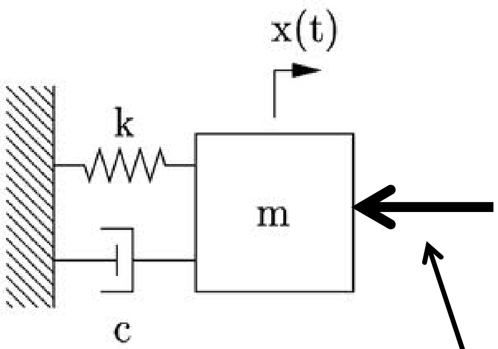
Intro to Frequency Analysis
Transfer Function Review

Dr. Mitch Pryor

Lesson Objective

- So far we have focused designing controllers in the time domain. This lesson will motivate why we also want to be able discuss controllers in the frequency domain.
- Understand the relationship between a system's state-space model and its transfer function
- Quick review of finding the transfer function via the Laplace Transform
- Solve and simulate systems given as a Transfer function T(s)
    - Quick review of partial fraction expansion
    - Simulating T(s) using MATLAB

Example: Mass – spring -damper



$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{z} + [0]\mathbf{u}$$

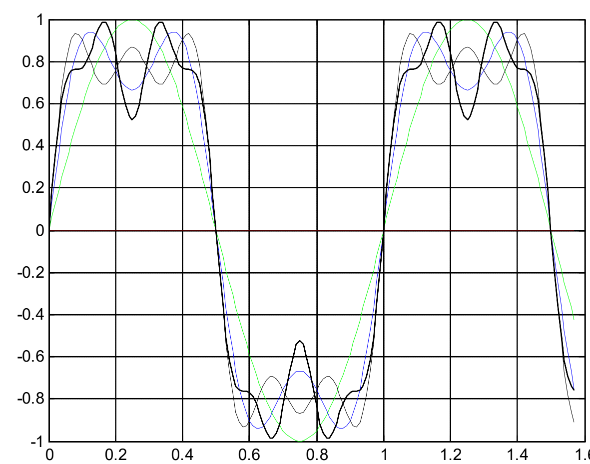
$$u(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t + \phi_k)$$

fundamental frequency of periodic input

note: no feedback!

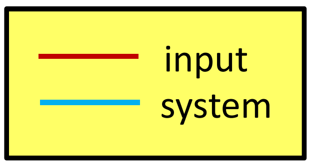
Example input: $u(t) = \sum_{k=0,2,4}^{\infty} 0 + \sum_{k=1,3,5}^{\infty} \frac{1}{k}\sin(k2t + 0)$

```
clear all;
t = [0.0:0.01:1.57];

for i=1:length(t)
    f1(i) = 0; %could use for offset
    f2(i) = sin(2*pi*t(i));
    f3(i) = (1/3)*sin(6*pi*t(i));
    f4(i) = (1/5)*sin(10*pi*t(i));
    f5(i) = (1/7)*sin(14*pi*t(i));
end

plot(t, f1,'r');
hold on; grid on;
plot(t, f1+f2, 'g');
plot(t, f1+f2+f3, 'b');
plot(t, f1+f2+f3+f4, 'k');
plot(t, f1+f2+f3+f5, 'k:');
```

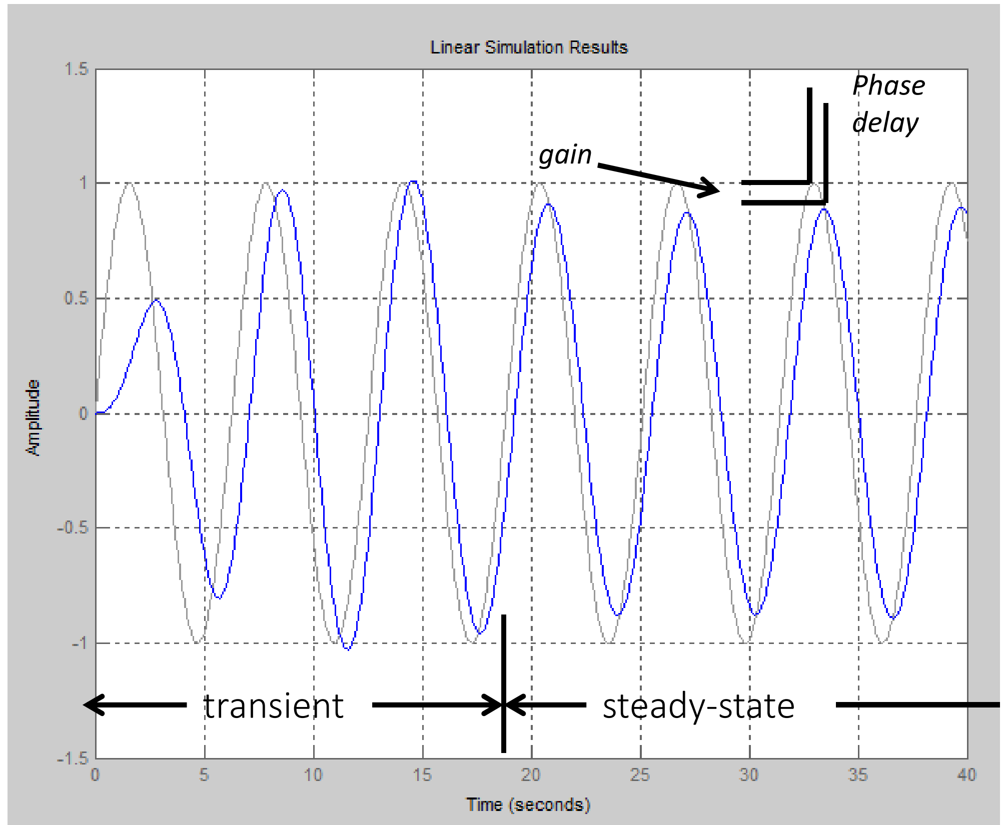


Example: Mass – spring -damper

$$u(t) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t + \phi_k)$$




```
clear all

m=2; c=.5; k=3;

A = [ 0 1; -(k/m) -(c/m)];
B = [ 0; 1/m ];
C = [ 1 0 ];
D = [ 0 ];

sys = ss( A, B, C, D );
t = 0:0.1:30;
u = 1.0*sin(1*t);
lsim(sys, u, t, [0 0 ]);
grid on;
```



Example: Mass – spring -damper

$$u(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t + \phi_k)$$

input
system

```
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B = [ 0; 1/m ];
C = [ 1 0 ];
D = [ 0 ];

sys = ss( A, B, C, D );
t = 0:0.1:30;
u = 1.0*sin(0.5*t);
lsim(sys, u, t, [0 0 ]);
grid on;
```
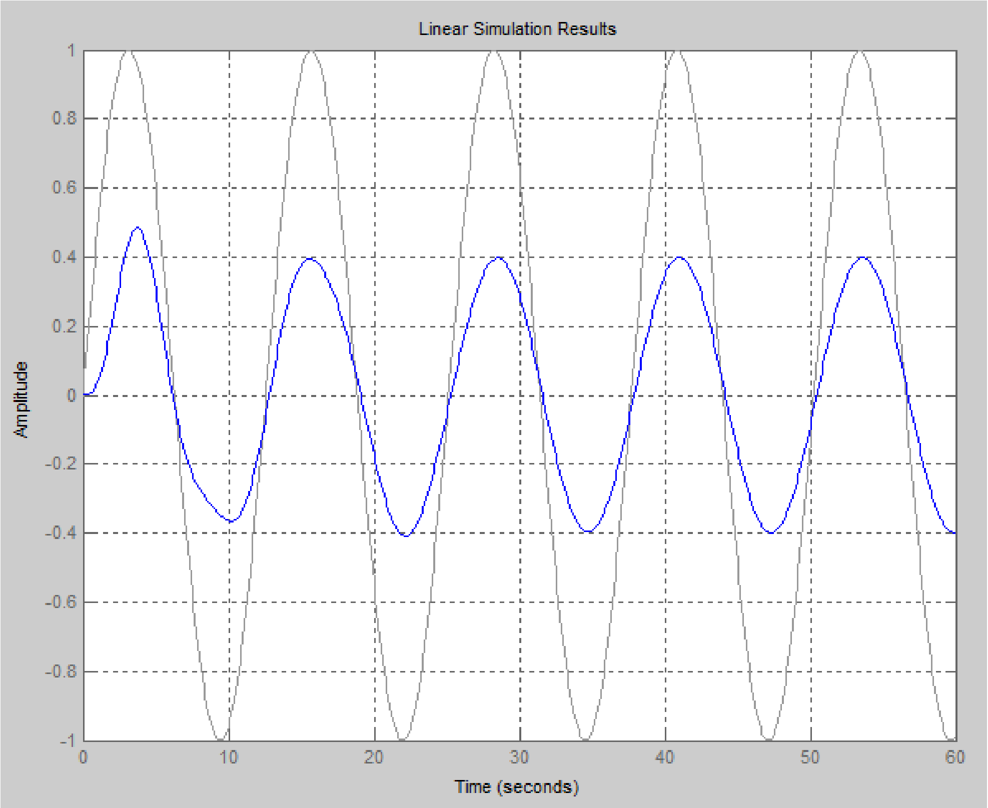
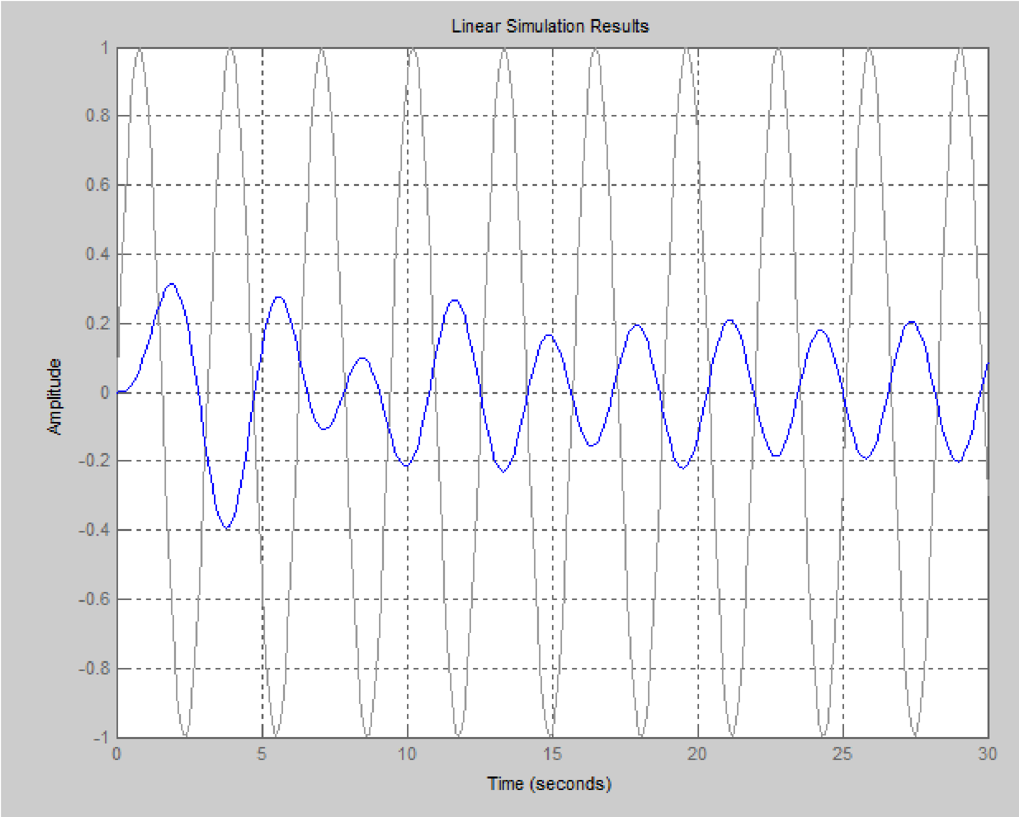



Example: Mass – spring -damper

$$u(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t + \phi_k)$$

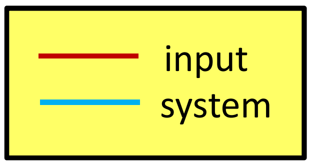
input
system

```
clear all

m=2; c=.5; k=3;

A = [ 0 1; -(k/m) -(c/m)];
B = [ 0; 1/m ];
C = [ 1 0 ];
D = [ 0 ];

sys = ss( A, B, C, D );
t = 0:0.1:30;
u = 1.0*sin(2*t);
lsim(sys, u, t, [0 0 ]);
grid on;
```

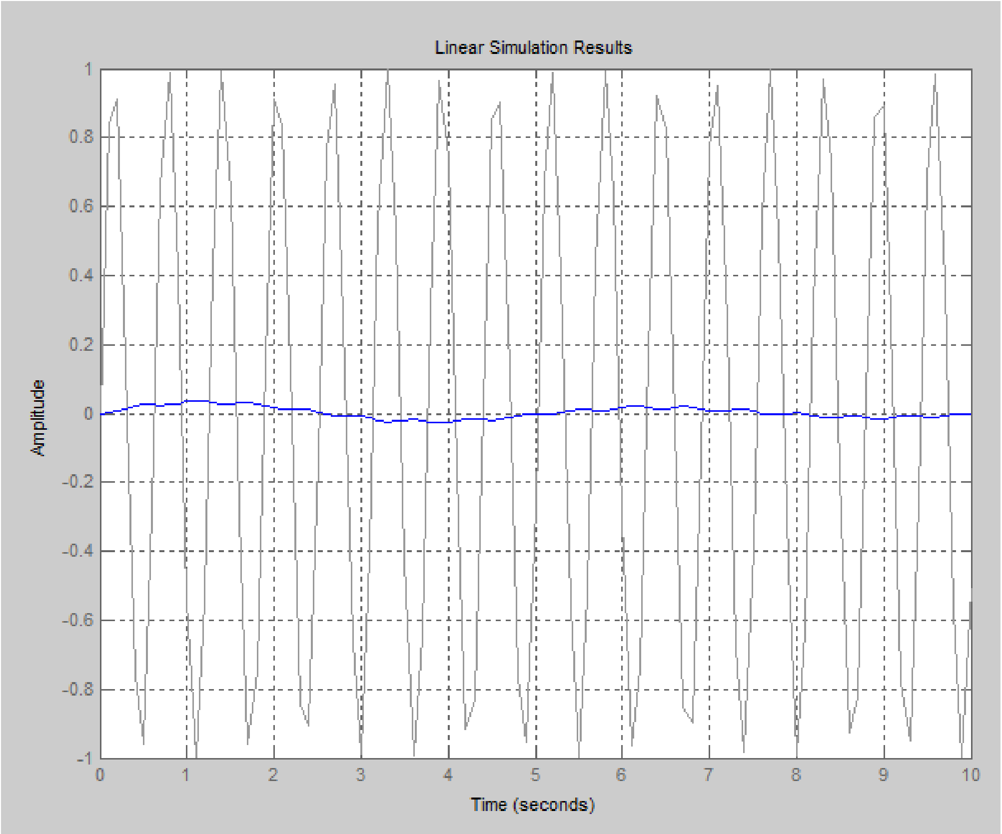


Example: Mass – spring -damper

$$u(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t + \phi_k)$$

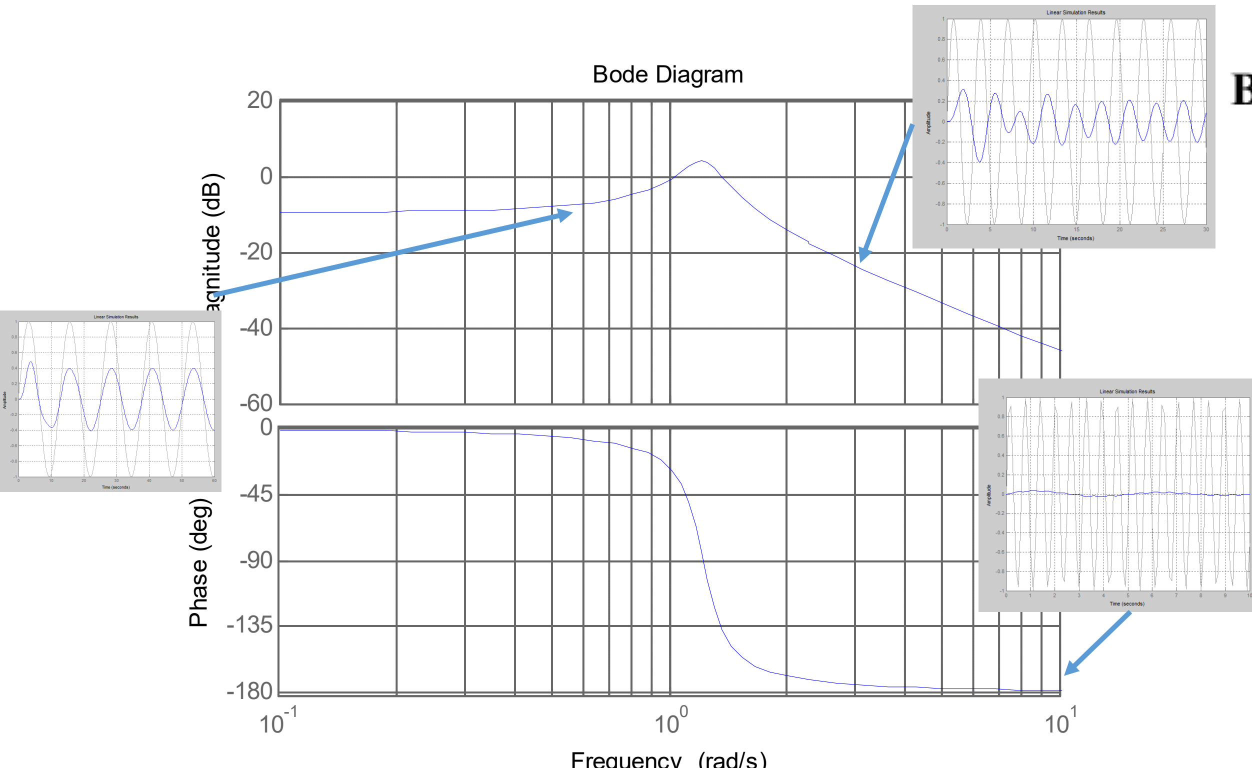



```
clear all

m=2; c=.5; k=3;

A = [ 0 1; -(k/m) -(c/m)];
B = [ 0; 1/m ];
C = [ 1 0 ];
D = [ 0 ];

sys = ss( A, B, C, D );
t = 0:0.1:10;
u = 1.0*sin(10*t);
lsim(sys, u, t, [0 0 ]);
grid on;
```



Frequency Response

Definition: The frequency response of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (also sinusoidal) output.



Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity $\Rightarrow$ can construct response to any input (via Fourier decomposition)

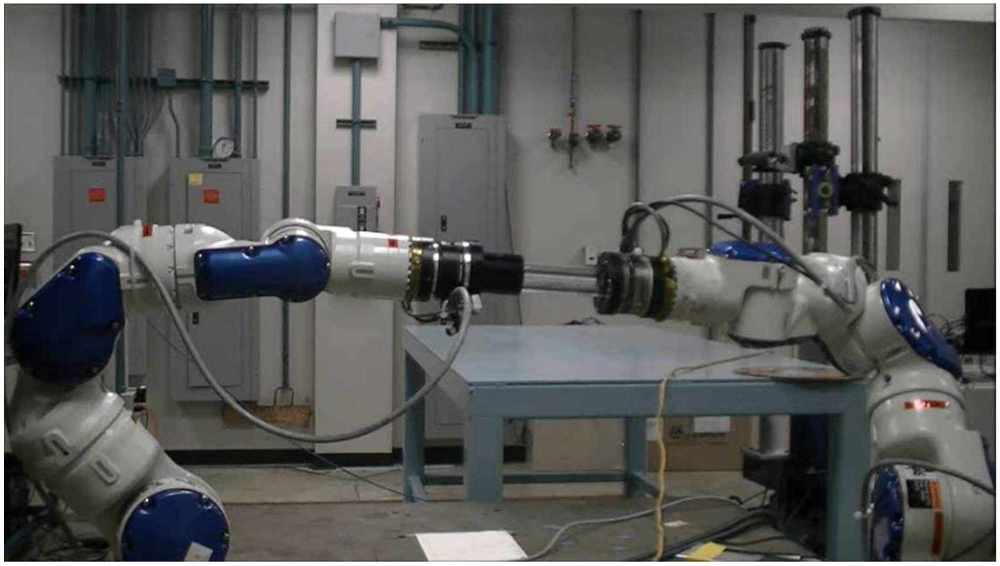
Objective: Clearly and succinctly present the performance capabilities of a system for a broad range of inputs.

Motivating examples

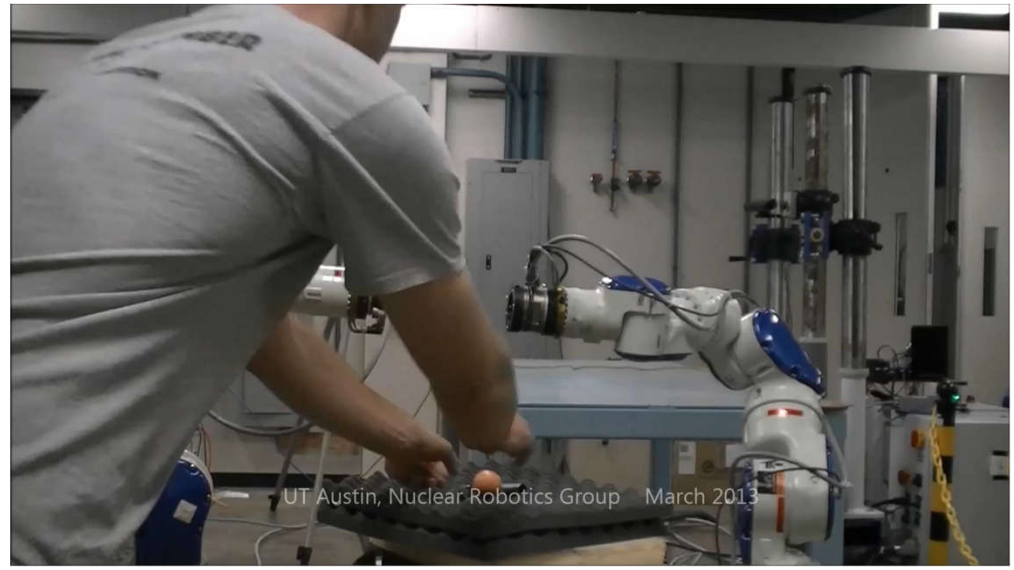
Component Level – Single input, Single output ((U.T. Austin HCRL)
http://www.youtube.com/watch?v=KaQ6lx3ifPU

System Level – Multi-input, Multi-output (U.T. Austin HCRL)
http://www.youtube.com/watch?v=tNTU5O5urmA

Motivating examples

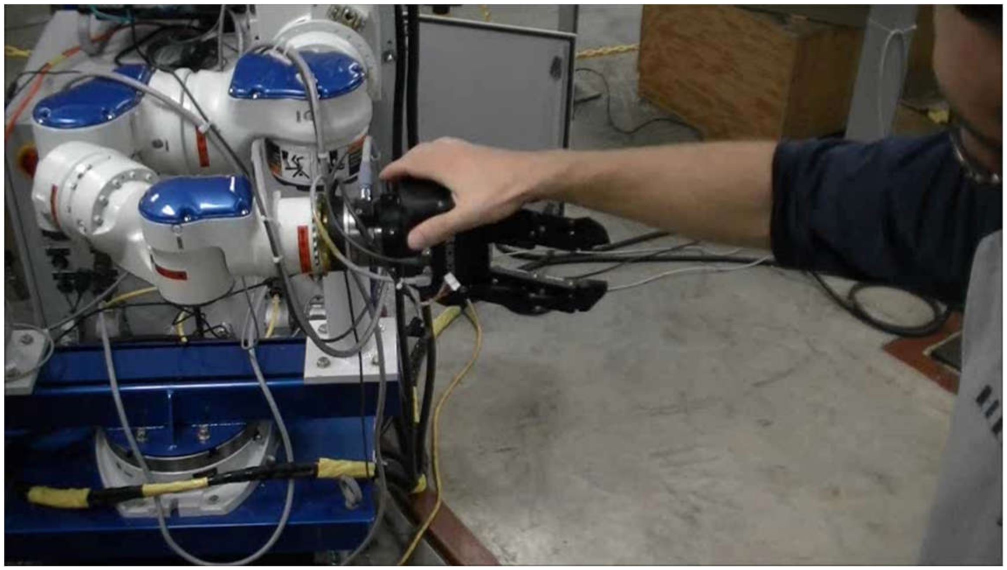


Multi-input, Multi-output Chirp Demonstration: The response to a chirp input is used to experimentally validate the system's response to a range of inputs (such as shown in a Bode Plot) to validate the range of inputs possible. (U.T. Austin, NRG)



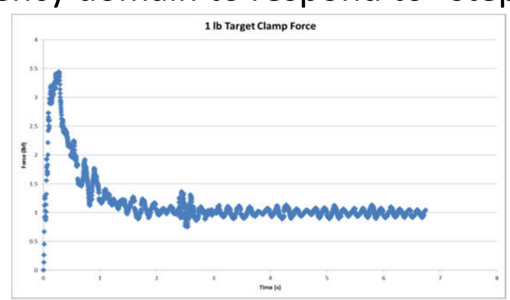
Safely manipulating objects: Date from the chirp experiment help determine the maximum allowable input without breaking or dropping the egg. (Testing a fuzzy logic controller at U.T. Austin, NRG)

Motivating Applications



Variable frequency input: In the real world we really are often interested in range of inputs anyway. (U.T. Austin, NRG)

Step input: Controllers can still be designed using analytical tools in the frequency domain to respond to "step" or "impulse" inputs.

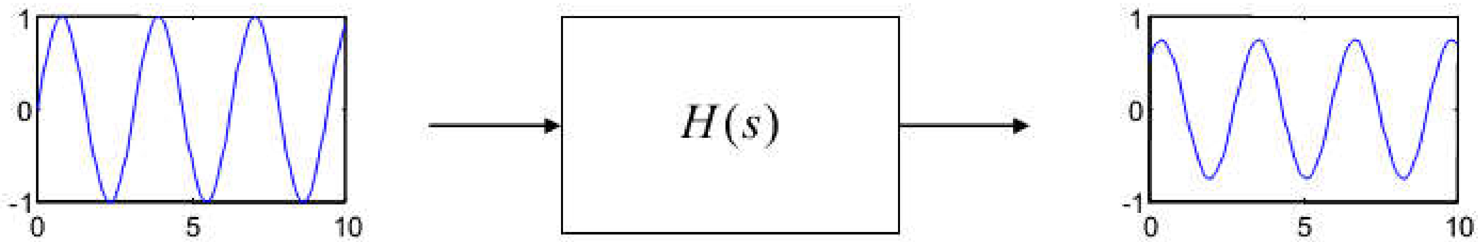


System/Controller Visualization

- We have numerous tools to represent our system's response, stability, input/output relationship, etc.
- Solve for output (y(t)) over time.
    - Convolution and the Matrix Exponential
    - Taylor method, Euler's Method, Runga-Kutta
    - MATLAB ode45, lsim, step, impulse
    - Laplace Transforms (partial fractions)
- System frequency response
    - Transfer Functions ← Next on our list
    - Polar Plots
    - Bode Diagrams (by hand, in MATLAB)
- System characteristics
    - Pole / Zero Maps, Root Locus
- System stability w.r.t. frequency
    - Nyquist Plots
- Many others we won't have time to cover…

Next up: Transfer Functions

Definition: The transfer function for a linear system ss=(A, B, C, D) is a function (say H(s)), that gives the gain and phase of the response to sinusoidal frequency ω.



$u = A\sin(\omega t)$ $\qquad$ $y = |H(j\omega)|A\,\sin(\omega t + \measuredangle H(j\omega))$

$$TransferFunction(\mathrm{s}) = H(s) = \frac{Output(s)}{Input(s)}$$

Objective: Find the transfer function for given system and/or controller in order to easily formulate the frequency response of a system.

Transfer functions from s-s models

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \int_o^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t)$$

Recall the convolution equation…

$$u(t) = \mathrm{e}^{st} = e^{(\sigma+\mathrm{j}\omega)\mathrm{t}} = e^{\sigma t}e^{\mathrm{j}\omega\mathrm{t}} = e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right)$$

And from earlier homework… so use as an oscillating input

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \int_o^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\,\mathrm{e}^{s\tau}d\tau + \mathbf{D}\,\mathrm{e}^{st}$$

Plug in for u(t)…

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \mathbf{C}e^{\mathbf{A}t}\int_o^t e^{-\mathbf{A}\tau}\mathbf{B}\,\mathrm{e}^{s\tau}d\tau + \mathbf{D}\,\mathrm{e}^{st}$$

Extract the 2 leftmost constant terms…

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \mathbf{C}e^{\mathbf{A}t}\int_o^t e^{(s\mathbf{I}-\mathbf{A})\tau}\mathbf{B}d\tau + \mathbf{D}\,\mathrm{e}^{st}$$

Combine the like terms…

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \mathbf{C}e^{\mathbf{A}t}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}e^{(s\mathbf{I}-\mathbf{A})\tau}\mathbf{B}\right]_0^t + \mathbf{D}\,\mathrm{e}^{st}$$

Integrate.

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{z}(0) + \mathbf{C}e^{\mathbf{A}t}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\left(e^{(s\mathbf{I}-\mathbf{A})t} - \mathbf{I}\right)\mathbf{B}\right] + \mathbf{D}\,\mathrm{e}^{st}$$

Solve.

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{z}(0) - \left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}\right)}_{\text{Transient solution!}} + \underbrace{\left[\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathrm{e}^{st}}_{\text{Steady state solution!}}$$

Combine similar terms and rearrange.

Transfer functions from s-s model

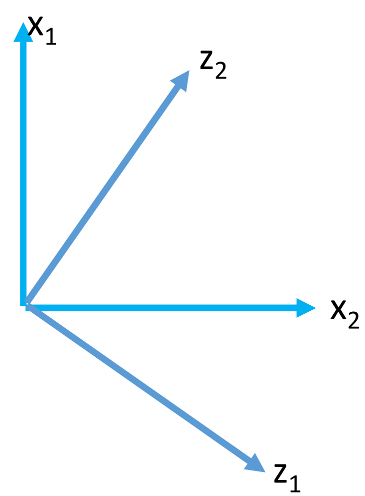
$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{z}(0) - \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\right)}_{\substack{\textit{Transient solution!} \\ \textit{(decays to 0 if stable)}}} + \underbrace{\left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]e^{st}}_{\substack{\textit{Steady state solution!} \\ \textit{(proportional to our input } u(t)=e^{st}\textit{!)}}}$$

$$\begin{aligned} y_{ss}(t) &= \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]e^{st} \\ &= H(s)u(t) \end{aligned}$$

Transfer function = H(s) = H(j\ω) = function that determines the steady state gains and phases for a given linear , time invariant system if the input is e^{st}.

$$u(t) = e^{st} \longrightarrow \boxed{\quad H(s) \quad} \longrightarrow y_{ss}(t)$$

Does the coordinate frame matter?



$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + \mathbf{D}u \end{cases} \Rightarrow \begin{cases} \dfrac{d\mathbf{z}}{dt} = \widetilde{\mathbf{A}}\mathbf{z} + \widetilde{\mathbf{B}}u \\ y = \widetilde{\mathbf{C}}\mathbf{z} + \mathbf{D}u \end{cases}$$

$$\text{where } \widetilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \widetilde{\mathbf{B}} = \mathbf{T}\mathbf{B} \text{ and } \widetilde{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$$

Solve for the output for each system representation....

$$y = \left[\widetilde{\mathbf{C}}(s\mathbf{I} - \widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{B}} + \widetilde{\mathbf{D}} \right] u$$

and

$$y = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \right] u$$

Since the output y is unchanged by the transformation…

$$y = \tilde{H}(s)u(t) = H(s)u(t) \Rightarrow \tilde{H}(s) = H\left(s\right)$$

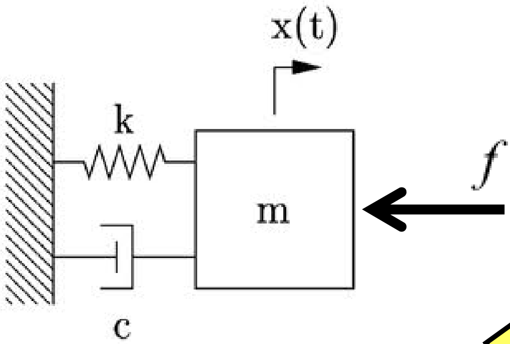
Therefore, Transfer Functions are invariant with respect to coordinate transformations.

Laplace Transform

$$y_{ss}(t) = \left[\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}\right]e^{st} = H(s)u(t)$$

- Traditionally, Feedback Control Theory starts by using the Laplace Transform of the differential equations to develop the Transfer Function
- The was one caveat: the initial conditions were assumed to be zero.
  - For most systems a simple coordinate change could effect this
  - If not, then a more complicated form using the derivative property of Laplace transforms had to be used which could lead to intractable forms
- While we derived the transfer function, G(s), using the convolution equation and the state space relationships, the **transfer function so derived is a Laplace Transform under zero initial conditions**

Find Transfer Functions using Laplace Transforms



$$m\ddot{x} + b\dot{x} + kx = f$$

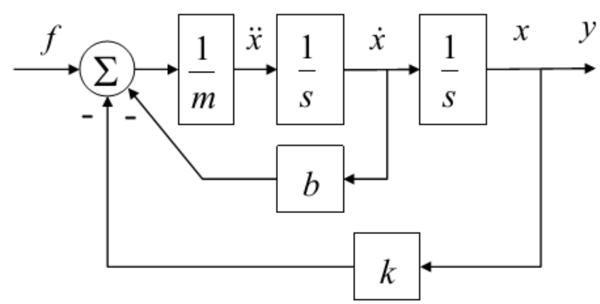
$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{z} + [0]u$$

$$ms^2 y + bsy + ky = f$$

$$y\left(ms^2 + bs + k\right) = f$$

$$H(s) = \frac{y}{f} = \frac{1}{ms^2 + bs + k}$$



Laplace Transform

- More mathematics rust to remove!
- The Laplace transform is defined as

For an analytic function $f(t)$

(i.e., integrable everywhere less than $e^{s_0 t}$ for finite s_0)

$$F(s) = \int_0^\infty e^{-st} f(t)dt = L\left(f(t)\right)$$

$F\left(s\right)$ is the Laplace transform of $f(t)$

s is a complex number

The Inverse Laplace transform is defined as

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st} F(t)ds = L^{-1}\left(F(t)\right)$$

Fortunately, we rarely have to use these integrals as there are other methods

Laplace Tables

Tables are available for determining the Laplace transform of most common functions

Table 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

Table 2–1 *(continued)*

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Modern Control Engineering by K. Ogata 4th ed., 2002

Laplace Transform

$$F(s) = \int_0^\infty e^{-st} f(t)dt = L\big(f(t)\big)$$

- Note that the index on the integral is 0:
  - it is assumed that no dynamics are considered prior to t=0

$$f(t) = 0 \qquad t < 0$$

- The Laplace is a linear transform:

$$L(af(t)) = \int_0^\infty e^{-st} af(t)dt = a\int_0^\infty e^{-st} f(t)dt = aL\big(f(t)\big)$$

$$\begin{aligned} L\big(af(t)+bg(t)\big) &= \int_0^\infty e^{-st}\big(af(t)+bg(t)\big)dt \\ &= \int_0^\infty e^{-st} af(t)dt + \int_0^\infty e^{-st} bg(t)dt \\ &= aL\big(f(t)\big) + bL\big(g(t)\big) \end{aligned}$$

Simple transfer functions

Differential Equation	Transfer Function	Name
$y = \dot{u}$	s	Differentiator
$\dot{y} = u$	$\frac{1}{s}$	Integrator
$\ddot{y} = u$	$\frac{1}{s^2}$	2nd order Integrator
$\dot{y} + ay = u$	$\frac{1}{s + a}$	1st order system
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n{}^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$	Damped Oscillator
$y = k_p e(t) + k_d \frac{de_i}{dt} + k_i \int e(t)dt$	$k_p + k_d s + \frac{k_i}{s}$	PID Controller

Laplace Transforms for Common Input Functions

- Laplace Transform of the Impulse Function

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{\varepsilon} & 0 \le t < \varepsilon \\ 0 & t \ge 0 \end{cases}$$

$$L\left(\delta(t)\right) = 1$$

- Laplace Transform of the Step Function

$$1(t) = \begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$$

$$L\left(1\left(t\right)\right) = \frac{1}{s}$$

- Laplace Transform of a Unit Ramp:

$$f(t) = \begin{cases} 0 & t<0 \\ t & t \ge 0 \end{cases}$$

$$L(f(t)) = \frac{1}{s^2}$$

- Laplace Transform of the n^{th} power of t:

$$f(t) = \begin{cases} 0 & t<0 \\ \dfrac{t^n}{n!} & t \ge 0 \end{cases}$$

$$L(f(t)) = \frac{1}{s^{n+1}}$$

Laplace Transforms for Common Input Functions

- Laplace Transform of exponentials:

$$f(t) = \begin{cases} 0 & \text{t<0} \\ e^{-at} & \text{t} \geq 0 \end{cases}$$

$$L(f(t)) = \frac{1}{s+a}$$

$$f(t) = \begin{cases} 0 & \text{t<0} \\ te^{-at} & \text{t} \geq 0 \end{cases}$$

$$L(f(t)) = \frac{1}{(s+a)^2}$$

$$f(t) = \begin{cases} 0 & \text{t<0} \\ t^n e^{-at} & \text{t} \geq 0 \end{cases}$$

$$L(f(t)) = \frac{n!}{(s+a)^{n+1}}$$

- Laplace Transform of trigonometric functions:

$$f(t) = \begin{cases} 0 & \text{t<0} \\ \sin \omega t & \text{t} \geq 0 \end{cases}$$

$$L(f(t)) = \frac{\omega}{s^2+\omega^2}$$

$$f(t) = \begin{cases} 0 & \text{t<0} \\ \cos \omega t & \text{t} \geq 0 \end{cases}$$

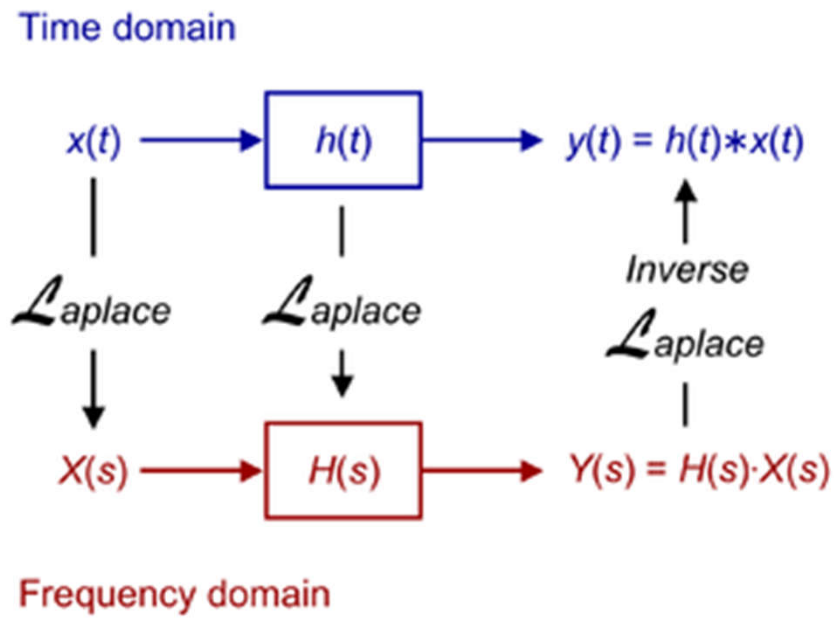
$$L(f(t)) = \frac{s}{s^2+\omega^2}$$

$$f(t) = \begin{cases} 0 & \text{t<0} \\ e^{-at} \sin \omega t & \text{t} \geq 0 \end{cases}$$

$$L(f(t)) = \frac{\omega}{(s+a)^2+\omega^2}$$

Break down the Transfer Function

Assumes Initial conditions are zero!



Convolution

$$H(s) = \frac{b_0 s^n + \cdots + b_{m-2}s^2 + b_{m-1}s + b_m}{a_0 s^n + \cdots + a_{n-2}s^2 + a_{n-1}s + a_n} = \frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}$$

Zeros

Poles

Poles or zeros may be complex!

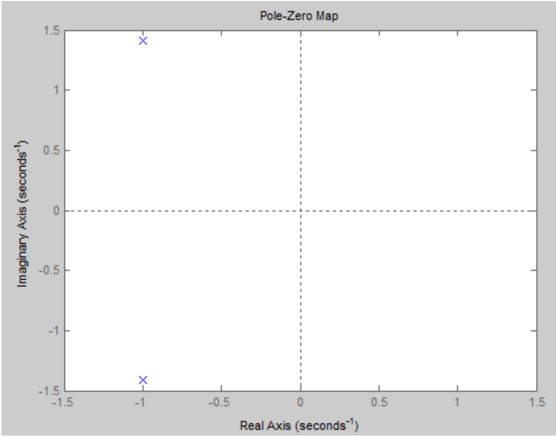
Pole / Zero Maps (either model)

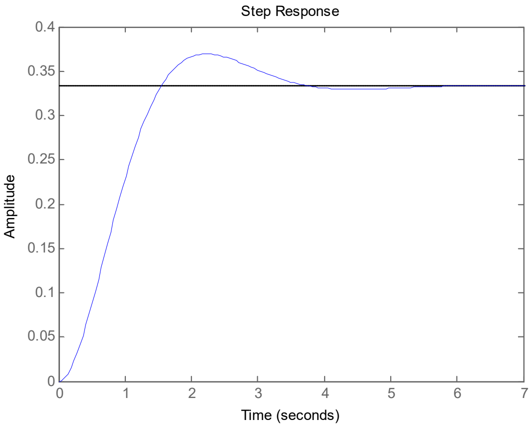
s-s model

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}u$$

$$u(t) = 1$$

```
clear all

m=1; c=2; k=3;

A = [ 0 1; -(k/m) -(c/m)];
B = [ 0; 1/m ];
C = [ 1 0 ];
D = [ 0 ];

sys = ss( A, B, C, D );
pzplot( sys )
axis( [-1.5 1.5 -1.5 1.5 ]);
step( sys )
```



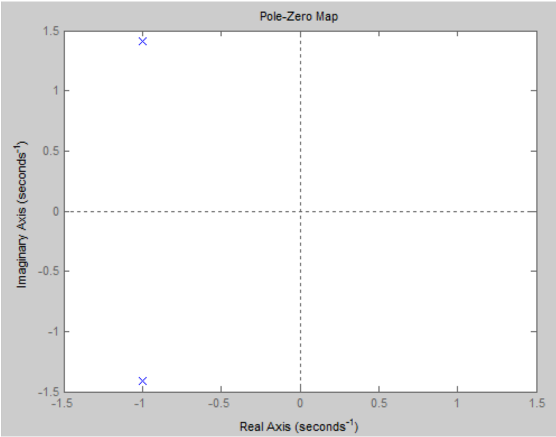


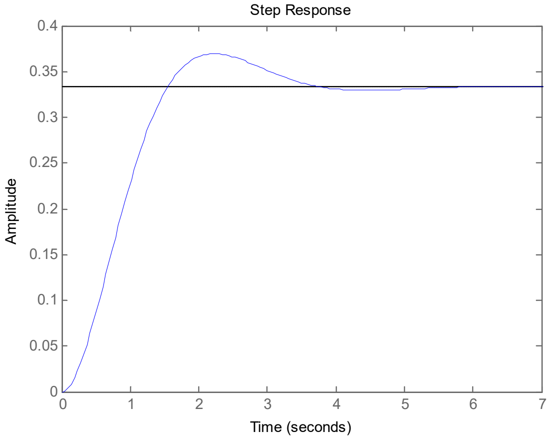
H(s) model

$$H(s) = \frac{1}{ms^2 + bs + k}$$

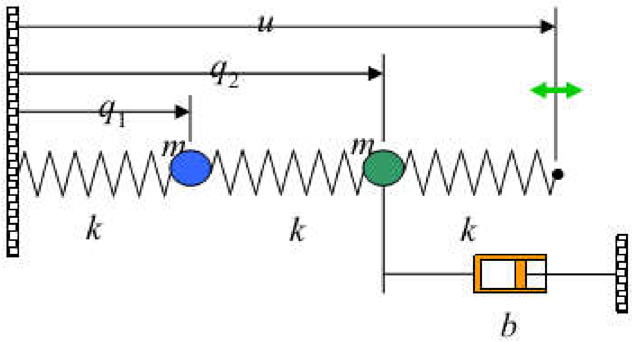
$$u(s) = \frac{1}{s}$$

```
clear all

m=1; c=2; k=3;

a = [ 1 ];
b = [ m c k];
sys = tf( a, b );

pzplot( sys )
axis( [-1.5 1.5 -1.5 1.5 ]);
step( sys )
```





Pole / Zero Maps



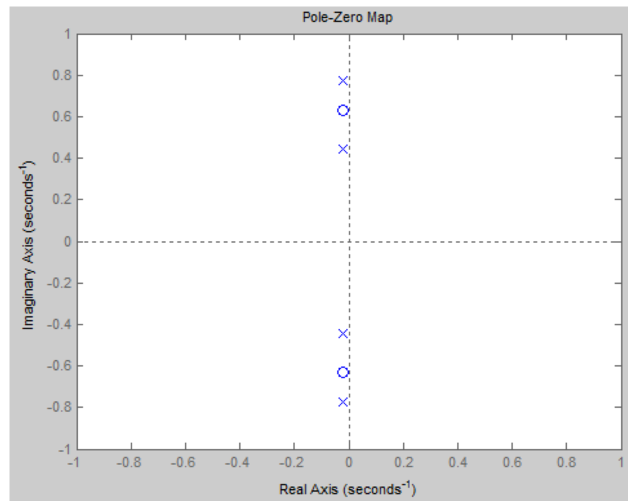
If the Transfer function derived from the EOMs is:

$$H(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

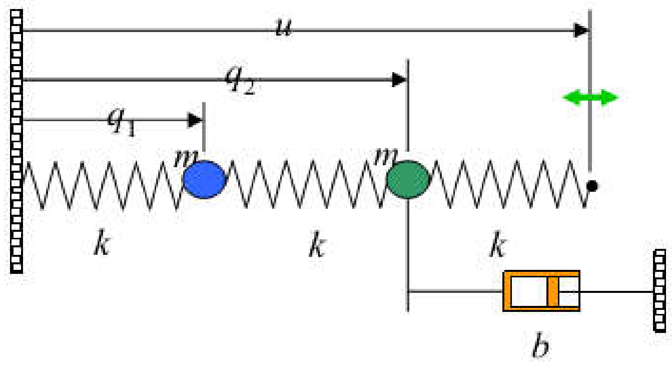
For example….

$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

```
clear all

num = [ .2 .008 0.08 ];
den = [ 1 0.08 0.8016 0.032 0.12];

sys = tf( num, den );

pzplot( sys );
axis( [-1.0 1.0 -1.0 1.0 ]);
```

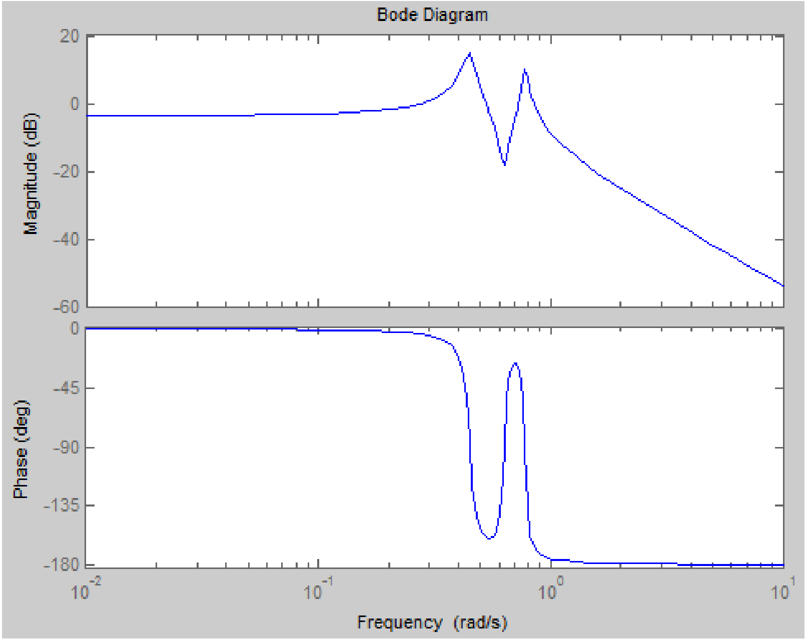


Bode Plot

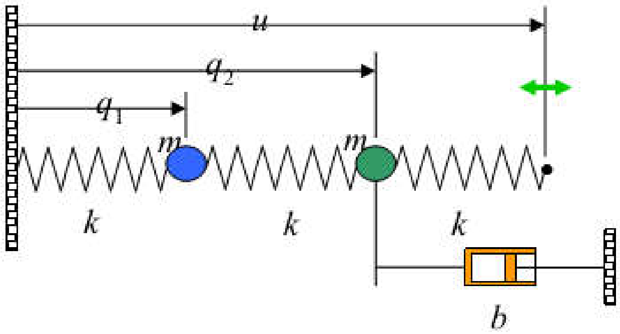


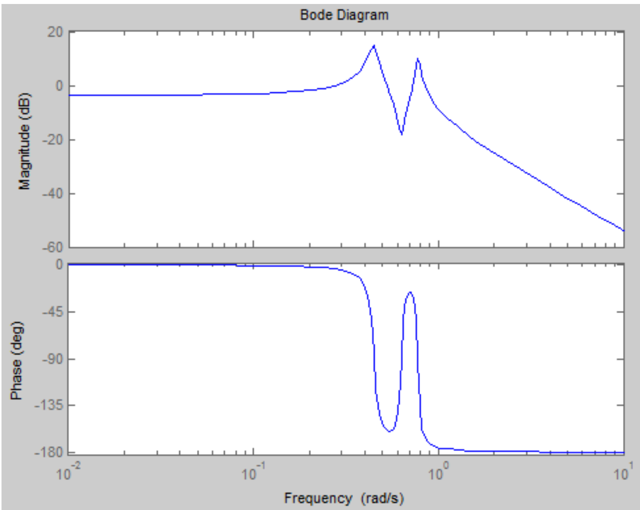
$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

```
a = [ .2 .008 0.08 ];
b = [ 1 0.08 0.8016 0.032 0.12];
sys = tf( a, b );

bode( sys )
```

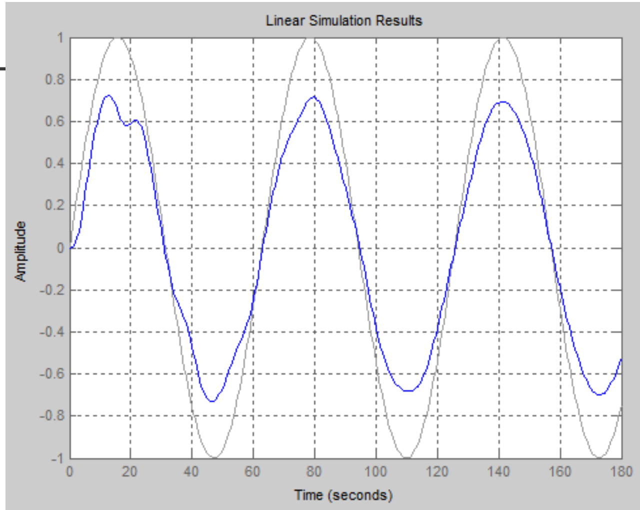


Bode Plot

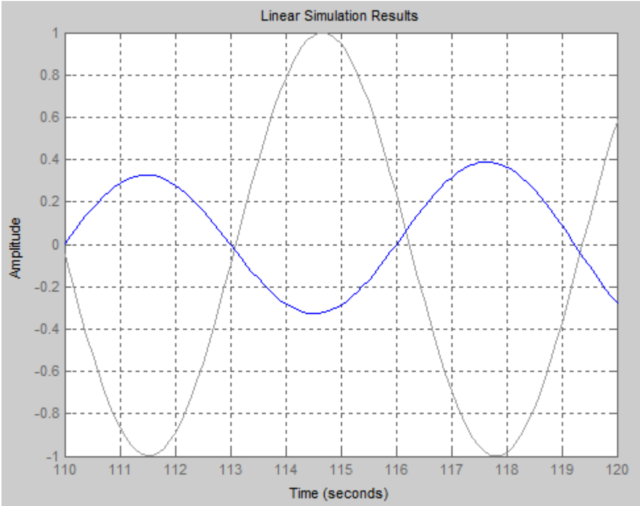




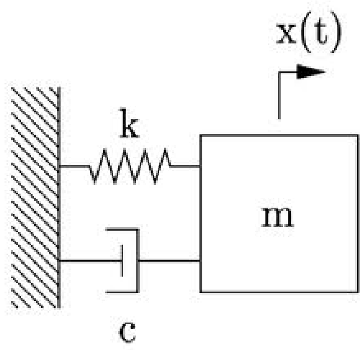

```
u = 1.0*sin(0.1*t);
```




```
u = 1.0*sin(1.0*t);
```



MSD Example, cont'd



$$\mathrm{H}(s) = \frac{m}{ms^2 + bs + k}$$

What about a sinusoidal input?

$$\mathrm{u(t)} = \sin \omega t$$

Given the identity….

$$\sin \omega t = \frac{i}{2}(e^{-i\omega t} - e^{i\omega t})$$

We let $u_1(t) = \frac{i}{2} e^{-i\omega t}$

$u_2(t) = \frac{i}{2} e^{i\omega t}$

And thus… $u(t) = u_1(t) - u_2(t)$

Using superposition….

$$y(t) = y_1(t) - y_2(t) = H(s)u_1(t)\Big|_{s=-i\omega} - H(s)u_2(t)\Big|_{s=i\omega}$$

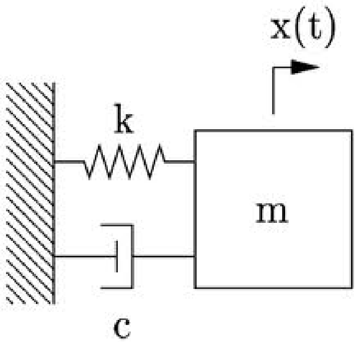
We know the transfer function for each input…

$$\begin{aligned}
\mathrm{H}(i\omega) &= \frac{m}{m(i\omega)^2 + b(i\omega) + k} \\
&= \frac{m}{k - m\omega^2 + ib\omega} \\
&= \left(\frac{m}{(k - m\omega^2) + ib\omega}\right)\left(\frac{(k - m\omega^2) - ib\omega}{(k - m\omega^2) - ib\omega}\right) \\
&= \frac{mk - m^2\omega^2 - imb\omega}{(k - m\omega^2)^2 + (b\omega)^2}
\end{aligned}$$

Complete the square…

Result is a complex number

MSD Example, cont'd



$$\mathrm{H}(s) = \frac{m}{ms^2 + bs + k}$$

What about a sinusoidal input?

$$\mathrm{u(t)} = \sin \omega t$$

From the previous slide…

$$\mathrm{H}(i\omega) = \frac{mk - m^2\omega^2 - imb\omega}{\left(k - m\omega^2\right)^2 + \left(b\omega\right)^2}$$

We can rewrite this complex number in imaginary form…

$$\mathrm{H}(i\omega) = Me^{i\theta}$$

where, $$M = \frac{1}{\left(k - m\omega^2\right)^2 + \left(b\omega\right)^2}\sqrt{\left(km - m^2\omega^2\right)^2 + \left(mb\omega\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$$

treat the second term similarly, and we can write the output in terms of the input frequency, gain, and phase lag.

phase lag

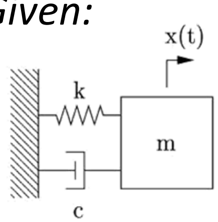
$$y(t) = M\left(\frac{i}{2}e^{-i(\omega t+\theta)} - \frac{i}{2}e^{i(\omega t+\theta)}\right) = M\sin\left(\omega t + \theta\right)$$

gain

input frequency

Generalized 2nd order system

Given:



$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^2 x = u(t)$$

where

$$y = x$$

$$u(t) = \underbrace{k\omega_o e^{st}}$$

Sinusoidal input w/ amplitude that is ratio of natural frequency.

Find: the Transfer function for this system from the s-s model.

Solve: First find the (normalized) s-s form is…

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -2\zeta\omega_o \end{bmatrix} z + \begin{bmatrix} 0 \\ k\omega_o \end{bmatrix} e^{st}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$

Plug into our equation derived from convolution….

$$H(s) = \left[C\left(s\mathbf{I} - A\right)^{-1} B + D \right]$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -\omega_o \\ \omega_o & s + 2\zeta\omega_o \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k\omega_o \end{bmatrix} + 0 = \frac{k\omega_0^2}{s^2 + 2\zeta\omega_o s + \omega_0^2}$$

Does this answer make sense?

Assume that s=0 (i.e. step input)

$$\Rightarrow u(t) = k\omega_o e^{0t} = k\omega_o$$

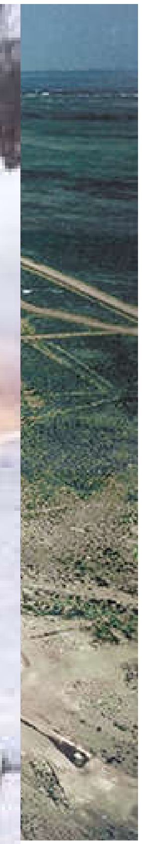
Which means our transfer function becomes….

$$H(s) = \frac{k\omega_0^2}{s^2 + 2\zeta\omega_o s + \omega_0^2} = \frac{k\omega_0^2}{\omega_0^2} = k$$

Which is simply the gain from a step input.

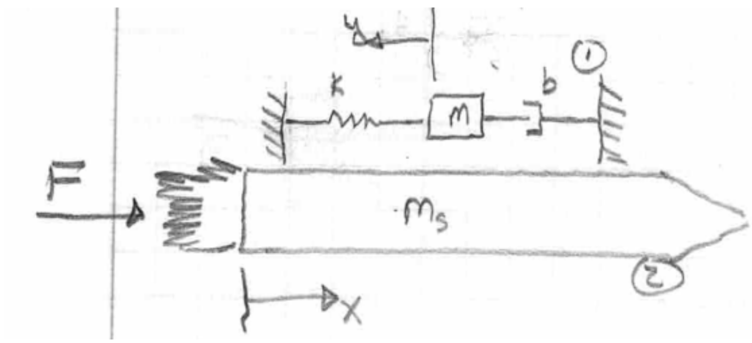
Solving system's using Transfer functions.

Start with a simple example: A rocket sled…



Rocket Sled

- Find the open loop transfer function T(s) for a rocket sled.



Note: y=0 when system is at rest.

$$\Sigma F = ma_y$$

$$m\frac{d^2y}{dt^2} = -ky - by + m\frac{d^2x}{dt^2} \qquad (1)$$

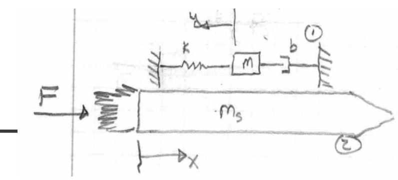
$$M_s\frac{d^2x}{dt^2} = F(\mathrm{t}) \qquad (2)$$

Plug (2) into (1)

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F(t)}{M_s}$$

← Note the force F(t) causes the inertial mass to move in the opposite direction of the sled.

Rocket Sled



a) Find the transfer function T(s) for the rocket sled.

b) Solve the system (i.e. find *y(t)*) using a Laplace transformation given the following system parameters and step input.

$$\frac{d^2y}{dt^2}+\frac{b}{m}\frac{dy}{dt}+\frac{k}{m}y=-\frac{F(t)}{M_s}$$

$$m=1 \qquad b=3 \qquad k=2$$

$$y(0)=-1 \qquad \dot{y}(0)=2$$

Not very realistic, but useful for now to better review solving ODEs using Laplace

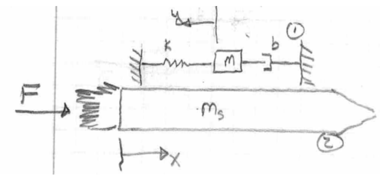
Solution to Part a)

Option 1) Let, $Q(s)=\dfrac{F(t)}{M_s}$

$$s^2Y(s)+3sY(\mathrm{s})+2Y(s)=Q(s) \quad \Rightarrow \quad T(s)=\frac{output}{input}=\frac{Y(s)}{Q(s)}=\frac{1}{s^2+3s+2}$$

Recall, T(s) is the ratio of the output/input for a system given all initial conditions are zero.

Rocket Sled



$$\frac{d^2 y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m} y = -\frac{F(t)}{M_s}$$

$$m = 1 \qquad b = 3 \qquad k = 2$$

$$\left.\begin{array}{l} y(0) = -1 \\ \dot{y}(0) = 2 \end{array}\right\}$$ *Not very realistic, but useful for now to better review solving ODEs using Laplace*

Solution to Part a), cont'd

Option 2) *Convert to state-space form*

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \end{bmatrix} u$$

where, $u = -\dfrac{F(t)}{M_s}$

Note, other options are possible, but should yield the same final result.

Then use the formula derived in the previous lesson

$$\mathbf{T}(s) = \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mathbf{0}$$

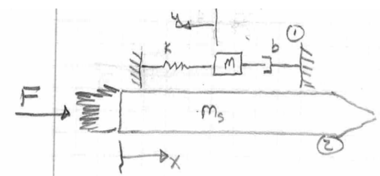
$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{T}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s^2 + 3s + 2} & 0 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{s^2 + 3s + 2} & \dfrac{1}{s^2 + 3s + 2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\frac{1}{s^2 + 3s + 2} = \mathbf{T}(s)}$$

Rocket Sled



b) Solve the system (i.e. find *y(t)*) using a Laplace transformation given the following system parameters and a step input.

$$\frac{d^2y}{dt^2}+\frac{b}{m}\frac{dy}{dt}+\frac{k}{m}y=-\frac{F(t)}{M_s}$$

$$m=1 \qquad b=3 \qquad k=2$$

$$\left.\begin{aligned} y(0)&=-1\\ \dot{y}(0)&=2 \end{aligned}\right\}$$

Not very realistic, but useful for now to better review solving ODEs using Laplace

Solve using Partial Fraction Expansion. Note, here we must account for the initial conditions.

$$\left[s^2Y(s)-\mathrm{sy}(0)-\dot{y}(0)\right]+3\left[sY(\mathrm{s})-\mathrm{y}(0)\right]+2\left[Y(s)\right]=Q(s)$$

$$\left[s^2Y(s)+\mathrm{s}-2\right]+3\left[sY(\mathrm{s})+1\right]+2\left[Y(s)\right]=Q(s)$$

Plug in I.C.'s

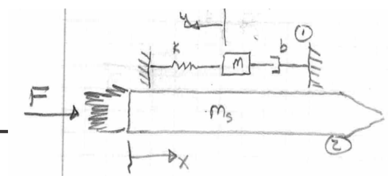
$$s^2Y(s)+3sY(s)+2Y(\mathrm{s})+s+1=Q(s)$$

The last two terms are aspects of the I.C.s

$$s^2Y(s)+3sY(s)+2Y(\mathrm{s})+s+1=-\frac{P}{s}$$

Assume a step input with magnitude P in the opposite direction of y (see graph above)

Rocket Sled



b) Solve the system (i.e. find *y(t)*) using a Laplace transformation given the following system parameters and a step input.

$$s^2Y(s) + 3sY(s) + 2Y(\mathrm{s}) + s + 1 = -\frac{P}{s}$$

From the bottom of the previous page

$$Y(s) = \frac{\frac{P}{s} - s - 1}{s^2 + 3s + 2}\left(\frac{s}{s}\right)$$

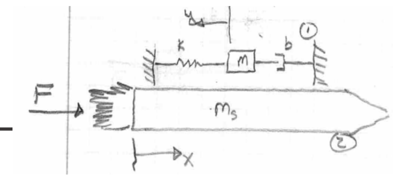
Solve for Y(s) and eliminate the fraction in the numerator.

$$Y(s) = \frac{-\left(s^2 + s - P\right)}{s(s+2)(s+1)}$$

Factor out the denominator and apply the partial fraction expansion method

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

Rocket Sled



$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

From the bottom of the previous page

$$k_1 = \left.\frac{-(s^2 + s - P)}{(s+1)(s+2)}\right|_{s=0} = \frac{P}{2}$$

Solve for the factors k.

$$k_2 = \left.\frac{-(s^2 + s - P)}{s(s+2)}\right|_{s=-1} = \frac{-(1-1+P)}{-1(-1+2)} = \frac{-P}{-1} = P$$

$$k_3 = \left.\frac{-(s^2 + s - P)}{s(s+1)}\right|_{s=-2} = \frac{-(4-2+P)}{-2(-2+1)} = \frac{-2-P}{2}$$

$$Y(s) = \frac{-P}{2s} + \frac{P}{s+1} + \frac{-P-2}{2(s+2)}$$

Plug in k factors

$$Y(s) = \frac{-P}{2} + Pe^{-t} - \frac{1}{2}(P+2)e^{-2t}$$

Inverse Laplace Transform

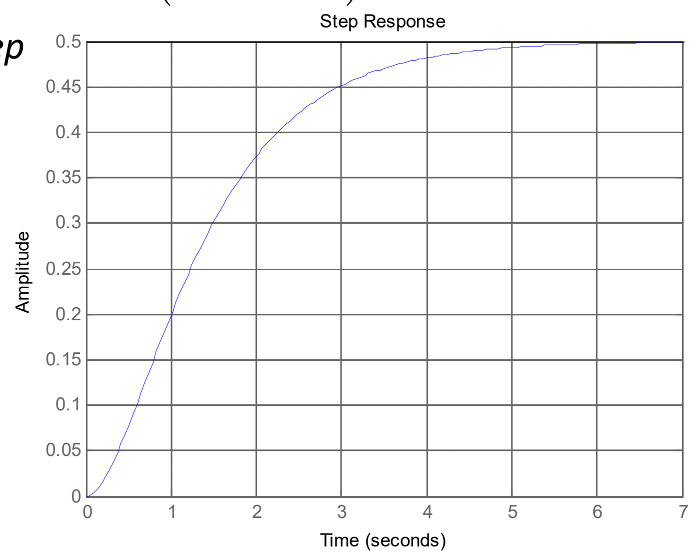
Rocket sled simulation

No initial conditions

$$\mathbf{T}(s) = \frac{\mathrm{Y}(\mathrm{s})}{Q(s)} = \frac{Y(s)}{\frac{P}{s}} = \frac{1}{s^2 + 3s + 2}$$

$$\mathrm{Y}(\mathrm{s}) = \frac{1}{\left(s^2 + 3s + 2\right)} \frac{s}{P}$$

Assume unit step




```
num = [1]
den = [ 1 3 2 ]

sys = tf( num, den )
step(sys, 'b')
```

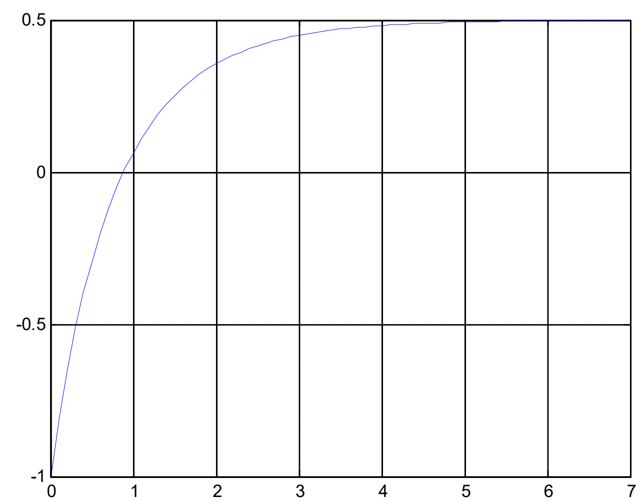
With initial conditions…

$$s^2 Y(s) + 3sY(s) + 2Y(\mathrm{s}) + s + 1 = -\frac{P}{s}$$

$$Y(\mathrm{t}) = \frac{-P}{2} + Pe^{-t} - \frac{1}{2}(P+2)e^{-2t}$$

$$y(0) = -1$$

$$\dot{y}(0) = 2$$




```
t=[0:.1:7];
P=-1;
for i=1:length(t)
    y(i)=(-P/2)+P*exp(-t(i))-0.5*(P+2)*exp(-2*t(i));
end
plot(t,y)
```

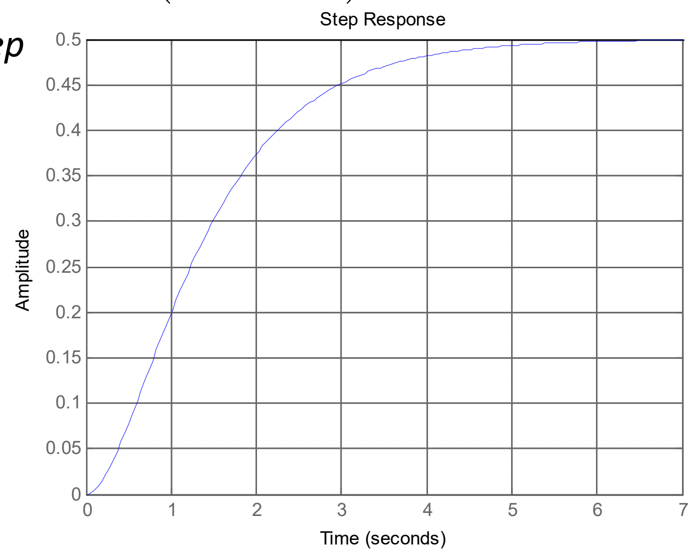
Rocket sled simulation

No initial conditions

$$\mathbf{T}(s) = \frac{\mathrm{Y(s)}}{Q(s)} = \frac{Y(s)}{\frac{P}{s}} = \frac{1}{s^2 + 3s + 2}$$

$$\mathrm{Y(s)} = \frac{1}{\left(s^2 + 3s + 2\right)} \frac{s}{P}$$

Assume unit step




```
num = [1]
den = [ 1 3 2 ]

sys = tf( num, den )
step(sys, 'b')
```

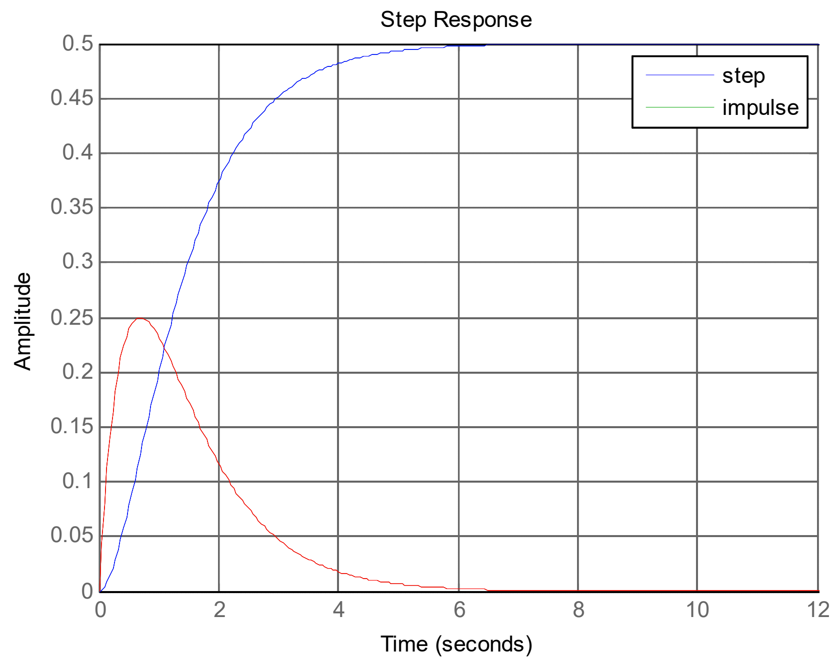
Note the steady state or "zero frequency gain" for the system.

$$\mathbf{T}(s) = \frac{1}{s^2 + 3s + 2}$$

$$\mathbf{T}(0) = \frac{1}{2}$$

Rocket Sled (step or impulse response)

```
num = [1]
den = [ 1 3 2 ]

sys = tf( num, den )


hold on;
impulse(sys, 'r')
step(sys, 'b')
legend('step', 'impulse');
```




```
[ A, B, C, D ] = tf2ss( num, den )

A =

   -3    -2
    1     0

B =

    1
    0

C =

    0     1

D =

    0
```

Summary

- We have a new way to represent a system
  - The Transfer Function

$$\mathrm{H}(s) = C(sI - A)^{-1} B + D = \frac{n(s)}{d(s)} = \frac{zeros}{poles}$$

- It is useful for examining the response of a system over a range of inputs
- We derived the gain transfer function from the convolution equation
  - Did a couple examples
- We reviewed how to find T(s) using Laplace Transforms
- We reviewed how we can use Laplace Transforms to find y(t).
- T(s) is one of many representations, but it is a very useful one.