Online Quickest Change Detection for Multiple Gaussian Sequences Using Stochastic Bandits

Joshua S. Kartzman[1] Benjamin D. Robinson[2] Matthew T. Hale[3]

[1]Stony Brook University
[2]Air Force Office of Scientific Research
[3]Georgia Institute of Technology

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Introduction

- Our work considers online multi-stream change-point detection where the goal is to flag a change in distribution of one of the streams as quickly as possible after the change has occurred. Moreover, the post-change distribution contains an unknown nuisance parameter which requires estimation
- An additional constraint in our problem formulation is that the agent can only observe a single stream at each time step. The goal is to identify which stream contains the change-point as soon as possible and observe it enough to flag a change-point, which introduces an exploration-exploitation tradeoff to the problem
- Based on ideas from stochastic bandits and reinforcement learning, we propose a new algorithm whose performance is within a constant factor of asymptotic optimality

Background: Quickest Change Detection

- In the traditional single-stream problem formulation, a sequence of data $X_1, X_2, \ldots$ is observed where the generating distribution before the change is f_0 and after the change is f_1. The σ-algebra generated by the observations up to time t is $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.
- A detection procedure is a stopping time τ (by definition, a random variable such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{N}$) based on some detection statistic T_t which is compared to a preset threshold λ:

$$\tau = \inf\left\{t \geq 1 : T_t \geq \lambda\right\}.$$

- Typically, f_0 is known but f_1 is either unknown or contains a nuisance parameter. This is since there is usually abundant pre-change data.

Background: Quickest Change Detection

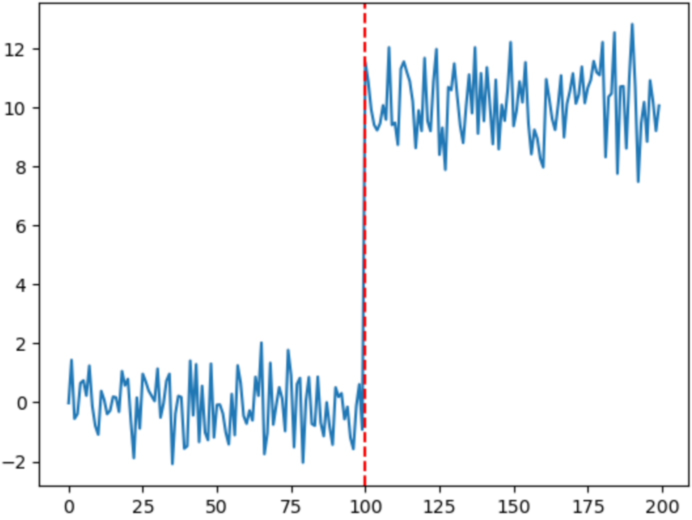


Figure: Single-Stream Change-point

Background: Single-Stream Problem Formulation

- Here $\mathbb{P}_\nu$ and $\mathbb{E}_\nu$ denote the probability measure and expectation when the change-point is located at ν. $\nu = \infty$ is the case when no change-point occurs.
- In the minimax formulation from Pollak (1985), the goal is to minimize the worst-case expected detection delay

$$\mathrm{CADD}(\tau) := \sup_{\nu \geq 0} \mathbb{E}_\nu[\tau - \nu | \tau > \nu]$$

  subject to the average run length (expected stopping time when no change occurs) being greater than a constant γ:

$$\mathrm{ARL}(\tau) := \mathbb{E}_\infty[\tau] \geq \gamma$$

- Siegmund and Venkatraman (1995) use the surrogate quantity $\mathbb{E}_0[\tau]$ for the worst-case expected detection delay, which we also use here.

Background: Single-Stream Detection Procedures

- It was shown in Theorem 1 of Lai (1998) that the asymptotic lower bound for the detection delay satisfies:

$$\inf \{\mathrm{CADD}(\tau) : \mathrm{ARL}(\tau) \geq \gamma\} \geq (1 + o(1)) \frac{\log \gamma}{D(f_1 || f_0)}$$

as $\gamma \to \infty$.

- An asymptotically optimal procedure τ is one which minimizes $\mathrm{CADD}(\tau)$ subject to $\mathrm{ARL}(\tau) \geq \gamma$ as $\gamma \to \infty$. CUSUM is one such procedure. The detection statistic for CUSUM is defined as

$$T_t^{\mathrm{CUSUM}} = \max_{0 \leq k < t} \sum_{i=k+1}^{t} \log \left(\frac{f_1(X_i)}{f_0(X_i)} \right)$$

However it requires prior complete knowledge of both f_0 and f_1.

Background: Generalized Likelihood Ratio Statistic

- An alternative procedure for an unknown post-change parameter which is asymptotically optimal is the Generalized Likelihood Ratio (GLR) statistic. It introduces a maximum likelihood estimator for the post-change parameter into the CUSUM detection statistic:

$$T_t^{\mathrm{GLR}} = \max_{0 \le k < t} \sup_{\theta \in \boldsymbol{\Theta}} \sum_{i=k+1}^{t} \log \left(\frac{f_\theta(X_i)}{f_0(X_i)} \right),$$

  where $\boldsymbol{\Theta}$ is a pre-defined set of possible parameter values.
- From Siegmund and Venkatraman (1995), for an unknown change in the mean of a normal distribution with known variance, the GLR statistic is

$$T_t^{\mathrm{GLR}} = \max_{0 \le k < t} \frac{\left(\sum_{i=k+1}^{t} X_i \right)^2}{2(t-k)}.$$

- Romano et al. (2023) provided an algorithm, known as FOCuS, which can compute the statistic at each iteration in $O(\log t)$.

Problem Formulation

- There are M random sequences. At each time t, the agent observes a single stream A_t and receives an observation X_t. At an unknown time step $\nu \in \mathbb{N}$ in an unknown stream (assumed without loss of generality to be stream 1), the generating distribution shifts from f_0 to f_1, such that if $A_t = 1$ and $t > \nu$ then $X_t \sim f_1$, and otherwise $X_t \sim f_0$. $\mathbb{P}_{M,\nu}$ and $\mathbb{E}_{M,\nu}$ denote the probability and expectation, respectively.
- We assume f_0 is a fully known Gaussian distribution, assumed without loss of generality to be $\mathcal{N}(0,1)$, and that f_1 is a Gaussian distribution $\mathcal{N}(\mu,1)$ with the same known variance but unknown mean μ.
- We use the same minimax formulation as Pollak (1985) and Siegmund and Venkatraman (1995) and seek to minimize

$$\mathrm{CADD}(\tau) := \sup_{\nu \geq 0} \mathbb{E}_{M,\nu}[\tau - \nu | \tau > \nu]$$

subject to

$$\mathrm{ARL}(\tau) := \mathbb{E}_{M,\infty}[\tau] \geq \gamma.$$

Multi-Stream Change Detection

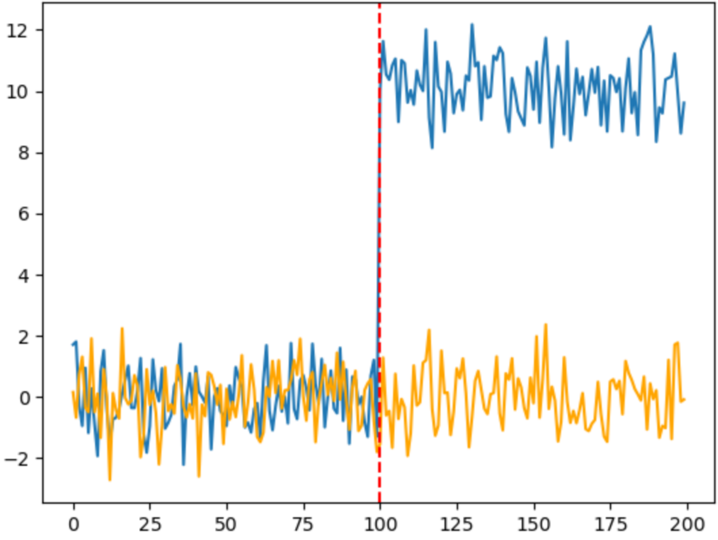


Figure: Multi-Stream Setting

ϵ-FOCuS Algorithm

- At each time t, each stream $m \in [M]$ has a local statistic

$$T_t^{(m)} = \max_{0 \le k < N_t^{(m)}} \frac{\left(\sum_{i=k+1}^{N_t^{(m)}} X_i^{(m)}\right)^2}{2\left(N_t^{(m)} - k\right)}.$$

- An exploration choice G_t is sampled from $\mathrm{Bernoulli}(\epsilon)$, where ϵ is the probability of exploration. If $G_t = 1$, the agent randomly samples a stream. Otherwise, the stream with the largest statistic is observed:

$$A_t = \operatorname*{argmax}_{m \in [M]} T_{t-1}^{(m)}.$$

- $T_t^{(A_t)}$ is calculated with FOCuS. The procedure stops if $\max_{m \in [M]} T_t^{(m)}$ exceeds the threshold λ and continues otherwise.

Analysis: Expected Detection Delay

- We first provide a theorem on the expected detection delay using the surrogate quantity from Siegmund and Venkatraman (1995):

Theorem 1

Consider an agent using ϵ-FOCuS on M streams, and suppose that stream 1 contains a change-point at time $\nu = 0$ that shifts its distribution from $\mathcal{N}(0,1)$ to $\mathcal{N}(\mu_1, 1)$, where $\mu_1 \neq 0$. For a detection threshold $\lambda > 0$ and an exploration parameter $\epsilon \in (0,1)$, the expected time until detection τ is bounded via

$$\mathbb{E}_{M,0}[\tau] \leq \frac{1}{1-\epsilon}\left(\frac{2\lambda(1+o(1))}{\mu_1^2} + C_{\epsilon,\mu_1,M}\right)$$

as $\lambda \to \infty$, where $C_{\epsilon,\mu_1,M} > 0$ is a constant determined by ϵ, μ_1, and M.

- Our analysis uses the Borel-Cantelli lemma to prove there a.s. exists a finite time step after which only stream 1 has the largest GLR statistic, implying it is observed during periods of exploitation.

Analysis: Average Run Length

- Our next theorem on the average run length follows from Theorem 1 in Siegmund and Venkatraman (1995) regarding the stopping time τ being exponentially distributed for the single-stream GLR procedure:

Theorem 2

Consider an agent using ϵ-FOCuS on M streams, where all streams are distributed according to f_0, i.e., no change has occurred. Then given a detection threshold $\lambda > 0$ and an exploration parameter $\epsilon \in (0, 1)$, the expected time until a false detection τ is bounded via

$$\mathbb{E}_{M,\infty}[\tau] \geq \frac{e^{\lambda}\sqrt{\pi}}{M\sqrt{\lambda}\int_0^{\infty} x g(x)^2 dx}$$

as $\lambda \to \infty$, where $g(x)$ is defined as

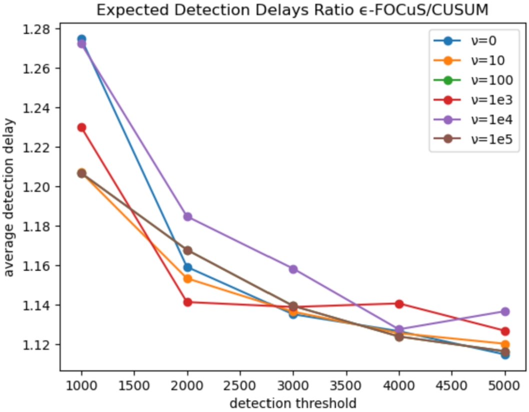
$$g(x) = 2x^{-2}\exp\left[-2\sum_{n=1}^{\infty} n^{-1}\Phi\left(-xn^{1/2}/2\right)\right], \; x > 0.$$

Analysis: Implications

- A consequence of the above theorems is that for a given ARL parameter $\gamma > 0$, we can derive a threshold such that $\mathbb{E}_{M,\infty}[\tau] \geq \gamma$ implies $\mathbb{E}_{M,0}[\tau] \leq \left(\frac{1+o(1)}{1-\epsilon}\right) \frac{\log \gamma}{D(f_1||f_0)}$ as $\gamma \to \infty$.
- This is within a constant factor $\frac{1}{1-\epsilon}$ of Lai's (1998) lower bound on the single-stream detection delay. This implies our algorithm is within a constant factor of the asymptotic performance of the single-stream CUSUM algorithm, despite having to search for the change-point over M streams and having no prior knowledge of the post-change parameter.

Simulations: ϵ-FOCuS vs. Single-Stream CUSUM

- We compare the expected detection delay across a set of change-points with the asymptotic expected detection delay for CUSUM. We set $M = 10$ and $\epsilon = 0.1$. The unknown post-change parameter is $\mu_1 = 1$. The asymptotic ratio in our experiments matches the theoretical results and approaches $1/(1-\epsilon) \approx 1.11$.



Conclusion

- In this talk we presented a new algorithm for multi-stream change-point detection with a controlled sampling constraint. Potential applications include industrial process management and fault detection.
- Future directions with this work will be focused on improving the detection delay by eliminating the $1/(1-\epsilon)$ constant. This can be done using a bandit algorithm with sub-linear regret such as UCB or Thompson Sampling.

Thank you!