Formula for Binomial Coefficients (Draft)

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Abstract. A simple formula for calculating binomial coefficients is described.

1 Formula

For $n > 0$ and $0 \leq k \leq n$:

$$\binom{n}{k} = \lfloor \frac{(1+2^n)^n}{2^{nk}} \rfloor - 2^n \lfloor \frac{(1+2^n)^n}{2^{nk+n}} \rfloor$$

2 Explanation

Theorem 2.1. If $P(x)$ is a polynomial with only non-negative integer coefficients, then it can be completely determined by the values $P(1)$ and $P(P(1))$.

Proof. If we let $q = P(1)$, then q gives the sum of the coefficients. Now think of $P(P(1)) = P(q)$ written in base q; one sees that the digits are exactly the coefficients of P. The only possible ambiguity comes if $P(q) = q * n$ for some n, but since the coefficients sum to q, one sees that $P = qx^{n-1}$ in this case. (Rupinski, A.) [1] [2]

2.2. Define $P(x)$ as the polynomial function:

$$P(x) = 1 + x$$

If n is a non-negative integer, then by the binomial theorem we have:

$$P(x)^n = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

2.3. Define $P(x)^n_k$ to represent the coefficient of the k-th degree term in the polynomial expansion of $P(x)^n$, such that:

$$P(x)^n = \sum_{k=0}^{n} P(x)^n_k x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Example

Let $n = 4$, then:

$$P(x)^4 = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + 1x^4$$

Choosing $k = 2$ gives:

$$P(x)^4_2 = \binom{4}{2} = 6$$

Since 6 is the coefficient of the x^2 term in the expansion of $P(x)^4$.

2.4. As the coefficients of the terms in $P(x)$ are precisely the binomial coefficients for the n-th row of Pascal's Triangle, we have:

$$P(x)^n_k = \binom{n}{k}$$

Remarks 2.5. The main idea behind our formula is this: because the coefficients of $P(x)^n$ are all binomial coefficients, and are thus are always non-negative integers, we can use Theorem 2.1. to recover the coefficients of $P(x)^n$ using only the values of $P(1)^n$ and $P(P(1)^n)^n$.

2.6. The value of $P(1)^n$ is simply the summation of the coefficients of $P(x)^n$. It is easy to see that:

$$P(1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k = (1+1)^n = 2^n$$

The value of $P(P(1)^n)^n$ can be determined by simple recursion to be:

$$P(P(1)^n)^n = \sum_{k=0}^{n} \binom{n}{k} (P(1)^n)^k = (1 + P(1)^n)^n$$

We have established that $P(1)^n = 2^n$. By replacement we get:

$$P(P(1)^n)^n = P(2^n)^n = \sum_{k=0}^{n} \binom{n}{k} (2^n)^k = (1 + 2^n)^n$$

2.7. The final step is to recover the coefficients of $P(x)^n$ by encoding $P(P(1)^n)^n$ in base $P(1)^n$. In this case, that means encoding $(1 + 2^n)^n$ in base 2^n.

The encoded representation of an integer n in base b, where n an b are both non-negative integers and $b > 1$, is the unique representation of n, such that:

$$n = \sum_{k=0}^{log_b(n)} b^k * (\lfloor n/b^k \rfloor \mod b)$$

From the above we can derive a simple formula to recover our polynomial $P(x)^n$:

$$P(x)^n = \sum_{k=0}^{n} (\lfloor P(P(1)^n)^n/(P(1)^n)^k \rfloor \mod P(1)^n) * x^k$$

It follows that, to recover the k-th coefficient of our polynomial $P(x)^n$, we can use:

$$P(x)^n_k = (\lfloor P(P(1)^n)^n/(P(1)^n)^k \rfloor \mod P(1)^n)$$

2.8. Putting it all together, we replace $P(1)^n$ and $P(P(1)^n)^n$ with the values we found earlier to see that:

$$P(x)^n_k = (\lfloor (1+2^n)^n/(2^n)^k \rfloor \mod 2^n) = \binom{n}{k}$$

Finally, we use the following well known identity to eliminate the reliance on modular arithmetic:

$$a \mod b = a - b\lfloor a/b \rfloor$$

To arrive at our original formula:

$$P(x)^n_k = \lfloor \frac{(1+2^n)^n}{2^{nk}} \rfloor - 2^n \lfloor \frac{(1+2^n)^n}{2^{nk+n}} \rfloor = \binom{n}{k}$$

References

[1] R. A., *Mathoverflow: application of polynomials with non-negative coefficients*, 2012.

[2] C. J., *Polynomial determined by two inputs*, 2012.