A Formula Proof for OEIS Sequence A007917

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1 Formula and Proof

A short proof of a formula for A007917(n) is given.

Theorem 1.1. *Define* A007917(n) *to be the integer sequence which returns the largest prime* $p \leq n$. *Then*

$$\forall n \in \mathbb{Z}_{\geq 2}, \quad \text{A007917}(n) = \left\lfloor \log\left(\sum_{k=2}^{n} e^{\varphi(k)+1}\right)\right\rfloor.$$

Proof. Let $n \in \mathbb{Z}_{\geq 2}$. Consider the polynomial

$$S_n(x) = \sum_{k=2}^{n} x^{\varphi(k)+1} \in \mathbb{Z}[x].$$

Substituting $x = e$ into $S_n(x)$ gives the expression $S_n(e) = \sum_{k=2}^{n} e^{\varphi(k)+1}$. Notice that the degree of $S_n(x)$ is determined by the largest $\varphi(k)+1$, which is always the largest prime $\leq n$. Thus, $\deg(S_n(x)) =$ A007917(n).

To complete the proof, we will show that

$$\text{A007917}(n) = \deg(S_n(x)) = \lfloor \log(S_n(e)) \rfloor = \left\lfloor \log\left(\sum_{k=2}^{n} e^{\varphi(k)+1}\right)\right\rfloor.$$

For $n = 2$, the formula holds since,

$$\left\lfloor \log(e^{\varphi(2)+1}) \right\rfloor = \left\lfloor \log(e^{1+1}) \right\rfloor = \left\lfloor \log(e^{2}) \right\rfloor = 2.$$

For $n > 2$, we consider the worst-case scenario, which occurs when n and $n-2$ are a twin prime pair. In such case, the leading terms in $S_n(e)$ are $e^{\varphi(n)+1}$ and $e^{\varphi(n-2)+1}$, so we need to ensure that

$$\forall n \in \mathbb{Z}_{>2}, \quad e^{\varphi(n)+1} > \sum_{k=2}^{n-1} e^{\varphi(k)+1}.$$

Rewriting this inequality, and noting that $\varphi(n) = n-1$ iff n is prime, we have

$$e^{\varphi(n)+1} > e^{\varphi(n-1)+1} + e^{\varphi(n-2)+1} + \sum_{k=2}^{n-3} e^{\varphi(k)+1},$$
$$e^{n} > e^{n-2} + e^{\varphi(n-1)+1} + \sum_{k=2}^{n-3} e^{\varphi(k)+1},$$

which simplifies to

$$e^{n} - e^{n-2} > e^{\varphi(n-1)+1} + \sum_{k=2}^{n-3} e^{\varphi(k)+1},$$
$$e^{n-2}(e^{2}-1) > e^{\varphi(n-1)+1} + \sum_{k=2}^{n-3} e^{\varphi(k)+1}.$$

We claim that, for $n > 2$, we have

$$e^{n-2}(e^2-1) > \sum_{k=2}^{n-3} e^k \geq \sum_{k=2}^{n-3} e^{\varphi(k)+1}.$$

Observe that the sum $\sum_{k=2}^{n-3} e^k$ is a geometric series with the closed form

$$\sum_{k=2}^{n-3} e^k = \frac{e^n - e^4}{e^2(e-1)}.$$

Substituting this into the previous inequality, we obtain

$$e^{n-2}(e^2-1) > \frac{e^n - e^4}{e^2(e-1)} \geq \sum_{k=2}^{n-3} e^{\varphi(k)+1},$$

which is certainly true, since

$$e^{n-2}(e^2-1) - \left(e^{n-1} + \frac{e^n - e^4}{e^2(e-1)}\right) = \frac{e^n(e-2)+e^2}{e-1} > 0.$$

The twin prime case is the worst possible case because, in such scenario, the second-largest term e^{n-2} is as close as possible to the largest term e^n, maximizing the contribution of lower-order terms to the sum $S_n(e)$. In other cases where n and $n-2$ are not both primes, the second-largest term is smaller, and thus the sum $S_n(e)$ is more heavily dominated by e^n. This means $\log(S_n(e))$ is closer to n and more easily approximated by n in non-twin prime cases.

Therefore, in all cases, the sum $S_n(e)$ remains dominated by the leading term e^n, and hence

$$\left\lfloor \log\left(\sum_{k=2}^{n} e^{\varphi(k)+1}\right) \right\rfloor = \deg(S_n(x)) = \text{A007917}(n).$$

$\square$