Constrained optimisation: Lagrange multipliers.

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Ex 4.1

We are going to minimise $f(x,y) = (x-1)^2 + y^2$ subject to $h(x,y) = -x + \beta y^2 \geq 0$ using necessary and sufficient conditions. $(\beta > 0)$

Necessary conditions:

$$L(x,\mu) = f(x,y) - \mu \cdot h(x,y) = (x-1)^2 + y^2 - \mu(-x+\beta y^2)$$

We need: $\nabla_x L(x,\mu) = 0$; $\mu \cdot h(x,y) = 0$ and $\mu \geq 0$:

$$\left.\begin{aligned} 2(x-1) + \mu &= 0 \\ 2y - 2\mu\beta y &= 0 \\ \mu(-x+\beta y^2) &= 0 \end{aligned}\right\}$$

- Suppose $\boxed{\mu = 0}$

Then, from the second equation

$$2y = 0 \iff y = 0 \quad \& \quad 2(x-1) = 0 \iff x = 1$$

- Suppose $\boxed{\mu \neq 0}$

Then, $[x = \beta y^2]$, from the second equation we obtain

$$2y(1-\mu\beta) = 0 \iff \begin{cases} y = 0,\ x = 0 \Rightarrow \mu = 2 \\ \mu = 1/\beta, \end{cases}$$

$$x = \frac{-\mu}{2} + 1 = -\frac{1}{2\beta} + 1 = \frac{2\beta - 1}{2\beta} \implies$$

$$\implies y = \pm\sqrt{\frac{2\beta - 1}{2\beta^2}}$$

So, the candidates are:

1) $x=1,\ y=0,\ \mu=0$

2) $x=0,\ y=0,\ \mu=2$

3) $x=\dfrac{2\beta-1}{2\beta},\ y=\sqrt{\dfrac{2\beta-1}{2\beta^2}},\ \mu=1/\beta$ (we need $\beta \geq 1/2$)

4) $x=\dfrac{2\beta-1}{2\beta},\ y=-\sqrt{\dfrac{2\beta-1}{2\beta^2}},\ \mu=1/\beta.$

Sufficient conditions

Note that the first candidate doesn't satisfy $h(x,y)\geq 0$ since $h(1,0)=-1\leq 0$. All the others do satisfy $h(x,y)\geq 0$ so we just have 3 candidates.

Let $z=(z_1,z_2)\in\mathbb{R}^2$, we want to satisfy

$$z\cdot\nabla h(x^*,y^*)=0 \iff (z_1\ z_2)\,(-1\ \ 2y\beta)^T=0 \iff$$

$$\iff z_1=-2y\beta\cancel{z_2}.$$

In the second case, $y^*=0$ so $z_1=0$.

Now, $z\,\nabla_x^2 L(x,\mu)\,z^T\geq 0$; (*)

(*) in order to get a minimum

$$\nabla_x^2 L(x,\mu)=\begin{pmatrix}2 & 0\\ 0 & 2(1-\mu\beta)\end{pmatrix}$$

So, $$(0,z_2)\begin{pmatrix}2 & 0\\ 0 & 2(1-2\beta)\end{pmatrix}\begin{pmatrix}0\\ z_2\end{pmatrix}=2z_2^2(1-2\beta)\geq 0 \iff$$

$\mu=2$

$\iff \beta\in(0,1/2]$

Regarding to the two last candidates note that $\nabla_x^2 L(x^*, \mu) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

so being $z = (z_1, z_2) \in \mathbb{R}^2$

$$z \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} z^T = 2z_1^2 \geq 0 \quad \forall z \in \mathbb{R}^2$$

In conclusion,

- If $\beta \in (0, 1/2]$, then we have a minima at $(x^*, y^*) = (0,0)$ with $f(0,0) = 1$ and $\mu = 0$.

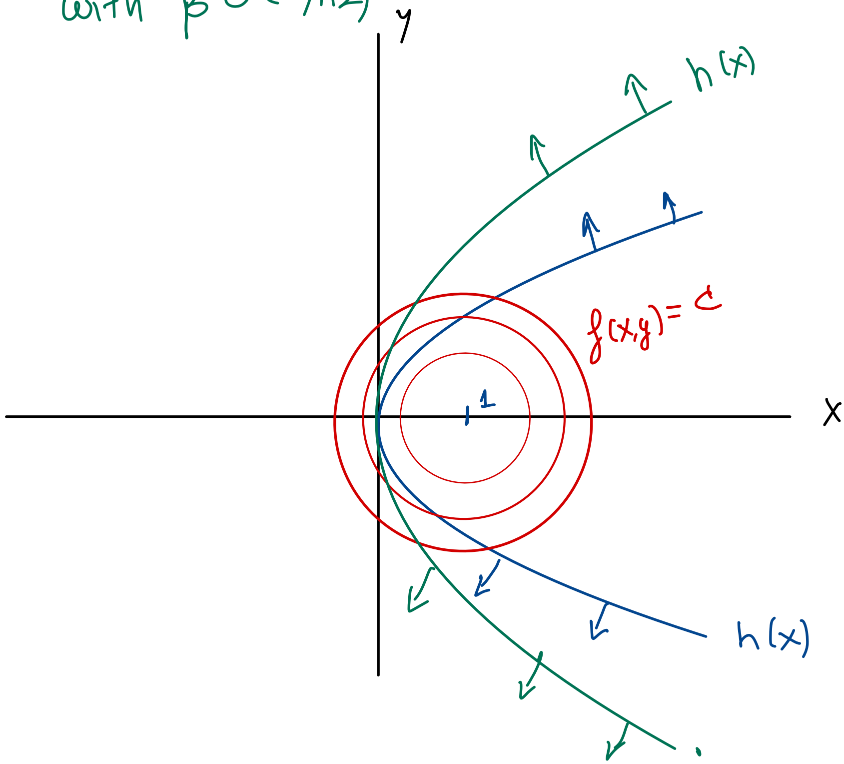
- If $\beta \geq 1/2$, then we have two minimums at $(x^*, y^*) = \left(\frac{2\beta-1}{2\beta}, \sqrt{\frac{2\beta-1}{2\beta^2}}\right)$ and $(x^*, y^*) = \left(\frac{2\beta-1}{2\beta}, -\sqrt{\frac{2\beta-1}{2\beta^2}}\right)$ with $f(x^*, y^*) = \left(\frac{2\beta-1}{2\beta} - 1\right)^2 + \frac{2\beta-1}{2\beta^2} =$

$$= \frac{1}{4\beta^2} + \frac{4\beta-2}{4\beta^2} = \frac{4\beta-1}{4\beta^2} \quad \text{and } \mu = 1/\beta.$$

Solution

$h(x)$ with $\beta \geq 1/2$

$h(x)$ with $\beta \in (0, 1/2)$



Ex 4.2.

We are going to minimise $f(x,y) = x$ subject to:

$$g(x,y) = (x-3)^2 + (y-2)^2 - 13 = 0 \quad \text{and}$$

$$h(x,y) = 16 - (x-4)^2 - y^2 \geq 0$$

using necessary and sufficient conditions.

Necessary conditions:

$$L(x,\lambda,\mu) = f(x,y) - \lambda g(x,y) - \mu \cdot h(x,y) =$$
$$= x - \lambda\left((x-3)^2 + (y-2)^2 - 13\right) - \mu\left(16 - (x-4)^2 - y^2\right)$$

We need $\nabla_x L(x,\lambda,\mu) = 0$, $\mu \cdot h(x,y) = 0$, $g(x,y) = 0$ and $\mu \geq 0$;

$$\left.\begin{array}{r} 1 - 2\lambda(x-3) + 2\mu(x-4) = 0 \\ -2\lambda(y-2) + 2\mu y = 0 \\ \mu\left(16 - (x-4)^2 - y^2\right) = 0 \\ (x-3)^2 + (y-2)^2 - 13 = 0 \\ \mu \geq 0 \end{array}\right\}$$

• Suppose $\boxed{\mu = 0}$ $\Longrightarrow$

$$\left.\begin{array}{r} 1 - 2\lambda(x-3) = 0 \\ -2\lambda(y-2) = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} x - 3 = \frac{1}{2\lambda} \Longleftrightarrow \left[x = \frac{1+6\lambda}{2\lambda}\right] \ (\lambda \neq 0) \\ y - 2 = 0 \Longleftrightarrow [y = 2] \end{array}$$

Hence, $\left(\frac{1}{2\lambda}\right)^2 + (0)^2 = 13 \Longleftrightarrow \frac{1}{13 \cdot 4} = \lambda^2 \Longleftrightarrow$

$$\Longleftrightarrow \left[\lambda = \pm \frac{1}{2\sqrt{13}}\right]$$

Which implies that

$$x_1 = \frac{1+6\left(\frac{1}{2\sqrt{13}}\right)}{2\cdot\left(\frac{1}{2\sqrt{13}}\right)} = \sqrt{13}+3 \quad \text{and} \quad x_2 = 3-\sqrt{13}$$

With this, the candidates are:

$$\left.\begin{array}{ll} 1) & x = 3+\sqrt{13},\ y=2,\ \lambda = \frac{1}{2\sqrt{13}},\ \mu = 0 \\ 2) & x = 3-\sqrt{13},\ y=2,\ \lambda = -\frac{1}{2\sqrt{13}},\ \mu=0 \end{array}\right\}$$

- Suppose now that $\boxed{\mu \neq 0}$

Then,

$$\left.\begin{array}{l} (x-4)^2 + y^2 = 16 \\ (x-3)^2 + (y-2)^2 = 13 \end{array}\right\} \Longleftrightarrow$$

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$$\Longleftrightarrow 8x - 6x + 16 - 9 - 4y - 4 = 3 \Longleftrightarrow \boxed{x = 2y}$$

$$\Longrightarrow (2y-4)^2 + y^2 = 16 \Longleftrightarrow 4y^2 - 16y + 16 + y^2 = 16 \Longleftrightarrow$$

$$\Longleftrightarrow 5y^2 - 16y = 0 \Longleftrightarrow y(5y-16) = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} y = 0 \Longrightarrow x = 0 \\ y = \frac{16}{5} \Longrightarrow x = \frac{32}{5} \end{cases}$$

1) If $x = y = 0$, $\begin{cases} 1 + 6\lambda - 8\mu = 0 \\ 4\lambda = 0 \end{cases} \Longleftrightarrow \begin{array}{l} \mu = 1/8 \\ \lambda = 0 \end{array}$

2) If $x = \frac{32}{5}$, $y = \frac{16}{5}$,

$$\begin{cases} 1 - 2\lambda\left(\frac{32}{5} - 3\right) + 2\mu\left(\frac{32}{5} - 4\right) = 0 \\ -2\lambda\left(16/5 - 2\right) + 2\mu\left(16/5\right) = 0 \end{cases} \Longrightarrow$$

$$\begin{cases} 5 - 34\lambda + 24\mu = 0 \\ -12\lambda + 32\mu = 0 \end{cases} \iff \begin{matrix} \mu = \frac{3\lambda}{8} \\ 5 - 34\lambda + 9\lambda = 0 \end{matrix}$$

$$\iff \quad \lambda = 1/5 \quad \mu = \frac{3}{40}$$

With this, all the candidates are:

1) $x = 3+\sqrt{13}$, $y = 2$, $\lambda = \frac{1}{2\sqrt{13}}$, $\mu = 0$

2) $x = 3-\sqrt{13}$, $y = 2$, $\lambda = -\frac{1}{2\sqrt{13}}$, $\mu = 0$

3) $x = 0$, $y = 0$, $\lambda = 0$, $\mu = 1/8$

4) $x = \frac{32}{5}$, $y = \frac{16}{5}$, $\lambda = 1/5$, $\mu = 3/40$

new ones.

Sufficient conditions

Note that the second candidate doesn't satisfy $h(x,y) \geq 0$, in fact, $h(3-\sqrt{13},2) = 16 - (3-\sqrt{13}-4)^2 + 0^2 \simeq -5.21 \leq 0$

Let $z = (z_1, z_2) \in \mathbb{R}^2$, we want to satisfy $z \cdot \nabla g(x^*, y^*)^T = 0$ and $z \nabla h(x^*, y^*) = 0$. and then for those z, if we want to find a minimum, we will verify if $z \cdot \nabla_x^2 L(x^*, \lambda^*, \mu^*) \cdot z^T \geq 0$.

Note that $\nabla_x^2 L(x, \lambda, \mu) = \begin{pmatrix} 2(\mu-\lambda) & 0 \\ 0 & 2(\mu-\lambda) \end{pmatrix}$

and that $z \nabla_x^2 L(x, \lambda, \mu) z^T = 2(\mu-\lambda)(z_1^2 + z_2^2)$

With this new result we don't really need to compute $z \cdot \nabla g(x^*, y^*)^T = 0$ and $z \cdot \nabla h(x^*, y^*)^T = 0$

Since $z_1^2 + z_2^2 \geq 0 \quad \forall z \in \mathbb{R}^2$.

Actually, what we need to satisfy is that $2(\mu - \lambda)(z_1^2 + z_2^2) \geq 0$, so we will just check if $\mu - \lambda \geq 0$. If not, it won't be a minimum.

1) $\mu - \lambda = 0 - \frac{1}{2\sqrt{3}} \leq 0 \Rightarrow$ We can discard it

3) $\mu - \lambda = 1/8 - 0 \geq 0 \Rightarrow$ A minimum!

4) $\mu - \lambda = 3/40 - 1/5 = \frac{-5}{40} \leq 0 \Rightarrow$ We can discard it.

So, in conclusion, there is only one minimum: $(x, y) = (0, 0)$ with $\mu = 1/8$ and $\lambda = 0$ and with $f(0,0) = 0$

Solution