

# On Arithmetic Terms for Number Theory

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## Abstract

We present new results on arithmetic terms related to the greatest common divisor function  $\gcd(a, b)$ . We also give elementary formulas for the factors of a semiprime  $n = p_1 p_2$  and the integer part of the  $n$ -th roots. The formulas presented require only the operations of addition, subtraction, multiplication, floored division, and exponentiation.

**Disclaimer.** This paper is a work in progress and will be continuously updated until the first version of the preprint is released (v1.0.0). In the meantime, consider all results to be conjectures.

## 1 Greatest Common Divisor

**Lemma 1** (Mazzanti).

$$\forall a, b \in \mathbb{Z}^+, \quad \gcd(a, b) = \left\lfloor \frac{(2^{a^2 b(b+1)} - 2^{a^2 b})(2^{a^2 b^2} - 1)}{(2^{a^2 b} - 1)(2^{ab^2} - 1)2^{a^2 b^2}} \right\rfloor \bmod 2^{ab}.$$

*Proof.* The lemma and proof belong to Mazzanti (2002) [1]. □

Applying Kronecker substitution techniques from our previous works [2, 3], we find that Mazzanti's formula can be simplified and expressed in a polynomial form.

**Theorem 2.**

$$\forall a, b \in \mathbb{Z}^+, \quad \gcd(a, b) = \left\lfloor \frac{x^{a+ab}}{(x^a - 1)(x^b - 1)} \right\rfloor \bmod x.$$

*Proof.* Consider Mazzanti's greatest common divisor formula (Lemma 1), which is given by

$$\gcd(a, b) = \left\lfloor \frac{(2^{a^2 b(b+1)} - 2^{a^2 b})(2^{a^2 b^2} - 1)}{(2^{a^2 b} - 1)(2^{ab^2} - 1)2^{a^2 b^2}} \right\rfloor \bmod 2^{ab}.$$

Observe that all integer powers in the arithmetic term are divisible by  $2^{ab}$ . Factoring these, we obtain

$$\gcd(a, b) = \left\lfloor \frac{((2^{ab})^{a(b+1)} - (2^{ab})^a)((2^{ab})^{ab} - 1)}{((2^{ab})^a - 1)((2^{ab})^b - 1)(2^{ab})^{ab}} \right\rfloor \bmod 2^{ab}.$$

Substituting with  $2^{ab} = x$  yields

$$\gcd(a, b) = \left\lfloor \frac{(x^{a(b+1)} - x^a)(x^{ab} - 1)}{(x^a - 1)(x^b - 1)x^{ab}} \right\rfloor \bmod x.$$

The substitution is valid, since  $2^{ab} > \gcd(a, b)$  and the substitution  $2^{ab} = x$  essentially inverts the Kronecker substitution with the base  $2^{ab}$  (See Theorem 1 in [2]).

Simplifying the fraction, we obtain

$$\gcd(a, b) = \left\lfloor \frac{x^{a-ab}(x^{ab} - 1)^2}{(x^a - 1)(x^b - 1)} \right\rfloor \bmod x.$$

This fraction can be expanded as the sum

$$\gcd(a, b) = \left\lfloor \frac{x^{a-ab}}{(x^a - 1)(x^b - 1)} + \frac{x^{a+ab}}{(x^a - 1)(x^b - 1)} + \frac{-2x^a}{(x^a - 1)(x^b - 1)} \right\rfloor \bmod x.$$

Since we are reducing the quotient mod  $x$ , we need only consider the term in the fraction which yields the constant term in the polynomial, which is  $\gcd(a, b)$ . We find

$$\gcd(a, b) = \left\lfloor \frac{x^{a+ab}}{(x^a - 1)(x^b - 1)} \right\rfloor \bmod x.$$

□

**Corollary 3.** *Let  $a, b, n \in \mathbb{Z}^+$  such that  $n > \gcd(a, b)$ . Then*

$$\gcd(a, b) = \left\lfloor \frac{n^{a+ab}}{(n^a - 1)(n^b - 1)} \right\rfloor \bmod n.$$

*Proof.* Consider the polynomial formula given by Theorem 2. Substituting with  $x = n$  yields the given formula. By Theorem 2 in [3], the substitution is valid since  $n > \gcd(a, b)$ . □

## 2 Semiprime Factors

Using our results on the greatest common divisor function (§ 1), as well as results from our earlier works [2, 3] and those of Mazzanti [1], Prunescu and Sauras-Altuzarra [4], we discover elementary formulas for the prime factors of a non-square semiprime  $n = p_1 p_2$ . We say these formulas are “elementary”, since they require only addition, subtraction, multiplication, floored division, and exponentiation.

**Theorem 4.** Let  $n \in \mathbb{Z}^+$  such that  $n = p_1 p_2$  is a non-square semiprime and  $p_1 < p_2$  are the prime factors of  $n$ .

Define

$$\omega = \left\lfloor \frac{(n^{2n} + 1)^{2n+1} \bmod (n^{4n} - n)}{(n^{2n} + 1)^{2n} \bmod (n^{4n} - n)} \right\rfloor - 1.$$

Then, set

$$\gamma = \left\lfloor \frac{2^{\omega(\omega+1)(\omega+2)}}{\left[ (2^{2^{\omega(\omega+1)(\omega+2)} - n} + 2^{-\omega})^{2^{\omega(\omega+1)(\omega+2)}} \right] \bmod 2^{\omega 2^{\omega(\omega+1)(\omega+2)}}} \right\rfloor.$$

Finally, we have

$$p_1 = \left\lfloor \frac{n^{n+n\gamma}}{(n^n - 1)(n^\gamma - 1)} \right\rfloor \bmod n.$$

*Proof.* From Shunia (2024) [3], for  $n$  that is not a square, we get the arithmetic term

$$\lfloor \sqrt{n} \rfloor = \left\lfloor \frac{(n^{2n} + 1)^{2n+1} \bmod (n^{4n} - n)}{(n^{2n} + 1)^{2n} \bmod (n^{4n} - n)} \right\rfloor - 1,$$

which matches our definition of  $\omega$ . Hence,  $\omega = \lfloor \sqrt{n} \rfloor$ .

From Prunescu and Sauras-Altuzarra (2024) [4], we also have the factorial formula

$$\begin{aligned} n! &= \left\lfloor 2^{n(n+1)(n+2)} / \binom{2^{(n+1)(n+2)}}{n} \right\rfloor \\ &= \left\lfloor \frac{2^{n(n+1)(n+2)}}{\left[ (2^{2^{(n+1)(n+2)} - n} + 2^{-n})^{2^{(n+1)(n+2)}} \right] \bmod 2^{2^{(n+1)(n+2)}}} \right\rfloor. \end{aligned}$$

Considering  $\omega!$ , this becomes

$$\omega! = \left\lfloor \frac{2^{\omega(\omega+1)(\omega+2)}}{\left[ (2^{2^{\omega(\omega+1)(\omega+2)} - n} + 2^{-\omega})^{2^{\omega(\omega+1)(\omega+2)}} \right] \bmod 2^{\omega 2^{\omega(\omega+1)(\omega+2)}}} \right\rfloor,$$

which matches the definition for  $\gamma$ . Hence,  $\gamma = \omega! = \lfloor \sqrt{n} \rfloor!$ .

Applying Corollary 3, we have

$$\gcd(n, \lfloor \sqrt{n} \rfloor!) = \gcd(n, \gamma) = \left\lfloor \frac{n^{n+n\gamma}}{(n^n - 1)(n^\gamma - 1)} \right\rfloor \bmod n.$$

Since  $n$  is a non-square semiprime and  $p_1 < p_2$ , we must have  $p_1 < \lfloor \sqrt{n} \rfloor$  and  $p_2 > \lfloor \sqrt{n} \rfloor$ . Hence,  $p_1 = \gcd(n, \lfloor \sqrt{n} \rfloor!)$ , which we showed is equivalent to the formula in the theorem.  $\square$

**Corollary 5.**

$$p_2 = \frac{n}{\left\lfloor \frac{n^{n+n\gamma}}{(n^n - 1)(n^\gamma - 1)} \right\rfloor \bmod n}.$$

*Proof.* The proof follows immediately from Theorem 4, since  $\frac{n}{p_1} = p_2$  in this case.  $\square$

## References

- [1] S. Mazzanti. Plain Bases for Classes of Primitive Recursive Functions. *Mathematical Logic Quarterly*, 48(1):93–104, 2002. ISSN 0942-5616.
- [2] Joseph M. Shunia. A Simple Formula for Binomial Coefficients Revealed Through Polynomial Encoding, 2023. URL <https://arxiv.org/abs/2312.00301>. Unpublished Preprint.
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- [4] M. Prunescu and L. Sauras-Altuzarra. An Arithmetic Term for the Factorial Function. *Examples and Counterexamples*, 5:100136, 2024. ISSN 2666-657X. URL <https://sciencedirect.com/science/article/pii/S2666657X24000028>.