The University of Western Ontario

Department of Mechanical and Materials Engineering

APPLIED COMPUTATIONAL FLUID MECHANICS
AND HEAT TRANSFER
Course MME 9614

Assignment No.1

"A comparison of Numerical solution with Exact Solution in case of 2-D steady state conduction through a rectangular domain"

Dated: 11 October 2016

Submitted to:
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1.Problem Description:

This problem involves 2-Dimesional steady state conduction analysis using numerical and analytical study through a plate. The results obtained with the help of numerical simulation techniques using ICEM CFD software will then further be compared with analytic solution of the given problem. For this particular problem analytical solution of Laplace equation is obtained with the help of Variable of separation method by applying the appropriate boundary conditions.

The main task of the problem is to compare numerical results with analytical solution in terms of temperature at x = 0.5 and y= 1, and flux at x =1 and y = 1.

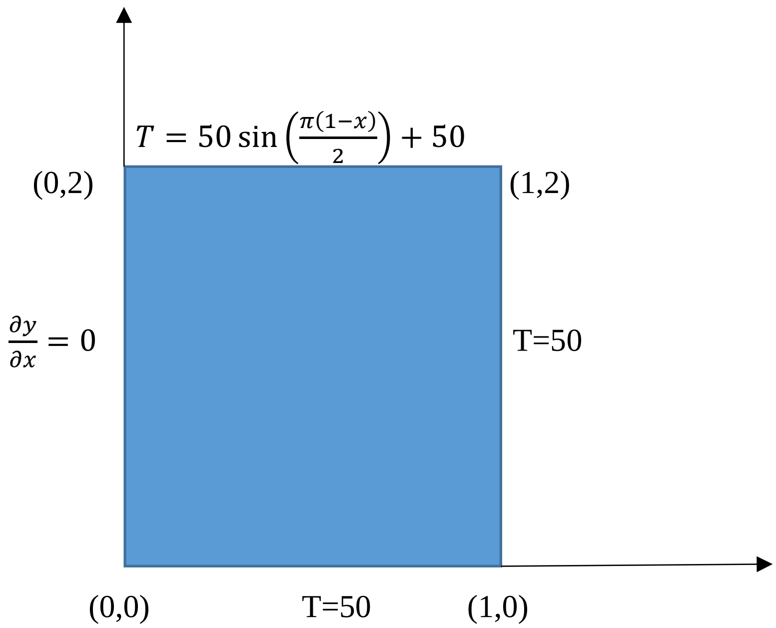

Fig. (1)

The Fig. (1) shows the dimensions of a rectangular plate and boundary conditions on different walls. The left and right wall is maintained at constant temperature of 50 degrees Celsius and temperature is a function of sine on the top wall. There is no heat flux on the left wall.

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2.Mathematical Models of the Problem:

[1]The general governing equation for the given below

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + S \qquad \text{Eq. (2.1)}$$

Where:
ρ = Density of solid
K = Thermal Conductivity; K= 0.75 W/m-K
T = Temperature
S = Source term; S=0

Assumptions for the problem:
Thermal conductivity is constant
Density of the solid is assumed to be constant
Steady state condition
There is no heat source term

The Eq. 1 can be deduced to the following equation by applying assumption;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{Eq. (2.2)}$$

The Eq. (2.2) is a second order linear homogeneous equation and can be solved by the method of separation of variables. To solve this equation analytically we need four boundary conditions which are stated in the problem:

a) At x = 0, $\frac{\partial T}{\partial x}$=0
b) At x = 1, T_1= 50
c) At y = 0, T_1 = 50
d) At y = 2, T = 50sin (π (1-x)/2) +50

Where temperature(T) is degree Celsius.

To make the boundary condition homogeneous for this simpler problem [2]shifting of co-ordinate system is required for driving the analytical solution shown in **Fig. (12)**

[3]Therefore, the final analytical solution of the above problem is given by:

$$T = T_m \left\{\frac{\sinh\left(\frac{\pi y}{L}\right)}{\sin\left(\frac{\pi H}{L}\right)}\right\} * \sin\left(\frac{\pi(1-x)}{2}\right) + T_1 \qquad \text{Eq. (2.3)}$$

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[3]The Eq. (2.3) shows the temperature distribution along the x-y direction.
The heat flux can be derived by doing partial differential w.r.t "x" of Eq. (2.3)

$$\emptyset_x = -K.\frac{\pi}{2} * T_m \left\{\frac{\sinh\left(\frac{\pi y}{L}\right)}{\sin\left(\frac{\pi H}{L}\right)}\right\} * \cos\left(\frac{\pi(1-x)}{2}\right) \qquad \text{Eq. (2.4)}$$

$$\emptyset_y = -K.\frac{\pi}{L} * T_m \left\{\frac{\cosh\left(\frac{\pi y}{L}\right)}{\sin\left(\frac{\pi H}{L}\right)}\right\} * \sin\left(\frac{\pi(1-x)}{2}\right) \qquad \text{Eq. (2.5)}$$

$$\emptyset_T = \emptyset_x + \emptyset_y \qquad \text{Eq. (2.6)}$$

[4]The analytical solution of Temperature Distribution is obtained by using the **MATLAB** code, which is given below:

```
Tc = 323.15; %given in the problem
Tm = 50;     % given in the problem
x = 0.5;     % Temperature monitoring point at x
y = 1;       % Temperature monitoring point at y
L = 2;       %Total length of x-co-ordinate after shifting
H = 2;       %given in the problem
T=0;
% Initialize rectangular grid
[x,y] = meshgrid(0:0.04:1,0:0.04:2);
% Initialize T
T = zeros(length(x),length(y));
% Calculate steady state temperature distribution
T = Tc + Tm*sinh(pi*y/L)/(sinh(pi*H/L)).*sin(pi*((1-x)/2));
% Plot temperature distribution
surface(x,y,T)
shading('interp')
colorbar
colormap gray
title('Steady-State Temperature Plot')
```

The analytical solution of Heat Flux is obtained by using the **MATLAB** code, which is given below:

```
T1 = 323.15;   %Given in the problem statement
 Tm = 50;      % Given in the Problem Statement
  x = 1;       %Flux Computation point
  y = 1 ;
    L = 2;     % Dimensions of rectangular Domain
    H = 2;
    K = 0.75;  % Thermal Conductivity of a Material
t = 0;

  for i=0:0.1:2;
```

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```
    fu = sinh(pi*i/L)/sinh(pi*H/L);
    fv =  cos(pi*(1-x)/2);

    ft =  -K*pi/2*Tm*(fu*fv)

end
```

3.Numerical Procedure:

This is a 2-D steady state conduction problem therefore only energy equation will be solved. Hybrid technique is used in fluent solver. However, this particular problem is calculated numerically using **central differencing** scheme because the Peclet Number is less than 2 which is the required condition for accuracy. We are dealing with 2-D steady state conduction problem only so there is no heat transport by convection. From Eq. (3.1), Peclet number is zero.

$$P_e = \frac{Heat\ transport\ by\ convection}{Heat\ transport\ by\ conduction} \qquad \text{Eq. (3.1)}$$

Discretization of partial differential equation for 2-D Steady state case by using central differencing scheme:

$$\nabla.(\rho u \emptyset) = \nabla(\zeta \nabla \emptyset) + s \qquad \text{Eq. (3.2)}$$

As described earlier, this is only 2-D steady state conduction. So there is no convection, no heat generation source and also the thermal conductivity is also constant. Therefore, we only need to discretize Laplace equation which is stated below:

$$\nabla(\nabla \emptyset) = 0 \qquad \text{Eq. (3.3)}$$

[5]The Eq. … can be written as:

$$\boldsymbol{\frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) = 0} \qquad \text{Eq. (3.4)}$$

To solve the eq. () discretization can be done by integrating over the volume in both x and y- direction.

$$\boldsymbol{\iint_{s\,w}^{n\,e} \frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) dxdy + \iint_{s\,w}^{n\,e} \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) dxdy = 0} \qquad \text{Eq. (3.5)}$$

$$\boldsymbol{\Delta y \frac{\emptyset_e - \emptyset_p}{\delta x_e} - \Delta y \frac{\emptyset_p - \emptyset_w}{\delta x_w} + \Delta x \frac{\emptyset_n - \emptyset_p}{\delta x_n} - \Delta y \frac{\emptyset_p - \emptyset_s}{\delta x_s} = 0} \qquad \text{Eq. (3.6)}$$

To solve this problem numerically ICEM CFD commercial package is used for geometry and meshing and FLUENT is used for post processing. It is assumed that properties are constant and value of thermal conductivity for the plate is, k=0.75 w/m k. The desired boundary conditions which are constant has been applied into FLUENT solver. However, the boundary for the top face is not constant and is function of sine. A user defined code is written for this function in c-language which is given below:

```
#include "udf.h"
#define PI 3.1415926535897932384626433832795
DEFINE_PROFILE(temperature,t,i)
{
face_t f;
real y;
real x[ND_ND];

  begin_f_loop(f,t)

   {
    F_CENTROID(x,f,t);
    y=x[0];
    F_PROFILE(f,t,i)=(50.*sin(PI/2.*(1.0-y))+50.)+273.15;
  }

   end_f_loop(f,t)

}
```

The Energy Criteria has been reduced for this particular to 1e-20 for higher accuracy. The temperature is monitored at point (0.5, 1) and the heat flux will be monitored at (1,1).

The quality of a mesh is checked with two factors namely skewness and aspect ratio. Shown in Fig. (13) and Fig. (14)

Mesh Quality calculated from FLUENT solver as shown below;

Skewness:
Minimum Orthogonal Quality = 1.00000e+00
Maximum Ortho Skew = 0.00000e+00
(Orthogonal Quality ranges from 0 to 1, where values close to 0 correspond to low quality.)

Maximum Ortho Skew = 0.00000e+00
(Ortho Skew ranges from 0 to 1, where values close to 1 correspond to low quality).

Aspect ratio:
Maximum Aspect Ratio = 1.43294e+00

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4.Grid/Time Step Independent Test:

The grid independent test is done on three different grid sizes, G_1=20 x 10, G_2= 40 X 20 and G_3=80 x 40. On analyzing the results as shown in Table 1 and **Table (2)** for the given grid sizes, the percentage difference in the solution for Temperature and Heat Flux distribution is less than 1% between G_1, G_2 and G_3 as shown in the **Table (3)**. Therefore, any of the mesh size is suitable to use for the determination of temperature distribution and heat flux accurately. No time independence test is performed due to the steady state condition in the given problem.

5.Results:

From Eq. (2.3), "x" is a function of sine so it clearly indicates that the temperature distribution in x-direction is periodic while "y" variable is a hyperbolic function which states that the temperature decay in y-direction should be exponential as per analytical solution and physics of boundary conditions. The temperature is decaying exponentially in y-direction and periodically in x- direction in numerical solution as well also, as shown in Fig. (6). The temperature for mesh size 80 x 40 at (x=0.5, y=1) is **330.1950 degree Celsius** as compared to exact value **330.1952** which is almost same as shown in Table (1) and Flux for mesh size 80 x 40, Table (2) at (x=1, y=1) is **-11.7417 k/w m** as compared with exact value which is **-11.7379 k/w m.**

6.Discussion of Results (Numerical and Theoretical):

The analytical solution of temperature distribution for given problem is represented in Eq. (2.3) and for Heat Flux is represented in Eq. (2.4) and Eq. (2.5) which is solved using variable of separation method. I have calculated the exact solution using Matlab Code and the results are shown in Table (1) and Table (2). On the other hand, the numerical solution is obtained with the help of ICEM CFD and Fluent solver and the solution values are shown in Table (1) and Table (2) for different meshes like 20 x 10, 40 x 20 and 80 x 40. If we look at Table (1) and Table (2) as we increase the mesh size the solution approaches to the exact value but it will never be attained the exact value regardless of the mesh size because it is computed numerically. The numerically computed values of temperature distribution and Heat Flux as shown in the Fig. (5) and Fig. (10) are coinciding with the exact solution it means there is very less difference between the values. Furthermore, Temperature is high on the Top surface, when y=2 and decaying exponentially in downward direction and attained a 323.15 kelvin at y=0 as shown in Table (1). These temperature lines as shown in Fig. (6) are called isotherm lines and total heat flow vector is perpendicular to these lines

7.Conclusions:

A comparison of Numerical study and analytical study was done on 2-D steady state conduction on a rectangular domain having constant properties. As we have seen in results the numerically obtained solution are very close to the exact solution therefore we can apt CFD solutions without any concern. On the other hand, it is a cumbersome task to solve the problems theoretically because of non-linear partial differential equations which are very difficult to solve. But the major drawback of CFD is validation of numerical results.

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One cannot rely on results without proper verification of results with analytical or experimental data.

8.Tables and Graphs

Comparison of Numerically computed temperatures with Exact Solution				
	Numerically computed values of Temperatures			EXACT solution
Y-Direction	MESH 20x10	MESH 40x20	MESH 80 X 40	
0	323.15	323.15	323.15	323.150000
0.1	323.632	323.633	323.633	323.6329
0.2	324.126	324.127	324.128	324.1277
0.3	324.645	324.646	324.647	324.6466
0.4	325.201	325.202	325.203	325.2026
0.5	325.808	325.809	325.809	325.8094
0.6	326.481	326.482	326.482	326.4818
0.7	327.235	327.237	327.237	327.2367
0.8	328.091	328.093	328.093	328.0926
0.9	329.068	329.07	329.071	329.0707
1	330.19	330.194	330.195	330.1952
1.1	331.485	331.492	331.494	331.4939
1.2	332.984	332.996	332.998	332.9989
1.3	334.724	334.743	334.746	334.7474
1.4	336.748	336.775	336.781	336.7826
1.5	339.104	339.144	339.152	339.1549
1.6	341.85	341.907	341.919	341.923
1.7	345.051	345.132	345.15	345.1551
1.8	348.787	348.899	348.924	348.9314
1.9	353.147	353.301	353.335	353.3451
2	358.237	358.445	358.491	358.5053

TABLE (1) (Temperature Distribution)

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	HEAT FLUX Variations			
	Numerically Computed Values			EXACT SOLUTION
Y-Direction	Heat Flux 20x10	Heat Flux 40x20	Heat Flux 80 x 20	Heat Flux
0	-0.433784	-0.208179	-0.102072	0
0.1	-0.829661	-0.805324	-0.804694	-0.8045
0.2	-1.63682	-1.63077	-1.62933	-1.6289
0.3	-2.50679	-2.49668	-2.4943	-2.4935
0.4	-3.43916	-3.42443	-3.42095	-3.4198
0.5	-4.45676	-4.43688	-4.4322	-4.4307
0.6	-5.58448	-5.55902	-5.55303	-5.5511
0.7	-6.84993	-6.81852	-6.81114	-6.8088
0.8	-8.28407	-8.24644	-8.23761	-8.2348
0.9	-9.92195	-9.87801	-9.86769	-9.8644
1	-11.8036	-11.7534	-11.7417	-11.7379
1.1	-13.975	-13.9189	-13.9058	-13.9016
1.2	-16.489	-16.4279	-16.4136	-16.409
1.3	-19.4069	-19.3422	-19.327	-19.3222
1.4	-22.7997	-22.7335	-22.718	-22.7131
1.5	-26.75	-26.6855	-26.6704	-26.6655
1.6	-31.3535	-31.2954	-31.2817	-31.2773
1.7	-36.7222	-36.6766	-36.6658	-36.6623
1.8	-42.9859	-42.9617	-42.9558	-42.9539
1.9	-48.6157	-50.3053	-50.307	-50.3075

Table (2) (Heat Flux Variations)

Percentage Difference Between G1, G2 and G3 Mesh			
Properties	**CFD Results on Mesh Size 20 x 10**	**CFD Results on Mesh Size 40 x 20**	**CFD Results on Mesh Size 80 x 40**
Heat Flux	-11.8036	-11.7534	-11.7417
Temperature	330.19	330.194	330.195
Temperature % Difference at x=0.5, y=1	**0.001211**		
		0.00030285	
Heat Flux % Difference at x=1, y=1	**0.4271105**		
		0.099644856	

Table 3(Comparison between three different Mesh Sizes)

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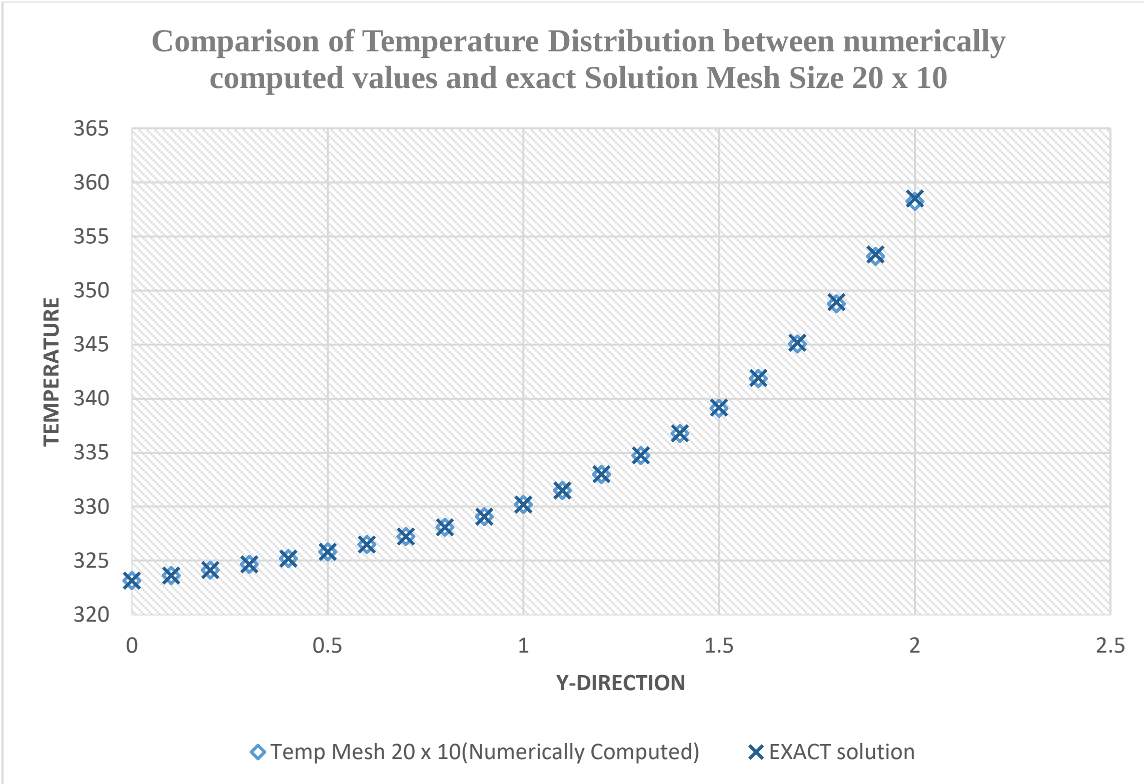

Fig. 1 (Temperature Distribution on Mesh Size 20 x 10)

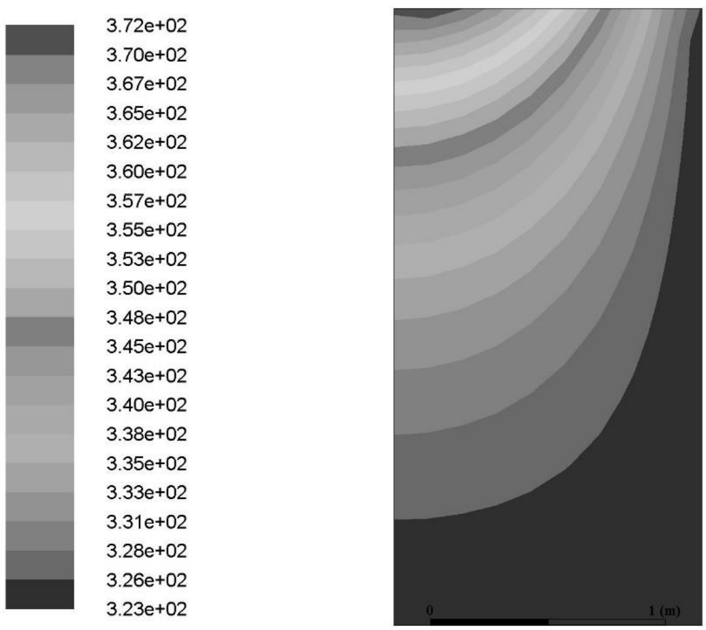

Fig. 2 (Temperature Contour On Mesh Size 20 x 10)

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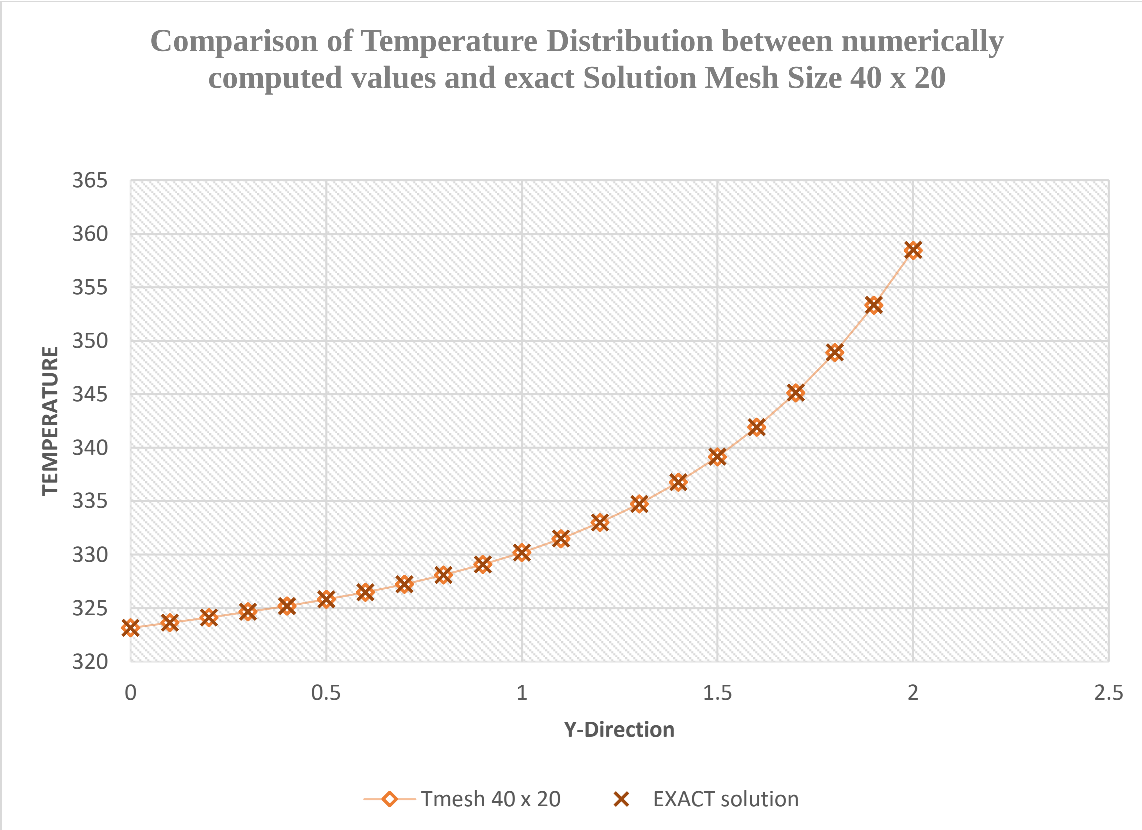

Fig. 3 (Temperature Distribution on Mesh Size 40 x 20)

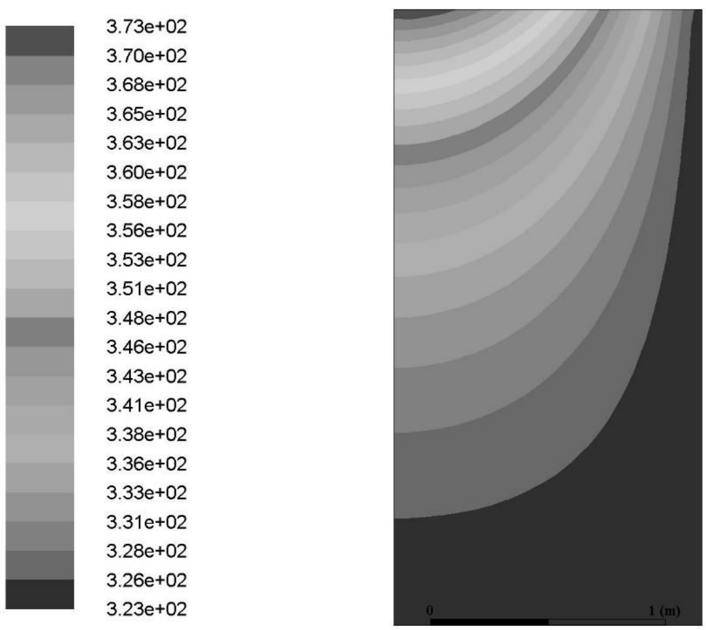

Fig. 4 (Temperature Contour On Mesh Size 40 x 20)

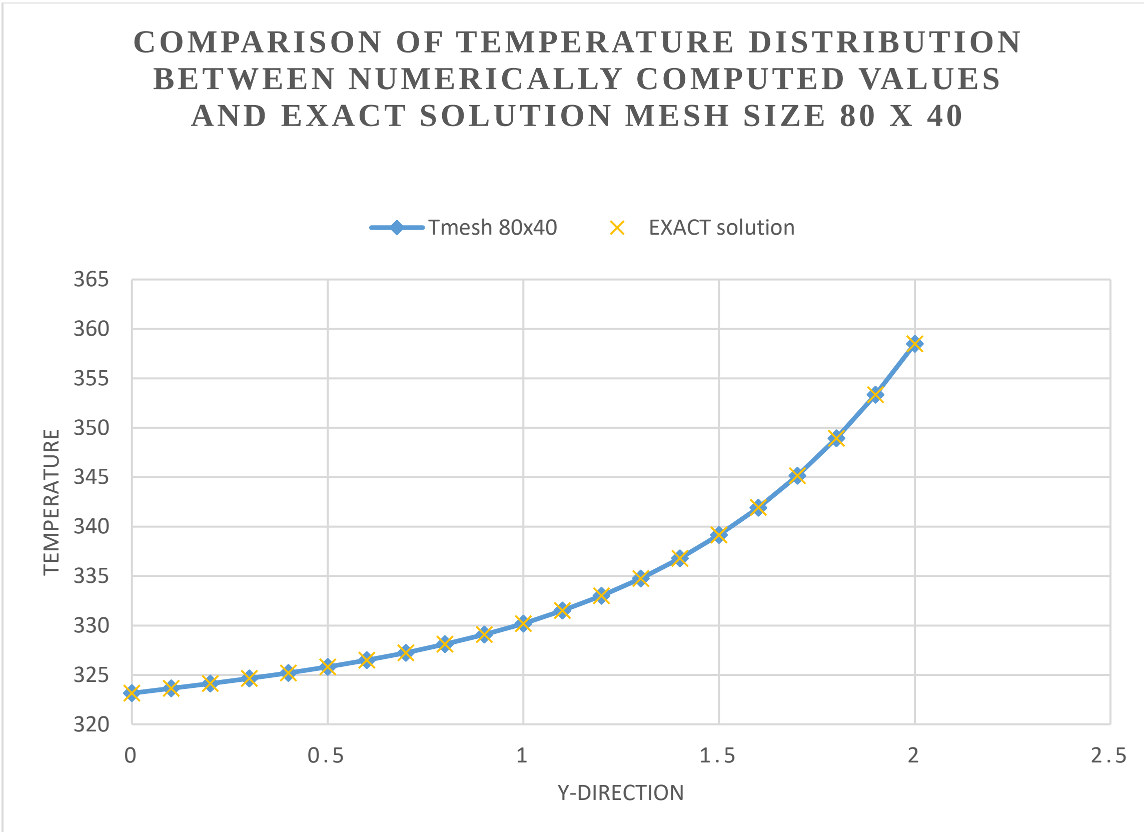

Fig. 5 (Temperature Distribution on Mesh Size 80 x 40)

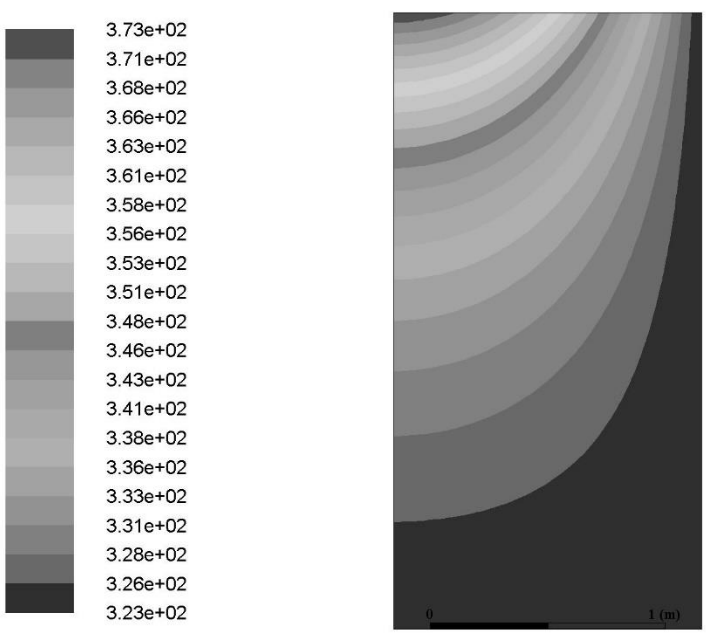

Fig. 6 (Temperature Contour On Mesh Size 80 x 40)

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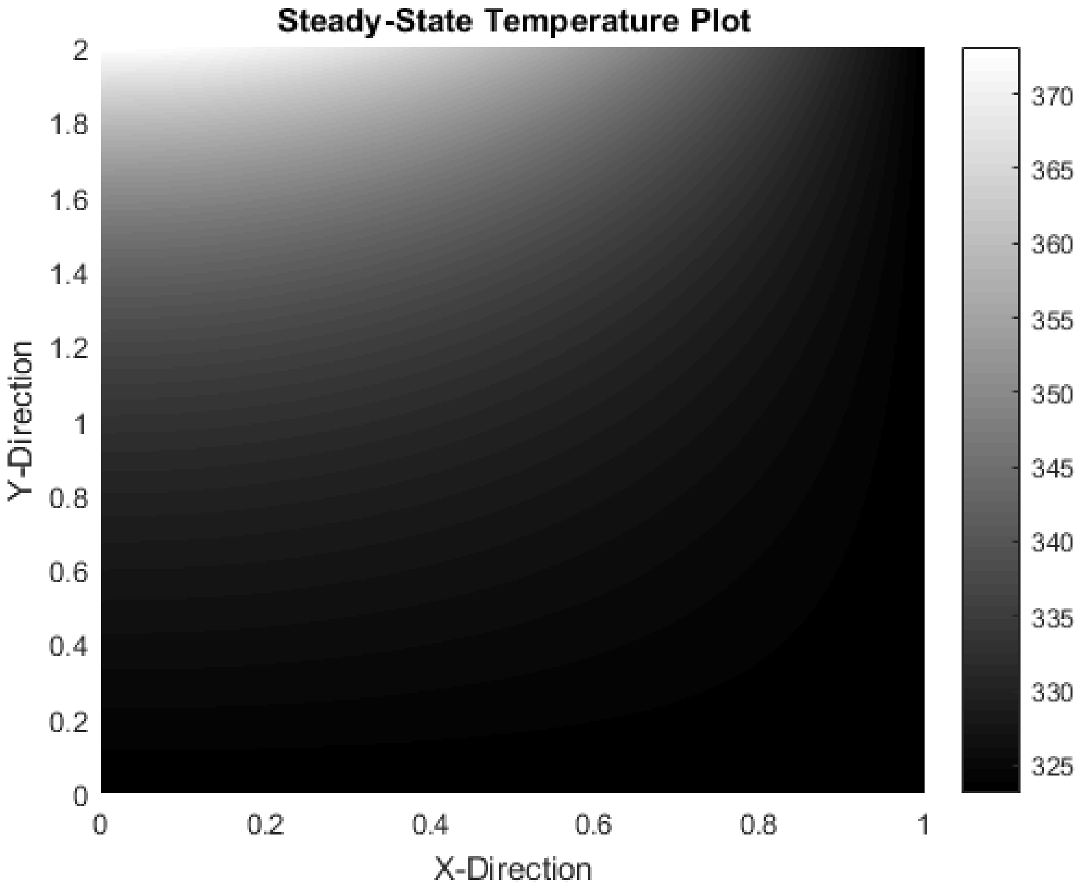

Fig. 7 (Temperature Contour of Exact Solution Obtained with Matlab Code)

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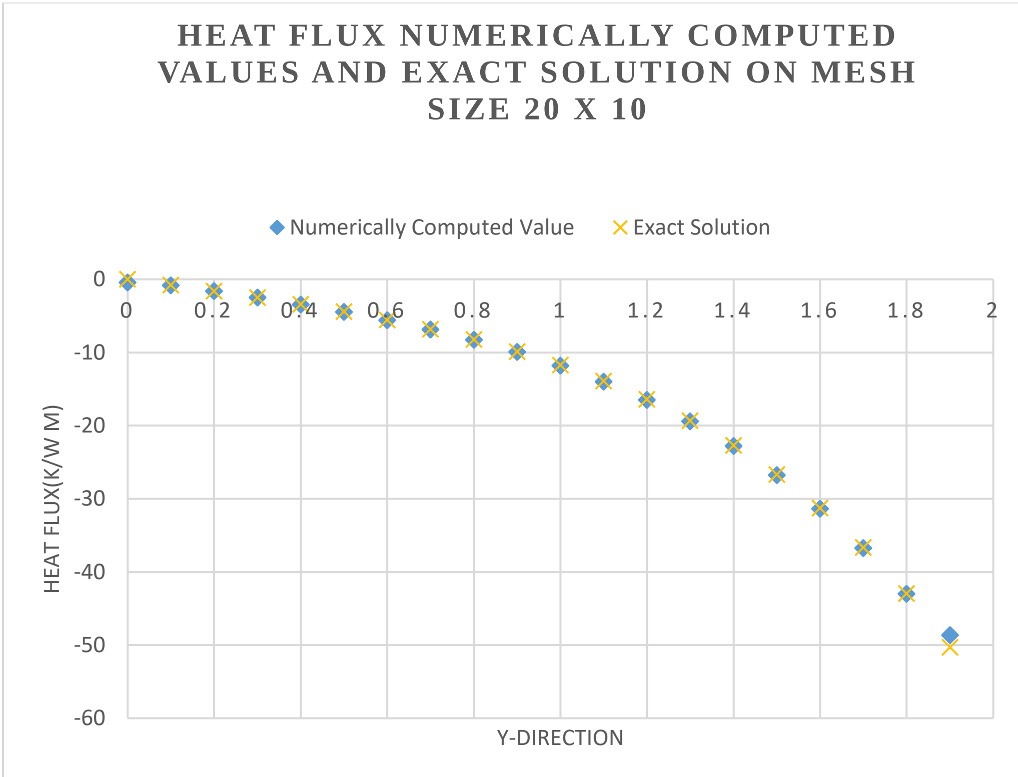

Fig. 8 (Heat Flux Distribution on Mesh Size 20 x 10)

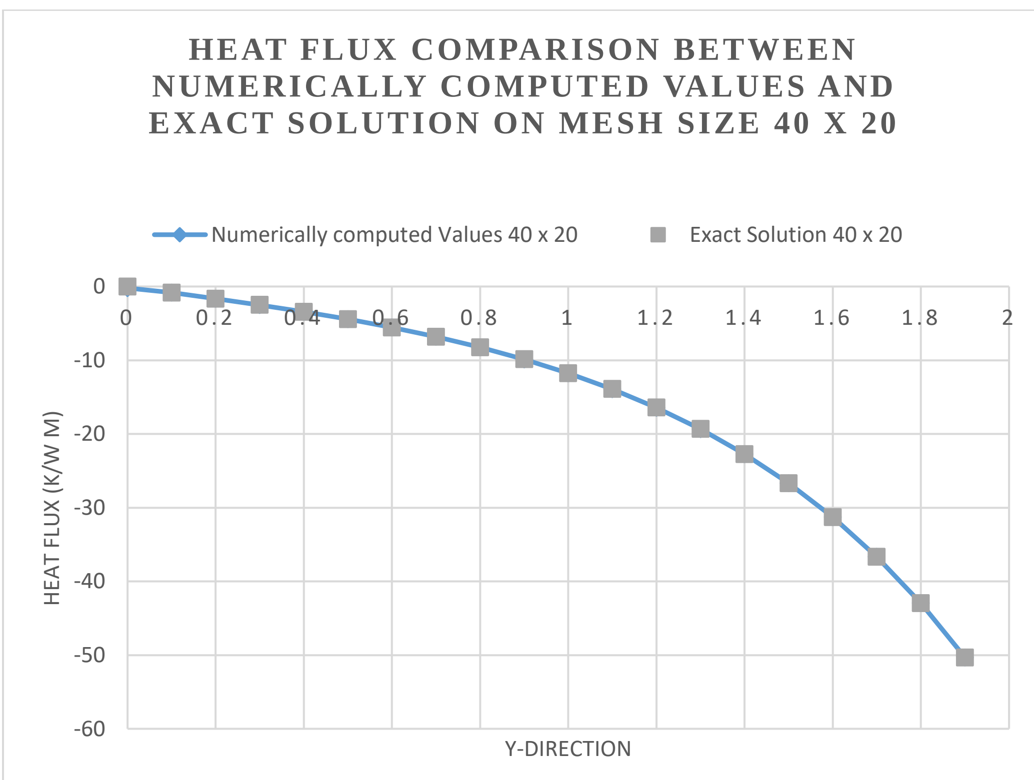

Fig. 9 (Heat Flux Distribution on Mesh Size 40 x 20)

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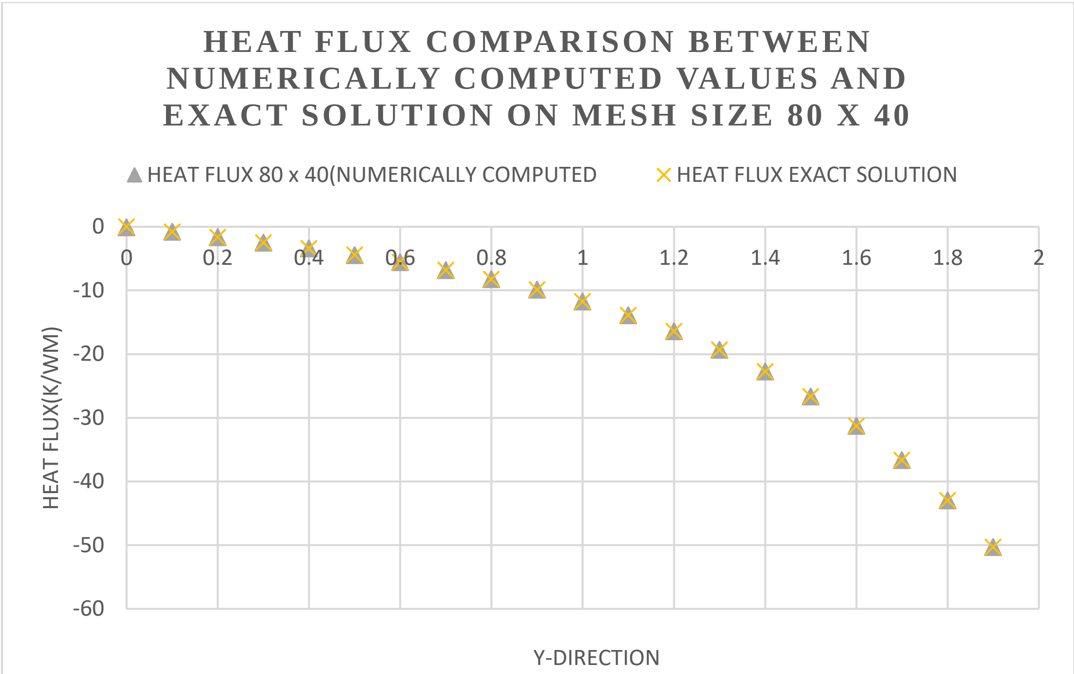

Fig. 10 (Heat Flux Distribution on Mesh Size 80 x 40)

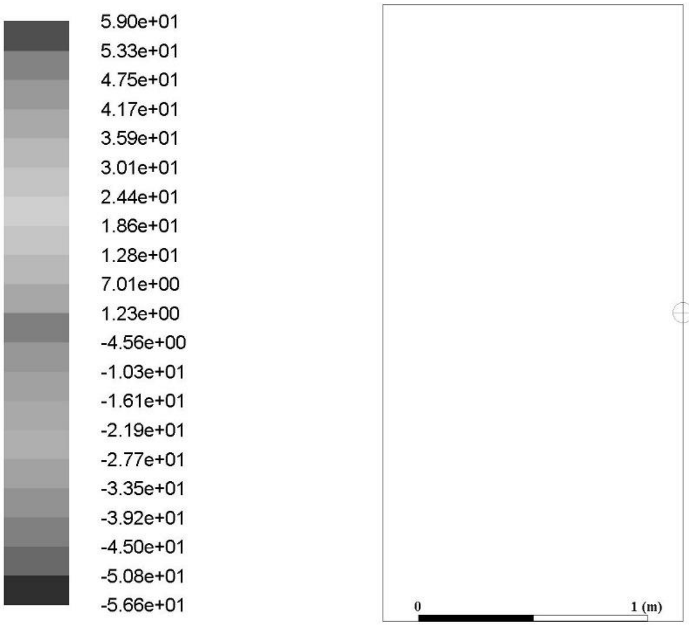

Fig. 11 (Heat Flux Point (x=1, y=1) on Mesh Size 80 x 40)

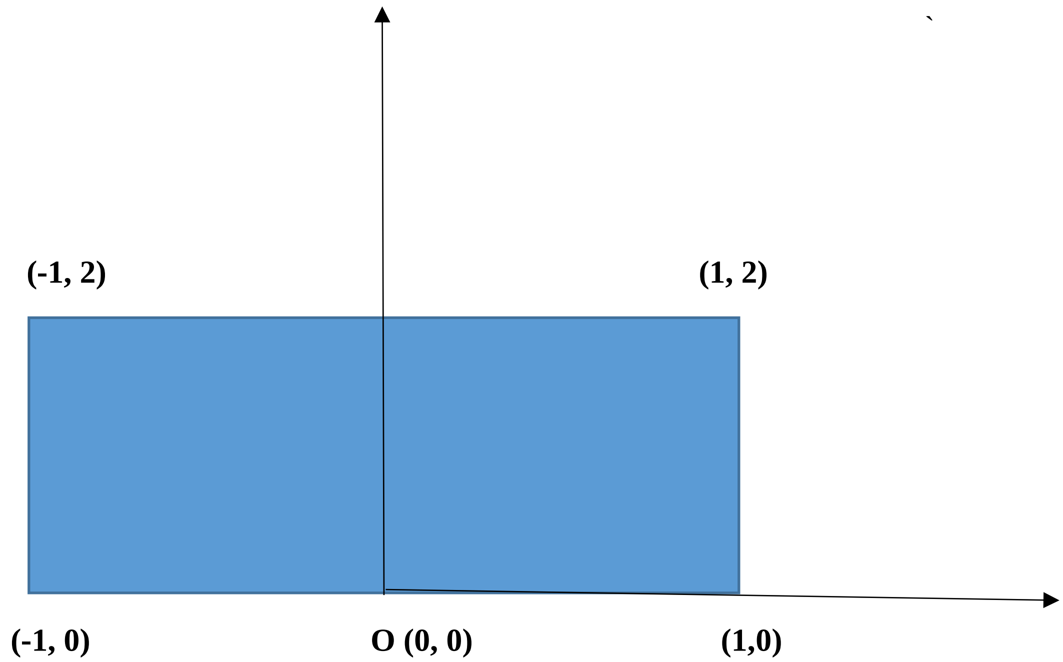

Fig. (12) Shifting of Co-ordinate Axis

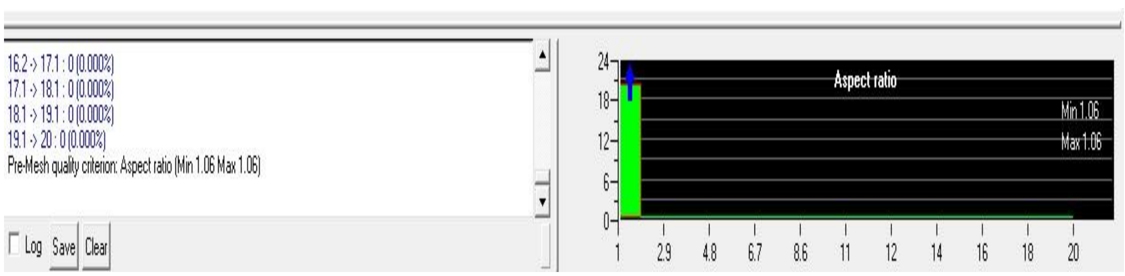

Fig. (13) Quality of a Mesh based on Aspect Ratio

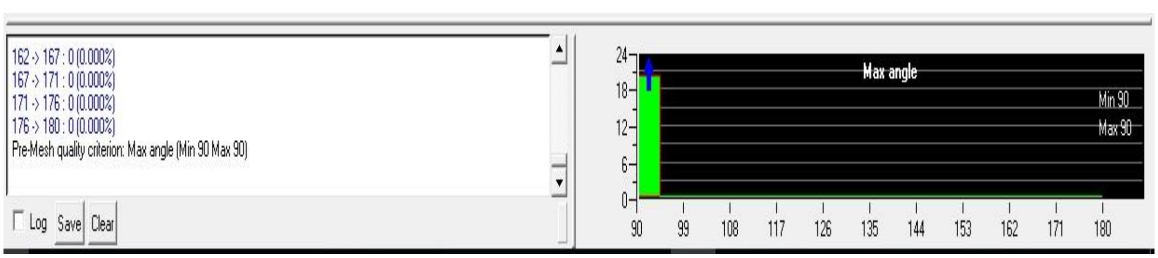

Fig. (14) Quality of a mesh based on Skewness

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References:

[1] Patankar, SV, "Numerical Heat Transfer and Fluid Flow", PP. 15-16.

[2] Latif M., Jiji, "Heat Conduction" Third Edition, Springer Link PP. 105-106.

[3] Hollman, JP, "Heat Transfer" 10^{th} Edition, PP-81.

[4] Jennifer C, Perrine Pepiot, Vadim Khayms. "MATLAB WorkBook" PP 14-15

[5] "https://ses.library.usyd.edu.au/bitstream/2123/376/3/adt-NU20010730.12021503_chapter_2.pdf" PP 14.

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EVALUATION SHEET

Name: ____________________________ Assignment NO.__________

	MARKS	MAX
General Organization		5
Quality of Presentation		5
Technical Writing		5
Problem Description		5
Mathematic Models		10
Numerical Procedure		10
Grid/Time Step Independent Tests		10
Analysis of Results		20
Discussion of Results		20
Conclusions		5
References		5
TOTAL		100