Homework 1 Solutions

Q1

10 points
Credit for a signed statement as per the instructions in the homework

Q2

10 points
Credit given for computing angular frequency (5 points) and experimental setup (5 points)

Q3

15 points
3 points for each feedback system (1 point each for sensing mechanism, actuation mechanism and control law)

Q4

15 points
5 points for correct construction of state space model for each set of ODEs

(a)

States:

- $z_1 = y$
- $z_2 = \dot{y}$
- $z_3 = \ddot{y}$

  This choice of states implies that $\dot{z_1} = z_2$ and $\dot{z_2} = z_3$.

  We get the state space representation:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & -7 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{z}$$

(b)

States:

- $z_1 = y_1$
- $z_2 = \dot{y}_1$
- $z_3 = y_2$
- $z_4 = \dot{y}_2$
- $z_5 = y_3$

The implied state relations are $\dot{z}_1 = z_2$ and $\dot{z}_3 = z_4$

We get the state space representation:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 22 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z}$$

(c)

States:

- $z_1 = y_1$
- $z_2 = \dot{y}_1$
- $z_3 = \ddot{y}_1$
- $z_4 = y_2$
- $z_5 = \dot{y}_2$

The implied state relations are $\dot{z}_1 = z_2$, $\dot{z}_2 = z_3$ and $\dot{z}_4 = z_5$

We get the state space representation:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -5 & 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ -2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z}$$

NOTE: In the state space models, y refers to the output. In (b) and (c), $y = y_1$.

Q5

10 points
2 points for correct derivation, 2 points for pointing out assumptions and 6 points for correct state space model

Assumptions:

1. Friction can be neglected.
2. The angles made by both bobs with the vertical, i.e., θ_1 and θ_2, are small.

The second assumption allows us to use the approximations: $\sin\theta \approx \theta, \cos\theta \approx 1$.
For the first bob, resolving forces along the direction perpendicular to the string, we get

$$m_1\ddot{x_1} = -m_1 g \sin\theta_1 + k(x_2 - x_1)$$

The small-angle approximation allows us to approximate x_1 and x_2 as the length of the arcs, i.e., $x_1 = d_1\theta_1$ and $x_2 = d_1\theta_2$. Substituting these expressions, and the sine and cosine approximations into the equation above gives us

$$m_1 d_1 \ddot{\theta_1} + g m_1 \theta_1 + k(d_1\theta_1 - d_2\theta_2) = 0$$

We can follow a similar approach to get the equation of motion for the other bob.

States:

- $z_1 = \theta_1$
- $z_2 = \dot{\theta}_1 \implies \dot{z_1} = z_2$
- $z_3 = \theta_2$
- $z_4 = \dot{\theta}_2 \implies \dot{z_3} = z_4$

The state space representation is

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\frac{g}{d_1} + \frac{k}{m_1}) & 0 & \frac{kd_2}{m_1 d_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{kd_1}{m_2 d_2} & 0 & -(\frac{g}{d_2} + \frac{k}{m_2}) & 0 \end{bmatrix} \mathbf{z}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{z}$$

The outputs we are considering here are θ_1 and θ_2.

Q6

20 points

States:

- $z_1 = x_1$
- $z_2 = \dot{x_1} \implies z_2 = \dot{z_1}$
- $z_3 = \ddot{x_1} \implies z_3 = \dot{z_2}$
- $z_4 = x_2$
- $z_5 = \dot{x_2} \implies z_5 = \dot{z_4}$

With these states, we get the state space representation:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{5}{2} & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{5}{2} & 0 \\ 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The outputs specified are $x_1, \dot{x_2}$ and $u_1(t)$. So, $y = \begin{bmatrix} z_1 & z_5 & u_1(t) \end{bmatrix}^T$.

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

This is an example of when the matrix D is non-zero, i.e., when the input features in the equation for the output.

Q7

5 points
1 point for determinant, 2 points each for eigenvalues and eigenvectors

```
clear all
close all
clc

A = [0 1 0 0 0; 0 0 1 0 0; 2 0 0 -1 1/2; 0 0 0 0 1; -5/2 0 -1/2 1 1/2];
[V, D] = eig(A);
```

Determinant:

```
detA = det(A);
detA
```

```
detA =

    0.5000
```

Eigenvalues:

```
eigval = diag(D)
```

```
eigval =

  -0.8956 + 1.1052i
  -0.8956 - 1.1052i
   1.5249 + 0.0000i
   0.3832 + 0.1234i
   0.3832 - 0.1234i
```

Corresponding eigenvectors (k-th column is the eigenvector with the k-th eigenvalue)

```
V
```

```
V =

  Columns 1 through 4

  -0.0666 + 0.3145i  -0.0666 - 0.3145i   0.0818 + 0.0000i  -0.3533 + 0.0175i
  -0.2879 - 0.3553i  -0.2879 + 0.3553i   0.1247 + 0.0000i  -0.1375 - 0.0369i
   0.6505 + 0.0000i   0.6505 + 0.0000i   0.1902 + 0.0000i  -0.0481 - 0.0311i
   0.2916 + 0.0492i   0.2916 - 0.0492i  -0.5321 + 0.0000i  -0.8559 + 0.0000i
  -0.3155 + 0.2782i  -0.3155 - 0.2782i  -0.8114 + 0.0000i  -0.3280 - 0.1056i

  Column 5

  -0.3533 - 0.0175i
  -0.1375 + 0.0369i
  -0.0481 + 0.0311i
  -0.8559 + 0.0000i
  -0.3280 + 0.1056i
```

Food for thought: Notice the conjugate pairs of eigenvalues and the corresponding eigenvectors. Can you explain this?

Q8

15 points

12 points for correct state space equations and 3 points for modification for different output

By KCL at node e_1, we get

$$\frac{V_1 - u}{R_1} + C_1\frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2} = 0$$
$$\implies \frac{dV_1}{dt} = -\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2})V_1 + \frac{1}{R_2C_1}V_2 + \frac{1}{R_1C_1}u$$

Looking at the current through capacitor C_2, we get

$$C_2\frac{dV_2}{dt} = \frac{V_1 - V_2}{R_2}$$
$$\implies \frac{dV_2}{dt} = \frac{1}{R_2C_2}V_1 - \frac{1}{R_2C_2}V_2$$

Choosing the states to be $z_1 = V_1, z_2 = V_2$, we get the state space representation:

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} -\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} u$$

The output, the voltage across R_2, is $V_1 - V_2$. We then get

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{z}$$

If we were to choose the current through R_1 as the output, we get $y = \frac{1}{R_1}u(t) - \frac{1}{R_1}V_1$. This simply changes the output equation to

$$y = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} \frac{1}{R_1} \end{bmatrix} u(t)$$

The other state space equation (describing the evolution of state variables) remains unchanged.