CS5002 Prof. Higger
Homework 7
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Problem 1
Solution:
We use induction on a sequence of statements:
$|A|=0$, $P(A)=2^0=1$ ......statement 0
$|A|=1$, $P(A)=2^1=2$ ......statement 1
$|A|=2$, $P(A)=2^2=4$ ......statement 2
...
$|A|=n$, $P(A)=2^n$ ......statement n

Base case:
$|A|=0$, $P(A)=2^0=1$, so that given a set A with $|A|=0$, we get $P(A)=2^0$.

Inductive Step:
Assume statement n is true, that is $|A|=n$, $P(A)=2^n$
Consider $|A_k|=k$, the subscript represent elements' number in set A.
For some k, we know $P(A_k) =2^k$ by our inductive hypothesis.
Consider $|A_{k+1}|=k+1$, we know $P(A_{k+1}) =2^k+N$, N is the number of every new subset containing the extra element in set A_{k+1}, which is just the number of the powerset of A_k.
(the new subset must contain the extra element, we can simply append the new element to every subset of A, so we get 2^k extra subset which contain the new appended element).
So, $P(A_{k+1}) =2^k+2^k=2\times2^k=2^{k+1}$, as we descired.
By induction, it is proved true for $\forall|A|=n$, $P(A)=2^n$

Problem 2

Solution:

We use induction on a sequence of statements:

n=1, $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 r^0 = a_1 \frac{1-r}{1-r}$ ......statement 1

n=2, $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 r^0 + a_1 r^1 = a_1 \frac{1-r^2}{1-r}$ ......statement 2

...

n=n, $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 r^0 + a_1 r^1 + \cdots + a_1 r^{n-1} = a_1 \frac{1-r^n}{1-r}$ ......statement n

Base case:

When n=1, we get statement 1, $\sum_{i=1}^{n} a_1 r^0 = a_1 \frac{1-r}{1-r}$.

Inductive Step:

Assume statement n is true, $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 r^0 + a_1 r^1 + \cdots + a_1 r^{n-1} = a_1 \frac{1-r^n}{1-r}$

So, for some k, we know $\sum_{i=1}^{k} a_1 r^{i-1} = a_1 r^0 + a_1 r^1 + \cdots + a_1 r^{k-1} = a_1 \frac{1-r^k}{1-r}$ is true by our inductive hypothesis.

Consider $\sum_{i=1}^{k+1} a_1 r^{i-1} = (a_1 r^0 + a_1 r^1 + \cdots + a_1 r^{k-1}) + a_1 r^k$, which is equal to

$$a_1 \frac{1-r^k}{1-r} + a_1 r^k = a_1 \frac{(1-r^k) + (1-r) r^k}{1-r} = a_1 \frac{1-r^k + r^k - r^{k+1}}{1-r} = a_1 \frac{1-r^{k+1}}{1-r}$$

by our inductive hypothesis.

So we have $\sum_{i=1}^{k+1} a_1 r^{i-1} = a_1 \frac{1-r^{k+1}}{1-r}$ as descired.

By induction, it is proved true $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$.

Problem 3

i 128,-64,32,-16,8,-4...

It is geometric. Because -64=128×-0.5, 32=-64×-0.5,-16=32×-0.5 ......

$a_i = 128 \times (-\frac{1}{2})^{i-1}$

$\sum a_n = 128 \times \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})}$, $\sum_{i=1}^{10} 128 \times \frac{1-(-\frac{1}{2})^{10}}{1-(-\frac{1}{2})} = \frac{341}{4}$

ii 1,4,13,28,49

It is quadratic. Because 4-1=3, 13-4=9, 28-13=15, 49-28=21, we get a new sequence 3,9,15,21, and 9-3=15-9=21-15=6, so it is quadratic.

$a_i = 3i^2 - 6i + 4$

iii -5,-2,1,4,7

It is arithmetic. Because -2-(-5)=1-(-2)=4-1=7-4=3

$a_i = 3i - 8$

$\sum a_n = \frac{3i^2-13i}{2}$, $\sum_{i=1}^{10} \frac{3i^2-13i}{2} = 85$

Problem 4

i T(n)=T(n-1)+1

T(n-1)=T(n-2)+1

T(n-2)=T(n-3)+1

...

T(n)=T(n-1)+1=T(n-2)+2=T(n-3)+3

=T(n-k)+k

let n-k=1, k=n-1

T(n)=T(1)+(n-1)=1+n-1

T(n)=n

ii T(n)=T(n-3)+4

T(n-3)=T(n-6)+4

T(n-6)=T(n-9)+4

...

T(n)=T(n-3)+4=T(n-6)+8=T(n-9)+12

=T(n-3k)+4k

let n-3k=1, k=$\frac{n-1}{3}$

T(n)=T(1)+4$\frac{n-1}{3}$=1+$\frac{4n-4}{3}$

T(n)= $\frac{4n-1}{3}$

Problem 5

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \dots \dots \frac{1}{n}$$

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} \int_{i}^{i+1} \frac{1}{x} dx = \int_{1}^{n+1} \frac{1}{x} + \frac{1}{|x|} - \frac{1}{x} dx$$

$$= \int_{1}^{n+1} \frac{1}{x} dx + \int_{1}^{n+1} \frac{1}{|x|} - \frac{1}{x} dx$$

$\cong \ln(n+1) + C$, C is Euler–Mascheroni constant