Canonical form for 2^{nd} order systems

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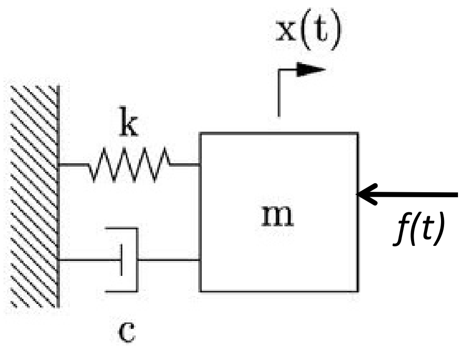
Recall the linear 2nd order MSD system



Our Equation of Motion (EOM)

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Let…

$$z_1 = x \qquad \dot{z}_1 = z_2$$

$$z_2 = \dot{x} \qquad \dot{z}_2 = -\frac{k}{m} z_1 - \frac{c}{m} z_2 + \frac{f(t)}{m}$$

And if the force is our input…

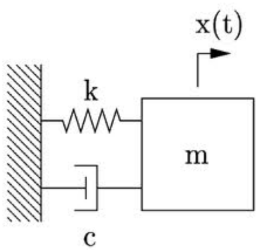
$$u_1 = f(t)$$

Thus in state-space form…

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}$$

Easy to solve… numerically

Assume some values….



Let…

$$k=3, c=2, m=1, \& f(t)=4$$

With IC's…

$$z(0)=\begin{cases}0\\0\end{cases}$$

Plug in…

$$\frac{dz}{dt}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{bmatrix}\mathbf{z}+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\mathbf{u}$$

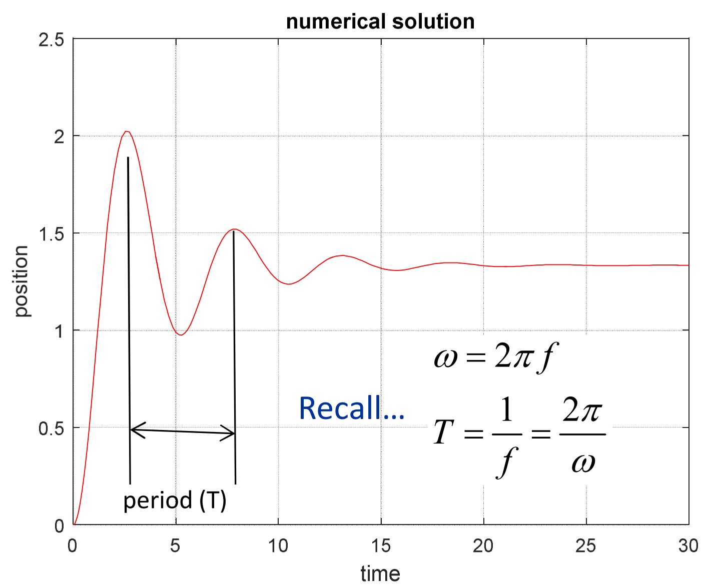
Solve with MATLAB's `ode45()` …

```
%m-s-d solution
[t, z] = ode45( @test, [0 30 ], [ 0 0 ]);
plot( t, z(:,1), 'r');
```

```
function zprime = test( t, z )
m = 2; c = 1; k = 3; F = 4;
zprime = [ 0 1; -(k/m) -c/m) ]*z + [ 0; 1/m ]*F;

%OR
% A = [ 0 1; -(k/m) -(c/m)];
% B = [ 0; 1/m ];
% u = F;
% zprime = A*z + B*u

%OR
%zprime = [ z(2);
%     -(c/m)*z(2) - (k/m)*z(1) + F/m; ];
```

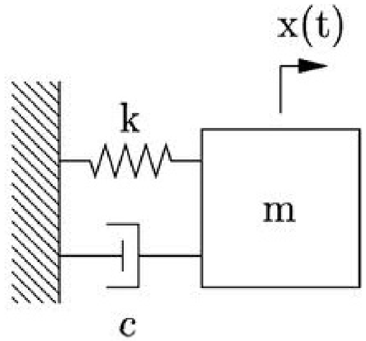


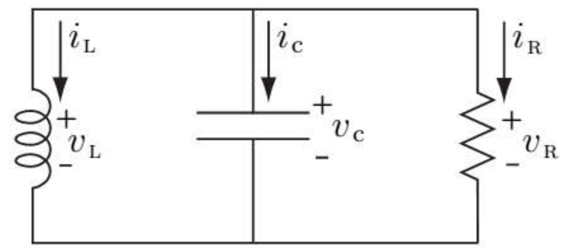
$$\omega = 2\pi f$$

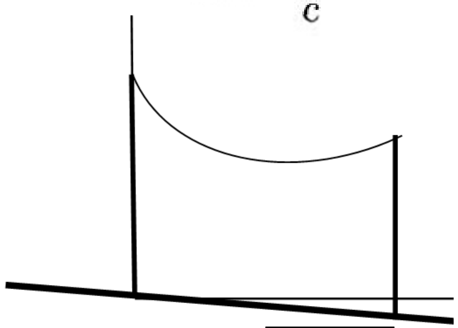
$$T=\frac{1}{f}=\frac{2\pi}{\omega}$$

The 2^{nd} order system

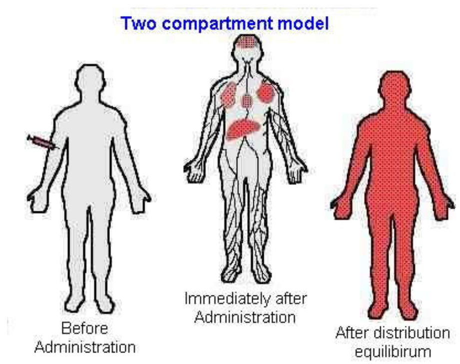
2^{nd} order systems are quite common….

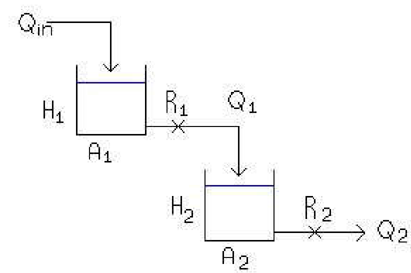






$$\frac{d^2y}{dx^2} = C_w\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$





… it is often possible to simplify complex systems to 2^{nd} order.

Numerical vs. Analytical Sol^n

- Benefits of numerical solutions to state-space systems
  - Many options. (Very good ones using MATLAB)
  - Works for any linear system (size does not matter)
  - Easy to change parameters
- Disadvantages of numerical solutions…
  - Not exact. Potential errors (round-off, small steps, etc.)
  - Introduction of programmatic complexity
  - Inability to analyze performance w.r.t. system properties.
  - Trial & Error analysis can be difficult and/or impractical for large systems
- Since 2nd order systems are so common, lets find the analytical (exact) solution.
  - Plus, let's knock a little more rust off our diffy-q skills!

General 2^{nd} order solution…

$$m\ddot{q} + c\dot{q} + kq = f(t) \quad \text{where,} \quad \begin{aligned} q(0) &= q_o \\ \dot{q}(0) &= v_o \end{aligned}$$

Step 1: Find the homogeneous solution (i.e. *f(t)*=0)

Assume a solution

$$q(t) = e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right)$$

Substitute into the ODE(s). Solve for constants using I.C.s

$$\dot{q}(t) = \alpha e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(B\omega\cos\omega t - A\omega\sin\omega t\right)$$

$$\ddot{q}(t) = \alpha^2 e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(-B\omega^2\sin\omega t - A\omega^2\cos\omega t\right)$$

2^{nd} order solution, cont'd

$$q(t) = e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right)$$
$$\dot{q}(t) = \alpha e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(B\omega\cos\omega t - A\omega\sin\omega t\right)$$
$$\ddot{q}(t) = \alpha^2 e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(-B\omega^2\sin\omega t - A\omega^2\cos\omega t\right)$$

Plug in. Collect common terms…

$$\begin{aligned} 0 &= m\ddot{q} + c\dot{q} + kq \\ &= e^{\alpha t}\left[\left(M\right)\cos\omega t + \left(N\right)\sin\omega t\right] \end{aligned}$$

Where,

$$\left(M\right) = \left(B\left(c + 2m\alpha\right)\omega + A\left(m\alpha^2 + c\alpha - m\omega^2 + k\right)\right)$$
$$\left(N\right) = Bm\alpha^2 + Bc\alpha - 2Am\omega\alpha - Bm\omega^2 + Bk - Ac\omega$$

2^{nd} order solution, cont'd

$$q(t) = e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right)$$
$$\dot{q}(t) = \alpha e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(B\omega\cos\omega t - A\omega\sin\omega t\right)$$
$$\ddot{q}(t) = \alpha^2 e^{\alpha t}\left(A\cos\omega t + B\sin\omega t\right) + e^{\alpha t}\left(-B\omega^2\sin\omega t - A\omega^2\cos\omega t\right)$$

Plug in initial conditions. At t=0,

$$\begin{aligned} q(0) = q_o &= e^{\alpha 0}\left(A\cos\left(\omega 0\right) + B\sin\left(\omega 0\right)\right) \\ &= 1(A(1) + B(0)) \\ &= A \end{aligned}$$

$$\begin{aligned} \dot{q}(0) &= \alpha e^{\alpha 0}\left(A\cos\left(\omega 0\right) + B\sin\left(\omega 0\right)\right) + e^{\alpha 0}\left(B\omega\cos\left(\omega 0\right) - A\omega\sin\left(\omega 0\right)\right) \\ &= \alpha\left(A + 0\right) + \left(B\omega - 0\right) \\ &= \alpha A + \omega B \end{aligned}$$

2nd order solution, cont'd

$$0 = m\ddot{q} + c\dot{q} + kq$$
$$= e^{\alpha t}\left[\left(B\left(c+2m\alpha\right)\omega + A\left(m\alpha^2 + c\alpha - m\omega^2 + k\right)\right)\cos\omega t + \left(Bm\alpha^2 + Bc\alpha - 2Am\omega\alpha - Bm\omega^2 + Bk - Ac\omega\right)\sin\omega t\right]$$

$$q_o = A$$

$$\dot{q}_o = v_o = A\alpha + B\omega$$

Therefore…

$$A = q_o$$

$$B = \frac{v_o - q_o\alpha}{\omega}$$

Plug in our results from the initial conditions…do some algebra….

$$q(t) = e^{-\zeta\omega_n t}\left[q_o\cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d}q_o + \frac{1}{\omega_d}v_o\right)\sin\omega_d t\right]$$

2^{nd} order solution, cont'd

Plug in our results from the initial conditions…do some algebra….

$$q(t) = e^{-\zeta\omega_n t}\left[q_o \cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d} q_o + \frac{1}{\omega_d} v_o \right) \sin\omega_d t \right]$$

Where…

$$\omega_n = \sqrt{\frac{k}{m}}$$ Angular (or natural) frequency

$$\zeta = \tfrac{1}{2}\sqrt{\frac{c^2}{km}}$$ Damping ratio

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$ Damped frequency

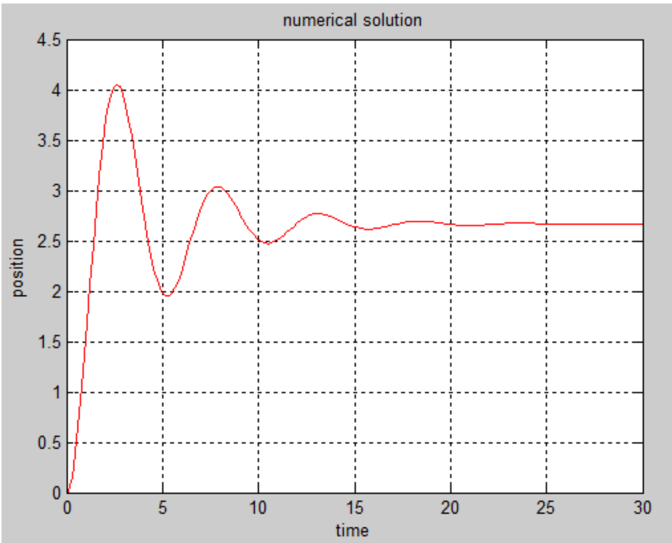
2^{nd} order solution, is it correct?

Numerical Solution (i.e. ODE45( ))

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}$$

Analytical Solution

$$q(t) = e^{-\zeta\omega_n t}\left[q_o \cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d} q_o + \frac{1}{\omega_d} v_o \right) \sin\omega_d t \right]$$



…seems like a good homework exercise…

Canonical 2nd order system

Canonical Form: The archetype or standard form – the default, "natural", or preferred form.

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \tfrac{1}{2}\sqrt{\frac{c^2}{km}}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

The damping coefficient.
What is its impact?

Natural frequency. I think we get this…

Oh yeah....

$$q(t) = e^{-\zeta\omega_n t}\left[q_o \cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d} q_o + \frac{1}{\omega_d} v_o\right)\sin\omega_d t\right]$$

...was the homogeneous *f(t)=0* solution. What about the input?

Step 2: Find the heterogeneous (particular) solution.

$$m\ddot{q} + c\dot{q} + kq = f(t) \quad \text{where,} \quad \begin{aligned} q(0) &= 0 \\ \dot{q}(0) &= 0 \end{aligned}$$

...recall that the final solution is the sum of the homogeneous and particular solution...

Complete 2nd order solution…

$$m\ddot{q} + c\dot{q} + kq = f(t) \quad \text{where,} \quad \begin{aligned} q(0) &= q_o = 0 \\ \dot{q}(0) &= v_o = 0 \end{aligned}$$

Can get messy quickly….. For example,

If *f(t)*=constant=F:

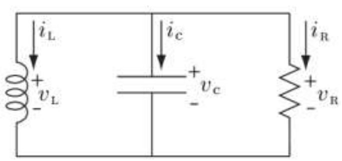
$$q(t) = \frac{F}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\cos\left(\omega_d t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)\right]$$

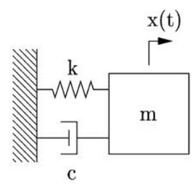
If *f(t)* = Asin(ωt), and I.C.s are all zero:

…we will get to this, but we still have numerical solving options.

Note: natural frequency and damping coefficient still play a primary role!

2nd order system Summary





$$m\ddot{q} + c\dot{q} + kq = f(t) \quad \text{where, } q(0) = q_o$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \dot{q}(0) = v_o$$

- 2nd order systems are common and thus worth additional focus.
  - Many systems are well modeled as second order systems
- Analytical solutions exist
  - Heterogeneous solutions can get a little messy.
  - We will look at additional methods to solve including linear algebra solutions and partial fractions expansion in Laplace space
- There is a clear (“natural”) relationship between the canonical form and observed behavior
  - Common physical characteristics (damping, natural frequency), and
  - Useful metrics for characterizing performance.
  - These are discussed in more detail in the next lesson.