2^{nd} Order Systems: Phase Portraits & Solving using Linear Algebra

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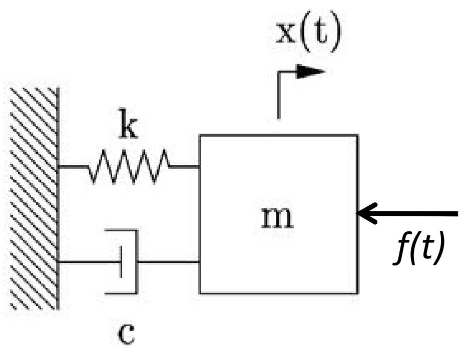
Recall our MSD (linear 2nd order) system

Our Equation of Motion (EOM)



$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Let…

$$u_1 = f(t)$$

$$z_1 = x \qquad \dot{z}_1 = z_2$$

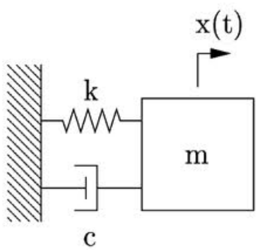
$$z_2 = \dot{x} \qquad \dot{z}_2 = -\frac{k}{m} z_1 - \frac{c}{m} z_2 + \frac{f(t)}{m}$$

Thus in state-space form…

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}$$

Easy to solve....numerically

Assume some values....



Let...

$$k = 3, c = 2, m = 1, \& f(t) = 4$$

With IC's...

$$z(0) = \begin{cases} 0 \\ 0 \end{cases}$$

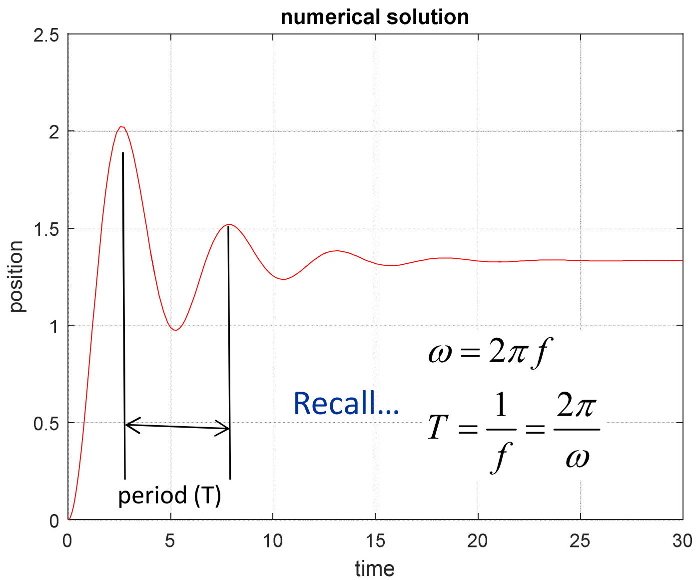
Plug in...

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}$$

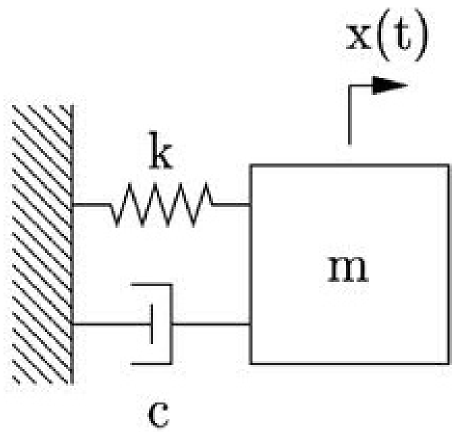
Solve with ode45 function in MATLAB...

```
%m-s-d solution
[t, z] = ode45( @test, [0 30 ], [ 0 0 ]);
plot( t, z(:,1), 'r');
```

```
function zprime = test( t, z )
m = 2; c = 1; k = 3; F = 4;
zprime = [ 0 1; -(c/m) -k/m) ]*z + [ 0; 1/m ]*F;

%OR
% A = [ 0 1; -(c/m) -(k/m)];
% B = [ 0; 1/m ];
% u = F;
% zprime = A*z + B*u

%OR
%zprime = [ z(2);
%     -(c/m)*z(2) - (k/m)*z(1) + F/m; ];
```



Another way to visualize: Phase Portraits

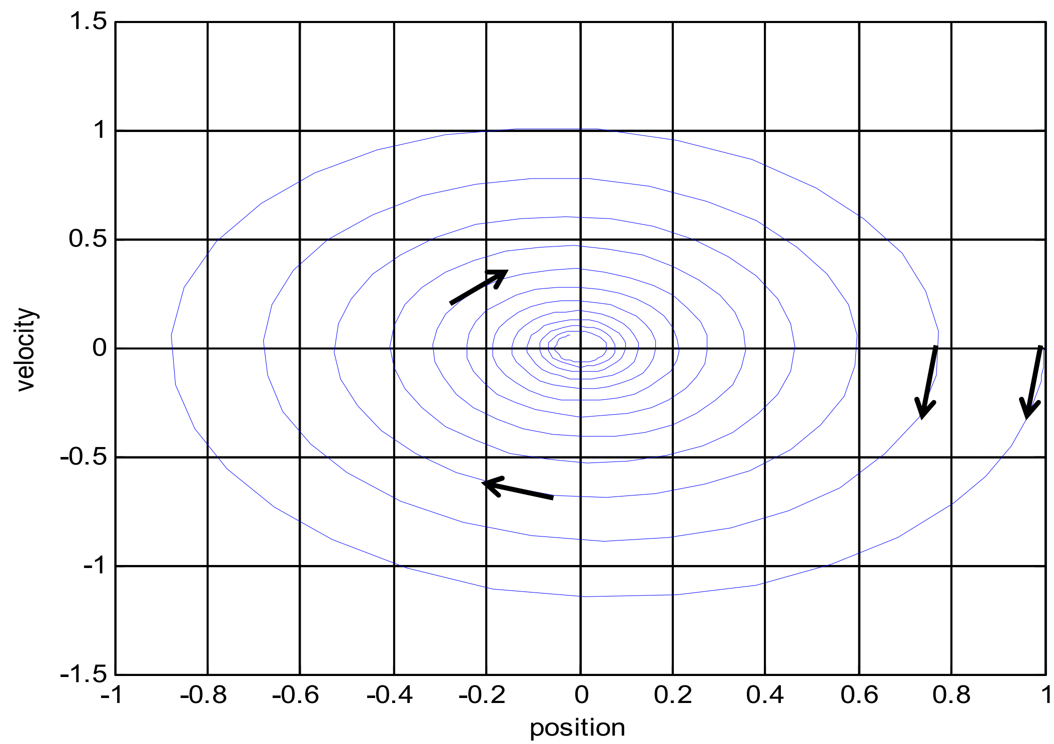


Let…

$$k = 3, \mathrm{b} = 0.2, m = 1, F = 0$$

Some initial conditions…

$$z(0) = \begin{cases} 1 \\ 0 \end{cases}$$



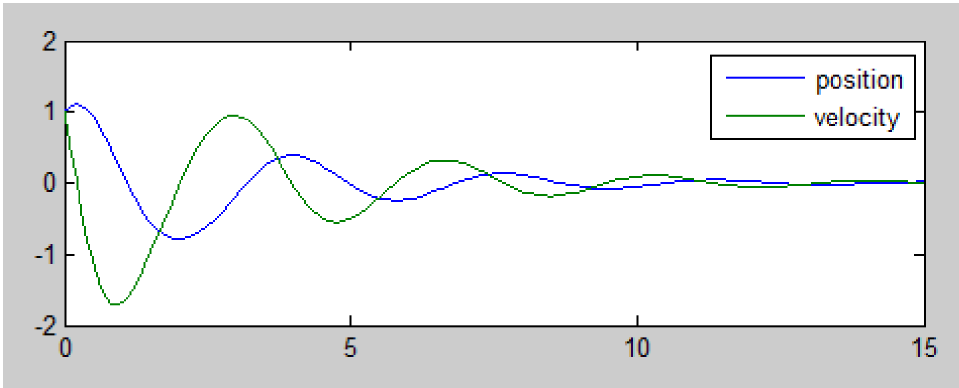

```
%m-s-d solution
[t, z] = ode45( @test, [0 60 ], [ 1 0 ]);
plot( z(:,1), z(:,2));
grid on;
```

```
function zprime = test( t, z )
m = 2; b = .2; k = 3; F = 0;
zprime = [
    z(2);
    -(b/m)*z(2) - (k/m)*z(1) + F; ];
```

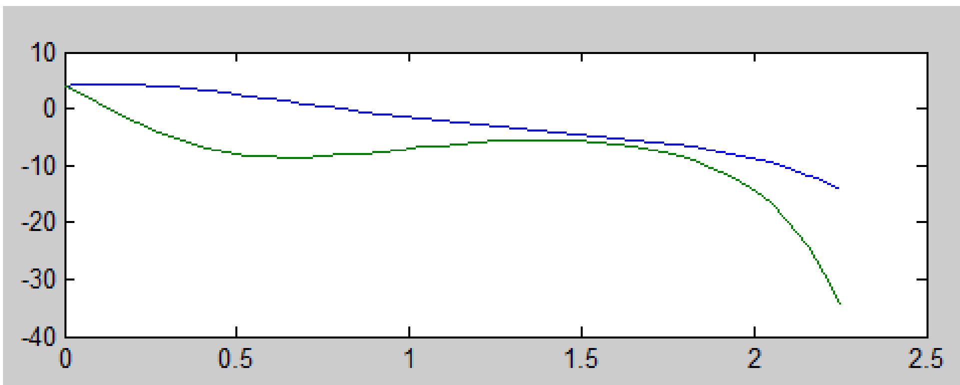
Phase portraits for nonlinear systems

Consider this 2nd order *nonlinear* system: $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$

$$x(0) = 1$$
$$\dot{x}(0) = 1$$

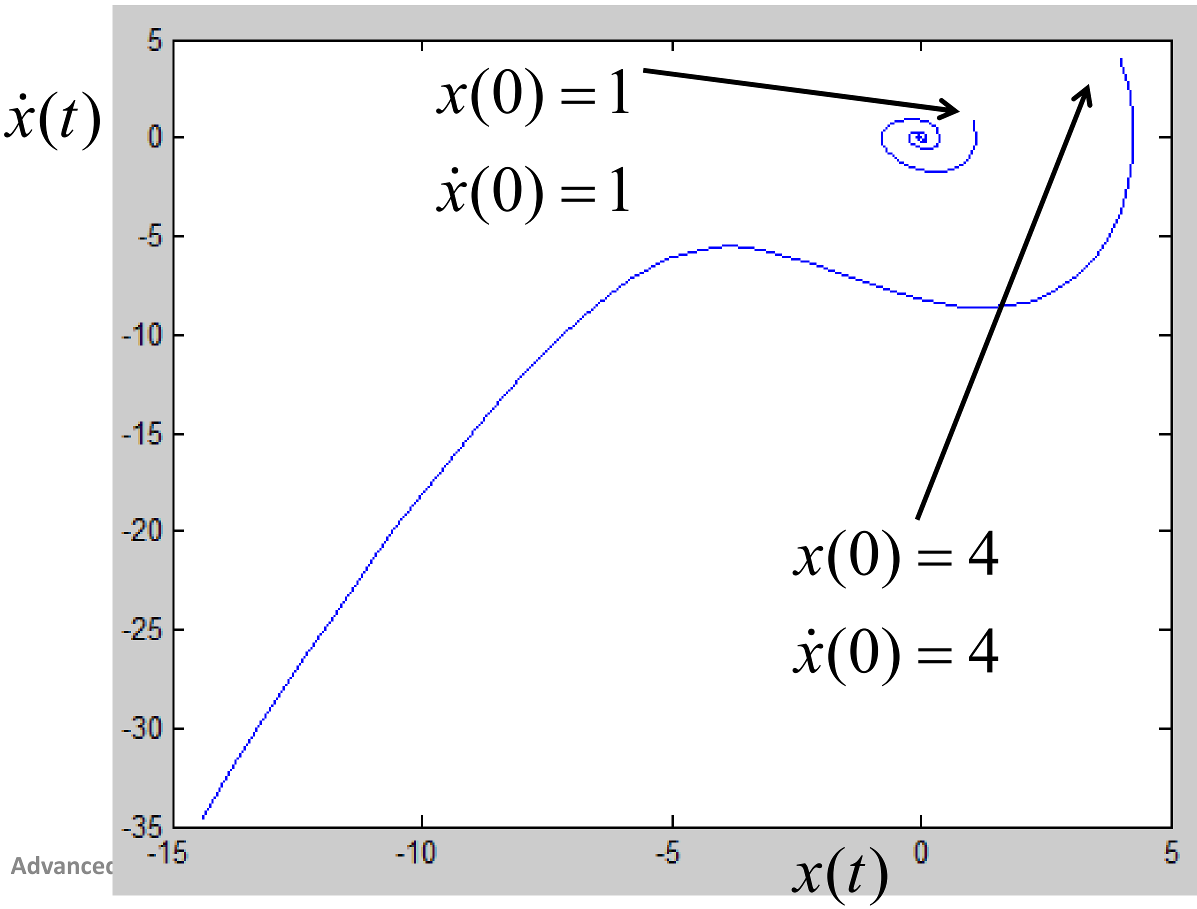


$$x(0) = 4$$
$$\dot{x}(0) = 4$$



Phase Portrait (note the Butterfly Effect)

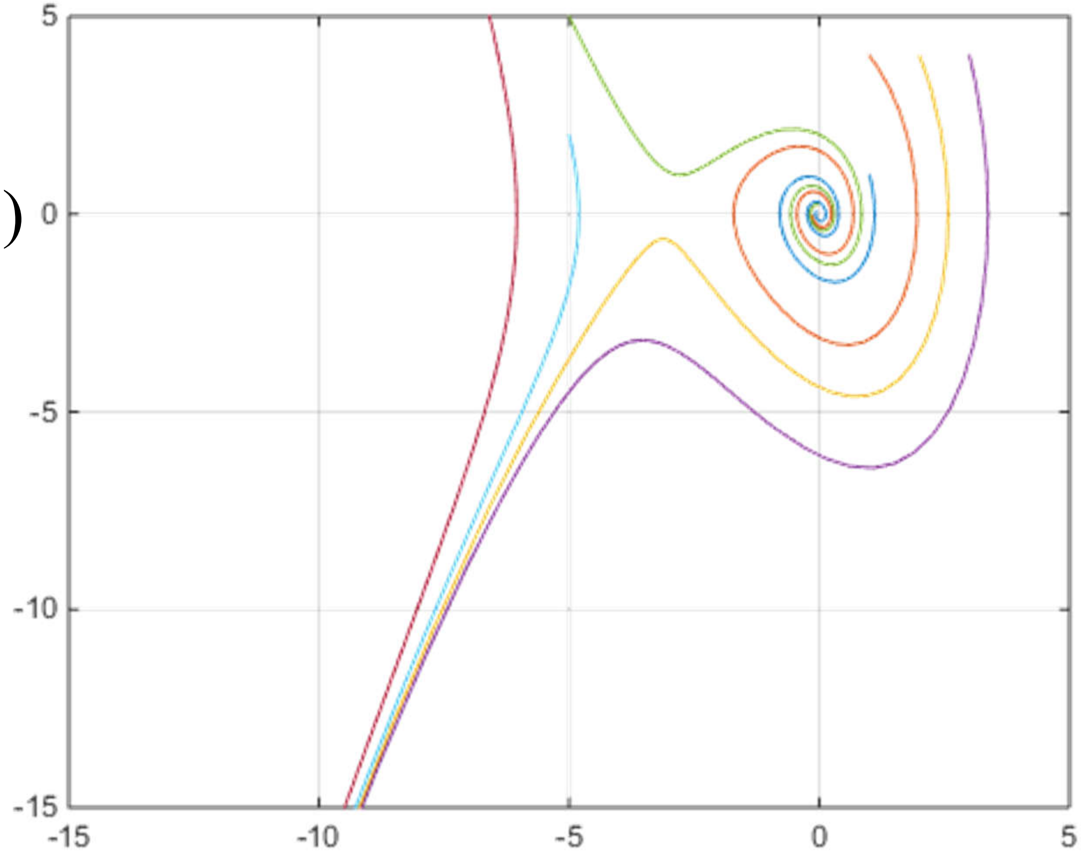
Consider lots of possible initial conditions… $\ddot{x}+0.6\dot{x}+3x+x^2=0$




```
clear all;

[t, z1] = ode45('ExPhase',[0 15], [1 1]);
[t, z2] = ode45('ExPhase',[0 2.25], [4 4]);

plot( z1(:,1), z1(:,2));
hold on;
plot( z2(:,1), z2(:,2));
```

Could I write a script to methodically consider initial conditions?

Phase portrait with multiple initial conditions



$\dot{x}(t)$

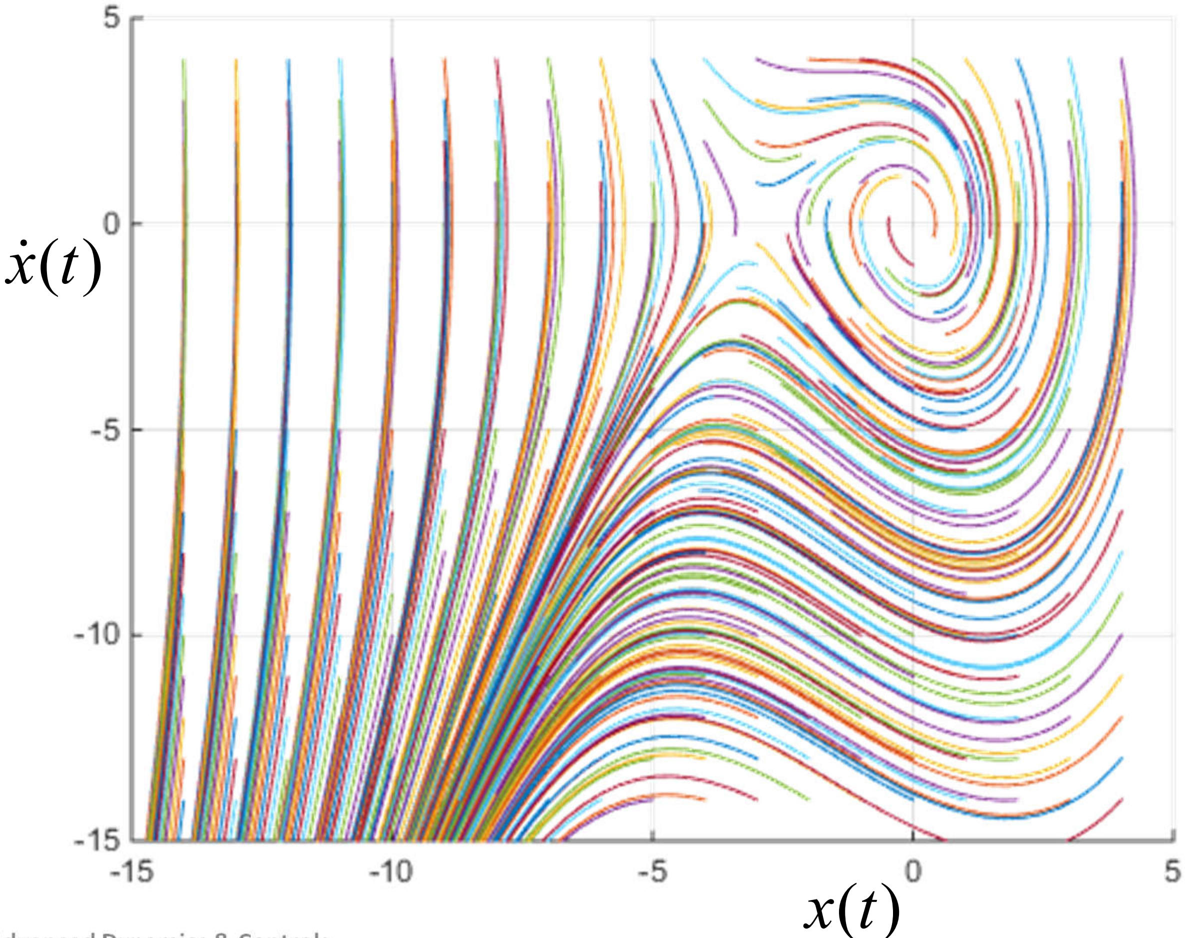
$x(t)$

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

```
clear all;
[t, z1] = ode45('ExPhase',[0 10], [1 1]);
[t, z2] = ode45('ExPhase',[0 10.0], [1 4]);
[t, z3] = ode45('ExPhase',[0 5.0], [2 4]);
[t, z4] = ode45('ExPhase',[0 3.0], [3 4]);
[t, z5] = ode45('ExPhase',[0 10.0], [-5 5]);
[t, z6] = ode45('ExPhase',[0 2.0], [-5 2]);
[t, z7] = ode45('ExPhase',[0 1.0], [-7 7]);

plot( z1(:,1), z1(:,2));
hold on;
plot( z2(:,1), z2(:,2));
plot( z3(:,1), z3(:,2));
plot( z4(:,1), z4(:,2));
plot( z5(:,1), z5(:,2));
plot( z6(:,1), z6(:,2));
plot( z7(:,1), z7(:,2));
axis( [-15 5 -15 5]);
grid on;
```

Sure, but can be dull and incomplete.

Phase Portrait



$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

```
clear all;
hold on;
for i=-14:4
    for j=-14:4
        [t, z] = ode45('ExPhase',[0 1], [i j]);
        plot( z(:,1), z(:,2));
        z=[0;0]; %reset z and t
        t=0;
    end
end

axis( [-15 5 -15 5]);
grid on;
```

Pretty, but MATLAB only gets us so far….

Calculate Phase Portraits in Two Steps

- Step 1 – Find the <u>equilibrium points</u>
  - Equilibrium Point := Set of states such that system is stationary

z_e *Is an equilibrium point of* $\dfrac{d\mathbf{z}}{dt} = f(\mathbf{z}, u)$ *if* $\dfrac{d\mathbf{z_e}}{dt} = 0$

  - At these points, the slope of the phase portrait will be indeterminate
  - A system can have 0, 1, or many equilibrium points.
    - (Another possibility is repeated periodic motion)

<u>In our example…</u>

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

We define our states

$$z_1 = x \Rightarrow \dot{z}_1 = z_2$$

$$z_2 = \dot{x} \Rightarrow \dot{z}_2 = -0.6 z_2 - 3z_1 - z_1^2$$

Equilibrium points exist when neither state is changing.

$$f_1(z_1, z_2) = z_2 = 0$$

$$f_2(z_1, z_2) = -0.6(0) - 3z_1 - z_1^2 = 0$$

$$0 = z_1(-3 - z_1)$$

We see that (0,0) and (-3,0) are equilibrium points;

Calculating Phase Portraits in Two Steps

- Step 2 – Examine behavior of solutions near the equilibria.

Let the construct $\mathbf{z}(t;\mathbf{a})$ *be a solution to* $\dfrac{d\mathbf{z}}{dt} = f(\mathbf{z}, u)$

with initial condition $\mathbf{z}(0) = \mathbf{a}$

> *A solution is stable if other solutions (i.e.* $\mathbf{z}(t;\mathbf{b})$ *) near* $\mathbf{a}$ *stays close to* $\mathbf{z}(t;\mathbf{a})$

How do we check that?

One option is trial and error….

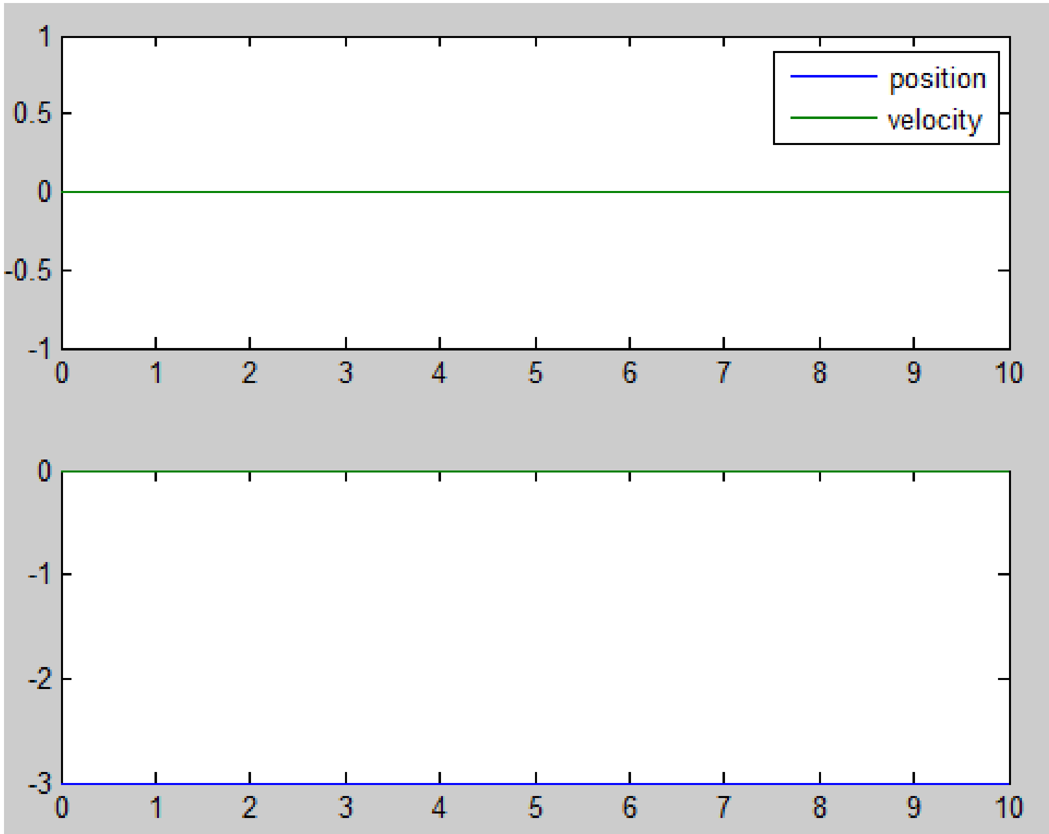
Phase Portrait

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

Examining the response at the equilibrium points is rather boring…

$$x(0) = 0$$
$$\dot{x}(0) = 0$$

$$x(0) = -3$$
$$\dot{x}(0) = 0$$



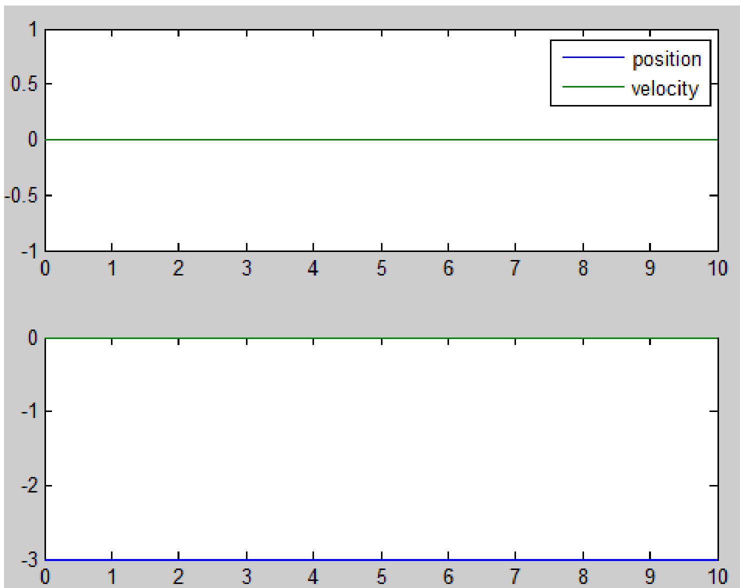
Phase Portrait

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

But looking at points nearby, we get some insight….

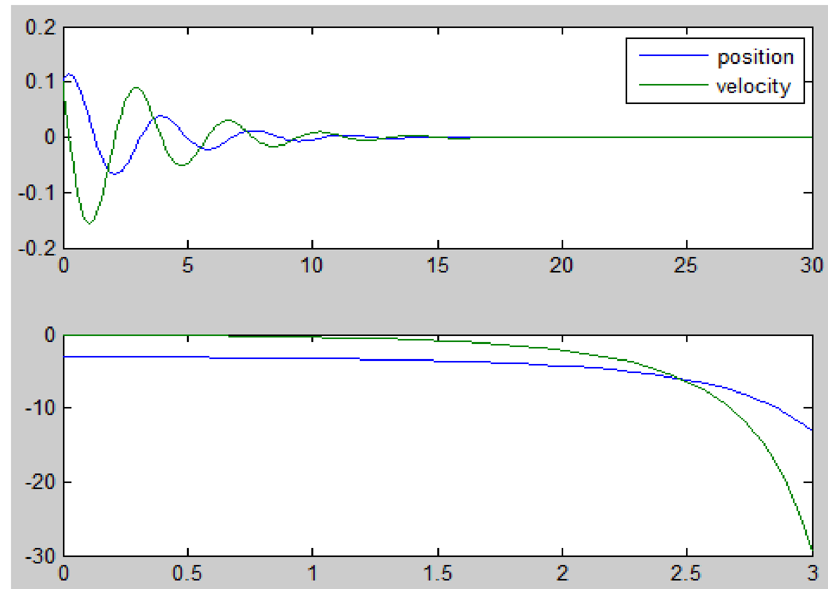
$$x(0) = 0 \qquad \dot{x}(0) = 0$$

$$x(0) = -3 \qquad \dot{x}(0) = 0$$



$$x(0) = 0.1 \qquad \dot{x}(0) = 0.1$$

$$x(0) = -3.1 \qquad \dot{x}(0) = 0$$

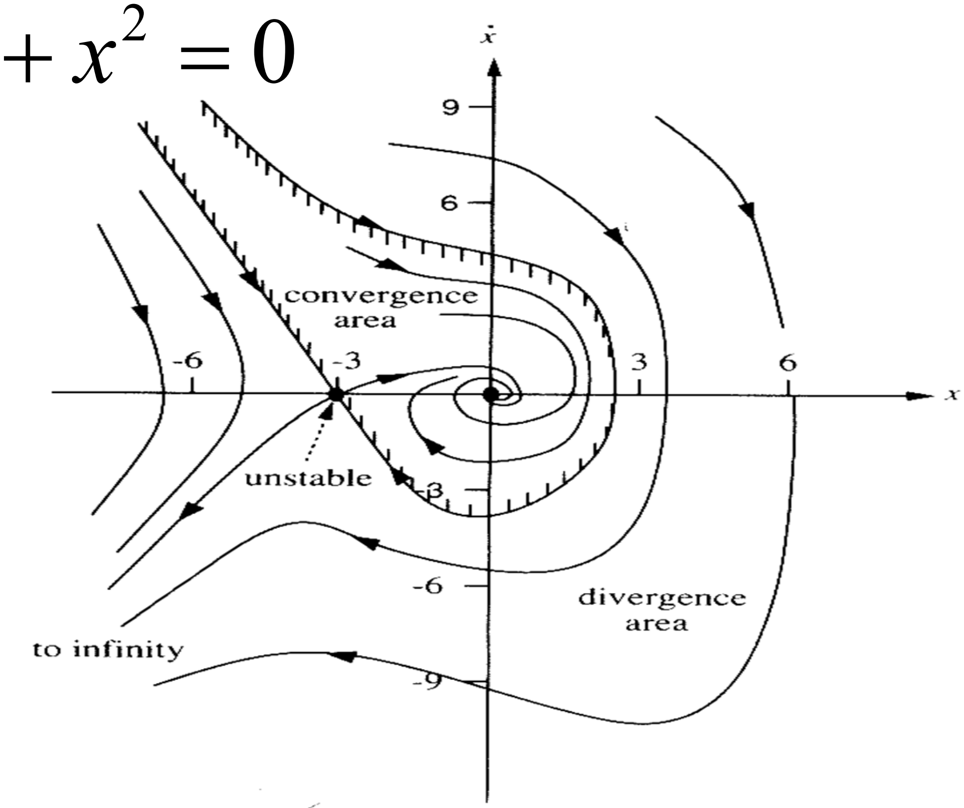


Phase Portrait

After enough testing, I might be able to make a sketch like this.

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

- *Equilibrium points at (0,0) and (-3,0)*
- *Trajectories move to (0,0)*
- *Trajectories move away from (-3,0)*



Phase Portrait Summary

- Phase Portraits are a great visualization tool for 2nd order systems.
  - Some MATLAB support
  - Only works for 2nd order (3rd?) systems.
  - Works for both linear and nonlinear systems
- There is a clear relationship between equilibrium points and stability.
  - This is true for any order system even if most easily visualized for 2nd order systems.
- Determining stability via trial & error (especially for higher order systems) may not be feasible.
  - We need something else…
- To develop another method let's review how to solve 1st order linear ODEs using linear algebra.

Solving systems using linear algebra

Equilibrium for <u>linear</u> 2^{nd} order systems

- Phase portraits work for both linear AND nonlinear systems
- But for linear 2^{nd} order systems, we have another, easier method to determine if the equilibria are stable.
- We can solve the system of ODEs by first finding its eigenvalues and eigenvectors for the system.
- Recall that….

$$\ddot{x} + a\dot{x} + bx = 0$$

Has the general solution:

$$x(t) = k_1 e^{\lambda_1 t} \boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t} \boldsymbol{\eta}_2$$

Recall for a linear 2nd order system

A general second order system... $\ddot{x} + a\dot{x} + bx = 0$

Has the general solution... $\mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$

Where the eigenvalues λ_1 and λ_2 are found by solving the **characteristic equation.**

$$s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0$$

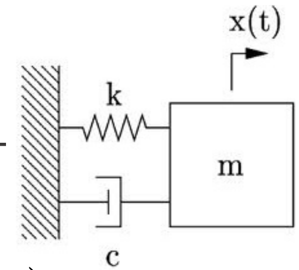
Which we can them find in general for 2nd order systems.

$$\ni \lambda_1 = \frac{(-a + \sqrt{a^2 - 4b})}{2}, \lambda_2 = \frac{(-a - \sqrt{a^2 - 4b})}{2}$$

The eigenvectors are found by solving $\left(\mathbf{A} - \lambda\mathbf{I}\right)\boldsymbol{\eta}_i = 0$

And constants k_i are found using the initial conditions.

Example 1 (real, distinct eigenvalues)



$$m\ddot{x} + c\dot{x} + kx = 0$$

Let…

$$k = 4, c = 6, m = 2, F = u = 0$$

and…

$$\mathbf{z}(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

In state space form…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{z}$$

Finding the eigenvalues

$$\det\left(\lambda \mathbf{I} - \mathbf{A}\right) = 0 \quad \text{or…} \quad \det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0$$

$$\det \begin{bmatrix} \lambda & -1 \\ 3 & \lambda + 2 \end{bmatrix} = 0 \qquad \det \begin{bmatrix} -\lambda & 1 \\ -3 & -\lambda - 2 \end{bmatrix} = 0$$

$$\lambda\left(\lambda + 2\right) + 3 = 0 \qquad -\lambda\left(-\lambda - 2\right) + 3 = 0$$

$$\lambda^2 + 2\lambda + 3 = 0 \qquad \lambda^2 + 2\lambda + 3 = 0$$

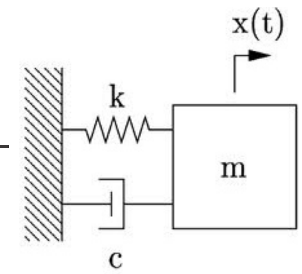
$$\left(\lambda + 1\right)\left(\lambda + 2\right) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Next, find the eigenvectors…

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right) \boldsymbol{\eta}_i = 0$$

Example 1 (real, distinct eigenvalues)



Next, find the eigenvectors…

$$\left(\mathbf{A}-\lambda_i\mathbf{I}\right)\boldsymbol{\eta}_i = 0$$

$$\Rightarrow \lambda = -1$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\boldsymbol{\eta}_1 = 0 \quad \Rightarrow \boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \lambda = -2$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\boldsymbol{\eta}_2 = 0 \quad \Rightarrow \boldsymbol{\eta}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So our solution to the linear 1st order ODEs is…

$$z(t) = k_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Use the initial conditions to find the constant values…

$$\mathbf{z}(0) = k_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} k_1 - k_2 &= 1 \\ -k_1 + 2k_2 &= 0 \quad \Rightarrow k_1 = 2k_2 \end{aligned}$$

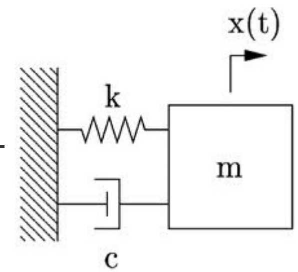
$$\left.\begin{aligned} 2k_2 - k_2 &= 1 \\ k_2 &= 1 \end{aligned}\right\}$$

$$\left. k_1 = 2 \right\}$$

$$\boxed{\mathbf{z}(t) = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

Let’s check our answer

Example 1 (real, distinct eigenvalues)



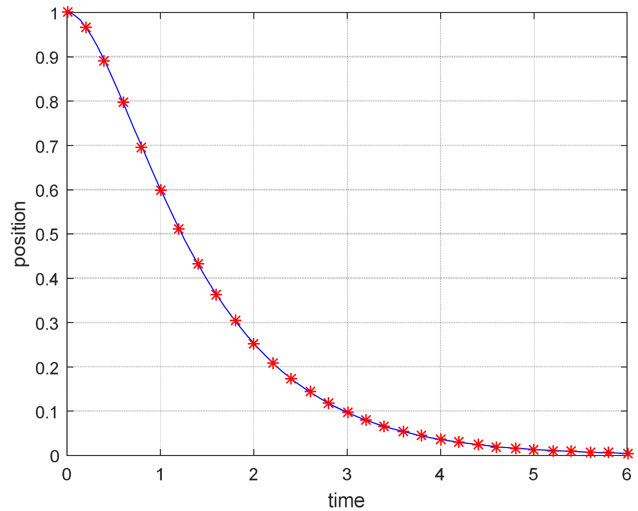
For our s-s system,

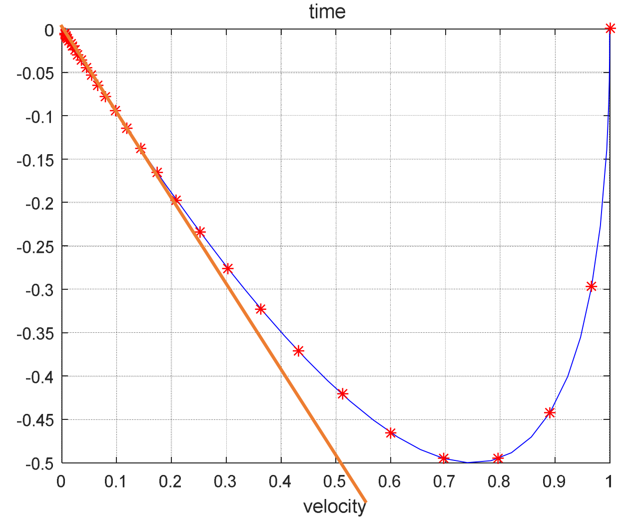
$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{z}$$

We found that,

$$\boxed{\mathbf{z}(t) = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

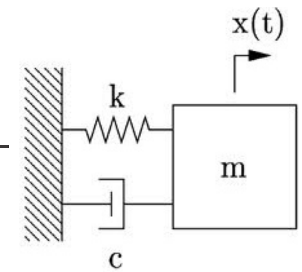
```
clear all;
global k; global c; global m;
k = 4; c = 6; m = 2; ta = [0:.2:6];

[tn, zn] = ode45( @test, [ 0 6 ], [ 1 0] );

for i=1:length(ta);
    za(i,:) = 2*exp(-ta(i))*[1;-1] + 1*exp(-2*ta(i))*[-1;2];
end

figure(1)
plot( tn, zn(:,1), 'b');
grid on; hold on;
plot( ta, za(:,1), 'r*' );
figure(2)
plot( zn(:,1), zn(:,2), 'b-' );
grid on; hold on;
plot( za(:,1), za(:,2), 'r*' );
```





Note the relationship between one of the eigenvectors and the phase portrait.

Example 2 (complex conjugate eigenvalues)



$$m\ddot{x} + c\dot{x} + kx = 0$$

Let…

$$k = 3, c = .2, m = 1, F = u = 0$$

and…

$$\mathbf{z}(0) = \begin{cases} 1 \\ 0 \end{cases}$$

In state space form…

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -0.2 \end{bmatrix} \mathbf{z}$$

Finding the eigenvalues

$$\det\left(\lambda \mathbf{I} - \mathbf{A}\right) = \det \begin{bmatrix} \lambda & -1 \\ 3 & \lambda + 0.2 \end{bmatrix} = 0$$

$$\lambda\left(\lambda + 0.2\right) + 3 = 0$$

$$\lambda^2 + 0.2\lambda + 3 = 0$$

$$\left(\lambda + \left(0.1 + 1.7292i\right)\right)\left(\lambda + \left(0.1 - 1.792i\right)\right) = 0$$

So the eigenvalues of the system are…

$$\lambda_{1,2} = -0.1 \pm 1.7292i$$

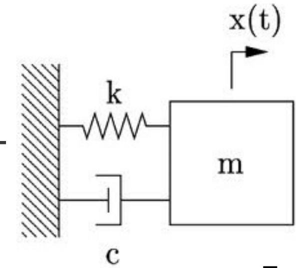
Let's confirm and get the eigenvalues using MATLAB.

```
A = [ 0 1; -3 -0.2 ];
[V,D] = eig(A)
```

```
V =
   -0.0289 - 0.4992i  -0.0289 + 0.4992i
    0.8660 + 0.0000i   0.8660 + 0.0000i
D =
   -0.1000 + 1.7292i   0.0000 + 0.0000i
    0.0000 + 0.0000i  -0.1000 - 1.7292i
```

Some examples for finding the eigenvectors by hand can be found here:
http://tutorial.math.lamar.edu/Classes/DE/ComplexEigenvalues.aspx

Example 2 (complex conjugate eigenvalues)



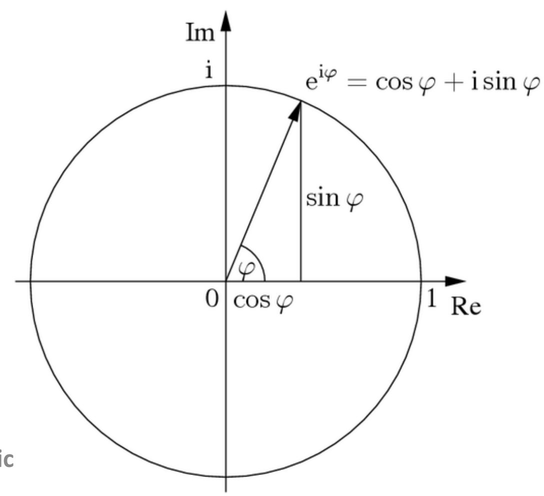
So our solution we get from the first eigenvalue is…

$$z_1(t) = e^{(-0.1+1.7292i)t}\begin{bmatrix}-0.0289-0.4992i\\0.8660\end{bmatrix}$$

We will deal with the second eigenvalue later. Separate the real and imaginary components.

$$z_1(t) = e^{-0.1t}e^{1.7292it}\begin{bmatrix}-0.0289-0.4992i\\0.8660\end{bmatrix}$$

And then apply Euler's formula to eliminate the complex numbers in the exponent and eigenvector.



$$z_1(t) = e^{-0.1t}\left(\cos(1.7292)+i\sin(1.7292)\right)\begin{bmatrix}-0.0289-0.4992i\\0.8660\end{bmatrix}$$

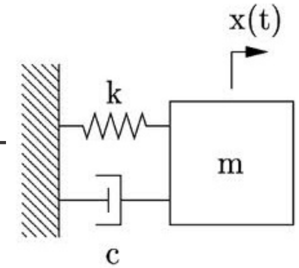
Multiply through to separate the real and imaginary terms.

$$\begin{aligned}
z_1(t) &= u(t)+iv(t)\\
&= e^{-0.1t}\begin{bmatrix}(-0.0289-0.4992i)\left(\cos(1.7292t)+i\sin(1.7292t)\right)\\0.8660\left(\cos(1.7292t)+i\sin(1.7292t)\right)\end{bmatrix}\\
&= e^{-0.1t}\begin{bmatrix}-0.0289\cos(1.7292t)-i0.4992\cos(1.7292t)-i0.0289\sin(1.7292t)-i^2 0.4992\sin(1.7292t)\\.8660\cos(1.7292t)+i0.8660\sin(1.7292t)\end{bmatrix}\\
&= e^{-0.1t}\begin{bmatrix}.4992\sin(1.7292t)-0.0289\cos(1.7292t)\\.8660\cos(1.7292t)\end{bmatrix}+ie^{-0.1t}\begin{bmatrix}-0.4992\cos(1.7292t)-0.0289\sin(1.7292t)\\0.8660\sin(1.7292t)\end{bmatrix}
\end{aligned}$$

(phew. Liking numerical methods more and more but want to make one point here.) Note *u* and *v* are linear and independent to each other AND linear dependent with the solutions we would find using the second eigenvalue/vector. So the general solution can be written as:

$$z(t) = k_1u(t)+k_2v(t)$$

Example 2 (complex conjugate eigenvalues)



So using our initial conditions to find the coefficients

$$\mathbf{z}(t) = k_1 u(t) + k_2 v(t)$$

$$\mathbf{z}(t) = k_1 e^{-0.1t} \begin{bmatrix} .4992\sin(1.7292t) - 0.0289\cos(1.7292t) \\ .8660\cos(1.7292t) \end{bmatrix} + k_2 e^{-0.1t} \begin{bmatrix} -0.4992\cos(1.7292t) - 0.0289\sin(1.7292t) \\ 0.8660\sin(1.7292t) \end{bmatrix}$$

$$\mathbf{z}(0) = k_1 \begin{bmatrix} -0.0289 \\ .8660 \end{bmatrix} + k_2 \begin{bmatrix} -0.4992 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$k_1 \begin{bmatrix} -0.0289 \\ .8660 \end{bmatrix} + k_2 \begin{bmatrix} -0.4992 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = z(0)$$

$$-0.0289k_1 - 0.4992k_2 = 1$$

$$0.8660k_1 + 0k_2 = 0$$

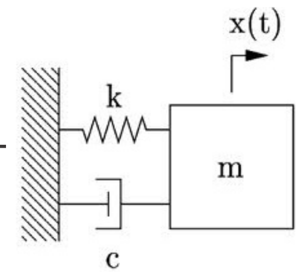
$$k_1 = 0$$

$$k_2 = 1/_{-0.4992} \approx -2$$

$$\Rightarrow \mathbf{z}(t) = 2e^{-0.1t} \begin{bmatrix} -0.4992\cos(1.7292t) - 0.0289\sin(1.7292t) \\ 0.8660\sin(1.7292t) \end{bmatrix}$$

So did “we” get all that math right?

Example 1 (real, distinct eigenvalues)



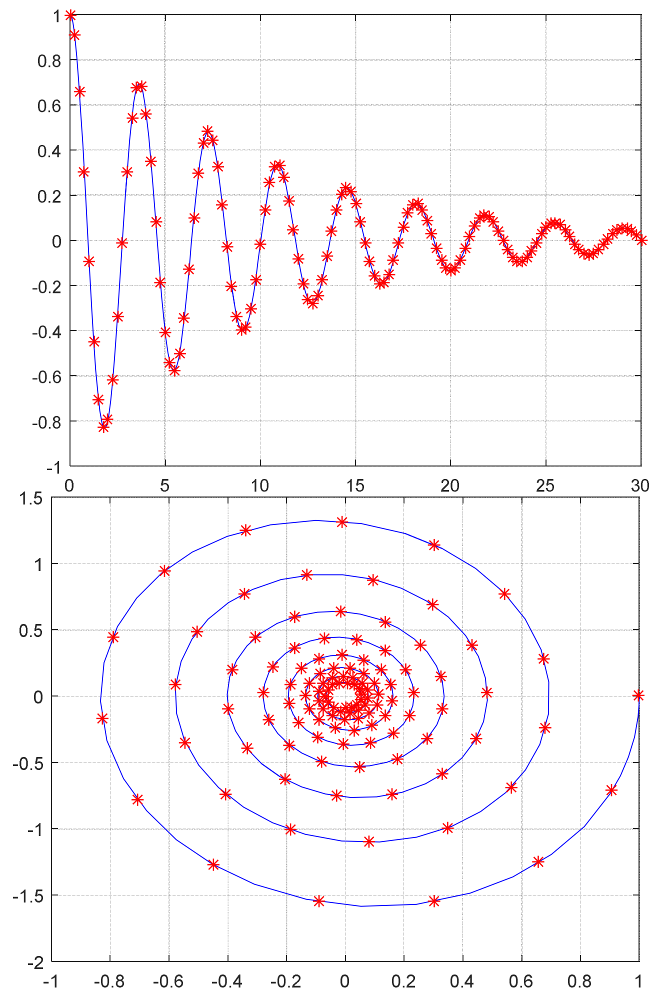
For our s-s system,

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ -3 & -0.2 \end{bmatrix}\mathbf{z}$$

We found that,

$$\mathbf{z}(t) = 2e^{-0.1t}\begin{bmatrix} -0.4992\cos(1.7292t) - 0.0289\sin(1.7292t) \\ 0.8660\sin(1.7292t) \end{bmatrix}$$

```
clear all;
global k; global c; global m;
k = 3; c = 0.2; m = 1; ta = [0:.25:30];

[tn, zn] = ode45( @test, [ 0 30 ], [ 1 0] );

for i=1:length(ta);
     za(i,:) = -2*exp(-0.1*ta(i))*[ -.4992*cos(1.7292*ta(i))-
0.0289*sin(1.7292*ta(i)); 0.8660*sin(1.7292*ta(i))];
end

figure(1)
plot( tn, zn(:,1), 'b');
grid on; hold on;
plot( ta, za(:,1), 'r*' );
figure(2)
plot( zn(:,1), zn(:,2), 'b-' );
grid on; hold on;
plot( za(:,1), za(:,2), 'r*' );
```



Phew!

Summary

- Reviewed how to solve linear sets of 1st order equations using linear algebra
  - Example with distinct real roots (overdamped)
  - Example with complex conjugate roots (underdamped)
- Compared the solutions to numerical solutions for both the output and phase portrait graphs
- We see some additional indications that there is a clear relationship between equilibrium points (and stability) and a systems eigenvalues and eigenvectors.

$$\ddot{x} + a\dot{x} + bx = 0 \qquad x(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$