PROOF. (i) Trivial.
(ii) By corollary 5.2.23.
(iii) $\mathfrak{M} \models \omega\text{-rule} \Rightarrow \mathfrak{M}^0 \models \omega$
$\Rightarrow \mathfrak{M}^0 \models \omega^0$
$\Rightarrow \mathfrak{M} \models \omega^0\text{-rule}$
$\Rightarrow \mathfrak{M} \models \omega$-rule by the proof of theorem 4.1.15(i). □

5.2.22. PROPOSITION. *Let $\mathfrak{T}$ be a λ-theory and $\boldsymbol{R}$ one of the rules. Then*

$$\mathfrak{T} \vdash \boldsymbol{R} \Leftrightarrow \mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}\textit{-rule}$$

$$\Leftrightarrow \mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}\text{-ax}.$$

PROOF. Let $\boldsymbol{R} = R_1 \Rightarrow R_2$ and suppose $\mathfrak{T} \models \boldsymbol{R}$. Then

$$\mathfrak{M}(\mathfrak{T}) \models R_1 \Rightarrow \mathfrak{T} \vdash R_1, \text{ by theorem 5.2.12(ii)},$$

$$\Rightarrow \mathfrak{T} \vdash R_2, \text{ by assumption},$$

$$\Rightarrow \mathfrak{M}(\mathfrak{T}) \models R_2.$$

Therefore $\mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}$-rule. Similarly one shows $\mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}\text{-rule} \Rightarrow \mathfrak{T} \vdash \boldsymbol{R}$. To show that $\mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}$-rule iff $\mathfrak{M}(\mathfrak{T}) \models \boldsymbol{R}$-ax is similar to the proof of fact 5.2.20(ii). □

5.2.23. COROLLARY. (i) *If $\mathfrak{T}$ is an extensional λ-theory, then $\mathfrak{M}(\mathfrak{T})$ is an extensional λ-model.*
(ii) *If $\mathfrak{T} \vdash \omega$, then $\mathfrak{M}^0(\mathfrak{T})$ is an extensional λ-model.*

PROOF. By propositions 5.2.21 and 5.2.22. □

5.3. Syntactical models

In this section a syntactical description of the λ-algebras and λ-models will be given, which is equivalent to the first order description in § 5.2. For some models, in particular the filter model of Barendregt et al. [1983], this syntactical description is more convenient than the first order one. The method is due to Hindley and Longo [1980].

5.3.1. DEFINITION. Let $\mathfrak{M} = \langle X, \cdot \rangle$ be an applicative structure.
(i) $\mathrm{Val}(\mathfrak{M})$ is the set of valuations in $\mathfrak{M}$.
(ii) A *syntactical interpretation* in $\mathfrak{M}$ is a map $I: \Lambda(\mathfrak{M}) \times \mathrm{Val}(\mathfrak{M}) \to X$ satisfying the following conditions; $I(M, \rho)$ is written as $[\![M]\!]_\rho$.

(1) $[\![x]\!]_\rho = \rho(x)$,
(2) $[\![c_a]\!]_\rho = a$,
(3) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \cdot [\![Q]\!]_\rho$,
(4) $[\![\lambda x.P]\!]_\rho \cdot a = [\![P]\!]_{\rho(x:=a)}$,
(5) $\rho \upharpoonright \mathrm{FV}(M) = \rho' \upharpoonright \mathrm{FV}(M) \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$.

Note that by the variable convention, (4) implies that for $y \notin \mathrm{FV}(M(x))$ one has

(4′) $[\![M(x)]\!]_{\rho(x:=a)} = [\![\lambda x.M(x)]\!]_\rho a$
$= [\![\lambda y.M(y)]\!]_\rho a = [\![M(y)]\!]_{\rho(y:=a)}$.

(iii) A *syntactical applicative* structure is of the form $\mathfrak{M} = \langle X, \cdot, [\![\ ]\!] \rangle$ where $[\![\ ]\!]$ is a syntactical interpretation in $\mathfrak{M}$.

5.3.2. Definition. Let $\mathfrak{M}$ be a syntactical applicative structure.

(i) The notion of satisfaction in $\mathfrak{M}$ is defined as usual:

$$\mathfrak{M}, \rho \models M = N \;\Leftrightarrow\; [\![M]\!]_\rho = [\![N]\!]_\rho$$

$$\mathfrak{M} \models M = N \;\Leftrightarrow\; \forall \rho\ \mathfrak{M}, \rho \models M = N$$

and this is extended to arbitrary first order formulas over the λ-calculus.

(ii) $\mathfrak{M}$ is a *syntactical λ-algebra* if

$$\lambda \vdash M = N \;\Rightarrow\; \mathfrak{M} \models M = N.$$

(iii) $\mathfrak{M}$ is a *syntactical λ-model* if

$$(\xi) \qquad \mathfrak{M} \models \forall x (M = N) \to \lambda x.M = \lambda x.N,$$

i.e.

$$\forall a\ [\![M]\!]_{\rho(x:=a)} = [\![N]\!]_{\rho(x:=a)} \;\Rightarrow\; [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho.$$

5.3.3. Lemma. *Let $\mathfrak{M}$ be a syntactical λ-model. Consider the statement*

$$\varphi(M, N) \equiv \forall \rho [\![M[x := N]]\!]_\rho = [\![M]\!]_{\rho(x:=[\![N]\!]_\rho)}.$$

Then for $M, N \in \Lambda(\mathfrak{M})$

(i) $z \notin \mathrm{FV}(M) \;\Rightarrow\; \varphi(M, z)$;
(ii) $\varphi(M, N) \;\Rightarrow\; \varphi(\lambda y.M, N)$;
(iii) $\varphi(M, N)$.

Proof. (i) Write $M \equiv M(x)$. Then

$$[\![M(z)]\!]_\rho = [\![M(z)]\!]_{\rho(z:=\rho(z))} = [\![M(x)]\!]_{\rho(x:=\rho(z))} \qquad \text{by (4′)}.$$

(ii) First assume $x \notin \mathrm{FV}(N)$. By the variable convention $y \not\equiv x$, $y \notin \mathrm{FV}(N)$. Then for $\rho^* = \rho(x := [\![N]\!]_\rho)$ and arbitrary $a \in \mathfrak{M}$

$$[\![M[x := N]]\!]_{\rho^*(y:=a)} = [\![M[x := N]]\!]_{\rho(y:=a)}$$

$$= [\![M]\!]_{\rho(y:=a)(x:=[\![N]\!]_\rho)}, \qquad \text{since } \varphi(M, N),$$

$$= [\![M]\!]_{\rho^*(y:=a)};$$

(note that $[\![N]\!]_\rho = [\![N]\!]_{\rho(y:=a)}$). Therefore by (ξ)

$$[\![\lambda y.M[x := N]]\!]_{\rho^*} = [\![\lambda y.M]\!]_{\rho^*}$$

and hence

$$[\![\lambda y.M[x := N]]\!]_\rho = [\![\lambda y.M[x := N]]\!]_{\rho^*} = [\![\lambda y.M]\!]_{\rho(x:=[\![N]\!]_\rho)}.$$

If $x \in \mathrm{FV}(N)$, then let z be a fresh variable. We have for $\tilde{M} \equiv \lambda y.M$

$$[\![\tilde{M}[x := N]]\!]_\rho = [\![\tilde{M}[x := z][z := N]]\!]_\rho$$

$$= [\![\tilde{M}[x := z]]\!]_{\rho(z:=[\![N]\!]_\rho)}$$

$$= [\![\tilde{M}]\!]_{\rho(z:=[\![N]\!]_\rho)(x:=[\![N]\!]_\rho)}, \qquad \text{by (i)},$$

$$= [\![\tilde{M}]\!]_{\rho(x:=[\![N]\!]_\rho)}.$$

(iii) Now $\varphi(M, N)$ follows by a simple induction on the structure of M □

5.3.4. **Theorem.** *Let $\mathfrak{M}$ be a syntactical λ-model. Then*

$$\lambda \vdash M = N \;\Rightarrow\; \mathfrak{M} \vDash M = N,$$

i.e. $\mathfrak{M}$ is a syntactical λ-algebra,

Proof. By induction on the length of proof.

The axiom $(\lambda x.M)N = M[x := N]$ is sound:

$$[\![(\lambda x.M)N]\!]_\rho = [\![\lambda x.M]\!]_\rho [\![N]\!]_\rho, \qquad \text{by (3)},$$

$$= [\![M]\!]_{\rho(x:=[\![N]\!]_\rho)}, \qquad \text{by (4)},$$

$$= [\![M[x := N]]\!]_\rho, \qquad \text{by lemma 5.3.3(iii)}.$$

Soundness of the rule $M = N \Rightarrow \lambda x.M = \lambda x.N$ follows from (ξ). The other rules are trivial. $\square$

5.3.5. DEFINITION. A *homomorphism* between syntactical λ-algebras is a map $\varphi: \mathfrak{M}_1 \to \mathfrak{M}_2$ such that for all $M \in \Lambda(\mathfrak{M})$ one has

$$\varphi[\![M]\!]^1_\rho = [\![\varphi(M)]\!]^2_{\varphi \circ \rho}$$

where in $\varphi(M)$ the c_a are replaced by $c_{\varphi(a)}$.

5.3.6. THEOREM. *The category of syntactical λ-algebras and homomorphisms and that of λ-algebras and homomorphisms are isomorphic. Moreover syntactical λ-models correspond exactly to λ-models under this isomorphism.*

PROOF. Easy. For a syntactical λ-algebra $\mathfrak{M} = \langle X, \cdot, [\![\ ]\!]\rangle$ define $F\mathfrak{M} = \langle X, \cdot, [\![K]\!], [\![S]\!]\rangle$; for $\varphi: \mathfrak{M}_1 \to \mathfrak{M}_2$ let $F\varphi = \varphi: F\mathfrak{M}_1 \to F\mathfrak{M}_2$. Then one has $[\![M]\!]^{F\mathfrak{M}}_\rho = [\![M]\!]^{\mathfrak{M}}_\rho$ for $M \in \Lambda(\mathfrak{M})$. Conversely for a λ-algebra $\mathfrak{A} = \langle X, \cdot, k, s\rangle$ define $G\mathfrak{A} = \langle X, \cdot, [\![\ ]\!]^{\mathfrak{A}}\rangle$ and $G\varphi = \varphi$ as above. Then F, with inverse G, is the required isomorphism. $\square$

5.3.7. REMARK. In view of theorem 5.3.6 we say that $\mathfrak{M} = (X, \cdot, [\![\ ]\!])$ is a λ-algebra (λ-model) if $\mathfrak{M}$ is a syntactical λ-algebra (λ-model).

5.3.8. CONVENTION. When working inside a λ-algebra $\mathfrak{M}$, we write equations valid in $\mathfrak{M}$ informally, e.g. for $a \in \mathfrak{M}$ one writes

$$(\lambda x.xx)a = aa$$

rather than the formal $[\![(\lambda x.xx)y]\!]_{\rho(y:=a)} = [\![yy]\!]_{\rho(y:=a)}$ or $[\![\lambda x.xx]\!]a = aa$.

5.4. Models in concrete cartesian closed categories

In this section the framework will be explained in which Scott constructed his non-syntactical λ-models. We will use the category of cpo's. But the method works for arbitrary concrete cartesian closed categories.

Recall that if D is a cpo, then $[D \to D]$ is the set of continuous maps considered as cpo by pointwise ordering.

5.4.1. DEFINITION. A cpo D is called *reflexive* if $[D \to D]$ is a retract of D, i.e. there are continuous maps

$$F: D \to [D \to D], \qquad G: [D \to D] \to D$$

such that $F \circ G = \mathrm{id}_{[D \to D]}$.