2a.

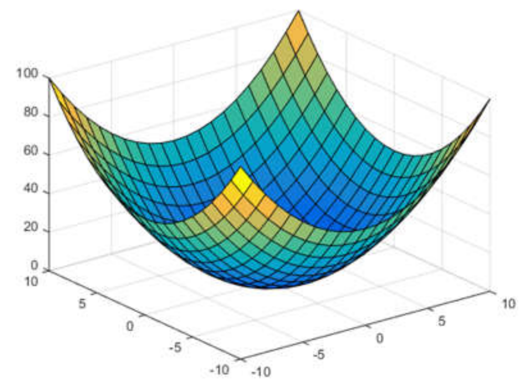
$$V_1(\mathbf{x}) = \frac{1}{2}x_1^{\ 2} + \frac{1}{2}x_2^{\ 2}$$

This is clearly positive definite. If we wanted to be clear about it, we could easily check in MATLAB. This is not required for the homework.

```
clear all;

[x1, x2] = meshgrid(-10:1:10,-10:1:10);
V = 0.5*x1.^2 + 0.5*x2.^2;

surf(x1, x2, V)
```



Next, we find the derivative

$$\begin{aligned}\dot{V}_1(\mathbf{x}) &= \frac{\partial V}{\partial x}\frac{dx}{dt} \\ &= x_1\dot{x} + x_x\dot{x}_2 \\ &= x_1(-ax_1) - bx_1x_2 - cx_2^2\end{aligned}$$

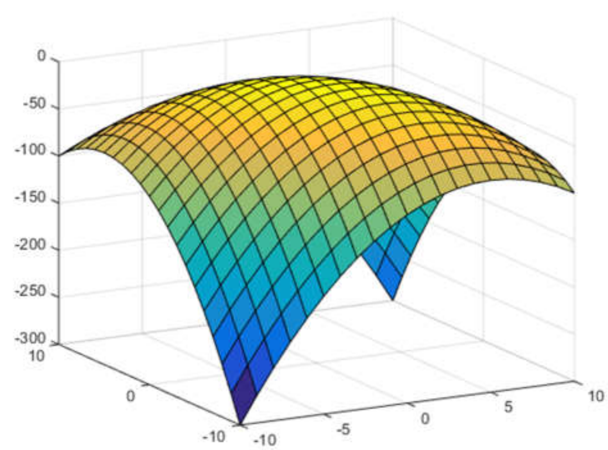
At this point, I was too quick to simply state that if b=0, $a, c > 0$ and then the candidate function was valid since the result was negative definite. But I then noticed that the problem states that $a, b, c > 0$. So....

We do note that the system is a quadratic equation. For example, if $a = b = c = 1$, then

```
clear all;

[x1, x2] = meshgrid(-10:1:10,-10:1:10);
Vdot = -x1.^2 - x1.*x2-x2.^2;

surf(x1, x2, Vdot)
```

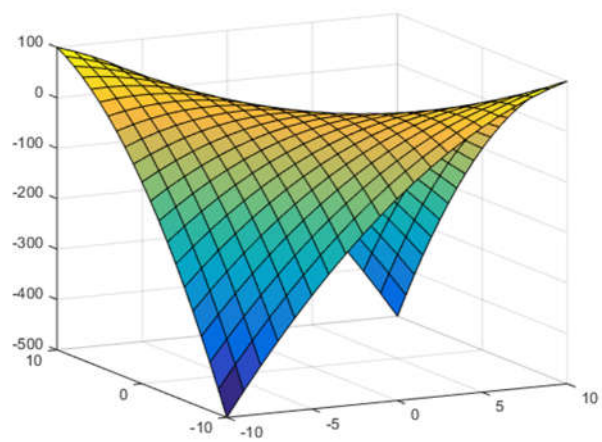


But if *b* instead is 3 as shown below, the Lyapunov function definitely doesn't work.

```
clear all;

[x1, x2] = meshgrid(-10:1:10,-10:1:10);
Vdot = -x1.^2 - x1.*x2-x2.^2;

surf(x1, x2, Vdot)
```



So over what ranged does it work? Trial and Error to determine a range of coefficients is not ideal. But this is a second order quadratic equation. So we can use the Complete the Squares[1] method to find the vertices.

Remember solving quadratics using this method?

$$x^2+4x+1=0$$
$$x^2+4x+4=-1+4$$
$$(x+2)^2=3$$
$$x+2=\pm\sqrt{3}$$
$$x=-2\pm\sqrt{3}$$

For a general equation

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

This was the method used to factor 2nd order polynomials in high school. It is also the basis for deriving the quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Remember?

[1] Never forget your high school math!

Set it up with one variable as the quadratic.

$$\dot{V} = x_1^{\ 2} + \frac{bx_2}{a} x_1 + \frac{cx_2^2}{a}$$

Solve using the derivation above

$$\begin{aligned}\dot{V} &= -a\left(x_1 + \frac{bx_2}{2a}\right)^2 - cx_2^2 + \frac{b^2 x_2^2}{4a} \\ &= -a\left(x_1 + \frac{bx_2}{2a}\right)^2 - \left(c - \frac{b^2}{4a}\right)x_2^2\end{aligned}$$

From this we see that the first term must always be negative since a>0 and b>0. The second term is only negative or zero if

$$c - \frac{b^2}{4a} \geq 0\,.$$

Which is a condition that matches with our observations above.