $t = s$ 'ap' u with $x \notin \mathrm{var}(u)$. The induction hypothesis is

$$\downarrow[x]'s = [x]\!\downarrow\! s \quad \text{and} \quad [x]'s\text{ 'ap' var } x \simeq_{C(A')} s \quad \text{and} \quad \mathrm{var}([x]'s) \subseteq \mathrm{var}(s).$$

Then we have

$$\downarrow[x]'t \overset{(c')}{=} \downarrow(\text{pure }\mathbf{C}\text{ 'ap' }[x]'s\text{ 'ap' }u) = \mathbf{C}(\downarrow[x]'s)(\downarrow u)$$
$$\overset{(\mathrm{IH})}{=} \mathbf{C}([x]\!\downarrow\! s)(\downarrow u) \overset{(c)}{=} [x](\downarrow s\downarrow u) = [x]\!\downarrow\! t$$

and

$$[x]'t'\text{ 'ap' var }x = \text{pure }\mathbf{C}\text{ 'ap' }[x]'s\text{ 'ap' }u\text{ 'ap' var }x \simeq_{C(A')} [x]'s\text{ 'ap' var }x\text{ 'ap' }u$$
$$\overset{(\mathrm{IH})}{\simeq}_{C(A')} s\text{ 'ap' }u = t.$$

$\square$

Now we can state the key obversation: The successful abstraction of all variables in an idiomatic term leaves a single pure term, per the homomorphism law. Moreover, that term is equivalent to the result of applying the same abstraction algorithm to the "unlifted term". In principle, this works with arbitrary rules, as long as the statements of Lemma 7 hold true.

Theorem 2. *In the following, bracket abstraction uses algorithms A and A' with a subset of the rules (i)–(t) and (i')–(t'), respectively. Let $t' \in \mathcal{I}'$ be a generic idiomatic term, and $x_1, \ldots, x_n$ a permutation of the variables $\mathrm{var}(t')$, or a superset thereof. If $f = [x_1]\cdots[x_n]\!\downarrow\! t'$ is defined for A, then*

a) $[x_1]'\cdots[x_n]'t'$ *consists only of applications of pure terms, and*

b) *the unique canonical form of $[x_1]'\cdots[x_n]'t'$ is pure f;*

c) pure f *'ap' var x_1 'ap' $\cdots$ 'ap' var x_n is a canonical form of t';*

d) *replacing all combinators from $C(A)$ in f with their definitions yields $f' =_{\beta\eta} \lambda x_1 \cdots x_n.\ \downarrow\! t'$.*

Proof. a) is due to $\downarrow([x_1]'\cdots[x_n]'t') = f$ (induction and Lemma 7).

b) It is not difficult to see that a pure-only term p has a unique canonical form, which is equal to pure $\downarrow p$.

c) We have pure f 'ap' var x_1 'ap' $\cdots$ 'ap' var $x_n \simeq_{C(A)} t'$ by induction, making repeated use of Lemma 7.

d) We show $[x_1]\cdots[x_n]\!\downarrow\! t' =_{\delta\beta} \lambda x_1 \cdots x_n.\ \downarrow\! t'$. Note that $([x_i]p)x_i =_{\delta\beta} p$ for all terms p, and hence $[x_i]p =_\eta \lambda x_i.\ ([x_i]p)x_i =_{\delta\beta} \lambda x_i.\ p$.

$\square$

Base	Abstraction	Example idioms
BI	(ibt)	state, list
BIC	$(ibtc)$	set
BIK	$(kibt)$	
BIW	$(ibtw)$	either
BCK	$(kibtc)$	
BKW	$(kibtw)$	
BICW	$(ibtcs)$	maybe
BCKW	$(kibtcs)$	stream, $\alpha \to$

Table 3: Substructures of BCKW.

Remember that we are interested in equations, which obviously consist of two idiomatic terms. We get to the base equation only if the same variable sequence is used for both terms, and the assumptions of Theorem 2 are satisfied. To complete the *generic lifting* approach, we need a procedure for determining the abstraction order. Since this procedure has to depend on the abstraction algorithm, we fix the combinator bases first. Hinze focuses on $\mathbf{SK} = \mathbf{BCKW}$ and $\mathbf{BICS} = \mathbf{BICW}$, noting that $\mathbf{BIC}$ is also relevant. The set $\{\mathbf{B}, \mathbf{I}, \mathbf{C}, \mathbf{K}, \mathbf{W}\}$ and its subsets seem to be a good starting point to cover relevant cases. The additional combinators play an intuitive role: $\mathbf{C}$ reorders variables, $\mathbf{W}$ duplicates them, and $\mathbf{K}$ permits abstraction over additional variables. Table 3 lists all distinct subsets containing $\mathbf{B}$ and $\mathbf{I}$, together with the abstraction algorithms we propose. We routinely ignore $\mathbf{T}$ when listing the combinators. This highlights the connection with combinatorial logic, where some of those bases have been studied.

The algorithms have been chosen in order to simplify the implementation, by using the rules conditionally depending on the available combinators: If $\mathbf{K}$ exists, start with k. Then, for all bases, perform *ibt*. If $\mathbf{C}$ (or $\mathbf{W}$) exists, add c (or w), respectively. However, if both do, use rule s instead. Below follows a detailed description of how the variable sequence can be found in each base, and we justify the abstraction algorithms, meaning that the preconditions of Theorem 2 are satisfied. The function $\mathrm{abs}_C(s, t)$ will denote the chosen sequence for terms s, t in the context of base C.

BI

This is the minimal base which is available for all idioms. We already know from Section 4.1 that there is only one canonical form with respect to $\simeq$. Therefore, there is exactly one permissible sequence:

$$\mathrm{abs}_{\mathbf{BI}}(s, t) = \overrightarrow{\mathrm{var}}(s) = \overrightarrow{\mathrm{var}}(t).$$

Equations where the two sequences differ are rejected. If any other sequence could be used, we would get a different canonical form per Theorem 2, thus contradicting the previous result on normal forms. Focusing on a single abstraction step $[x_i]t_i$, x_i must occur once in t_i, and it is the right-most variable. If t_i is an application, there are two cases: x_i occurs in the right subterm, and rule (b) is used successfully. Otherwise, there can be no variable in the right subterm, so (t) applies. This confirms that algorithm (ibt) is indeed acceptable for this base.

BIC

Bunder shows [13] that **BIC**-definable lambda terms are those where each bound variable occurs excatly once, irrespective of their order. His definition of a C-definable term t (with combinator base C) implies that there exists an abstraction algorithm such that $[x]s$ is defined if $t = \lambda x.\, s$. In particular, $(i\eta bc)$ is a valid algorithm. From this it follows that we can choose the order in which we abstract, as long as the corresponding variable occurs exactly one. Note that our special **T** combinator is not considered there. But as it can be simulated by the more powerful **CI**, adding it to the combinator base does not change anything.

In this base, we can work with all equations where the variable sequence of one term can be reordered to the sequence of the second. The order used for the abstraction is irrelevant, but it will be reflected in the quantifier order of the base equation. A simple choice is $\mathrm{abs}_{\mathbf{BIC}}(s, t) = \overrightarrow{\mathrm{var}}(s)$. However note that we do not use the (η) rule, but add the (t) rule. (η) could be a considered as an optimization, since $\lambda x.\, yx =_\beta \mathbf{B}y\mathbf{I}$, so (b) and (i) suffice. Conversely, (t) is a special case of (c) with (i), and adding it to the algorithm next to (c) just results in a slightly different combinator representation. We use this particular algorithm in order to simplify the implementation, such that as much code as possible can be shared between the bases.

BCK and BICW

These cases were also analyzed by Bunder. For **BCK**, the definable terms are those where each bound variable occurs at most once, again ignoring the order. In terms of bracket abstraction, $[x]y$ is then also defined if x is not free in y. This allows us to extend the sequence with other variables. We make use of this to deal with variables which only occur on one side of an equation—it does not hurt to abstract over those too. Hence, $\mathrm{abs}_{\mathbf{BCK}}(s, t)$ can be any arrangement of the set $\mathrm{var}(s) \cup \mathrm{var}(t)$. The implementation uses a total order on the set of variables. On the contrary, the definable terms of **BICW** have at least one occurence of each bound variable; $[x]y$ can therefore be used if x occurs multiple times in y, and x will not be free in $[x]y$. Comparing this with **BIC**, we loosen the restriction that variables must not be repeated. $\mathrm{abs}_{\mathbf{BICW}}(s, t)$ must be an arrangement of $\mathrm{var}(s) = \mathrm{var}(t)$. We can compute this by sorting the sequences again, and then trimming duplicates which now are next to each other.

Bunder proposes the $(i\eta kbc)$ algorithm for **BCK**, and $(i\eta bcs)$ for **BICW**; our rules are $(kibtc)$ and $(ibtcs)$, respectively. The comments on (η) and (t) from above apply here as well. The only other difference is the position of (k) in the list. But (k) on one side and (i) and (η) on the sider work with disjoint sets of terms, so this is not an issue.

BCKW

This base, which is logically equivalent to **SK**, has some useful properties: $[x]y$ can always be defined, making it possible to use any abstraction sequence, and in turn handle all equations. In conjunction with Theorem 2, we want to abstract all free variables, though. Therefore we determine the sequence as with **BCK**, but the term restriction is rescinded. As for the abstraction algorithm, we use a variation of $(ik\eta bcs)$ from [12].

BIK, BIW and BKW

These bases do not appear to be significantly covered in the literature. Since there is at least one useful example of an idiom with **BIW** combinators, we still shall implement and discuss them. By adding the **K** combinator to **BI**, additional variables may be abstracted, but the other restrictions (order, single occurrence) remain. Accordingly, we demand that $\mathrm{abs}_{\mathbf{BIK}}(s, t)$ has $\overrightarrow{\mathrm{var}}(s)$ and $\overrightarrow{\mathrm{var}}(t)$ as subsequences. For definiteness, we can limit it to the shortest sequence where variables only in s appear before those only in t, whenever there is an ambiguity: $\mathrm{abs}_{\mathbf{BIK}}(s, t) = \mathrm{abs}'_{\mathbf{BIK}}(\overrightarrow{\mathrm{var}}(s), \overrightarrow{\mathrm{var}}(t))$,

$$
\mathrm{abs}'_{\mathbf{BIK}}(a, b) = \begin{cases} x \cdot \mathrm{abs}'_{\mathbf{BIK}}(a', b') & \text{if } a = x \cdot a', \ b = x \cdot b'; \\ x \cdot \mathrm{abs}'_{\mathbf{BIK}}(a', b) & \text{if } a = x \cdot a', \\ x \cdot \mathrm{abs}'_{\mathbf{BIK}}(a, b') & \text{if } b = x \cdot b', \\ \langle \rangle & \text{if } a = b = \langle \rangle. \end{cases}
$$

Trying rule (k) in the beginning of the abstraction algorithm obviously takes care of unused variables.

For the other bases, we simply take the abstraction algorithms as granted. Based on that we analyze the set of defined bracket abstractions. Since **W** duplicates variables, one might conjecture that a single variable may now occur repeatedly. No other variable may be interspersed in the lexical order, though, because none of the available combinators reorder their arguments. The following lemma proves that this intuition is correct. In order to express it on the level of lambda terms, we overload $\overrightarrow{\mathrm{var}}(x)$ in the obvious way. We use the notation x^n with sequence x to stand for n concatenated copies of x.

Lemma 8. *Let $x \in \mathcal{V}$ and $t \in \mathcal{T}$. With algorithm (ibtw), $[x]t$ is defined iff there is a natural number $n \geq 1$ and a variable sequence v such that $\overrightarrow{\mathrm{var}}(t) = v @ \langle x \rangle^n$, where x does not appear in v. The same statement holds true for algorithm (kibtw), except that n may be zero as well.*

Proof. The first part of the proof is concerned with the direction where $[x]t$ is assumed to be defined. We perform induction over the steps of the abstraction algorithm. In the evaluation of $[x]s$ for a subterm s of t, there are the following cases: If (k) is used, then x must not be free in s, thus $n = 0$ and $v = \overrightarrow{\mathrm{var}}(s)$. For (i), we have $n = 1$ and $v = \langle \rangle$. For (b), there must be terms u and w with $s = uw$, and x is not free in u. From the induction hypothesis we get adequate n' and v' for $[x]w$. Then n' and $\overrightarrow{\mathrm{var}}(u) @ v'$ satisfy the conditions for $[x]s$. Rules (t) and (w) are analogous: For (t), the side condition implies that $\overrightarrow{\mathrm{var}}(w) = \langle \rangle$; for (w) we have $\overrightarrow{\mathrm{var}}(w) = \langle x \rangle^k$ for some $k \geq 1$, thus $\overrightarrow{\mathrm{var}}(u) @ \overrightarrow{\mathrm{var}}(w) = v' @ \langle x \rangle^{n'+k}$, where n' and v' are from $[x]u$.

Now we show the other direction. Assume that suitable n and v exist. If $n = 0$, rule (k) is used, because x cannot be free in t. Otherwise we assume $n \geq 1$ during an induction on the structure of t. If t is a variable, it must be x, so rule (i) applies. If it is an application $s = uw$ instead, there are three cases: $x \notin \mathrm{var}(u)$, hence $v = \overrightarrow{\mathrm{var}}(u) @ v'$ for some v', and $[x]w$ is defined by the induction hypothesis so we can use rule (b). If $\mathrm{var}(w) = \emptyset$, $[x]u$ is defined and rule (t) applies. Otherwise, $\overrightarrow{\mathrm{var}}(w) = \langle x \rangle^k$ with $k \leq n$ must hold, so (w) can be used. $\square$

It follows that the abstraction sequences for **BIW** and **BKW** should be chosen like those of **BI** and **BIK**, respectively; but for each variable, a single span of repetitions in $\overrightarrow{\mathrm{var}}(s)$ and $\overrightarrow{\mathrm{var}}(t)$ is allowed and treated like a single instance.

5.3 Implementation

The implementation of generic unlifting essentially has to compute bracket abstractions, i.e., recursively select the correct rule for a term. Similar to what we did with the normal form conversion, the goal is to prove a theorem representing

$$t \simeq ([x_1]' \cdots [x_n]'t) \text{ `ap` } x_1 \text{ `ap` } \cdots \text{ `ap` } x_n.$$

However, we also add a heuristic that tries to determine how variables have been instantiated in the supplied equation. First, arbitrary opaque terms take the place of variables. We can work directly with these, but it becomes less clear which subterms are the same, meaning that they should be abstracted simultaneously. Consider a base equation with two, universally quantified variables x and y. If the lifted version can be proven, it will also have two variables. Now, the user might want to prove a proposition which is equivalent to the lifted equation, except that both variables have been replaced by the (same) term z. This modified statement is obviously true, so we might expect that the lifting tool can handle it. Remember that we prohibited multiple occurrences in certain bases, e.g. **BI**, when we defined bracket abstraction. What would happend if we normalize the equation? Normalization treats each instance of a opaque subterm separately, thus we will get something of the form *pure $f \diamond z \diamond z = $ pure $g \diamond z \diamond z$.* By magic, the functions f and g have been equipped with two arguments! In the base **BI**, generic unlifting is not more powerful than normal form conversion, but we expect it not to be weaker either. We solve the problem by handling each opaque subterm as if it were a different variable. For other bases, we need to be more clever, or we would lose the advantage that generic unflifting was supposed to give us. For example, idempotency of an operator $\circ$, $x \circ x = x$ can be lifted if **W** exists; when proving a lifted instance like $f(x) \hat{\circ} f(x) = f(x)$, the two copies of $f(x)$ on the left-hand side must be identified. Therefore, the base of the idiom influences the assignment of "virtual" variables to opaque terms. In general, it is best if we identify as many terms as possible: The resulting base equation is certainly provable if it were so without grouping some pair of variables. However, we will see that the situation can sometimes be ambiguous.

```
1  datatype apterm =
2    Pure of term | Var of int * term | Ap of apterm * apterm;
```

Figure 6: The datatype to keep track of idiomatic terms.

We use a ML datatype to make the term structure explicit. `apterm` (see Figure 6) has a similar structure as $\mathcal{I}$. Opaque terms `Var` are tagged with an integer, which is unique for each position in an equation. Sets of these positions play the role of variables in the previous section. The function `eliminate` takes an `apterm` t and such a set x, computing the pair $([x]'t, \phi)$, where ϕ is the theorem $t \simeq [x]'t \text{ `ap` } x$. One important building block for this function is

```
1  fun rewr_subst_ap ctxt af rewr thm1 thm2 =
2    let
3      val funT = thm1 |> Thm.lhs_of |> Thm.typ_of_cterm;
4      val ap_inst = Thm.cterm_of ctxt (ap ctxt af funT);
5      val rule1 = Drule.binop_cong_rule ap_inst thm1 thm2;
6      val rule2 = Conv.rewr_conv rewr (Thm.rhs_of rule1);
7    in Thm.transitive rule1 rule2 end;
```

Figure 7: The `rewr_subst_ap` function.

```
1  fun extract_comb n thm = Pure (thm |> Thm.rhs_of |>
2    funpow n Thm.dest_arg1 |> Thm.term_of);
3  fun comb2_step def (tt1, thm1) (tt2, thm2) =
4    let val thm = rewr_subst_ap def thm1 thm2;
5    in (Ap (Ap (extract_comb 3 thm, tt1), tt2), thm) end;
6  val B_step = comb2_step (#B_intro rules);
```

Figure 8: Implementation of rule (b').

`rewr_subst_ap`, whose code is shown in Figure 7. It combines three rewrite rules. Two of them, say $x = x'$ and $y = y'$, get merged to a congruence rule of the idiomatic application operator, $x \diamond y = x' \diamond y'$, by the standard `Drule.binop_cong_rule`. However, the application operator's types need to be instantiated before. The resulting equation is then chained with the third rule. Similar to our use of conversion combinators in Section 4.2, we employ `rewr_subst_ap` to handle the recursive nature of most abstraction rules. As an example refer to Figure 8. It takes as arguments two pairs of an idiomatic term and a theorem, the results of abstracting the two subterms of an application. Here, the key transformation is again provided by `B_intro` from Table 1. We must also update the `apterm` structure. New combinator terms are introduced, which also need proper type instantiation. Because these combinator are part of the equation we have computed, it seems to be the easiest method to extract the term from that equation. The integer supplied to `extract_comb` indicates how deeply nested to the left the combinator is.

The function `consolidate` determines the common abstraction sequence for two idiomatic terms. It begins with creating a left-to-right list of the variables for each term. Then an additional tag is added such that two list elements have the same tag iff we consider the instantiated terms equal. In the current implementation, the equality predicate is plain alpha convertibility. The next steps depend on the combinator base:

a) If **C** exists, each list is sorted according to the tag. This groups equal terms together.

b) If **W** exists, adjacent copies of the same term (again as indicated by the tag) are merged in each list. The positions of the involved variables are collected into a list. Otherwise, each position is put in an singleton list, so we have a uniform type after this step.

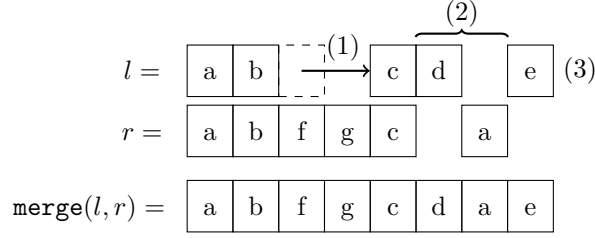c) If **K** exists, both lists are merged in a peculiar way: If the heads of them

Figure 9: Example of the merge operation for **K** in `consolidate`. (1) Variable c can be aligned with the remainder of list r, which is not possible with f and list l. (2) Neither d nor the copy of a can be aligned, so we intersperse them. The previous occurrence of a is not considered, because the algorithm is greedy and works from left to right. (3) A single tail is preserved.

> have the same tag, we merge their positions to obtain a single element and add it to the result, then proceeding with the tails. Otherwise, we try to align one head with the other list and vice versa. A head is aligned if its tag occurs in the list after some prefix. If alignment is possible in one direction, but not the other, the prefix and the aligned element become part of the result. Otherwise, the two heads are emitted one after the other. A potentially remaining single tail is kept as-is. Figure 9 displays an example.
>
> If **K** does not exist, we check that both lists have the same length and contain the same sequence of tags, or an error condition is raised. Again, the position lists within both lists are zipped.

We should investigate whether this algorithm is consistent with the theoretical development in Section 5.2, and also with the criteria regarding variable instantiation.

- With **BI**, there is not much flexibility—we can only abstract over a single position at a time. Aborting due to a sequence mismatch is basically the single difference to normal form conversion. However, there we will just get a likely unprovable result, so this behavior is justified.

- **BIC**: As before; sorting produces a suitable sequence if one exists.

- **BIW**: We have shown that simultaneous abstraction is only possible for coherent spans of a variable. Merging all adjacent copies of a variable is the best we can hope for.

- **BICW**: Sorting puts equal terms next to each other, where they will be merged. Thus, all possible terms are identified, making this the most general solution. If the sanity check at the end fails, then the set of variables for both sides of the equation must differ, and the precondition for this base is violated.

- **BCKW** differs from the previous case in the last step. As expected, we now accept every pair of terms. It should not happen that we will get a duplicate variable tag, in other words, if a tag appears in both lists,

there must be a single element in the output. This is easily shown by induction. Especially if alignment is used, the "skipped" prefix can only contain variables which do not occur in both terms, because the lists are sorted.

- **BCK**: The lack of **W** implies that all positions have to be abstracted individually. However, it is interesting to see what the last step does with respect to consolidating both sides of the equation. Again by induction we see that as many pairings as possible between the two sides of the equation are created. The difference to **BCKW** is that the same term may occur repeatedly in the lists. In case of different repetition counts, the algorithm is biased to the left. This is a potentially destructive choice, but the user can always resort to using new variables in the desired manner, and then do the instantiation in a second step.

- **BIK** and **BKW** are not perfectly solvable either. For example, consider the lists (tags only) $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$. If we try to consolidate, then $\langle 1, 2, 1 \rangle$ or $\langle 2, 1, 2 \rangle$ are the only possible sequences. It is not clear which is the correct one. The algorithm bails out and selects $\langle 1, 2, 2, 1 \rangle$ (which is probably useless). Likewise, discretion on behalf of the user is required. One could try more sophisticated algorithms like minimizing the edit distance. Due to a lack of example idioms for these bases, we decided against this extra effort.

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