Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

Problem 1

a.) What is the probability that x = 1?

We find the marginal from the joint.

$$p(x) = \sum_y p(x,y)$$
$$= \frac{1}{4} + \frac{1}{6} \quad ; x = 0$$
$$and \quad \frac{1}{4} + \frac{1}{3} \quad ; x = 1$$

And we can see,

$$Pr(x = 1) = \frac{7}{12}$$

b.) What is the probability that x = 1, given y = 1?

From the definition,

$$Pr(x = 1|y = 1) = \frac{Pr(x = 1 \cap y = 1)}{Pr(y = 1)}$$

From the given table, we can see,

$$= \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}$$

And so $Pr(x = 1|y = 1) = \frac{2}{3}$

c.) What is the variance of the random variable X?

From the definition,

$$var(X) = E[X^2] - E[X]^2$$

And so we calculate the expectation of X from its definition,

$$E[X] = \sum x * f(x)$$
$$= 0 * \frac{5}{12} + 1 * \frac{7}{12}$$

$$E[X] = \frac{7}{12}$$

Using an alternate definition to calculate variance:

$$var(X) = \sum (x - \mu)^2 * p(x)$$
$$= \left(0 - \frac{7}{12}\right)^2 * \frac{5}{12} + \left(1 - \frac{7}{12}\right)^2 * \frac{7}{12}$$

$$= \frac{35}{144} = 0.243$$

So then, $var(X) = 0.243$

d.) What is the variance of X, given that y = 1?

We find the conditional expectation of X given that Y = y = 1.

$$E[X|Y = y = 1] = 0 * \frac{1}{3} + 1 * \frac{2}{3} = \frac{2}{3}$$

Then,

$$var(X|Y = y = 1) = \left(0 - \frac{2}{3}\right)^2 * \frac{1}{3} + \left(1 - \frac{2}{3}\right)^2 * \frac{2}{3}$$
$$= \frac{2}{9}$$

e.) What is $E[X^3 + X^2 + 3Y^7|Y = 1]$?

From the linearity of expectation, we have,

$$E[X^3|y = 1] + E[X^2|y = 1] + 3 * E[Y^7|y = 1]$$

Where the last term is equal to 3. Then we can calculate the moment generating function of the random variable $W = g(X)$ where $g(X) = X|(Y = y = 1)$.

$$M_W(s) = E[e^{sW}] = \sum_w e^{sW} * f_W(w) = e^{s0} * f_W(0) + e^{s1} * f_W(1)$$

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$$M_W(s) = \frac{1}{3} + \frac{2}{3}e^s$$

We can find the second and third moments by taking successive derivatives of the m.g.f., and evaluating them at $\mathrm{s} = 0$.

$$\frac{dM}{ds} = \frac{2}{3} * e^s \rightarrow \frac{2}{3}\mathrm{e^s}|_{s=0} = \frac{2}{3} = \mu_1 = E[W^1]$$

$$\frac{d^2M}{ds^2} = \frac{d}{ds}\left(\frac{dM}{ds}\right) = \frac{2}{3} * e^s \rightarrow \frac{2}{3}\mathrm{e^s}|_{s=0} = \frac{2}{3} = \mu_2 = E[W^2]$$

$$\frac{d^3M}{ds^3} = \frac{d}{ds}\left(\frac{d^2M}{ds^2}\right) = \frac{2}{3} * e^s \rightarrow \frac{2}{3}\mathrm{e^s}|_{s=0} = \frac{2}{3} = \mu_3 = E[W^3]$$

Rewriting our original expression,

$$E[W^3] + E[W^2] + 3 * E[Y^7|y = 1] = \frac{2}{3} + \frac{2}{3} + 1$$
$$= \frac{7}{3}$$

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Problem 2

We have a subspace $W \subseteq \mathbb{R}^3$ defined by the span of v_1 and v_2, where

$$\begin{aligned} v_1 &= \langle 1,1,1 \rangle \\ v_2 &= \langle 1,0,0 \rangle \end{aligned}$$

We wish to find the projection of 3 points (vectors) onto this subspace. Our points are:

$$\begin{aligned} p_1 &= \langle 3,3,3 \rangle \\ p_2 &= \langle 1,2,3 \rangle \\ p_3 &= \langle 0,0,1 \rangle \end{aligned}$$

We can call the projections of these points $\widehat{p_1},\ \widehat{p_2},\ and\ \widehat{p_3}$. If we have an orthogonal basis for W such as $\{u_1, u_2\}$, then we can find these projections by the following formula,

$$p_i = \frac{p_i \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{p_i \cdot u_2}{u_2 \cdot u_2} u_2 \quad (1)$$

We apply Gram-Schmidt to find the orthogonal basis for W. First, we let $u_1 = v_1$. Then to find u_2, we use,

$$u_2 = v_2 - proj_{v_1} v_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

We find that

$$u_2 = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3} \right\rangle$$

With our orthogonal basis in hand, we simply apply formula 1 to each p vector. Doing the calculations in python gives the following results:

$$\begin{aligned} \widehat{p_1} &= \langle 3,3,3 \rangle \\ \widehat{p_2} &= \langle 2.22, 1.5, 1.5 \rangle \\ \widehat{p_3} &= \langle 5.55, 5, 5 \rangle \end{aligned}$$

```
In [25]: import numpy as np

p1 = np.ones(3, dtype=np.float)*3
p2 = np.arange(3, dtype=np.float)
p3 = np.zeros(3, dtype=np.float) + [0, 0, 1]
u1 = np.ones(3, dtype=np.float)
u2 = np.zeros(3, dtype=np.float) + [2/3, -1/3, -1/3]

p1hat = np.dot(np.dot(p1,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p1,u2)/np.dot(u2, u2), u2)
p2hat = np.dot(np.dot(p2,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p2,u2)/np.dot(u2, u2), u2)
p3hat = np.dot(np.dot(p3,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p3,u2)/np.dot(u2, u2), u2)
print(p1hat, p2hat, p3hat)

[3. 3. 3.] [2.22044605e-16 1.50000000e+00 1.50000000e+00] [5.55111512e-17 5.00000000e-01 5.00000000e-01]
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Problem 3

We know that $Pr(Heads) = \frac{2}{3}$ and therefore $Pr(Tails) = \frac{1}{3}$. We wish to know the probability of getting 50 heads or fewer in 100 tosses.

This can be represented in a binomial distribution, as the probability of 50 or fewer successes in 100 trials.

Then we know that the expected value of the distribution is:

$$n * p \; = \; 66.67.$$

From this, the standard deviation is given by:

$$\sqrt{np(1-p)} = \sqrt{100 * \frac{2}{3} * \frac{1}{3}} = 4.714$$

Using a normal approximation and applying the central limit theorem:

$$Pr(heads < 50) = Pr\left(2 < \frac{50 - 66.67}{4.714}\right) = Pr(2 < -3.53)$$
$$= \; 0.0001$$

To check our answer, you can also sum the binomial distributions evaluated at 1 through 50:

$$Pr(heads \leq 50) = \sum_{i=1}^{50} \binom{100}{i} \left(\frac{2}{3}\right)^{i} \left(1 - \frac{2}{3}\right)^{100-i}$$

This calculation was done in python and resulted in the same answer.

```
In [8]: import numpy as np
        from scipy import special as sp

        p = 2/3
        n = 100
        probability_array = np.zeros(50)

        for i in range (50):
            probability_array[i] = sp.comb(n, i)*(p**i)*((1-p)**(n-i))
            #print(probability_array[i])

        print("The sum of all these probabilities is: {}".format(np.sum(probability_array)))

        The sum of all these probabilities is: 0.0001989326425396602
```