Impact of Non-Constant Volatility (σ) on Hedging Profits

Sebastian Jaramillo

In this research project, we critically analyze the implications of employing non-constant volatility (σ) within hedging frameworks. Our objective is to quantify how deviations from constant volatility, typically assumed in classical Black-Scholes settings, affect hedging outcomes. To achieve this, we simulate stock price dynamics under both constant volatility and stochastic volatility conditions, particularly employing the GARCH and Heston models. Furthermore, we implement and evaluate a vega-hedging strategy, specifically designed to neutralize volatility exposure.

Methodology

We simulate stock prices under three volatility assumptions:

- Constant volatility (Black-Scholes framework)
- Stochastic volatility via the Heston model
- Time-varying volatility using a GARCH(1,1) process

For each simulated price path, we compute the option price and its Greeks, particularly delta and vega, required to implement hedging strategies.

Delta Hedging: The delta-hedged portfolio consists of a short call option position and a dynamically updated position in the underlying asset. The hedge ratio (delta) is:

$$\Delta_C = \frac{\partial C}{\partial S} = \Phi(d_1)$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution, and d_1 is given by:

$$d_1 = \frac{\ln(S/K) + (r + 0.5\sigma\hat{}2)(T-t)}{\sigma\sqrt{T-t}}$$

Vega Hedging: Vega hedging aims to make the portfolio insensitive to small changes in volatility. This is achieved by adding a second option (often a different strike or maturity) to the portfolio such that the total vega is zero:

$$\text{Vega}_{\text{portfolio}} = \text{Vega}_1 + \lambda \text{Vega}_2 = 0$$

Solving for λ:

$$\lambda = -\frac{\text{Vega}_1}{\text{Vega}_2}$$

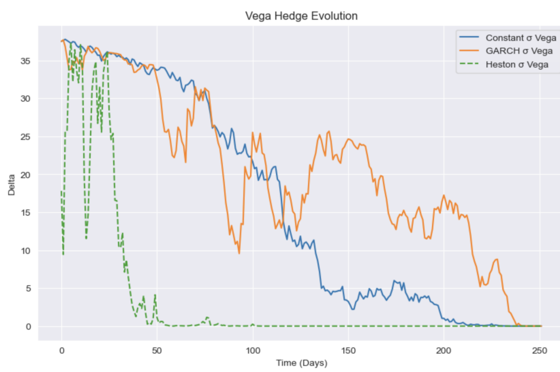
This tells us how many units of the second option to hold to offset the volatility sensitivity of the first

The vega of a European option is:

$$\text{Vega} = S\phi(d_1)\sqrt{T-t}$$

where $\phi(d_1)$ is the standard normal probability density function.

Each strategy is applied across 1000 Monte Carlo paths, and the resulting profit and loss (P/L) distributions are analyzed statistically.



Empirical Results & Strategy Comparison

We compute profit distributions from six strategy-model combinations:

- Delta hedging under constant volatility
- Delta hedging under Heston volatility
- Delta hedging under GARCH volatility
- Vega hedging under constant volatility
- Vega hedging under Heston volatility
- Vega hedging under GARCH volatility

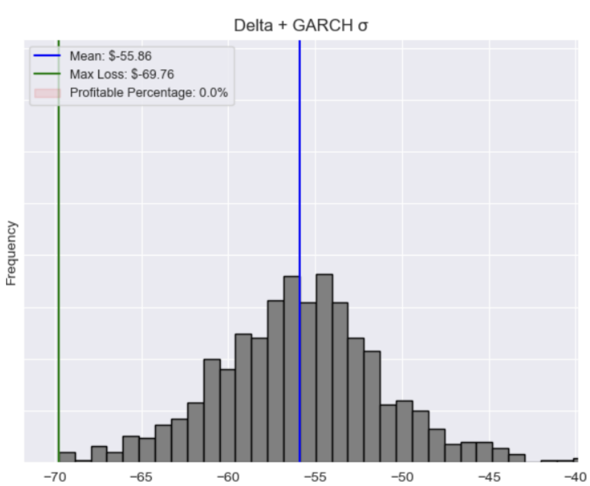


Figure 1: Delta + GARCH σ

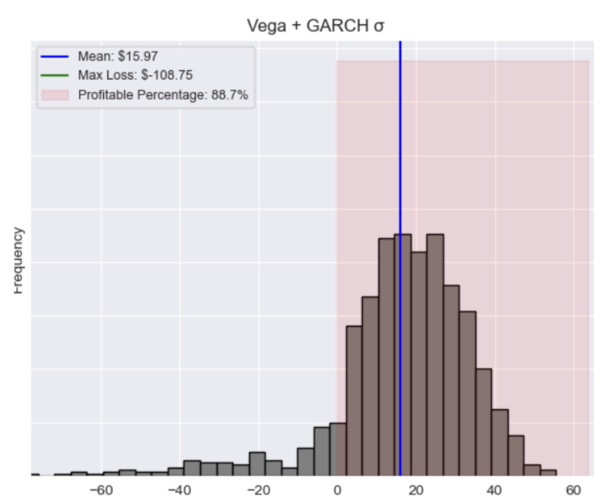


Figure 2: Vega + GARCH σ

Figure 3: Profit Distributions: Delta vs Vega under GARCH Volatility

Histograms illustrate wide variation in profit outcomes depending on the volatility model and strategy. Summary statistics indicate:

- Delta + GARCH performs the worst, with high losses and wide dispersion.

- Vega + GARCH shows the highest average profit but with fat tails.
- Constant volatility scenarios are more stable but yield modest profits.

These observations highlight that hedging effectiveness strongly depends on the interplay between strategy choice and the assumed volatility dynamics.

Advanced Statistical Evaluation of Hedging Strategies

- **Sharpe Ratio Calculation**: We compute the Sharpe ratio for each of the six strategy-model configurations as the ratio of the mean profit to the standard deviation of profits. This metric serves as a first-order measure of risk-adjusted return. A higher Sharpe ratio indicates more desirable performance from a reward-to-risk perspective.
- **Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)**: To quantify downside risk, we compute the 5% historical Value-at-Risk for each configuration, defined as the 5th percentile of the profit distribution. We also compute the Conditional Value-at-Risk (CVaR), the expected loss conditional on falling below the 5% threshold. These measures provide critical insight into potential extreme losses.
- **Q-Q Plots**: Quantile-Quantile (Q-Q) plots are generated for each profit distribution to assess how closely the empirical data follow a normal distribution. Deviations from the straight line in the tails indicate skewness or heavy tails that may not be captured by standard models.
- **Statistical Tests**: We apply the Kolmogorov–Smirnov test to evaluate the normality assumption for each profit distribution. Additionally, we perform pairwise t-tests comparing delta and vega hedging strategies under each volatility model to assess the statistical significance of observed differences in means.

  These advanced diagnostics strengthen our empirical conclusions and validate the robustness of our findings.

Interpretation and Synthesis of Advanced Statistical Findings

- **Sharpe Ratios – Risk-Adjusted Performance**: Among all strategy-model combinations, Vega + GARCH σ stands out with the highest Sharpe ratio (≈ 0.812), suggesting superior risk-adjusted returns under volatile market conditions. In contrast, Delta + GARCH σ shows an extremely negative Sharpe ratio ($\approx$ -11.62), indicating that using delta hedging under GARCH volatility leads to consistent losses and high dispersion. Delta + Constant σ and Delta + Heston σ maintain Sharpe ratios close to zero, showing weak risk-reward tradeoffs.
- **VaR and CVaR – Tail Risk Sensitivity**: The Vega + GARCH σ strategy, despite showing strong mean returns, exhibits considerable tail risk, with a 5% VaR of -$20.17 and CVaR of -$45.20. This highlights exposure to rare but extreme losses. Conversely, Vega + Constant σ and Delta + Constant σ have much tighter loss bounds, reflecting greater predictability but reduced potential upside. Interestingly, Delta + GARCH σ shows moderate tail risk (CVaR $\approx$ -65.9) but with very low profitability, reinforcing its unsuitability.
- **Normality Diagnostics – Q-Q Plots and KS Test**: Q-Q plots reveal notable deviations from normality, especially in the tails. Vega + GARCH σ and Vega + Heston σ both exhibit significant departure from normality, supported by KS test p-values ≈ 0, rejecting the null hypothesis of normality. On the other hand, Delta + Heston σ and Delta + GARCH σ exhibit better alignment with the normal assumption, reflected in higher KS p-values (0.26–0.40), though not perfect.
- **Statistical Significance – T-Tests for Delta vs Vega**: The pairwise t-tests show that the differences in mean profits between delta and vega hedging are statistically significant for all three volatility models (p-values ≈ 0.0). This indicates that the choice of hedging strategy fundamentally alters expected outcomes, especially under model misspecification or stochastic volatility.

Strategy	Mean	Std Dev	Sharpe	VaR 5%	CVaR 5%	KS p-value
Delta + Constant σ	0.003	0.426	0.007	-0.665	-1.007	0.0049
Delta + Heston σ	-0.095	1.093	-0.087	-2.087	-2.654	0.4036
Delta + GARCH σ	-55.858	4.807	-11.621	-63.773	-65.939	0.2602
Vega + Constant σ	-1.497	0.551	-2.718	-2.508	-2.752	0.1833
Vega + Heston σ	-2.706	1.485	-1.822	-4.766	-5.157	0.0000
Vega + GARCH σ	15.971	19.672	0.812	-20.169	-45.205	0.0000

Table 1: Advanced Risk and Performance Statistics for Each Strategy

Overall, the analysis highlights that while vega hedging under GARCH volatility achieves the highest average returns and the best Sharpe ratio, it does so at the cost of severe downside risk and significant deviation from normality. In contrast, delta hedging under GARCH is not only ineffective but also yields large, consistent losses, emphasizing its vulnerability to volatility misspecification. Constant volatility models offer more stable, though modest, returns. The sharp statistical contrasts between delta and vega strategies across all volatility regimes underscore the necessity of tailoring hedging methods to the underlying volatility dynamics. This reinforces the importance of model selection and strategy robustness in practical risk management.