Lecture Notes on Denotational Semantics

Jeremy G. Siek

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Contents

Figure 1: Language of binary arithmetic

Syntax

$$
\begin{array}{lrl}
\text{digit} & d ::= & 0 \mid 1 \\
\text{binary numeral} & b^n \in Bin^n ::= & \epsilon^n \mid (b^n d)^{n+1} \\
\text{expression} & e \in \mathbb{E} ::= & b^{64} \mid e + e \mid e \times e
\end{array}
$$

Semantics

$$
\begin{array}{rl}
 & E : \mathbb{E} \rightharpoonup \mathbb{N}_{64} \\
 & E(b) = N(b) \\
N : Bin^{64} \to \mathbb{N}_{64} & E(e_1 + e_2) = \begin{cases} n_1 + n_2 & \text{if } n_1 + n_2 < 2^{64} \\ \text{undefined} & \text{otherwise} \end{cases} \\
N(\epsilon) = 0 & \text{where } n_1 = E(e_1), n_2 = E(e_2) \\
N(bd) = 2N(b) + d & E(e_1 \times e_2) = \begin{cases} n_1 n_2 & \text{if } n_1 n_2 < 2^{64} \\ \text{undefined} & \text{otherwise} \end{cases} \\
 & \text{where } n_1 = E(e_1), n_2 = E(e_2)
\end{array}
$$

1 *Binary Arithmetic*

Reading: Schmidt [1986] Chapter 4
Exercises: 4.2, 4.6

Borrowing and combining elements from Chapter 4 of Schmidt [1986], we consider the language of binary arithmetic specified in Figure 1. The Syntax defines the form of the programs and the Semantics specifies the behavior of running the program. In this case, the behavior is simply to output a number (in decimal). In our grammar for binary numerals, ϵ represents the empty string of digits and bd is a number whose least-significant digit is d and whose more-significant digits are given by b. (This is the reverse of Schmidt [1986] but the same direction as Stoy [1977]). We place superscripts on our binary numerals to enfore a bound on their maximum length, restricting them 64 bits. We ommit the superscripts when their length is not of interest.

The main purpose of a semantics is communicate in a precise way with other people, primarily language implementers and programmers. Thus, it is incredibly important for the semantics to be written in a way that will make it most easily understood, while still being completely precise. Here we have chosen to give the semantics of binary arithmetic in terms of decimal numbers because people are generally much more familiar with decimal numbers. We write $\mathbb{N}_{64}$ for the set of numbers from 0 to $2^{64} - 1$.

When writing down a semantics, one is often tempted to consider the efficiency of a semantics, as if it were an implementation. Indeed,

it would be straightforward to transcribe the definitions of E and N in Figure 1 into your favorite programming language and thereby obtain an interpreter. All other things being equal, it is fine to prefer a semantics that is suggestive of an implementation, but one should prioritize ease of understanding first. As a result, some semantics that we study may be more declarative in nature. This is not to say that one should not consider the efficiency of implementations when designing a language. Indeed, the semantics should be co-designed with implementations to avoid accidental designs that preclude the desired level of efficiency. Thus, a recurring theme of these notes will be to consider implementations of languages alongside their semantics.

Figure 2 presents an interpreter for binary arithmetic. This interpreter, in a way reminiscent of real computers, operates on the binary numbers directly. The auxiliary functions *add* and *mult* implement the algorithms you learned in grade school, but for binary instead of decimals.

Exercise 1 Prove that $N(bin^2(n)) = n$ for any $n < 4$.

Exercise 2 Prove that $N(add(b_1, b_2, c)) = N(b_1) + N(b_2) + c$.

Exercise 3 Prove that $N(mult(b_1, b_2)) = N(b_1)N(b_2)$.

Exercise 4 Prove that the interpreter is correct. That is

$$N(I(e)) = E(e)$$

2 *Loops and Mutable Variables*

Reading: Schmidt [1986] Chapter 5
Exercises: 5.4, 5.5 a, 5.9

Figure 3 defines a language with mutable variables, a variant of the IMP [Plotkin, 1983, Winskel, 1993, Amadio and Pierre-Louis, 1998] and WHILE [Hoare, 1969] languages that often appear in textbooks on programming languages. As a bonus, we include the `while` loop, even though Schmidt [1986] does not cover loops until Chapter 6. The reason for his delayed treatment of `while` loops is that their semantics is typically expressed in terms of fixed points of continuous functions, which takes some time to explain. However, it turns out that the semantics of `while` loops can be defined more simply.

2.1 *Semantics*

To give meaning to mutable variables, we use a *Store* which is partial function from variables to numbers.

$$Store = \mathbb{X} \rightharpoonup \mathbb{N}_{64}$$

Figure 2: Binary arithmetic interpreter.

Interpreter

$$
\begin{aligned}
I : \mathbb{E} &\rightharpoonup Bin^{64} \\
I(b) &= b \\
I(e_1 + e_2) &= add(I(e_1), I(e_2), 0) \\
I(e_1 \times e_2) &= mult(I(e_1), I(e_2))
\end{aligned}
$$

Auxiliary Functions

$$
bin^2(n) = \begin{cases} 00 & \text{if } n = 0 \\ 01 & \text{if } n = 1 \\ 10 & \text{if } n = 2 \\ 11 & \text{if } n = 3 \end{cases}
$$

$$
add(\epsilon, \epsilon, c) = \begin{cases} \epsilon & \text{if } c = 0 \\ \epsilon 1 & \text{if } c = 1 \end{cases}
$$

$$
\begin{aligned}
add(b_1 d_1, b_2 d_2, c) &= b_3\, d_3 \\
&\quad \text{if } bin^2(d_1 + d_2 + c) = c' d_3, \\
&\quad\quad add(b_1, b_2, c') = b_3^n, \text{ and } n < 64 \\
add(b_1 d_1, \epsilon, c) &= add(b_1 d_1, \epsilon 0, c) \\
add(\epsilon, b_2 d_2, c) &= add(\epsilon 0, b_2 d_2, c)
\end{aligned}
$$

$$
\begin{aligned}
mult(b_1, \epsilon) &= \epsilon \\
mult(b_1, b_2 0) &= b_3 0 \text{ if } mult(b_1, b_2) = b_3^n \text{ and } n < 64 \\
mult(b_1, b_2 1) &= add(b_1, b_3 0, 0) \\
&\quad \text{if } mult(b_1, b_2) = b_3^n \text{ and } n < 64
\end{aligned}
$$

Syntax

$$
\begin{array}{lll}
\text{variables} & x \in \mathbb{X} & \\
\text{expressions} & e \in \mathbb{E} ::= \ldots \mid x & \\
\text{conditions} & b \in \mathit{Cnd} ::= \mathtt{true} \mid \mathtt{false} \mid e = e \mid & \\
 & \quad \neg b \mid b \vee b \mid b \wedge b & \\
\text{commands} & c \in \mathit{Cmd} ::= \mathtt{skip} \mid x := e \mid c\,;c \mid & \\
 & \quad \mathtt{if}\ b\ \mathtt{then}\ c\ \mathtt{else}\ c \mid & \\
 & \quad \mathtt{while}\ b\ \mathtt{do}\ c &
\end{array}
$$

$$\mathit{loop} : (\mathit{Store} \to \mathbb{B}) \to \mathcal{P}(\mathit{Store} \times \mathit{Store}) \to \mathcal{P}(\mathit{Store} \times \mathit{Store})$$

$$\frac{m_b\, s = \mathtt{false}}{(s,s) \in \mathit{loop}(m_b, m_c)}$$

$$\frac{m_b\, s_1 = \mathtt{true} \quad (s_1,s_2) \in m_c \quad (s_2,s_3) \in \mathit{loop}(m_b, m_c)}{(s_1,s_3) \in \mathit{loop}(m_b, m_c)}$$

Semantics

$$P : \mathit{Cmd} \to \mathcal{P}(\mathbb{N}_{64} \times \mathbb{N}_{64})$$
$$P\, c = \{(n, s'\,\mathtt{Z}) \mid (\{\mathtt{A} \mapsto n\}, s') \in C\, c\}$$
$$E : \mathbb{E} \to \mathit{Store} \to \mathbb{N}_{64}$$
$$E\, x\, s = s\, x$$
$$\vdots$$
$$B : \mathit{Cnd} \to \mathit{Store} \to \mathbb{B}$$
$$B\,(e_1 = e_2)\, s = (E\, e_1\, s) = (E\, e_2\, s)$$
$$\vdots$$
$$C : \mathit{Cmd} \to \mathcal{P}(\mathit{Store} \times \mathit{Store})$$
$$C\,\mathtt{skip} = \mathit{id}$$
$$C(x := e) = \{(s, [x \mapsto E\, e\, s]s) \mid s \in \mathit{Store}\}$$
$$C(c_1\,;c_2) = C\, c_2 \circ C\, c_1$$
$$C\begin{pmatrix}\mathtt{if}\ b\ \mathtt{then}\ c_1 \\ \mathtt{else}\ c_2\end{pmatrix} = \left\{ (s,s') \;\middle|\; \begin{array}{l} \mathit{if}\ B\, b\, s\ \mathit{then}\ (s,s') \in C\, c_1 \\ \mathit{else}\ (s,s') \in C\, c_2 \end{array} \right\}$$
$$C(\mathtt{while}\ b\ \mathtt{do}\ c) = \mathit{loop}(B\, b, C\, c)$$

Figure 3: An Imperative Language: IMP

We write $[x \mapsto n]s$ for removing the entry for x is s (if there is one) and then adding the entry $x \mapsto n$, that is, $\{x \mapsto n\} \cup (s|_{\mathrm{dom}(s)-\{x\}})$.

The syntax of expressions is extended to include variables, so the meaning of expressions must be parameterized on the store. The meaning of a variable x is the associated number in the store s. A program takes a number as input and it may produce a number or it might diverge. Traditional denotational semantics model this behavior with a partial function ($\mathbb{N}_{64} \rightharpoonup \mathbb{N}_{64}$). Here we shall use the alternate, but equivalent, approach of using a relation, a subset of $\mathbb{N}_{64} \times \mathbb{N}_{64}$. A program is a command and the meaning of commands is given by the semantic function C, which maps a command to a relation on stores, that is, to a subset of $\mathit{Store} \times \mathit{Store}$. The meaning of a program is given by the function P, which initializes the store with the input number in variable `A`. When the program completes, the output is obtained from variable `Z`.

We define the `while` loop using an auxiliary relation named *loop*, which we define inductively in Figure 3. Its two parameters are for the meaning of the condition b and the body c of the loop. If the meaning of the condition m_b is `false`, then the loop is already finished so the starting and finishing stores are the same. If the condition m_b is `true`, then the loop executes the body, relating the starting

$\boxed{k, s \longrightarrow k', s'}$

$$
\begin{array}{rll}
\texttt{skip} \,;\, k,\ s & \longrightarrow k,\ s & \\
(x := e) \,;\, k,\ s & \longrightarrow k,\ [x \mapsto E(e)(s)]s & \\
(c_1 \,;\, c_2) \,;\, k,\ s & \longrightarrow c_1 \,;\, (c_2 \,;\, k),\ s & \\
(\texttt{if}\, b\, \texttt{then}\, c_1\, \texttt{else}\, c_2) \,;\, k,\ s & \longrightarrow c_1 \,;\, k,\ s & \text{if } B\, b\, s = \texttt{true} \\
(\texttt{if}\, b\, \texttt{then}\, c_1\, \texttt{else}\, c_2) \,;\, k,\ s & \longrightarrow c_2 \,;\, k,\ s & \text{if } B\, b\, s = \texttt{false} \\
(\texttt{while}\, b\, \texttt{do}\, c) \,;\, k,\ s & \longrightarrow c \,;\, ((\texttt{while}\, b\, \texttt{do}\, c) \,;\, k),\ s & \text{if } B\, b\, s = \texttt{true} \\
(\texttt{while}\, b\, \texttt{do}\, c) \,;\, k,\ s & \longrightarrow k,\ s & \text{if } B\, b\, s = \texttt{false}
\end{array}
$$

$$
\mathit{eval}(c) = \{(n, n') \mid (c \,;\, \texttt{skip}), \{A \mapsto n\} \longrightarrow^* \texttt{skip}, s' \text{ and } n' = s'(Z)\}
$$

Figure 4: Abstract Machine for IMP

store s_1 to an intermediate store s_2, and then the loop continues, relating s_2 to the finishing store s_3.

2.2 *Implementation*

We define an implementation of the imperative language, in terms of an abstract machine, in Figure 4. The machine executes one command at a time, similar to how a debugger such as `gdb` can be used to view the execution of a C program one statement at a time. Each command causes the machine to transition from one state to the next, where a state is represented by a control component and the store. The control component is the sequence of commands that need to be executed, which is convenient to represent as a command of the following form.

$$
\text{control} \quad k \quad ::= \quad \texttt{skip} \mid c \,;\, k
$$

The partial function *eval*, also defined in Figure 4 in the main entry point for the abstract machine.

Notation: given a relation R, we write $R(a)$ for the image of $\{a\}$ under R. For example, if $R = \{(0,4),(1,2),(1,3),(2,5)\}$, then $R(1) = \{2,3\}$ and $R(0) = \{4\}$.

Exercise 5 Prove that if $k, s \longrightarrow k', s'$, then $C\, k\, s = C\, k'\, s'$.

Exercise 6 Prove that $P(c) = \mathit{eval}(c)$.

Exercise 7 Is the IMP language Turing-complete? Why or why not?

Is the semantic function C sound wrt. contextual equivalence? Is it complete? –Jeremy

2.3 Semantics via Least Fixed Points

We shall revisit the semantics of the imperative language, this time taking the traditional but more complex approach of defining `while` with a recursive equation and using least fixed points to solve it. Recall that the meaning of a command is a partial function from stores to stores, or more precisely,

$$C\,c : \mathit{Store} \rightharpoonup \mathit{Store}$$

The meaning of a loop $(\texttt{while}\,b\,\texttt{do}\,c)$ is a solution to the equation

$$w = \lambda s.\, \mathit{if}\ B\,b\,s\ \mathit{then}\ w(C\,c\,s)\ \mathit{else}\ s \tag{1}$$

In general, one can write down recursive equations that do not have solutions, so how do we know whether this one does? When is there a unique solution? The theory of least fixed points provides answers to these questions.

Definition 1. *A* **fixed point** *of a function is an element that gets mapped to itself, i.e.,* $x = f(x)$.

In this case, the element that we're interested in is w, which is itself a function, so our f will be higher-order function. We reformulate Equation 1 as a fixed point equation by abstracting away the recursion into a parameter r.

$$w = F_{b,c}(w) \qquad \text{where } F_{b,c}\,r\,s = \mathit{if}\ B\,b\,s\ \mathit{then}\ r(C\,c\,s)\ \mathit{else}\ s \tag{2}$$

There are quite a few theorems in the literature that guarantee the existence of fixed points, but with different assumptions about the function and its domain [Lassez et al., 1982]. For our purposes, the CPO Fixed-Point Theorem will do the job. The idea of this theorem is to construct an infinite sequence that provides increasingly better approximations of the least fixed point. The union of this sequence will turn out to be the least fixed point.

The CPO Fixed-Point Theorem is quite general; it is stated in terms of a function F over partially ordered sets with a few mild conditions. The ordering captures the notion of approximation, that is, we write $x \sqsubseteq y$ if x approximates y.

Definition 2. *A* **partially ordered set (poset)** *is a pair* $(L, \sqsubseteq)$ *that consists of a set* L *and a partial order* $\sqsubseteq$ *on* L.

For example, consider the poset $(\mathbb{N} \rightharpoonup \mathbb{N}, \subseteq)$ of partial functions on natural numbers. Some partial function f is a better approximation than another partial function g if it is defined on more inputs, that is, if the graph of f is a subset of the graph of g. Two partial functions are incomparable if neither is a subset of the other.

The sequence of approximations will start with the worst approximation, a bottom element, written $\bot$, that contains no information. (For example, $\emptyset$ is the $\bot$ for the poset of partial functions.) The sequence proceeds to by applying F over and over again, that is,

$$\bot \quad F(\bot) \quad F(F(\bot)) \quad F(F(F(\bot))) \quad \cdots \quad F^i(\bot) \quad \cdots$$

But how do we know that this sequence will produce increasingly better approximations? How do we know that

$$F^i(\bot) \sqsubseteq F^{i+1}(\bot)$$

We can ensure this by requiring the output of F to improve when the input improves, that is, require F to be monotonic.

Definition 3. *Given two partial orders $(A, \sqsubseteq)$ and $(B, \sqsubseteq)$, $F : A \to B$ is* **monotonic** *iff for any $x, y \in A$, $x \sqsubseteq y$ implies $F(x) \sqsubseteq F(x)$.*

Proposition 1. *The functional $F_{b,c}$ for* `while` *loops* (2) *is monotonic.*

Proof. Let $f, g : \mathit{Store} \rightharpoonup \mathit{Store}$ such that $f \subseteq g$. We need to show that $F_{b,c}(f) \subseteq F_{b,c}(g)$. Let s be an arbitrary state. Suppose $B\,b\,s = \texttt{true}$.

$$F_{b,c}\,f\,s = f(C\,c\,s) \subseteq g(C\,c\,s) = F_{b,c}\,g\,s$$

So we have $F_{b,c}(f) \subseteq F_{b,c}(g)$. Next suppose $B\,b\,s = \texttt{false}$.

$$F_{b,c}\,f\,s = s = F_{b,c}\,g\,s$$

So again we have $F_{b,c}(f) \subseteq F_{b,c}(g)$. Having completed both cases, we conclude that $F_{b,c}(f) \subseteq F_{b,c}(g)$. □

We have $\bot \sqsubseteq F(\bot)$ because $\bot$ is less or equal to everything. Then we apply monotonicity to obtain $F(\bot) \sqsubseteq F(F(\bot))$. Continuing in this way we obtain the sequence of increasingly better approximations in Figure 6. If at some point the approximation stops improving and F produces an element that is equal to the last one, then we have found a fixed point. However, because we are interested in elements that are partial functions, which are infinite, the sequences of approximations will also be infinite. So we'll need some other way to go from the sequences of approximations to the actual fixed point.

The solution is to take the union of all the approximations. The analogue of union for an arbitrary partial order is least upper bound.

Definition 4. *Given a subset S of a partial order $(L, \sqsubseteq)$, an* **upper bound** *of S is an element y such that for all $x \in S$ we have $x \sqsubseteq y$. The* **least upper bound (lub)** *of S, written $\bigsqcup S$, is the least of all the upper bounds of S, that is, given any upper bound z of S, we have $\bigsqcup S \sqsubseteq z$.*

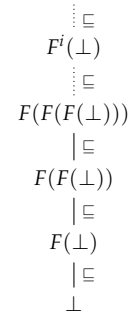


Figure 6: Ascending sequence of F.

$$\bigsqcup \left\{ \begin{array}{l} \{2 \mapsto 4, 3 \mapsto 9\}, \\ \{3 \mapsto 9, 4 \mapsto 16\} \end{array} \right\} = \left\{ \begin{array}{l} 2 \mapsto 4, \\ 3 \mapsto 9, \\ 4 \mapsto 16 \end{array} \right\}$$

Figure 7: The lub of partial functions.

In arbitrary posets, a least upper bound may not exist for a subset. In particular, for the poset $(\mathbb{N} \rightharpoonup \mathbb{N}, \subseteq)$, two partial functions do not have a lub if they are inconsistent, that is, if they map the same input to different outputs, such as $\{3 \mapsto 8\}$ and $\{3 \mapsto 9\}$. However, the CPO Fixed-Point Theorem will only need to consider totally ordered sequences, and all the elements in such a sequence are consistent.

Definition 5. *A* **complete partial order (cpo)** *has a least upper bound for every sequence of totally ordered elements.*

Proposition 2. *The poset of partial functions $(A \rightharpoonup B, \subseteq)$ is a cpo.*

Proof. Let S be a totally ordered sequence in $(A \rightharpoonup B, \subseteq)$. We claim that the lub of S is the union of all the elements of S, that is $\bigcup S$. Recall that $\forall xy, (x, y) \in \bigcup S$ iff $\exists f \in S, (x, y) \in f$. We first need to show that $\bigcup S$ is an upper bound of S. Suppose $f \in S$. We need to show that $f \subseteq \bigcup S$. Consider $(x, y) \in f$. Then $(x, y) \in \bigcup S$. So indeed, $\bigcup S$ is an upper bound of S. Second, consider another upper bound g of S. We need to show that $\bigcup S \subseteq g$. Suppose $(x, y) \in \bigcup S$. Then $(x, y) \in h$ for some $h \in S$. Because g is an upper bounf of S, we have $h \subseteq g$ and therefore $(x, y) \in g$. So we conclude that $\bigcup S \subseteq g$. □

The last ingredient required in the proof of the fixed point theorem is that the output of F should only depend on a finite amount of information from the input, that is, it should be continuous. For example, if the input to F is itself a function g, F should only need to apply g to a finite number of different values. This requirement is at the core of what it means for a function to be computable [Gunter et al., 1990]. So applying F to the lub of a sequence S should be the same as taking the lub of the set obtained by mapping F over each element of S.

Definition 6. *A monotonic function $F : A \to B$ on a cpo is* **continuous** *iff for all totally ordered sequences S of A*

$$F(\bigsqcup S) = \bigsqcup \{F(x) \mid x \in S\}.$$

Proposition 3. *The functional $F_{b,c}$ for* `while` *loops* (2) *is continuous.*

Proof. Let S be a totally ordered sequence in $(A \rightharpoonup B, \subseteq)$. We need to show that

$$F_{b,c}(\bigcup S) = \bigcup \{F_{b,c}(f) \mid f \in S\}$$

We shall prove this equality by showing that each graph is a subset of the other. We assume $(s, s'') \in F_{b,c}(\bigcup S)$. Suppose $B\, b\, s = \texttt{true}$. Then $(s, s'') \in (\bigcup S) \circ (C\, c)$, so $(s, s') \in C\, c$ and $(s', s'') \in g$ for some s' and $g \in S$. So $(s, s'') \in g \circ (C\, c)$. Therefore $(s, s'') \in \bigcup \{F_{b,c}(f) \mid f \in S\}$. So we have shown that $F_{b,c}(\bigcup S) \subseteq \bigcup \{F_{b,c}(f) \mid f \in S\}$. Next suppose $B\, b\, s = \texttt{false}$. Then $F_{b,c}(\bigcup S) = id = \bigcup \{F_{b,c}(f) \mid f \in S\}$.

For the other direction, we assume $(s, s'') \in \bigcup\{F_{b,c}(f) \mid f \in S\}$. So $(s, s'') \in F_{b,c}(g)$ for some $g \in S$. Suppose $B\,b\,s = \texttt{true}$. Then $(s, s') \in C\,c$ and $(s', s'') \in g$ for some s'. So $(s', s'') \in \bigcup S$ and therefore $(s, s'') \in F_{b,c}(\bigcup S)$. So we have shown that $\bigcup\{F_{b,c}(f) \mid f \in S\} \subseteq F_{b,c}(\bigcup S)$. Next suppose $B\,b\,s = \texttt{false}$. Then again we have both graphs equal to the identity relation. □

We now state the fixed point theorem for cpos.

Theorem 4 (CPO Fixed-Point Theorem)**.** *Suppose $(L, \sqsubseteq)$ is a cpo and let $F : L \to L$ be a continuous function. Then F has a least fixed point, written* fix F*, which is the least upper bound of the ascending sequence of F:*

$$\operatorname{fix} F = \bigsqcup\{F^n(\bot) \mid n \in \mathbb{N}\}$$

Proof. Note that $\bot$ is an element of L because it is the lub of the empty sequence. We first prove that fix F is a fixed point of F.

$$\begin{aligned}
F(\operatorname{fix} F) &= F(\bigsqcup\{F^n(\bot) \mid n \in \mathbb{N}\}) \\
&= \bigsqcup\{F(F^n(\bot)) \mid n \in \mathbb{N}\} && \text{by continuity} \\
&= \bigsqcup\{F^{n+1}(\bot)) \mid n \in \mathbb{N}\} \\
&= \bigsqcup\{F^n(\bot)) \mid n \in \mathbb{N}\} && \text{because } F^0(\bot) = \bot \sqsubseteq F^1(\bot) \\
&= \operatorname{fix} F
\end{aligned}$$

Next we prove that fix F is the least of the fixed points of F. Suppose e is an arbitrary fixed point of F. By the monotonicity of F we have $F^i(\bot) \sqsubseteq F^i(e)$ for all i. And because e is a fixed point, we also have $F^i(e) = e$, so e is an upper bound of the ascending chain, and therefore fix $F \sqsubseteq e$. □

Returning to the semantics of the `while` loop, we give the least fixed-point semantics of an imperative language in Figure 8. We have already seen that the poset of partitial functions is a cpo and that $F_{b,c}$ is continuous, so fix $(F_{b,c})$ is well defined and is the least fixed-point of $F_{b,c}$. Thus, we can define the meaning of the `while` loop as follows.

$$C(\texttt{while}\ b\ \texttt{do}\ c)\,s = \operatorname{fix}(F_{b,c})\,s$$

Exercise 8 Prove that the semantics in Figure 3 is equivalent to the least fixed-point semantics in Figure 8, i.e., $(n, n') \in P\,c$ iff $P\,c\,n = n'$.

In the literature there are two schools of thought regarding how to define complete partial orders. There is the one presented above, that requires lubs to exists for all totally ordered sequences [Plotkin, 1983, Schmidt, 1986, Winskel, 1993]. The other requires that the poset be directed complete, that is, lubs exists for all directed sets [Gunter

Figure 8: Least Fixed-Point Semantics of an Imperative Language

$$P' : Cmd \to \mathbb{N} \rightharpoonup \mathbb{N}$$

$$P'\, c\, n = (C'\, c\, \{A \mapsto n]\})\, Z$$

$$C' : Cmd \to Store \rightharpoonup Store$$

$$\begin{aligned}
C'\, \texttt{skip}\, s &= s \\
C'(x := e)\, s &= [x \mapsto E\, e\, s]s \\
C'(c_1\, ;\, c_2)\, s &= C'\, c_2 (C'\, c_1\, s) \\
C' \begin{pmatrix} \texttt{if}\, b\, \texttt{then}\, c_1 \\ \texttt{else}\, c_2 \end{pmatrix} s &= \begin{cases} C'\, c_1\, s & \text{if } B\, b\, s = \texttt{true} \\ C'\, c_2\, s & \text{if } B\, b\, s = \texttt{false} \end{cases} \\
C'(\texttt{while}\, b\, \texttt{do}\, c)\, s &= \text{fix}(F_{b,c})\, s
\end{aligned}$$

et al., 1990, Mitchell, 1996, Amadio and Pierre-Louis, 1998]. The two schools of thought are equivalent, i.e., a poset P with a least element is directed-complete iff every totally ordered sequence in P has a lub [Davey and Priestley, 2002] (Theorem 8.11).

3 *Simply-Typed Functions*

Reading: Gunter [1992] Chapter 2, Chlipala [2007]

The appropriately-named simply-typed λ-calculus (STLC) provides a simple setting in which to study first-class functions. That is, functions that can be used like any other data. In the limited context of the STLC, this just means that they can be passed as parameters and returned from functions. Figure 9 defines the syntax and type system of the STLC. Whenever we discuss expression of the STLC, we always assume that they are well-typed, that is, that they satisfy the type system defined in Figure 9. The types include natural numbers and functions $A \to B$ where A is the type of the input (there is just one) and B is the type of the output.

The purpose of the variables in the STLC is for refering to the parameters of functions. The expression $\lambda x : A.\, e$ creates a function of one parameter named x of type A. The expression e is the body of the function and it may refer to parameter x. The value of e is the return value of the function. The expression $e_1\, e_2$ *applies* the function produced by e_1 to the value of e_2. The type system ensures that we never apply numbers (as if they were functions) or perform arith-

Syntax

types	$A, B ::=$	$Nat \mid A \to B$
numbers	$n \in$	$\mathbb{N}$
variables	$x \in$	$\mathbb{X}$
arithmetic	$\oplus ::=$	$+ \mid \times$
expressions	$e ::=$	$n \mid e \oplus e \mid x \mid \lambda x{:}A.\, e \mid e\, e$
type env.	$\Gamma ::=$	$\emptyset \mid \Gamma, x{:}A$

Type System

$$\frac{}{\Gamma \vdash n : Nat} \qquad \frac{\Gamma \vdash e_1 : Nat \quad \Gamma \vdash e_2 : Nat}{\Gamma \vdash e_1 \oplus e_2 : Nat}$$

$$\frac{\Gamma_n = x{:}A}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x{:}A \vdash e : B}{\Gamma \vdash \lambda x{:}A.\, e : A \to B} \qquad \frac{\Gamma \vdash e_1 : A \to B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1\, e_2 : B}$$

Figure 9: Syntax and types of the simply-typed λ-calculus

metic on functions (as if they were numbers). The type system also prevents the use of undefined variables.

3.1 *Semantics*

The denotational semantics of the STLC is simple because STLC functions are total, they always terminate with an answer. So they can be straightforwardly modeled by mathematical (total) functions. But there is one twist, describing the type of the semantic function is a bit interesting. The set of values that can be produced by an STLC expression *depends* on the type of the expression. For example, if an expression has type *Nat*, then its meaning will be in $\mathbb{N}$. If an expression has type $Nat \to Nat$, then its meaning will be in $\mathbb{N}^{\mathbb{N}}$ [1]. Thus, we define a mapping E from each type into a set.

[1] The set-theoretic notation B^A refers to the set of all total functions from set A to set B.

$$E\, Nat = \mathbb{N}$$
$$E\, (A \to B) = (E\, B)^{E\, A}$$

Likewise, we define a mapping from type environments to sets, in particular, to a cartesian product of the variable bindings.

$$E\, \emptyset = Unit$$
$$E\, (\Gamma, x{:}A) = (\mathbb{X} \times E\, A) \times E\, \Gamma$$

With these mapping in hand, we can describe the semantic function E for the STLC. Because of the dependency described above, the type of E is interesting. It is a dependent function whose first parameter is some well-typed expression e, so we have $\Gamma \vdash e : A$.

Figure 10: Semantics of the simply-typed λ-calculus

$$E : \prod(e : \mathbb{E}), E\,\Gamma \to E\,A \qquad \text{where } \Gamma \vdash e : A$$

$$\begin{aligned}
E\,n\,\rho &= n \\
E\,(e_1 \oplus e_2)\,\rho &= \delta(\oplus, n_1, n_2) \\
&\qquad \text{if } n_1{=}E\,e_1\,\rho \text{ and } n_2{=}E\,e_2\,\rho \\
E\,x\,\rho &= \rho(x) \\
E\,(\lambda x{:}A.\,e)\,\rho &= \{(d, d') \mid d \in E\,A \text{ and } d' = E\,e\,\rho(x{\mapsto}d)\} \\
E\,(e_1\,e_2)\,\rho &= (E\,e_1\,\rho)\,(E\,e_2\,\rho)
\end{aligned}$$

Figure 11: Equational Theory of the simply-typed λ-calculus

$$(\text{refl})\frac{}{\vdash e = e} \qquad (\text{sym})\frac{\vdash e_2 = e_1}{\vdash e_1 = e_2} \qquad (\text{trans})\frac{\vdash e_1 = e_2 \quad \vdash e_2 = e_3}{\vdash e_1 = e_3}$$

$$(\text{cong})\frac{\vdash e_1 = e_3 \quad \vdash e_2 = e_4}{\vdash (e_1\,e_2) = (e_3\,e_4)} \qquad (\xi)\frac{\vdash e = e'}{\vdash \lambda x.\,e = \lambda x.\,e'}$$

$$(\beta)\frac{}{\vdash (\lambda x{:}A.\,e)\,e' = [x{:=}e']e} \qquad (\eta)\frac{x \notin \mathrm{FV}(e)}{\vdash (\lambda x{:}A.\,e\,x) = e}$$

$$(\text{cong-}\oplus)\frac{\vdash e_1 = e_3 \quad \vdash e_2 = e_4}{\vdash (e_1 \oplus e_2) = (e_3 \oplus e_4)} \qquad (\delta)\frac{n_3 = \delta(\oplus, n_1, n_2)}{n_1 \oplus n_2 = n_3}$$

The second parameter is a type environment drawn from $E\,\Gamma$ and the result is in $E\,A$. The definition of the semantic function E is given in Figure 10. We interpret each λ as a function that maps every element d in its domain $E\,A$ to the element produced by the body e of the λ, with d bound to parameter x. We interpret application as mathematical function application. Regarding variables, their meaning is given by the environment ρ.

3.2 *Equational Theory*

Another way to specify the meaning of a language is with an equational theory. Figure 11 gives the standard equational rules for the STLC, and the equational theory is the set of all equalities that can be deduced from these rules. Note that these rules are closely related to the reduction rules of the STLC; they basically specify a bidirectional version of those rules.

3.3 Soundness of Equational Theory

Next we check whether the equational theory is sound with respect to the model. That is, for each equation that holds in the theory, does it also hold in the model?

The most intersting of the equations is β, which involves substitution. So we shall need the following lemma regarding the interaction between substitution and the semantic function E.

Lemma 5 (Substitution). *Suppose* $\Gamma, x : B \vdash e : A$ *and* $\Gamma \vdash e' : B$.
$E\,([x:=e']e)\,\rho = E\,e\,\rho(x\mapsto E\,e'\,\rho)$

Proof. The proof is by induction on e.

- Case $e = n$:

$$E\,([x:=e']n)\,\rho = n = E\,n\,\rho(x\mapsto E\,e'\,\rho)$$

- Case $e = e_1 \oplus e_2$:

$$\begin{aligned}
E\,([x:=e'](e_1 \oplus e_2))\,\rho &= E\,([x:=e']e_1 \oplus [x:=e']e_2))\,\rho && \text{def. substitution}\\
&= E\,([x:=e']e_1)\,\rho \oplus E\,([x:=e']e_2)\,\rho && \text{def. } E\\
&= E\,e_1\,\rho(x\mapsto E\,e'\,\rho) \oplus E\,e_2\,\rho(x\mapsto E\,e'\,\rho) && \text{I.H.}\\
&= E\,(e_1 \oplus e_2)\,\rho(x\mapsto E\,e'\,\rho) && \text{def. } E
\end{aligned}$$

- Case $e = y$: We consider two cases, whether $y = x$ or not.
  - Subcase $y = x$:

$$E\,([x:=e']x)\,\rho = E\,e'\,\rho = E\,x\,\rho(x\mapsto E\,e'\,\rho)$$

  - Subcase $y \neq x$:

$$E\,([x:=e']y)\,\rho = E\,y\,\rho = E\,y\,\rho(x\mapsto E\,e'\,\rho)$$

- Case $e = \lambda y{:}A.\,e_1$:
  By α-conversion we can make sure that $y \notin \mathrm{fv}(e')$ and $y \neq x$.

$$\begin{aligned}
E\,[x:=e'](\lambda y{:}A.\,e_1)\,\rho &= E\,(\lambda y{:}A.\,[x:=e']e_1)\,\rho && \text{by def. substitution}\\
&= \{(d,d') \mid d \in E\,A \text{ and } d'{=}E\,([x:=e']e_1)\,\rho(y\mapsto d)\} && \text{by def. } E\\
&= \{(d,d') \mid d \in E\,A \text{ and } d'{=}E\,e_1\,\rho(y\mapsto d)(x\mapsto E\,e'\,\rho(y\mapsto d))\} && \text{by I.H.}\\
&= \{(d,d') \mid d \in E\,A \text{ and } d'{=}E\,e_1\,\rho(y\mapsto d)(x\mapsto E\,e'\,\rho)\} && y \notin \mathrm{FV}(e')\\
&= \{(d,d') \mid d \in E\,A \text{ and } d'{=}E\,e_1\,\rho(x\mapsto E\,e'\,\rho)(y\mapsto d)\} && y \neq x\\
&= E\,(\lambda y{:}A.\,e_1)\,\rho(x\mapsto E\,e'\,\rho) && \text{by def. } E
\end{aligned}$$

- Case $e = e_1\,e_2$:

$$\begin{aligned}
E\,[x:=e'](e_1\,e_2)\,\rho &= E\,([x:=e']e_1\;[x:=e']e_2)\,\rho && \text{by def. substitution}\\
&= (E\,[x:=e']e_1\,\rho)\,(E\,[x:=e']e_2\,\rho) && \text{by def. } E\\
&= (E\,e_1\,\rho(x\mapsto E\,e'\,\rho))\,(E\,e_2\,\rho(x\mapsto E\,e'\,\rho)) && \text{by I.H.}\\
&= E\,(e_1\,e_2)\,\rho(x\mapsto E\,e'\,\rho) && \text{by def. } E
\end{aligned}$$

□

Definition 7. *Suppose M is a semantic function for the STLC. A Γ-environment ρ maps each variable $x : A \in \Gamma$ to an element of the set $M\,A$, i.e. to an equivalence class of expressions of type A.*

$$\rho \models \Gamma \quad \stackrel{\text{def}}{=} \quad \forall x : A \in \Gamma, \rho(x) \in M\,A$$

Definition 8. *Suppose $\Gamma \vdash e : A$, $\Gamma \vdash e' : A$, and M is a semantic function for STLC.*

$$M \models e = e' \quad \stackrel{\text{def}}{=} \quad \forall \rho,\ \rho \models \Gamma \implies M\,e\,\rho = M\,e'\,\rho$$

Theorem 6 (Soundness of Theory wrt. the Model)**.**
If $\vdash e = e'$, then $E \models e = e'$.

Proof. Let ρ be an environment such that $\rho \models \Gamma$. We need to show that $E\,e\,\rho = E\,e'\,\rho$. The proof is by induction on the derivation of $\vdash e_1 = e_2$.

- (refl), (sym), (trans) are immediate.
- (cong) Let ρ be an arbitrary environment. By the induction hypothesis, we have $E\,e_1\,\rho = E\,e_3\,\rho$ and $E\,e_2\,\rho = E\,e_4\,\rho$. Therefore $(E\,e_1\,\rho)\,(E\,e_2\,\rho) = (E\,e_3\,\rho)\,(E\,e_4\,\rho)$.
- (ξ) Let ρ be an arbitrary environment. We need to show that $E\,(\lambda x.\,e)\,\rho = E\,(\lambda x.\,e')\,\rho$. Thus, it suffices to show for an arbitrary d that $E\,e\,\rho(x{\mapsto}d) = E\,e'\,\rho(x{\mapsto}d)$, but that follows from the induction hypothesis.
- (β) Let ρ be an arbitrary environment.

$$\begin{aligned} E\,((\lambda x{:}A.\,e)\,e')\,\rho &= (E\,(\lambda x{:}A.\,e)\,\rho)(E\,e'\,\rho) \\ &= E\,e\,\rho(x{\mapsto}E\,e'\,\rho) \\ &= E\,[x{:=}e']e\,\rho && \text{by Lemma 5} \end{aligned}$$

- (η) Let ρ be an arbitrary environment.

$$\begin{aligned} E\,(\lambda x.\,e\,x)\,\rho &= \lambda d.\,E\,(e\,x)\,\rho(x{\mapsto}d) \\ &= \lambda d.\,(E\,e\,\rho(x{\mapsto}d))\,d \\ &= \lambda d.\,(E\,e\,\rho)\,d && \text{because } x \notin \mathrm{FV}(e) \\ &= E\,e\,\rho \end{aligned}$$

- (cong-$\oplus$) TODO
- (δ) TODO

□

3.4 *Completeness of the Equational Theory*

Next we turn to the question of completeness of the equational theory with respect to the model. That is, we seek to prove that

$$E \models e = e' \implies \vdash e = e'$$

We shall prove this in two steps, depicted below, with a detour through another model of STLC, the Term Model T.

$$\begin{array}{c} E \models e = e' \\ \Downarrow 1 \\ T \models e = e' \\ \Downarrow 2 \\ \vdash e = e' \end{array}$$

The Term Model is built from the equational theory, so the proof of step 2 is straightforward. The proof of step 1 will be accomplished by constructing a homomorphism from E to T and proving that equality is preserved from the source to the target of a homomorphism.

The Term Model We construct a model for STLC that is built from sets of expressions of STLC modulo the equational theory.

We partition the expressions into equivalence classes, each of which contains all the expressions that are equal to each other according to the equational theory. Each equivalence class is identified by a representative expression as follows.

$$[e] \stackrel{\text{def}}{=} \{e' \mid \vdash e = e'\}$$

We define the semantic function T on types as follows.

$$T\,A \stackrel{\text{def}}{=} \{[e] \mid \exists \Gamma, \Gamma \subset \mathcal{G} \text{ and } \Gamma \vdash e : A\}$$

The purpose of the type environment $\mathcal{G}$ is to make sure that each variable is used at a single type. So throughout this discussion, we choose a particular $\mathcal{G}$ that contains an infinite number of distinct variable bindings, including an infinite number of variables for each type.

Given $[e_1] \in T\,(A \to B)$ and $[e_2] \in T\,A$, we define an application operator as follows:

$$[e_1] \cdot [e_2] \stackrel{\text{def}}{=} [(e_1\, e_2)]$$

In the term model, a Γ-environment ρ maps each variable $x : A \in \Gamma$ to an element of the set $T\,A$, i.e. to an equivalence class of expressions of type A.

$$\rho \models \Gamma \stackrel{\text{def}}{=} \forall x : A \in \Gamma, \rho(x) \in T\,A$$

A substitution σ maps each variable to a single expression. A substitution $\mathsf{rep}(\rho)$ represents ρ if it maps each variable to the representative in ρ.

$$\mathsf{rep}(\rho)(x) \stackrel{\text{def}}{=} e \quad \text{if } \rho(x) = [e]$$

The meaning of expressions in the term model is defined as follows.

$$T\, e\, \rho \stackrel{\text{def}}{=} [\mathsf{rep}(\rho)(e)]$$

Frames We shall formalize part of what it means to be a model of the STLC with the notions of pre-frame and frame. Then we will speak of homomorphisms between frames.

Definition 9. *A structure $(M, \cdot)$ is a* pre-frame *if extensionality holds, i.e. if two functions $f, g \in M(A \to B)$ map every argument $a \in M\,A$ to equal results, $f \cdot a = g \cdot a$, then the two functions are the same function, $f = g$.*

Lemma 7. *The term model $(T, \cdot)$ is a pre-frame.*

Proof. Let $[f], [g] \in T\,(A \to B)$. We assume $[f] \cdot [a] = [g] \cdot [a]$ for any $[a] \in T\,A$. We need to show that $[f] = [g]$, so it suffices to show $\vdash f = g$. Find some $x : A \in \mathcal{G}$ s.t. $x \notin \mathrm{FV}(f) \cup \mathrm{FV}(g)$. Then

$$[f\,x] = [f] \cdot [x] = [g] \cdot [x] = [g\,x]$$

and so $\vdash f\,x = g\,x$. Then by the (ξ) rule we have

$$\vdash \lambda x\!:\!A.\, f\,x = \lambda x\!:\!A.\, g\,x$$

By the (η) rule we have $\vdash \lambda x\!:\!A.\, f\,x = f$ and $\vdash \lambda x\!:\!A.\, g\,x = g$, so we conclude that $\vdash f = g$. □

Definition 10. *A structure $(M, \cdot)$ is a* frame *if $(M, \cdot)$ is a pre-frame and M is also a mapping on expressions that satisfies the following equations*

$$\begin{aligned} M\,n\,\rho &= n \\ M\,(e_1 \oplus e_2)\,\rho &= M\,e_1\,\rho \oplus M\,e_2\,\rho \\ M\,x\,\rho &= \rho(x) \\ M\,(e_1\,e_2)\,\rho &= (M\,e_1\,\rho) \cdot (M\,e_2\,\rho) \\ (M\,(\lambda x\!:\!A.\, e_1)\,\rho) \cdot d &= M\,e_1\,\rho(x \mapsto d) \end{aligned}$$

Lemma 8. *The term model $(T, \cdot)$ is a frame.*

Proof. Let $\sigma = \mathsf{rep}(\rho)$.

- nts. $T\,x\,\rho = \rho(x)$.

$$T\,x\,\rho = [\sigma(x)] = \rho(x)$$

- nts. $T\,(e_1\,e_2)\,\rho = (T\,e_1\,\rho)\cdot(T\,e_2\,\rho)$

  $$T\,(e_1\,e_2)\,\rho = [\sigma(e_1\,e_2)] = [\sigma(e_1)]\cdot[\sigma(e_2)] = (T\,e_1\,\rho)\cdot(T\,e_2\,\rho)$$

- nts. $(T\,(\lambda x\!:\!A.\,e_1)\,\rho)\cdot d = T\,e_1\,\rho(x{\mapsto}d)$. Let $d = [e]$.

  $$\begin{aligned}(T\,(\lambda x\!:\!A.\,e_1)\,\rho)\cdot[e] &= [\lambda x\!:\!A.\,\sigma(e_1)]\cdot[e] && \text{by def. } T\\ &= [(\lambda x\!:\!A.\,\sigma(e_1))\,e] && \text{by def. } \cdot\\ &= [\sigma(x{:=}e)(e_1)] && \text{by } (\beta)\\ &= T\,e_1\,\rho(x{\mapsto}[e]) && \text{by def. } T\end{aligned}$$

□

Lemma 9. *The structure $(E,\cdot)$ is a frame, where the application operator is just mathematical function application:*

$$f\cdot d \stackrel{\text{def}}{=} f(d)$$

Proof. First, to show that $(E,\cdot)$ is a pre-frame, we need to prove that functions in the model are extensional. But this is straightforward because the elements of $E\,(A\to B)$ are just functions, i.e., elements of $(E\,B)^{E\,A}$. Suppose $f,g\in(E\,B)^{E\,A}$ and $f\cdot a = g\cdot a$ for any $a\in E\,A$. We need to show that $f=g$. By extensionality of functions, it suffices to show that $f(a)=g(a)$ for all $a\in E\,A$. We conclude by unfolding application in the assumption $f\cdot a = g\cdot a$.

Next we show that $(E,\cdot)$ satisfies the frame equations.

$$\begin{aligned}E\,n\,\rho &= n && \text{by def. of } E\\ E\,(e_1\oplus e_2)\,\rho &= E\,e_1\,\rho\oplus E\,e_2\,\rho && \text{by def. of } E\\ E\,x\,\rho &= \rho(x) && \text{by def. of } E\\ E\,(e_1\,e_2)\,\rho &= (E\,e_1\,\rho)\cdot(E\,e_2\,\rho) && \text{by def. of } E\\ (M\,(\lambda x\!:\!A.\,e_1)\,\rho)\cdot d &= \{(d,d')\mid d\in E\,A, d' = E\,e_1\,\rho(x{\mapsto}d)\}\cdot d && \text{by def. of } E\\ &= E\,e_1\,\rho(x{\mapsto}d) && \text{by def. of } \cdot\end{aligned}$$

□

Lemma 10 (Frames are Sound)**.** *Suppose $(M,\cdot)$ is a frame.*
If $\vdash e = e'$, then $M\models e = e'$

Proof. The proof is similar to the one for Theorem 6. □

The following theorem establishes step 2 of our proof (and more)

$$\begin{array}{c} T\models e = e' \\ \Downarrow 2 \\ \vdash e = e' \end{array}$$

Theorem 11 (Term Model is Sound & Complete wrt. Eqn. Theory)**.**
Suppose $\Gamma \vdash e : A$ *and* $\Gamma \vdash e' : A$.
$\vdash e = e'$ *iff* $T \models e = e'$.

Proof. We consider each direction of the iff separately.

- $\vdash e = e' \implies T \models e = e'$
  The term model is a frame (Lemma 8), and frames are sound (Lemma 10).
- $T \models e = e' \implies \vdash e = e'$
  Wlog. we assume $\Gamma \subset \mathcal{G}$. Let ρ be the identity environment

$$\rho = \{x \mapsto [x] \mid x \in \mathrm{dom}(\Gamma)\}$$

  So we have $\rho \models \Gamma$ and therefore $T\, e\, \rho = T\, e'\, \rho$. We then calculate

$$[e] = [\mathsf{rep}(\rho)(e)] = T\, e\, \rho = T\, e'\, \rho = [\mathsf{rep}(\rho)(e')] = [e']$$

  from which we conclude that $\vdash e = e'$.

□

Homomorphisms Between Frames Step 1 of the proof of completeness relies on the construction of a homomorphism between the models E and T. But first we review the general notion of homomorphism.

A homomorphism is a function between two instances of an algebraic structure that respects the operations of the structure. For example, a monoid is an algebraic structure that consists of some set D plus a binary operator $\bullet : D \times D \to D$ and an identity element $id \in D$. The binary operator must be associative:

$$(a \bullet b) \bullet c = a \bullet (b \bullet c)$$

and the identity element must act like one:

$$a \bullet id = a = id \bullet a$$

A monoid homomorphism is a function h from one monoid $(D_1, \bullet_1, id_1)$ to another one $(D_2, \bullet_2, id_2)$ such that

$$h(id_1) = id_2$$
$$h(a \bullet_1 b) = h(a) \bullet_2 h(b)$$

Consider the two instances of monoid:

- lists of Booleans with concatenation $(\mathbb{B}^*, @, \epsilon)$, and
- integers with addition $(\mathbb{Z}, +, 0)$.

Then the length function defined as follows is a monoid homomorphism between lists and integers.

$$\begin{aligned} length(\epsilon) &= 0 \\ length(ls_1 @ ls_2) &= length(ls_1) + length(ls_2) \end{aligned}$$

For our present purposes, we are interested in homomorphisms between frames.

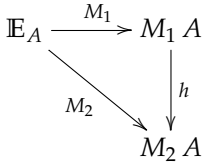
Definition 11. *Suppose $(M_1, \cdot)$ and $(M_2, \cdot)$ are frames. A* partial homomorphism $h : M_1 \to M_2$ *(morphism for short) is a type-indexed set of surjective partial functions $h_A : M_1\, A \rightharpoonup M_2\, A$ such that*

1. *for every A, B and $f \in M_1(A \to B)$, either*
   - *for all a in the domain of h_A*
   $$h_B(f \cdot a) = h_{A\to B}(f) \cdot h_A(a)$$
   - *or $h_{A\to B}(f)$ is undefined and there is no $g \in M_2\,(A \to B)$ that satisfies the above equation.*
2. *For any $n \in \mathbb{N}$ and any ρ, ρ'*
   $$h_{Nat}(M_1\, n\, \rho) = M_2\, n\, \rho'$$
3. *for every $a, b \in M_1\, Nat$*
   $$h_{Nat}(a \oplus b) = h_{Nat}(a) \oplus h_{Nat}(b)$$

Recall that step 1 of our proof is to show that

$$\begin{array}{c} E \models e = e' \\ \Downarrow 1 \\ T \models e = e' \end{array}$$

We shall accomplish this by showing that whenever there is a partial homomorphism from one frame to another, then equality in the first frame implies equality in the second frame. But first we require a lemma that says applying M_1 to any expression e and then a morphism h is the same as just applying M_2 to e. In other words, the following diagram commutes. Let $\mathbb{E}_A$ stand for the set of expressions of type A.



Lemma 12. *Suppose $h : M_1 \to M_2$ is a partial homomorphism, $\Gamma \vdash e : A$, $\rho \models \Gamma$, $\rho' \models \Gamma$, and $h(\rho(x)) = \rho'(x)$ for every $x \in \mathrm{dom}(\Gamma)$.*

$$h(M_1\, e\, \rho) = M_2\, e\, \rho'$$

Proof. The proof is by induction on e.

- Case $e = n$:

$$h(M_1\, n\, \rho) = M_2\, n\, \rho' \qquad\qquad h \text{ is a morphism}$$

- Case $e = e_1 \oplus e_2$:

$$\begin{aligned} h(M_1\, (e_1 \oplus e_2)\, \rho) &= h(M_1\, e_1\, \rho \oplus M_1\, e_2\, \rho) && M_1 \text{ is a frame} \\ &= h(M_1\, e_1\, \rho) \oplus h(M_1\, e_2\, \rho) && h \text{ is a morphism} \\ &= M_2\, e_1\, \rho' \oplus M_2\, e_2\, \rho' && \text{by I.H.} \\ &= M_2\, (e_1 \oplus e_2)\, \rho' && M_2 \text{ is a frame} \end{aligned}$$

- Case $e = x$:

$$h(M_1\, x\, \rho) = h(\rho(x)) = \rho'(x) = M_2\, x\, \rho'$$

- Case $e = \lambda y{:}A_1.\, e_1$ and $A = A_1 \to A_2$:
  We need to prove

$$h(M_1\, (\lambda y{:}A_1.\, e_1)\, \rho) = M_2\, (\lambda y{:}A_1.\, e_1)\, \rho'$$

  but because M_2 is a pre-frame, we have extensionality, so it suffices to show that for any $d' \in M_2\, A_1$

$$h(M_1\, (\lambda y{:}A_1.\, e_1)\, \rho) \cdot d' = (M_2\, (\lambda y{:}A_1.\, e_1)\, \rho') \cdot d'$$

  Let $d = h^{-1}(d')$.

$$\begin{aligned} h(M_1\, (\lambda y{:}A_1.\, e_1)\, \rho) \cdot h(d) &= h(M_1\, (\lambda y{:}A_1.\, e_1)\, \rho \cdot d) && h \text{ is a morphism} \\ &= h(M_1\, e_1\, \rho(y{\mapsto}d)) && M_1 \text{ is a frame} \\ &= M_2\, e_1\, \rho'(y{\mapsto}d') && \text{by I.H.} \\ &= M_2\, (\lambda y{:}A_1.\, e_1)\, \rho' \cdot d' && M_2 \text{ is a frame} \end{aligned}$$

- Case $e = e_1\, e_2$:

$$\begin{aligned} h(M_1\, (e_1\, e_2)\, \rho) &= h(M_1\, e_1\, \rho \cdot M_1\, e_2\, \rho) && M_1 \text{ is a frame} \\ &= h(M_1\, e_1\, \rho) \cdot h(M_1\, e_2\, \rho) && h \text{ is a morphism} \\ &= M_2\, e_1\, \rho' \cdot M_2\, e_2\, \rho' && \text{by I.H.} \\ &= M_2\, (e_1\, e_2)\, \rho' && M_2 \text{ is a frame} \end{aligned}$$

□

With Lemma 12 in hand it is now straightforward to prove that if there is a morphism from one frame to another, then equality in the source frame implies equality in the target frame.

Lemma 13. *Suppose there is a partial homomorphism $h : M_1 \to M_2$. Also, suppose $\Gamma \vdash e : A$ and $\Gamma \vdash e' : A$. If $M_1 \models e = e'$, then $M_2 \models e = e'$.*

Proof. Let ρ' be an environment s.t. $\rho' \models \Gamma$. We need to show that $M_2\, e\, \rho' = M_2\, e'\, \rho'$. Define $\rho(x) = h^{-1}(\rho'(x))$, so $\rho'(x) = h(\rho(x))$. (h is a surjection, so h^{-1} is total.) Note that $\rho \models \Gamma$.

$$\begin{aligned} M_2\, e\, \rho' &= h(M_1\, e\, \rho) && \text{by Lemma 12} \\ &= h(M_1\, e'\, \rho) && \text{because } M_1 \models e = e' \\ &= M_2\, e'\, \rho' && \text{by Lemma 12} \end{aligned}$$

□

We have proved a lemma about morphisms between arbitrary frames, but we are particularly interested in E and T. In the following we show how to construct a homomorphism from E to any other frame, including T.

Lemma 14 (Homomorphisms from E to other frames)**.** *Suppose $(M, \cdot)$ is a frame. Then there exists a partial homomorphism $h : E \to M$.*

Proof. For natural numbers, the definition of h is straightforward. Recall that $E\, Nat = \mathbb{N}$.

$$\begin{aligned} h_{Nat} &: E\, Nat \rightharpoonup M\, Nat \\ h_{Nat}(n) &= M\, n\, \emptyset \end{aligned}$$

For h on a function type $A \to B$, we have the following signature

$$h_{A \to B} : E\,(A \to B) \rightharpoonup M\,(A \to B)$$

We need to satisfy the following equation, for any $a \in E\, A$ and $f \in E\,(A \to B)$.

$$h_B(f \cdot a) = h_{A \to B}(f) \cdot h_A(a)$$

As a first step, let us construct a function $g : M\, A \to M\, B$ that satisfies the equation

$$h_B(f \cdot a) = g(h_A(a)) \tag{3}$$

We define g in terms of f as follows

$$g(d) = h_B(f \cdot h_A^{-1}(d))$$

The inverse h_A^{-1} is a total function because h_A is surjective (see below). Now it is easy to check that g satisfies equation (3).

$$g(h_A(a)) = h_B(f \cdot h_A^{-1}(h_A(a))) = h_B(f \cdot a)$$

We have a function $g : M\,A \to M\,B$, but $h_{A\to B}$ needs to map each $f \in E\,(A \to B)$ into $M\,(A \to B)$. So we rummage around in $M\,(A \to B)$ until we find some g' that behaves just like g, or we fail to find such g', in which case $h(f)$ is undefined. If there is such a g', then we know it is unique because of extensionality.

This is rather hand-wavy! –Jeremy

$$h_{A\to B}(f) = \begin{cases} g' & \text{if } \forall a \in \mathrm{dom}(h_A), g' \cdot h_A(a) = h_B(f \cdot a) \\ \bot & \text{there is no such } g' \end{cases}$$

But we still need to prove that h is surjective. We proceed by induction on its type index. It is trivial for *Nat*, so we consider $A \to B$. Suppose $g' \in M\,(A \to B)$. We need to show that $h_{A\to B}(f) = g'$ for some $f \in E\,(A \to B)$. Let f be defined as follows.

$$f(a) = h_B^{-1}(g' \cdot h_A(a))$$

The inverse h_B^{-1} is total because, by the induction hypothesis, we may assume that h_B is a surjection. Now for any $a' \in E\,A$ we have

$$h(f) \cdot h(a') = h(f \cdot a') = h(h^{-1}(g' \cdot h(a'))) = g' \cdot h(a')$$

so by extensionality, we conclude that $h_{A\to B}(f) = g'$.

□

As promised, we prove completeness in two steps.

Theorem 15 (Completeness of Theory wrt. the Model)**.**
If $E \models e = e'$, then $\vdash e = e'$.

Proof. There is a partial homomorphism $h : E \to T$ (Lemma 14), so from $E \models e = e'$ we have $T \models e = e'$ (Lemma 13). And then from $T \models e = e'$ we conclude $\vdash e = e'$ (Theorem 11). □

4 *Untyped Functions*

We now turn to an untyped functional language, the λ-calculus. The syntax is nearly the same as that of the simply-typed λ-calculus, except that λ abstractions no longer come with a type annotation on their parameter. Because the λ-calculus is untyped, we consider any syntactically well-formed expression to be in the language (no type system necessary) and any closed expression is a valid program. We study here the call-by-value (CBV) version of the λ-calculus; its syntax is presented in Figure 12 . As we did for the STLC, we include arithmetic on natural numbers.

Reading: Siek [2017]

numbers	$n \in \mathbb{N}$	
variables	$x \in \mathbb{X}$	
arithmetic	$\oplus ::=$	$+ \mid \times$
expressions	$e \in \mathbb{E} ::=$	$n \mid e \oplus e \mid x \mid \lambda x.\, e \mid (e\, e)$

Figure 12: Syntax of the λ-calculus with arithmetic.

4.1 Semantics

In the λ-calculus, functions are first-class entities, so they can be passed to other functions and returned from them. This introduces a difficulty in trying to give a semantics in terms of regular mathematical sets and functions. It seems that we need a set $\mathbb{D}$ that solves the following equation.

$$\mathbb{D} = \mathbb{N} + (\mathbb{D} \rightharpoonup \mathbb{D}) \tag{4}$$

But such a set cannot exist because the size of $\mathbb{D} \rightharpoonup \mathbb{D}$ is necessarily larger than $\mathbb{D}$!

There are several ways around this problem. One approach is to consider only continuous functions, which cuts down the size enough to make it possible to solve the equation. The first model of the λ-calculus, D_∞ of Scott [1970], takes this approach. We shall study D_∞ in Section 4.8.

There is a different approach that does not require solving the above equation, but recognizes that when passing a function to another function, one doesn't need to pass the entirety of the function (which is infinite), but one can instead pass a finite subset of the function's graph. The trouble of deciding which finite subset to pass can be sidestepped by trying all possible subsets. This would of course be prohibitive if our current goal was to implement the λ-calculus, but this "inefficiency" does not pose a problem for a semantics, a *specification*, of the λ-calculus. The various semantics that take this approach are called *graph models*. This first such model was T_C^* of Plotkin [1972]. Then came $\mathcal{P}(\omega)$ of Scott [1976] and the simplified D_A of Engeler [1981]. We shall study these three models in Section 4.6. The discussion of $\mathcal{P}(\omega)$ will also show how to recover a solution to Equation (4). But first, we study the graph model that I discovered [Siek, 2017].

As mentioned above, the graph models work with finite subsets of a function's graph, which enables the use of finite values (or elements). For my model, the elements are literally just the integers and finite graphs (association tables), so they can be defined by the following grammar:[2]

$$d \in \mathbb{D} ::= n \mid \{(d_1, d_1'), \ldots, (d_n, d_n')\}$$

[2] This definition of element originates from the work on semantic subtyping of Frisch et al. [2008].

Equivalently, we can defined $\mathbb{D}$ as the least solution of the following equation

$$\mathbb{D} = \mathbb{Z} + \mathcal{P}_f(\mathbb{D} \times \mathbb{D})$$

Let t range over the finite graphs, that is, $t \in \mathcal{P}_f(\mathbb{D} \times \mathbb{D})$ and D range over subsets of $\mathbb{D}$ (sometimes finite, sometimes infinite depending, on the cirmcumstances).

Relational Semantics We shall study several different presentations of my graph model. We start with the relational semantics because it is similar to a big-step operational semantics (aka. natural semantics [Kahn, 1987]), which hopefully is familiar to many readers. We shall see that an important difference is that, unlike big-step semantics, this semantics is not operational, i.e., it is not suggestive of an implementation.

We write $\rho \vdash e \Rightarrow d$ for the relation that holds when e evaluates to d in environment ρ. An environment ρ is a mapping of variables to elements and $\emptyset$ is the empty environment. The following is an example and a non-example for this relation.

$$\emptyset \vdash (20 + 1) \times 2 \Rightarrow 42 \quad \text{and} \quad \neg\, \emptyset \vdash 21 \times (2 + 3) \Rightarrow 42.$$

We represent a function by a finite table, which means that the user of the semantics can ask whether a function maps certain inputs to certain outputs. For example, we have

$$\begin{aligned} \emptyset &\vdash \lambda x.\, x + 1 \Rightarrow \{(41, 42), (-10, -9)\}, \\ \emptyset &\vdash \lambda x.\, x + 1 \Rightarrow \{(1, 2), (41, 42)\}, \text{ and} \\ \neg\, \emptyset &\vdash \lambda x.\, x + 1 \Rightarrow \{(1, 1)\}. \end{aligned}$$

For deterministic programming languages, a relational semantics typically specifies a unique output value for a program. As you can see here, our declarative semantics instead relates a λ abstraction to many different tables. The tables should be interpreted as providing only positive information and not negative, that is, the absence of an entry in a table does not mean anything.

For programs whose output is a function, as in the above example, it helps to think of the parameter of the function as a query to the user of the semantics. In the above, the semantics asks for a value of x and gets several different responses from the user (41, -10, and 1). So the query-response nature of the semantics gets flipped when dealing with functions.

One of the main features of the λ-calculus is that functions are first class and can therefore be passed to other functions. For example, if we let $t_0 = \{(1, 2)\}$ and $t_1 = \{(0, 1), (1, 2)\}$, then we have

$$\emptyset \vdash \lambda x.\, x \Rightarrow \{(t_0, t_0), (t_1, t_1)\} \text{ and } \neg\, \emptyset \vdash \lambda x.\, x \Rightarrow \{(t_0, t_1)\}.$$

$$\boxed{\rho \vdash e \Rightarrow d}$$

$$\frac{}{\rho \vdash n \Rightarrow n} \qquad \frac{\rho \vdash e_1 \Rightarrow n_1 \quad \rho \vdash e_2 \Rightarrow n_2}{\rho \vdash e_1 \oplus e_2 \Rightarrow n_1 \oplus n_2} \qquad \frac{d \sqsubseteq \rho(x)}{\rho \vdash x \Rightarrow d}$$

$$\frac{\forall (d, d') \in t.\ \rho(x{:=}d) \vdash e \Rightarrow d'}{\rho \vdash \lambda x.\, e \Rightarrow t} \qquad \frac{\begin{array}{cc}\rho \vdash e_1 \Rightarrow t & \rho \vdash e_2 \Rightarrow d_2 \\ (d_1, d_1') \in t & d_1 \sqsubseteq d_2 \quad d \sqsubseteq d_1'\end{array}}{\rho \vdash (e_1\, e_2) \Rightarrow d}$$

Figure 13: A relational semantics for CBV λ-calculus with arithmetic.

The way in which this program can return a function brings up an interesting question. What if instead of asking whether, given input t_1, the output is a function that maps 0 to 1 and 1 to 2, we only ask if the output maps 1 to 2.

$$\emptyset \vdash \lambda x.\, x \Rightarrow \{(t_1, t_0)\}$$

One would think that the answer should still be "yes". It turns out the answer had better be yes! Otherwise the semantics cannot handle self application, which would be a deal breaker for the λ-calculus. Consider the following example of self application. Is there a table t_3 such that we get 1 as a response.

$$\emptyset \vdash \lambda f.\, f\, f \Rightarrow \{(t_3, 1)\})$$

It seems that the application of f to itself would require that $(t_3, 1) \in t_3$, but such circularity is not possible because we have inductively defined our values[3]. But if we allow a "yes" response when the query is a subset of the answer, then for example, we could let $t_3 = \{(\emptyset, 1)\}$ and then answering t_3 for the query $\emptyset$ is fine because $\emptyset \subseteq t_3$. This can be thought of as a kind of subsumption: one can use a more-defined table in places where a less-defined table is expected.

[3] Could we instead coinductively define the values?

To make our definitions uniform, we lift the subset operation to all elements, writing $d \sqsubseteq d'$, which we define as follows:

$$n \sqsubseteq n \qquad \frac{t \subseteq t'}{t \sqsubseteq t'}$$

Proposition 16 ($\sqsubseteq$ is a partial order)**.**

1. *$d \sqsubseteq d$ (reflexive) and*
2. *if $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3$, then $d_1 \sqsubseteq d_3$ (transitive).*
3. *if $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1$, then $d_1 = d_2$ (antisymmetric).*

Motivated by the above discussion, we give an inductive definition of the relation $\rho \vdash e \Rightarrow d$ in Figure 13. The rules for λ abstraction

and application are the most important and quite different from prior relational semantics. The rule for $\lambda x.\, e$ states that the result is some table t if, for every input-output pair (d, d') in t, extending the environment with $x := d$ and evaluating the body e results in d'. We like to think of the table t as being produced by an oracle that looks into the future and chooses a table whose domain is big enough to handle all of the subsequent applications of it in the program execution. The premise of the rule makes sure that the table t is a subset of the function described by $\lambda x.\, e$.

The rule for a function application $(e_1\, e_2)$ performs table lookup. Expression e_1 evaluates to a table t and e_2 evaluates to d_2. The rule then finds a match for the argument d_2 in the table t and obtains the associated output value. The twist is that when finding a match, instead of using equality on values we use $\sqsubseteq$ to allow for the subsumption that we discussed above. Also, we use $\sqsubseteq$ a second time to relate the output value from the table and the result value d, to make sure that subsumption is admissible (Proposition 19).

The rules for integers and arithmetic are straightforward. Regarding variables, the twist is that a variable evaluates to any value d such that $d \sqsubseteq \rho(x)$, again to make sure that subsumption is admissible.

Example of Recursion In our earlier discussion we showed how the declarative semantics can support self application, but the reader may yet be wondering whether it supports recursion. We ultimately answer the question by proving that the semantics is equivalent to the operational semantics (Theorem 21 and 24). Nevertheless, it may be helpful to see an example of a recursive function to see how it works. The main idea is that of a Matryoshka doll: we shall nest ever-smaller versions of a recursive function inside of itself.

Recall the Z combinator (the version of the Y combinator for strict languages):

$$M \equiv \lambda x.\, f\, (\lambda y.\, (x\, x)\, y) \qquad Z \equiv \lambda f.\, M\, M$$

The factorial function is defined as follows, with a parameter r for calling itself recusively. We give names (F and H) to the λ abstractions because we shall define tables for each of them.

$$F \equiv \lambda n.\, \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n \times r\, (n-1) \qquad H \equiv \lambda r.\, F \qquad \mathit{fact} \equiv Z\, H$$

We begin with the tables for F: we define a function F_t that gives the table for just one input n; it simply maps n to n factorial.

$$F_t(n) = \{(n, n!)\}$$

The tables for H map a factorial table for $n-1$ to a table for n. We

environments $\quad \rho \in \mathbb{X} \rightharpoonup \mathbb{D}$

Ordering

$$n \sqsubseteq n \qquad \frac{t \subseteq t'}{t \sqsubseteq t'}$$

Application

$$\_ \cdot \_ : \mathcal{P}(\mathbb{D}) \times \mathcal{P}(\mathbb{D}) \to \mathcal{P}(\mathbb{D})$$

$$D_1 \cdot D_2 \stackrel{\text{def}}{=} \left\{ d_3 \;\middle|\; \begin{array}{l} \exists t d_2 d d', t \in D_1, d_2 \in D_2, \\ (d, d') \in t, d \sqsubseteq d_2, d_3 \sqsubseteq d' \end{array} \right\}$$

Semantics

$$P\, e \stackrel{\text{def}}{=} \{d \mid d \in E\, e\, \emptyset\}$$

$$E : \mathbb{E} \to (\mathbb{X} \rightharpoonup \mathbb{D}) \to \mathcal{P}(\mathbb{D})$$

$$E\, n\, \rho = \{n\}$$

$$E\, (e_1 \oplus e_2)\, \rho = \{n_1 \oplus n_2 \mid n_1 \in E\, e_1\, \rho, n_2 \in E\, e_2\, \rho\}$$

$$E\, x\, \rho = \{d \mid d \sqsubseteq \rho\, x\}$$

$$E\, (\lambda x.\, e)\, \rho = \{t \mid \forall (d, d') \in t, d' \in E\, e\, \rho(x \mapsto d))\}$$

$$E\, (e_1\, e_2)\, \rho = (E\, e_1\, \rho) \cdot (E\, e_2\, \rho)$$

Figure 14: Semantics of the λ-calculus

invite the reader to check that $\emptyset \vdash H \Rightarrow H_t(n)$ for any n.

$$H_t(n) = \{(\emptyset, F_t(0)), (F_t(0), F_t(1)), \ldots, (F_t(n-1), F_t(n))\}$$

Next we come to the most important part: describing the tables for M. Recall that M is applied to itself; this is where the analogy to Matryoshka dolls comes in. We define M_t by recursion on n. Each $M_t(n)$ extends the smaller version of itself, $M_t(n-1)$, with one more entry that maps the smaller version of itself to the table for factorial of $n-1$.

$$M_t(0) = \emptyset$$
$$M_t(n) = M_t(n-1) \cup \{(M_t(n-1), F_t(n-1))\}$$

The tables M_t are valid meanings for M because $f{:=}H_t(k) \vdash M \Rightarrow M_t(n+1)$ for any $n \leq k$. The application of M to itself is OK, that is, $f{:=}H_t(k) \vdash M\, M \Rightarrow F_t(n)$, because we have $(M_t(n), F_t(n)) \in M_t(n+1)$, $M_t(n) \sqsubseteq M_t(n+1)$, and $F_t(n) \sqsubseteq F_t(n)$.

To finish things up, the tables for Z map the H_t's to the factorial tables.

$$Z_t(n) = \{(H_t(n), F_t(n))\}$$

Then we have $\emptyset \vdash Z \Rightarrow Z_t(n)$ for all n. So $\emptyset \vdash Z\, H \Rightarrow F_t(n)$ for any n and therefore $\emptyset \vdash \mathit{fact}\, n \Rightarrow n!$.

Denotational Semantics Next we present a second semantics based on the graph model, but this time it is a function and not a relation. That is, it is a denotational semantics. This denotational semantics is straightforward to derive from the relational semantics because it is well known that relations are isomorphic to set-valued functions. In particular

$$\mathcal{P}(\mathbb{E} \times (\mathbb{X} \rightharpoonup \mathbb{V}) \times \mathbb{V}) \quad \cong \quad \mathbb{E} \to (\mathbb{X} \rightharpoonup \mathbb{V}) \to \mathcal{P}(\mathbb{V})$$

Figure 14 shows the results of converting the semantics from a relation into a function. The meaning of an abstraction $(\lambda x.\, e)$ is the set

Figure 15: Interpreter for the λ-calculus

$$
\begin{array}{lll}
\text{values} & v \in \mathbb{V} ::= & n \mid \langle \lambda x.\, e, \varrho \rangle \\
\text{env.} & \varrho \in \mathbb{X} \rightharpoonup \mathbb{V} &
\end{array}
$$

$$
\begin{aligned}
I\, n\, \varrho &= n \\
I\, (e_1 \oplus e_2)\, \varrho &= n_1 \oplus n_2 \\
&\quad \text{if } I\, e_1\, \varrho = n_1, I\, e_2\, \varrho = n_2 \\
I\, x\, \varrho &= \varrho(x) \\
I\, (\lambda x.\, e)\, \varrho &= \langle \lambda x.\, e, \varrho \rangle \\
I\, (e_1\, e_2)\, \varrho &= I\, e\, [x \mapsto v] \varrho' \\
&\quad \text{if } I\, e_1\, \varrho = \langle \lambda x.\, e, \varrho' \rangle, I\, e_2\, \varrho = v
\end{aligned}
$$

of all tables t such that each input-output entry $(d, d') \in t$ makes sense for the λ. That is, $d' \in E\, e\, \rho(x \mapsto d)$. To define the meaning of application, we introduce an auxilliary application operator $D_1 \cdot D_2$, which applies all the tables in D_1 to all the elements (arguments) in D_2, collecting all of the results into a set.

4.2 *Interpreter*

We turn to an implementation of the λ-calculus, the interpreter in Figure 15. The interpreter is based on the notion of a closure, which pairs a λ abstraction with its environment to ensure that the free variables get their definitions from the lexical scope.

We prove the correctness of the interpreter in two steps. First we show that if the semantics says the result of a program should be an number n, then the interpreter also produces n. The second part is to show that if the interpreter produces an answer n, then the semantics agrees that n should be the answer.

For the first part, we need to relate denotations (elements d) to the values v used by the interpreter.

$$
\begin{aligned}
\mathcal{G}(n) &= \{n\} \\
\mathcal{G}(t) &= \left\{ \langle \lambda x.\, e, \varrho \rangle \;\middle|\; \begin{array}{l} \forall (d, d') \in t, v \in \mathcal{G}(d), \\ \exists v', I\, e\, \varrho(x \mapsto v) = v' \text{ and } v' \in \mathcal{G}(d') \end{array} \right\}
\end{aligned}
$$

Similarly, we relate semantic environments to the interpreter's environments.

$$
\frac{}{\mathcal{G}(\emptyset, \emptyset)} \qquad \frac{v \in \mathcal{G}(d) \qquad \mathcal{G}(\rho, \varrho)}{\mathcal{G}([x \mapsto d]\rho, [x \mapsto v]\varrho}
$$

Leading up to the first theorem, we establish the following lemmas.

Figure 16: Reduction semantics of the CBV λ-calculus

$$\begin{array}{llll} \text{values} & v & ::= & n \mid \lambda x.\, e \\ \text{eval. frames} & F & ::= & \Box \oplus e \mid v \oplus \Box \mid (\Box\, e) \mid (v\, \Box) \end{array}$$

$$(\delta)\frac{\delta(\oplus, n_1, n_2) = n_3}{n_1 \oplus n_2 \longrightarrow n_3} \qquad (\beta_v)\frac{}{(\lambda x.\, e)\, v \longrightarrow [x := v]e}$$

$$(\text{cong})\frac{e \longrightarrow e'}{F[e] \longrightarrow F[e']}$$

Lemma 17 ($\mathcal{G}$ is downward closed)**.** *If $v \in \mathcal{G}(d)$ and $d' \sqsubseteq d$, then $v \in \mathcal{G}(d')$.*

Lemma 18. *If $\mathcal{G}(\rho, \varrho)$, then $\varrho(x) \in \mathcal{G}(\rho(x))$*

$$X \vdash \rho \sqsubseteq \rho' \stackrel{\text{def}}{=} \forall x, x \in X \implies \rho(x) \sqsubseteq \rho'(x)$$

Proposition 19 (Subsumption)**.**
If $e \in E\, e\, \rho$, $d' \sqsubseteq d$, and $\mathrm{FV}(e) \vdash \rho \sqsubseteq \rho'$, then $d' \in E\, e\, \rho'$.

Lemma 20. *If $d \in E\, e\, \rho$ and $\mathcal{G}(\rho, \varrho)$, then $I\, e\, \varrho = v$, $v \in \mathcal{G}(d)$ for some v.*

Theorem 21 (Adequacy)**.** *If $E\, e\, \emptyset = E\, n\, \emptyset$, then $I\, e\, \emptyset = n$.*

4.3 Reduction Semantics

Figure 16 defines the reduction semantics for the λ-calculus.

Lemma 22 (Invariance under Substitution)**.**

$$E\, [x := v]e\, \rho = \bigcup_{d' \in E\, v\, \emptyset} E\, e\, \rho(x \mapsto d')$$

Proposition 23 (Invariance under reduction)**.**
If $e \longrightarrow e'$, then $\forall \rho, E\, e\, \rho = E\, e'\, \rho$.

Theorem 24 (Completeness wrt. reduction)**.**
If $e \longrightarrow^* n$, then $E\, e\, \emptyset = E\, n\, \emptyset$.

Definition 12 (Contexts)**.**

$$C ::= \Box \mid C + e \mid e + C \mid C \times e \mid e \times C \mid \lambda x.\, C \mid C\, e \mid e\, C$$

Proposition 25 (Congruence)**.**
If $\forall \rho, E\, e\, \rho = E\, e'\, \rho$, then $\forall \rho, E\, C[e]\, \rho = E\, C[e']\, \rho$ for any C.

We write $e \Downarrow$ to mean that e terminates, i.e., it reaches a value or gets stuck after some number of reduction steps.

Theorem 26 (Sound wrt. contextual equivalence)**.**
If $\forall \rho, E\, e\, \rho = E\, e'\, \rho$, then $C[e] \Downarrow$ iff $C[e'] \Downarrow$ for any closing context C.

4.4 *Equational Theory and Models*

Figure 17 defines the equational theory of two variants of the λ-calculus, the full λ-calculus with the rule β, and the call-by-value calculus with the rule β_v.

Figure 17: Equational Theory of the λ-calculus

$$(\text{refl}) \vdash e = e \quad (\text{sym}) \frac{\vdash e_1 = e_2}{\vdash e_2 = e_1} \quad (\text{trans}) \frac{\vdash e_1 = e_2 \quad \vdash e_2 = e_3}{\vdash e_1 = e_3}$$

$$(\text{cong}) \frac{\vdash e_1 = e_3 \quad \vdash e_2 = e_4}{\vdash e_1\, e_2 = e_3\, e_4} \quad (\xi) \frac{\vdash e = e'}{\vdash \lambda x.\, e = \lambda x.\, e'}$$

$$(\beta) \vdash (\lambda x.\, e_1)\, e_2 = [x := e_2] e_1$$

$$(\beta_v) \vdash (\lambda x.\, e)\, v = [x := v] e$$

The following notion of λ-model is used in the literature that captures a set of conditions that are sufficient for a model to satisfy the equational theory of the λ-calculus.

Definition 13 (λ-models)**.** *An* R-model of the λ-calculus *is a triple* $\langle D, \cdot, [\![\ ]\!] \rangle$ *such that* D *is a set,* $\_ \cdot \_ : D \times D \to D$*, and* $[\![\_]\!] : \mathbb{E} \to (\mathbb{X} \rightharpoonup D) \to D$*, and the following conditions hold.*

1. $[\![x]\!]\rho = \rho(x)$,
2. $[\![e_1\, e_2]\!]\rho = [\![e_1]\!]\rho \cdot [\![e_2]\!]\rho$,
3. *if* $(\lambda x.\, e)\, e'$ *is a R-redex,* $[\![(\lambda x.\, e)]\!]\rho \cdot [\![e']\!]\rho = [\![e]\!]\rho(x \mapsto [\![e']\!]\rho)$, *alternative:* $[\![(\lambda x.\, e)]\!]\rho \cdot d = [\![e]\!]\rho(x \mapsto d)$
4. $\forall x \in \mathrm{FV}(e)$, *if* $\rho(x) = \rho'(x)$, *then* $[\![e]\!]\rho = [\![e]\!]\rho'$.
5. *If* $\forall d \in D, [\![e]\!]\rho(x \mapsto d) = [\![e']\!]\rho(x \mapsto d)$, *then* $[\![\lambda x.\, e]\!]\rho = [\![\lambda x.\, e']\!]\rho$.

The semantic function E given in Figure 14 does not directly fit the above definition of λ-model because its environment maps variables to $\mathbb{D}$ instead of $\mathcal{P}(\mathbb{D})$. However, this difference can be bridged with the following alternative semantic function, based on an analogous construction for filter models [Alessi et al., 2006], which we discuss in Section 4.5.

$$E'\, e\, \rho' = \{d \mid \exists \rho, d \in E\, e\, \rho \text{ and } \rho \models \rho'\}$$

where

$$\rho \models \rho' \stackrel{\text{def}}{=} \forall x \in \mathrm{dom}(\rho), \rho(x) \in \rho'(x)$$

Actually, for the proof to go through, we can't let environments map variables to arbitrary sets, many of which couldn't possibly be the denotation of an expression. Such sets are missing two important properties of actual denotations; they are 1) downward closed with respect to $\sqsubseteq$, and 2) closed under $\sqcup$. A set with these closure conditions is called an *ideal*. We write $\mathcal{I}(\mathbb{D})$ for the ideals over $\mathbb{D}$. So E' has the following type.

$$E' : \mathbb{E} \to (\mathbb{X} \rightharpoonup \mathcal{I}(\mathbb{D})) \to \mathcal{I}(\mathbb{D})$$

Lemma 27. *$\langle \mathcal{I}(\mathbb{D}), \cdot, E' \rangle$ is a β_v-model of the λ-calculus.*

Proof.

1. We need to show that $E'\, x\, \rho' = \rho'(x)$.

$$\begin{aligned} E'\, x\, \rho' &= \{d \mid \exists \rho, d \in E\, x\, \rho, \rho \models \rho'\} \\ &= \{d \mid \exists \rho, \rho(x) = d, d \in \rho'(x)\} \\ &= \{d \mid d \in \rho'(x)\} \\ &= \rho'(x) \end{aligned}$$

2. We need to show that $E'\, (e_1\, e_2)\, \rho' = (E'\, e_1\, \rho') \cdot (E'\, e_2\, \rho')$.

$$\begin{aligned} E'\, (e_1\, e_2)\, \rho' &= \{d' \mid \exists \rho, d' \in E\, (e_1\, e_2)\, \rho, \rho \models \rho'\} \\ &= \{d' \mid \exists \rho, d' \in (E\, e_1\, \rho) \cdot (E\, e_2\, \rho), \rho \models \rho'\} \\ &= \{d' \mid \exists t d d_2 \rho, t \in E\, e_1\, \rho, d_2 \in E\, e_2\, \rho, (d, d') \in t, d \sqsubseteq d_2, \rho \models \rho'\} \\ &= \{d' \mid \exists t d d_2, (\exists \rho, t \in E\, e_1\, \rho, \rho \models \rho'), (\exists \rho, d_2 \in E\, e_2\, \rho, \rho \models \rho'), (d, d') \in t, d \sqsubseteq d_2\} \\ &\quad \text{by Proposition 19 and Lemma 28} \\ &= \{d' \mid \exists t d d_2, t \in E'\, e_1\, \rho', d_2 \in E'\, e_2\, \rho', (d, d') \in t, d \sqsubseteq d_2\} \\ &= (E'\, e_1\, \rho') \cdot (E'\, e_2\, \rho') \end{aligned}$$

3.

$$\begin{aligned} E'\, (\lambda x.\, e)\, \rho' \cdot D &= \{t \mid \exists \rho, \rho \models \rho', \forall (d, d) \in t, d' \in E\, e\, \rho(x \mapsto d)\} \cdot D \\ &= \{d_3 \mid \exists d_2 d d' \rho, \rho \models \rho', d' \in E\, e\, \rho(x \mapsto d), d_2 \in D, d \sqsubseteq d_2, d_3 \sqsubseteq d'\} \\ &= \{d \mid \exists \rho, d \in E\, e\, \rho, \rho \models \rho'(x \mapsto D)\} \\ &\quad \text{because } \rho(x \mapsto d) \models \rho'(x \mapsto D), d_3 \in E\, e\, \rho(x \mapsto d) \\ &= E'\, e\, \rho'(x \mapsto D) \end{aligned}$$

4. TODO

5. TODO

□

Lemma 28. *If $\rho_1 \models \rho'$, $\rho_2 \models \rho'$, then $\rho_1 \sqcup \rho_2 \models \rho'$.*

Proof. TODO This is where we probably need the requirements of being closed under union (ideal), dual to filters. □

Lemma 29 (Replace Variable)**.** *Suppose $\langle D, \cdot, [\![\ ]\!] \rangle$ is an R-model of the λ-calculus and $y \notin \mathrm{FV}(e)$.*

$$[\![e]\!]\rho(x \mapsto d) = [\![[x := y]e]\!]\rho(y \mapsto d)$$

Proof.

$$\begin{aligned} [\![e]\!]\rho(x \mapsto d) &= [\![\lambda x.\, e]\!]\rho \cdot d && \text{by condition 3} \\ &= [\![\lambda y.\, [x := y]e]\!]\rho \cdot d && \alpha \text{ conversion} \\ &= [\![[x := y]e]\!]\rho(y \mapsto d) && \text{by condition 3} \end{aligned}$$

□

Lemma 30 (Substitution)**.** *(5.3.3 of Barendregt [1984]) Suppose $\langle D, \cdot, [\![\ ]\!] \rangle$ is an R-model of the λ-calculus.*

1. *If $z \notin \mathrm{FV}(e)$, then $\forall \rho, [\![[x := z]e]\!]\rho = [\![e]\!]\rho(x \mapsto [\![z]\!]\rho)$.*
2. *If $\forall \rho, [\![[x := e']e]\!]\rho = [\![e]\!]\rho(x \mapsto [\![e']\!]\rho)$,*
   then $\forall \rho, [\![[x := e']\lambda y.\, e]\!]\rho = [\![\lambda y.\, e]\!]\rho(x \mapsto [\![e']\!]\rho)$.
3. *$\forall \rho, [\![[x := e']e]\!]\rho = [\![e]\!]\rho(x \mapsto [\![e']\!]\rho)$.*

Proof. We prove part 1, then part 2 using part 1, and finally part 3 using part 2 of this lemma.

1. Assume $z \notin \mathrm{FV}(e)$. Let $e \equiv e(x)$.

$$\begin{aligned} [\![e(z)]\!]\rho &= [\![e(z)]\!][z := \rho(z)]\rho && \text{by condition 4} \\ &= [\![e(x)]\!][x := \rho(z)]\rho && \text{by Lemma 29} \\ &= [\![e(x)]\!][x := [\![z]\!]\rho]\rho && \text{by condition 1} \end{aligned}$$

2. To prove this implication, we assume its premise:

$$\forall \rho, [\![[x := e']e]\!]\rho = [\![e]\!]\rho(x \mapsto [\![e']\!]\rho) \tag{5}$$

   Let ρ be an arbitrary environment. We consider two cases, whether $x \in \mathrm{FV}(e')$ or not.

   - Assume $x \in \mathrm{FV}(e')$. Let z be a fresh variable.

$$\begin{aligned} [\![[x := e']\lambda y.\, e]\!]\rho &= [\![[z := e'][x := z]\lambda y.\, e]\!]\rho \\ &= [\![[x := z]\lambda y.\, e]\!]\rho(z \mapsto [\![e']\!]\rho) && ?? \\ &= [\![\lambda y.\, e]\!]\rho(z \mapsto [\![e']\!]\rho)(x \mapsto [\![e']\!]\rho) && \text{by part 1} \\ &= [\![\lambda y.\, e]\!]\rho(x \mapsto [\![e']\!]\rho) \end{aligned}$$

- Assume $x \notin \mathrm{FV}(e')$. Let $d \in D$ be an arbitrary element.

$$\begin{aligned}
[\![[x:=e']e]\!]\rho(x\mapsto[\![e']\!]\rho)(y\mapsto d) &= [\![[x:=e']e]\!]\rho(y\mapsto d) && \text{cond. 4}\\
&= [\![e]\!]\rho(y\mapsto d)(x\mapsto[\![e']\!]\rho) && \text{by eq. (5)}\\
&= [\![e]\!]\rho(x\mapsto[\![e']\!]\rho)(y\mapsto d) && x \neq y
\end{aligned} \tag{6}$$

$$\begin{aligned}
[\![[x:=e']\lambda y.\, e]\!]\rho &= [\![[x:=e']\lambda y.\, e]\!]\rho(x\mapsto[\![e']\!]\rho) && \text{condition 4}\\
&= [\![\lambda y.\, [x:=e']e]\!]\rho(x\mapsto[\![e']\!]\rho)\\
&= [\![\lambda y.\, e]\!]\rho(x\mapsto[\![e']\!]\rho) && \text{cond. 5 and eq. (6)}
\end{aligned}$$

3. The proof is by induction on e.

   Case $e = y$: If $y \neq x$, then $[\![[x:=e']y]\!]\rho = \rho(y) = [\![y]\!]\rho(x\mapsto[\![e']\!]\rho)$.
   If $y = x$, then $[\![[x:=e']y]\!]\rho = [\![e']\!]\rho = [\![y]\!]\rho(x\mapsto[\![e']\!]\rho)$.

   Case $e = \lambda y.\, e_1$: By the IH we have

$$\forall\rho, [\![[x:=e']e_1]\!]\rho = [\![e_1]\!]\rho(x\mapsto[\![e']\!]\rho)$$

   Then by part 2 of this lemma we conclude that

$$\forall\rho, [\![[x:=e']\lambda y.\, e_1]\!]\rho = [\![\lambda y.\, e_1]\!]\rho(x\mapsto[\![e']\!]\rho)$$

   Case $e = e_1\, e_2$: Let ρ be an arbitrary environment.

$$\begin{aligned}
[\![[x:=e'](e_1\, e_2)]\!]\rho &= [\![[x:=e']e_1\ [x:=e']e_2]\!]\rho\\
&= [\![[x:=e']e_1]\!]\rho \cdot [\![[x:=e']e_2]\!]\rho && \text{by cond. 2}\\
&= [\![e_1]\!]\rho(x\mapsto[\![e']\!]\rho) \cdot [\![e_2]\!]\rho(x\mapsto[\![e']\!]\rho) && \text{by IH}\\
&= [\![e_1\, e_2]\!]\rho(x\mapsto[\![e']\!]\rho) && \text{by cond. 2}
\end{aligned}$$

□

Lemma 31 (Soundess of R-models wrt. λ theory)**.** *Suppose $\langle D, \cdot, [\![\ ]\!]\rangle$ is an R-model of the λ-calculus.*

$$\textit{if } R \vdash e = e',\ \textit{then } \forall\rho, [\![e]\!]\rho = [\![e']\!]\rho$$

Proof. The proof is by induction on $R \vdash e = e'$.

Case (refl): We have $\forall\rho, [\![e]\!]\rho = [\![e]\!]\rho$ because $[\![\ ]\!]$ is a function.

Case (sym): By the IH, we have $\forall\rho, [\![e_1]\!]\rho = [\![e_2]\!]\rho$. So it trivially follows that $\forall\rho, [\![e_2]\!]\rho = [\![e_1]\!]\rho$.

Case (trans): By the IH, we have $\forall\rho, [\![e_1]\!]\rho = [\![e_2]\!]\rho$ and $\forall\rho, [\![e_2]\!]\rho = [\![e_3]\!]\rho$. Thus, we immediately have $\forall\rho, [\![e_1]\!]\rho = [\![e_3]\!]\rho$.

Case (cong): Let ρ be an arbitrary environment. We need to show that $[\![(e_1\, e_2)]\!]\rho = [\![(e_3\, e_4)]\!]\rho$.

$$\begin{aligned} [\![e_1\, e_2]\!]\rho &= [\![e_1]\!]\rho \cdot [\![e_2]\!]\rho && \text{condition 2} \\ &= [\![e_3]\!]\rho \cdot [\![e_4]\!]\rho && \text{by IH} \\ &= [\![e_3\, e_4]\!]\rho && \text{condition 2} \end{aligned}$$

Case ξ: We need to show $\forall\rho, [\![\lambda x.\, e]\!]\rho = [\![\lambda x.\, e']\!]\rho$. Let ρ be an arbitrary environment. By condition 5 it suffices to show that $[\![e]\!]\rho(x{\mapsto}d) = [\![e']\!]\rho(x{\mapsto}d)$, and that follows from the IH.

Case R: Let ρ be an arbitrary environment. We need to show that $[\![(\lambda x.\, e)\, e']\!]\rho = [\![[x{:=}e']e]\!]\rho$, where $(\lambda x.\, e)\, e'$ is an R-redex.

$$\begin{aligned} [\![(\lambda x.\, e)\, e']\!]\rho &= [\![\lambda x.\, e]\!]\rho \cdot [\![e']\!]\rho && \text{condition 2} \\ &= [\![e]\!]\rho(x{\mapsto}[\![e']\!]\rho) && \text{condition 3} \\ &= [\![[x{:=}e']e]\!]\rho && \text{by Lemma 30} \end{aligned}$$

□

Theorem 32. *$\langle \mathcal{I}(\mathbb{D}), \cdot, E' \rangle$ is a model of the CBV λ-calculus. That is,*

$$\text{if } \beta_v \vdash e = e',\ \text{then } \forall\rho, E'\, e\, \rho = E'\, e'\, \rho$$

Resources:

- Barendregt [1984] (Section 2.1 for equational theory and Section 5.3 for (syntactical) λ-models).
- Alessi et al. [2006]
- Hindley and Longo [1980]

4.5 *Intersection Types and Filter models*

Reading: Alessi et al. [2006]

The addition of intersection types to a type system dramatically increases its precision. In fact, intersection types are fully precise; an intersection type system specifies the dynamic semantics of a program. There are a large number of different but related intersection type systems. Alessi et al. [2006] give a survey of them and prove general results that relate them to equational theories of λ-calculi. Here we focus on just one of them, a variant of the $\mathcal{EHR}$ system for the call-by-value λ-calculus of Egidi et al. [1992]. Figure 18 defines types and the subtyping and type equivalence rules. With respect to Egidi et al. [1992], here we have added a singleton type n for natural numbers.

The intuition behind the typing rules for intersection types is that if we think of types as sets of values, subtyping acts like set inclusion

($\subseteq$), and intersection is like set intersection. So, for example, the rule (incl-L) corresponds to an obviously true statement about sets, that $A \cap B \subseteq A$.

The type ν characterizes all functions, so the (ν) subtyping rule says that any function type is a subtype of ν. The rule ($\to\ -\cap$) captures the intuition that if you call a function of intersection type $(A \to B) \cap (A \to C)$ with an argument of type A, we can use $A \to B$ to deduce that the result is in B and use $A \to C$ to deduce that the result is also in C. Thus, the result must be in $B \cap C$. The (η) subtyping rule is the standard one for function types.

Figure 18: Types, subtyping, and equivalence

$$\text{types} \quad A, B, C \in \mathbb{T} ::= \quad n \mid \nu \mid A \to B \mid A \cap B$$

$$\text{(refl)}\ A <: A \qquad \text{(idem)}\ A <: A \cap A$$

$$\text{(incl-L)}\ A \cap B <: A \qquad \text{(incl-R)}\ A \cap B <: B$$

$$\text{(mon)}\ \frac{A <: A' \quad B <: B'}{A \cap B <: A' \cap B'} \qquad \text{(trans)}\ \frac{A <: B \quad B <: C}{A <: C}$$

$$(\nu)\ A \to B <: \nu$$

$$(\to -\cap)\ (A \to B) \cap (A \to C) <: A \to (B \cap C) \qquad (\eta)\ \frac{A' <: A \quad B <: B'}{A \to B <: A' \to B'}$$

$$A \sim B \stackrel{\text{def}}{=} A <: B \text{ and } B <: A$$

Figure 19 defines the intersection type system. It includes the introduction rule for intersection, ($\cap$-intro), which assigns expression e the type $A \cap B$ if e can be typed with both A and B. The introduction rule for ν says that every λ has type ν, even if the body of the λ is ill typed! The subsumpton rule (sub), as usual, enables implicit up-casts with respect to subtyping. The rules for variables (var), λ abstraction ($\to$-intro), and application ($\to$-elim), are the same as in the simply-typed λ-calculus. The (nat) rule is typical of singleton types.

Figure 19: An Intersection Type System

$$\text{(var)}\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \text{(nat)}\frac{}{\Gamma \vdash n : n} \qquad (\nu\text{-intro})\frac{}{\Gamma \vdash \lambda x.\, e : \nu}$$

$$(\to\text{-intro})\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.\, e : A \to B} \qquad (\to\text{-elim})\frac{\Gamma \vdash e_1 : A \to B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1\, e_2 : B}$$

$$(\cap\text{-intro})\frac{\Gamma \vdash e : A \quad \Gamma \vdash e : B}{\Gamma \vdash e : A \cap B} \qquad \text{(sub)}\frac{\Gamma \vdash e : A \quad A <: B}{\Gamma \vdash e : B}$$

Proposition 33 (Subject Reduction and Expansion)**.**

- *If* $e \longrightarrow e'$ *and* $\Gamma \vdash e : A$, *then* $\Gamma \vdash e' : A$.
- *If* $e \longrightarrow e'$ *and* $\Gamma \vdash e' : A$, *then* $\Gamma \vdash e : A$.

The intersection type system can be made into a λ-model similar to the way we made E into a λ-model by constructing E'. Subtyping plays the same role as $\sqsubseteq$, but inverted. So instead of working with ideals, we work with filters.

Definition 14 (Filter)**.** *A subset* S *of types* $\mathbb{T}$ *is a* filter *if it is closed with respect to subtyping and intersection. That is,*

- *if* $A \in S$ *and* $A <: B$, *then* $B \in S$, *and*
- *if* $A \in S$ *and* $B \in S$, *then* $A \cap B \in S$.

The set of all filters is $\mathbb{F}$. *We define the upward closure of a set* $\uparrow S$ *as follows.*

$$\frac{A \in S}{A \in \uparrow S} \qquad \frac{A \in \uparrow S \quad A <: B}{B \in \uparrow S} \qquad \frac{A \in \uparrow S \quad B \in \uparrow S}{A \cap B \in \uparrow S}$$

Is the upward closure really needed in the definition of application? We are missing that for E and E'. TODO: mechanize this λ-model stuff to make sure one way or the other. -Jeremy

Definition 15. *The* filter λ-structure *is the triple* $\langle \mathbb{F}, \cdot, F \rangle$ *where*

$$\begin{aligned}
_ \cdot _ &: \mathbb{F} \times \mathbb{F} \to \mathbb{F} \\
S_1 \cdot S_2 &\stackrel{\text{def}}{=} \uparrow \{B \mid \exists A \in S_2, A \to B \in S_1\} \\
\Gamma \models \rho &\stackrel{\text{def}}{=} \forall x : B \in \Gamma, B \in \rho(x) \\
F &: \mathbb{E} \to (\mathbb{X} \rightharpoonup \mathbb{F}) \to \mathbb{F} \\
F\, e\, \rho &\stackrel{\text{def}}{=} \{A \mid \exists \Gamma, \Gamma \models \rho \text{ and } \Gamma \vdash e : A\}
\end{aligned}$$

Lemma 34. *The filter* λ*-structure* $\langle \mathbb{F}, \cdot, F \rangle$ *is a* β_v*-model of the* λ*-calculus.*

Theorem 35. *The filter* λ*-structure* $\langle \mathbb{F}, \cdot, F \rangle$ *is a model of the CBV* λ*-calculus. That is,*

$$\text{if } \beta_v \vdash e = e', \text{ then } \forall \rho, F\, e\, \rho = F\, e'\, \rho$$

4.6 Graph models of λ-calculus (T_C^*, D_A, and $\mathcal{P}(\omega)$)

The T_C^* *model of Plotkin [1972]* Let D range over sets of elements from T_C (finite or infinite). Figure 20 defines the semantics.

The D_A *model of Engeler [1981]* Let d range over elements of B_A. Let D range over sets of elements from B_A (finite or infinite). Figure 21 gives the semantics, following the presentation of Barendregt [1984] (Section 5.4).

D_A ordered by set inclusion $\subseteq$ is a cpo.

Figure 20: The T_C^* model

$$
\begin{aligned}
T_C &= C + \mathcal{P}_f(T_C) \times \mathcal{P}_f(T_C) \\
T_C^* &= \mathcal{P}(T_C) \\
\lambda^P &: [T_C^* \to T_C^*] \to T_C^* \\
\lambda^P f &= \{(D, D') \mid D' \subseteq f\, D\} \\
- \cdot_P - &: T_C^* \to T_C^* \to T_C^* \\
D_1 \cdot_P D_2 &\stackrel{\text{def}}{=} \bigcup \{D' \mid \exists D, (D, D') \in D_1, D \subseteq D_2\}
\end{aligned}
$$

$$
\begin{aligned}
E_P &: \mathbb{E} \to (\mathbb{X} \rightharpoonup T_C^*) \to T_C^* \\
E_P\, x\, \rho &= \rho(x) \\
E_P\, (\lambda x.\, e)\, \rho &= \lambda^P (\lambda D.\, E_P\, e\, [x \mapsto D] \rho) \\
E_P\, (e_1\, e_2)\, \rho &= (E_P\, e_1\, \rho) \cdot_P (E_P\, e_2\, \rho)
\end{aligned}
$$

Figure 21: The D_A model

$$
\begin{aligned}
B_A &= A + \mathcal{P}_f(B_A) \times B_A \\
D_A &= \mathcal{P}(B_A) \\
\lambda^E &: [D_A \to D_A] \to D_A \\
\lambda^E f &= \{(D, d') \mid d' \in f\, D\} \\
- \cdot_E - &: D_A \to D_A \to D_A \\
D_1 \cdot_E D_2 &= \{d' \mid \exists D, D \subseteq D_2 \wedge (D, d') \in D_1\}
\end{aligned}
$$

$$
\begin{aligned}
E_E &: \mathbb{E} \to (\mathbb{X} \rightharpoonup D_A) \to D_A \\
E_E\, x\, \rho &= \rho(x) \\
E_E\, (\lambda x.\, e)\, \rho &= \lambda^E (\lambda D.\, E_E\, e\, [x \mapsto D] \rho) \\
E_E\, (e_1\, e_2)\, \rho &= (E_E\, e_1\, \rho) \cdot_E (E_E\, e_2\, \rho)
\end{aligned}
$$

Definition 16. *A cpo D is* reflexive *if $[D \to D]$ is a retract of D. That is to say, there are continuous maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ such that for any $f \in [D \to D]$,*

$$(F \circ G)\, f = f$$

Lemma 36. *D_A is a reflexive cpo by choosing $F = \lambda xy.\, x \cdot_E y$ and $G = \lambda^E$. Therefore D_A is a λ-model.*

Proof. We need to show that $(F \circ G)\, f = f$ for any $f \in [D_A \to D_A]$.

$$\begin{aligned}
(F \circ G)\, f &= \lambda x.(\lambda^E f) \cdot_E x \\
&= \lambda x.\{d' \mid \exists D, D \subseteq x \wedge d' \in f\, D\} \\
&= \lambda x.\bigcup\{f\, D \mid D \subseteq x\} \\
&= \lambda x.\, f\, x && \text{by continuity of } f \\
&= f
\end{aligned}$$

□

Resources about D_A:

- Barendregt [1984] (Section 5.4),
- Gunter [1992] (Section 8.1), and
- Engeler [1981].

The $\mathcal{P}(\omega)$ model of Scott [1976] Figure 22 defines a semantics for the λ-calculus using the $\mathcal{P}(\omega)$ model of Scott [1976]. However, here we use the formulation of $\mathcal{P}(\omega)$ given by Barendregt [1984].

Figure 22: λ-calculus in $\mathcal{P}(\omega)$

$$\begin{aligned}
m, n &\in \mathbb{N} \\
\langle n, m\rangle &= \frac{1}{2}(n+m)(n+m+1) + m \\
e_n &= \{k_0, k_1, \ldots, k_{m-1}\} \quad \text{where } k_i < k_{i+1} \text{ and } n = \sum_{i<m} 2^{k_i}
\end{aligned}$$

$$\begin{aligned}
E_S &: \mathbb{E} \to (\mathbb{X} \rightharpoonup \mathcal{P}(\mathbb{N})) \to \mathcal{P}(\mathbb{N}) \\
E_S\, x\, \rho &= \rho\, x \\
E_S\, (\lambda x.\, e)\, \rho &= \{\langle n, m\rangle \mid m \in E_S\, e\, [x \mapsto e_n]\rho\} \\
E_S\, (e_1\, e_2)\, \rho &= \{m \mid \exists e_n.\, \langle n, m\rangle \in E_S\, e_1\, \rho \text{ and } e_n \subseteq E_S\, e_2\, \rho\}
\end{aligned}$$

4.7 Untyped as a uni-typed

$$
\begin{array}{lrcl}
\text{types} & A, B & ::= & \mathit{nat} \mid A \to B \mid A \times B \mid \mathit{dyn} \\
\text{type tag} & t & ::= & \mathit{fun} \mid \mathit{nat} \\
\text{coercion} & c & ::= & t! \mid t? \\
\text{expr.} & e \in \mathbb{E} & ::= & n \mid e + e \mid e \times e \mid x \mid \lambda x\!:\!A.\, e \mid (e\, e) \mid e\langle c \rangle
\end{array}
$$

$$
\begin{aligned}
U(n) &= n\langle \mathit{nat}! \rangle \\
U(e_1 + e_2) &= (U(e_1)\langle \mathit{nat}? \rangle + U(e_2)\langle \mathit{nat}? \rangle)\langle \mathit{nat}! \rangle \\
U(x) &= x \\
U(\lambda x.\, e) &= (\lambda x\!:\!\mathit{dyn}.\, U(e))\langle \mathit{fun}! \rangle \\
U(e_1\, e_2) &= U(e_1)\langle \mathit{fun}? \rangle\, U(e_2)
\end{aligned}
$$

Equational theory: STLC's +

$$e\langle t! \rangle \langle t? \rangle = e$$

Theorem 37. *If* $\vdash e = e'$, *then* $\vdash U(e) = U(e')$.

Proof. The proof is by induction on the derivation of $\vdash e = e'$.

Case $\vdash (\lambda x.\, e_1)\, e_2 = [x := e_2]e_1$

$$
\begin{aligned}
U((\lambda x.\, e_1)\, e_2) &= (U(\lambda x.\, e_1))\langle \mathit{fun}? \rangle\, U(e_2) \\
&= (\lambda x.\, U(e_1))\langle \mathit{fun}! \rangle \langle \mathit{fun}? \rangle\, U(e_2) \\
&= (\lambda x.\, U(e_1))\, U(e_2) \\
&= [x := U(e_2)]U(e_1) \\
&= U([x := e_2]e_1)
\end{aligned}
$$

Case $\vdash n_1 + n_2 = n_3$ where n_3 is $n_1 + n_2$

$$
\begin{aligned}
U(n_1 + n_2) &= (n_1\langle \mathit{nat}! \rangle \langle \mathit{nat}? \rangle + n_2\langle \mathit{nat}! \rangle \langle \mathit{nat}? \rangle)\langle \mathit{nat}! \rangle \\
&= (n_1 + n_2)\langle \mathit{nat}! \rangle \\
&= n_3\langle \mathit{nat}! \rangle \\
&= U(n_3)
\end{aligned}
$$

TODO

□

4.8 D_∞ model of λ-calculus

UNDER CONSTRUCTION

Answers to Exercises

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