Link Functions, Dose-Response Modeling

1 Introduction

- Objective: Examine effect of dose on probability of event
- Notation:

covariate, D_i, $p_i = p(D_i) = P(Y_i = 1|D_i)$

dependent variable $Y_i = 0, 1$

1.1 Link Functions

- $\log(\frac{p_i}{1-p_i}) = x_i^T\beta$
- $\Phi^{-1}(p_i) = x_i^T\beta$
- $\log\{-\log(1-p_i)\} = x_i^T\beta$
- $-\log\{-\log(p_i)\} = x_i^T\beta$

- Logit and probit are symmetric
- All continous on $(0, 1)$
- Note that **only** the logit link will consistently estimate $\hat{OR}$ in a retrospective sampling design

i.e. case-control

Probit Model
As the logistic tolerance distribution implies a logistic link function, so too does the normal tolerance distribution imply the use of a probit function
Here we let $f(s)$ be the tolerance distribution, and $p(t) = \int_{-\infty}^{t} f(s)ds$ be the cumulative tolerance function.

$$f(s) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{\frac{(s-\mu)^2}{2\sigma^2}\right\}$$
$$P(T \le t) = \Phi(\frac{t-\mu}{\sigma}) = p(t) \iff \Phi^{-1}(t) = \frac{t-\mu}{\sigma} = \beta + \beta_1 t$$

This is equivalent to using the probit function as the link function for our model

$$\Phi^{-1}(p(x)) = \beta_0 + \beta_1 x$$
$$\beta_0 = -\frac{\mu}{\sigma}$$
$$\beta_1 = \frac{1}{\sigma}$$

Complementary Log-Log Model Alternatively, if the tolerance follows the *extreme value distribution* then the link function is derived as follows.

$$f(s) = \beta_1 \exp\left\{(\beta_0 + \beta_1 s) - \exp\{\beta_0 + \beta_1 s\}\right\} \iff$$
$$p(t) = 1 - \exp\{-\exp\{\beta_0 + \beta_1 t\}\}$$
$$\log\{-\log\{(1-p_i)\}\} = \beta_0 + \beta_1 x$$

Note that this function is related to the hazard function (1) and hazard ratio(2)

$$h(t) = P(T = t | T \geq t) = \frac{f(t)}{1 - p(t)} \tag{1}$$

$$\frac{h(t+1)}{h(t)} = \exp\{\beta_1\} \tag{2}$$