Soundness of the rule $M = N \Rightarrow \lambda x.M = \lambda x.N$ follows from (ξ). The other rules are trivial. □

5.3.5. DEFINITION. A *homomorphism* between syntactical λ-algebras is a map $\varphi: \mathfrak{M}_1 \to \mathfrak{M}_2$ such that for all $M \in \Lambda(\mathfrak{M})$ one has

$$\varphi[\![M]\!]^1_\rho = [\![\varphi(M)]\!]^2_{\varphi \circ \rho}$$

where in $\varphi(M)$ the c_a are replaced by $c_{\varphi(a)}$.

5.3.6. THEOREM. *The category of syntactical λ-algebras and homomorphisms and that of λ-algebras and homomorphisms are isomorphic. Moreover syntactical λ-models correspond exactly to λ-models under this isomorphism.*

PROOF. Easy. For a syntactical λ-algebra $\mathfrak{M} = \langle X, \cdot, [\![\ ]\!]\rangle$ define $F\mathfrak{M} = \langle X, \cdot, [\![K]\!], [\![S]\!]\rangle$; for $\varphi: \mathfrak{M}_1 \to \mathfrak{M}_2$ let $F\varphi = \varphi: F\mathfrak{M}_1 \to F\mathfrak{M}_2$. Then one has $[\![M]\!]^{F\mathfrak{M}}_\rho = [\![M]\!]^{\mathfrak{M}}_\rho$ for $M \in \Lambda(\mathfrak{M})$. Conversely for a λ-algebra $\mathfrak{A} = \langle X, \cdot, k, s\rangle$ define $G\mathfrak{A} = \langle X, \cdot, [\![\ ]\!]^{\mathfrak{A}}\rangle$ and $G\varphi = \varphi$ as above. Then F, with inverse G, is the required isomorphism. □

5.3.7. REMARK. In view of theorem 5.3.6 we say that $\mathfrak{M} = (X, \cdot, [\![\ ]\!])$ is a λ-algebra (λ-model) if $\mathfrak{M}$ is a syntactical λ-algebra (λ-model).

5.3.8. CONVENTION. When working inside a λ-algebra $\mathfrak{M}$, we write equations valid in $\mathfrak{M}$ informally, e.g. for $a \in \mathfrak{M}$ one writes

$$(\lambda x.xx)a = aa$$

rather than the formal $[\![(\lambda x.xx)y]\!]_{\rho(y:=a)} = [\![yy]\!]_{\rho(y:=a)}$ or $[\![\lambda x.xx]\!]a = aa$.

5.4. Models in concrete cartesian closed categories

In this section the framework will be explained in which Scott constructed his non-syntactical λ-models. We will use the category of cpo's. But the method works for arbitrary concrete cartesian closed categories.

Recall that if D is a cpo, then $[D \to D]$ is the set of continuous maps considered as cpo by pointwise ordering.

5.4.1. DEFINITION. A cpo D is called *reflexive* if $[D \to D]$ is a retract of D, i.e. there are continuous maps

$$F: D \to [D \to D], \qquad G: [D \to D] \to D$$

such that $F \circ G = \mathrm{id}_{[D \to D]}$.

It will be shown that every reflexive cpo defines in a natural way a λ-model.

5.4.2. DEFINITION. Let D be a reflexive cpo via the maps F, G.

(i) For $x, y \in D$ define

$$x.y = F(x)(y).$$

(ii) Let ρ be a valuation in D. Define the interpretation $[\![\ ]\!]_\rho : \Lambda \to D$ by induction as follows.

$$[\![x]\!]_\rho = \rho(x), \qquad [\![c_a]\!]_\rho = a,$$

$$[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho,$$

$$[\![\lambda x.M]\!]_\rho = G(\lambda\!\!\!\lambda d.[\![M]\!]_{\rho(x:=d)}).$$

5.4.3. LEMMA. *$\lambda\!\!\!\lambda d.[\![M]\!]_{\rho(x:=d)}$ is continuous; hence $[\![\lambda x.M]\!]_\rho$ is well-defined.*

PROOF. By induction on M one shows that $[\![M]\!]_{\rho(x:=d)}$ depends for all ρ continuously on d. The only nontrivial case is $M \equiv \lambda y.P$. Then

$$\begin{aligned}[\![\lambda y.P]\!]_{\rho(x:=d)} &= G(\lambda\!\!\!\lambda e.[\![D]\!]_{\rho(x:=d)(y:=e)}) \\ &= G(\lambda\!\!\!\lambda e.f(d,e)), \qquad \text{say} \\ &= g(d), \qquad \text{say}.\end{aligned}$$

By the induction hypothesis f is continuous in d and e separately, hence by lemma 1.2.12 continuous. Therefore, by proposition 1.2.14(i) and the continuity of G, the map $g = G \circ \hat{f}$ is continuous. □

5.4.4. THEOREM. *Let D be a reflexive cpo via F, G and let $\mathfrak{M} = (D, \cdot, [\![\ ]\!])$. Then*

(i) *$\mathfrak{M}$ is a λ-model.*

(ii) *The functions representable are exactly the continuous functions.*

(iii) *$\mathfrak{M}$ is extensional iff $G \circ F = \mathrm{id}_D$, i.e. $G = F^{-1}$ and $D \cong [D \to D]$ via F, G.*

PROOF. (i) We verify the conditions in definition 5.3.1. (1), (2) and (3) are trivial. As to (4)

$$\begin{aligned}[\![\lambda x.P]\!]_\rho.a &= G(\lambda\!\!\!\lambda d.[\![P]\!]_{\rho(x:=d)}).a \\ &= F(G(\lambda\!\!\!\lambda d.[\![P]\!]_{\rho(x:=d)}))(a) \\ &= (\lambda\!\!\!\lambda d.[\![P]\!]_{\rho(x:=d)})(a) = [\![P]\!]_{\rho(x:=a)}\end{aligned}$$

Condition (5) follows by an easy induction on M.

Therefore $\mathfrak{M}$ is a syntactical applicative structure. Moreover $\mathfrak{M}$ satisfies (ξ):

$$\begin{aligned}
\forall d\ [\![M]\!]_{\rho(x:=d)} = [\![N]\!]_{\rho(x:=d)} &\Rightarrow \lambda\!\!\lambda d.[\![M]\!]_{\rho(x:=d)} = \lambda\!\!\lambda d.[\![N]\!]_{\rho(x:=d)} \\
&\Rightarrow G\big(\lambda\!\!\lambda d.[\![M]\!]_{\rho(x:=d)}\big) = G\big(\lambda\!\!\lambda d.[\![N]\!]_{\rho(x:=d)}\big) \\
&\Rightarrow [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho.
\end{aligned}$$

It follows that $\mathfrak{M}$ is a λ-model; see remark 5.3.7.

(ii) Application $\cdot$ is continuous, since F is; therefore all representable functions are continuous. Conversely, a continuous $f: D \to D$ is represented by $G(f)$:

$$G(f)a = F(G(f))(a) = f(a).$$

In general, a continuous $f: D^n \to D$ is represented by

$$\lambda^G d_1 \cdots \lambda^G d_n . f(d_1, \ldots, d_n)$$

where

$$\lambda^G d. \cdots = G(\lambda\!\!\lambda d. \cdots).$$

(iii) If $G \circ F = \mathrm{id}_D$, then

$$\begin{aligned}
\forall e\ de = d'e &\Rightarrow \forall e F(d)(e) = F(d')(e) \\
&\Rightarrow F(d) = F(d') \\
&\Rightarrow d = d', \qquad \text{by applying } G.
\end{aligned}$$

Therefore $\mathfrak{M}$ is extensional.

Conversely, suppose $\mathfrak{M}$ is extensional. Let $d \in D$ and $d' = G(F(d))$. Then for all $e \in D$

$$d'e = F(d')(e) = F(G(F(d)))(e) = F(d)(e) = de.$$

Hence $d' = d$ i.e. $G \circ F = \mathrm{id}_D$. $\square$

To give an idea of how a reflexive cpo can be defined, we will describe the models D_A introduced by Engeler [1981] as a simplification of the graph model $P\omega$ introduced in § 18.1.

5.4.5. DEFINITION. Let A be a set.

(i) $B \supseteq A$ is the least set such that

$$\beta \subseteq B,\ \beta \text{ finite and } b \in B \ \Rightarrow\ (\beta, b) \in B.$$

(Assume that A does not contain such pairs).

(ii) $D_A = P(B)$, the powerset of B partially ordered by inclusion. This is a cpo (even an algebraic lattice).

(iii) For $x, y \in D_A$ and $f \in [D_A \to D_A]$ define

$$x \cdot y = \{ b \in B | \exists \beta \subseteq y (\beta, b) \in x \},$$

$$\lambda^G x.f(x) = \{ (\beta, b) \in B | \beta \text{ finite } \subseteq B \text{ and } b \in f(\beta) \}.$$

5.4.6. THEOREM. *D_A becomes a reflexive cpo by defining $F(x) = \lambda\!\!\!\lambda y.xy$, $G(f) = \lambda^G x.f(x)$. Therefore D_A defines a λ-model.*

PROOF. The continuity of F, G follows easily from propositions 1.2.24 and 1.2.31(i).

$$\begin{aligned} F \circ G(f) &= F(\{ (\beta, b) | b \in f(\beta) \}) \\ &= \lambda\!\!\!\lambda y.\{ b | \exists \beta \subseteq y\, b \in f(\beta) \} \\ &= \lambda\!\!\!\lambda y.\bigcup \{ f(\beta) | \beta \subseteq y \} \\ &= \lambda\!\!\!\lambda y.f(y), \qquad \text{by continuity of } f, \\ &= f. \quad \Box \end{aligned}$$

See exercises 5.7.7, 18.5.29 and 18.4.31 for more information on D_A.

5.5. Models in arbitrary cartesian closed categories

In this section it will be shown that in arbitrary cartesian closed categories reflexive objects give rise to λ-algebras and to all of them. The λ-models are then those λ-algebras that come from categories "with enough points". The method is due to Koymans [1982] and is based on work of Scott. In exercise 5.8.9 a categorial description of combinatory algebras is given.

5.5.1. DEFINITION. Let $\mathbb{C}$ be a category. The identity map on an object $A \in \mathbb{C}$ is denoted by id_A.

(i) $\mathbb{C}$ is *a cartesian closed category* (ccc) iff

(1) $\mathbb{C}$ has a *terminal object* T such that for every object $A \in \mathbb{C}$ there exists a unique map $!_A : A \to T$.