Observer, Example

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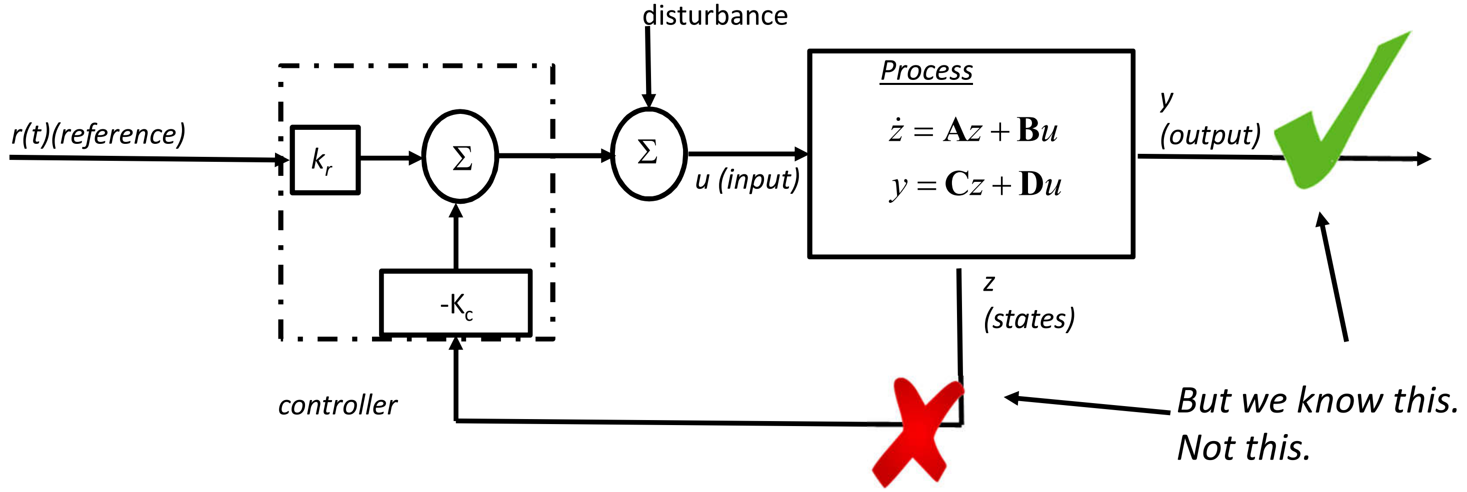
Objective: Use output not states

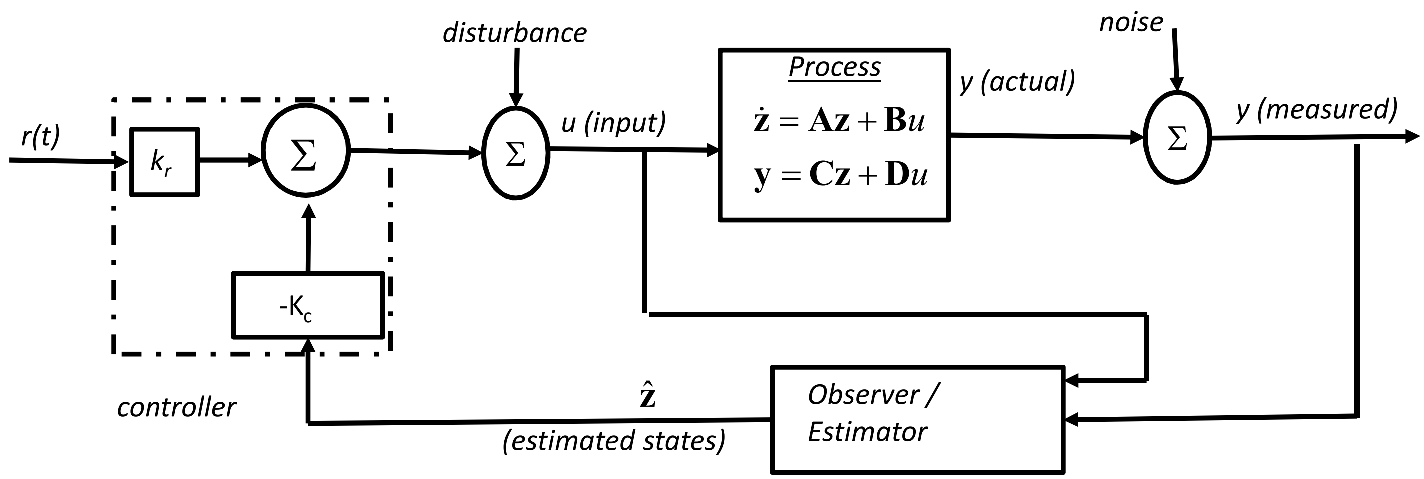
Given a system with the following dynamic model and output:

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$
$$\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}u$$

Design a linear controller with a single input which is stable at an equilibrium point we define as z_e=0.



Observer/Estimator

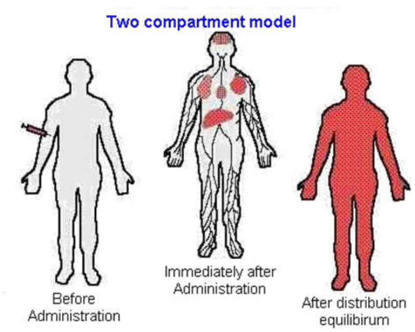


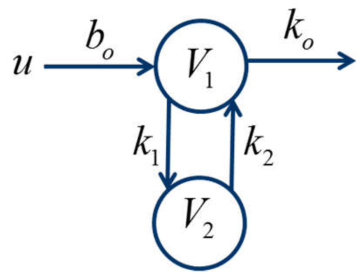
Our objectives

- Determine if a system is observable
- Define the Observable Canonical Form (OCF)
- Create estimates of the states that allow us to continue to implement state feedback

Example, Determining observer gains

Given:





$$\frac{dc}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} c + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Find: Observer gains that converge quickly on the actual state values

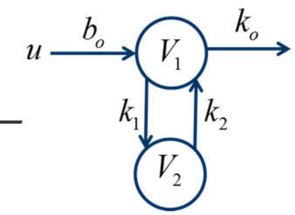
Solution:

Step 1: Is the system observable?

$$w_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{bmatrix}$$

Yes. As long as k_1 is not zero.

Example, Determining observer gains



Step 2: What are the actual states (so we can see how well our example works)

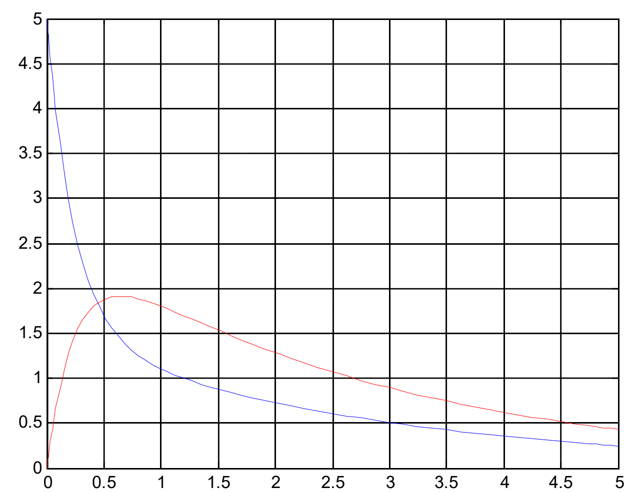
$$\frac{dc}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} c + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1-2 & 1.5 \\ 2.0 & -1.5 \end{bmatrix} c + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u$$

$$b_0 = 1$$
$$k_1 = 2$$
$$k_2 = 2.5$$
$$k_0 = 1$$

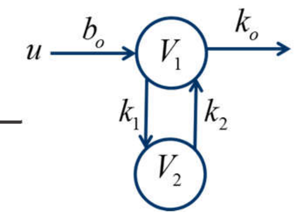
```
clear all;
global u;
u=0;

k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

[t,z] = ode45( @twoVolume, [0 5], [5 0] );

plot(t, z(:,1), 'b');
hold on; grid on;
plot(t, z(:,2), 'r');
```

```
function cprime = twoVolume( t, c )
global u;

k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

A = [ -k0-k1 k2; k1 -k2 ];
B = [ b0; 0 ];
C = [ 1 0 ];
D = [0];

cprime = A*c + B*u;
```



Does this result make sense?

Example, Determining observer gains



Step 2: What happens if we assume the observer gains are 1?

$$\frac{d\mathbf{z}}{dt} = \left(\mathbf{Az} + \mathbf{B}u\right)$$

$$\frac{d\hat{\mathbf{z}}}{dt} = \mathbf{A}\hat{\mathbf{z}} + \mathbf{L}\left(y - \mathbf{C}\hat{\mathbf{z}}\right)$$

$$= \mathbf{A}\hat{\mathbf{z}} + \mathbf{L}\left(\mathbf{Cz} - \mathbf{C}\hat{\mathbf{z}}\right)$$

$$\frac{dc}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} c + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u$$

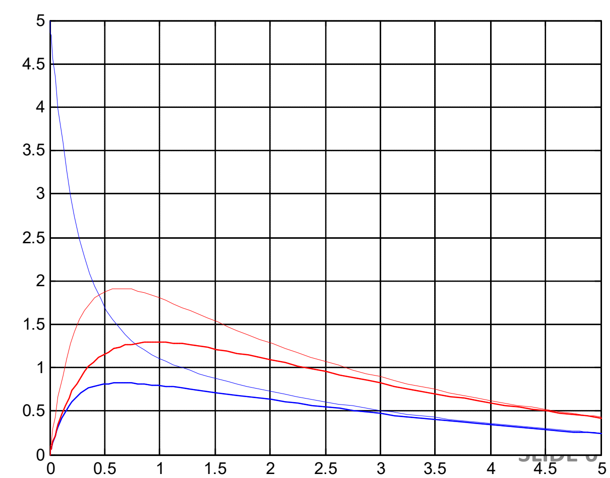
$$\frac{d\hat{c}}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} \hat{c} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \end{bmatrix} c - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{c} \right]$$

Set up the system so we can simulation the actual and estimated states at the same time...

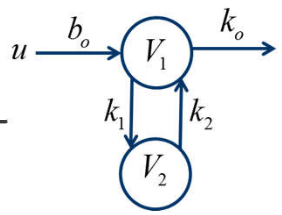
$$\frac{d}{dt}\begin{bmatrix} \mathbf{z} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{LC} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \hat{\mathbf{z}} \end{bmatrix}$$

```
clear all;
global u;
u=0;

[ta,za]=ode45(@twoVolObs,[0 5],[5 0 0 0]);
plot(ta, za(:,1), 'b');
hold on; grid on;
plot(ta, za(:,2), 'r');
plot(ta, za(:,3), 'b--');
plot(ta, za(:,4), 'r--')
```

```
function cprime = twoVolumeObserver( t, c )
global u;

k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

A = [ -k0-k1 k2; k1 -k2 ]; B = [ b0; 0 ];
C = [ 1 0 ]; D = [0];

L = [ 1; 1 ];

Aa = [ A zeros(2, 2); L*C A-L*C ];
Ba = [ B(1); B(2); 0; 0 ];

cprime = Aa*c + Ba*u;
```



Example, Determining observer gains



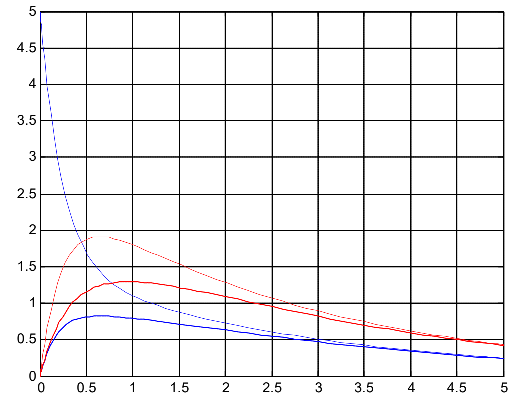
Step 3: Now let's pick some gains.

$$\frac{d\mathbf{z}}{dt} = \left(\mathbf{A}\mathbf{z} + \mathbf{B}u\right) \qquad \frac{dc}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} c + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u$$

$$\frac{d\hat{\mathbf{z}}}{dt} = \mathbf{A}\hat{\mathbf{z}} + \mathbf{L}\left(\mathbf{C}\mathbf{z} - \mathbf{C}\hat{\mathbf{z}}\right) \qquad \frac{d\hat{c}}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} \hat{c} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \end{bmatrix} c - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{c} \right]$$

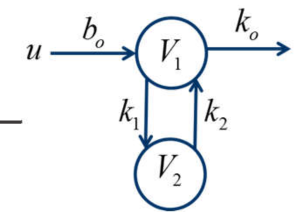
The gains for the actual system are (-4.1357 and -0.3625) and the eigenvalues for our current observer are (-4.5 and -1):

```
k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

A = [ -k0-k1 k2; k1 -k2 ]; B = [ b0; 0 ];
C = [ 1 0 ]; D = [0];
L = [ 1; 1 ];

eig(A)
eig(A-L*C)
```



Let's find L such that the eigenvalues are -10 and -5, and then see how that system performs.

Example, Determining observer gains



Let's find L such that the eigenvalues are -10 and -5, and then see how that system performs.

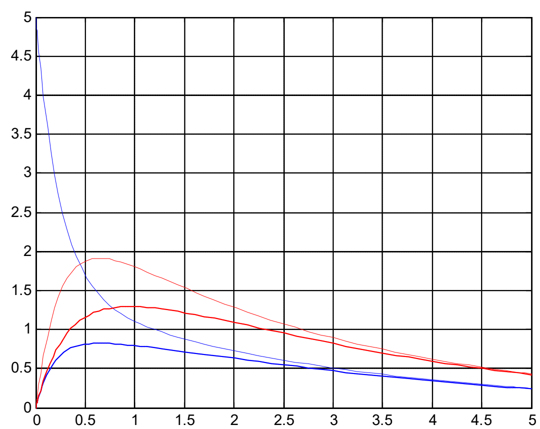
$$(s+10)(s+5)$$
$$(s^2+15s+50)$$

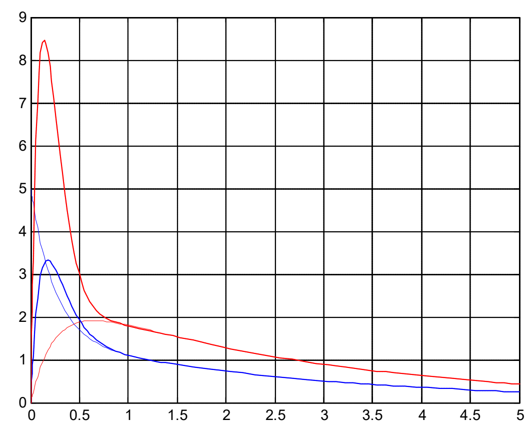
$$\det\left(\lambda\mathbf{I}-[\mathbf{A}\text{ - }\mathbf{LC}]\right)$$
$$\lambda^2+(4.5+l_1)\lambda+1.5l_2-3$$

$$l_1 = 10.5 \qquad l_2 = \frac{53}{1.5} = 35\tfrac{1}{3}$$

So let's compare our two observers:

```
function cprime = twoVolObs( t, c )
global u;

k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

A = [ -k0-k1 k2; k1 -k2 ];
B = [ b0; 0 ];
C = [ 1 0 ]; D = [0];

L = [ 1; 1 ];

Aa = [ A zeros(2, 2); L*C A-L*C ];
Ba = [ B(1); B(2); 0; 0 ];

cprime = Aa*c + Ba*u;
```





```
function cprime = twoVolObs( t, c )
global u;

k0 = 1.0; k1 = 2.0;
k2 = 1.5; b0 = 1.0;

A = [ -k0-k1 k2; k1 -k2 ];
B = [ b0; 0 ];
C = [ 1 0 ]; D = [0];

L = [ 10.5; 33.5 ];

Aa = [ A zeros(2, 2); L*C A-L*C ];
Ba = [ B(1); B(2); 0; 0 ];

cprime = Aa*c + Ba*u;
```

Can also find the L matrix using Acker's method. $L = W_o^{-1}\tilde{W}_0 \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \cdots \\ p_n - a_n \end{bmatrix}$

Summary

- Observability
  - We say the system is observable if for any time T>0 it is possible to determine the state vector, **z**, through the measurements of the output, y(t), as the result of input, u(t), over the period between t=0 and t=T.
- Observability Matrix

$$\mathbf{W_0} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA^2} \\ \dots \\ \mathbf{CA^{n-1}} \end{bmatrix}$$

- Observable Canonical Form

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ -a_3 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & 0 & 1 \\ -a_5 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} u$$

- How to compare observer to actual states

$$\frac{d}{dt}\begin{bmatrix} \mathbf{z} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{LC} & \mathbf{A-LC} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \hat{\mathbf{z}} \end{bmatrix}$$

- Use of Observers/Estimators

$$\frac{d}{dt}\begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A-BK} & \mathbf{BK} \\ 0 & \mathbf{A-LC} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{B}k_r \\ 0 \end{bmatrix} y_d$$