Introduction to
Automatic Controls

Phase Portraits for 2^{nd} order linear systems

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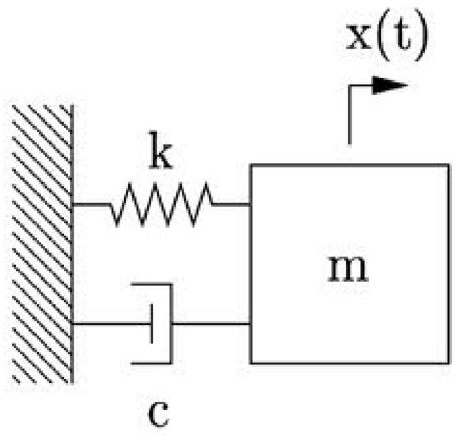
Phase Portraits

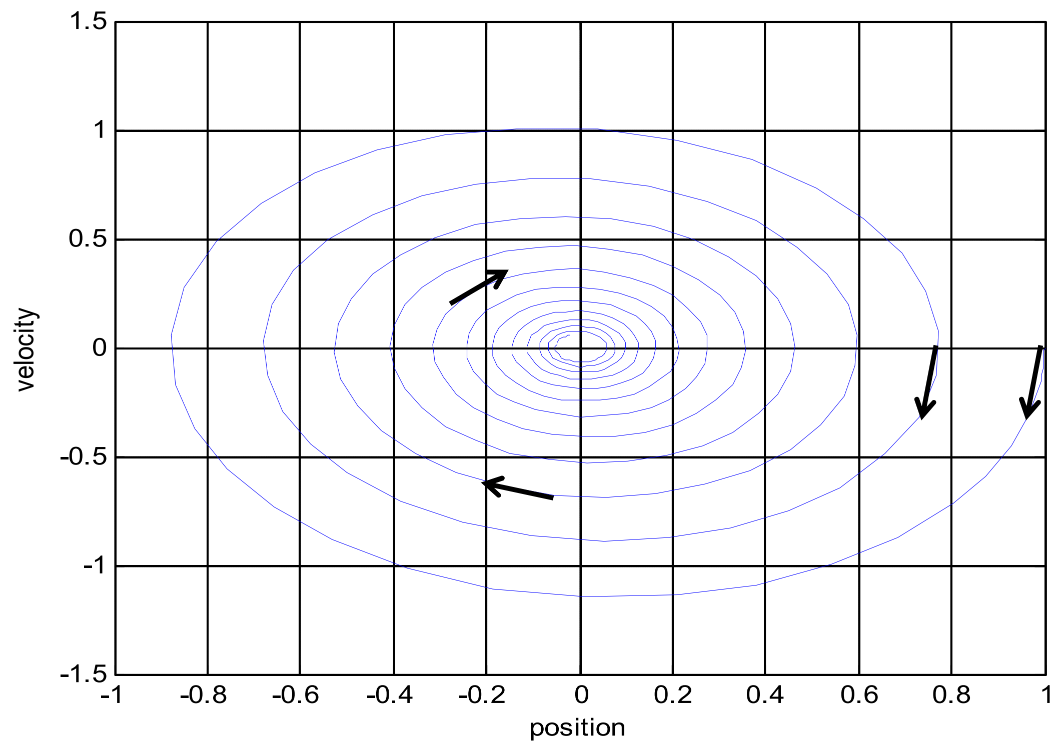


Let…

$$k = 3, \mathrm{b} = 0.2, m = 1, F = 0$$

Some initial conditions…

$$z(0) = \begin{cases} 1 \\ 0 \end{cases}$$




```
%m-s-d solution
[t, z] = ode45( @test, [0 60 ], [ 1 0 ]);
plot( z(:,1), z(:,2));
grid on;
```

```
function zprime = test( t, z )
m = 2; b = .2; k = 3; F = 0;
zprime = [
    z(2);
    -(b/m)*z(2) - (k/m)*z(1) + F; ];
```

A general Linear 2nd order system

Linear systems have only one equilibrium point where $\frac{d\mathbf{z}_e}{dt} = 0$

$$\dot{z}_1 = f_1(z_1, z_2) = az_1 + bz_2 = 0$$
$$\dot{z}_2 = f_2(z_1, z_2) = cz_1 + dz_2 = 0$$

<u>Only solution at (0,0), the Origin!</u>

True for all linear systems.

There are 4 possible solutions to a second order system.

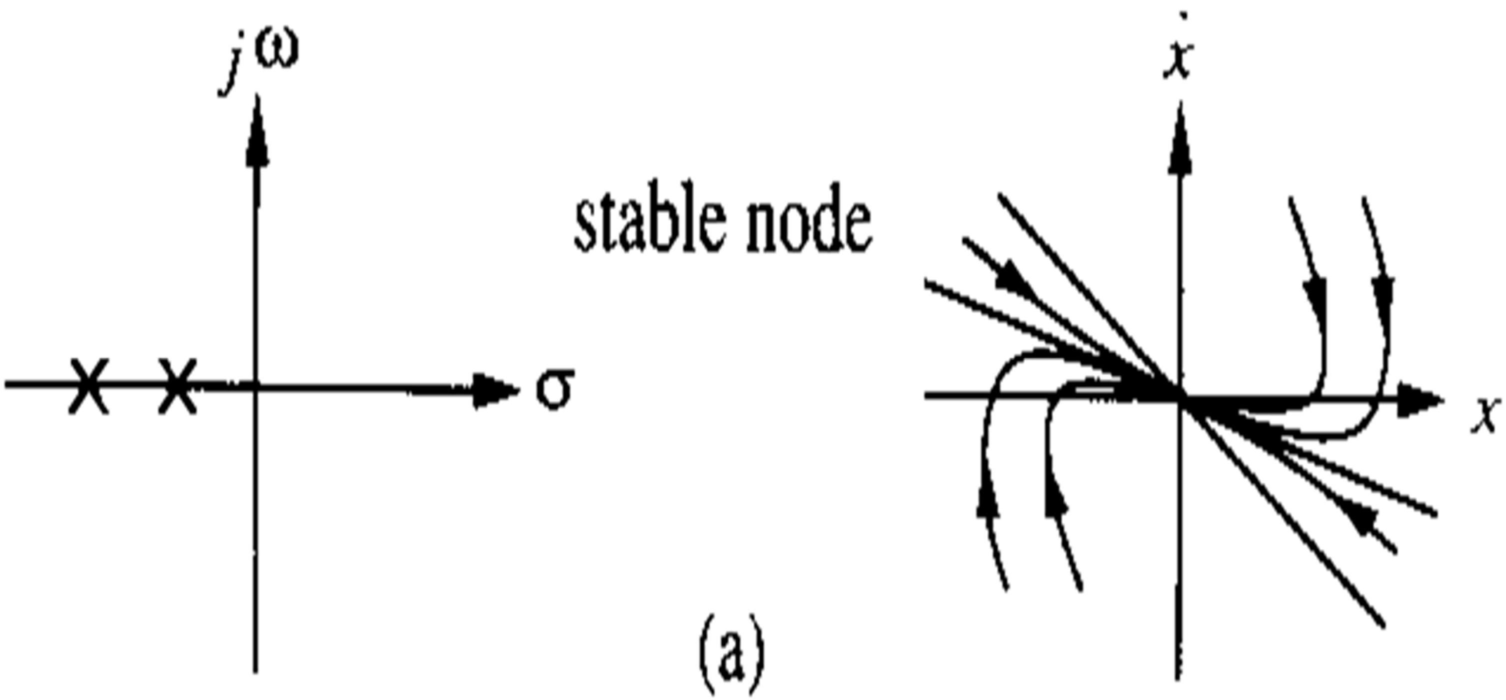
$$\mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$

1. *The eigenvalues are both real and have the same sign.*
2. *The eigenvalues are both real and have opposite signs.*
3. *The eigenvalues are complex conjugates with nonzero real parts.*
4. *The eigenvalues are complex conjugates with real parts equal to zero*

"What do the phase portraits look like for each scenario?"

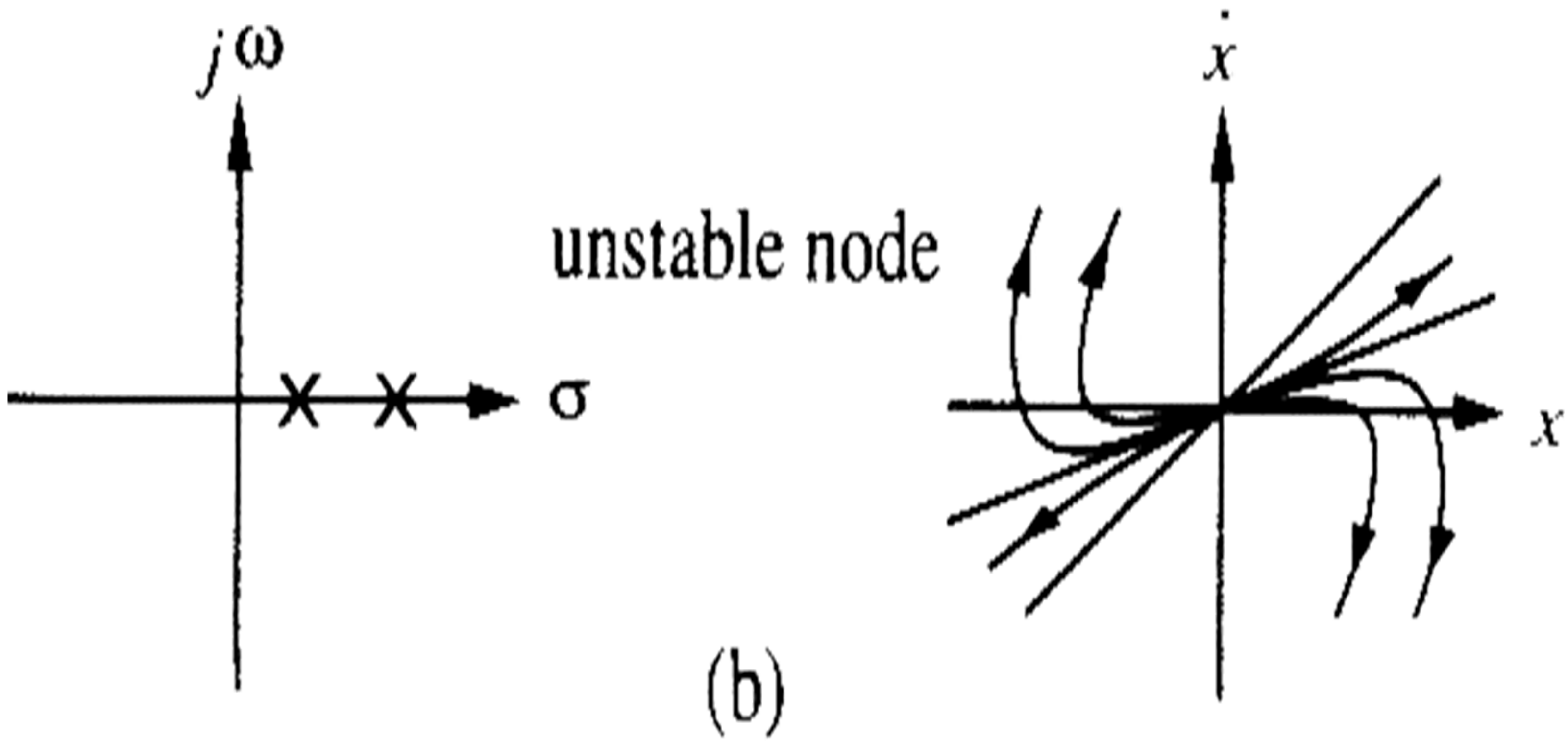
Both eigenvalues are real and negative

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



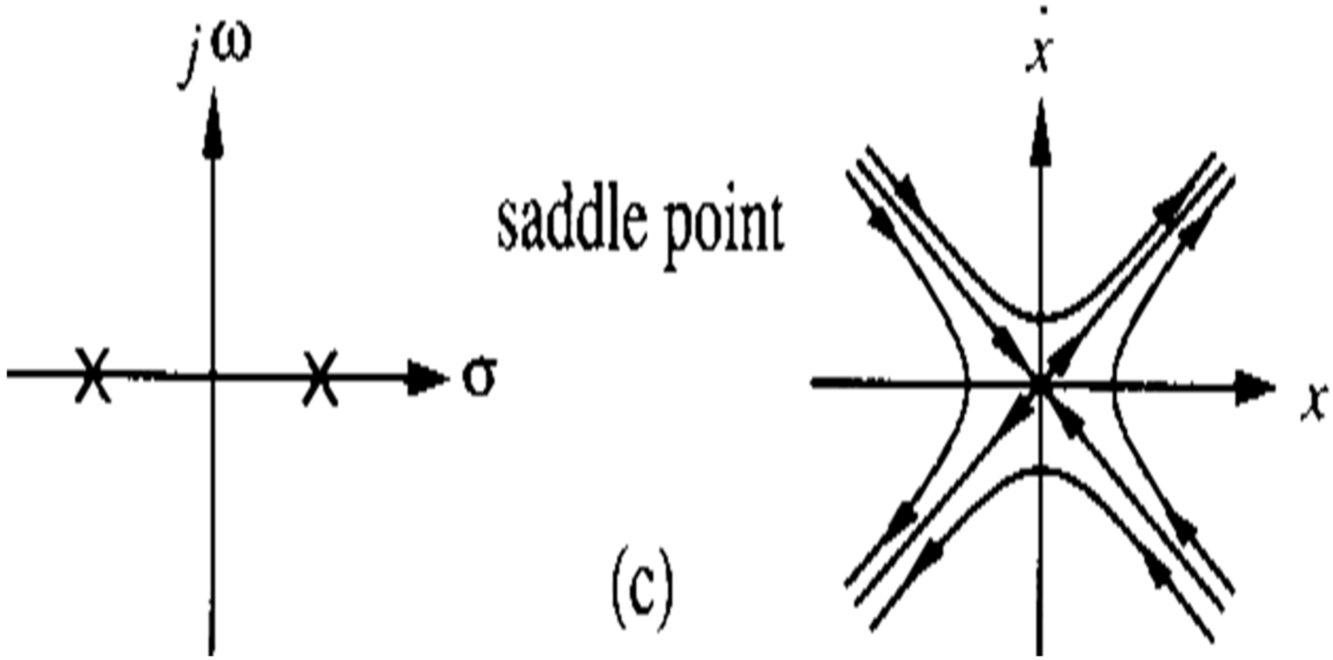
Both eigenvalue are real and positive

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



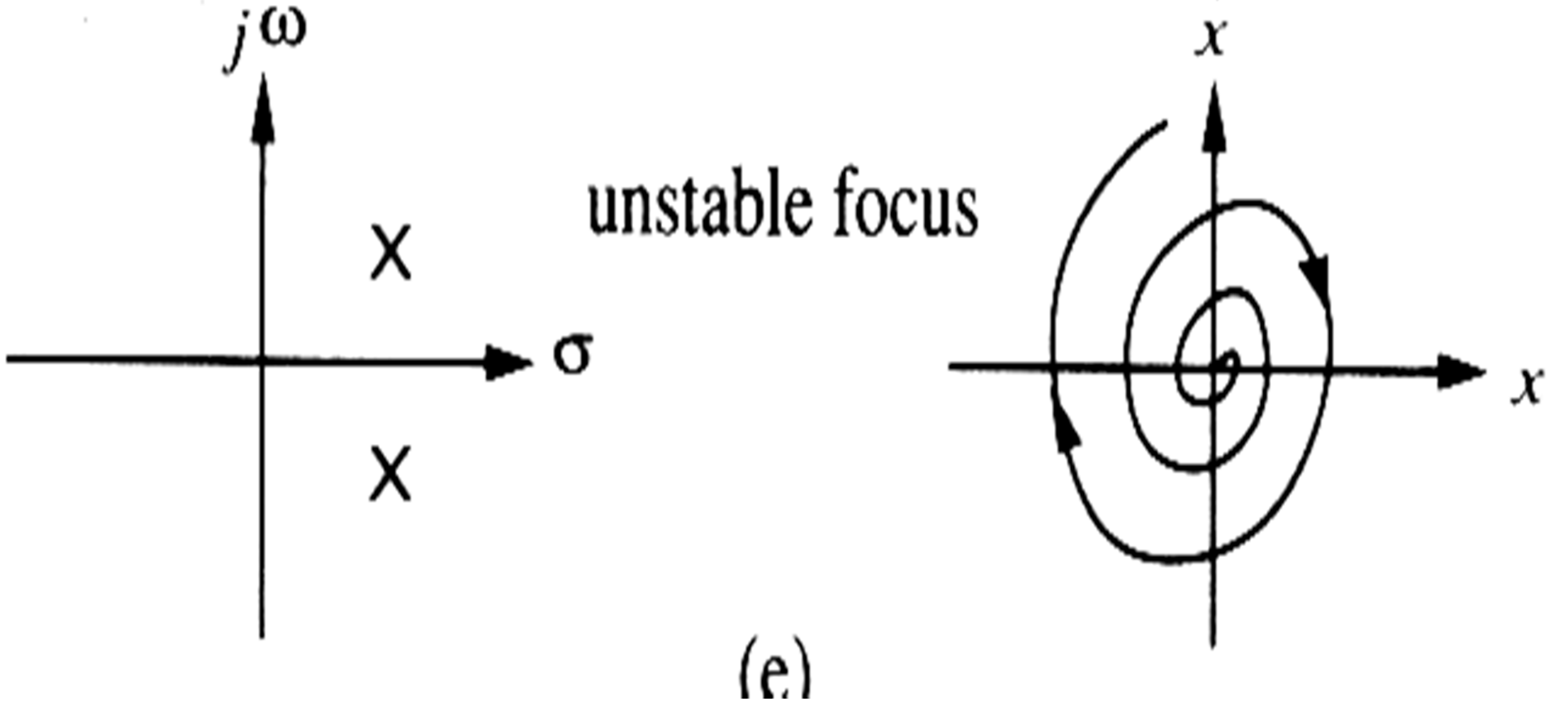
Both eigenvalues are real with different signs

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



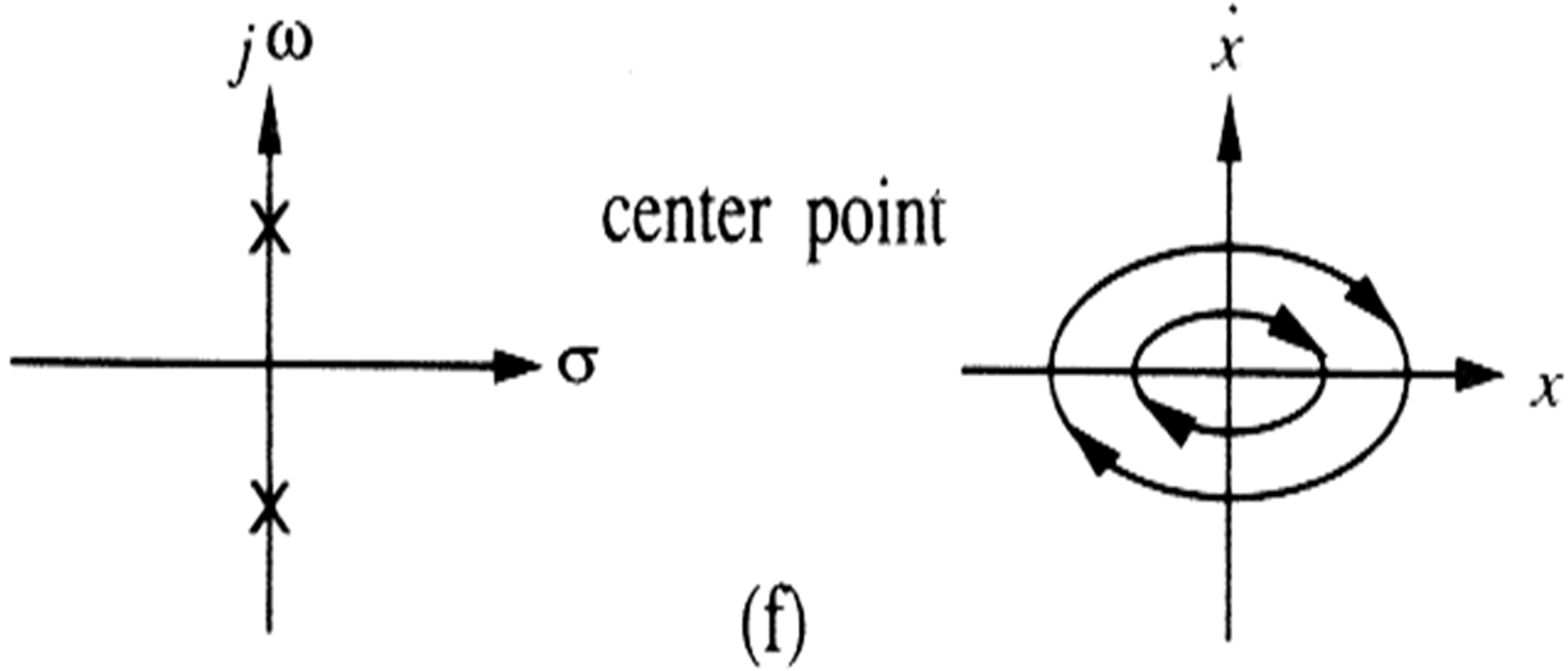
Both eigenvalue positive complex conjugates

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



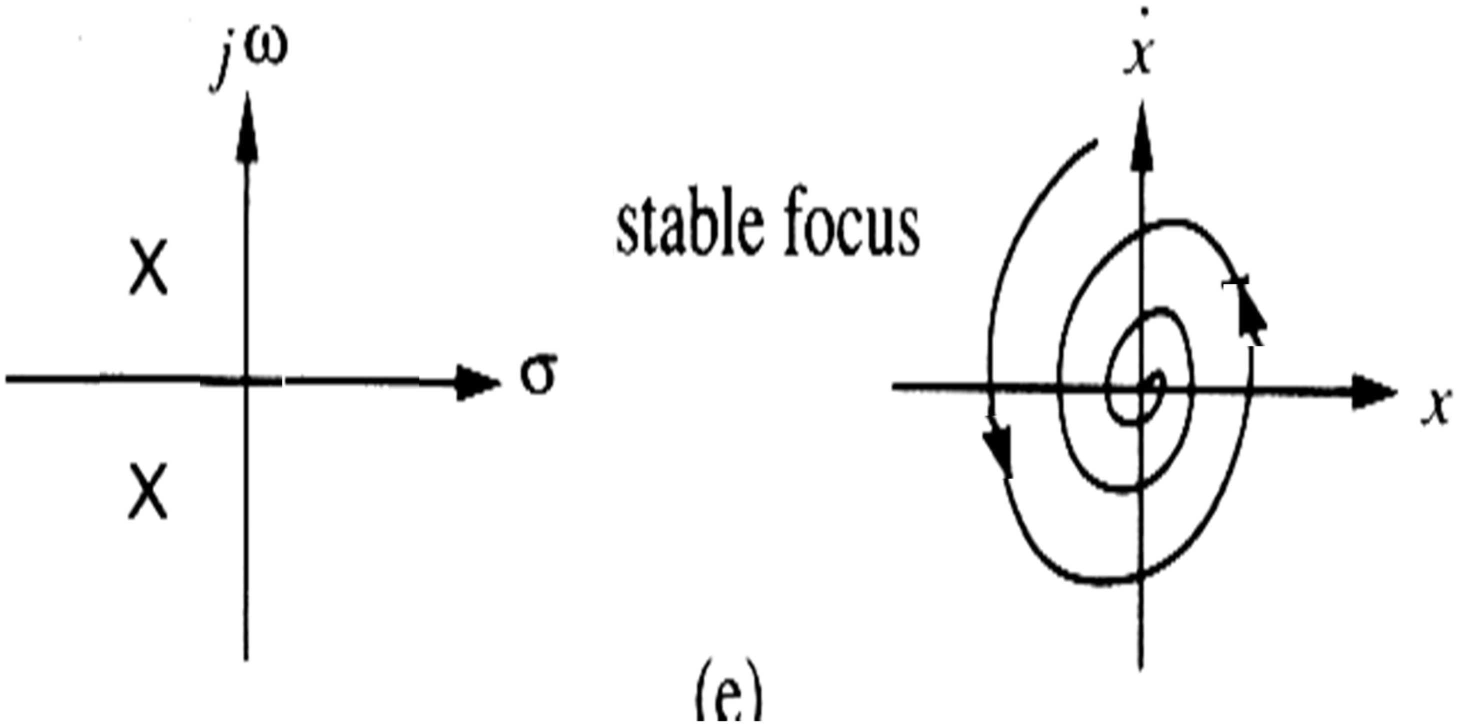
Both eigenvalues have no real component

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



Both eigenvalues are complex conjugates with negative real components

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \Rightarrow \qquad \mathbf{z}(t) = k_1 e^{\lambda_1 t}\boldsymbol{\eta}_1 + k_2 e^{\lambda_2 t}\boldsymbol{\eta}_2$$



Back to the nonlinear example…

In our example…

$$\ddot{x}+0.6\dot{x}+3x+x^2=0$$

$$z_1=x\Rightarrow\dot{z}_1=z_2$$

$$z_2=\dot{x}\Rightarrow\dot{z}_2=-0.6z_2-3z_1-z_1^2$$

$$f_1(z_1,z_2)=z_2=0$$

$$f_2(z_1,z_2)=-0.6(0)-3z_1-z_1^2=0$$

$$0=z_1(-3-z_1)$$

(0,0) and (-3,0) are equilibrium points;

We can linearize about the equilibrium points and examine stability in near z_e

Find the Jacobian for the system….

$$\mathbf{J}=\begin{bmatrix}\frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2}\\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3-2z_1 & -0.6\end{bmatrix}$$

Like a gradient, but for multiple variables

At (0,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ -3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}-0.3000+1.7059i\\ -0.3000-1.7059i\end{bmatrix}$$

so stable near (0,0)

At (-3,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ 3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}1.4578\\ -2.0578\end{bmatrix}$$

so unstable saddle point at (-3,0)

Our solution…

At (0,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ -3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}-0.3000+1.7059i\\ -0.3000-1.7059i\end{bmatrix}$$

so stable near (0,0)

At (-3,0)

$$\mathbf{J}=\begin{bmatrix}0 & 1\\ 3 & -0.6\end{bmatrix}\Rightarrow\boldsymbol{\lambda}=\begin{bmatrix}1.4578\\ -2.0578\end{bmatrix}$$

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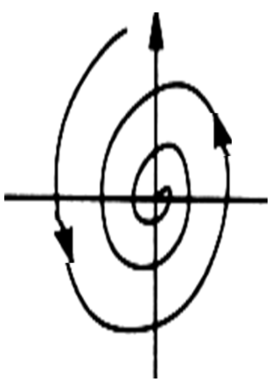
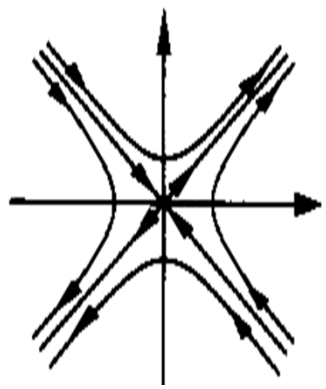
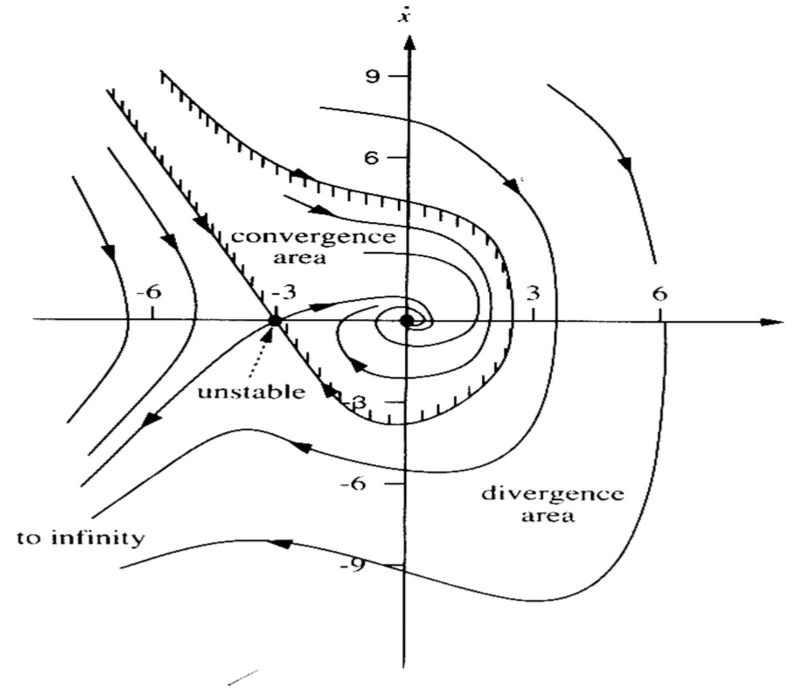
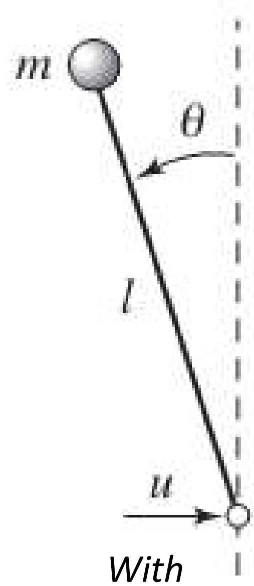






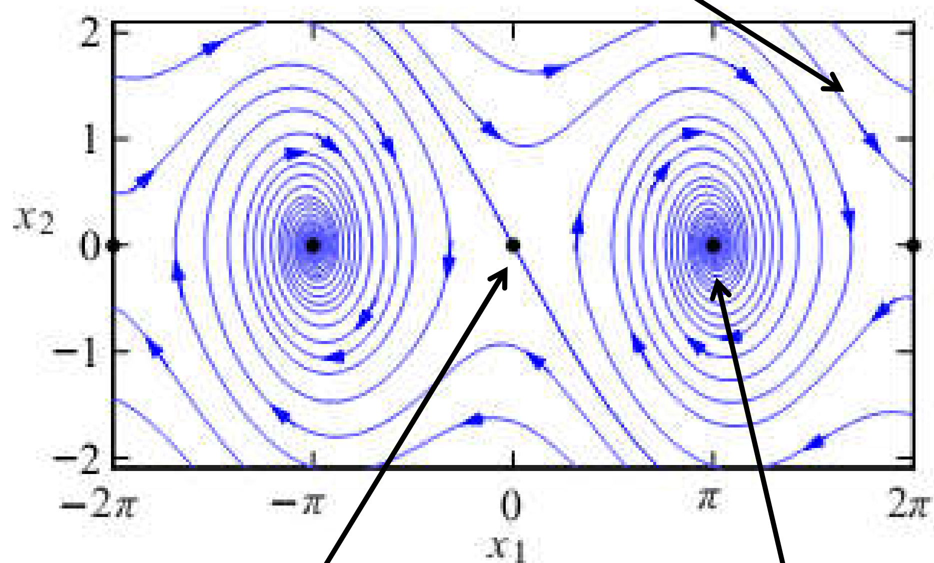
Tada!

Final Example, Pendulum



With friction

360°+1° is 361°, not 0°



unstable equilibrium point

Stable equilibrium point

Summary

- Summarized the relationship between eigensystems and phase portraits for second order systems
  - From linear algebra, we see the clear relationship between eigenvalues and stability.
  - We see that there is always 1 (and only 1) equilibrium point at the origin.
  - We also see how eigenvectors impact a systems behavior.
- We can find the Jacobian (i.e. linearize) nonlinear systems at identified equilibrium points to determine system behavior near them.
- From this we can infer, that a controller that stabilizes a system or changes its dynamic behavior must modify the system's underlying eigenvalues and eigenvectors.