Diagonalization, the Matrix Exponential and Jordan Canonical Form

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Lesson Objectives

- Observe that diagonalized state-space systems are easier solve.
- Review how to diagonalize systems using coordinate transformations
- Define the Matrix Exponential

Recall solution to 1st order ODEs w/ linear algebra

Example from a previous lesson:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$k = 4, c = 6, m = 2, F = u = 0$$

$$\mathbf{z}(0) = \begin{cases} 1 \\ 0 \end{cases}$$

In state space form…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}u$$

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{z}$$

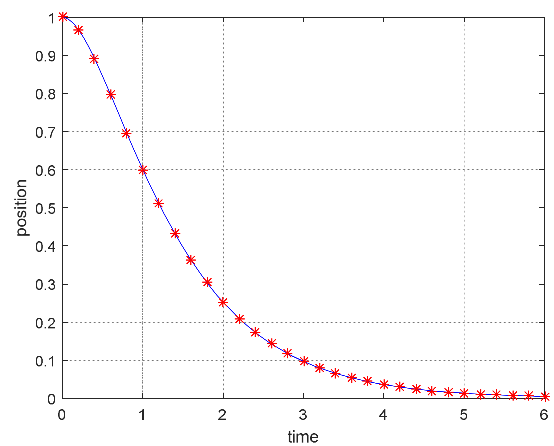
Had the general solution…

$$\mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$

which we solved for these parameters and got…

$$\boxed{\mathbf{z}(t) = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

which we verified using MATLAB…



Note that this procedure is significantly simpler if the **A** matrix is diagonal.

Diagonal example

Consider this system...

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{z} \quad \mathbf{z}(0) = \begin{cases} 1 \\ 1 \end{cases}$$

It still has the general solution…

$$\mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$

But the solution is much simpler…

$$\det\left(\lambda\mathbf{I} - \mathbf{A}\right) = 0$$

$$\left(\lambda + 2\right)\left(\lambda + 3\right) = 0$$

$$\Rightarrow \lambda = -2, -3$$

But we already knew the diagonal elements of a diagonal matrix are the eigenvalues right?

So the eigenvectors are

$$\left(\mathbf{A} - \lambda_i\mathbf{I}\right)\boldsymbol{\eta}_i = 0$$

$$\Rightarrow \lambda = -2 \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\boldsymbol{\eta}_1 = 0 \quad \Rightarrow \boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \lambda = -3 \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{\eta}_2 = 0 \quad \Rightarrow \boldsymbol{\eta}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{z}(0) = k_1 e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow k_1 = k_2 = 1$$

$$\mathbf{z}(\mathrm{t}) = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So diagonal systems are significantly easier to solve using linear algebra.

$z_j(\mathrm{t}) = z_j(0)e^{\lambda_j t}$ Can we use this observation?

Coordinate Transformations

Maybe there is a coordinate frame we can view our system from where the system is diagonal…

$$\frac{d\mathbf{z}_n}{dt} = \mathbf{A}_n\mathbf{z}_n \quad \text{where} \quad \mathbf{z}_n = \mathbf{M}\mathbf{z}_o$$

where *n* and *o* refer to the new and old coordinate systems respectively and $\mathbf{A_n}$ is a diagonal matrix. Thus,

$$\mathbf{M}\frac{d\mathbf{z}_o}{dt} = \mathbf{A}\mathbf{M}\mathbf{z}_o$$

$$\frac{d\mathbf{z}_o}{dt} = \underbrace{\mathbf{M}^{-1}\mathbf{A}\mathbf{M}}_{\text{Similarity Transform}}\mathbf{z}_o$$

Theorem: If $\mathbf{A}_n$=$\mathbf{M^{-1}AM}$ then $\mathbf{A_n}$ and $\mathbf{A}$ have the eigenvalues and an eigenvalue η of $\mathbf{A}$ corresponds to the $\mathbf{M^{-1}}$η of $\mathbf{A}_n$.

Proof:

$$\mathbf{A}_n = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} \qquad \Rightarrow \mathbf{A} = \mathbf{M}\mathbf{A}_n\mathbf{M}^{-1}$$

Since,

$$\mathbf{A}\boldsymbol{\eta} = \lambda\boldsymbol{\eta}$$

$$\mathbf{M}\mathbf{A}_n\mathbf{M}^{-1}\boldsymbol{\eta} = \lambda\boldsymbol{\eta}$$

$$\mathbf{A}_n\mathbf{M}^{-1}\boldsymbol{\eta} = \mathbf{M}^{-1}\lambda\boldsymbol{\eta}$$

$$\mathbf{A}_n\mathbf{M}^{-1}\boldsymbol{\eta} = \underbrace{\lambda}_{\text{same eigenvalue}}\underbrace{\mathbf{M}^{-1}\boldsymbol{\eta}}_{\text{new eigenvector}}$$

the theorem is correct.

Example

<u>Find</u>: the transformation **T** such that **Λ** in **A=TΛT⁻¹** is a diagonal matrix.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

<u>Solve</u>:

$$\det\left(\lambda \mathbf{I} - \mathbf{A}\right) = 0$$

$$\det \begin{bmatrix} \lambda + 2 & -1 \\ -1 & \lambda + 2 \end{bmatrix} = 0$$

$$\lambda^2 + 4\lambda + 4 - 1 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = -3, -1$$

Next, find the eigenvectors…

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right)\boldsymbol{\eta}_i = 0$$

$$\Rightarrow \lambda = -3$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\boldsymbol{\eta}_1 = 0 \qquad \Rightarrow \boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \lambda = -1$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\boldsymbol{\eta}_2 = 0 \qquad \Rightarrow \boldsymbol{\eta}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Consequence of simplification

<u>Given</u>: the following 1st order ODES

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{z} \qquad \mathbf{z}(0) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

<u>Solve</u>: Recall that

$$z_j(\mathrm{t}) = z_j(0)e^{\lambda_j t}$$

and,

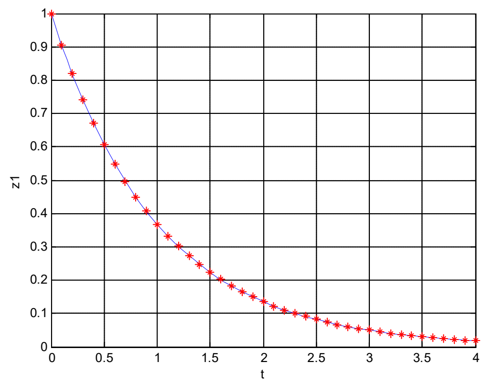
$$\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

therefore,

$$\mathbf{z}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\mathbf{z}(0)$$

$$\mathbf{z}(t) = \frac{1}{2}\begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \end{bmatrix}\mathbf{z}(0)$$

Did this give us the right answer (at least compared to ODE45()?




```
global A; A = [ -2 1; 1 -2 ];
[t,zn] = ode45('mySystem', [0 4], [1 1]);

z0=[1;1]; ta=0:.1:4;
for i=1:length(ta)
    A = [ exp(-ta(i))+exp(-3*ta(i)) exp(-ta(i))-exp(-3*ta(i));
          exp(-ta(i))+exp(-3*ta(i)) exp(-ta(i))-exp(-3*ta(i)) ];
    za(i,:)=0.5*A*z0;
end

plot(t, zn(:,1),'b'); hold on;
plot(ta, za(:,2),'r*');
```

yep. Note that...

$$\mathbf{z}(t) = \frac{1}{2}\begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

Matrix Exponential

To simplify the notation, let us define, the Matrix Exponential. Similar to the Taylor expansion of the scalar exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

Let the Matrix Exponential be defined as:

$$e^{\mathbf{X}} = \mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \frac{\mathbf{X}^4}{4!} + \dots.$$

In MATLAB, use `expm(X)`. We use this function just like we would any other function.

So, $\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z}$ Can be solved with $\mathbf{z}(\mathrm{t}) = e^{\mathbf{A}t}\mathbf{z}(0)$ So we don't have to think about each $z_j(\mathrm{t}) = z_j(0)e^{\lambda_j t}$

Which for our entire state-space system $\begin{aligned}\frac{d\mathbf{z}}{dt} &= \mathbf{A}\mathbf{z} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{z} + \mathbf{D}u\end{aligned}$ yields the homogeneous solution $\begin{aligned}\mathbf{z}(\mathrm{t}) &= e^{\mathbf{A}t}\mathbf{z}(0) \\ y &= \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0)\end{aligned}$

In MATLAB, use `initial(A,B,C,D,z0)`. To solve a problem in this form.

Matrix Exponential, Properties

As it should be, the Matrix Exponential result is invariant with respect to coordinate transformations.

$$
\begin{aligned}
e^{\mathbf{A}t} &= e^{\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}t} \\
&= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \ldots. \\
&= \mathbf{T}\mathbf{T}^{-1} + \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}t + \frac{\mathbf{T}\mathbf{\Lambda}^2\mathbf{T}^{-1}t^2}{2!} + \frac{\mathbf{T}\mathbf{\Lambda}^3\mathbf{T}^{-1}t^3}{3!} + \frac{\mathbf{T}\mathbf{\Lambda}^4\mathbf{T}^{-1}t^4}{4!} + \ldots. \\
&= \mathbf{T}\mathbf{T}^{-1} + \mathbf{T}\mathbf{\Lambda}t\mathbf{T}^{-1} + \mathbf{T}\frac{\mathbf{\Lambda}^2t^2}{2!}\mathbf{T}^{-1} + \mathbf{T}\frac{\mathbf{\Lambda}^3t^3}{3!}\mathbf{T}^{-1} + \mathbf{T}\frac{\mathbf{\Lambda}^4t^4}{4!}\mathbf{T}^{-1} + \ldots. \\
&= \mathbf{T}\left(1 + \mathbf{\Lambda}t + \frac{\mathbf{\Lambda}^2t^2}{2!} + \frac{\mathbf{\Lambda}^3t^3}{3!} + \frac{\mathbf{\Lambda}^4t^4}{4!} + \ldots.\right)\mathbf{T}^{-1} \\
&= \mathbf{T}e^{\mathbf{\Lambda}t}\mathbf{T}^{-1}
\end{aligned}
$$

So we could simply have written our example as:

$$\mathbf{z}(t) = \mathbf{T}e^{\mathbf{\Lambda}t}\mathbf{T}^{-1}\mathbf{z}(0)$$

And other properties of exponents apply as well.

$$e^{x+y} = e^x e^y$$

$$\left(e^x\right)^p = e^{xp}$$

$$\frac{de^x}{dt} = e^x$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$\sqrt[p]{e^x} = e^{\frac{x}{p}}$$

$$etc.$$

Diagonalization Summary

- Diagonal systems are relatively simple to solve.
- We can often diagonalize a system via a coordinate transformation and an appropriately applied similarity transformation to the system parameter matrix.
- These transformations do not change the underlying system behavior.
- The solution to diagonalized systems are often easier to visualize. MATLAB minimizes the resulting linear algebra drudgery.
- The matrix exponential allows us to succinctly write and manipulate examined systems using linear algebra.
  - And thus (as we will see soon) allows us to examine state-space systems using other solution methods.
- But not all systems, can be diagonalized….

Jordan Canonical Form

Lesson Objectives

- Diagonalized state-space systems are easier to solve.
- But not all systems can be diagonalized
  - Define **Jordan Canonical Form** as a system in its most ‘diagonal’ form.
- Solve systems in Jordan Canonical Form

So…

Diagonalized systems are easier to solve…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \mathbf{z}$$

Since…

$$z_j(\mathrm{t}) = z_j(0) e^{\lambda_j t}$$

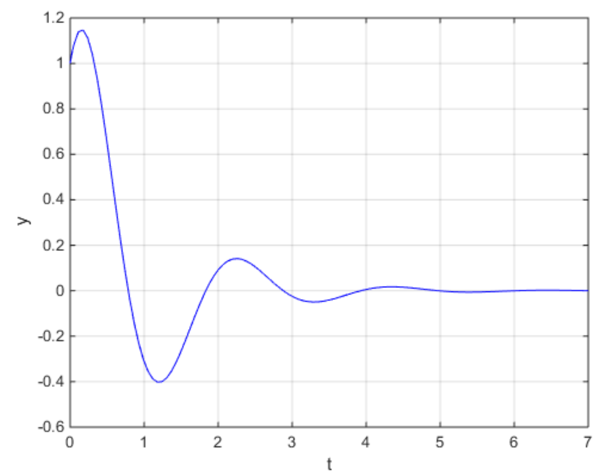
But not all systems can be converted to a diagonal matrix. For example,

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} = \begin{bmatrix} -1 & 3 & 0 \\ -3 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{z} \quad \mathbf{z}(0) = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$y = \mathbf{C}\mathbf{z} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{z}$$

This system has the eigenvalues $\lambda_i = -1 \pm 3i, -1$

and has the solution,



But there is no coordinate frame that fully diagonalizes the **A** matrix. In fact, the matrix is already in its most diagonal form.

A matrix in its most diagonal form is said to be in its **Jordan Canonical Form.**

Example: System with only complex eigenvalues.

A system with only complex eigenvalues has the form:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} \sigma_1 & \omega_1 & 0 & 0 & & & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 & & & 0 \\ 0 & 0 & \sigma_2 & \omega_2 & & & 0 \\ 0 & 0 & -\omega_2 & \sigma_2 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & \sigma_m & \omega_m \\ 0 & 0 & 0 & 0 & \cdots & -\omega_m & \sigma_m \end{bmatrix} \mathbf{z}$$

where,

$$n = 2m \quad \mathbf{A} = \mathbb{R}^{n\times n} = \mathbb{R}^{2m\times 2m}$$

if,

$$i = 1, 2, \cdots, n$$

$$j = 1, 2, \cdots, m$$

Then the eigenvalues are:

$$\lambda_i = \sigma_j + i\omega_j$$

$$\lambda_{i+1} = \sigma_j - i\omega_j$$

And the solution **Z**(t) is:

$$z_{2j-1}(t) = e^{\sigma_j t}\left(z_{2j-1}(0)\cos\left(\omega_j t\right) + z_{2j}(0)\sin\left(\omega_j t\right)\right)$$

$$z_{2j}(t) = e^{\sigma_j t}\left(-z_{2j-1}(0)\sin\left(\omega_j t\right) + z_{2j}(0)\cos\left(\omega_j t\right)\right)$$

So if a coordinate transform exists that puts a system into a diagonal form, a complex diagonal form (like this one), or a hybrid of the two (like the previous example), then the systems is still relatively easy to solve analytically. (and maybe worth it, if you have MATLAB)

What about repeated eigenvalues?

If there are repeated eigenvalues, there *may* not be a sufficient number of linearly independent eigenvectors to construct a coordinate transformation.

To illustrate this, let's first use MATLAB to find the Jordan Canonical Form of the matrix below.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

```
>> eig(A)

>> jordan(A)
```

these functions return:

ans = [2 2 2 3]'

$$\text{ans} = \begin{matrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{matrix}$$

This system cannot be diagonalized since there are not three *linearly independent* eigenvectors for the eigenvalue of 2.

Recall how we find these. Solve

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right)\eta_i = 0$$

for each eigenvalue.

Repeated Eigenvalues, cont'd

$$jordan(\mathbf{A}) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

For the eigenvalue 3, we solve the system as we did (would) in any linear algebra course…

$$(\mathbf{A} - 3\mathbf{I})\eta_1 = \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & -1 & 10 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \eta_1 = 0 \qquad \eta_1 = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

For the eigenvalue of 2, we can find two independent eigenvalues.

$$(\mathbf{A} - 2\mathbf{I})\eta_i = \begin{bmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 10 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \eta_i = 0$$

$$\underbrace{\eta_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \eta_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Linear independent eigenvalues}}$$

or, $\eta_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or other options…

Repeated eigenvalues cont'd

$$jordan(\mathbf{A}) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

So to generate another eigenvector, we take the square of the calculated matrix …

$$(\mathbf{A} - 2\mathbf{I})^2 \eta_4 = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \eta_4 = 0$$

Several options including… $\eta_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

So looking at these 4 eigenvectors…

$$\eta_i = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The results from MATLAB to produce the similarity transform to JCF should seem reasonable.

$$\mathbf{A} = \mathbf{TJT^{-1}}$$

```
A = [ 2 0 1 -3;
    0 2 10 4;
    0 0 2 0;
    0 0 0 3 ];

[V, J] = jordan(A)
```

J =

3	0	0	0
0	2	1	0
0	0	2	0
0	0	0	2

V =

-3.0000	-0.4000	3.0000	8.5000
4.0000	-4.0000	-4.0000	0
0	0	-0.4000	0
1.0000	0	0	0

*the goal here is NOT to relearn how to find eigenvectors (linearly independent or not) using pen and paper, but rather to review when and why the Jordan Canonical Form may not be diagonal (or not) when eigenvalues are repeated.

Solving systems in Jordan Canonical Form

If the matrix is diagonal, then we can solve with…

$$\mathbf{z}(\mathrm{t}) = e^{\Lambda t}\mathbf{z}(0)$$

A system in Jordan Canonical Form can be solved using

$$\mathbf{z}(\mathrm{t}) = e^{\mathbf{J}t}\mathbf{z}(0)$$

where

$$e^{\mathbf{J}} = \begin{bmatrix} e^{\mathbf{J}_1} & 0 & \cdots & 0 \\ 0 & e^{\mathbf{J}_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{\mathbf{J}_k} \end{bmatrix}$$

And in this case, each Jordan Block has the solution

$$e^{\mathbf{J}_i t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & t \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} e^{\lambda_i t}$$

For most dynamical systems, the Jordan Blocks are rarely large.

If you have fond memories from diff-q, you may see how this relates to other methods you may have learned to solve ODEs.

Solving Systems in JCF, Back to our example

Let's create a first order system with the matrix we have been examining…

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} = \begin{bmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\mathbf{z}$$

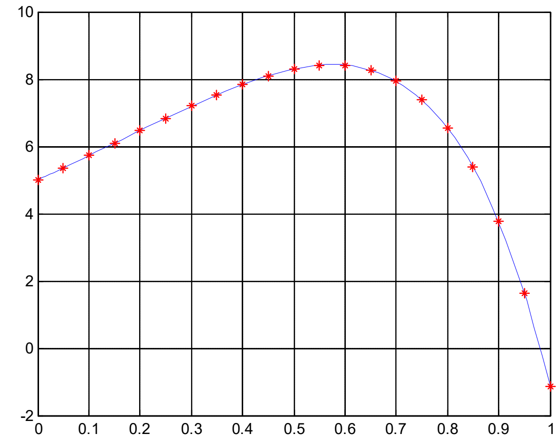
A system in Jordan Canonical Form can be solved using

$$\mathbf{z}(\mathrm{t}) = e^{\mathbf{A}t}\mathbf{z}(0) = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{z}(0)$$

The values of **T** and **J** are on a previous slide found with MATLAB. So the solution can be found with:

$$e^{\mathbf{A}t} = \mathbf{T}\begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{2t} & t & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}\mathbf{T}^{-1}$$

Let's check it….



Analytical and numerical solutions match

```
clear all;

global A;
A = [ 2 0 1 -3;
    0 2 10 4;
    0 0 2 0;
    0 0 0 3 ];

[V, J1] = jordan(A);
z0 = [ 5; 0; 0; 1];
t=[0:.05:1];
```

```
[tn, zn] = ode45('mySystem', [0 1],
[5 0 0 1]);
for i=1:length(t)
    J = [exp(3*t(i)) 0 0 0;
        0 exp(2*t(i)) t(i) 0
        0 0 exp(2*t(i)) 0;
        0 0 0 exp(2*t(i)) ];
    za(i,:)=V*J*inv(V)*z0;
end

plot( tn, zn(:,1) ); hold on;
plot( t, za(:,1), 'r*'); grid on;
```

Summary

- Diagonal systems are relatively simple to solve.
  - But if the eigenvectors are not linearly independent, then there is not similarity transform that will diagonalize the system
- The best we can do is Jordan Canonical Form
- Systems in JCF can still be solved in a fashion similar to using the Matrix Exponential (ME)
- MATLAB can provide us with the Jordan form and the Jordan basis for the similarity transform,
  - but we did a quick review to help you recall roughly how the basis (and thus the transformation) is found.