CS5002 Discrete and Data Structures	Prof Higger
Fall 2020	Dec 7, 2020

Final Exam Practice Problems

Due: DO NOT SUBMIT

Instructions:

- These practice problems are helpful to warm-up for the homework itself. They're great to discuss with course staff or class mates because **these problems are not to be handed in** so you may discuss ideas freely without worrying about academic integrity issues. Solutions will be provided immediately with the problems themselves.

Problem 1 Bayes

A license plate reader is a system which identifies the text of a license plate given a camera's image of the back of a car (useful in parking enforcement). When its not raining, the system is able to identify 99% of the license plates correctly. Unfortunately, when it rains the system struggles to see clearly and only identifies 90% of the license plates correctly. In a particularly rainy parking lot, there is a $\frac{1}{4}$ chance of rain each day.

i What is the probability that its raining and a license isn't read correctly?

ii What is the probability that its not raining and a license isn't read correctly?

iii What is the probability that a license isn't read correctly?

iv Suppose that you receive a parking ticket for someone else's car mailed by mail because their license plate was incorrectly read as yours. What is the probability that it was raining when this other car's license plate was read incorrectly?

In your solutions, let $R = 1$ be the event its raining (0 otherwise) and $C = 1$ be the event that a license plate is read correctly (0 otherwise).

HINT: Though you've seen conditional distributions defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

you may find the equivalent formulation more useful in this problem:

$$P(A \cap B) = P(A|B)P(B)$$

(Generally, I think this second form is as useful as the first as you do algebra with probabilities. Get comfortable with both!)

Solution:

i
$$P(R=1, C=0) = P(C=0|R=1)P(R=1) = .1 * .25 = .025$$

ii
$$P(R=0, C=0) = P(C=0|R=0)P(R=0) = .01 * .75 = .0075$$

iii
$$P(C=0) = P(C=0, R=0) + P(C=0, R=1) = .025 + .0075 = .0325$$

iv
$$\begin{aligned} P(R=1|C=0) &= \frac{P(C=0|R=1)P(R=1)}{P(C=0)} \\ &= \frac{(.1)(.25)}{.0325} \\ &\approx .77 \end{aligned}$$

Problem 2 Expected Value and Variance
A basketball player makes shots according to the following table:

	Distance to net	**Points earned (if made)**	**Prob shot made (%)**
P	In Paint	2	45
J	Mid Range Jump	2	40
T	3 pt	3	35

Where we use random variables P, J or T to indicate the number of points earned when a player takes each of the corresponding shots.

i Compute the expected value and variance of: P

ii Compute the expected value and variance of: J

iii Compute the expected value and variance of: T

iv If your team was down by 1 point with time for 1 more shot, which shot should this player prefer to maximize their chance of winning?

v In the early game, which shot should this player prefer?

Solution:

i

$$\begin{aligned} E[P] &= .45 * 2 + .55 * 0 = .9 \\ \mathtt{Var}(P) &= .45(2 - .9)^2 + .55 * (0 - .9)^2 = .99 \end{aligned}$$

ii

$$E[J] = .4 * 2 + .6 * 0 = .8$$
$$\texttt{Var}(J) = .4(2 - .8)^2 + .6 * (0 - .8)^2 = .96$$

iii

$$E[T] = .35 * 3 + .65 * 0 = 1.05$$
$$\texttt{Var}(T) = .35(3 - 1.05)^2 + .65 * (0 - 1.05)^2 = 2.0475$$

iv Any made shot wins the game. This player is most likely to hit a shot in the paint: P.

v In the early game, we're interested in scoring as much as possible. This player has the highest expected points earned with their three point shot.

Problem 3 Sequences and Series

For each of the following sequences:

- identify if the sequence is arithmetic, geometric or quadratic[1]. Justify your response.
- assuming the first item of each sequence is a_1, give an expression for a_i. (In other words, find a formula for the i-th term in the sequence).
- if the sequence is arithmetic or geometric, compute the sum of the first 10 terms in the sequence

i $4, 9, 14, 19, \ldots$

ii $4, -12, 36, -108, \ldots$

iii $0, 5, 12, 21, 32, \ldots$

<u>*Solution:*</u>

i
- This sequence is arithmetic as each term is 5 more than the term before it (constant difference)
- $a_i = a_1 + (i - 1) * d = 4 + (i - 1) * 5$
-

$$\begin{aligned}\sum_{i=1}^{10} a_i &= n\frac{a_1 + a_10}{2} \\ &= \frac{10}{2}(4 + (4 + (10 - 1) * 5)) \\ &= 265\end{aligned}$$

[1]its possible a sequence can be neither arithmetic, geometric or quadratic, but each of these examples is of one of these three types

ii
- This is geomteric as each term is -3 of the term before it.
- $a_i = a_1 r^{i-1} = 4 * (-3)^{i-1}$
- $$\begin{aligned}\sum_{i=1}^{10} &= a_1 \frac{1-r^n}{1-r} \\ &= 4\frac{1-(-3)^{10}}{1-(-3)} \\ &= -59048\end{aligned}$$

iii
- This sequence is quadratic as the second difference is constant (2).
- We must solve the system:

$$\begin{aligned}a * 1^2 + b * 1 + c = 0 \\ a * 2^2 + b * 2 + c = 5 \\ a * 3^2 + b * 3 + c = 12\end{aligned}$$

which yields $a = 1, b = 2, c = -3$ so that our i-th term is:

$$a_i = 1 * i^2 - 2 * i + c$$

- (the instructions ask us to skip the sum of the quadratic)

Problem 4 Induction

i Identify a simple expression for the sum of the first n positive odd integers:

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + 7 + 9 + \dots$$

ii Show that your expression is valid by proving it equals the sum above by using induction.

<u>*Solution:*</u>

i For $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} (2k-1) = n^2$$

ii Proof of the equality above: (induction)

Base Case (n=1)[2]: If $n = 1$ then:

$$\sum_{k=1}^{1} (2k-1) = 1 = 1^2 = n^2$$

[2]The base case is where we show equality from part i holds for some initial case

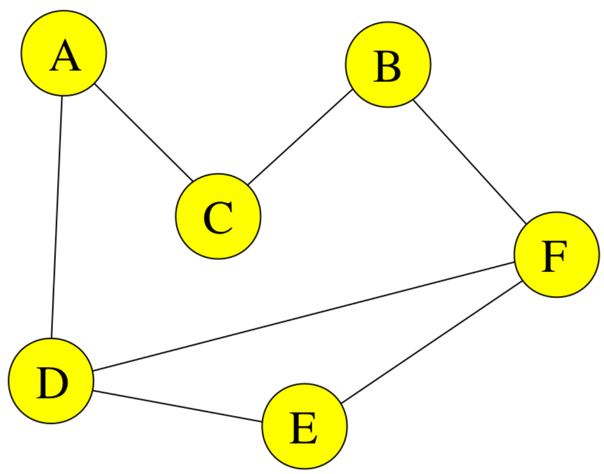
Inductive Step[3]: For $n \in \mathbb{N}$, let us assume:

$$\sum_{k=1}^{n}(2k-1) = n^2$$

Then:

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= (2(n+1)-1) + \sum_{k=1}^{n}(2k-1)\\ &= (2(n+1)-1) + n^2\\ &= n^2 + 2n + 2\\ &= (n+1)^2\end{aligned}$$

Problem 5 Breadth First Search / Depth First Search



Give the order of nodes in each of the searches below, starting at the indicated node. When a method may select from among many next candidate nodes, prefer the node which is alphabetically first.

i Breadth First Search starting at node A

ii Depth First Search starting at node A

iii Breadth First Search starting at node E

iv Depth First Search starting at node E

v Breadth First Search starting at node F

[3]The inductive step shows that if the equality from part i holds for n then the equality from part i holds for n + 1

vi Depth First Search starting at node F

<u>*Solution:*</u>

i $[A, C, D, B, E, F]$

ii $[A, C, B, F, D, E]$

iii $[E, D, F, A, B, C]$

iv $[E, D, A, C, B, F]$

v $[F, B, D, E, C, A]$

vi $[F, B, C, A, D, E]$

Problem 6 Function Growth

Let:

$$\begin{aligned} a(n) &= 3n + 2 \\ b(n) &= 2\log_{10} n \\ c(n) &= 3\log_2 n + 2 \\ d(n) &= 2n^2 + n! \\ e(n) &= n^{10} \\ f(n) &= 4n^2 + 5n^4 \end{aligned}$$

In all but the final sub-problem below, tell whether the statements is true or false:

i $a(n) = O(e(n))$

ii $e(n) = O(a(n))$

iii $b(n) = \Theta(c(n))$

iv If some function $f(n) = O(g(n))$ then $f(n) > g(n)$ for every value of n.

v $c(n) = \Omega(c(n))$

vi Find the simplest, most informative function $g(n)$ such that $f(n) = O(g(n))$. Justify the big-O relation by finding the n_0 and c values required to satisfy the definition.

<u>*Solution:*</u>

i "$e(n)$ grows more quickly than $a(n)$" True

ii "$a(n)$ grows more quickly than $e(n)$" False

iii "$b(n)$ and $c(n)$ grow equally quickly" True. Don't forget that:

$$\log_a b * \log_b x = \log_a x$$

So that we can change the base of the logarithm by multiplying by a scalar. Therefore, just like n and $3n$ grow equally quickly, we may also say that $\log_a n$ and $\log_b n$ grow equally.

iv False. Remember that $n = O(.5n)$ (choose $n_0 = 1$ and $c = 3$) and yet $n > .5n$ for all $n > 0$.

v True

vi $f(n) = O(n^4)$ Let $n_0 = 1$ and $c = 10$ so that, for $n > n_0$, we have:

$$\begin{aligned} f(n) &= 4n^2 + 5n^4 \\ &\leq 4n^4 + 5n^4 \\ &= 9n^4 \\ &< 10n^4 \\ &= g(n) \end{aligned}$$

Problem 7 Proofs

Prove each of the following statements:

i (Direct) If x is even and y is odd then $x + y$ is odd

ii (Contradiction) If k objects are placed in m groups than one group must have at least $\lceil \frac{k}{m} \rceil$ objects.

<u>*Solution:*</u>

i Proof (direct): Assume that x is even and y is odd. Then, there exists some integer z_1 with:

$$x = 2z_1$$

and some, potentially different, integer z_2 with:

$$y = 2z_2 + 1$$

Then:

$$x + y = 2z_1 + 2z_2 + 1 = 2(z_1 + z_2) + 1$$

So that $x + y$ is odd since there exists an integer $z = z_1 + z_2$ with $x + y = 2z + 1$.

ii Proof (contradiction): Let us place k objects in m groups. Assume that no group has $\frac{k}{m}$ or more objects in it. Then every group has less than $\frac{k}{m}$ objects. This is impossible, since then the total number of objects is less than $\frac{k}{m} * m = k$. Therefore, some group must have at least $\frac{k}{m}$ objects in it. Of course, we cannot place a fraction of an object in a group, so this group must have at least $\lceil \frac{k}{m} \rceil$ objects in it.