Output Feedback cont'd

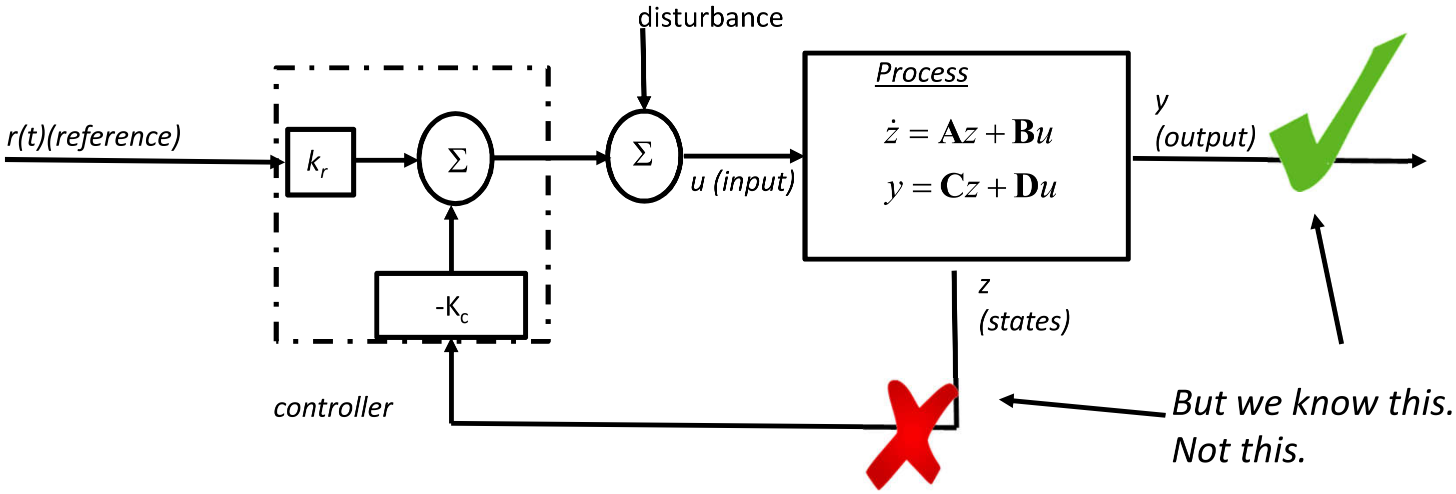
Mitch Pryor

But do we know all the states?

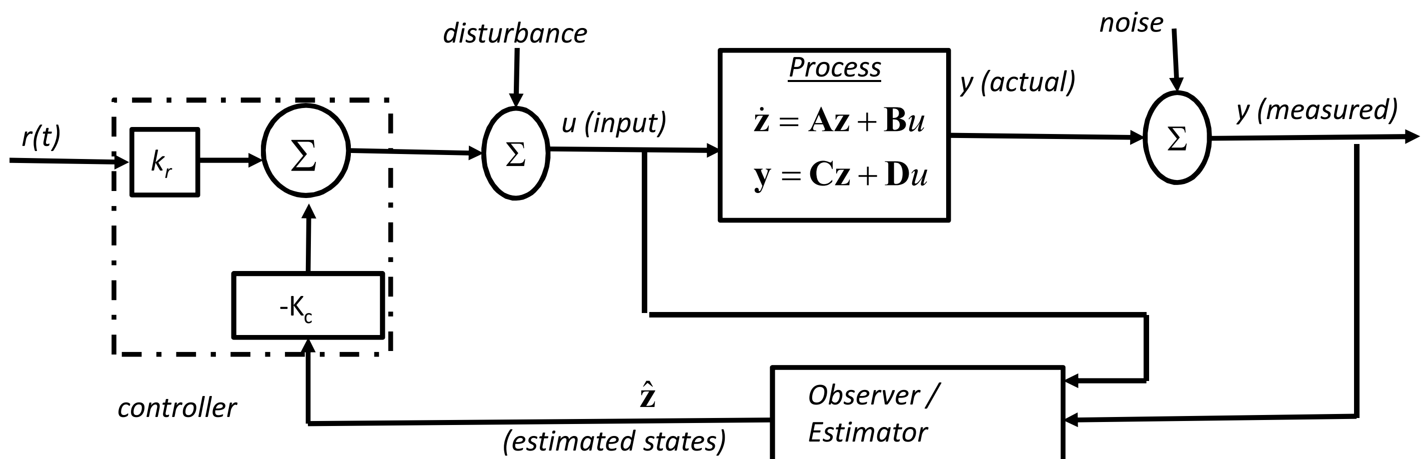
Given a system with the following dynamic model and output:

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$
$$\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}u$$

Design a linear controller with a single input which is stable at an equilibrium point we define as z_e=0.



Observer/Estimator Summary



<u>Our objectives</u>

- Determine if a system is observable
- Define the Observable Canonical Form (OCF)
- Create Estimators that allow us to continue to implement state feedback

Observable Canonical Form

If a system is observable, it can be put into Observable Canonical Form.

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & 0 & & 0 \\ \vdots & & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & 0 & & 1 \\ -a_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} u$$

Where a_i are the coefficients of the characteristic equation:

$$\left| \lambda \mathbf{I} - \mathbf{A} \right|$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{z} + \mathbf{D} u$$

Observable Canonical Form

If a system is in OCF, then we can calculate its observability matrix symbolically.

Consider a system with 5 states:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ -a_3 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & 0 & 1 \\ -a_5 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} + \mathbf{D}u$$

w_o has the form…

$$\widetilde{W}_0 = \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^2 \\ \vdots \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 & 0 \\ (-a_1)^2 - a_2 & -a_1 & 1 & 0 & 0 \\ (-a_1)^3 - (-a_2)^2 - a_3 & (-a_1)^2 - a_2 & -a_1 & 1 & 0 \\ (-a_1)^4 - (-a_2)^3 - (-a_3)^2 - a_4 & (-a_1)^3 - (-a_2)^2 - a_3 & (-a_1)^2 - a_2 & -a_1 & 1 \end{bmatrix}$$

There exists a Transformation T such that for $\mathbf{z} = \mathbf{T}\mathbf{x}$ *where* $\mathbf{T} = \mathbf{w}_o^{-1}\tilde{\mathbf{w}}_o$

Note a system in OCF may be poorly conditioned.

Duality of Canonical Forms…

Controllable/Reachable Canonical Form is…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \dots & \dots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$= \mathbf{A_c z} + \mathbf{B_c u}$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \end{bmatrix} \mathbf{z} + \mathbf{D}u$$

$$= \mathbf{C_c} z + \mathbf{D}u$$

And that Observable Canonical Form is…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} u$$

$$= \mathbf{A_o z} + \mathbf{B_o} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{z} + \mathbf{D}u$$

$$= \mathbf{C_o z} + \mathbf{D}u$$

We note that for a given system

$$\frac{d\mathbf{z}}{dt} = \mathbf{Az} + \mathbf{B}u \quad y = \mathbf{Cz}$$

The following relationships are true

$$\mathbf{A_c} = \mathbf{A_o}^T \quad \mathbf{B_c} = \mathbf{C_o}^T \quad \textbf{and} \quad \mathbf{C_c} = \mathbf{B_o}^T$$

$$\tilde{\mathbf{W}}_r = \tilde{\mathbf{W}}_o^T$$

$$\mathbf{T}_o = \mathbf{W}_o^{-1} \tilde{\mathbf{W}}_o \qquad \mathbf{T} = \mathbf{T}_r = \tilde{\mathbf{W}}_r \mathbf{W}_r^{-1}$$

Thus the system elements in all canonical forms are related by the transformation T.

$$\mathbf{A_c} = \mathbf{TAT^{-1}} \quad \mathbf{B_c} = \mathbf{TB} \quad \mathbf{C_c} = \mathbf{CT^{-1}}$$

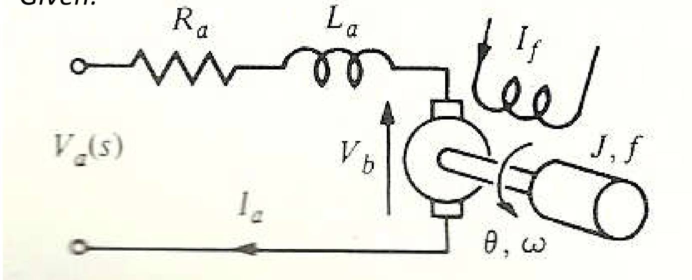
$$\mathbf{A_0} = \mathbf{A_c}^T = \left(\mathbf{TAT^{-1}}\right)^T = \mathbf{T}^{-1^T} \mathbf{A}^T \mathbf{T}^T$$

$$\mathbf{B_0} = \mathbf{C_c}^T = \left(\mathbf{CT^{-1}}\right)^T = \mathbf{T}^{-1^T} \mathbf{C}^T$$

$$\mathbf{C_0} = \mathbf{B_c}^T = \left(\mathbf{TB}\right)^T = \mathbf{B}^T \mathbf{T}^T$$

Example, OCF for DC Motor

Given:



The input is the V_a and the output of interest is the angular velocity. Assume there is a friction force proportional to the angular speed.

$$f_f = b\dot{\theta}$$

Find the state-space model for this system and then convert it to observable canonical form.

$$\begin{cases} \text{Voltage Loop:} & L_a \dfrac{di_a}{dt} + R_a i_a + V_b = V_a \\ \text{Back Voltage:} & V_b = K_b \dfrac{d\theta}{dt} \\ \text{Motor Torque:} & T = K i_a \\ \text{Mech. System:} & J \dfrac{d^2\theta}{dt} + b\dfrac{d\theta}{dt} = T \end{cases}$$

$$\begin{cases} \dfrac{di_a}{dt} = -\dfrac{R_a i_a}{L_a} - \dfrac{K_b}{L_a}\dfrac{d\theta}{dt} + \dfrac{V_a}{L_a} \\ \dfrac{d^2\theta}{dt} = \dfrac{K i_a}{J} - \dfrac{b}{J}\dfrac{d\theta}{dt} \end{cases}$$

Define the state vector as the current and angular velocity as well as the output as the angular velocity.

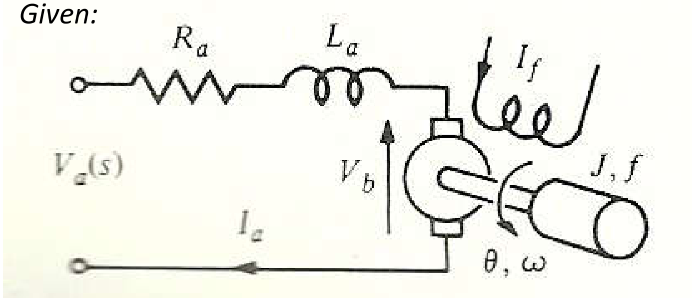
$$\frac{d}{dt}\begin{bmatrix} i_a \\ \omega \end{bmatrix} = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{K_b}{L_a} \\ \dfrac{K}{J} & -\dfrac{b}{J} \end{bmatrix} \begin{bmatrix} i_a \\ \omega \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_a} \\ 0 \end{bmatrix} V_a$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_a \\ \omega \end{bmatrix}$$

Very simple mechatronics example.

Same example, different assumptions…

Given:



The input is the V_a and the output of interest is the angular position. ***Assume the system inductance can be neglected*** *and there is a friction force proportional to the angular speed.*

$$f_f = b\dot{\theta}$$

Find the state-space model for this system and then convert it to observable canonical form.

$$\begin{cases} \text{Voltage Loop:} & R_a i_a + V_b = V_a \\ \text{Back Voltage:} & V_b = K_b \dfrac{d\theta}{dt} \\ \text{Motor Torque:} & T = K i_a \\ \text{Rotations NSL:} & J\dfrac{d^2\theta}{dt} + b\dfrac{d\theta}{dt} = T \end{cases}$$

$$\begin{cases} i_a = -\dfrac{K_b}{R_a}\dfrac{d\theta}{dt} + \dfrac{V_a}{R_a} \\ \dfrac{d^2\theta}{dt} = \dfrac{Ki_a}{J} - \dfrac{b}{J}\dfrac{d\theta}{dt} \end{cases} \Rightarrow \frac{d^2\theta}{dt} = -\left(\frac{b}{J} + \frac{KK_b}{JR_a}\right)\frac{d\theta}{dt} + \frac{KV_a}{JR_a}$$

Instead of two first order differential equations, I got 1 second order differential equation.

$$\mathbf{z} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}$$

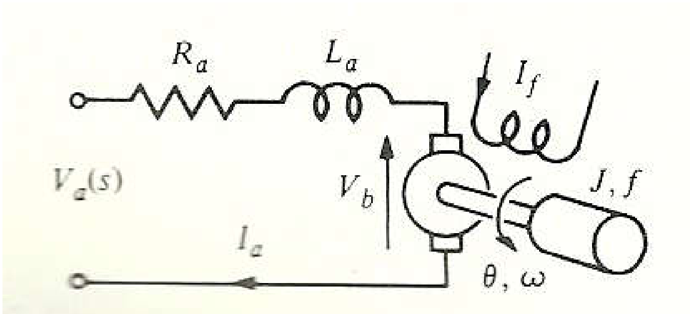
$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -\dfrac{bR_a + KK_b}{JR_a} & 0 \\ 1 & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} \dfrac{K}{JR_a} \\ 0 \end{bmatrix} V_a$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{z}$$

Note: State space models can vary for the same system depending on the assumptions made and in the input/outputs of interest.

Note: Did the order of the states in the z state vector really matter?

Example, OCF for DC Motor



Start with our s-s model

$$\frac{d}{dt}\begin{bmatrix}\omega\\ \theta\end{bmatrix} = \begin{bmatrix}-\dfrac{bR_a + KK_b}{JR_a} & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\omega\\ \theta\end{bmatrix} + \begin{bmatrix}\dfrac{K}{JR_a}\\ 0\end{bmatrix}V_a$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\omega\\ \theta\end{bmatrix}$$

Assume some values.

$$R_a = 0.2\Omega$$
$$J = 1\times10^{-5} \text{ kg-m}^2$$
$$K = 6\times10^{-5} \text{ N-m/amp}$$
$$K_b = 5.5\times10^{-2} \text{ V-s/rad}$$
$$b = 4\times10^{-2} \text{ N-m/rad/s}$$

First we verify the system is observable.

$$\mathbf{C} = \begin{bmatrix}0 & 1\end{bmatrix}$$

$$\mathbf{CA} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-\dfrac{bR_a + KK_b}{JR_a} & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\mathbf{w}_o = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

It is constant no matter what values we picked. The characteristic equation is.

$$|\lambda I - A| = \lambda^2 + 4001.37\lambda$$

Which we use to rewrite the system in observable canonical form.

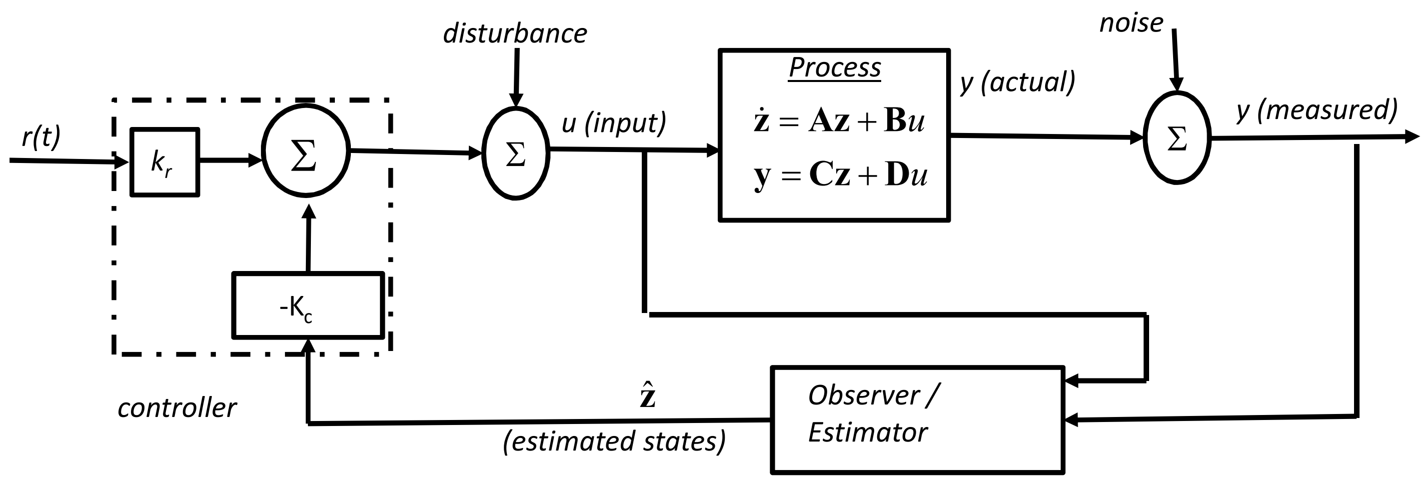
$$\mathbf{A}_o = \begin{bmatrix}0 & 1\\ -4001.37 & 0\end{bmatrix} \quad \mathbf{C}_o = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\boxed{\frac{d\mathbf{z}}{dt} = \begin{bmatrix}0 & 1\\ -4001.37 & 0\end{bmatrix}\mathbf{z} + \mathbf{B}_o V_a \quad y = \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{z}}$$

What can we say about the stability of this system?

Observer/Estimator Summary

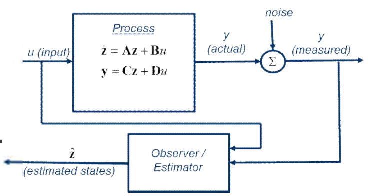
Knowing the system is observable, how do we estimate the states?



Our objectives

- ✓ Determine if a system is observable
- ✓ Define the Observable Canonical Form (OCF)
- Create Estimators that allow us to continue to implement state feedback

Observer/Estimator



Given: y and u find an estimate of the states ($\hat{\mathbf{z}}$ *) such that*

$$\frac{d\hat{\mathbf{z}}}{dt} = \mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u \qquad y = \mathbf{C}\hat{\mathbf{z}} + \mathbf{D}u$$

is a good estimate of

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}u \qquad y = \mathbf{C}\mathbf{z} + \mathbf{D}u$$

Let,

$$\tilde{\mathbf{z}} = estimation\_\text{error} = \mathbf{z} - \hat{\mathbf{z}}$$

And we see that….

$$\frac{d\mathbf{z}}{dt} - \frac{d\hat{\mathbf{z}}}{dt} = \left(\mathbf{A}\mathbf{z} + \mathbf{B}u\right) - \left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u\right)$$

$$\frac{d\tilde{\mathbf{z}}}{dt} = \mathbf{A}\mathbf{z} - \mathbf{A}\hat{\mathbf{z}}$$

$$\frac{d\tilde{\mathbf{z}}}{dt} = \mathbf{A}\tilde{\mathbf{z}}$$

So the error goes to zero as time goes to zero if and only if the system is stable!

But, note we only have the measured signal y AND the system must be stable for this to work.

So assume ***D*** *is zero and instead let us try this observer*

$$\frac{d\hat{\mathbf{z}}}{dt} = \left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u\right) + \mathbf{L}\left(y - \hat{y}\right)$$

$$= \underbrace{\left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u\right)}_{\substack{\text{prediction} \\ \text{(from estimate of} \\ \text{dynamic model)}}} + \mathbf{L}\underbrace{\left(\mathbf{C}\mathbf{z} - \mathbf{C}\hat{\mathbf{z}}\right)}_{\substack{\text{correction} \\ \text{(based on output error)}}}$$

Recalculate the error…

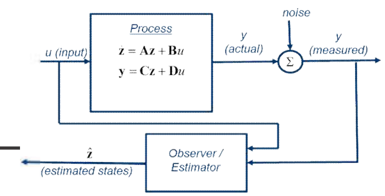
$$\frac{d\mathbf{z}}{dt} - \frac{d\hat{\mathbf{z}}}{dt} = \left(\mathbf{A}\mathbf{z} + \mathbf{B}u\right) - \left(\left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u\right) + \mathbf{L}\left(y - \mathbf{C}\hat{\mathbf{z}}\right)\right)$$

$$= \left(\mathbf{A}\mathbf{z}\right) - \left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{L}\left(\mathbf{C}\mathbf{z} - \mathbf{C}\hat{\mathbf{z}}\right)\right)$$

$$\frac{d\tilde{\mathbf{z}}}{dt} = \mathbf{A}\tilde{\mathbf{z}} - \mathbf{L}\mathbf{C}\tilde{\mathbf{z}}$$

$$= (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{z}}$$

Observer/Estimator



Theorem: If a pair (A,C) are observable, then we can place the eigenvalues of A-LC arbitrarily through and appropriate choice of L

Proof: The Transpose of A-LC is A^T-$C^T L^T$ and in Canonical form this is the same as the eigenvalue problem for state space controllers A-BK

For a given system….

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}u \quad y = \mathbf{C}\mathbf{z}$$

The dual relationship of the Canonical form shows that….

$$\mathbf{A_c} = \mathbf{A_o}^T \ \mathbf{B_c} = \mathbf{C_o}^T \ \textbf{and} \ \boxed{\mathbf{C_c} = \mathbf{B_o}^T}$$

So our objective is to pick L such that our state estimate <u>converges quickly</u> so that are state feedback controller uses accurate estimates.

We use the same methodology as we did to find that feedback gain matrix, and typically would like our observer to <u>converge 5-10x faster</u> than the response of our feedback controller.

Finding the estimation gain matrix, L

So to compute L in our observer/estimator…

$$\frac{d\hat{\mathbf{z}}}{dt} = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{z}}$$

We first compute the observability matrix for the system for the system in Observable Canonical Form (OCF)

$$W_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \ldots \\ \mathbf{CA}^{n-1} \end{bmatrix} \text{ and } \tilde{W}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 & 0 \\ a_1^2 - a_2 & -a_1 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & a_1^2 - a_2 & -a_1 & 1 \end{bmatrix}$$

*We then find our observer gain matrix **L** using a method similar to the one used to find the gain matrix **K.***

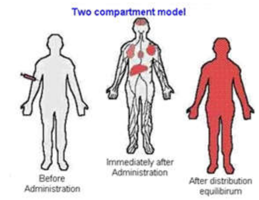
$$\mathbf{L} = \tilde{\mathbf{w}}_o{}^{-1}\tilde{\mathbf{w}}_0 \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \ldots \\ p_n - a_n \end{bmatrix} = \mathbf{T}_o \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \ldots \\ p_n - a_n \end{bmatrix}$$

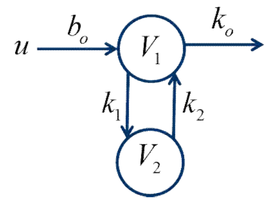
where the Characteristic Equation of the original system and desired system are respectively…

$$\left|\lambda \mathbf{I} - \mathbf{A}\right| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n \qquad \lambda^n + p_1\lambda^{n-1} + \ldots + p_{n-1}\lambda + p_n$$

Estimator Example

$$\frac{dc}{dt} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} c + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$





Objective: *Create a state estimator for our two vessel system.*

First, find the Characteristic Equation for our actual system.

$$CE(\mathbf{A}) = \det \begin{bmatrix} \lambda + k_o + k_1 & -k_1 \\ -k_2 & \lambda + k_2 \end{bmatrix} = 0$$

$$= \lambda^2 + (k_0 + k_1 + k_2)\lambda + k_0 k_2$$

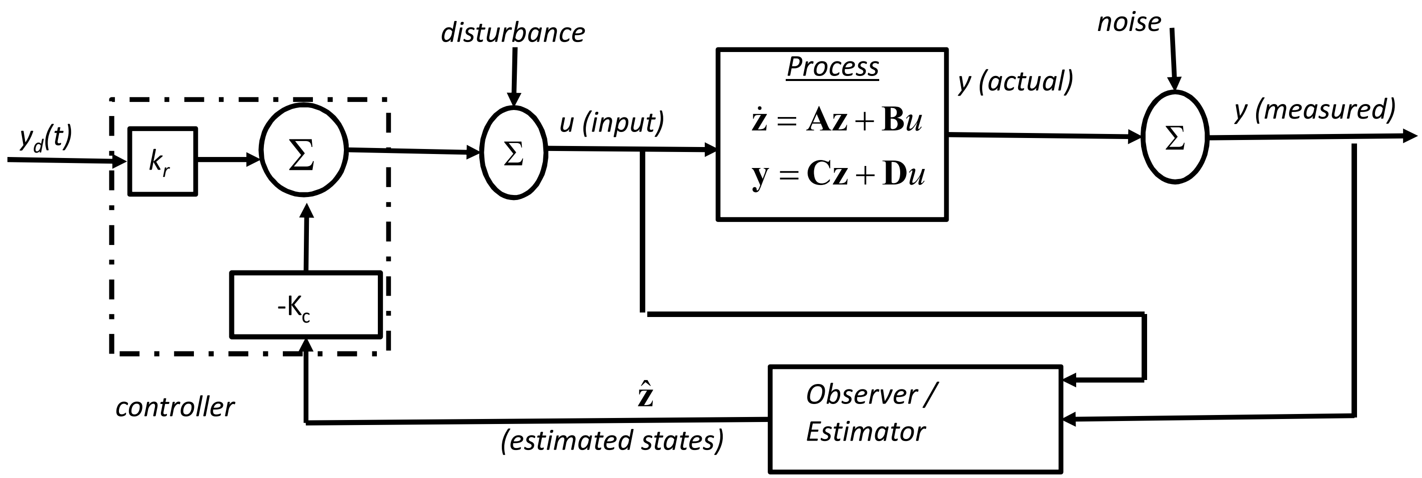
Second, find the desired Characteristic Equation that reflect the desired eigenvalues.

$$CE(\mathbf{A} - \mathbf{LC}) = \lambda^2 + p_1\lambda + p_2 k_2$$

Third, Solve for L

$$L = W_o^{-1}\widetilde{W}_0 \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \dots \\ p_n - a_n \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -a_1 & 1 \end{bmatrix} \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix} = \begin{bmatrix} p_1 - k_o - k_1 - k_2 \\ \left(p_2 - p_1 k_2 + k_1 k_2 + k_2^2\right) k_1^{-1} \end{bmatrix}$$

Observer/Estimator



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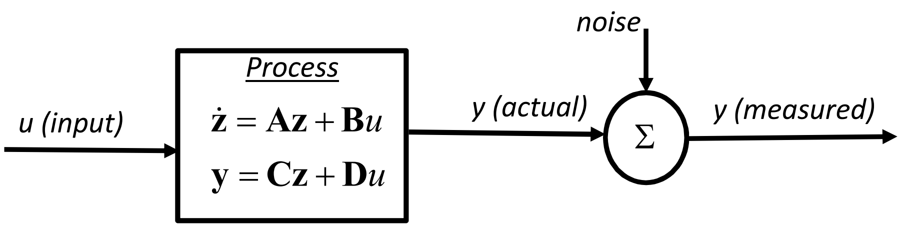
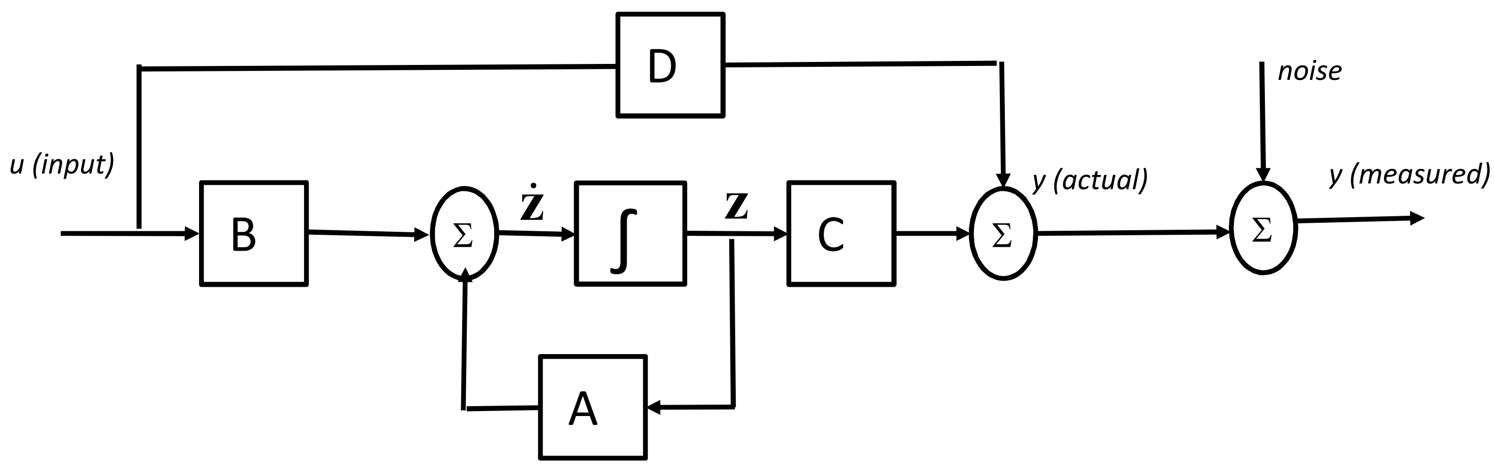
Process



noise
Process
$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$
$\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}u$
u (input)
y (actual)
Σ
y (measured)

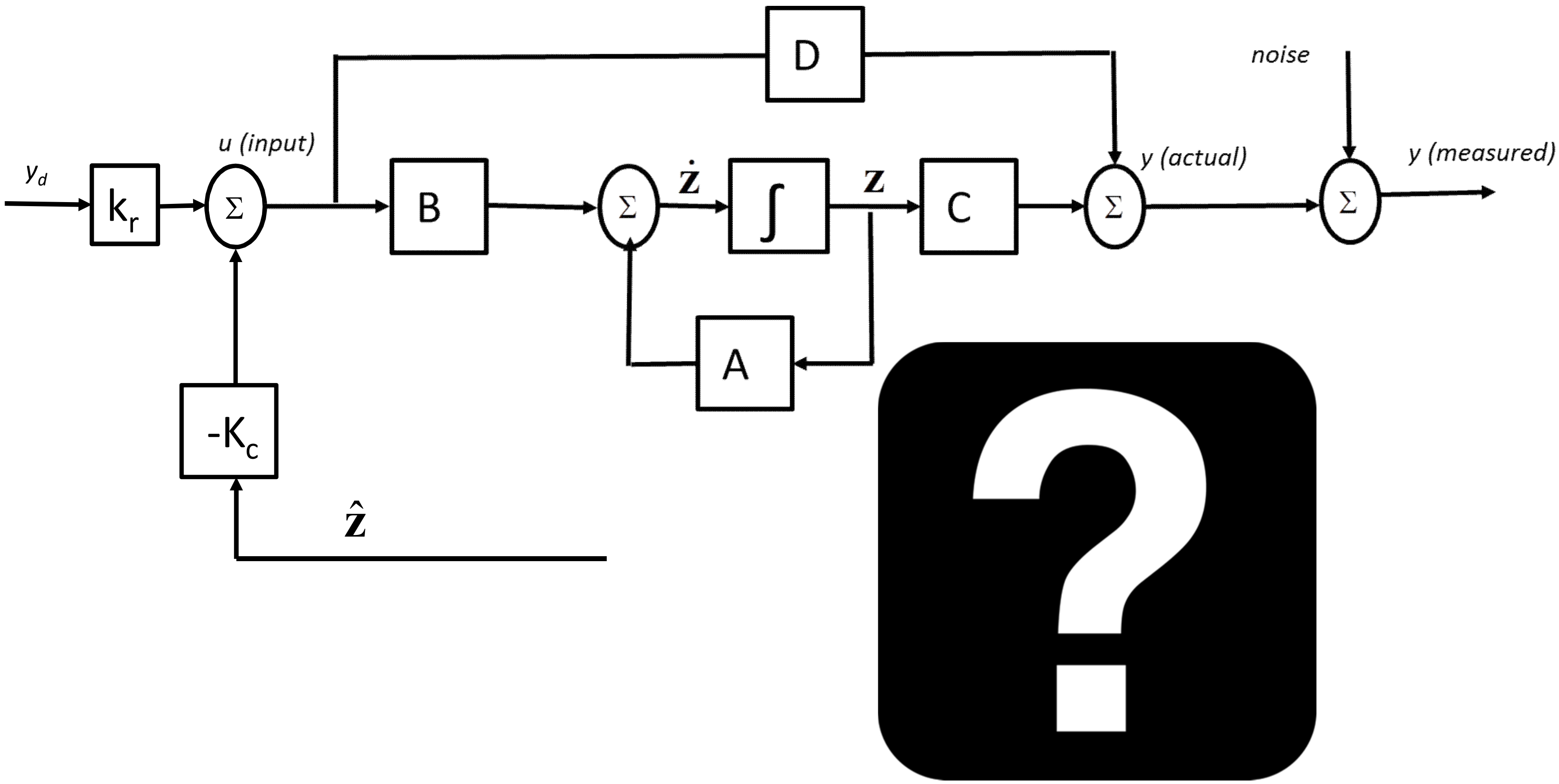


D
noise
u (input)
B
Σ
$\dot{\mathbf{z}}$
∫
$\mathbf{z}$
C
Σ
y (actual)
Σ
y (measured)
A

Broken out block diagram for controller

$$\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{B}(-\mathbf{K_c z} + k_r y_d) = (\mathbf{A} - \mathbf{BK_c})\mathbf{z} + k_r y_d$$

$$\mathbf{y} = \mathbf{Cz} + \mathbf{D}u$$



State Feedback with Estimator.

So we have our two systems.

$$\frac{d\hat{\mathbf{z}}}{dt} = \left(\mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u\right) + \mathbf{LC}\left(\mathbf{z} - \hat{\mathbf{z}}\right)$$

Estimated states

$$\frac{d\mathbf{z}}{dt} = \mathbf{Az} + \mathbf{B}u$$

Actual system.

If we subtract the two systems, we get

$$\frac{d\tilde{\mathbf{z}}}{dt} = \left(\mathbf{A} - \mathbf{LC}\right)\tilde{\mathbf{z}}$$

Drive error estimate to 0.

Recalling that…

$$\begin{aligned}\tilde{\mathbf{z}} &= estimation\_\text{error}\\ &= \mathbf{z} - \hat{\mathbf{z}}\end{aligned}$$

If we use the same input we designed for our state feedback controller

$$u = -K\mathbf{z} + k_r y_d$$

We have a controller using our estimates of the states

$$\frac{d\mathbf{z}}{dt} = \mathbf{Az} + \mathbf{B}\left(-\mathbf{K}\hat{\mathbf{z}} + k_r r\right)$$

Which we can rewrite in terms of our actual states and estimation error.

$$\begin{aligned}\frac{d\mathbf{z}}{dt} &= \mathbf{Az} + \mathbf{B}\left(-\mathbf{K}\hat{\mathbf{z}} + k_r y_d\right)\\ &= \mathbf{Az} - \mathbf{BK}\hat{\mathbf{z}} + \mathbf{B}k_r y_d\\ &= \mathbf{Az} - \mathbf{BK}(\mathbf{z} - \tilde{\mathbf{z}}) + \mathbf{B}k_r y_d\end{aligned}$$

$$\frac{d\mathbf{z}}{dt} = (\mathbf{A}\text{ - }\mathbf{BK})\mathbf{z} + \mathbf{BK}\tilde{\mathbf{z}} + \mathbf{B}k_r y_d$$

Our actual states and estimation error – both go to zero!

Combine this system with our observer and we get:

$$\frac{d}{dt}\begin{bmatrix}\mathbf{z}\\ \tilde{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\mathbf{A}\text{ - }\mathbf{BK} & \mathbf{BK}\\ 0 & \mathbf{A}\text{ - }\mathbf{LC}\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \tilde{\mathbf{z}}\end{bmatrix} + \begin{bmatrix}\mathbf{B}k_r\\ 0\end{bmatrix}y_d$$

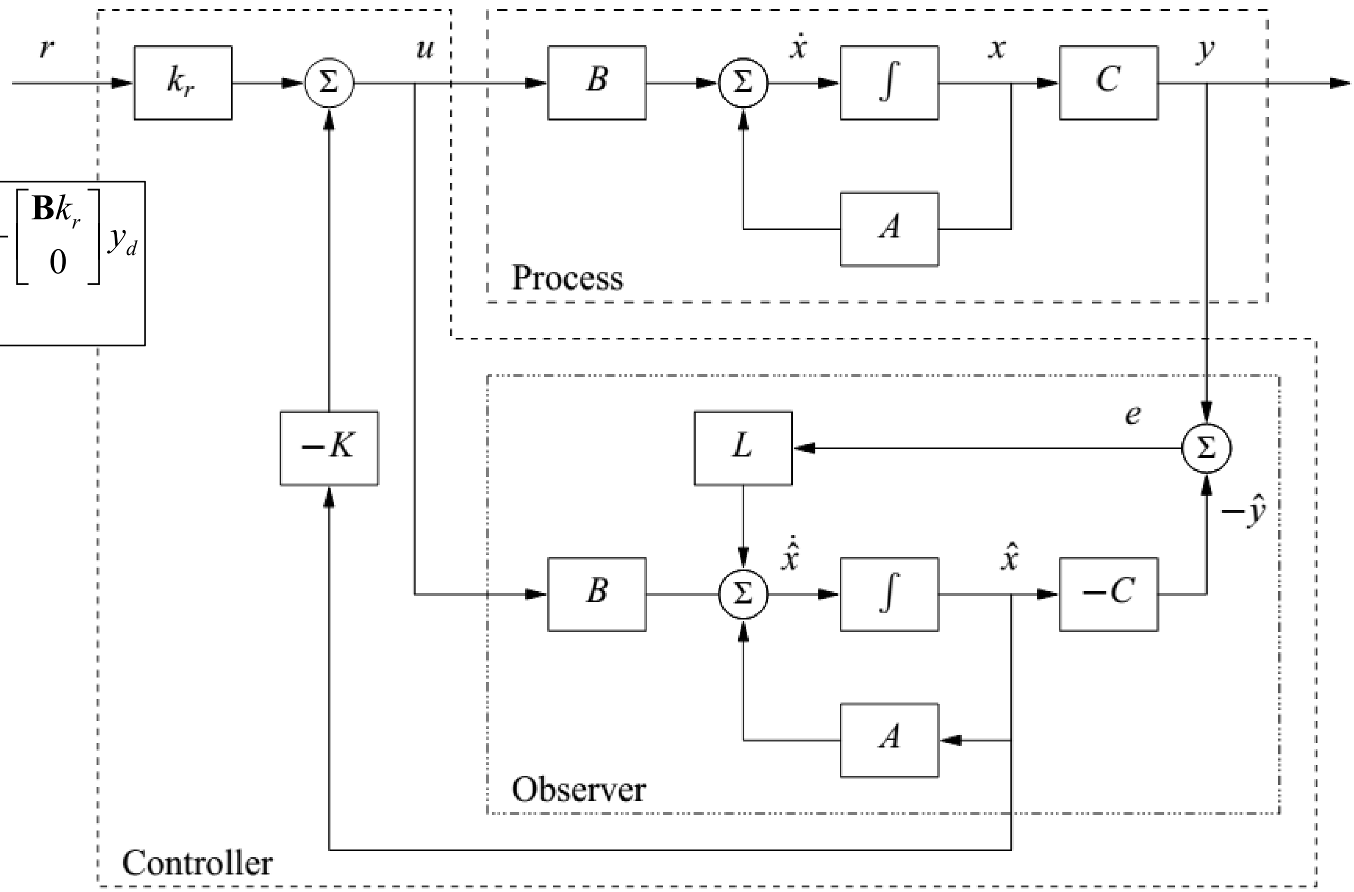
Where the characteristic equation is given by.

$$f\left(\lambda\right) = \det(\lambda I - A + BK)\det(\lambda I - A + LC)$$

Add the estimator

$$\frac{d}{dt}\begin{bmatrix}\mathbf{z}\\ \tilde{\mathbf{z}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{BK}\\ 0 & \mathbf{A}-\mathbf{LC}\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \tilde{\mathbf{z}}\end{bmatrix}+\begin{bmatrix}\mathbf{B}k_r\\ 0\end{bmatrix}y_d$$

$$y=\mathbf{Cz}+\mathbf{D}u=\mathbf{Cz}$$



**Note figure uses x instead of z and r instead of y_d*

Designing Controllers with Observers

$$\frac{d}{dt}\begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ 0 & \mathbf{A}-\mathbf{LC} \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{B}k_r \\ 0 \end{bmatrix} y_d$$

$$y = \mathbf{Cz}$$

$$f(\lambda) = \det(\lambda\mathbf{I}-\mathbf{A}+\mathbf{BK})\det(\lambda\mathbf{I}-\mathbf{A}+\mathbf{LC})$$

Since the eigenvalues of the state feedback are separate from the eigenvalues of the observer we can design each part of the system separately.

It makes no difference whether the state feedback is designed first (using the methods of the previous lecture) or you design the observer first

Summary

- Observability
  - We say the system is observable if for any time T>0 it is possible to determine the state vector, **z**, through the measurements of the output, y(t), as the result of input, u(t), over the period between t=0 and t=T.
- Observability Matrix

$$\mathbf{W_0} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA^2} \\ \cdots \\ \mathbf{CA^{n-1}} \end{bmatrix}$$

- Observable Canonical Form

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ -a_3 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & 0 & 1 \\ -a_5 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} u$$

- Use of Observers/Estimators

$$\frac{d}{dt}\begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A-BK} & \mathbf{BK} \\ 0 & \mathbf{A-LC} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{B}k_r \\ 0 \end{bmatrix} y_d$$