[Draft] On Kalmar Numbers and Arithmetic Terms

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Abstract

From the classes of Kalmar functions and arithmetic terms, we describe the class of Kalmar numbers, which is the subset of computable real numbers that can be represented as a limit of the quotient of two Kalmar functions. We give Kalmar numbers for many important mathematical constants and functions, such as e, π, $\sqrt[r]{n}$, $\log(n)$, $\exp(n)$, $\sin(n)$, $\cos(n)$, $\Gamma_k(q)$, $(q)_{n,k}$, $\psi^{(k)}(q)$.

DISCLAIMER: This paper is a work in progress. Many proofs and results are currently missing.

1 Kalmar Numbers

We define a **Kalmar number** as a computable real number which can be represented as a limit of the quotient of two Kalmar functions. That is, $r \in \mathbb{R}$ is a Kalmar number if there exists a limit

$$r = \lim_{n\to+\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n\to-\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n\to 0} \frac{f(n)}{g(n)}$$

where $f(n), g(n)$ are Kalmar functions. This definition is due to Lorenzo Sauras-Altuzarra, who described it in a private correspondence.

2 Characterizing the Kalmar Numbers

An interesting question from Mihai Prunescu is:

Question 2.1. *What is the subset of computable real numbers that are Kalmar numbers?*

We offer a partial solution to the question.

Theorem 2.1. *Let $r \in \mathbb{R}$ such that the ratios of consecutive terms for both the numerators and denominators of the n-th convergents in its generalized continued fraction representation are bounded by a tower of exponentials $2^{2^{\cdots^{2^n}}}$ with height n. Then, there exist Kalmar functions $f(n)$ and $g(n)$ such that*

$$r = \lim_{n\to\infty} \frac{f(n)}{g(n)}.$$

Proof. The generalized continued fraction representation of r is given by

$$r = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}},$$

where a_i and b_i are sequences of integers. The n-th convergents of the generalized continued fraction for r can be expressed using the recurrence relations

$$P_n = a_n P_{n-1} + b_n P_{n-2},$$
$$Q_n = a_n Q_{n-1} + b_n Q_{n-2},$$

with initial starting conditions $P_{-1} = 1$, $P_0 = a_0$, $Q_{-1} = 0$, and $Q_0 = 1$.

The sequence of convergents $\frac{P_n}{Q_n}$ of the generalized continued fraction for r will approach r as $n \to \infty$. That is,

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = r.$$

By the theorem's assumption, we have

$$\frac{P_n}{P_{n-1}} \leq 2^{2^{2^{\cdots^n}}} \text{ and } \frac{Q_n}{Q_{n-1}} \leq 2^{2^{2^{\cdots^n}}}.$$

This implies that for all $n \geq 1$,

$$\begin{aligned} P_n \leq P_{n-1} 2^{2^{2^{\cdots^n}}} \leq P_{n-2} 2^{2^{2^{\cdots^{n-1}}}} 2^{2^{2^{\cdots^n}}} \leq \cdots \leq \\ P_0 2^{2^{2^1}} 2^{2^{2^2}} \cdots 2^{2^{2^{\cdots^n}}} = P_0 2^{2^1+2^2+\cdots+2^{2^{\cdots^n}}} = P_0 2^{2^{2^{\cdots^{n+1}}}-2} \\ = O(2^{2^{2^{\cdots^n}}}), \end{aligned}$$

and similarly for Q_n.

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus, for any $r \in \mathbb{R}$ and any $\epsilon > 0$, there exists a rational number $q \in \mathbb{Q}$ such that $|r - q| < \epsilon$.

Define $\epsilon_n = \left| r - \frac{P_n}{Q_n} \right|$. Since $\lim_{n\to\infty} \frac{P_n}{Q_n} = r$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $\epsilon_n < \frac{P_n}{Q_n}$ and $\epsilon_{n+1} < \epsilon_n$. Furthermore, since P_n and Q_n are bounded by $O(2^{2^{\cdots^{2^n}}})$ and their recursions require only elementary arithmetic operations to compute, there exists a pair of Kalmar functions $f(n)$ and $g(n)$ such that $f(n) = P_n$ and $g(n) = Q_n$. Therefore, we conclude

$$\lim_{n\to\infty} \left| r - \frac{f(n)}{g(n)} \right| = 0 \implies \lim_{n\to\infty} \frac{f(n)}{g(n)} = r.$$

□

3 Pochhammer Symbol

The **Pochhammer k-symbol** is defined as the integer-valued function

$$(n)_{a,b} = \prod_{k=1}^{n} (a + (b-1)k).$$

Lemma 3.1. *Let $a, b \in \mathbb{Z}$. For large n, the product*

$$P_n = \prod_{k=0}^{n} (a + bk)$$

is approximated by

$$P_n \approx \frac{b^n n! (a/b + n)^n}{(a/b)^n}.$$

Proof. We begin by expressing the product P_n as

$$P_n = b^n \prod_{k=0}^{n} \left(\frac{a}{b} + k \right).$$

This can be rewritten using the Gamma function as

$$P_n = b^n \frac{\Gamma(a/b + n + 1)}{\Gamma(a/b)}.$$

Stirling's approximation for the Gamma function gives

$$\Gamma(z) \sim \sqrt{2\pi z} \left(\frac{z}{e} \right)^z \quad \text{as } z \to \infty.$$

Applying the approximation, we have

$$\frac{\Gamma(a/b+n+1)}{\Gamma(a/b)} \sim \frac{\sqrt{2\pi(a/b+n+1)}\left(\frac{a/b+n+1}{e}\right)^{a/b+n+1}}{\sqrt{2\pi a/b}\left(\frac{a/b}{e}\right)^{a/b}}.$$

Simplifying this expression

$$\frac{\Gamma(a/b+n+1)}{\Gamma(a/b)} \sim \sqrt{\frac{a/b+n+1}{a/b}}\left(\frac{a/b+n+1}{a/b}\right)^{a/b}(a/b+n+1)^n e^{-n-1}.$$

For large n and a/b, $\frac{a/b+n+1}{a/b} \approx 1 + \frac{n+1}{a/b}$, so

$$\frac{\Gamma(x+n+1)}{\Gamma(x)} \sim n!\left(1+\frac{n}{a/b}\right)^n\left(\frac{x+n}{x}\right)^n.$$

Hence, we find

$$P_n \approx b^n \frac{n!(a/b+n)^n}{(a/b)^n}.$$

□

Theorem 3.1. *Let $n, a, b \in \mathbb{Z}$ such that $n > 0$ and $\log(n) > a$.*
Define $\sigma(n,a,b) = \frac{b^n n!(a/b+n)^n}{(a/b)^n}$.
Set $\ell = \lfloor \log_b(\sigma(n,a,b) \rfloor + 1$.
Set $m = b^\ell + 1$.
Set $c = (ab^{-1}) \bmod m$.
Then

$$(n)_{a,b} = \left\lfloor \frac{\left(\binom{n+c}{n+1} b^{n+1}(n+1)!\right) \bmod m}{a+nb} \right\rfloor.$$

Further, $(n)_{a,b}$ is an arithmetic term.

Proof. From [6], for $a \equiv bc \pmod{m}$, we have

$$\prod_{k=0}^{n}(a+bk) \equiv \binom{n+c}{n+1} b^{n+1}(n+1)! \pmod{m}. \tag{1}$$

We are given $m = b^\ell + 1$, thus m and b are coprime and $b^{-1} \pmod{m}$ exists. Therefore, $c = ab^{-1} \bmod m$ is valid. Hence, we have established a congruence of $\prod_{k=0}^{n}(a+bk)$ modulo m.

Now, since

$$\left(\prod_{k=0}^{n}(a+bk)\right)^2 > m = b^\ell = b^{\lfloor \log_b((n)_{a,b}) \rfloor + 1} > \prod_{k=0}^{n}(a+bk),$$

it follows that

$$(n)_{a,b} = \left\lfloor \frac{\left(\binom{n+c}{n+1} b^{n+1}(n+1)!\right) \bmod m}{a+nb} \right\rfloor.$$

Finally, due to Prunescu and Sauras-Altuzarra [3], we have an arithmetic term for the modular inverse $b^{-1} \pmod{m}$. Arithmetic terms for the factorial function, binomial coefficients, and roots are also known [2, 5, 1, 3]. Therefore, it follows that $(n)_{a,b}$ is an arithmetic term. □

4 Generalized Binomial Coefficients

We define the **generalized binomial coefficients** as the coefficients of the form

$$\binom{\frac{a}{b}}{k} = \frac{\left(\frac{a}{b}\right)\left(\frac{a}{b}-1\right)\left(\frac{a}{b}-2\right)\cdots\left(\frac{a}{b}-(k-1)\right)}{k!}. \tag{2}$$

Lemma 4.1.

$$\forall a,b,k \in \mathbb{Z}^+, \quad \binom{\frac{a}{b}}{k} = \frac{(k)_{a,-b}}{b^k k!}.$$

Proof. We begin with our definition

$$\binom{\frac{a}{b}}{k} = \frac{\left(\frac{a}{b}\right)\left(\frac{a}{b}-1\right)\left(\frac{a}{b}-2\right)\cdots\left(\frac{a}{b}-(k-1)\right)}{k!}.$$

Expanding the terms and simplifying, we obtain

$$\begin{aligned}\binom{\frac{a}{b}}{k} &= \frac{\frac{a}{b}\frac{a-b}{b}\frac{a-2b}{b}\cdots\frac{a-(k-1)b}{b}}{k!}\\ &= \frac{a(a-b)(a-2b)\cdots(a-(k-1)b)}{b^k k!}\\ &= \frac{(k)_{a,-b}}{b^k k!}.\end{aligned}$$

□

5 Beta Function

The **beta function** is defined as

$$\mathbf{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3}$$

For all $a,b \in \mathbb{Z}^+$, we have

$$\mathbf{B}(a,b) = \frac{a+b}{ab} \bigg/ \binom{a+b}{a}. \tag{4}$$

6 Gamma Function

The k**-Gamma function** [4] is defined as

$$\Gamma_k(q) = \lim_{n\to\infty} \frac{n!k^n(nk)^{q/k-1}}{(q)_{n,k}}.$$

Theorem 6.1. *The function $\Gamma_k(q)$, where $q \in \mathbb{Q}$, is a Kalmar number.*

Proof. From [4], we have

$$\Gamma_k(q) = \lim_{n\to\infty} \frac{n!k^n(nk)^{q/k-1}}{(q)_{n,k}}.$$

By Theorem 3.1, $(q)_{n,k}$ is an arithmetic term. Arithmetic terms for the factorial function and roots are also known [3, 1]. Therefore, it follows that $\Gamma_k(q)$ is a Kalmar number. □

7 Polygamma Function

From our Kalmar number for $\Gamma(q)$, we can calculate $\psi(q)$.

$$\psi^{(0)}(q) = \psi(q) = \frac{\Gamma'(q)}{\Gamma(q)}.$$

8 Useful Kalmar Numbers

Roots

$$\sqrt[n]{a} = \lim_{k\to\infty} \frac{(k^{kn}+1)^{kn+1} \bmod (k^{kn^2}-a)}{(k^{kn}+1)^{kn} \bmod (k^{kn^2}-a)} - 1. \tag{5}$$

Derivatives

$$f'(x) = \lim_{b\to 0} \frac{f(a+b)-f(a)}{b} \tag{6}$$

Exponential Function

$$e = \lim_{k\to\infty} \frac{(k+1)^k}{k^k}$$

$$e^n = \lim_{k\to\infty} \frac{(k+1)^{nk}}{k^{nk}}$$

$$\exp(in) = \lim_{k\to\infty} \left(\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1) \right)$$

Natural Logarithm

$$\log(n) = \lim_{k\to\infty} k(\sqrt[k]{n}-1).$$

Trigonometric Functions

$$\cos(n) = \lim_{k\to\infty} \left(\left(\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1) \right) \bmod x \right),$$
$$\sin(n) = \lim_{k\to\infty} \frac{\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1)}{x}.$$

Mod One

$$a \bmod b = b\left(\frac{a}{b} \bmod 1\right) = b\left(\frac{a}{b} - \left\lfloor \frac{a}{b} \right\rfloor\right)$$

Pi

$$\sqrt{2\pi} = \lim_{n\to\infty} \frac{n!}{\sqrt{n}n^n e^{-n}}$$

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{4^n n!^2}{\sqrt{n}(2n)!}$$

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n} n!^4}{(2n)!^2(2n+1)}$$

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