Stability

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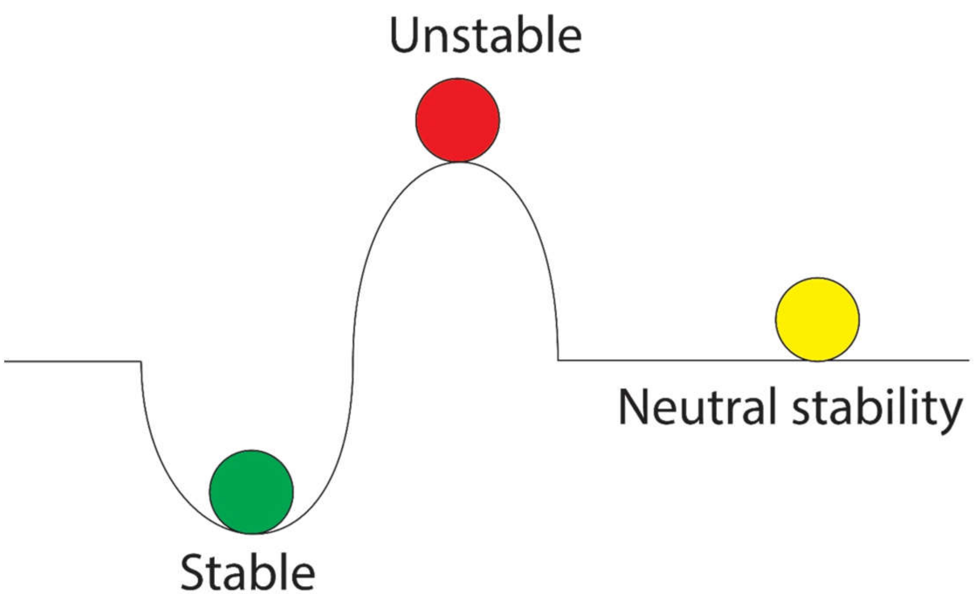
Lesson Objectives

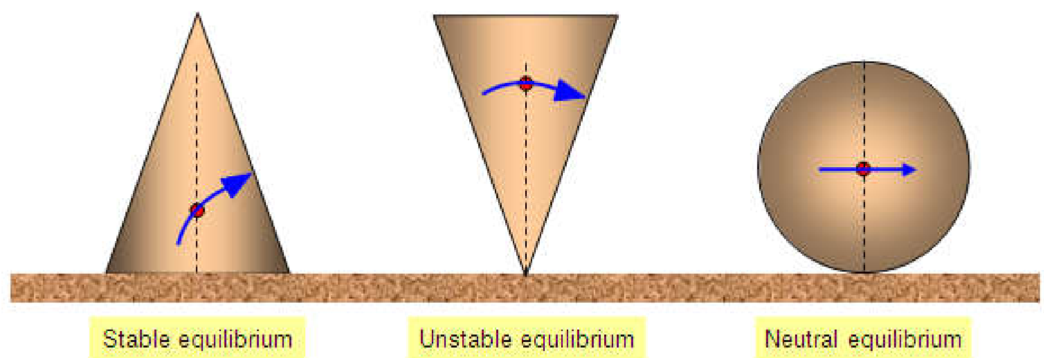
- Formally define stability
- For linear systems
  - We have already seen a strong pattern between a system's eigenvalues. In this lesson, we will:
    - Review system response with respect to stability
    - Provide additional insight for determining which systems are stable (or under what conditions).
- For nonlinear systems
  - Linearization and stability
  - Lyapunov functions for evaluating stability

Formal definition of stability

A system is <u>stable</u> if that system's response stays arbitrarily near some value, $\mathbf{z}_a$, for all of time greater than some value, t_f.

$$\left\| \mathbf{z}_a - \mathbf{z}_b \right\| < \delta \Rightarrow \left\| \mathbf{z}(t;\mathbf{z}_b) - \mathbf{z}(t;\mathbf{z}_a) \right\| < \varepsilon \quad \text{for all } t > 0$$

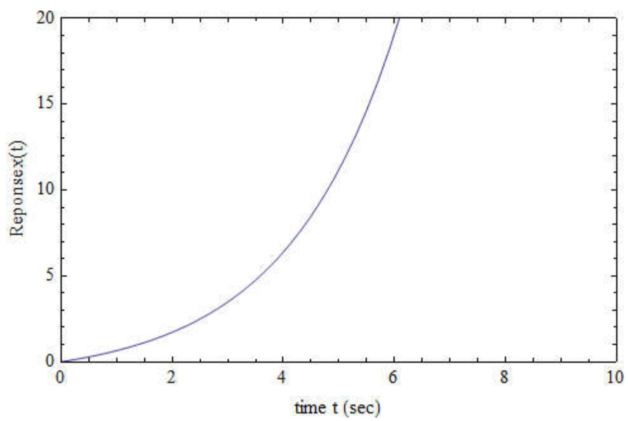




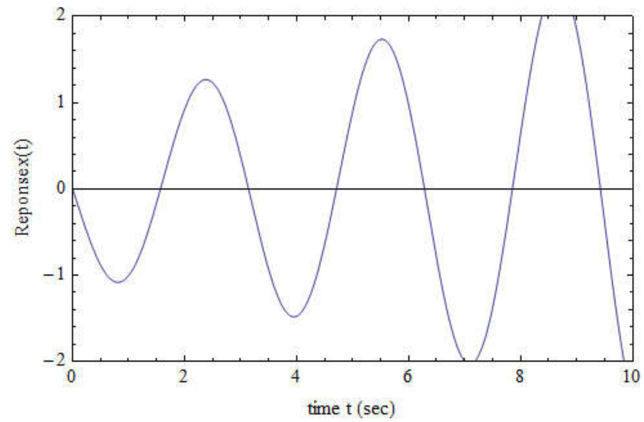
http://www.schoolphysics.co.uk/age11-14/Mechanics/Statics/text/Stability_/index.html

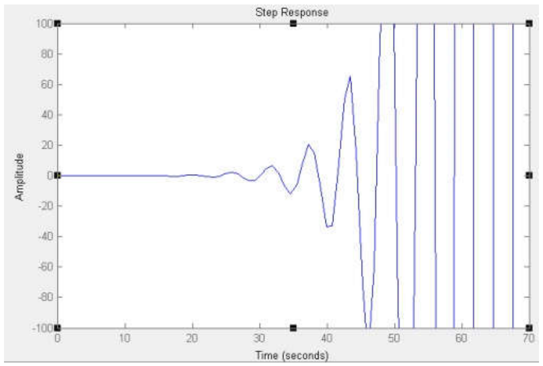
Unstable, neutrally stable responses

$$e^{+\alpha t}$$

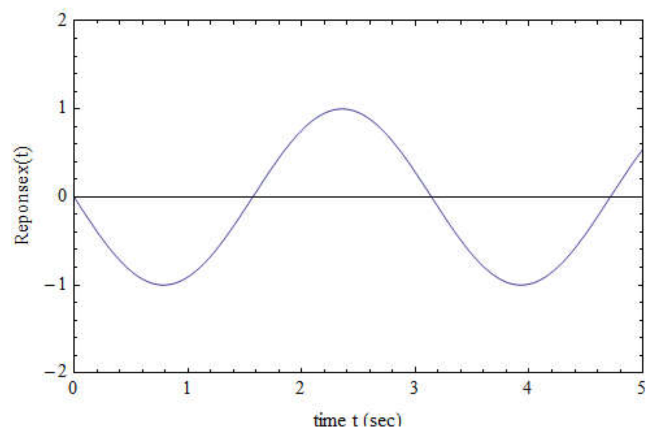


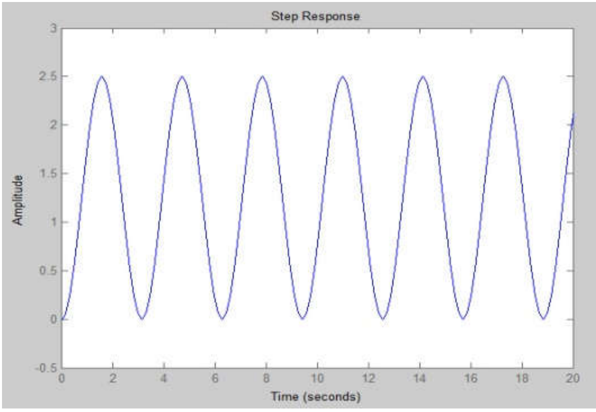
$$e^{+\alpha t}\left(c_1 \sin bt + c_2 \cos bt\right)$$





$$e^{0t}\ \left(c_1 \sin bt + c_2 \cos bt\right)$$





Common 2nd order example

Given:

$$\ddot{x} + a_1\dot{x} + a_2 x = bu(t)$$

Where:

$$x(0) = w \quad b = 1$$

$$\dot{x}(0) = v \qquad u = \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \begin{matrix} t < t_0 \\ t \geq t_0 \end{matrix}$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Find:

what happens as a_1 and a_2 vary?

Solve:

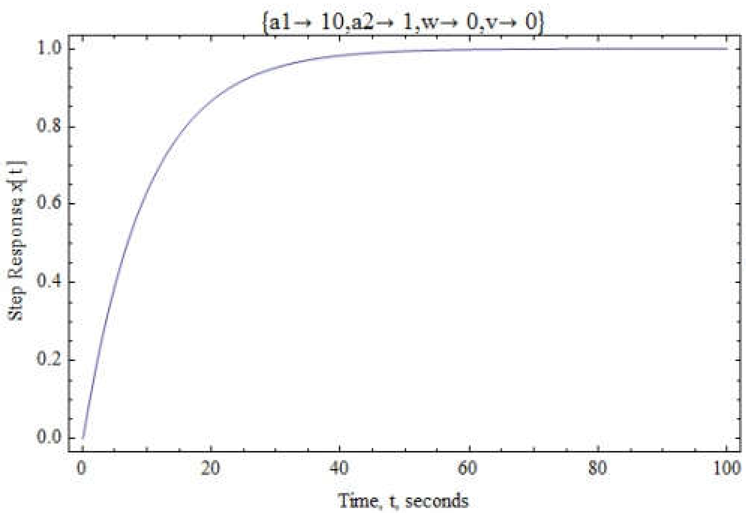
Using methods from previous lessons:

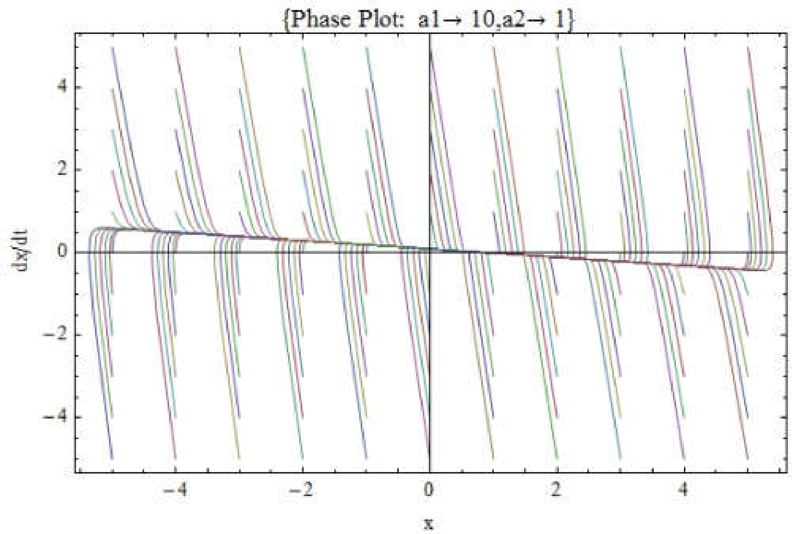
$$y(t) = \frac{1}{a_2} + C_1 e^{\frac{1}{2}(-a_1 - \sqrt{a_1^2 - 4a_2})t} + C_2 e^{\frac{1}{2}(-a_1 + \sqrt{a_1^2 - 4a_2})t}$$

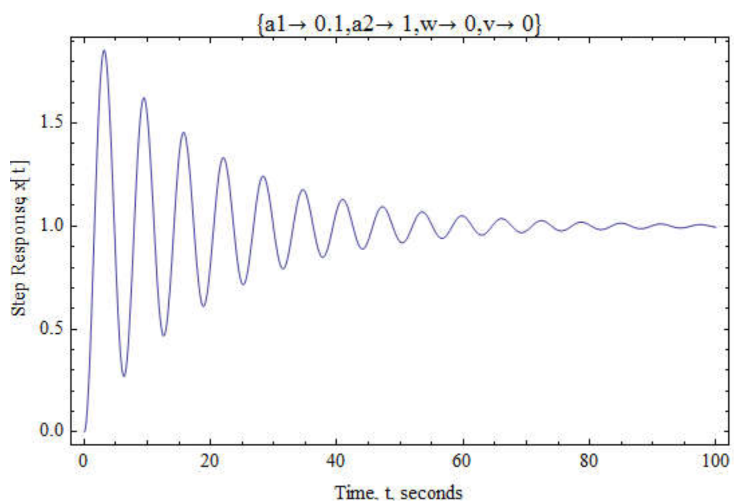
Which is used to generate the following examples for a variety of system parameters and initial conditions that illustrate common stability modalities.

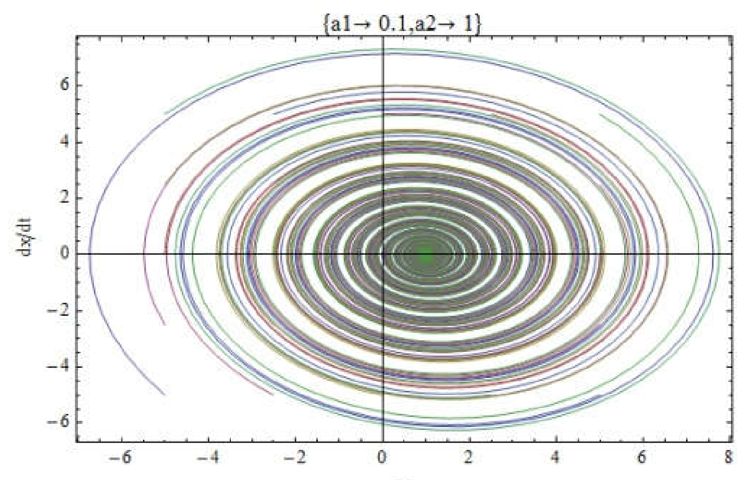
Asymptotically Stable Examples

$$x(t)=\frac{1}{a_2}+C_1\mathrm{e}^{\frac{1}{2}(-a_1-\sqrt{a_1^{2}-4a_2})t}+C_2\mathrm{e}^{\frac{1}{2}(-a_1+\sqrt{a_1^{2}-4a_2})t}$$



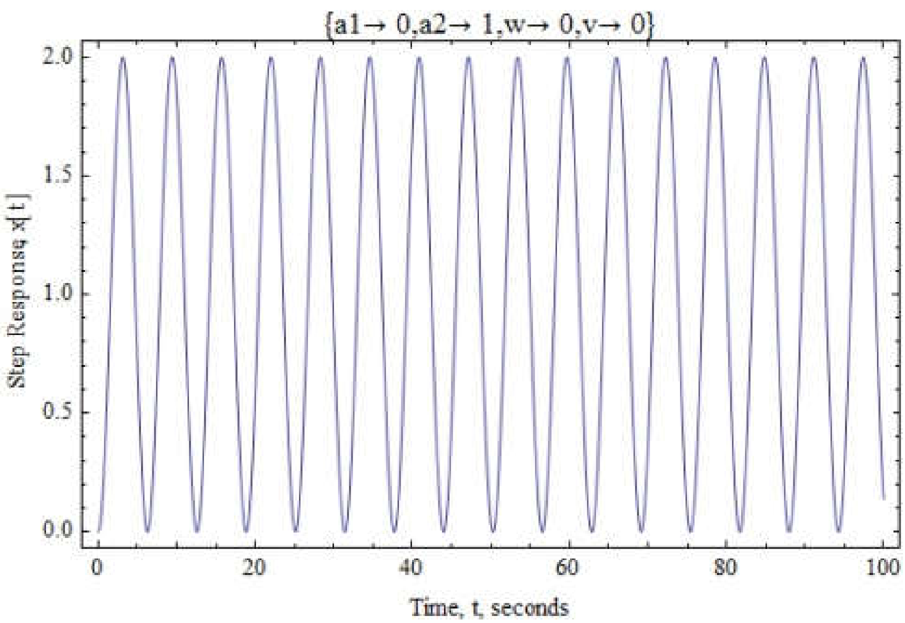


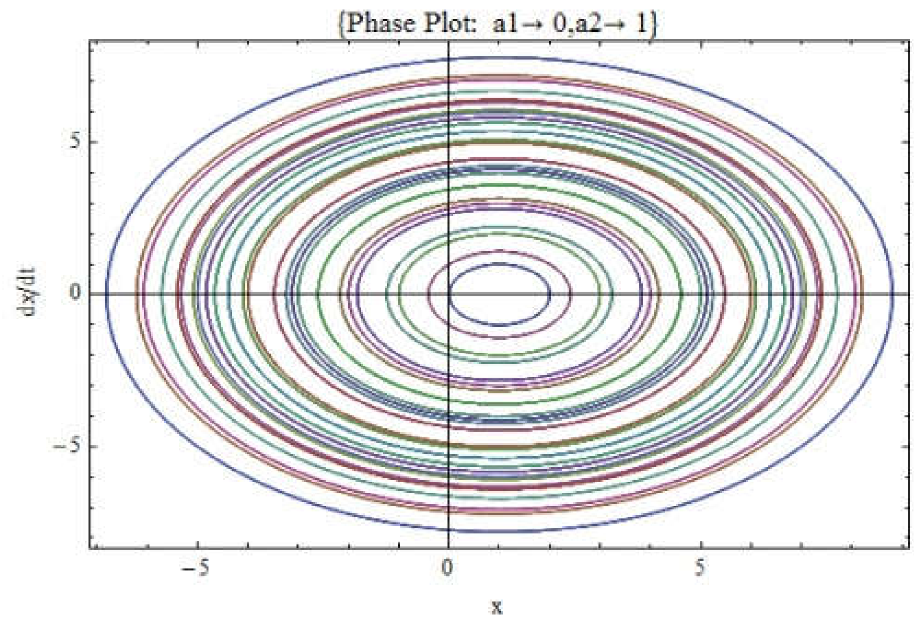




Neutrally stable examples

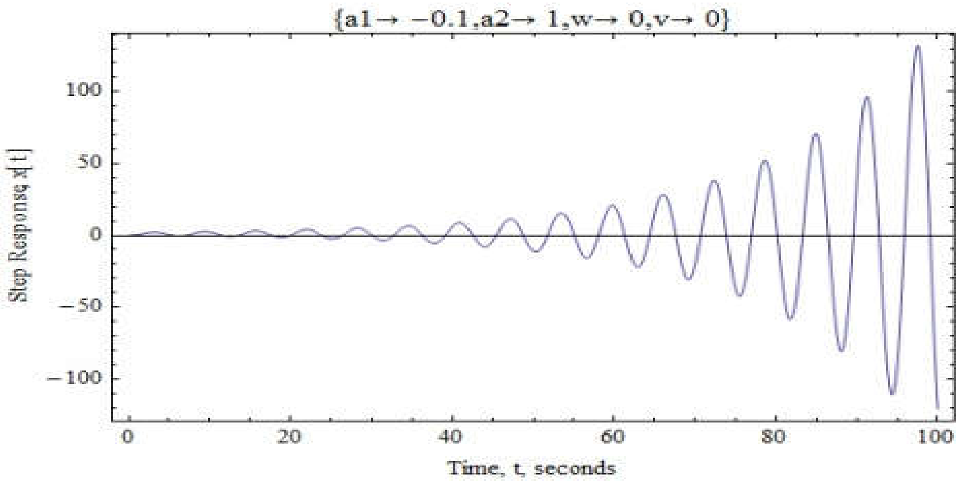
$$x(t)=\frac{1}{a_2}+C_1 e^{\frac{1}{2}(-a_1-\sqrt{a_1^2-4a_2})t}+C_2 e^{\frac{1}{2}(-a_1+\sqrt{a_1^2-4a_2})t}$$

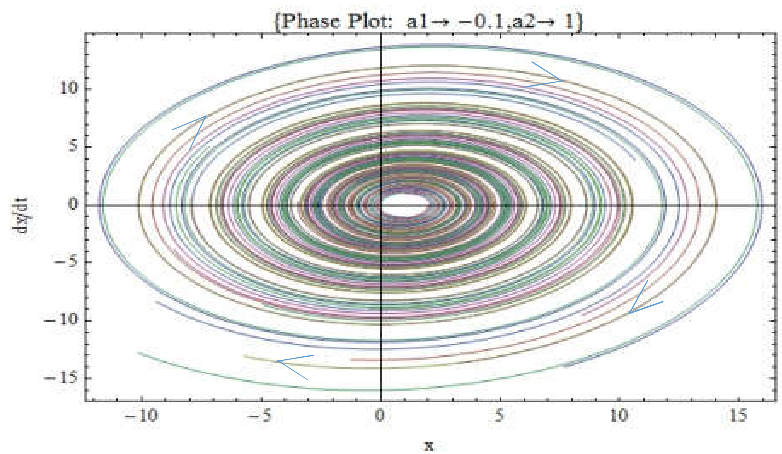


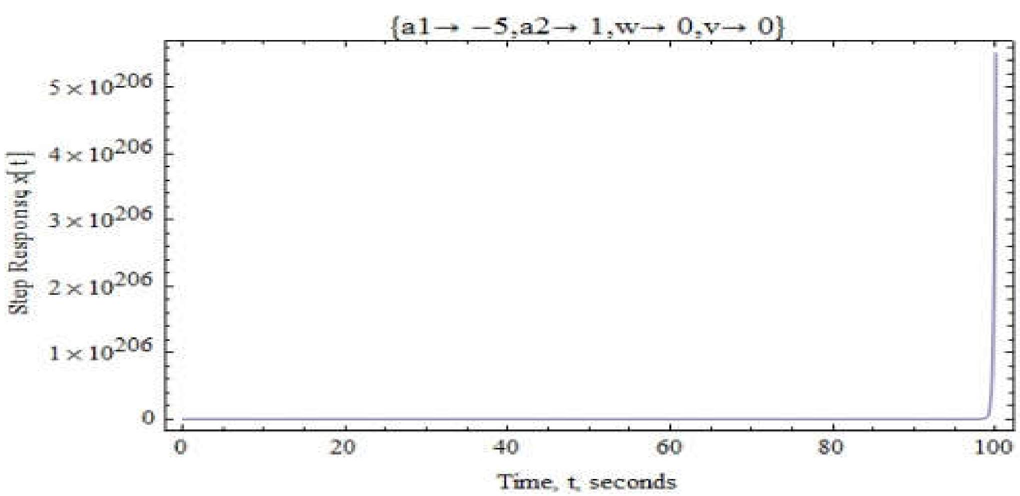


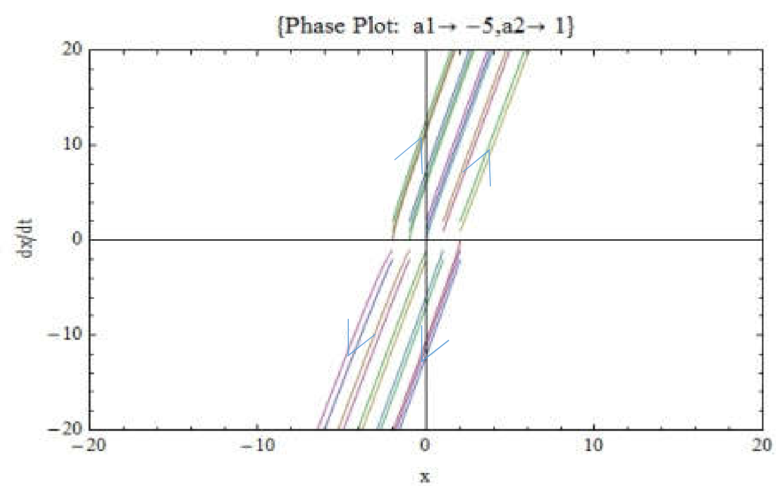
Unstable examples

$$x(t)=\frac{1}{a_2}+C_1\mathrm{e}^{\frac{1}{2}(-a_1-\sqrt{a_1^2-4a_2})t}+C_2\mathrm{e}^{\frac{1}{2}(-a_1+\sqrt{a_1^2-4a_2})t}$$









Stability in higher order systems

Example: For what values of α (if any) is the following system stable?

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \alpha & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 4 \end{bmatrix}u$$

Solve:

$$\det(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}) = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -\alpha & \lambda+2 \end{bmatrix}\mathbf{z}$$

$$\lambda\left(\lambda^2 + 2\lambda - \alpha\right) + 0 = 0$$

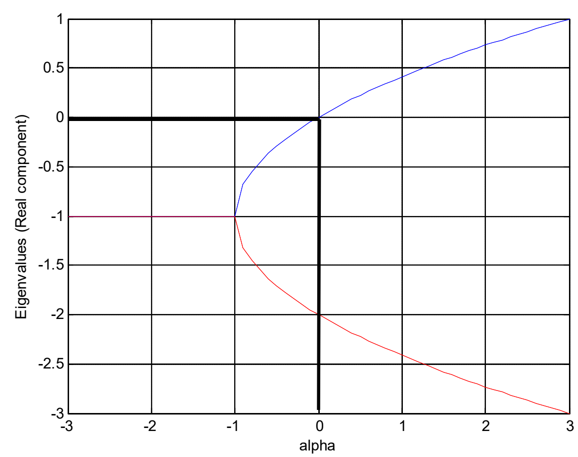
Therefore,

$$\lambda_1 = 0 \qquad \lambda_{2,3} = \frac{-2 \pm \sqrt{4+4\alpha}}{2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm 2\sqrt{1+\alpha}}{2}$$

$$= -1 \pm \sqrt{1+\alpha}$$

The system is – at best – neutrally stable. Is there a range where the system is unstable?




```
a=[-3:.1:3];
for i=1:length(a)
    eig1(i) = real(-1+sqrt(1+a(i)));
    eig2(i) = real(-1-sqrt(1+a(i)));
end
```

So the system becomes unstable if α > 0.

So far…

- Presented formal definition of stability
- For Linear Systems
  - We have seen many examples.
  - Stability can be determined with respect to system parameters.
    - But method can get burdensome.
    - Note that "***all coefficients of the Characteristic Equation must be nonzero and have the same sign***" in order for the system to be stable.
    - This is a necessary, but NOT sufficient condition for stability.

$$6\lambda^5 - 5\lambda^4 + 3\lambda^3 + 2\lambda^2 + 2\lambda + 3 = 0 \leftarrow NOT\_stable$$

$$6\lambda^5 + 5\lambda^4 + 3\lambda^3 + 2\lambda^2 + 3 = 0 \leftarrow NOT\_stable$$

$$6\lambda^5 + 5\lambda^4 + 3\lambda^3 + 2\lambda^2 + 2\lambda + 3 = 0 \leftarrow MAYBE\_stable$$

- Still need to deal with stability of nonlinear systems.

Nonlinear Systems: Multiple Options

- Determining stability for nonlinear systems using linearization.
- Exploit assumption that a system is properly controlled.
  - This allows us to treat some nonlinear systems as linear.
- Apply Lyapunov stability analysis to determine if a solution to a nonlinear dynamical system is stable.

Alexandr Lyapunov (1857-1918)

Previous example

In our example…

$$\ddot{x}+0.6\dot{x}+3x+x^2=0$$

$$z_1=x\Rightarrow\dot{z}_1=z_2$$

$$z_2=\dot{x}\Rightarrow\dot{z}_2=-0.6z_2-3z_1-z_1^2$$

$$f_1(z_1,z_2)=z_2=0$$

$$f_2(z_1,z_2)=-0.6(0)-3z_1-z_1^2=0$$

$$0=z_1(-3-z_1)$$

(0,0) and (-3,0) are equilibrium points;

We can linearize about the equilibrium points and examine stability in near z_e

Find the Jacobian for the system….

$$\mathbf{J}=\begin{bmatrix}\frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2}\\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3-2z_1 & -0.6\end{bmatrix}$$

Like a gradient, but for multiple variables

At (0,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ -3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}-0.3000+1.7059i\\ -0.3000-1.7059i\end{bmatrix}$$

so stable near (0,0)

At (-3,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ 3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}1.4578\\ -2.0578\end{bmatrix}$$

so unstable saddle point at (-3,0)

Stability in the 'region' of an equilibrium point.

At (0,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ -3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}-0.3000+1.7059i\\ -0.3000-1.7059i\end{bmatrix}$$

so stable near (0,0)

At (-3,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ 3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}1.4578\\ -2.0578\end{bmatrix}$$

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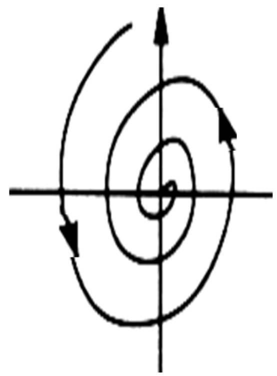
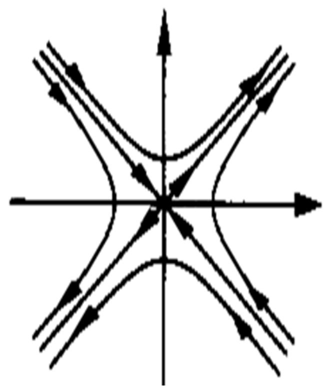
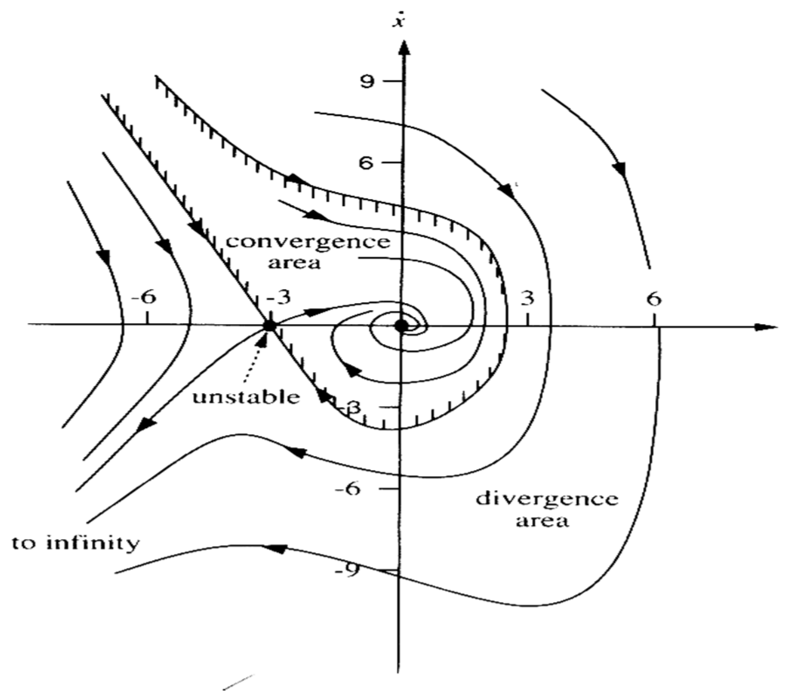


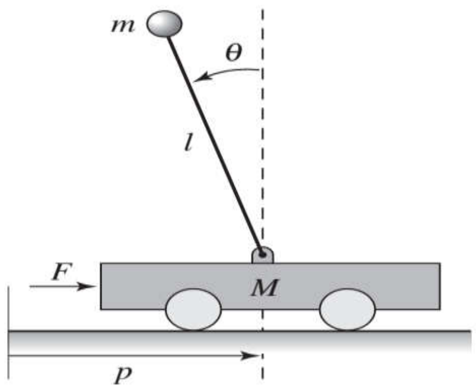




Tada!

Inverted Pendulum Example

Given: A inverted pendulum on a moving cart:



Determine: If the inverted pendulum system shown is <u>stable</u> if the pendulum is initially perpendicular to the ground.

Solution:

$$\sum F_i = (M + m)\ddot{x}$$
$$\sum \tau_i = I\ddot{\theta}$$

$$(M + m)\ddot{x} = ml\cos(\theta)\ddot{\theta} - c\dot{x} - ml\sin(\theta)\dot{\theta}^2 + F$$
$$(J + ml^2)\ddot{\theta} = ml\cos(\theta)\ddot{x} - \gamma\dot{\theta} + mgl\sin(\theta)$$

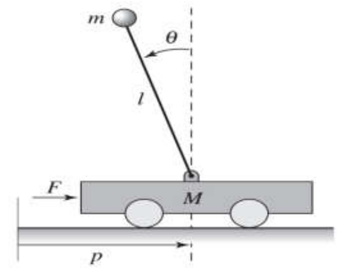
F is the input, linearize at θ =0º (i.e. cos(θ)=1 & sin(θ)= θ.)

$$(M + m)\ddot{x} = ml(1)\ddot{\theta} - (0)\dot{x} - ml\theta\dot{\theta}^2 + u$$
$$(J + ml^2)\ddot{\theta} = ml(1)\ddot{x} - (0)\dot{\theta} + mgl\theta$$

Put in matrix form…

$$\begin{bmatrix} (M + m) & -ml \\ -ml & (J + ml^2) \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -ml\theta\dot{\theta}^2 \\ mgl\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix}$$

Inverted pendulum example



$$\begin{bmatrix} (M+m) & -ml \\ -ml & (J+ml^2) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -ml\theta\dot{\theta}^2 \\ mgl\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix}$$

When <u>*controlled*</u>*, the angular velocity should be close to zero, so we can ignore terms quadratic and higher angular velocity terms.*

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} (M+m) & -ml \\ -ml & (J+ml^2) \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 \\ mgl\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \right]$$

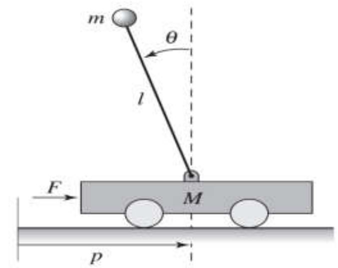
$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{(M+m)(J+ml^2) - m^2 l^2} \begin{bmatrix} (J+ml^2) & ml \\ ml & (M+m) \end{bmatrix} \left[\begin{bmatrix} 0 \\ mgl\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \right]$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{\mu} \begin{bmatrix} (J+ml^2) & -ml \\ ml & (M+m) \end{bmatrix} \left[\begin{bmatrix} 0 \\ mgl\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \right]$$

Let's define the states as.

$$\mathbf{z} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$$

Inverted Pendulum Example



Note, in this case that: $\mathbf{y} = \mathbf{C}\,\mathbf{z} + \mathbf{D}\,u = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{z}$

And our system is….

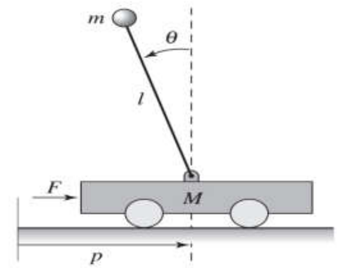
$$\mathbf{z} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{\mu}\begin{bmatrix} (J + m\,l^2) & -m\,l \\ m\,l & (M + m) \end{bmatrix}\left[\begin{bmatrix} 0 \\ m\,g\,l\theta \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix}\right]$$

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{B}\,u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & 0 & 0 \\ 0 & \frac{(M+m)\,m\,g\,l}{\mu} & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J + m\,l^2}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

This system is linearized at θ=0 *assuming that the angular velocity is small. So is the system stable?*

Inverted Pendulum Example



$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & 0 & 0 \\ 0 & \frac{(M+m)mgl}{\mu} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

This system is linearized at θ=0 assuming that the angular velocity is small. So is the system stable?

$$\det(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & -\frac{m^2 l^2 g}{\mu} & \lambda & 0 \\ 0 & -\frac{(M+m)mgl}{\mu} & 0 & \lambda \end{bmatrix}$$

$$CE = \lambda\left(\lambda\left(\lambda^2\right) - 1\left(-\frac{(M+m)mgl}{\mu}\lambda\right)\right) - 1(0)$$

$$= \lambda^4 - \lambda^2 \frac{(M+m)mgl}{\mu}$$

From this we get that the system's eigenvalues at this equilibrium point are:

$$\lambda = 0, 0, \pm\sqrt{\frac{(M+m)mgl}{\mu}}$$

Therefore the system is unstable for any mechanical system qualifying as a pendulum!

Wouldn’t mind something a little more… general.

Lyapunov Functions

Lyapunov Function: $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^1$ is an "energy-like" function that is <u>nonnegative</u> and always decreasing along trajectories of a given system. If this is true, then we can conclude that the minimum of the function *V* is a stable equilibrium point.

Before presenting a more formal definition, we need a few definitions.

A function is ***positive definite*** if: $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ AND $V(\mathbf{0}) = 0$

A function is ***negative definite*** if: $V(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$ AND $V(\mathbf{0}) = 0$

A function is ***positive semi-definite*** if: $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq 0$ AND $V(\mathbf{0}) = 0$

(i.e. the function can be 0 at pts other than **x**=0.)

Examples where $\mathbf{x} \in \mathbb{R}^2$

$$V_1 = x_1^2 \qquad\qquad V_2 = x_1^2 + x_2^2$$

Is only positive semi-definite since its value can be 0 when **x**=(0,a)

Is only positive definite since its value can only be 0 when **x**=(0,0)

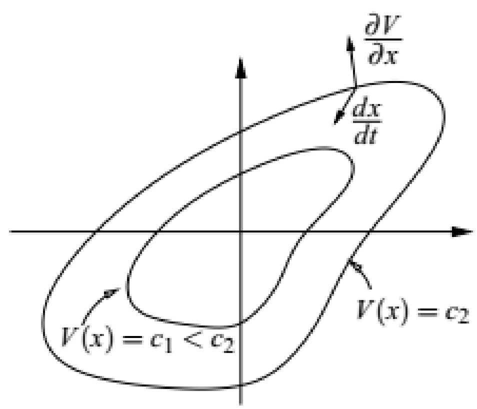
Lyapunov Stability Theorem

Let V be a nonnegative function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics.

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \frac{\partial V}{\partial \mathbf{x}} F(\mathbf{x})$$

Let $B_r = B_r(0)$ be a ball of radius r around the origin. If there exists an $r>0$ such that V is positive definite and $\dot{V}$ is negative semi-definite for all $\mathbf{x} \in B_r$

Then, $\boldsymbol{x}$=0 is locally stable in the "sense of Lyapunov." If V is positive definite and $\dot{V}$ is negative definite, in B_r, then $\boldsymbol{x}$=0 is locally asymptotically stable.



If we are interested in examining an area B_r around an equilibrium point that is *not* at the origin, then we can can rewrite the equations in a new set of coordinates (i.e. $\mathbf{z}=\mathbf{x}-\mathbf{x}_e$)

Example

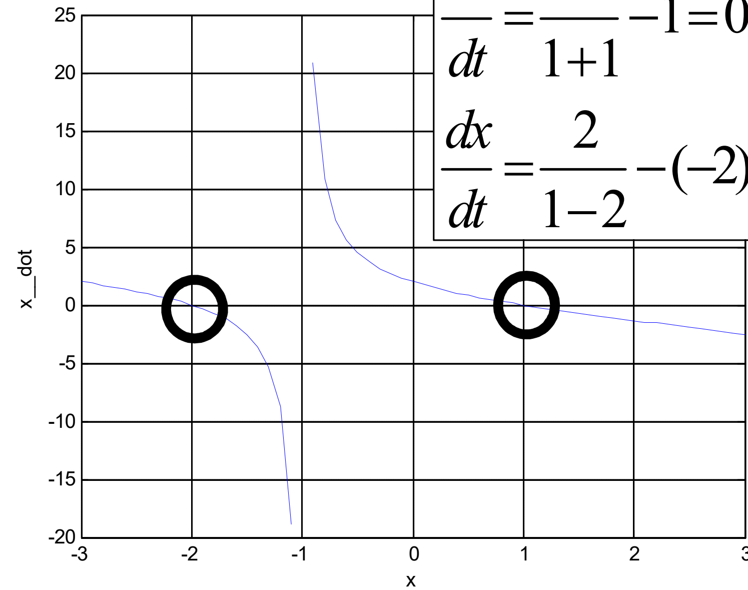
Given: Simple scalar nonlinear system

$$\frac{dx}{dt}=\frac{2}{1+x}-x$$

Note, this system has two equilibrium points (at 1 and -2)

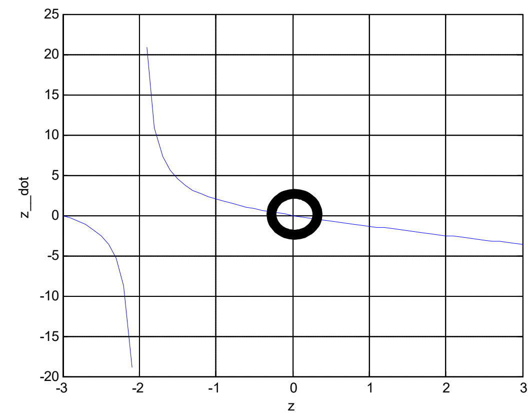
$$\frac{dx}{dt}=\frac{2}{1+1}-1=0$$

$$\frac{dx}{dt}=\frac{2}{1-2}-(-2)=0$$



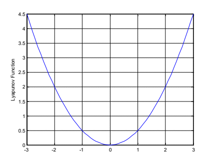
Let's consider the equilibrium point x=1 by rewriting the dynamic equations in a coordinate frame that moves this equilibrium point to the origin (i.e. $z=x-1$ thus $x=z+1$)

$$\frac{dz}{dt}=\frac{2}{2+z}-z-1$$



Let's try the candidate Lyapunov function

$$V(z)=\frac{1}{2}z^2$$



Which is globally positive definite. The derivative of V along the trajectories is given by:

$$\dot{V}(z)=\frac{\partial V}{\partial z}\dot{z}=z\left(\frac{2}{2+z}-z-1\right)=\frac{2z}{2+z}-z^2-z$$

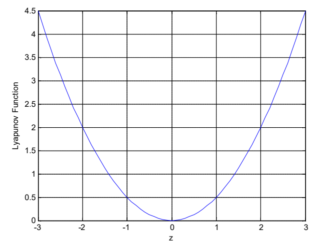
Consider an interval (1 dimensional region) B_r where $r<2$.

Example cont'd

$$\frac{dx}{dt} = \frac{2}{1+x} - x$$

Candidate Lyapunov function

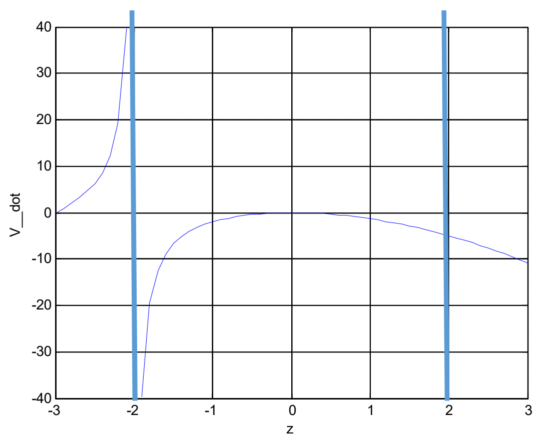
$$V(z) = \frac{1}{2} z^2$$



…is globally positive definite.

A its derivative is negative definite over B_r if r<2.

$$\dot{V}(z) = \frac{\partial V}{\partial z}\dot{z} = z\left(\frac{2}{2+z} - z - 1\right) = \frac{2z}{2+z} - z^2 - z$$



Lyapunov's method requires that one <u>choose</u> *a positive definite Lyapunov function (candidate) and then <u>prove</u> that its derivative is negative (semi) definite in the region of interest.*

For proof in this case, let us consider all values of z>-2, or z+2>0. Since z+2>0 and V_dot must be negative, then

$$(z+2)\left(\frac{2z}{2+z} - z^2 - z\right) < 0$$

$$2z - (z^2 + z)(z+2) < 0$$

$$2z - z^3 + z^2 + 2z^2 + 2z < 0$$

$$-z^3 + 3z^2 < 0$$

$$-z^2(z+3) < 0$$

Which provides more formal proof than the graph to the left. So we proved that our original function is asymptotically stable at x_e=1.

Pendulum Example

Given: Normalized model for a hanging pendulum

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\sin x_1$$

Where x_1 is the angle between the pendulum and the vertical and a positive value occurring with a clockwise rotation.

There is an equilibrium when

$$x_1 = x_2 = 0$$

when the pendulum is at rest and hanging straight down.

One candidate Lyapunov function is

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \approx \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

Where the Taylor series approximation is positive definite.

Next we find the derivative of the Lyapunov Function…

$$\begin{aligned} \dot{V}(x) &= \dot{x}_1 \sin x_1 + \dot{x}_2 x_2 \\ &= x_2 \sin x_1 - x_2 \sin x_1 \\ &= 0 \end{aligned}$$

Which is negative semi-definite, proving the system is stable, but not asymptotically stable. (which make sense since there is no frictional term in our dynamic equations)

Summary

- We formally defined stability
- We can determine the stability of linear systems including as a function of a system's parameters.
- We can use linearization to ascertain the stability of a nonlinear system around a given equilibrium point
  - The analysis required to evaluate stability can be extensive.
- Lyapunov Functions can be utilized the determine the stability of nonlinear systems.
  - This approach can provide additional insight into the region of stability about an equilibrium (more so than linearization)
  - Provides a method to discuss the stability for nonlinear controllers (not a subject we will have time for in this course)
  - But functions can be hard to identify and/or prove their validity.