An Analytical and Numerical Investigation of Theta and Delta Dynamics in Black–Scholes European Options

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Objective

This project investigates how the Black–Scholes prices of European call and put options respond to changes in (1) time to expiration and (2) underlying spot price. By computing and plotting the partial derivatives (the Greeks "theta" and "delta"), we gain quantitative insight into an option's time value decay and price sensitivity as market conditions evolve.

Methodology

- **Implementation**: Using Python (NumPy, SciPy, Matplotlib), we coded the closed-form Black–Scholes formulas for calls and puts.
- **Theta Analysis**: For fixed $S_0 = 100, K = 110, \sigma = 30\%, r = 0$, we evaluated $\frac{\partial C}{\partial t}$ and $\frac{\partial P}{\partial t}$ over maturities from 1 month to 5 years, approximating derivatives numerically via finite differences.
- **Delta Analysis**: For a one-year expiration, we swept S_0 from 50 to 150 and plotted $\frac{\partial C}{\partial S_0}$ (call delta) and $\frac{\partial P}{\partial S_0}$ (put delta), comparing them to their analytic forms $\Phi(d_1)$ and $\Phi(d_1) - 1$

Findings

- Theta (Time Sensitivity): Both call and put theta exhibit a steep $O(t\hat{}-0.5)$ spike as $t \to 0\hat{}+$, reflecting the explosion of time value for near-expiration options. Beyond a few months, theta decays rapidly and then plateaus near zero for long maturities. In non-zero-rate scenarios, one would observe put theta eventually cross zero and turn negative for very long maturities.
- Delta (Spot Sensitivity): Call delta rises in an S-shaped curve from ~ 0 (deep OTM) through ~ 0.5 at-the-money to ~ 1 (deep ITM). Put delta mirrors this from $\sim \breve{}1$ (deep ITM) through $\sim \breve{}0.5$ at-the-money to ~ 0 (deep OTM). The steepest slope occurs at the strike, coinciding with peak gamma and the region of greatest hedging risk.

Conclusion

This exercise underscores the dual nature of option Greeks:

- Theta quantifies how rapidly time decay erodes extrinsic value—paramount when managing short-dated positions.
- Delta measures directional exposure and drives delta-hedging costs, with the ATM region demanding the highest hedging attention due to maximal curvature.