

# [Draft] On Kalmar Numbers and Arithmetic Terms

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## Abstract

From the classes of Kalmar functions and arithmetic terms, we describe the class of Kalmar numbers, which is the subset of computable real numbers that can be represented as a limit of the quotient of two Kalmar functions. We give Kalmar numbers for many important mathematical constants and functions, such as  $e$ ,  $\pi$ ,  $\sqrt[n]{n}$ ,  $\log(n)$ ,  $\exp(n)$ ,  $\sin(n)$ ,  $\cos(n)$ ,  $\Gamma_k(q)$ ,  $(q)_{n,k}$ ,  $\psi^{(k)}(q)$ .

**DISCLAIMER:** This paper is a work in progress. Many proofs and results are currently missing.

## 1 Kalmar Numbers

We define a **Kalmar number** as a computable real number which can be represented as a limit of the quotient of two Kalmar functions. That is,  $r \in \mathbb{R}$  is a Kalmar number if there exists a limit

$$r = \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n \rightarrow -\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n \rightarrow 0} \frac{f(n)}{g(n)}$$

where  $f(n), g(n)$  are Kalmar functions. This definition is due to Lorenzo Sauras-Altuzarra, who described it in a private correspondence.

## 2 Characterizing the Kalmar Numbers

An interesting question from Mihai Prunescu is:

**Question 2.1.** *What is the subset of computable real numbers that are Kalmar numbers?*

We offer a partial solution to the question.

**Theorem 2.1.** *Let  $r \in \mathbb{R}$  such that the ratios of consecutive terms for both the numerators and denominators of the  $n$ -th convergents in its generalized continued fraction representation are bounded by a tower of exponentials  $2^{2^{\dots 2^n}}$  with height  $n$ . Then, there exist Kalmar functions  $f(n)$  and  $g(n)$  such that*

$$r = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}.$$

*Proof.* The generalized continued fraction representation of  $r$  is given by

$$r = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ddots}}},$$

where  $a_i$  and  $b_i$  are sequences of integers. The  $n$ -th convergents of the generalized continued fraction for  $r$  can be expressed using the recurrence relations

$$\begin{aligned} P_n &= a_n P_{n-1} + b_n P_{n-2}, \\ Q_n &= a_n Q_{n-1} + b_n Q_{n-2}, \end{aligned}$$

with initial starting conditions  $P_{-1} = 1$ ,  $P_0 = a_0$ ,  $Q_{-1} = 0$ , and  $Q_0 = 1$ .

The sequence of convergents  $\frac{P_n}{Q_n}$  of the generalized continued fraction for  $r$  will approach  $r$  as  $n \rightarrow \infty$ . That is,

$$\lim_{n \rightarrow \infty} \frac{P_n}{Q_n} = r.$$

By the theorem's assumption, we have

$$\frac{P_n}{P_{n-1}} \leq 2^{2^{\dots^n}} \quad \text{and} \quad \frac{Q_n}{Q_{n-1}} \leq 2^{2^{\dots^n}}.$$

This implies that for all  $n \geq 1$ ,

$$\begin{aligned} P_n &\leq P_{n-1} 2^{2^{\dots^n}} \leq P_{n-2} 2^{2^{\dots^{n-1}}} 2^{2^{\dots^n}} \leq \dots \leq \\ &P_0 2^{2^{2^1}} 2^{2^{2^2}} \dots 2^{2^{2^{\dots^n}}} = P_0 2^{2^1 + 2^2 + \dots + 2^{2^{\dots^n}}} = P_0 2^{2^{2^{\dots^{n+1}}} - 2} \\ &= O(2^{2^{2^{\dots^n}}}), \end{aligned}$$

and similarly for  $Q_n$ .

The set of rational numbers  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Thus, for any  $r \in \mathbb{R}$  and any  $\epsilon > 0$ , there exists a rational number  $q \in \mathbb{Q}$  such that  $|r - q| < \epsilon$ .

Define  $\epsilon_n = \left| r - \frac{P_n}{Q_n} \right|$ . Since  $\lim_{n \rightarrow \infty} \frac{P_n}{Q_n} = r$ , there exists an  $N \in \mathbb{N}$  such that for all  $n > N$ , we have  $\epsilon_n < \frac{P_n}{Q_n}$  and  $\epsilon_{n+1} < \epsilon_n$ . Furthermore, since  $P_n$  and  $Q_n$  are bounded by  $O(2^{2^{\dots^{2^n}}})$  and their recursions require only elementary arithmetic operations to compute, there exists a pair of Kalmar functions  $f(n)$  and  $g(n)$  such that  $f(n) = P_n$  and  $g(n) = Q_n$ . Therefore, we conclude

$$\lim_{n \rightarrow \infty} \left| r - \frac{f(n)}{g(n)} \right| = 0 \implies \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = r.$$

□

### 3 Pochhammer Symbol

The **Pochhammer  $k$ -symbol** is defined as the integer-valued function

$$(n)_{a,b} = \prod_{k=1}^n (a + (b-1)k).$$

**Lemma 3.1.** *Let  $a, b \in \mathbb{Z}$ . For large  $n$ , the product*

$$P_n = \prod_{k=0}^n (a + bk)$$

*is approximated by*

$$P_n \approx \frac{b^n n! (a/b + n)^n}{(a/b)^n}.$$

*Proof.* We begin by expressing the product  $P_n$  as

$$P_n = b^n \prod_{k=0}^n \left( \frac{a}{b} + k \right).$$

This can be rewritten using the Gamma function as

$$P_n = b^n \frac{\Gamma(a/b + n + 1)}{\Gamma(a/b)}.$$

Stirling's approximation for the Gamma function gives

$$\Gamma(z) \sim \sqrt{2\pi z} \left( \frac{z}{e} \right)^z \quad \text{as } z \rightarrow \infty.$$

Applying the approximation, we have

$$\frac{\Gamma(a/b + n + 1)}{\Gamma(a/b)} \sim \frac{\sqrt{2\pi(a/b + n + 1)} \left(\frac{a/b + n + 1}{e}\right)^{a/b + n + 1}}{\sqrt{2\pi a/b} \left(\frac{a/b}{e}\right)^{a/b}}.$$

Simplifying this expression

$$\frac{\Gamma(a/b + n + 1)}{\Gamma(a/b)} \sim \sqrt{\frac{a/b + n + 1}{a/b}} \left(\frac{a/b + n + 1}{a/b}\right)^{a/b} (a/b + n + 1)^n e^{-n-1}.$$

For large  $n$  and  $a/b$ ,  $\frac{a/b + n + 1}{a/b} \approx 1 + \frac{n+1}{a/b}$ , so

$$\frac{\Gamma(x + n + 1)}{\Gamma(x)} \sim n! \left(1 + \frac{n}{a/b}\right)^n \left(\frac{x + n}{x}\right)^n.$$

Hence, we find

$$P_n \approx b^n \frac{n!(a/b + n)^n}{(a/b)^n}.$$

□

**Theorem 3.1.** Let  $n, a, b \in \mathbb{Z}$  such that  $n > 0$  and  $\log(n) > a$ .

Define  $\sigma(n, a, b) = \frac{b^n n!(a/b + n)^n}{(a/b)^n}$ .

Set  $\ell = \lfloor \log_b(\sigma(n, a, b)) \rfloor + 1$ .

Set  $m = b^\ell + 1$ .

Set  $c = (ab^{-1}) \bmod m$ .

Then

$$(n)_{a,b} = \left\lfloor \frac{\left(\binom{n+c}{n+1} b^{n+1} (n+1)!\right) \bmod m}{a + nb} \right\rfloor.$$

Further,  $(n)_{a,b}$  is an arithmetic term.

*Proof.* From [6], for  $a \equiv bc \pmod{m}$ , we have

$$\prod_{k=0}^n (a + bk) \equiv \binom{n+c}{n+1} b^{n+1} (n+1)! \pmod{m}. \quad (1)$$

We are given  $m = b^\ell + 1$ , thus  $m$  and  $b$  are coprime and  $b^{-1} \pmod{m}$  exists. Therefore,  $c = ab^{-1} \bmod m$  is valid. Hence, we have established a congruence of  $\prod_{k=0}^n (a + bk)$  modulo  $m$ .

Now, since

$$\left(\prod_{k=0}^n (a + bk)\right)^2 > m = b^\ell = b^{\lfloor \log_b((n)_{a,b}) \rfloor + 1} > \prod_{k=0}^n (a + bk),$$

it follows that

$$(n)_{a,b} = \left\lfloor \frac{\left(\binom{n+c}{n+1} b^{n+1} (n+1)!\right) \bmod m}{a + nb} \right\rfloor.$$

Finally, due to Prunescu and Sauras-Altuzarra [3], we have an arithmetic term for the modular inverse  $b^{-1} \pmod{m}$ . Arithmetic terms for the factorial function, binomial coefficients, and roots are also known [2, 5, 1, 3]. Therefore, it follows that  $(n)_{a,b}$  is an arithmetic term. □

## 4 Generalized Binomial Coefficients

We define the **generalized binomial coefficients** as the coefficients of the form

$$\binom{\frac{a}{b}}{k} = \frac{\left(\frac{a}{b}\right) \left(\frac{a}{b} - 1\right) \left(\frac{a}{b} - 2\right) \cdots \left(\frac{a}{b} - (k-1)\right)}{k!}. \quad (2)$$

**Lemma 4.1.**

$$\forall a, b, k \in \mathbb{Z}^+, \quad \binom{\frac{a}{b}}{k} = \frac{(k)_{a,-b}}{b^k k!}.$$

*Proof.* We begin with our definition

$$\binom{\frac{a}{b}}{k} = \frac{\left(\frac{a}{b}\right) \left(\frac{a}{b} - 1\right) \left(\frac{a}{b} - 2\right) \cdots \left(\frac{a}{b} - (k-1)\right)}{k!}.$$

Expanding the terms and simplifying, we obtain

$$\begin{aligned} \binom{\frac{a}{b}}{k} &= \frac{\frac{a}{b} \frac{a-b}{b} \frac{a-2b}{b} \cdots \frac{a-(k-1)b}{b}}{k!} \\ &= \frac{a(a-b)(a-2b) \cdots (a-(k-1)b)}{b^k k!} \\ &= \frac{(k)_{a,-b}}{b^k k!}. \end{aligned}$$

□

## 5 Beta Function

The **beta function** is defined as

$$\mathbf{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (3)$$

For all  $a, b \in \mathbb{Z}^+$ , we have

$$\mathbf{B}(a, b) = \frac{a+b}{ab} \bigg/ \binom{a+b}{a}. \quad (4)$$

## 6 Gamma Function

The  **$k$ -Gamma function** [4] is defined as

$$\Gamma_k(q) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{q/k-1}}{(q)_{n,k}}.$$

**Theorem 6.1.** *The function  $\Gamma_k(q)$ , where  $q \in \mathbb{Q}$ , is a Kalmar number.*

*Proof.* From [4], we have

$$\Gamma_k(q) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{q/k-1}}{(q)_{n,k}}.$$

By Theorem 3.1,  $(q)_{n,k}$  is an arithmetic term. Arithmetic terms for the factorial function and roots are also known [3, 1]. Therefore, it follows that  $\Gamma_k(q)$  is a Kalmar number. □

## 7 Polygamma Function

From our Kalmar number for  $\Gamma(q)$ , we can calculate  $\psi(q)$ .

$$\psi^{(0)}(q) = \psi(q) = \frac{\Gamma'(q)}{\Gamma(q)}.$$

## 8 Useful Kalmar Numbers

### Roots

$$\sqrt[n]{a} = \lim_{k \rightarrow \infty} \frac{(k^{kn} + 1)^{kn+1} \bmod (k^{kn^2} - a)}{(k^{kn} + 1)^{kn} \bmod (k^{kn^2} - a)} - 1. \quad (5)$$

### Derivatives

$$f'(x) = \lim_{b \rightarrow 0} \frac{f(a+b) - f(a)}{b} \quad (6)$$

### Exponential Function

$$e = \lim_{k \rightarrow \infty} \frac{(k+1)^k}{k^k}$$

$$e^n = \lim_{k \rightarrow \infty} \frac{(k+1)^{nk}}{k^{nk}}$$

$$\exp(in) = \lim_{k \rightarrow \infty} \left( \left( 1 + \frac{xn}{k} \right)^k \bmod (x^2 + 1) \right)$$

### Natural Logarithm

$$\log(n) = \lim_{k \rightarrow \infty} k(\sqrt[k]{n} - 1).$$

### Trigonometric Functions

$$\cos(n) = \lim_{k \rightarrow \infty} \left( \left( \left( 1 + \frac{xn}{k} \right)^k \bmod (x^2 + 1) \right) \bmod x \right),$$

$$\sin(n) = \lim_{k \rightarrow \infty} \frac{\left( 1 + \frac{xn}{k} \right)^k \bmod (x^2 + 1)}{x}.$$

### Mod One

$$a \bmod b = b \left( \frac{a}{b} \bmod 1 \right) = b \left( \frac{a}{b} - \left\lfloor \frac{a}{b} \right\rfloor \right)$$

### Pi

$$\sqrt{2\pi} = \lim_{n \rightarrow \infty} \frac{n!}{\sqrt{n} n^n e^{-n}}$$

$$\sqrt{\pi} = \lim_{n \rightarrow \infty} \frac{4^n n!^2}{\sqrt{n} (2n)!}$$

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{2^{4n} n!^4}{(2n)!^2 (2n+1)}$$

## References

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