Homework 6
EE 362K

Problem 1

Tuesday, October 31, 2017 2:13 PM

Find the transfer function between u and y from s.s. model and from Laplace Transforms. Check that they agree.

System: $\dddot{x} - 3x + 1\ddot{x} - 2\dot{x} = u$
$y = 2x + u$

S.S. Derivation

$$z_1 = x,\quad z_2 = \dot{x},\quad z_3 = \ddot{x} \quad \Rightarrow \quad \dot{z}_1 = z_2,\quad \dot{z}_2 = z_3,\quad \dot{z}_3 = 3z_1 + 2z_2 - z_3 + u$$

$$y = 2z_1 + u$$

$$\dot{\bar{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \bar{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{u}$$

$$y = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \bar{z} + [1]\bar{u}$$

By convolution Equation and matrix algebra:

$$y_{ss}(t) = \underbrace{[C(sI-A)^{-1}B + D]}_{H(s)} e^{st} \rightarrow Y(s) = H(s)U(s)$$

Evaluating in MATLAB gives: $\boxed{H(s) = 1 - \dfrac{2}{-s^3 - s^2 + 2s + 3}}$

Laplace Transforms

$$\dddot{x} + \ddot{x} - 2\dot{x} - 3x = u \rightarrow s^3 + s^2 - 2s - 3 = u$$

$$y = 2x + u$$

$$\frac{y}{u} = \frac{2+u}{u}$$

$$\frac{y}{u} = H(s) = \frac{2}{u} + 1$$

$$\boxed{= 1 + \frac{2}{s^3 + s^2 - 2s - 3}}$$

The two results agree

The code that I ran to get evaluate the matrix algebra is:

```
function problem1

A = [0 1 0 ; 0 0 1; 3 2 -1];
B = [0;0;1];
C = [2 0 0];
D = 1;
syms s;
sI = [s 0 0; 0 s 0; 0 0 s];

disp('H(s) is: ');
disp(C*inv(sI-A)*B + D);
```

Problem 2

Part A)

$$Y(s) = \frac{15(s+50)}{s^2+110s+1000} R(s)$$

$r(t)$ is desired position

a.) Consider $r(t) = \begin{cases} 1 & ; 0 \le t \\ 0 & ; \text{else} \end{cases} \rightarrow \mathcal{L}(\{r(t)\}) = \frac{1}{s}$

$$Y(s) = \frac{15(s+50)}{s(s^2+110s+1000)} \quad ; \text{ need to find } \mathcal{L}^{-1}(\{Y(s)\}) = y(t)$$

$$Y(s) = \frac{A}{s} + \frac{B}{(s+10)} + \frac{C}{(s+100)}$$

$$A = Y(s)\Big|_{s=0} = \frac{15 \cdot 50}{20\cancel{1000}} = \frac{3}{4}$$

$$B = Y(s)\Big|_{s=-10} = \frac{15(-10+50)}{-10(-10+100)} = \frac{15(40)}{-10(90)} = -\frac{3}{2} \cdot \frac{4}{9} = -2/3$$

$$C = Y(s)\Big|_{s=-100} = \frac{15(-100+50)}{-100(-100+10)} = \frac{-50(15)}{-100(90)} = \frac{1}{2} \cdot \frac{1}{4} = 1/8$$

$$Y(s) = \left(\frac{3}{4}\right)\left(\frac{1}{s}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{s+10}\right) + \left(\frac{1}{8}\right)\left(\frac{1}{s+100}\right)$$

$$\mathcal{L}^{-1}(Y(s)) = \frac{3}{4}u(t) - \frac{2}{3}e^{-10t} + \frac{1}{8}e^{-100t}$$

Both exponential terms $\rightarrow 0$ as $t \rightarrow \infty$, so, $\boxed{y_{ss} = \frac{3}{4}u(t)}$

Part B)

Here is the MATLAB code used to calculate the S.S. model and confirm the eigenvalues of A.

```
function problem2
Num = [15 15*50];
Den = [1 110 1000];
[A, B, C, D] = tf2ss(Num, Den);
disp('A is:');
disp(A);
disp('B is:');
disp(B);
disp('C is:');
disp(C);
disp('D is:');
disp(D);
disp('Eigenvalues of A are:');
disp(eig(A));
```

The results of running the code:

```
>> problem2
A is:
       -110        -1000
          1            0

B is:
     1
     0

C is:
    15   750

D is:
     0

Eigenvalues of A are:
  -100
   -10
```

By factoring, the denominator of Y(s)/R(s) becomes (s + 10)(s + 100), and by inspection the poles are -10 and -100. This is reflected in the eigenvalue calculation.

Part C)

c) Use convolution to obtain the t.f. from s.s model.
(I am interpreting this as, show they are equivalent.)
From MATLAB:

$$A=\begin{bmatrix}-110 & -1000\\ 1 & 0\end{bmatrix} \quad B=\begin{bmatrix}1\\0\end{bmatrix} \quad C=[15 \ \ 75] \quad D=0$$

This system has solution:

$$y(t)=Ce^{At}y(0)+\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau+\cancel{Du(t)} \quad \text{; By Convolution}$$

To obtain the transfer function, we let $u(t)=e^{st}$.

$$y(t)=Ce^{At}y(0)+\int_0^t \underbrace{Ce^{A(t-\tau)}}_{\text{constant}}Be^{s\tau}d\tau$$

$$=Ce^{At}y(0)+Ce^{At}\int_0^t e^{(sI-A)\tau}Bd\tau$$

$$=Ce^{At}y(0)+Ce^{At}\left[(sI-A)^{-1}(e^{(sI-A)t}-I)B\right]$$

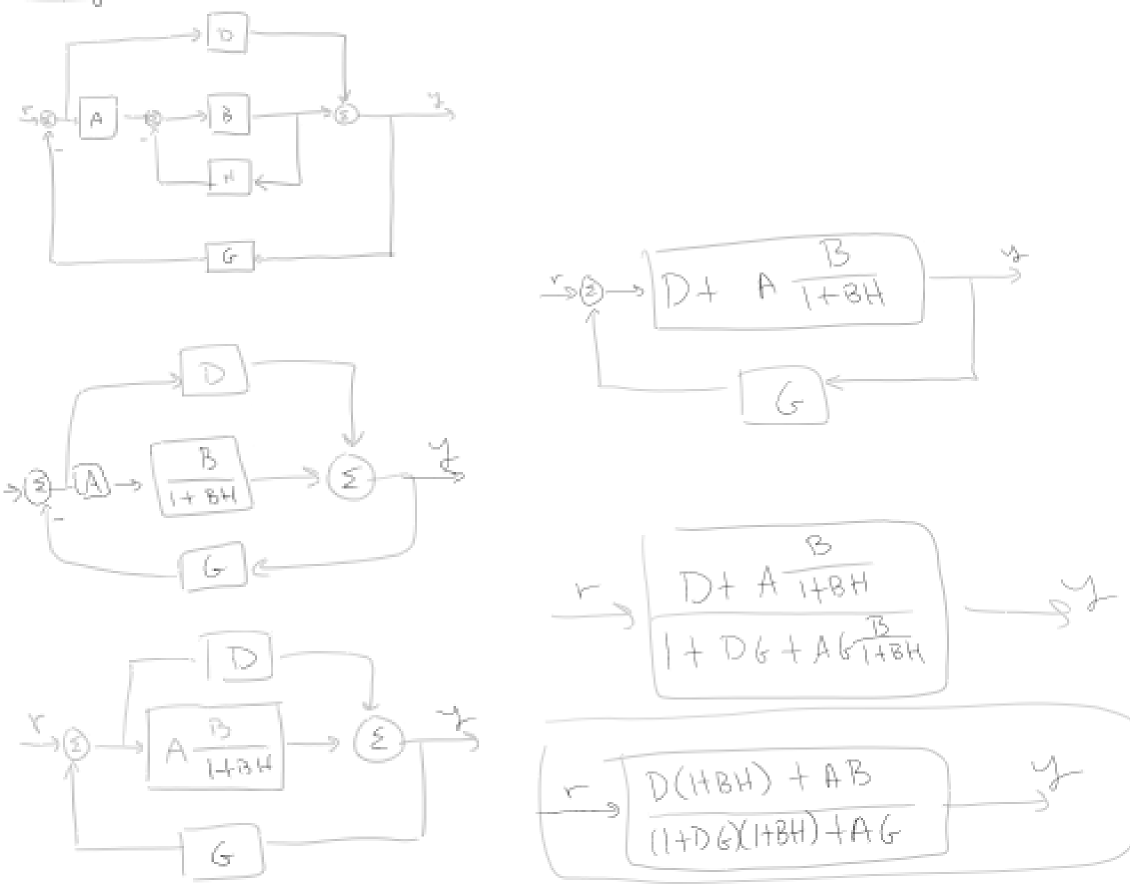
$$=\underbrace{Ce^{At}\left(y(0)-(sI-A^{-1})B\right)}_{\text{Goes to zero}}+\underbrace{\left[C(sI-A)^{-1}B\right]e^{st}}_{\text{Steady State Solution}}$$

$$y(t)=\left[C(sI-A)^{-1}B\right]e^{st} \leftarrow \text{this is } u(t)!$$

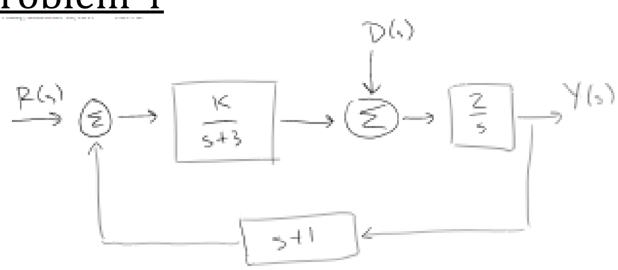
$$\frac{y(t)}{u(t)}=\boxed{C(sI-A)^{-1}B=H(s) \quad \text{by definition of } H(s)}$$

Problem 3

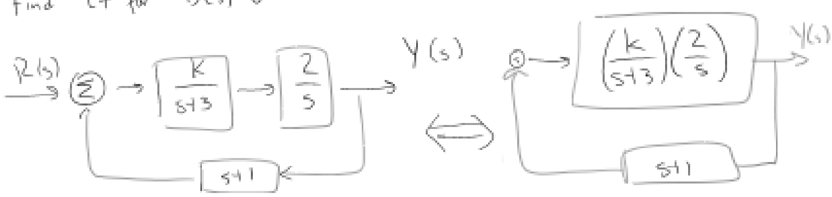
Simplify:



Problem 4



a) Find tf for D(s) = 0

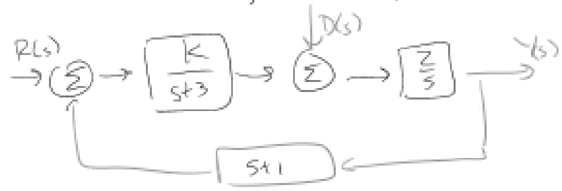


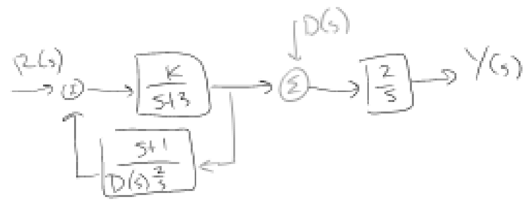
$$H(s)=\frac{Y(s)}{R(s)}=\frac{2k}{(s+3)(s)+(s+1)(2k)}$$

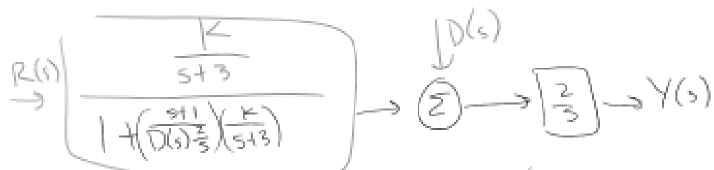
b.) If $R(s)=0$, there is no transfer function representation for the system. A transfer function is a system's response to a sinusoidal input. If there is no input,

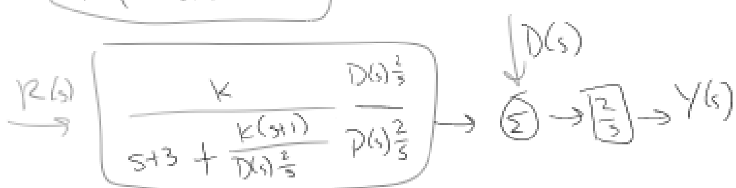
$$H(s)=\frac{Y(s)}{R(s)}=\frac{Y(s)}{0}$$ which is not a valid t.f.

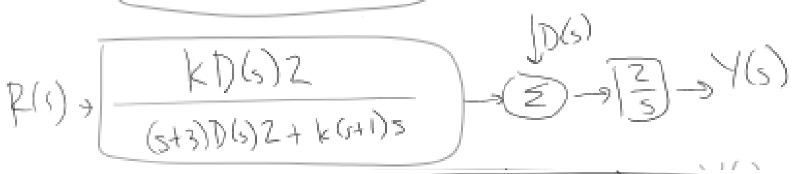
c.) Consider $R(s)\neq 0$ and $D(s)\neq 0$

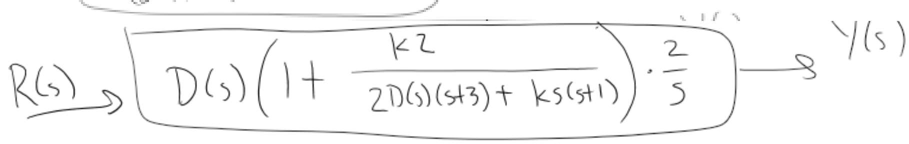








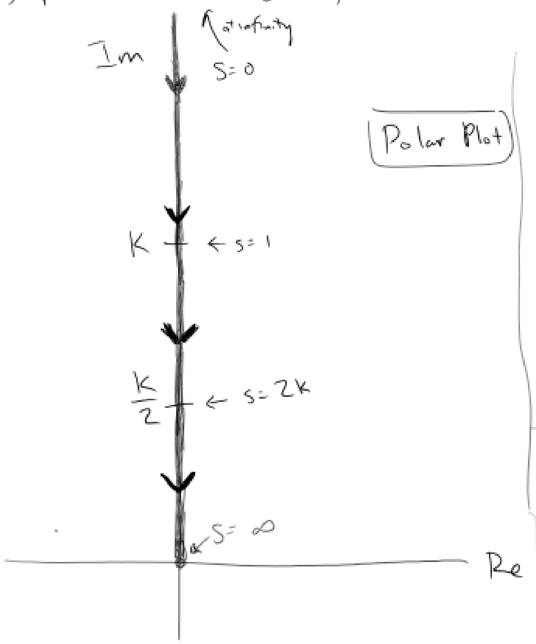




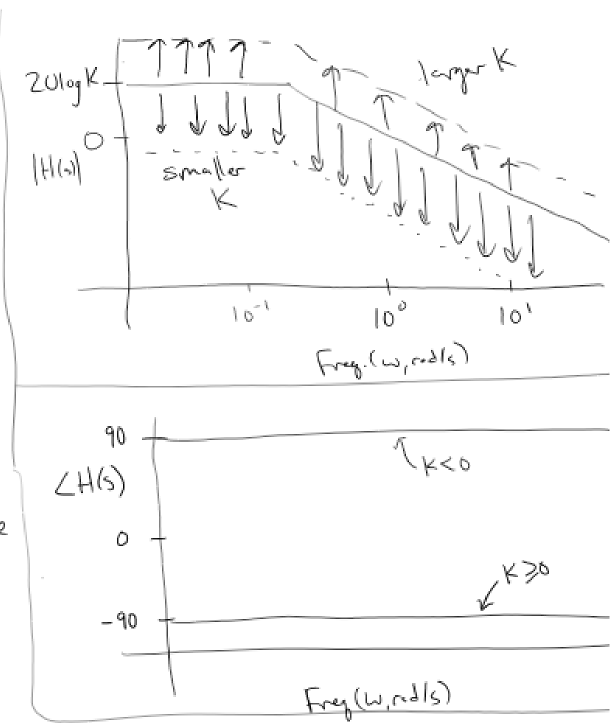
Problem 5

Part A)

a.) For $H(s) = \frac{K}{s}$; sketch Bode/Polar Plots



Polar Plot



Part B)

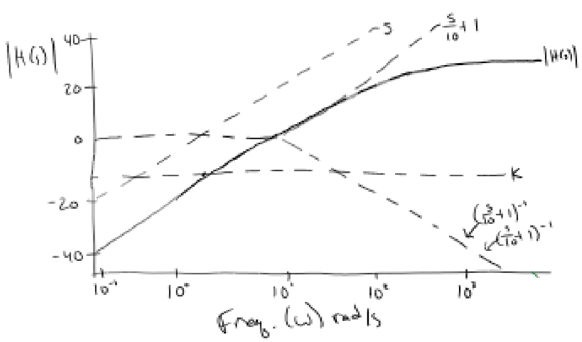
b.) $H(s) = \frac{20s(0.1s+1)}{(s+10)^2} \rightarrow \frac{20s\left(\frac{s}{10}+1\right)}{(s+10)(s+10)} \rightarrow \frac{20s\left(\frac{s}{10}+1\right)}{100\left(\frac{s}{10}+1\right)\left(\frac{s}{10}+1\right)}$

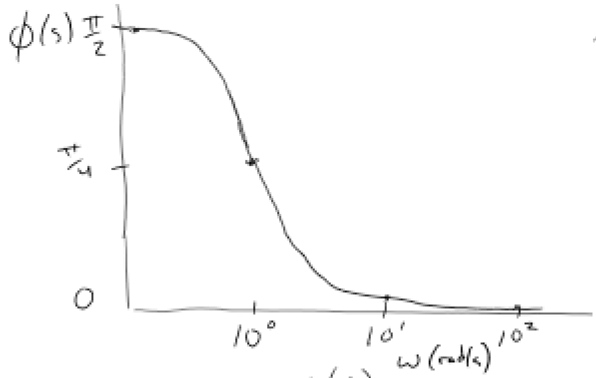
$= \frac{0.2s\left(\frac{s}{10}+1\right)}{\left(\frac{s}{10}+1\right)\cancel{\left(\frac{s}{10}+1\right)}} = \frac{0.2s}{\left(\frac{s}{10}+1\right)}$

1.) $\tilde{K} = 20\log 0.2$

2.) $s: 20\log|j\omega|$

3.) $\left[\frac{s}{10}+1\right]^{-1} = -20\log\left|\frac{j\omega}{10}+1\right|$





So, $\arg(H(s)) = \tan^{-1}\left(\frac{1}{\omega^2}\right)$

$\arg(H(0)) = \tan^{-1}\left(\frac{1}{0}\right) = \pi/2$

$\arg(H(\infty)) = \tan^{-1}\left(\frac{1}{\infty}\right) = 0$

$\arg(H(1)) = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$

$\arg(H(10)) = \tan^{-1}\left(\frac{1}{100}\right) \approx 0$

$\arg(H(100)) = \tan^{-1}\left(\frac{1}{10000}\right) \approx 0$

$$\frac{0.2j\omega\left(\frac{j\omega}{10}-1\right)}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{10}-1\right)} = \frac{0.2\frac{j^2\omega^2}{10} - 0.2j\omega}{\frac{j^2\omega^2}{100}-1} = \frac{-\frac{1}{50}\omega^2 - \frac{1}{5}j\omega}{-1-\frac{\omega^2}{100}}$$

$$Re\{H(s)\} = \frac{\frac{1}{50}\omega^2}{1+\frac{\omega^2}{100}} \cdot \frac{50}{50} = \frac{\omega^2}{50+\frac{\omega^2}{2}}$$

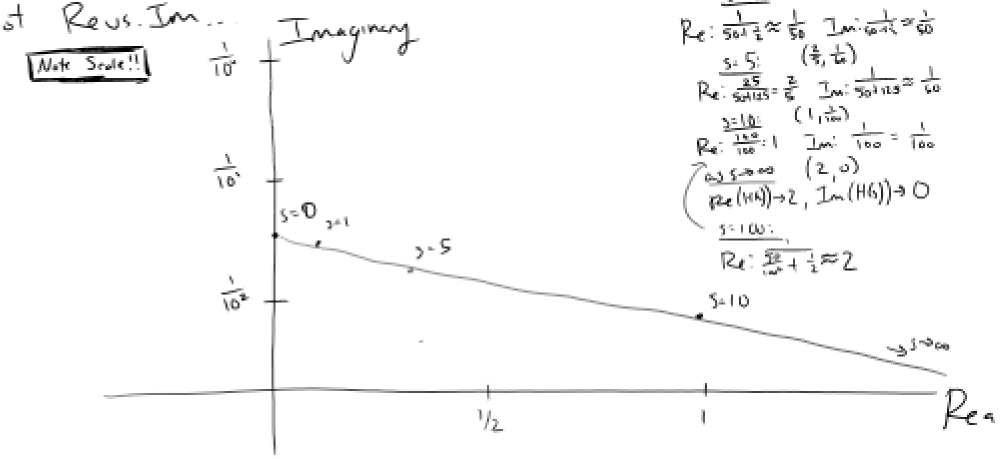
$$Im\{H(s)\} = \frac{\frac{1}{5}}{1+\frac{\omega^2}{100}} \cdot \frac{5}{5} = \frac{1}{50+\frac{1}{2}\omega^2}$$

$$\tan^{-1}\left(\frac{Im}{Re}\right) = \frac{\frac{1}{50+\frac{\omega^2}{2}}}{\frac{\omega^2}{50+\frac{\omega^2}{2}}} = \frac{1}{\omega^2}$$

Polar Plot

$$Re\{H(s)\} = \frac{\omega^2}{50+\frac{\omega^2}{2}} \left(\frac{1}{\frac{50}{\omega^2}+\frac{1}{2}}\right) \qquad Im\{H(s)\} = \frac{1}{50+\frac{\omega^2}{2}}$$

Plot Re vs. Im...



$s=0$: $(0, \frac{1}{50})$

Re: $\frac{0}{50+0}$ Im: $\frac{1}{50+0}$

$s=1$: $(\frac{1}{50}, \frac{1}{50})$

Re: $\frac{1}{50+\frac{1}{2}} \approx \frac{1}{50}$ Im: $\frac{1}{50+\frac{1}{2}} \approx \frac{1}{50}$

$s=5$: $(\frac{2}{5}, \frac{1}{60})$

Re: $\frac{25}{50+12.5} = \frac{2}{5}$ Im: $\frac{1}{50+12.5} \approx \frac{1}{60}$

$s=10$: $(1, \frac{1}{100})$

Re: $\frac{100}{100} = 1$ Im: $\frac{1}{100} = \frac{1}{100}$

as $s \to \infty$ $(2, 0)$

$Re(H(s)) \to 2$, $Im(H(s)) \to 0$

$s=100$:

Re: $\frac{1}{\frac{50}{100^2}+\frac{1}{2}} \approx 2$

Problem 6

First, sketch root locus by hand following step by step. Assume the loci converge back to real axis to the left of the plot (roughly -20.67).

Saturday, November 11, 2017 11:49 AM

Create Root Locus for: $C(s)G(s) = \frac{k(s+10)(s+6)}{s(s+2)(s+4)}$

Step 1: Write so k is the multiplier.
Already done.

Step 2: Factor into zeroes and poles.
Already done.

Step 3: Locate open-loop zeroes and poles
Zeroes $= \{-10, -6\}$ Poles $= \{0, -2, -4\}$

Step 4: Locate segments on real axis that are root loci

Step 5: Determine SL (SL=1)

Step 6: Symmetry

Step 7: Find asymptotes

$$\sigma_1 = \frac{\sum Re(\text{poles}) - \sum Re(\text{zeros})}{|n-m|} \Rightarrow \boxed{\sigma_1 = 10}$$

$$= \frac{(-2-4)-(-10-6)}{3-2} = \frac{-6-(-16)}{1} = 10$$

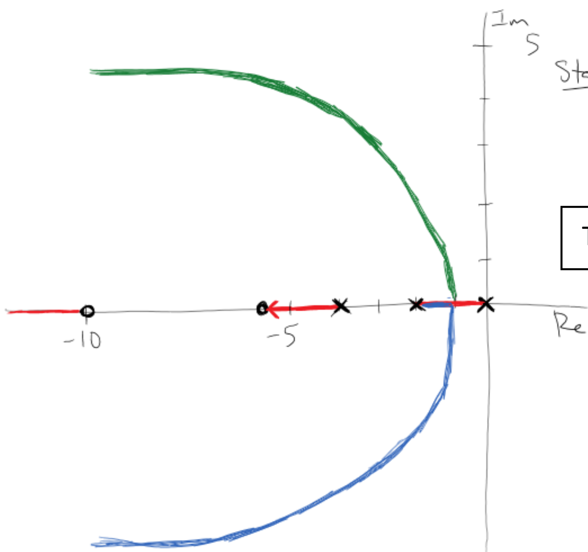
$$\theta_1 = \frac{2i+1}{|n-m|}\cdot\pi = \frac{2i+1}{1}\cdot\pi, \text{ let } i=0 \rightarrow \boxed{\theta_1 = \pi}$$

Step 8: N/A

Step 9: Solve: $\sum \frac{1}{s+z_i} = \sum \frac{1}{s+p_i} \rightarrow \frac{1}{s-10} + \frac{1}{s-6} = \frac{1}{s} + \frac{1}{s-2} + \frac{1}{s-4}$

After some messy calculations, this was solved in MATLAB.
Breakaway points are $-1, \sim -20.67$

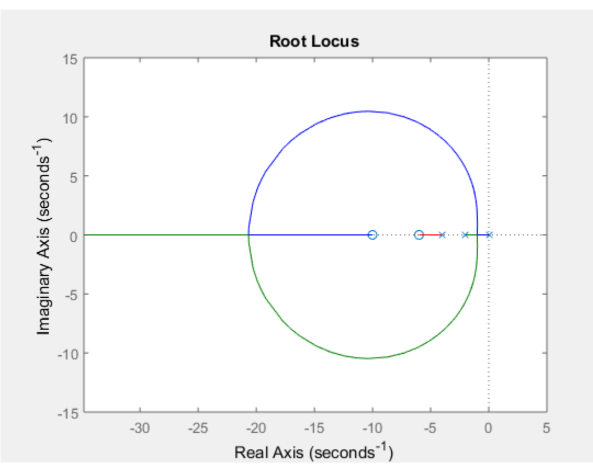
Step 10:
N/A



This function was used to plot the breakaway points

```
function problem6c

syms s;
N(s) = (s+10)*(s+6);
D(s) = s*(s+2)*(s+4);
K(s) = -(D(s)/N(s));
dK(s) = diff(K, s)
fplot(dK(s), [-21 -20]);
```

Then the root locus graph was confirmed via MATLAB. The code is shown below in the bode plot code.



Then the bode plots are sketched for the closed loop system with gains K = -1, and K = -20.67. The hand drawn plots are for the exact breakaway points. Sketching separate plots for just before and after the points would result in nearly identical graphs, because in each case the only change would be a small vertical shift. This would not be noticeable in a hand sketch, and so hand plots are shown only for the points themselves. There are no changes in the phase plot for any K < 0 (all "k" values being considered satisfy this requirement), and so the phase plot remains identical for all four cases.

Hand Drawn Sketches

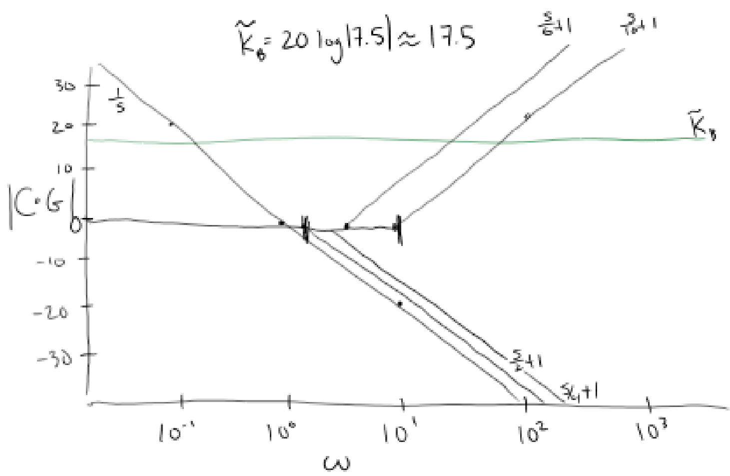
Sketch Bode Plots for $K = -1$, $K = -20.67$

$$C(s)G(s) = \frac{k(s+10)(s+6)}{s(s+2)(s+4)}$$

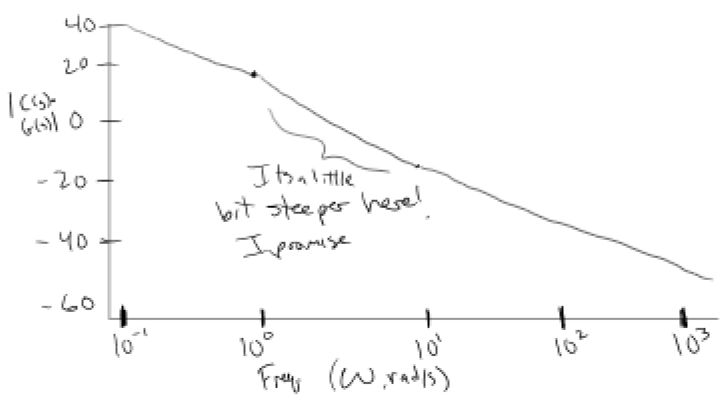
Plot 1: $$C \cdot G = -\frac{(s+10)(s+6)}{s(s+2)(s+4)} \rightarrow -\frac{60\left(\frac{s}{10}+1\right)\left(\frac{s}{6}+1\right)}{8s\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}$$

Standard form: $$\frac{7.5\left(\frac{s}{10}+1\right)\left(\frac{s}{6}+1\right)}{s\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}$$

$\tilde{K}_B = 20\log|7.5| \approx 17.5$



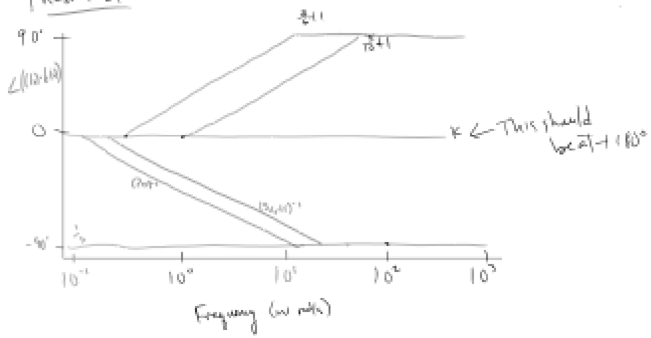
Magnitude of C(s)·G(s) vs. frequency

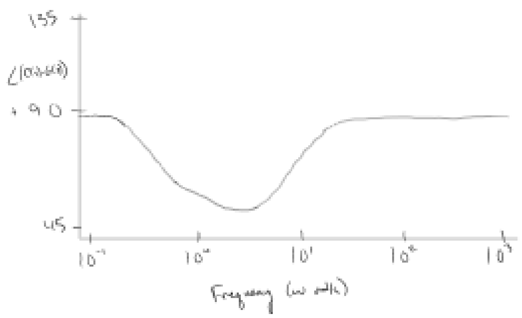


<u>NOTE:</u> These hand sketches are all incorrect. After attempting problem 7 for some time I realized that the C(s)G(s) given must be the <u>open loop system</u>. I reproduced the correct bode plots for the closed loop system in MATLAB.

Additionally, the reasoning given is only valid assuming the transfer function given was the closed loop system. The reasoning holds for the magnitude plots, but not for the phase plots. The new MATLAB plots clearly show the changes.

Phase Plot



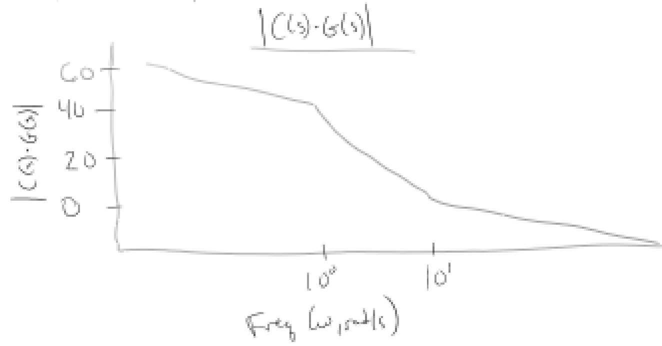


Plot for K = -20.67 :

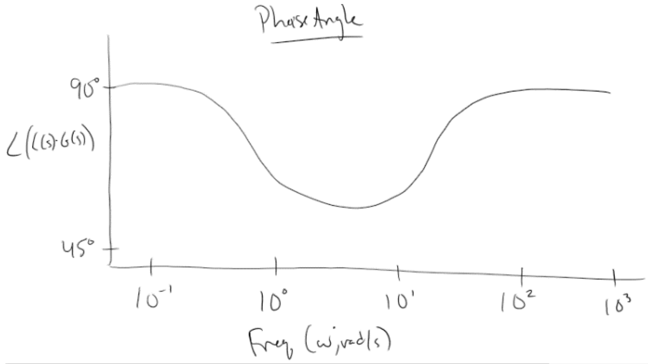
Standard form becomes $\dfrac{(7.5)(-20.67)\left(\frac{s}{10}+1\right)\left(\frac{s}{6}+1\right)}{s\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}$

$\hat{K} \rightarrow 43.8$; otherwise both plots are same

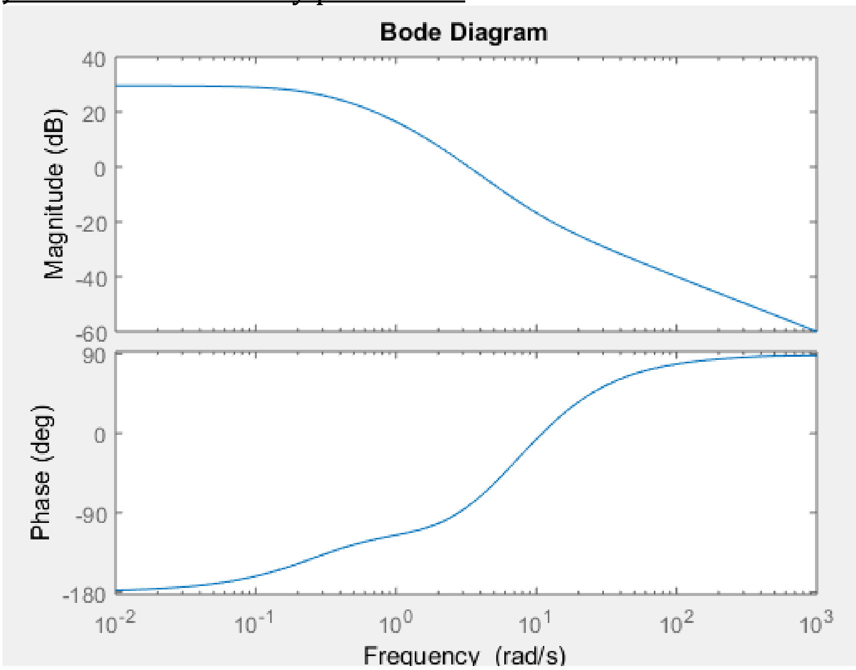
$|C(s)\cdot G(s)|$



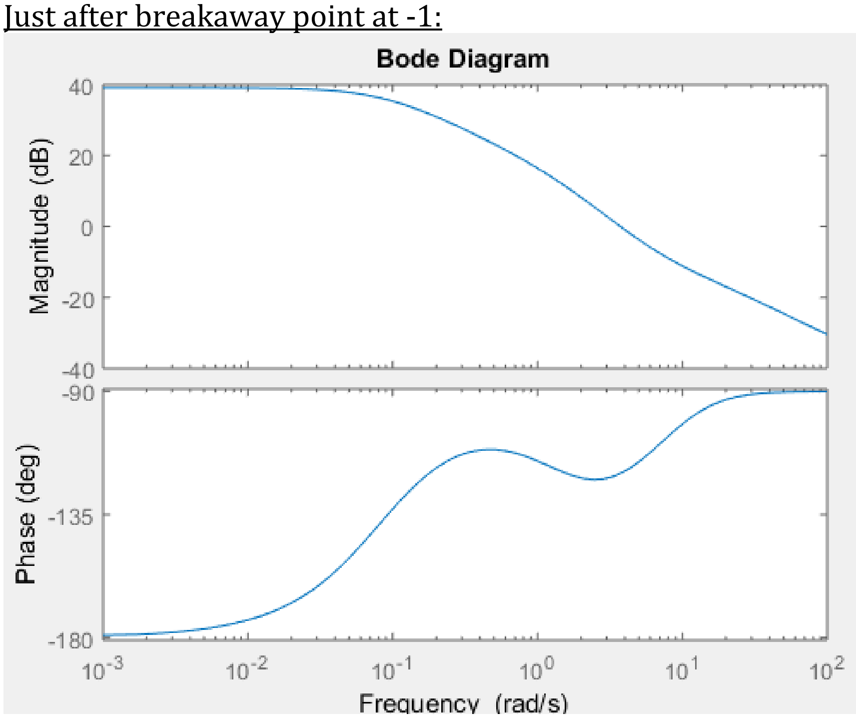
Phase Angle



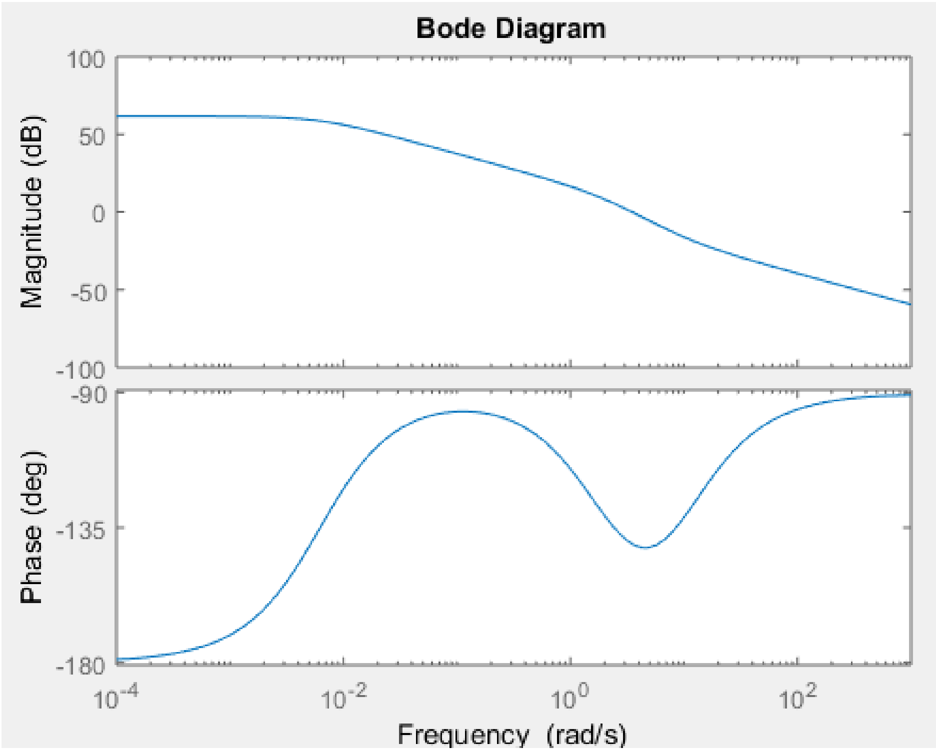
Just before breakaway point at -1:



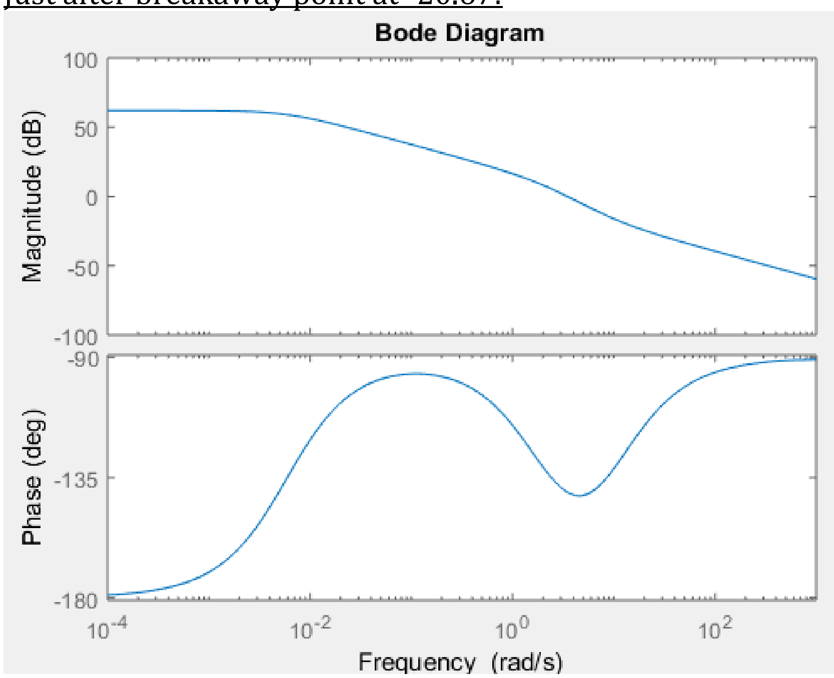
Just after breakaway point at -1:



Just before breakaway point at -20.67:



<u>Just after breakaway point at -20.67:</u>

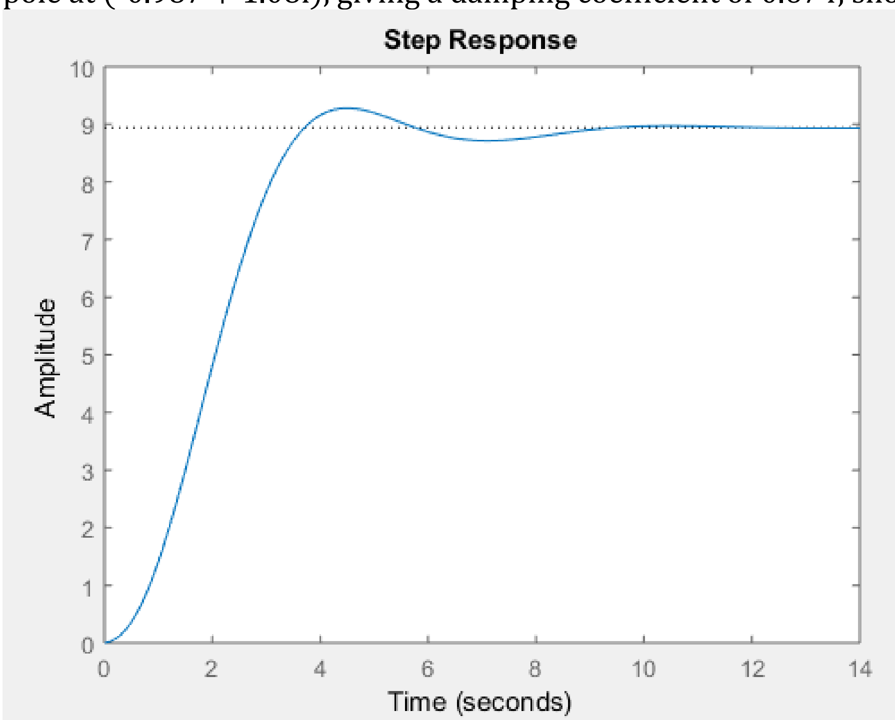


The following code was used to plot all 4 bode plots.

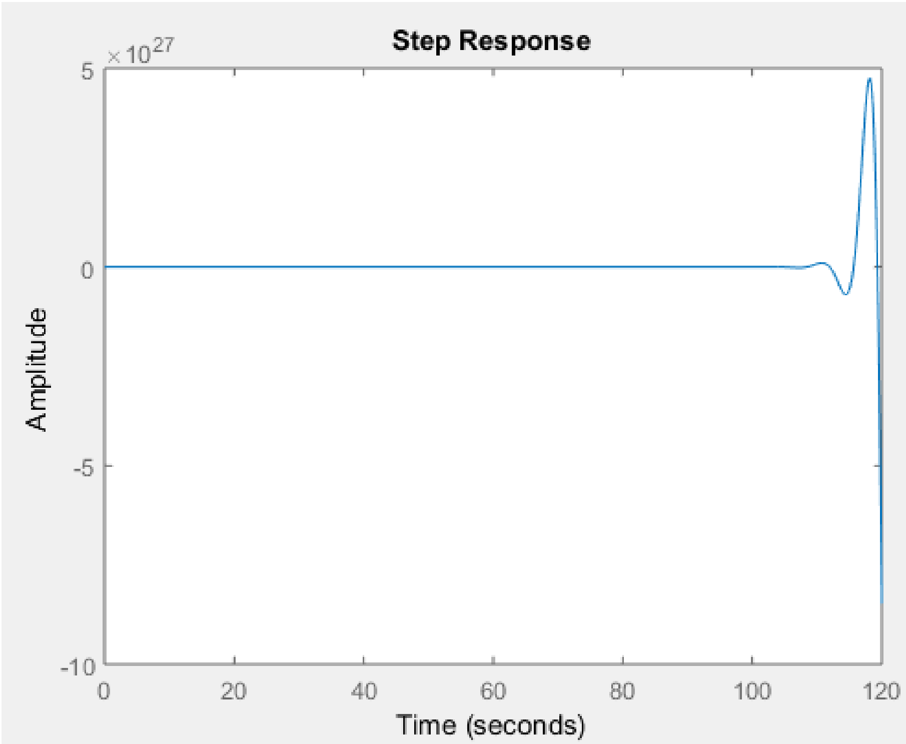
```
function problem6
dk = 0.5;
k = -20.67 - dk;

num = k*[1 16 60];
den = k*[1 6 8 0];

h = tf(num, 1+ den);
%rlocus(h);
bode(h);
```

Problem 7

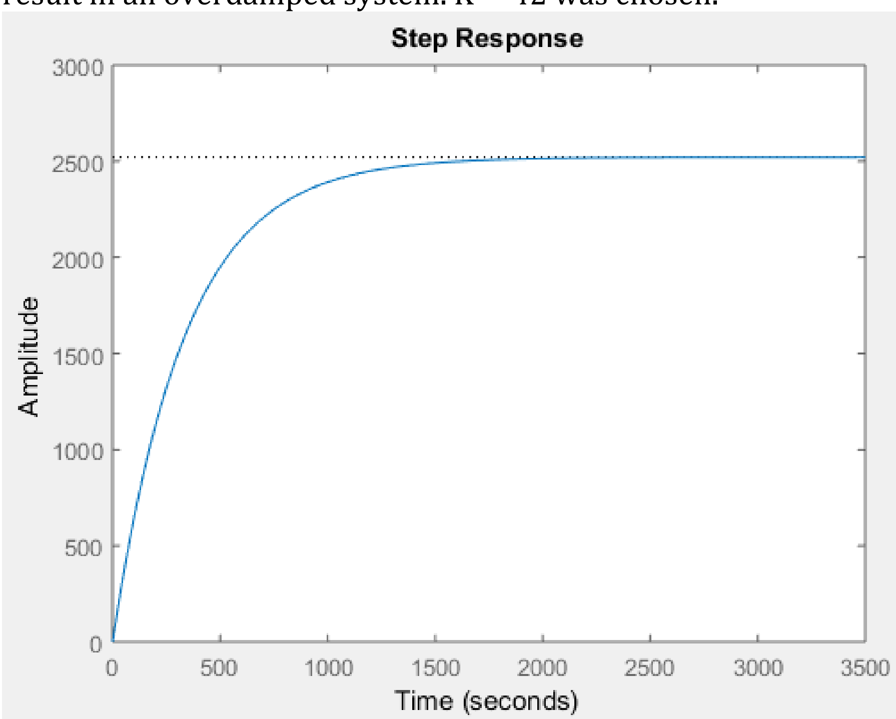
The underdamped plot is the simplest. By the root locus plot, a gain of 0.149 would place a pole at (-0.987 + 1.08i), giving a damping coefficient of 0.674, shown below.



The root locus plot from problem 6 shows that there are no values of K greater than zero for which the system is unstable. However, the root locus is symmetric about the imaginary axis for values of K < 0, which would create a positive feedback loop. Using the same gain from before, multiplied by -1, confirms that a gain of -0.149 makes the system unstable.



Choosing any positive gain on the real axis that coincides with a root locus branch will result in an overdamped system. K = 42 was chosen.



Problem 8

After reconsideration, I have decided the answer to this question is a no. If you were to interpret it literally as a “single” pole in the left-hand plane (with other poles not in the left-hand plane), there is still the qualifier that *that* pole is the one that makes the system unstable as $k \to \infty$. Disallowing $k \le 0$ and adhering to the qualifier that k must approach positive infinity, this means that all possible poles and zeros are shown on the root locus. Inspection of the root locus shows that as $k \to \infty$, all poles and zeros stay in the left-hand plane. I left my previous work below, however it is not my answer to the question.

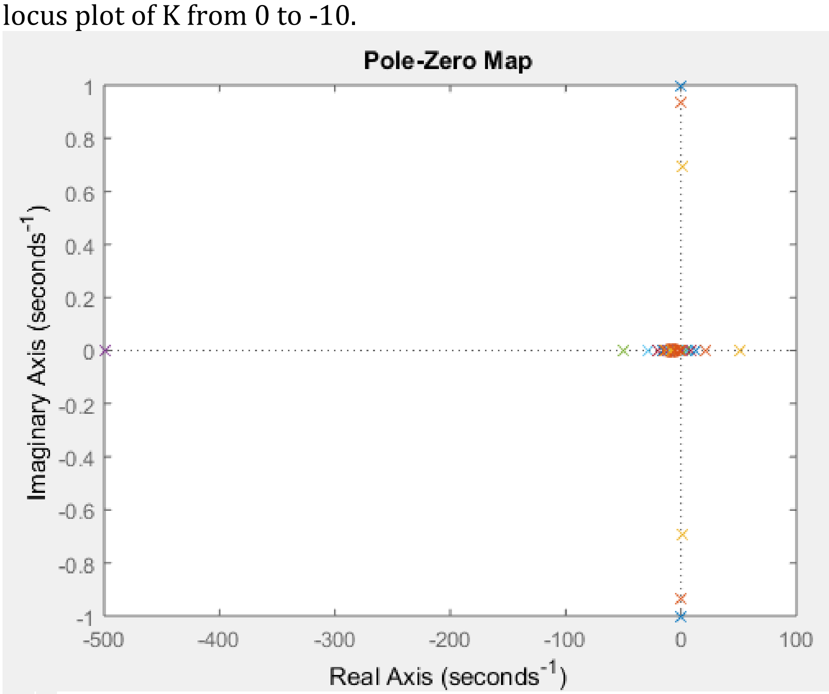
The root locus plot provided by MATLAB plots the system’s poles and zeros as the gain goes from 0 to infinity. All possible gains shown are greater than zero and in the left-hand plane, resulting in stable negative feedback. ~~There are two possible scenarios.~~

Case 1: Values of K are restricted to K >= 0

In this case, there are no values of K that can be chosen to place a pole in the left-hand plane to make the system unstable. This can be seen by tracing each loci along its branch as K goes from 0 to infinity in the root locus plot in Problem 6.

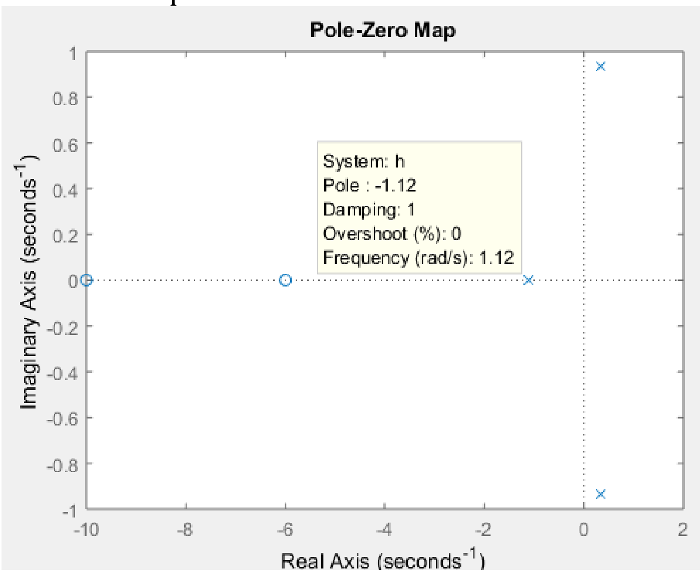
Case 2: Values of K are not restricted to K >= 0

In this case, K can be chosen to be a negative number which allows a pole in the left-hand plane that makes the system unstable. MATLAB was used to generate a pole-zero map equivalent to a root locus plot of K from 0 to -10.



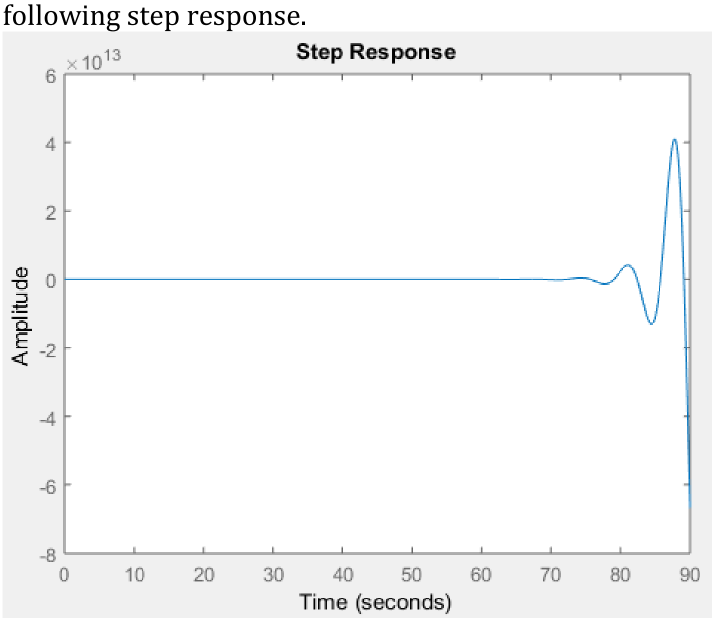

```
function problem8

gain = linspace(0, 10);
for i =1:length(gain)
    n = -gain(i)*[0 1 16 60];
    d = -gain(i)*[1 6 8 0];
    h = tf(n, 1 + d);
    pzmap(h);
    hold on;
end
```

After inspection of this map, I re-ran a single pole-zero map with a gain of -0.1 to choose a pole in the left-hand plane.




```
function problem8

n = -0.1*[0 1 16 60];
d = -0.1*[1 6 8 0];
h = tf(n, 1+d);
pzmap(h);
```

Choosing K to be -0.1, I place a pole at -1.12, resulting in positive feedback. This results in the following step response.




```
function problem8

n = -0.1*[0 1 16 60];
d = -0.1*[1 6 8 0];
h = tf(n, 1+d);
%pzmap(h);
step(h);
```

This system is clearly unstable, although a pole was chosen in the left-hand plane.
(Note- My answer assumes that the question is asking about the closed loop system, NOT the given open loop system of C(s)*G(s)).

Problem 9

Initial inspection shows there is a gain value between 0 and 1, two zeros and two poles. Working from left to right, the first zero is encountered at the origin, as it breaks at $\omega = 1$. Going from a 20 dB slope to a -40 dB slope means there are 3 poles at that breakaway frequency, which is 10. Another zero counteracts one of those poles just past that point, likely around 20. Then to calculate the gain, we estimate the resulting value is around -10.

Our equation is,

$$-10 = 20log_{10}(K) \quad \text{by which we get that} \quad K = \frac{1}{\sqrt{10}} \approx 0.32$$

The separate terms are:

$K = 0.32$ $\quad Z_1\ term\!: s$ $\quad Z_2\ term\!: (\frac{s}{20} + 1)$ $\quad P_1 = P_2 = P_3 = \left(\frac{s}{10} + 1\right)^{-1}$

Putting it all together, the transfer function is:

$$T(s) = \frac{0.32s(\frac{s}{20} + 1)}{(\frac{s}{10} + 1)^3}$$

Inspection of the phase plot confirms the calculations.