Simple Formulas for Univariate Multinomial Coefficients

Joseph M. Shunia

September 2023
(Revised: July 2024, Version 4)

Abstract

We examine an underexplored property of polynomials, which states that under certain conditions, a polynomial can be completely determined by only two evaluations. Using this result, we derive new elementary formulas for univariate multinomial coefficients and their partial sums. Notably, we present what is potentially the first closed-form expression for partial sums of binomial coefficients.

1 Introduction

In this paper, we begin by proving a theorem stating that under certain conditions, a polynomial can be completely determined by only two evaluations (Theorem 1). This property allows us to recover the coefficients of a polynomial using a formula involving floor and modulo operations.

Subsequently, we revisit an unconventional formula for binomial coefficients, initially given by Julia Robinson (1952) [1] and later by Adi Shamir (1978) [2]:

$$\binom{n}{k} = \left\lfloor \frac{(2^n + 1)^n}{2^{nk}} \right\rfloor \bmod 2^n.$$

In § 3, we present elementary formulas for the partial sums of binomial coefficients, for which it has been alleged that no explicit formula is known [3, 4]. A formula of ours is

$$\sum_{k=0}^{j} \binom{n}{k} = \left\lfloor \frac{(2^n + 1)^n}{2^{n(n-j)}} \right\rfloor \bmod (2^n - 1).$$

We proceed in § 4 by generalizing Robinson's binomial coefficient formula, to calculate the coefficients in the expansion of univariate unit polynomials of the form

$$[x^k](1 + x + \cdots + x^{r-1})^n.$$

These coefficients are the univariate multinomial coefficients denoted as $\binom{n}{k}_{r-1}$, which are a generalization of the binomial coefficients. Conventional techniques for computing univariate multinomial coefficients involve factorials and summations over specific criteria. In contrast, our formula uses only elementary arithmetic operations and is given by

$$\binom{n}{k}_{r-1} = \left\lfloor \left(\frac{r^{rn} - 1}{r^{n+k} - r^k} \right)^n \right\rfloor \bmod r^n,$$

where $n > 0$ and $0 \le k \le n(r-1)$ (Theorem 6).

2 Polynomial Interpolation with Two Evaluations

We begin by presenting a theorem which shows how to recover a polynomial in $\mathbb{Z}[x]$ completely using only two carefully chosen evaluation points.

Theorem 1. *Let $b, r \in \mathbb{Z}$ such that $b > 0$, $r \geq 0$. Consider a polynomial $f(x) \in \mathbb{Z}[x]$ of degree r, which has the form*

$$f(x) = a_d x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0.$$

Suppose $f(b) \neq 0$ and that all coefficients of $f(x)$ are non-negative. Then, $f(x)$ can be completely determined by the evaluations $f(b)$ and $f(f(b))$. Furthermore, $\forall k \in \mathbb{Z} : 0 \leq k \leq r$, the coefficient a_k can be recovered explicitly from the formula:

$$a_k = \left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor \bmod f(b).$$

Proof. By assumption, $\forall k \in \mathbb{Z} : 0 \leq k \leq r$, the coefficient a_k of $f(x)$ can be recovered from the given formula. To prove the validity of the formula, we proceed by examining its arithmetic operations step-by-step.

Suppose we choose some k in the range $0 \leq k \leq r$. Now, let's consider the expansion of $f(f(b))$, which can be written as

$$f(f(b)) = a_d f(b)^r + a_{r-1} f(b)^{r-1} + \cdots + a_k f(b)^k + a_{k-1} f(b)^{k-1} + \cdots + a_1 f(b) + a_0.$$

The first step in the formula is to divide $f(f(b))$ by $f(b)^k$. This results in the quotient

$$
\begin{aligned}
\frac{f(f(b))}{f(b)^k} &= f(b)^{-k} (a_d f(b)^r + \cdots + a_k f(b)^k + a_{k-1} f(b)^{k-1} + \cdots + a_0) \\
&= a_d f(b)^r f(b)^{-k} + \cdots + a_k f(b)^k f(b)^{-k} + a_{k-1} f(b)^{k-1} f(b)^{-k} + \cdots + a_0 f(b)^{-k} \\
&= a_d f(b)^{r-k} + \cdots + a_k f(b)^{k-k} + a_{k-1} f(b)^{k-k-1} + \cdots + a_0 f(b)^{-k} \\
&= a_d f(b)^{r-k} + \cdots + a_k f(b)^0 + a_{k-1} f(b)^{-1} + \cdots + a_0 f(b)^{-k} \\
&= a_d f(b)^{r-k} + \cdots + a_k + a_{k-1} f(b)^{-1} + \cdots + a_0 f(b)^{-k}.
\end{aligned}
$$

The next step is to take the floor of the quotient $\frac{f(f(b))}{f(b)^k}$. In doing so, we effectively isolate the terms ranging from $a_k x^k$ up to and including $a_d x^r$. The result is

$$\left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor = a_d f(b)^{r-k} + \cdots + a_k.$$

The final step is to take the floored result $\left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor$ modulo $f(b)$. Given $r \geq k \geq 0$, we have two possibilities: The first is that $k = r$, in which case we have a monomial and it is not necessary to carry out the mod operation, since $a_d f(b)^{r-r} = a_d f(b)^0 = a_d$. Thus, we are done. On the other hand, if $k < r$, we must perform the mod $f(b)$ operation. Carrying it out, noting that the mod operation is distributive over addition, we see

$$
\begin{aligned}
\left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor \bmod f(b) &= (a_d f(b)^{r-k} \bmod f(b)) + \cdots + (a_k \bmod f(b)) \\
&= (0) + \cdots + (a_k \bmod f(b)) \\
&= 0 + (a_k \bmod f(b)) \\
&= a_k \bmod f(b).
\end{aligned}
$$

By assumption, all coefficients of $f(x)$ are positive. Therefore, the modular reduction by $f(b)$ leaves the coefficient a_k unchanged. Thus, we arrive at

$$a_k = \left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor \bmod f(b).$$

Which is the formula we wanted to prove. In proving the formula, we have shown that it is possible to recover all the coefficients $a_0, a_1, \ldots, a_d$ using only the values $f(b)$ and $f(f(b))$ under the given conditions. Since we can recover the coefficient a_k given its degree k, we can determine the degree of the term corresponding to the coefficient recovered. Hence, we can reconstruct the polynomial $f(x)$, with the correct degrees and coefficients, using only the evaluations $f(b)$ and $f(f(b))$. Thus, we can reconstruct the polynomial one-to-one.

In conclusion, under the given conditions, $f(x)$ can be completely determined by the evaluations $f(b)$ and $f(f(b))$ using the provided formula. □

Remark 1. *The polynomial property described in Theorem 1 appears to be underexplored in the literature. However, it has been the subject of some online discussions [5, 6] and at least one blog post [7]. Despite these mentions, the property has been treated mostly as a novelty or curiosity, and its applications have not been thoroughly examined.*

To provide an intuitive example of how Theorem 1 can be used, we prove an unconventional binomial coefficient formula as a corollary. This formula was initially described by Robinson (1952) [1] and later by Shamir (1978) [2].

Corollary 2. *Let $n, k \in \mathbb{Z} : 0 \le k \le n$. Then*

$$\binom{n}{k} = \left\lfloor \frac{(2^n + 1)^n}{2^{nk}} \right\rfloor \bmod 2^n.$$

Proof. Consider the polynomial $f(x) := (x + 1)^n \in \mathbb{Z}[x]$. The binomial theorem gives the polynomial expansion

$$f(x) = (x + 1)^n = \sum_{j=0}^{n} \binom{n}{j} x^j 1^{n-j} = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

By expanding out the inner terms of sum, we can see

$$f(x) = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n.$$

Hence, $f(x)$ is a polynomial with terms whose coefficients are the binomial coefficients for row n of Pascal's triangle.

If we evaluate at $x = 1$, we get the coefficient sum. Applying this to $f(x)$, the evaluation $f(1)$ is equal to the sum of the coefficients of the n-th row of Pascal's triangle. This sum is well-known to be equal to 2^n [8]. Carrying out the evaluation, we get

$$\begin{aligned} f(1) &= \binom{n}{0} 1^0 + \binom{n}{1} 1^1 + \cdots + \binom{n}{n-1} 1^{n-1} + \binom{n}{n} 1^n \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} \\ &= 2^n. \end{aligned}$$

Let $b = 1$, so that $f(b) = f(1) = 2^n$. By Theorem 1, for all $0 \leq k \leq n$, we can recover the coefficient $\binom{n}{k}$ using only the evaluations $f(b)$ and $f(f(b))$ by way of the formula

$$a_k = \left\lfloor \frac{f(f(b))}{f(b)^k} \right\rfloor \bmod f(b).$$

In this case, $a_k = \binom{n}{k}$. Substituting $f(b) = 2^n$ and $a_k = \binom{n}{k}$ into the formula, we get

$$\binom{n}{k} = \left\lfloor \frac{f(2^n)}{(2^n)^k} \right\rfloor \bmod 2^n.$$

Finally, by expanding $f(2^n) = (2^n + 1)^n$ and simplifying, we arrive at

$$\binom{n}{k} = \left\lfloor \frac{(2^n + 1)^n}{2^{nk}} \right\rfloor \bmod 2^n.$$

Proving the formula. □

3 Partial Sums of Binomial Coefficients

Boardman (2004) asserted that "it is well-known that there is no closed form (that is, direct formula) for the partial sum of binomial coefficients" [3]. This statement has been cited in the Wikipedia article on binomial coefficients to suggest the impossibility of a closed-form expression for these partial sums [4]. However, this interpretation appears to misrepresent Boardman's intended meaning. In his paper, Boardman references a theorem by Petkovšek, Wilf, and Zeilberger from their book $A{=}B$ (1996), which proves the non-existence of a closed-form expression for the partial sums of binomial coefficients specifically as a hypergeometric function [9]. It is more likely that Boardman was citing this result to indicate the absence of a known formula, rather than asserting the impossibility of any such formula. If indeed no closed-form expression has been previously established, the formula we present here may constitute the first of its kind.

Theorem 3. *Let $n, j \in \mathbb{Z}_{>-1}$ such that $j \leq n$. Then*

$$\sum_{k=0}^{j} \binom{n}{k} = \left\lfloor \frac{(2^n + 1)^n}{2^{n(n-j)}} \right\rfloor \bmod (2^n - 1)$$

and

$$\sum_{k=0}^{j} \binom{n}{k} = \left((2^n + 1)^n \bmod 2^{nj+1} \right) \bmod (2^n - 1).$$

Proof. First, we note that $(2^n - 1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{nk}$. Next, we divide the sum by 2^{nj} to get

$$\left\lfloor \frac{(2^n + 1)^n}{2^{n(n-j)}} \right\rfloor = \sum_{k=0}^{j} \binom{n}{k} 2^{nk}.$$

Taking this sum mod $(2^n - 1)$ is the same as replacing all instances of 2^n with 1. Thus

$$\sum_{k=0}^{j} \binom{n}{k} 2^{nk} \bmod (2^n - 1) = \sum_{k=0}^{j} \binom{n}{k} (1)^k = \sum_{k=0}^{j} \binom{n}{k}.$$

4

Due to the symmetry of the binomial coefficients in row n, we see that taking $(2^n - 1)^n \bmod 2^{nj+1}$, then mod $(2^n - 1)$, yields the same result. Consider

$$\left((2^n + 1)^n \bmod 2^{nj+1}\right) = \sum_{k=0}^{j} \binom{n}{n-k} 2^{n(n-k)} = \sum_{k=0}^{j} \binom{n}{k} 2^{nk}$$

By taking this sum mod $(2^n - 1)$, we once again obtain $\sum_{k=0}^{j} \binom{n}{k}$. $\qquad\square$

Corollary 4. *Let $n, a, b \in \mathbb{Z}_{>-1}$ such that $a < b \leq n$. Then*

$$\sum_{k=a}^{b} \binom{n}{k} = \left(\left((2^n + 1)^n \bmod 2^{nb+1}\right) - \left((2^n + 1)^n \bmod 2^{na}\right)\right) \bmod (2^n - 1)$$

Proof. The proof follows trivially from Theorem 3, since

$$\left(\left((2^n + 1)^n \bmod 2^{nb+1}\right) \bmod (2^n - 1)\right) - \left(\left((2^n + 1)^n \bmod 2^{na}\right) \bmod (2^n - 1)\right)$$
$$= \left(\left((2^n + 1)^n \bmod 2^{nb+1}\right) - \left((2^n + 1)^n \bmod 2^{na}\right)\right) \bmod (2^n - 1)$$
$$= \sum_{k=0}^{b} \binom{n}{k} - \sum_{k=0}^{a-1} \binom{n}{k}$$
$$= \sum_{k=a}^{b} \binom{n}{k}.$$

$\qquad\square$

We now provide an alternative formula for the partial sums of binomial coefficients, using results from Boardman (2004).

Theorem 5. *Let $n, j \in \mathbb{Z}_{>-1}$ such that $j \leq n$. Then*

$$\sum_{k=0}^{j} \binom{n}{k} = 1 + \left(\left\lfloor \frac{(2^n + 1)^n - 1}{2^{nj}(1 - 2^n)} \right\rfloor \bmod 2^n\right)$$

Proof. From Boardman (2004) [3], we have

$$\frac{(x+1)^n - 1}{1 - x} = \binom{n}{1}x + \left(\binom{n}{1} + \binom{n}{2}\right)x^2 + \cdots + \left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right)x^n.$$

By substituting $x = 2^n$, we get

$$\frac{(2^n + 1)^n - 1}{1 - 2^n} = \binom{n}{1}(2^n)^1 + \left(\binom{n}{1} + \binom{n}{2}\right)(2^n)^2 + \cdots + \left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right)(2^n)^n.$$

This substitution is valid, since the coefficients are at most $(2^n - 1)$.

Now, to recover the partial sum of binomial coefficients from $\binom{n}{1}$ up to $\binom{n}{j}$, we must isolate the j-th term in the sum. To do this, we can divide by 2^{nj} and then mod by 2^n (Theorem 1). This yields

$$\left\lfloor \frac{(2^n + 1)^n - 1}{2^{nj}(1 - 2^n)} \right\rfloor \bmod 2^n = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Finally, we add $\binom{n}{0} = 1$ to get the desired sum, which is

$$1 + \left(\left\lfloor \frac{(2^n + 1)^n - 1}{2^{nj}(1 - 2^n)} \right\rfloor \bmod 2^n\right) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{j} \binom{n}{j}.$$

$\qquad\square$

4 Univariate Multinomial Coefficients

Applying Theorem 1, we derive a generalized formula for calculating coefficients within the multinomial expansion of arbitrary degree univariate unit polynomials. These are polynomials of the form

$$\left(\frac{x^r - 1}{x - 1}\right)^n = (1 + x + \cdots + x^{r-1})^n.$$

The conventional approach to determine these coefficients utilizes conditional summations of multivariate multinomial coefficients, which represent the number of ways specific choices can be made to yield the term x^k [10]. The standard formula for multivariate multinomial coefficients is

$$\binom{n}{k_0, k_1, \cdots, k_{r-1}} = \frac{n!}{k_0! k_1! \cdots k_{r-1}!}.$$

In the context of our univariate polynomial, for each power of x in the expansion, the coefficient will come from all the combinations of powers that sum up to that specific power. Specifically, the coefficient of x^k in the expansion of our polynomial is [11]

$$\binom{n}{k}_{r-1} = [x^k](1 + x + \cdots + x^{r-1})^n = \sum \binom{n}{k_0, k_1, \cdots, k_{r-1}},$$

where the summation criteria are

$$k_0 + k_1 + \cdots + k_{r-1} = n,$$
$$0k_0 + 1k_1 + \cdots + (r-1)k_{r-1} = k.$$

Theorem 6. *Let $n, k, r \in \mathbb{Z}$ such that $n > 0$ and $0 \leq k \leq n(r-1)$. Then*

$$\binom{n}{k}_{r-1} = \left\lfloor \left(\frac{r^{rn} - 1}{r^{n+k} - r^k}\right)^n \right\rfloor \bmod r^n.$$

Proof. Consider the polynomial function

$$f_r(x)^n = \left(\frac{x^r - 1}{x - 1}\right)^n = (1 + x + \cdots + x^{r-1})^n.$$

In this case, it is clear that

$$f_r(1)^n = r^n.$$

Therefore, we have

$$f_r(f_r(1)^n)^n = (1 + r^n + \cdots + r^{n(r-1)})^n.$$

Observe that the inner sum is equivalent to the summation of the powers of r^n from 0 to $(r-1)$. We note the following identity from the OEIS [12]:

$$\sum_{k=0}^{n-1} n^k = \frac{n^n - 1}{n - 1}.$$

By substitution, we have

$$f_r(f_r(1)^n)^n = \left(\sum_{k=0}^{r-1} r^{nk}\right)^n = \left(\frac{r^{rn} - 1}{r^n - 1}\right)^n.$$

In Theorem 1, we showed that

$$[x^k]f(x)^n = \left\lfloor \frac{f(f(1)^n)^n}{f(1)^{nk}} \right\rfloor \bmod f(1)^n.$$

By equivalence, we have

$$\binom{n}{k}_{r-1} = [x^k]f_r(x)^n = \left\lfloor \frac{f_r(f_r(1)^n)^n}{f_r(1)^{nk}} \right\rfloor \bmod f_r(1)^n.$$

Replacing the values of $f_r(1)^n$ and $f_r(f_r(1)^n)^n$ and simplifying, we arrive at our original formula

$$\binom{n}{k}_{r-1} = [x^k]\left(\frac{x^r-1}{x-1}\right)^n = \left\lfloor \left(\frac{r^{rn}-1}{r^n-1}\right)^n r^{-nk} \right\rfloor \bmod r^n = \left\lfloor \left(\frac{r^{rn}-1}{r^{n+k}-r^k}\right)^n \right\rfloor \bmod r^n.$$

To prove why the equation holds only for $n > 0$ and $0 \leq k \leq n(r-1)$, we consider the possible cases in relation to

$$\binom{n}{k}_{r-1} = \left\lfloor \left(\frac{r^{rn}-1}{r^{n+k}-r^k}\right)^n \right\rfloor \bmod r^n = \left\lfloor \left(\frac{r^{rn}-1}{r^n-1}\right)^n r^{-nk} \right\rfloor \bmod r^n.$$

Cases:

(i) $n \leq 0$: In this case, the modulus becomes r^{-n} and the original equation does not make sense.

(ii) $k < 0$: In this case, we have $r^{-(-nk)} = r^{nk}$ and thus the original expression is always equal to 0, since $r^n \mid r^{nk}$.

(iii) $k > n(r-1)$: In this case, $r^{nk} > \left(\frac{r^{rn}-1}{r^n-1}\right)^n > r^{n^2}$, so the value of the expression is always equal to 0.

In conclusion, we have shown that the formula is valid and that it holds only for $n > 0$ and $0 \leq k \leq n(r-1)$. This completes the proof. $\qquad \square$

4.1   Partial Sums

We conclude with a formula for the partial sums of univariate multinomial coefficients, taking a similar approach as in § 3.

Corollary 7. *Let* $n, j, r \in \mathbb{Z}^+$ *such that* $n > 0$ *and* $0 \leq j \leq n(r-1)$. *Then*

$$\sum_{k=0}^{j} \binom{n}{k}_{r-1} = \left(\left(\frac{r^{rn}-1}{r^n-1}\right)^n \bmod r^{n(j+1)}\right) \bmod (r^n-1).$$

Proof. First, we note that $\left(\frac{r^{nr}-1}{r^n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}_{r-1} r^{nk}$. Reducing the sum mod $r^{n(j+1)}$, we get

$$\sum_{k=0}^{j} \binom{n}{k}_{r-1} r^{nk}.$$

Finally, reducing this sum mod (r^n-1) is the same as replacing all instances of r^n with 1. Thus

$$\sum_{k=0}^{j} \binom{n}{k}_{r-1} r^{nk} \bmod (r^n-1) = \sum_{k=0}^{j} \binom{n}{k}_{r-1} (1)^k = \sum_{k=0}^{j} \binom{n}{k}_{r-1}.$$

$\qquad \square$

References

[1] J. Robinson. Existential Definability in Arithmetic. *Transactions of the American Mathematical Society*, 72(3):437–449, 1952. ISSN 0002-9947.

[2] A. Shamir. Factoring Numbers in O(log n) Arithmetic Steps. *Information Processing Letters*, 8(1): 28–31, 1979. ISSN 0020-0190. URL https://sciencedirect.com/science/article/pii/0020019079900875.

[3] M. Boardman. The Egg-Drop Numbers. *Mathematics Magazine*, 77(5):368–372, 2004. URL https://doi.org/10.1080/0025570X.2004.11953281.

[4] Wikipedia Contributors. Binomial Coefficient, 2024. URL https://en.wikipedia.org/wiki/Binomial_coefficient. Wikipedia Article.

[5] MathOverflow Users. Application of Polynomials with Non-Negative Coefficients, 2012. URL https://mathoverflow.net/questions/91827. MathOverflow Discussion.

[6] Reddit Users. Determine a Polynomial from Just Two Inputs, 2023. URL https://www.reddit.com/r/math/comments/yx0i7r/determine_a_polynomial_from_just_two_inputs. Reddit Discussion.

[7] J. D. Cook. Polynomial Determined by Two Inputs, 2012. URL https://johndcook.com/blog/2012/03/27/polynomial-trick. Blog Post.

[8] OEIS Foundation Inc. Powers of 2 - Entry A000079 in The On-Line Encyclopedia of Integer Sequences, 2024. URL https://oeis.org/A000079.

[9] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A=B*. A K Peters/CRC Press, 1996. ISBN 978-1568810638.

[10] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation For Computer Science*. Addison-Wesley Professional, 2nd edition, 1994. ISBN 978-0201558029.

[11] R. A. Brualdi. *Introductory Combinatorics*. Pearson, 5th edition, 2017. ISBN 978-0134689616.

[12] OEIS Foundation Inc. Entry A023037 in The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A023037, 2024.