CS5002 Discrete and Data Structures	Prof Higger
Fall 2020	Nov 20, 2020

HW 09 Practice Problems

Due: DO NOT SUBMIT

Instructions:

- These practice problems are helpful to warm-up for the homework itself. They're great to discuss with course staff or class mates because **these problems are not to be handed in** so you may discuss ideas freely without worrying about academic integrity issues. Solutions will be provided immediately with the problems themselves.

Problem 1 Sorting Intermediate Steps

i **Insertion Sort**
Below is an intermediate step of during insertion sort. The square in the middle separates the sorted and the unsorted array.

$$1\,3\,8\,\square\,2\,7\,6$$

How many comparisons (start from the left) will there be before inserting the next element to the sorted sub-array? How is the array going to look like after the next iteration?

Solution: 2 comparisons (first 2 with 1, $2 > 1$. Then compare 2 with 3, $2 < 3$, we stop).
Next iteration: $1\,2\,3\,8\,\square\,7\,6$

ii **Merge Sort** You are about to merge the two sorted lists below into one sorted list.

$$L_1 : 1\,2\,7\,12\,15$$
$$L_2 : 0\,3\,4\,18\,20$$

List the indices of the first 5 comparisons in python notation (e.g. $L_2[3] = 18$) Write the merged list after the merge operation is complete.

Solution:
1^{st} comparison: $L_1[0]$ and $L_2[0]$
2^{nd} comparison: $L_1[0]$ and $L_2[1]$
3^{rd} comparison: $L_1[1]$ and $L_2[1]$
4^{th} comparison: $L_1[2]$ and $L_2[1]$
5^{th} comparison: $L_1[2]$ and $L_2[2]$
After merge: $0\,1\,2\,3\,4\,7\,12\,15\,18\,20$

Problem 2 Function Growth
There are three postage options in a town:

1. **Snail mail**: it takes 3 days for each letter to arrive by snail mail, only a single letter may be carried at one time.
2. **Carrier pigeon**: a single carrier pigeon takes 8 days to deliver some mail. A carrier pigeon may carry half of the remaining mail rounded down or a single letter, whichever is larger. For example, if 14 letters are sent then:
   - the carrier pigeon will carry 7 letters on the first trip (7 remaining)
   - the carrier pigeon will carry 3 letters on the next trip (4 remaining)
   - the carrier pigeon will carry 2 letters on the next trip (2 remaining)
   - the carrier pigeon will carry 1 letter on the next trip (1 remaining)
   - the carrier pigeon will carry 1 letter on the next trip (0 remain)

   So it will take 5 trips (40 days) to deliver all the mail.
3. **Pony express**: In 60 days the pony express can deliver any amount of letters.

a) Which shipping method offers the quickest delivery of 7 letters? How many days will this method take?

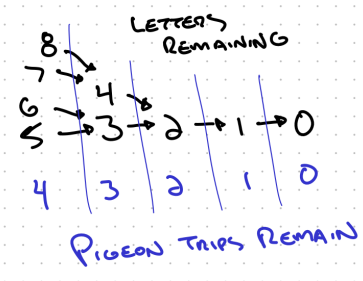
b) Which shipping method offers the quickest delivery of 29 letters? How many days will this method take?

c) What is the smallest amount of letters which must be sent such that the pony express becomes faster than snail mail?

d) What is the smallest amount of letters which must be sent such that the carrier pigeon becomes faster than snail mail?

e) What is the smallest amount of letters which must be sent such that the pony express becomes faster than the carrier pigeon mail?

Solution: Lets define functions which describe how long each method takes to deliver n letters:
Snail mail: $S(n) = 3n$
Carrier pigeon:



a Pigeon takes 8 days to deliver a single letter $C(1) = 8$. For larger n: $C(n) = 8(\lceil \log_2 n \rceil + 1)$
Pony express: $P(n) = 60$

a) For $n = 7$, delivery by snail mail would take 21 days:

$$\begin{aligned} S(n) &= 3n \\ S(7) &= 3 \cdot 7 = 21 \end{aligned}$$

Delivery by carrier pigeon would take 24 days:

$$\begin{aligned} C(n) &= 8(\lceil \log_2 n \rceil + 1) \\ C(7) &= 8(\lceil \log_2 7 \rceil + 1) \\ &= 8 * 4 = 32 \end{aligned}$$

Delivery by pony express always takes $P(7) = 60$ days.

The best option is Snail mail.

b) For $n = 29$, delivery by snail mail would take 87 days:

$$\begin{aligned} S(n) &= 3n \\ S(29) &= 3 \cdot 29 = 87 \end{aligned}$$

Delivery by carrier pigeon would take 40 days:

$$\begin{aligned} C(n) &= 8(\lceil \log_2 n \rceil + 1) \\ C(29) &= 8(\lceil \log_2 29 \rceil + 1) \\ &= 8 * 6 = 48 \end{aligned}$$

Delivery by pony express would take $P(29) = 60$ days.

The best option is the carrier pigeon.

c) Here we are asking for the value of n such that $P(n) < S(n)$:

$$\begin{aligned} P(n) &< S(n) \\ 60 &< 3n \\ 20 &< n \end{aligned}$$

Thus, at least 21 letters must be sent so the pony express is faster than snail mail.

d) We want to find the smallest number of letters, n, where the pigeons are faster:

$$\begin{aligned} C(n) &< S(n) \\ 8(\lceil \log_2 n \rceil + 1) &< 3n \end{aligned}$$

Equations of this form are difficult to solve algebraically, but we can use python to guess and check for us. The loop below tries to run from $n = 1$ letters up to 1000. At each iteration, it checks if the carrier pigeons are faster than snail mail for that particular number of letters. If they are faster, it prints a statement and breaks out of the loop early.

```
import numpy as np

def carrier_pigeon(n):
    if n == 1:
        return 8
    return 8 * (np.ceil(np.log2(n)) + 1)

def snail_mail(n):
    return 3 * n

def pony_express(n):
    return 60

for n in range(1, 1000):
    if carrier_pigeon(n) < snail_mail(n):
        print(f'pigeon faster than snail for n={n}')
        break
```

```
pigeon faster than snail for n=14
```

e) Eventually, as one sends enough mail, the pony express will be quicker than either of the other two methods. It may be simplest to use the python again:

[12]:
```
import numpy as np

def carrier_pigeon(n):
    if n == 1:
        return 8
    return 8 * (np.ceil(np.log2(n)) + 1)

def snail_mail(n):
    return 3 * n

def pony_express(n):
    return 60

for n in range(1, 1000):
    if pony_express(n) < carrier_pigeon(n):
        print(f'pony faster than pigeon for n={n}')
        break
```

```
pony faster than pigeon for n=65
```