Modeling a Vibrating Manifold Within a Confined Space Using Runge-Kutta 4

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1 Abstract

The purpose of this experiment is to explore the application of Runge-Kutta as a method for finite difference algorithms. A wave function was sued to find a specific solution for the general wave equation. Usinge Runge-Kutta 4 the manifold properly iterated through time, rebounded against the boundaries of our plane in a confined space, and interference dampened the amplitude as the wave spread along the plane through the space as predicted.

2 Introduction

A wave equation defines the degrees of freedom to some set of observables. Waves are naturally occurring everywhere. Waves can clearly seen with everything from a ripple in water to relativistic quantum mechanics and to Newtonian spring motion, to name a few examples. The general wave equation appears as:

$$c^2\nabla^2 z \tag{1}$$

where c is a fixed constant and is directly proportional to wave speed and inversely proportional to the density of the medium. ∇ is the Laplacian and z is a general wave function. For this experiment the wave function was chosen as:

$$z = e^{-\frac{(x\Delta x)^2+(y\Delta y)^2}{r^2}} - e^{-1} \tag{2}$$

For our wave function, x and y indicate a position along a mesh grid cast along the x and y axes and r refers to the radius of our manifold.

In order to proceed through time the finite difference method Runge-Kutta 4 (RK4) is used. In numerical analysis, RK4 is among a family of implicit and iterative methods. RK4 also takes advantage of including Euler's Method within the algorithm, another stand-alone finite difference method.

3 Methods and Modeling

A plane of one meter (m) by one m with a center for a circular manifold of diameter .25 m is the goal. In order to accomplish this initial conditions for the wave function, using the z-axis, to represent amplitude along a x, y axes grid. All points outside radius of the manifold are initialized at amplitude of zero.

Using z to represent some unknown positional wave function of time t, and vz to be the wave function of time and position, or velocity. And assign τ as a time step, Runge-Kutta 4 (RK4) can be viewed as follows:

$$z_{t+\tau} = z_t + \frac{\tau}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3}$$

, and the derivative with respect to time gives us vz:

$$vz_{t+\tau} = vz_t + \frac{\tau}{6}(a_1 + 2a_2 + 2a_3 + a_4) \tag{4}$$

k is calculated from finding the slope at the beginning (Euler's method), midpoint (twice), and end of the interval. A mathematical explanation can be shown below describe the relationship of a and k, k essentially is $vz_{t+\tau}$:

$$k_1 = vz_t + a_1\tau \tag{5}$$

$$k_2 = vz_{t+\tau} + a_2\frac{\tau}{2} \tag{6}$$

$$k_3 = vz_{t+\tau} + a_3\frac{\tau}{2} \tag{7}$$

$$k_4 = vz_{t+\tau} + a_4\tau \tag{8}$$

Where a is found using three sequential position points and a Δ position squared all multiplied by the speed of a wave phenomenon through some medium. For purposes of this experiment the speed of sound through air at sea level is used.

4 Results

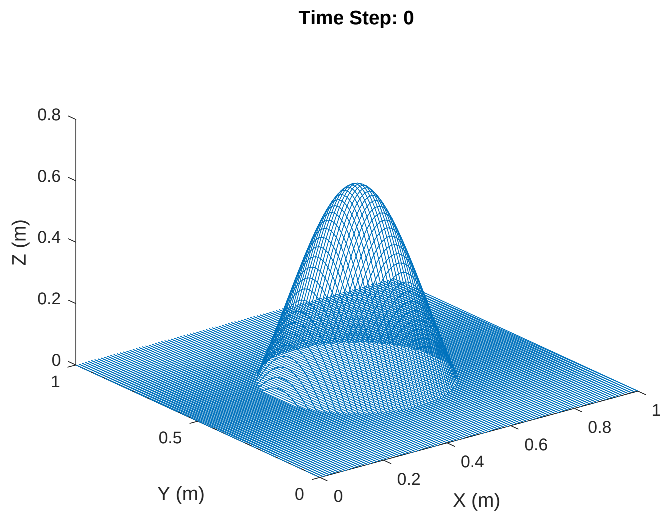

Figure 1:
Initial conditions for the manifold at maximum amplitude.

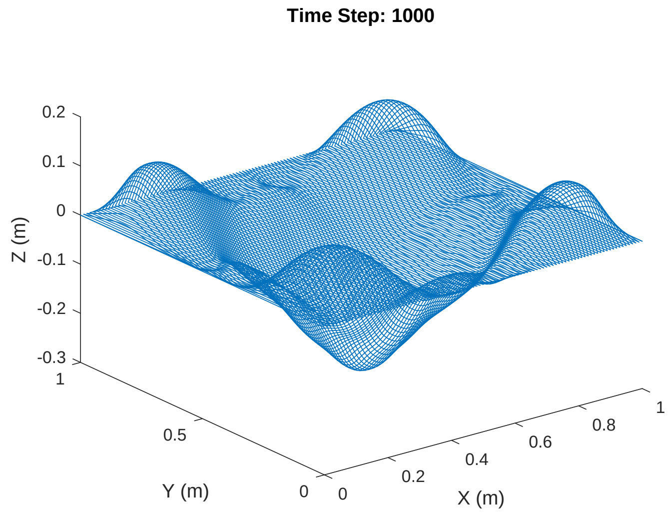

Figure 2:
Waves have been reverberating against boundaries causing reflections and interference.

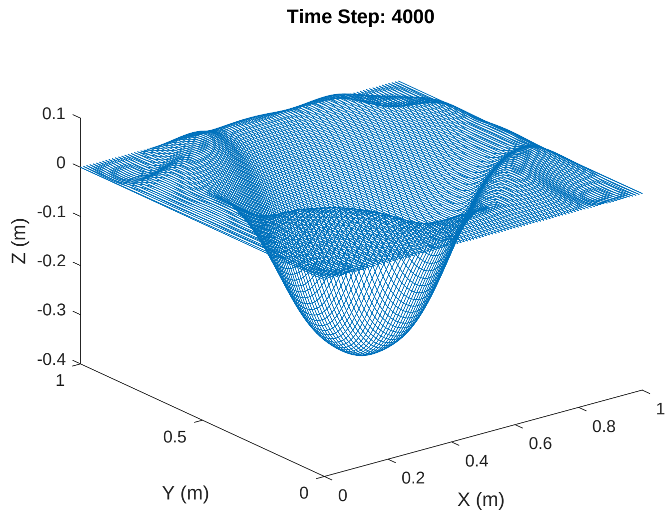

Figure 3:
Wave continues to decrease in amplitude at the center due to interference.

5 Conclusion

The purpose of this experiment is to explore the application of Runge-Kutta as a method for finite difference algorithms. The manifold properly iterated through time, rebounded against the boundaries of our plane in a confined space, and interference dampened the amplitude as the wave spread along the plane through the space.

References

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