EE 362K
Homework 4 Solutions

Q1

8 points

- *2 points for showing that V is positive definite*
- *2 points for showing that $\dot{V}$ is negative (semi) definite*
- *3 points for calculations to attain final form*
- *1 point for condition*

$V_1(x)$ is clearly positive definite as $V_1(x) = 0 \implies x_1 = x_2 = 0$ and $V_1(x) \geq 0$.

$$\begin{aligned}
\dot{V_1} &= x_1\dot{x_1} + x_2\dot{x_2} \\
&= x_1(-ax_1) + x_2(-bx_1 - cx_2) \\
&= -ax_1^2 - cx_2^2 - bx_1x_2
\end{aligned}$$

We want that $\dot{V_1}$ is either negative semi-definite or negative definite.

$$\begin{aligned}
\dot{V_1} \leq 0 &\implies -ax_1^2 - cx_2^2 - bx_1x_2 \leq 0 \\
&\implies ax_1^2 + cx_2^2 + bx_1x_2 \geq 0 \\
&\implies b^2 - 4ac \leq 0
\end{aligned}$$

Hence, for V_1 to be a Lyapunov function for this system, we require $b^2 \leq 4ac$.

$V_2(x)$ is also positive definite as $V_2(x) = 0 \implies x_1 = x_2 = 0$ and $V_2(x) \geq 0$.

$$\begin{aligned}
\dot{V_2} &= x_1\dot{x_1} + (x_2 + \frac{b}{c-a}x_1)(\dot{x_2} + \frac{b}{c-a}\dot{x_1}) \\
&= x_1(-ax_1) + (x_2 + \frac{b}{c-a}x_1)[-bx_1 - cx_2 + \frac{b}{c-a}(-ax_1)] \\
&= -ax_1^2 + (x_2 + \frac{b}{c-a}x_1)(-\frac{bc}{c-a}x_1 - cx_2) \\
&= -ax_1^2 - c(x_2 + \frac{b}{c-a}x_1)^2 \leq 0
\end{aligned}$$

$\dot{V_2} = 0 \implies x_1 = x_2 = 0$, so $\dot{V_2}$ is negative definite for all a,b,c. So, V_2 is a Lyapunov function for this system for all $a, b, c > 0$ as long as $a \neq c$. The condition is required for the fraction $\frac{b}{c-a}$ to be finite.

Q2

14 points

- *2 points for convolution equation*
- *1 point for using expm*
- *2 points for step response*
- *1 point for plot + 1 point for labels*
- *2 points for identifying inputs to be superposed*
- *2 points for pulse response*
- *1 point for labels*
- *2 points if pulse response rises faster than step response*

$$A = \begin{bmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & -0.5 \end{bmatrix}$$
$$B = \begin{bmatrix} b_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$D = 0$$

With zero initial conditions, the convolution theorem gives that $y(t) = CA^{-1}e^{At}B - CA^{-1}B + D$. The eigenvalues of A are $\lambda_1 = \frac{-7-\sqrt{29}}{20}$ and $\lambda_1 = \frac{-7+\sqrt{29}}{20}$.
Using expm in MATLAB,

$$e^{At} = \begin{bmatrix} (\frac{1}{2} - \frac{3}{2\sqrt{29}})e^{\lambda_1 t} + (\frac{1}{2} + \frac{3}{2\sqrt{29}})e^{\lambda_2 t} & -\frac{1}{\sqrt{29}}e^{\lambda_1 t} + \frac{1}{\sqrt{29}}e^{\lambda_2 t} \\ -\frac{5}{\sqrt{29}}e^{\lambda_1 t} + \frac{5}{\sqrt{29}}e^{\lambda_2 t} & (\frac{1}{2} + \frac{3}{2\sqrt{29}})e^{\lambda_1 t} + (\frac{1}{2} - \frac{3}{2\sqrt{29}})e^{\lambda_2 t} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -10 & -2 \\ -10 & -4 \end{bmatrix}$$

So,

$$\begin{aligned} CA^{-1}e^{At}B &= \begin{bmatrix} -10 & -4 \end{bmatrix} \begin{bmatrix} (\frac{3}{4} - \frac{9}{4\sqrt{29}})e^{\lambda_1 t} + (\frac{3}{4} + \frac{9}{4\sqrt{29}})e^{\lambda_2 t} \\ -\frac{15}{2\sqrt{29}}e^{\lambda_1 t} + \frac{15}{2\sqrt{29}}e^{\lambda_2 t} \end{bmatrix} \\ &= (\frac{105}{2\sqrt{29}} - \frac{15}{2})e^{\lambda_1 t} - (\frac{105}{2\sqrt{29}} + \frac{15}{2})e^{\lambda_2 t} \end{aligned}$$

$CA^{-1}B = -15$, so

$$y(t) = (\frac{105}{2\sqrt{29}} - \frac{15}{2})e^{\lambda_1 t} - (\frac{105}{2\sqrt{29}} + \frac{15}{2})e^{\lambda_2 t} + 15$$

The step response is shown below.

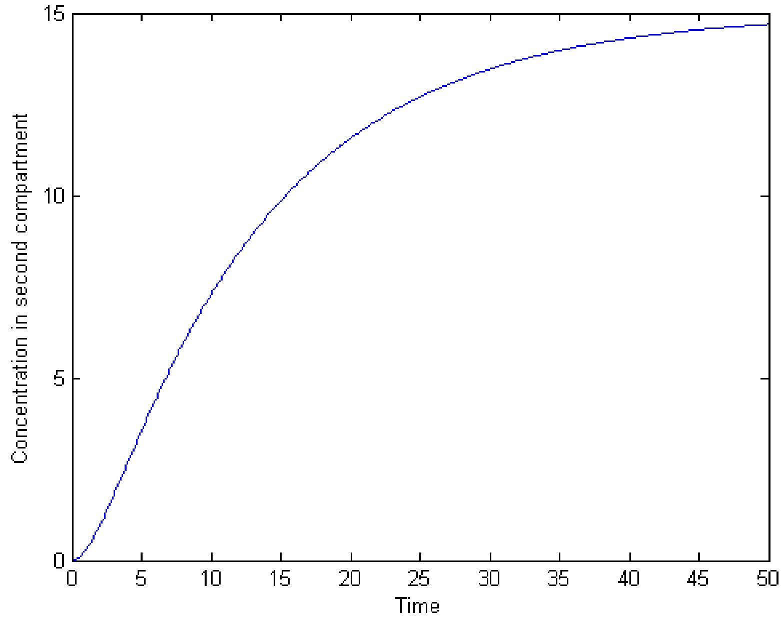

Figure 1: Step response

The 5s pulse input can be viewed as a summation of a step input starting at t = 0 (height = 3) and another step input starting at t = 5 (of height = -2). The system is LTI, so with a step response $f(t)$, we get that the response to the pulse is $3f(t) - 2f(t-5)$. This has been plotted below.

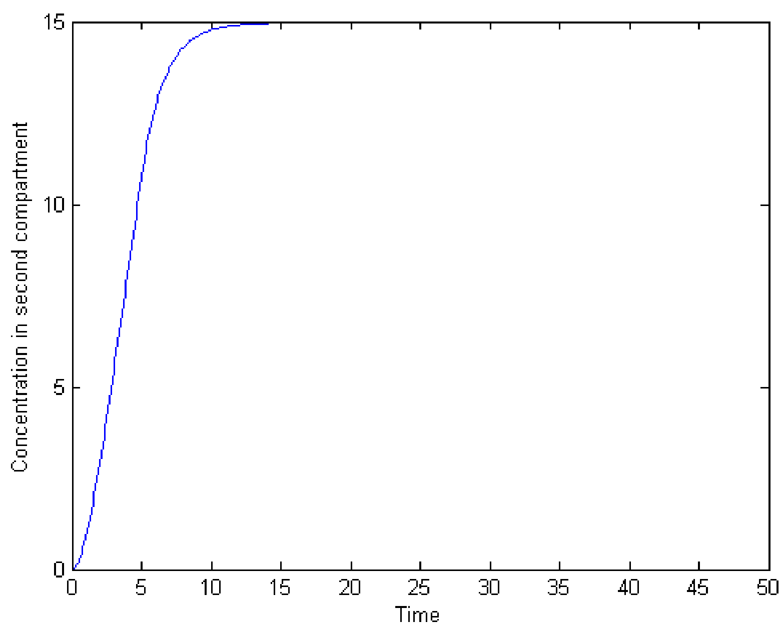

Figure 2: Response to 5s pulse

These figures resemble those in Figure 5.10 in the book with the response to the pulse having a ”shorter” transient response.
Code:

```
%% Q2

clear all
close all

k1 = (-7-sqrt(29))/20;
k2 = (-7+sqrt(29))/20;

t = 0:0.1:50;

% Step response
ys = (105/(2*sqrt(29))-7.5)*exp(k1*t) - (105/(2*sqrt(29))+7.5)*exp(k2*t) + 15;
figure;
plot(t,ys)
xlabel('Time')
ylabel('Concentration in second compartment')

% Response to pulse - 4.5 for first 5s, 1.5 thereafer
td = ((t-5)>=0).*(t-5); % Delayed time axis
ysd = 2*((105/(2*sqrt(29))-7.5)*exp(k1*td) - (105/(2*sqrt(29))+7.5)*exp(k2*td) + 15);
yp = 3*ys - ysd;
figure;
plot(t,yp)
xlabel('Time')
ylabel('Concentration in second compartment')
```

Q3

11 points

- *1 point for formula of reachability matrix*
- *3 points for obtaining reachability matrix*
- *2 points for rank of reachability matrix to determine controllability*
- *2 points for characteristic equation*
- *3 points for RCF*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 5 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

The reachability matrix is given by

$$w_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{5}{2} & 14 \end{bmatrix}$$

The rank of w_r is 3, so the system is controllable.

The characteristic equation of the system is given by $det(\lambda I - A) = \lambda^3 - 5\lambda^2 - 3\lambda + 2$. This implies that the reachable canonical form is

$$\tilde{A} = \begin{bmatrix} 5 & 3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Q4

15 points

- *2 points for equation to get step response for states*
- *1 point for using expm*
- *3 points for plot + 1 point for using log scale + 1 point for labels*
- *2 points for equation to get impulse response*
- *3 points for plot + 1 point for using log scale + 1 point for labels*

For a step input, we know that $z(t) = e^{At}z(0) + A^{-1}e^{At}B - A^{-1}B$. This gives us the following result.

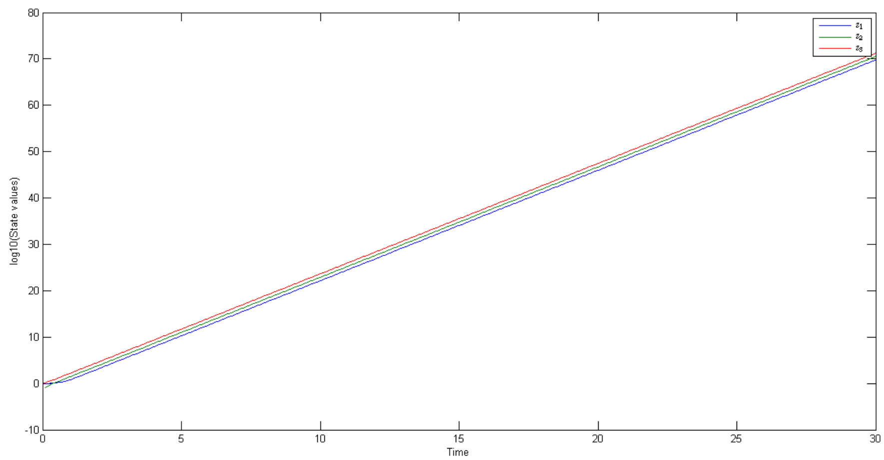

Figure 3: Step response with non-zero initial conditions

For an impulse input of value 3 (assumed to be at $t = 0^+$), we have $z(t) = e^{At}z(0) + 3e^{At}B$. The response looks like

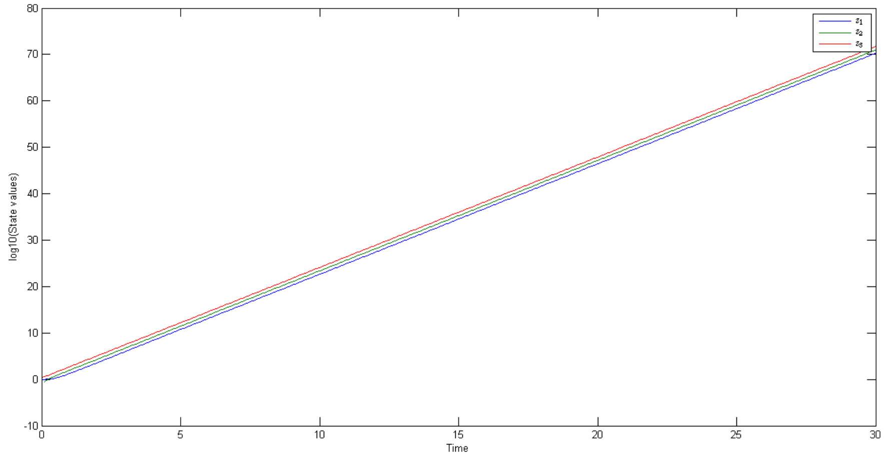

Figure 4: Impulse response with non-zero initial conditions

Note that a log scale has been used due to the wide range of state values. The rapid increase in state values is not surprising given that A has some positive eigenvalues.

Q5

9 points

- *1 point for identifying B in canonical space*
- *1 point for formula for reachability matrix in canonical space*
- *3 points for obtaining reachability matrix in canonical space*
- *2 points for formula for T*
- *2 points for correct T*

The reachability matrix in the canonical space is

$$\begin{aligned}\tilde{w_r} &= \begin{bmatrix}\tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B}\end{bmatrix} \\ &= \begin{bmatrix}1 & 5 & 28 \\ 0 & 1 & 5 \\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

The transformation matrix

$$\begin{aligned}T &= \tilde{w_r} w_r^{-1} \\ &= \begin{bmatrix}0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0\end{bmatrix}\end{aligned}$$

The coordinate system for the canonical space is $\tilde{x} = Tz$.

Q6

30 points

- *2 points for identifying how to find characteristic equation in terms of K*
- *4 points for obtaining characteristic equation in terms of K For each part:*
- *2 points for obtaining characteristic equation*
- *3 points for obtaining K*
- *2 points for obtaining k_r*
- *2 points for computing input*
- *2 points for plot + 1 point for labels*

For eigenvalues -1, -3 and -6, the characteristic equation is $p(\lambda) = (\lambda+1)(\lambda+3)(\lambda+6) = \lambda^3 + 10\lambda^2 + 27\lambda + 18$.
If we take the gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, the characteristic equation is

$$\begin{aligned} p(\lambda) &= det(\lambda I - (A - BK)) \\ &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 + \frac{k_1}{2} & \frac{k_2}{2} - 3 & \lambda + \frac{k_3}{2} - 5 \end{vmatrix} \\ &= \lambda^3 + (\frac{k_3}{2} - 5)\lambda^2 + (\frac{k_2}{2} - 3)\lambda + \frac{k_1}{2} + 2 \end{aligned}$$

Comparing the above equations, we get $k_1 = 32$, $k_2 = 60$ and $k_3 = 30$. So, $K = \begin{bmatrix} 32 & 60 & 30 \end{bmatrix}$. We now proceed to find $k_r = -\frac{1}{C(A-BK)^{-1}B} = 36$.
The input to the system is $k_r y_r - Kz$. This has been shown below.

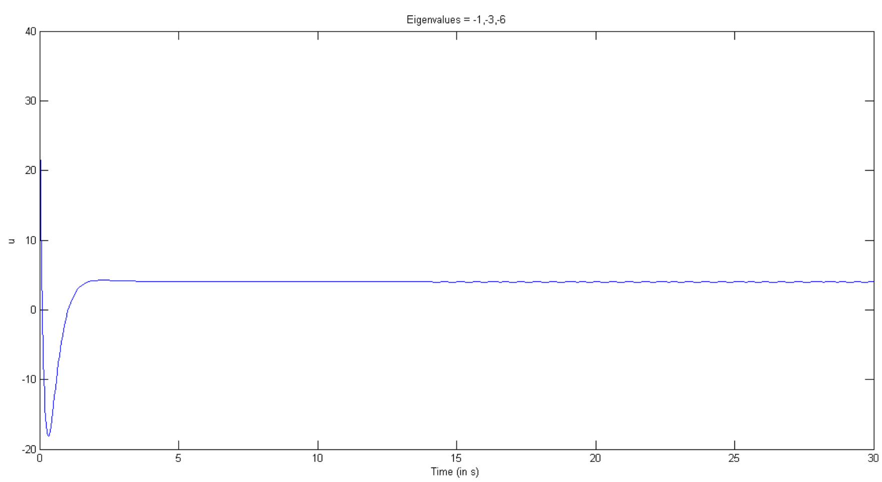

Figure 5: Input u for eigenvalues -1, -3, -6

For eigenvalues $= -2, -1 \pm 5j$, the characteristic equation is $p(\lambda) = \lambda^3 + 4\lambda^2 + 30\lambda + 52$. Comparing with the general form of the characteristic equation derived earlier, we find that $k_1 = 100$, $k_2 = 66$ and $k_3 = 18$, i.e., $K = \begin{bmatrix} 100 & 66 & 18 \end{bmatrix}$. Then, $k_r = -\frac{1}{C(A-BK)^{-1}B} = 104$.

You can verify the correctness of these solutions by plotting the state values against time. The output is the value of the first state and can be seen to settle to 1 (the reference input) in both cases.

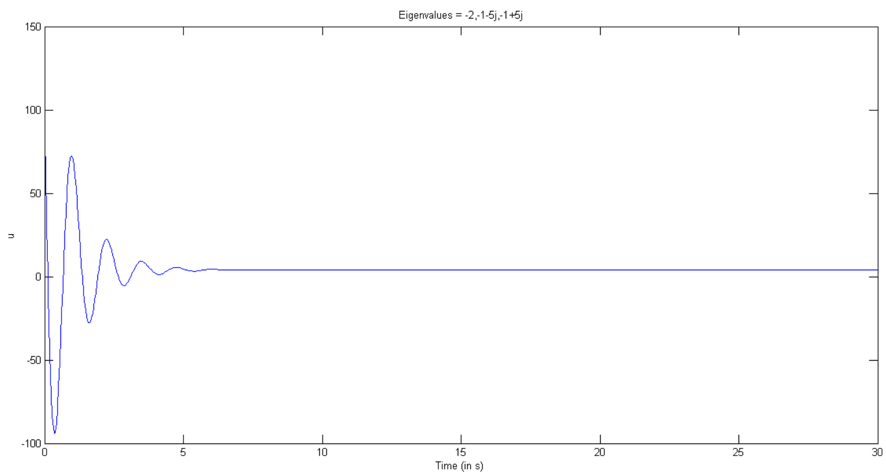

Figure 6: Input u for eigenvalues -2, $-1 \pm 5j$

Q7

5 points

- *3 points for stability (including condition on eigenvalues, else 2 points)*
- *2 points on eigenvalues occuring in conjugate pairs*

Ackerman's solution can be used to determine state feedback gains if the desired complex eigenvalues satisfy the following conditions

1. They have a non-positive real part. This is required for the system to be stable.
2. The eigenvalues are in conjugate pairs. This condition stems from the fact that the characteristic equation has only real coefficients.

Q8

8 points

- *1 point for formula for controllability matrix*
- *3 points for reachability matrix*
- *2 points for condition on beta*
- *2 points for condition on alpha*

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 1 & 1 \end{bmatrix} \\
B &= \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} \\
w_r &= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \\
&= \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & \alpha\beta \\ 0 & \alpha\beta & \alpha\beta \end{bmatrix} \\
&= \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & \alpha \end{bmatrix}
\end{aligned}
$$

w_r should be full rank for the system to be controllable. It can be seen that this is only possible is $\alpha \neq 0$ and $\beta \neq 0$.

Bonus

20 points

- *4 points for eigenvalues*
- *4 points - 2 points for each eigenvector*
- *2 points for V^-1*
- *2 points for identifying D*
- *2 points for using property of matrix exponential*
- *2 points for using that e^D is equivalent to exponentiation of diagonal elements*
- *4 points for computation*

Let

$$
\begin{aligned}
A &= \begin{bmatrix} -\zeta\omega_0 t & \omega_d t \\ -\omega_d t & -\zeta\omega_0 t \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, a = -\zeta\omega_0 t, b = \omega_d t
\end{aligned}
$$

We can see that $a^2 + b^2 = \omega_0^2 t^2$. To find the eigenvalues of A, we set

$$
\begin{aligned}
&det(\lambda I - A) = 0 \\
\implies &(\lambda - a)^2 + b^2 = 0 \\
\implies &\lambda = a \pm jb
\end{aligned}
$$

Taking $\lambda_1 = a + jb$ and $\lambda_2 = a - jb$, we refer to the corresponding eigenvectors as v_1 and v_2. We can see that since A is real and $\lambda_2 = \bar{\lambda_1}$, $v_2 = \bar{v_1}$. To find v_1,

$$
\begin{aligned}
& (A - \lambda_1 I)v_1 = 0 \\
\implies & \begin{bmatrix} -jb & b \\ -b - jb & \end{bmatrix} v_1 = 0 \\
\implies & v_1 = \begin{bmatrix} j \\ -1 \end{bmatrix} \\
\implies & v_2 = \begin{bmatrix} j \\ 1 \end{bmatrix}
\end{aligned}
$$

Note that v_2 has been taken to be $-\bar{v_1}$ which is also correct as any scalar multiple of an eigenvector is also an eigenvector. Hence,

$$
\begin{aligned}
V &= \begin{bmatrix} j & j \\ -1 & -1 \end{bmatrix} \\
\implies \quad V^{-1} &= \frac{1}{2} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix}
\end{aligned}
$$

We know that $A = VDV^{-1}$ where $D = \begin{bmatrix} a + jb & 0 \\ 0 & a - jb \end{bmatrix}$, so

$$
\begin{aligned}
e^A &= Ve^D V^{-1} \\
&= \frac{1}{2} \begin{bmatrix} j & j \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{a+jb} & 0 \\ 0 & e^{a-jb} \end{bmatrix} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{a+jb} + e^{a-jb} & j(e^{a-jb} - e^{a+jb}) \\ j(e^{a+jb} - e^{a-jb}) & e^{a+jb} + e^{a-jb} \end{bmatrix} \\
&= \begin{bmatrix} e^a \cos b & e^a \sin b \\ -e^a \sin b & e^a \cos b \end{bmatrix} \\
&= \begin{bmatrix} e^{-\zeta\omega_0 t} \cos \omega_d t & e^{-\zeta\omega_0 t} \sin \omega_d t \\ -e^{-\zeta\omega_0 t} \sin \omega_d t & e^{-\zeta\omega_0 t} \cos \omega_d t \end{bmatrix}
\end{aligned}
$$