Advanced Dynamics
& Automatic Control

Visualizing Frequency Response
Nyquist (polar) plots

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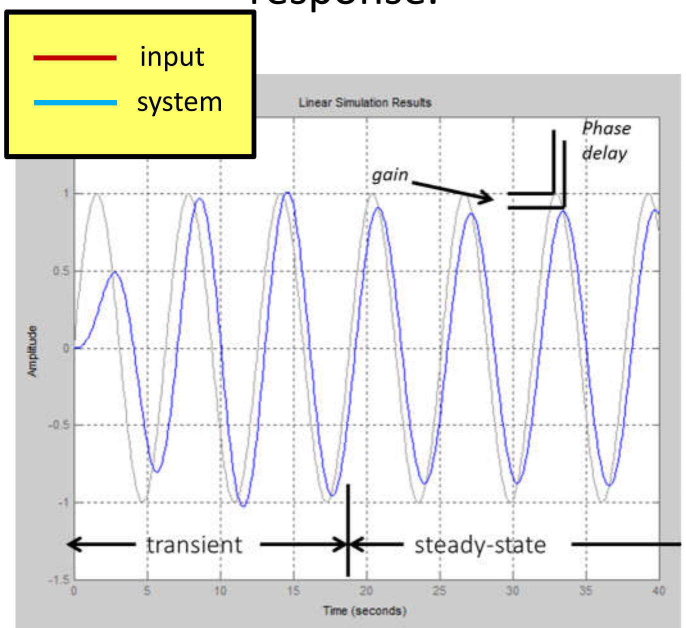
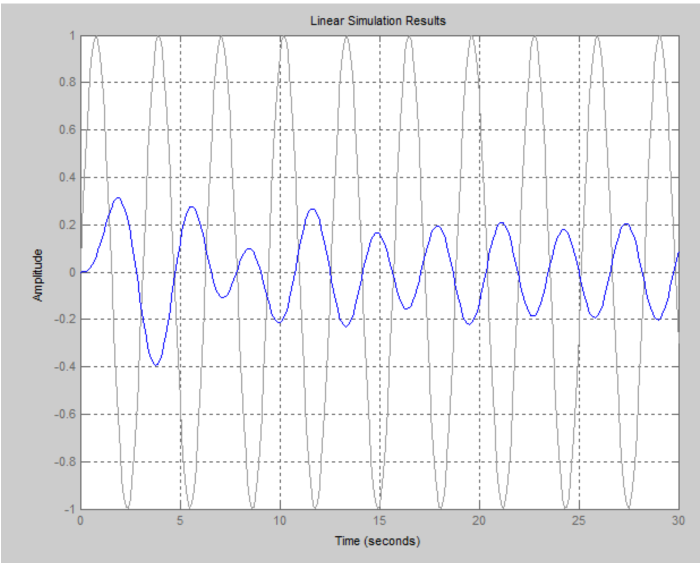
Lesson Objective

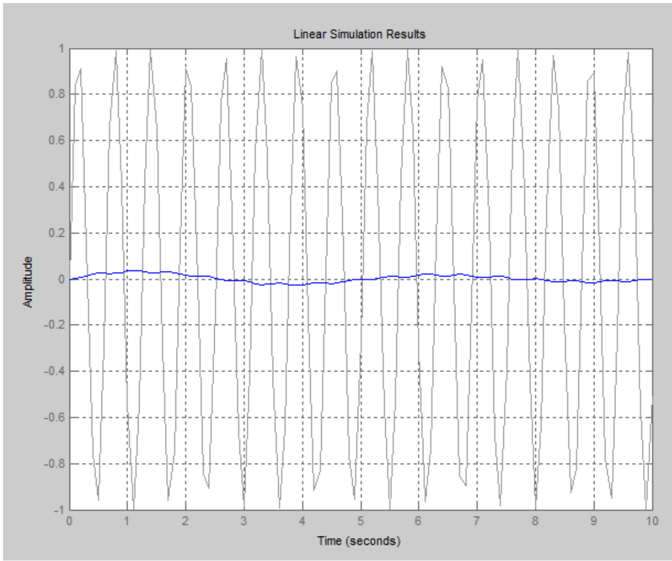
- Now that we can find the transfer function for any system, let’s learn to visualize the output of the system over a range of inputs.
- Learn how to construct Bode plots using MATLAB and by hand.
  - Manual construction provides insight into how various aspects of T(s) impact system response.



`u = 1.0*sin(1*t);`



`u = 1.0*sin(2*t);`



`u = 1.0*sin(10*t);`

Review: frequency response

Previously…

$$y(\mathrm{s}) = \frac{s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots + b_{n-2} s^2 + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-2} s^2 + a_{n-1} s + a_n} u(\mathrm{s}) = \frac{b(s)}{a(s)} u(\mathrm{s}) = \frac{\mathrm{num}(s)}{\mathrm{den}(s)} u(\mathrm{s}) = \boxed{G(s)u(t)}$$

Assume a sinusoidal input with frequency ω…

$$u = e^{st} = e^{\sigma + j\omega t} = e^{0 + j\omega t} = \cos\left(j\omega\right) + j\sin\left(j\omega\right)$$

Gives us the output…

$$y(t) = G(j\omega)e^{j\omega t} = Me^{\left(j\omega+\phi\right)t}$$

Where…

$$M = \left|G(j\omega)\right|$$

M is the magnitude or gain of *G(jω), and*

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}\left(G\left(j\omega\right)\right)}{\mathrm{Re}\left(G\left(j\omega\right)\right)}\right)$$

Phi is the phase angle of *G(jω).*

We can use these functions to determine the response of a system to an input at any frequency. What would be nice is a way to visualize the response over a range of frequencies?

Option 1: Polar Plots

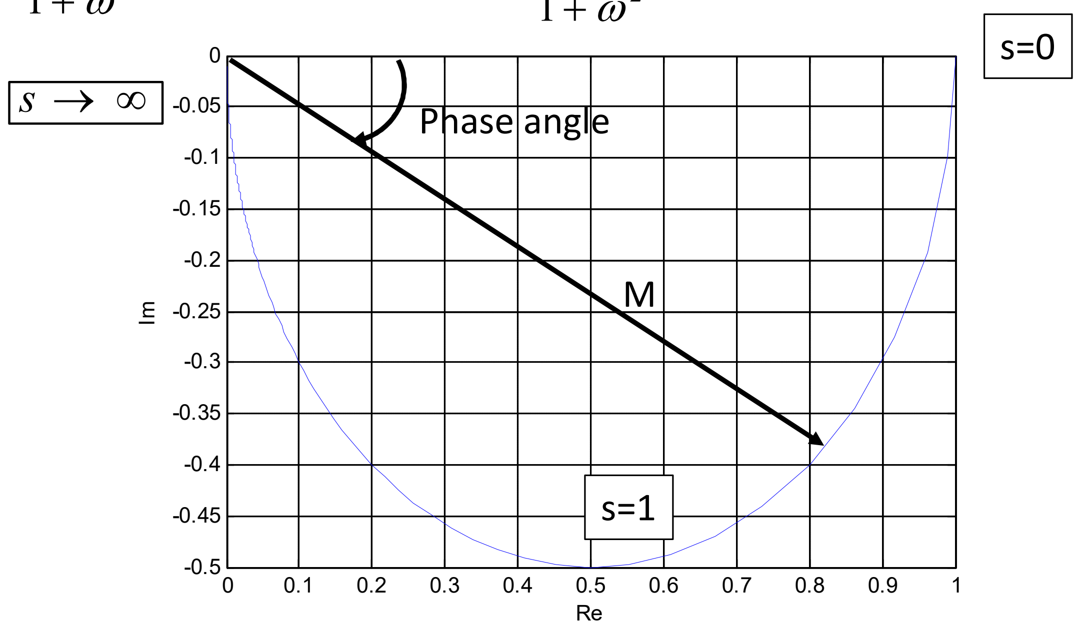
A simple example where $u = e^{st} = e^{(\sigma + j\omega)t} = e^{(0+j\omega)t} = e^{0t}e^{(j\omega)t} = e^{(j\omega)t}$

$$G(s) = \frac{1}{s+1} \qquad G(j\omega) = \frac{1}{j\omega + 1} = \frac{1}{j\omega+1}\left(\frac{1-j\omega}{1-j\omega}\right) = \frac{1-j\omega}{1+\omega^2} = \frac{1}{1+\omega^2} + j\frac{-\omega}{1+\omega^2}$$

Separate this into the real and imaginary components…

$$\mathrm{Re}(G(j\omega)) = \frac{1}{1+\omega^2} \qquad \mathrm{Im}(G(j\omega)) = \frac{-\omega}{1+\omega^2}$$

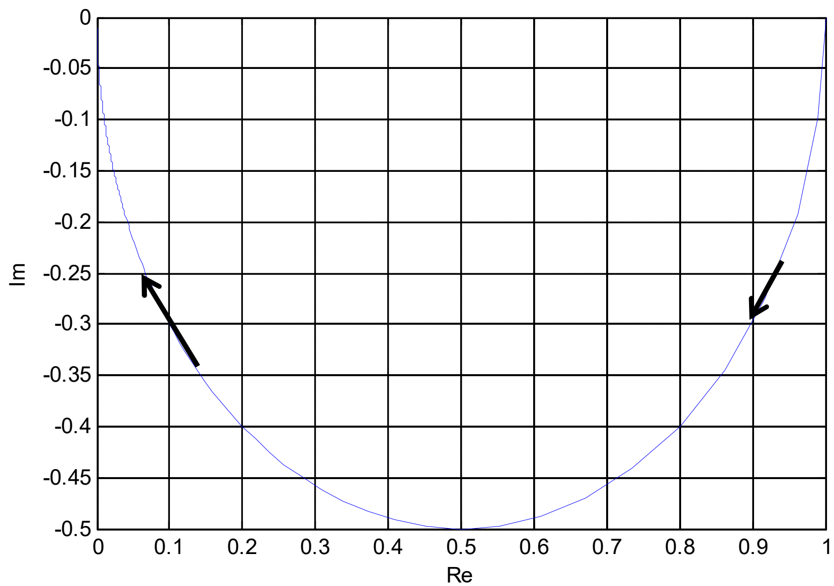
```
w = [0:.1:100];

for i=1:length(w)
    re(i) = 1/(1+w(i)*w(i));
    im(i) = -w(i)/(1+w(i)*w(i));
end

figure(1)
plot( re, im );
xlabel('Re'), ylabel('Im')
grid on;
```



Do the values make sense?

Recall our example….

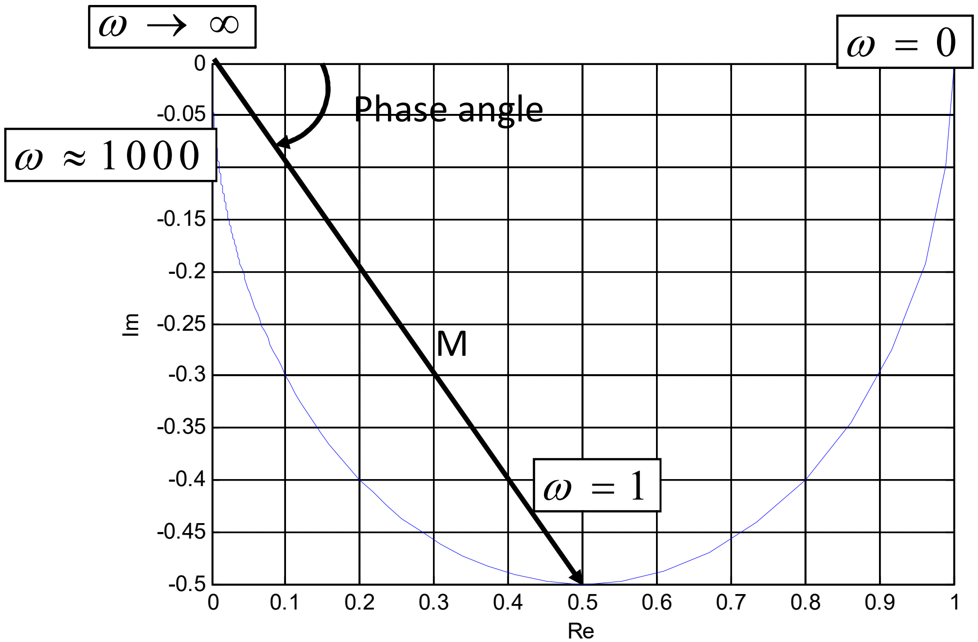
$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega + 1} = \frac{1 - j\omega}{1 + \omega^2}$$

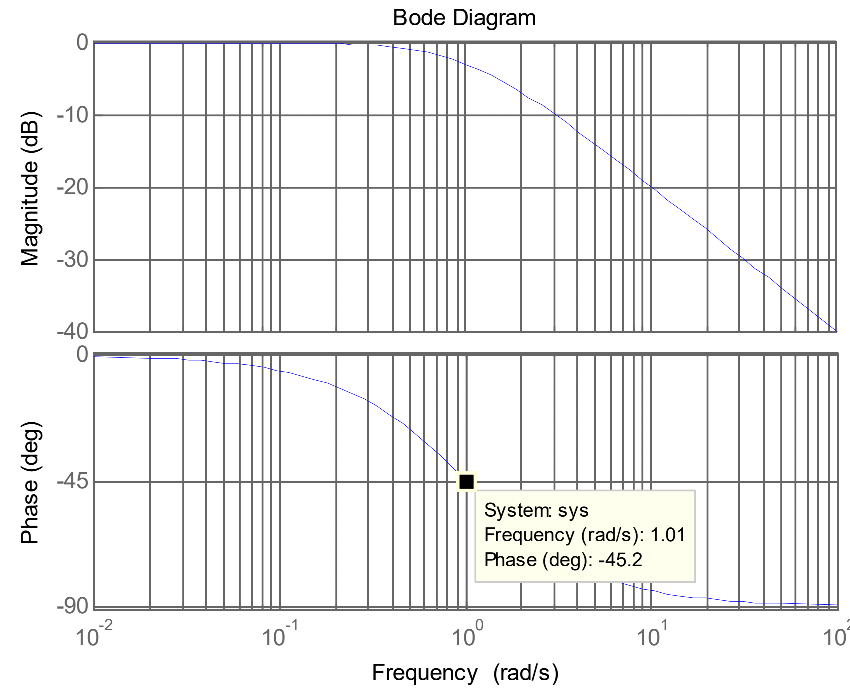


- At low frequencies, the plot is near (1, 0)
  - Makes sense as systems will follow input with little delay.
- At high frequencies, the plot is near (0, 0)
  - We noted earlier for the mass-spring-damper system that the magnitude of the output goes to zero as the frequency goes to infinity.
- At high frequencies the polar plot shows the phase angle approaches -90°.
- Arrows should be added to manually generated graphs to make the direction of increasing frequency clear.

Polar plots vs Bode Plots

$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2}$$





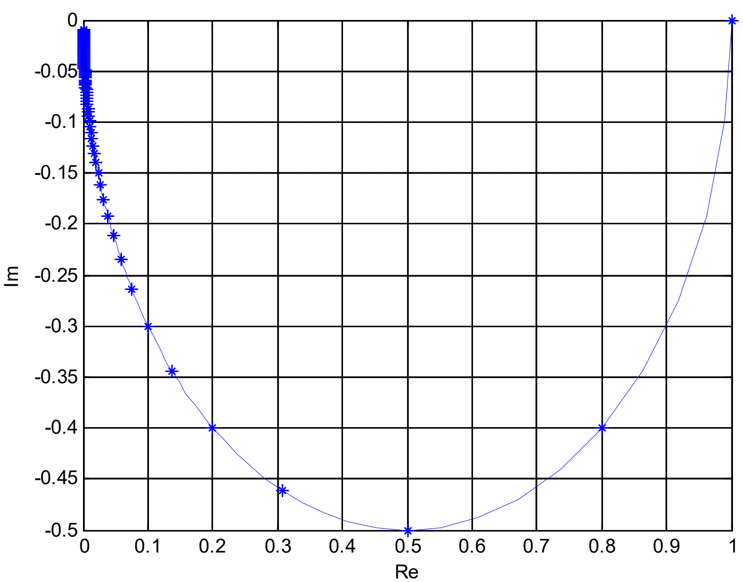
- They show the same data! (log-log vs. Real/Imag)
  - Bode Plot is a bit easier to read since the frequency is on an axis and there is no need for a secondary calculation

Nyquist (or Polar) plots

$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2}$$

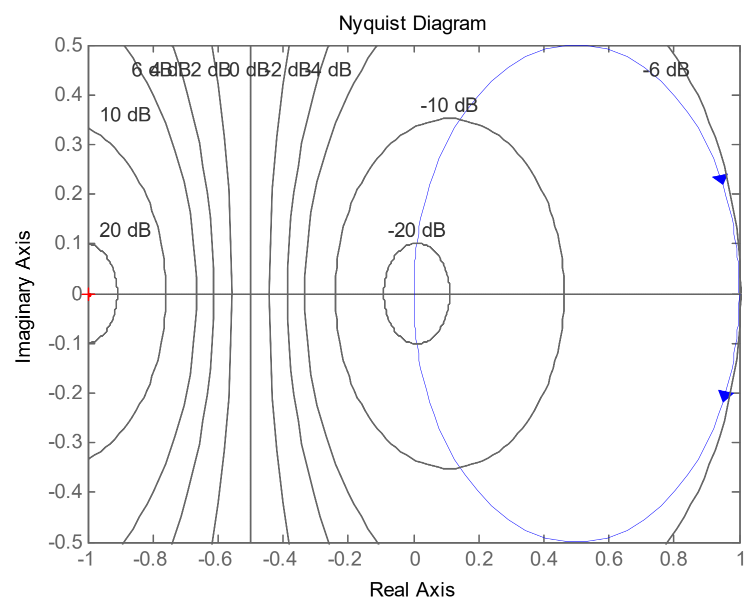
Manually

```
w = [0:.1:100];
for i=1:length(w)
    re(i) = 1/(1+w(i)*w(i));
    im(i) = -w(i)/(1+w(i)*w(i));
end
plot( re, im ); hold on;

w = [0:.5:100];
for i=1:length(w)
    re2(i) = 1/(1+w(i)*w(i));
    im2(i) = -w(i)/(1+w(i)*w(i));
end
plot( re2, im2, '*' );
```



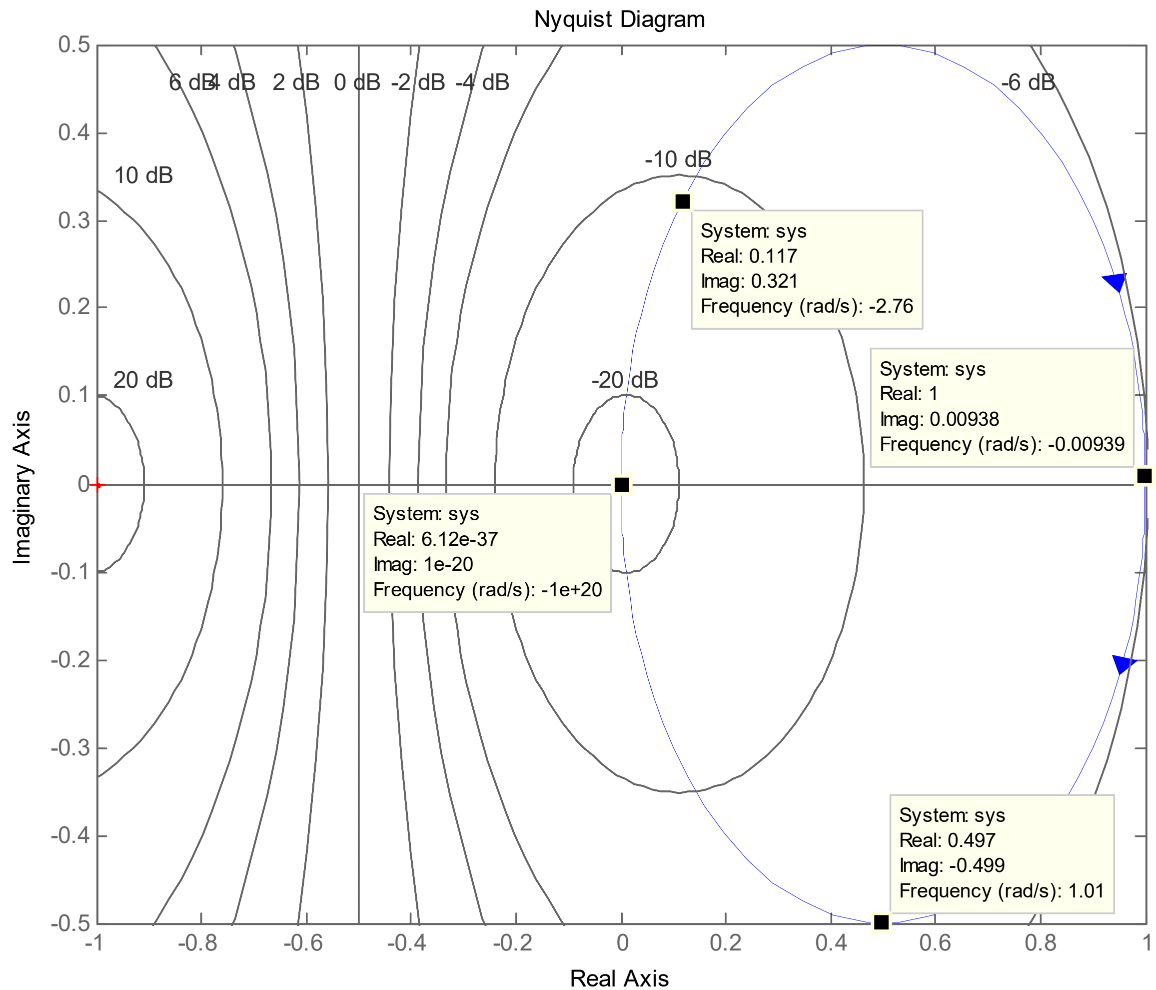
Using MATLAB

```
num = [1]
den = [ 1 1 ]
sys = tf( num, den)
figure(2)
nyquist(sys)
```



Closer look?

$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2}$$



Nyquist for higher order systems

A 2nd order example….

$$G(j\omega)=\frac{1}{(s+1)^2}=\frac{1}{(j\omega+1)^2}$$

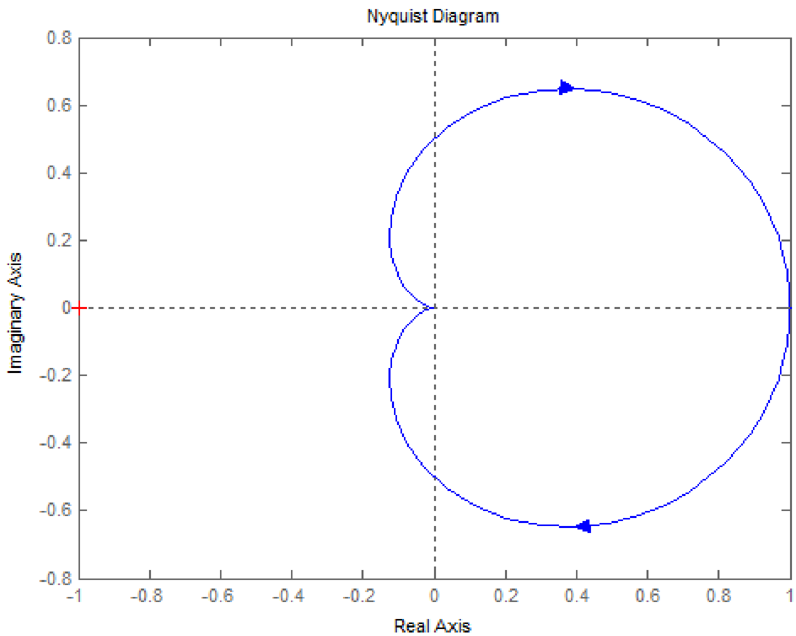
$$=\frac{1}{(j\omega+1)^2}\left(\frac{(1-j\omega)^2}{(1-j\omega)^2}\right)$$

$$=\frac{1-2j\omega-\omega^2}{\omega^4+2\omega^2+1}$$

Separate this into the real and imaginary components…

$$\mathrm{Re}(G(j\omega))=\frac{1-\omega^2}{\omega^4+2\omega^2+1}$$

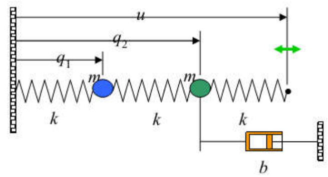
$$\mathrm{Im}(G(j\omega))=\frac{-2j\omega}{\omega^4+2\omega^2+1}$$



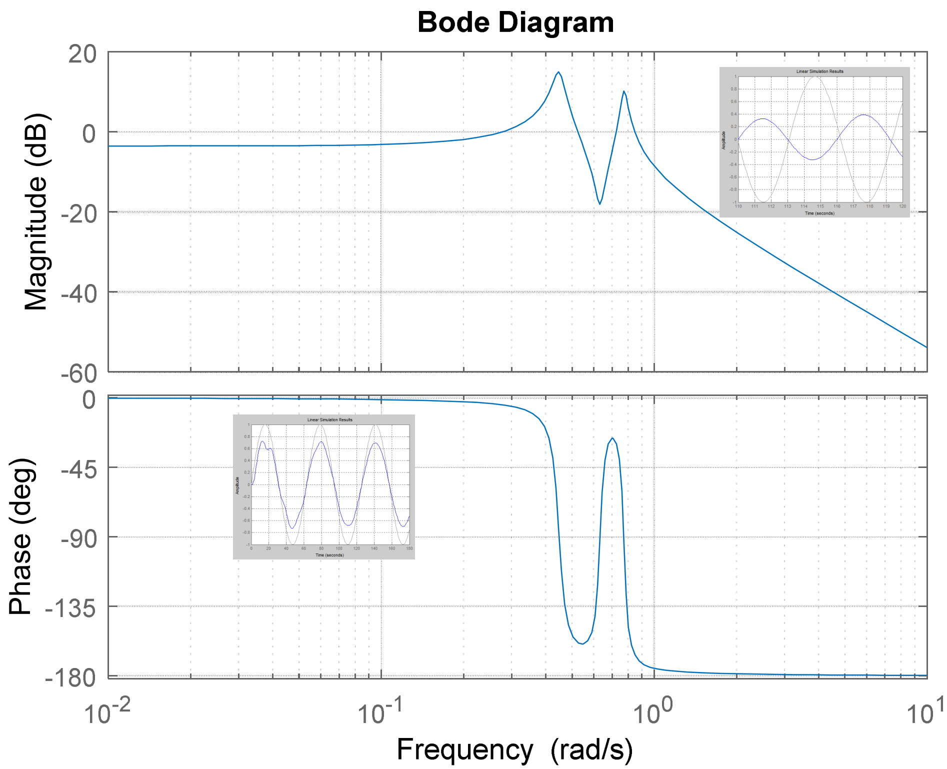
…without MATLAB it can start to get a little complicated…

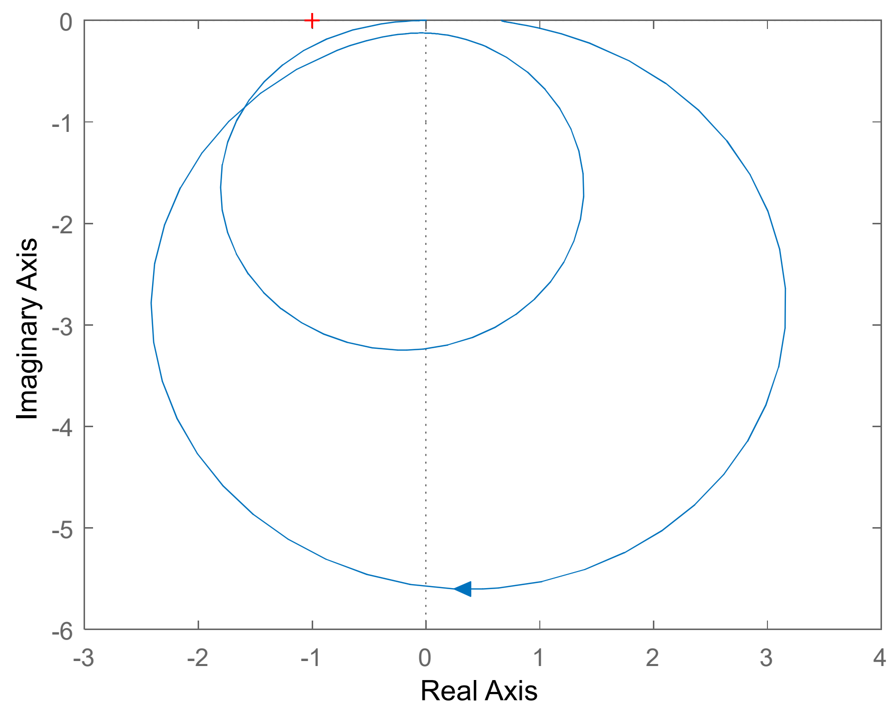
MATLAB has functions `real(f)` and `imag(f)`

Bode vs. Nyquist (complex system)

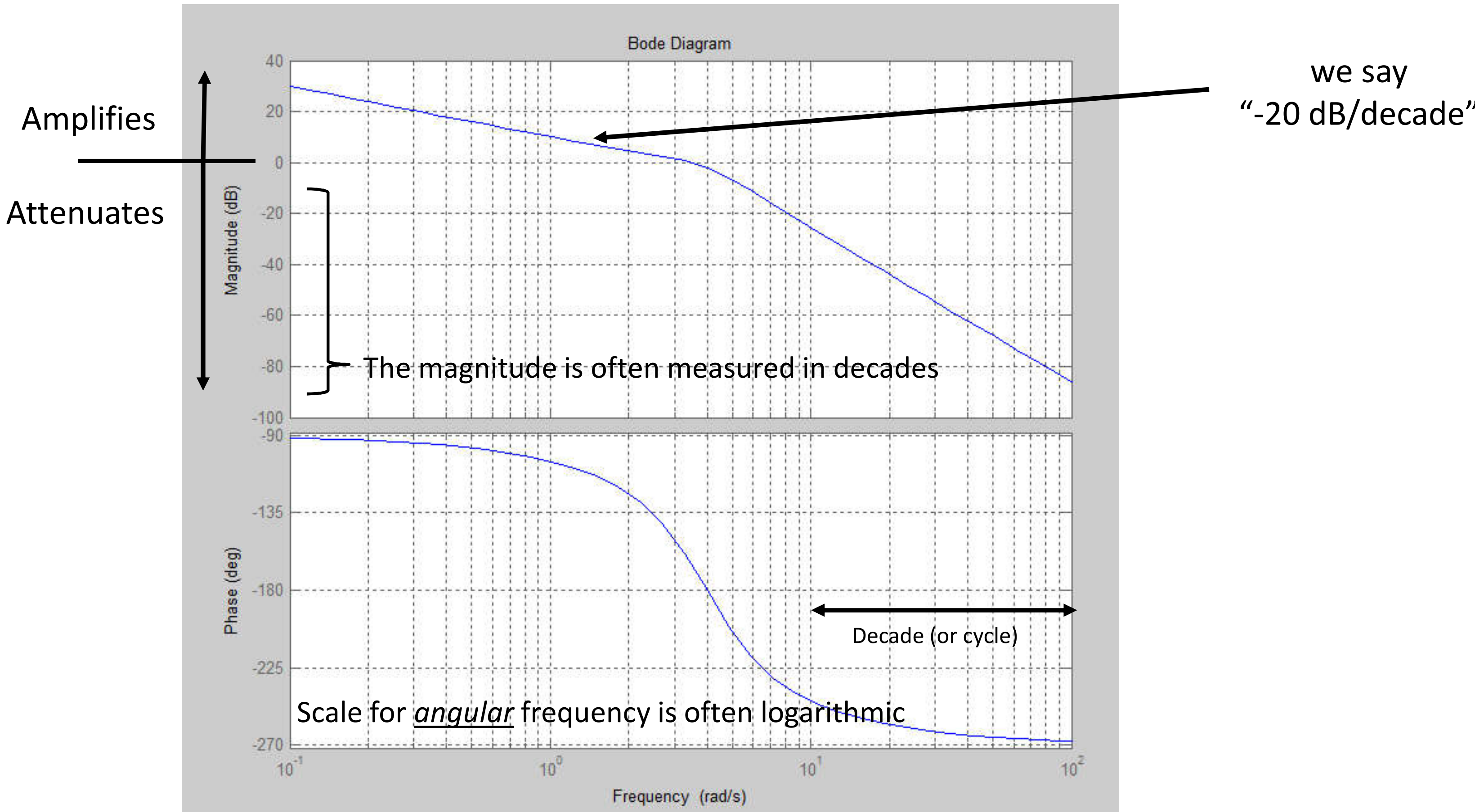


$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$





Reading a Bode Plot



Decibel Scale

- The **decibel** (**dB**) is a logarithmic unit for the ratio of a physical quantity relative to a reference level (such as an input).
  - *IEEE Standard 100 Dictionary of IEEE Standards Terms, Seventh Edition*, The Institute of Electrical and Electronics Engineering, New York, 2000; ISBN 0-7381-2601-2;
- Some common conversions
  - 20 $\text{Log}_{10}(1)=0$ dB (no amplification or attenuation)
  - 20 $\text{Log}_{10}(100)=40$ decibels (amplified 100 times of original value)
  - 20 $\text{Log}_{10}(0.01)=-40$ decibels (attenuated to 1/100 of original value)
- Why $20\log_{10}$?
  - First 10 $\log_{10}$ allows us to express a large range of inputs in a moderately sized and intuitive plot.
  - i.e. if x is 10 times x_{input}, we say "x is 10 dB greater than the input."
  - The 20 is an artifact from determining the power developed by a circuit with a constant resistance.

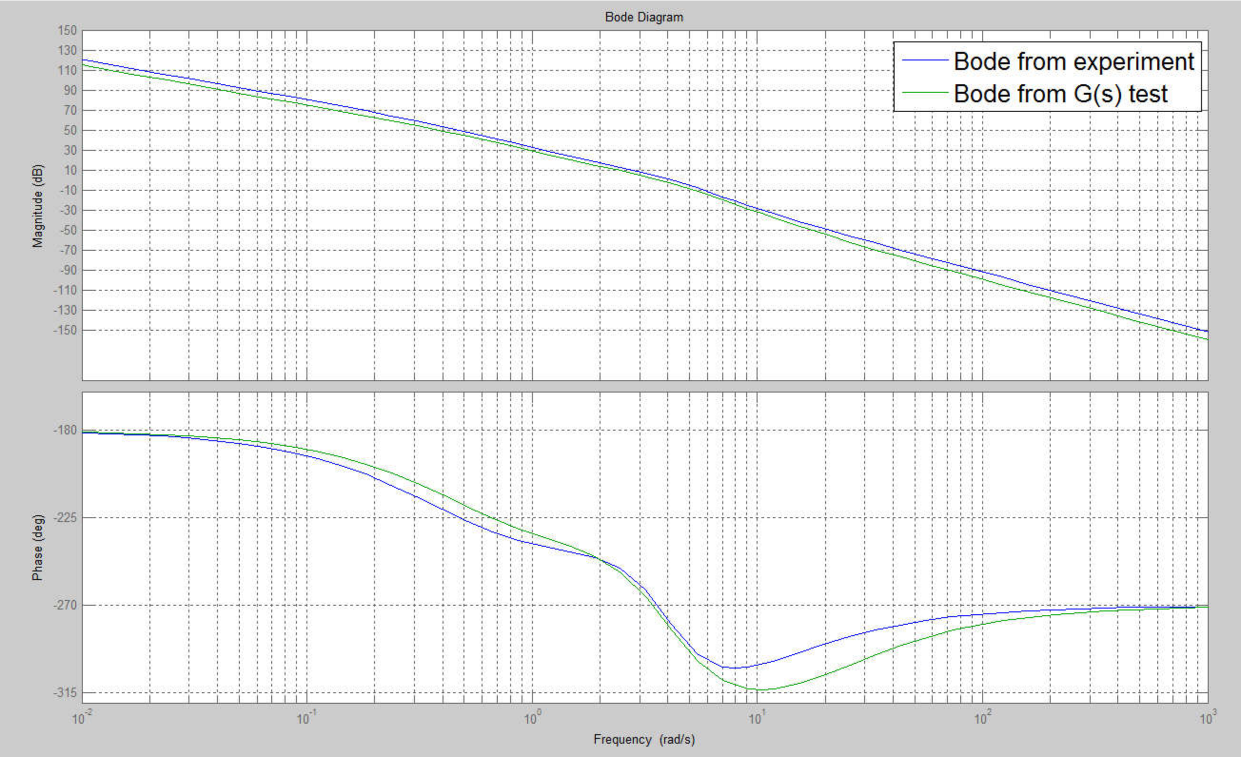
$$V_{dB} = 10\log_{10}\left(\frac{V_{out}^2}{V_{in}^2}\right) = 20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$

Example Bode plot

$$G(s)_{test} = 10.63\frac{(s+2.2)(s+20)}{s^2(s+0.52)(s^2+4.8s+16)}$$

$$G(s)_{actual} = 25\frac{(s+3)(s+9)}{s^2(s+0.4)(s^2+4s+16)}$$

Note the similar structure for similar transfer functions, but dB scale can subtly deemphasize performance differences.



<u>Finding the Bode Plot (3 Options)</u>

- MATLAB
- Using superposition
- Using tabulation (not covered)

1) Bode by MATLAB

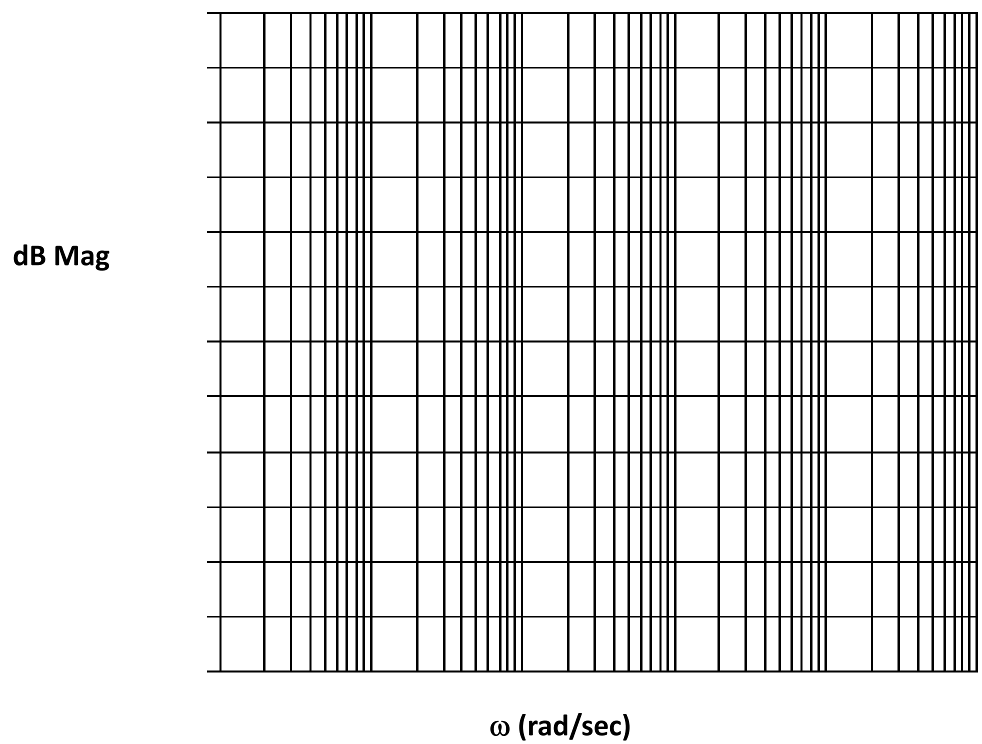
```
>> bode(sys)
```

2) Creating Bode plots by hand

- Better understand how each pole impacts the frequency response
- Create plots (magnitude and phase) separately
  - Both have logarithmic scales on the x-axis
  - The y-axis of the magnitude (M) of G(s) is in dB $Decibels = Db = 20\log_{10} M$
  - The y-axis of the phase angle plot is in degrees
- First, find the Transfer function for the system G(s) and then follow the procedure illustrated in the following example.

~~Creating~~ Understanding Bode Plots

$$G(s) = \frac{(s+8)(s+14)}{s(s+4)(s+10)}$$



~~Creating~~ Understanding Bode Plots

Start with a Transfer function and find the roots

$$G(s) = \frac{(s+8)(s+14)}{s(s+4)(s+10)}$$

Rewrite the system in *standard form.*

$$G(s) = \frac{2.8(\frac{s}{8}+1)(\frac{s}{14}+1)}{s(\frac{s}{4}+1)(\frac{s}{10}+1)}$$

Only 5 Options if.... $s = j\omega$

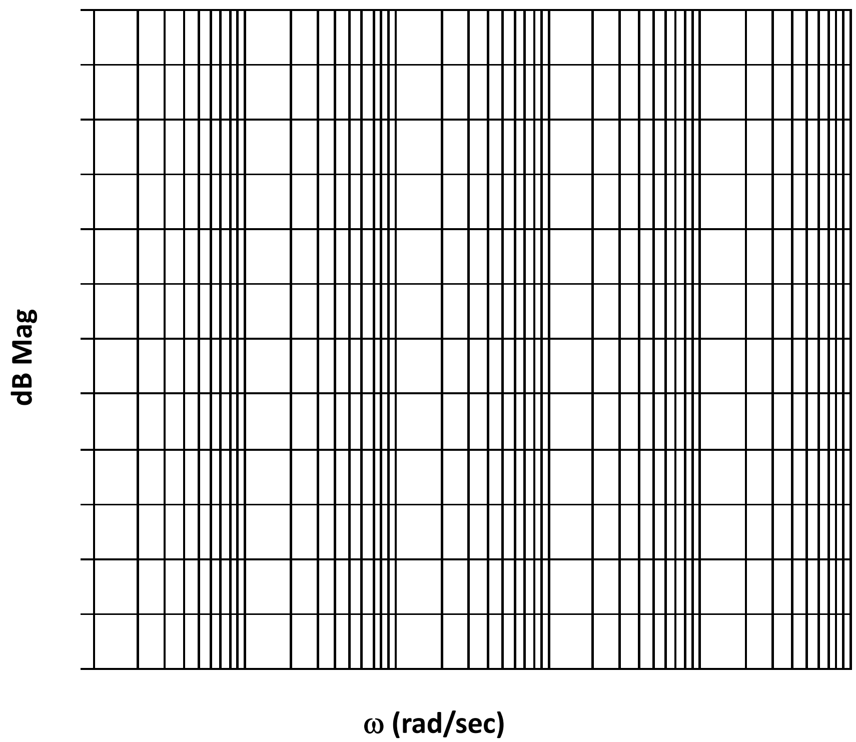
$$\tilde{K}_B = 20\log K_B$$

$$\frac{1}{s} = -20\log\left|j\omega\right|$$

$$s = 20\log\left|j\omega\right|$$

$$(\frac{s}{z}+1) = 20\log\left|\frac{j\omega}{z}+1\right|$$

$$\frac{1}{(\frac{s}{p}+1)} = -20\log\left|\frac{j\omega}{p}+1\right|$$



~~Creating~~ Understanding Bode Plots

$$G(s) = \frac{2.8(\frac{s}{8}+1)(\frac{s}{14}+1)}{s(\frac{s}{4}+1)(\frac{s}{10}+1)}$$

Option 1….

$$\tilde{K}_B = 20\log K_B$$

Just a straight line.

In our example… $\tilde{K}_B = 20\log 2.8 \approx 8.94$

Option 2….

$$\frac{1}{s} = -20\log|j\omega|$$

Line sloping at -20db/decade

Magnitude of 0 at ω=1

In our example…

Just one pole due to root at zero.

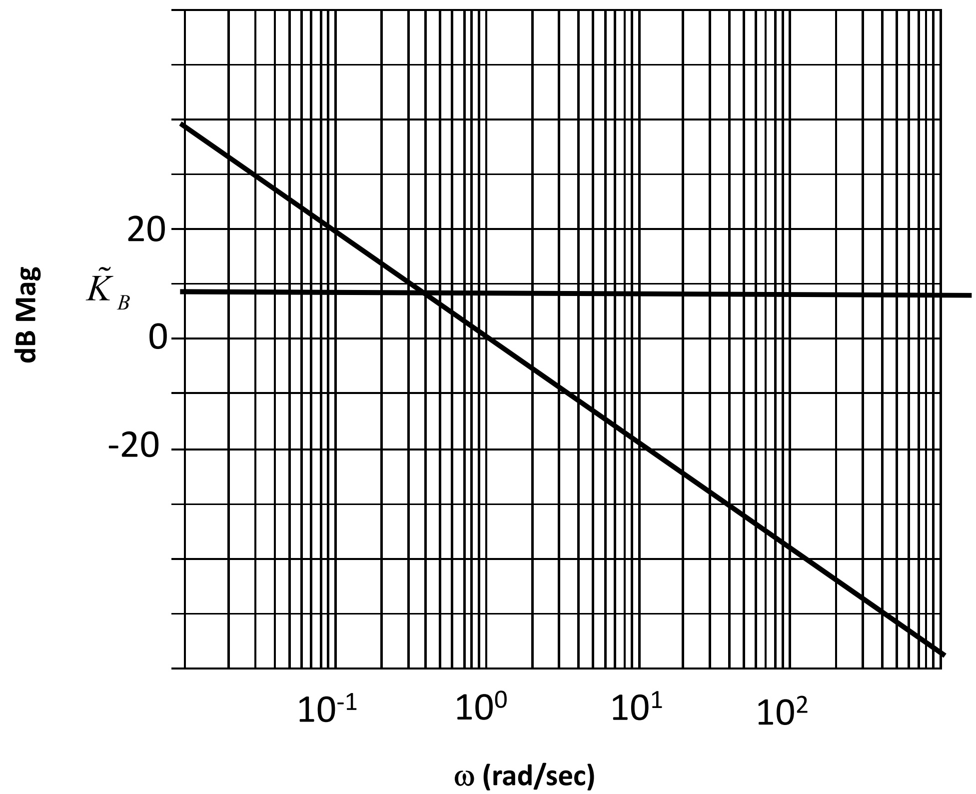
Option 3….

$$s = 20\log|j\omega|$$

Line sloping at 20db/decade

Magnitude of 0 at ω=1

Not Applicable in our example.



~~Creating~~ Understanding Bode Plots

$$G(s) = \frac{2.8(\frac{s}{8}+1)(\frac{s}{14}+1)}{s(\frac{s}{4}+1)(\frac{s}{10}+1)}$$

Option 4….

$$\frac{1}{(\frac{s}{p}+1)} = -20\log\left|\frac{j\omega}{p}+1\right|$$

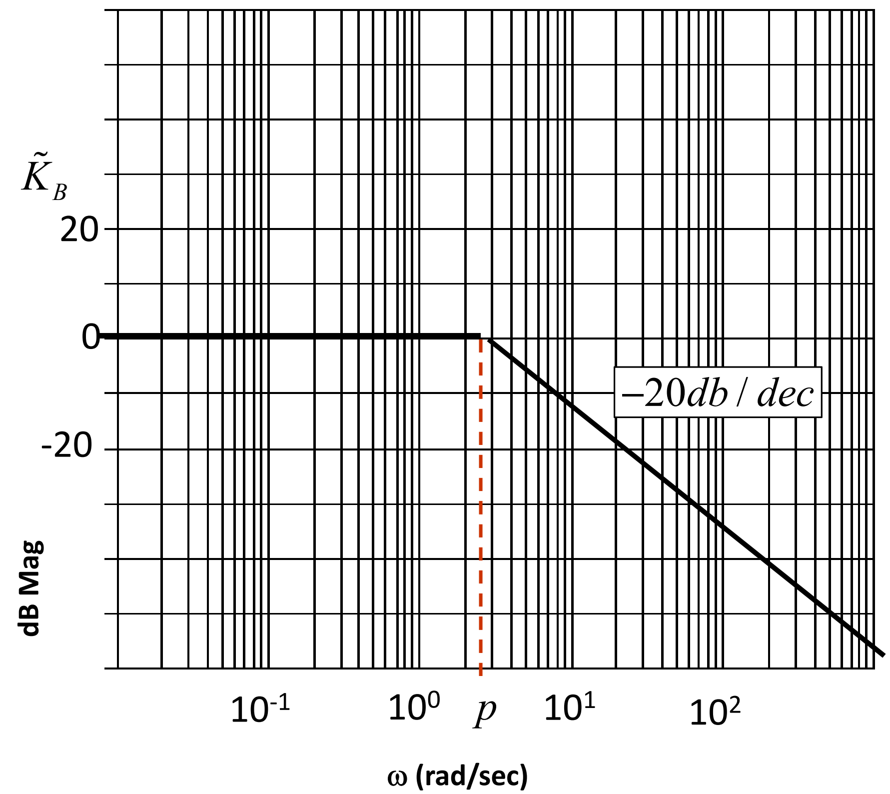
An approximation.

The error is about

- -3dB at ω=p
- -1dB at ω=p/2
- -1dB at ω=2p

In our example…

Two poles which would break at 4 and 10.



~~Creating~~ Understanding Bode Plots

$$G(s) = \frac{2.8(\frac{s}{8}+1)(\frac{s}{14}+1)}{s(\frac{s}{4}+1)(\frac{s}{10}+1)}$$

Option 5….

$$(\frac{s}{z}+1) = 20\log\left|\frac{j\omega}{z}+1\right|$$

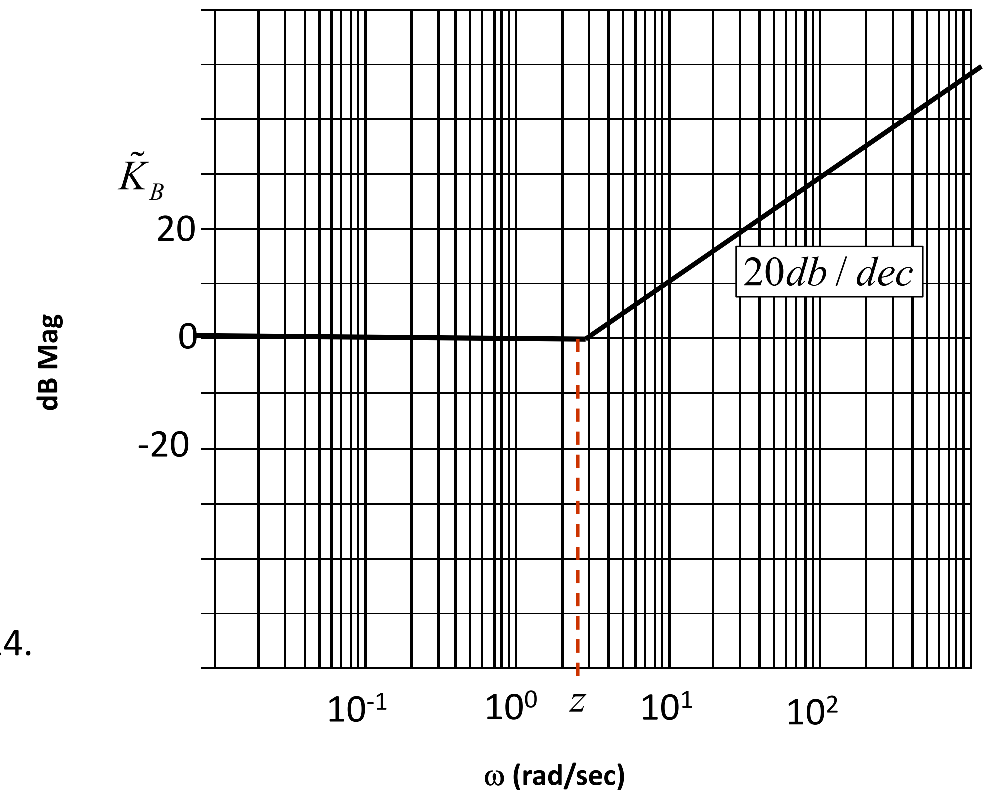
An approximation.

The error is about

- 3dB at ω=p
- 1dB at ω=p/2
- 1dB at ω=2p

In our example…

Two zeros which would break at 8 and 14.



~~Creating~~ Understanding Bode Plots

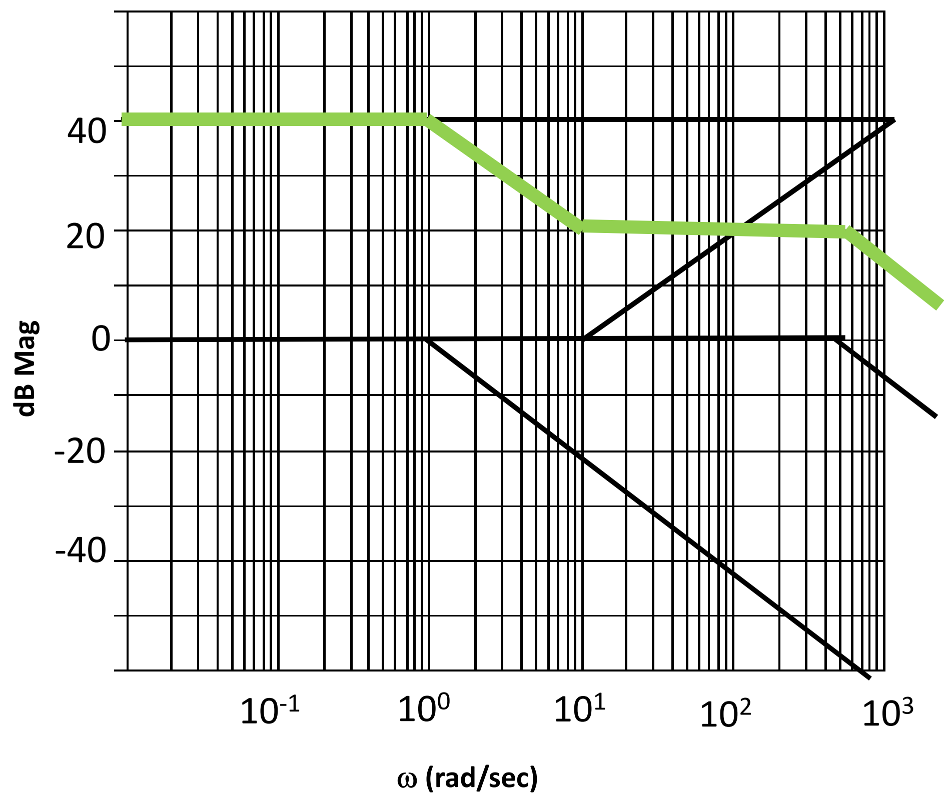
Putting it all together
- Linear system!
- Principle of Superposition!

Example

$$G(j\omega) = \frac{100(\frac{jw}{10}+1)}{(\frac{jw}{1}+1)(\frac{jw}{500}+1)}$$

Note….

$$\tilde{K}_B = 20\log_{10}(100) = 40$$

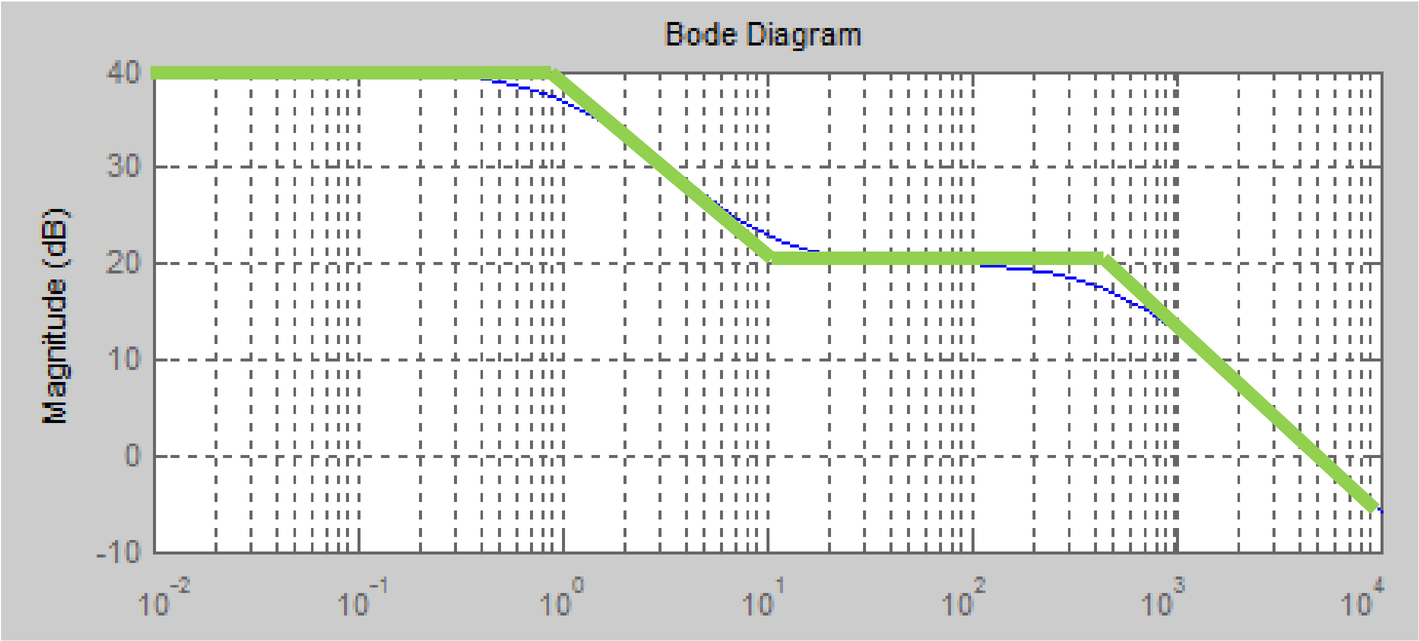


The Secret
- Sketch them in order
- Look at when each pole/zero breaks

~~Creating~~ Understanding Bode Plots

$$G(j\omega) = \frac{100(\frac{jw}{10}+1)}{(\frac{jw}{1}+1)(\frac{jw}{500}+1)}$$

```
a = [ 5000 50000 ];
b = [ 1 501 500];
sys = tf( a, b );
bode( sys )
```



Break point for 2nd order elements

- *If the system is 2nd order (or the system has complex conjugate poles)*

$$20Log_{10}\left|(i\omega)^2+2\zeta\omega_n(i\omega)+\omega_n^{\ 2}\right|=20Log_{10}\sqrt{\left(\omega_n^{\ 2}-\omega^2\right)^2+4\zeta^2\omega_n^{\ 2}\omega^2}$$

- ω_n is also known as the *break frequency*

- For frequencies of less than ω_n rad/sec, this is plotted as a horizontal line at the level of $40Log_{10}\,\omega_n$,

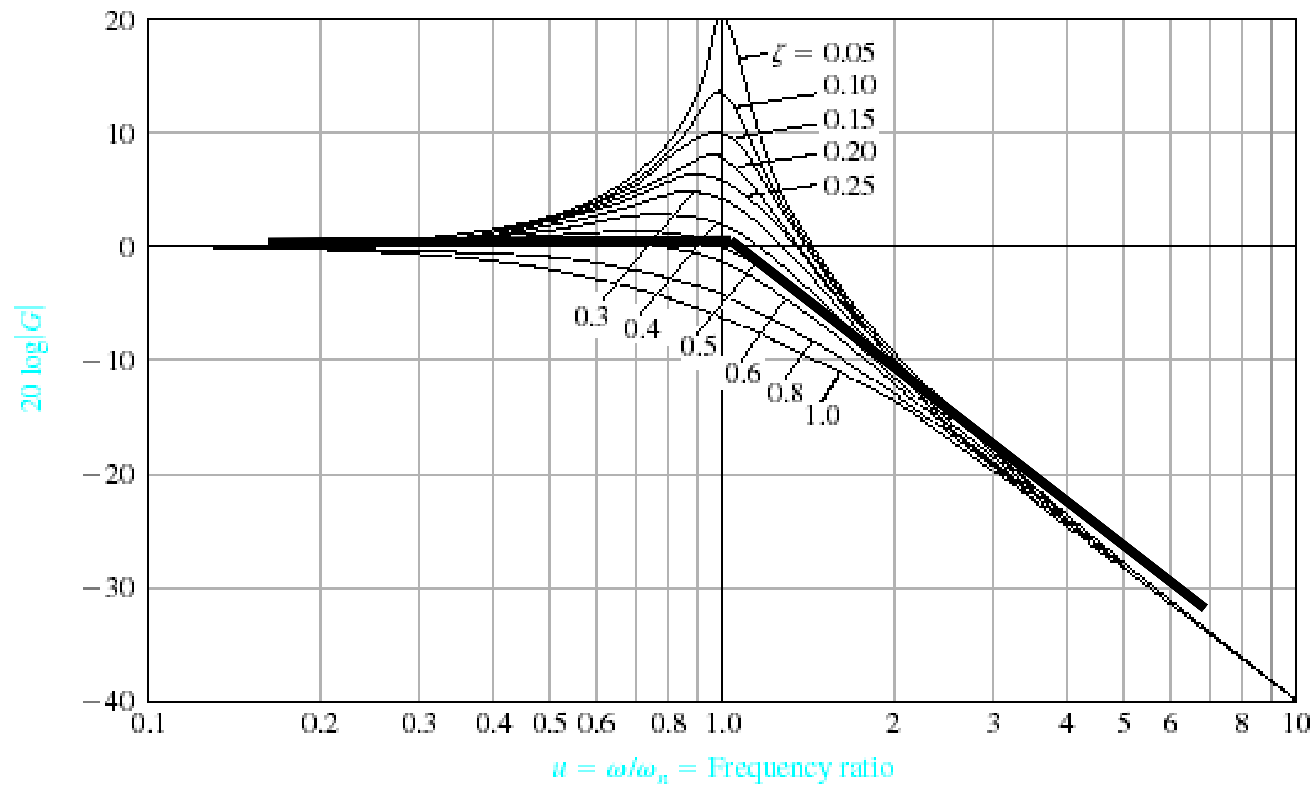
$$\omega \ll \omega_n \quad 20Log_{10}\sqrt{\left(\omega_n^{\ 2}-\omega^2\right)^2+4\zeta^2\omega_n^{\ 2}\omega^2}\approx 20Log_{10}\omega_n^{\ 2}=40Log_{10}\omega_n=\text{constant}$$

- For frequencies greater than ω_n rad/sec, this is plotted as a line with a slope of ± 40 dB/decade, the sign determined by position in G(s)

$$\omega \gg \omega_n \quad 20Log_{10}\sqrt{\left(\omega_n^{\ 2}-\omega^2\right)^2+4\zeta^2\omega_n^{\ 2}\omega^2}\approx 20Log_{10}\omega^2=40Log_{10}\omega$$

~~Creating~~ Understanding Bode Plots

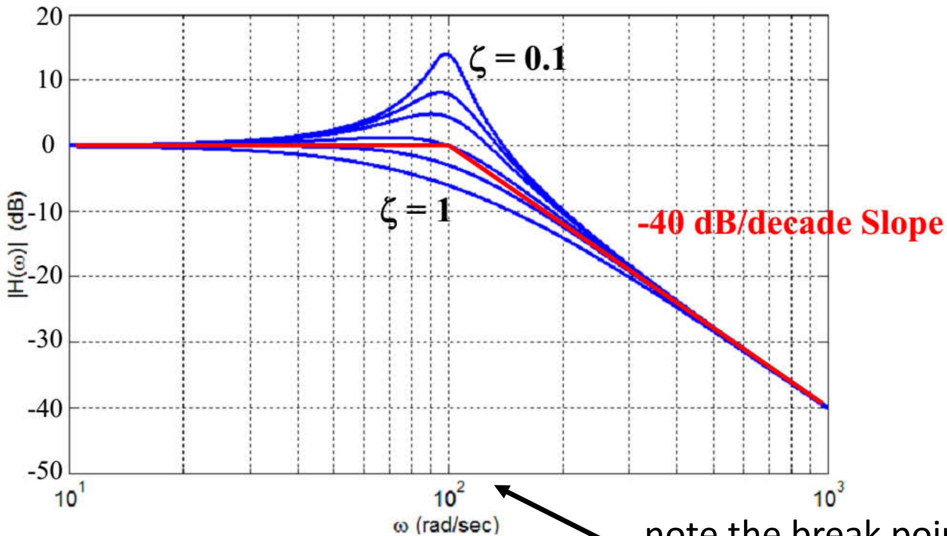
Canonical 2nd order system (including imaginary poles) $T(j\omega) = \dfrac{1}{(j\omega)^2 + (2\zeta\omega_n)j\omega + \omega_n^2}$

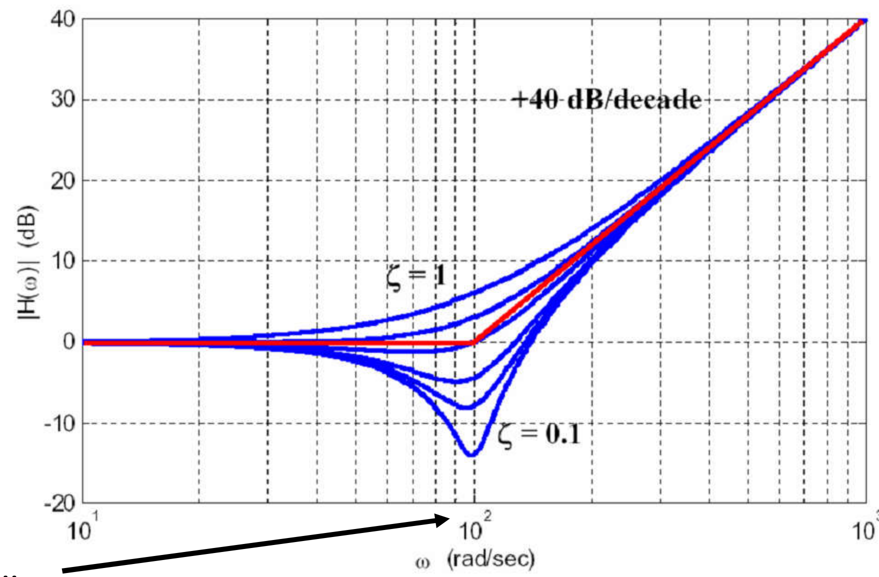


Remember all imaginary poles and/or zeroes will come in complex conjugate pairs! Thus it is simplest to leave these pairs as quadratics when finding the bode plot!

~~Creating~~ Understanding Bode Plots

For example, consider a Canonical 2nd order system with ω_o = 100 (both as poles or zeros).



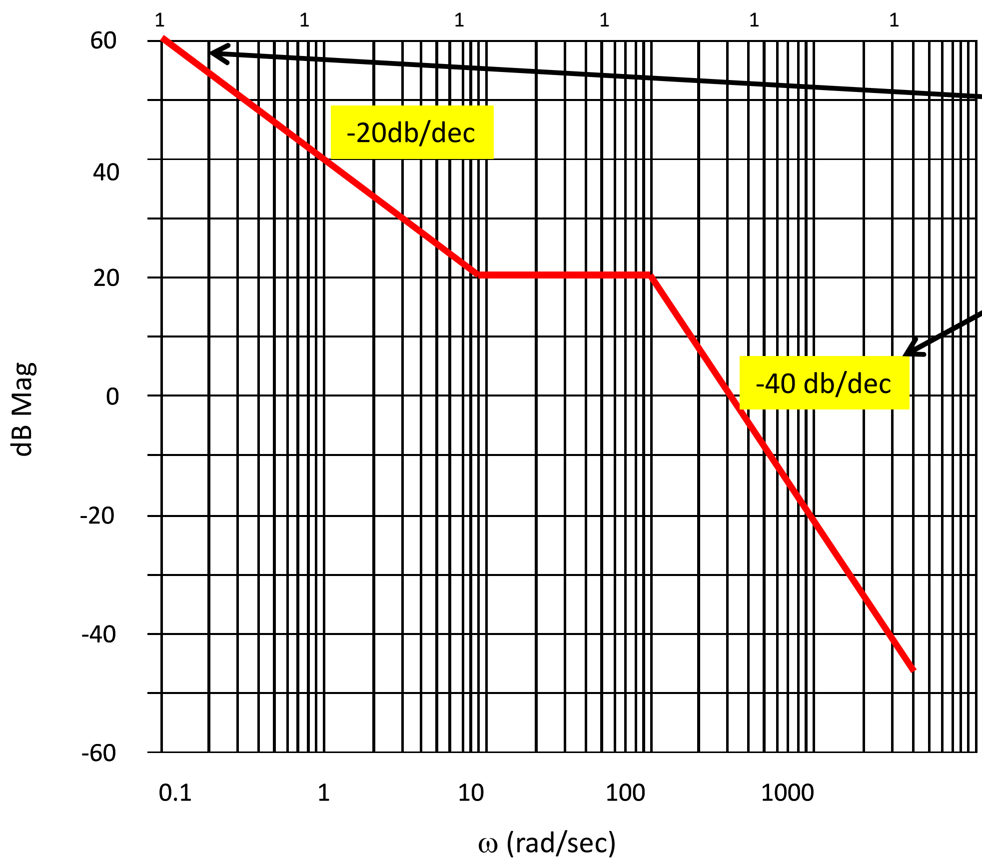


note the break point…

Use these “resonant corrections” when the damping coefficient is < 0.5.

ζ value	Adjustment
0.1	14 dB
0.2	8 dB
0.3	5 dB
0.4	3 dB
0.5	1 dB

Another example (duplicate poles)



$$G(s) = \frac{100(1+s/10)}{s(1+s/100)^2}$$

Phase (deg)

Finding the phase plot

$$G(j\omega)=\frac{100(\frac{jw}{10}+1)}{(\frac{jw}{1}+1)(\frac{jw}{500}+1)}$$

Finding the Phase

- Superposition of the angle w.r.t to each zero and pole.
- For this example

$$\angle G(s)=\phi=\tan^{-1}\left(\frac{\mathrm{Im}\big(G(i\omega)\big)}{\mathrm{Re}\big(G(i\omega)\big)}\right)$$

$$\angle G(jw)=\tan^{-1}(\tfrac{\omega}{10})-\tan^{-1}(\tfrac{\omega}{1})-\tan^{-1}(\tfrac{\omega}{500})$$

calculate select points "by hand"

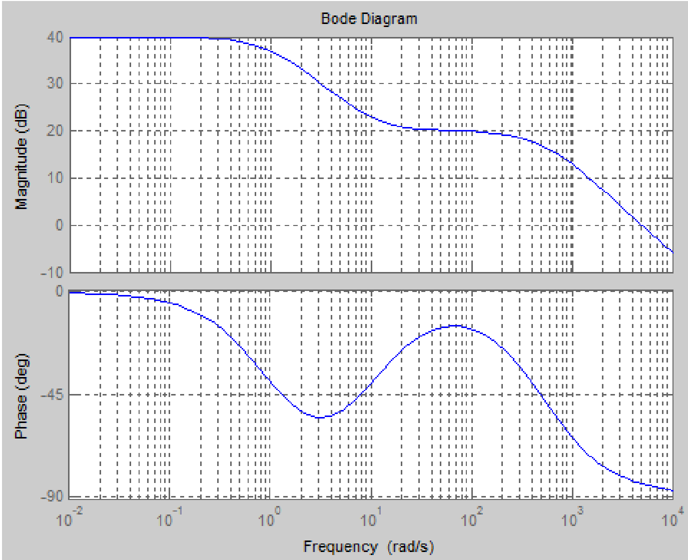
$$\angle G(0)=\tan^{-1}(0/10)-\tan^{-1}(0/1)-\tan^{-1}(0/500)=0$$

$$\angle G(\infty)=\tan^{-1}(\infty/10)-\tan^{-1}(\infty/1)-\tan^{-1}(\infty/500)=90-90-90=-90$$

$$\angle G(1)=\tan^{-1}(1/10)-\tan^{-1}(1/1)-\tan^{-1}(1/500)\approx 0-45-0=-45$$

$$\angle G(10)=\tan^{-1}(10/10)-\tan^{-1}(10/1)-\tan^{-1}(10/500)\approx 45-85-0=-40$$

$$\angle G(500)=\tan^{-1}(500/10)-\tan^{-1}(500/1)-\tan^{-1}(500/500)\approx 90-90-45=-45$$



Note:

$$\tan^{-1}(0)=0$$

$$\tan^{-1}(1)=45$$

$$\tan^{-1}(10)\approx 85$$

$$\tan^{-1}(100)\approx 89.5$$

$$\tan^{-1}(1000)\approx 89.9$$

Finding the phase plot

$$G(j\omega) = \frac{100(\frac{jw}{10}+1)}{(\frac{jw}{1}+1)(\frac{jw}{500}+1)}$$

Finding the Phase

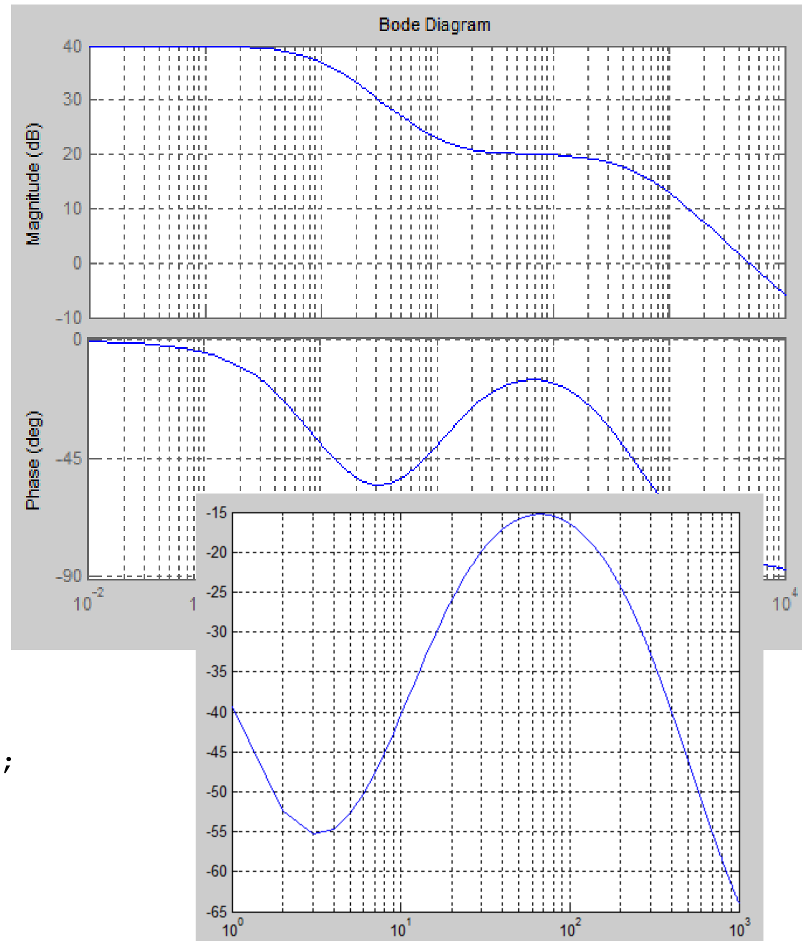
- Superposition of the angle w.r.t to each zero and pole.
- For this example

$$\angle G(s) = \phi = \tan^{-1}\left(\frac{\mathrm{Im}\left(G\left(i\omega\right)\right)}{\mathrm{Re}\left(G\left(i\omega\right)\right)}\right)$$

$$\angle G(jw) = \tan^{-1}(\tfrac{\omega}{10}) - \tan^{-1}(\tfrac{\omega}{1}) - \tan^{-1}(\tfrac{\omega}{500})$$

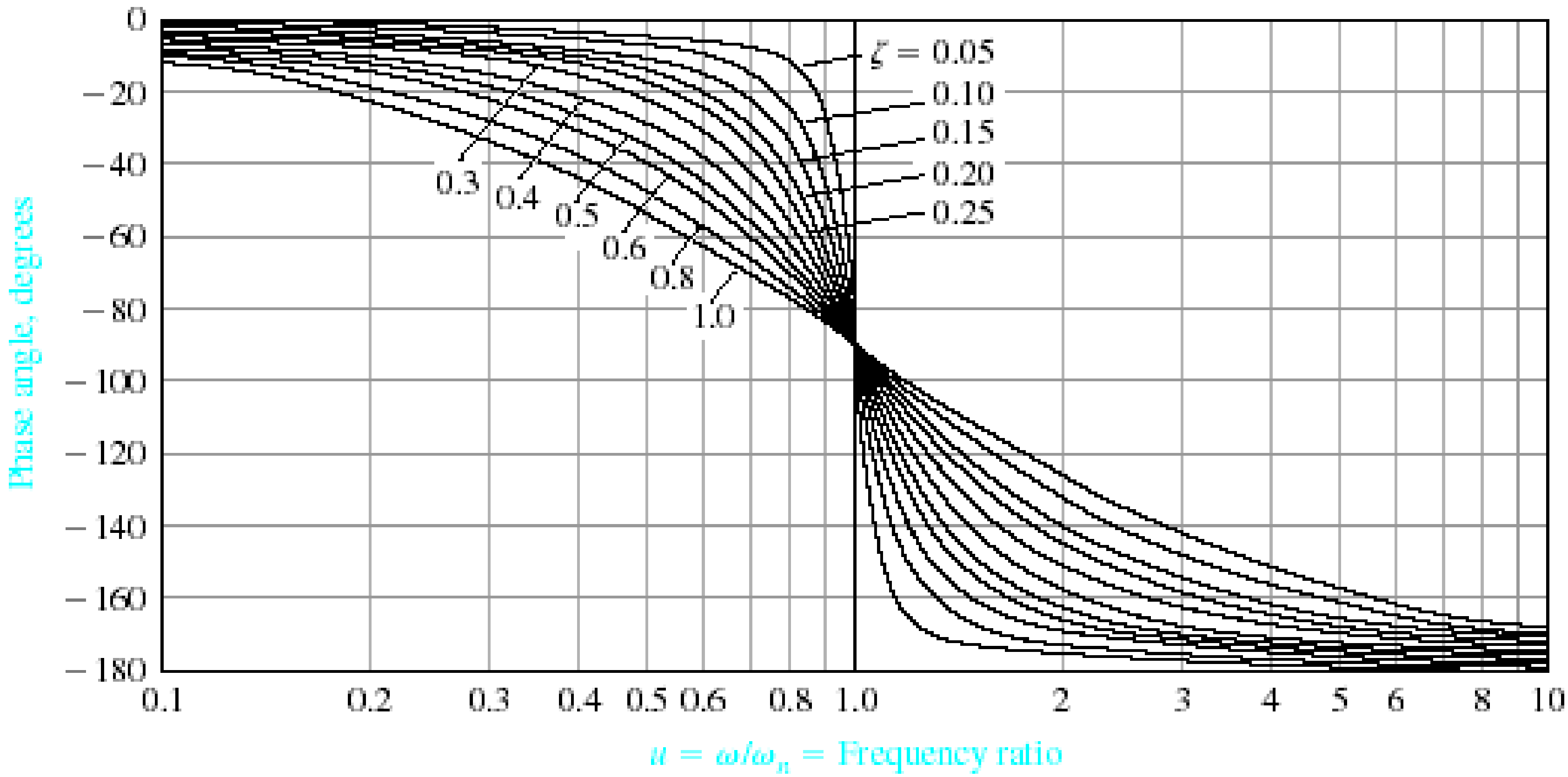
Using MATLAB (but not Bode)

```
w = [1: 1: 1000];
for i=1:length(w)
    p=atan(w/10)-atan(w/1)-atan(w/500);
end

semilogx( w, p*180/pi() )
grid on;
```



2^{nd} order system phase angle

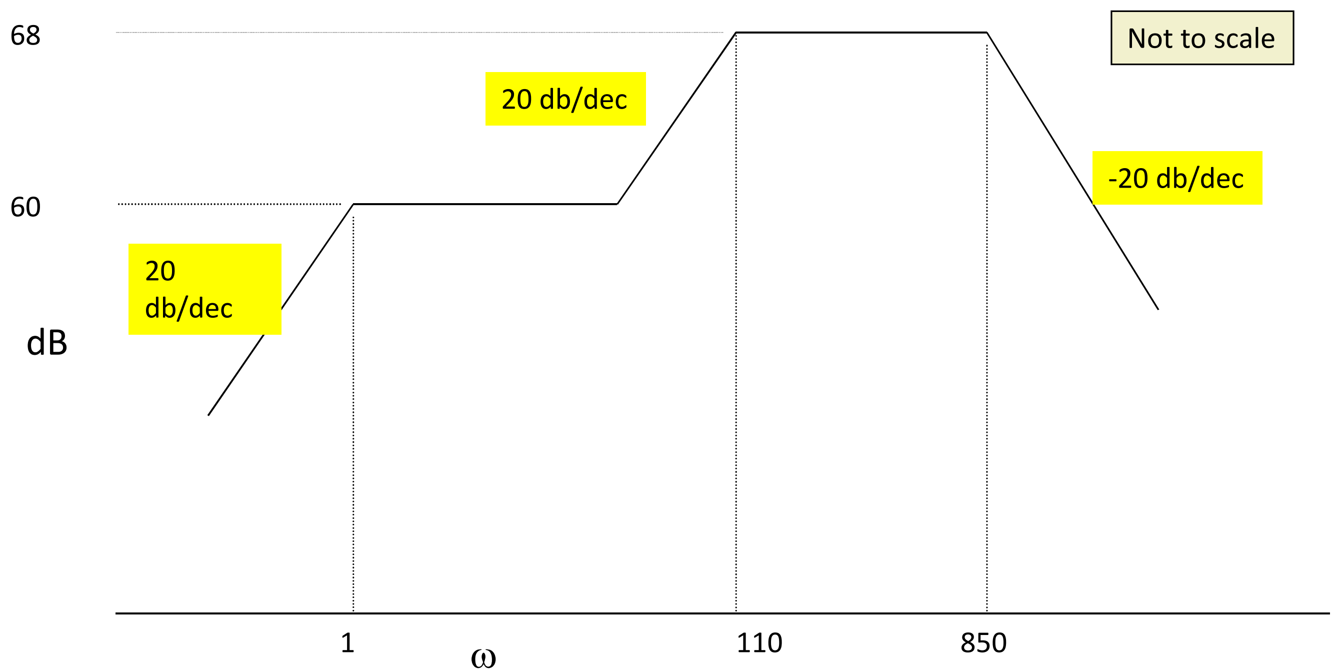
Canonical 2^{nd} order system: $$T(j\omega) = \frac{1}{(j\omega)^2 + (2\zeta\omega_n)\, j\omega + \omega_n^2}$$



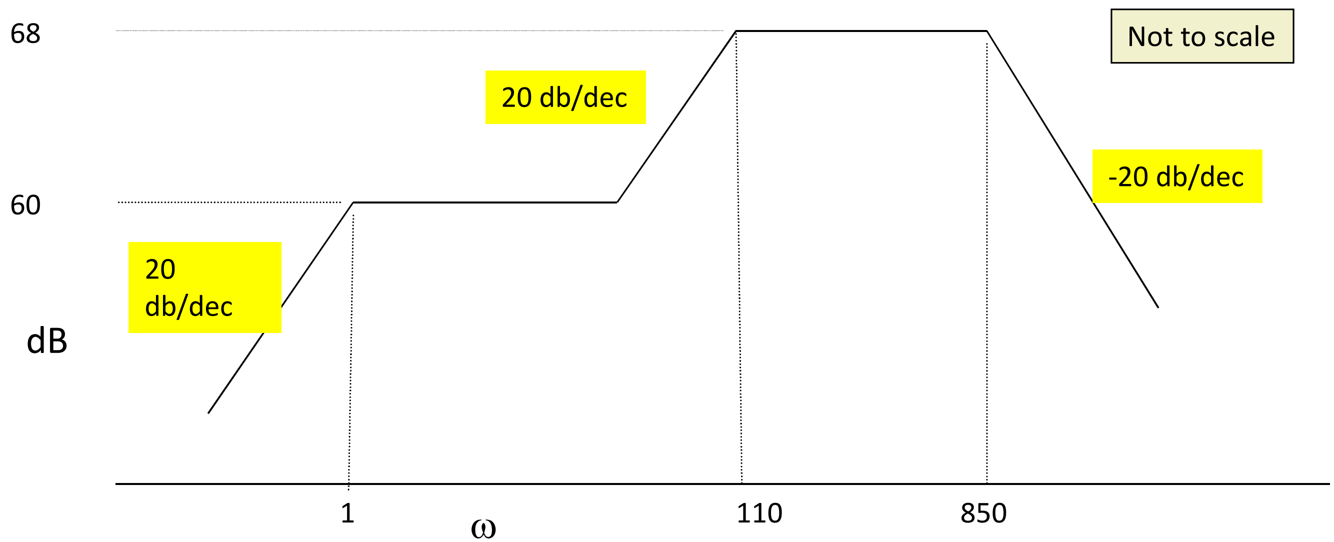
~~Creating~~ Understanding Bode Plots

If we know the Bode Plot (say by experimentation)

- Now we can model the transfer function of a system even if we don’t have the model!

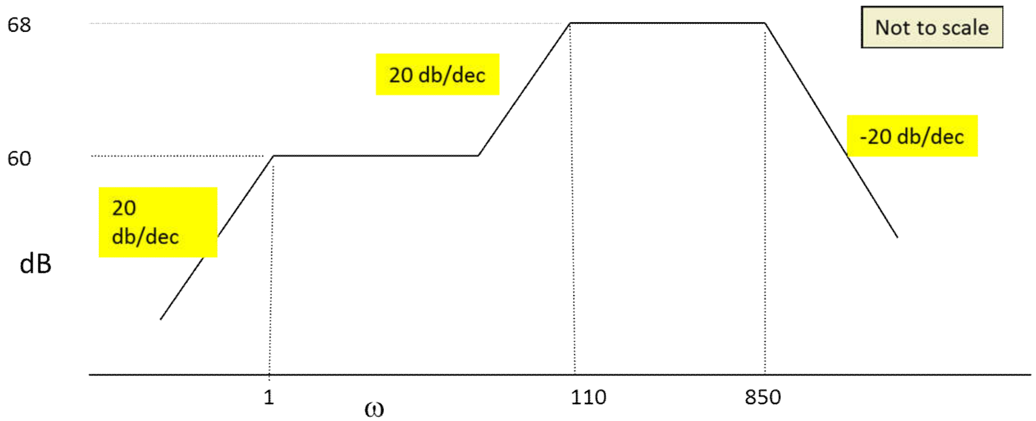


~~Creating~~ Understanding Bode Plots



- Zero at the origin, pole at ω=1, zero at 1<z<110, pole at 110, and pole at 850
- We know at the zero at 1 should pass through 0 dB. Thus the zero frequency gain is $20\log_{10}(K)=60$.
  - Thus the gain of the system *in standard form* is $10^3=1000$.

~~Creating~~ Understanding Bode Plots



- To find the zero that breaks between 1 and 110, we recognize the semi-log linear relationship:

$$slope = \frac{rise}{run}$$

$$20 = \frac{68 - 60}{\log_{10}(110) - \log_{10} z}$$

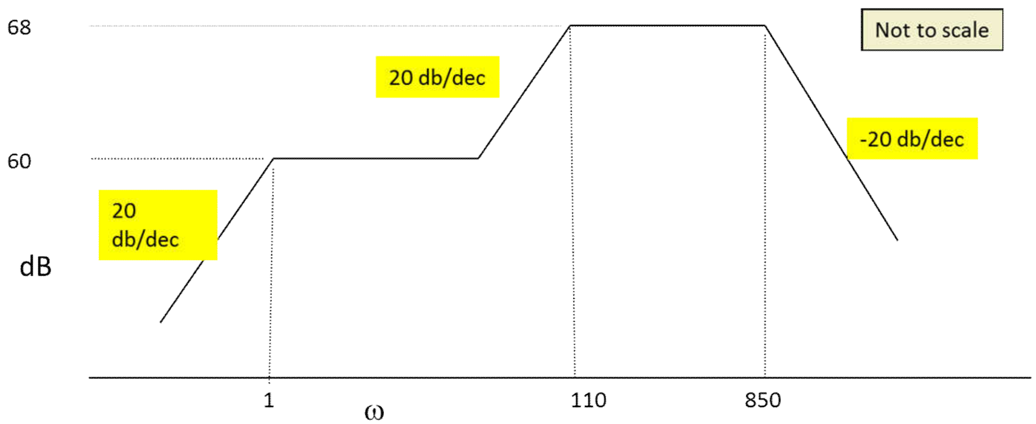
$$-20\log_{10} z = 8 - 20\log_{10}(110)$$

$$\log_{10} z = \frac{8 - 40.82}{-20}$$

$$z = 10^{1.6414}$$

$$z = 43.7920$$

~~Creating~~ Understanding Bode Plots



With that we determine the Transfer function for the system.

$$T(s) = \frac{1000s\left(\dfrac{s}{43.8}+1\right)}{\left(\dfrac{s}{1}+1\right)\left(\dfrac{s}{110}+1\right)\left(\dfrac{s}{850}+1\right)}$$

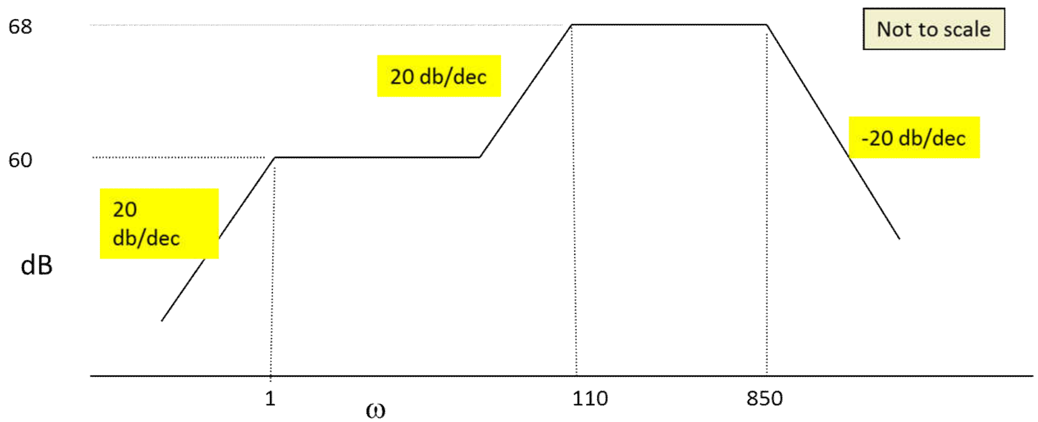
$$= \frac{\dfrac{1000(110)(850)}{43.8}s(s+43.8)}{(s+1)(s+110)(s+850)} = \frac{2134700s(s+43.8)}{(s+1)(s+110)(s+850)}$$

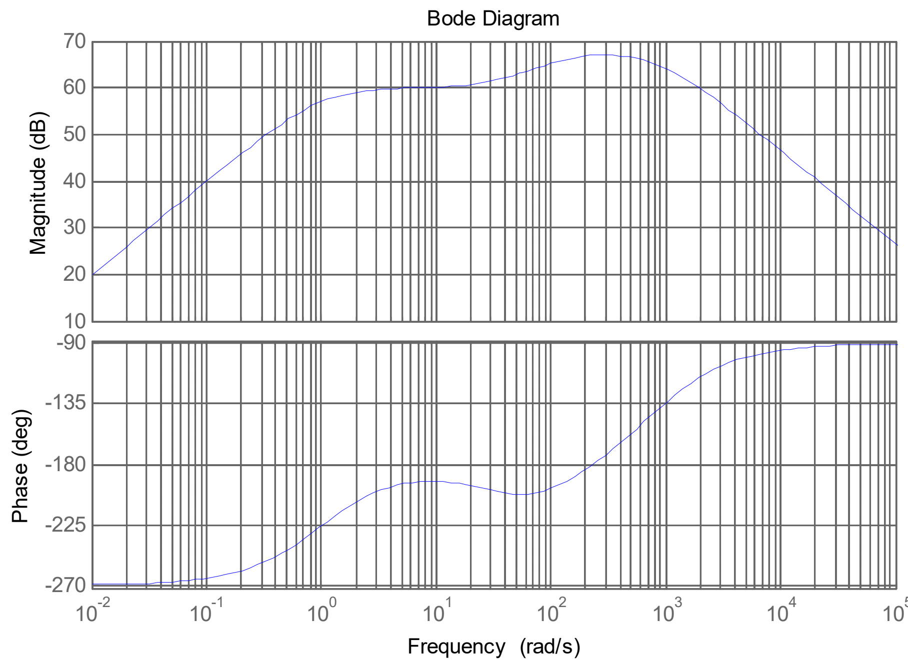
Did we get the right answer?

~~Creating~~ Understanding Bode Plots

$$T(s) = \frac{2134700s(s+43.8)}{(s+1)(s+110)(s+850)}$$

```
zt = 43.8;
z = [ 0 zt ];
p = [ 1 110 850 ];
k = 1000*110*850*(1/zt);
sys = zpk( z, p, k );

bode( sys )
grid
```





Graphical Representation Summary

- Multiple ways to visualize frequency response graphically
  - Succinctly represent gain and phase over a wide range of frequencies
- Polar (Nyquist) plots
  - Harry Nyquist (1889-1976)
  - Easy to plot for simple systems and using MATLAB's `nyquist(sys)`
  - Will be useful to discuss system stability.
  - Can get complicated for complex systems.
- It is possible to find the Transfer function given a Bode plot
  - Recognize that the bode plot *may* be written on semi-log paper.
- We can also find the transfer function from an experimentally generated transfer function