Laplacian formula for Gaussian curvature and volume measurement, Gauss-Bonnet theorem

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We want to further familiarize the reader with properties of 2-dimensional closed Riemannian manifolds. Therefore we first provide a mean to compute the Gaussian curvature with a differential operator. Then we will give a definition of the total area of such a manifold and prove that it is well defined. We will also state the Gauss-Bonnet theorem which provides a link between the topology and the geometry of such manifolds.

1 Prerequisites

Last week we learned that the first fundamental form assigns an inner product to each tangent plane $T_p\Sigma$ of a 2-dimensional manifold by restricting the inner product of the ambient $\mathbb{R}^3$. We will now generalise this by defining the *Riemannian metric* which does not depend on the inner product of the ambient space.

Definition 1.1 (Riemannian metric)**.** Let $\Sigma \subseteq \mathbb{R}^3$ be a 2-dimensional manifold. A Riemannian metric is a map $g : \Sigma \ni p \mapsto \mathcal{G}_p(\cdot,\cdot)$ that assigns an inner product $\mathcal{G}_p(\cdot,\cdot)$ to the tangent plane $T_p\Sigma$ at each point $p \in \Sigma$ such that for each chart $(U, \varphi : U \to V \subseteq \mathbb{R}^2)$ the functions

$$g_{ij}(v_1, v_2) := \mathcal{G}_{\varphi^{-1}(v)}\left(\frac{\partial\varphi^{-1}(v)}{\partial v^i}, \frac{\partial\varphi^{-1}(v)}{\partial v^j}\right) \tag{1}$$

are smooth. Then $(\Sigma, \mathcal{G})$ is called a 2-dimensional Riemannian manifold.

We already know that $\{\frac{\partial\varphi^{-1}(v)}{\partial v^i}\}_{i=1,2}$ is a basis of $T_{\varphi^{-1}(v)}\Sigma$ for some chart (U,φ), now let $\{dv^i\}_{i=1,2}$ denote a basis of the dual space $T^*_{\varphi^{-1}}(v)\Sigma$ that is dual to $\{\frac{\partial\varphi^{-1}(v)}{\partial v^i}\}_{i=1,2}$

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i.e.

$$dv^i \left(\frac{\partial \varphi^{-1}(v)}{\partial v^j} \right) = \delta^i_j \tag{2}$$

Using this we can express $\mathcal{G}$ in terms of the dual basis

$$\mathcal{G}_{\varphi^{-1}(v)}(\cdot, \cdot) = g_{ij}(v) dv^i(\cdot) dv^j(\cdot) \tag{3}$$

or for short

$$\mathcal{G} = g_{ij} dv^i dv^j \tag{4}$$

Previously the Christoffel symbols and the Gaussian curvature were defined in terms of the first fundamental form, now we define them in terms of the Riemannian metric by the same formulas

$$\Gamma^k_{ij} = \frac{1}{2} g^{lk}(g_{il;j} + g_{jl;i} - g_{ij;l}) \tag{5}$$

$$K = \frac{2}{g_{11}}(\Gamma^2_{1[1;2]} + \Gamma^j_{1[1}\Gamma^2_{2]j}) \tag{6}$$

One has to check that this definition does not depend on the coordinate chart.

2 Laplacian formula for Gaussian curvature

In this section we will prove that for a metric in so called *isothermal coordinates* the Gaussian curvature can be computed essentially by applying a differential operator to the conformal factor in these coordinates. This can be used to shorten the calculation of Gaussian curvature. For further reference cf. [5].

Definition 2.1. (Isothermal coordinates) Let $(\Sigma, \mathcal{G})$ be a 2-dimensional Riemannian manifold then we call a chart $(U, \varphi \ni p \mapsto (x, y) \in \mathbb{R}^2)$ *isothermal coordinates* if the metric becomes conformal to the standard flat metric in these coordinates i.e. $\mathcal{G} = \lambda(x, y)(dx^2 + dy^2)$, $\lambda > 0$.

Definition 2.2 (Laplace-Beltrami operator)**.** The *Laplace-Beltrami* operator for a metric $\lambda(x, y)\delta_{ij}$ in isothermal coordinates is given by

$$\Delta_{LB} = \frac{1}{\lambda(x, y)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{7}$$

Corollary 2.3. *For a metric $g_{ij} = \lambda(x, y)\delta_{ij}$ in isothermal coordinates the following holds*

$$K = -\frac{1}{2} \Delta_{LB} \log \lambda(x, y) \tag{8}$$

here K denotes the Gaussian curvature.

Proof. Let $g_{ij} = \lambda(x,y)\delta_{ij}$ be a metric in isothermal coordinates. Since $\lambda(x,y) > 0$ we can choose a function $\mu(x,y)$ such that

$$\mu(x,y) = \frac{1}{2}\log\lambda(x,y) \tag{9}$$

hence we can write the metric as follows

$$g_{ij} = e^{2\mu(x,y)}\delta_{ij}\ . \tag{10}$$

Now we compute the Christoffel symbols of such a metric by using formula (5)

$$\Gamma^k_{ij} = \frac{1}{2}g^{lk}(g_{il;j} + g_{jl;i} - g_{ij;l}) \tag{11}$$

The g^{ij} denote the components of the inverse matrix of g_{ij}. First we note

$$g_{ij;l} = 2\mu_l e^{2\mu}\delta_{ij} = 2\mu_{;l}g_{ij} \quad \text{for} \quad i,j,l = 1,2 \tag{12}$$

here $\mu_{;l}$ denotes partial differentiation of with respect to the l'th coordinate. Using this we can write

$$\Gamma^k_{ij} = \frac{1}{2}g^{lk}(2\mu_{;j}g_{il} + 2\mu_{;i}g_{jl} - 2\mu_{;l}g_{ij}) \tag{13}$$

$$= \mu_{;j}\delta^k_i + \mu_{;i}\delta^k_j - \mu_{;k}\delta_{ij} \tag{14}$$

From this formula we can read off the Christoffel symbols. We can use these to calculate

Γ^1_{ij}	$j=1$	$j=2$
$i=1$	$\mu_{;1}$	$\mu_{;2}$
$i=2$	$\mu_{;2}$	$-\mu_{;1}$

Γ^2_{ij}	$j=1$	$j=2$
$i=1$	$-\mu_{;2}$	$\mu_{;1}$
$i=2$	$\mu_{;1}$	$\mu_{;2}$

the Gaussian curvature in terms of the derivatives of the μ's, we start with formula (6)

$$K = \frac{2}{g_{11}}(\Gamma^2_{1[1;2]} + \Gamma^j_{1[1}\Gamma^2_{2]j}) \tag{15}$$

Writing the summation and the antisymmetrizer in the second term explicitly yields

$$\Gamma^j_{1[1}\Gamma^2_{2]j} = \Gamma^1_{1[1}\Gamma^2_{2]1} + \Gamma^2_{1[1}\Gamma^2_{2]2} \tag{16}$$

$$= \frac{1}{2}(\Gamma^1_{11}\Gamma^2_{21} - \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{12}) \tag{17}$$

$$= \frac{1}{2}(\mu_1\mu_1 + \mu_2\mu_2 - \mu_2\mu_2 - \mu_1\mu_1) \tag{18}$$

$$= 0 \tag{19}$$

Since $g_{ij} = \lambda\delta_{ij}$ we have $g_{11} = \lambda$, which leaves us with

$$\begin{aligned} K &= \frac{2}{\lambda}(\Gamma^2_{1[1;2]}) && (20)\\ &= \frac{1}{\lambda}(\Gamma^2_{11;2} - \Gamma^2_{12;1}) && (21)\\ &= -\frac{1}{\lambda}(\mu_{;11} + \mu_{;22}) && (22)\\ &= -\frac{1}{\lambda}(\frac{\partial^2\mu}{\partial x^2} + \frac{\partial^2\mu}{\partial y^2}) && (23)\\ &= -\Delta_{LB}\mu && (24) \end{aligned}$$

Using equation (10) we obtain the desired result

$$K = -\frac{1}{2}\Delta_{LB} log\lambda \tag{25}$$

□

In the following example (cf. [2]) we want to show how one can use this theorem to compute the Gaussian curvature.

Example 2.4. Let S^2 be the unit sphere, which we parametrize with spherical coordinates (φ, θ) for $0 < \theta < \pi$ and $0 < \varphi < 2\pi$. The metric in these coordinates reads as follows

$$\mathcal{G} = d\theta^2 + \sin^2\theta d\varphi^2 \tag{26}$$

Now we perform a transformation to isothermal coordinates (x, y)

$$\begin{aligned} \varphi &= x && (27)\\ \theta &= 2\arctan(e^y) && (28) \end{aligned}$$

for $0 < x < 2\pi$ and $-\infty < y < \infty$. We calculate the metric in the (x, y) chart

$$\frac{d\theta}{dy} = 2\frac{1}{1+e^{2y}}e^y = \frac{2}{e^y + e^{-y}} = \frac{1}{\cosh y} \tag{29}$$

$$dx = d\varphi \tag{30}$$

Using the following indentity

$$\sin(2z) = \frac{2\tan z}{1+\tan^2 z} \tag{31}$$

we get

$$\sin^2\theta = \sin^2(2\arctan(e^y)) = \left(\frac{2e^y}{1+e^{2y}}\right)^2 = \frac{1}{\cosh^2 y} \tag{32}$$

Then the metric in the (x, y) chart reads

$$\mathcal{G} = \frac{1}{\cosh^2 y}(dx^2 + dy^2) \tag{33}$$

So (x, y) indeed are isothermal coordinates and the conformal factor is $\lambda(x, y) = \cosh^{-2} y$ this now allows us to use (2.3)

$$K = -\frac{1}{2}\Delta_{LB} \log \lambda = -\frac{\cosh^2 y}{2}\frac{\partial^2}{\partial y^2} \log(\cosh^{-2} y) \tag{34}$$

$$= \cosh^2 y \frac{\partial}{\partial y} \tanh y = \cosh^2 y \frac{1}{\cosh^2 y} = 1 \tag{35}$$

Thus we see that the Gaussian curvature is equal to 1 in the whole domain of the chart (x, y). Since we can repeat this whole argument with spherical coordinates with a different domain such that the domains of both coordinate charts cover the whole S^2 we can conclude that the Gaussian curvature is constant on the whole S^2, i.e. $K \equiv 1$.

3 Volume measurement

In this section we will define the total area of a closed 2-dimensional Riemannian manifold and prove that it is well defined. As an application we will compute the total area of S^2. For further reference cf. [5].

Definition 3.1 (Partition of unity)**.** For a topological space X, a partition of unity is a set of continuous functions $\rho = \{\rho_i(x)\}_{i \in I}$

$$\rho_i : X \to [0, 1] \tag{36}$$

such that

$$\sum_{i \in I} \rho_i(x) = 1 \quad \forall x \in X \tag{37}$$

and for all $x \in X$ there exists a neighborhood $U \ni x$ such that only a finite number of the ρ_i is nonzero on U. (cf. [6])

Definition 3.2 (Total area)**.** We define the *total area* of a closed 2-dimensional Riemannian manifold $(\Sigma, \mathcal{G})$ by the following formula

$$\text{area}(\Sigma, \mathcal{G}) = \sum_{a \in A} \int_{\varphi_a(U_a)} \rho_a(\varphi_a^{-1}(v_a)) \sqrt{\det(g_{ij}(v_a))} dv_a^1 dv_a^2 \tag{38}$$

here A is a finite index set and $\rho = \{\rho_a\}_{a \in A}$ is a partition of unity such that $\text{supp}(\rho_a) \subseteq U_a$. $U = \{U_a\}_{a \in A}$ denotes a finite open cover of Σ consisting of the domains of coordinate charts $(U_a, \varphi_a : U_a \to V_a \subseteq \mathbb{R}^2)$.

Theorem 3.3. *The total area of a closed 2-dimensional Riemannian manifold* $(\Sigma, \mathcal{G})$ *does not depend on the choice of the cover* U *and the partition of unity* ρ*.*

Proof. Consider two charts $(U, \varphi : U \to V)$ and $(U', \varphi' : U' \to W)$ such that $U \cap U' \neq \emptyset$, then we have the smooth transition maps

$$(\varphi' \circ \varphi^{-1}) : \varphi(U \cap U') \to \varphi'(U \cap U') \tag{39}$$

$$(\varphi \circ \varphi'^{-1}) : \varphi'(U \cap U') \to \varphi(U \cap U') \tag{40}$$

for convenience we define

$$w(v) := (\varphi' \circ \varphi^{-1})(v) \tag{41}$$

$$v(w) := (\varphi \circ \varphi'^{-1})(w) \tag{42}$$

We will now show that the following equation holds

$$\int_{\varphi'(U\cap U')} f(\varphi'^{-1}(w))\sqrt{\det(g_{ij}(w))}dw^1dw^2 = \int_{\varphi(U\cap U')} f(\varphi'^{-1}(w(v)))\sqrt{\det(g_{kl}(v))}dv^1dv^2 \tag{43}$$

In order to do so we first have to work out the relation between the coefficients of the metric in the φ' chart and in the φ chart, namely the relation between $g_{ij}(w)$ and $g_{kl}(v)$. We start with equation (1)

$$g_{ij}(w) = \mathcal{G}_{\varphi'^{-1}(w)}\left(\frac{\partial\varphi'^{-1}(w)}{\partial w^i}, \frac{\partial\varphi'^{-1}(w)}{\partial w^j}\right) \tag{44}$$

Using

$$\varphi'^{-1}(w) = \varphi^{-1}((\varphi \circ \varphi'^{-1})(w)) = \varphi^{-1}(v(w)) \tag{45}$$

we obtain

$$g_{ij}(w) = \mathcal{G}_{\varphi^{-1}(v)}\left(\frac{\partial\varphi^{-1}(v(w))}{\partial w^i}, \frac{\partial\varphi^{-1}(v(w))}{\partial w^j}\right) \tag{46}$$

$$= \mathcal{G}_{\varphi^{-1}(v)}\left(\frac{\partial\varphi^{-1}(v)}{\partial v^k}\frac{\partial v^k}{\partial w^i}, \frac{\partial\varphi^{-1}(v)}{\partial v^l}\frac{\partial v^l}{\partial w^j}\right) \tag{47}$$

Using the linearity of the inner product we obtain

$$= \frac{\partial v^k}{\partial w^i}\frac{\partial v^l}{\partial w^j}\,\mathcal{G}_{\varphi^{-1}(v)}\left(\frac{\partial\varphi^{-1}(v)}{\partial v^k}, \frac{\partial\varphi^{-1}(v)}{\partial v^l}\right) \tag{48}$$

$$= \frac{\partial v^k}{\partial w^i}\frac{\partial v^l}{\partial w^j}\,g_{kl}(v) \tag{49}$$

We can view this as a matrix equation, then taking the determinant and using that the determinant is multiplicative yields

$$\det(g_{ij}(w)) = \det(g_{kl}(v))\det\left(\frac{\partial(v^1,v^2)}{\partial(w^1,w^2)}\right)^2 \tag{50}$$

Here the Jacobian matrix of the transition map (42) is denoted by

$$\frac{\partial(v^1,v^2)}{\partial(w^1,w^2)} \tag{51}$$

From calculus we know that for a domain $D \subset \mathbb{R}^2$ we have the identity

$$\int_D g(w)dw^1dw^2 = \int_D g(w(v))\det\left(\frac{\partial(w^1,w^2)}{\partial(v^1,v^2)}\right)dv^1dv^2 \tag{52}$$

Applying both this and the former result to the left hand side of equation (43) gives

$$\int_{\varphi'(U\cap U')} f(\varphi'^{-1}(w))\sqrt{\det(g_{ij}(w))}\; dw^1 dw^2$$

$$\begin{aligned}
&= \int_{\varphi(U\cap U')} f(\varphi'^{-1}(w(v)))\sqrt{\det(g_{ij}(w(v)))}\det\left(\frac{\partial(w^1,w^2)}{\partial(v^1,v^2)}\right) dv^1 dv^2\\
&= \int_{\varphi(U\cap U')} f(\varphi'^{-1}(w(v)))\sqrt{\det(g_{kl}(v))}\det\left(\frac{\partial(v^1,v^2)}{\partial(w^1,w^2)}\right)\det\left(\frac{\partial(w^1,w^2)}{\partial(v^1,v^2)}\right) dv^1 dv^2\\
&= \int_{\varphi(U\cap U')} f(\varphi'^{-1}(w(v)))\sqrt{\det(g_{kl}(v))} dv^1 dv^2
\end{aligned}$$

this proves equation (43). In the last step we used that inverse maps have inverse Jacobians.

Now are able to prove that the total area is independent of the choice of the covering U and partition of unity ρ. Let $U = \{U_a\}_{a\in A}$ and $U' = \{U'_b\}_{b\in B}$ be two finite coverings of domains of corresponding coordinate charts $\varphi_a : U_a \to V_a$ and $\varphi'_b : U'_b \to W_b$. Also let $\rho = \{\rho_a\}$ and $\rho' = \{\rho'_b\}$ be two partitions of unity such that $\mathrm{supp}(\rho_a) \subseteq U_a$ and $\mathrm{supp}(\rho'_b) \subseteq U_b$ hold for all $a \in A$ and $b \in B$. Then

$$\begin{aligned}
area(\Sigma,\mathcal{G}) &= \sum_{a\in A}\int_{\varphi_a(U_a)} \rho_a(\varphi_a^{-1}(v_a))\sqrt{\det(g_{ij}(v_a))}\; dv_a^1 dv_a^2\\
&= \sum_{a\in A}\int_{\varphi_a(U_a)} \rho_a(\varphi_a^{-1}(v_a))\left(\sum_{b\in B}\rho'_b({\varphi'}_b^{-1}(w_b(v_a)))\right)\sqrt{\det(g_{ij}(v_a))} dv_a^1 dv_a^2\\
&= \sum_{a\in A}\sum_{b\in B}\int_{\varphi_a(U_a\cap U'_b)} \rho_a(\varphi_a^{-1}(v_a))\rho'_b({\varphi'}_b^{-1}(w_b(v_a)))\sqrt{\det(g_{ij}(v_a))}\; dv_a^1 dv_a^2
\end{aligned}$$

now we apply equation (43)

$$\begin{aligned}
&= \sum_{a\in A}\sum_{b\in B}\int_{\varphi'(U_a\cap U'_b)} \rho_a(\varphi_a^{-1}(v_a(w_b)))\rho'_b({\varphi'}_b^{-1}(w_b))\sqrt{\det(g_{kl}(w_b))}\; dw_b^1 dw_b^2\\
&= \sum_{b\in B}\int_{\varphi'(U'_b)}\left(\sum_{a\in A}\rho_a(\varphi_a^{-1}(v_a(w_b)))\right)\rho_b({\varphi'}_b^{-1}(w_b))\sqrt{\det(g_{kl}(w_b))}\; dw_b^1 dw_b^2\\
&= \sum_{b\in B}\int_{\varphi'(U'_b)} \rho'_b({\varphi'}_b^{-1}(w_b))\sqrt{\det(g_{kl}(w_b))}\; dw_b^1 dw_b^2 = area'(\Sigma,\mathcal{G})
\end{aligned}$$

which shows that the definition is indeed independent of the choice of a covering an a partition of unity. □

Example 3.4. Let $\Sigma = S^2$ and $\mathcal{G} = d\theta^2 + \sin^2\theta d\varphi^2$ be the standard metric in spherical coordinates, then

$$\det\left(g_{ij}(\varphi,\theta)\right) = \sin^2\theta \tag{53}$$

$$\begin{aligned} area(S^2, \mathcal{G}) &= \int_{\varphi(U)=(0,\pi)\times(0,2\pi)} \sqrt{\det g_{ij}(\varphi,\theta)}\; d(\theta,\varphi) = \int_0^{\pi}\int_0^{2\pi} \sin\theta\; d\varphi d\theta \\ &= 2\pi \int_0^{\pi} \sin\theta\; d\theta = -2\pi\cos\theta\Big|_0^{\pi} = -2\pi(-1-1) = 4\pi \end{aligned}$$

4 Gauss-Bonnet Theorem

In this section our goal is to state the Gauss-Bonnet theorem and to give all definitions that are necessary to understand its assertion. The Gauss-Bonnet theorem provides us with a link between the topology and the geometry of a 2-dimensional closed Riemannian manifold. Namely it relates the Euler characteristic and the Gaussian curvature of a 2-dimensional Riemannian manifold. For further reference on this topic cf. [1].

Definition 4.1 (Polyhedron)**.** A *triangle* Δ is the closed convex hull of three affine independent points $v_0, v_1, v_2 \in \mathbb{R}^3$

$$\Delta = \overline{\text{convhull}(v_1, v_2, v_3)} \tag{54}$$

A polyhedron $X \subset \mathbb{R}^3$ is finite union of triangles

$$X = \bigcup_{j=1}^{k} \Delta_j \tag{55}$$

such that if for any two triangles $\Delta_i \cap \Delta_j \neq \emptyset$ then they intersect in exactly one common edge or vertex.

For a polyhedron X we introduce the following bit of notation

$$V(X) := \#\text{vertices of } X \tag{56}$$
$$E(X) := \#\text{edges of } X \tag{57}$$
$$F(X) := \#\text{faces of } X \tag{58}$$

Definition 4.2 (Euler characteristic)**.** The integer

$$\chi(X) := V(X) - E(X) + F(X) \tag{59}$$

is called the Euler characteristic of the polyhedron X.

Definition 4.3 (Triangulation)**.** Let $(\Sigma, \mathcal{G})$ be a closed 2-dimensional Riemannian manifold then a *triangulation* of $(\Sigma, \mathcal{G})$ is a pair (Φ, X) consisting of a polyhedron X and a homeomorphism

$$\Phi : X \to \Sigma \tag{60}$$

Theorem 4.4 (Gauss-Bonnet)**.** *Let $(\Sigma, \mathcal{G})$ be a closed 2-dimensional Riemannian manifold and (Φ, X) a triangulation of $(\Sigma, \mathcal{G})$, then the following identity holds*

$$\int_\Sigma K\sqrt{\det(g_{ij}(v))}dv^1dv^2 = 2\pi\chi(X) \tag{61}$$

(without proof; for a proof cf. [1])

Remark 4.5. The left-hand side obviously does not depend on a triangulation, hence we can conclude from the fact that Gauss-Bonnet holds for any triangulation that the Euler characteristic does not depend on the triangulation. We can therefore speak of *the* Euler characteristic of a closed 2-dimensional Riemannian manifold.

Example 4.6. In order to get a better feeling for the Euler characteristic we will exploit the following formula in a few cases.

$$\chi(X) = 2 - 2g \tag{62}$$

where X is a closed orientable 2-dimensional Riemannian manifold and g its genus. We state this formula without proof cf. [4]. The genus can intuitively be understood as 'number of holes', which is illustrated by the following pictures [3].

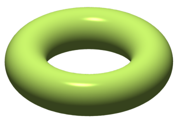

Figure 1: torus: $g = 1$ $\chi = 0$

Figure 2: doubletorus: $g = 2$ $\chi = -2$

Remark 4.7. The Euler characteristic is a topological invariant which does not depend on the metric, therefore the total curvature is also a topological invariant by virtue of Gauss-Bonnet. In particular this means that the total curvature does not depend on the metric and also not on the curvature which is determined by the metric anyway.

Remark 4.8. Another immediate consequence of the Gauss-Bonnet theorem is, that for a closed 2-dimensional Riemannian manifold of negative Euler characteristic, such as the doubletorus cf. Example 4.6, the Gaussian curvature can not be positive in every point of the manifold. This is because otherwise the total curvature would be positive which would contradict Gauss-Bonnet.

We will now check the Gauss-Bonnet theorem in a simple case.

Example 4.9. Let $\Sigma = S^2$ and $\mathcal{G} = d\theta^2 + \sin^2\theta d\varphi^2$ be the metric in spherical coordinates. From Example 2.4 we already know that the Gaussian curvature of S^2

is constant $K = 1$. Thus the calculation of the left-hand side in the Gauss-Bonnet theorem reduces to calculating the total area of S^2, which we already did in Example 3.4. So we have for the left-hand side

$$\int_{\Sigma} K\sqrt{\det(g_{ij}(v))}dv^1dv^2 = \int_{\Sigma} \sqrt{\det(g_{ij}(v))}dv^1dv^2 = area(S^2, \mathcal{G}) = 4\pi \tag{63}$$

In order to calculate the Euler characteristic we choose the tetrahedron as a triangulation for S^2. We leave it as an exercise to show that it actually is a triangulation of S^2. We can easily count the number of vertices, edges and faces

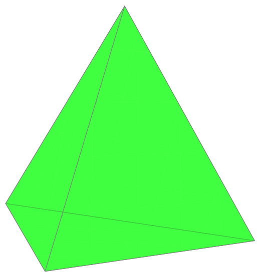

Figure 3: A tetrahedron

$$V(X) = 4 \quad E(X) = 6 \quad F(X) = 4 \tag{64}$$

$$\Rightarrow \quad \chi(X) = V(X) - E(X) + F(X) = 4 - 6 + 4 = 2 \tag{65}$$

Now we see that the right-hand side $2\pi\chi(X)$ also equals 4π we have therefore confirmed the Gauss-Bonnet theorem for S^2.

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