Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

Problem 1

a.) What is the probability that x = 1?

We find the marginal from the joint.

$$p(x) = \sum_y p(x, y)$$

¿ $\frac{1}{4} + \frac{1}{6}; x = 0$

¿ $\frac{1}{4} + \frac{1}{3}; x = 1$

And we can see,

$$Pr(x=1) = \frac{7}{12}$$

b.) What is the probability that x = 1, given y = 1?

From the definition,

$$Pr(x=1|y=1) = \frac{Pr(x=1\ y=1)}{Pr(y=1)}$$

From the given table, we can see,

¿ $\frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = \frac{2}{3}$

And so $Pr(x=1|y=1) = \frac{2}{3}$

c.) What is the variance of the random variable X?

From the definition,

$$var(X) = E[X^2] - E[X]^2$$

And so we calculate the expectation of X from its definition,

$$E[X] = \sum x * f(x)$$
$$¿ \frac{0*5}{12} + \frac{1*7}{12}$$

$$E[X] = \frac{7}{12}$$

Using an alternate definition to calculate variance:

$$var(X) = \sum (x-\mu)^2 * p(x)$$
$$¿ \frac{\left(0-\frac{7}{12}\right)^2 *5}{12} + \frac{\left(1-\frac{7}{12}\right)^2 *7}{12}$$

$$¿ \frac{35}{144} = 0.243$$

So then, $var(X) = 0.243$

d.) What is the variance of X, given that y = 1?

We find the conditional expectation of X given that Y = y = 1.

$$E[X|Y=y=1] = \frac{0*1}{3} + \frac{1*2}{3} = \frac{2}{3}$$

Then,

$$var(X|Y=y=1) = \frac{\left(0-\frac{2}{3}\right)^2 *1}{3} + \frac{\left(1-\frac{2}{3}\right)^2 *2}{3}$$
$$¿ \frac{2}{9}$$

e.) What is $E\left[X^3 + X^2 + 3Y^7 | Y=1\right]$?

From the linearity of expectation, we have,

$$E\left[X^3|y=1\right] + E\left[X^2|y=1\right] + 3*E\left[Y^7|y=1\right]$$

Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

Where the last term is equal to 3. Then we can calculate the moment generating function of the random variable $W=g(X)$ where $g(X)=X \vee ¿ \; Y=y=1$) .

$$M_W(s)=E\left[e^{sW}\right]=\sum_w e^{sW} * f_W(w)=e^{s0} * f_W(0)+e^{s1} * f_W(1)$$

$$M_W(s)=\frac{1}{3}+\frac{2}{3}e^s$$

We can find the second and third moments by taking successive derivatives of the m.g.f., and evaluating them at s = 0.

$$\frac{dM}{ds}=\frac{2}{3} * e^s \rightarrow \frac{2}{3}e^s|_{s=0}=\frac{2}{3}=\mu_1=E\left[W^1\right]$$

$$\frac{d^2M}{ds^2}=\frac{d}{ds}\left(\frac{dM}{ds}\right)=\frac{2}{3} * e^s \rightarrow \frac{2}{3}e^s|_{s=0}=\frac{2}{3}=\mu_2=E\left[W^2\right]$$

$$\frac{d^3M}{ds^3}=\frac{d}{ds}\left(\frac{d^2M}{ds^2}\right)=\frac{2}{3} * e^s \rightarrow \frac{2}{3}e^s|_{s=0}=\frac{2}{3}=\mu_3=E\left[W^3\right]$$

Rewriting our original expression,

$$E\left[W^3\right]+E\left[W^2\right]+3 * E\left[Y^7|y=1\right]=\frac{2}{3}+\frac{2}{3}+1$$

$$¿\frac{7}{3}$$

Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

Problem 2

We have a subspace $W \subseteq R^3$ defined by the span of v_1 and v_2, where

$v_1 = \langle 1,1,1 \rangle$

$$v_2 = \langle 1,0,0 \rangle$$

We wish to find the projection of 3 points (vectors) onto this subspace. Our points are:

$p_1 = \langle 3,3,3 \rangle$
$p_2 = \langle 1,2,3 \rangle$

$$p_3 = \langle 0,0,1 \rangle$$

We can call the projections of these points $\widehat{p_1}$, $\widehat{p_2}, \wedge \widehat{p_3}$. If we have an orthogonal basis for W such as $\backslash\{u_1, u_2\backslash\}$, then we can find these projections by the following formula,

$$p_i = \frac{p_i \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{p_i \cdot u_2}{u_2 \cdot u_2} u_2 \qquad (1)$$

We apply Gram-Schmidt to find the orthogonal basis for W. First, we let $u_1 = v_1$. Then to find u_2 , we use,

$u_2 = v_2 - proj_{v_1} v_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$

We find that

$u_2 = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3} \right\rangle$

With our orthogonal basis in hand, we simply apply formula 1 to each p vector. Doing the calculations in python gives the following results:

$\widehat{p_1} = \langle 3,3,3 \rangle$
$\widehat{p_2} = \langle 2.22, 1.5, 1.5 \rangle$
$\widehat{p_3} = \langle 5.55, 5, 5 \rangle$

Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

```
In [25]: import numpy as np

         p1 = np.ones(3, dtype=np.float)*3
         p2 = np.arange(3, dtype=np.float)
         p3 = np.zeros(3, dtype=np.float) + [0, 0, 1]
         u1 = np.ones(3, dtype=np.float)
         u2 = np.zeros(3, dtype=np.float) + [2/3, -1/3, -1/3]

         p1hat = np.dot(np.dot(p1,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p1,u2)/np.dot(u2, u2), u2)
         p2hat = np.dot(np.dot(p2,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p2,u2)/np.dot(u2, u2), u2)
         p3hat = np.dot(np.dot(p3,u1)/np.dot(u1, u1), u1) + np.dot(np.dot(p3,u2)/np.dot(u2, u2), u2)
         print(p1hat, p2hat, p3hat)

         [3. 3. 3.] [2.22044605e-16 1.50000000e+00 1.50000000e+00] [5.55111512e-17 5.00000000e-01 5.00000000e-01]
```

Problem 3

We know that $Pr(Heads)=\frac{2}{3}$ and therefore $Pr(Tails)=\frac{1}{3}$. We wish to know the probability of getting 50 heads or fewer in 100 tosses.

This can be represented in a binomial distribution, as the probability of 50 or fewer successes in 100 trials.

Then we know that the expected value of the distribution is:

$$n*p=66.67 \quad .$$

From this, the standard deviation is given by:

$$\sqrt{np(1-p)}=\sqrt{\frac{\frac{100*2}{3}*1}{3}}=4.714$$

Using a normal approximation and applying the central limit theorem:

$$Pr(heads<50)=Pr\left(z<\frac{50-66.67}{4.714}\right)=Pr(z\leftarrow 3.53)$$

$$¿0.0001$$

To check our answer, you can also sum the binomial distributions evaluated at 1 through 50:

$$Pr(heads\leq 50)=\sum_{i=1}^{50}\binom{100}{i}\left(\frac{2}{3}\right)^{i}\left(1-\frac{2}{3}\right)^{100-i}$$

This calculation was done in python and resulted in the same answer.

Lab 1 Written Questions
By John Sigmon (js85773) and Amanraj Mahal (asm3539)
1/19/18

In [8]:
```
import numpy as np
from scipy import special as sp

p = 2/3
n = 100
probability_array = np.zeros(50)

for i in range (50):
    probability_array[i] = sp.comb(n, i)*(p**i)*((1-p)**(n-i))
    #print(probability_array[i])

print("The sum of all these probabilities is: {}".format(np.sum(probability_array)))
```

```
The sum of all these probabilities is: 0.0001989326425396602
```