6.19 Arnolds five-minute problem

$$\int_{-\pi}^{\pi} |\cos^N(x)|\,dx = 2\int_{-\pi/2}^{\pi/2} |\cos^N(x)|\,dx \approx 2\int_{-\pi/2}^{\pi/2} \left(1-\frac{x^2}{2}\right)^N dx$$

this is a good approximation as $\cos(x) \approx 1 - x^2/2$ is good for small $x$ and for larger $x$ the power of $N$ large makes it irrelevant

$$\approx 2\int_{-\pi/2}^{\pi/2} \exp\left(-\frac{x^2}{2}\right)^N \approx 2\int_{-\infty}^{\infty} \exp\left(-\frac{Nx^2}{2}\right)$$

large $N$ makes this a very good approximation.

again for small $x$ the approximation $\exp(-x^2/2) = 1 - \frac{x^2}{2}$ is good and for $x$ closer to $\pm \pi/2$ the large power of $N$ makes it irrelevant

$$= 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{N/2}} \exp(-u^2)\,du = \frac{2\sqrt{2}}{\sqrt{N}} \int_{-\infty}^{\infty} \exp(-u^2) = \frac{2\sqrt{2}\sqrt{\pi}}{\sqrt{N}}$$

well known gauss integral

$$\int_{-\pi}^{\pi} |\sin^N(x)|\,dx = \int_{-\pi}^{\pi} |\cos^N(x)|\,dx = \frac{2\sqrt{2}\sqrt{\pi}}{\sqrt{N}}$$

$\sin(x)$ is simply a translation of $\cos(x)$

$$\Rightarrow \int_{-\pi}^{\pi} \sin^{100}(x)\,dx = \frac{2\sqrt{2}\sqrt{\pi}}{\sqrt{100}}$$

To find the average we divide by $2\pi$ giving an average of $\frac{\sqrt{2}}{\sqrt{100}\sqrt{\pi}} = 0.079788\ldots$

Compared to the correct answer (from mathematica) of $0.079588\ldots$

Well within the 10 % tolerance