Performance of 2nd order systems

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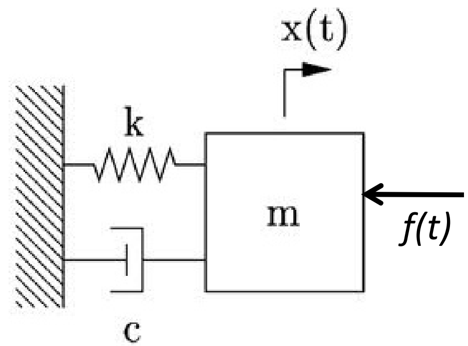
Recall the linear 2nd order MSD system



Our Equation of Motion (EOM)

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Let…

$$z_1 = x \qquad \dot{z}_1 = z_2$$

$$z_2 = \dot{x} \qquad \dot{z}_2 = -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{f(t)}{m}$$

And if the force is our input…

$$u_1 = f(t)$$

Thus in state-space form…

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}\mathbf{u}$$

Canonical 2nd order system

Canonical Form: The archetype or standard form – the default, “natural”, or preferred form.

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \tfrac{1}{2}\sqrt{\frac{c^2}{km}} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

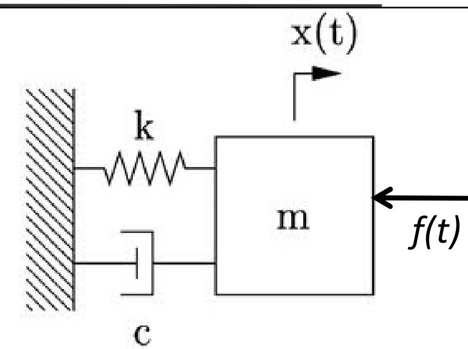
$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u}$$

Which we can also write in state-space form

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega_n{}^2 & 2\zeta\omega_n \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Why these parameters?

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = u$$



(Values for a 2^{nd} order mechanical system)

$$\omega_n = \sqrt{\frac{k}{m}}$$

Natural frequency: The frequency at which a system oscillates when not subjected to a continuous or repeated external force. (The frequency at which it will oscillate *after* a disturbance.)

$$\zeta = \tfrac{1}{2}\sqrt{\frac{c^2}{km}}$$

Damping ratio: A dimensionless measure that describes how a system's motion will decay after a disturbance due to frictional losses.

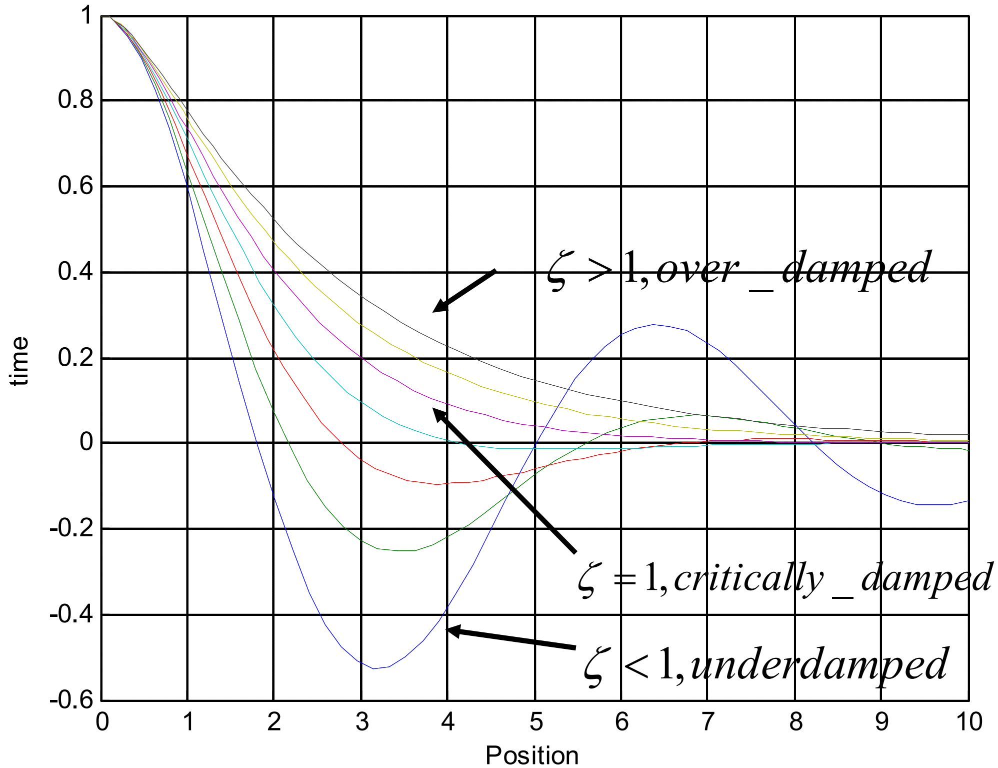
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Damped frequency: The frequency at which an underdamped system will oscillate.

Damping

Damping ratio: A dimensionless measure that describes how a system's motion will decay after a disturbance due to frictional losses.

$$\zeta = \tfrac{1}{2}\sqrt{\frac{c^2}{km}}$$




```
function zprime = msd(t, z);
global dc;
w = 1; %natural frequency

zprime = [
  z(2);
  -2*dc*w*z(2) - w^2*z(1)
];
```

```
global dc
dc = .1

hold all;
for dc=.2:.2:1.4
 [t,z]=ode45('msd', [0 10], [1 0]);
 plot(t, z(:,1));
end;

xlabel('Position');
ylabel('time');
grid on;
```

Canonical system performance…

$$m\ddot{q} + b\dot{q} + kq = f(t)$$

$$m = 1\ kg$$

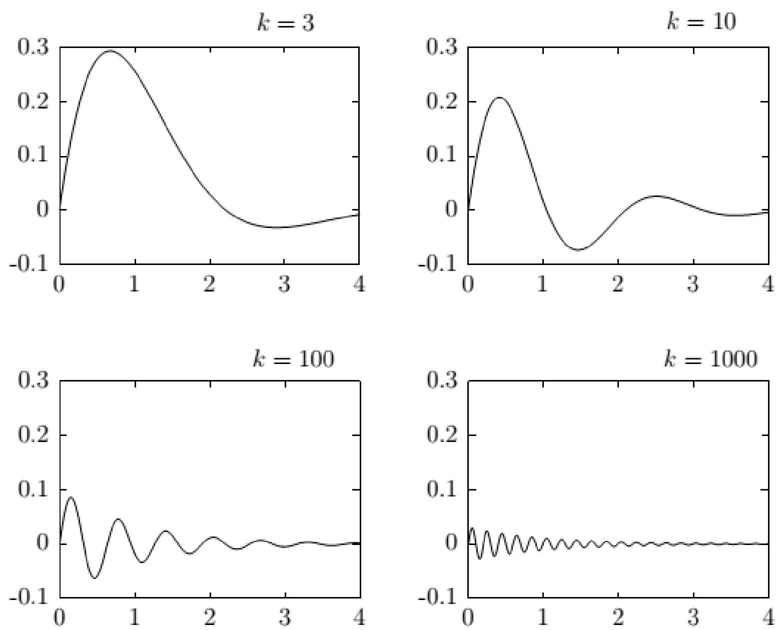
$$b = 2\,\tfrac{N-\text{sec}}{m}$$

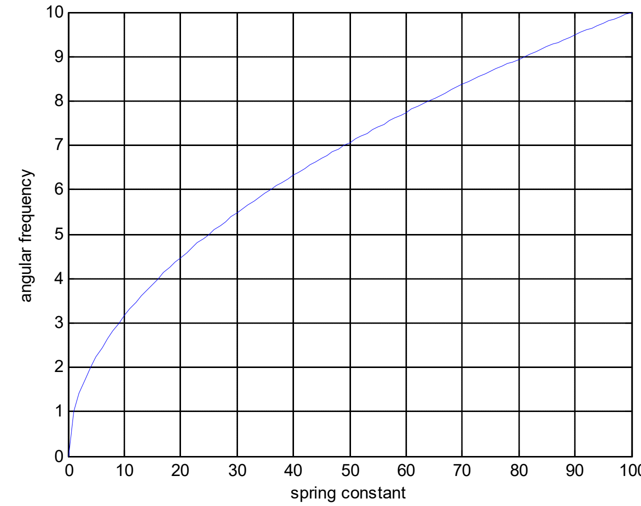
$$q_0 = 0$$

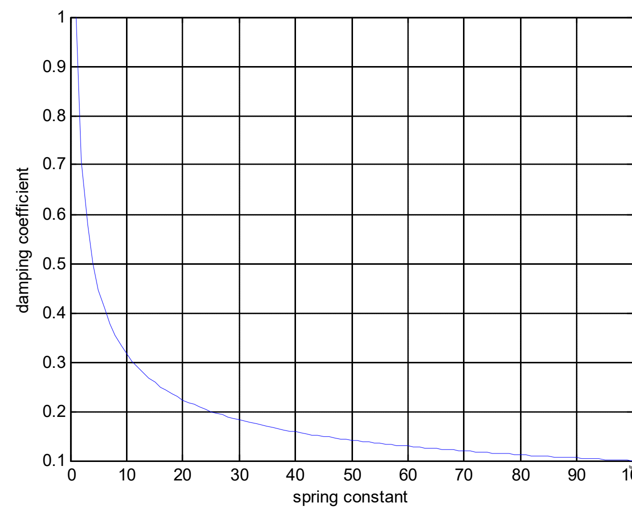
$$v_o = 1\,\tfrac{m}{s}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \tfrac{1}{2}\sqrt{\frac{b^2}{km}}$$







Canonical system performance…

$$m\ddot{q} + b\dot{q} + kq = f(t)$$

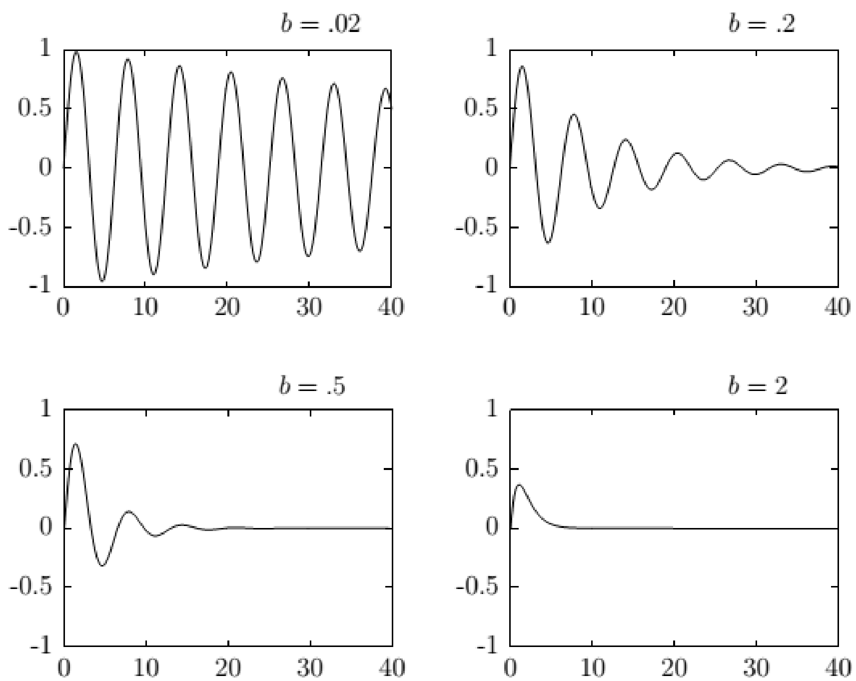
$$m = 1\ kg$$

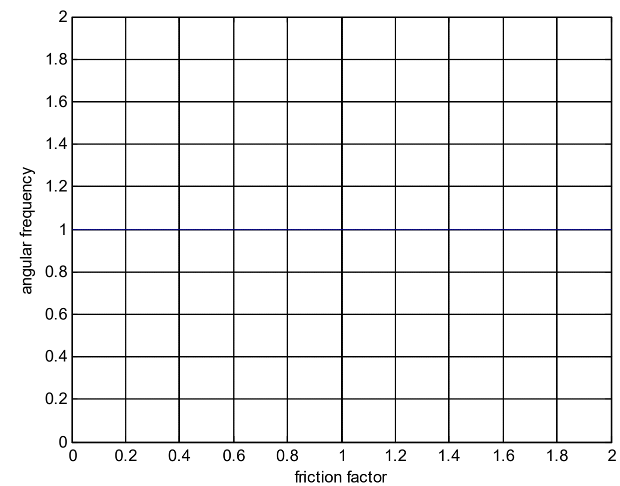
$$k = 1\,\tfrac{N}{m}$$

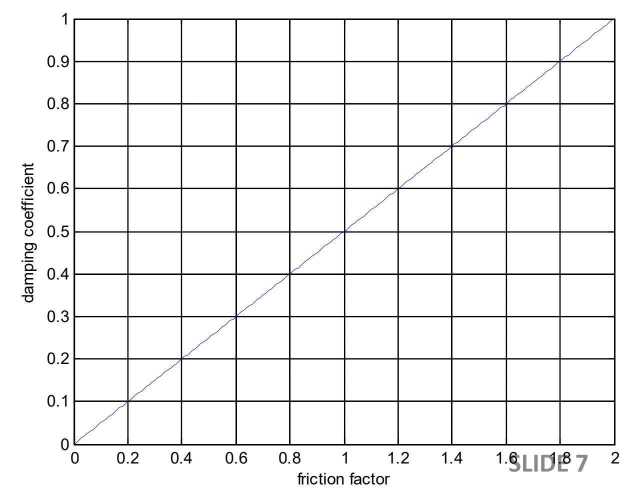
$$q_0 = 0$$

$$v_o = 1\,\tfrac{m}{s}$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \tfrac{1}{2}\sqrt{\frac{b^2}{km}}$$







Canonical system performance…

$$m\ddot{q} + b\dot{q} + kq = f(t)$$

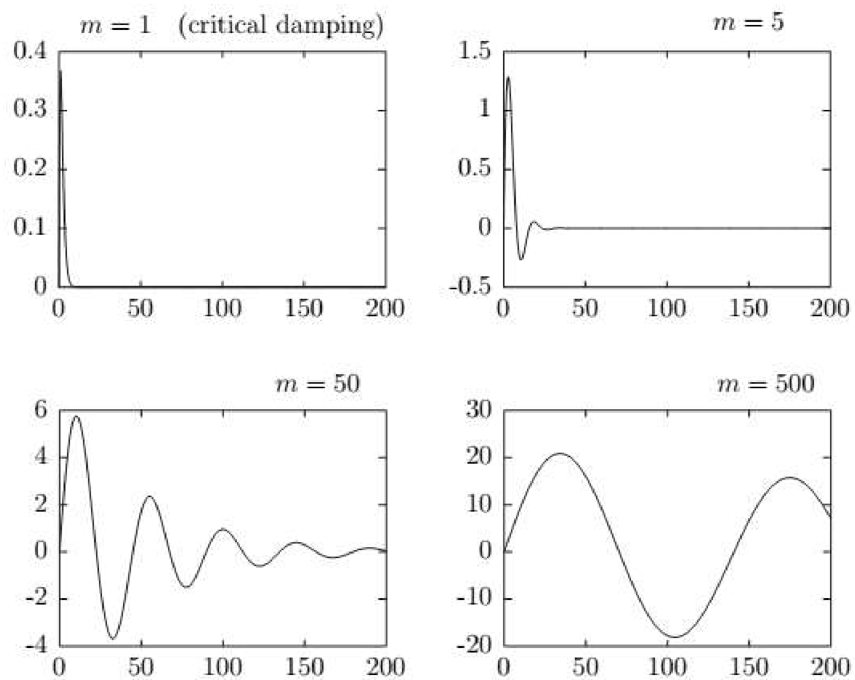
$$k = 1 \ \frac{N}{m}$$

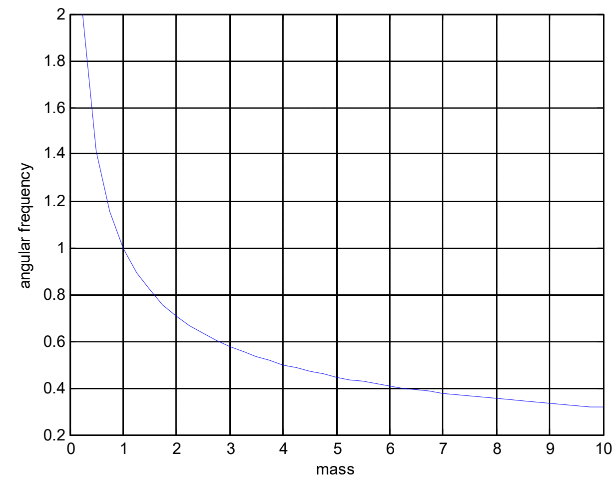
$$b = 2\,\frac{N\text{-sec}}{m}$$

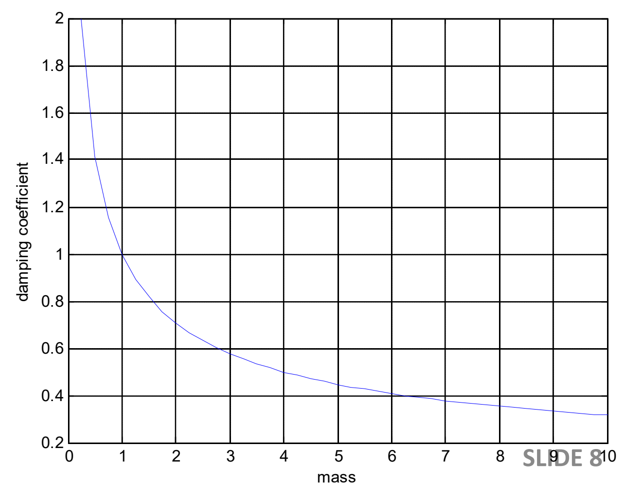
$$q_0 = 0$$

$$v_o = 1\,\frac{m}{s}$$

$$\omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \tfrac{1}{2}\sqrt{\frac{b^2}{km}}$$



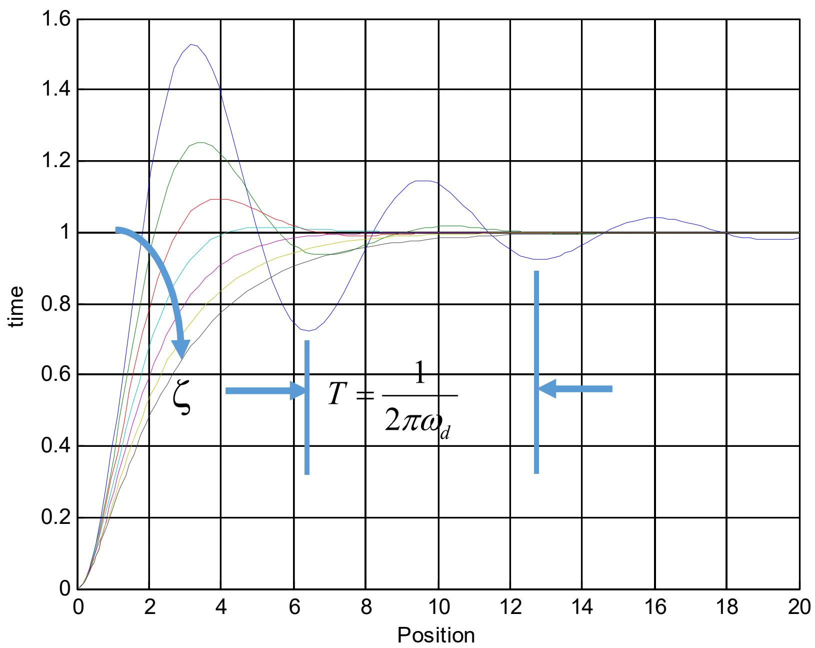




Step Input Response

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^{\,2} q = u \Rightarrow \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^{\,2} & -2\zeta\omega_n \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

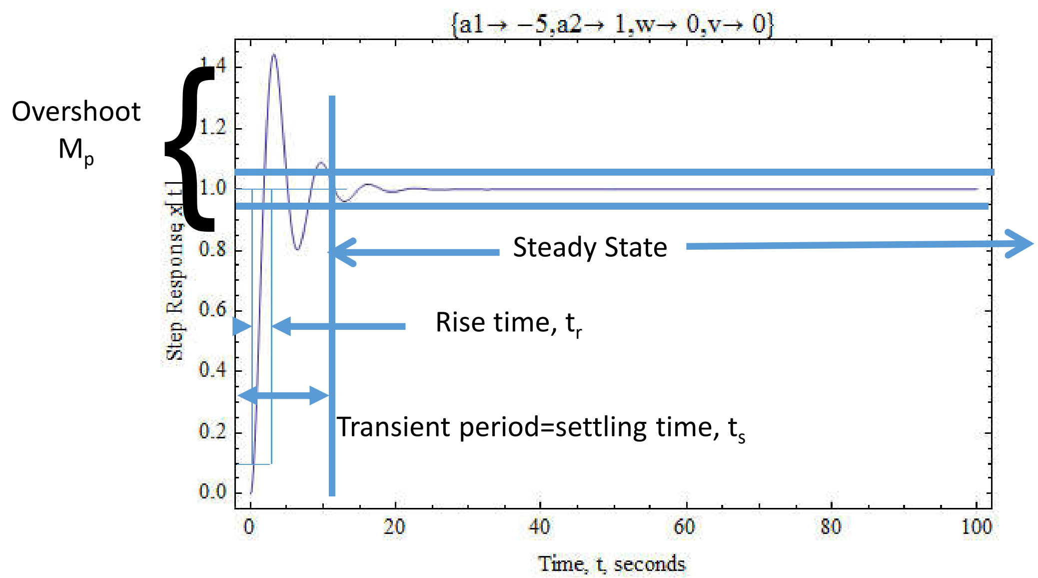
A step input




```
global dc
dc = .1

hold all;
for dc=.2:.2:1.4
  [t,z] = ode45('msd', [0 20], [0 0] );
    plot(t, z(:,1));
end;

xlabel('Position');
ylabel('time');
grid on;

function zprime = msd( t, z );

global dc;
w = 1; %natural frequency
F = 1; %unit step input

zprime = [
    z(2);
    -2*dc*w*z(2) - w^2*z(1) + F
    ];
```

Quantifying the step (u=1) input response

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n{}^2 q = u \Rightarrow \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n{}^2 & -2\zeta\omega_n \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



$$\mathrm{M}_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

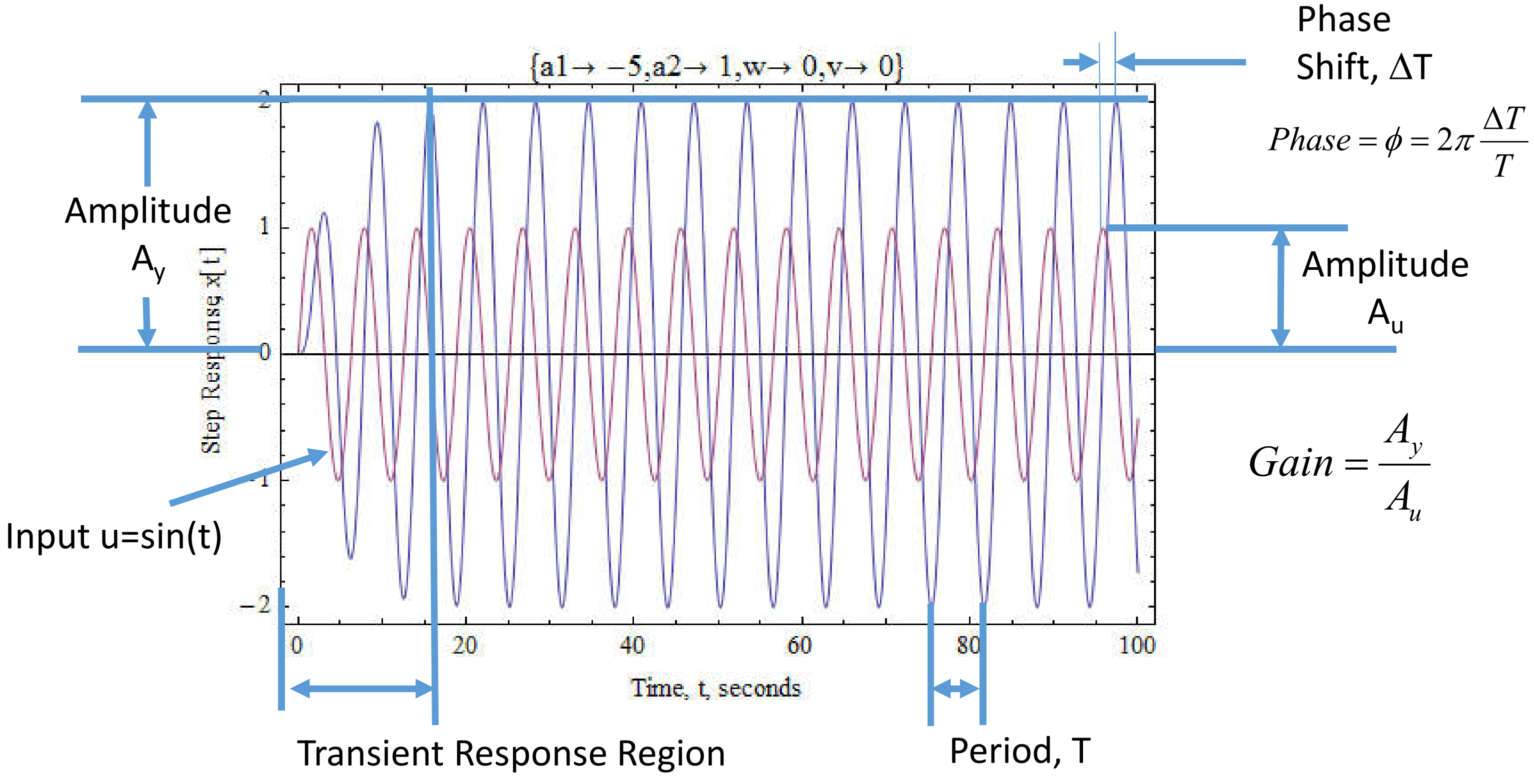
$$t_r = \frac{\pi - \beta}{\omega_d}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n} = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

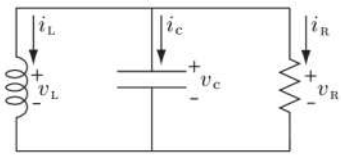
$$t_s = \frac{3}{\zeta\omega_n} \quad \text{(5\% allowable tolerance)}$$

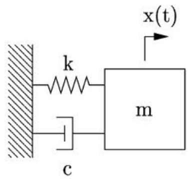
Quantify the sinusoidal input (u=sin(t)) response

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^{\ 2} q = u \Rightarrow \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^{\ 2} & -2\zeta\omega_n \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



2^{nd} order system Summary





$$m\ddot{q} + b\dot{q} + kq = f(t) \quad \text{where, } q(0) = q_o$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \dot{q}(0) = v_o$$

- 2^{nd} order systems are common and thus worth additional focus.
  - Many systems are well modeled as second order systems
- For a 2^{nd} order system in Canonical form there are intuitive responses to changes in the natural frequency and damping coefficient
- Performance metrics can, in turn, be derived from these parameters