CS5002 Discrete and Data Structures	Prof Higger
Fall 2020	Oct 30, 2020

HW 07 Practice Problems

Due: DO NOT SUBMIT

Instructions:

- These practice problems are helpful to warm-up for the homework itself. They're great to discuss with course staff or class mates because **these problems are not to be handed in** so you may discuss ideas freely without worrying about academic integrity issues. Solutions will be provided immediately with the problems themselves.

Problem 1 Induction: Factorial Inequality

Using induction, show that $n! < n^n$ for all $n > 1$. (You may find it helpful to see the practice problem on induction to see what is required of your proof).

Solution: We use induction on a sequence of statements:

- $2! < 2^2$ (statement 2)
- $3! < 3^3$ (statement 3)

  $\vdots$
- $n! < n^n$ (statement n)

Base Case (statement 2):

$$2! = 2 * 1 = 2 < 4 = 2^2$$

so that $2! < 2^2$

Inductive Step: Assume statement n is true, that is $n! < n^n$. Then:

$$\begin{aligned}(n+1)! &= n! * (n+1) \\ &< n^n * (n+1) \\ &< (n+1)^n * (n+1) \\ &= (n+1)^{n+1}\end{aligned}$$

Where we use statement n in the second line and the fact that $n^n < (n+1)^n$ in the third (since n is positive). This statement shows that $(n+1)! < (n+1)^{n+1}$.

By induction, $n! < n^n$ for all $n > 1$.

Problem 2 Recursion

Solve eac of the following recurrences by substitution. Assume a base case of $T(1) = 1$. As part of your solution, you will need to establish a pattern for what the recurrence looks like after the k-th substitution. Check that this pattern is consistent with your substitutions, but you do not need to formally prove it is correct via induction.

i $T(n) = T(n-2) * 7$

<u>*Solution:*</u>

i Our scratch work:

$$\begin{aligned} T(\square) &= T(\square - 2) * 7 \\ T(n-2) &= T(n-2-2) * 7 \\ T(n-4) &= T(n-4-2) * 7 \end{aligned}$$

$$\begin{aligned} T(n) &= T(n-2) * 7 \\ &= T(n-4) * 7 * 7 \\ &= T(n-4) * 7^2 \\ &= T(n-6) * 7 * 7^2 \\ &= T(n-6) * 7^3 \\ &\vdots \\ &= T(n-2*k) * 7^k \end{aligned}$$

Remember, we're looking to continue the above equalities until we get to the point where we plug in $T(1)$. Why this value? Because we know $T(1) = 1$, and this will finally break the recursion. For $T(n-2*k)$ to be $T(1)$ then $n - 2k = 1 \implies k = \frac{n-1}{2}$. Plugging this in for k in the last line above:

$$\begin{aligned} T(n) &= T(n - 2\frac{n-1}{2}) * 7^{\frac{n-1}{2}} \\ &= T(1) * 7^{\frac{n-1}{2}} \\ &= 7^{\frac{n-1}{2}} \end{aligned}$$