[Draft] On Kalmar Numbers and Arithmetic Terms

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Abstract

From the classes of Kalmar functions and arithmetic terms, we describe the class of Kalmar numbers, which is the subset of computable real numbers that can be represented as a limit of the quotient of two Kalmar functions. We give Kalmar numbers for many important mathematical constants and functions, such as e, π, $\sqrt[r]{n}$, $\log(n)$, $\exp(n)$, $\sin(n)$, $\cos(n)$, $\Gamma_k(q)$, $(q)_{n,k}$, $\psi^{(k)}(q)$.

DISCLAIMER: This paper is a work in progress. Many proofs and results are currently missing.

1 Kalmar Numbers

We define a **Kalmar number** as a computable real number which can be represented as a limit of the quotient of two Kalmar functions. That is, $r \in \mathbb{R}$ is a Kalmar number if there exists a limit

$$r = \lim_{n\to+\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n\to-\infty} \frac{f(n)}{g(n)} \quad \text{or} \quad r = \lim_{n\to 0} \frac{f(n)}{g(n)}$$

where $f(n), g(n)$ are Kalmar functions. This definition is due to Lorenzo Sauras-Altuzarra, who described it in a private correspondence.

2 Characterizing the Set of Kalmar Numbers

An interesting question from Mihai Prunescu is:

Question 2.1. *What is the subset of computable real numbers that are Kalmar numbers?*

We offer a partial solution to the question.

Theorem 2.1. *Let $r \in \mathbb{R}$ such that the ratios of consecutive terms for both the numerators and denominators of the n-th convergents in its generalized continued fraction representation are bounded by a tower of exponentials $2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$ with height n. Then, there exist Kalmar functions $f(n)$ and $g(n)$ such that*

$$r = \lim_{n\to\infty} \frac{f(n)}{g(n)}.$$

Proof. The generalized continued fraction representation of r is given by

$$r = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}},$$

where a_i and b_i are sequences of integers. The n-th convergents of the generalized continued fraction for r can be expressed using the recurrence relations

$$P_n = a_n P_{n-1} + b_n P_{n-2},$$
$$Q_n = a_n Q_{n-1} + b_n Q_{n-2},$$

with initial starting conditions $P_{-1} = 1$, $P_0 = a_0$, $Q_{-1} = 0$, and $Q_0 = 1$.

The sequence of convergents $\frac{P_n}{Q_n}$ of the generalized continued fraction for r will approach r as $n \to \infty$. That is,

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = r.$$

By the theorem's assumption, we have

$$\frac{P_n}{P_{n-1}} \leq 2^{2^{2^{\cdots^{n}}}} \text{ and } \frac{Q_n}{Q_{n-1}} \leq 2^{2^{2^{\cdots^{n}}}}.$$

This implies that for all $n \geq 1$,

$$\begin{aligned} P_n &\leq P_{n-1}2^{2^{2^{\cdots^{n}}}} \leq P_{n-2}2^{2^{2^{\cdots^{n-1}}}}2^{2^{2^{\cdots^{n}}}} \leq \cdots \leq \\ &P_0 2^{2^{2^1}}2^{2^{2^2}}\cdots 2^{2^{2^{\cdots^{n}}}} = P_0 2^{2^1+2^2+\cdots+2^{2^{\cdots^{n}}}} = P_0 2^{2^{2^{\cdots^{n+1}}}-2} \\ &= O(2^{2^{2^{\cdots^{n}}}}), \end{aligned}$$

and similarly for Q_n.

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus, for any $r \in \mathbb{R}$ and any $\epsilon > 0$, there exists a rational number $q \in \mathbb{Q}$ such that $|r - q| < \epsilon$.

Define $\epsilon_n = \left|r - \frac{P_n}{Q_n}\right|$. Since $\lim_{n\to\infty} \frac{P_n}{Q_n} = r$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $\epsilon_n < \frac{P_n}{Q_n}$ and $\epsilon_{n+1} < \epsilon_n$. Furthermore, since P_n and Q_n are bounded by $O(2^{2^{\cdots^{2^n}}})$ and their recursions require only elementary arithmetic operations to compute, there exists a pair of Kalmar functions $f(n)$ and $g(n)$ such that $f(n) = P_n$ and $g(n) = Q_n$. Therefore, we conclude

$$\lim_{n\to\infty} \left|r - \frac{f(n)}{g(n)}\right| = 0 \implies \lim_{n\to\infty} \frac{f(n)}{g(n)} = r.$$

□

3 Pochhammer Function

We define the integer-valued function

$$(n)_{a,b} = \prod_{k=0}^{n}(a + bk)$$

Theorem 3.1. *Let $n, a, b \in \mathbb{Z}$ such that $n > 0$. Set $\ell = \lfloor \log_b((n)_{a,b}) \rfloor + 1$, $m = b^\ell + 1$, and $c = (ab^{-1}) \bmod m$. Then*

$$(n)_{a,b} = \left(\binom{n+c}{n+1} b^{n+1}(n+1)!\right) \bmod m.$$

Further, $(n)_{a,b}$ is an arithmetic term.

Proof. From [6], for $a \equiv bc \pmod{m}$, we have

$$\prod_{k=0}^{n}(a + bk) \equiv \binom{n+c}{n+1} b^{n+1}(n+1)! \pmod{m}. \tag{1}$$

We are given $m = b^\ell + 1$, thus m and b are coprime and $b^{-1} \pmod{m}$ exists. Therefore, $c = ab^{-1} \bmod m$ is valid. Hence, we have established a congruence of $(n)_{a,b}$ modulo m.

Now, since

$$(n + 1)_{a,b} > m = b^\ell = b^{\lfloor \log_b((n)_{a,b}) \rfloor + 1} > (n)_{a,b},$$

it follows that

$$(n)_{a,b} = \left(\binom{n+c}{n+1} b^{n+1}(n+1)!\right) \bmod m.$$

Due to Prunescu and Sauras-Altuzarra [3], we have an arithmetic term for the modular inverse $b^{-1} \pmod{m}$. Arithmetic terms for the factorial function, binomial coefficients, and roots are also known [2, 5, 1, 3]. Therefore, it follows that $(n)_{a,b}$ is an arithmetic term. □

4 Gamma Function

Theorem 4.1. *The function $\Gamma_k(q)$, where $q \in \mathbb{Q}$, is a Kalmar number.*

Proof. From [4], we have

$$\Gamma_k(q) = \lim_{n\to\infty} \frac{n!k^n(nk)^{q/k-1}}{(q)_{n,k}}.$$

By Theorem 3.1, $(q)_{n,k}$ is an arithmetic term. Arithmetic terms for the factorial function and roots are also known [3, 1]. Therefore, it follows that $\Gamma_k(q)$ is a Kalmar number. □

5 Polygamma Function

From our Kalmar number for $\Gamma(q)$, we can calculate $\psi(q)$.

$$\psi^{(0)}(q) = \psi(q) = \frac{\Gamma'(q)}{\Gamma(q)}.$$

6 Useful Kalmar Numbers

Roots

$$\sqrt[n]{a} = \lim_{k\to\infty} \frac{(k^{kn}+1)^{kn+1} \bmod (k^{kn^2}-a)}{(k^{kn}+1)^{kn} \bmod (k^{kn^2}-a)} - 1. \tag{2}$$

Derivatives

$$f'(x) = \lim_{b\to 0} \frac{f(a+b)-f(a)}{b} \tag{3}$$

Exponential Function

$$e = \lim_{k\to\infty} \frac{(k+1)^k}{k^k}$$

$$e^n = \lim_{k\to\infty} \frac{(k+1)^{nk}}{k^{nk}}$$

$$\exp(in) = \lim_{k\to\infty} \left(\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1)\right)$$

Natural Logarithm

$$\log(n) = \lim_{k\to\infty} k(\sqrt[k]{n}-1).$$

Trigonometric Functions

$$\cos(n) = \lim_{k\to\infty} \left(\left(\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1)\right) \bmod x\right),$$
$$\sin(n) = \lim_{k\to\infty} \left\lfloor \frac{\left(1+\frac{xn}{k}\right)^k \bmod (x^2+1)}{x} \right\rfloor.$$

Mod One

$$a \bmod b = b\left(\frac{a}{b} \bmod 1\right) = b\left(\frac{a}{b} - \left\lfloor \frac{a}{b} \right\rfloor\right)$$

Pi

$$\sqrt{2\pi} = \lim_{n\to\infty} \frac{n!}{\sqrt{n}n^n e^{-n}}$$

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{4^n n!^2}{\sqrt{n}(2n)!}$$

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n} n!^4}{(2n)!^2(2n+1)}$$

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