Root Locus, Part 2

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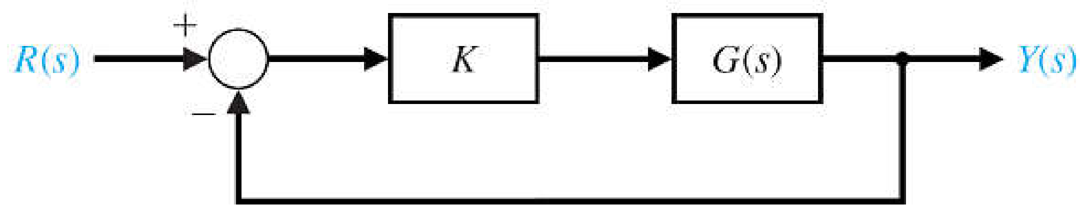
Lesson Objective

- Learn to create a Root Locus plot using pen and paper.
- Quick review of some common Root Locus systems
  - found in tables in most textbooks and web sites.
- Utilize Root Locus to find a desired controller gain.

The root locus plot

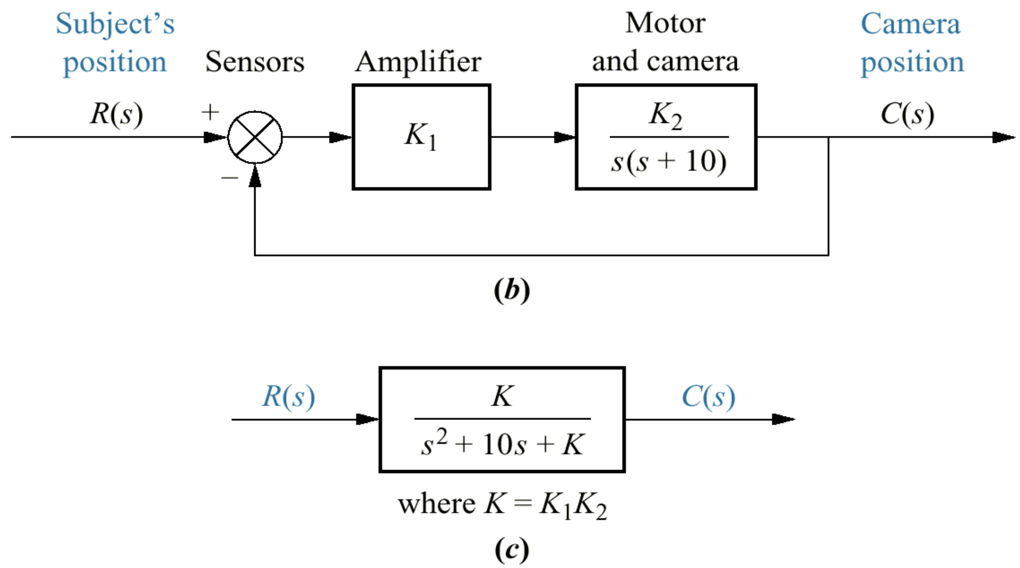


- The root locus plots system response as a function of positive real values of K.
- With *unity* negative feedback, the closed loop transfer function is

$$\frac{Output(s)}{Input(s)} = \frac{KG(s)}{1+KG(s)} = \frac{K\dfrac{N_G(s)}{D_G(s)}}{1+K\dfrac{N_G(s)}{D_G(s)}} = \frac{KN_G(s)}{D_G(s)+KN_G(s)}$$

- The characteristic equation is $\Delta(s) = 1 + KG(s) = D_G(s) + KN_G(s)$
- MATLAB makes the rlocus relatively easy to find
  - rlocus, rlocfind, rltool, zpk, pzmap, pzplot, etc.
- It is (relatively) easy *and useful* to sketch the Root Locus.

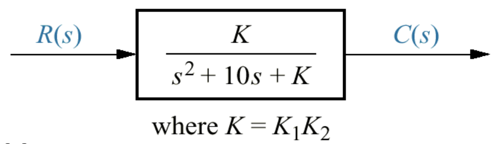
A simple example (camera control system)



How the dynamics of the camera changes as K is varied ?

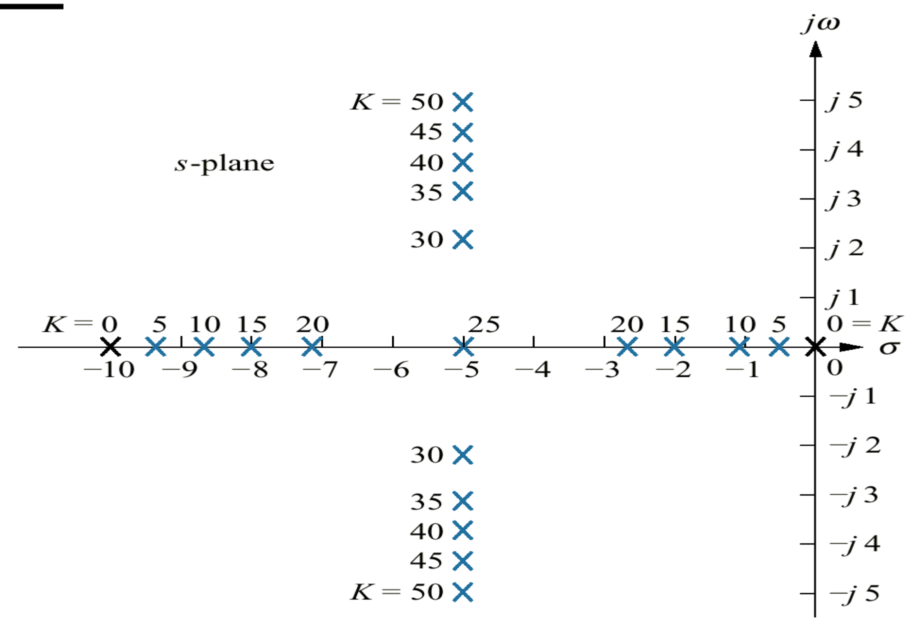
One option is a "brute force" solution. (i.e. calculate the poles for a bunch of different values of K and plot them.)

One option….



- Recalculate the polynomials for various values of K.
- Then plot them…..

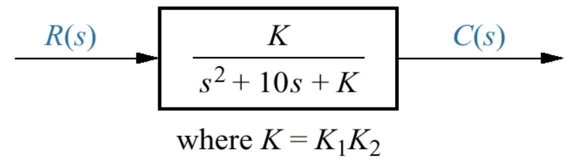
K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

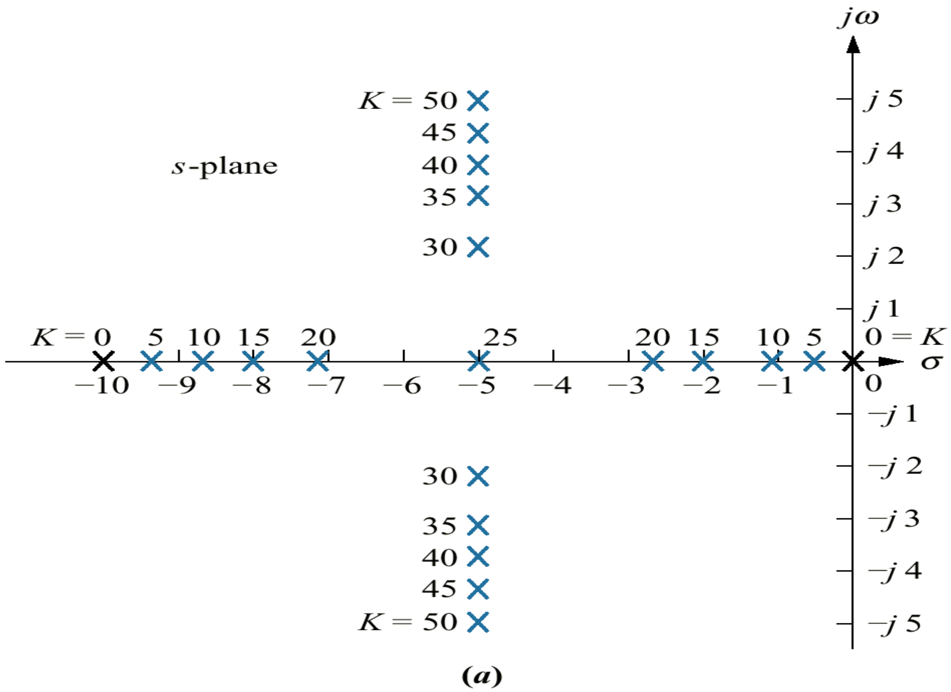


(a)

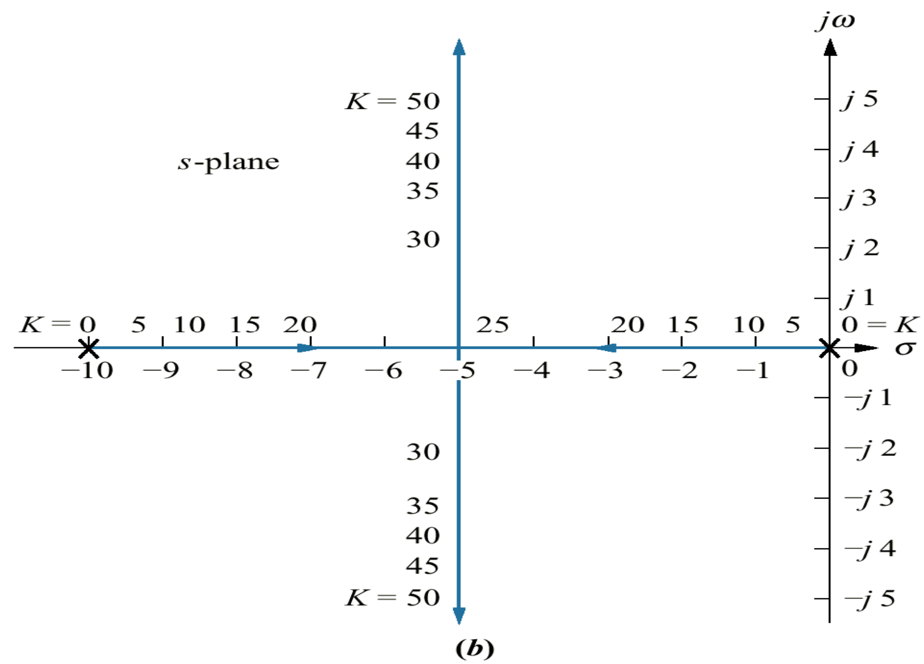
Tedious…. Also, note the equivalent to open loop transfer function is when K=0…

A better option (Root Locus)…





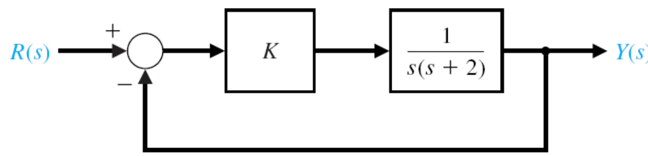
(a) Pole plots from the table.



(b) Root locus.

2^{nd} order system, in general

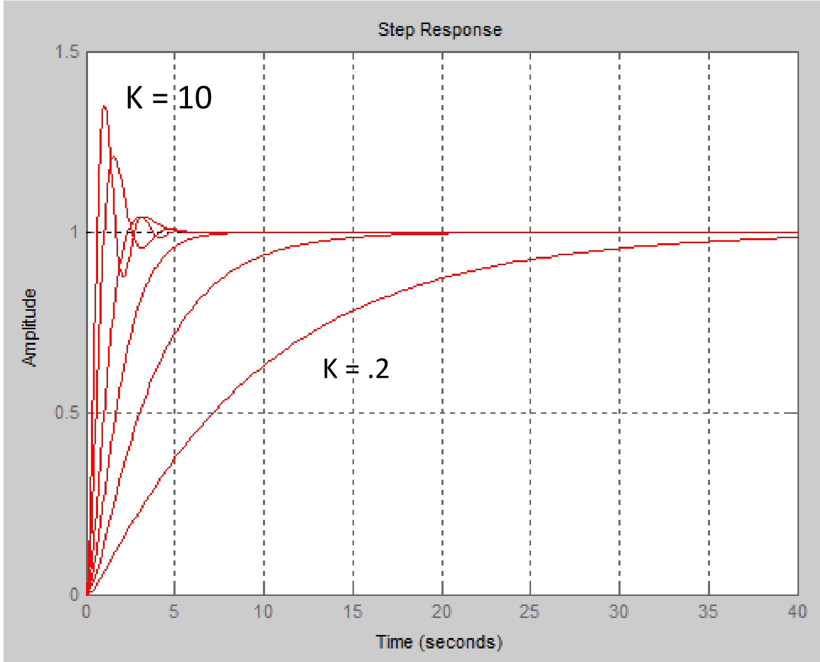
- Similar to the above example…



- The characteristic equation is:

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0$$
$$= \frac{s(s+2)}{s(s+2)} + \frac{K}{s(s+2)} = 0$$
$$= s^2 + 2s + K$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

```
clear all;
K = [.2 .5 1 2 5 10 ];

for i=1:length(K)
    A = [ K(i) ];
    B = [ 1 2 K(i) ];
    sys = tf( A, B );
    step( sys, 'r');
    hold on;
end
grid on;
axis( [0 40 0 1.5 ]);
```

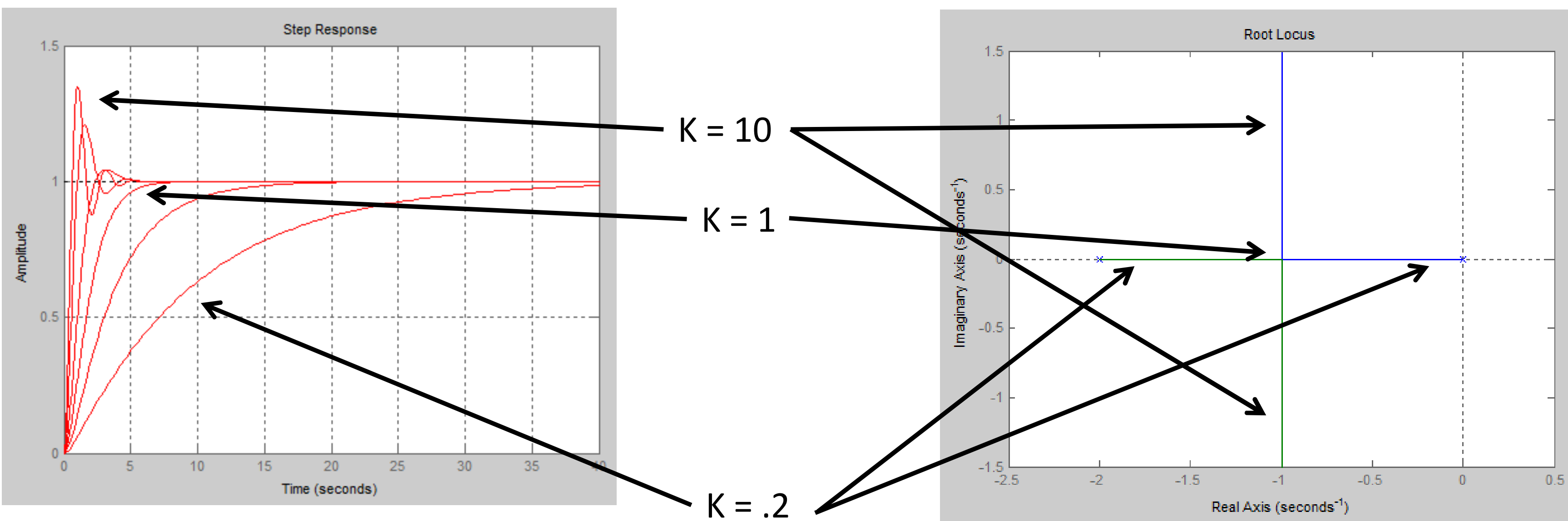


Step input responses

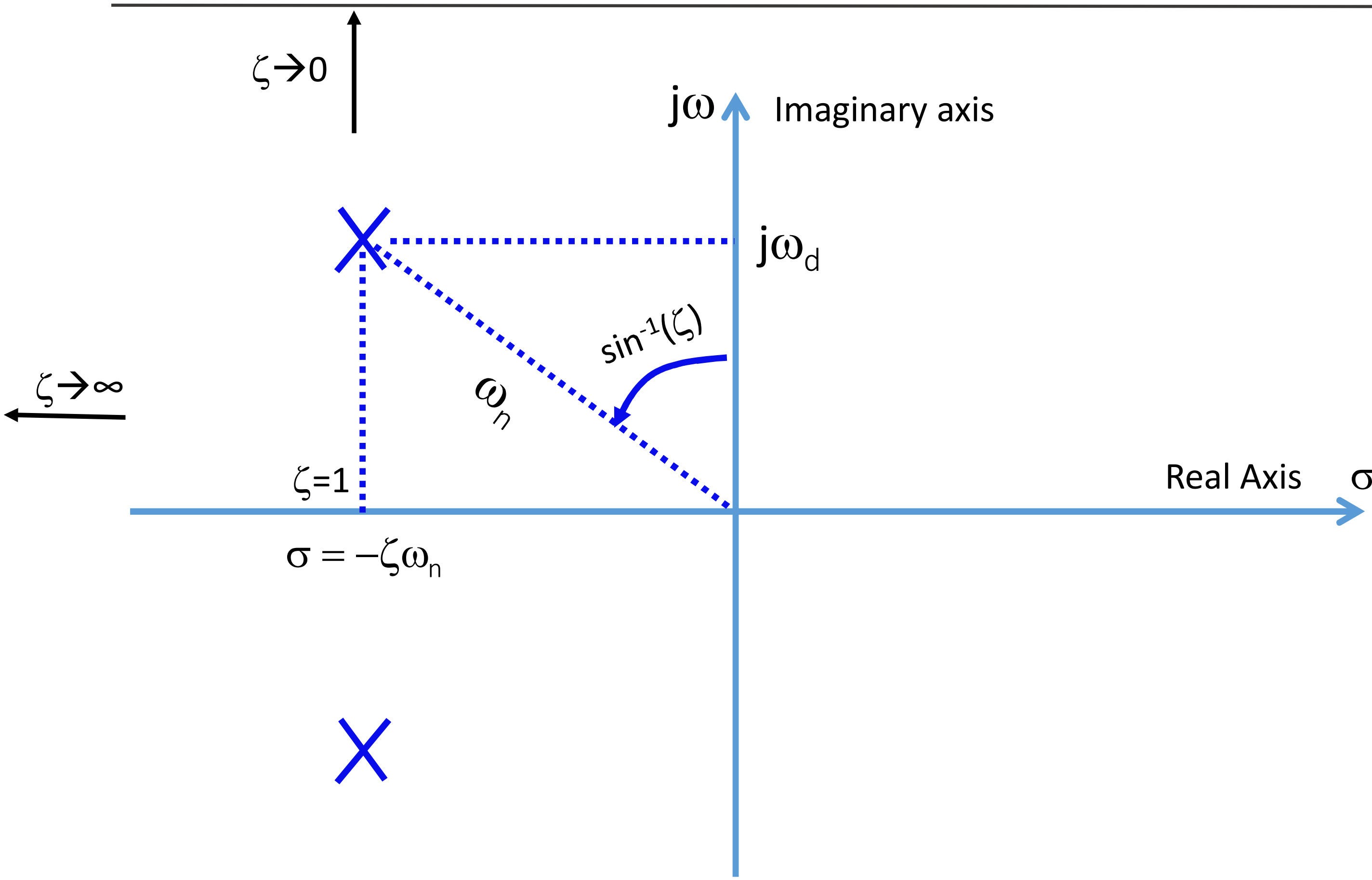
$$G(s) = \frac{1}{s(s+2)}$$

$$T_{cl}(s) = \frac{KG(s)}{1+KG(s)} = \frac{KN_G(s)}{D_G(s)+KN_G(s)} = \frac{K}{s^2+2s+K}$$

```
clear all;
hold off;

K = [.2 .5 1 2 5 10 ];

for i=1:length(K)
    A = [ K(i) ];
    B = [ 1 2 K(i) ];
    sys = tf( A, B );
    step( sys, 'r');
    hold on;
end
axis( [0 40 0 1.5 ]); grid on;

figure(2);
sys = tf( [1], [1 2 0] );
rlocus(sys);
```



Pole placement in Root Locus

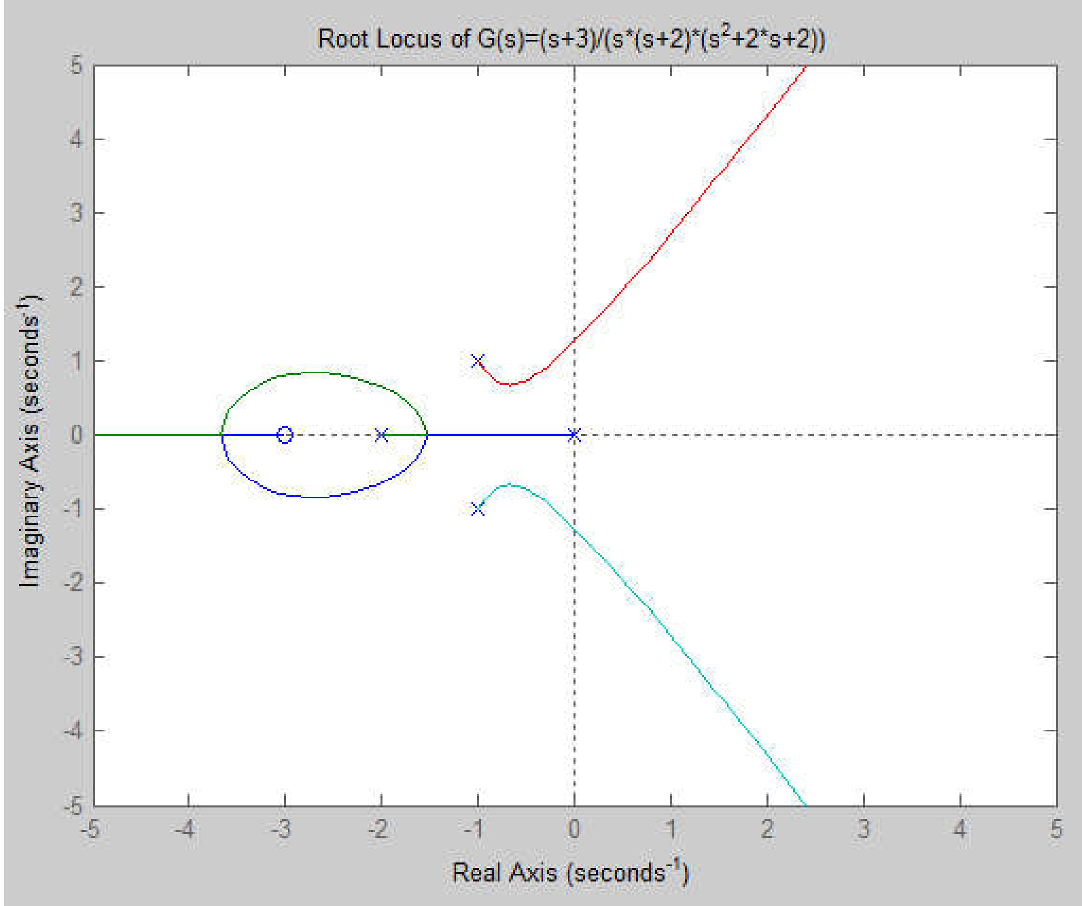


- The magnitude of the vector to pole location is the natural frequency of the response, ω_n
- The vertical component (the imaginary part) is the damped frequency, ω_d
- The vertical is the inverse sine of the damping ratio, ζ
- What if the damping coefficient is less than one?

Root locus key components

$$G(s) = \frac{s+3}{s(s+2)(s^2+2s+2)}$$

- Poles marked with X
- Zeros with an O
- There is a zero for each pole
- Those not shown are at infinity
- 's' is a complex number
- Each path represents a branch of the transfer function in the complex plane
- All paths
  - start at poles (no K)
  - end at zeros (K=∞)
  - symmetry across the real axis



Creating Root Locus

- First in MATLAB
  - rlocus(), etc.

- Create by hand
  - Have seen MATLAB do some funky things (ambiguous functions)
  - ***Understand the significance of adding/removing poles/zeros.***
  - Ability to examine components other than the feedforward gain.
  - *Homework / tests ;)

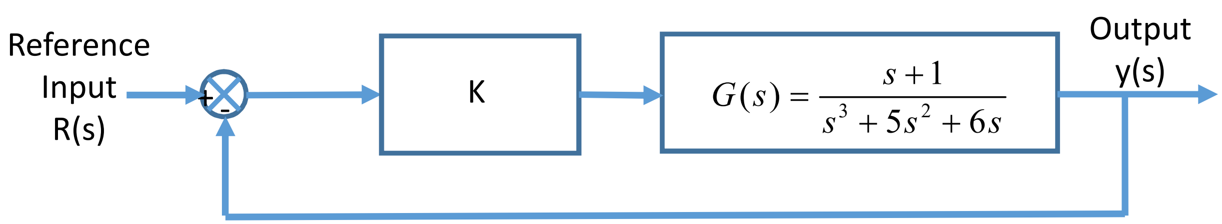
- Root Locus Tables
  - Most books have a table of RL plots for common systems.

Create by hand: 10 Step plan...

1. Write the characteristic equation so that the parameter of interest, *K*, is the multiplier.

$$\Delta(s) = 1 + KP(s)$$

Example:



$$T_{LC}(\mathrm{s}) = \frac{G}{1 + KG}$$

$$\Delta(s) = 1 + K\frac{s+1}{s^3 + 5s^2 + 6s}$$

10 Step plan…

$$G(s) = \frac{s+1}{s^3 + 5s^2 + 6s}$$

2. Factor P(s) into zeros and poles

$$\Delta(s) = 1 + KP(s) = 1 + K\,\frac{zeros}{poles}$$

Example:

$$\Delta(s) = 1 + K\,\frac{s+1}{s^3 + 5s^2 + 6s}$$

$$\Delta(s) = 1 + K\,\frac{s+1}{s(s+2)(s+3)}$$

10 Step plan…

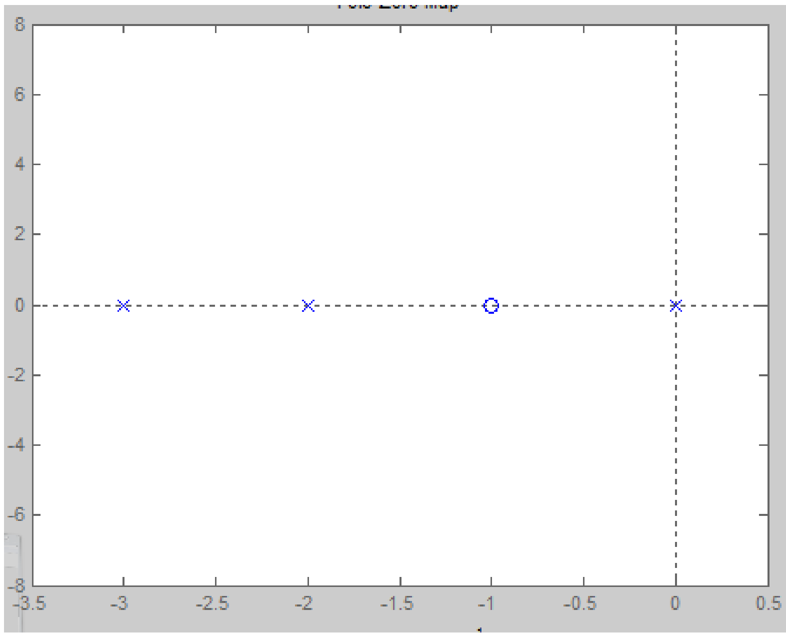
$$G(s) = \frac{s+1}{s^3 + 5s^2 + 6s}$$

3. Locate the open loop poles and zeros in the s-plane.

$$\Delta(s) = 1 + KP(s) = 1 + K\frac{zeros}{poles}$$

Example:

$$\boxed{\Delta(s) = 1 + K\frac{s+1}{s(s+2)(s+3)}}$$



10 Step plan...

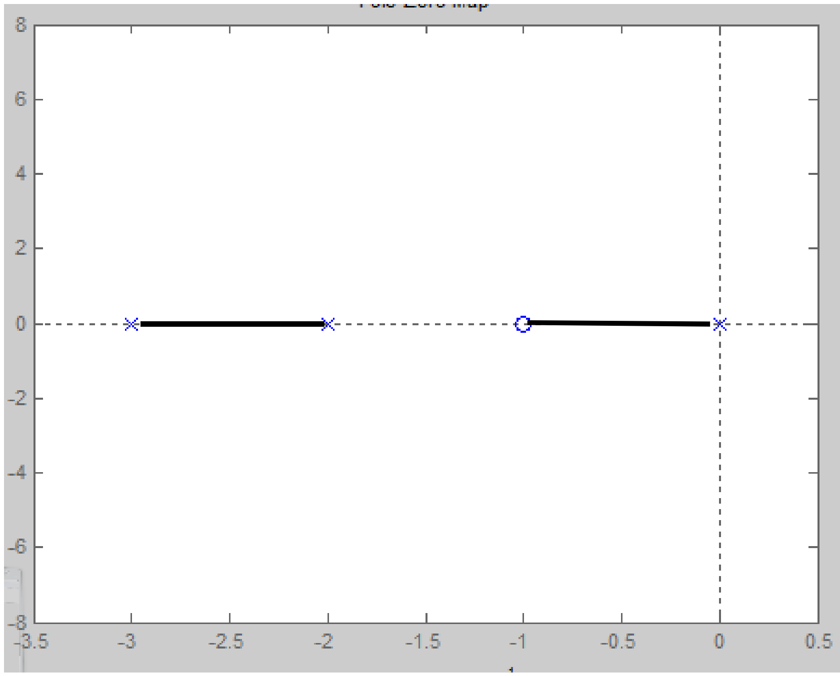
$$G(s) = \frac{s+1}{s^3 + 5s^2 + 6s}$$

4. Locate the segments on the real axis that are root loci.

"Locus lies to the left of odd number of poles and zeros"

Example:

$$\boxed{1 + K\frac{s+1}{s(s+2)(s+3)} = 0}$$



10 Step plan…

$$G(s) = \frac{s+1}{s^3 + 5s^2 + 6s}$$

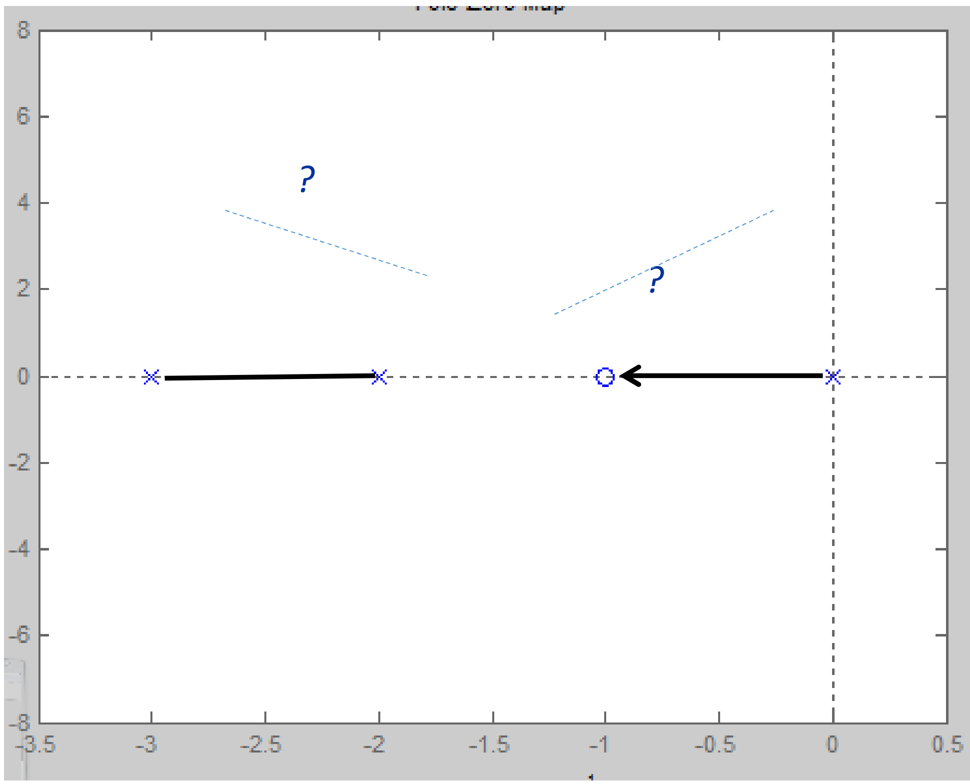
5. Determine the number of *Separate Loci, SL*. Each SL will be a line that tracks from a pole to a zero at infinity

"SL=number of poles when $n_p > n_z$ (in controls, this is almost always true)"

Example:

$$\boxed{1 + K\frac{s+1}{s(s+2)(s+3)} = 0}$$

$$\begin{aligned} SL &= n_p - n_z \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

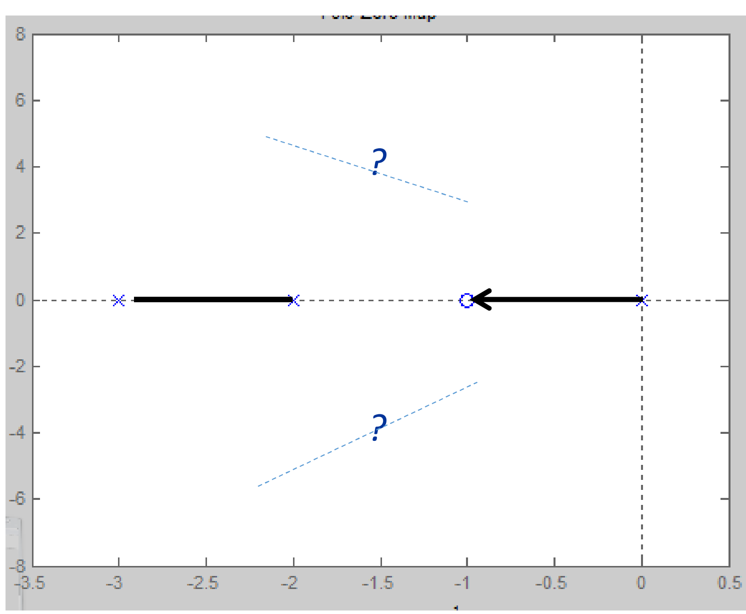


10 Step plan…

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

6. The root loci are symmetrical with respect to the real axis.

Example:



10 Step plan…

$$\Delta(s) = 1 + K\frac{s+1}{s(s+2)(s+3)}$$

7. The root loci proceed to "zeros" at infinity along asymptotes intersecting the real axis at at σ_A and at an angle ϕ_A

$$\sigma_A = \frac{\sum(p_j) - \sum(z_j)}{n_p - n_z}$$

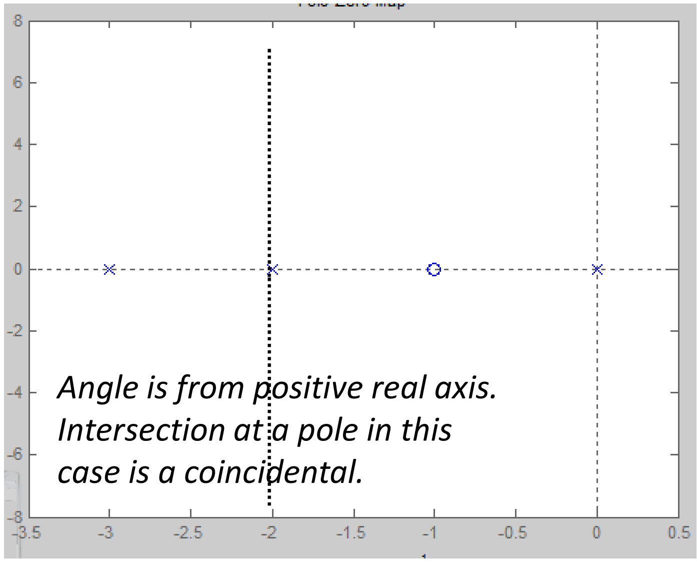
$$\phi_A = \frac{2q+1}{n_p - n_z}180°, q = 0,1,2,\ldots(n_p - n_z - 1)$$

Example:

$$\sigma_A = \frac{\sum(-0-2-3) - \sum(-1)}{3-1} = \frac{-4}{2} = -2$$

$$n_p - n_z - 1 = 3 - 1 - 1 = 1 \Rightarrow q = [0,1]$$

$$\phi_A = \frac{2([0,1]) + 1}{3-1}180° = 90°, 270°$$



10 Step plan…

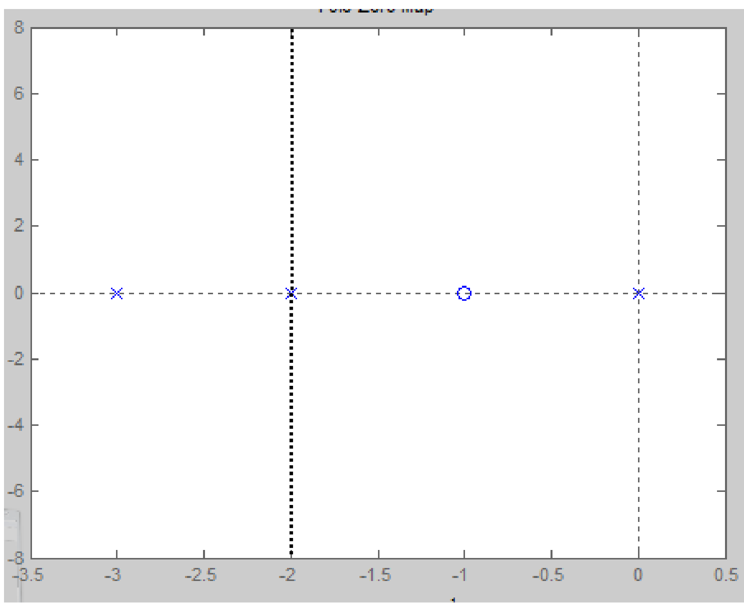
$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

8. Determine (if necessary) the point where a branch crosses the imaginary axis.

If necessary use Hurwitz or Routh Criteria discussed in previous lecture.

Example:

Not applicable. System is stable for all K as our root loci indicate the poles will never make the system unstable between zero and infinity.



10 Step plan

$$1+K\frac{s+1}{s(s+2)(s+3)}=0$$

9. Determine (if necessary) the breakaway points on the real axis.

"Breakaway points occur where two poles have the same value."

a) *Set* $K\frac{N_G}{D_G}=-1 \Rightarrow K=-\frac{D_G}{N_G}=p(s)$

b) *Determine* $\frac{dp(s)}{ds}=0$

c) *Find roots for the equation found in part b.)*

d) *Loci always leave or enter the real axis at 90°*

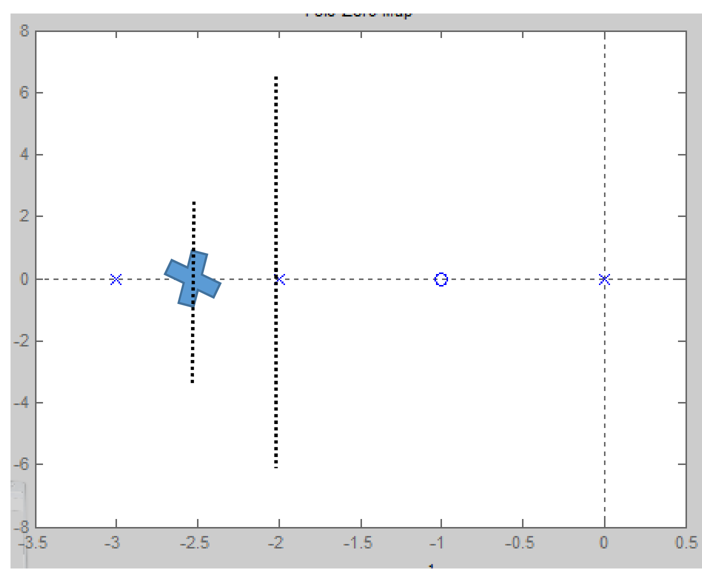
Example:

$$p(s)=\frac{-s(s+2)(s+3)}{s+1}=\frac{-(s^3+5s^2+6s)}{s+1}$$

$$\frac{dp}{ds}=2s^3+8s^2+10s+6=0$$

```
>>roots( [2 8 10 6])
ans = [ -2.4656 -0.7672±0.7926i ]
```

Often do not need to find exactly and the general area of the break is sufficient for manual analysis.



10 Step plan…

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

10. Determine (if necessary) the angles of departure from complex poles or angle of arrival for complex zeros.

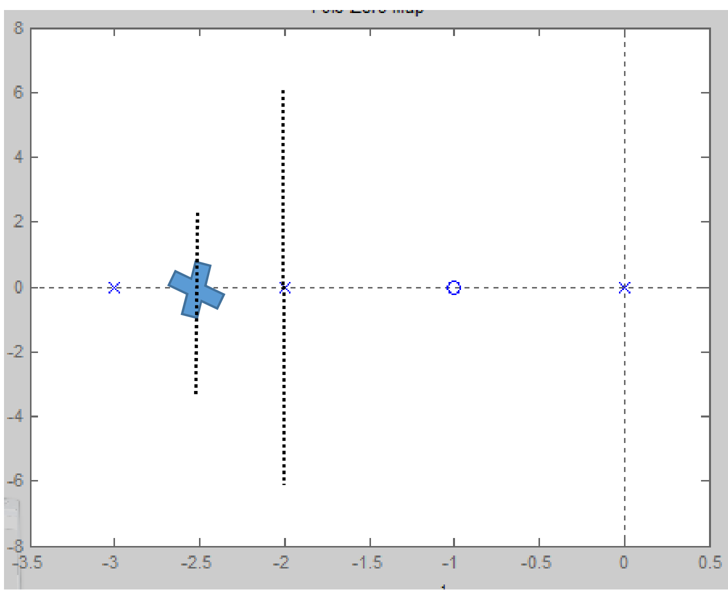
"Angle of departure/arrival is the difference between the net angle due to all other poles and zeros."

$$\theta_{p_k,departure} = 180 + \sum_{i=1}^{n} \measuredangle\left(p_k - z_i\right) - \sum_{j=1}^{m} \measuredangle\left(p_k - p_j\right)$$

$$\theta_{z_k,arrival} = 180 + \sum_{j=1}^{m} \measuredangle\left(z_k - p_j\right) - \sum_{j=1}^{m} \measuredangle\left(z_k - z_i\right)$$

Our Example:

Not applicable. System has no complex poles of zeros.



10 Step plan…

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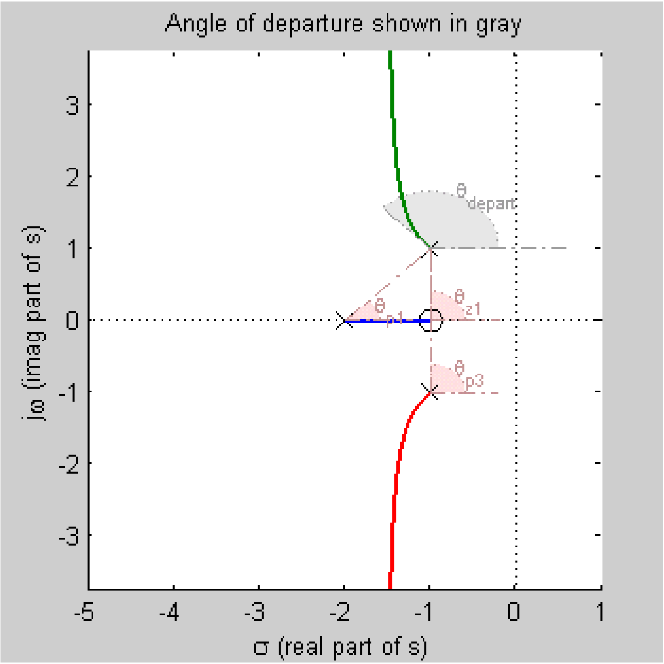
$$\theta_{z_k,arrival} = 180 + \sum_{j=1}^{m} \measuredangle\left(z_k - p_j\right) - \sum_{j=1}^{m} \measuredangle\left(z_k - z_i\right)$$



A different example:

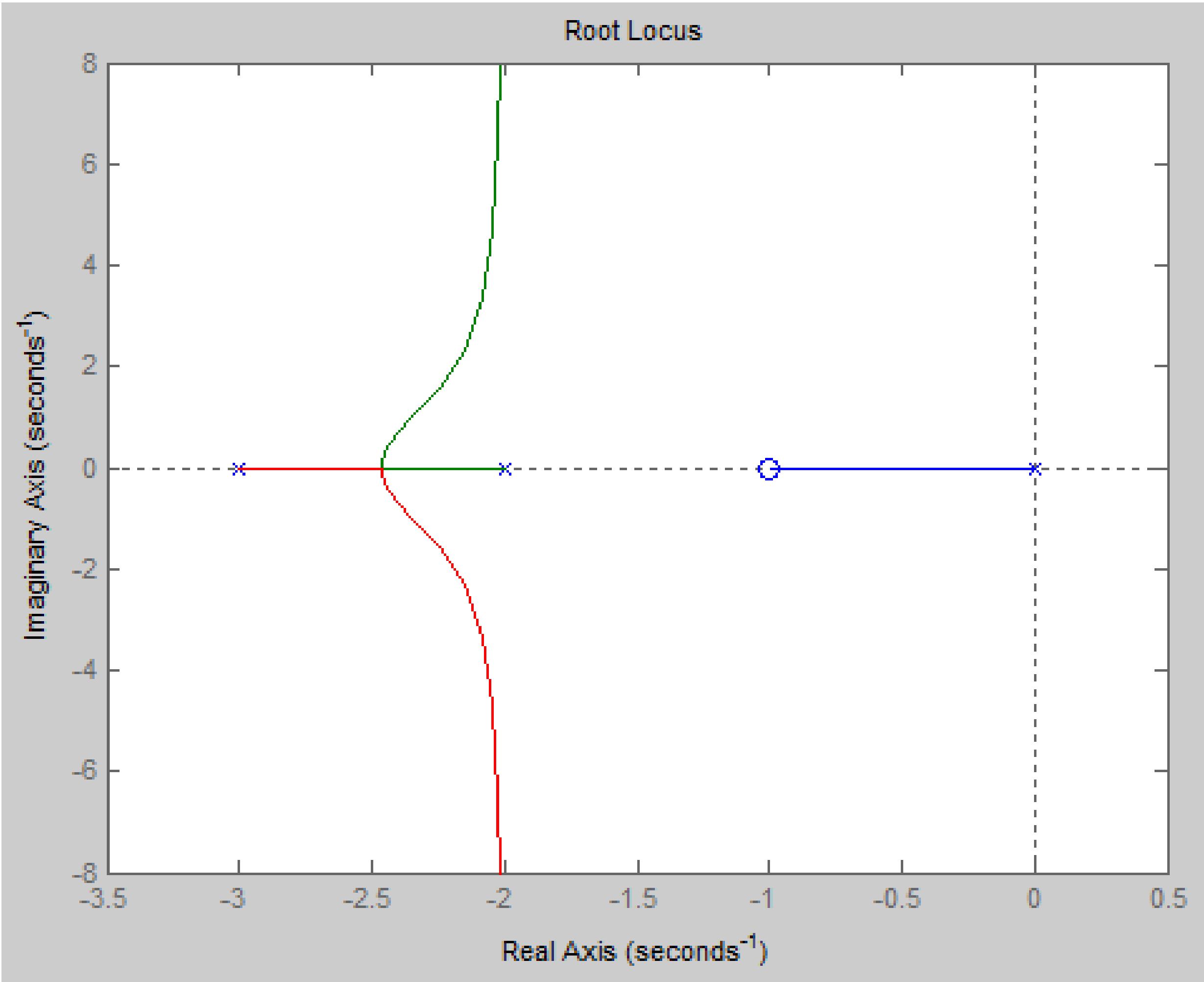
$$\boxed{1 + K\frac{s+1}{s^3 + 4s^2 + 6s + 4} = 0}$$

$$\boxed{z_1 = -1;\ p_1 = -2;\ p_{2,3} = -1 \pm j}$$

$$\boxed{\theta_{-1+j} = 180 + 90 - \left(45 + 90\right)}$$

Complete Root Locus

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$



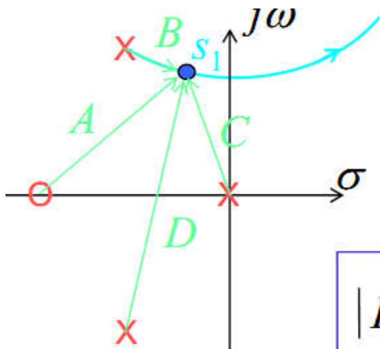
Finding K

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

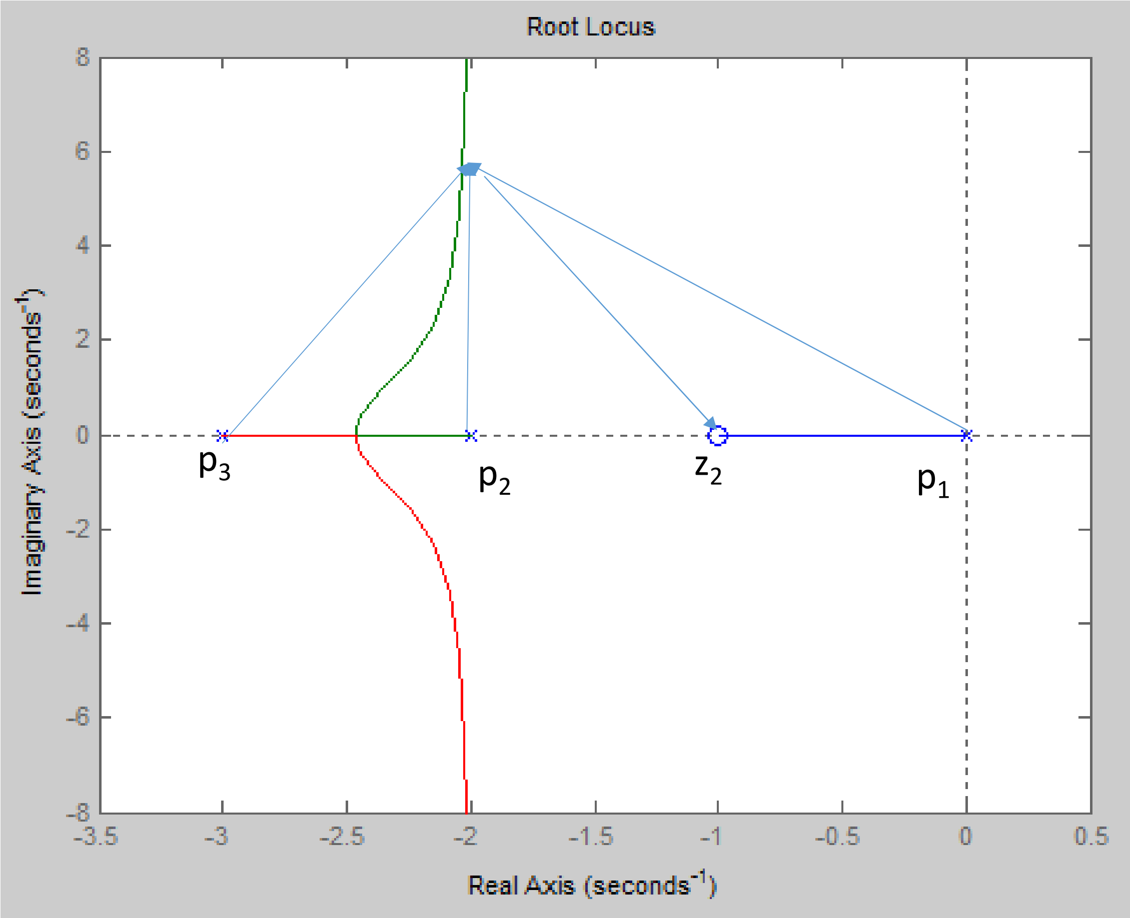
Mathematically,

$$K_x = \left. \frac{\prod_{j=1}^{n_p} |s + p_i|}{\prod_{i=1}^{n_z} |s + z_i|} \right|_{s=s_x}$$

Or graphically (for example),



$$|K| = \frac{BCD}{A}$$



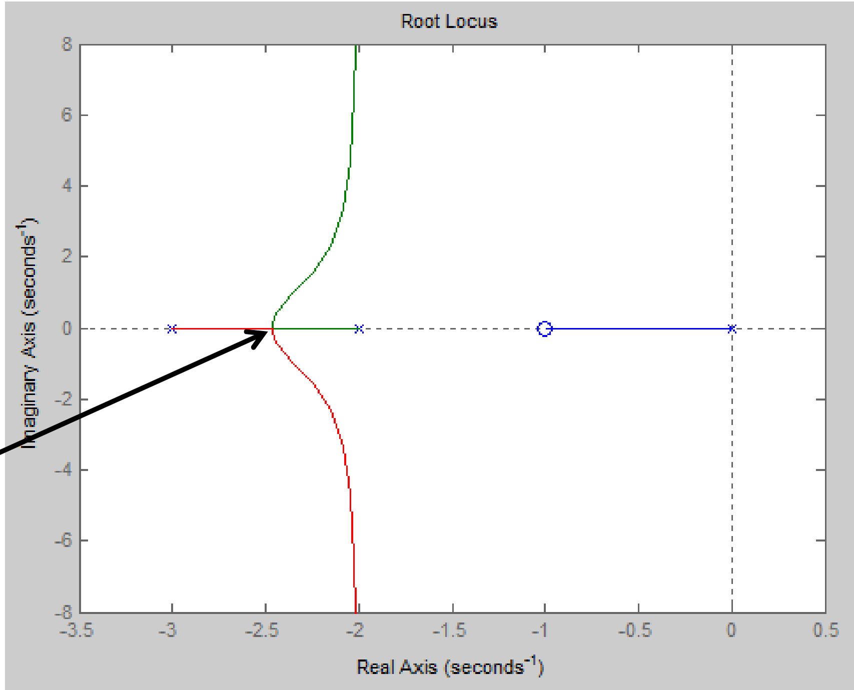
Finding K

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

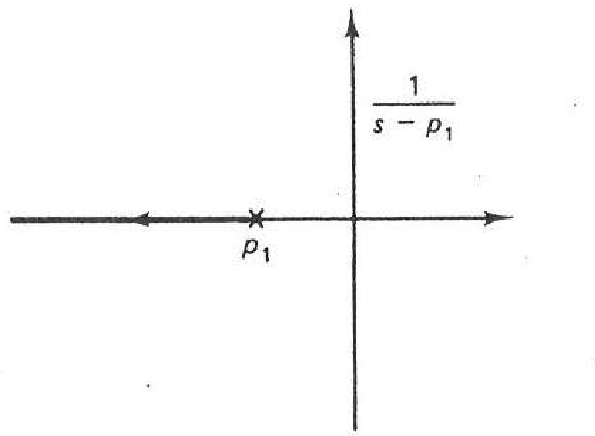
Or using MATLAB, Interactively….

```
%test
clear all;
hold off;

p = [ 1 1];
q = [ 1 5 6 0 ];

%rlocus( p, q)
rlocfind( p, q )
```

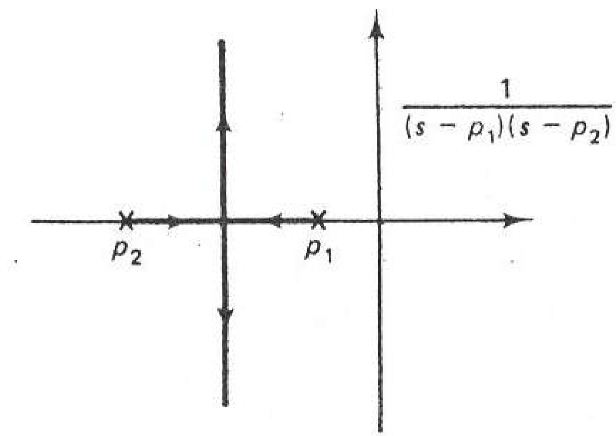
```
>>test
Select a point in the
graphics window

selected_point =

  -2.4621 - 0.0248i

ans =
    0.4196
```



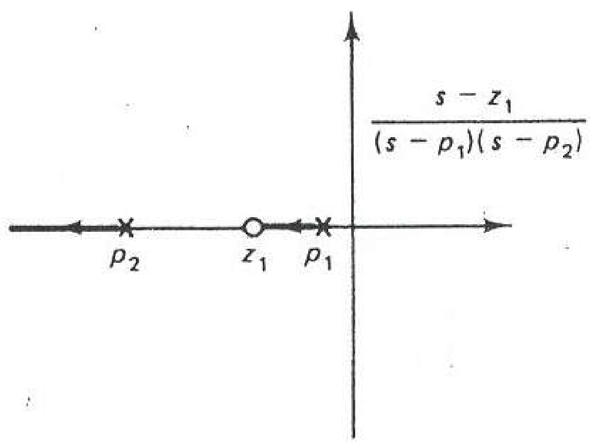
Root locus tables



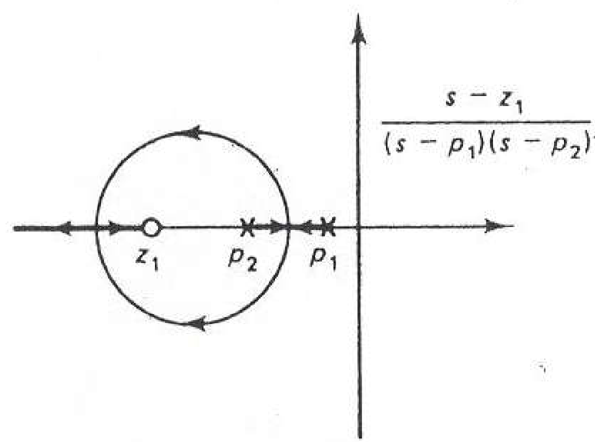
(a)



(b)

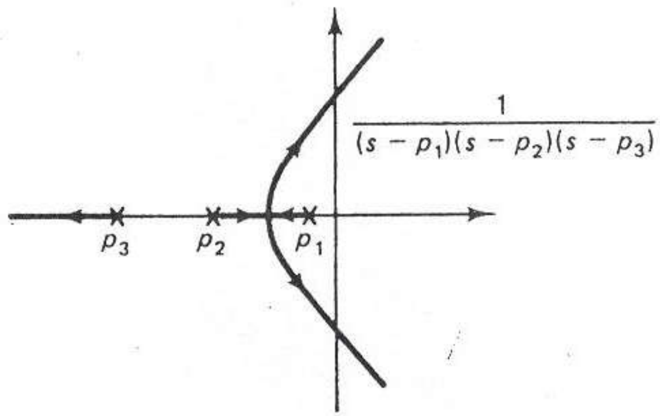


(c)

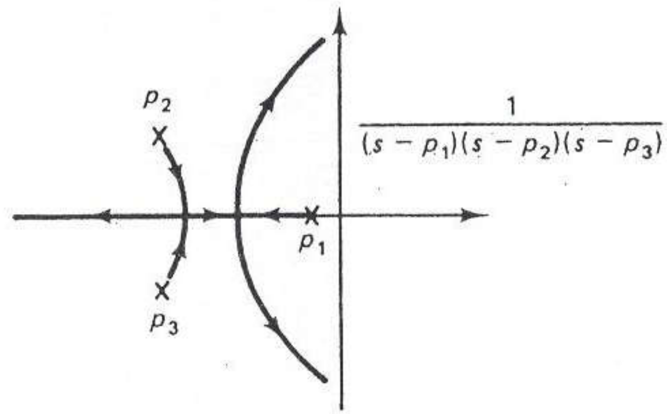


(d)

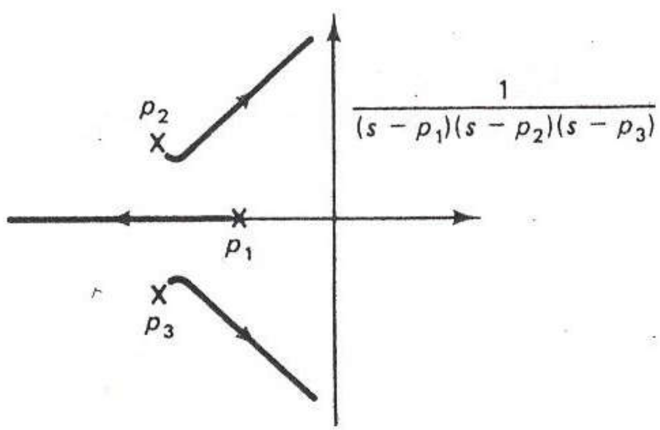
Root locus tables



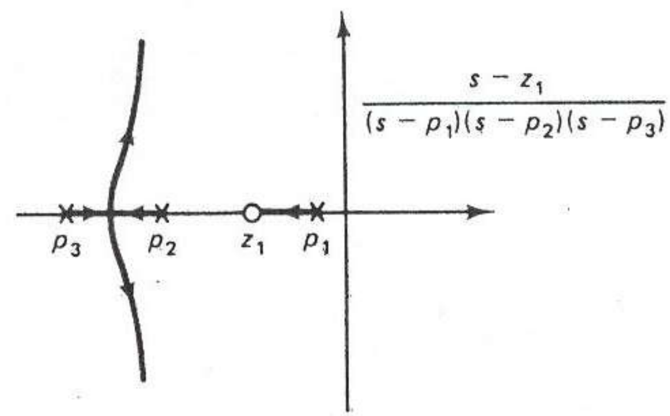
(e)



(f)

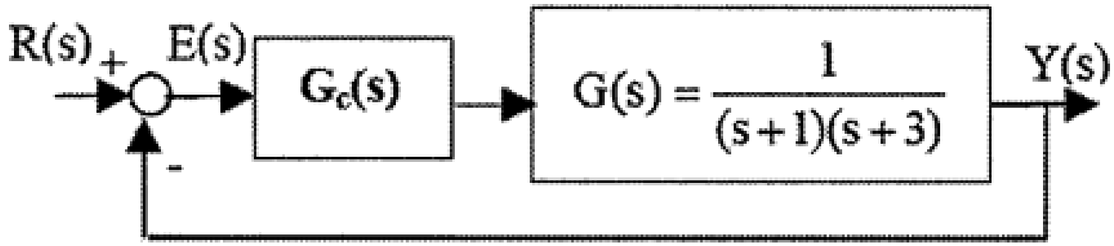


(g)



(h)

Example

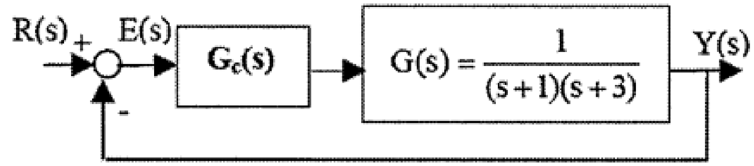


a) Draw the root locus for the system if $G_c(s)=K$.

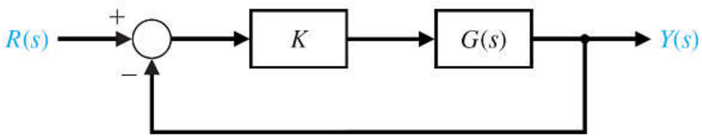
b) Assuming a proportional controller ($G_c(s)=K$), select a gain value that yields a closed loop time constant of $\tau<0.5$ seconds and a damping ratio between 0.4 and 0.8.

c) What is the resulting steady state error for the closed loop system for a step input?

Example, cont’d

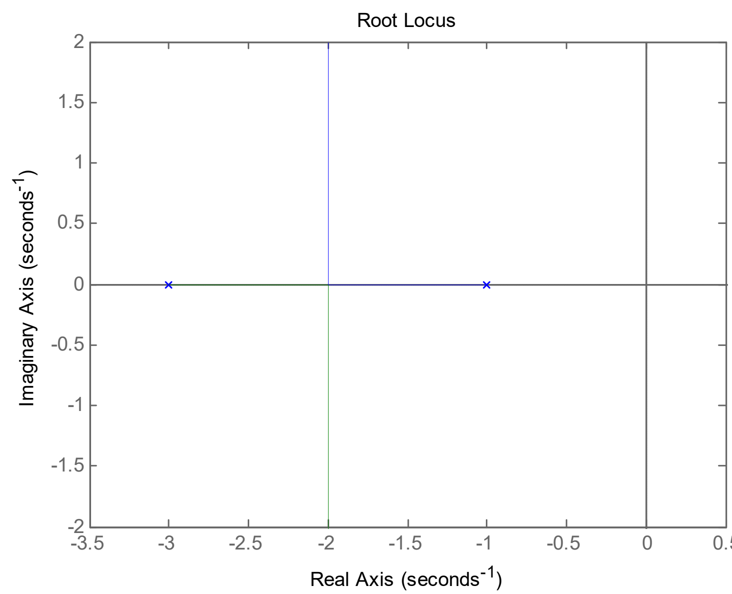


a) Draw the root locus for the system.

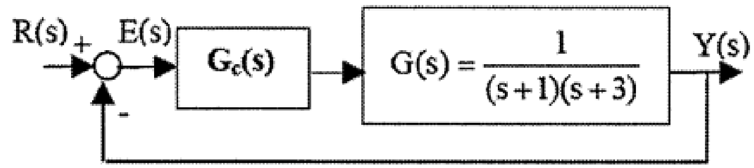


$$\Delta(s) = 1 + KG(s) = 1 + K\left[(s+1)(s+3)\right]$$

```
clear all;
z = [ ];
p = [ -1 -3 ];
k = 1;
sys = zpk( z, p, k );
rlocus( sys )

or

num = [ 1 ];
den = [ 1 4 3 ];
sys = tf( num, den );
rlocus( sys )
```



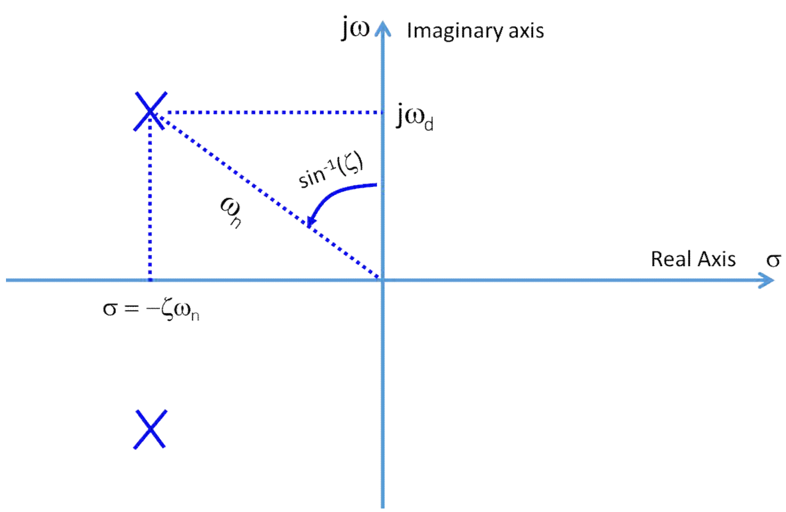
Example, cont'd



b) Assuming a proportional controller $G_c(s)=K$, select a gain value that yields a closed loop time constant $\tau<0.5$ seconds and damping ratio between 0.4 and 0.8.

<u>Solution</u>:

Since the damping coefficient is less than one, the system is underdamped, therefore system will have complex poles.



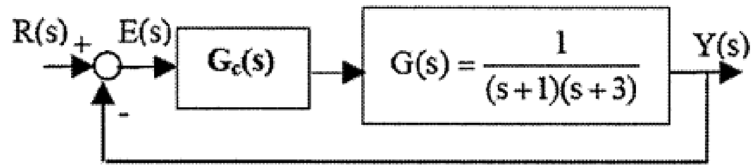
$$y(t) = y_o e^{-\frac{t}{\tau}} \qquad \Rightarrow \mathrm{Re}(\lambda) = -\frac{1}{\tau} = -2$$

$$\omega_n = -\frac{-2}{.4} = 5 \qquad \omega_d = 5\sqrt{1-.4^2} = 4.58$$

$$\omega_n = -\frac{-2}{.8} = 2.5 \qquad \omega_d = 2.5\sqrt{1-.8^2} = 1.5$$

ζ	s_d	K
0.4	-2+4.58j	?
0.8	-2+1.5j	?

Example, cont'd



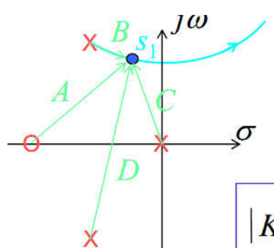
b) Assuming a proportional controller $G_c(s)=K$, select a gain value that yields a closed loop time constant $\tau<0.5$ seconds and damping ratio between 0.4 and 0.8.

<u>Solution</u>:

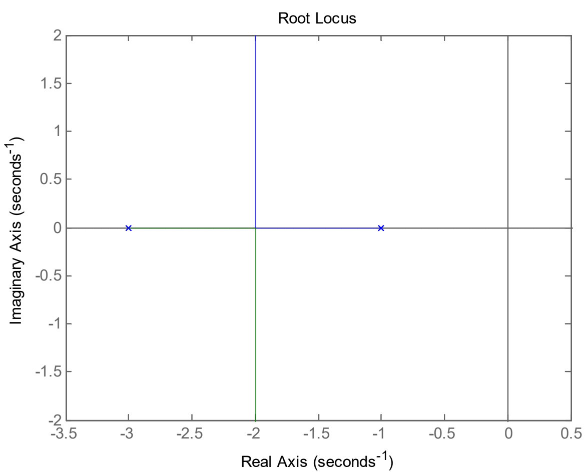
At any point the on the Root Locus, the gain can be calculated as the product of the distances from the point to the poles divided by the product of the distances from the points to the zeros.

Recall,

$$K_x = \left. \frac{\prod_{j=1}^{n_p} |s+p_i|}{\prod_{i=1}^{n_z} |s+z_i|} \right|_{s=s_x}$$



$$|K| = \frac{BCD}{A}$$



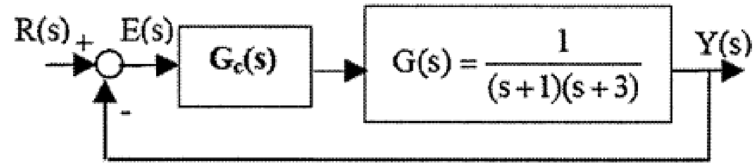
Therefore, in our case…

$$K_{0.4} = 1^2 + 4.58^2 = 21.97$$

$$K_{0.8} = 1^2 + 1.5^2 = 3.25$$

ζ	s_d	K
0.4	-2+4.58j	22
0.8	-2+1.5j	3.25

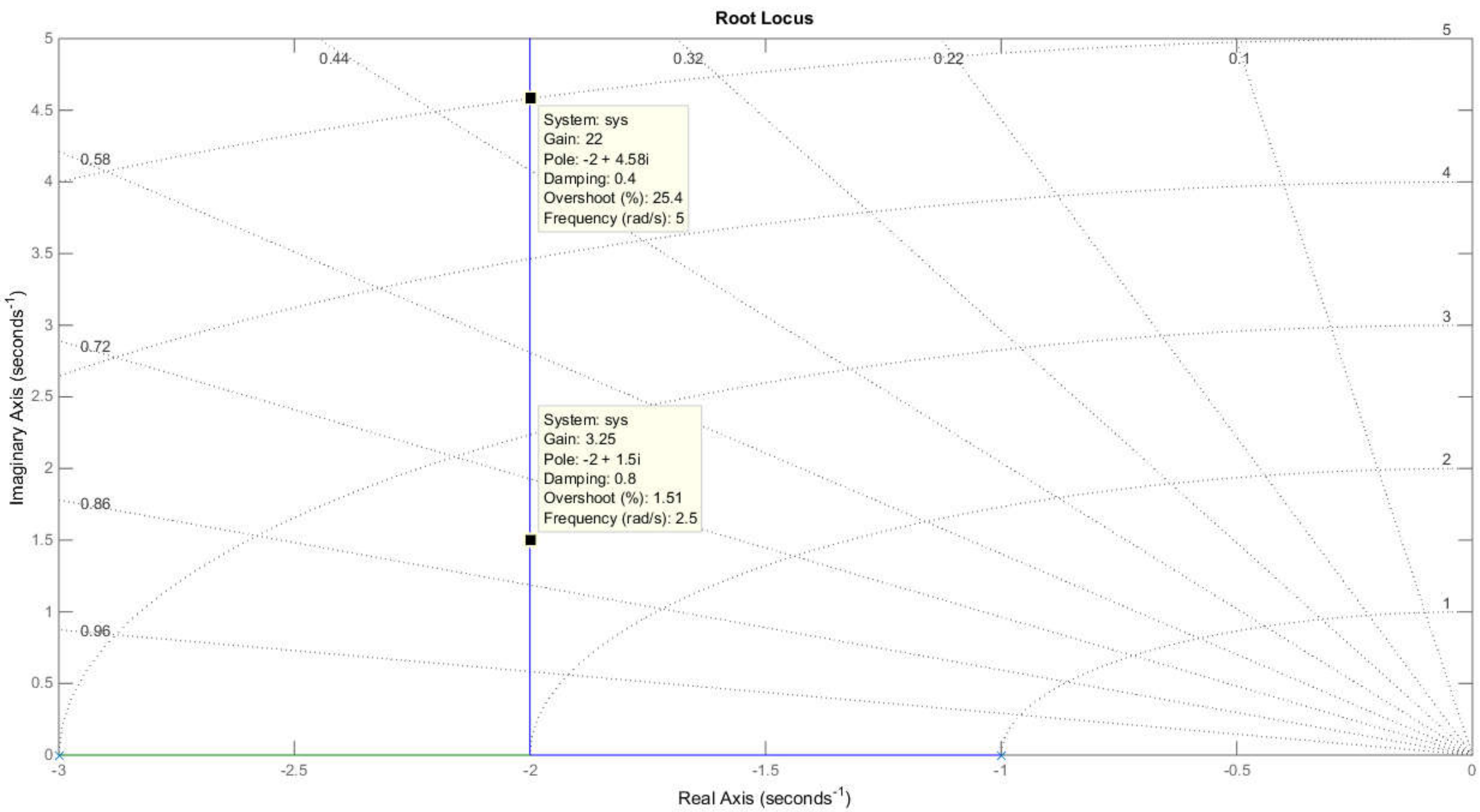
Example, cont'd



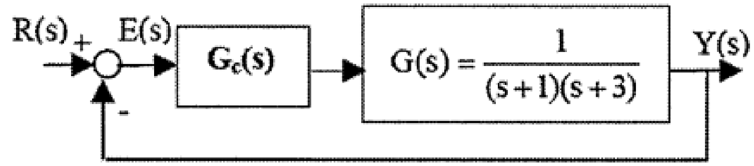
b) Assuming a proportional controller $G_c(s)$=K, select a gain value that yields a closed loop time constant τ<0.5 seconds and damping ratio between 0.4 and 0.8.

<u>MATLAB agrees</u>:

```
clear all;

sys = zpk( [], [-1 -3], 1 )

rlocus( sys )
axis( [-3 0 0 5]);
grid;
```



Example, cont'd



c) What is the resulting steady state error for the closed loop system for a step input?

Solution:

We can now find the closed loop Transfer Function

$$T(s) = \frac{G_c G}{1 + G_c G}$$

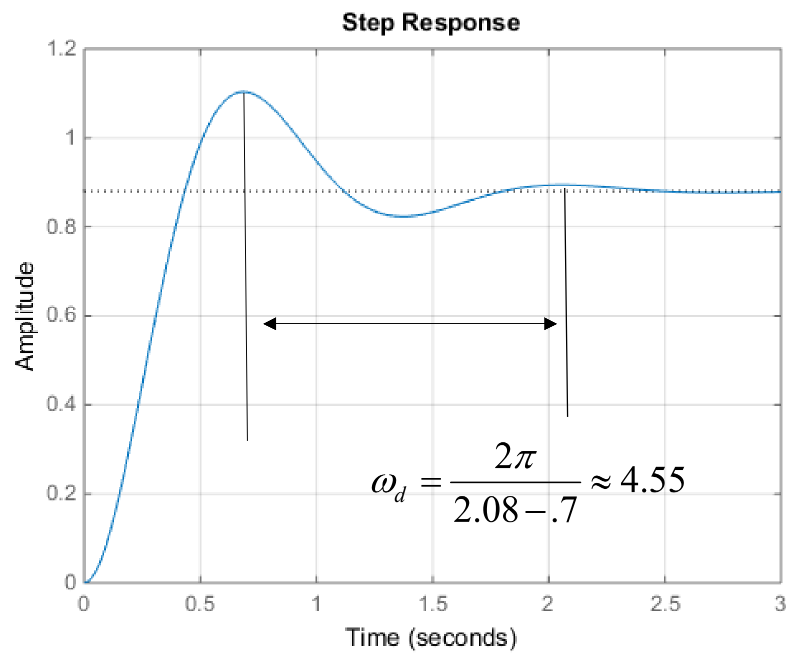
$$= \frac{K \frac{1}{(s+1)(s+3)}}{1 + K \frac{1}{(s+1)(s+3)}}$$

$$= \frac{22}{(s+1)(s+3) + 22}$$

$$= \frac{22}{s^2 + 4s + 25}$$

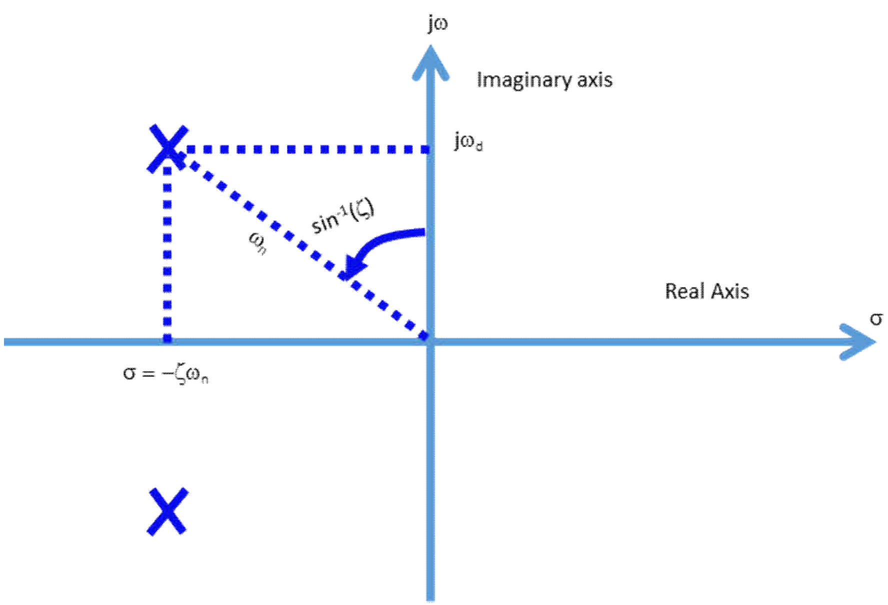
$$T(0) = \frac{22}{25} = 0.88$$

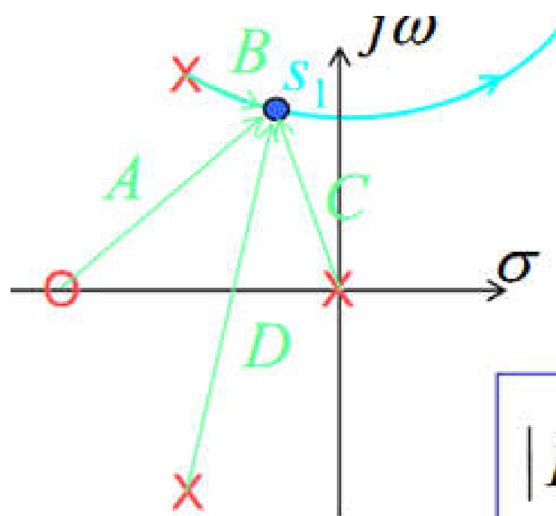
...and verify using MATLAB.



Summary

- If we are primarily interest in stability, using MATLAB's rlocus( ) is often sufficient.
- But understanding how to interpret a Root Locus plot can help us design systems to behave how we want them to.





$$|K| = \frac{BCD}{A}$$