Nyquist Plots and Stability Margins

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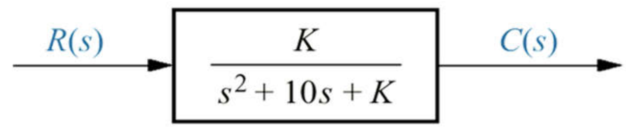
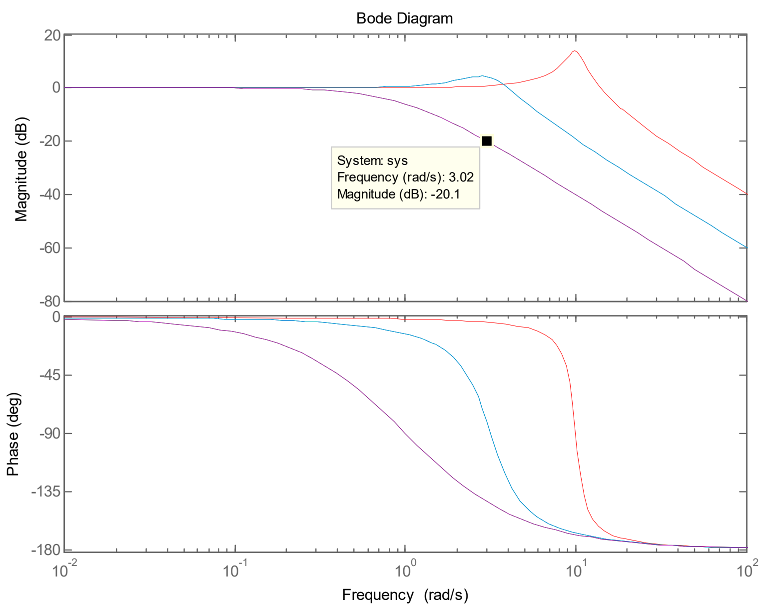
Lesson Objective

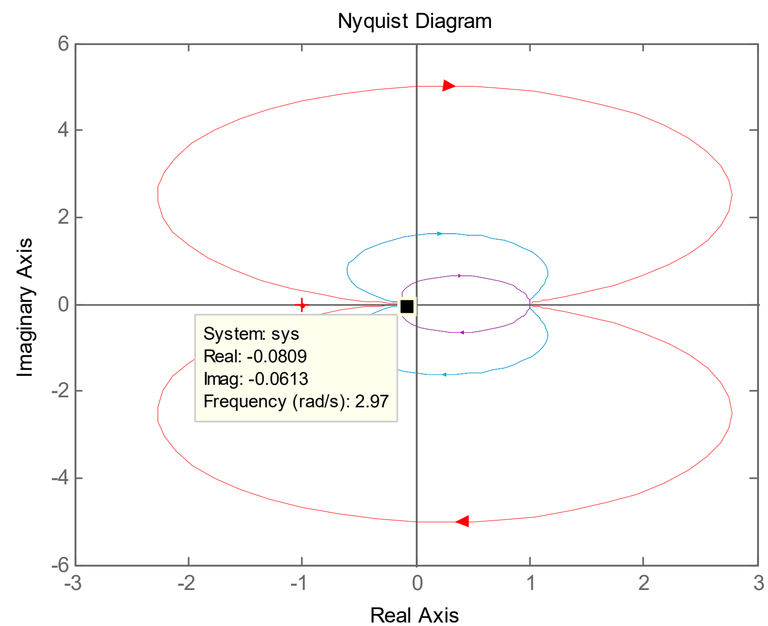
- Define Nyquist Plots
- Review how to create them
  - Learn how complex poles impact the Nyquist Plot
  - Learn how poles at the origin impact the Nyquist Plot
  - Learn how zeros at the origin impact the Nyquist Plot
- Summarize why they are useful
  - Define the Loop Transfer Function
  - Define the Stability Margins (gain and phase)

Nyquist plots vs. Bode Plots



- **Bode Plots** show the frequency vs amplitude and phase on different plots
- **Nyquist (Polar) Plots** display the frequency and amplitude on same plot
  - frequency (0 to infinity (or –infinity to infinity)) is the parameter that is plotted.
- **Nyquist Stability Criterion** defines a system’s stability in the frequency domain.
  - The Nyquist plot changes with respect to a system variable in a way that controller designers can better see its impact on stability.
- Example plots K = 100 (red), 10 (green), 1 (blue)





Creating a Nyquist Plot

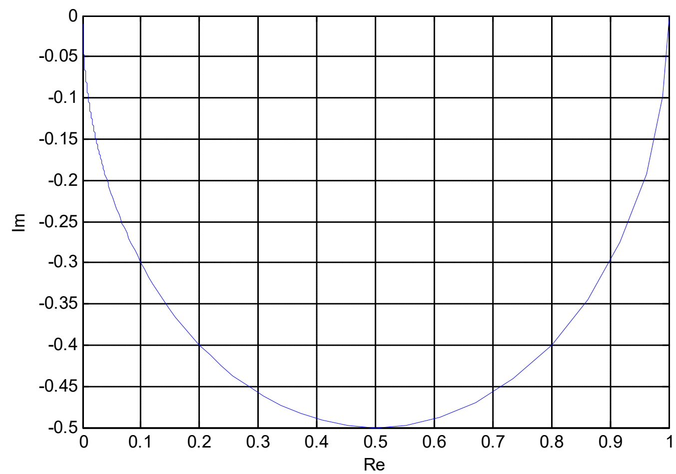
A simple example….

$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega+1} = \frac{1}{j\omega+1}\left(\frac{1-j\omega}{1-j\omega}\right) = \frac{1-j\omega}{1+\omega^2}$$

Separate this into the real and imaginary components…

$$\mathrm{Re}(G(j\omega)) = \frac{1}{1+\omega^2} \qquad \mathrm{Im}(G(j\omega)) = \frac{-\omega}{1+\omega^2}$$

```
w = [0:.1:100];

for i=1:length(w)
    re(i) = 1/(1+w(i)*w(i));
    im(i) = -w(i)/(1+w(i)*w(i));
end

figure(1)
plot( re, im );
xlabel('Re'), ylabel('Im')
grid on;
```



Nyquist plots

$$G(s) = G(j\omega) = \frac{1}{s+1} = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2}$$

Manually

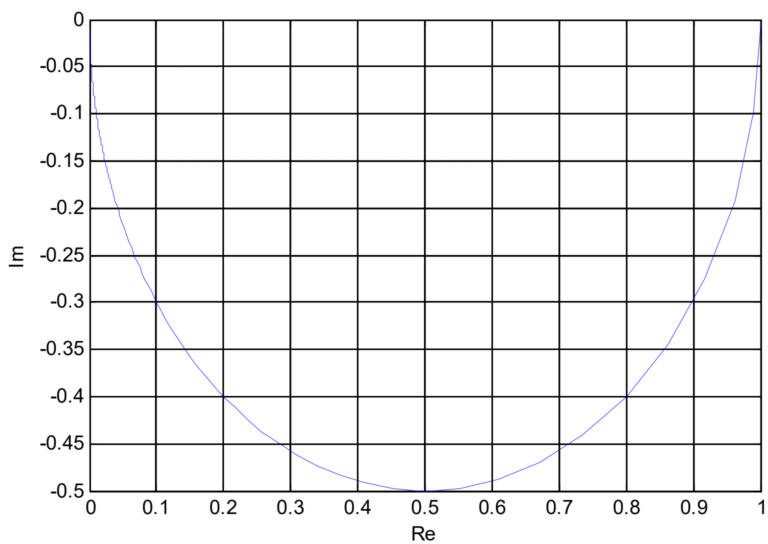
```
w = [0:.1:100];

for i=1:length(w)
    re(i) = 1/(1+w(i)*w(i));
    im(i) = -w(i)/(1+w(i)*w(i));
end

plot( re, im );
xlabel('Re'), ylabel('Im')
```

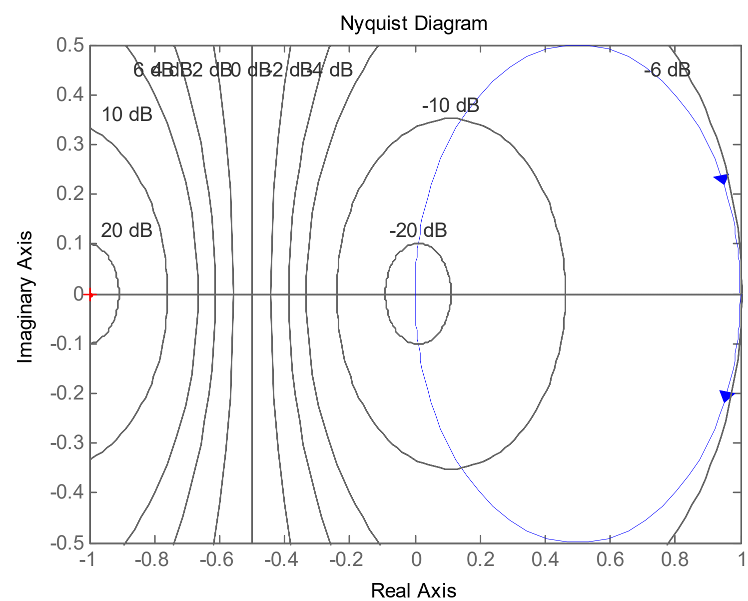
Or

```
w = [0:.1:100];

for i=1:length(w)
    re(i) = real(1/(w(i)+1));
    im(i) = imag(1/(w(i)_1));
end

plot( re, im );
xlabel('Re'), ylabel('Im')
```

Using MATLAB

```
num = [1]
den = [ 1 1 ]
sys = tf( num, den)
figure(2)
nyquist(sys)
```





Nyquist for higher order systems

A 2nd order example….

$$G(j\omega)=\frac{1}{(s+1)^2}=\frac{1}{(j\omega+1)^2}$$

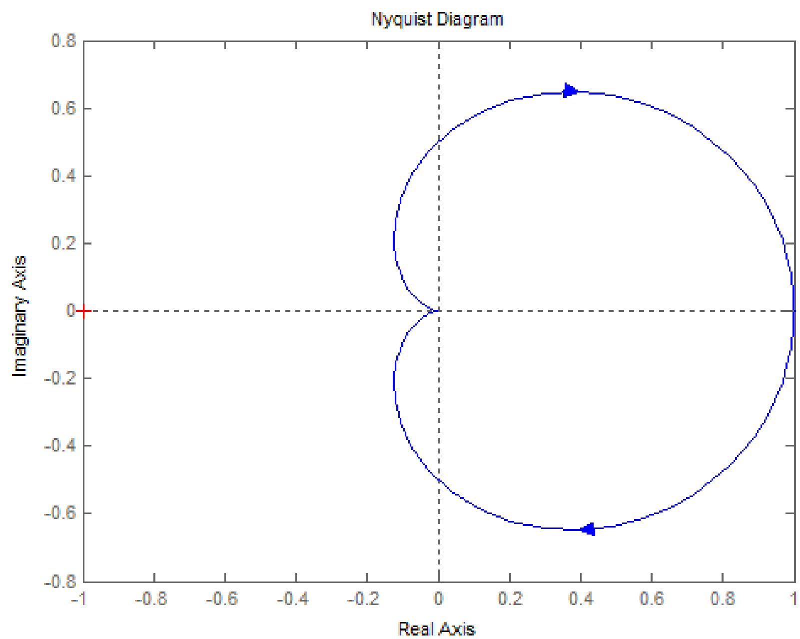
$$=\frac{1}{(j\omega+1)^2}\left(\frac{(1-j\omega)^2}{(1-j\omega)^2}\right)$$

$$=\frac{1-2j\omega-\omega^2}{\omega^4+2\omega^2+1}$$

Separate this into the real and imaginary components…

$$\mathrm{Re}(G(j\omega))=\frac{1-\omega^2}{\omega^4+2\omega^2+1}$$

$$\mathrm{Im}(G(j\omega))=\frac{-2j\omega}{\omega^4+2\omega^2+1}$$



…without MATLAB it can start to get a little complicated…

But there are some simple rules/patterns.

Nyquist for Higher Order Systems

Consider a general transfer function…

$$G(s) = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

with *m* zeroes and *n* poles. Assume (for the moment) that there are no poles at the origin..

$$G(j\omega) = \frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega - p_1)\cdots(j\omega - p_n)}$$

As the frequency goes to zero, the value of G will be a finite real number (aka the *zero frequency gain*).

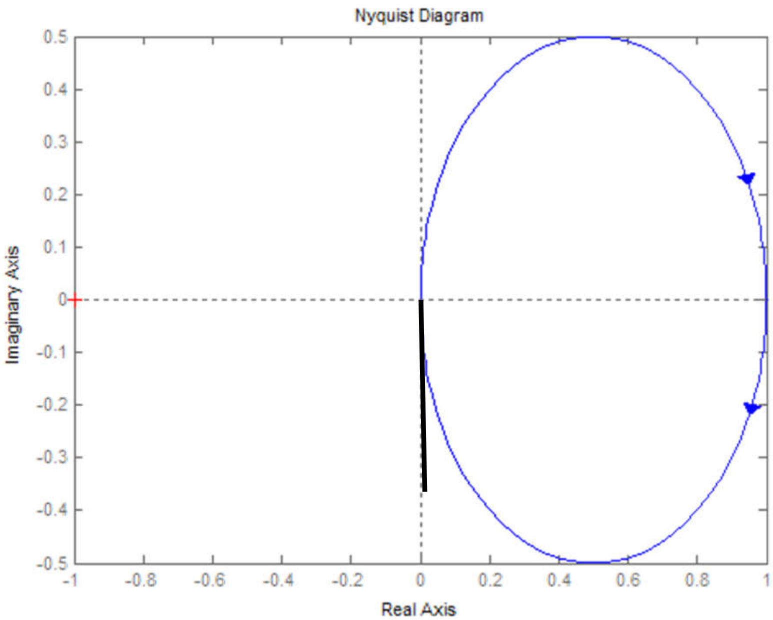
$$G(j\omega) = \frac{(-z_1)(-z_2)\cdots(-z_m)}{(-p_1)(-p_2)\cdots(-p_n)}$$

As the frequency becomes very large, the frequency terms will dominate.

$$G(j\omega) \approx \frac{(j\omega)^m}{(j\omega)^n} = \frac{1}{(j\omega)^{n-m}}$$

Systems of this form (where *n>m*) will approach 0 as frequency increases. The angle of approach is determined by *n-m* and is given by -90*(n-m)*° For example:

$$G(s) = \frac{1}{s+1} \Rightarrow n - m = 1$$



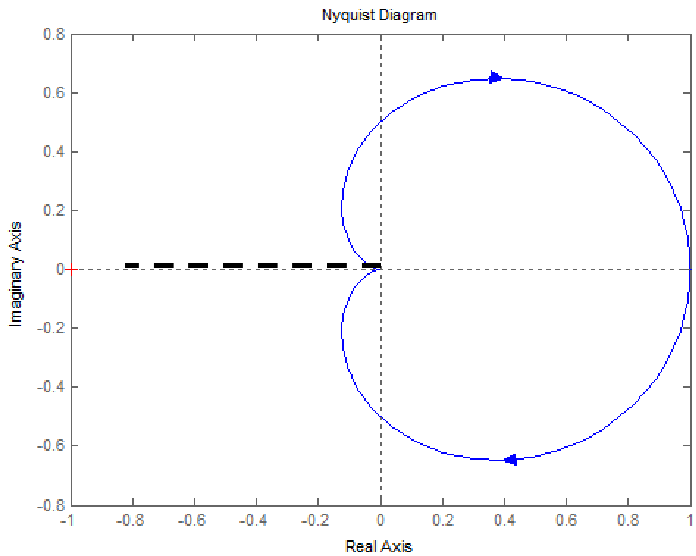
Nyquist for Higher Order Systems

Some more examples showing the angle of approach is given by -90*(n-m)*°.

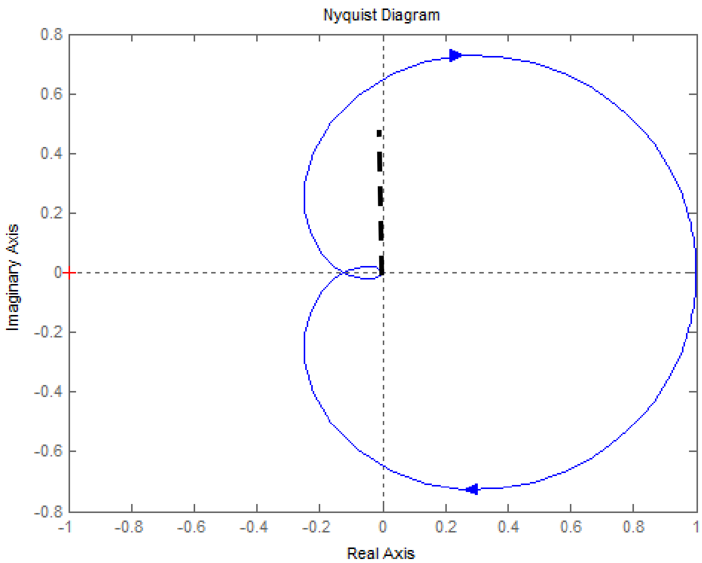
$$G(s)=\frac{1}{(s+1)^2}\Rightarrow n-m=2$$

$$G(s)=\frac{1}{(s+1)^3}\Rightarrow n-m=3$$

-90*(2-0)*°=-180°



-90*(3-0)*°=-270°

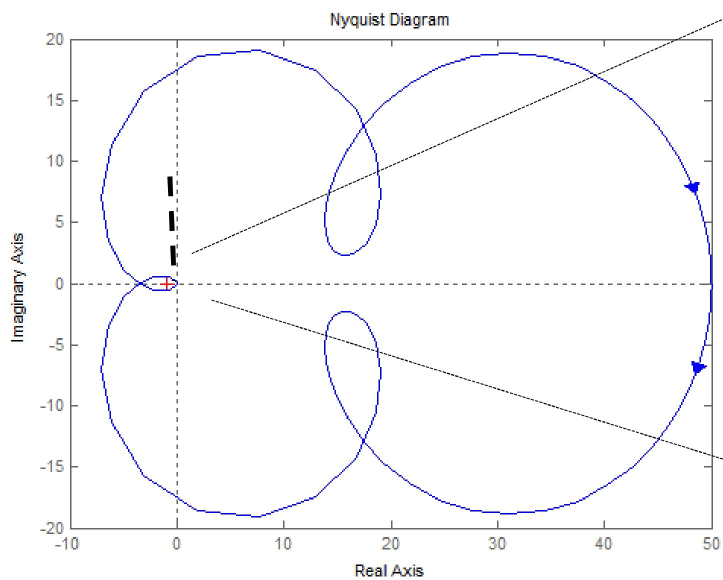


Nyquist for Higher Order Systems

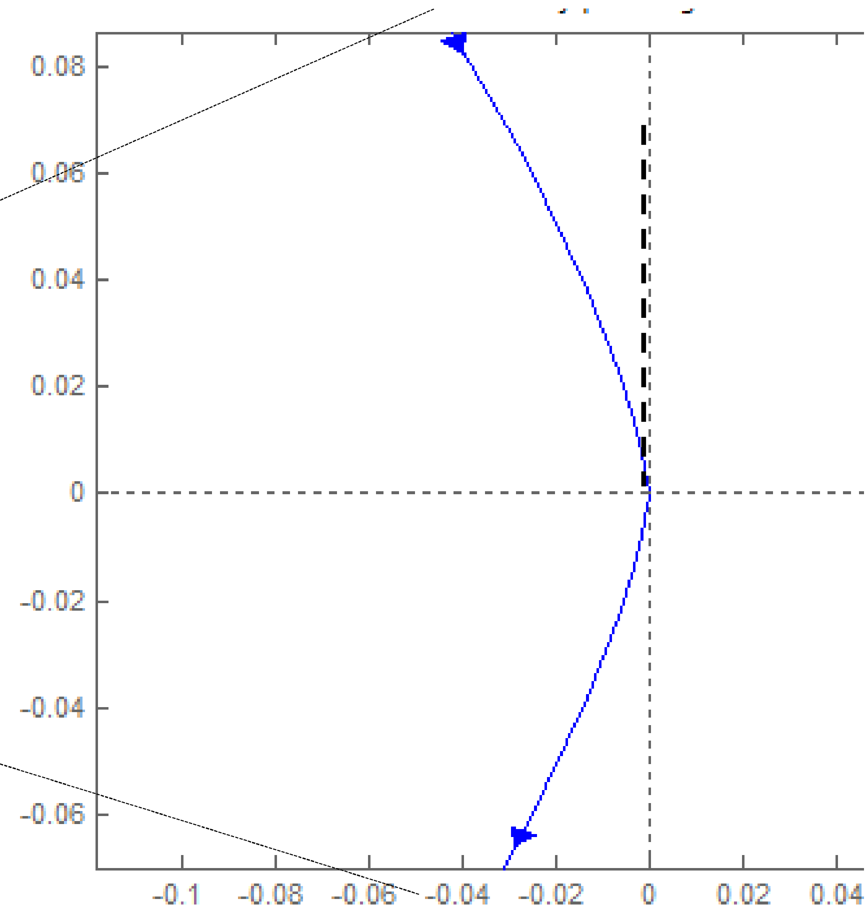
One last example showing the angle of approach is given by -90*(n-m)*°.

$$G(s) = \frac{25\times10^6(s+5)(s+10)}{(s+1)(s+50)^2(s+100)^2}$$

-90*(5-2)*°=-270°

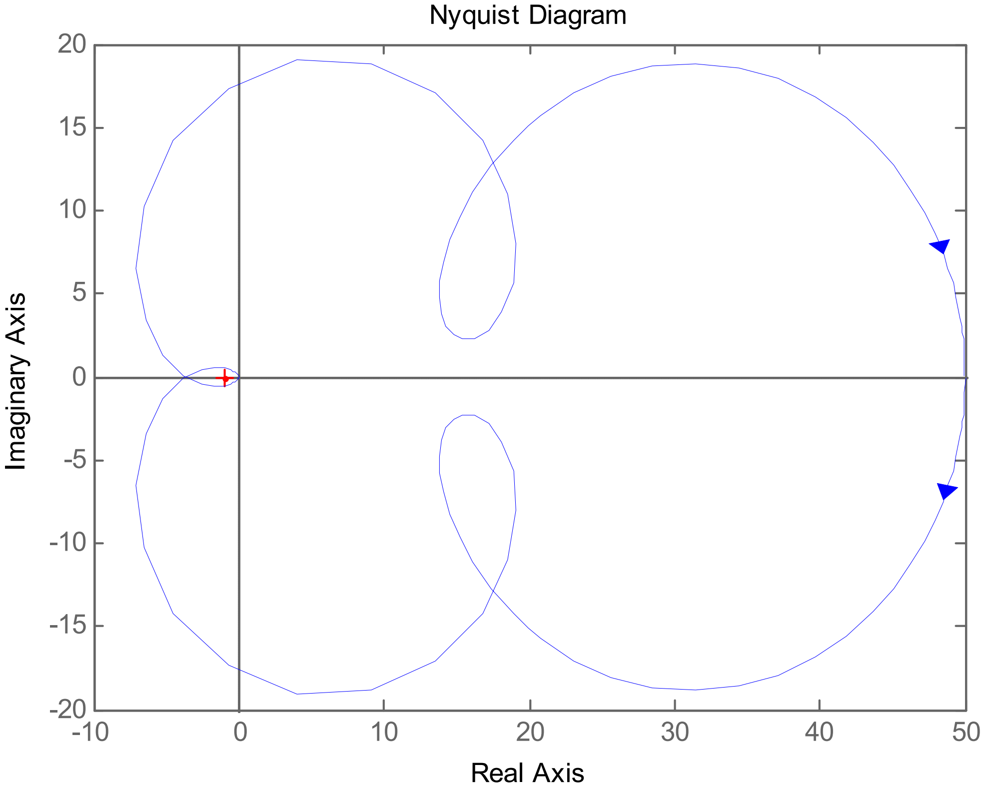


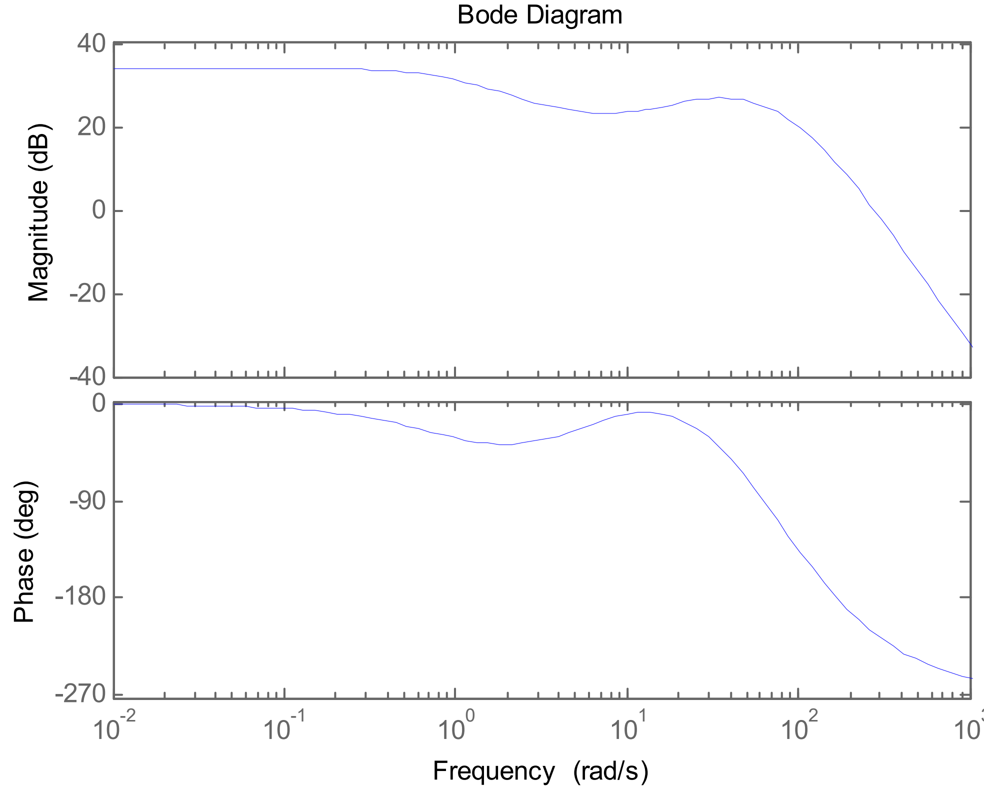
Note the arrows…



Nyquist vs. Bode

$$G(s)=\frac{25\times10^{6}(s+5)(s+10)}{(s+1)(s+50)^{2}(s+100)^{2}}$$





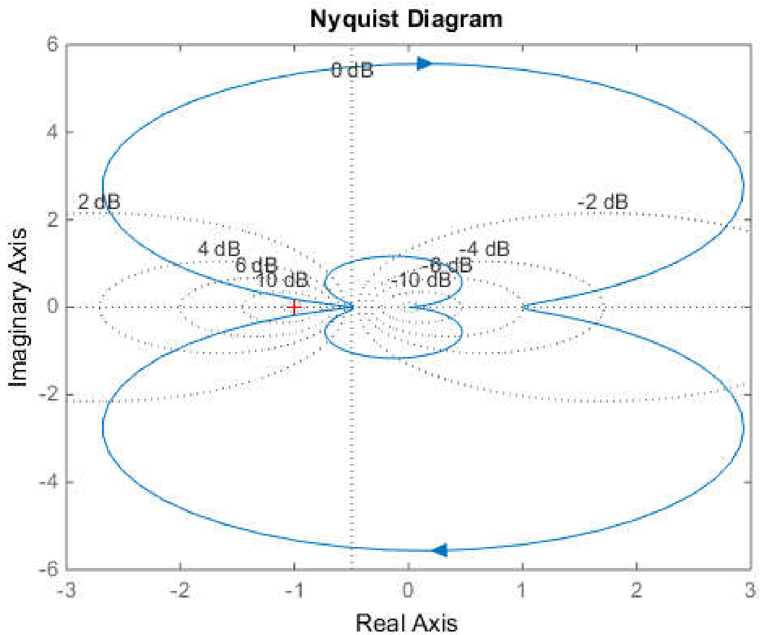

```
z = [ -5 -10 ];
p = [ -1 -50 -50 -100 -100 ]
k = 25000000;
sys = zpk( z, p, k )

nyquist(sys)
bode(sys)
```

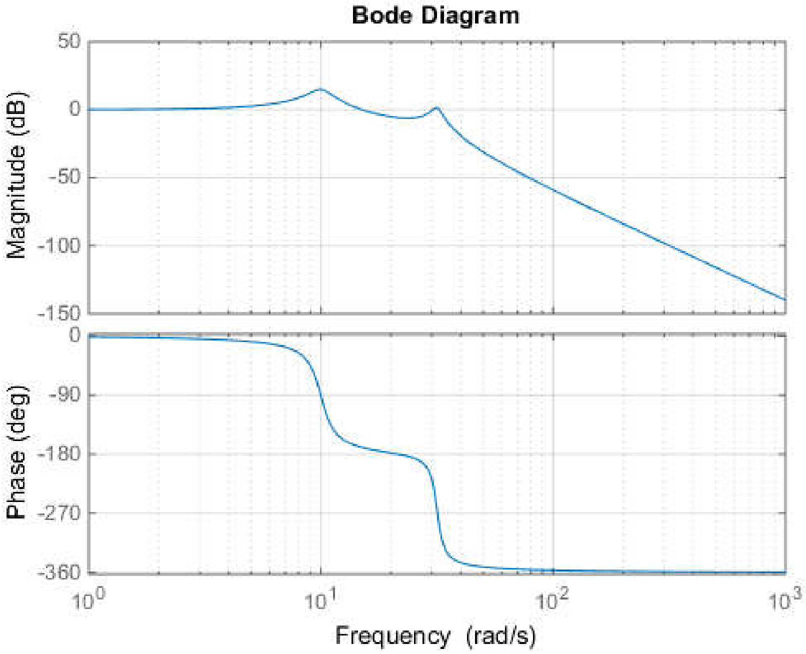
-90*(5-2)*°= -270°

Special cases: Complex roots

$$G(s) = \frac{10^5}{(s^2 + 2s + 100)(s^2 + 3s + 1000)}$$

Looking at the Bode Plot for the system…





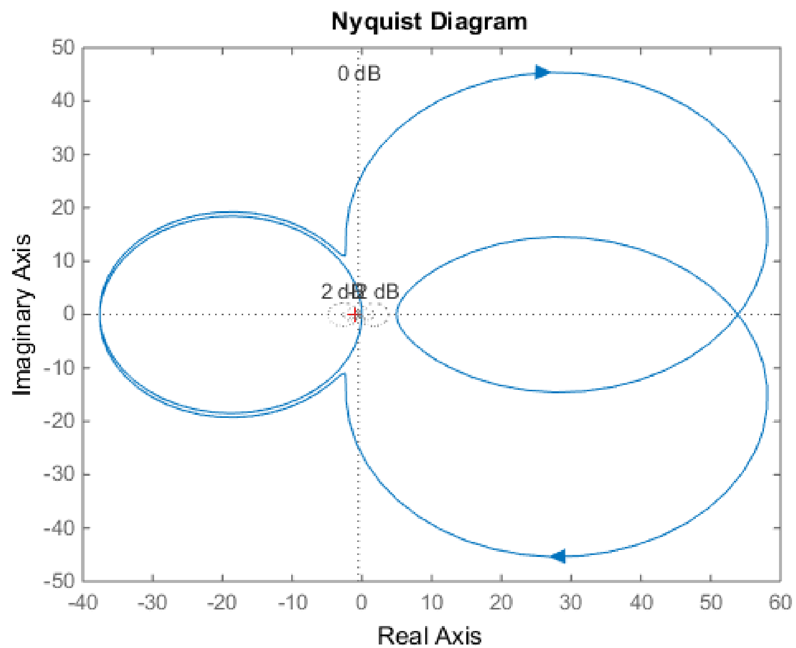
Still observe the increases in magnitude at times when the frequency increases.

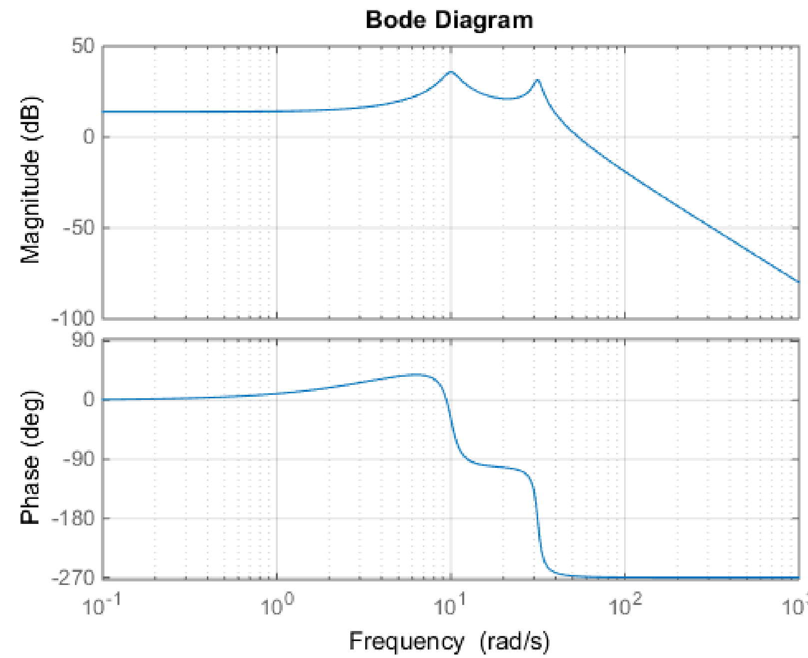
See the final -360° phase lag as the frequency goes to infinity as expected.

Special cases: Complex roots

$$G(s) = \frac{10^5(s+5)}{(s^2+2s+100)(s^2+3s+1000)}$$

The impact of a zero we should recognize on the Bode plot, but the response becomes harder to follow on the Nyquist Plot





Now considering more complex systems, it is less clear why we care about Nyquist plots. We are getting to that.

You see the bulge, the rapid phase change and the -270° phase lag as the frequency goes to infinity.

Special cases: Zero(s) at Origin

Zero at the origin

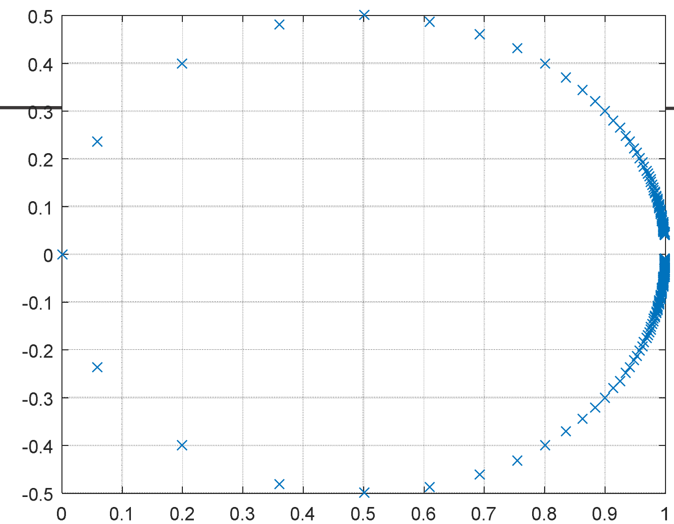
Similar procedure, but note the frequency of the real and imaginary components when the frequency is 0.

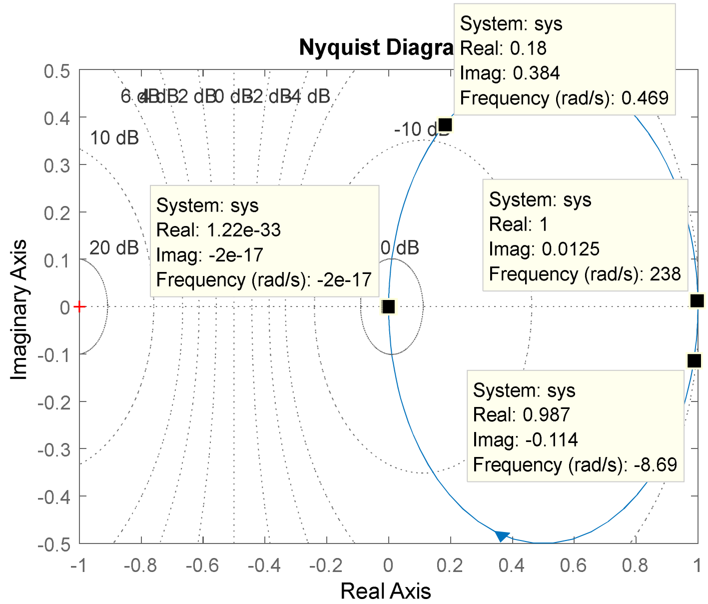
$$G(j\omega) = \frac{j\omega}{(j\omega+1)} \frac{(j\omega-1)}{(j\omega-1)}$$

$$= \frac{-\omega^2 - j\omega}{-\omega^2 - 1}$$

$$= \frac{\omega^2}{\omega^2+1} + j\frac{\omega}{\omega^2+1}$$

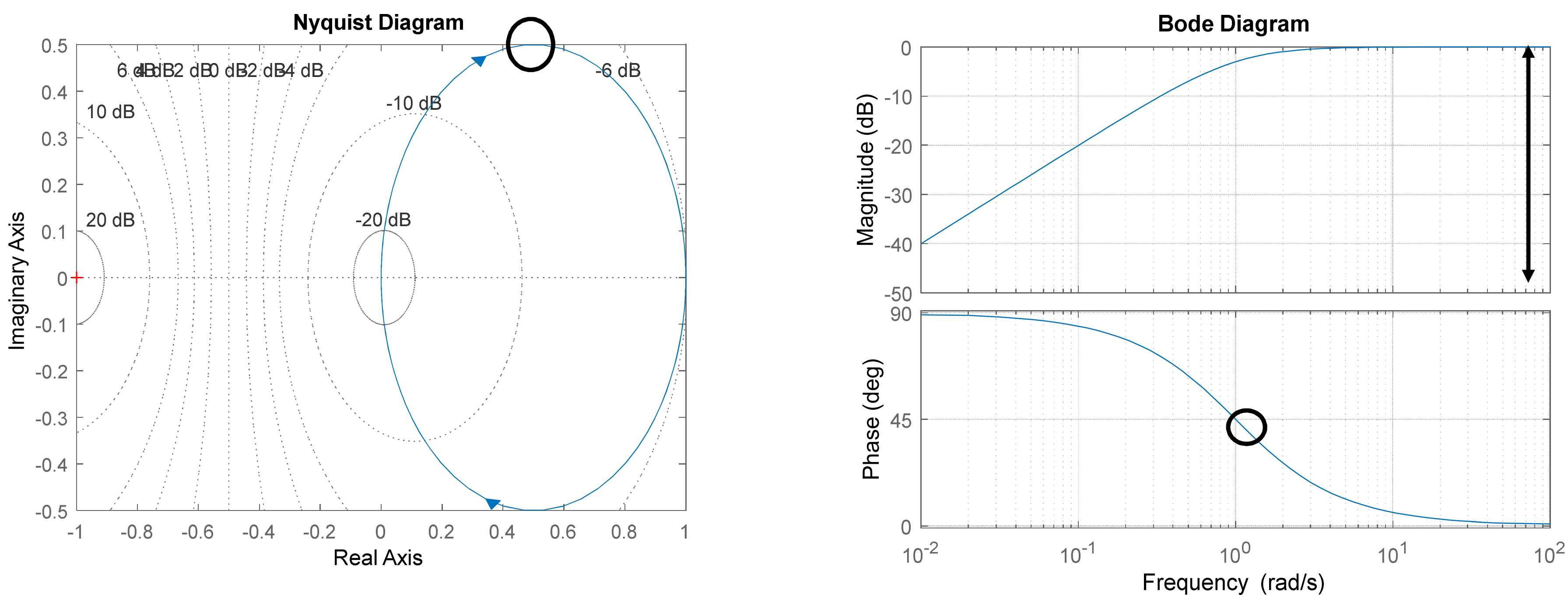
```
clear all;
w(1) = 0-100i;
for c=2:500
  w(c) = w(c-1) + (0+0.25*i);
end

for i=1:length(w)
  re(i) = real( w(i)/(w(i)+1 ) );
  im(i) = imag( w(i)/(w(i)+1 ) );
end
figure(1); plot( re, im, 'x'); grid on;

sys = zpk( [0], [-1], 1 );
figure(2); nyquist( sys ); grid on;
```





Special Case – Zero(s) at the origin

$$G(s) = \frac{s}{(s+1)} \Rightarrow n - m = 1$$



Special cases – poles at origin

$$G(s)=\frac{1}{s(s+1)}\Rightarrow n-m=2$$

Now our gain approaches infinity as the frequency approaches 0 instead of a finite value

$$G(j\omega)=\frac{1}{j\omega(j\omega+1)}$$

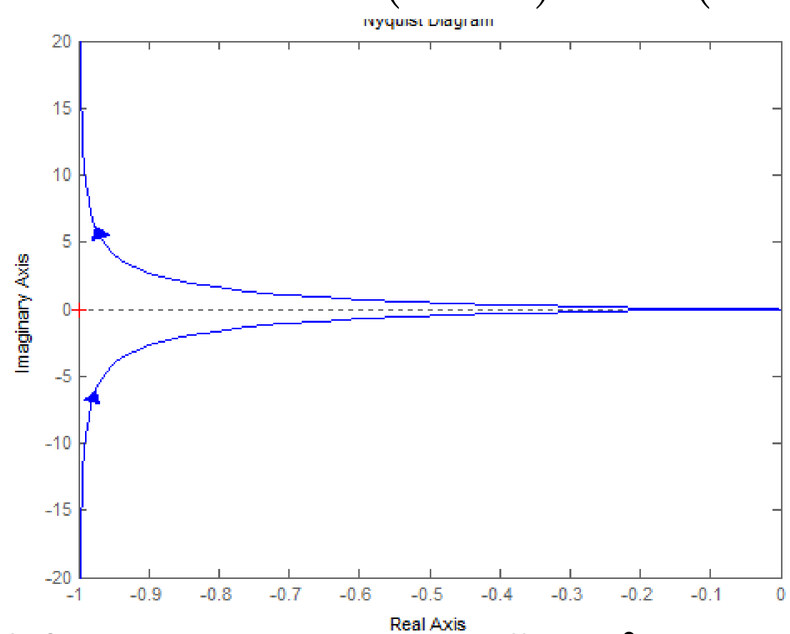
$$=\frac{1}{(j\omega-\omega^2)}\left(\frac{(j\omega+\omega^2)}{(j\omega+\omega^2)}\right)$$

$$=\frac{j\omega+\omega^2}{-\omega^4-\omega^2}$$

This one we can separate into real and imaginary components…

$$\mathrm{Re}(G(j\omega))=\frac{\omega^2}{-\omega^2(\omega^2+1)}=\frac{1}{-(\omega^2+1)}$$

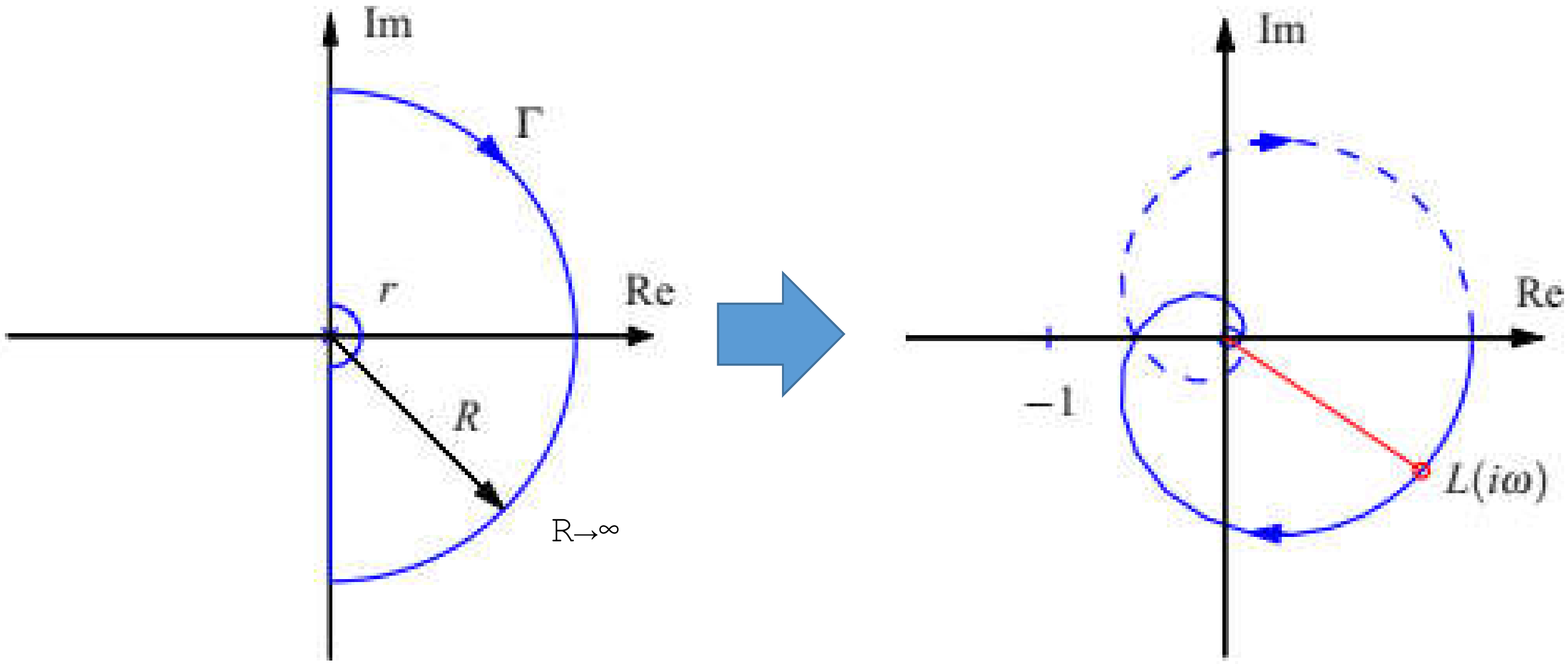
$$\mathrm{Im}(G(j\omega))=\frac{\omega}{-\omega^2(\omega^2+1)}=\frac{1}{-\omega(\omega^2+1)}$$



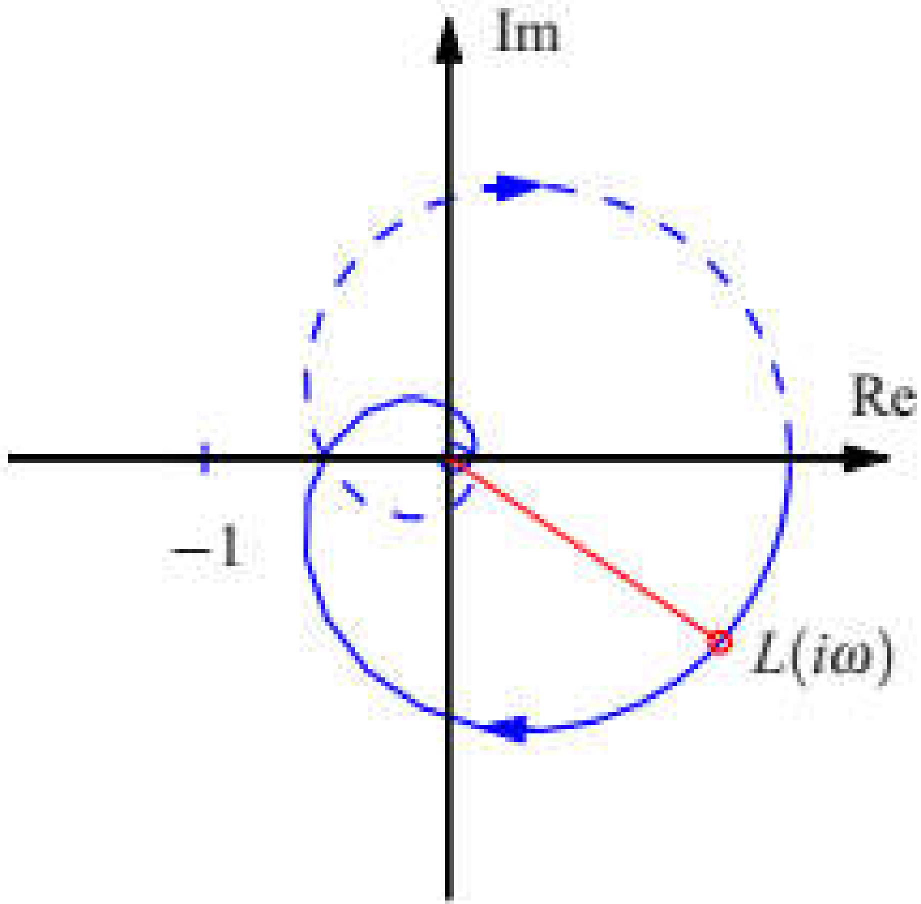
The high frequency asymptote is still -180°

The Nyquist D Contour

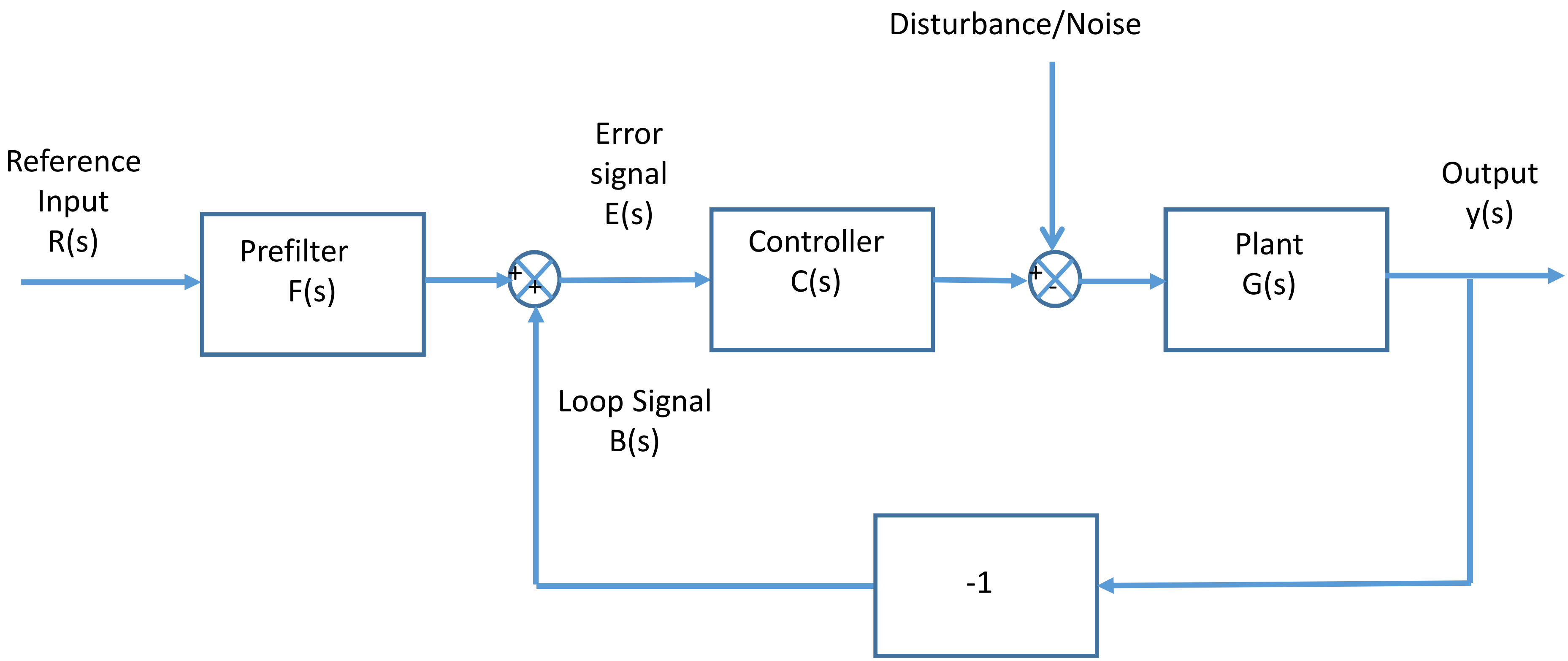
It has been helpful in when drawing the Nyquist to not just think of the frequencies from 0 to ∞, but from -∞ to ∞.



So why are these useful?



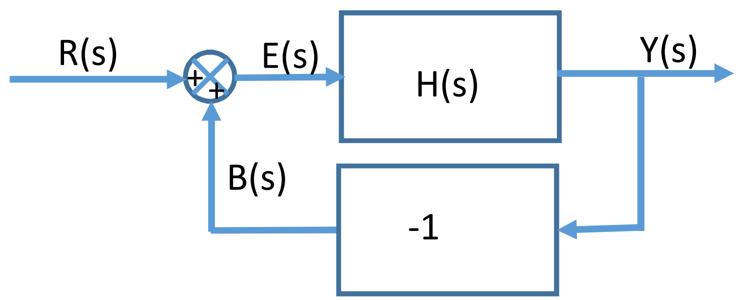
Recall our closed system nomenclature...



*not standardized. Nomenclature varies from textbook to textbook

Recall our closed loop transfer function

Make a couple assumptions (no prefilter, no disturbance, etc.)



<u>Typical Control Blocks and Signals</u>

R(s) – Reference signal or desired output
E(s) – Error signal
H(s) – Often C(s)*P(s)
C(s) – The control law that produces a u for the model or plant
P(s) – The model of the system to be controlled or plant
Y(s) – The output of the system
B(s) – Loop transfer signal (last signal value prior to feedback)

We get…

$$\boxed{T_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)}}$$

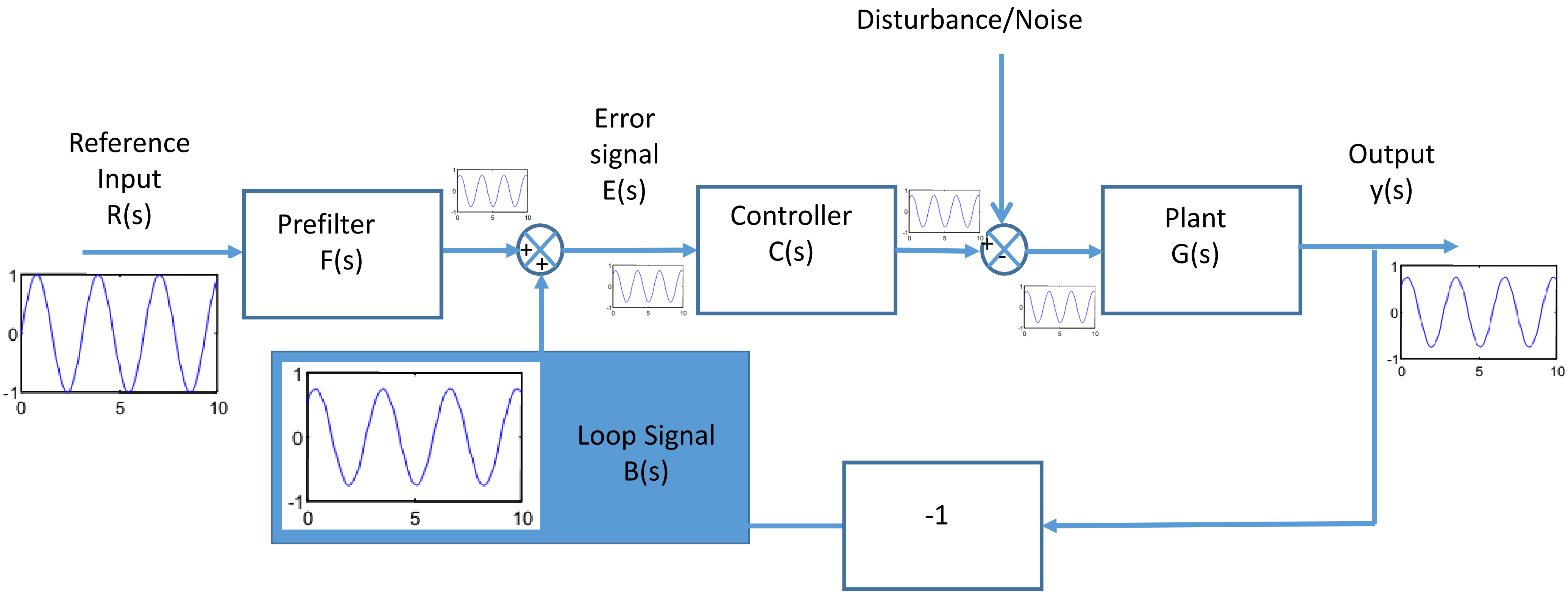
The system will be stable if all the roots of the CE are negative.

$$CE = \Delta(s) = 1 + C(s)P(s) = d_C(s)d_P(s) + n_C(s)n_P(s)$$

Given the calculation there is little insight into how to (re)design a system for stability.

This is where the *Loop Transfer Function* **B(s)** can be helpful.

Loop Transfer Function



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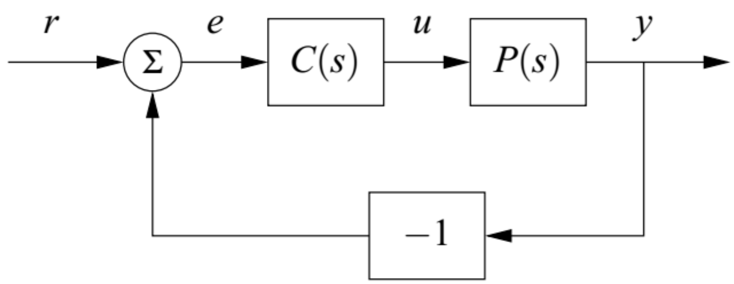
Previous method to determine stability…



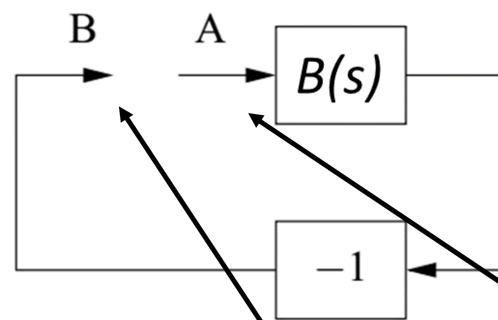
$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)}$$

Find the Characteristic Equation…

$$\lambda(s) = d_p(s)d_c(s) + n_p(s)n_c(s)$$

Ensure all the roots have a negative real part.

If it is not stable, there is little information here that helps us pick a better C(s)



If we inject a signal with frequency ω_o here…

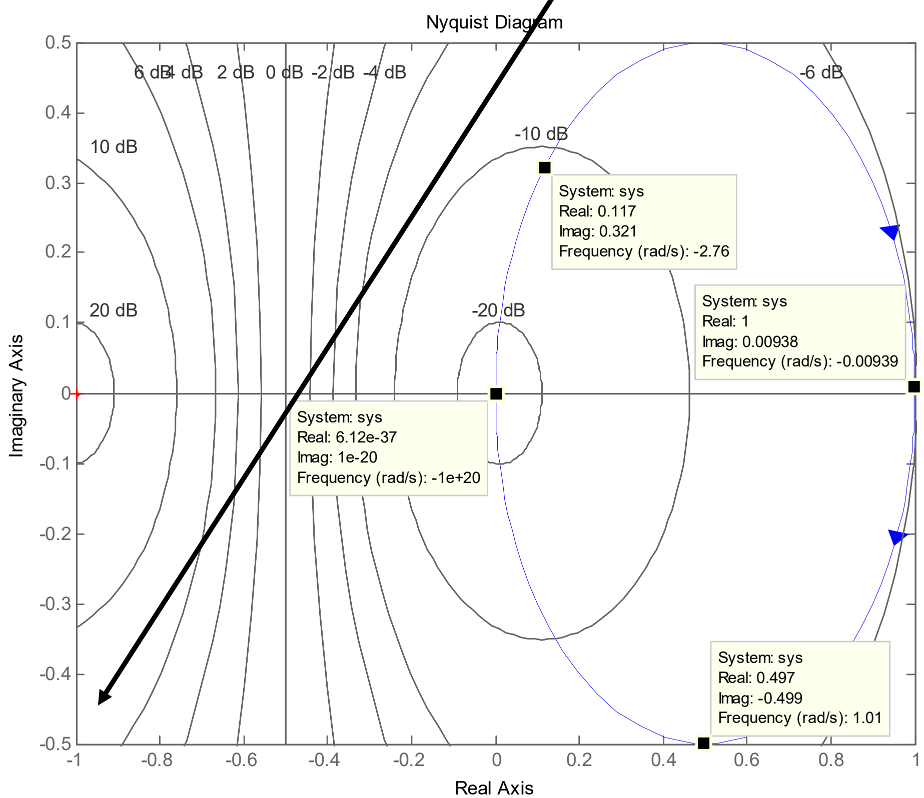
For a linear system we should get an output frequency ω_o here…

It is reasonable that an oscillation can be *maintained* if the amplitude at B is the same as the amplitude at A. Thus…

$$B(i\omega) = -1$$

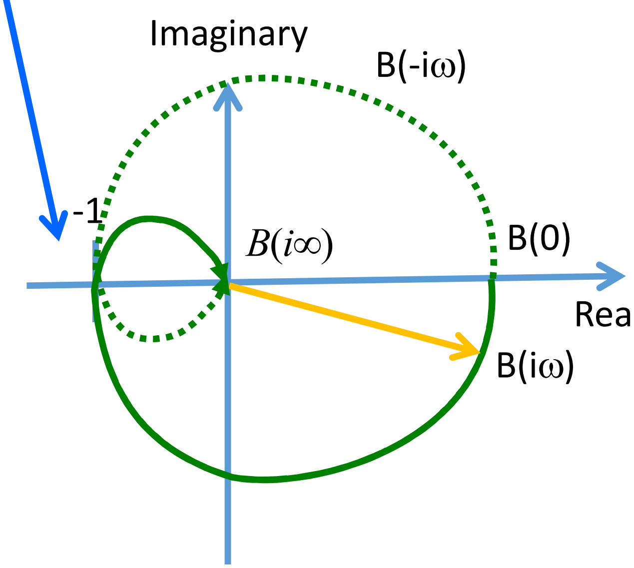
Back to our Nyquist Plot…

Notice that MATLAB ensures -1 is shown on the Nyquist Plot… even if that didn't seem to make much sense…



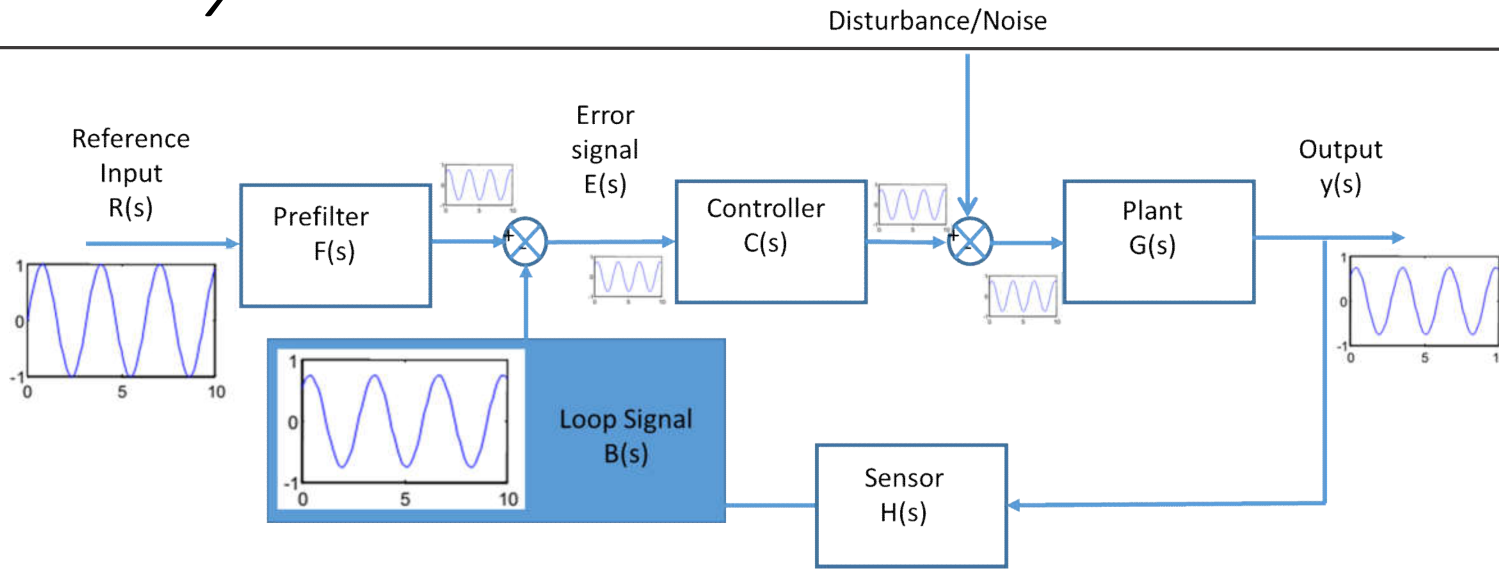
Plane of the Open Loop Transfer Function

$$B(i\omega_0) = \frac{n_c(s)}{d_c(s)} \frac{n_p(s)}{d_p(s)} = -1$$



-1 is called the critical point

Loop *Analysis*



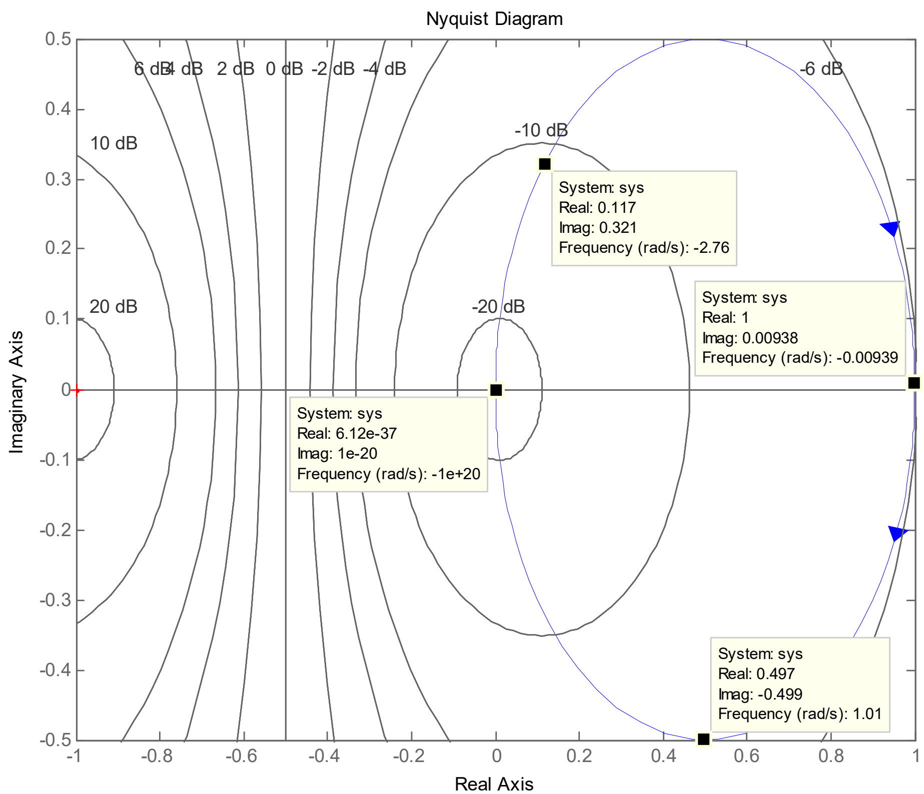
Loop analysis: trace how a sinusoidal signal propagates in the feedback loop and investigate if the propagating signal grows or decays.

This is the key concept supporting the **Nyquist Stability Theorem**.

Its key advantage over eigenanalysis and Lyapunov stability is it also provides insight into *how* stable we are since we can use it to define the **gain margin** and **phase margin** (aka the **stability margins**.)

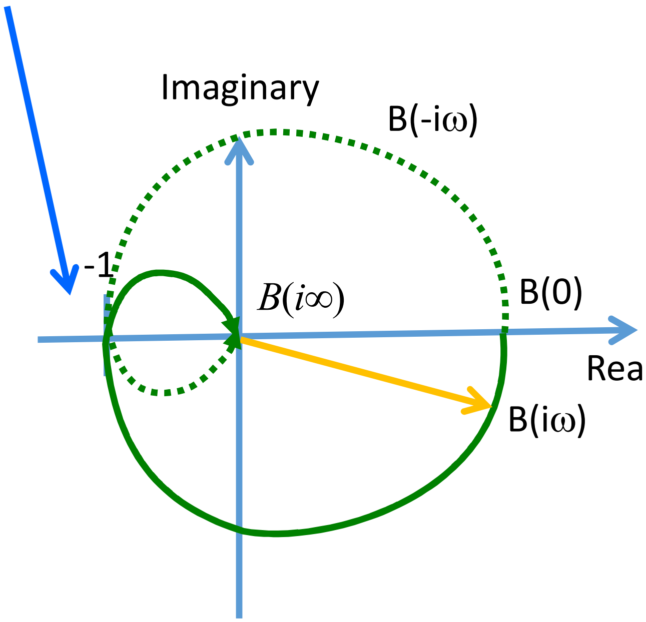
Back to our Nyquist Plot…

Notice that MATLAB ensures -1 is shown on the Nyquist Plot… even if that didn't seem to make much sense…



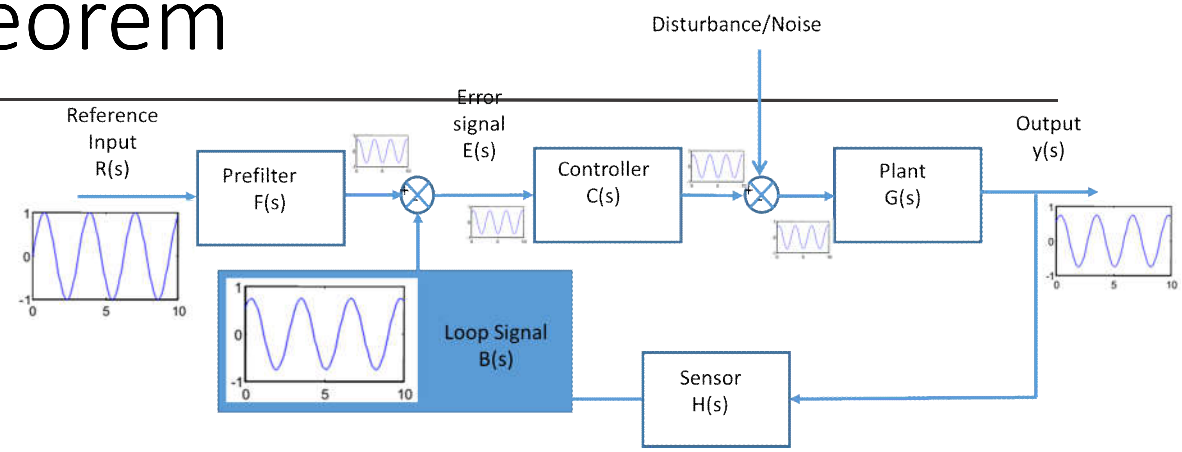
Plane of the Open Loop Transfer Function

$$B(i\omega_0) = \frac{n_c(s)}{d_c(s)} \frac{n_p(s)}{d_p(s)} = -1$$

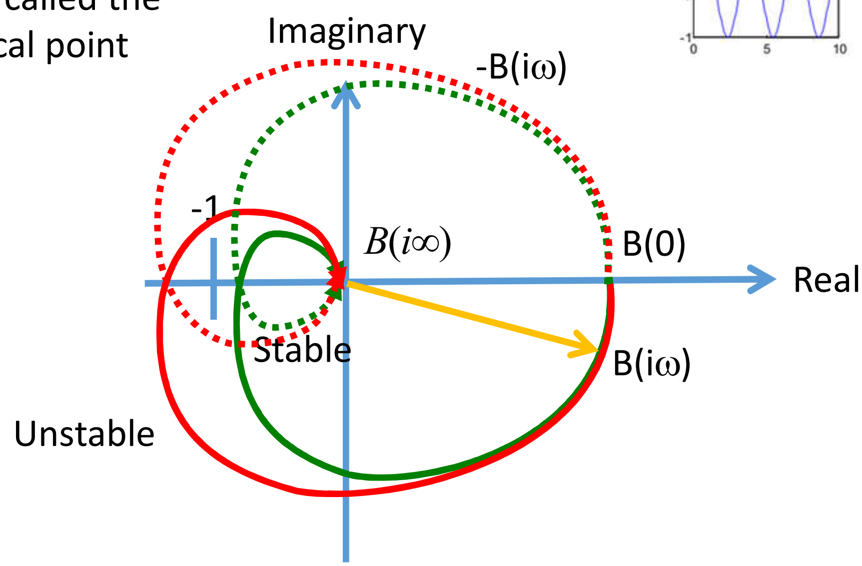


-1 is called the critical point

Nyquist (simple)Theorem



-1 is called the critical point



Nyquist (simple) Theorem: If the Loop Transfer Function *B(iω) has no poles in the right hand side* (except for simple poles on the imaginary axis) then the system is stable iff there are no encirclements of the critical point, -1.

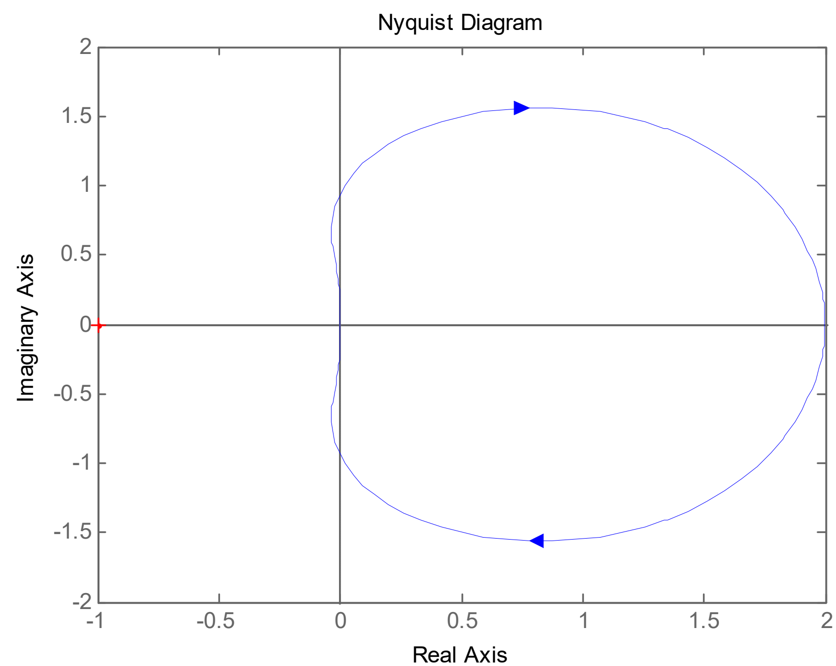
Example, stable system

$$B(s) = \frac{K(s^2 + 2s + 2)}{s^3 + 2s^2 + 2s + 1}$$

$$B(j\omega) = \frac{1(j\omega^2 + j\omega s + 2)}{j\omega^3 + 2j\omega^2 + 2j\omega + 1}$$

$$B(0) = 2$$

$$B(\infty) = B(-\infty) = 0$$



Will increasing K make this system unstable?

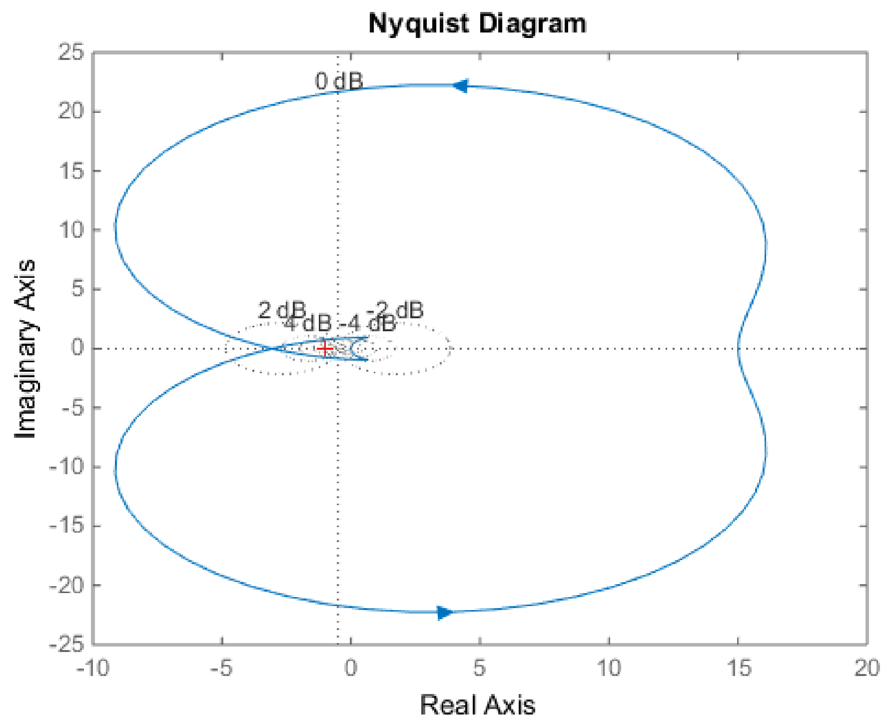
Example, unstable system

$$\mathrm{B}(s) = \frac{K(s^2 + s + 3)}{s^3 + 6s^2 - 2s + 2}$$

$$\mathrm{B}(j\omega) = \frac{10(j\omega^2 + j\omega + 3)}{j\omega^3 + 6j\omega^2 - 2j\omega + 2}$$

$$\mathrm{B}(0) = 15$$

$$B(\infty) = \mathrm{L}(-\infty) = 0$$

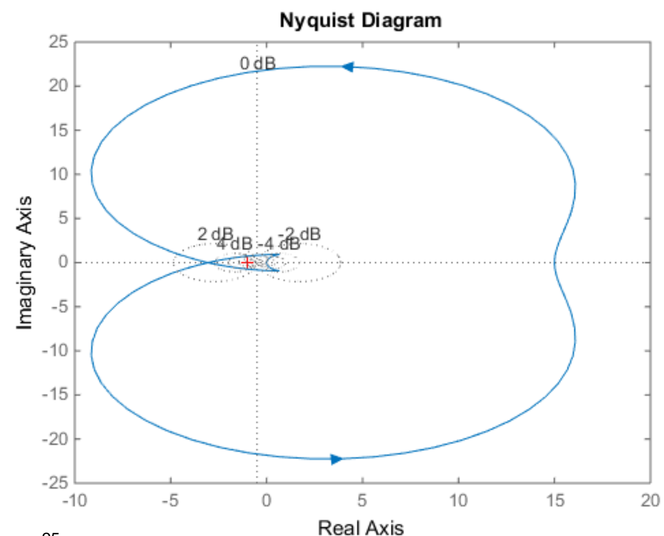


Will decreasing K make this system stable?

Example, Unstable System (Old School)

$$\mathrm{B}(s) = \frac{K(s^2 + s + 3)}{s^3 + 6s^2 - 2s + 2}$$

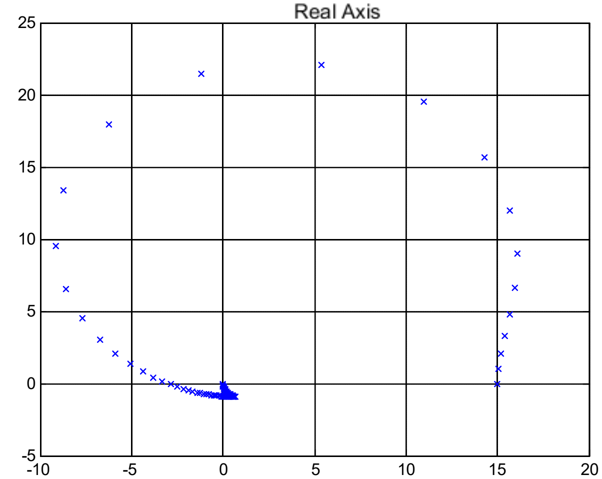
$$\mathrm{B}(j\omega) = \frac{10(j\omega^2 + j\omega + 3)}{j\omega^3 + 6j\omega^2 - 2j\omega + 2}$$




```
clear all;
f=[0:.05:500];
K = 10;

for j=1:length(f)
    w = 0 + 1i*f(j);
    Bnum = K*(w^2 + w + 3 );
    Bden = w^3 + 6*w^2 - 2*w + 2;
    B = Bnum/Bden;

    re(j) = real(B);
    im(j) = imag(B);
end
plot( re, im, 'bx' );
```

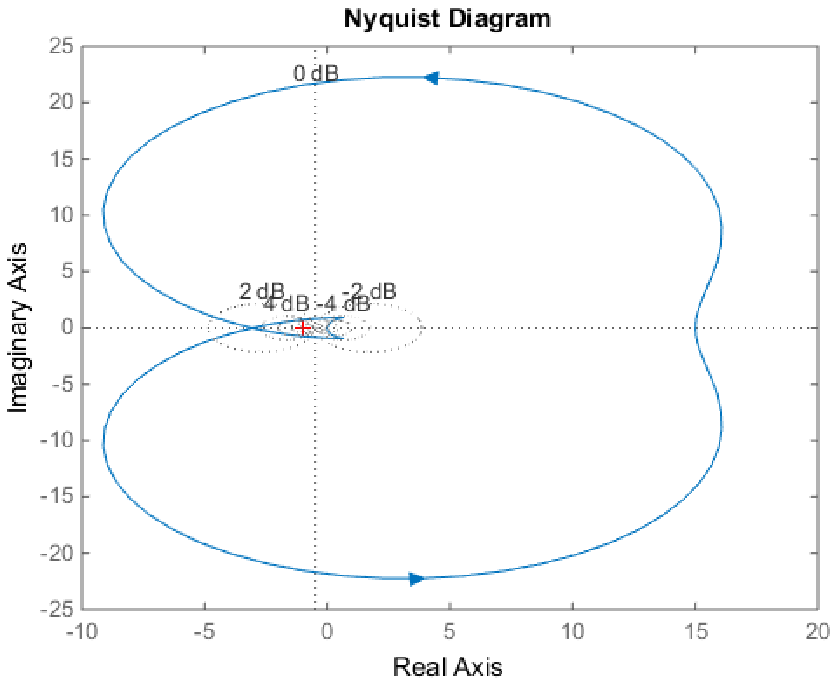


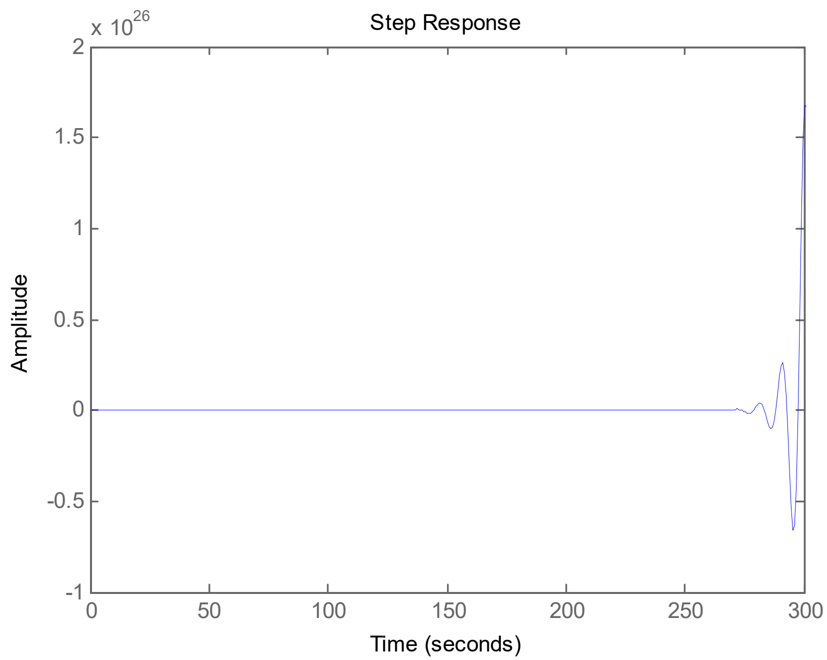
Example, Unstable System

$$\mathrm{B}(s) = \frac{K(s^2 + s + 3)}{s^3 + 6s^2 - 2s + 2}$$

$$\mathrm{B}(j\omega) = \frac{10(j\omega^2 + j\omega + 3)}{j\omega^3 + 6j\omega^2 - 2j\omega + 2}$$

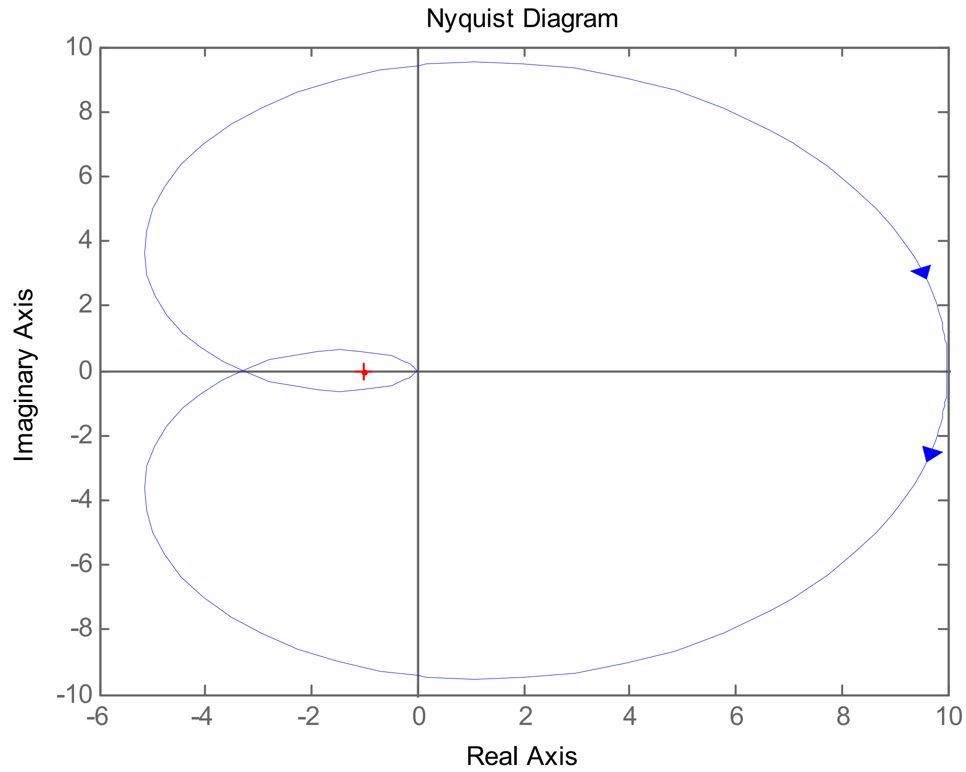
```
sys = tf( [10 10 30], [ 1 6 -2 3 ] );
step( sys )
```





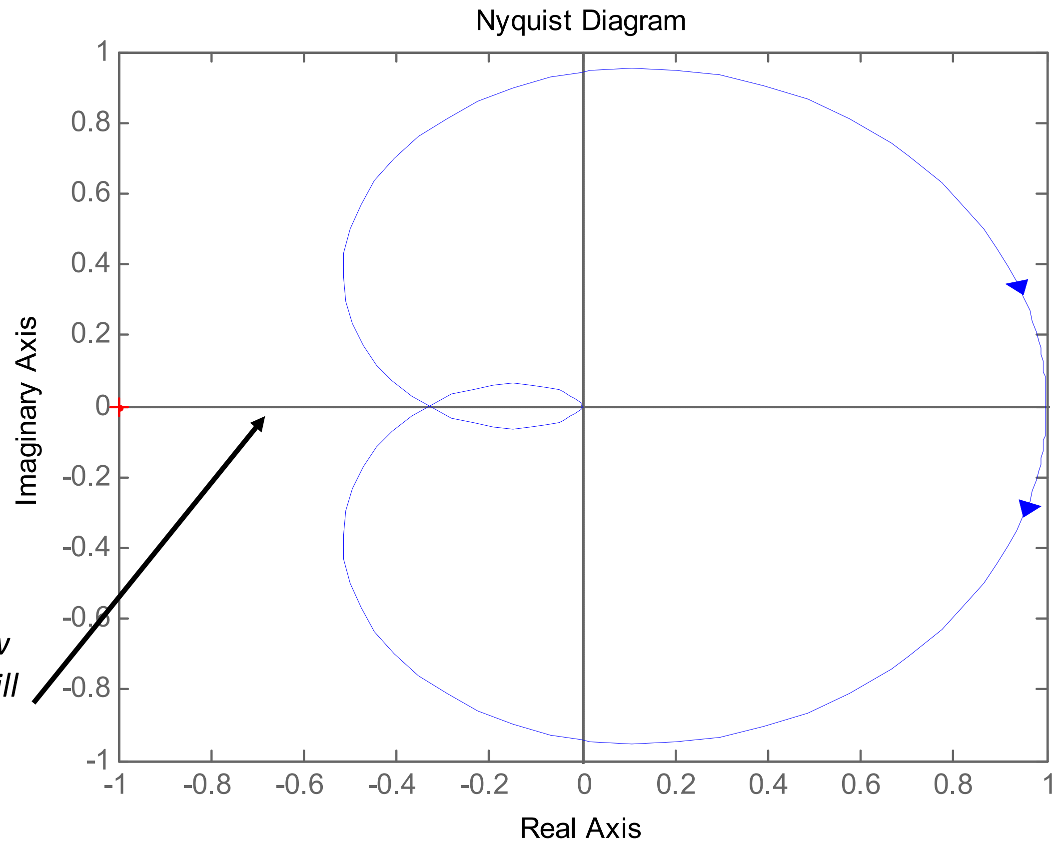
Example, unstable system

$$B(s) = \frac{10}{s\left(s^2 + 2s + 2\right)}$$



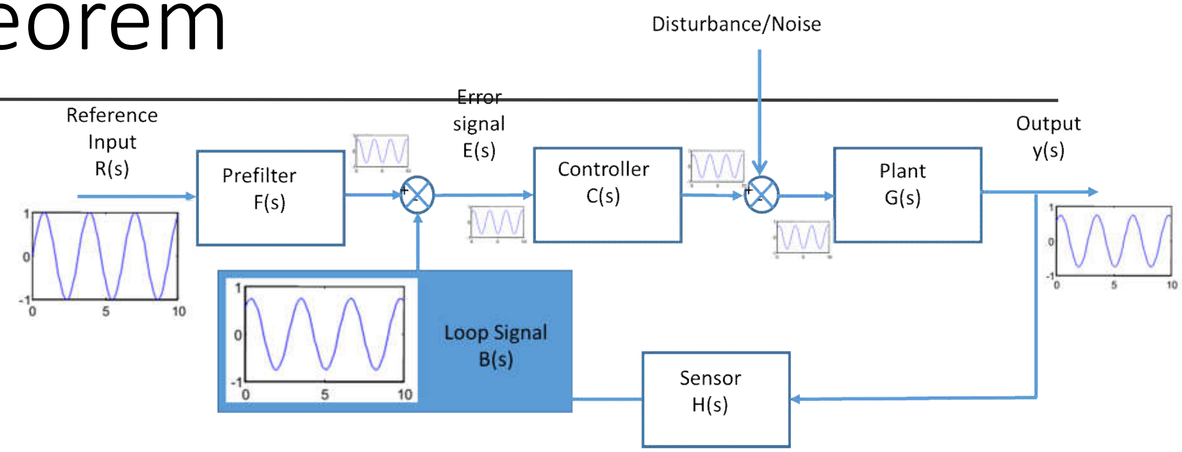
But made stable by reducing the gain

$$B(s) = \frac{1}{s\left(s^2 + 2s + 2\right)}$$

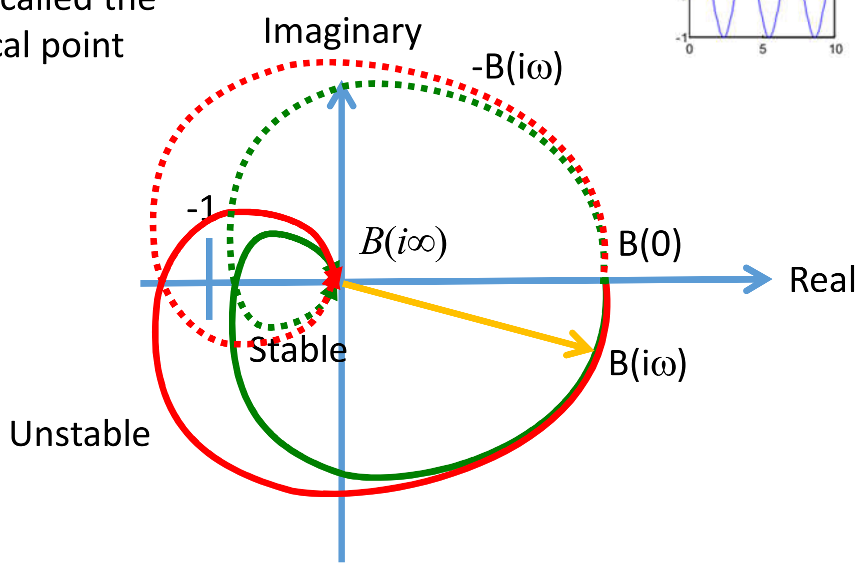


Foreshadows the notion of *stability gain margins. How far can we push the gains for a given controller and still be stable?*

Nyquist (simple)Theorem



-1 is called the critical point



Nyquist (simple) Theorem: If the Loop Transfer Function *B(iω) has no poles in the right hand side* (except for simple poles on the imaginary axis) then the system is stable iff there are no encirclements of the critical point, -1.

Nyquist's (full) Stability Theorem

Nyquist (simple) stability Theorem required that B(s) have NO poles in the right half plane. In some cases this will not be true *even* when the closed loop transfer function T(s) is stable.

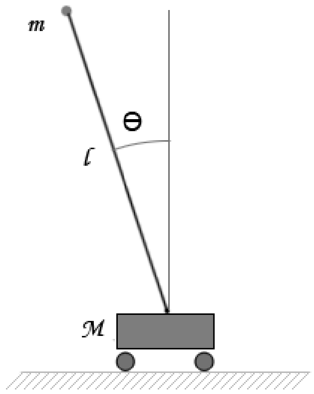
Thus, we need a more general (but slightly more complicated) theorem.

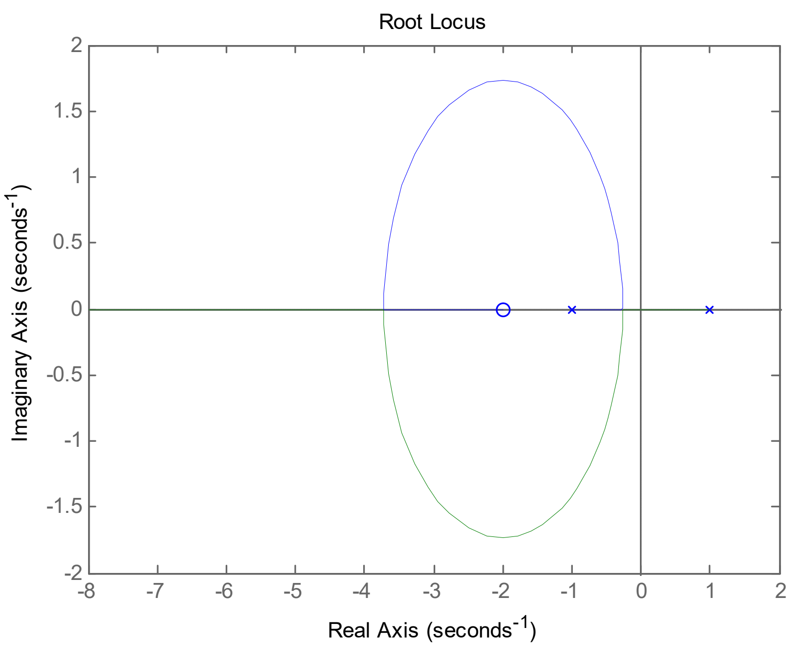
Nyquist Stability Theorem: Given a closed loop system with the loop transfer function B(s) with P poles in the right hand plane.

- Let N be the number of clockwise encirclements of -1 by $B(i\omega)$ minus the counterclockwise encirclements of -1 by $B(i\omega)$.Then the closed loop system has Z=N+P zeros in the right half plane.

This requires that if a B(s) with P poles in the right hand plane, there must be *P counter-clockwise encirclements of -1*.

An example, Inverted Pendulum





The old fashion way…

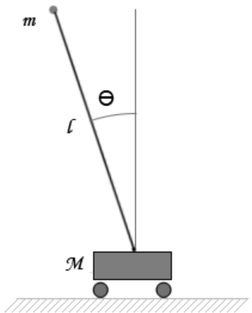
$$B(s) = \frac{k(s+2)}{s^2 - 1}$$

Note there is one pole in the RHP.

The Nyquist stability criterion dictates we should expect 1 counterclockwise encirclements of -1 if the system is stable…

…for what values of *k is the system stable?*

An example, Inverted Pendulum



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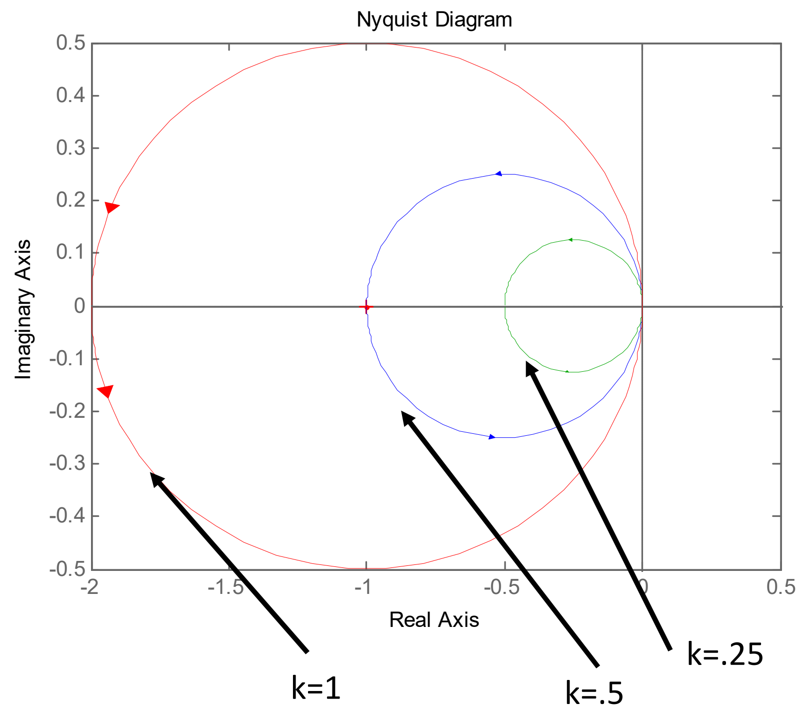
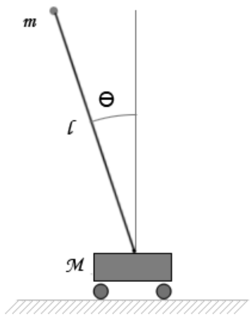
Note there is one pole in the RHP, we should expect 1 counterclockwise encirclement of -1 if the system is stable….

…for what values of *k is the system stable?*



Does this result make physical sense for this system?

An example, Inverted Pendulum

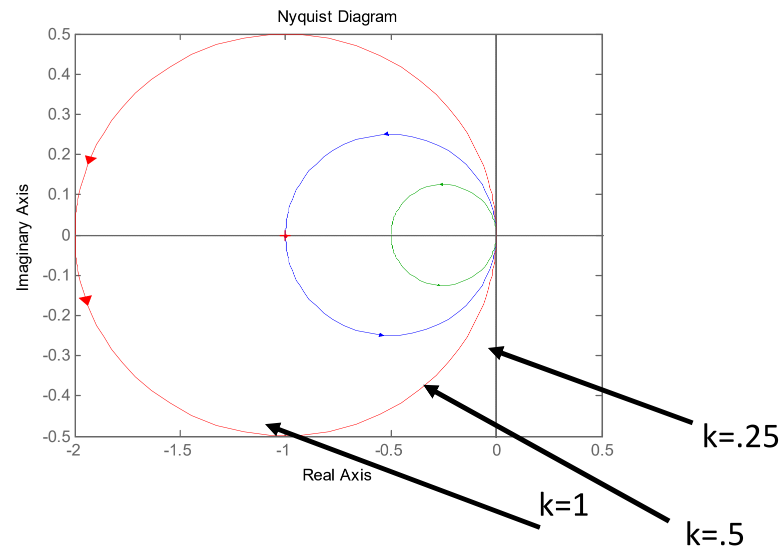


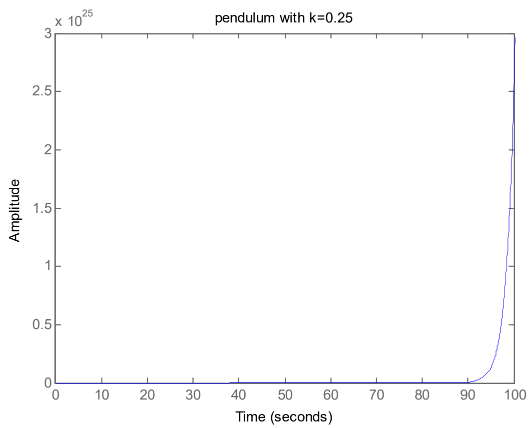
$$B(s) = \frac{k(s+2)}{s^2 - 1}$$

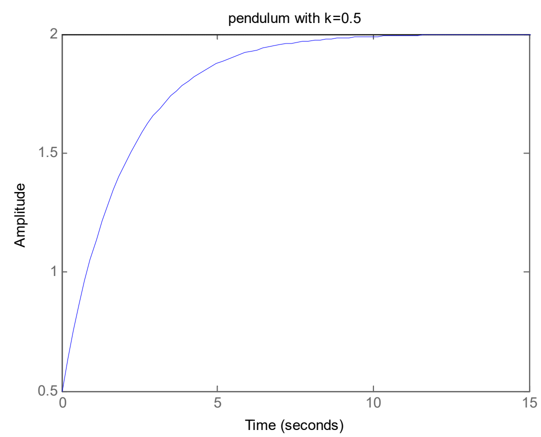
$$T_{cl}(s) = \frac{ks + 2k)}{s^2 + ks + (2k-1)}$$

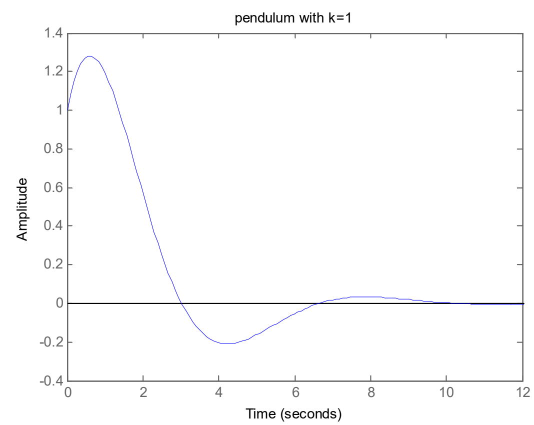
The old fashion way…

```
k=1;
sys = tf( [ k 2*k ], [ 1 k 2*k-1 ] );
impulse( sys );
title('pendulum with k=0.25');
```





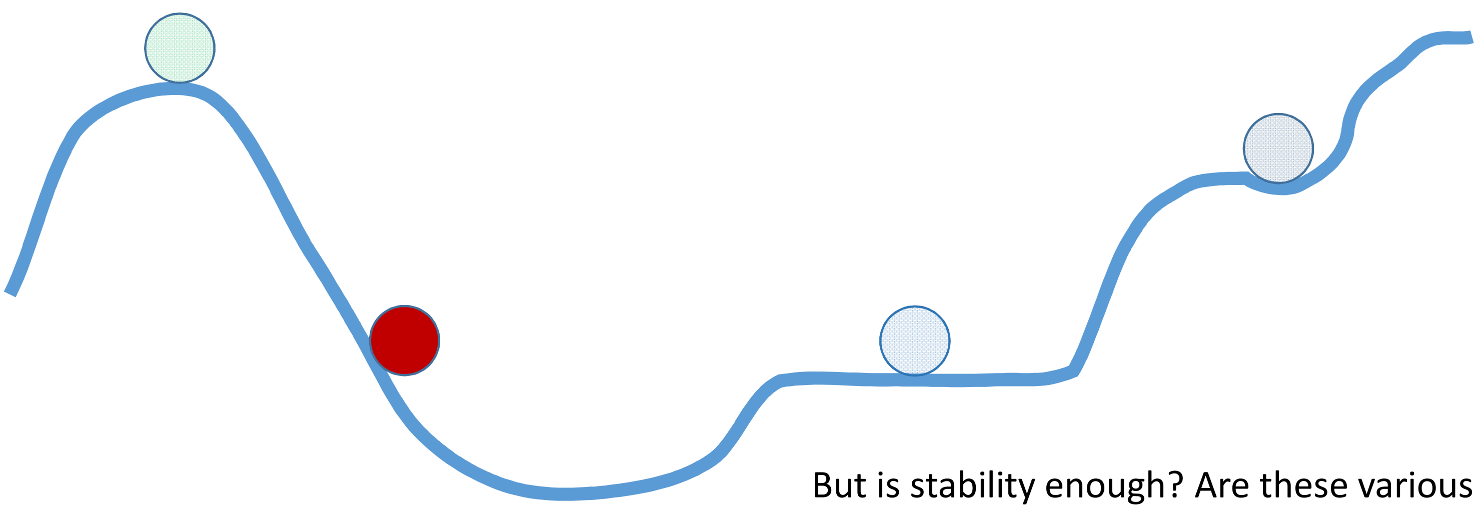




Stability Margins

Recall the definition of stability: A system's response to an input results in an output that stays arbitrarily near some value, a, for all of time greater than some value, t_f.

$$\|b-a\|<\delta \Rightarrow \|x(t;b)-x(t;a)\|<\varepsilon \quad \text{for all } t>0$$

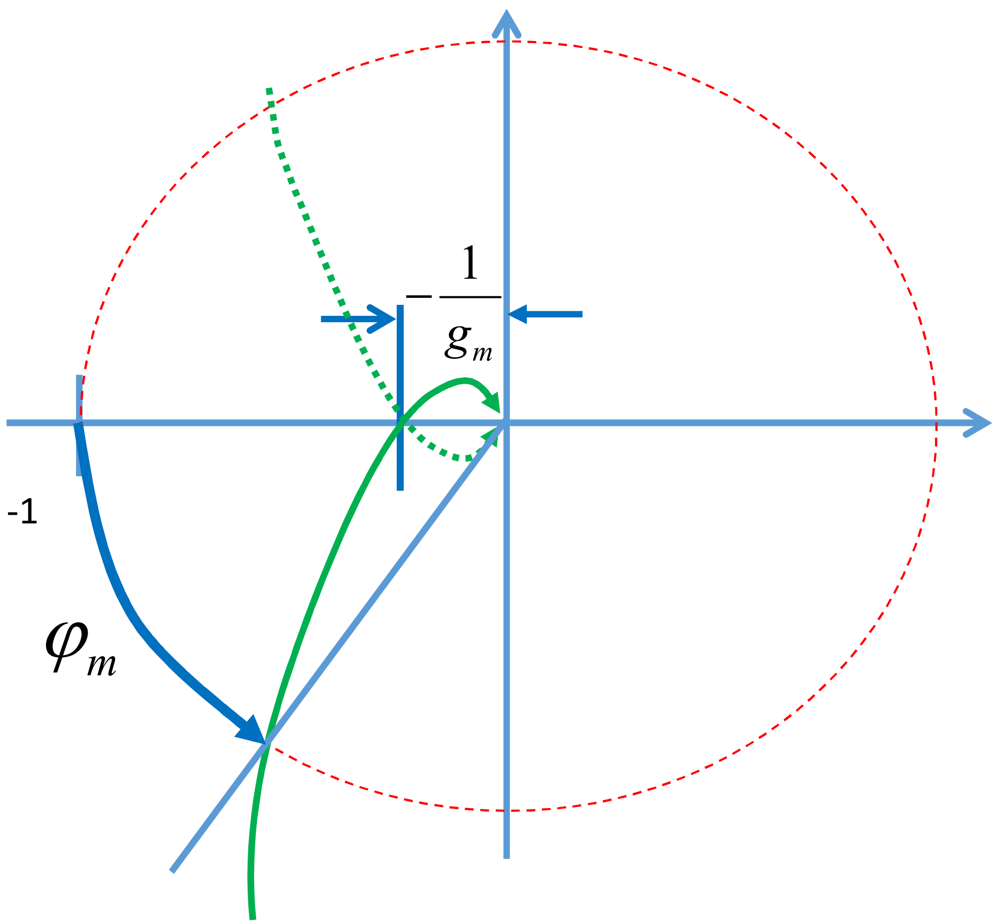


But is stability enough? Are these various stable points all equally stable?

Stability Margins Summary

- Margins are the range from the current system design to the edge of instability. We will determine…
- Gain Margin
  - How much can gain be increased?
  - Formally: the smallest multiple amount the gain can be increased before the closed loop response is unstable.
- Phase Margin
  - How much further can the phase be shifted?
  - Formally: the smallest amount the phase can be increased before the closed loop response is unstable.
- Stability Margin
  - How far is the system from the critical point?

Gain & Phase Margins

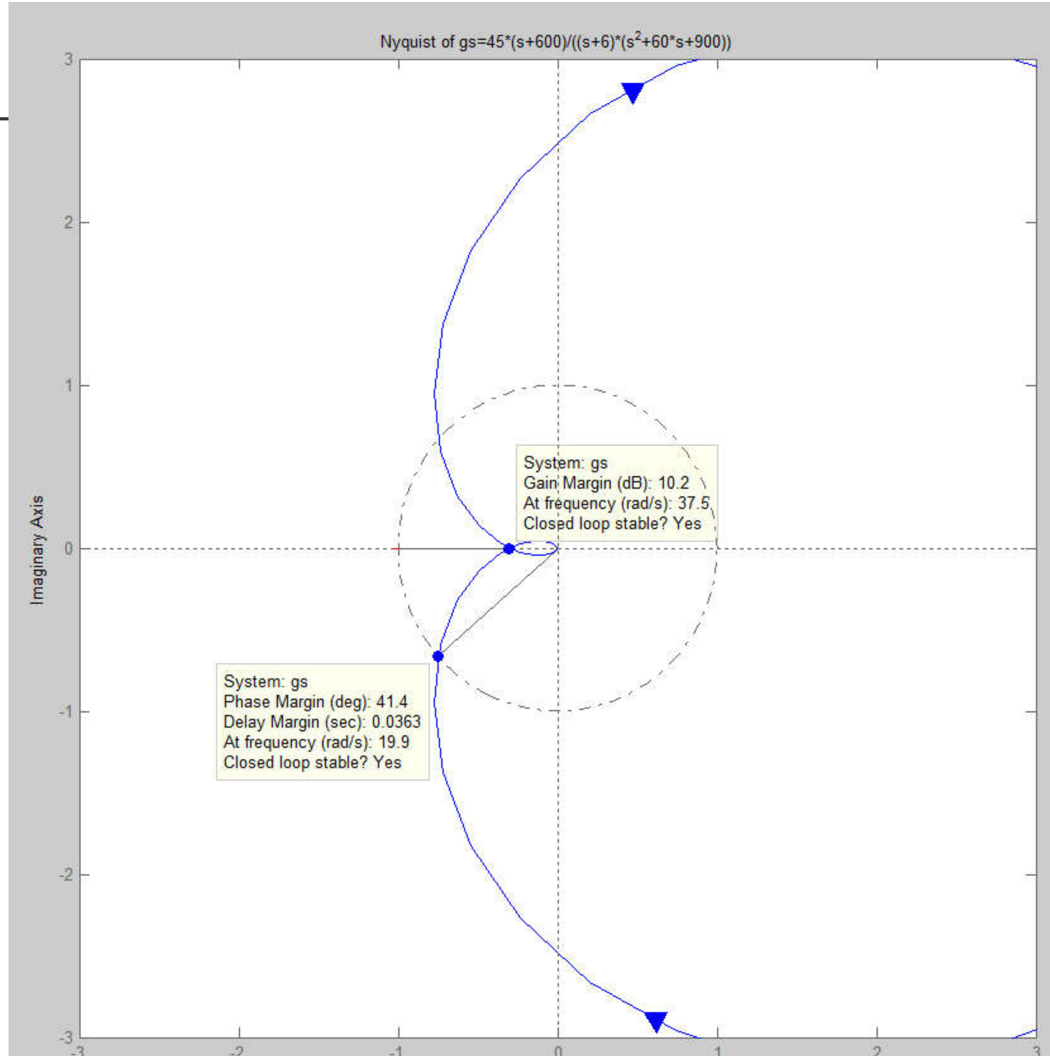


Using MATLAB

$$B(s) = \frac{45(s+600)}{(s+6)(s^2+60s+900)}$$

From the plot, the gain margin is 10.2 dB

The gain multiple is $G = 10^{0.05 g_m} = 10^{0.05*10.2} = 3.2359$

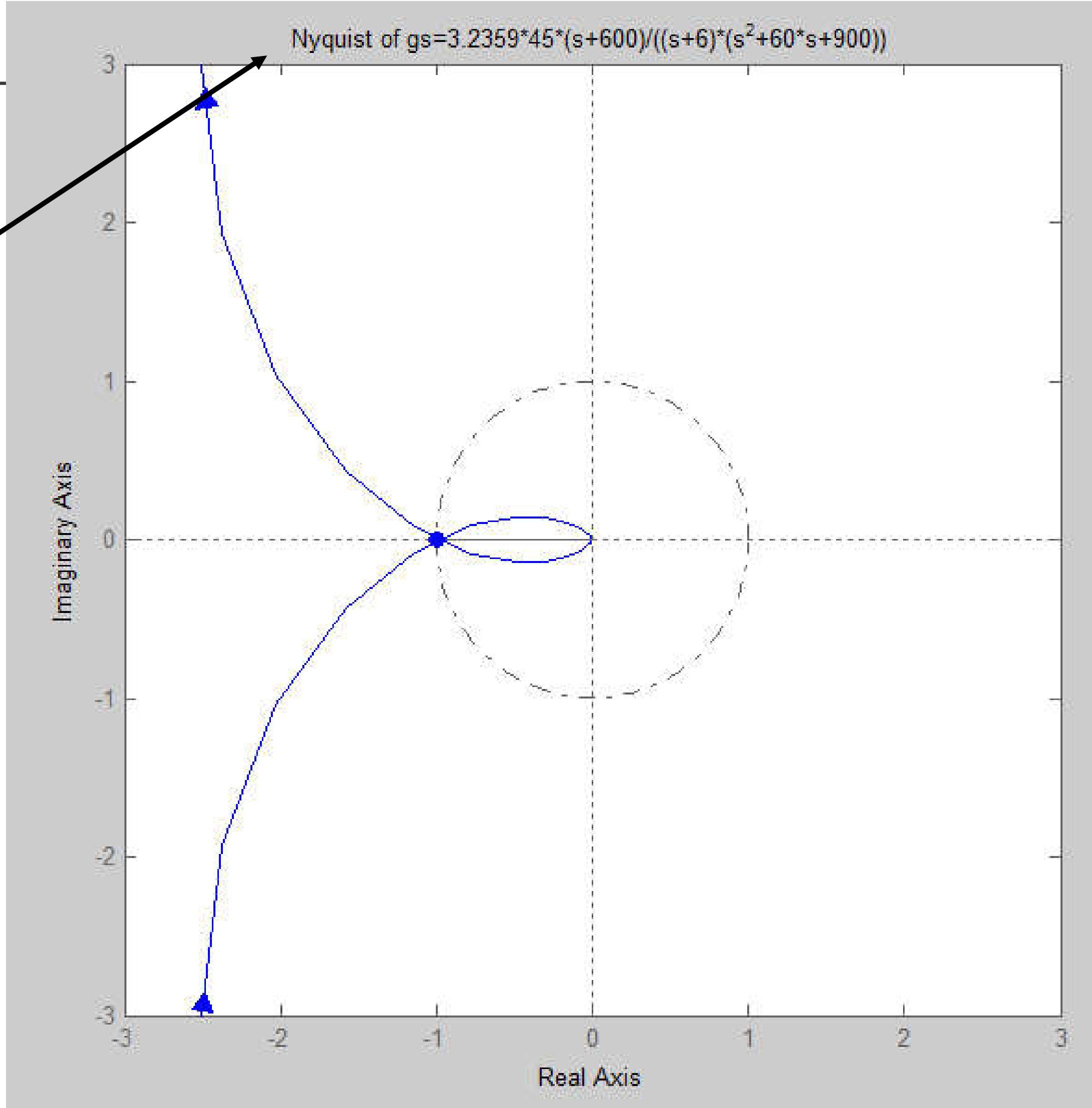


nyquist(sys) is an interactive graph

Using the gain margin

$$B(s) = \frac{(3.2359 * 45)(s+600)}{(s+6)(s^2+60s+900)}$$

Here the gain from the previous plot has been multiplied by 3.2359



With this gain, the system is neutrally stable.

Using the gain margin

$$B(s) = \frac{(270)(s+600)}{(s+6)(s^2+60s+900)}$$

Here the gain has been changed to 270.

With this gain, the system is unstable.

With this gain margin is now -5.36 dB

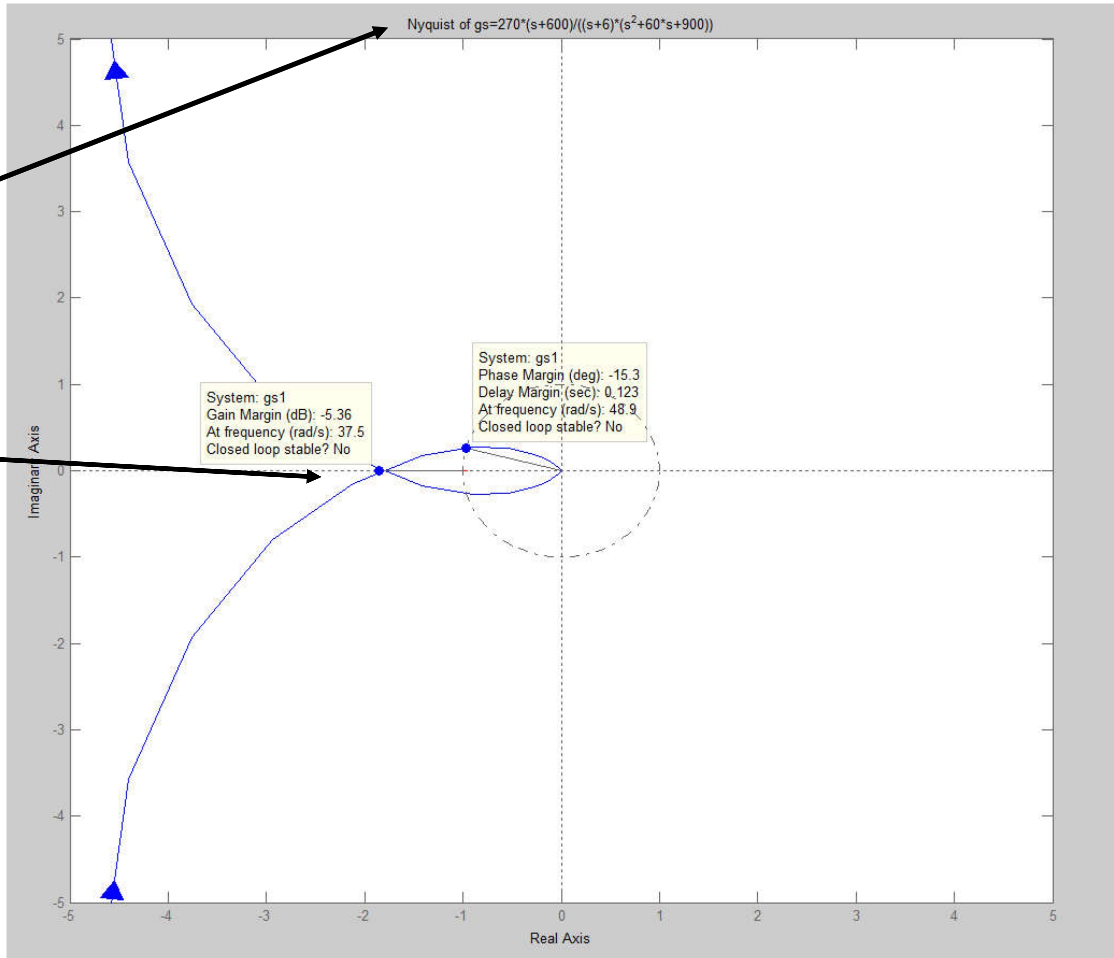
$$-5.36 = 20\log_{10} x$$

$$10^{\frac{-5.36}{20}} = x$$

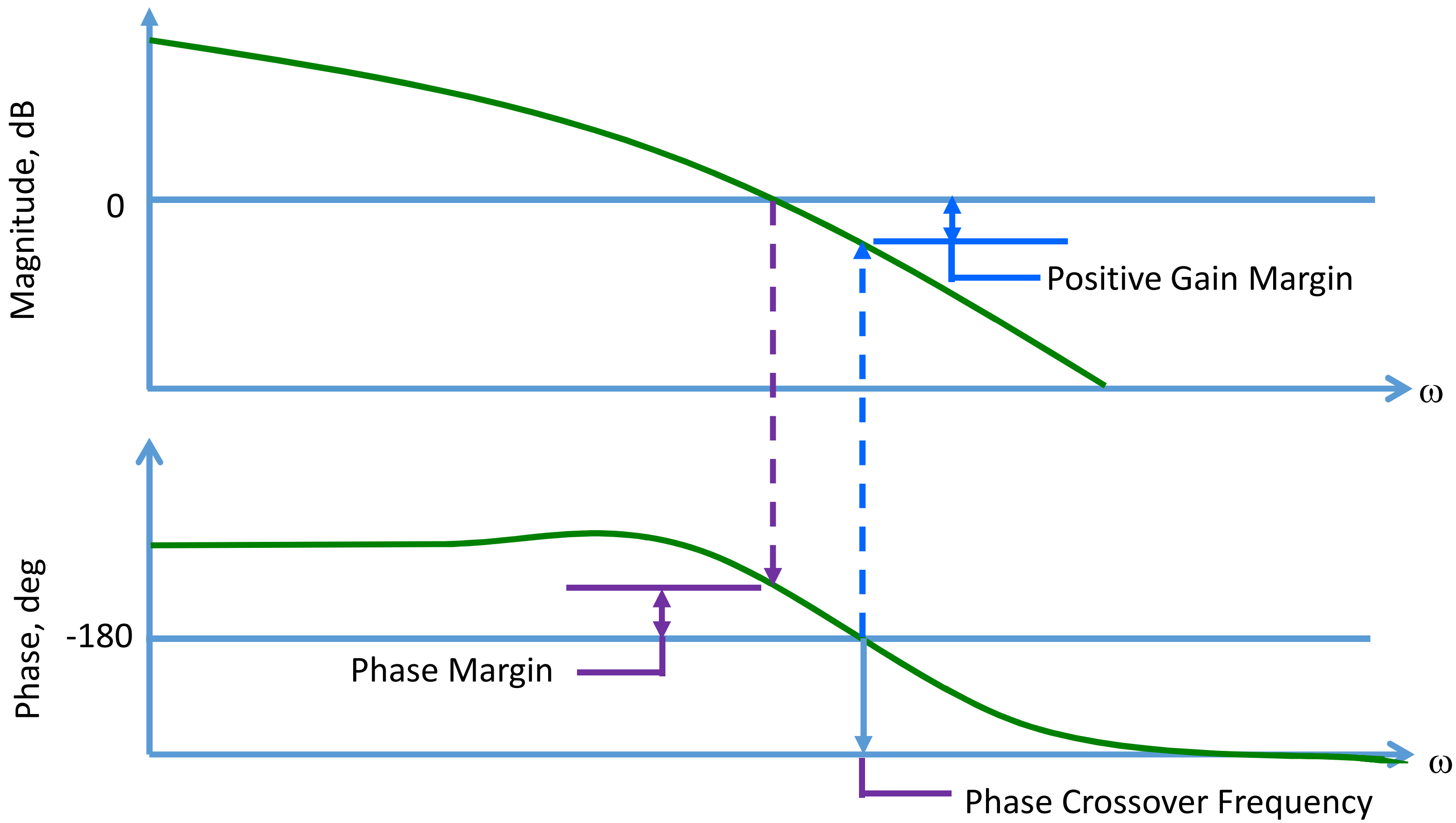
$$0.54 = x$$

And this is the correct gain multiplier, since

$$0.5393 = \frac{45 * 3.2359}{270} = \frac{K_{neutralstable}}{K}$$

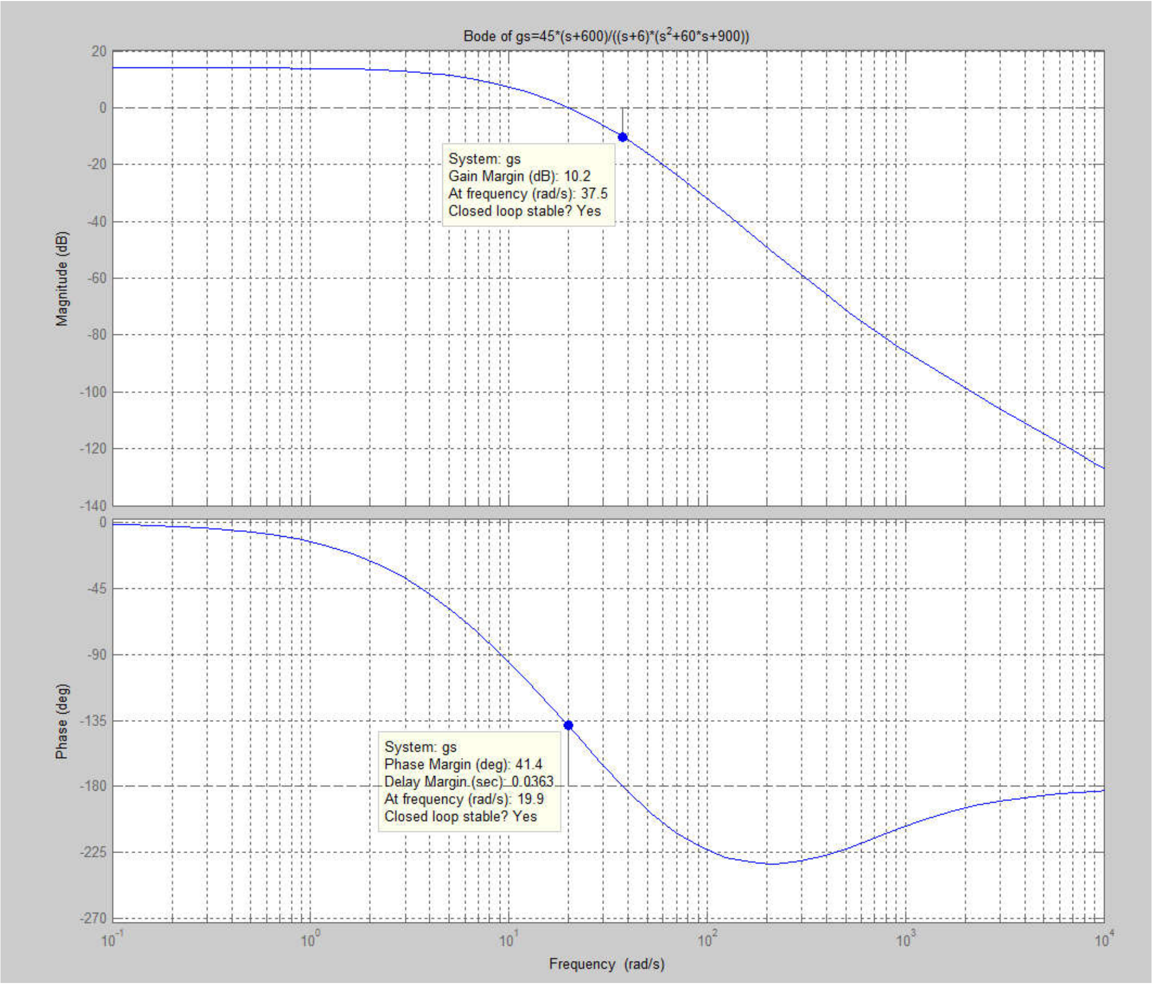


Margins on a Bode Plot (for B(s)!)



Back to our example...

$$\mathrm{B}(s) = \frac{45(s+600)}{(s+6)(s^2+60s+900)}$$



Summary

- Examining the Loop Transfer Function B(s) is a common sense way to look at the response of a feedback system.
  - Can simplify stability analysis. Don't have to calculate $T_{cl}(S)$
- By graphing real and imaginary components of the response as a function of frequency (Nyquist plot) we can determine the stability of the system.
- Nyquist plots can get complicated, but a combination of MATLAB, hand computations and symmetry make most problems solvable
- Using Nyquist plots we can roughly determine how stable we are by calculating the gain and phase margins.