5.16

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \frac{1}{\pi}\left[\frac{(n^2x^2-2)\sin(nx) + 2nx\cos(nx)}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi n\cos(n\pi) + 2\pi n\cos(-n\pi)}{n^3}\right] = \frac{4}{n^2}(-1)^n \quad \text{for } n>0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(\cancel{2}\, nx)\,dx = 0$$ as $x^2\sin(nx)$ is odd

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

$$\Rightarrow \quad x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx)$$

Set $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n(-1)^n = \frac{\pi^2}{3}\sum_{n=1}^{\infty}\frac{4}{n^2} \Rightarrow \pi^2 = 6\sum_{n=1}^{\infty}\frac{1}{n^2} \quad (*)$$

Now find a series for $f(x) = x^4$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\cos(nx)\,dx = \frac{8\cancel{\pi n}(\pi^2n^2-6)\cos(\pi n)}{\cancel{\pi}\, n^{\cancel{8}4}} \quad n>0 \qquad a_0 = \frac{2\pi^4}{5}$$

$$\cancel{\Rightarrow x^4 \Rightarrow} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\sin(nx)\,dx = 0$$ again as $x^4\sin(nx)$ is odd

$$\Rightarrow \pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty}\frac{8(\pi^2n^2-6)}{n^4}\underbrace{\cos(\pi n)}_{(-1)^n}\underbrace{\cos(\pi n)}_{(-1)^n} = \frac{\pi^4}{5} + 8\left(\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2} - 6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\pi^4 \underset{(*)}{=} \frac{\pi^4}{5} + 8\cdot\pi^2\cdot\frac{\pi^2}{6} - 8\cdot 6\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^4}{15} = 6\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \pi^4 = 90\sum_{n=1}^{\infty}\frac{1}{n^4}$$