Applicative Functors in Isabelle/HOL

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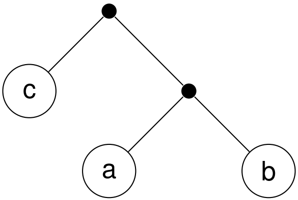
Introduction: Tree Labels

```
data Tree a = Leaf a | Node (Tree a) (Tree a)
```



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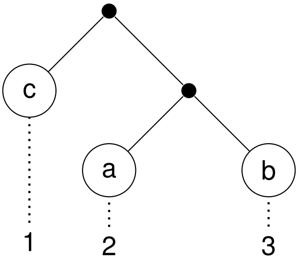

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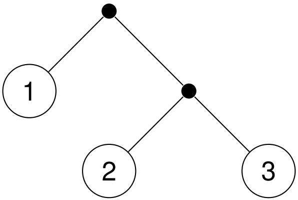

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Composing Stateful Computations

Standard solution:
state monad

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Applicative style

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class Applicative m => Monad m where
  ...
class Functor f => Applicative f where
  pure  :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
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Renaming Trees and Lists

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labels (Leaf x)   = [x]
labels (Node l r) = labels l ++ labels r
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numberList []     = pure []
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Proposition

```
pure labels <*> numberTree t = numberList (labels t)
```

Proof by induction on *t*. Leaf case:

```
pure labels <*> (pure Leaf <*> fresh) = pure (:) <*> fresh <*> pure []
```

Compare with

```
     labels     (     Leaf        x  ) =      (:)        x            []
```

A Proof Method for Isabelle

Project Goal

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- Applicative functor or *idiom* given by
  - type constructor
  - constants pure, $\diamond$ (`<*>` in Haskell)
  - proofs of applicative functor laws

A Proof Method for Isabelle

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Implement a proof method for Isabelle/HOL which lifts equations to applicative functors.

base equation $\Longrightarrow$ lifted equation

- Proof method: User interface for goal state transformation
- Applicative functor or *idiom* given by
  - type constructor
  - constants pure, $\diamond$ (<*> in Haskell)
  - proofs of applicative functor laws
- Lifting an equation (x is a variable): $[\,] \mathbin{@} x = x$

$$\begin{array}{llcccccc}
\text{base equation:} & & \mathit{append} & & [\,] & x & = & x \\
\Longrightarrow & \text{pure} & \mathit{append} & \diamond & \text{pure}\,[\,] \diamond & x & = & x
\end{array}$$

Overview of Operation

Input: Lifted equation

$$e_1 = e_2$$

Transform into *canonical forms*

$$\Longleftarrow \quad \text{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n = \text{pure}\, g \diamond x_1 \diamond \ldots x_n$$

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Reduce by congruence

$$\Longleftarrow \quad f = g$$

Extensionality

$$\Longleftarrow \quad \forall y_1 \ldots y_n. \quad f y_1 \ldots y_n = g y_1 \ldots y_n$$

Hinze's Lemmas

Lemma (Normal Form)

Let e be an idiomatic term with variables $x_1, \ldots, x_n$, from left to right. There exists f such that

$$e = \mathrm{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n.$$

Can lift equations

1. where both sides have the same list of variables, and
2. no variable is repeated.

R. Hinze, "Lifting Operators and Laws." 2010. Retrieved June 6, 2015, `http://www.cs.ox.ac.uk/ralf.hinze/Lifting.pdf`

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Lemma (Lifting Lemma, modified)

Let e' be the lifted term of e, with variables $x_1, \ldots, x_n$. If the idiom satisfies additional properties, then

$$e' = \text{pure}\, e \diamond x_1 \diamond \ldots \diamond x_n.$$

Lifts any equation, but not to all applicative functors.

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Combinatory Logic

- Eliminate variables from terms, introduce combinator constants
- BCKW system is equivalent to lambda calculus

$$
\begin{aligned}
\mathbf{B}gfx &= g(fx)\\
\mathbf{C}fxy &= fyx\\
\mathbf{K}xy &= x\\
\mathbf{W}fx &= fxx
\end{aligned}
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H. B. Curry et. al., *Combinatory Logic,* vol. 1. North-Holland, Amsterdam, 1968.

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- If not all combinators are available, not all terms can be represented
- Conversion by *bracket abstraction* algorithms

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Fancier Idioms

- Some idioms satisfy additional laws, one or more of

$$
\begin{array}{lrl}
\text{(c)} & \mathsf{pure}\,\mathbf{C} \diamond f \diamond x \diamond y = f \diamond y \diamond x & \mathbf{C} f x y = f y x \\
\text{(k)} & \mathsf{pure}\,\mathbf{K} \diamond x \diamond y = x & \mathbf{K} x y = x \\
\text{(w)} & \mathsf{pure}\,\mathbf{W} \diamond f \diamond x = f \diamond x \diamond x & \mathbf{W} f x = f x x
\end{array}
$$

- Hinze's Lifting Lemma requires all three laws
- Examples

	(c)	(w)	(k)
state monad	–	–	–
set (application via Cartesian product)	(c)	–	–
sum type, e.g. `Either`	–	(w)	–
option/`Maybe`	(c)	(w)	–
environment functor, streams	(c)	(w)	(k)

Lifting Bracket Abstraction

- Assume an applicative functors satisfies (c)

$$
\begin{array}{rlrl}
 & \lambda xy.\, x(fy) & & x \diamond (\mathsf{pure}\, f \diamond y) \\
= & \mathbf{CB}f & = & \mathsf{pure}\, \mathbf{C} \diamond \mathsf{pure}\, \mathbf{B} \diamond \mathsf{pure}\, f \diamond x \diamond y \\
 & & = & \mathsf{pure}\, (\mathbf{CB}f) \diamond x \diamond y \\
 & & = & \mathsf{pure}\, (\lambda xy.\, x(fy)) \diamond x \diamond y
\end{array}
$$

- We obtain a canonical form if the base term is representable

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- We obtain a canonical form if the base term is representable
- Variable order for idiomatic terms? Remember

$$\mathrm{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n = \mathrm{pure}\, g \diamond x_1 \diamond \ldots x_n$$

Usage (1)

```
applicative state
for
  pure: Pair
  ap: "ap_state :: ('a ⇒ 'b, 's) state ⇒ ('a, 's) state ⇒ ('b, 's) state"
unfolding ap_state_def
by (auto split: split_split)
```

```
goal (4 subgoals):
 1. ⋀x. pure (λx. x) ◇ x = x
 2. ⋀g f x. pure (λg f x. g (f x)) ◇ g ◇ f ◇ x = g ◇ (f ◇ x)
 3. ⋀f x. pure f ◇ pure x = pure (f x)
 4. ⋀f x. f ◇ pure x = pure (λf. f x) ◇ f
```

Usage (2)

```
lemma "Pair labels ◇ number_tree t = number_list (labels t)"
proof (induction t)
  case (Leaf x)
  have "Pair labels ◇ (Pair Leaf ◇ fresh) = Pair op # ◇ fresh ◇ Pair []"
    by applicative_lifting simp
  thus ?case by simp
next
  case (Node l r)
  let ?ll = "Pair labels ◇ number_tree l"
  let ?lr = "Pair labels ◇ number_tree r"
  have "Pair labels ◇ (Pair Node ◇ number_tree l ◇ number_tree r) = Pair op @ ◇ ?ll ◇ ?lr"
    by applicative_lifting simp
  thus ?case using Node.IH by (simp add: label_append)
qed
```

Conclusion

- Implemented applicative lifting in Isabelle/HOL
- Extended Hinze's results with bracket abstraction
- Use case: Algebra lifted to streams and infinite trees

Questions?

Applicative Functor Laws

(identity)	$\mathsf{pure}\,\mathbf{I} \diamond u = u$
(composition)	$\mathsf{pure}\,\mathbf{B} \diamond u \diamond v \diamond w = u \diamond (v \diamond w)$
(homomorphism)	$\mathsf{pure}\,f \diamond \mathsf{pure}\,x = \mathsf{pure}\,(fx)$
(interchange)	$u \diamond \mathsf{pure}\,x = \mathsf{pure}\,(\lambda f.\,fx) \diamond u$

Bracket Abstraction Rules

(i)	$[x]x = \mathbf{I}$,	
(k)	$[x]t = \mathbf{K}t$	if x not free in t,
(η)	$[x]tx = t$	if x not free in t,
(b)	$[x]st = \mathbf{B}s([x]t)$	if x not free in s,
(c)	$[x]st = \mathbf{C}([x]s)t$	if x not free in t,
(s)	$[x]st = \mathbf{S}([x]s)([x]t)$;	

Special rules for idioms:

(t)	$[x]st = \mathbf{T}t([x]s)$	if t contains no variables,
(w)	$[x]st = \mathbf{W}(\mathbf{B}(\mathbf{T}[x]t)(\mathbf{BB}[x]s))$	if $[x]t$ contains no variables.

What are the Variables?

- Remember that both canonical forms need the same variable lists:

$$\mathsf{pure}\, f \diamond x_1 \diamond \ldots \diamond x_n = \mathsf{pure}\, g \diamond x_1 \diamond \ldots x_n$$

- Must be able to represent terms with available combinators
- Instantiation:

$$\begin{aligned} & \forall xy.\ \mathsf{pure}\, f \diamond x \diamond y = \ldots \\ \Longrightarrow\quad & \forall z.\quad \mathsf{pure}\, f \diamond z \diamond z = \ldots \end{aligned}$$

  What if we want to prove the latter, but can only represent the former?
- Algorithm depends on available combinators, partially a heuristic