# The Smith Chart and its Applications

by

Anthony A. R. Townsend

1995

## PREFACE

The five chapters of this book have been primarily written for students studying microwave engineering at the diploma and degree levels, and practising microwave engineers. It is intended that this book bridge the theoretical and the practical. To attain this basic objective, theory is developed and where possible, the theoretical solution to the various problems which are posed is given. The solution of these problems using the Smith chart is then provided. This permits an intuitive understanding of the Smith chart to be developed. The book advances from chapter to chapter, and as each chapter progresses, it builds on the work of the previous chapters.

The first two chapters review transmission line theory and develop the Smith and admittance charts. The third chapter considers single frequency applications to various matching problems which the Smith chart is adept at solving. The fourth chapter builds on the previous chapters by considering wide band matching problems using the various tools developed in the third chapter. The fifth chapter considers the design of microwave amplifiers using S-parameters and the Smith chart. It relies on the knowledge of the earlier chapters to design the input and output matching circuits of the microwave amplifiers. My colleague Mr. Deryk J. McNeill, developed the Smith chart used throughout this book.

This book is intended to bring together in one work, the subject of Smith charts and its applications. This is because texts on microwave engineering normally provide no more than one chapter on this subject, due to the breadth of the discipline. In recent times, several software programs have appeared on the market which are designed to assist in the solution of various problems, to which the Smith chart lends itself. This book can provide the principles behind the use of these programs as well as provide a foundation for the design of the more advanced microwave frequency multi-stage transistor circuits, and oscillator circuits.

Material in chapters 1, 2 and part of chapter 3 has been used for some years in the courses Microwaves and Radar and Electrical Principles C which I have been teaching for the BTEC HND here in Brunei .

I wish to acknowledge my colleague, Dr. Terence J. Fairclough for his continual assistance in the mathematical aspects of this manuscript. The many discussions I had with him were most valuable.

Anthony A.R. Townsend

**TABLE OF CONTENTS**

1. Transmission Lines 1

1.1 Distribution constants and travelling waves 1

1.2 Distribution constants of the lines 2
1.2.1 Balanced to unbalanced line conversion 3

1.3 Primary coefficients of a line 4
1.3.1 Resistance 4
1.3.2 Inductance 5
1.3.2.1 Two-wire or Twin 5
1.3.2.2 Coaxial 5
1.3.3 Capacitance 5
1.3.3.1 Two-wire or Twin 5
1.3.3.2 coaxial 6
1.3.4 Conductance (Leakance) 6

1.4 Differential equations for the uniform line 6

1.5 Travelling waves on a lossless line 7

1.6 Secondary coefficients of a line 9
1.6.1 Characteristic impedance $Z_o$ 1a*9
1.6.1.1 Twin line 12
1.6.1.2 Coaxial line 12
1.6.2 Propagation Coefficient γ 12
1.6.2.1 Very low frequencies - where R′ >> ωL′ and G′ >> ωC′ 15
1.6.2.2 Low frequencies - where R′ >> ωL′ and ωC′ >> G′ 15
1.6.2.3 High frequencies - where ωL′ >> R′ and ωC′ >> G′ 15

1.7 Velocity of propagation 15

1.8 Terminated transmission line 17
1.8.1 Correctly terminated lines or lines terminated in $Z_o$ 17
1.8.2 Mismatched transmission lines 18
1.8.2.1 Reflection coefficient ρ 1b*19
1.8.2.2 voltage standing wave ratio, Return Loss and Reflection Loss 21
1.8.2.2.1 Voltage standing wave ratio 21
1.8.2.2.2 Return Loss 23
1.8.2.2.3 Reflection Loss 24
1.8.2.3 The impedance of a mismatch line 24
1.8.2.3.1 The input impedance of a low-loss line 25
1.8.2.3.2 The input impedance of a lossless transmission line 25
1.8.2.4 Examples of transmission line impedance transformations 26
1.8.2.4.1 Short circuited transmission line 26
1.8.2.4.2 Open circuited transmission line 27
1.8.2.4.3 λ/4 Length of transmission line 27
1.8.2.4.4 λ/2 Length of transmission line 28
1.8.2.4.5 λ/8 Length of transmission line 29
1.8.2.5 Transmission line used as components 30
1.8.2.5.1 Lossless transmission line used as components 30
1.8.2.5.2 Low-loss transmission line used as components 1c*31

2. The Smith chart 33

2.1 Impedance and Reflection Charts 33

2.2 Conformal Transformation 35

2.3 Admittance chart 37

2.4 Relationship between impedance and admittance charts 2a*39

2.5 Properties of the Smith Chart 40
2.5.1 Some properties of the admittance chart 41

2.6 The use of the Smith chart for single frequency impedance matching 42
2.6.1 L-Type impedance transformation circuits 43
2.6.1.1 Constant conductance circles 44
2.6.1.2 Constant resistance circles 45

2.7 The use of boundary circles for design 2b*47

3. Transmission line circuit components 49

3.1 The transmission used as a reactance (or susceptance) 49
3.1.1 Short circuit transmission line 49
3.1.1.1 Inductance 49
3.1.1.2 Capacitance 50
3.1.2 Open circuited transmission line 52
3.1.3 The stub line as a series or shunt matching element 54
3.1.3.1 Series matching element 54
3.1.3.2 Shunt matching element 56
3.1.3.2.1 Single stub matching 58
3.1.3.2.1.1 Analytical approach to single stub matching 60
3.1.3.2.2 Double stub matching 61
3.1.3.2.2.1 Analytical approach to double stub matching 65
3.1.3.2.3 Triple stub matching 67
3.1.3.2.3.1 Analytical approach to triple stub matching 3a*71

3.2 The transmission line used as a transformer 73
3.2.1 The short transformer 73
3.2.1.1 Analysis of short transformer for Smith chart application 74
3.2.2 Quarter-wave transformers 76
3.2.2.1 Bandwidth of a quarter-wavelength transformer 77
3.2.3 Two short transformers in tandem 81
3.2.3.1 Analysis of short transformer for Smith chart application 82
3.2.3.2 Procedure for designing a double short-transformer matching network on a Smith chart 3b*85

4. matching over a frequency band 87

4.1 Variation of reactance and susceptance with frequency 87
4.1.1 Variation of reactance 87
4.1.1.1 Capacitive reactance 87
4.1.1.2 Addition of series capacitive reactance to an impedance 88
4.1.1.3 Inductive reactance 90
4.1.1.4 Addition of series inductive reactance to an impedance 91
4.1.2 Variation of susceptance 93
4.1.2.1 Capacitive susceptance 93
4.1.2.2 Addition of shunt capacitive susceptance to an admittance 94
4.1.2.3 Inductive susceptance 96
4.1.2.4 Addition of shunt inductive susceptance to an admittance 97

4.2 Variation of impedance and admittance with frequency 98
4.2.1 Load impedance converted to a load admittance 98
4.2.2 Load admittance converted to a load impedance 4a*99

4.3 Matching a mismatched load using L-type circuits 102

4.3.1 L-type impedance transformation circuits 102
4.3.1.1 Constant conductance circles 102
4.3.1.2 Constant resistance circles 4b*113

4.4 Matching to within a specified VSWR 125
4.4.1 Narrow-band system 125
4.4.1.1 Use to L-type circuits in matching 125
4.4.1.2 The use of s short-transformer in matching 129
4.4.1.3 The use of the short-circuit and open circuit stub in matching 134
4.4.1.3.1 The short-circuit quarter-wavelength stub 134
4.4.1.3.2 The open-circuit quarter-wavelength stub 137
4.4.1.4 The use of two quarter-wavelength transformers in tandem in matching 4c*139
4.4.1.5 The use of the various components to match a band of impedances 145
4.4.1.5.1 Capacitors and inductors 145
4.4.1.5.1.1 Series capacitors and inductors 145
4.4.1.5.1.2 Shunt capacitors and inductors 146
4.4.1.5.2 The series and parallel tuned circuit 147
4.4.1.5.2.1 The series tuned circuit 147
4.4.1.5.2.2 The parallel tuned circuit 148
4.4.1.5.3 Summary of short-circuit and open-circuit quarter and half-wavelength stubs for matching 157
4.4.1.5.3.1 Short-circuit stub 157
4.4.1.5.3.2 Open-circuit stub 157
4.4.1.6 The use of the single, double and triple stubs in matching 164
4.4.1.6.1 Single stub matching 164
4.4.1.6.2 Double stub matching 4d*168
4.4.1.6.3 Triple stub matching 4e*177
4.4.1.7 Velocity factor 184
4.4.2 Wide-band systems 4f*185

5. microwave amplifier design 189

5.1 The scattering matrix 189
5.1.1 Properties of S parameters for lossless passive junction 193
5.1.1.1 Symmetry 193
5.1.1.2 Unitary matrix identity 193
5.1.1.2.1 Unit property 193
5.1.1.2.2 Zero property 194
5.1.1.3 Phase shift 194
5.1.2 Two-port Power-Gain equations for active networks 194
5.1.2.1 The power-gain $G_p$, 195
5.1.2.1.1 Perfect matching 195
5.1.2.1.2 Conjugate matching 195
5.1.2.2 The transducer gain $G_t$ 196
5.1.2.2.1 Perfect matching 196
5.1.2.2.2 Conjugate matching 196
5.1.2.2.3 Maximum unilateral transducer power gain [2] 197
5.1.2.3 The available power gain $G_a$, 197
5.1.2.4 The maximum power gain using conjugate impedance matching 198

5.2 Amplifier stability 198
5.2.1 Stability circles 199
5.2.1.1 Load reflection coefficient stability circle 199
5.2.1.2 Source reflection coefficient stability circle 204
5.2.1.3 Stability design strategy 208
5.2.1.4 S-parameters and the transistor data sheet 208
5.2.1.5 Some definitions of frequencies used in data sheets 210
5.2.1.5.1 Current Gain Bandwidth Product ( $f_t$ ) 210
5.2.1.5.2 Power Gain Bandwidth Product ( $f_s$ ) 210
5.2.1.5.3 Maximum Unilateral Transducer Power Gain Bandwidth Product ( $f_{\max}$ ) 210

5.3 Constant power gain circles 215
5.3.1 Constant power gain circle equations 215
5.3.2 Constant power gain circles for stable amplifiers 216
5.3.2.1 Stability circle not enclosing the Smith chart 216
5.3.2.1.1 Constant power gain circles which lie within the Smith chart 216
5.3.2.1.2 Constant power gain circles which lie within the stability circle 217
5.3.2.1.3 Constant power gain circles which lie between the Smith chart and the stability circle 217
5.3.2.2 Stability circle enclosing the Smith chart 5a*217
5.3.3 Constant power gain circles for conditionally stable amplifiers 220
5.3.3.1 Intersection of load stability circle with the Smith chart 220
5.3.3.1.1 Constant gain circles where stability circle encloses the origin $|S_{11}| < 1$ 221
5.3.3.1.2 Constant gain circles where stability circle encloses the origin $|S_{11}| > 1$ 221
5.3.3.1.3 Constant gain circles where stability circle does not encloses the origin $|S_{11}| < 1$ 221
5.3.3.1.4 Constant gain circles where stability circle does not encloses the origin $|S_{11}| > 1$ 5b*222

5.4 Conjugate input matching circles 226
5.4.1 A brief overview of the reflection coefficient domains 226
5.4.1.1 Properties of the conjugate input reflection coefficient circle $\rho_{in}^{*}$ 227

5.5 Constant Noise Figure Circles 230

5.6 Constant Input Mismatch Circles 233

5.7 Constant Output Mismatch Circles 5c*236

Appendix A Derivation of power equations

Appendix B Bilinear transforms

Appendix C Derivation of constant spot noise factor circles

Appendix D Derivation of constant mismatch circles

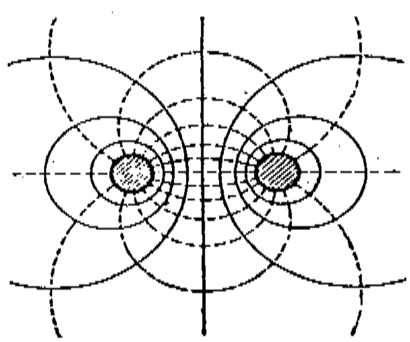
# 1. TRANSMISSION LINES

To understand the Smith Chart, the theory of electric waves along a uniform transmission line is first considered. A *uniform* transmission line can be defined as a line which has *identical* dimensions and electrical properties in all planes *transverse to the direction of propagation*. Unlike the waveguide, the transmission line consists of two conductors separated by a dielectric. If the two conductors are identical and placed along side each other then a twin line is formed. These lines are usually used at the lower frequencies. Examples of twin lines are the familiar TV twin-lead and the open two-wire line. If one conductor is placed inside the hollowed tube of the other, then a coaxial line is formed. Coaxial lines are used for frequencies up to 30 GHz. At frequencies from 3 to 300 GHz, hollow waveguides are used. Transmission lines provide a method of transmitting electrical energy between two points in space, similar to antennas. Their *behavior* is normally described in terms of the current and voltage waves which propagate along it. The *performance* of a transmission line is normally described in terms of the secondary coefficients, as discussed below.

## *1.1 DISTRIBUTED CONSTANTS AND TRAVELLING WAVES*

In a coaxial line the centre conductor may be held in place by dielectric spacers or by a continuous solid dielectric fill which supports the inner conductor keeping it central to the outer conductor. A coaxial cable is self-shielded and has no external field except possibly near terminations. For this reason it is widely used throughout the radio frequency range and well within the microwave region; which is the name given to the radio spectrum at wavelengths below half a metre.

Twin lines, used as open wire lines, are normally balanced with respect to ground, but as a coaxial cable is unsymmetrical it cannot be balanced with respect to ground.



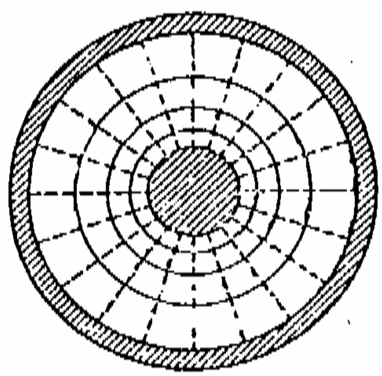


**Figure 1(a) Open two-wire line** **Figure 1(b) Coaxial cable**

When the transmission-line is analyzed by means of electromagnetic field theory, it is found that the principal-mode of transmission of the electromagnetic wave is not the only one that can exist on a set of parallel conductors. Our discussion will consider only the *principal-mode* in which the electric and the magnetic fields are perpendicular to each other and to the direction of the conductors. The principal-mode wave is called the *transverse electromagnetic, or TEM*, wave, and is the only kind that can exist on a transmission line at the lower frequencies. When the frequencies become so high that the wavelength is comparable with the distance between conductors, other types of waves, of the kind utilized in hollow waveguides become possible. Except in very special cases, these “higher modes” are considered undesirable on the transmission systems that we are considering; therefore, whenever possible, the spacing between conductors is kept much smaller than a quarter wavelength. Another reason for a small spacing is that, when the distance between the wires of an unshielded line approaches a quarter wavelength, the line acts as an antenna and radiates a considerable portion of the energy that it carries. In this chapter, we will assume a very small spacing and shall neglect radiation losses altogether.

## 1.2 DISTRIBUTED CONSTANTS OF THE LINE

Transmission lines are most easily analyzed by an extension of lumped-constant theory. The same theory will apply to the lines shown in Figure 1 above.

The most important constants of the line are its distributed inductance and capacitance. When a current flows in the conductors of a transmission line, a magnetic flux is set up around the conductors. Any change in this flux will induce a voltage (L di/dt). The inductance of the transmission-line conductors is smoothly distributed throughout their length. The distributed inductance, representing the net effect of all the line conductors and is associated with the magnetic flux linking these conductors, is given the symbol L′ and is expressed in henries per unit length.

Between the conductors of the line there exists a uniformly distributed capacitance C′ which is associated with the charge on the conductors. This will be measured in farads per unit length of the line. The distributed inductance and capacitance are illustrated in Figure 2. When a line is viewed in his way, it is not hard to see that the voltage and current can vary from point to point on the line, and that resonance may exist under certain conditions.

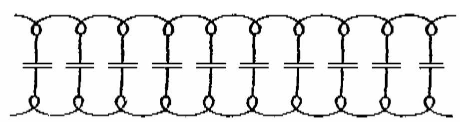


**Figure 2** **Schematic representation of the distributed inductance and capacitance of a transmission line**

In addition to inductance and capacitance, the conductors also have a resistance R′ ohms per unit length. This includes the effect of all the conductors. Finally, the insulation of the line may allow some current to leak from one conductor to the other. This is denoted by a conductance G′, measured in siemens per unit length of line. The quantity R′ represents the imperfection of the conductor and is related to its dimensions and conductivity, while G′ represents the imperfection of the insulator and is related to the loss tangent of the insulating material between the conductors. Note however, that G′ does not represent the reciprocal of R′. When solid insulation is used at very high frequencies, the dielectric loss may be considerable. This has the same effect on the line as true ohmic leakage and forms the major contribution to G′ at these frequencies.

Even though the line constants are uniformly distributed along the line, we can gain a rough idea of their effect by imagining they line to be made up of short sections of length $\Delta z$, as shown in Figure 3. If L′ is the series inductance per unit length (H/m), the inductance of a short section will be $L'\Delta z$ henrys. Similarly, the resistance of the section will be $R'\Delta z$ ohms, the capacitance will be $C'\Delta z$ farads, and the leakage conductance will be $G'\Delta z$ siemens.

Although the inductance and resistance are shown lumped in one conductor in Figure 3, they actually represent the net effect of both conductors in the short section $\Delta z$. As the section lengths $\Delta z$ are made smaller and approach zero length, the “lumpy” line of Figure 3 will approach the actual smooth line and since $\Delta z$ can be made small compared to the operating wavelength, an individual section of line can be analysed using lumped circuit theory. As $\Delta z$ may approach zero, the results of the derivation to follow are valid at all frequencies.

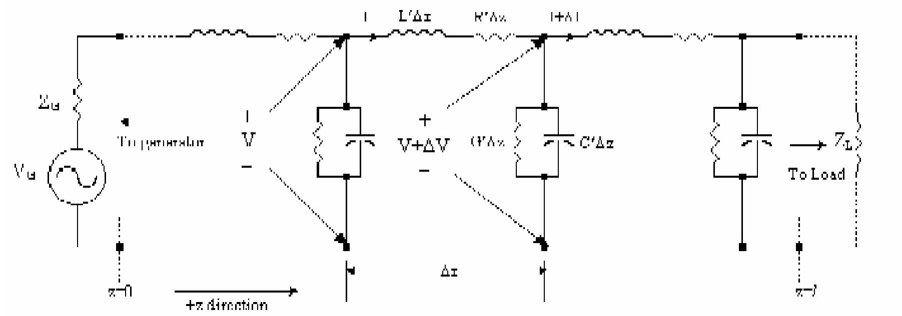



**Figure 3** **Lumped circuit representation of a uniform transmission line**

### 1.2.1 BALANCED TO UNBALANCED LINE CONVERSION

As may have been noticed when considering Figure 2 and Figure 3, the schematic representation of the distributed inductance and capacitance of a balanced transmission line has been changed to the lumped circuit representation of an unbalanced uniform line. This section considers how this transition is justified.

Figure 4 shows how the transmission line of Figure 2 can be represented by lumped circuit representation of impedances, where $Z_1 = (j\omega L' + R')\Delta z$ and $Z_2 = 1/(j\omega C' + G')\Delta z$, where $\Delta z$ is an infinitesimal section of the line. As the line is balanced half of the distributed inductance and resistance, which could be measured, is placed in each of the upper and lower arms.

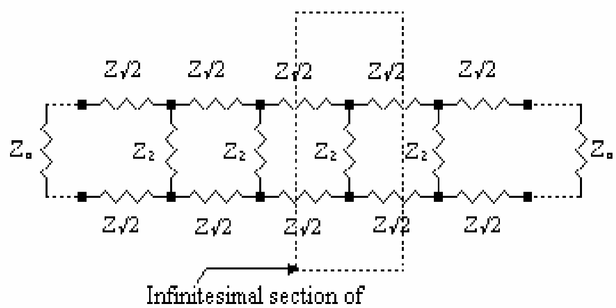



**Figure 4** **Lumped circuit representation of a balanced transmission line**

However, because in the following analysis, we want to deal with T-networks, the balanced lumped circuit model can be divided into a number of cascaded T-sections, each of length $\Delta z$, and each of which are contained within the dotted region shown in Figure 4. To do this each of the $Z_1/2$ impedances must be split into two $Z_1/4$ series impedances as shown in Figure 5. This permits each of the $Z_1/4$ series impedances to be balanced around each of the $Z_2$ branch impedances. The dotted region shown in figure 5 also represents the infinitesimal length $\Delta z$. The remaining section of the transmission line at either end of the section is shown replaced by the impedance $Z_o$ which, as will be discussed in more detail below, is known as the "characteristic impedance" of the line and is normally taken as being *resistive*.

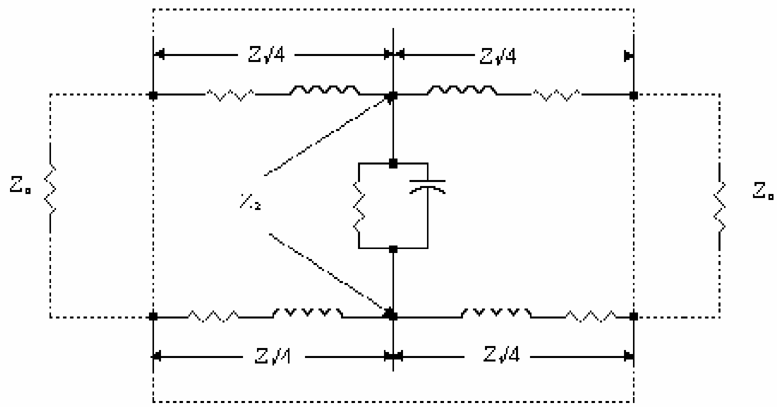



**Figure 5** **Balanced lumped circuit representation of a transmission line**

If the balanced circuit is considered to work into a resistive impedance $Z_o$ then, the lower arm $Z_1/4$ series impedances can be moved into the upper arm and added to the existing $Z_1/4$ series impedances to give each upper arm an impedance of $Z_1/2$ and provide the unbalanced T-section required for analysis. This is shown in Figure 6.

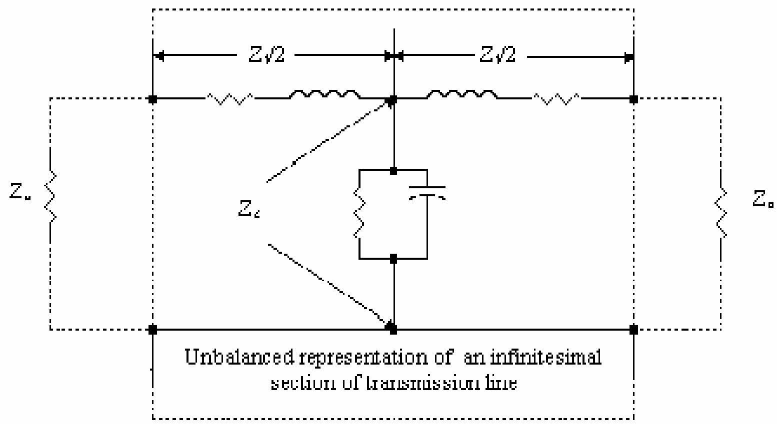



**Figure 6** **Unbalanced lumped circuit representation of a transmission line**

## *1.3 PRIMARY COEFFICIENTS OF A LINE*

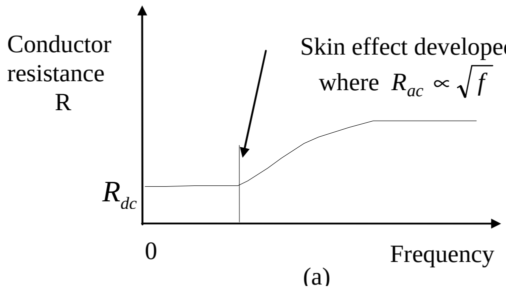
### 1.3.1 RESISTANCE

The resistance R′ of a line is the sum of the resistances of the two conductors comprising a pair. At zero frequency the resistance of a pair per unit length is merely the d.c. resistance R′ but at frequencies greater than a few kilohertz a phenomenon known as skin effect comes into play. This effect ensures that current flows only in a thin layer or "skin" at the surface of the conductors. The thickness of this layer reduces as the frequency is increased and this means that the effective cross-sectional area of the conductor is reduced. Since resistance is equal to ρl/A (resistivity x length / cross-sectional area), the a.c. resistance of a conductor will increase with increase in frequency. While the skin effect is developing, the relationship between a.c. resistance and the frequency is rather complicated., but once skin effect is fully developed (around 12 kHz) the ac resistance becomes directly proportional to the square root of the frequency, that is:

$$R_{ac} = K_1 .\sqrt{f}$$

where $K_1$ is a constant.

Figure 7(a) below shows how the resistance of a line varies with the frequency. Initially, little variation from the d.c. value is observed but at higher frequencies the shape of the graph is determined by $R_{ac} = K_1 .\sqrt{f}$



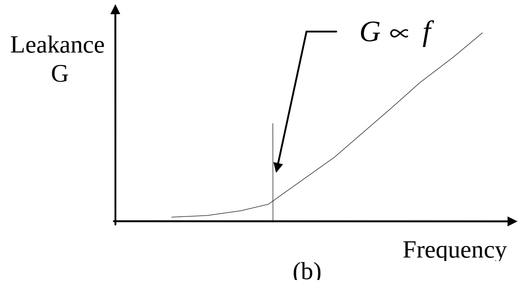




**Figure 7** **(a) Variation of line resistance with frequency**

**(b) Variation of line leakance with frequency**

## 1.3.2 INDUCTANCE

The inductance of a line L′, usually expressed in henrys/kilometre, depends upon the dimensions and the spacing of its two conductors. The equations for the inductance of both two-wire and coaxial pairs are given below. The permeability of free space $\mu_o$, is given as $4\pi \times 10^{-7}$ H/m.

### *1.3.2.1 TWO-WIRE OR TWIN*

L′ = $\frac{\mu_o}{4\pi} + \frac{\mu}{\pi}\log_e \frac{(d-r)}{r}$ H/m ≈ $\frac{\mu_o}{4\pi} + \frac{\mu}{\pi}\log_e d/r$ H/m when d >> r or $0.1 + 0.92\mu_r \log_{10} d/r$ μH/m, where "d" is the spacing between the two conductors, each of radius "r", μ is the permeability (H/m), $\mu_r = \mu/\mu_o$, is the relative permeability and the component $\mu_o/4\pi$ arises because of the flux linkages inside the conductors themselves.

### *1.3.2.2 COAXIAL*

L′ = $\frac{\mu_o}{8\pi} + \frac{\mu}{2\pi}\log_e w/r$ H/m = $0.1 + 0.46\mu_r \log_{10} w/r$ μH/m where the component $\mu_o/8\pi$ arises due to the interior flux linkages inside the two conductors, and where "w" is the radius of the outer conductor and "r" is the radius of the inner conductor. Neither the dimensions of a line nor the absolute permeability are functions of frequency and consequently the inductance of a line is not a frequency-dependent parameter.

## 1.3.3 CAPACITANCE

The capacitance C′, usually expressed in farads/kilometre, for both two-wire and coaxial pairs are given below: The permittivity is given by $\varepsilon = \varepsilon_o \varepsilon_r$ where $\varepsilon_o$ is the permittivity of free-space and is given by $8.854187818 \times 10^{-12}$ F/m or approximately, by $10^{-9}/36\pi$ F/m, and $\varepsilon_r$ is the relative permittivity or dielectric constant, which is a dimensionless quantity.

### *1.3.3.1 TWO-WIRE OR TWIN*

C′ = $\frac{\pi\varepsilon}{\log_e (d-r)/r}$ F/m ≈ $\frac{\pi\varepsilon}{\log_e d/r}$ F/m, when d >> r, or $\frac{12.08\varepsilon_r}{\log_{10} d/r}$ pF/m

where "d" is the spacing between the two conductors, each of radius "r".

### *1.3.3.2 COAXIAL*

$C' = \dfrac{2\pi\varepsilon}{\log_e {w}/{r}}$ F/m or $\dfrac{24.16\varepsilon_r}{\log_{10} {w}/{r}}$ pF/m

where "w" is the radius of the outer conductor and "r" is the radius of the inner conductor. Neither the dimensions of a line nor the permittivity are functions of frequency and consequently the capacitance of a line is not a frequency-dependent parameter.

## 1.3.4 CONDUCTANCE (Leakance)

The conductance of a line G′, usually expressed in siemens (mhos) per kilometer, represents the leakage of current between the conductors via the dielectric separating them. The leakage current has two components: one of these passes through the insulation resistance between the conductors, while the other supplies the energy losses in the dielectric as the line capacitance repeatedly charges and discharges.

The leakance increases with an increase in frequency and at the higher frequencies it becomes directly proportional to frequency:

$G = K_2 f$ where $K_2$ is a constant.

Figure 7(b) above, shows how the leakance of a line varies with frequency.

Below are some typical figures for the primary coefficients of a line.

**Table 1 Typical values of the primary coefficients of a line**

| **Line Type** | **R′(Ω/km)** | **L′(mH/km)** | **C′(μF/km)** | **G′(μS/km)** |
|---|---|---|---|---|
| Twin (800 Hz) | 55 | 0.6 | 0.033 | 0.6 |
| Coaxial (1 MHz) | 34 | 0.28 | 0.05 | 1.4 |

## *1.4 DIFFERENTIAL EQUATIONS FOR THE UNIFORM LINE*

Consider an infinitesimal section of a line as shown in Figure 3, and consider the time varying instantaneous voltage v(z,t) and the instantaneous current i(z,t). The series inductance of the section will be L′Δz Henrys, and the series resistance will be R′Δz ohms. Similarly, the shunt capacitance will be C′Δz Farads and the shunt conductance will be G′Δz siemens. If the positive z direction is taken as horizontal and to the right, or from the generator toward the load, the difference between the instantaneous line-to-line voltages at the two ends of the section will be (∂v(z,t)/∂z)Δz. The partial derivative is taken because there are two independent variables, time t and distance z. The voltage difference (∂v(z,t)/∂z)Δz is caused by the current i(z,t) flowing through the resistance R′Δz and changing at the rate ∂i(z,t)/∂t in the inductance L′Δz. Thus, we can write:

$$-\frac{\partial v(z,t)}{\partial z}\Delta z = (R'\Delta z)i(z,t) + (L'\Delta z)\frac{\partial i(z,t)}{\partial t}$$

The negative sign is used because positive values of i(z,t) and ∂*i(z,t)*/∂t cause v(z,t) to decrease with increasing z. Upon dividing through by Δz,

$$-\frac{\partial v(z,t)}{\partial z} = R'i(z,t) + L'\frac{\partial i(z,t)}{\partial t} \qquad \textbf{(1-1)}$$

This differential equation indicates the manner in which the instantaneous line-to-line voltage v(z,t) changes along the line. In a similar manner, the difference in current between the two ends of the section (∂*i(z,t)*/∂z).Δz, will be made up of two parts. The first is the current caused by the voltage v(z,t), acting on the shunt conductance G′Δz and second, the displacement current through the capacitance C′.Δz caused by the voltage changing at the rate ∂v(z,t)/∂t. We can, therefore, write:

$$-\frac{\partial i(z,t)}{\partial z}\Delta z = (G'\Delta z)v(z,t) + (C'\Delta z)\frac{\partial v(z,t)}{\partial t}$$

Dividing through by Δz, we obtain a differential equation that indicates the manner in which the current i(z,t) changes along the line:

$$-\frac{\partial i(z,t)}{\partial z} = G'v(z,t) + C'\frac{\partial v(z,t)}{\partial t} \qquad \textbf{(1-2)}$$

The two partial differential equations (1-1) and (1-2), contain the independent variables, v(z,t), i(z,t), z, and t. These equations, together with the boundary conditions relating to the two ends, will in principle yield both the steady-state and the transient solutions. We shall concentrate mainly on the steady-state a.c. solution.

By taking ∂/∂z of equation (1-1) and ∂/∂t of equation (1-2) and then eliminating ∂i(z,t)/∂z and $\partial^2 i(z,t)/\partial z\partial t$, a second-order differential equation for voltage is obtained.

$$\frac{\partial^2 v(z,t)}{\partial z^2} = L'C'\frac{\partial^2 v(z,t)}{\partial t^2} + (R'C' + G'L')\frac{\partial v(z,t)}{\partial t} + R'G'v(z,t) \qquad \textbf{(1-3)}$$

Solving for current in a similar manner yields

$$\frac{\partial^2 i(z,t)}{\partial z^2} = L'C'\frac{\partial^2 i(z,t)}{\partial t^2} + (R'C' + G'L')\frac{\partial i(z,t)}{\partial t} + R'G'i(z,t) \qquad \textbf{(1-4)}$$

The solution of either of these second-order equations (1-3) or (1-4) and equations (1-1) and (1-2), together with the electrical properties of the generator and load, permit us to determine the instantaneous voltage and current at any time t and any place z along the uniform transmission line.

## 1.5 TRAVELLING WAVES ON A LOSSLESS LINE

It is informative to consider the hypothetical case of a line without loss, for which R′ = 0 and G′ = 0. This approximation is reasonably good when the line losses are much smaller than the energy which travels along the line. Physically short radio-frequency lines can often be treated satisfactorily by this method. Also, the approximation affords a simple and useful, although rather idealized, method for calculating the propagation of surges such as those caused by lightning strokes on power lines. The operation of delay lines and of pulse-forming lines is most easily understood by using the lossless theory. If the line is not considered to be lossless, then the solution of the differential equation produces a dispersive term which alters the shape of the waveform travelling down the line.

For the lossless conditions equations (1-1) and (1-2) reduce to Equations (1-5)

$$-\frac{\partial v(z,t)}{\partial z} = L'\frac{\partial i(z,t)}{\partial t} \quad \textbf{and} \quad -\frac{\partial i(z,t)}{\partial z} = C'\frac{\partial v(z,t)}{\partial t} \qquad \textbf{(1-5)}$$

Equations (1-3) and (1-4) reduce to equations (1-6):

$$\frac{\partial^2 v(z,t)}{\partial z^2} = L'C'\frac{\partial^2 v(z,t)}{\partial t^2} \quad \textbf{and} \quad \frac{\partial^2 i(z,t)}{\partial z^2} = L'C'\frac{\partial^2 i(z,t)}{\partial t^2} \qquad \textbf{(1-6)}$$

which are forms of the wave equation, well known in physics.

Equations (1-6) represent the differential equations for a lossless line, which can be taken as an idealization which represents, in many cases, an excellent approximation. They are the one dimensional forms of the wave equation, the solution of which are known to consist of waves that can travel in either direction, without change in form or magnitude, at the velocity v = 1/√(L′C′). To show this, we shall formulate a mathematical expression for such a travelling wave and then demonstrate that this expression satisfies the first of the differential equations given in equation (1-6).

In general, equations (1-6) are satisfied by single-valued functions of the form f[t ± z√(L′C′)], where the plus sign indicates propagation in the *negative z* direction and the minus sign propagation in the *positive z* direction and f represents any single-valued function of the argument [t ± z√(L′C′)]. The following are specific illustrations of such functions: $e^{a(t-z\sqrt{L'C'})}$, $\sin\omega(t-z\sqrt{L'C'})$, and $K(t-z\sqrt{L'C'})^2$, where a, ω and K are constants. To understand the meanings of these solutions, assume v(z,t) = f[t − z√(L′C′)]. At the point

z = 0, the voltage versus time function is given by v(0,t) = f(t). Further down the z axis at z = $z_1$, v($z_1$,t) = f[t – $z_1$√(L′C′)], which is exactly the same as f(t) except that f(t) has been delayed by $t_d$ = $z_1$√(L′C′)]. This indicates that the voltage at z = 0 has moved to z = $z_1$ with a velocity v = $z_1$/ $t_d$ = 1/√(L′C′) as stated in the paragraph above. Similarly, the solution v($z_1$,t) = f[t + $z_1$√(L′C′)] represents a voltage function travelling in the negative z direction. The solutions of the current equation may be also interpreted as forward and reverse travelling current functions having the same velocity as the voltage.

Figure 8 shows the travelling wave at two successive instants in time $t_1$ and $t_2$ along the x-axis, for the function f[x√(L′C′) – t]. If the observer travels with the wave in such a way that he stays with a particular point on the wave, for example, the point marked "P", he would notice that the function would not change shape, but remain constant in value. This means that the argument x√(L′C′) – t is constant. That is, x√(L′C′) – t = K′. As he is moving, he must be moving with some velocity v. This is found by taking the derivative term-by-term with respect to time of x√(L′C′) – t = K′. That is,

$\frac{dx}{dt}\sqrt{L'C'}-1=0$ from which the velocity v, is found to be dx/dt = 1/√(L′C′).

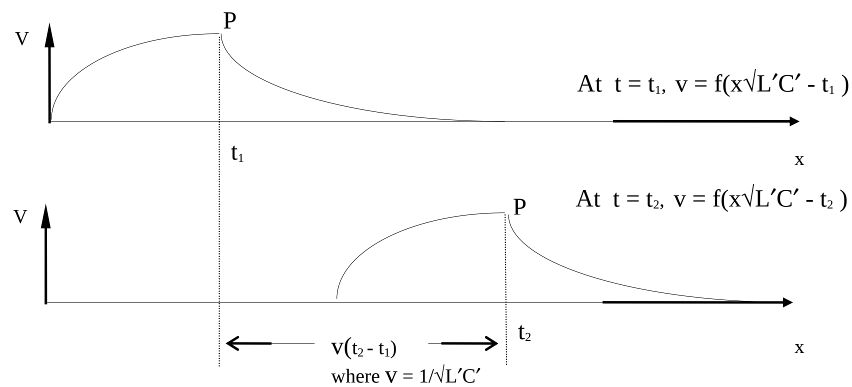



**Figure 8 Voltage travelling wave shown at two successive instants of time**

The differential equation for the instantaneous current is similar to that for the instantaneous voltage, and so we expect its solution to be a corresponding travelling wave. The solution for i(z,t) that corresponds to the example function v(z,t) = f[t − z√(L′C′)] used above, is :

$$i(z,t)=\left(\frac{1}{\sqrt{L'/C'}}\right)f\left[t-z\sqrt{L'C'}\right]$$

Notice that the ratio of v(z,t) to i(z,t) for a travelling wave on a lossless transmission line is a constant, given by [√(L′/C′)]. This constant is called the *characteristic impedance* ($Z_o$) of the line and its units are in Ohms. It could be more meaningfully called the characteristic resistance because for a *lossless line* the impedance would always be purely resistive.

$$Z_o=\sqrt{L'/C'} \quad \text{ohms} \tag{1-7}$$

To verify this expression, let v(z,t) = $f_1$(u) and i(z,t) = $f_2$(u), where u = t – z√(L′C′)]. Using the chain rule,

$$\frac{\partial v(z,t)}{\partial z} = \frac{df_1(u)}{du}\frac{\partial u}{\partial z} = -\sqrt{L'C'}\frac{df_1(u)}{du} \quad \text{and} \quad \frac{\partial i(z,t)}{\partial t} = \frac{df_2(u)}{du}\frac{\partial u}{\partial t} = \frac{df_2(u)}{du}$$

Substitution of these equations into the first equation of equation (1-5), $-\frac{\partial v(z,t)}{\partial z} = L'\frac{\partial i(z,t)}{\partial t}$ produces $\sqrt{L'C'}\frac{df_1(u)}{du} = L'\frac{df_2(u)}{du}$. Integration with respect to *u* and simplifying, results in $\frac{f_1(u)}{f_2(u)} = \frac{v(z,t)}{i(z,t)} = \sqrt{L'/C'}$ which is equation (1-7) and which is purely resistive.

In contrast to the lossless line, impedance cannot be defined when using an *arbitrary complex* waveform ratio of voltage to current, such as $\frac{v(z,t)}{i(z,t)}$, as shown in Figure 8, when considering a *lossy* line. This is because the solution of the differential equations does not produce a ratio of voltage to current which is a constant and independent of time and thus frequency. It is because of this, impedance can *only* be defined for a *single* sinusoidal waveform rather than a spectrum of frequencies. A lossy transmission line in the *sinusoidal steady state*, exhibits a characteristic impedance which is generally a complex function of frequency, and, as will be shown below, which is given by $Z_o \cong \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$.

The remainder of this book, whether stated or not, will infer that *only any single sinusoidal frequency is used* on a *lossy* or on a *lossless* transmission line and all equations are valid for only a single sinusoid. This also includes the characteristic impedance of the lossy transmission line.

The *characteristic* impedance of a transmission line is a function of the cross-sectional dimensions of the line as well as the electrical properties of the insulating material between the conductors. This will be shown below when considering the specific examples of the twin-pair and coaxial transmission lines.

## *1.6 SECONDARY COEFFICIENTS OF A LINE*

The secondary coefficients of a transmission line are its; *characteristic impedance* $Z_o$ *propagation coefficient* γ and *velocity of propagation* $v_p$.

The propagation coefficient γ, has both a real part and an imaginary part; the real part is known as the *attenuation coefficient* α, and the imaginary part is known as the *phase-change coefficient* β.

### 1.6.1 CHARACTERISTIC IMPEDANCE $Z_o$

The characteristic impedance of the lossless line is a real quantity, that is, it is a resistance and is independent of frequency. For lines with loss, the characteristic impedance is generally complex and is not independent of frequency. When a symmetrical T network is terminated in its characteristic impedance $Z_o$, the input impedance of the network is also equal to $Z_o$ . Similarly, if more identical T networks are connected in cascade, the input impedance of the combination will also be equal to $Z_o$. Since a transmission line can be considered to consist of the tandem connection of a very large number of very short lengths Δz of line, as shown in Figure 9 below, the concept of extending T-network sections to represent the characteristic impedance of a transmission line is applicable.

If a uniform transmission line of infinite length is considered, a signal would travel down the line toward infinity. Because there is no end to the line, the signal would never be reflected, and the line can be thought of as absorbing all of the energy fed to it. If the line were to be a **lossless** line, we would be able to measure the impedance of the line and find that it was a pure resistance. This particular resistance is called the characteristic impedance of the line. We could cut the infinite length of lossless line and terminate a finite section of it in this pure resistance. On measuring the impedance of this section we would again find the line to be equal to this characteristic impedance, and that this resistance, would represent the remaining section of the infinite length of line, as it has absorbed all of the energy fed into the line. This is the case where the line is matched to its characteristic impedance. For any value of terminating resistance other than that equal to the characteristic impedance of the line, some or, in the extreme case, all of the signal fed to the line would be reflected. The open and short circuited line represents the extreme case.

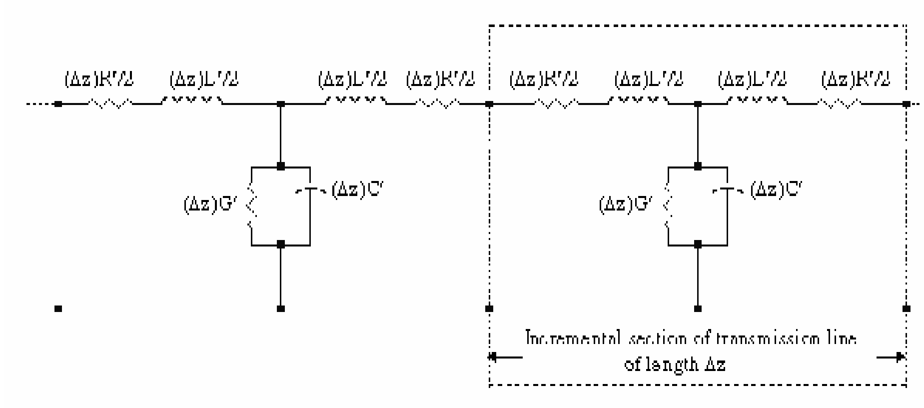



**Figure 9 Representation of a transmission line by a cascade of sections of length $\Delta z$**

If the line is not lossless, but contains a series resistive component $R'$ and a shunt conductance $G'$, then the characteristic impedance is not a pure resistance, but takes on the characteristics of impedance. However, there still would not be any reflections for an infinite length of line. The difference would be, in this case, that the current and the voltage waves would not be in phase. If the line was not of infinite length and terminated in its characteristic impedance, then there would still be no reflections and the conditions would remain the same as for the infinite length of line. If the line is not terminated in its characteristic impedance, then some or all of the incident wave travelling down the transmission line would be reflected back to the generator. For a complex characteristic impedance, the generator impedance is the complex conjugate of this characteristic impedance if maximum power is to be transferred to the line. It is easier to think of the transmission line as being transparent to power transmitted to the load from the generator for maximum power transfer considerations for it is not the transmission line behaving as a generator and delivering power to the load.

Each elemental length $\Delta z$ of line has a total series impedance of $(R' + j\omega L')\Delta z$ and a total shunt admittance of $(G' + j\omega C')\Delta z$. The characteristic impedance $Z_o$ of a line can therefore be defined as being the input impedance of a line of infinite length. Figure 10(a) below, shows a very long length of line; its input impedance is the ratio of the voltage $V_i$, impressed across the input-end terminals of the line to the current $I_i$, that flows into those terminals, that is,

$Z_o = V_i/I_i$ ohms

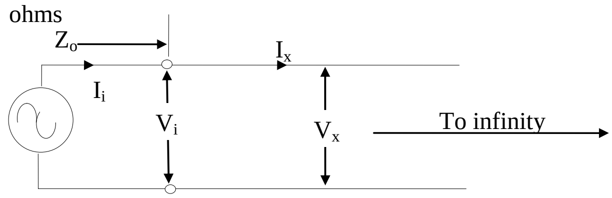



**Figure 10(a) The concept of the characteristic impedance of a transmission line**

Similarly, at any point x along the line the ratio $V_x/I_x$ is always equal to $Z_o$. Suppose that the line is cut a finite distance from its sending-end terminals as shown in the Figure 10(b) below. The remainder of the line is still very long and so the impedance measured at terminals 2-2 is equal to the characteristic impedance. Thus, before the line was cut, terminals 1-1 were effectively terminated in impedance $Z_o$. The condition at the input terminals will not be changed if the terminals 1-1 are terminated in a physical impedance equal to $Z_o$ as shown in Figure 10(c) below. Thus, the characteristic impedance of a transmission line is the input impedance of a line that is itself terminated in its characteristic impedance. This definition ` can be used to derive an expression for the characteristic impedance. A line that is terminated in its characteristic impedance is generally said to be correctly terminated.

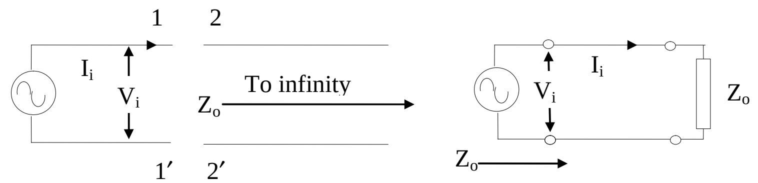



**Figure 10 (b) Characteristic impedance (c) Characteristic impedance**

The characteristic impedance of a single T-network is given by

$$Z_{oT} = \sqrt{\left\{\frac{Z_1^2}{4} + Z_1 Z_2\right\}} \qquad \textbf{(1-8)}$$

This can be derived from Figure 11, below

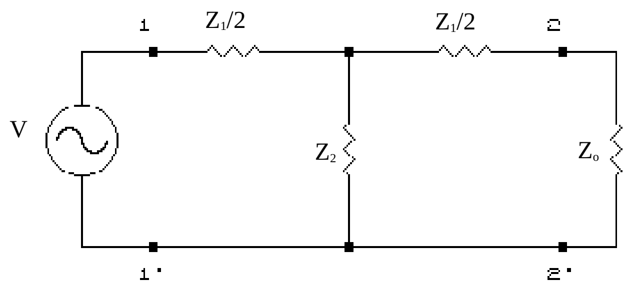



**Figure 11 Single T-network with generator and load**

In the case of an elemental length of transmission line, as shown in Figure 12,

$Z_1 = (R' + j\omega L')\Delta z$ and $Z_2 = 1/[(G' + j\omega C')\Delta z]$

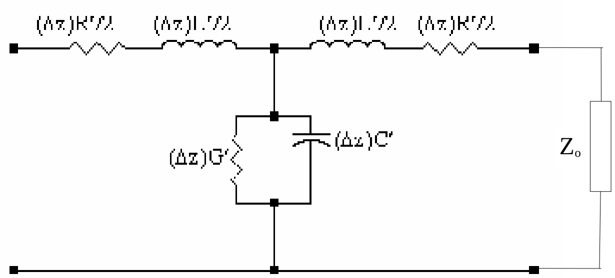



**Figure 12 Calculation of $Z_o$**

Therefore,

$$Z_o = \sqrt{\left\{\frac{\left(R' + j\omega L'\right)^2 \Delta z^2}{4} + \frac{R' + j\omega L'}{G' + j\omega C'}\right\}} \cong \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad \textbf{(1-9)}$$

since $\Delta z^2$ is very small.

*At zero, and very low frequencies* R′ >> ωL′ and G′ >> ωC′ giving $Z_o \cong \sqrt{R'/G'}$ ohms. With an *increase in frequency* both the magnitude and the phase angle of $Z_o$ decrease in value and, when a frequency is reached at which the statements ωL′ >> R′ and ωC′ >> G′ become valid, then as shown above for the lossless line, $Z_o \cong \sqrt{L'/C'}$ ohms.

Figure 13 shows how the magnitude of the characteristic impedance of a transmission line varies with frequency

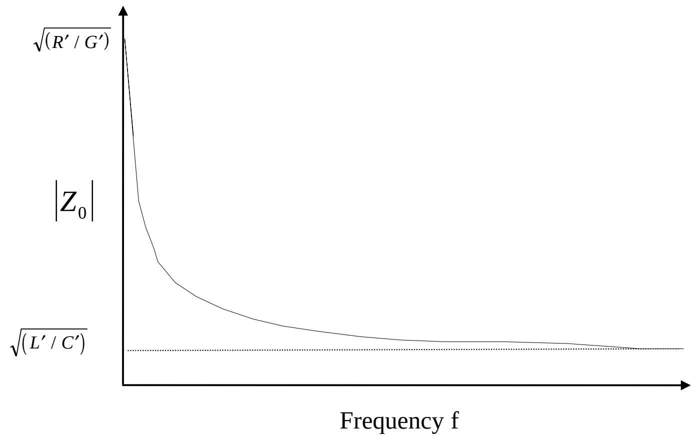



**Figure 13 Variation of $Z_o$ with frequency**

As will be shown for the twin line and the coaxial cable at the higher frequencies, the characteristic impedance of a transmission line is a function of the physical dimensions of the line and the permittivity of the dielectric. At higher frequencies when the approximate expression for $Z_o$ is given by $Z_o \cong \sqrt{L'/C'}$ ohms, the characteristic impedance can be written as:

### *1.6.1.1 TWIN LINE*

$$Z_o \cong \sqrt{L'/C'} = \sqrt{\left\{\frac{\mu_o \log_e d/r}{\pi}\frac{\log_e d/r}{\pi\varepsilon}\right\}} = \frac{276\log_{10} d/r}{\sqrt{\varepsilon_r}} \qquad \textbf{(1-10)}$$

where "d" is the separation of the two conductors, each of radius "r" and the extra component to the inductance due to flux linkages inside the conductors has been ignored..

### *1.6.1.2 COAXIAL LINE*

$$Z_o \cong \sqrt{L'/C'} = \sqrt{\left\{\frac{\mu_o \log_e w/r}{2\pi}\frac{\log_e w/r}{2\pi\varepsilon}\right\}} = \frac{138\log_{10} w/r}{\sqrt{\varepsilon_r}} \qquad \textbf{(1-11)}$$

where "w" is the radius of the outer conductor and "r" is the radius of the inner conductor, and again the extra component to the inductance due to the flux linkages inside the conductors has been ignored.

## 1.6.2 PROPAGATION COEFFICIENT $\gamma'$

The propagation coefficient $\gamma'$, is a special case of the IMAGE TRANSFER COEFFICIENT $\gamma'$ of a network. Referring to Figure 14, the *image transfer coefficient* $\gamma'$, is defined by

$$\gamma' = 1/2\log_e \frac{I_i V_i}{I_L V_L} = \log_e \frac{I_i}{I_L}\sqrt{\frac{Z_i}{Z_L}} \qquad \textbf{(1-12)}$$

where $I_i$ and $V_i$ are the current and voltage at the input terminals and $I_L$ and $V_L$ are the current and voltage at the output terminals. The image transfer coefficient is a complex quantity. The real part of $\gamma'$ is known as the *image attenuation coefficient* $\alpha'$, expressed in Nepers. To

convert Nepers N, into decibels dB, use is made of the relationship,
$x = 20\log_{10} e^N \, dB = 20N \log_{10} e \; dB = 8.6859 N \; dB$. The imaginary part of $\gamma'$ is known as the *image phase-change coefficient* $\beta'$, expressed in radians. Thus, for the general case,
$\gamma' = \alpha' + j\beta'$

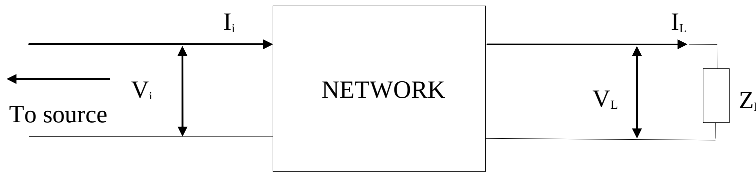



**Figure 14 Calculation of the Image Transfer Coefficient**

For the special case of a symmetrical network, the source impedance is equal to the load impedance, and so equation 1-12, can be written as

$$\gamma' = 1/2 \log_e \frac{I_i V_i}{I_L V_L} = \log_e \frac{I_i}{I_L}\sqrt{\frac{Z_i}{Z_L}} \qquad \textbf{(1-13)}$$

$\gamma'$ is then known as the *propagation coefficient* of the network. Note that the propagation coefficient is dimensionless. If the propagation coefficient is expressed in terms of length, then equation 1-13 becomes

$$\gamma l = 1/2 \log_e \frac{I_i V_i}{I_L V_L} = \log_e \frac{I_i}{I_L} = \log_e \frac{V_i}{V_L} \qquad \textbf{(1-14)}$$

where $l$ is the length of line, or any point along the line from the sending end, usually in kilometers, and $\gamma$ in units of /Km. Equation 1-14 permits the receiving end voltage $V_L$, to be expressed in terms of the sending end voltage $V_i$, on a transmission line, by

$$V_L = V_s e^{-\gamma l} = V_s e^{-(\alpha + j\beta)l} = V_s e^{-\alpha l} \angle -\beta l \qquad \textbf{(1-15)}$$

and similarly for the receiving end current

$$I_L = I_s e^{-\gamma l} = I_s e^{-(\alpha + j\beta)l} = I_s e^{-\alpha l} \angle -\beta l \qquad \textbf{(1-16)}$$

Equation 1-15 shows that the magnitude of the line voltage decreases exponentially with the distance $l$ from the sending end of the line. The phase angle of the line voltage is always a *lagging* angle and this angel increases in direct proportion to the length of the line from the sending end. Similar remarks apply to the magnitude of the line current, as given in equation 1-16.

Consider a uniform lossy transmission line excited by a general instantaneous sinusoidal voltage which is attenuated as it travels down the line away from the source, that is, $v(z,t) = Ve^{-\alpha z}\cos(\omega t - \beta z)$ and current $i(z,t) = I\,e^{-\alpha z}\cos(\omega t - \beta z + \varphi)$. If we express these equations in phasor form

$$v(z,t) = \mathrm{Re}\left[Ve^{j(\omega t - \beta z + j\alpha z)}\right] = \mathrm{Re}\left[\tilde{V}\right] = \mathrm{Re}\left[Ve^{j\omega t}e^{-z(\alpha + j\beta)}\right] = \mathrm{Re}\left[Ve^{j\omega t}e^{-z\gamma}\right]$$

and similarly,

$$i(z,t) = \mathrm{Re}\left[Ie^{j(\omega t - \beta z + j\alpha z + \varphi)}\right] = \mathrm{Re}\left[\tilde{I}\right] = \mathrm{Re}\left[Ie^{j(\omega t + \varphi)}e^{-z(\alpha + j\beta)}\right] = \mathrm{Re}\left[Ie^{j(\omega t + \varphi)}e^{-z\gamma}\right]$$

If we had chosen a sine wave instead of a cosine wave, we would take instead the imaginary part of the phasor $\tilde{V}$ or $\tilde{I}$. So, for a more general form of the instantaneous sinusoidal excitation on the transmission line, which we can take to consider the steady-state solution of the differential equations, we can employ the use of the phasors $\tilde{V}$ and $\tilde{I}$.

Writing equations 1-1 and 1-2 in phasor form, we get

$$-\frac{\partial \tilde{V}}{\partial z} = R'\tilde{I} + L'\frac{\partial \tilde{I}}{\partial t} = (R' + j\omega L')\tilde{I} = Z_1'\tilde{I} \qquad \textbf{(1-17)}$$

and

$$-\frac{\partial \tilde{I}}{\partial z} = G'\tilde{V} + C'\frac{\partial \tilde{V}}{\partial t} = (G' + j\omega C')\tilde{V} = Y_2'\tilde{V} \qquad \textbf{(1-18)}$$

Where $Z_1' \equiv (R' + j\omega L')$ is the series impedance per unit length and $Y_2' \equiv (G' + j\omega C')$ is the shunt admittance per unit length.
Differentiating equation 1-17 with respect to z and substituting $-Y_2'\tilde{V}$ for $\frac{\partial \tilde{I}}{\partial z}$ produces the second-order differential equation

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = Z'Y'\tilde{V} \qquad \textbf{(1-19)}$$

A phasor solution may be written as

$$\tilde{V} = (V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z})e^{j\omega t} \qquad \textbf{(1-20)}$$

where the propagation coefficient γ, is given by

$$\gamma = \sqrt{Z'Y'} = \sqrt{(R' + j\omega L')(G' + j\omega C')} \qquad \textbf{(1-21)}$$

The propagation coefficient is in general complex and in general is given by

$$\gamma = \alpha + j\beta \qquad \textbf{(1-22)}$$

where the propagation coefficient per unit length γ, is given in units of per unit length, the attenuation coefficient per unit length α, in units of Nepers per unit length and the phase-change coefficient β, in radians per unit length.
The term $V_o^+ e^{-\gamma z}$ represents the forward voltage wave on the line and the term $V_o^- e^{\gamma z}$ represents the reflected wave which may exist if the line is improperly terminated and not infinite in length. If the line is infinite in length and thus the reflected wave zero, then $V_o^+ e^{-\gamma z}$ is the same as $V_s$ of equation 1-15. In this case, where the line is not infinite and is improperly terminated there will be exist the reflected voltage wave component $V_o^- e^{\gamma z}$. In the steady state condition, at the sending end of the line (z=0), $\tilde{V} = V_s e^{j\omega t} = (V_o^+ + V_o^-)e^{j\omega t}$ and thus, $V_s = V_o^+ + V_o^-$. That is, the excitation voltage at the sending end of the line must be equal to the sum of the forward and reflected wave. By substituting equation 1-20 into equation 1-17, we find that $\gamma(V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z})e^{j\omega t} = Z_1'\tilde{I}$ and from the definition of γ given by equation 1-21, we find that

$$\tilde{I} = (I_o^+ e^{-\gamma z} - I_o^- e^{\gamma z})e^{j\omega t} \qquad \textbf{(1-23)}$$

Where $I_o^+ = V_o^+ / Z_o$ *and* $I_o^- = V_o^- / Z_o$ and the characteristic impedance $Z_o$ is again given for travelling waves, as shown in equation 1-9, by $Z_o = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$. Note that this expression is not an approximation. This is due to the fact that the differential equations are set up for the limiting case where Δz → , and the lumped " -network" a roach is not used in this derivation.

The reciprocal of $Z_o$ is defined as the characteristic admittance ($Z_o$) of the line and therefore

$$Y_o = 1/Z_o = \sqrt{Y'/Z'} \quad \text{siemens} \qquad \textbf{(1-24)}$$

Since the voltages and currents in equations 1-20 and 1-23 are phasor quantities, the are usually complex quantities and are dependent on the conditions of the generator and load.

Approximate expressions for the attenuation coefficient and the phase-change coefficient can be obtained for different frequency ranges. That is from equation 1-21;

#### *1.6.2.1 Very low frequencies - where R′ >> ωL′ and G′ >> ωC′*

$$\gamma = \sqrt{(R'G')} \tag{1-25}$$

#### *1.6.2.2 Low frequencies - where R′ >> ωL′ and ωC′ >> G′*

$$\gamma \cong \sqrt{(j\omega C'R')} = \sqrt{(\omega C'R' \angle 90^o)} = \sqrt{(\omega C'R')}\angle 45^o$$

which if expressed in rectangular coordinates becomes

$$\gamma \cong \sqrt{\frac{\omega C'R'}{2}} + j\sqrt{\frac{\omega C'R'}{2}} \tag{1-26}$$

#### *1.6.2.3 High frequencies - where ωL′ >> R′ and ωC′ >> G′*

In the microwave region, the above inequalities are almost always true.

Considering, $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$ **(1-21)**

which, after the squaring of $\gamma = \alpha + j\beta$ and equation 1-21 and equating the real and imaginary parts and then solving for β, using the conditions above and the condition that $R'G' << \omega^2 L'C'$, we find

$$\beta \cong \omega\sqrt{L'C'} \quad \text{radians/unit length} \tag{1-27}$$

$$\alpha \cong \frac{R'}{2}\sqrt{\frac{C'}{L'}} + \frac{G'}{2}\sqrt{\frac{L'}{C'}} = \frac{R'}{2Z_o} + \frac{G'Z_o}{2} \quad \text{nepers/unit length} \tag{1-28}$$

For practical transmission lines, and referring to Table 1, $\frac{R'}{2Z_o} >> \frac{G'Z_o}{2}$ and so

$$\alpha \cong \frac{R'}{2Z_o} \quad \text{nepers/unit length} \tag{1-29}$$

$$\gamma \cong \frac{R'}{2}\sqrt{\frac{C'}{L'}} + \frac{G'}{2}\sqrt{\frac{L'}{C'}} + j\omega\sqrt{L'C'} = \frac{R'}{2Z_o} + \frac{G'Z_o}{2} + j\omega\sqrt{L'C'} \tag{1-30}$$

## *1.7 VELOCITY OF PROPAGATION*

Taking the unit of length as the kilometer in this case, the phase velocity $v_p$ in kilometers/second of wave on a line is the velocity with which a sinusoidal wave travels along that line. This may be explained by the fact that a peak of the wave moves at a speed of one wavelength per period or λ kilometers/period, and since a wave may be oscillating at f periods in one second it will travel in one second a distance equal to λf kilometers. That is the peak of the wave moves with a velocity of λf kilometers per second, that is

$v_p = \lambda f$ kilometres/second

In one wavelength, a phase change of 2π radians occurs. Hence the phase change for a length of line, *l* kilometers long, where there are (*l*/λ) wavelengths, will be 2π(*l*/λ) radians. This is equal to the phase change over a length of line *l*. The phase change ***coefficient*** β, of the line is the change in phase which would occur, in this case, in one kilometer. Thus

$$\beta = \frac{2\pi}{\lambda} \text{ radians/kilometer} \tag{1-31}$$

or

$$\lambda = \frac{2\pi}{\beta} \text{ kilometers} \tag{1-32}$$

and

$$v_p = \frac{2\pi f}{\beta} = \omega/\beta \qquad \text{kilometers/second} \tag{1-33}$$

Since, $\beta \cong \omega\sqrt{L'C'}$

$$v_p \cong \frac{1}{\sqrt{L'C'}} \qquad \text{kilometers/second} \tag{1-34}$$

Considering the equations for L′ and C′ in sections 1.3.2 and 1.3.3, where there is magnetic material contained in the dielectric, that is $L' = \frac{\mu_o}{8\pi} + \frac{\mu}{2\pi}\log_e(w/r)$, and where $\mu = \mu_o\mu_r$, where $\mu_r$ is the dimensionless relative permeability of the medium separating the conductors, and μ is the permeability of the medium (Henries/kilometer). It can be seen that the velocity of propagation for a wave travelling down a coaxial cable in kilometers per second is given by

$$v_p = \frac{1}{\sqrt{\mu_o\varepsilon_o}}\frac{1}{\sqrt{\varepsilon_r\mu_r\left[1+\frac{1}{4\mu_r\log_e(w/r)}\right]}} = \frac{c}{\sqrt{\mu_r\varepsilon_r\left[1+\frac{1}{4\mu_r\log_e(w/r)}\right]}} \tag{1-35}$$

where the speed of light c is given by $c = \frac{1}{\sqrt{\mu_o\varepsilon_o}}$ in kilometers per second and where the permittivity ε (farads/meter) is given by $\varepsilon = \varepsilon_o\varepsilon_r$, that is the permittivity of free space $\varepsilon_o$ (farads/kilometer) times the relative permittivity (or dielectric constant) $\varepsilon_r$.

Equation 1.35 shows that the velocity of propagation of a wave travelling down a transmission line can never be equal to the speed of light. This is due to the dielectric constant of the material and the flux linkages set up inside the conductor as well as the permeability of the medium separating the conductors. Since $v_p = \lambda f$ and $c = f\lambda_o$, we find by eliminating f, that the wavelength of the wave travelling down a transmission line, or being ***guided*** by the transmission line, is given by

$$\lambda = \lambda_o\frac{v_p}{c} \cong \frac{\lambda_o}{c\sqrt{L'C'}} = \frac{\lambda_o}{\sqrt{\varepsilon_r\mu_r\left[1+\frac{1}{4\mu_r\log_e(w/r)}\right]}} \qquad \text{kilometers} \tag{1-36}$$

Where $\lambda_o$ is the wavelength of the wave if it were travelling in free space measured in kilometers.

Since $\beta = \omega/v_p$, we find the phase-delay coefficient to be approximated by

$$\beta \cong \frac{\omega}{c}\sqrt{\varepsilon_r\mu_r\left[1+\frac{1}{4\mu_r\log_e(w/r)}\right]} \qquad \text{radians/kilometer} \tag{1-37}$$

showing that the phase-delay coefficient increases with increasing sinusoidal frequency and increasing dielectric constant and permeability of the medium, separating the conductors.

The phase delay of a line is the line length divided by the phase velocity. In other words, the phase delay of a line is the time taken for the peak point on a wave, or some other reference point on the wave, to travel the length of the line

$$\Phi_p = \beta l/\omega \quad \text{seconds} \tag{1-38}$$

That is, for a coaxial cable, $\Phi_p \cong \frac{l}{c}\sqrt{\varepsilon_r\mu_r\left[1+\frac{1}{4\mu_r\log_e(w/r)}\right]}$ seconds **(1-39)**

Any periodic ***non-sinusoidal*** waveform contains sinusoidal components at a number of different frequencies, amplitudes and phase. Each of these components will be propagated along a line with a phase velocity given by equation 1-34. For all of these components to travel with the same velocity and arrive at the far end of the line together, it is necessary for β to be a linear function of frequency. Unfortunately, it is only at microwave frequencies where ωL′ >> R′ and ωC′ >> G′ is satisfied that this can approximately occur. When the component frequencies of a complex wave travel with different velocities, their relative phase relationships will be altered and the resultant waveform will be changed, or distorted. It is customary to consider the ***group velocity*** of a complex wave rather than the phase velocities of its individual components.

**Group velocity** is the velocity with which the envelope of a complex wave is propagated along a line. The group velocity $v_g$ of a line is given by

$$v_g = \frac{d\omega}{d\beta} \quad \text{kilometres/second} \qquad \textbf{(1-40)}$$

The **group delay** of a transmission line is the length *l* kilometers, of the line divided by the group velocity. That is

$$\Phi_g = l\frac{d\omega}{d\beta} \quad \text{seconds} \qquad \textbf{(1-41)}$$

Group delay measures the time taken for the envelope of a complex wave to propagate down the length of the transmission line.

## *1.8 TERMINATED TRANSMISSION LINES*

### 1.8.1 CORRECTLY TERMINATED LINES OR LINES TERMINATED IN $Z_o$

If an infinite line is driven by a single sinusoidal source, having an open-circuit voltage $V_g$ and an internal impedance $Z_g$ only forward travelling waves exist and therefore at z=0, equations 1-20 and 1-23 reduce to $V_s = V_o^+$ and $I_s = I_o^+ = V_o^+ / Z_o$ where the subscript “ ” indicates that z , and the subscript “s” indicates the voltage and current at the sending end, as shown in Figure 15(a). As the line is infinite its impedance is equal to the characteristic impedance of the line.

If the transmission line is broken at z = *l* and the remaining infinite line to the right is replaced by a load impedance equal to its characteristic impedance $Z_o$ , the section of the transmission line to the left of the break will not know the difference. This is because the same conditions apply, in terms of impedance, before and after the break occurs

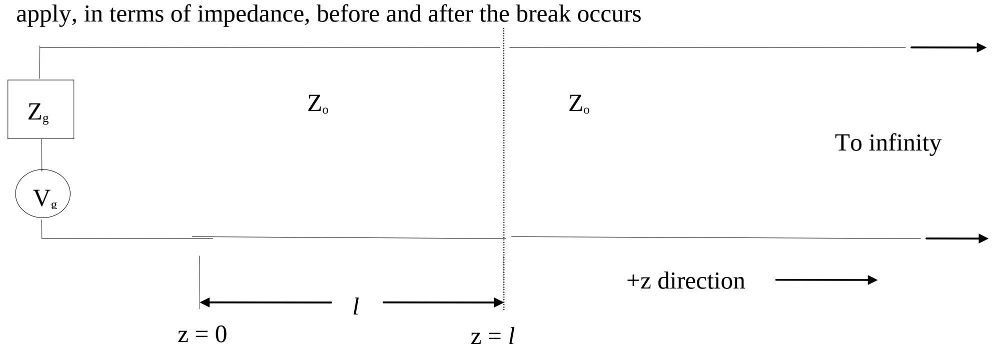



**Figure 15(a) An infinite transmission line**

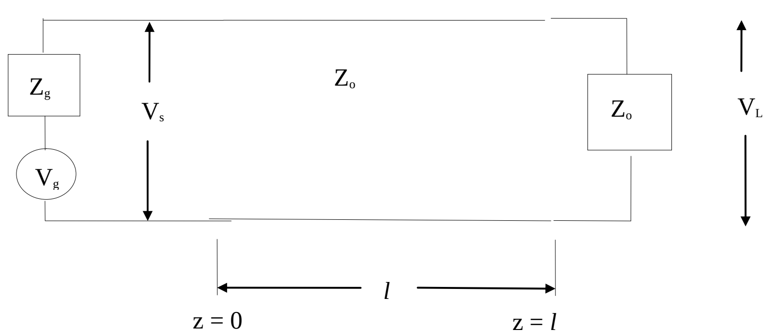



**Figure 15(b) Finite transmission line with load equal to characteristic impedance**

Thus, *a finite length line which is terminated in its characteristic impedance is equivalent to an infinite length line.* The input impedance of the finite length line when terminated in a load impedance equal to $Z_o$ is also $Z_o$ . his condition is often referred to as a "correctly terminated line". f the line is not ter inated in $_o$ , but some other impedance, then the line is said to be incorrectly terminated. It is the solution of this incorrect termination problem which gives rise to much of the material in the remainder of this book. Figure 15(b) above shows the equivalent of the infinite length of transmission line which has been cut at length *l*, and terminated in $Z_o$.. As mentioned previously, the voltage and current phasors given by equations 1-20 and 1-23 are usually complex quantities and are dependent on the conditions of the generator and load. For the reflectionless line, or properly terminated line, the current flowing into the transmission line $I_s$, and the voltage across the input of the transmission line $V_s$ at z = 0, and the current $I_L$, flowing into the load $Z_o$ at z = *l*, as well as the voltage across the load $V_L$, can easily be determined From figure 15(b),
at z = 0

$$V_s = V_o^+ e^{-\gamma z} = V_o^+ = \frac{Z_o}{Z_g + Z_o} V_g \tag{1-42}$$

and $$I_s = I_o^+ e^{-\gamma z} = \frac{V_o^+}{Z_o} e^{-\gamma z} = \frac{V_g}{Z_g + Z_o} \tag{1-43}$$

Whereas, at the load, z = *l*

$$V_L = V_o^+ e^{-\gamma z} = V_o^+ e^{-\gamma l} = \frac{Z_o}{Z_g + Z_o} e^{-\alpha l} V_g \angle -\beta l \tag{1-44}$$

and $$I_L = I_o^+ e^{-\gamma z} = \frac{V_o^+}{Z_o} e^{-\gamma l} = \frac{V_g}{Z_g + Z_o} e^{-\alpha l} \angle -\beta l \tag{1-45}$$

Note that the line length *l*, is arbitrary, so equations 1-44 and 1-45 can represent the voltage and current at any point on the line, because whatever the line length chosen, the impedance to the right of this point on the line is always $Z_o$ .

The **peak** power absorbed by the load is given by $\mathrm{Re}(V_L I_L^*)$, where $I_L^*$ represents the complex conjugate of the load current, and since the characteristic impedance for a lossy infinite length of line is resistive, the power factor is unity. Thus, the **r.m.s.** power is given by

$$P_L = \frac{Z_o}{2} \left| \frac{V_g}{Z_g + Z_o} \right|^2 e^{-2\alpha l} \tag{1-46}$$

where the modulus of the impedance ratio is taken, because $Z_g$ could be complex. For a lossless line $\alpha = 0$, and the exponential term in equation 1-46 reduces to unity.

## 1.8.2 MISMATCHED TRANSMISSION LINES

When a line is mismatched, the load impedance is not equal to the characteristic impedance of the line. When this occurs the load impedance is unable to absorb all of the power incident upon it and so a fraction of this power is reflected back towards the sending end of the line. If the sending-end terminals are matched to the source impedance, all of the reflected energy will be dissipated in the source impedance and there will be no further reflections. If however, the sending end of the line is also mismatched, a fraction of the energy reflected by the load will be further reflected at the sending-end terminals and will be sent towards the load again, causing multiple reflections to take place.

Consider figure 16 below which shows a lossy line whose output terminals are terminated in a load impedance $Z_L$ which is not equal to the characteristic impedance of the line. For ease in description, the generator impedance is taken to equal the characteristic impedance of the line and thus is matched. From equations 1-42 and 1-43, the voltage into the line $V_s$ is half of the generator e.m.f $V_g$ , and the current into the line $I_s$ is given by $V_g/(2Z_o)$. As the voltage and current waves propagate along the line, they experience a change in amplitude and phase. At the end of the line, the forward travelling waves are given by equations 1-44 and 1-45. That is their amplitudes have been reduced by the factor $e^{-\gamma l}$.

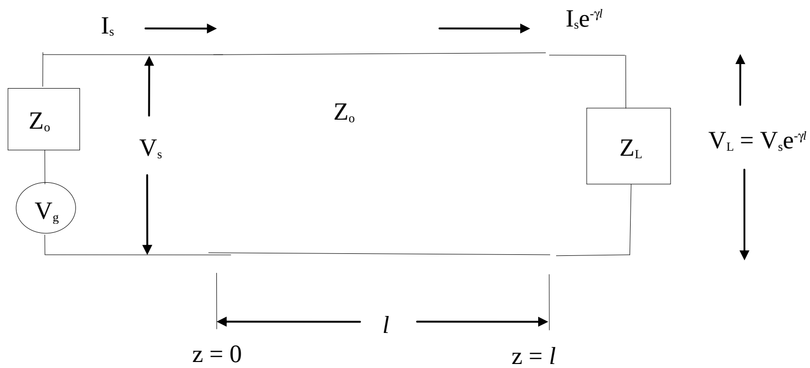



**Figure 16** **Forward currents and voltages on a lossy mismatched transmission line**

### 1.8.2.1 *Reflection coefficient $\rho$*

Before we can describe further what is happening to the reflected waves from the mismatched load, the reflection coefficient ρ is first defined. The reflection coefficient is defined as

$$\rho \equiv \frac{\text{Reflected Voltage or Current at some point z}}{\text{Forward Voltage or Current at some point z}} \quad \textbf{(1-47)}$$

There are two reflection coefficients, one for voltage $\rho_v$, and one for current $\rho_I$. At the load, which is a specific point of z, some fraction of the voltage or current which has arrived at the load is being sent back towards the source in the negative direction This is given by $\rho_{vL}$ the fraction of the voltage and $\rho_{iL}$ the fraction of current. The voltage reflection coefficient, at the load, can be defined with the aid of $V_r = V_o^- e^{+\gamma l}$, the amount of reflected voltage at the load, and $V_L = V_o^+ e^{-\gamma l}$, the amount of voltage arriving at the load, by

$$\rho_{vL} = \rho_L = \frac{V_r}{V_L} = \frac{V_o^- e^{+\gamma l}}{V_o^+ e^{-\gamma l}} = \frac{V_o^-}{V_o^+} e^{2\gamma l} \quad \textbf{(1-48)}$$

Equation 1-48 shows the reflection coefficient $\rho_{vL}$ to be complex due to γ. As the load is complex for a mismatched line, there will also be a phase shift between the forward and reflected voltages $V_o^+$ and $V_o^-$. This phase shift is given by the argument of $\frac{Z_L - Z_o}{Z_L + Z_o}$ as will be shown below.

The current reflection coefficient, at the load, can be similarly be defined from $I_r = I_o^- e^{+\gamma l}$, the amount of reflected current at the load and $I_L = I_o^+ e^{-\gamma l}$ the amount of current arriving at the load, by

$$\rho_{iL} = \frac{I_r}{I_L} = \frac{I_o^- e^{+\gamma l}}{I_o^+ e^{-\gamma l}} = \frac{I_o^-}{I_o^+} e^{2\gamma l} \quad \textbf{(1-49)}$$

The phasor solutions for voltage and current on the transmission line, at any distance z from the source, to the differential equations, were given by equations 1-20 and 1-23, and are again given for convenience below.

$$\tilde{V} = (V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}) e^{j\omega t} \quad \textbf{(1-20)}$$

$$\tilde{I} = (I_o^+ e^{-\gamma z} - I_o^- e^{+\gamma z}) e^{j\omega t} \quad \textbf{(1-23)}$$

These equations show that the reflected current is in antiphase with the reflected voltage. To reconcile this the reflection coefficient of the current is taken as the negative of the reflection coefficient of the voltage

$$\rho_i = -\rho_v \quad \textbf{(1-50)}$$

If equations 1-20 and 1-23 are taken with the voltage reflection coefficients inserted and also, the distance is taken from the load end in the negative z direction by the insertion of d = l - z, then, with the aid of equation 1-47

$$\tilde{V} = (V_o^+ e^{-\gamma l})(e^{\gamma d} + \rho_L e^{-\gamma d}) e^{j\omega t} \quad \textbf{(1-51)}$$

$$\tilde{I} = (I_o^+ e^{-\gamma l})(e^{\gamma d} - \rho_L e^{-\gamma d}) e^{j\omega t} \quad \textbf{(1-52)}$$

Therefore, the phasor solutions for voltage and current on the transmission line, at any distance d from the load are given by equations 1-51 and 1-52.
From equations 1-51 and 1-52, at any point on the line, at some distance d, from the load, the forward voltage is given by $V^+ = V_o^+ e^{-\gamma l} e^{\gamma d}$ and the reflected voltage is given by $V^- = V_o^+ \rho_L e^{-\gamma l} e^{-\gamma d}$, thus in general, the voltage reflection coefficient at any arbitrary point d on the line can be written as

$$\rho = \rho_v = \frac{V^-}{V^+} = \rho_L e^{-2\gamma d} = \rho_L e^{-2\alpha d} \angle -2\beta d \qquad \textbf{(1-53)}$$

which can be represented by

$$\rho = \rho_v = \left|\rho_L\right| e^{-2\alpha d} \angle \varphi - 2\beta d \text{ where } \varphi = \arg(\frac{Z_L - Z_o}{Z_L + Z_o}) \qquad \textbf{(1-54)}$$

Equation 1-53 shows that the reflection coefficient ρ is complex, and that when $\rho_L$ is known, the reflection coefficient ρ can be determined at any point on the line d, from the load.
Figure 17 shows the forward and the reflected currents and voltages at the extreme ends of the line

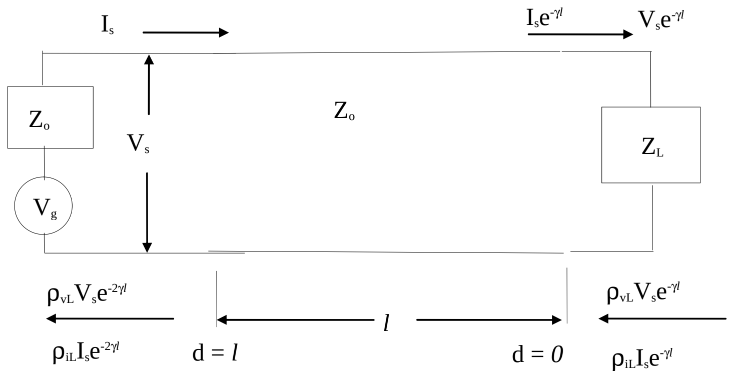



**Figure 17 Reflected currents and voltages on a lossy mismatched transmission line, referenced from the load at some distance d = *l* - z**

From equations 1-51 and 1-52 together with equation 1-54, the phasor solution at any point on the line d, from the load is given by

$$\tilde{V} = (V_o^+ e^{\gamma(d-l)})(1+\rho)e^{j\omega t} \qquad \textbf{(1-55)}$$

$$\tilde{I} = (\frac{V_o^+}{Z_o} e^{\gamma(d-l)})(1-\rho)e^{j\omega t} \qquad \textbf{(1-56)}$$

The load is at d=0, so from equations 1-55 and 1-56, the division of the phasor voltage by the phasor current at d=0 must equal the load impedance, thus

$$Z_L = Z_0 \frac{1+\rho_L}{1-\rho_L} \qquad \textbf{(1-57)}$$

therefore,

$$\rho_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\frac{Z_L}{Z_o} - 1}{\frac{Z_L}{Z_o} - 1} = \frac{\overline{Z}_L - 1}{\overline{Z}_L + 1} = -\rho_{iL} \qquad \textbf{(1-58)}$$

where $\overline{Z}_L$ is defined as the normalized load impedance, which will be used extensively when dealing with Smith charts. Equation 1-58 is particularly useful if the mismatched impedance and characteristic impedance of the line is known, as the voltage reflection coefficient and current reflection coefficient at the load can be determined. Note that the reflection coefficients are usually complex, because the load impedance is complex. If the following admittances are defined, then the voltage reflection coefficient can be defined in terms of admittances. That is $Y_L \equiv 1/Z_L$, $Y_o \equiv 1/Z_o$ and $\overline{Y}_L = Y_L / Y_o$, then

$$\rho_L = \frac{Y_o - Y_L}{Y_o + Y_L} = \frac{1 - \overline{Y}_L}{1 + \overline{Y}_L} \qquad \textbf{(1-59)}$$

the general form for the reflection coefficient at any point on the line can be found from equations 1-55 and 1-56, so that

$$\rho = \frac{Y_o - Y}{Y_o + Y} = \frac{Z - Z_o}{Z + Z_o} \qquad \textbf{(1-60)}$$

### *1.8.2.2 Voltage standing wave ratio, Return Loss and Reflection Loss*

In Figure 18, below the first diagram shows a travelling wave $v(z,t) = V\sin(\omega t - \beta z) = \mathrm{Im}(\widetilde{V})$ moving down a transmission line with no reflections. The distance axis z, and the time axis t, are both shown to visually describe how the wave the wave propagates. The remaining diagrams show cases where reflected waves of different amplitudes exist and combine with the forward travelling wave to form standing waves, as given by $v(z,t) = V_o^+ \sin(\omega t - \beta z) + V_o^- \sin(\omega t + \beta z) = \mathrm{Im}(\widetilde{V})$.

#### 1.8.2.2.1 Voltage standing wave ratio

Figure 18 shows as the amplitude of the reflected wave is increased the maximum and minimum amplitude of the standing wave at a particular point on the line varies with time, that is, at a particular point on the line, the standing wave pulsates. This is particularly apparent in the last of the diagrams, where the reflected wave is the same amplitude as the forward wave. What is not apparent from the diagrams is the scale of the amplitude of the maximum and minimum amplitudes. Again the last of the diagrams shown indicates that there is complete cancellation at one time, where no voltage would exist along the line, and at another time, complete wave addition occurs, where an amplitude exists at some point that is twice the amplitude of the forward wave. For reflections which are not of the same amplitude as the forward wave, a maximum standing wave voltage less than twice the amplitude of the forward wave exists and a minimum standing wave voltage of greater than zero exists. At the point where the forward wave maximum (or minimum) adds to the reflected wave maximum (or minimum), that is where both waves are in phase, we find from

$\widetilde{V} = V^+ + V^-$ that

$$\widetilde{V} = V_o^+ e^{-\gamma l} e^{\gamma d} + V_o^+ \rho_L e^{-\gamma l} e^{-\gamma d} = V_o^+ e^{-\gamma l} e^{\gamma d}(1 + \rho) = V^+(1 + \rho)$$

and so, as $\rho$ is complex, taking its amplitude only,

$$V_{\max(pk)} = V^+(1 + |\rho|) \qquad \textbf{(1-61)}$$

At the point where the forward wave maximum (or minimum) adds to the reflected wave minimum (or maximum), that is where both waves are out of phase,

$$V_{\min(pk)} = V^+(1 - |\rho|) \qquad \textbf{(1-62)}$$

The voltage standing wave ratio (VSWR) is defined as

$$VSWR = S = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\rho|}{1 - |\rho|} \qquad \textbf{(1-63)}$$

This definition of VSWR shows that VSWR takes values between unity and infinity. If no reflections exist, then $|\rho| = 0$, and the VSWR is unity. For full reflection of the travelling wave, $|\rho| = 1$, and the VSWR is infinite. From equation 1-63, $|\rho|$ can be found as

$$|\rho| = \frac{VSWR - 1}{VSWR + 1} = \frac{S - 1}{S + 1} \qquad \textbf{(1-64)}$$

For a lossy line $\rho = \rho_v = \dfrac{V^-}{V^+} = \rho_L e^{-2\gamma d} = \rho_L e^{-2\alpha d} \angle -2\beta d$ **(1-53)**

showing that the reflection coefficient is not constant but reduces with the distance from the load by an amount $e^{-2\alpha d}$. Normally, when measuring VSWR, a lossless line is assumed, so the reflection coefficient is of constant amplitude and equal to the reflection coefficient no matter what the value of d, the distance from the load along the line. As the phase-change coefficient of the reflection coefficient changes by $2\beta$, compared to the travelling wave, which changes only by β, from equation 1-31, the wavelength of the standing wave is half that of the forward travelling wave.

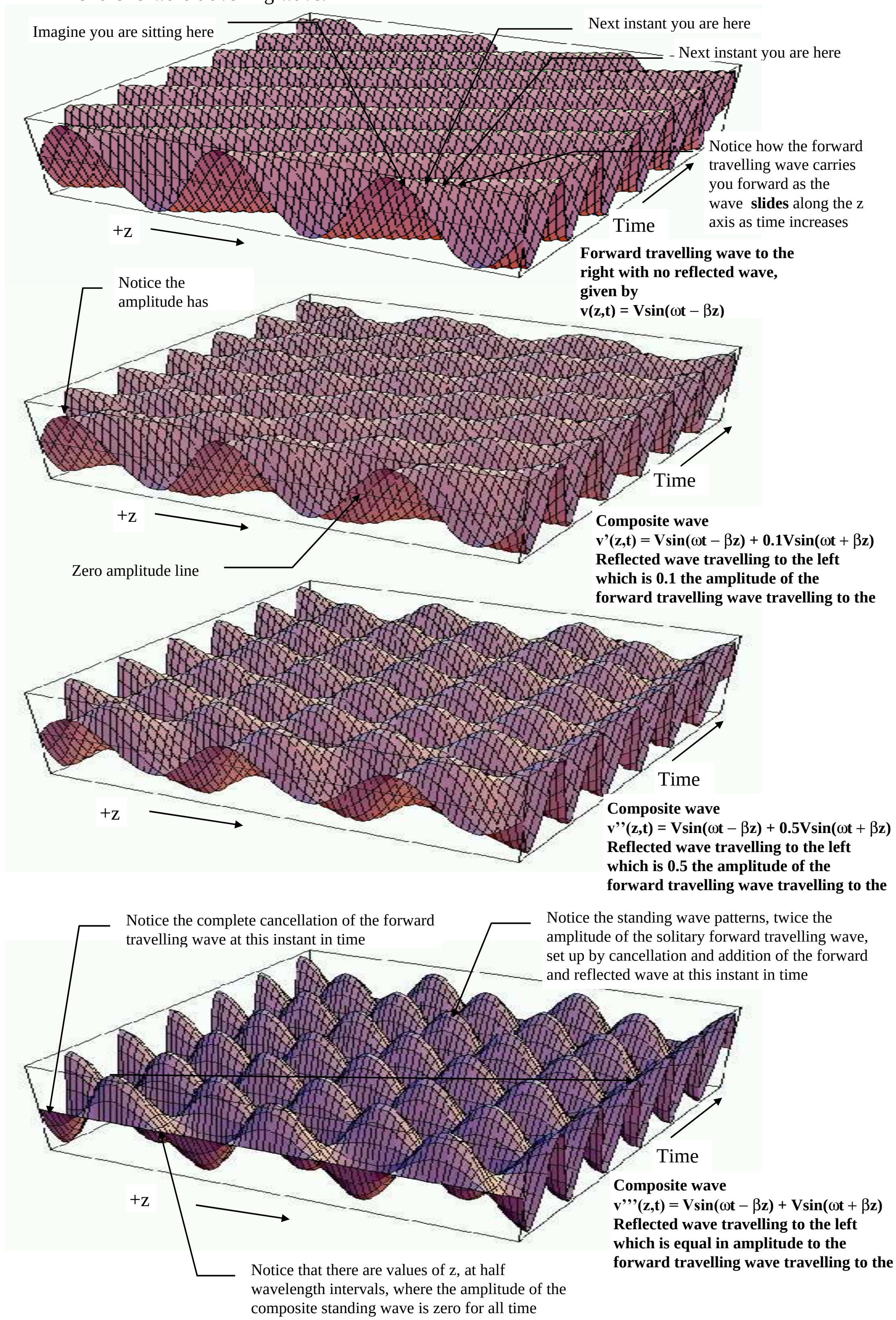



**Figure 18 Composite wave formed from forward travelling wave and its reflected wave**

This is more easily seen looking at the last diagram of figure 18, and following the peaks of the standing wave diagonally across the diagram, as time and distance increases, rather than trying to consider the wave at one instant in time.
When measuring VSWR, the r.m.s. voltage of the standing wave is measured. The r.m.s. voltage is measured over at least one, and usually many wavelengths, so that the measurement becomes independent of time, and dependent on position only. Therefore, when measuring VSWR, the r.m.s. voltage detector is placed where the r.m.s. voltage is a maximum, that is when the forward and reflected wave are in phase, and then moved to a position where the r.m.s. voltage is a minimum, that is where the forward and reflected wave are 180$^{o}$ out of phase.

#### 1.8.2.2.2 Return Loss

Return loss is defined in terms of power. It is the forward wave power divided by the reflected wave power, expressed in dB. That is

$$\text{Return Loss} = 10\log_{10}\left[\frac{\text{incident power}}{\text{reflected power}}\right]\text{dB} \qquad \textbf{(1-65)}$$

The average power flow into an impedance Z is given by

$$P = \mathrm{Re}\left(\frac{\tilde{V}\tilde{I}^*}{2}\right) = VI\cos\theta \qquad \text{watts} \qquad \textbf{(1-66)}$$

Where V and I are peak voltages, θ is the power factor angle and the asterisk denotes the complex conjugate. From equations 1-55 and 1-56

$$\tilde{V} = (V_o^+ e^{\gamma(d-l)})(1+\rho)e^{j\omega t} \qquad \textbf{(1-55)}$$

$$\tilde{I} = \left(\frac{V_o^+}{Z_o} e^{\gamma(d-l)}\right)(1-\rho)e^{j\omega t} \qquad \textbf{(1-56)}$$

$$P = \mathrm{Re}\left[\frac{(V_o^+)^2}{2Z_o} e^{2\alpha(d-l)}(1+\rho-\rho^*-|\rho|^2)\right] \qquad \text{watts} \qquad \textbf{(1-67)}$$

as $\rho\rho^* = |\rho|^2$ . Since $\rho-\rho^*$ is imaginary equation 1-67 reduces to

$$P = \left[\frac{(V_o^+)^2}{2Z_o} e^{2\alpha(d-l)}(1-|\rho|^2)\right] \qquad \text{watts} \qquad \textbf{(1-68)}$$

Although the average power flow along the transmission line is from the generator towards the load, the power at any point along the transmission line of length *l*, is determined from equation 1-68 at the distance d, from the load.

Also as equation 1-68 can be expressed as the multiplication of the sum and difference of the forward and reflected waves, the power dissipated in the load can be expressed in terms of VSWR

$$P = \frac{1}{Z_o}\left[\frac{V_o^+}{\sqrt{2}} e^{\alpha(d-l)}(1-|\rho|)\frac{V_o^+}{\sqrt{2}} e^{\alpha(d-l)}(1+|\rho|)\right] \qquad \text{watts}$$

$$= \mathrm{Re}\left[\frac{1}{Z_o}\left[\frac{V_o^+}{\sqrt{2}} e^{\gamma(d-l)}(1-|\rho|)\frac{V_o^+}{\sqrt{2}} e^{\gamma(d-l)}(1+|\rho|)\right]\right] \qquad \text{watts}$$

which from equations 1-61 and 1-62

$$= \mathrm{Re}\left[\frac{1}{Z_o}\frac{V_{\max(pk)}}{\sqrt{2}}\frac{V_{\min(pk)}}{\sqrt{2}}\right] = \mathrm{Re}\left[\frac{V^2_{\max(pk)}}{2\ \text{VSWR}\ Z_o}\right] \qquad \textbf{(1-69)}$$

As the average forward power on the transmission line is given by

$$P_f = \left[ \frac{(V_o^+)^2}{2Z_o} e^{2\alpha(d-l)} \right] \quad \text{watts} \qquad \textbf{(1-70)}$$

And the reflected power is given by

$$P_r = \left[ |\rho|^2 \frac{(V_o^+)^2}{2Z_o} e^{2\alpha(d-l)} \right] \quad \text{watts} \qquad \textbf{(1-71)}$$

equation 1-68 simply states that the power anywhere on a transmission line, measured from the distance from the load, is the difference between the forward power and the reflected power. Thus the return loss, at any point along the line measured from the load, can be determined from equations 1-70 and 1-71 as

$$\text{Return loss} = 20\log_{10}\left[\frac{1}{|\rho|}\right] \text{dB} \qquad \textbf{(1-72)}$$

For the return loss measured at the load, $|\rho| = |\rho_L|$ in equation 1-72.

As

$$P_r = |\rho|^2 P_f \qquad \textbf{(1-73)}$$

From equation 1-64, the reflected power expressed in terms of the VSWR is given with the aid of equation 1-73 as

$$P_r = \left[\frac{S-1}{S+1}\right]^2 P_f \qquad \textbf{(1-74)}$$

which allows the return loss to be expressed in terms of the VSWR, that is

$$\text{Return loss} = 20\log_{10}\left[\frac{S+1}{S-1}\right] \quad \text{dB} \qquad \textbf{(1-75)}$$

#### 1.8.2.2.3 Reflection loss

Reflection loss is also defined in terms of power. It is the forward wave power divided by the power dissipated in the load, expressed in dB when the generator impedance is equal to the . characteristic impedance of the line, that is

$$\text{Reflection Loss} = 10\log_{10}\left[\frac{\text{incident power}}{\text{power dissipated in the load}}\right] \text{dB} \qquad \textbf{(1-76)}$$

The power which is dissipated in the load, is the difference in forward power and reflected power arriving at the load. Thus, reflection loss can be determined from equations 1-64, 1-70 and 1-68, where $|\rho| = |\rho_L|$

$$\text{Reflection Loss} = 10\log_{10}\left[\frac{1}{1-|\rho_L|^2}\right] \text{dB} = 10\log_{10}\left[\frac{(S+1)^2}{4S}\right] \text{dB} \qquad \textbf{(1-77)}$$

### *1.8.2.3 The impedance of a mismatched line*

The impedance at any point along a mismatched line at a distance d, from the load, is the ratio of the voltage phasor to the current phasor at that point. From equations 1-53, 1-55 and 1-56, we find

$$Z = Z_o \frac{1+\rho}{1-\rho} = Z_o \frac{1+\rho_L e^{-2\gamma d}}{1-\rho_L e^{-2\gamma d}} \quad \text{ohms} \qquad \textbf{(1-78)}$$

As

$$\cosh\gamma d = \frac{e^{\gamma d}+e^{-\gamma d}}{2} = \frac{e^{\gamma d}(1+e^{-2\gamma d})}{2},\ \text{and}\ \sinh\gamma d = \frac{e^{\gamma d}-e^{-\gamma d}}{2} = \frac{e^{\gamma d}(1-e^{-2\gamma d})}{2}$$ a

nd

$$\rho_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\frac{Z_L}{Z_o}-1}{\frac{Z_L}{Z_o}-1} = \frac{\overline{Z}_L - 1}{\overline{Z}_L + 1} = -\rho_{iL} \qquad \textbf{(1-58)}$$

$$Z = Z_o\,\frac{(Z_L+Z_o)+(Z_L-Z_o)e^{-2\gamma d}}{(Z_L+Z_o)-(Z_L-Z_o)e^{-2\gamma d}} = Z_o\,\frac{Z_L(1+e^{-2\gamma d})+Z_o(1-e^{-2\gamma d})}{Z_L(1-e^{-2\gamma d})+Z_o(1+e^{-2\gamma d})}\ \text{ohms}$$

giving

$$Z = Z_o\,\frac{Z_L\cosh\gamma d + Z_o\sinh\gamma d}{Z_L\sinh\gamma d + Z_o\cosh\gamma d} \qquad \text{ohms} \qquad \textbf{(1-79)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L\cosh\gamma d + \sinh\gamma d}{\overline{Z}_L\sinh\gamma d + \cosh\gamma d} \qquad \textbf{(1-80)}$$

The admittance of the mismatched transmission line is given as

$$Y = Y_o\,\frac{Y_L\cosh\gamma d + Y_o\sinh\gamma d}{Y_L\sinh\gamma d + Y_o\cosh\gamma d} \qquad \text{siemens} \qquad \textbf{(1-81)}$$

and its normalized form is given as

$$\overline{Y} = \frac{\overline{Y}_L\cosh\gamma d + \sinh\gamma d}{\overline{Y}_L\sinh\gamma d + \cosh\gamma d} \qquad \textbf{(1-82)}$$

where $Y_L \equiv 1/Z_L$, $Y_o \equiv 1/Z_o$ and $\overline{Y}_L = Y_L/Y_o$,.

The input impedance of the transmission line can be found by setting d = $l$.

#### 1.8.2.3.1 The input impedance of a low-loss line

A low-loss line is one whose attenuation αd, is low enough to permit the following approximations: $\cosh\alpha d \approx 1$, and $\sinh\alpha d \approx \alpha d$.
As $\cosh j\beta d = \cos\beta d$ and $\sinh j\beta d = j\sin\beta d$, equations 1-79 and 1-80 reduce to

$$Z = Z_o\,\frac{Z_L(\cos\beta d + j\alpha d\sin\beta d) + Z_o(\alpha d\cos\beta d + j\sin\beta d)}{Z_L(\alpha d\cos\beta d + j\sin\beta d) + Z_o(\cos\beta d + j\alpha d\sin\beta d)} \quad \text{ohms} \qquad \textbf{(1-83)}$$

$$\overline{Z} = \frac{\overline{Z}_L(\cos\beta d + j\alpha d\sin\beta d) + (\alpha d\cos\beta d + j\sin\beta d)}{\overline{Z}_L(\alpha d\cos\beta d + j\sin\beta d) + (\cos\beta d + j\alpha d\sin\beta d)} \qquad \textbf{(1-84)}$$

#### 1.8.2.3.2 The input impedance of a lossless transmission line

A special case of equations 1-79 and 1-80 occurs when the transmission line is considered to be lossless, because α = 0 giving γ = jβ, which is purely imaginary.

As $\cosh j\beta d = \cos\beta d$ and $\sinh j\beta d = j\sin\beta d$, equations 1-79 and 1-80 reduce to

$$Z = Z_o\,\frac{Z_L\cos\beta d + jZ_o\sin\beta d}{Z_o\cos\beta d + jZ_L\sin\beta d} \qquad \text{ohms} \qquad \textbf{(1-85)}$$

$$\overline{Z} = \frac{\overline{Z}_L \cos\beta d + j\sin\beta d}{\cos\beta d + j\overline{Z}_L \sin\beta d} \qquad \textbf{(1-86)}$$

and similarly for the admittance equations 1-81 and 1-82

$$Y = Y_o \frac{Y_L \cos\beta d + jY_o \sin\beta d}{Y_o \cos\beta d + jY_L \sin\beta d} \quad \text{siemens} \qquad \textbf{(1-87)}$$

$$\overline{Y} = \frac{\overline{Y}_L \cos\beta d + j\sin\beta d}{\cos\beta d + j\overline{Y}_L \sin\beta d} \qquad \textbf{(1-88)}$$

Equations 1-85 through 1-88 show that the impedance and admittance are functions of the phase change coefficient β, and as equation 1-33 shows $\beta = \omega / v_p$, where ω = 2πf, β is a function of frequency f. Therefore, the impedance and admittance of a lossless mismatched line are a function of frequency. It is this property which permits small sections of transmission line to be added to the main mismatched transmission line in order to change its impedance so that it can become matched at a particular frequency. The discussion below, deals with these small sections of the transmission line which are used for matching purposes. his form of matching is called “stub tuning”.

### 1.8.2.4 *Examples of transmission line impedance transformations*

#### 1.8.2.4.1 Short circuited transmission line

When the load is short-circuited $Z_L = 0$, and for the

- **Lossy transmission line**

From equations 1-79 and 1-80

$$Z = Z_o \tanh\gamma d \quad \text{or} \quad \overline{Z} = \tanh\gamma d \qquad \textbf{(1-89)}$$

- **Low-loss transmission line**

From equations 1-83 and 1-84

$$Z = Z_o \frac{(\alpha d\cos\beta d + j\sin\beta d)}{(\cos\beta d + j\alpha d\sin\beta d)} \quad \text{ohms} \qquad \textbf{(1-90)}$$

$$\overline{Z} = \frac{(\alpha d\cos\beta d + j\sin\beta d)}{(\cos\beta d + j\alpha d\sin\beta d)} \qquad \textbf{(1-91)}$$

- **Lossless transmission line**

From equations 1.85 and 1.86

$$Z = jZ_o \tan\beta d \quad \text{or} \quad \overline{Z} = j\tan\beta d \qquad \textbf{(1-92)}$$

and as $\beta = \omega / v_p$,

$$Z = jZ_o \tan(\omega d/v_p) = jZ_o \tan(2\pi d/\lambda) \quad \text{ohms} \qquad \textbf{(1-93)}$$

indicating that as the length of the line d, increases from the terminating short circuit, the pure reactance Z, seen from the input to the transmission line, changes from an inductance to a parallel tuned circuit for the first quarter wavelength. For the next quarter wavelength , the input impedance changes from a capacitance to a series tuned circuit. This cycle is repeated for every half wavelength or for $d = n\lambda/2$, where n goes from 1 to infinity. Thus, it can be seen that the length of a transmission line is more conveniently expressed in wavelengths. This is exactly how the transmission line length is expressed when using the Smith Chart, and in addition, the Smith Chart is expressed in half-wavelength multiples, taking account of the cycling of the reactance changes every half wavelength.
The input impedance of a transmission line with its load short-circuited is a pure reactance whose magnitude and sign are determined by the characteristic impedance and the length of the line.

#### 1.8.2.4.2 Open circuited transmission line

When the load is open-circuited $Z_L \to \infty$, and for the

- **Lossy transmission line**

From equations 1-79 and 1-80

$$Z = Z_o \coth \gamma d \quad \text{or} \quad \overline{Z} = \coth \gamma d \qquad \textbf{(1-94)}$$

- **Low-loss transmission line**

From equations 1-83 and 1-84

$$Z = Z_o \frac{(\cos \beta d + j\alpha d \sin \beta d)}{(\alpha d \cos \beta d + j \sin \beta d)} \quad \text{ohms} \qquad \textbf{(1-95)}$$

$$\overline{Z} = \frac{(\cos \beta d + j\alpha d \sin \beta d)}{(\alpha d \cos \beta d + j \sin \beta d)} \qquad \textbf{(1-96)}$$

- **Lossless transmission line**

From equations 1.85 and 1.86

$$Z = -jZ_o \cot \beta d \quad \text{or} \quad \overline{Z} = -j \cot \beta d \qquad \textbf{(1-97)}$$

and as $\beta = \omega/v_p$ ,

$$Z = -jZ_o \cot(\omega d/v_p) = -jZ_o \cot(2\pi d/\lambda) \quad \text{ohms} \qquad \textbf{(1-98)}$$

showing that as the length of the line d, increases from the terminating short circuit, the pure reactance Z, seen from the input to the transmission line, changes from an inductance to a parallel tuned circuit for the first quarter wavelength. For the next quarter wavelength , the input impedance changes from a capacitance to a series tuned circuit. This cycle is repeated for every half wavelength or for $d = n\lambda/2$, where n goes from 1 to infinity. Thus, it can be seen that the length of a transmission line is more conveniently expressed in wavelengths. This is exactly how the transmission line length is expressed when using the Smith Chart, and in addition, the Smith Chart is expressed in half-wavelength multiples, taking account of the cycling of the reactance changes every half wavelength.
The input impedance of a transmission line with its load short-circuited is a pure reactance whose magnitude and sign are determined by the characteristic impedance and the length of the line.

#### 1.8.2.4.3 λ/4 Length of transmission line

When the load is not correctly terminated, that is $Z_L \neq Z_o$, and $d = \lambda/4$, for the

- **Lossy transmission line**

From equations 1-22, 1-79 and 1-80, as $\cosh j\beta d = \cos \beta d$ and $\sinh j\beta d = j \sin \beta d$ ,
$\gamma d = (\alpha + j\beta)d = (\alpha\lambda/4 + j\pi/2)$

$$Z = Z_o \frac{Z_o \cosh(\alpha\lambda/4) + Z_L \sinh(\alpha\lambda/4)}{Z_o \sinh(\alpha\lambda/4) + Z_L \cosh(\alpha\lambda/4)} \quad \text{ohms} \qquad \textbf{(1-99)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L \sinh(\alpha\lambda/4) + \cosh(\alpha\lambda/4)}{\overline{Z}_L \cosh(\alpha\lambda/4) + \sinh(\alpha\lambda/4)} \qquad \textbf{(1-100)}$$

- **Low loss transmission line**

From equations 1-99 and 1-100, and as $\cosh \alpha d \approx 1,$ and $\sinh \alpha d \approx \alpha d$.

$$Z = Z_o \frac{Z_L(\alpha\lambda/4) + Z_o}{Z_L + Z_o(\alpha\lambda/4)} \quad \text{ohms} \qquad \textbf{(1-101)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L(\alpha\lambda/4)+1}{\overline{Z}_L + \alpha\lambda/4} \qquad \textbf{(1-102)}$$

- **Lossless transmission line**

A special case of equations 1-101 and 1-102 occurs when the transmission line is considered to be lossless, because α = 0 giving

$$Z = \frac{Z_o^2}{Z_L} \quad \text{ohms} \qquad \textbf{(1-103)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{1}{\overline{Z}_L} \qquad \textbf{(1-104)}$$

Equations 1-103 and 1-104 show that a quarter wavelength line transforms a small or large value of load impedance into a large or small value of impedance as seen at the input of the line. If the load is purely capacitive, then the quarter wavelength line appears to be inductive. Similarly, if the load is inductive, the line appears capacitive. If the load is a parallel tuned circuit, then the input appears as a series tuned circuit, and similarly, if the load is a series tuned circuit, then the input to the line appears as a parallel tuned circuit. The quarter wavelength line has the properties of an impedance transformer and is ideal for matching a resistive load to a generator at a specific frequency. Equation 1-103 permits the calculation of the required characteristic impedance of a line $Z_o$, which is to be used to match a generator resistance R, to a load resistance $R_L$, that is

$$Z_o = \sqrt{RR_L} \quad \text{ohms} \qquad \textbf{(1-105)}$$

#### 1.8.2.4.4 λ/2 Length of transmission line

When the load is not correctly terminated, that is $Z_L \neq Z_o$, and $d = \lambda/2$, for the

- **Lossy transmission line**

From equations 1-22, 1-79 and 1-80, as $\cosh j\beta d = \cos\beta d$ and $\sinh j\beta d = j\sin\beta d$, $\gamma d = (\alpha + j\beta)d = (\alpha\lambda/2 + j\pi)$

$$Z = Z_o\frac{Z_o \sinh h(\alpha\lambda/4) + Z_L\cosh(\alpha\lambda/4)}{Z_o\cosh(\alpha\lambda/4) + Z_L\sinh(\alpha\lambda/4)} \quad \text{ohms} \qquad \textbf{(1-106)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L\cosh(\alpha\lambda/4) + \sinh(\alpha\lambda/4)}{\overline{Z}_L\sinh(\alpha\lambda/4) + \cosh(\alpha\lambda/4)} \qquad \textbf{(1-107)}$$

- **Low loss transmission line**

From equations 1-106 and 1-107, and as $\cosh\alpha d \approx 1$, and $\sinh\alpha d \approx \alpha d$.

$$Z = Z_o\frac{Z_o(\alpha\lambda/4) + Z_L}{Z_o + Z_L(\alpha\lambda/4)} \quad \text{ohms} \qquad \textbf{(1-108)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{(\alpha\lambda/4) + \overline{Z}_L}{1 + \overline{Z}_L(\alpha\lambda/4)} \qquad \textbf{(1-109)}$$

- **Lossless transmission line**

A special case of equations 1-108 and 1-109 occurs when the transmission line is considered to be lossless, because α = 0 giving

$$Z = Z_L \quad \text{ohms} \qquad \textbf{(1-110)}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \overline{Z}_L \tag{1-111}$$

Equations 1-110 and 1-111 show that for a half wavelength line the input impedance equals the load impedance. For a lossless transmission line, it can be seen that the impedance repeats itself, whatever it is, each half wavelength along the line. This is shown by substituting $\beta d = n\pi$, where n is any number from 1 to infinity, into equation 1-85. The result is equation 1-110.

### 1.8.2.4.5 λ/8 Length of transmission line

When the load is not correctly terminated, that is $Z_L \neq Z_o$, and $d = \lambda/8$, for the

- **Lossy transmission line**

From equations 1-22, 1-79 and 1-80, as $\cosh j\beta d = \cos\beta d$ and $\sinh j\beta d = j\sin\beta d$, $\gamma d = (\alpha + j\beta)d = (\alpha\lambda/8 + j\pi/4)$

$$Z = Z_o \frac{Z_o[\sinh h(\alpha\lambda/8) + j\cosh(\alpha\lambda/8)] + Z_L[\cosh(\alpha\lambda/8) + j\sinh(\alpha\lambda/8)]}{Z_L[\sinh h(\alpha\lambda/8) + j\cosh(\alpha\lambda/8)] + Z_o[\cosh(\alpha\lambda/8) + j\sinh(\alpha\lambda/8)]} \text{ ohms} \tag{1-112}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L[\cosh(\alpha\lambda/8) + j\sinh(\alpha\lambda/8)] + [\sinh h(\alpha\lambda/8) + j\cosh(\alpha\lambda/8)]}{\overline{Z}_L[\sinh h(\alpha\lambda/8) + j\cosh(\alpha\lambda/8)] + [\cosh(\alpha\lambda/8) + j\sinh(\alpha\lambda/8)]} \tag{1-113}$$

**Low loss transmission line**

From equations 1-112 and 1-113, and as $\cosh\alpha d \approx 1$, and $\sinh\alpha d \approx \alpha d$.

$$Z = Z_o \frac{Z_o[(\alpha\lambda/8) + j] + Z_L[1 + j\alpha\lambda/8]}{Z_L[\alpha\lambda/8 + j] + Z_o[1 + j\alpha\lambda/8]} \quad \text{ohms} \tag{1-114}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L[1 + j\alpha\lambda/8] + [(\alpha\lambda/8) + j]}{\overline{Z}_L[\alpha\lambda/8 + j] + [1 + j\alpha\lambda/8]} \tag{1-115}$$

- **Lossless transmission line**

A special case of equations 1-114 and 1-115 occurs when the transmission line is considered to be lossless, because α = 0 giving

$$Z = Z_o \frac{Z_L + jZ_o}{Z_o + jZ_L} = Z_o \angle \tan^{-1}\left(\frac{Z_o}{Z_L} - \frac{Z_L}{Z_o}\right) \quad \text{ohms} \tag{1-116}$$

the normalized impedance for a mismatched transmission line is given as

$$\overline{Z} = \frac{\overline{Z}_L + j}{1 + j\overline{Z}_L} = \angle \tan^{-1}\left(\frac{1}{\overline{Z}_L} - \overline{Z}_L\right) \tag{1-117}$$

Equations 1-116 and 1-117 show that for a lossless eighth wavelength line, the magnitude of the input impedance equals the magnitude of the load impedance.

### *1.8.2.5 Transmission lines used as components*

The design of circuits using discrete components becomes difficult as the frequency is increased into the ultra-high frequency (UHF) band and above, due to stray capacitance effects. These effects make the design of inductors and precision capacitors difficult and place a limit on the physical size of resistors and active components. Use of a transmission line can be made to simulate a capacitors, an inductor, a series tuned circuit or a parallel tuned circuit.

#### 1.8.2.5.1 Lossless transmission lines used as components

The input impedance Z of a lossless transmission line short-circuited at its load terminals is given by equation 1-93, that is $Z = jZ_o \tan(2\pi d/\lambda)$. This line has an input impedance which is a pure reactance. If the electrical length d, of the line is less than $\lambda/4$, the line will simulate an inductor, because, as can be seen from Figure 19, $\tan(2\pi d/\lambda)$ is a real positive number.

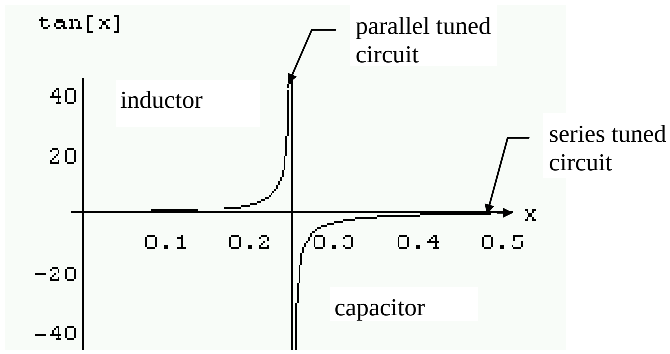



**Figure 19** **Plot of $\tan(2\pi d/\lambda)$ against $d/\lambda$ for a length of line $d \le \lambda/2$ where x =$d/\lambda$.**

If the length of line d, is less than $\lambda/2$, but greater than $\lambda/4$ a capacitor will be simulated because $\tan(2\pi d/\lambda)$ is a real negative number.
If the line length is $\lambda/4$, a parallel tuned circuit is simulated, and similarly, if the line length is $\lambda/2$, a series tuned circuit is similated.

**EXAMPLE 1**

Given a reactance $Z^*$, with which a lossless short circuited transmission line is to series resonate, the length of line can be calculated. To series resonate with the given reactance, which is say a capacitor, the line must have the complex conjugate reactance $Z$, which in this case would be inductive. From equation 1-93, that is $j|Z| = jZ_o \tan(2\pi d/\lambda)$, we find

$\frac{|Z|}{Z_o} = x = \tan(2\pi d/\lambda)$, giving

$$2\pi d\Big/\lambda = \arctan(x) = \frac{\pi}{2} - \frac{1}{x}\sum_{k=o}^{\infty}\frac{(-1)^k}{(2k+1)x^{2k}}, \qquad |x| \ge 1$$

Hence,

$$d = \frac{\lambda}{2\pi}\left(\frac{\pi}{2} - \frac{1}{x}\sum_{k=o}^{\infty}\frac{(-1)^k}{(2k+1)x^{2k}}\right) \approx \frac{\lambda}{4}\left(1 - \frac{2}{\pi x}\right) = \frac{\lambda}{4}\left(1 - \frac{2Z_o}{\pi|Z|}\right) \qquad \textbf{(1-118)}$$

If the line was to be capacitive instead of inductive, as in the case above, the value of x would be negative in equation 1-118. Notice that the length of the line is $2Z_o/\pi|Z|$ less than a quarter wavelength line. For the line being capacitive, it would be $2Z_o/\pi|Z|$ greater than a quarter wavelength line.

#### 1.8.2.5.2 Low loss transmission line used as components

Normally, lengths of line longer than $\lambda/2$ are not employed due to the line losses not being negligibly small.
From equation 1-90, for a short-circuited transmission line

$$Z = Z_o \frac{(\alpha d + j\tan 2\pi d/\lambda)}{(1 + j\alpha d \tan 2\pi d/\lambda)}, \text{ for } \alpha d \ll 1,$$

$$\overline{Z} = \frac{\alpha d(1+\tan^2 2\pi d/\lambda)}{1+(\alpha d)^2 \tan^2 2\pi d/\lambda} + j\frac{[1-(\alpha d)^2]\tan 2\pi d/\lambda}{1+(\alpha d)^2 \tan^2 2\pi d/\lambda}$$

$$Z \approx \frac{Z_o}{\dfrac{\cot^2 2\pi d/\lambda}{\alpha d} + \alpha d} + \left[\frac{Z_o}{1+(\alpha d \tan 2\pi d/\lambda)^2}\right] j\tan 2\pi d/\lambda \qquad \textbf{(1-119)}$$

showing that if equation 1-119 is compared with equation 1-93, the equation for a lossless line, for large $\tan 2\pi d/\lambda$ and small $\alpha d$, there is a significant resistive component with the pure reactance and a reduction in the size of the reactance,. Therefore, for very low loss lines, the attenuation factor cannot simply be ignored.
*At a quarter wavelength*, the ideal line is a parallel resonant circuit. For the non-ideal case, there exists an equivalent parallel resistance placed across the circuit. This limits the Q of the parallel resonant circuit. From equation 1-104, for a quarter wavelength line, the equivalent parallel resistance of the parallel tuned circuit appears as a series resistance of a series tuned circuit looking into the input of the line, therefore inductive reactance divided by the series resistance gives the Q of the circuit. From equation 1-119, the series resistance can be found from the real term, and the inductive reactance from the imaginary term, giving

$$Q \approx \frac{1}{\alpha d \tan 2\pi d/\lambda} \qquad \textbf{(1-120)}$$

Equation 1-120 shows that even for a low loss line where $\alpha d$ is small, if the line is to resonate with a large capacitive reactance, which makes $\tan 2\pi d/\lambda$ become large, the Q of the circuit can be reduced to a low value.

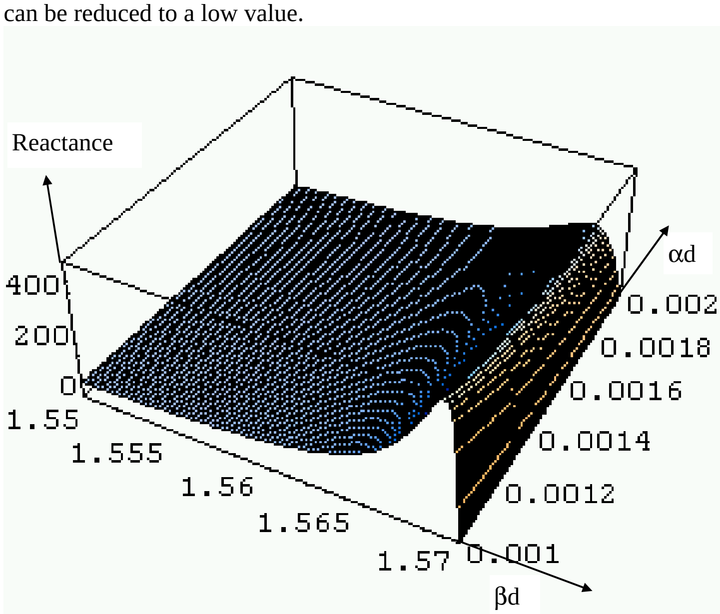



**Figure 20 Reactance for different values of αd and βd**

Figure 20 shows how the reactive component of equation 1-119 maximizes at particular values of βd and remains almost independent of the value of αd, although αd does determine the maximum value which the reactance will take. When α is zero, the reactance goes to infinity for the short-circuited quarter-wavelength line, as expected.

By differentiating the normalized reactance part of the equation of equation 1-119, with respect to the distance d, the condition for finding the maximum value of the reactance can be found. The assumptions make are $\alpha d << 1$ and $\alpha d \tan\frac{2\pi d}{\lambda} \approx 1$ and $\pi/2 > \beta d > 1.565$, giving.

$$X \approx \frac{Z_o}{2\alpha d}\cdot\frac{1}{\alpha d \tan 2\pi d/\lambda} \approx \frac{Z_o}{2\alpha d} \qquad \textbf{(1-121)}$$

It is interesting to note that when maximizing the reactance, the value of Q in equation 1-120 becomes approximately unity, inferring that the resistive component increases to the same value as the magnitude of the reactance. To obtain a value of Q higher than unity, the reactance value must be less than maximum.

For a short-circuited transmission line, where the length is *less than a quarter of a wavelength,* the Q of the circuit is not given by equation 1-20, because $\tan\beta d$, is not large. This can be seen from figure 20, as $\beta d < 1.55$ or $d < 0.246\lambda$. In this case, the line starts to look inductive, as seen from figure 19, rather than a parallel tuned circuit. The Q of the inductance can be determined from equation 1-119,

$$\overline{Z} = \frac{\alpha d(1+\tan^2 2\pi d/\lambda)}{1+(\alpha d)^2\tan^2 2\pi d/\lambda} + j\frac{[1-(\alpha d)^2]\tan 2\pi d/\lambda}{1+(\alpha d)^2\tan^2 2\pi d/\lambda} \qquad \textbf{(1-119)}$$

as

$$Q = \frac{[1-(\alpha d)^2]\tan 2\pi d/\lambda}{\alpha d(1+\tan^2 2\pi d/\lambda)} \qquad \textbf{(1-122)}$$

For the case where $\alpha d << 1$, $\tan 2\pi d/\lambda \approx 2\pi d/\lambda$ and $(\tan 2\pi d/\lambda)^2 << 1$, equation 1-122 reduces to

$$Q \approx \frac{2\pi}{\alpha\lambda} = \frac{\beta}{\alpha} \qquad \textbf{(1-123)}$$

Equation 1-123 shows that for small $\alpha$, and high frequencies (small $\lambda$), the Q of the simulated inductor can become very large.
The value of the simulated inductor is found to be from equation 1-119 and 1-33 as

$$L = \frac{\beta d Z_o}{\omega} = \frac{dZ_o}{v_p} \qquad \textbf{(1-124)}$$

and its equivalent series resistance

$$R_s = \alpha d Z_o \qquad \textbf{(1-125)}$$

Similar results can be derived for the capacitance, where the line length is greater than a quarter wavelength and less than half a wavelength.

# 2. THE SMITH CHART

The Smith chart, shown in Figure 22 below, provides a graphical procedure for the solution of transmission line problems, such as the determination of the impedance or impedances and their placement position along a mismatched transmission line, in order to match that line. Developed by P.H.Smith[1,2,3] , the chart is a plot of normalized impedance $\overline{Z}$ , and/or normalized admittance $\overline{Y}$ , against the polar coordinate plot of magnitude and angle of the voltage reflection coefficient ρ. Smith developed the present form of the chart through an interative process in his attempts to simplify the tedious work of matching open-wire telephone and telegraph lines. Prior to its development, transmission line impedance analysis was carried out using the rigorous hyperbolic transmission line equations, presented in chapter 1, and experimental work focussed on a trial and error approach perhaps supplemented with transmission line measurements of standing wave ratios. Thus, the ability to represent and manipulate an impedance vector on a single graphical chart was a significant step forward at that time.

The pair of coordinates used for plotting the complex values of impedance and admittance accommodates all possible values, and permit impedance to be converted to its equivalent admittance, and vice-versa. The advantages of using a Smith chart for the solution of transmission line and waveguide problems is the ease in arriving at a graphical solution over the alternative computational methods and the intuitive understanding of the effect that individual variables have on the final solution.

The Smith chart consists of a real axis with values which vary from zero to infinity, with unity at the centre, and a *series of circles* centred on the real axis which represent the *real* parts of the normalized impedances, and a *series of arcs* of circles that start from the infinity point on the real axis which represent the *imaginary* parts of the normalized impedances. In addition to the circles and arcs, the edge of the chart is marked with scales of the angle of the reflection coefficient (in degrees), and the distance (in wavelengths) along the line from either the source or the generator. Movement around the edge of the chart in the *clockwise* direction corresponds to movement along the transmission line towards the *generator* . Movement around the edge of the chart in the *anti-clockwise* direction corresponds to movement along the transmission line towards the *load*. One complete circle around the chart represents a movement of half a wavelength along the transmission line.

This chapter will discuss the advantages of the form which the Smith chart takes over the other possible forms of charts. It will develop the mathematics involved in its design, and it will describe how it is to be used by developing its construction.

## 2.1 Impedance and Reflection Charts

Both the impedance Z, and the reflection coefficient ρ, are expressed in complex numbers

$$Z = R + jX \qquad \rho = \rho_r + j\rho_j$$

where R and X are the resistive and reactive parts of the complex impedance and $\rho_r$ and $\rho_j$ are the real and imaginary parts of the complex reflection coefficient. Figure 21 shows the polar and the rectangular representations of the two complex quantities Z and ρ.

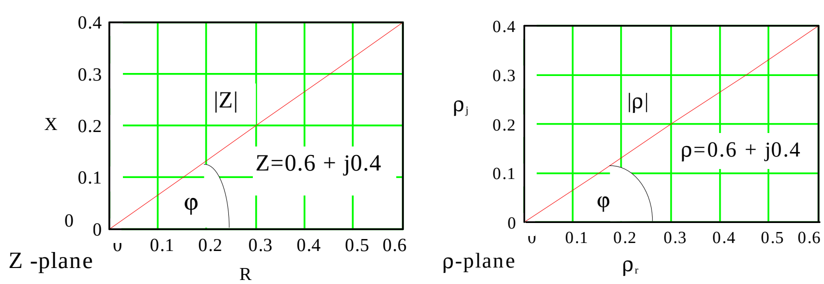



**Figure 21 Rectangular and polar planes of the impedance and reflection coefficient**

Although the impedance and the reflection coefficient look similar on their complex planes there is a difference. To represent all values of impedances, the complex plane must extend to -∞ to +∞ in both the real and imaginary directions. Whereas, for the reflection coefficient, to represent all values of reflection coefficients, the complex plane only needs to extend to +1 and -1 in both the imaginary and real directions. This is because the magnitude of the reflection coefficient never exceeds unity

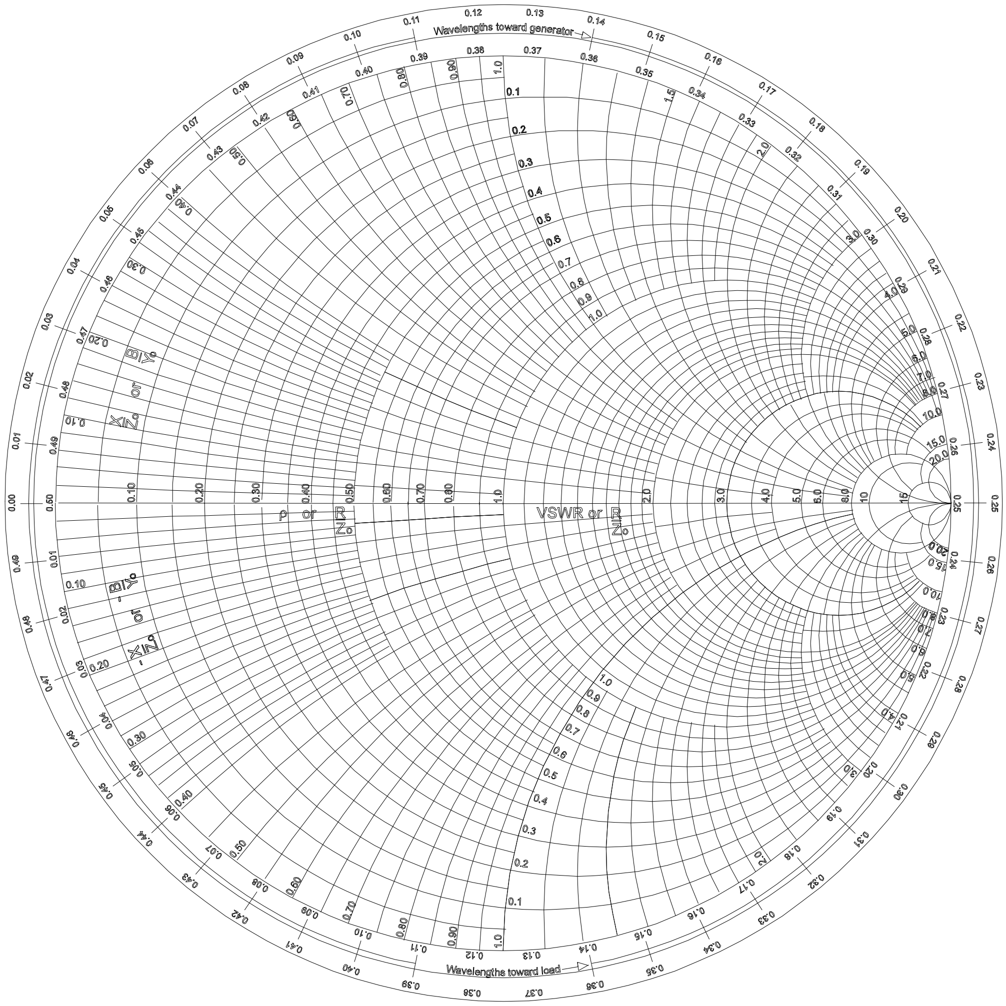



**Figure 22** **The Smith Chart**

The use of the complex reflection coefficient plane in polar form, as shown in Figure 23, on a single chart is an advantage, as was pointed out by P.H. Smith. However, there is still the problem of how to represent the range of complex impedance values on the same chart as the reflection coefficient. The solution of this problem forms the mathematical basis of the Smith chart

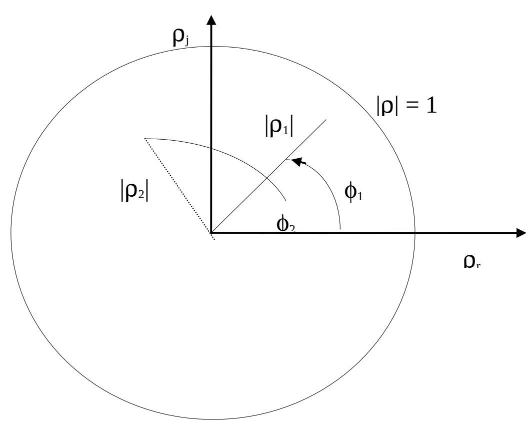



**Figure 23** **The reflection coefficient plane, where all values of the reflection coefficient are contained within a circle of radius** |ρ|

## 2.2 Conformal Transformation

The analytic basis of the Smith chart is a conformal bilinear transformation for which all data values approaching infinity on the complex impedance plane map to a single uniquely defined point. Thus the rectangular impedance plane extending to plus and minus infinity on both the resistive and reactive axis transforms into two families of circles, as shown in figure 24. The conformal bilinear transformation is used because of equation 1.78, which shows that the impedance of a mismatched line can be related to the reflection coefficient through this transformation. That is,

$$\overline{Z} = \frac{1+\rho}{1-\rho} = \frac{Z}{Z_0} = \overline{R} + j\overline{X}$$

(1-78)

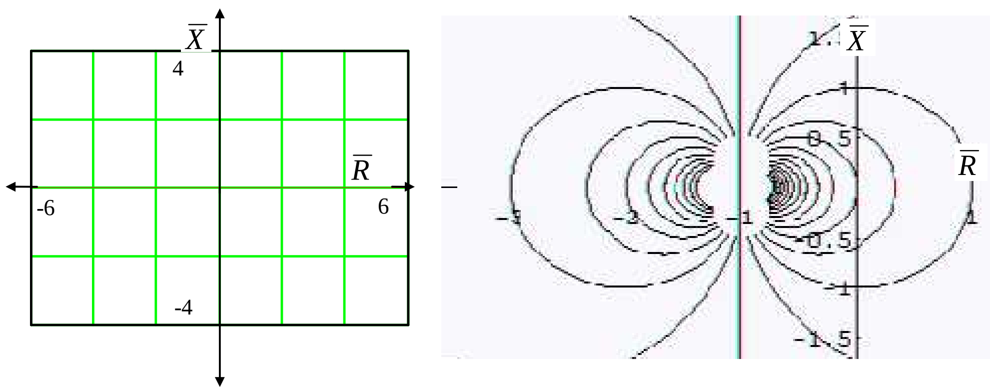



**Figure 24 Rectangular Impedance Plane and Bilinear Transformed Impedance Plane**

As $\rho = \rho_r + j\rho_j$ where $\rho_r$ and $\rho_j$ are ther real and imaginary parts of ρ, and $\overline{R}$ and $\overline{X}$ are the normalized resistance and reactance of the input impedance looking into the transmission line,

$$\overline{R} + j\overline{X} = \frac{1+\rho}{1-\rho} = \frac{1+\rho_r + j\rho_j}{1-(\rho_r + j\rho_j)} \qquad \textbf{(2-1)}$$

By separating the real and imaginary parts of equation 2.1,

$$\overline{R} = \frac{1-\rho_r^2 - \rho_j^2}{(1-\rho_r)^2 + \rho_j^2} \qquad \textbf{(2-2)}$$

$$\overline{X} = \frac{2\rho_j}{(1-\rho_r)^2 + \rho_j^2} \qquad \textbf{(2-3)}$$

Treating equations 2-2 and 2-3 as parametric equations in $\rho_r$ and $\rho_j$, the bilinear transformed impedance plane is plotted, as shown in Figure 24. However, this diagram is still an impedance diagram and is difficult to relate to the reflection coefficient ρ. What is required is a reflection coefficient diagram in which normalized resistance $\overline{R}$ , and normalized reactance $\overline{X}$ can be directly related to the reflection coefficient by superimposing Figure 23.
To plot values of normalized impedances on the reflection coefficient diagram (Figure 23), we need to separate out normalized resistance from normalized reactance, that is, to identify constant resistance and constant reactance curves, and separately associated each with $\rho_r$ and $\rho_j$. This is obtained using equations 2-2 and 2-3, by rearrangement of each of these equations, giving

$$\left\{\rho_r - \frac{\overline{R}}{\overline{R}+1}\right\}^2 + \rho_j^2 = \left\{\frac{1}{\overline{R}+1}\right\}^2 \qquad \textbf{(2-4)}$$

which shows a family of circles with centre $\left\{\frac{\overline{R}}{\overline{R}+1},0\right\}$, and radius $\left\{\frac{1}{\overline{R}+1}\right\}$ associated with $\rho_r$ and $\rho_j$,

These are the constant resistance circles, because if $\overline{R}$ is set to a constant, a circle can be drawn. For different values of $\overline{R}$ , different circles with different radii and different centres along the $\rho_r$ axis can be drawn.

$$\left\{\rho_r - 1\right\}^2 + \left\{\rho_j - \frac{1}{\overline{X}}\right\}^2 = \left\{\frac{1}{\overline{X}}\right\}^2 \qquad \textbf{(2-5)}$$

which shows a family of circles with centre $\left\{1,\frac{1}{\overline{X}}\right\}$, and radius $\left\{\frac{1}{\overline{X}}\right\}$ associated with $\rho_r$ and $\rho_j$,

These are the constant reactance circles, because if $\overline{X}$ is set to a constant, a circle can be drawn. For different values of $\overline{X}$ , different circles of different radii and with different centres along the $\rho_j$ axis at $\rho_r$ = 1 can be drawn. As $\overline{X}$ can be positive or negative, circles along the positive $\rho_j$ axis will represent circles of capacitance, where as circles along the negative $\rho_j$ axis will represent circles of inductance.

Figure 25 shows equations 2-4 and 2-5 plotted on the reflection coefficient plane.

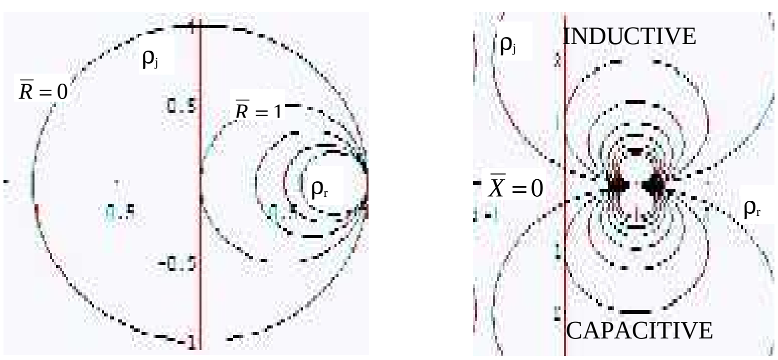



**Figure 25 Plot of constant resistance circles and constant reactance circles**

It can be noticed from Figure 25 that both constant resistance and constant reactance circles have a point of convergence at $\rho_r$ = 1, that is, the coordinate point where all the circles vanish and where all the circular arcs share the common intersection point ($\overline{Z} \to \infty$, and $\rho = 1e^{j0}$). By superimposing the constant resistance and reactance circles onto one chart and restricting their rectangular coordinate points $\rho_j$ and $\rho_r$, to fall within the range of ±1, the final Smith Chart is obtained. This is shown in Figure 26. This means that both the constant resistance and reactance circles are constrained to be within the region of |ρ| = 1 allowing the range of values which boths sets of circles can take, to be from zero to infinity. The right hand-side of the reactance chart from $\rho_r$ = 1 to infinity is a duplication of its left-hand side and is not required. Notice that the outer constant resistance circle is $\overline{R} = 0$, and as the value of $\overline{R}$ increases, the radius decreases until the innermost inner circle becomes a point at $\overline{R} \to \infty$. For the constant reactance circles, the $\rho_r$ axis is the circle of infinite radius, where $\overline{X} = 0$, and as the circles decrease in radius, $\overline{X} \to \pm\infty$.
To avoid congestion on the chart, the polar coordinates reflection coefficient plane, as shown in Figure 23, with |ρ| = 0, which is located at the centre of the Smith chart is not superimposed onto the constant resistance and reactance circles. This does not pose a problem, due to the ease of imagining the existance of these circles, centred at the middle of the chart, and radiating to the edge of the chart where |ρ| = 1. Manual construction of one or more of these circles, as required, is usual.

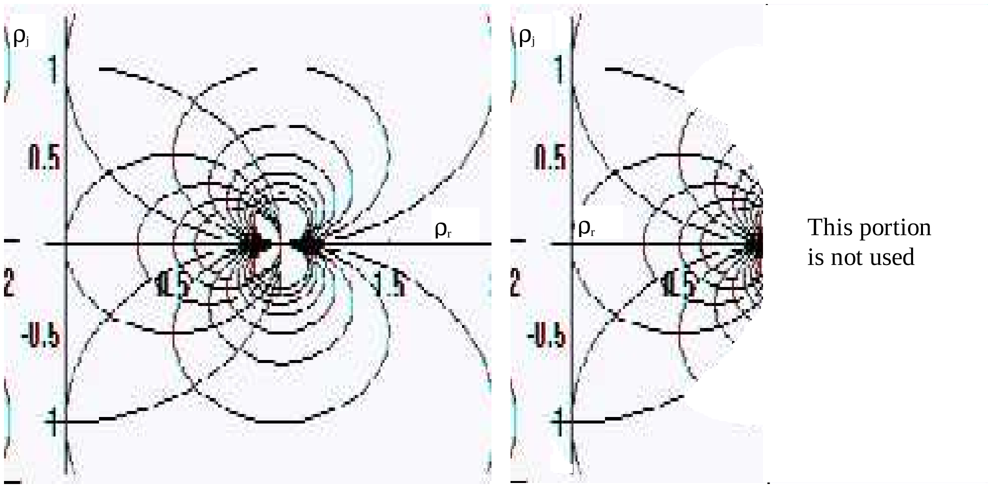



**Figure 26 Combined constant reactance and resistance circles**

The relationship between voltage standing wave ratio (VSWR) and the modulus of the reflection coefficient |ρ| is given by equation 1-63, that is

$$VSWR = S = \frac{1+|\rho|}{1-|\rho|} \quad \textbf{(1-63)}$$

This permits the determination of the VSWR from the Smith chart, by determing the value of the modulus of the reflection coefficient, as shown in the examples below. The more usual method of determination of VSWR is by directly reading off the values on the $\overline{X} = 0$ line, right of the chart centre, where the constant refelection coefficient circle intersects. This is discussed below in section 2.4. Note that the VSWR ranges from unity at the centre of the constant |ρ| circles (|ρ| = 0), to infinity at the edge of the chart (|ρ| = 1). The circles of constant reflection coefficient magnitude can be represented as circles of constant VSWR because of the relationship between reflection coefficient magnitude and VSWR given by equation 1-63.

When the impedance or the VSWR is measured along a transmission line, a repetitive behaviour is exhibited. On the Smith chart this repetition is equivalent to a full rotation around the chart. The repetitions occur every half wavelength as one moves along the transmission line. Because of this, the entire perimeter of the Smith chart is calibrated uniformly in ±0.25 wavelengths relative to a reference location. Moving anti-clockwise around the edge of the chart is equivalent to moving towards the load, whereas moving clockwise is equivalent to moving towards the generator.

## *2.3 Admittance chart*

Normalized admittance $\overline{Y}$ is defined as the inverse of normalized impedance $\overline{Z}$, thus, from equation 2-1,

$$\frac{1}{\overline{Z}} = \overline{Y} = \overline{G} + j\overline{B} = \frac{1-\rho}{1+\rho} = \frac{1-(\rho_r + j\rho_j)}{1+(\rho_r + j\rho_j)} \quad \textbf{(2-6)}$$

Again by separating the real and imaginary parts of equation 2-6,

$$\overline{G} = \frac{1-\rho_r^2-\rho_j^2}{(1+\rho_r)^2+\rho_j^2} \quad \textbf{(2-7)}$$

$$\overline{B} = \frac{-2\rho_j}{(1+\rho_r)^2+\rho_j^2} \quad \textbf{(2-8)}$$

Again there is a need to separate out normalized resistance from normalized reactance, that is, to identify constant conductance and constant susceptance curves, and separately associated each with

$\rho_r$ and $\rho_j$. This is obtained using equations 2-7 and 2-8, by rearrangement of each of these equations, giving

$$\left\{\rho_r + \frac{\overline{G}}{\overline{G}+1}\right\}^2 + \rho_j^2 = \left\{\frac{1}{\overline{G}+1}\right\}^2 \qquad \textbf{(2-9)}$$

which shows a family of circles with centre $\left\{\frac{-\overline{G}}{\overline{G}+1}, 0\right\}$, and radius $\left\{\frac{1}{\overline{G}+1}\right\}$ associated with $\rho_r$ and $\rho_j$,

These are the constant-G circles, because if $\overline{G}$ is set to a constant, a circle can be drawn. For different values of $\overline{G}$ **,** different circles with different radii and different centres along the $\rho_r$ axis can be drawn.

$$\{\rho_r + 1\}^2 + \left\{\rho_j + \frac{1}{\overline{B}}\right\}^2 = \left\{\frac{1}{\overline{B}}\right\}^2 \qquad \textbf{(2-10)}$$

which shows a family of circles with centre $\left\{-1, \frac{-1}{\overline{B}}\right\}$, and radius $\left\{\frac{1}{\overline{B}}\right\}$ associated with $\rho_r$ and $\rho_j$,

These are the constant susceptance circles, because if $\overline{B}$ is set to a constant, a circle can be drawn. For different values of $\overline{B}$, different circles of different radii and with different centres along the $\rho_j$ axis at $\rho_r$ = -1 can be drawn. As $\overline{B}$ can be positive or negative, circles along the positive $\rho_j$ axis will represent circles of inductance, where as circles along the negative $\rho_j$ axis will represent circles of capacitance.

Figure 27 shows equations 2-9 and 2-10 plotted on the reflection coefficient plane.

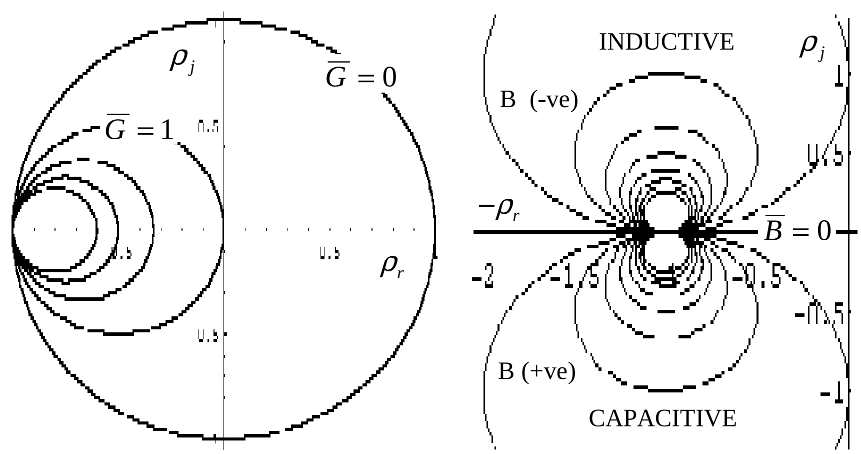



**Figure 27 Plot of constant conductance circles and constant susceptance circles**

From Figure 27, it can be seen that both constant conductance and constant susceptance circles have a point of convergence at $\rho_r = -1$, that is, the coordinate point where all the circles vanish and where all the circular arcs share the common intersection point. This is the opposite to that of the impedance chart, in the sense that, it is what would be seen if the impedance chart was printed on clear cellophane and then turned over. By superimposing the constant conductance and susceptance circles onto one chart and restricting their rectangular coordinate points $\rho_j$ and $\rho_r$, to fall within the range of $\pm 1$, the final admittance form of the Smith Chart is obtained. This is shown in Figure 28. Again, both circles, this time the constant conductance and susceptance circles, are constrained to be within the region of $|\rho| = 1$ allowing the range of values which both sets of circles can take, to be from zero to infinity. The left-hand-side of the reactance chart from $\rho_r$ = -1 to minus infinity is a duplication of its right-hand side and is not required. Notice that the outer constant conductance circle is $\overline{G} = 0$, and as the value of $\overline{G}$ increases, the radius decreases until the innermost inner circle becomes a point at $\overline{G} \to \infty$. For the constant susceptance circles, the $\rho_r$ axis is the circle of infinite radius, where $\overline{B} = 0$, and as the circles decrease in radius, $\overline{B} \to \infty$.

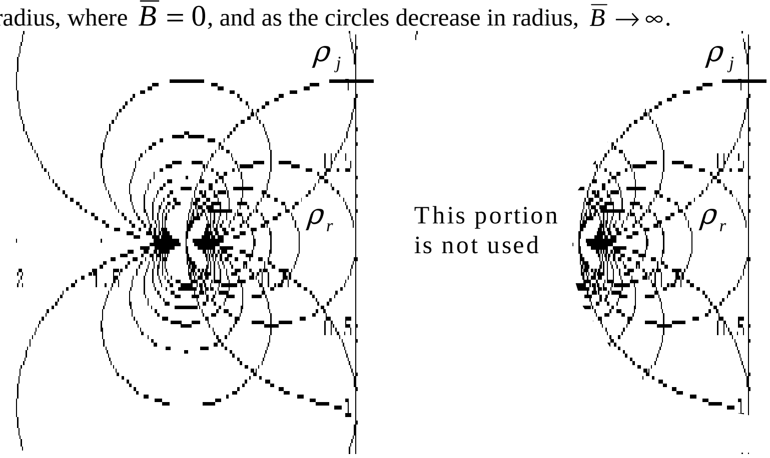



**Figure 28 Combined constant conductance and susceptance circles**

### *2.4 Relationship between impedance and admittance charts*

From the intersection of a circle of constant resistance with a circle of constant reactance, a point of normalized impedance $[\overline{R}, \overline{X}]$ can be found on the Smith chart, where

$$\overline{R} = \frac{1-\rho_r^2 - \rho_j^2}{(1-\rho_r)^2 + \rho_j^2} \quad \textbf{(2-2)}$$

$$\overline{X} = \frac{2\rho_j}{(1-\rho_r)^2 + \rho_j^2} \quad \textbf{(2-3)}$$

Similarly, from the intersection of a circle of constant conductance with a circle of constant susceptance, a point of normalized admittance $[\overline{G}, \overline{B}]$ can be found on the admittance chart, where

$$\overline{G} = \frac{1-\rho_r^2 - \rho_j^2}{(1+\rho_r)^2 + \rho_j^2} \quad \textbf{(2-7)}$$

$$\overline{B} = \frac{-2\rho_j}{(1+\rho_r)^2 + \rho_j^2} \quad \textbf{(2-8)}$$

As both the points $[\overline{R}, \overline{X}]$ and $[\overline{G}, \overline{B}]$, are both functions of $\rho_r$ and $\rho_j$ and are plotted on the $\rho_r$-$\rho_j$ plane, there exists a relationship between them. By changing the sign of both $\rho_r$ and $\rho_j$ in equations 2-2 and 2-3, we find that we obtain equations 2-7 and 2-8. Graphically, the relationship is that the point $[\overline{G}, \overline{B}]$ lies in the opposite quadrant from $[\overline{R}, \overline{X}]$, at the same distance from the centre of the chart $[\rho_r, \rho_j] = [0,0]$. Alternatively, if a point $[\overline{R}, \overline{X}]$ is plotted on the Smith chart and the chart is then rotated about an axis through its centre, $[\rho_r, \rho_j] = [0,0]$, by 180°, the point $[\overline{G}, \overline{B}]$ is obtained. This is shown in Figure 29.

Equations 2-2, 2-3, 2-7 and 2-8 show their dependence on the circles of constant reflection coefficient, whose equation is given by

$$|\rho|^2 = \rho_r^2 + \rho_j^2 \quad \textbf{(2-11)}$$

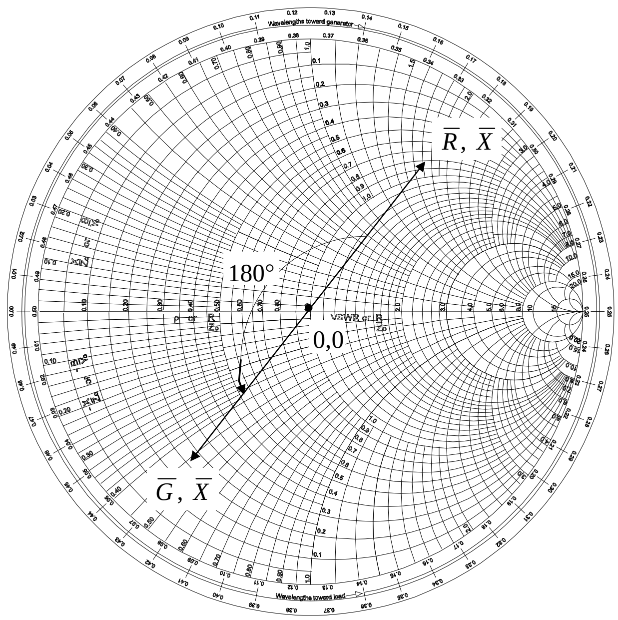



**Figure 29 Impedance point and equivalent admittance point**

## 2.5 Properties of the Smith Chart

Below is a listing of some of the properties of the Smith chart which have been developed so far:

- Impedances repeat every half-wavelength. One full tour around the Smith chart represents half a wavelength. Usually the chart is drawn so that from the left-hand side of the chart $\left(\overline{Z}=0\right)$, an arrow pointing ***upwards***, representing a maximum of a quarter wavelength ***clockwise*** rotation around the upper semi-circle ***towards the generator,*** whereas, an arrow pointing ***downwards***, represents a maximum of a quarter wavelength ***anticlockwise*** rotation around the lower semi-circle ***towards the load***. For example, if an impedance looking into a transmission line is measured, the impedance of the load at the far end of the line can be approximately determined, knowing the length of the line $l$. This load impedance is found by first expressing the length of the line in wavelengths, and considering only the remainder after integer half-wavelengths have been subtracted, the measured impedance point plotted on the chart, is then rotated "towards the load" along a constant reflection coefficient or circle, through this remainder wavelength, and the required impedance of the load is then read from the chart. The constant reflection coefficient circle is determined by drawing a circle through the measured impedance point. If the line was not considered to be lossless, then the rotation would not be along a constant VSWR circle, but along a spiral.
- The impedance point lying anywhere on the Smith chart will have a corresponding admittance point that is diametrically opposite at an equal distance from the centre of the chart. This is shown in Figure 29,
- By rotating the chart through 180° about an axis through its centre, a point of impedance on the chart will become a point of admittance, and a point of admittance will become a point of impedance,
- As a full cycle around the Smith chart represents a quarter wavelength, the impedance and admittance points are separated by a quarter wavelength,
- The top half of the Smith chart represents a reactance which is inductive and the bottom half of the chart represents a reactance which is capacitive. The same applies for the admittance chart,
- ***Capacitive reactance*** when added in series with a load impedance anywhere on the chart, will move the plotted impedance point ***anticlockwise*** around circles of constant resistance,
- ***Inductive reactance*** when added in series with a load impedance anywhere on the chart, will move the plotted impedance point ***clockwise*** around circles of constant resistance,

[1] Smith, P.H., Transmission Line Calculator. *Electronics,* 12, January 1939, pp. 29-31.

[2]Smith,P.H., An Improved Transmission Line Calculator. *Electronics,* 17, January 1944, pp 130-133, 318-325.

[3]Smith,P.H., *Electronic Applications of the Smith Chart,* McGraw-Hill Book Co., New York, 1983.

- ***Capacitive susceptance*** when added in shunt with a load admittance anywhere on the Smith chart (following lines of constant conductance), will move the plotted impedance point ***clockwise*** around circles of constant resistance,
- ***Inductive susceptance*** when added in shunt with a load admittance anywhere on the chart, will move the plotted impedance point ***anticlockwise*** around circles of constant resistance.
- The VSWR can be determined directly from the Smith chart from the constant reflection coefficient circles crossings on the "resistance line", that is the line bisecting the mith chart from left to right ($\overline{X} = 0$). The constant reflection coefficient circle crosses the resistance line at two points; left of the chart centre and right of the chart centre. The VSWR is read from the crossing, right of centre. This can be easily proved by considering equation 2-2 and 1-63. From equation 2-3, as $\overline{X} = 0$ we must have $\rho_j = 0$ that is

$$\overline{R} = \frac{1-\rho_r^2-\rho_j^2}{(1-\rho_r)^2+\rho_j^2} \qquad \textbf{(2-2)}$$

reduces to $\overline{R} = \frac{1-\rho_r^2}{(1-\rho_r)^2} = \frac{1-|\rho|^2}{\left(1-|\rho|\right)^2} = \frac{1+|\rho|}{1-|\rho|}$, for $\rho_r$ positive, that is on the right side of the centre of the chart. From equation 1-63,

$$VSWR = S = \frac{1+|\rho|}{1-|\rho|} \qquad \textbf{(1-63)}$$

Showing that the values of $\overline{R}$ read from the chart right of centre, is equivalent to the VSWR. An alternative expression for VSWR is the inverse of that given by equation 1-63. This is the case where $\rho_r$ is negative. These values of VSWR are read directly from values of $\overline{R}$ read from the chart left of centre.

### 2.5.1 Some properties of the admittance chart

- ***Capacitive reactance*** when added in series with a load impedance anywhere on the Smith chart, will move the plotted admittance point ***anticlockwise*** around circles of constant conductance on the admittance chart,
- ***Inductive reactance*** when added in series with a load impedance anywhere on the Smith chart, will move the plotted admittance point ***clockwise*** around circles of constant conductance on the admittance chart,
- ***Capacitive susceptance*** when added in shunt with a load admittance anywhere on the admittance chart, will move the plotted admittance point ***clockwise*** around circles of constant conductance,
- ***Inductive susceptance*** when added in shunt with a load admittance anywhere on the admittance chart, will move the plotted admittance point ***anticlockwise*** around circles of constant conductance.

- The VSWR can be determined directly from the admittance chart from the constant reflection coefficient circles crossings on the "conductance line", that is the line bisecting the admittance chart from left to right ($\overline{B} = 0$). The constant reflection coefficient circle crosses the conductance line at two points; left of the chart centre and right of the chart centre. The VSWR is read from the crossing, ***left*** of centre, as $\rho_r$ must be negative in equation 2-11 to obtain equation 1-63.

- The admittance chart is used to add shunt elements and the Smith chart is used to add series elements,
- On both the Smith chart and the admittance chart the upper semicircle is inductive and the lower semicircle is capacitive,
- On both the Smith chart and the admittance chart, from left to right on the upper semicircle is “towards the generator” and from left to right on the lower semicircle is “towards the load”,
- On the admittance chart, the upper semicircle is negative susceptance (inductive) and the lower semicircle is positive susceptance (capacitive),
- On the Smith chart, the upper semicircle is positive reactance (inductive) and the lower semicircle is negative reactance (capacitive)

Figure 30 below shows a summary of the addition of series and shunt capacitance and inductance to the Smith chart and admittance chart (constant G circles).

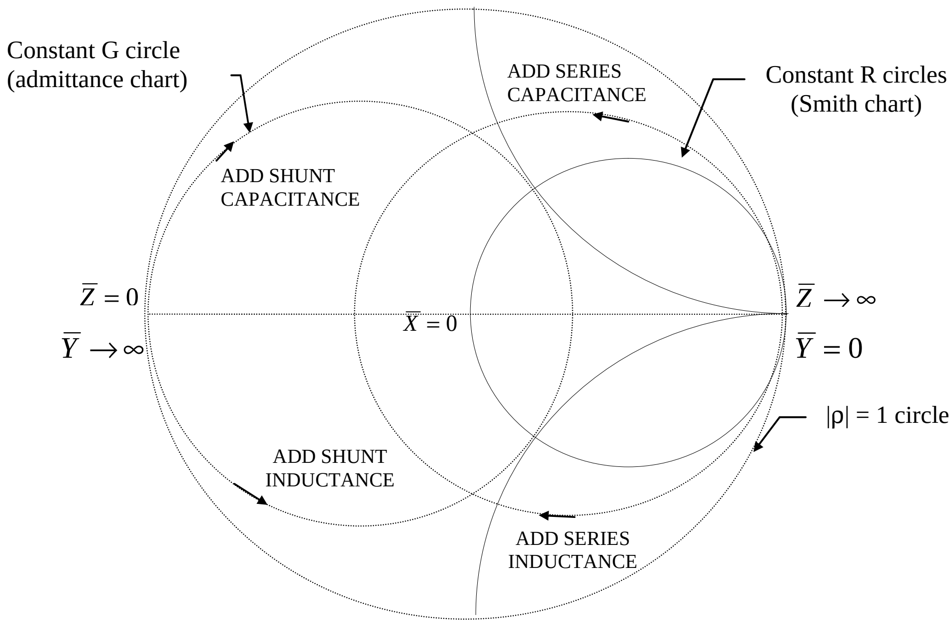



**Figure 30** **Direction around the constant G circles and constant R circles for shunt and series addition of capacitance or inductance.**

## *2.6 The use of the Smith chart for single frequency impedance matching*

One of the main uses of the Smith chart and its associated admittance chart is that it can be used to assist in the design of matching networks. It permits the value of a component or the values of components to be determined, which are to be added near to the load of a mismatched transmission line. The purpose of the addition of theses components, added in a series or in shunt or in a combination of both is to cancel the effect of the mismatch. Thus, when looking into the transmission line at its source end, the transmission line appears as its characteristic impedance.
To achieve matching, a mismatch impedance plotted on the Smith chart must be moved around the chart in such a way, by the addition of elements of the network, that the final impedance point is the centre of the chart, that is where $\overline{R} = 1$ and $\overline{X} = 0$.
Many of the single frequency impedance matching problems can be resolved using an L-type matching network[i,ii]. P.H. Smith developed eight matching networks that each use only two reactances, where one of the reactances is either in shunt or in series with the load. These networks are shown in Figure 30.
The position of the mismatched impedance on the Smith chart determines which pair or pairs of the eight networks is to be used. Only when the resistive component of a mismatched load is equal to the characteristic impedance, can a single reactive element be used for perfect matching (VSWR=1). This is because the mismatched impedance will lie on the $\overline{R} = 1$ or the $\overline{G} = 1$ circle, and the addition of a matching susceptance will rotate the impedance towards the centre of the chart. However, if a perfect match is not required, and a maximum value of VSWR is specified, the

mismatched load resistive component does not necessarily have to lie on the $\overline{R} = 1$. or the $\overline{G} = 1$ circle, but must lie within the range of constant resistance circles that pass through the constant VSWR circle as shown in the example of Figure 32 and discussed in section 2.7.
If the mismatched resistive component does not lie on one of the constant resistance circles which pass through the specified VSWR circle, then a single element cannot be used for matching. A two reactance component network (L-type circuit) or a short section of transmission line used in conjunction with a reactance component or a transmission line stub must be considered.
The selection of a L-type network from the pairs available, according to the position on the Smith chart of the mismatch impedance, depends on the availability of components and their size (inductors, transformers and capacitors), as well as the convenience of the use of transmission lines used as components, etc.

### 2.6.1 L-Type Impedance Transformation Circuits

Figure 30 shows the eight L-type circuits which can be used transforming a mismatched load, which in this case is an antenna, into the characteristic impedance of the transmission line.
Each of the L-type circuits are labelled with the first component from the left of the diagram, followed by whether it is series (S) or shunt (SH), then followed by the second component and finally, whether it is series or shunt. For example circuit [1], is an inductance (L) in series (S) followed by a capacitor (C) in shunt (SH) giving LSCSH. Circuits [1] and [2] are complementary pairs, as are [3] and [4], [5] and [6], and [7] and [8].
Figures 31 to 38 show how a mismatched load impedance on the Smith chart can be be moved to the centre of the chart for perfect match. Any component which is to be placed in shunt with the load (in this case an antenna), such as circuits [1], [3], [5] and [7], will first move along a constant conductance circle to meet the $\overline{R} = 1$ circle. This will determine the value of the shunt component. Once on the $\overline{R} = 1$ circle the impedance is rotated along this circle to meet the centre of the chart at $Z = Z_o$. This is because the centre of the chart is given by $Z/Z_o = \overline{R} + 0j\overline{X} = 1$. The amount of rotation on the $\overline{R} = 1$ circle will determine the value of the series reactance. Any component which is to be placed in series with the load , such as circuits [2], [4], [6] and [8], will first move along a constant resistance circle to meet the $\overline{G} = 1$ circle. This will determine the value of the series component. Once on the $\overline{G} = 1$ circle, the impedance is rotated along this circle to meet the centre of the chart at $Z = Z_o$. The amount of rotation on the $\overline{G} = 1$ circle will determine the value of the shunt reactance.

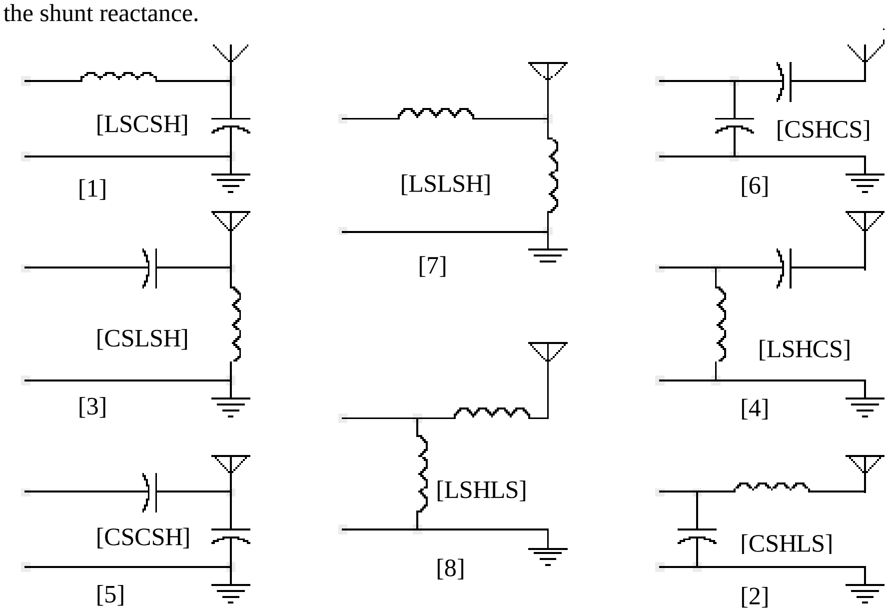



**Figure 31 The eight possible L-type circuits**

As mentioned above, when to use one circuit in preference to others is dependent on the availability of components and their size (inductors, transformers and capacitors), as well as the convenience of the use of transmission lines used as components, etc.

#### *2.6.1.1 Constant conductance circles*

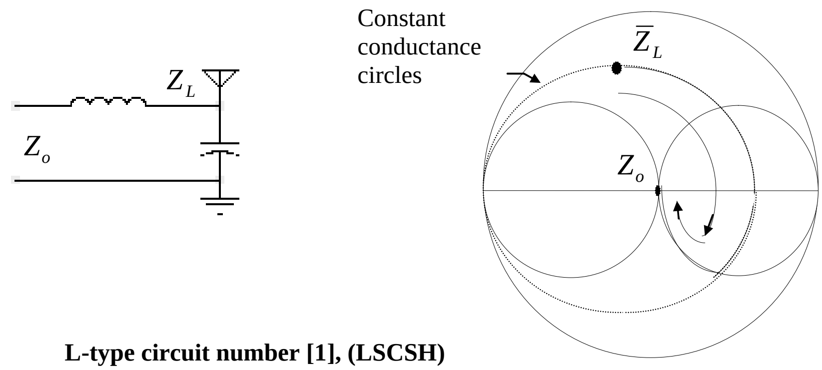



**Figure 32 L-type circuit number [1], (LSCSH)**

Referring to Figures 30 and 32, as the capacitor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle clockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_c$, around the admittance chart will permit the value of the shunt capacitance to be determined, as $\Delta\overline{B}_c = \omega C = 2\pi fC$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 3 , we see that we are following one of the "add series inductance" circles. he amount of rotation on the Smith chart permits the determination of the series inductance, from $\Delta X_L = \omega L = 2\pi fL$.

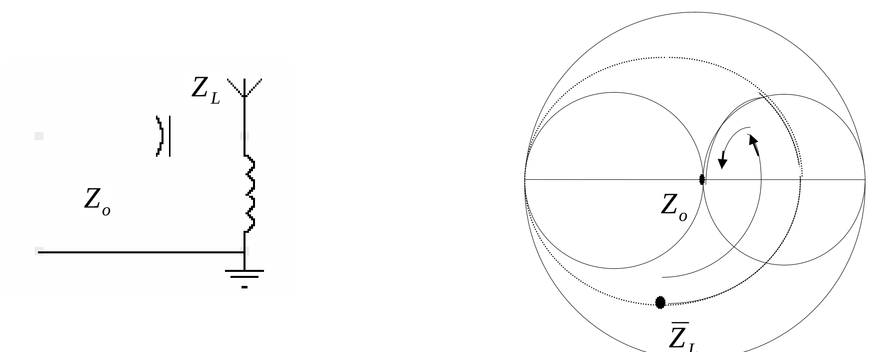
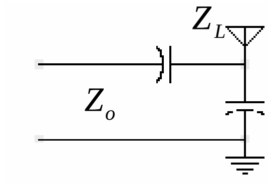



**Figure 33 L-type circuit number [3], (CSLSH)**

Referring to Figures 30 and 33, as the inductor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle anticlockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_L$, around the admittance chart will permit the value of the shunt inductance to be determined, as $\Delta\overline{B}_L = 1/\omega L = 1/2\pi fL$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add series capacitance" circles. he amount of rotation on the mith chart permits the determination of the series capacitance, from $\Delta\overline{X}_C = 1/\omega C = 1/2\pi fC$.



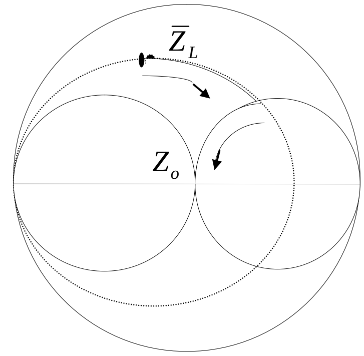
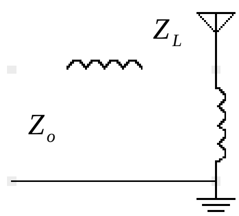




**Figure 34 L-type circuit number [5], (CSCSH)**

Referring to Figures 30 and 34, as the capacitor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle clockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_c$, around the admittance chart will permit the value of the shunt capacitance to be determined, as $\Delta\overline{B}_c = \omega C = 2\pi fC$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the "add series capacitance" circles. The amount of rotation on the Smith chart permits the determination of the series capacitance, from $\Delta\overline{X}_C = 1/\omega C = 1/2\pi fC$.



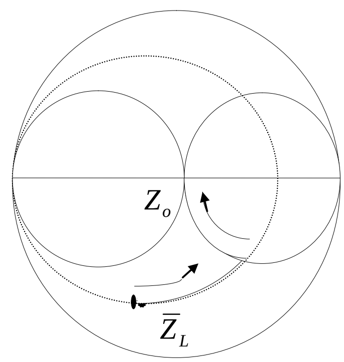
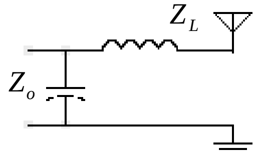




**Figure 35 L-type circuit number [7], (LSLSH)**

Referring to Figures 30 and 35, as the inductor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle anticlockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_L$, around the admittance chart will permit the value of the shunt inductance to be determined, as $\Delta\overline{B}_L = 1/\omega L = 1/2\pi fL$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle clockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the "add series inductance" circles. he a ount of rotation on the ith chart er its the deter ination of the series inductance, from $\Delta X_L = \omega L = 2\pi fL$.

### 2.6.1.2 *Constant resistance circles*



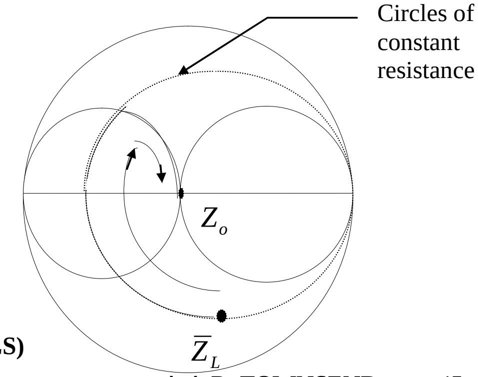




**Figure 36 L-type circuit number [2], (CSHLS)**

Referring to Figures 30 and 36, as the inductor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle clockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_L$, around the Smith chart will permit the value of the series inductance to be determined, as $\Delta\overline{X}_L = \omega L = 2\pi f L$. From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle clockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the "add shunt capacitance" circles. The amount of clockwise rotation on the admittance chart permits the determination of the shunt capacitance, from $\Delta\overline{B}_C = \omega C = 2\pi f C$.

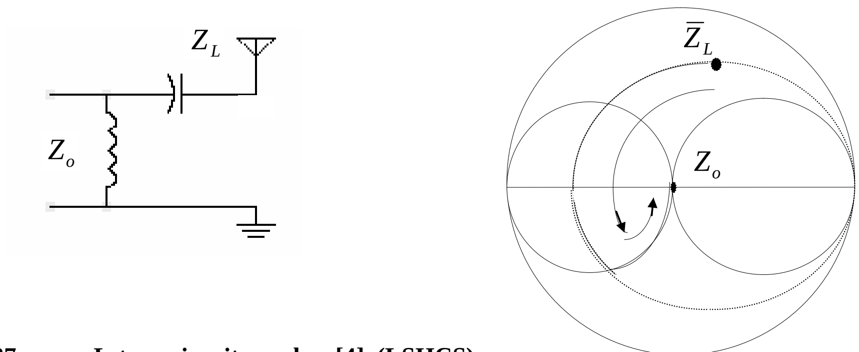



**Figure 37 L-type circuit number [4], (LSHCS)**

Referring to Figures 30 and 37, as the capacitor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle anticlockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_C$, around the Smith chart will permit the value of the series capacitance to be determined, as $\Delta\overline{X}_C = 1/\omega C = 1/2\pi f C$. From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the "add shunt inductance" circles. he a ount of anticlockwise rotation on the ad ittance chart er its the determination of the shunt inductance, from $\Delta\overline{B}_L = 1/\omega L = 1/2\pi f L$.

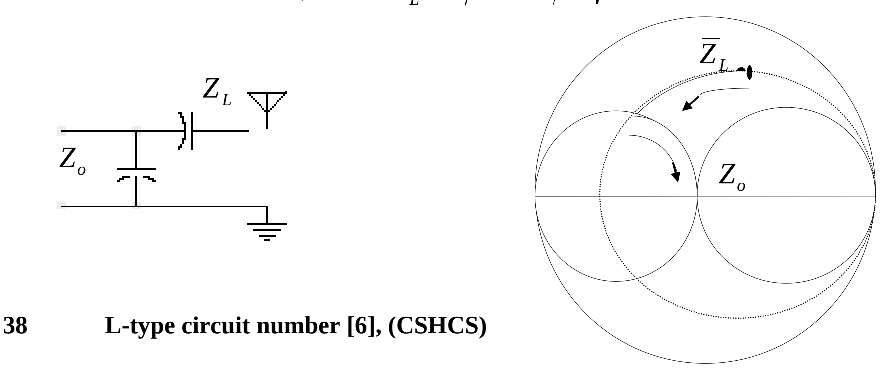
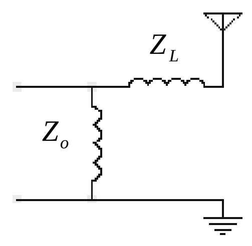



**Figure 38 L-type circuit number [6], (CSHCS)**

Referring to Figures 30 and 38, as the capacitor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle anticlockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_C$, around the Smith chart will permit the value of the series capacitance to be determined, as $\Delta\overline{X}_C = 1/\omega C = 1/2\pi f C$. From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle clockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the "add shunt capacitance" circles. The amount of clockwise rotation on the admittance chart permits the determination of the shunt capacitance, from $\Delta\overline{B}_C = \omega C = 2\pi f C$.



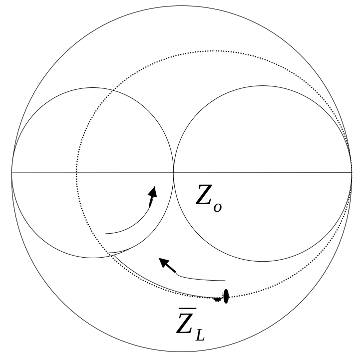




**Figure 39 L-type circuit number [8], (LSHLS)**

Referring to Figures 30 and 39, as the inductor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle clockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_L$, around the Smith chart will permit the value of the series inductance to be determined, as $\Delta\overline{X}_L = \omega L = 2\pi f L$. From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the “add shunt inductance” circles. he a ount of anticlockwise rotation on the ad ittance chart er its the determination of the shunt inductance, from $\Delta\overline{B}_L = 1/\omega L = 1/2\pi f L$.

## *2.7 The use of boundary circles for design*

There are additional geometric constructions which can be added to or used in conjunction with the constant resistance and reactance circles which simplify or decrease the design process time. One of these construction is the use of “boundary circles”, which for a given reflection coefficient (or VSWR) are two related pairs of circles. Each pair are tangent to the target constant reflection coefficient magnitude (or VSWR) circle and doubly tangent to the $|\rho| = 1$ chart periphery circle. One pair of circles has the double tangency on the periphery of the left-hand side of the chart at $\overline{Z} = 0$, and the other pair has the double tangency on the periphery of the right-hand side of the chart at $\overline{Z} \to \infty$. All circles have their centres on the $\overline{X} = 0$ line, which is an arc of a circle of infinite radius.

Both pairs of circles are shown in Figure 40, for $|\rho| = 1/3$ or VSWR = 2.

Also shown on Figure 40 are four labelled regions which indicate that either a shunt or series capacitance or inductance can be used to narrowband match out a mismatched transmission line. If the specified component type is used on a transmission line with a mismatched impedance falling in this region, then the mismatched transmission line operating at a particular single frequency would be forced to a perfect match at the centre of the chart. It may be observed that pairs of regions overlap, permitting dual choices to occur.

In addition, there are four regions which indicate that a single element cannot be used to obtain a match into the centre of the chart for a mismatched transmission line, as defined by the target VSWR circle. For these cases, at least two elements are required for matching.

As can be seen from Figure 40, the four different branch types, series or shunt capacitance or inductance, for accomplishing single element narrowband matching are determined by the sense of rotation for each along the constant-R and the inverse constant-G circles . That is, series-L rotates clockwise on a constant-R circle, shunt-L rotates anticlockwise on a constant G circle, series-C rotates anti-clockwise on a constant-R circle and shunt-C rotates clockwise along a constant-G circle.

The boundary circles permit the implementation of a visual design of the combination of transmission line components required to match a transmission line over *a band of frequencies* so that over this band the value of VSWR is restricted to lie within a certain value, which in the case shown in Figure 40, is VSWR=2. Specific examples will be given in later chapters to illustrate the use of the boundary circle method for multiple frequency impedance matching to obtain a specified VSWR.

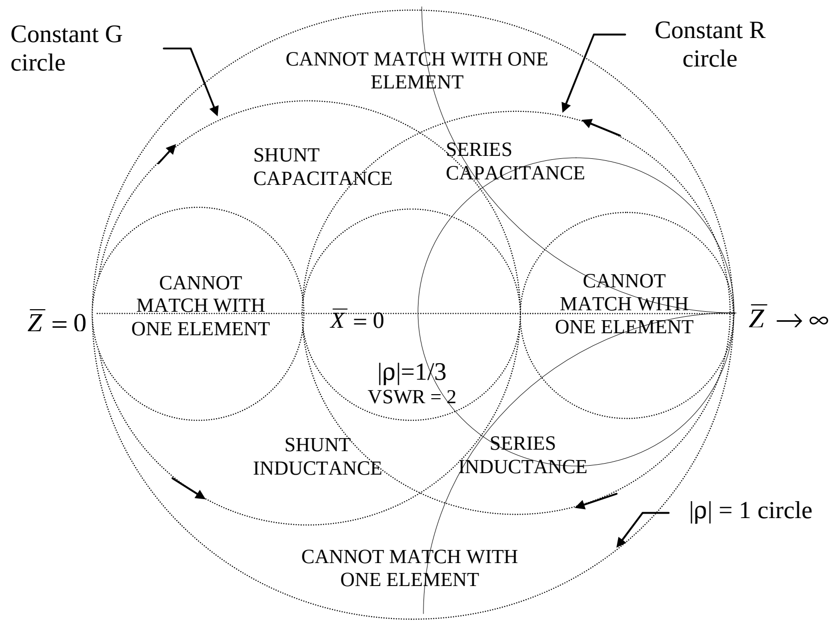



**Figure 40 Sets of boundary circle pairs for |ρ| = 1/3 or VSWR=2**

## References

---

[i]Smith, P.H., L-Type Impedance Transform Circuits. *Electronics,* March 1942
[ii]Smith,P.H.,*Electronic Applications of the Smith Chart in Waveguide, Circuit, and Component Analysis,* Robert E. Krieger Publishing Company, Malabar, Fl, 1983, pp. 116-127

# 3. TRANSMISSION LINE CIRCUIT COMPONENTS

This chapter will describe how circuit components can be formed from transmission lines and used for matching purposes using the Smith chart as the design medium. Above frequencies of 100 MHz, lengths of transmission line are commonly used as circuit components, in preference to discrete capacitors or inductors.. This is due to the transmission line lengths being shorter, and thus more practical, at the higher frequencies than at the lower frequencies. In this chapter, only single frequency matching will be considered. The next chapter will deal with matching to a specific VSWR over a band of frequencies.

## *3.1. The transmission line used as a reactance (or susceptance)*

### 3.1.1. Short circuited transmission line

#### *3.1.1.1.Inductance*

When the load is short-circuited $Z_L = 0$, and for the

- **Low-loss transmission line**

$$\overline{Z} = \frac{(\alpha d \cos\beta d + j\sin\beta d)}{(\cos\beta d + j\alpha d\sin\beta d)} \qquad \textbf{(1-91)}$$

From equation 1-91, for $\alpha d << 1$,

$$\overline{Z} = \frac{\alpha d(1+\tan^2 2\pi d/\lambda)}{1+(\alpha d)^2\tan^2 2\pi d/\lambda} + j\frac{[1-(\alpha d)^2]\tan 2\pi d/\lambda}{1+(\alpha d)^2\tan^2 2\pi d/\lambda} \qquad \textbf{(1-119)}$$

giving

$$Q = \frac{[1-(\alpha d)^2]\tan 2\pi d/\lambda}{\alpha d(1+\tan^2 2\pi d/\lambda)} \qquad \textbf{(1-122)}$$

**FOR A LINE CLOSE TO A QUARTER WAVELENGTH**

*At a quarter wavelength*, the ideal line is a parallel resonant circuit. For the non-ideal case, there exists an equivalent parallel resistance placed across the circuit. This limits the Q of the parallel resonant circuit. From equation 1-119, the series resistance can be found from the real term, and the inductive reactance from the imaginary term, giving

$$Q \approx \frac{1}{\alpha d\tan 2\pi d/\lambda} \qquad \textbf{(1-120)}$$

where

$\alpha d << 1, \quad \tan 2\pi d/\lambda >> 1$ and $(\alpha d\tan 2\pi d/\lambda) \approx 1$

**FOR A LINE MUCH SHORTER THAN A QUARTER WAVELENGTH**

For a short-circuited transmission line, where the length is ***far less** than a quarter of a wavelength*, the Q of the inductance is given by

$$Q \approx \frac{2\pi}{\alpha\lambda} = \frac{\beta}{\alpha} \qquad \textbf{(1-123)}$$

where $\alpha d << 1, \quad \tan 2\pi d/\lambda \approx 2\pi d/\lambda$ and $(\tan 2\pi d/\lambda)^2 << 1$,

Equation 1-123 shows that for small $\alpha$, and high frequencies (small $\lambda$), the Q of the simulated inductor can become very large.

$$L = \frac{Z_o}{f}\frac{d}{\lambda} \qquad \textbf{(1-124)}$$

and its equivalent series resistance

$$R_s = \alpha d Z_o \qquad \textbf{(1-125)}$$

- **Lossless transmission line**

$$Z = jZ_o \tan \beta d \quad \text{or} \quad \overline{Z} = j \tan \beta d \qquad \textbf{(1-92)}$$

and as $\beta = 2\pi/\lambda$,

$$\overline{Z} = j \tan 2\pi d/\lambda \qquad \textbf{(1-93)}$$

In order to draw circles of constant VSWR on the Smith chart, the value of $\alpha$ is taken to be zero. If $\alpha$ is not taken to be zero, then the reflection coefficient would be a locus of a spiral, spiraling in towards the centre of the chart as the line length increases, rather than that of a circle.

Figure 41 shows how the value of inductance varies around the Smith chart as the length of the short-circuited line varies according to equation 1.93.

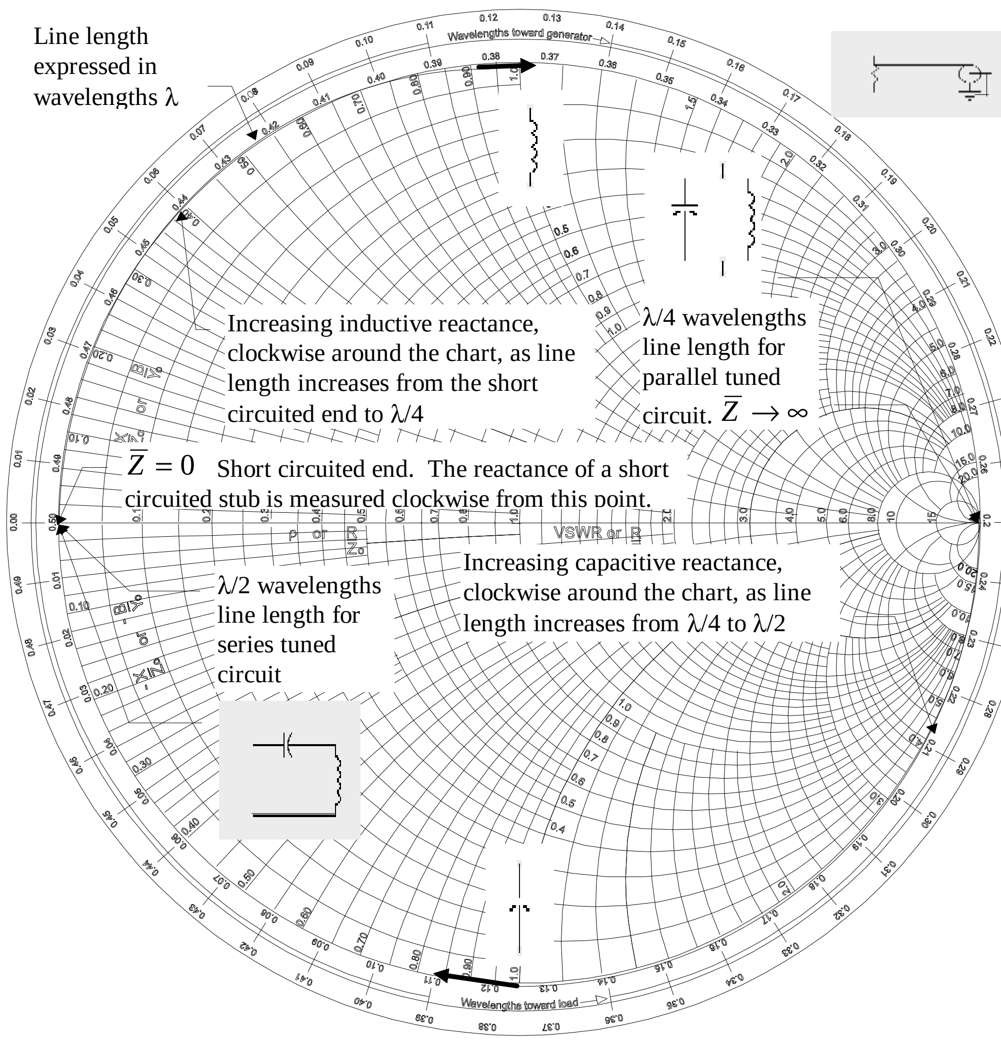



**Figure 41** **Inductance and capacitance on Smith chart as short-circuited line length increases**

### *3.1.1.2.Capacitance*

When the load is short-circuited $Z_L = 0$,and for the

- **Low-loss transmission line**

From equation 1-91, for $\alpha d << 1$,

$$\overline{Z} = \frac{\alpha d(1 + \tan^2 2\pi d/\lambda)}{1 + (\alpha d)^2 \tan^2 2\pi d/\lambda} + j\frac{[1 - (\alpha d)^2] \tan 2\pi d/\lambda}{1 + (\alpha d)^2 \tan^2 2\pi d/\lambda} \qquad \textbf{(1-119)}$$

Where the length is *far greater than a quarter of a wavelength but less than half a wavelength ($0.5\lambda > d >> 0.25\lambda$)*, and where $\alpha d << 1$, $\tan 2\pi d/\lambda \approx -2\pi d/\lambda$ and $(\tan 2\pi d/\lambda)^2 << 1$, the value of capacitance is found to be

$$C = \frac{1}{4\pi^2 f Z_o}\frac{\lambda}{d} \qquad \textbf{(3-1)}$$

- **Lossless transmission line** $(\lambda/4 < d < \lambda/2)$

The value of capacitive reactance is given by

$$\overline{Z} = j\tan 2\pi d/\lambda \qquad \textbf{(1-93)}$$

Figure 41 shows how the value of reactance varies around the Smith chart as the length of the short-circuited line varies according to equation 1.93.

Figures 42 and 43 show how the transmission line circuit components can be used in a series or parallel configuration with the transmission line.

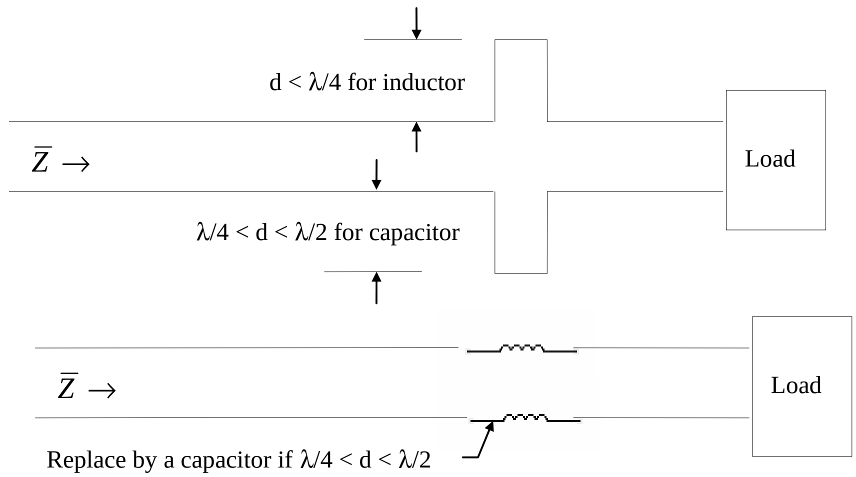



**Figure 42 A series stub used as an inductor or capacitor**

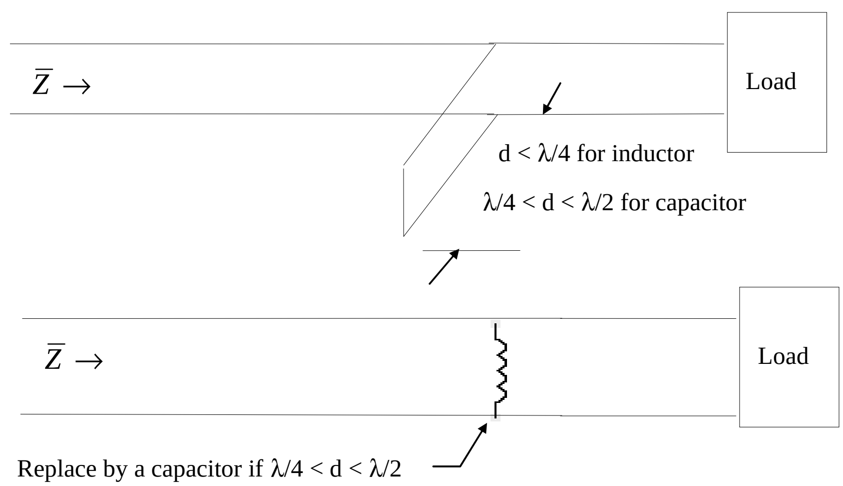
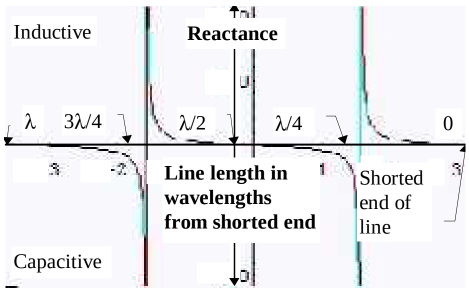


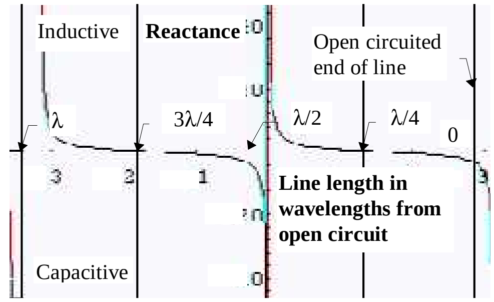

**Figure 43 A shunt stub used as an inductor or capacitor**

Figure 44 shows plots of the short circuited line normalized impedance and the open circuited line normalized impedance, as would be seen from the short or open circuited end of the line. The first of the diagrams is given by $\overline{Z} = -j\tan 2\pi d/\lambda$, the negative of equation 1-93, whereas the second diagram is given by $\overline{Z} = j\cot 2\pi d/\lambda$, the negative of equation 1-98.

The negative of these equations is taken because from the short or open circuited end of the line, we are travelling in the negative direction.




Short circuited line




Open circuited line

**Figure 44 Short circuited and open circuited transmission line stub behavior**

## 3.1.2. Open Circuited Transmission Line

When the load is open-circuited $Z_L \to \infty$,
and for the

- **Low-loss transmission line**

$$\overline{Z} = \frac{(\cos\beta d + j\alpha d \sin\beta d)}{(\alpha d \cos\beta d + j \sin\beta d)} \quad \textbf{(1-96)}$$

$$\overline{Z} = \frac{\alpha d(1+\cot^2 2\pi d/\lambda)}{(\alpha d)^2 \cot^2 2\pi d/\lambda + 1} - j\frac{[1-(\alpha d)^2]\cot 2\pi d/\lambda}{(\alpha d)^2 \cot^2 2\pi d/\lambda + 1} \quad \textbf{(3-2)}$$

giving

$$Q = \left|\frac{\alpha d(1+\cot^2 2\pi d/\lambda)}{[(\alpha d)^2 - 1]\cot 2\pi d/\lambda}\right| \quad \textbf{(3-3)}$$

**FOR A LINE CLOSE TO A QUARTER WAVELENGTH**

*At a quarter wavelength*, the ideal line is a series resonant circuit. For the non-ideal case, there exists an equivalent series resistance placed in series with the circuit. This limits the Q of the series resonant circuit. From equation 3-3, the series resistance can be found from the real term and the inductive reactance from the imaginary term, giving

$$Q \approx -\alpha d \cot 2\pi d/\lambda \quad \textbf{(3-4)}$$

where

$\alpha d << 1, \quad \cot(2\pi d/\lambda - \pi/2) \approx (2\pi d/\lambda - \pi/2)$ and $(\alpha d \cot 2\pi d/\lambda) \approx -1$

showing that open circuited transmission lines cannot be used as inductors unless their length is greater than quarter of a wavelength and less than half a wavelength, and that the inductor Q is approximately unity for stubs a little longer than a quarter of a wavelength.

**FOR A LINE MUCH SHORTER THAN A QUARTER WAVELENGTH**

For a short-circuited transmission line, where the length is ***far less** than a quarter of a wavelength*, the line is capacitive and given by equation 3-2.

- **Lossless transmission line**

The equation for the reactance of a stub transmission line is given by

$$\overline{Z} = -j\cot(2\pi d/\lambda) \quad \textbf{(1-98)}$$

Figure 45 shows how the value of reactance varies around the Smith chart as the length of the open-circuited line varies according to equation 1.98.

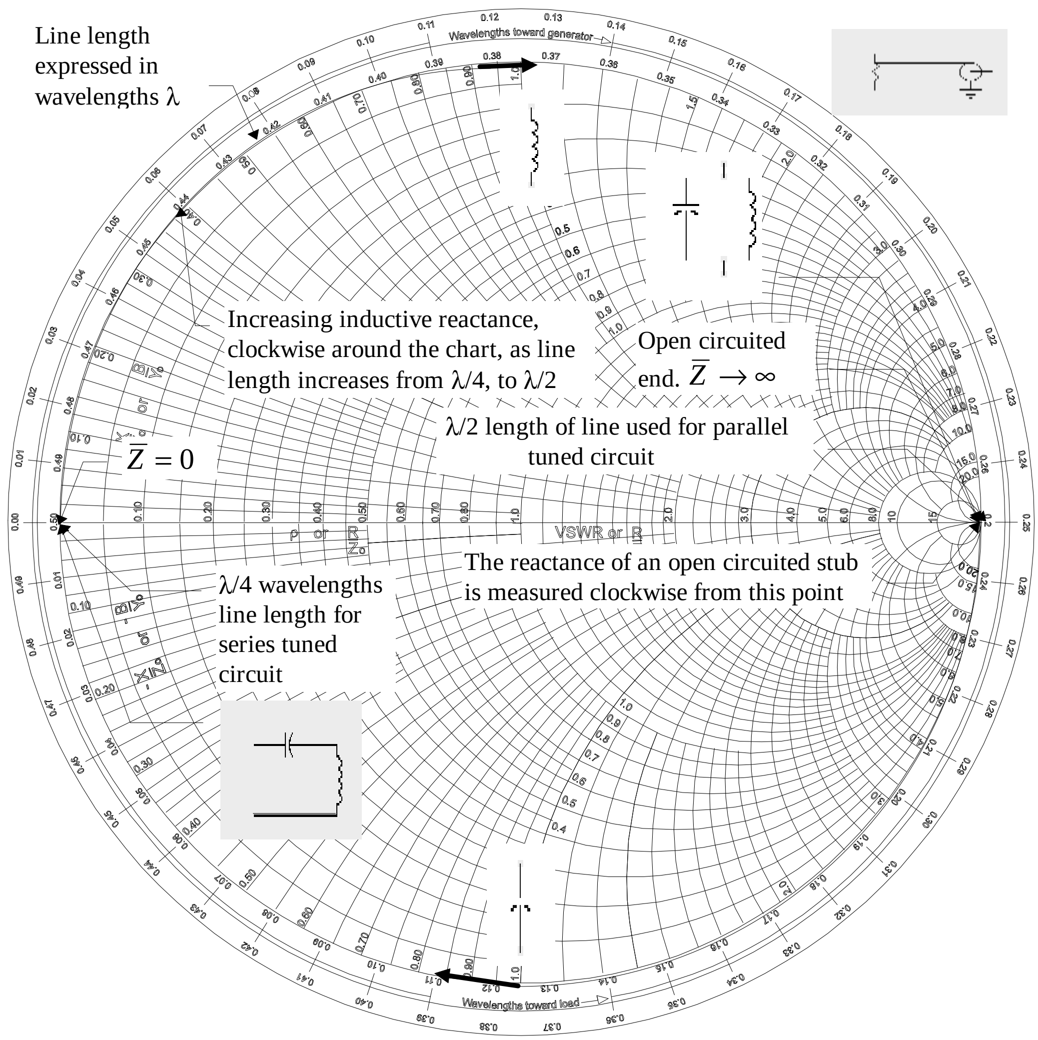



**Figure 45** **Inductance and capacitance on Smith chart as open-circuited line length increases**

Figures 46 and 47 show how the transmission line circuit components can be used in a series or parallel configuration with the transmission line.

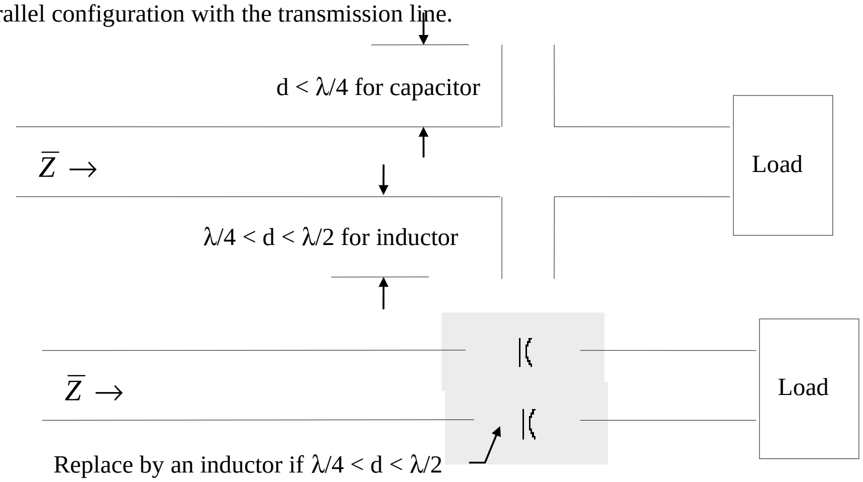



**Figure 46** **A series stub used as a capacitor or inductor**

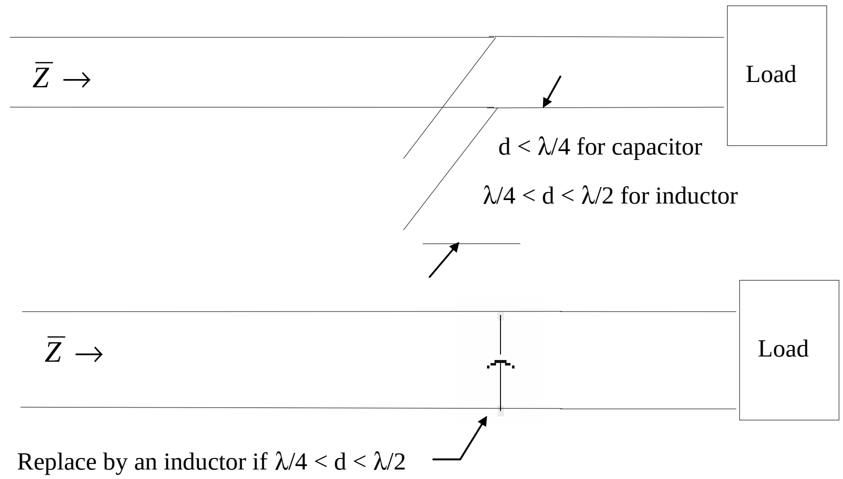



**Figure 47 A shunt stub used as an inductor or capacitor**

## 3.1.3. The stub line as a series or shunt matching element

### *3.1.3.1.Series matching element*

If an equal and opposite reactance is added in series with that of the load, at the single frequency in question, there will be an improvement in performance of the transmission line system. This performance will not necessarily be optimum unless the resistive component of the load is equal to the characteristic impedance of the line. Because the matching element in this case is taken to be in series with the line, only the Smith chart is considered. Let the mismatched impedance of the load be $\overline{Z}_L = \overline{R}_L + j\overline{X}_L$, then by adding an equal and opposite reactance $-j\overline{X}_L$, in series with the line will produce an effective load impedance of $\overline{Z}_L = \overline{R}_L + j\overline{X}_L - j\overline{X}_L = \overline{R}_L$. This will not reduce the VSWR to unity unless $R_L = Z_o$. The limitations of adding an equal and opposite reactance in series with the line are, that it will only cancel out the load reactance at one frequency, it will not necessarily reduce the VSWR to unity and the transmission line must be broken between it and the load to add the series element. If a band of frequencies is considered, then the problem must be defined in such a way, that after matching has occurred, the effective load impedance over this band produces a defined VSWR.

EXAMPLE 1

Figure 48 shows how, for example a load of $\overline{Z}_L = 0.5 + j1.5$ can be matched to lie on a VSWR circle of 2.0 by the addition of a series capacitance of reactance -j1.5, which can be formed from an open circuited coaxial cable stub of length 0.093λ long. The use of the Smith chart to design the capacitor is shown in figure 49.

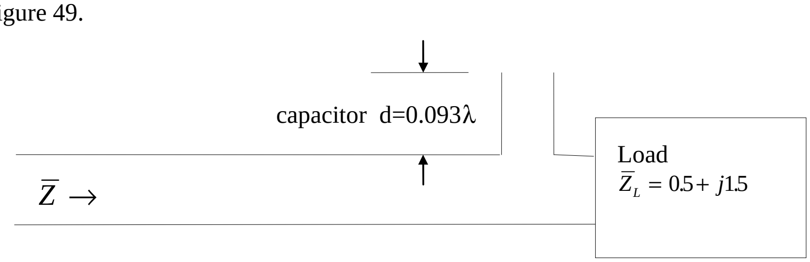



**Figure 48 A series capacitive stub used to match out a load**

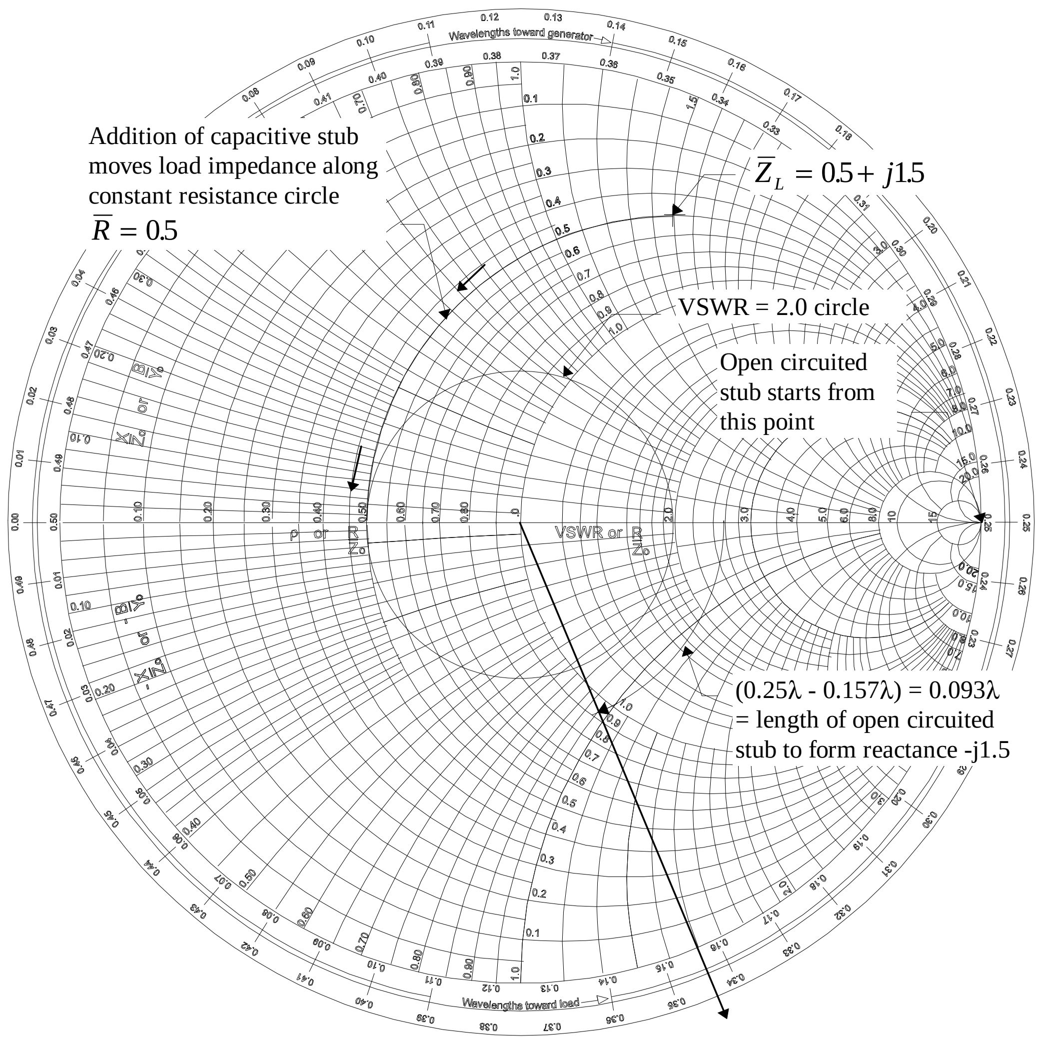



**Figure 49 A series capacitive stub used to match out a load**

Again, because the matching element in this case is taken to be in series with the unbalanced transmission line, only the Smith chart is considered. Let the mismatched impedance of the load be $\overline{Z}_L = \overline{R}_L - j\overline{X}_L$, then by adding an equal and opposite reactance $+j\overline{X}_L$, in series with the line will produce an effective load impedance of $\overline{Z}_L = \overline{R}_L - j\overline{X}_L + j\overline{X}_L = \overline{R}_L$. Similarly, if a band of frequencies is considered, then the problem must be defined in such a way, that after matching has occurred, the effective load impedance over this band produces a defined VSWR.

EXAMPLE 2

Figure 50 shows how, for example a load of $\overline{Z}_L = 0.5 - j1.5$ can be matched to lie on a VSWR circle of 2.0 by the addition of a series inductance of reactance j1.5, which can be formed from a short circuited coaxial cable stub of length 0.156λ long. The use of the Smith chart to design the capacitor is shown in figure 51.

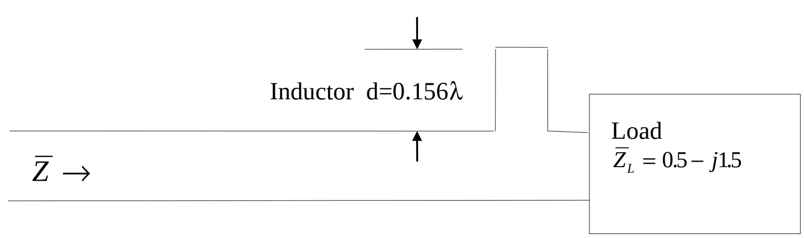



**Figure 50 A series inductive stub used to match out a load**

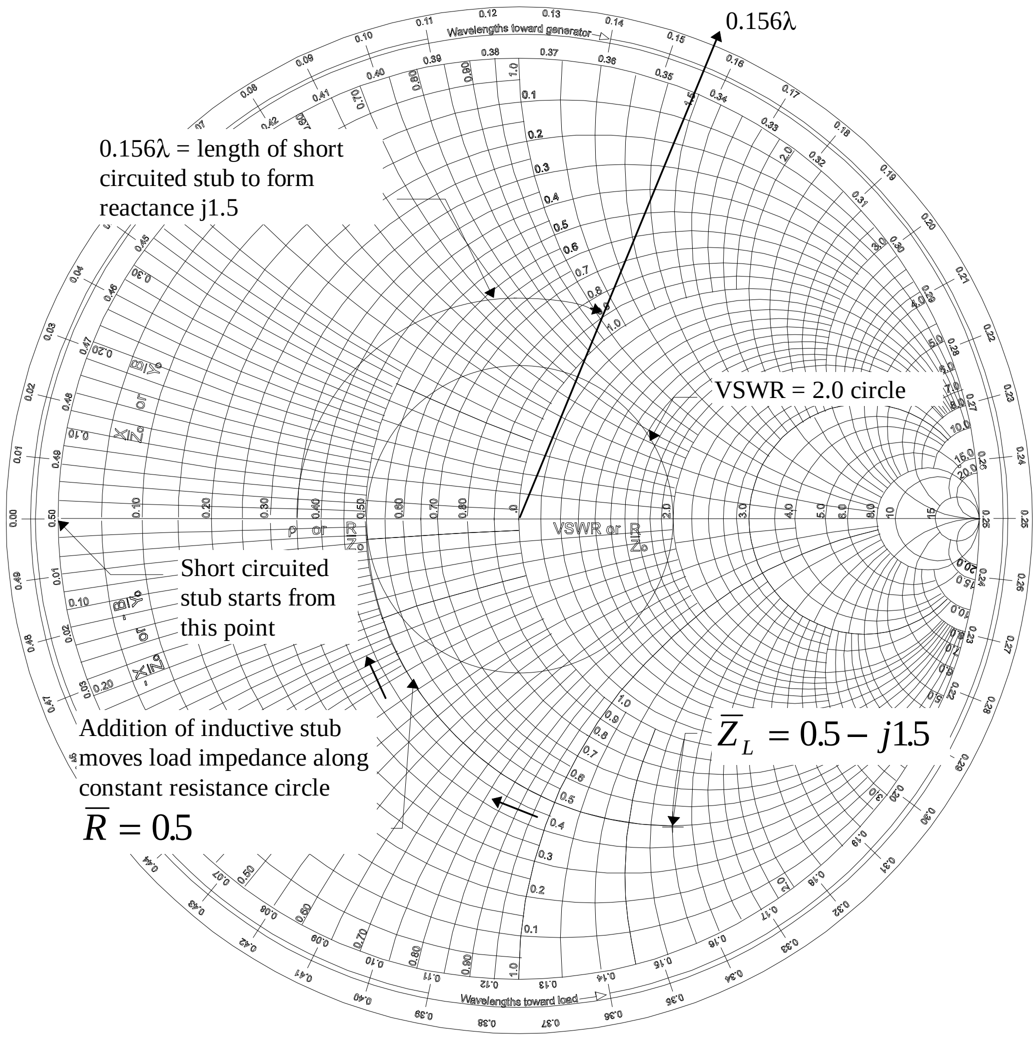



**Figure 51 A series inductive stub used to match out a load**

### *3.1.3.2.Shunt matching element*

Because the matching element in this case is taken to be in shunt with the unbalanced transmission line, the admittance chart is considered. Let the mismatched impedance of the load be $\overline{Z}_L = \overline{R}_L - j\overline{X}_L$, then by converting this to its admittance form

$\overline{Y}_L = \dfrac{\overline{R}_L}{\overline{R}_L^2 + \overline{X}_L^2} + j\dfrac{\overline{X}_L}{\overline{R}_L^2 + \overline{X}_L^2}$, and adding a negative susceptance (inductive) of value

$\overline{B} = -\dfrac{\overline{X}_L}{\overline{R}_L^2 + \overline{X}_L^2}$ in shunt with the line will produce an effective load impedance of

$\overline{Z}_L = \dfrac{\overline{R}_L^2 + \overline{X}_L^2}{\overline{R}_L}$. Similarly, if a band of frequencies is considered, then the problem must be defined in such a way, that after matching has occurred, the effective load impedance over this band produces a defined VSWR. The difficulty in this type of problem lies in the alternation between the admittance chart and the Smith chart, although for convenience, for simple problems, the Smith chart alone can be used..

EXAMPLE 3

Figure 52 shows how, for example a load of $\overline{Z}_L = 0.5 - j1.5$ can be matched to lie on a VSWR circle of 2.0 by the addition of a shunt inductance of susceptance -j0.6, which can be formed from a short circuited coaxial cable stub of length 0.164λ long. The use of the Smith chart to design the inductor is shown in figure 53. Note that the admittance points are plotted on the Smith Chart. Also from figure 53 it can be seen how an impedance can be converted to an admittance on the Smith chart by extending a line from the impedance point on the chart through the centre of the chart to an equal distance from the centre of the chart. Whenever, shunt components are used, admittances must be used. For a short circuited stub, the admittance is infinite. The position on the Smith chart is labeled accordingly, and the starting point for the length of the stub is taken from this point in a clockwise direction, so that a negative susceptance is obtained on the Smith chart, The inverse of equation 1-93 provides the equation for determining the length of the line for a normalized susceptance $\overline{B}$. As $\overline{B} = -0.6,\ d = 0.164\lambda$, as shown in Figure 53.

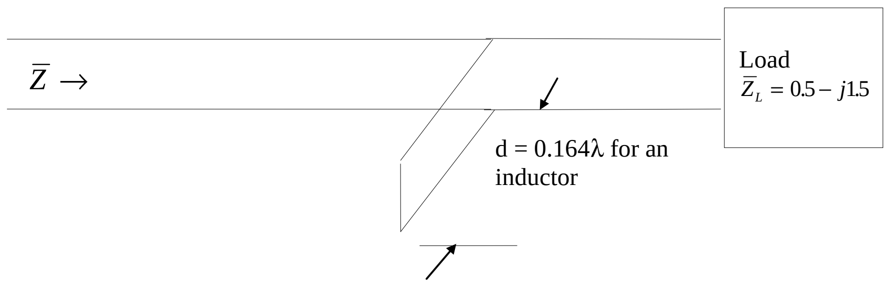



**Figure 52 A shunt inductive stub used to match out a load**

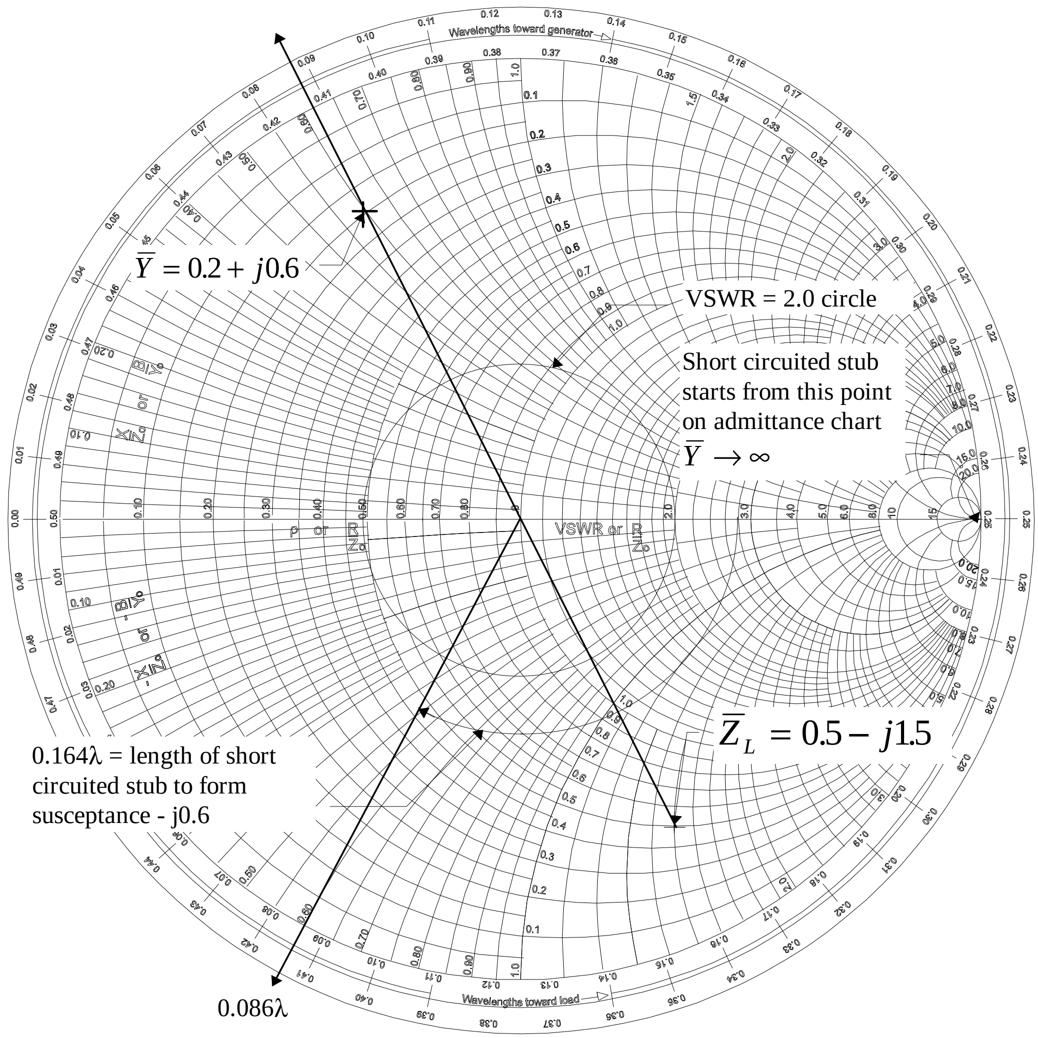



**Figure 53 A shunt inductive stub used to match out a load**

Figure 54 shows the admittance chart and the plot of the normalized admittance for the problem used in example 3. This chart should be referred to for an understanding of how admittance is plotted on the Smith chart of figure 53. Note how rotating Figure 53 by 180 degrees will produce the admittance chart shown in figure 54

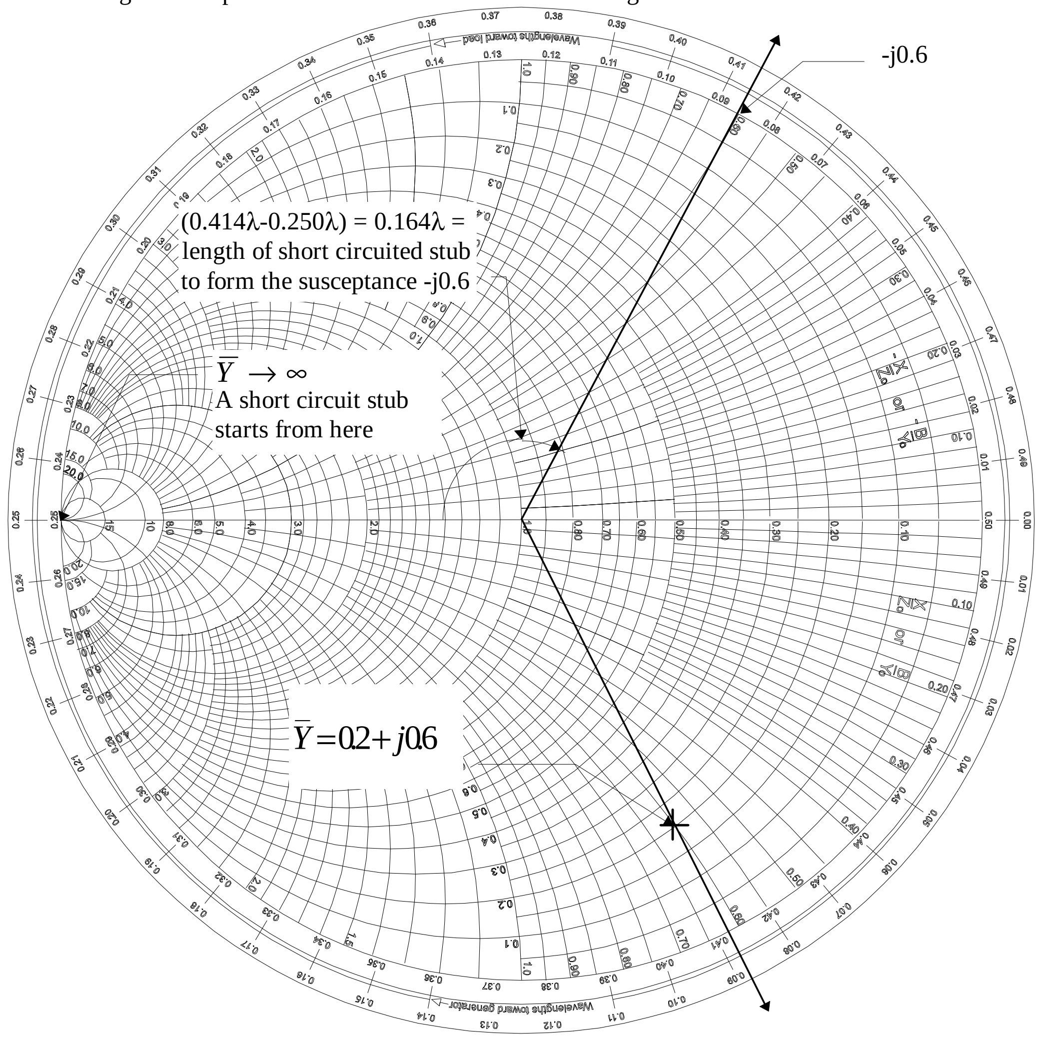



**Figure 54** **A shunt inductive stub used to match out a load using the admittance chart**

Normally, the shunt inductive stub is not placed directly across the load as given in example 3, but is placed some distance from it. This is because it may not be practically possible, and also because it does not reduce the value of the VSWR to unity. Placing the stub some distance from the load means that standing waves are still present between the matching stub and the load, but not between the source and the stub. The simple problem of example 3 becomes a little more complex, because if the distance from the load is to be found, in order to permit a VSWR of unity to be achieved, then the length of the stub must change to accommodate the change in load impedance at this distance along the transmission line from the termination to provide the necessary matching. xample illustrates how "single shunt stu atching" can e achie ed.

### 3.1.3.2.1. Single stub matching

As can be seen from example 4, the admittance of a mismatched line varies with the distance from the load. At some distance from the load, the load admittance will comprise the characteristic admittance $Y_0$, in parallel with some susceptance B. At this point $\overline{Y} = 1 \pm j\overline{B}$. If the normalized susceptance $\overline{B}$, can be canceled by the addition of another normalized

susceptance of the same magnitude but of opposite sign, the line will become matched at this point, as $\overline{Y} = 1$, and therefore $Y = Y_o$.
The normalized cancelling susceptance used can be provided by a short-circuited transmission line called a "stub line", which is placed in parallel with the main transmission line at some distance from the load. Not all values of load impedance can be matched using the single stub technique, and at times it is necessary to use two or more stubs. A two stub problem is provided in example 5.

EXAMPLE 4
Figure 55 shows how, for example a load of $\overline{Z}_L = 0.5 - j1.5$ can be matched to lie on a VSWR circle of 1.0 by the addition of a shunt inductance of susceptance -j2.2 at a distance 0.104 λ from the mismatched load. The shunt inductance of susceptance -j2.2 can be formed from a short circuited coaxial cable stub of length 0.0685λ long. The use of the Smith chart to design the inductor is shown in figure 56. Note how the VSWR of unity is obtained by rotating along a constant VSWR circle of 7.0 (VSWR=7.0 from the admittance point) from the load admittance to the unity normalized resistance circle. At the intersection of the unity circle, we find a transformed load susceptance given by 1 + j2.2. This is at a distance 0.104 λ from the load itself. In order to match the line, a -j2.2 susceptance is required to be added in parallel to the line at this point. The -j2.2 susceptance is obtained from the short circuit stub, which is found to be 0.0685λ long by moving from the infinite admittance on the Smith chart clockwise into the negative susceptance region of the Smith chart (lower semi-circle). Figure 56 shows the design on the Smith chart and each step is labeled accordingly.

The design procedure for single-stub matching can be summarized as follows:

- Plot the normalized load admittance on the Smith chart by extending an equidistant line through the centre of the chart from the plot of the normalized impedance,
- Draw the VSWR circle of such a radius that it passes through the admittance point,
- Move around this circle towards the generator, until the unity resistance circle is reached (which is actually the unity conductance circle if the chart is rotated to become the admittance chart), and note the distance in wavelengths between the load admittance and the point where the unity circle intersects the VSWR circle. This length will be the distance the stub is to be placed from the load.
- Determine from the chart the susceptance at the point where the unity circle intersects the VSWR circle, as this is the susceptance which is to be cancelled.
- Determine the length of the stub by plotting on the Smith chart the negative of the susceptance found in the step above and noting the distance in wavelengths, in a clockwise direction from the $\overline{Y} \rightarrow \infty$ point to this susceptance point. This length will be the required length of the stub.

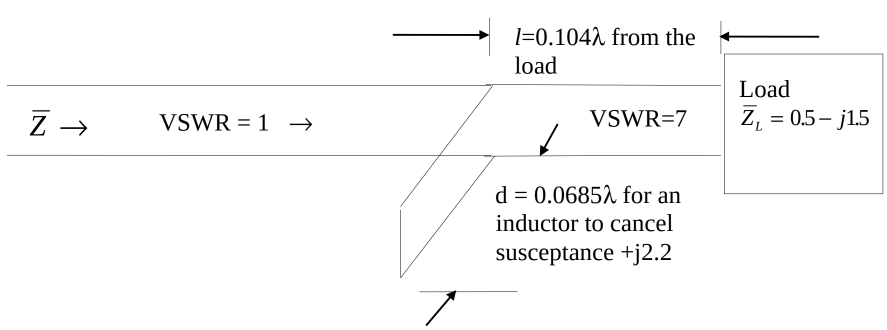



**Figure 55 A shunt inductive stub used to match out a load to a VSWR=1**

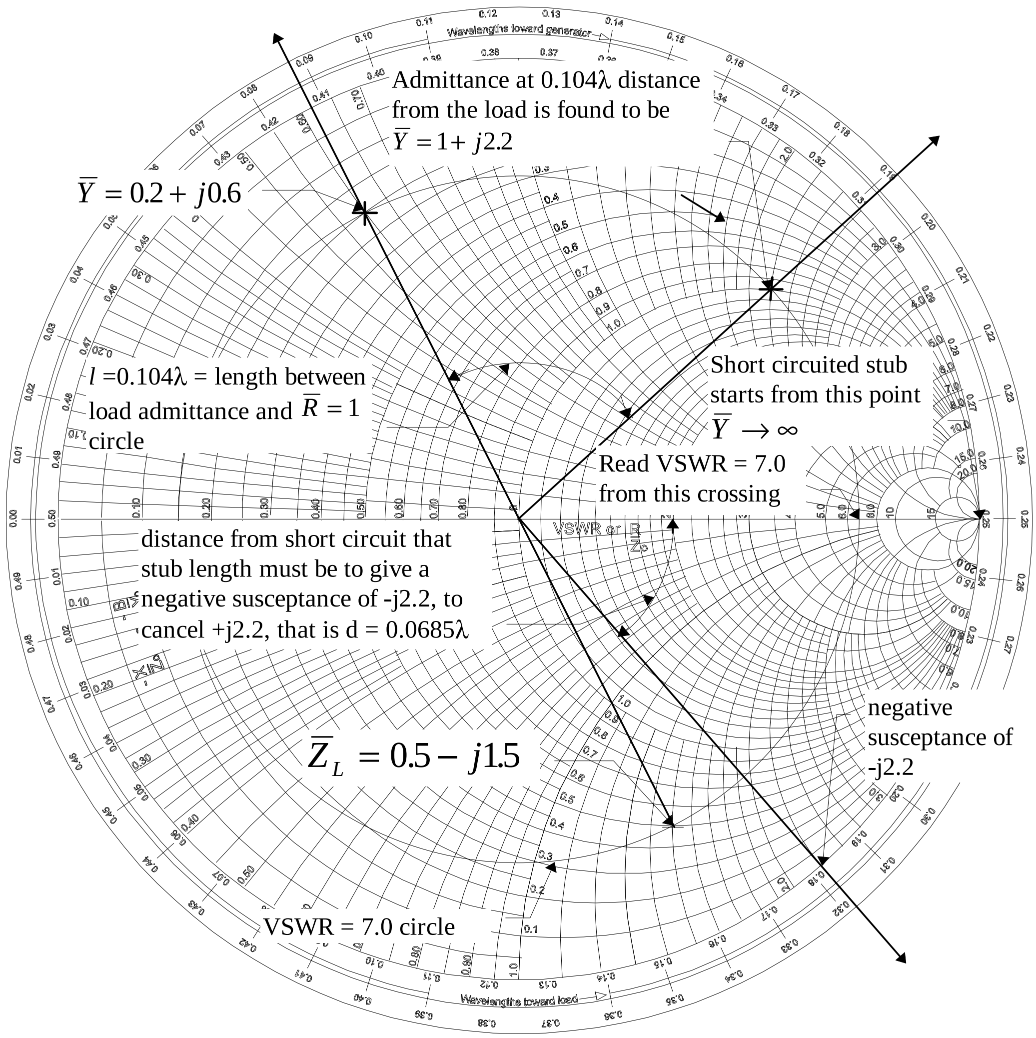



**Figure 56 A shunt inductive stub used to match out a load to a VSWR =1.**

### *3.1.3.2.1.1. Analytical approach to single stub matching*

In order to demonstrate how easy it is to use the Smith chart in the solution of transmission line problems, in comparison with the analytical approach, the analytical approach to solving the distance from the load and the length of the short-circuited stub is presented below. Using the transmission line equation for the input admittance of a mismatched transmission line

$$\overline{Y} = \frac{\overline{Y}_L + \tanh \gamma d}{1 + \overline{Y}_L \tanh \gamma d} \qquad \textbf{(1-88)}$$

**Determining the distance $l_1$, from the load**

For a lossless line, $\tanh \gamma d = j \tan \beta d$, and if the point of attachment of the stub is at a distance $l_1$ from the load, then from equation 1-88,

$$\overline{Y} = \frac{\overline{Y}_L + j \tan \beta l_1}{1 + j\overline{Y}_L \tan \beta l_1} \qquad \textbf{(3-5)}$$

Rationalizing into the form of $\overline{Y} = \overline{G} + j\overline{B}$ , where $\overline{Y}_L = g + jb$ gives

$$\overline{Y} = \frac{g[1 + \tan^2 \beta l_1]}{(g^2 + b^2)\tan^2 \beta l_1 - 2b \tan \beta l_1 + 1} - j\frac{[(g^2 + b^2)\tan \beta l_1 - b(1 - \tan^2 \beta l_1) - \tan \beta l_1 ]}{(g^2 + b^2)\tan^2 \beta l_1 - 2b \tan \beta l_1 + 1}$$

**(3-6)**

As it is required that the conductance at some distance $l_1$ be equal to the characteristic conductance $\overline{G} = 1$, and therefore

$$\overline{G} = \frac{g[1+\tan^2 \beta l_1]}{(g^2+b^2)\tan^2 \beta l_1 - 2b\tan\beta l_1 + 1} = 1$$

Solving this for $\tan\beta l_1$,

$$\tan\beta l_1 = \tan {}^{2\pi l_1}\!/_{\lambda} = \frac{b \pm \sqrt{g(g^2+b^2+1-2g)}}{g^2+b^2-g} \qquad \textbf{(3-7)}$$

As the load admittance in example 4 is given by $\overline{Y} = 0.2 + j0.6$ substituting g = 0.2 and b = 0.6, we find $l_1$ = 0.1038λ or 0.2200λ.

The Smith chart method given in example 4 gave $l_1$ = 0.104λ,

**Determining the length of the matching stub**

The input reactance X, of a lossless short-circuited stub of length $l_2$, is given by

$$j\overline{X} = j\tan\beta l_2$$

the susceptance $\overline{B}$ is thus found to be

$$j\overline{B} = \frac{1}{j\overline{X}} = \frac{-j}{\tan\beta l_2}$$

To obtain a match, the susceptance of the short-circuited stub must cancel the susceptance at the distance $l_1$, from the load, therefore

$$\frac{1}{\tan\beta l_2} = \frac{[(g^2+b^2)\tan\beta l_1 - b(1-\tan^2 \beta l_1) - \tan\beta l_1]}{(g^2+b^2)\tan^2 \beta l_1 - 2b\tan\beta l_1 + 1}$$

giving

$$\tan\beta l_2 = \tan {}^{2\pi l_2}\!/_{\lambda} = \frac{(g^2+b^2)\tan^2 \beta l_1 - 2b\tan\beta l_1 + 1}{[(g^2+b^2)\tan\beta l_1 - b(1-\tan^2 \beta l_1) - \tan\beta l_1]} \qquad \textbf{(3-8)}$$

For $l_1$ = 0.1038λ, g = 0.2 and b = 0.6,

$l_2$ = 0.0669λ, or -0.0669λ

The Smith chart method given in example 4 gave $l_2$ = 0.0685λ,

The second solution, $l_1$ = 0.2200λ and $l_2$ = -0.0669λ indicates that if the distance of the stub placement is further out, $l_1$ = 0.2200λ, and that the stub must be capacitive and of length (0.5 - 0.0669)λ = 0.4331λ.

The complexity of the mathematics required to get results against the ease of the Smith chart for reasonable accuracy becomes apparent in this example.

### 3.1.3.2.2. Double stub matching

As mentioned above some load impedances cannot be matched with a single stub. An extreme case may arise if the distance a matching single stub is to be placed from the load is zero, in other words, the stub is placed at the load, which due to practical considerations is not possible. It is convenient if we can develop some matching condition which allows us to move the stub to some other position. However, if we move the single stub to another position a mismatched condition would again occur. The way to overcome this problem is to introduce a second stub. If we state that the first stub is to be some distance from the load and that the second stub is to be separated by some distance from the first stub, then we have produced matching system which is practically possible. Once these distances have been specified and are thus fixed, the problem lies in determining the length of each of the two stubs. In addition to the situation of single stub matching which may occur at the load, there is the situation where the load impedance may need to be changed. Single stub matching would mean that the distance the stub is to be placed from the load would have to be changed. This again would present a practical problem if the load was an antenna at the top of a high tower. It is easier to change the length of two stubs rather than change the distance that a single stub must be from the load Thus, the double-stub matching technique permits matching to changes in load impedances by varying the stub lengths and not by varying the distance along the transmission line where the stubs are to be placed. Example 5 gives an example of how this can be done.

EXAMPLE 5

Figure 57 shows how, for example a load of $\overline{Z}_L = 0.133' + j0.266'$, or $\overline{Y}_L = 1.5 - j3.0$ can be matched to give a VSWR of 1.0 by the addition of a shunt stub of susceptance j0.412 at a distance 0.15λ from the mismatched load and by the addition of a shunt stub of susceptance -j2.26 at a distance 0.3λ from the stub closest to the load. The shunt stub of susceptance j0.412 can be formed from a short circuited coaxial cable stub of length 0.3115λ long and the shunt stub of susceptance j2.26 can be formed from a short circuited coaxial cable stub of length 0.433λ long. The use of the Smith chart to design the stubs is shown in figure 58.

The design procedure for double-stub matching can be summarized as follows:

- Plot the normalized load admittance $\overline{Y}_{LA} = 1.5 - j3.0$ , on the Smith chart, at point A, by extending an equidistant line through the centre of the chart from the plot of the normalized impedance, $\overline{Z}_L = 0.133' + j0.266'$
- Draw a matching circle, that is a $\overline{R} = 1$ circle, 0.3λ *towards* the *load* from the original position of the matching circle.
- Move on a constant reflection circle, (VSWR=8.1) through the load admittance, a distance of 0.15λ towards the generator to point B. This gives the new normalized load admittance at the position of the first stub as $\overline{Y}_{LB} = 0.135 - j0.372$
- At position B we insert the first stub. Because the admittance just before the stub is the same as just after it and is only the conductance 0.135, we therefore, rotate from point B along the circle of constant conductance towards the generator until we intersect the rotated circle at point C. The normalized admittance at point C is $\overline{Y}_{LC} = 0.135 + j0.039$
- Because the admittance at point C just after the first stub and the admittance at point D, just before the second stub, is a 0.30λ section of lossless transmission line, the admittance at point D may be obtained by rotating from point C along a constant VSWR circle (VSWR=7) a distance of 0.3λ. Point D may be found also by the intersection of the constant VSWR=7 circle and the matching circle. The normalized admittance at point D is found to be 1 - j2.26,
- The susceptance of the second stub is required to cancel the -j2.26 at point D to that at the matching point at the centre of the Smith chart,
- The normalized susceptance of the first stub is given by the difference in the susceptances between points C and B, that is 0.135+j0.04 - (0.135 - j0.372) = j0.412. This allows the susceptance at point B to be changed to that required at point C, and therefore is the actual value of susceptance required of the first stub,
- The normalized susceptance of the second stub is given by the difference in susceptances at the centre of the chart and point D, that is 1.0 + j0.0 - (1 - j2.26) = j2.26
- The length of each stub is determined by rotating from the short-circuited end of each stub on the Smith chart, that is the point where the admittance tends to infinity, along the edge of the Smith chart towards the generator, to the value of the susceptances +j0.412 and +j2.26. That is the length of the first stub is 0.3115λ, and that of the second stub is 0.433λ.

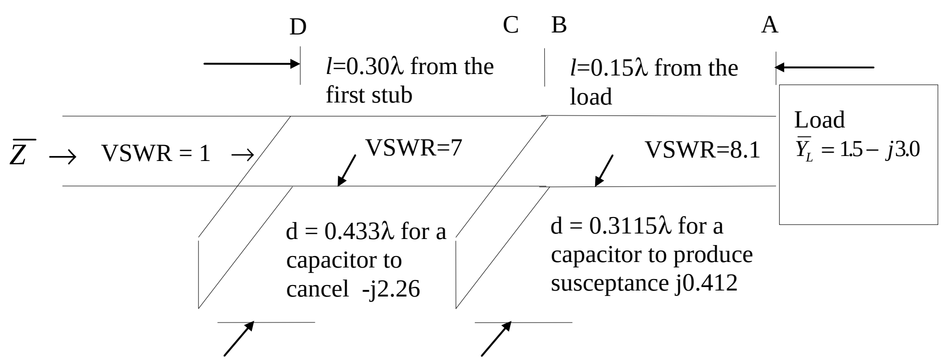



**Figure 57 Double stub matching used to match out a load to a VSWR =1.**

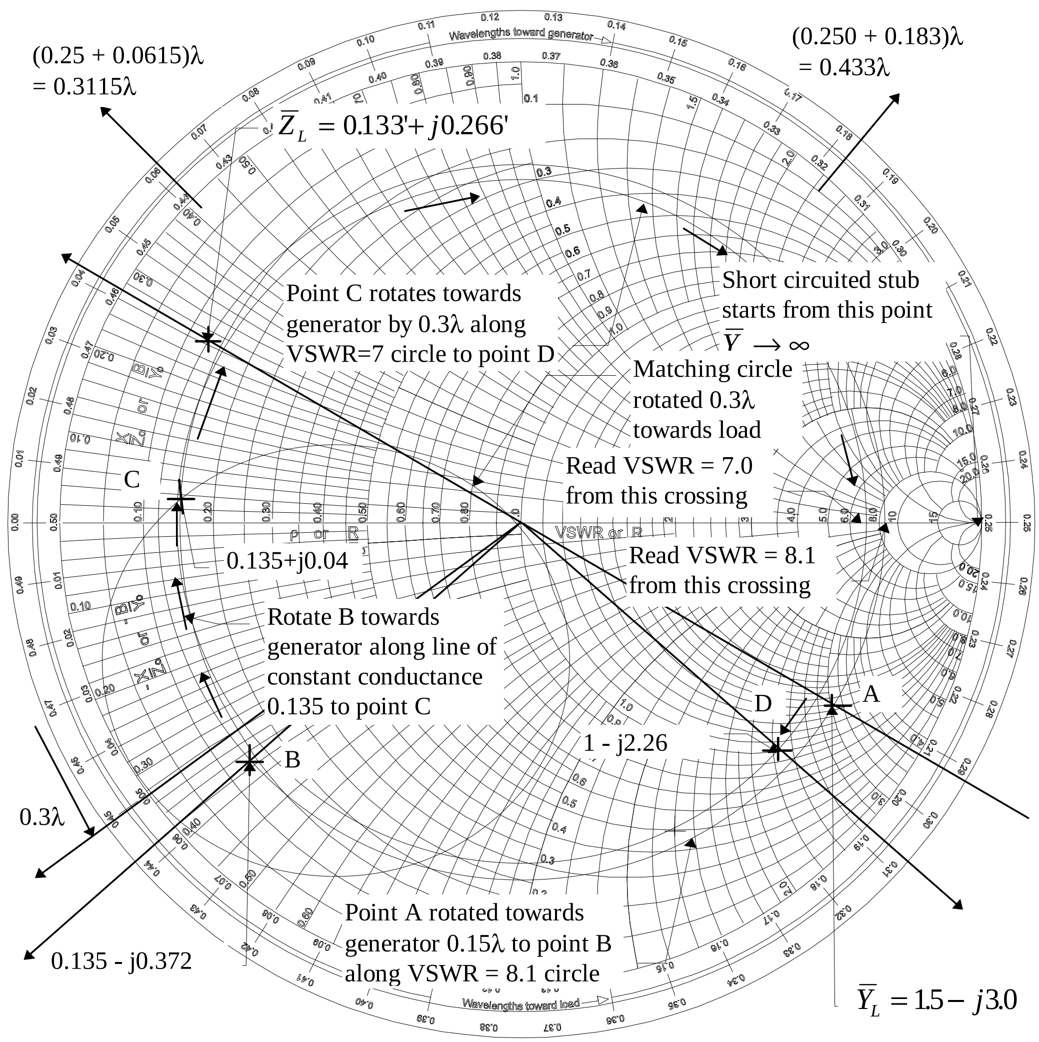



**Figure 58 Double stub matching used to match out a load to a VSWR =1.**

EXPLANATION OF THE DESIGN PROCEDURE

In understanding how the design procedure works, it is informative if we work the procedure from the solution back to the problem. The main feature of the design procedure is the rotation of the matching circle through the portion of a wavelength λ, which equals the separation of the two stubs, which in the case of example 5 is 0.3λ. This is done because the final result ***must*** be a point lying on the matching circle, namely point D in the above example. By rotating the matching circle *towards the load* 0.3λ, which is back towards the first stub, point D becomes point C, which is the admittance required to produce the VSWR circle which eventually will permit point D to be determined. As the first stub produces a susceptance only, the value of the first stub must be the difference between the value of the susceptance at point C and the translated load admittance at point B, and must be such that it moves point B along the line of constant conductance from point B to point C. Thus, the value of the susceptance of the first stub which has to be added to the susceptance at point B is the difference between the susceptance at point C and point B. The movement along the line of constant conductance from point B to point C is made because the stub, being susceptance only, does not change the value of conductance. Point C must be determined as it is the pivot point in determining the solution to the problem. Point C permits the susceptance of the first stub to be determined and permits the required VSWR circle to be determined so that rotation around it will produce point D, a point on the matching circle. Point B, which is the translated load admittance, is the admittance of the load after it has been rotated along a constant VSWR circle, which in the problem is 8.1, a distance 0.15λ. The value of each of the first stub length can be found by noting the length of line in wavelengths from the $\overline{Y} \rightarrow \infty$ point to the value of susceptance representing the negative value of the difference in susceptance values between point C and B, and the value of the second stub length can be found from the negative value of the susceptance found at point D.

CASES WHERE MATCHING CANNOT BE ACHIEVED

If the load admittance $\overline{Y}_L = 1.5 - j3.0$ given in example 5 was not translated to $\overline{Y}_{LB} = 0.135 - j0.372$ by the 0.15λ length of transmission line between the load and the first stub, then the admittance could not be matched using the double stub method of matching. That is if the first stub was directly across the load admittance $\overline{Y}_L = 1.5 - j3.0$ and the intersection of the rotated matching circle was required to determine point C by following along the circle of constant conductance $\overline{G} = 1.5$, there could not possibly be any intersection. This is because the circle of constant conductance lies outside of the rotated matching circle. This is shown in figure 59.

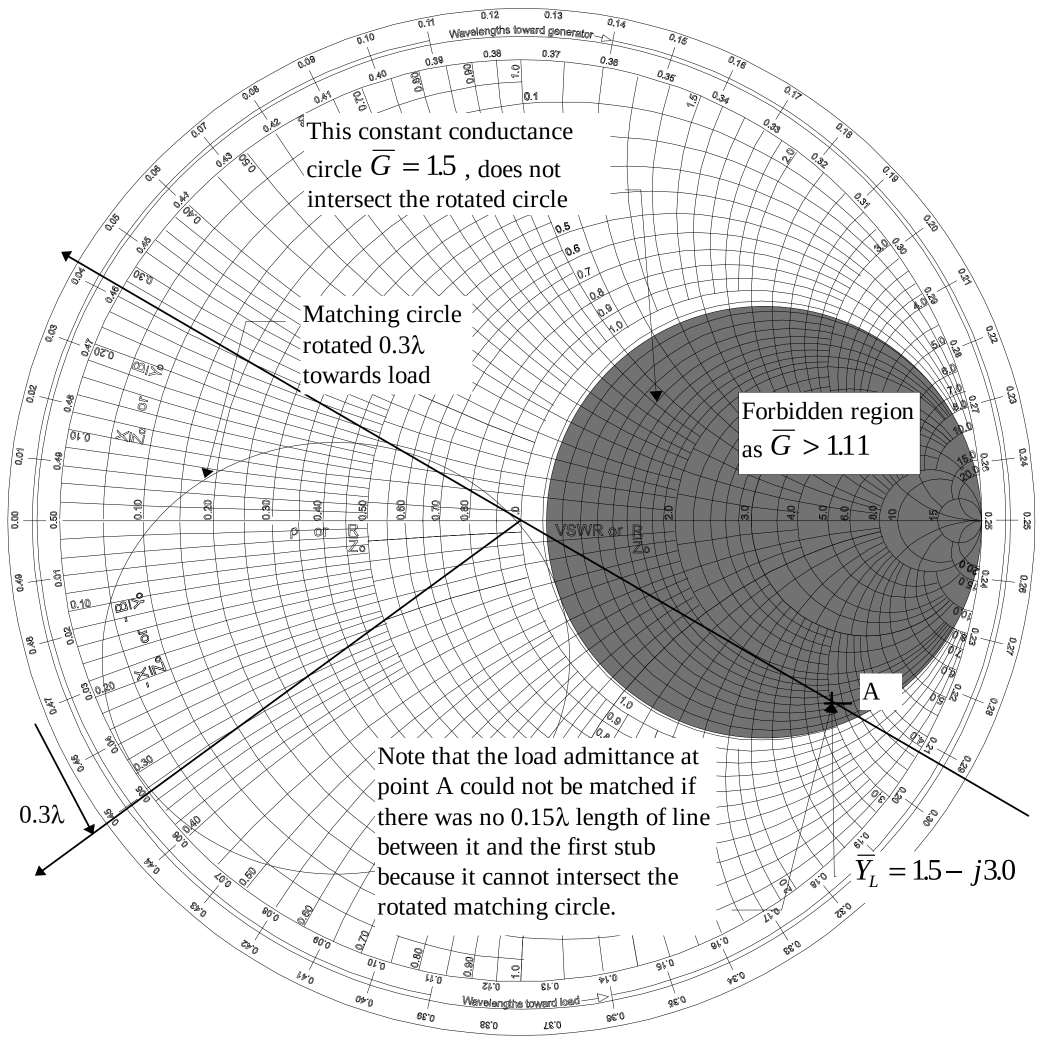



**Figure 59** **Double stub matching prohibited region.**

Following is the discussion on the analytical approach to determine the condition for the forbidden region and the matching of a lossless transmission line using the double-stub matching method.

###### 3.1.3.2.2.1. Analytical approach to double stub matching

**Forbidden Region**

Referring to figure 57 and 58. Let the transformed admittance at point B be $\overline{Y}_{LB}$, then just to the left of the first stub the admittance will be $\overline{Y}_{LC} = \overline{Y}_{LB} + j\overline{B}_S$ where $\overline{B}_S$ is the increase in susceptance to move the admittance from point B to the intersection of the rotated circle at point C on figure 58. In other words, $\overline{B}_S$ is the susceptance of the first stub. As the susceptance at point C moves along a constant VSWR circle through a distance between the two stubs of d (wavelengths), the admittance change along the lossless transmission line $\overline{Y}_{LD}$, is given by equation 3-5 as

$$\overline{Y}_{LD} = \frac{\overline{Y}_{LC} + j\tan\beta d}{1 + j\overline{Y}_{LC}\tan\beta d} \qquad \textbf{(3-5)}$$

Letting $x = \tan\beta d = \tan 2\pi d/\lambda$, and since at the point D, we must have $\overline{Y}_{LD} = 1 + j\overline{B}$, where $\overline{B}$ is the susceptance which has to be cancelled by the second stub,

$$1 + j\overline{B} = \frac{\overline{G}_{LB} + j\overline{B}_{LB} + j\overline{B}_S + jx}{1 + jx(\overline{G}_{LB} + j\overline{B}_{LB} + j\overline{B}_S)} \qquad \textbf{(3-9)}$$

By equating the real part to unity, we obtain the quadratic equation in $\overline{G}_{LB}$

$$\overline{G}_{LB}^2 - \overline{G}_{LB}\frac{1+x^2}{x^2} + \frac{\left(1 - x\overline{B}_{LB} - x\overline{B}_S\right)^2}{x^2} = 0 \qquad \textbf{(3-10)}$$

$$\overline{G}_{LB} = \frac{1+x^2}{2x^2}\left[1 \pm \sqrt{1 - \frac{4x^2\left(1 - x\overline{B}_{LB} - x\overline{B}_S\right)^2}{\left(1+x^2\right)^2}}\right] \qquad \textbf{(3-11)}$$

Equation 3-11 must be real, because $\overline{G}_{LB}$ must be real, this means that the expression under the square root sign must be either positive or zero. This in turn means that the algebraic fraction ranges from zero to unity because it can never be negative, and thus the expression under the square root sign ranges from unity to zero. Therefore,

$$0 \le \overline{G}_{LB} \le \frac{1+x^2}{x^2} = \frac{1}{\sin^2 2\pi d/\lambda} \qquad \textbf{(3-12)}$$

Equation 3-12 provides the condition for the forbidden region on the Smith chart as shown in Figure 59 and all values of transferred load admittance outside of or on the $\overline{G}_{LB}$ circle can be matched . Substitution of the value d = 0.3λ into equation 3-12 gives the limit for $\overline{G}_{LB}$ as

$$0 \le \overline{G}_{LB} \le \frac{1}{\sin^2 2\pi d/\lambda} = 1.1056$$

which corresponds to that found on the Smith chart of figure 59.

This theory is developed assuming that the line attenuation is zero and from it, it appears that all load impedances could be matched if the distance between the two stubs was chosen to be close to zero or λ/2. In practice the maximum value of stub that can be obtained is limited by the finite attenuation of the transmission line used to make the stub. If jβ was replaced by α + jβ, it would be found that matching of all values of load impedance could not be obtained, even with a λ/2 spacing. In addition, stub spacings close to λ/2 produce frequency sensitive matching networks, so that in practice λ/8 or 3λ/8 spacings are preferred. The 3λ/8 spacing is used at the higher frequencies, where the use of the λ/8 is prohibited by the closeness of the two stubs.

**Determination of the susceptance of the two stubs**

FIRST STUB (CLOSEST TO THE LOAD)

Solution of equation 3-10 for $\overline{B}_S$ , the susceptance of the first stub, is found to be

$$\overline{B}_S = -\overline{B}_{LB} + \frac{1 \pm \sqrt{\left(1+x^2\right)\overline{G}_{LB} - x^2\overline{G}_{LB}^2}}{x} \tag{3-13}$$

where $\overline{B}_{LB}$, $\overline{G}_{LB}$ and $x = \tan 2\pi d/\lambda$ are all known.
The Smith chart determined $\overline{G}_{LB}$ to be 0.135λ, whereas using equation 3.6, the value of $\overline{G}_{LB}$ is found to be 0.142.
Substituting
$\overline{B}_{LB} = -0.374$, $\overline{G}_{LB} = 0.142$ and $x = \tan 0.6\pi = -3.078$ we find $\overline{B}_S = 0.419$
which compares with $\overline{B}_S$ = 0.412 from example 5 using the Smith chart in figure 58. The alternative solution to $\overline{B}_S$ is -0.3207 . This means that from point B, the line of constant conductance would have to be followed around the Smith chart until it intersected the rotated circle to the right of point B to intersect at an admittance of 0.142 - j0.6868, so that the value of the susceptance of the first stub is (0.142 - j0.6947) - B(0.142 - j0.374) = $\overline{B}_S$ = -0.3207.
The length of the stub to produce the susceptance of -0.3207 is found to be 0.049λ.

SECOND STUB (FURTHEST FROM THE LOAD)
If the imaginary part of equation 3-9 is equated to $\overline{B}$ , then

$$\overline{B} = \frac{\left(1 - x\overline{B}_{LB} - x\overline{B}_S\right)\left(\overline{B}_{LB} + \overline{B}_S + x\right) - x\overline{G}_{LB}^2}{\left(1 - x\overline{B}_{LB} - x\overline{B}_S\right)^2 + x^2\overline{G}_{LB}^2} \tag{3-14}$$

By substituting equation 3-13 for $\overline{B}_S$ into equation 3-14, we get

$$\overline{B} = -\left(\frac{\overline{G}_{LB} \pm \sqrt{\overline{G}_{LB}\left(1+x^2\right) - x^2\overline{G}_{LB}^2}}{x\overline{G}_{LB}}\right) \tag{3-15}$$

The negative value of $\overline{B}$ given by equation 3-15 must be taken in order to determine the susceptance of the second stub for obtaining a matched condition.
Substituting $\overline{G}_{LB} = 0.142$ and $x = \tan 0.6\pi = -3.078$ we find $\overline{B} = -2.280$ or $2.930$
the value of $\overline{B}$ = -2.280 compares favourably with the value of -2.26 obtained using the Smith Chart. The length of the stub to cancel the susceptance of -2.26 is found to be 0.434λ.
Again notice that the Smith chart determined $\overline{G}_{LB}$ to be 0.135λ, whereas using equation 3.6, the value of $\overline{G}_{LB}$ was found to be 0.142.
The alternative value of $\overline{B}$ = 2.930 is also obtained if the constant conductance from point B to the intersection of the rotated circle had followed an anti-clockwise direction. The length of the stub to produce this susceptance is found to be 0.052λ.
In summary, the Smith chart and the analytical method permits two solutions for each of the stub lengths to be found. The choice of which solution to take is determined by practical considerations. The Smith chart results compare favourably with the analytical method of obtaining the length of the two stubs, given the distance of the first stub from the load and the distance between the two stubs. If open circuit stubs had been used instead of short circuit stubs, their lengths would be determined by starting at the left hand side edge of the chart $\left(\overline{Y} = 0\right)$ and determining the distance in wavelengths by going clockwise around the edge of the chart (towards the generator) to the value of the susceptance determined by equation 3-13 for the first stub, and negative of the susceptance found by equation 3-15 for the second stub.
The disadvantage of not being able to match all of the load admittances encountered, with a double stub tuner may be overcome if a triple stub tuner is used. This is discussed in the next section.

### 3.1.3.2.3. Triple stub matching

Figure 60 and figure 61 show the case where an admittance point $\overline{Y}_{LA} = 3.7 + j1.8$, which has been transferred from the load $\overline{Y}_L = 0.74 - j0.344$, by an 0.125λ length of transmission line to the position of the first stub and which lies in the forbidden circle region, cannot be matched. This is because the lines of constant conductance of this reflected admittance do not intersect the 0.125λ rotated circle.
The apparent obvious way of overcoming this problem would be to change the length of the transmission line between the stub and the load from 0.282λ to say, 0.154λ as shown in figure 63.

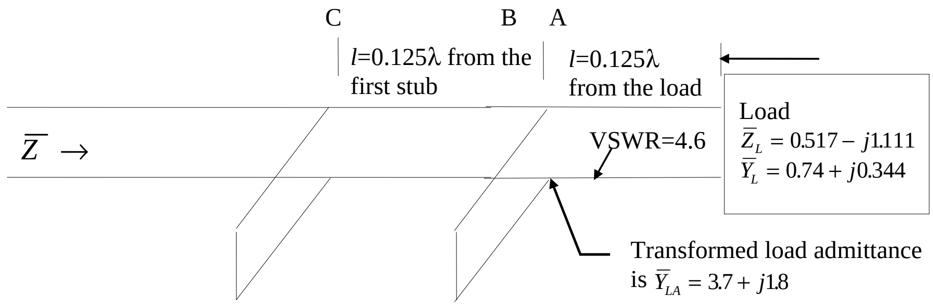



**Figure 60** **Case where double stub matching cannot be achieved**

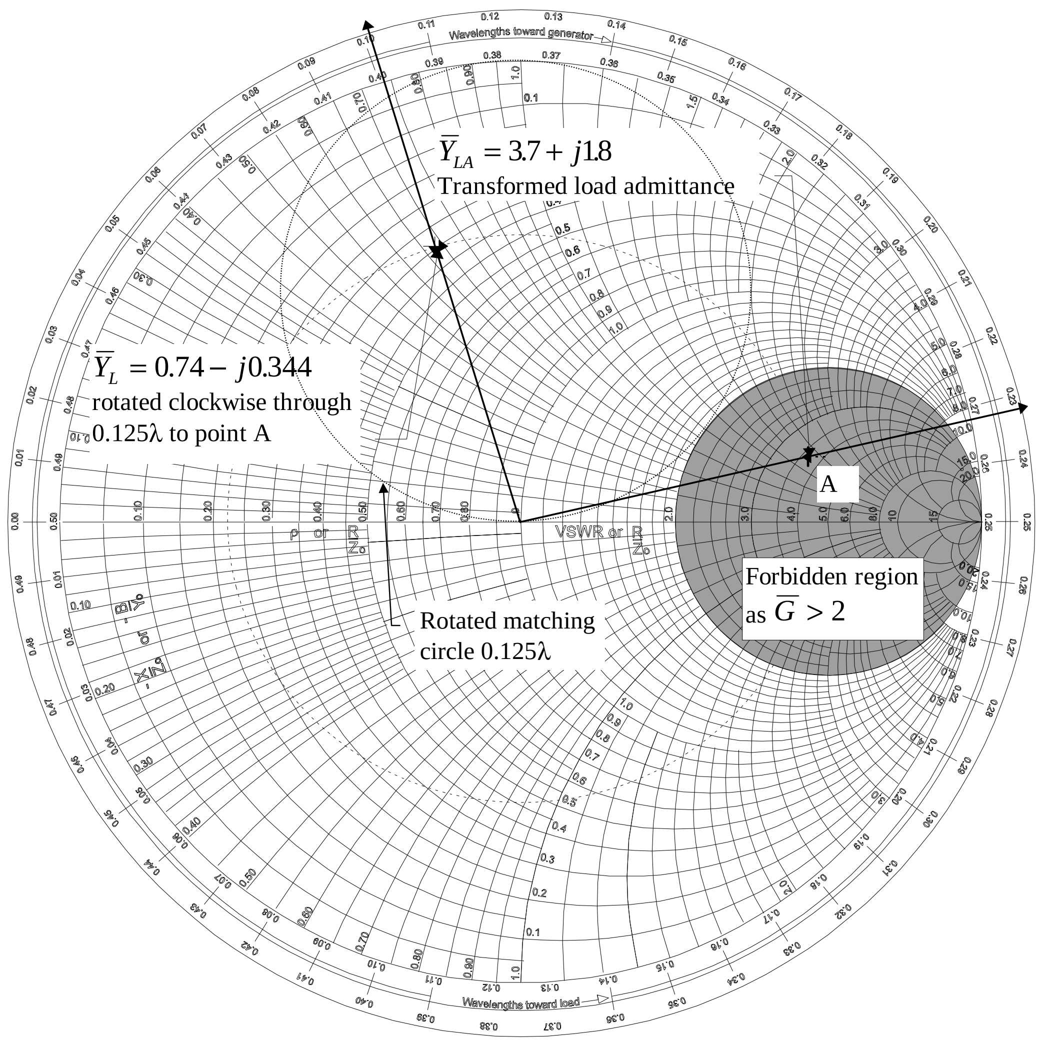



**Figure 61** **Case where double stub matching cannot be achieved**

However, changing the admittance point to $\overline{Y}_{LA} = 0.35 + j0.7$ also has its problems as discussed in the design procedure of example 6, below.
Decreasing the distance between the first and second stub, according to equation 3-12, will reduce the radius of the forbidden circle region, and allow a greater range of admittances which could be matched, but the practical limit of the stub separation is given by λ/8 or 3λ/8. As shown in Figure 60 and 61, with the constraint that the stub separation is fixed at λ/8 or 3λ/8, and where the distance of the first stub to the load is also constrained, there may exist load admittances which cannot be matched. The distance of the first stub to the load may be constrained, for example, after installation of the double-stub tuner and the load impedance then changed for some reason, such as the changing out of a smaller antenna with a larger antenna.
To maintain a fixed distance between the two stubs and match over all possible load admittances, where the distance between the load and the first stub may also be constrained, the triple stub tuner may be employed. This may seem to be an expensive way of doing things, but it is practically expedient when considering that you may be working at the top of a 150 m tower and wish to make as few adjustments as possible For matching, all that is required is to slide in or out tuning rods on each of the three stubs to vary the length of each of the stubs.

EXAMPLE 6
Design a shunt triple-stub tuner to match a load impedance of $\overline{Z}_L = 1.64 + j1.97$. The distances between the stubs are 0.125λ, and the distance of the first stub from the load is 0.154λ.
As all stubs are in shunt, admittances must be used for the determination of the solution. Figures 62 to 65 show how, for example a load of $\overline{Z}_L = 1.64 + j1.97$, or $\overline{Y}_L = 0.25 - j3.0$ can be matched to give a VSWR of 1.0 by the addition of a shunt stub of susceptance j0.9 or -j0.3 at a distance 0.154λ from the mismatched load and by the addition of a second shunt stub of susceptance j0.41, j5.39, -j1.37 or j3.61 at a distance 0.125λ from the stub closest to the load and also by the addition of a third shunt stub of susceptance j1.63 or j0.39 at a distance 0.125λ from the second stub. The first stub can be formed from a short circuited coaxial cable stub of length 0.367λ or 0.203λ, the second stub by a short-circuited cable 0.312λ, 0.471λ, 0.100λ or 0.475λ long and the third short-circuited shunt stub of length 0.088λ or 0.191λ. The use of the Smith chart to design the stubs is shown in figures 63 and 65.

The design procedure for triple-stub matching can be summarized as follows:

- Plot the normalized load admittance $\overline{Y}_{LA} = 0.35 + j0.7$ , on the Smith chart, at point A, by rotating the load admittance $\overline{Y}_L = 0.25 - j0.3$ clockwise though a distance of 0.154λ, as shown in Figure 63.
- Draw a forbidden region circle from equation 3-12, that is $\overline{G} > \dfrac{1}{\sin^2 2\pi d/\lambda}$ where d is the distance between the stubs, which is 0.125λ . This gives $\overline{G} > 2$

Figure 62 and 63 show the first steps in the determination of the first stub susceptance.

- Rotate the forbidden circle anti-clockwise (towards the load) a distance of 0.125λ as shown in Figure 63. This will permit us to determine whether the transformed load admittance lies within the forbidden circle. If it does, it means that if there was no first stub, point A would on rotation by another 0.125λ to reach the second stub, lie in the forbidden region. In this case, point A, $\overline{Y}_{LA} = 0.35 + j0.7$, lies within the forbidden region.
- In order to bring the transformed load admittance at point A outside of the rotated forbidden circle, a susceptance must be added to $\overline{Y}_{LA} = 0.35 + j0.7$. By rotating point A along a line of constant conductance $\overline{G}_{LA} = 0.35$ such that it lies outside of the forbidden circle, will ensure that the distance to the second stub 0.125λ does not cause a "no solution" condition to occur. oints and are chosen in this exa le, as shown on Figure 63, giving admittances B: $\overline{Y}_{LB} = 0.35 + j0.4$ and C: $\overline{Y}_{LC} = 0.35 + j01.6$
- The difference in susceptances between point A and point B, or point C determines the length of the first stub. The susceptance to be added to reach point C is j(1.6 - 0.7) = j0.9 and the susceptance to be added to reach point B is j(0.4 - 0.7) = -j0.3. Thus, the length of the first short-circuited stub is either 0.367λ (+j0.9) or 0.203λ.(-j0.3).
- As shown in Figure 65, the admittances at points B and C are now rotated towards the generator through 0.125λ, to points and , as the distance between the first stub and the second stub is given as 0.125λ. Because the forbidden circle was rotated and the

points and selected outside of this rotated circle, the points and lie outside of the forbidden circle region.

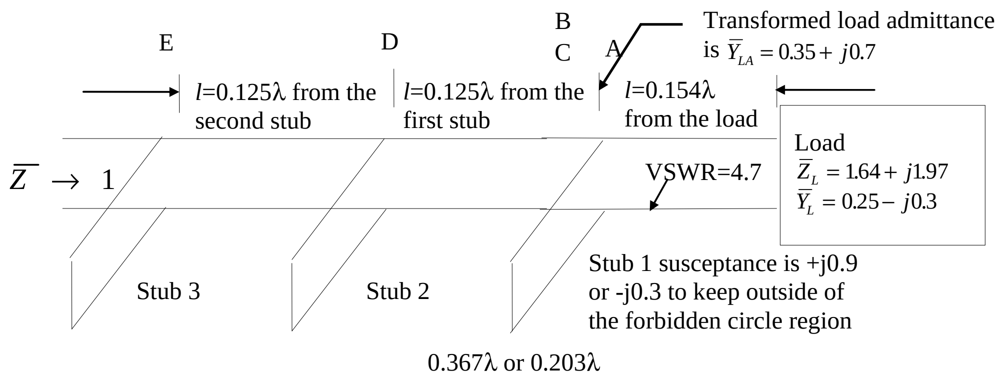



**Figure 62 Determination of first stub susceptance**

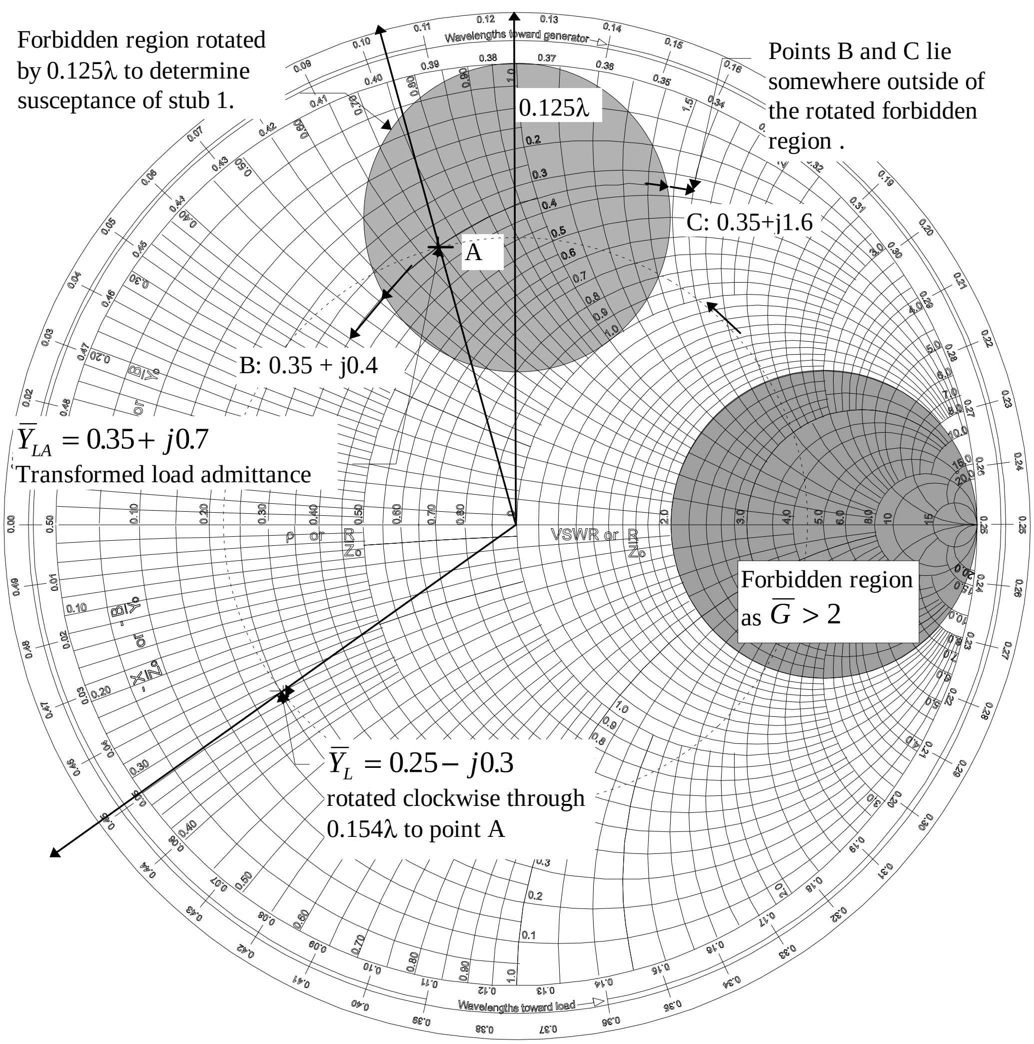



**Figure 63 Determination of first stub susceptance**

- From here on, the problem becomes that of the double-stub matching problem and point . . follows along lines of constant conductance to intersect the rotated matching circle at points and . imilarly, point . -j3.49) is rotated along circles of constant conductance to intersect the matching circle at points D (1.45 + j1.90) and . . .

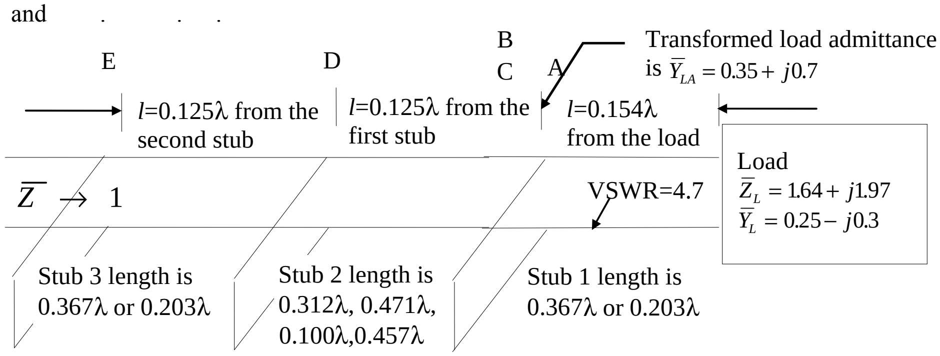



**Figure 64 Determination of second and third stub susceptance**

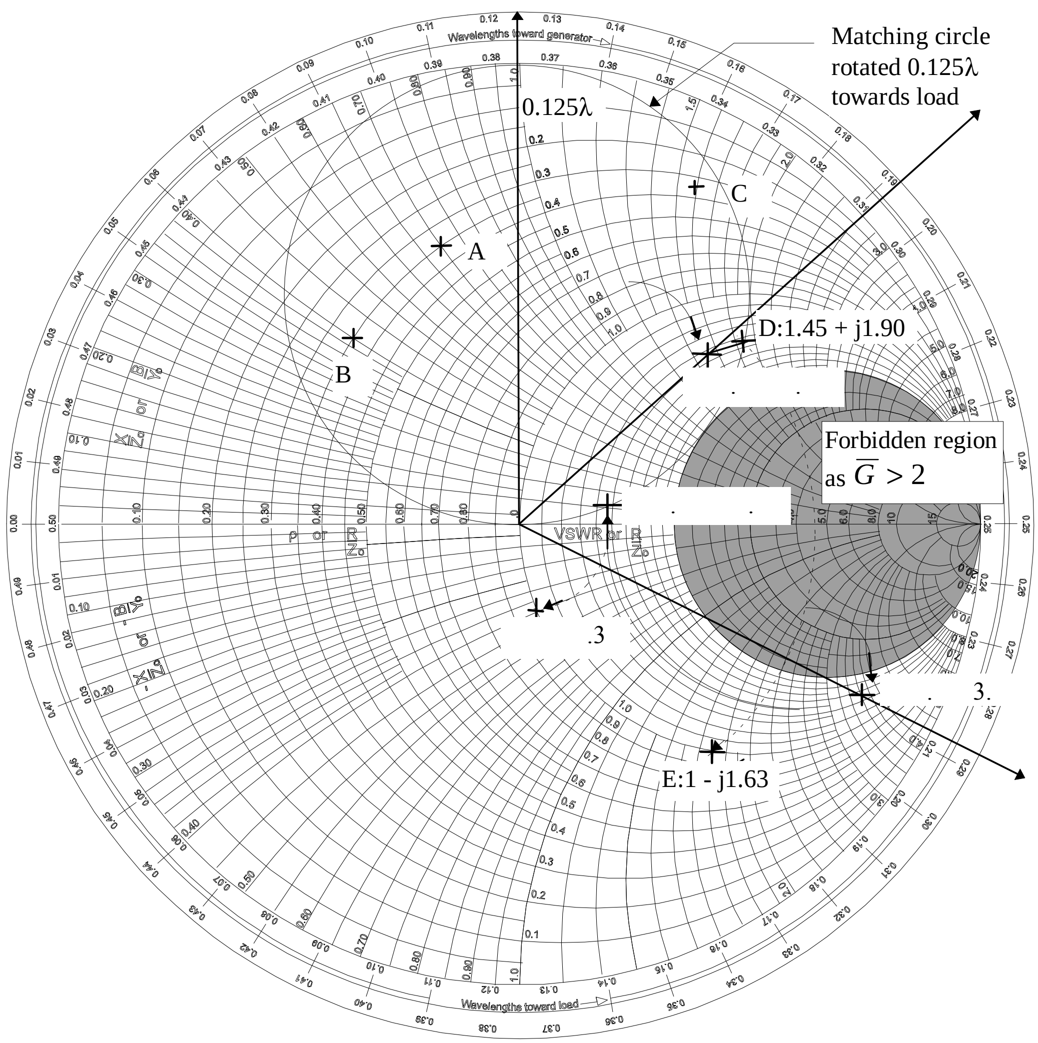



**Figure 65 Determination of second and third stub susceptance**

- The length of the second stub is found from the susceptance difference between the susceptance at point and and , as well as the susceptance difference between the susceptance at point to point and . hat is

D - . , - .3 , - - .3 , - 3. . he corresponding lengths are found to be:

D - .3 λ, D - . λ, - . λ, - . λ.

- The rotation of points and through . λ along circles of constant VSWR to intersect the matching circle, allows the admittance at the third stub to be determined. The length of the stub is to provide an opposite susceptance to cancel the admittance at the third stub. The admittances are found to be:
  For D: 1- . 3 and for -j0.39. Thus. the susceptance of the third stub is to be +j1.63 or +j0.39, which means that the length of the third stub is D: 0.088λ or . λ.

### *3.1.3.2.3.1. Analytical approach to triple-stub matching*

Using equations 3-13 and 3-15 for the double stub matching, we find

- The susceptance of the second stub is found to be:

D - . , - .3 3, - - .3 3, - 3.

- The susceptance of the third stub is determined to be: +j1.616 or +j0.384.

All values found using the Smith chart compare favourably with the calculated values.

The only difference between the analysis of the triple stub tuner and the double stub tuner is in the determination of the susceptances $j\overline{B}_1$ and $-j\overline{B}_2$, which are used to determine the length of the first stub. Once these have been determined, the analysis follows that of the second stub tuner, with two admittances to be matched rather than one.
The first stub provides a susceptance which transforms the already transformed load admittance $\overline{Y}_{LA}$ (transformed by the distance along the transmission line from the load admittance $\overline{Y}_L$ to the first stub position) into two admittances $\overline{Y}_{LB}$ and $\overline{Y}_{LC}$, by the addition of susceptances $j\overline{B}_1$ and $-j\overline{B}_2$, that is, $\overline{Y}_{LC} = \overline{Y}_{LA} + j\overline{B}_1$ and $\overline{Y}_{LB} = \overline{Y}_{LA} - j\overline{B}_2$. These two new admittances are chosen to lie outside of the forbidden circle region, so that when they are transformed to their new admittance values at the position of the second stub, they can be matched using the method described for double stub tuning . In terms of $\overline{Y}_{LB}$ and $\overline{Y}_{LC}$, where $\overline{Y}_{LB} = \overline{G}_{LB} + j(-\overline{B}_{LB})$ and $\overline{Y}_{LC} = \overline{G}_{LC} + j\overline{B}_{LC}$, equations 3-13 and 3-15 become,

**Second stub**

$$\overline{B}_{S1} = -\overline{B}_{LB} + \frac{1 \pm \sqrt{\left(1+x^2\right)\overline{G}_{LB} - x^2\overline{G}_{LB}^2}}{x} \qquad \textbf{(3-16a)}$$

$$\overline{B}_{S2} = -\overline{B}_{LC} + \frac{1 \pm \sqrt{\left(1+x^2\right)\overline{G}_{LC} - x^2\overline{G}_{LC}^2}}{x} \qquad \textbf{(3-16b)}$$

where $x = \tan\beta d = \tan 2\pi d/\lambda$, and d is the distance between the second and third stub.

**Third stub**

$$\overline{B}_{T1} = -\left(\frac{-\overline{G}_{LB} \pm \sqrt{\overline{G}_{LB}\left(1+x^2\right) - x^2\overline{G}_{LB}^2}}{x\overline{G}_{LB}}\right) \qquad \textbf{(3-17a)}$$

$$\overline{B}_{T2} = -\left(\frac{-\overline{G}_{LC} \pm \sqrt{\overline{G}_{LC}\left(1+x^2\right) - x^2\overline{G}_{LC}^2}}{x\overline{G}_{LC}}\right) \qquad \textbf{(3-17b)}$$

**EXERCISES**

**Double stub tuning**

1. Design a double stub tuner to match a normalized load impedance $\bar{Z}_L = 0.4 - j0.2$, which has its first short circuit stub at a distance of 0.1λ from the load and a distance between the first and second short circuited stub of 0.25λ.

*Answer:* Stub closest to the load 0.277λ or 0.385λ, second stub 0.137λ or 0.362λ.

2. Design a double stub tuner to match a normalized load impedance $\bar{Z}_L = 2 + j0.5$, which has its first short circuit stub at a distance of 0.2λ from the load and a distance between the first and second short circuited stub of 0.6λ. Complete the design using shunt stubs and then series stubs

*Answer:*

**Shunt stubs:**

Stub closest to the load 0.424λ or 0.1421λ, stub furthest from the load 0.4365λ or 0.2855λ

**Series stubs**

Stub closest to the load 0.085λ or 0.196λ, stub furthest from the load 0.403λ or 0.294λ.

3. Design a shunt double stub matching system to match a normalized load impedance $\bar{Z}_L = 1.1 - j2.0$, which has its first stub at a distance of 0.07λ from the load and its second stub 0.125λ from the first stub. Design the stubs to be open circuit or short circuit.

*Answer:*

**Open circuit stubs**

Stub closest to the load 0.107λ or 0.393λ, stub furthest from the load 0.199λ or 0.375λ,

**Short circuit stubs**

Stub closest to the load 0.143λ or 0.357λ stub furthest from the load 0.449λ or 0.125λ

### 3.2. *The transmission line used as a transformer*

Another widely used impedance matching technique uses one or more transmission line sections in cascade with the load impedance. This section will consider the design procedure for the short-transformer and the quarter-wavelength transformer. The multisection quarter-wavelength transformers and the tapered-line transformer used for multi-frequency applications, will be considered in chapter 4..

#### 3.2.1. The short transformer

This transformer, derived from a transmission line which has a characteristic impedance different from that of the main transmission line, can, under certain conditions, be used to match the main transmission line to a load impedance which is other than the characteristic impedance of the line. The length of the line is usually less than a quarter wavelength, and thus the term *short transformer* is used.
From equation 1-85, the input impedance $Z_{in}$, looking into a transmission line with a characteristic impedance of $Z_{oA}$, of length *l*, and terminated with a load impedance $Z_L$, is given by

$$Z_{in} = Z_{oA}\frac{Z_L + jZ_{oA}\tan\beta l}{Z_{oA} + jZ_L\tan\beta l} \quad \text{ohms} \tag{3-18}$$

If it is required that a section of this transmission line (the short transformer) with the load attached, is to be attached to the main transmission line of characteristic impedance $Z_o$, as shown in figure 66, then for perfect matching a set of design equations can be found for $Z_{in} = Z_o$ , where the general length *l* becomes a fixed length d, from

$$Z_o = Z_{oA}\frac{Z_L + jZ_{oA}\tan\beta d}{Z_{oA} + jZ_L\tan\beta d} \tag{3-19}$$

Given that $Z_L = R + jX$ , we find

$$Z_o\left(Z_{oA} - X\tan\beta d\right) + jZ_o R\tan\beta d = RZ_{oA} + jZ_{oA}\left(X + Z_{oA}\tan\beta d\right) \tag{3-20}$$

By equating the real and imaginary parts,

$$\tan\beta d = \tan 2\pi d/\lambda = \frac{Z_{oA}\left(Z_o - R\right)}{XZ_o} = \frac{\overline{Z}_{oA}\left(1 - \overline{R}\right)}{\overline{X}} \tag{3-21}$$

and

$$Z_{oA} = \sqrt{RZ_o - \frac{X^2 Z_o}{Z_o - R}} \quad \text{or} \quad \overline{Z}_{oA} = \sqrt{\overline{R} - \frac{\overline{X}^2}{1 - \overline{R}}} \tag{3-22}$$

In order for a solution to equation 3-22 to exist,

$$\overline{R} > \frac{\overline{X}^2}{1 - \overline{R}}, \text{ giving } \overline{R}(1 - \overline{R}) > \overline{X}^2 \text{ and } 1 \neq \overline{R}, \text{ but } \overline{R} < 1 \text{ or } \overline{R} > 1 \tag{3-23}$$

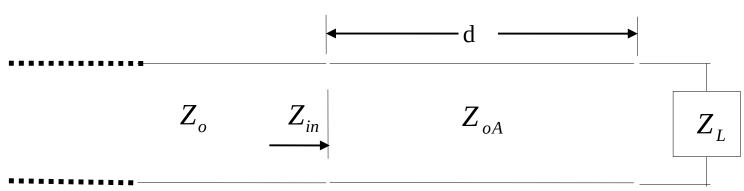



**Figure 66 A short transformer**

EXAMPLE 7
Design a short transformer to match a load of (30 + j20)Ω, given that the characteristic impedance of the transmission line is 50Ω.
The conditions in equation 3-23 are satisfied, and the characteristic impedance of the transmission line used for the transformer is found from equation 3-22, as 22.36 Ω, and the length of the transmission line used for the transformer, found from equation 3-21, is 0.067 λ.
Notice that the length of the transmission line is less than a quarter-wavelength..

### *3.2.1.1. Analysis of short transformer for Smith chart application*

From equation 1-58 and given that $\overline{Z}_L = \overline{R} + j\overline{X}$ and $\rho_L = \rho_r + j\rho_j$ we find

$$\rho_r = \frac{\overline{X}^2 + \overline{R}^2 - 1}{\overline{X}^2 + (\overline{R}+1)^2} \qquad \textbf{(3-24)}$$

$$\rho_r = \frac{2\overline{X}}{\overline{X}^2 + (\overline{R}+1)^2} \qquad \textbf{(3-25)}$$

Equations 3-24 and 3-25 can be expressed in terms of $Z_{oA}$ using equation 3-22, to give

$$\rho_r = \frac{(\overline{Z}_{oA}^2 + 1)(\overline{R}-1)}{(\overline{Z}_{oA}^2 + 1)(\overline{R}-1) + 2(\overline{R}+1)} \qquad \textbf{(3-26)}$$

$$\rho_j = \frac{2\overline{X}}{(\overline{Z}_{oA}^2 + 1)(\overline{R}-1) + 2(\overline{R}+1)} \qquad \textbf{(3-27)}$$

Rearranging equation 3-22 for $\overline{X}$, equation 3-27 becomes

$$\rho_r = \frac{2\sqrt{(\overline{Z}_{oA}^2 - \overline{R})(\overline{R}-1)}}{(\overline{Z}_{oA}^2 + 1)(\overline{R}-1) + 2(\overline{R}+1)} \qquad \textbf{(3-28)}$$

Equations 3-26 and 3-28 allow the reflection coefficient at the input to the short transformer to be determined, with the short transformer terminated with a load $Z_L$, and with a short transformer characteristic impedance as determined from equation 3-22.
Moving along the short transformer transmission line, a constant VSWR must exist. That is a circle on the Smith chart must exist on which the load impedance must lie. As the transformer is matched to the main transmission line, the circle must also pass through the centre of the Smith chart. In order to find the equation of this circle, which is the locus of the translated value of load impedance along the transformer as seen from the main transmission line, and thus its radius and its centre, equations 3-26 and 3-28 must be arranged so that $\overline{R}$ is eliminated. On eliminating $\overline{R}$, we find that two circles exist on the Smith chart, namely,

$$\rho_r^2 - \rho_r\left(\frac{\overline{Z}_{oA}^2 - 1}{\overline{Z}_{oA}^2 + 1}\right) + \rho_j^2 = 0 \qquad \textbf{(3-29)}$$

and

$$\rho_r^2 - \rho_r\left(\frac{1 - \overline{Z}_{oA}^2}{1 + \overline{Z}_{oA}^2}\right) + \rho_j^2 = 0 \qquad \textbf{(3-30)}$$

showing that it doesn t matter whether the value of $\overline{Z}_{oA}^2$ is greater or less than unity, as there will always be a circle either left or right of the Smith chart centre to accommodate the value of $\overline{Z}_{oA}^2$. The circle which is applicable is that which cuts the load impedance points $\overline{R}, \overline{X}$.
Notice that the circle lies on the $\rho_r$ axis with centre at ($\rho_{rC}$,0), where

$$\rho_{rC} = \frac{1}{2}\left(\frac{\overline{Z}_{oA}^2 - 1}{\overline{Z}_{oA}^2 + 1}\right) \text{or } \frac{1}{2}\left(\frac{1 - \overline{Z}_{oA}^2}{1 + \overline{Z}_{oA}^2}\right) \qquad \textbf{(3-31)}$$

and

$$radius = \frac{1}{2}\left(\frac{\overline{Z}_{oA}^2 - 1}{\overline{Z}_{oA}^2 + 1}\right) \text{or } \frac{1}{2}\left(\frac{1 - \overline{Z}_{oA}^2}{1 + \overline{Z}_{oA}^2}\right) \qquad \textbf{(3-32)}$$

To construct the circle on the Smith chart, the centre is first determined and then from this centre a circle is drawn which passes through the centre of the chart and which also passes through the plotted load impedance.
To determine the centre of the transformer circle on the Smith chart, either a graphical method can be used, knowing that the distance from the load to the centre is the same as the distance from the centre to the centre of the Smith chart, or an analytical method can be used, by making use of equation 1-78 and 3-31, that is,

$$\overline{R}_C = \left( \frac{3\overline{Z}_{oA}^2 + 1}{\overline{Z}_{oA}^2 + 3} \right) \text{ for } Z_{oA} > Z_o \text{ or } Z_{oA} < Z_o \qquad \textbf{(3-33)}$$

$$\overline{R}_C = \left( \frac{\overline{Z}_{oA}^2 + 3}{3\overline{Z}_{oA}^2 + 1} \right) \text{for } Z_{oA} > Z_o \text{ or } Z_{oA} < Z_o \qquad \textbf{(3-34)}$$

The choice of either equation 3-33 or 3-34 is determined by which circle has the load impedance $\overline{Z}_L$, passing through it.
When matched, the short-transformer circle will pass through the centre of the Smith chart and cut the $\rho_r$ axis at twice its radius. That is, in terms of resistance on the Smith chart,

$$\overline{R}_{Diam} = \overline{Z}_{oA}^2 = \left( \frac{Z_{oA}}{Z_o} \right)^2 \text{ for } Z_{oA} > Z_o \text{ or for } Z_{oA} < Z_o \qquad \textbf{(3-35)}$$

or

$$\overline{R}_{Diam} = \frac{1}{\overline{Z}_{oA}^2} = \left( \frac{Z_o}{Z_{oA}} \right)^2 \text{ for } Z_{oA} > Z_o \text{ or for } Z_{oA} < Z_o \qquad \textbf{(3-36)}$$

Again, the equation to choose is that which has its circle passing through the load impedance $\overline{Z}_L$.
Equations 3-27 to 3-36 have been developed from the position of the main transmission line looking into the transmission line short transformer and not from within the short transformer itself. As a consequence of this the Smith chart used is normalized to the main transmission line. This Smith chart will *not* permit the length of the short-transformer to be determined directly. The only way to do this is to construct a Smith chart normalized to $Z_{oA}$, and from the load impedance on this chart, normalized to $Z_{oA}$, read off the angle to the $\overline{X} = 0$ line. This angle in wavelengths is the length of the short-transformer.
The main transmission line Smith chart (normalized to $Z_o$) is useful in determining the characteristic impedance of the short-transformer $Z_{oA}$, as shown by equations 3-35 and 3-36. Once the centre of the transformer circle has been graphically determined for a given load impedance, the intersection of the transformer circle with the $\overline{X} = 0$ line permits the value of $Z_{oA}$ to be determined from $\overline{Z}_{oA}^2$, and hence, a Smith chart normalized to $Z_{oA}$ can be constructed on the same chart to determine the length of the short-transformer, as shown in example 8, below.

EXAMPLE 8
Using the Smith chart, design a short-transformer to match a load of (30+ j20) $\Omega$, given that the characteristic impedance of the main transmission line is 50 $\Omega$.

Figure 67 shows the Smith chart with the load impedance normalized to the main transmission line characteristic impedance (50 $\Omega$), and the short transformer circle determined graphically. The graphical construction is made by first determining the centre of the short transformer circle by having equal distances from the centre of the Smith chart to the centre of the transformer circle and from the load to the centre of the transformer circle. As the circle drawn from this radius does not extend outside of the Smith chart boundaries ( $\rho_r < \pm 0.5$,) that is, the radius is less than $\overline{R} = 0.33$, a solution to this problem exists. The boundary circles enclosing permissable values of load resistance and reactance for a solution to exist are shown on Figure 67. Equation 3-22 and 3-34 indicate that $\overline{R}_C = 0.5$, and equation 3-36 shows that $\overline{R}_{Diam} = 0.2$. The Smith chart gives $\overline{R}_{Diam} = 0.2$ or $\overline{Z}_{oA} = \sqrt{0.2} = 0.447$ and the construction for the Smith chart normalized to $Z_{oA}$ on the same chart, to determine the length d (in wavelengths), of the transformer, gives d=0.067 $\lambda$, which is the same as the calculated value given in Example 7.

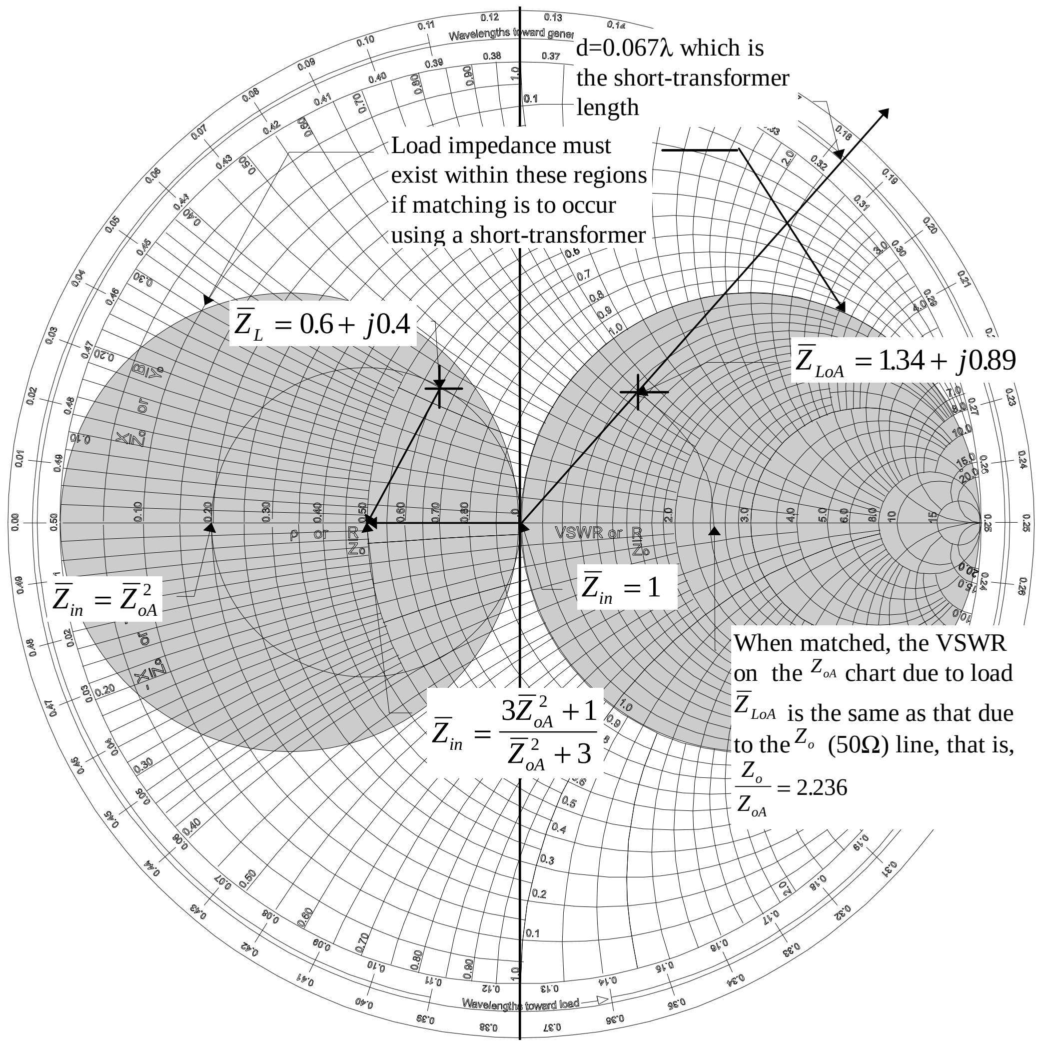



**Figure 67 Design of short-transformer of Example 8.**

### 3.2.2. Quarter-wave transformers

As discussed in section 3.2.3.2.2., for a λ/4 length of transmission line ($d = \lambda/4$), when the load is not correctly terminated, ($Z_L \neq Z_o$), the normalized impedance looking into the line for various types of line is given by,

- **Lossy transmission line**

$$\overline{Z} = \frac{\overline{Z}_L \sinh(\alpha\lambda/4) + \cosh(\alpha\lambda/4)}{\overline{Z}_L \cosh(\alpha\lambda/4) + \sinh(\alpha\lambda/4)} \quad \textbf{(1-100)}$$

- **Low loss transmission line**

$$\overline{Z} = \frac{\overline{Z}_L(\alpha\lambda/4) + 1}{\overline{Z}_L + \alpha\lambda/4} \quad \textbf{(1-102)}$$

- **Lossless transmission line**

A special case of equations 1-101 and 1-102 occurs when the transmission line is considered to be lossless, because α = 0 giving

$$Z_{\text{int}} = \frac{Z_{oAt}^2}{Z_L} \qquad \text{ohms} \quad \textbf{(1-103)}$$

The normalized impedance, that is the impedance normalized to the ideal quarter-wavelength transformer characteristic impedance $Z_{oA}$, for a mismatched transmission line is given as

$$\overline{Z}_{\text{int}} = \frac{1}{\overline{Z}_L} \qquad \textbf{(1-104)}$$

Equations 1-103 and 1-104 show that a quarter wavelength line transforms a small or large value of load impedance into a large or small value of impedance as seen at the input of the quarter-wavelength line. Because it inverts the normalized load impedance, it is sometimes called an impedance inverter. The quarter wavelength line has the properties of an impedance transformer of turns ratio $\sqrt{\frac{Z_{\text{int}}}{Z_L}}$ and is ideal for matching a resistive load to a generator at a *specific frequency*, which is a necessary condition for delivering all the available generator power to the load. That specifice frequency being where the transformer is a quarter-wavelength or $n\lambda/2+\lambda/4$ long. Quarter wavelength transformers are primarily used as matching sections where two transmission lines of different characteristic impedance are required to be matched. To obtain a match over a broad band of frequencies, two or more quarter-wave sections are used, as will be discussed in chapter 4. Equation 1-103 permits the calculation of the required characteristic impedance of the quarter-wavelength transformer line $Z_{oA}$, which is to be used to match another transmission line, or a generator impedance, of impedance $Z_{in}$, to another transmission line, the load impedance $Z_L$, that is

$$Z_{oA} = \sqrt{Z_{\text{int}} Z_L} \quad \text{ohms} \qquad \textbf{(1-105)}$$

### *3.2.2.1. Bandwidth of a quarter-wavelength transformer*[i]

Equation 3-18 gives the input impedance $Z_{in}$, into a transmission line with a characteristic impedance of $Z_{oA}$, of length *l*, and terminated with a load impedance $Z_L$, at any frequency f, from the position of the main transmission line

$$Z_{in} = Z_{oA}\frac{Z_L + jZ_{oA}\tan\beta l}{Z_{oA} + jZ_L\tan\beta l} \quad \text{ohms} \qquad \textbf{(3-18)}$$

Assuming that $Z_{int}$ and $Z_L$ are independent of frequency, from equation 1-60, at any frequency, and making use of equation 1-103

$$\rho = \frac{Z_{in} - Z_{\text{int}}}{Z_{in} + Z_{\text{int}}} = \frac{Z_{oA}(Z_L - Z_{\text{int}}) + jx(Z_{oA}^2 - Z_{\text{int}}Z_L)}{Z_{oA}(Z_L + Z_{\text{int}}) + jx(Z_{oA}^2 + Z_{\text{int}}Z_L)} = \frac{Z_L - Z_{\text{int}}}{Z_L + Z_{\text{int}} + jx2\sqrt{Z_{\text{int}}Z_L}}$$

**(3-37)**

where $x = \tan {}^{2\pi l}\!/_{\lambda(f)}$ and λ(f) denotes the wavelength changes with frequency.

The magnitude of equation 3-37, is found to be

$$|\rho| = \frac{|Z_L - Z_{\text{int}}|}{\left[(Z_L + Z_{\text{int}})^2 + 4x^2 Z_{\text{int}}Z_L\right]^{1/2}} = \frac{1}{\left[1 + \left(\frac{2\sqrt{Z_{\text{int}}Z_L}}{Z_L - Z_{\text{int}}}\sec {}^{2\pi l}\!/_{\lambda(f)}\right)^2\right]^{1/2}}$$

**(3-38)**

Using the Taylor series, we find that

$$\frac{1}{\left(1 + \frac{1}{p^2}\right)^{1/2}} \cong p - \frac{p^3}{2} + \frac{3p^5}{8} - \frac{5p^7}{16} + \ldots \qquad \textbf{(3-39)}$$

thus for $2\pi l/\lambda(f)$ close to $\pi/2$, p in equation 3-39 is very small and

$$|\rho| \cong \frac{|Z_L - Z_{\text{int}}|}{2\sqrt{Z_{\text{int}}Z_L}}\left|\cos {}^{2\pi l}\!/_{\lambda(f)}\right| \qquad \textbf{(3-40)}$$

From equation 3-38, it can be seen that when $2\pi l/\lambda(f)$ equals $(2n-1)\pi/2$, where n = 1,2,..∞, the value of $|\rho|$ is zero, and when $2\pi l/\lambda(f)$ equals nπ, again where n = 1,2,..∞, $|\rho| = \left|\dfrac{Z_L - Z_{int}}{Z_L + Z_{int}}\right|$.

The variation of $|\rho|$ is periodic because of the periodic nature of the input impedance with line length, that is, the impedance returns to the same value each time the quarter-wave transformer transmission line electrical length changes by π.

The plot of equation 3-40 is that of a fully rectified cosine wave in which the negative region of the cosine is made positive. The abscissa is $2\pi l/\lambda(f)$ and the ordinate is ρ.

For a given maximum value of VSWR, S, that can be tolerated, a maximum value of ρ can be derived, and from this the bandwidth that can be provided by the transformer can be derived.

From equation 1-64, the maximum value of $|\rho|$ which can be tolerated is given by

$$|\rho_m| = \frac{S-1}{S+1} \qquad \textbf{(1-64)}$$

$$2\pi l/\lambda(f) = \cos^{-1}\left|\frac{2\rho_m\sqrt{Z_{int}Z_L}}{(Z_L - Z_{int})\sqrt{1-\rho_m^2}}\right| = \cos^{-1}\left|\frac{2(S-1)\sqrt{Z_{int}Z_L}}{\sqrt{S}(Z_L - Z_{int})}\right| \qquad \textbf{(3-41)}$$

From $\beta l = \dfrac{2\pi l}{\lambda(f)}$, a relationship for an exact quarter-wave line length $\left(l = \lambda_o/4\right)$ can be derived, where $\lambda_o = v_p/f_o$. The frequency $f_o$ is that frequency which produces the quarter-wavelength for a line length $l$, at some propagation velocity $v_p$.

Thus, $\dfrac{2\pi l}{\lambda_o} = \dfrac{\pi}{2}$ giving $l = \dfrac{\lambda_o}{2\pi}\dfrac{\pi}{2} = \dfrac{v_p}{2\pi f_o}\dfrac{\pi}{2}$.

For some general frequency f, with the transformer length still fixed at $l$, and $\lambda(f) = v_p/f$

$$\beta l = \frac{2\pi l}{\lambda(f)} = \frac{\pi}{2}\frac{f}{f_o}$$

and $$f = \frac{2}{\pi} f_o \frac{2\pi l}{\lambda(f)} \qquad \textbf{(3-42)}$$

Using equation 3-41, and letting $f = f_m$, in equation 3-42, that is the general frequency f becomes the maximum allowable frequency, the bandwidth is defined as

$$\Delta f = 2(f_o - f_m) = 2f_o\left(1 - \frac{(2\pi l)}{\lambda(f)}\frac{2}{\pi}\right) = 2f_o\left(1 - \frac{2}{\pi}\cos^{-1}\left|\frac{2(S-1)\sqrt{Z_{int}Z_L}}{\sqrt{S}(Z_L - Z_{int})}\right|\right) \qquad \textbf{(3-43)}$$

The use of the Smith chart to accommodate a band of frequencies will be covered in Chapter 4.

EXAMPLE 9

- Determine the bandwidth ratio $\Delta f / f_o$ ,of an ideal quarter-wavelength transformer which matches a 600Ω line to a 50Ω line if the VSWR is to be no greater than 1.8.
- Determine the solution if the VSWR is to be no greater than 1.1.
- Determine the characteristic impedance of the transformer quarter-wavelength line.
- Plot, on a Smith chart, the input impedance looking into the transformer from the main transmission line

Let $Z_{int} = 50\Omega$ and $Z_L = 600\Omega$.

As S=1.8, from equation 3-43,

$\Delta f / f_o = 0.490$

As S=1.1, from equation 3-43,

$\Delta f / f_o = 0.077$

From $Z_{oA} = \sqrt{Z_{int}Z_L}$ ohms **(1-105)**

The characteristic impedance of the transformer $Z_{oA} = \sqrt{(50)(600)} = 173.205\Omega$

The equations for the short-transformer, presented in section 3.2.1 hold for a quarter-wave transformer. In this case, $\overline{X} = 0$ and for the quarter-wave line in general, $\tan \beta d \rightarrow \infty$ in equation 3-21. From equation 3-22, $Z_{oA} = \sqrt{Z_{\text{int}} Z_L}$ , which is the same as presented as equation 1-105. The centre of the two circular loci of the transformer are given by equations 3-33 and 3-34, and the diameters by equations 3-35 and 3-36.
For $Z_{oA} = 173.205\Omega$ and thus, $\overline{Z}_{oA} = 3.4641$, we find $\overline{Z}_{oA}^{\,2} = 12$
Thus $\overline{R}_C = 2.467$ and $\overline{R}_{Diam} = 12$. The locus of the input impedance passes through $\overline{R}_L = 12$ which can be seen to be equal to the locus diameter, and also passes through the centre of the chart $\overline{R} = 1$.
If the Smith chart was normalized to $Z_{oA} = 173.205\Omega$, then the load impedance would be normalized to $\overline{Z}_{LoA} = {}^{600}\!/_{173.205} = 3.4641$ and travel along a constant VSWR circle equal to 3.4641 through an angle of λ/4, towards the generator, to the 0.2887 mark on the centre line of the chart. This would correspond to a load resistance of 0.2887 x 173.205 Ω = 50Ω, which is the characteristic impedance of the main transmission line, showing that the transformer has sucessfully matched the 600Ω load and used the property given by equation 1-104. Unlike the short-transformer, note that because both charts have the same *resistive* load, both require 0.25λ to complete their paths. A distance of 0.25λ travelling from the load in the transformer corresponds to an 0.25λ change looking into the transformer from the main transmission line. This is shown on the Smith chart normalized to 50Ω.

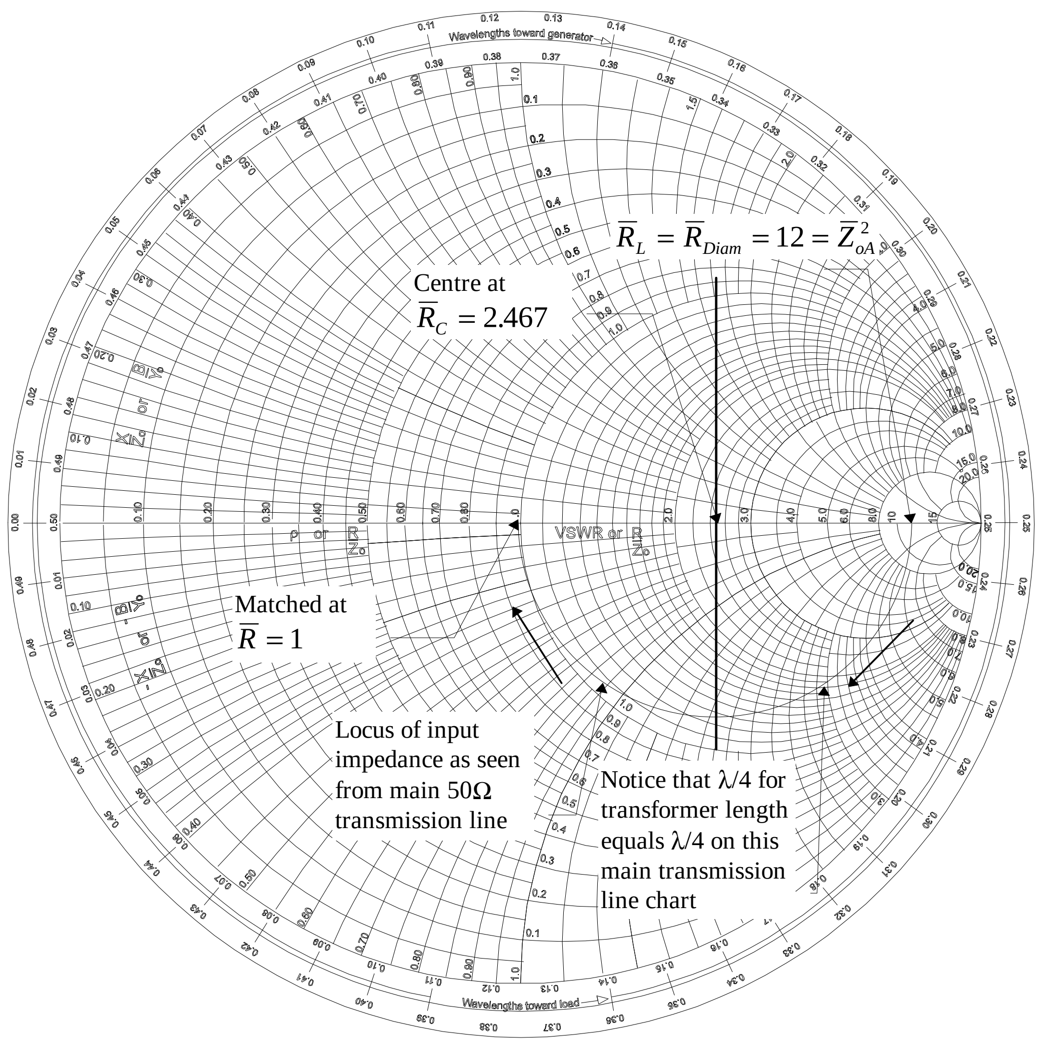



**Figure 68** **50Ω Smith chart for quarter-wave transformer of Example 9**

Quarter-wave matching sections, for the case of a twin-wire line, may be made from conductors that have different diameters or have a different spacing between the conductors from those of the main transmission line. Coaxial cable may be constructed by using conducting sleeves. Either the sleeve is placed over the inner conductor keeping the transformer outer sleeve at the same radius as the main coaxial cable, or a quarter-wavelength transformer cable is formed which has an outer sleeve that fits snugly into the outer sleeve of the main cable. In either case, the transformer will have a characteristic impedance which is less than that of the main cable. Alternatively, if the cable is air-cored, a slug of dielectric material may be inserted for a distance of a quarter-wavelength, but as indicated by equation 1-35, the introduction of a dielectric will reduce the phase velocity. Because the frequency will remain constant, the quarter-wavelength measured at this reduced velocity will be less than that in the main transmission line, making the quarter-wavelength transformer shorter than it would be calculated using the main transmission line wavelength..

EXAMPLE 10
Using a Smith chart, design a quarter-wavelength transformer which will match a load of (35 +j44)Ω to a 50Ω main transmission line.
The solution of this problem centres around having a length of 50 Ω transmission line between the load and the quarter-wavelength transformer, so that the complex load impedance may be transformed by the length of the line to a resistance. Once this resistance is found, the problem reduces to that of using the Smith chart to determine the value of $\overline{Z}_{oA}^2$ and thus $Z_{oA}$ for the transformer.
Figure 69 shows the solution in line diagram form and Figure 70 shows the solution using the Smith chart.

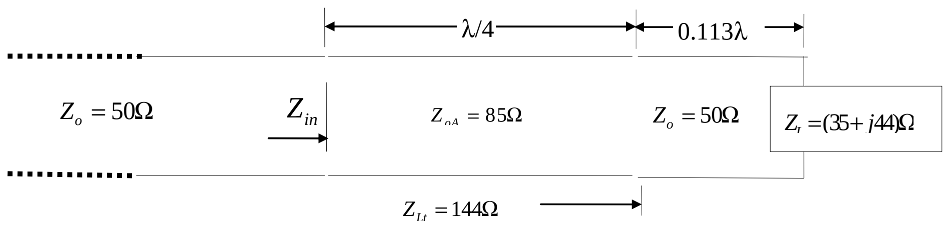



**Figure 69 Quarter-wavelength transformer matching a complex load**

Normalizing the load impedance $Z_L$, to the main transmission line, we find $\overline{Z}_L = 0.7 + j0.88$. This is shown plotted on the Smith chart on Figure 70. Rotating along a constant VSWR circle, seen to be equal to 2.88, until reaching the centre of the chart, we find the angle rotated is equal to 0.113λ, and the value of $\overline{Z}_{oA}^2$ = 2.88, which is also the value of the transformed load impedance $\overline{Z}_{Lt}$. This gives the normalized characteristic impedance of the quarter-wavelength transformer $\overline{Z}_{oA}$ = 1.697. Thus, the characteristic impedance of the quarter-wavelength line is 84.85Ω or approximately 85Ω.

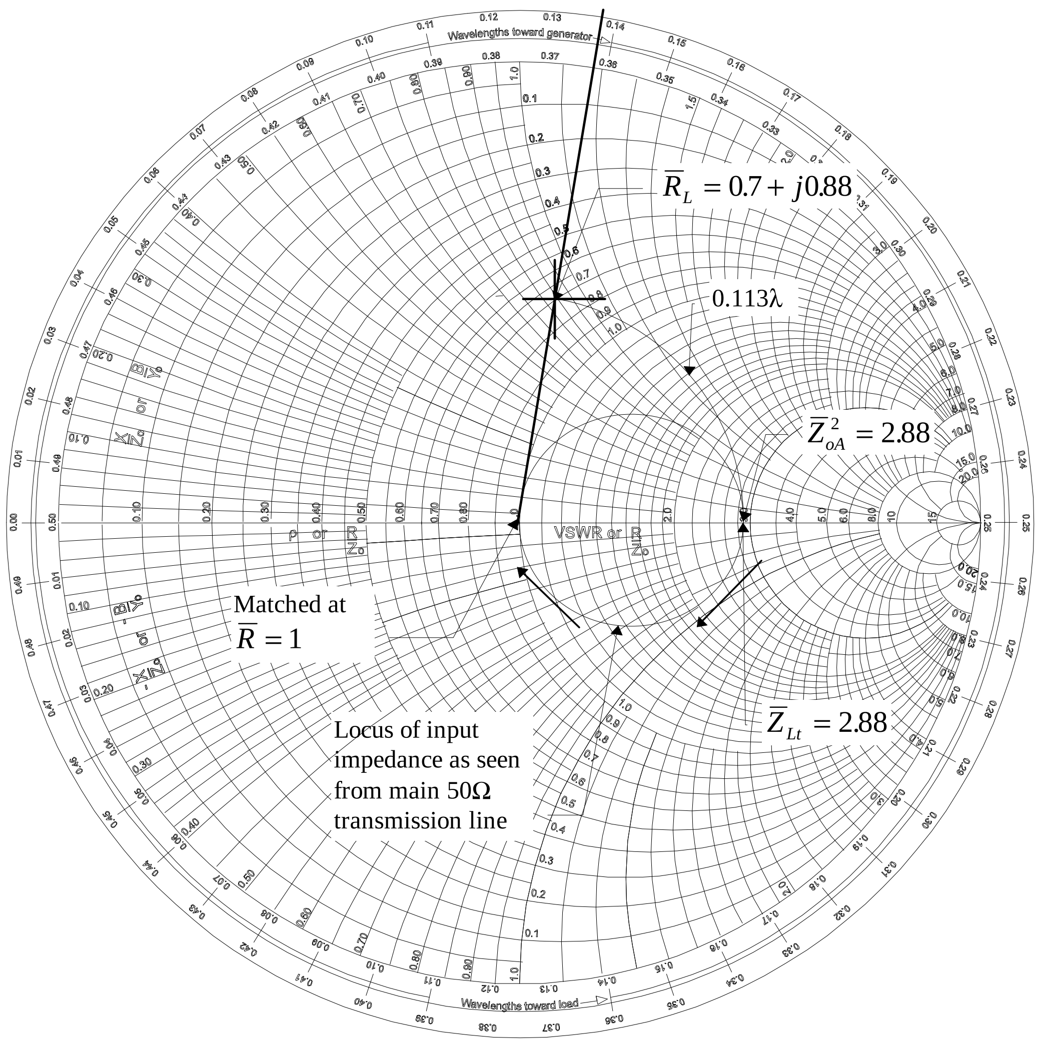



**Figure 70 Quarter-wavelength transformer matching a complex load**

### 3.2.3. Two short transformers in tandem

From equation 1-85, the input impedance $Z_{in}$, looking into a transmission line with a characteristic impedance of $Z_{oA}$, of length $l_1$, and terminated with a load impedance $Z_L$, is given by

$$Z_{inA} = Z_{oA} \frac{Z_L + jZ_{oA} \tan \beta_1 l_1}{Z_{oA} + jZ_L \tan \beta_1 l_1} \quad \text{ohms} \tag{3-44}$$

If it is required that a section of this transmission line (the short transformer) with the load attached, is to be attached to another short transformer with a characteristic impedance of $Z_{oB}$, of length $l_2$, and this then connected to the main transmission line of characteristic impedance $Z_o$, as shown in figure 71, then for perfect matching, a set of design equations can be found for $Z_{in} = Z_o$, from

$$Z_{in} = Z_{oB} \frac{Z_{inA} + jZ_{oB} \tan \beta_2 l_2}{Z_{oB} + jZ_{inA} \tan \beta_2 l_2} = Z_o \tag{3-45}$$

Given that $\overline{Z}_L = \overline{R} + j\overline{X}$, and normalizing equations 3-44 and 3-45 to $Z_o$, we find on eliminating $\overline{Z}_{inA}$ and then equating the real and imaginary parts, that there exist two equations

$$-\bar{X}(\bar{Z}_{oB}\tan\beta_1 l_1+\bar{Z}_{oA}\tan\beta_2 l_2)-\tan\beta_1 l_1\tan\beta_2 l_2(\bar{Z}_{oA}^2-\bar{Z}_{oB}^2)+\bar{Z}_{oA}\bar{Z}_{oB}(1-\bar{R})=0 \quad \textbf{(3-46)}$$

$$-\bar{Z}_{oB}\tan\beta_1 l_1(\bar{Z}_{oA}^2-\bar{R})+\bar{Z}_{oA}\tan\beta_2 l_2(\bar{R}-\bar{Z}_{oB}^2)-\bar{X}\bar{Z}_{oB}(\bar{Z}_{oA}-\tan\beta_1 l_1\tan\beta_2 l_2\bar{Z}_{oB})=0 \quad \textbf{(3-47)}$$

From equations 3-46 and 3-47, we find the conditions for the characteristic impedance $\bar{Z}_{oB}$, given first $\bar{Z}_{oA}$. This is given by equations 3-48 and 3-49.

$$\bar{Z}_{oB}=\bar{Z}_{oA}\sqrt{\frac{\left(\bar{R}-\bar{Z}_{oA}^2\right)}{\bar{X}^2+\bar{R}\left(\bar{R}-\bar{Z}_{oA}^2\right)}} \quad \textbf{(3-48)}$$

$$\bar{Z}_{oB}=\sqrt{\bar{R}+\frac{\bar{X}^2}{\bar{R}-1}} \quad \textbf{(3-49)}$$

Note that one of the solutions, namely equation 3-49 is the same as the short-transformer equation 3-22.

Should $\bar{Z}_{oB}$.be given first, then $\bar{Z}_{oA}$ can be found from equations 3-46 and 3-47, as

$$\bar{Z}_{oA}=\frac{\sqrt{\bar{R}\left(\bar{Z}_{oB}^2+1\right)\pm\sqrt{\left(\left(\bar{Z}_{oB}^2-1\right)\bar{R}-2\bar{Z}_{oB}\bar{X}\right)\left(\left(\bar{Z}_{oB}^2-1\right)\bar{R}+2\bar{Z}_{oB}\bar{X}\right)}}}{\sqrt{2}} \quad \textbf{(3-50)}$$

Substituting equation 3-48 back into equations 3-46 and 3-47, we can determine the lengths of the transformers $l_1$ and $l_2$ given $\bar{Z}_{oA}$ , $\bar{R}$ and $\bar{X}$ from,

$$\tan\beta_2 l_2=-\sqrt{\frac{\left(\bar{Z}_{oA}^2-\bar{R}\right)\left(\bar{R}\left(\bar{Z}_{oA}^2-\bar{R}\right)-\bar{X}^2\right)}{\bar{X}^2\bar{Z}_{oA}^2}} \quad \textbf{(3-51a)}$$

$$\tan(\pi-\beta_2 l_2)=\sqrt{\frac{\left(\bar{Z}_{oA}^2-\bar{R}\right)\left(\bar{R}\left(\bar{Z}_{oA}^2-\bar{R}\right)-\bar{X}^2\right)}{\bar{X}^2\bar{Z}_{oA}^2}} \quad \textbf{(3-51b)}$$

$$\tan\beta_1 l_1=\frac{\left(\bar{Z}_{oA}^2-\left(\bar{X}^2+\bar{R}^2\right)\right)}{2\bar{Z}_{oA}\bar{X}} \quad \textbf{(3-52)}$$

Note that equation 3-51a is negative. This indicates that the line furthest from the load is greater than a quarter-wavelength in order to match the load, as shown by equation 3-51b.

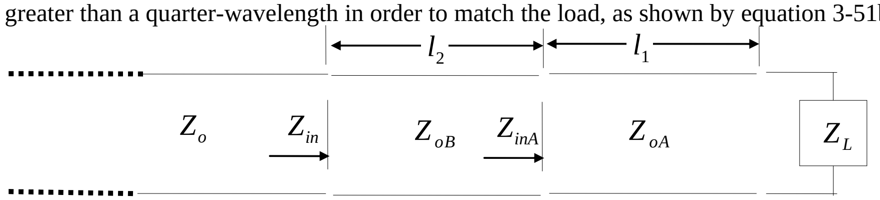



**Figure 71 Two short transformers in tandem**

### *3.2.3.1. Analysis of short transformer for Smith chart application*

From equation 1-58 and given that $\bar{Z}_L=\bar{R}+j\bar{X}$ and $\rho_L=\rho_r+j\rho_j$ we find

$$\rho_r=\frac{\bar{X}^2+\bar{R}^2-1}{\bar{X}^2+(\bar{R}+1)^2} \quad \textbf{(3-24)}$$

$$\rho_j=\frac{2X}{\bar{X}^2+(\bar{R}+1)^2} \quad \textbf{(3-25)}$$

Using equation 3-48 with equations 3-24 and 3-25, and with the use of a personal computer symbolic algebra program, the equations of the following circles are obtained,

$$\left[\rho_r \pm \left(\frac{\overline{Z}_{oB}^2 - \overline{Z}_{0A}^4}{\left(\overline{Z}_{oB}^2 + \overline{Z}_{0A}^2\right)\left(\overline{Z}_{0A}^2 + 1\right)}\right)\right]^2 + \rho_j^2 = \left(\frac{\overline{Z}_{oA}^2\left(\overline{Z}_{oB}^2 - 1\right)}{\left(\overline{Z}_{oB}^2 + \overline{Z}_{0A}^2\right)\left(\overline{Z}_{0A}^2 + 1\right)}\right)^2 \quad \textbf{(3-53)}$$

Similarly, using equation 3-49 with equations 3-24 and 3-25, another set of circles are obtained, which are the same as those for the single short-transformer, given by equations 3-29 and 3-30.

$$\left[\rho_r \pm \frac{1}{2}\left(\frac{\overline{Z}_{oB}^2 - 1}{\overline{Z}_{oB}^2 + 1}\right)\right]^2 + \rho_j^2 = \left[\frac{1}{2}\left(\frac{\overline{Z}_{oB}^2 - 1}{\overline{Z}_{oB}^2 + 1}\right)\right]^2 \quad \textbf{(3-54)}$$

Both sets of circles have their centres placed on the $\rho_r$ axis or $\overline{X} = 0$ line, and if the circle of equation 3-53 is not to collapse into that of equation 3-54, it must straddle the $\rho_j$. axis. Both of these sets of circles when plotted on the Smith chart indicate how the matching of a load with two short-transformers is achieved. Unlike the L-type circuits described in section 2.61, the two short-transformers do not move along circles of constant conductance and resistance.

From equations 3-53 and 3-54, the diameters provide an easy way of determining the conditions for the existence of a solution to the determination of the characteristic impedance of each of the two transformers. Figure 72 and figure 73 show the arrangement of the circles and the values of their diameter extremities.

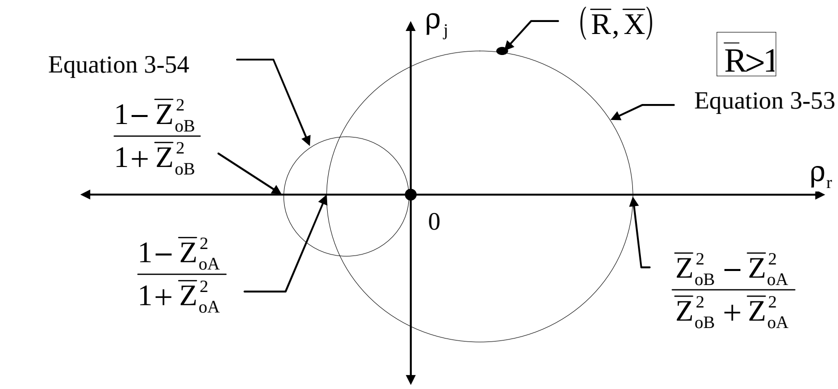



**Figure 72** **Smith chart plot of short-transformer circles** $(\overline{R} > 1)$

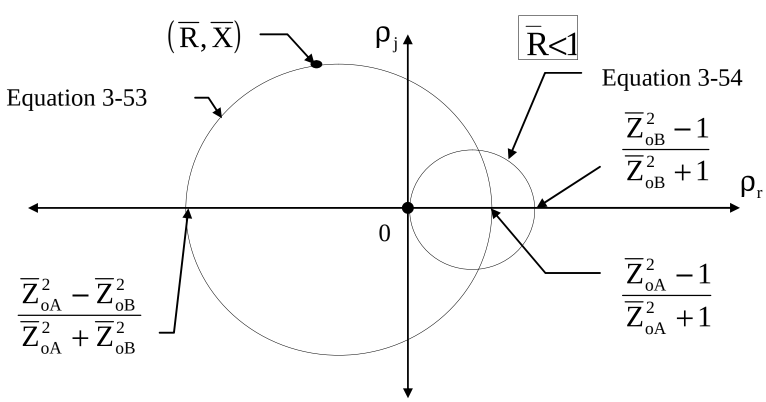



**Figure 73** **Smith chart plot of short-transformer circles** $(\overline{R} < 1)$

The conditions for existence of equations 3-53 and 3-54, determined from Figures 72 and 73 are:

**For** $\overline{R} > 1$

$$\overline{Z}_{oA}^2 > 1,\ \ \overline{Z}_{oB}^2 > 1,\ \ \overline{Z}_{oB} > \overline{Z}_{oA} \tag{3-55}$$

As the position of the circle of equation 3-53 (the larger circle in Figure 72) is not uniquely determined, a condition which will allow it to be place in a region where it will have a solution, is to ensure that diameter on the positive $\rho_r$ axis is greater than $\overline{R}^2 + \overline{X}^2$. The use of equation 1-78, shows that in terms of resistance read off on the Smith chart, this diameter point on the positive $\rho_r$ axis is given by:

$$\overline{R}_{D2} = \frac{1+\rho_r}{1-\rho_r} = \frac{1+\dfrac{\overline{Z}_{oB}^2 - \overline{Z}_{oA}^2}{\overline{Z}_{oB}^2 + \overline{Z}_{oA}^2}}{1-\dfrac{\overline{Z}_{oB}^2 - \overline{Z}_{oA}^2}{\overline{Z}_{oB}^2 + \overline{Z}_{oA}^2}} = \frac{\overline{Z}_{oB}^2}{\overline{Z}_{oA}^2} \tag{3-56}$$

The diameter point on the negative $\rho_r$ axis is given by:

$$\overline{R}_{D!} = \frac{1+\rho_r}{1-\rho_r} = \frac{1+\dfrac{1-\overline{Z}_{oA}^2}{1+\overline{Z}_{oA}^2}}{1-\dfrac{1-\overline{Z}_{oA}^2}{1+\overline{Z}_{oA}^2}} = \frac{1}{\overline{Z}_{oA}^2} \tag{3-57}$$

This provides a means of determining from the Smith chart a method of determining $\overline{Z}_{oA}$, if we take a condition that

$$\frac{\overline{Z}_{oB}^2}{\overline{Z}_{oA}^2} > \overline{R}^2 + \overline{X}^2 \tag{3-58}$$

**For** $\overline{R} < 1$

$$\overline{Z}_{oA}^2 > 1,\ \ \overline{Z}_{oB}^2 > 1,\ \ \overline{Z}_{oB} > \overline{Z}_{oA} \tag{3-59}$$

and by similar reasoning

$$\overline{R}_{D2} = \frac{1+\rho_r}{1-\rho_r} = \frac{1+\dfrac{\overline{Z}_{oA}^2 - 1}{\overline{Z}_{oA}^2 + 1}}{1-\dfrac{\overline{Z}_{oA}^2 - 1}{\overline{Z}_{oA}^2 + 1}} = \overline{Z}_{oA}^2 \tag{3-60}$$

$$\overline{R}_{D1} = \frac{1+\rho_r}{1-\rho_r} = \frac{1+\dfrac{\overline{Z}_{oA}^2 - \overline{Z}_{oB}^2}{\overline{Z}_{oA}^2 + \overline{Z}_{oB}^2}}{1-\dfrac{\overline{Z}_{oA}^2 - \overline{Z}_{oB}^2}{\overline{Z}_{oA}^2 + \overline{Z}_{oB}^2}} = \frac{\overline{Z}_{oA}^2}{\overline{Z}_{oB}^2} \tag{3-61}$$

This provides a means of determining from the Smith chart a method of determining $\overline{Z}_{oA}$, if we take a condition that

$$\frac{\overline{Z}_{oA}^2}{\overline{Z}_{oB}^2} < \overline{R}^2 + \overline{X}^2 \tag{3-62}$$

Figure 74 and 75 show the conversion of the diameter points of the circles of equations 3-56, 3-57, 3-60 and 3-61 to those of normalized resistance, normalized to the main line characteristic impedance $Z_o$, using equation 1-78. This allows the normalized resistance found to be converted into the main line normalized characteristic impedances of the short-transformer sections ( $\overline{Z}_{oA}$ and $\overline{Z}_{oB}$).

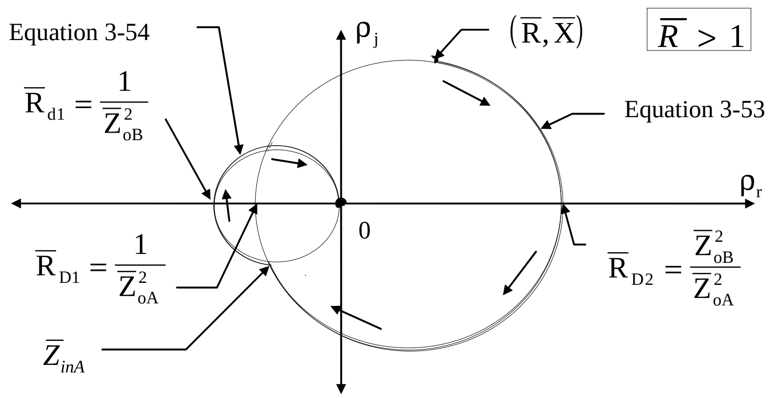



**Figure 74 Smith chart plot of short-transformer circles** $(\overline{R} > 1)$

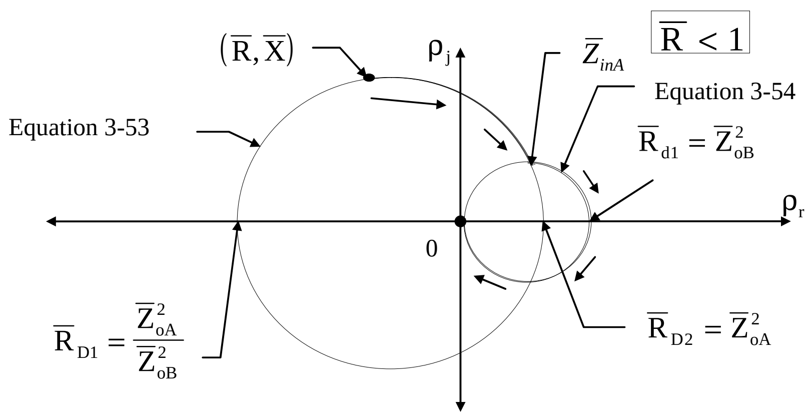



**Figure 75 Smith chart plot of short-transformer circles** $(\overline{R} < 1)$

### *3.2.3.2. Procedure for designing a double short-transformer matching network on a Smith chart*

On the Smith chart draw a circle through the plotted load impedance, ensuring that this circle straddles the centre of the chart as shown in Figures 74 and 75. For $\overline{R} > 1$ , the intersection of this circle with the negative $\rho_r$ axis determines the normalized characteristic impedance of the short-transformer closest to the load $\overline{Z}_{oA}$.
The large circle intersection on the right hand side of the chart, allows the characteristic impedance of the short-transformer furthest from the load to be determined $\overline{Z}_{oB}$, as we now know $\overline{Z}_{oA}$. This allows the second circle to be drawn with a diameter extending from the centre of the chart $\rho_r = 0$ or $\overline{R} = 1$, to $1/\overline{Z}^2_{oB}$ . For a solution to exist, there must be an intersection of both circles..

The determination of the length of each of the transformers can only be found from equations 3-51b and 3-52.

EXAMPLE 11
Using the Smith chart and equations 3-51b and 3-52, design a two short-transformer network to match a load of (30 + j20)Ω, given that the impedance of the main transmission line is 50Ω.

Plotting the normalized impedance (0.6 + j0.4), on the 50Ω Smith chart, we find that the load resistance is less than unity. Hence Figure 75 applies in this case. As any circle which straddles the centre of the chart and which passes through the load impedance should provide a solution, a circle with centre located at 1.6 on the Smith chart should suffice. This circle crosses the chart at approximately 6.8, giving $\overline{Z}_{oA}^2$=6.8, and hence $\overline{Z}_{oA}$= 2.6. The characteristic impedance of this short-transformer is thus found to be $Z_{oA} = 130\Omega$. The opposite diameter crossing cuts the horizontal centre line of the Smith chart at approximately 0.57 giving $\frac{\overline{Z}_{oA}^2}{\overline{Z}_{0B}^2}$=0.57, and thus, $\overline{Z}_{oB}$=3.454. The characteristic impedance of this short-transformer is thus found to be $Z_{oB} = 173\Omega$. This permits the second circle to be drawn where the diameter crosses the centre of the chart and 11.93.
Figure 76 shows the solution in diagrammatic form and Figure 77 shows the solution on the Smith chart.

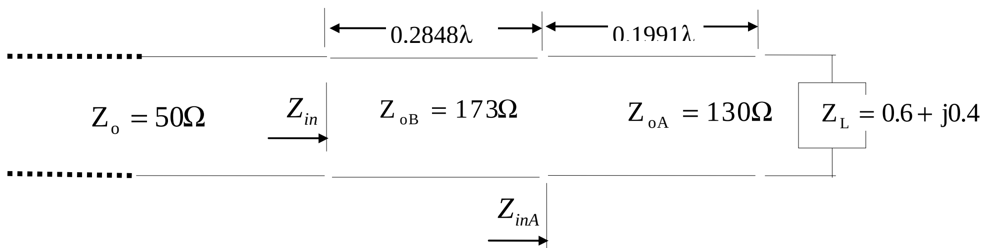



**Figure 76** **Solution to Example 11**

For $\overline{Z}_{oA}$= 2.6, equation 3-48 gives $\overline{Z}_{oB}$= 3.4312 against the 3.454 found using the Smith chart. For $\overline{Z}_{oA}$= 2.6, equation 3-51b gives the length of the short-transformer furthest from the load as 0.2848λ and equation 3-52 gives the length of the short-transformer closest to the load as 0.1991λ.

## References

[i] Robert E. Collin., *Foundations for Microwave Engineering,* McGraw-Hill, Inc., Singapore, 1992

# 4. MATCHING OVER A FREQUENCY BAND

This chapter will consider matching a load to a transmission line or network over a band of frequencies, where instead of a single impedance or admittance point existing on the Smith chart, there exists a range of impedances or admittances.
This range exists because if the mismatched load is non-resistive, then at each different frequency in the band, there will be a different value of the load impedance and thus a different value of VSWR. Thus, it is not possible to match to a single VSWR value, which ideally is unity. The objective when dealing with a band of frequencies is to match the line to the load so that the VSWR values over the band all lie within a specified VSWR circle.

## *4.1. Variation of reactance and susceptance with frequency*

As the frequency changes the values of reactance and susceptance will change. Although the value of resistance remains independent of frequency, there will be a change in impedance or admittance with frequency due to the variation of reactance and susceptance. This section will quantify this change and lay the ground rules for using the Smith chart.

### 4.1.1. Variation of reactance

#### *4.1.1.1.Capacitive reactance*

If the reactance at a band centre frequency $f_o$ is given by $X_o = 1/j\omega_o C$ , then, at an incremental frequency $\delta\omega = 2\pi\delta f$ , higher or lower than the centre frequency $\omega_o = 2\pi f_o$, that is, $\omega_o \pm \delta\omega$ , the reactance becomes,

$X_o \mp \Delta X = \dfrac{1}{j(\omega_o \pm \delta\omega)C} = \dfrac{1}{j\omega C}$ giving, $\dfrac{X_o \mp \Delta X}{X_o} = \dfrac{f_o}{f_o \pm \delta f}$ which reduces to

$X_o \mp \Delta X = X_o \dfrac{f_o}{f_o \pm \delta f}$ , or more explicitly:

$$X_{hi} = X_o \frac{f_o}{f_{hi}} \qquad \textbf{(4-1a)}$$

where $X_{hi} = X_o - \Delta X$ , $f_{hi} = f_o + \delta f$ and $\Delta X = X_o \dfrac{\delta f}{f_{hi}}$

$$X_{lo} = X_o \frac{f_o}{f_{lo}} \qquad \textbf{(4-1b)}$$

where $X_{lo} = X_o + \Delta X$ , $f_{lo} = f_o - \delta f$ and $\Delta X = X_o \dfrac{\delta f}{f_{lo}}$

To avoid obtaining the capacitive reactance at d.c., $f_o > \delta f$ .
It should be noted that for equal frequency steps $\delta f$ , throughout the frequency band under consideration, the capacitive reactance does not change by linearly proportional amounts. Equation 4-1 shows that the frequencies lower than the band centre frequency $f_o$, produce a greater change in the capacitive reactance than the frequencies higher in the band. That is, the lower frequencies produce a greater change in reactance than the higher frequencies. Dividing equation 4-1b by equation 4-1a gives a measure of how much greater the capacitive reactance at the lower frequencies is compared to the capacitive reactance at the higher frequencies, that is,

$$X_{lo} = X_{hi} \frac{f_{hi}}{f_{lo}} \qquad \textbf{(4-2)}$$

Equation 4-2 can be generalized to equation 4-3 to predict values of capacitive reactance $X_{c1}(f_1)$, at a particular frequency $f_1$, given a value of a known capacitive reactance $X_{c1}(f_2)$ and its frequency $f_2$. That is.

$$X_{c1}(f_1) = X_{c1}(f_2)\frac{f_2}{f_1} \tag{4-3}$$

If an impedance is formed from a capacitive reactance, as the resistive component is independent of frequency, the impedance will vary over band of frequencies due only to its capacitive reactance component. That is,

$$Z(f) = R - j\frac{1}{2\pi C f_{hi/lo}} \tag{4-4}$$

#### *4.1.1.2.Addition of series capacitive reactance to an impedance*

In most cases where matching occurs, the emphasis is placed on adding a reactance to an impedance. In this section, the addition of a series capacitive reactance to an impedance will be considered.
At a particular frequency $f_1$, the addition of a series capacitive reactance $X_{c1}(f_1)$, to an impedance $Z_1(f_1)$, forms a new impedance $Z_f(f_1)$ given by,

$$Z_f(f_1) = Z_1(f_1) + X_{c1}(f_1) = R_1 + X_1(f_1) + X_{c1}(f_1)$$

At a different frequency $f_2$, the addition of a series capacitive reactance $X_{c1}(f_2)$, to an impedance $Z_1(f_2)$, forms a new impedance $Z_f(f_2)$, and with the aid of equation 4-3, is given by,

$$Z_f(f_2) = Z_1(f_2) + X_{c1}(f_2) = R_1 + \frac{f_1}{f_2}\left(X_1(f_1) + X_{c1}(f_1)\right) \tag{4-5}$$

The change in impedance due to the change in frequency is given by

$$\Delta Z_f(f_1, f_2) = X_1(f_1) - X_1(f_2) + X_{c1}(f_1) - X_{c1}(f_2)$$

or

$$\Delta Z_f(f_1, f_2) = X_1(f_2) - X_1(f_1) + X_{c1}(f_2) - X_{c1}(f_1)$$

$$\left|\Delta Z_f(f_1, f_2)\right| = \left(X_1(f_1) + X_{c1}(f_1)\right)\left|1 - \frac{f_1}{f_2}\right| \tag{4-6}$$

Equation 4-6 can be modified in terms of high and low frequencies, to give

$$\left|\Delta Z_f(f_{lo}, f_{hi})\right| = \left(X_1(f_{lo}) + X_{c1}(f_{lo})\right)\left|\frac{f_{hi} - f_{lo}}{f_{hi}}\right| \tag{4-7a}$$

$$\left|\Delta Z_f(f_{hi}, f_{lo})\right| = \left(X_1(f_{hi}) + X_{c1}(f_{hi})\right)\left|\frac{f_{hi} - f_{lo}}{f_{lo}}\right| \tag{4-7b}$$

The addition of a series capacitance will move an impedance anticlockwise around the Smith chart with the lower frequency moving the centre-band impedance more anticlockwise than the higher frequency. In other words, a load impedance with a capacitive reactance will form a line around the constant resistance circle $R_1$, rather than a point, with the lower frequency point having a greater length from the centre-band point than the higher frequency point. On the addition of additional capacitance this difference in lengths from the band-centre impedance point will be more pronounced.
Although in theory, there should not be any change in the value of the resistance component of the impedance as the frequency changes, this is not usually the case in practice. There usually is a change in resistance, due to different attenuations of the medium at different frequencies, reflections, etc., which means that the band impedance line crosses onto different constant resistance circles, with the band-centre impedance point lying on the constant resistance circle $R_1$.

EXAMPLE 1

Given a normalized load impedance $Z_{300} = 1.0 - j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the impedance band between 150 MHz and 450 MHz. Assume that the resistive component of the impedance remains constant over the band.

SOLUTION

From equation 4-1a, $X_{hi} = X_o \dfrac{f_o}{f_{hi}} = -j2.5\dfrac{300}{450} = -j1.667$ so $Z_{450} = 1.0 - j1.667$

From equation 4-1b, $X_{lo} = X_o \dfrac{f_o}{f_{lo}} = -j2.5\dfrac{300}{150} = -j5.000$ so $Z_{150} = 1.0 - j5.000$

Figure 77 shows the plot of the impedance band on the Smith chart. Note how the lower frequency impedance in the frequency band is further away from the impedance at the band centre than the impedance at the higher frequency of the band.

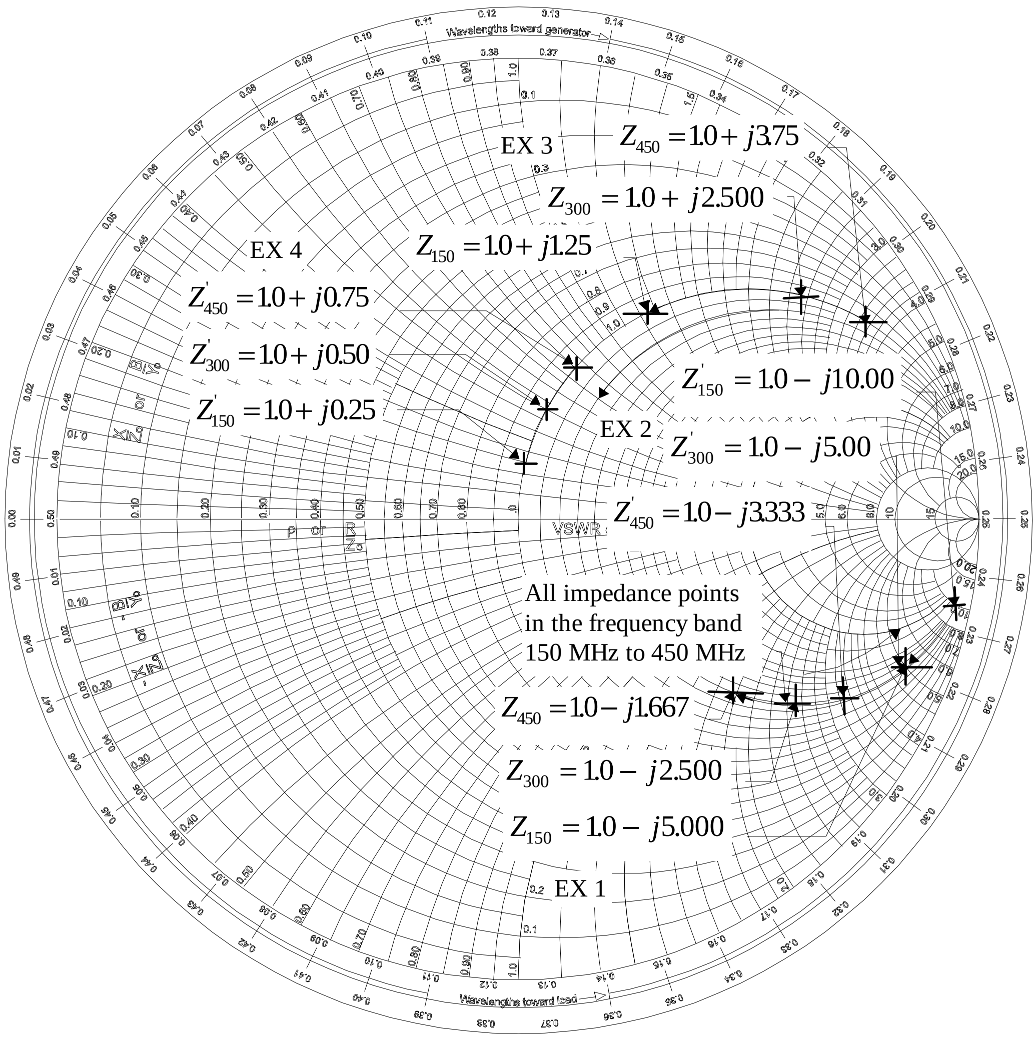



**Figure 77** **Examples 1 to 4** **Impedance curves due to a band of frequencies**

EXAMPLE 2

Given a normalized load impedance $Z_{300} = 1.0 - j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the impedance band between 150 MHz and 450 MHz when a capacitance of -j2.5 at a frequency of 300 MHz, is added in series to the load impedance. Again assume that the resistive component of the impedance remains constant over the band.

SOLUTION
At the centre of the band, the load impedance with the added capacitance is given by,
$Z'_{300} = 1.0 - j2.5 - j2.5 = 1.0 - j5.0$ . Using equation 4-5 and

from equation 4-1a, $X_{hi} = X_o \dfrac{f_o}{f_{hi}} = -j5.0 \dfrac{300}{450} = -j3.333$ so $Z_{450} = 1.0 - j3.333$

from equation 4-1b, $X_{lo} = X_o \dfrac{f_o}{f_{lo}} = -j5.0 \dfrac{300}{150} = -j10.000$ so $Z_{150} = 1.0 - j10.000$

Figure 77 shows the new plot of the impedance band on the Smith chart. Note how the addition of the capacitance has rotated the whole band unequally anticlockwise along the circle of constant resistance, which in this case is 1.0. The lower frequency of the band having the greatest effect on the reactive component as seen by the change from -j5.0 in example 1 to -j10.0 in this example. The higher frequency of the band moving only from -j1.667 in example 1 to -j3.333 in this example.

Summarizing the results of sections 4.1.1.1. and 4.1.1.2.,

- The lower frequency in a band of frequencies moves the band-centered impedance more anticlockwise from the band-centered impedance position than does the higher frequency impedance,
- The addition of a capacitive reactance, over a band of frequencies, to an impedance band will rotate the impedance band anticlockwise along a circle, or circles, of constant resistance. The lower frequency impedance being further extended from the band centered impedance than without the added capacitive reactance.

### *4.1.1.3.Inductive reactance*

If the reactance at a band centre frequency $f_o$ is given by $X_o = j\omega_o L$ , then, at an incremental frequency $\delta\omega = 2\pi\delta f$ , higher or lower than the centre frequency $\omega_o = 2\pi f_o$, that is, $\omega_o \pm \delta\omega$ , the reactance becomes,

$X_o \pm \Delta X = j(\omega_o \pm \delta\omega)L = j\omega L$ giving, $\dfrac{X_o \pm \Delta X}{X_o} = \dfrac{f_o \pm \delta f}{f_o}$ which reduces to

$X_o \pm \Delta X = X_o \pm X_o \dfrac{\delta f}{f_o}$, or more explicitly:

$$X_{hi} = X_o \frac{f_{hi}}{f_o} \qquad \textbf{(4-8a)}$$

where $X_{hi} = X_o + \Delta X$ , $f_{hi} = f_o + \delta f$ and $\Delta X = X_o \dfrac{\delta f}{f_o}$

$$X_{lo} = X_o \frac{f_{lo}}{f_o} \qquad \textbf{(4-8b)}$$

where $X_{lo} = X_o - \Delta X$ , $f_{lo} = f_o - \delta f$ and $\Delta X = X_o \dfrac{\delta f}{f_o}$

To avoid obtaining the inductive reactance at d.c., $f_o > \delta f$ .

It should be noted that for equal frequency steps throughout the frequency band under consideration, the inductive reactance **does** change by linearly proportional amounts. Equation 4-8 shows that the frequencies higher than the band centre frequency $f_o$, produce the same change in the inductive reactance than the frequencies lower in the band. That is, the higher frequencies produce the same change in reactance as the lower frequencies. This effect is different to that for capacitive reactance, as described in section 4.1.1.1. Dividing equation 4-8a by equation 4-8b gives a measure of how much greater the inductive reactance at the higher frequencies is when compared to the inductive reactance at the lower frequencies, that is,

$$X_{hi} = X_{lo}\frac{f_{hi}}{f_{lo}} \qquad \textbf{(4-9)}$$

Equation 4-9 can be generalized to equation 4-10 to predict values of inductive reactance $X_{L1}(f_2)$, at a particular frequency $f_2$, given a value of a known inductive reactance $X_{L1}(f_1)$ and its frequency $f_1$. That is.

$$X_{L1}(f_2) = X_{L1}(f_1)\frac{f_2}{f_1} \qquad \textbf{(4-10)}$$

If an impedance is formed from a inductive reactance, as the resistive component is independent of frequency, the impedance will vary over band of frequencies due only to its inductive reactance component. That is,

$$Z(f) = R + j2\pi L f_{hi/lo} \qquad \textbf{(4-11)}$$

#### *4.1.1.4. Addition of series inductive reactance to an impedance*

In this section, the addition of a series inductive reactance to an impedance will be considered.
At a particular frequency $f_1$, the addition of a series inductive reactance $X_{L1}(f_1)$, to an impedance $Z_1(f_1)$, forms a new impedance $Z_f(f_1)$ given by,

$$Z_f(f_1) = Z_1(f_1) + X_{L1}(f_1) = R_1 + X_1(f_1) + X_{L1}(f_1)$$

At a different frequency $f_2$, the addition of a series inductive reactance $X_{L1}(f_2)$, to an impedance $Z_1(f_2)$, forms a new impedance $Z_f(f_2)$, and with the aid of equation 4-10, is given by,

$$Z_f(f_2) = Z_1(f_2) + X_{L1}(f_2) = R_1 + \frac{f_2}{f_1}\left(X_1(f_1) + X_{L1}(f_1)\right) \qquad \textbf{(4-12)}$$

The change in impedance due to the change in frequency is given by

$$\Delta Z_f(f_1, f_2) = X_1(f_2) - X_1(f_1) + X_{L1}(f_2) - X_{L1}(f_1)$$

or

$$\Delta Z_f(f_1, f_2) = X_1(f_1) - X_1(f_2) + X_{L1}(f_1) - X_{L1}(f_2)$$

$$\left|\Delta Z_f(f_1, f_2)\right| = \left(X_1(f_1) + X_{L1}(f_1)\right)\left|\frac{f_2}{f_1} - 1\right| \qquad \textbf{(4-13)}$$

Equation 4-13 can be modified in terms of high and low frequencies, to give

$$\left|\Delta Z_f(f_{lo}, f_{hi})\right| = \left(X_1(f_{lo}) + X_{L1}(f_{lo})\right)\left|\frac{f_{hi} - f_{lo}}{f_{lo}}\right| \qquad \textbf{(4-14a)}$$

$$\left|\Delta Z_f(f_{hi}, f_{lo})\right| = \left(X_1(f_{hi}) + X_{L1}(f_{hi})\right)\left|\frac{f_{hi} - f_{lo}}{f_{hi}}\right| \qquad \textbf{(4-14b)}$$

The addition of a series inductance will move an impedance clockwise around the Smith chart with the lower frequency moving the centre-band impedance the same amount clockwise as the higher frequency. In other words, a load impedance with a inductive reactance will form a line around the constant resistance circle $R_1$, rather than a point, with the lower frequency point having the same impedance change from the centre-band point as the higher frequency point.
Although in theory, there should not be any change in the value of the resistance component of the impedance as the frequency changes, again this is not usually the case in practice. This means that the band impedance line crosses onto different constant resistance circles, with the band-centre impedance point lying on the constant resistance circle $R_1$.

EXAMPLE 3
Given a normalized load impedance $Z_{300} = 1.0 + j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the impedance band between 150 MHz and 450 MHz. Assume that the resistive component of the impedance remains constant over the band.

SOLUTION
From equation 4-8a, $X_{hi} = X_o \frac{f_{hi}}{f_o} = j2.5\frac{450}{300} = j3.75$ so $Z_{450} = 1.0 + j3.75$

From equation 4-8b, $X_{lo} = X_o \frac{f_{lo}}{f_o} = j2.5\frac{150}{300} = j1.25$ so $Z_{150} = 1.0 + j1.25$

Figure 77 shows the plot of the impedance band on the Smith chart. Note how the lower frequency impedance in the frequency band is same value away from the impedance at the band centre as the impedance at the higher frequency of the band.

EXAMPLE 4
Given a normalized load impedance $Z_{300} = 1.0 + j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the impedance band between 150 MHz and 450 MHz when a capacitance of -j2.5 at a frequency of 300 MHz, is added in series to the load impedance. Again assume that the resistive component of the impedance remains constant over the band.

SOLUTION
At the centre of the band, the load impedance with the added capacitance is given by,
$Z'_{300} = 1.0 + j2.5 - j2 = 1.0 + j0.5$. Using equation 4-12 and

from equation 4-8a, $X_{hi} = X_o \frac{f_{hi}}{f_o} = j0.5\frac{450}{300} = j0.75$ so $Z_{450} = 1.0 + j0.75$

from equation 4-8b, $X_{lo} = X_o \frac{f_{lo}}{f_o} = j0.5\frac{150}{300} = j0.25$ so $Z_{150} = 1.0 + j0.25$

Figure 77 shows the new plot of the impedance band on the Smith chart. Again note how the addition of the capacitance has rotated the whole band anticlockwise along the circle of constant resistance, which in this case is 1.0..

Summarizing the results of sections 4.1.1.3. and 4.1.1.4.,

- For a series inductive reactance, the higher frequency in a band of frequencies moves the band-centered impedance clockwise from the band-centered impedance position, whereas the lower frequency in the band moves the band-centered impedance an equal amount anticlockwise from the band-centered position,
- The addition of a series inductive reactance, over a band of frequencies, to an impedance band will rotate the impedance band clockwise along a circle, or circles, of constant resistance. The lower frequency impedance will move the same impedance increment around the chart as the high frequency impedance.

Summary of results of series capacitive or series inductive reactance,

- The lower frequencies move to the higher frequencies by the impedance band moving clockwise around the Smith chart,
- Addition of series capacitance to an impedance will rotate the band of impedances anticlockwise around the Smith chart,
- Addition of series inductance to an impedance will rotate the band of frequencies clockwise around the Smith Chart,
- For a capacitive impedance band, the high frequency of the band represents the lowest capacitive reactance,
- For an inductive impedance band, the high frequency of the band represents the highest inductive reactance.

These effects can be observed for each of the four examples given above drawn on the Smith chart of Figure 77.

The next two sections consider the variation of susceptance on the Smith chart.

### 4.1.2. Variation of susceptance

#### *4.1.2.1.Capacitive susceptance*

If the susceptance at a band centre frequency $f_o$ is given by $B_o = j\omega_o C$ , then, at an incremental frequency $\delta\omega = 2\pi\delta f$ , higher or lower than the centre frequency $\omega_o = 2\pi f_o$, that is, $\omega_o \pm \delta\omega$ , the susceptance becomes,

$B_o \pm \Delta B = j(\omega_o \pm \delta\omega)C = j\omega C$ giving, $\dfrac{B_o \pm \Delta B}{B_o} = \dfrac{f_o \pm \delta f}{f_o}$ which reduces to

$B_o \pm \Delta B = B_o \pm B_o \dfrac{\delta f}{f_o}$ , or more explicitly:

$$B_{hi} = B_o \frac{f_{hi}}{f_o} \qquad \textbf{(4-15a)}$$

where $B_{hi} = B_o + \Delta B$ , $f_{hi} = f_o + \delta f$ and $\Delta B = B_o \dfrac{\delta f}{f_o}$

$$B_{lo} = B_o \frac{f_{lo}}{f_o} \qquad \textbf{(4-15b)}$$

where $B_{lo} = B_o - \Delta B$ , $f_{lo} = f_o - \delta f$ and $\Delta B = B_o \dfrac{\delta f}{f_o}$

To avoid obtaining the capacitive susceptance at d.c., $f_o > \delta f$ .

It should be noted that for equal frequency steps throughout the frequency band under consideration, the capacitive susceptance **does** change by linearly proportional amounts. Equation 4-15 shows that the frequencies higher than the band centre frequency $f_o$, produce the same change in the capacitive susceptance than the frequencies lower in the band. That is, the higher frequencies produce the same change in susceptance as the lower frequencies. Dividing equation 4-15a by equation 4-15b gives a measure of how much greater the capacitive susceptance at the higher frequencies is when compared to the capacitive susceptance at the lower frequencies, that is,

$$B_{hi} = B_{lo} \frac{f_{hi}}{f_{lo}} \qquad \textbf{(4-16)}$$

Equation 4-16 can be generalized to equation 4-17 to predict values of capacitive susceptance $B_{C1}(f_2)$, at a particular frequency $f_2$, given a value of a known capacitive susceptance $B_{C1}(f_1)$ and its frequency $f_1$. That is.

$$B_{C1}(f_2) = B_{C1}(f_1)\frac{f_2}{f_1} \qquad \textbf{(4-17)}$$

If an admittance is formed from a capacitive susceptance, as the conductance component is independent of frequency, the impedance will vary over band of frequencies due only to its capacitive susceptance component. That is,

$$Y(f) = G + j2\pi C f_{hi/lo} \qquad \textbf{(4-18)}$$

#### *4.1.2.2. Addition of shunt capacitive susceptance to an admittance*

In this section, the addition of a shunt capacitive susceptance to an admittance will be considered.

At a particular frequency $f_1$, the addition of a shunt capacitive susceptance $B_{C1}(f_1)$, to an admittance $Y_1(f_1)$, forms a new admittance $Y_f(f_1)$ given by,

$$Y_f(f_1) = Y_1(f_1) + B_{C1}(f_1) = G_1 + B_1(f_1) + B_{C1}(f_1)$$

At a different frequency $f_2$, the addition of a shunt capacitive susceptance $B_{C1}(f_2)$, to an admittance $Y_1(f_2)$, forms a new admittance $Y_f(f_2)$, and with the aid of equation 4-17, is given by,

$$Y_f(f_2) = Y_1(f_2) + B_{B1}(f_2) = G_1 + \frac{f_2}{f_1}\left(B_1(f_1) + B_{C1}(f_1)\right) \qquad \textbf{(4-19)}$$

The change in admittance due to the change in frequency is given by

$$\Delta Y_f(f_1, f_2) = B_1(f_2) - B_1(f_1) + B_{B1}(f_2) - B_{B1}(f_1)$$

or

$$\Delta Y_f(f_1, f_2) = B_1(f_1) - B_1(f_2) + B_{C1}(f_1) - B_{C1}(f_2)$$

$$\left|\Delta Y_f(f_1, f_2)\right| = \left(B_1(f_1) + B_{C1}(f_1)\right)\left|\frac{f_2}{f_1} - 1\right| \qquad \textbf{(4-20)}$$

Equation 4-20 can be modified in terms of high and low frequencies, to give

$$\left|\Delta Y_f(f_{lo}, f_{hi})\right| = \left(B_1(f_{lo}) + B_{C1}(f_{lo})\right)\left|\frac{f_{hi} - f_{lo}}{f_{lo}}\right| \qquad \textbf{(4-21a)}$$

$$\left|\Delta Y_f(f_{hi}, f_{lo})\right| = \left(B_1(f_{hi}) + B_{C1}(f_{hi})\right)\left|\frac{f_{hi} - f_{lo}}{f_{hi}}\right| \qquad \textbf{(4-21b)}$$

The addition of a shunt capacitance will move an admittance clockwise around the Smith chart with the lower frequency moving the centre-band admittance the same amount clockwise as the higher frequency. In other words, a load admittance with a capacitive susceptance will form a line around the constant conductance circle $G_1$, rather than just a point, with the lower frequency point having the same admittance change from the centre-band point as the higher frequency point.

Although in theory, there should not be any change in the value of the conductance component of the admittance as the frequency changes, again this is not usually the case in practice. This means that the band admittance line crosses onto different constant conductance circles, with the band-centre admittance point lying on the constant conductance circle $G_1$.

EXAMPLE 5

Given a normalized load admittance $Y_{300} = 1.0 + j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the admittance band between 150 MHz and 450 MHz. Assume that the conductance component of the admittance remains constant over the band.

SOLUTION

From equation 4-15a, $B_{hi} = B_o \dfrac{f_{hi}}{f_o} = j2.5\dfrac{450}{300} = j3.75$ so $Y_{450} = 1.0 + j3.75$

From equation 4-15b, $B_{lo} = B_o \dfrac{f_{lo}}{f_o} = j2.5\dfrac{150}{300} = j1.25$ so $Y_{150} = 1.0 + j1.25$

Figure 78 shows the plot of the admittance band on the Smith chart. Note how the lower frequency admittance in the frequency band is same value away from the admittance at the band centre as the admittance at the higher frequency of the band.

EXAMPLE 6

Given a normalized load admittance $Y_{300} = 1.0 + j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the admittance band between 150 MHz and 450 MHz when an inductance of -j2.5 at a frequency of 300 MHz, is added in shunt to the load admittance. Again assume that the conductance component of the admittance remains constant over the band.

SOLUTION

At the centre of the band, the load admittance with the added shunt inductance is given by,

$Y'_{300} = 1.0 + j2.5 - j2 = 1.0 + j0.5$. Using equation 4-19 and

from equation 4-15a, $B_{hi} = B_o \dfrac{f_{hi}}{f_o} = j0.5\dfrac{450}{300} = j0.75$ so $Y_{450} = 1.0 + j0.75$

from equation 4-15b, $B_{lo} = B_o \dfrac{f_{lo}}{f_o} = j0.5\dfrac{150}{300} = j0.25$ so $Y_{150} = 1.0 + j0.25$

Figure 78 shows the new plot of the admittance band on the Smith chart. Again note how the addition of the shunt inductance has rotated the whole band anticlockwise along the circle of constant conductance, which in this case is 1.0.

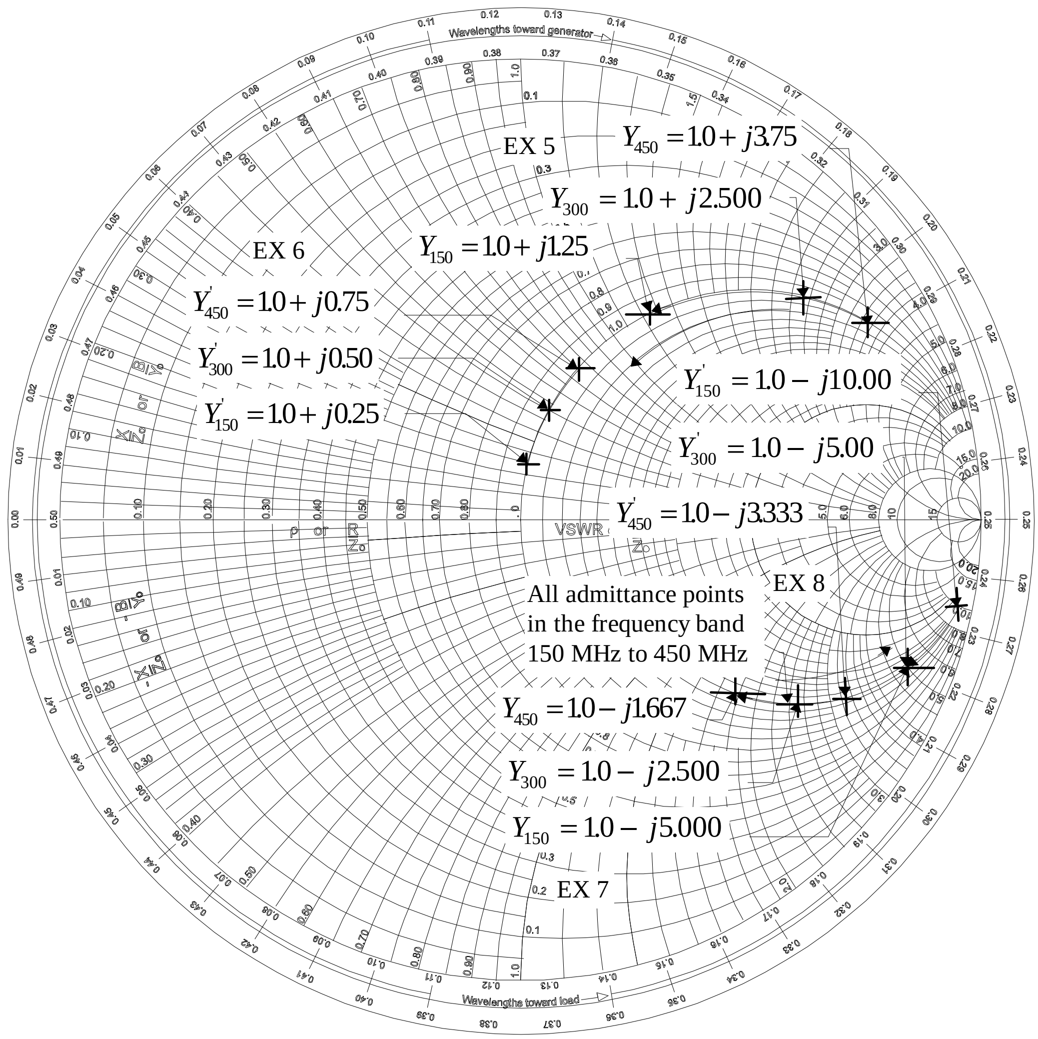



**Figure 78** **Examples 5 to 8** **Admittance curves due to a band of frequencies**

Summarizing the results of sections 4.1.2.1. and 4.1.2.2.,

- For a shunt capacitive susceptance, the higher frequency in a band of frequencies moves the band-centered admittance clockwise from the band-centered admittance position, whereas the lower frequency in the band moves the band-centered admittance an equal amount anticlockwise from the band-centered position,
- The addition of a shunt capacitive susceptance, over a band of frequencies, to an admittance band will rotate the admittance band clockwise along a circle, or circles, of constant conductance. The

lower frequency admittance will move the same admittance increment around the chart as the high frequency admittance.

### *4.1.2.3.Inductive susceptance*

If the susceptance at a band centre frequency $f_o$ is given by $B_o = 1/j\omega_o L$ , then, at an incremental frequency $\delta\omega = 2\pi\delta f$ , higher or lower than the centre frequency $\omega_o = 2\pi f_o$, that is, $\omega_o \pm \delta\omega$ , the susceptance becomes,

$B_o \mp \Delta B = \dfrac{1}{j(\omega_o \pm \delta\omega)L} = \dfrac{1}{j\omega L}$ giving, $\dfrac{B_o \mp \Delta B}{B_o} = \dfrac{f_o}{f_o \pm \delta f}$ which reduces to

$B_o \mp \Delta B = B_o \dfrac{f_o}{f_o \pm \delta f}$ , or more explicitly:

$$B_{hi} = B_o \frac{f_o}{f_{hi}} \qquad \textbf{(4-22a)}$$

where $B_{hi} = B_o - \Delta B$ , $f_{hi} = f_o + \delta f$ and $\Delta B = B_o \dfrac{\delta f}{f_{hi}}$

$$B_{lo} = B_o \frac{f_o}{f_{lo}} \qquad \textbf{(4-22b)}$$

where $B_{lo} = B_o + \Delta B$ , $f_{lo} = f_o - \delta f$ and $\Delta B = B_o \dfrac{\delta f}{f_{lo}}$

To avoid obtaining the inductive susceptance at d.c., $f_o > \delta f$ .

It should be noted that for equal frequency steps $\delta f$ , throughout the frequency band under consideration, the inductive susceptance does not change by linearly proportional amounts. Equations 4-22 show that the frequencies lower than the band centre frequency $f_o$, produce a greater change in the inductive susceptance than the frequencies higher in the band. That is, the lower frequencies produce a greater change in susceptance than the higher frequencies. Dividing equation 4-22b by equation 4-22a gives a measure of how much greater the inductive susceptance at the lower frequencies is compared to the inductive susceptance at the higher frequencies, that is,

$$B_{lo} = B_{hi} \frac{f_{hi}}{f_{lo}} \qquad \textbf{(4-23)}$$

Equation 4-23 can be generalized to equation 4-24 to predict values of inductive susceptance $B_{L1}(f_1)$ , at a particular frequency $f_1$, given a value of a known inductive susceptance $B_{L1}(f_2)$ and its frequency $f_2$. That is.

$$B_{L1}(f_1) = B_{L1}(f_2)\frac{f_2}{f_1} \qquad \textbf{(4-24)}$$

If an admittance is formed from a inductive susceptance, as the conductance component is independent of frequency, the admittance will vary over band of frequencies due only to its inductive susceptance component. That is,

$$Y(f) = G - j\frac{1}{2\pi L f_{hi/lo}} \qquad \textbf{(4-25)}$$

#### *4.1.2.4. Addition of shunt inductive susceptance to an admittance*

In most cases where matching occurs, the emphasis is placed on adding a susceptance to an admittance. In this section, the addition of a shunt inductive susceptance to an admittance will be considered.
At a particular frequency $f_1$, the addition of a shunt inductive susceptance $B_{L1}(f_1)$, to an admittance $Y_1(f_1)$, forms a new admittance $Y_f(f_1)$ given by,

$$Y_f(f_1) = Y_1(f_1) + B_{L1}(f_1) = G_1 + B_1(f_1) + B_{L1}(f_1)$$

At a different frequency $f_2$, the addition of a shunt inductive susceptance $B_{L1}(f_2)$, to an admittance $Y_1(f_2)$, forms a new admittance $Y_f(f_2)$, and with the aid of equation 4-24, is given by,

$$Y_f(f_2) = Y_1(f_2) + B_{L1}(f_2) = G_1 + \frac{f_1}{f_2}\left(B_1(f_1) + B_{L1}(f_1)\right) \qquad \textbf{(4-26)}$$

The change in admittance due to the change in frequency is given by

$$\Delta Y_f(f_1, f_2) = B_1(f_1) - B_1(f_2) + B_{L1}(f_1) - B_{L1}(f_2)$$

or

$$\Delta Y_f(f_1, f_2) = B_1(f_2) - B_1(f_1) + B_{L1}(f_2) - B_{L1}(f_1)$$

$$\left|\Delta Y_f(f_1, f_2)\right| = \left(B_1(f_1) + B_{L1}(f_1)\right)\left|1 - \frac{f_1}{f_2}\right| \qquad \textbf{(4-27)}$$

Equation 4-27 can be modified in terms of high and low frequencies, to give

$$\left|\Delta Y_f(f_{lo}, f_{hi})\right| = \left(B_1(f_{lo}) + B_{L1}(f_{lo})\right)\left|\frac{f_{hi} - f_{lo}}{f_{hi}}\right| \qquad \textbf{(4-28a)}$$

$$\left|\Delta Y_f(f_{hi}, f_{lo})\right| = \left(B_1(f_{hi}) + B_{L1}(f_{hi})\right)\left|\frac{f_{hi} - f_{lo}}{f_{lo}}\right| \qquad \textbf{(4-28b)}$$

The addition of a shunt inductance will move an admittance anticlockwise around the Smith chart with the lower frequency moving the centre-band admittance more anticlockwise than the higher frequency. In other words, a load admittance with a inductive susceptance will form a line around the constant conductance circle $G_1$, rather than a point, with the lower frequency admittance point having a greater admittance change from the centre-band point than the higher frequency point. On the addition of additional shunt inductance the difference in the admittance change from the band-centre admittance point will be more pronounced.
Although in theory, there should not be any change in the value of the conductance component of the admittance as the frequency changes, this is not usually the case in practice. There usually is a change in conductance, due to different attenuations of the medium at different frequencies, reflections, etc., which means that the band admittance line crosses onto different constant conductance circles, with the band-centre admittance point lying on the constant conductance circle $G_1$.

EXAMPLE 7
Given a normalized load admittance $Y_{300} = 1.0 - j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the admittance band between 150 MHz and 450 MHz. Assume that the conductance component of the admittance remains constant over the band.

SOLUTION
From equation 4-22a, $B_{hi} = B_o \dfrac{f_o}{f_{hi}} = -j2.5\dfrac{300}{450} = -j1.667$ so $Y_{450} = 1.0 - j1.667$

From equation 4-22b, $B_{lo} = B_o \dfrac{f_o}{f_{lo}} = -j2.5\dfrac{300}{150} = -j5.000$ so $Y_{150} = 1.0 - j5.000$

Figure 78 shows the plot of the admittance band on the Smith chart. Note how the lower frequency admittance in the frequency band is further away from the admittance at the band centre than the admittance at the higher frequency of the band.

EXAMPLE 8
Given a normalized load admittance $Y_{300} = 1.0 - j2.5$, at a centre frequency of 300 MHz, plot on the Smith chart the admittance band between 150 MHz and 450 MHz when a an inductive susceptance of -j2.5 at a frequency of 300 MHz, is added in shunt to the load admittance. Again assume that the conductance component of the admittance remains constant over the band.

SOLUTION
At the centre of the band, the load admittance with the added shunt inductance is given by,
$Y_{300}^{'} = 1.0 - j2.5 - j2.5 = 1.0 - j5.0$. Using equation 4-26 and

from equation 4-22a, $B_{hi} = B_o \dfrac{f_o}{f_{hi}} = -j5.0 \dfrac{300}{450} = -j3.333$ so $Y_{450} = 1.0 - j3.333$

from equation 4-22b, $B_{lo} = B_o \dfrac{f_o}{f_{lo}} = -j5.0 \dfrac{300}{150} = -j10.000$ so $Y_{150} = 1.0 - j10.000$

Figure 78 shows the new plot of the admittance band on the Smith chart. Note how the addition of the shunt inductance has rotated the whole band unequally anticlockwise along the circle of constant conductance, which in this case is 1.0. The lower frequency of the band having the greatest effect on the susceptive component as seen by the change from -j5.0 in example 1 to -j10.0 in this example. The higher frequency of the band moving only from -j1.667 in example 7 to -j3.333 in this example.

Summarizing the results of sections 4.1.2.3. and 4.1.2.4.,

- The lower frequency in a band of frequencies moves the band-centered admittance more anticlockwise from the band-centered admittance position than it does the higher frequency admittance,
- The addition of an inductive susceptance, over a band of frequencies, to an admittance band will rotate the admittance band anticlockwise along a circle, or circles, of constant conductance. The lower frequency admittance increment being further extended from the band centered admittance by the addition of an additional inductive susceptance.

## *4.2. Variation of impedance and admittance with frequency*

It has been stated and assumed in section 4.1 that the resistance and the conductance of a load remains independent of frequency. If the load impedance or admittance were to be measured at some point down-line then due to some practical considerations, which may be due to the variation of the primary coefficients of the line, multiple reflections occurring in the line due to a mismatched load, then the resistance or conductance would not be constant, but would be frequency dependent. Apart from this scenario, there is a much more important reason why, under certain circumstances, the resistance or conductance will be frequency dependent. This is the subject of discussion in this section, and will have a bearing on our work with Smith charts.

### 4.2.1. Load impedance converted to a load admittance

If at a certain frequency f, a load impedance $Z_L(f)$, is measured and it is found to have a resistive component $R_L$, and a reactance component $jA_L(f)$, then the impedance would be given by $Z_L(f) = R_L + jA_L(f)$. However, its equivalent admittance would be given by $Y_L(f) = \dfrac{1}{Z_L(f)}$, which is given by,

$$Y_L(f) = \frac{1}{R_L + jA_L(f)} = \frac{R_L}{R_L^2 + A_L^2(f)} - j\frac{A_L(f)}{R_L^2 + A_L^2(f)} = G_L(f) - jD_L(f) \qquad \textbf{(4-29)}$$

showing that the conductance $G_L(f)$ and the susceptance $-\,jD_L(f)$ of the load admittance are both dependent on frequency. The conductance becoming dependent on frequency due to the inclusion of the modulus of the reactance, $A_L(f)$. This means that the plotting of a load admittance over a frequency band on the Smith chart, which is derived from a load impedance over a frequency band, cannot be directly plotted on the Smith chart to follow circles of constant conductance. This follows also for the single frequency case where the reactance changes due to the changing of the capacitance or inductance rather than the changing of frequency. The only time that circles of constant conductance are utilized is when the load is directly measured as an admittance.

EXAMPLE 9
A normalized load impedance is given as 0.2 -j0.4 at a frequency of 150 MHz. Determine the load impedances at 50 MHz, 300 MHz and 450 MHz and plot these points on a Smith chart with their corresponding admittance points.

SOLUTION

From equation 4-3, $X_{c1}(50) = X_{c1}(150)\dfrac{150}{50} = -j0.4\dfrac{150}{50} = -j1.2$ so $Z_{300} = 0.2 - j1.2$

From equation 4-3, $X_{c1}(300) = X_{c1}(150)\dfrac{150}{300} = -j0.4\dfrac{150}{300} = -j0.2$ so $Z_{300} = 0.2 - j0.2$

From equation 4-3, $X_{c1}(450) = X_{c1}(150)\dfrac{150}{450} = -j0.4\dfrac{150}{450} = -j0.1333$ so $Z_{450} = 0.2 - j0.1333$

Figure 79 shows the plot of the impedance band on the Smith chart. The admittance points are plotted by drawing a line from each impedance point through the centre of the chart to the same distance from the centre of the chart. These admittance points are also plotted on figure 79. The values of the calculated admittances are:

$$Y_{50} = 0.135 + j0.81,\;\; Y_{150} = 1 + j2.0,\;\; Y_{300} = 2.5 + j2.5,\;\; Y_{450} = 3.46 + j2.31.$$

If an admittance chart were to be superimposed on the Smith chart of figure 79, then the admittance points on the underlying Smith chart would be converted into impedance points on the overlay admittance chart and would follow a circle of constant resistance of 0.2.

### 4.2.2. Load admittance converted to a load impedance

If at a certain frequency f, a load impedance $Y_L(f)$, is measured and it is found to have a conductance component $G_L$, and a susceptance component $jM_L(f)$, then the admittance would be given by $Y_L(f) = G_L + jM_L(f)$. However, its equivalent impedance would be given by

$Z_L(f) = \dfrac{1}{Y_L(f)}$, which is given by,

$$Z_L(f) = \frac{1}{G_L + jM_L(f)} = \frac{G_L}{G_L^2 + M_L^2(f)} - j\frac{M_L(f)}{G_L^2 + M_L^2(f)} = F_L(f) - jE_L(f) \qquad \textbf{(4-30)}$$

showing that the resistance $F_L(f)$ and the reactance $-\,jE_L(f)$ of the derived load resistance are both dependent on frequency. The resistance becoming dependent on frequency due to the inclusion of the modulus of the susceptance, $M_L(f)$. This means that the plotting of a load impedance over

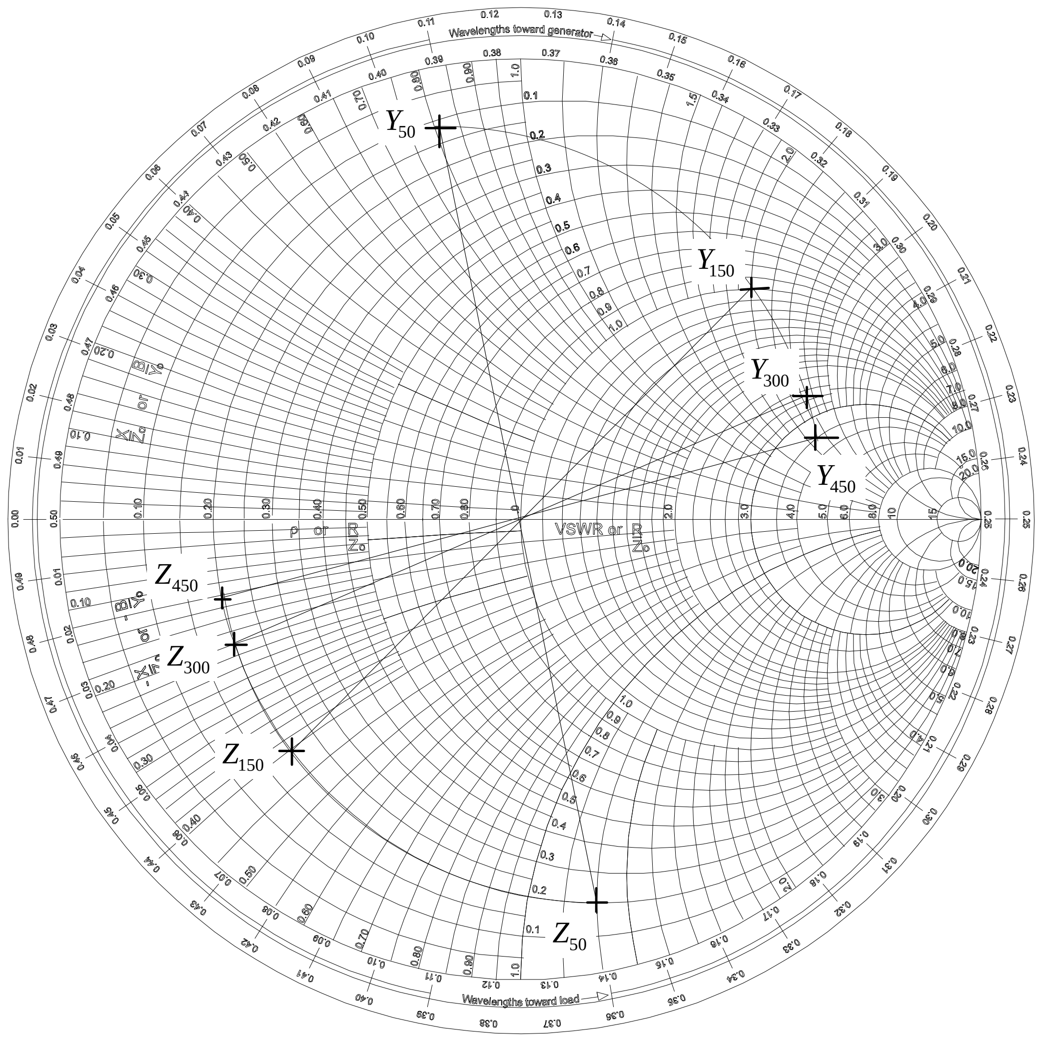



**Figure 79 Example 9 Impedance and Admittance curves due to a band of frequencies**

a frequency band on the Smith chart, which is derived from a load admittance over a frequency band, cannot be directly plotted on the Smith chart to follow circles of constant resistance. This follows for the single frequency case where the susceptance changes due to the changing of the capacitance or inductance rather than the changing of frequency. The only time circles of constant resistance will be used is where the load is directly measured as an impedance.

EXAMPLE 10

A normalized load admittance is given as 1 +j2.0 at a frequency of 150 MHz. Determine the load admittances at 50 MHz, 300 MHz and 450 MHz and plot these points on a Smith chart with their corresponding impedance points.

SOLUTION

The solution to this question is shown on figure 80 and use is made of equation 4-17 to obtain the following values of susceptance and admittance,

$$B_{C1}(50) = B_{C1}(150)\frac{50}{150} = j2.0\frac{50}{150} = j0.667 \text{ so } Y_{50} = 1 + j0.667$$

$$B_{C1}(300) = B_{C1}(150)\frac{300}{150} = j2.0\frac{300}{150} = j4.0 \text{ so } Y_{300} = 1 + j4.0$$

$$B_{C1}(450) = B_{C1}(150)\frac{450}{150} = j2.0\frac{450}{150} = j6.0 \text{ so } Y_{450} = 1 + j6.0$$

The values of the calculated impedances are:

$Z_{50} = 0.6923 - j0.461$, $Z_{150} = 0.2 - j0.4$, $Z_{300} = 0.059 - j0.2353$, $Z_{450} = 0.027 - j0.162$.

Figure 80 shows the plot of the impedance band on the Smith chart. The admittance points are plotted by drawing a line from each impedance point through the centre of the chart to the same distance from the centre of the chart. These admittance points are also plotted on figure 80. The values of the calculated admittances are:

$Y_{50} = 0.135 + j0.81$, $Y_{150} = 1 + j2.0$, $Y_{300} = 2.5 + j2.5$, $Y_{450} = 3.46 + j2.31$.

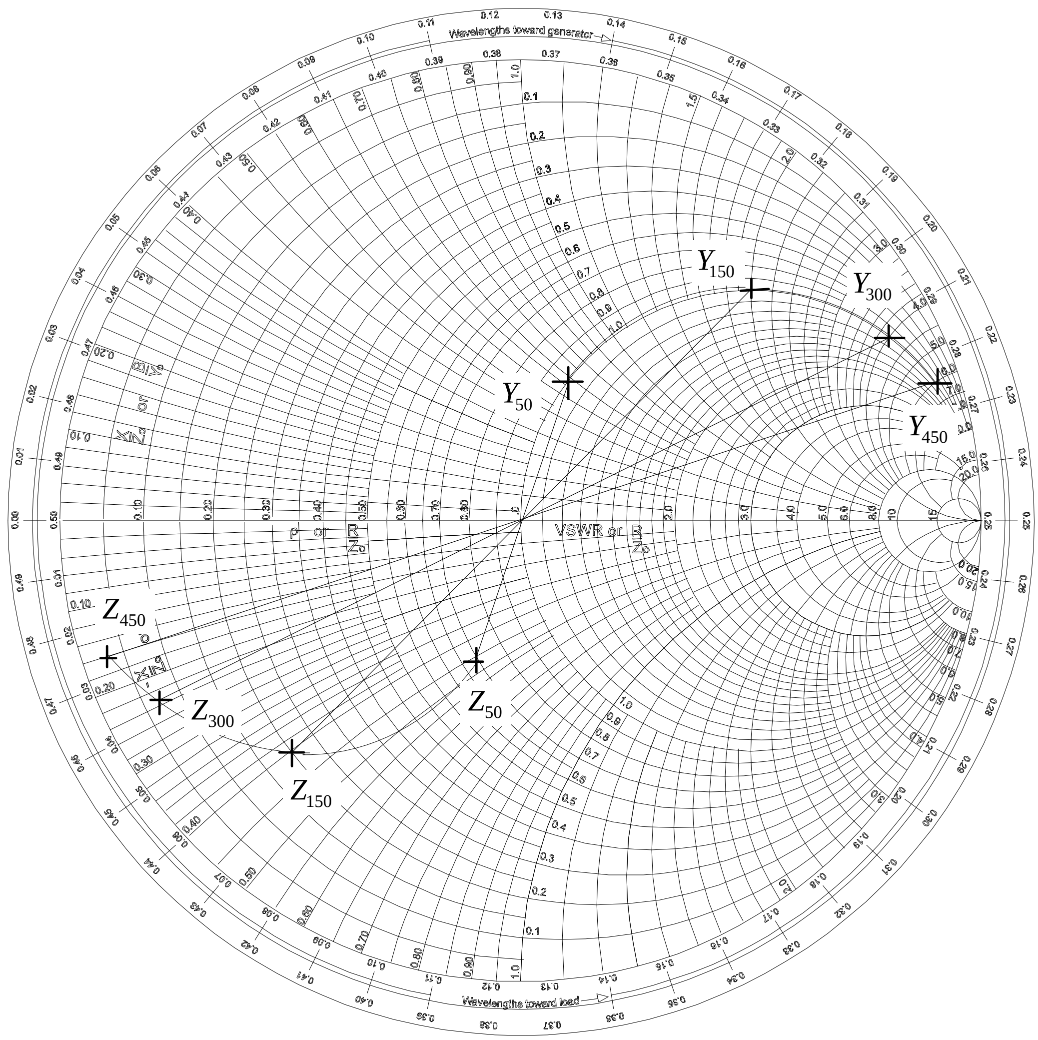



**Figure 80 Example 10 - Admittance and Impedance curves due to a band of frequencies**

## 4.3 Matching a mismatched load using L-type circuits

Perfect matching of a mismatched load impedance over a band of frequencies by changing the impedance so that it becomes the characteristic impedance of the line, that is, by adding additional circuits to bring the load impedance to the centre of the Smith chart and making its VSWR unity, is not practically possible when using a band of fre uencies. he term “matching” when applied to a band of frequencies is to reduce the mismatched load VSWR so that is lies within a specified VSWR circle. The smaller the VSWR circle which the load impedance VSWR can be brought to lie within, the closer to perfect matching over the frequency band the load becomes. Usually, a VSWR circle is specified over the frequency band of interest, and the ob ect is to “match” the load impedance to within this specified circle. his is the sense of the word “matching” in this chapter.
In this section the eight L-type impedance transformation circuits, discussed in section 2.6.1, will be used to demonstrate how the VSWR of a mismatched load over a band of frequencies can be reduced, using the concepts developed in sections 4.2 and 4.3. These L-type circuits individually, do not permit a mismatched load to be matched by bringing the load impedance or admittance band to lie **within a specified voltage standing wave ratio** (VSWR) circle, but do allow the band centre frequency impedance or admittance to be matched to the centre of the chart (VSWR =1). However, the use of single L-type circuits is a step in the right direction towards the final objective of changing the mismatched load VSWR so that it lies within a specified VSWR circle. Section 4.4, will consider more complicated circuits which can be designed to permit the mismatched load VSWR to be reduced so it will lie within a specified VSWR circle over the frequency band used.

### 4.3.1 L-Type Impedance Transformation Circuits

This section will consider the eight single L-type circuits and provide examples on their use in reducing the VSWR of the mismatched load.

#### *4.3.1.1 Constant conductance circles*

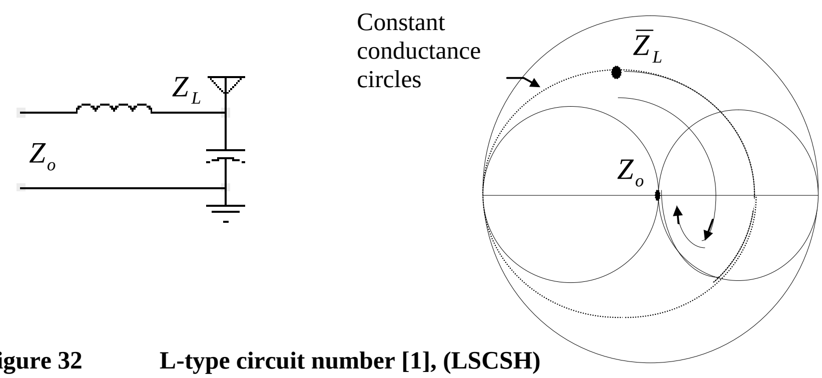



**Figure 32 L-type circuit number [1], (LSCSH)**

Referring to Figure 32, as the capacitor is in shunt with the load, we take the plotted load impedances $\overline{Z}_{Lf}$ on the Smith chart and using an overlay admittance chart, taking the impedance at the centre of the band, follow the constant G circle clockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_{cf_o}$, around the admittance chart will permit the value of the shunt capacitance to be determined, as $\Delta\overline{B}_{cf_o} = j\omega_o\overline{C} = j2\pi f_o\overline{C}$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle around to the $R/Z_o = 1$ point, or centre of the mith chart. From Figure 3 , we see that we are following one of the “add series inductance” circles. he a ount of rotation on the ith chart er its the determination of the series inductance, from $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$.

EXAMPLE 11
A normalized load impedance is given as 0.2 +j0.5 at a centre frequency of 200 MHz. Determine the load impedances at the extremes of the band, which are 100 MHz and 300 MHz and plot these points on a Smith chart with their corresponding admittance points. Using an overlay admittance chart, rotate

the admittance points until the centre band impedance point (which are admittance points on the admittance chart) cuts the unity resistance circle at the lower point , as shown in Figure 32, and determine the value of the shunt capacitance required. By rotating the centre frequency impedance along the unity resistance circle until the centre of the Smith chart is reached, determine the value of the series inductance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

From equation 4-10, $X_{L1}(100) = X_{L1}(200)\dfrac{100}{200} = +j0.5\dfrac{100}{200} = j0.25$ so $Z_{100} = 0.2 + j0.25$

From equation 4-10, $X_{L1}(300) = X_{L1}(200)\dfrac{300}{200} = j0.5\dfrac{300}{200} = j0.75$ so $Z_{300} = 0.2 + j0.75$

The values of the calculated admittances are:

$Y_{100} = 1.951 - j2.439$ , $Y_{200} = 0.69 - j1.724$ , $Y_{300} = 0.332 - j1.245$

The rotation of . $Y_{200} = 0.69 - j1.724$ along a circle of constant conductance $\overline{G} = 0.69$ on the admittance chart, until it reaches the $\overline{R} = 1$ circle on the Smith chart permits the value of capacitive susceptance to be determined. This value of capacitive susceptance at the centre frequency can be translated to values of capacitive susceptance at the other two frequencies using equation 4-17 which are then plotted on the Smith chart. The admittance chart overlayed on the Smith chart together with the plotted impedance (corresponding admittance points) and the rotation clockwise along the $\overline{G} = 0.69$ circle is shown on Figure 81. From Figure 81, it can be seen that the intersection of the $\overline{G} = 0.69$ circle with the $\overline{R} = 1$ circle occurs at the admittance point $Y_{R=1} = 0.69 + j0.46$ .

The value of $\Delta\overline{B}_{cf_o}$ is found by subtracting $Y_{200} = 0.69 - j1.724$ from $Y_{R=1} = 0.69 + j0.46$ giving $\Delta\overline{B}_{cf_o} = j2.18$. The load capacitive susceptances at 100 MHz and 300 MHz move around their respective conductance circles by an amount which is related to $\Delta\overline{B}_{cf_o}$ . The relationship to $\Delta\overline{B}_{cf_o}$ is given by equation 4.17. Hence,

$B_C(100) = \Delta\overline{B}_{cf_o}\dfrac{100}{200} = j1.09$, and so

$$Y'_{100} = Y_{100} + \Delta\overline{B}_{cf_o} = 1.951 - j2.439 + j1.09 = 1.951 - j1.349$$

$B_C(300) = \Delta\overline{B}_{cf_o}\dfrac{300}{200} = j3.27$, and so

$$Y'_{300} = Y_{300} + \Delta\overline{B}_{cf_o} = 0.332 - j1.245 + j3.27 = 0.332 + j2.025$$

The value of shunt capacitance is found from the relation $\Delta\overline{B}_{cf_o} = j\omega_o\overline{C} = j2\pi f_o\overline{C}$ , so at the band centre of 200 MHz, with $\Delta\overline{B}_{cf_o} = j2.18$, and C = 1.735$Z_o$ nF.

As the centre frequency admittance $Y_{R=1} = 0.69 + j0.46$ , is to be changed to its equivalent impedance for matching at the centre of the Smith chart, we find that the impedance to be, $Z_{G=0.69} = 1 - j0.67$. This means that an inductive reactance of $\overline{X}_{Lf_o} = j0.67$ is required to be added in series with $Z_{G=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in a clockwise direction of $Z_{G=0.69}$ to the centre of the Smith chart is shown in Figure 82. The outer band frequencies however, do not rotate clockwise through a reactance of j0.67, but have to be modified because of their different frequencies.

The values of the outer band impedances are obtained from the Smith/Admittance chart as,

$Z'_{100} = 0.35 + j0.24$ and $Z'_{300} = 0.08 - j0.48$

Equation 4-10 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$$X_L(100) = j0.67\frac{100}{200} = j0.335 \text{, so}$$

$$Z_{100\,final} = 0.35 + j0.24 + j0.335 = 0.35 + j0.575 \approx 0.35 + j0.58$$

Converting this to an admittance for ease in plotting on the admittance chart

$$Y_{100\,final} = 0.772 - j1.269$$

$$X_L(300) = j0.67\frac{300}{200} = j1.005 \text{, so}$$

$$Z_{300\,final} = 0.0788 - j0.481 + j1.005 = 0.08 + j0.524 \approx 0.08 + j0.52$$

Again converting this to an admittance for ease in plotting,

$$\mathrm{Y}_{300\mathrm{final}} = 0.236 - \mathrm{j}1.536$$

These admittance points are shown plotted as impedances on Figure 81.

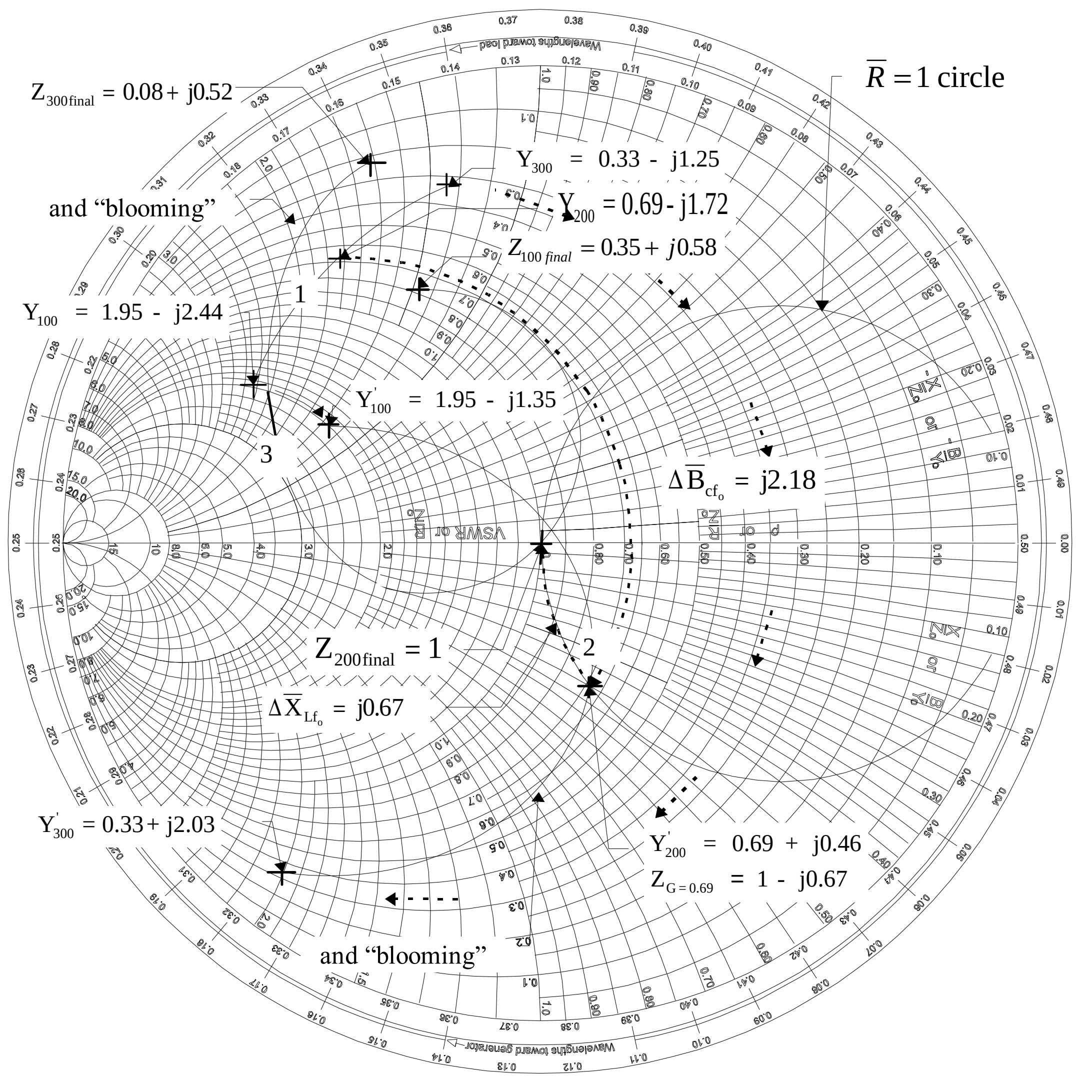
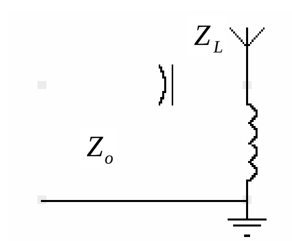



**Figure 81** **Example 11, Matching with shunt capacitance and series inductance**

The VSWR is found to be 12.5. This was found be drawing a circle around the impedance point which was the furthest out from the centre of the Smith chart, $Z_{300\,final}$, and reading the VSWR where the circle crossed the horizontal axis. Note that this VSWR is worse at 300 MHz than the VSWR of the original load impedance point $Z_{300}$, where the . . Note the "blooming" of the impedance

band. It is wider than the original unmatched band. However at the lower frequency, there is some improvement in VSWR, and at the centre frequency of 200 MHz we have perfect matching.
If the impedance or admittance bandwidth is small to begin with, the blooming will not be so prominent.

The value of inductance is found from $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ and as $\overline{X}_{Lf_o} = j0.67$, we find L= $0.533Z_o$ nH.
Overlaying the admittance chart onto the Smith chart produces a chart which is exceptionally complex and difficult to work with. In these matching cases, this problem is alleviated by using only the unit circle of the admittance or Smith chart with the Smith or admittance chart. That is, if the problem involves a shunt component closest to the load, then use an admittance chart with a unity normalized resistance circle from the Smith chart. If the problem involves a series component closest to the load, use a Smith chart with a unity normalized conductance circle from the admittance chart.



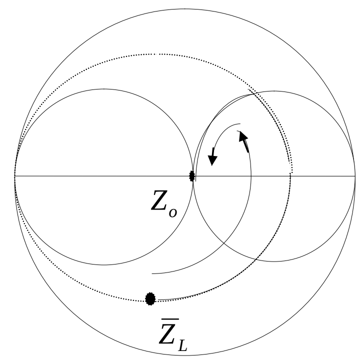




**Figure 33 L-type circuit number [3], (CSLSH)**

Referring to Figures 30 and 33, as the inductor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle anticlockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_{Lf_o}$, around the admittance chart will permit the value of the shunt inductance to be determined, as $\Delta\overline{B}_{Lf_o} = 1/j\omega_o L = 1/j2\pi f_o L$. From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add series capacitance" circles. he amount of rotation on the Smith chart permits the determination of the series capacitance, from $\Delta X_{Cf_o} = 1/j\omega_o C = 1/j2\pi f_o C$.

EXAMPLE 12
A normalized load impedance is given as 0.2 -j0.5 at a centre frequency of 200 MHz. Determine the load impedances at the extremes of the band, which are 100 MHz and 300 MHz and plot these points on a Smith chart with their corresponding admittance points. Using an overlay admittance chart, rotate the admittance points until the centre band impedance point (which are admittance points on the admittance chart) cuts the unity resistance circle at the higher point , as shown in Figure 33, and determine the value of the shunt inductance required. By rotating the centre frequency impedance along the unity resistance circle until the centre of the Smith chart is reached, determine the value of the series capacitance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

From equation 4-3, $X_{c1}(100) = X_{c1}(200)\dfrac{200}{100} = -j0.5\dfrac{200}{100} = -j1.0$ so $Z_{100} = 0.2 - j1.0$

From equation 4-3, $X_{c1}(300) = X_{c1}(200)\dfrac{200}{300} = -j0.5\dfrac{200}{300} = -j0.33$ so $Z_{300} = 0.2 - j0.33$

The values of the calculated admittances are:

$Y_{100} = 0.192 + j0.962$ , $Y_{200} = 0.69 + j1.724$ , $Y_{300} = 1.324 + j2.206$

The rotation of . $Y_{200} = 0.69 + j1.724$ along a circle of constant conductance $\overline{G} = 0.69$ on the admittance chart, until it reaches the $\overline{R} = 1$ circle on the Smith chart permits the value of inductive susceptance to be determined. This value of inductive susceptance at the centre frequency can be translated to values of inductive susceptance at the other two frequencies using equation 4-24 which are then plotted on the Smith chart. The admittance chart overlayed on the Smith chart together with the plotted impedance (corresponding admittance points) and the rotation anti-clockwise along the $\overline{G} = 0.69$ circle is shown on Figure 82. From Figure 82, it can be seen that the intersection of the $\overline{G} = 0.69$ circle with the $\overline{R} = 1$ circle occurs at the admittance point $Y_{R=1} = 0.69 - j0.46$ .
The value of $\Delta\overline{B}_{Lf_o}$ is found by subtracting $Y_{200} = 0.69 + j1.724$ from $Y_{R=1} = 0.69 - j0.46$ giving $\Delta\overline{B}_{Lf_o} = -j2.18$ . The load inductive susceptances at 100 MHz and 300 MHz move around their respective conductance circles by an amount which is related to $\Delta\overline{B}_{Lf_o}$ . The relationship to $\Delta\overline{B}_{Lf_o}$ is given by equation 4.24. Hence,

$B_L(100) = \Delta\overline{B}_{Lf_o}\dfrac{200}{100} = -j4.36$ , and so

$Y'_{100} = Y_{100} + \Delta\overline{B}_{Lf_o} = 0.192 + j0.962 - j4.36 = 0.192 - j3.398$

$B_C(300) = \Delta\overline{B}_{Lf_o}\dfrac{200}{300} = -j1.453$ , and so

$Y'_{300} = Y_{300} + \Delta\overline{B}_{Lf_o} = 1.324 + j2.206 - j1.453 = 1.324 + j0.753$

The value of shunt inductance is found from the relation $\Delta\overline{B}_{Lf_o} = 1/ j\omega_o L = 1/ j2\pi f_o L$ , so at the band centre of 200 MHz, with $\Delta\overline{B}_{Lf_o} = -j2.18$ , and C = 0.365$Z_o$ nH.
As the centre frequency admittance $Y_{R=1} = 0.69 - j0.46$ , is to be changed to its equivalent impedance for matching at the centre of the Smith chart, we find that the impedance to be, $Z_{G=0.69} = 1 + j0.67$ . This means that an capacitive reactance of $\overline{X}_{Cf_o} = -j0.67$ is required to be added in series with $Z_{G=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$ . The rotation in a anti-clockwise direction of $Z_{G=0.69}$ to the centre of the Smith chart is shown in Figure 82. The outer band frequencies however, do not rotate clockwise through a reactance of -j0.67, but have to be modified because of their different frequencies.
The values of the outer band impedances are obtained from the Smith/Admittance chart as,
$Z'_{100} = 0.02 + j0.29$ and $Z'_{300} = 0.58 - j0.33$
Equation 4-3 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$X_C(100) = -j0.67\dfrac{200}{100} = -j1.34$ , so $Z_{100\,final} = 0.02 + j0.29 - j1.34 = 0.02 - j1.05$

Converting this to an admittance for ease in plotting on the admittance chart
$Y_{100\,final} = 0.02 + j0.95$

$X_C(300) = -j0.67\dfrac{200}{300} = -j0.447$ , so $Z_{300\,final} = 0.58 - j0.33 - j0.447 = 0.58 - j0.78$

Again converting this to an admittance for ease in plotting,
$Y_{300\,final} = 0.61 + j0.83$

These admittance points are shown plotted as impedances on Figure 82.

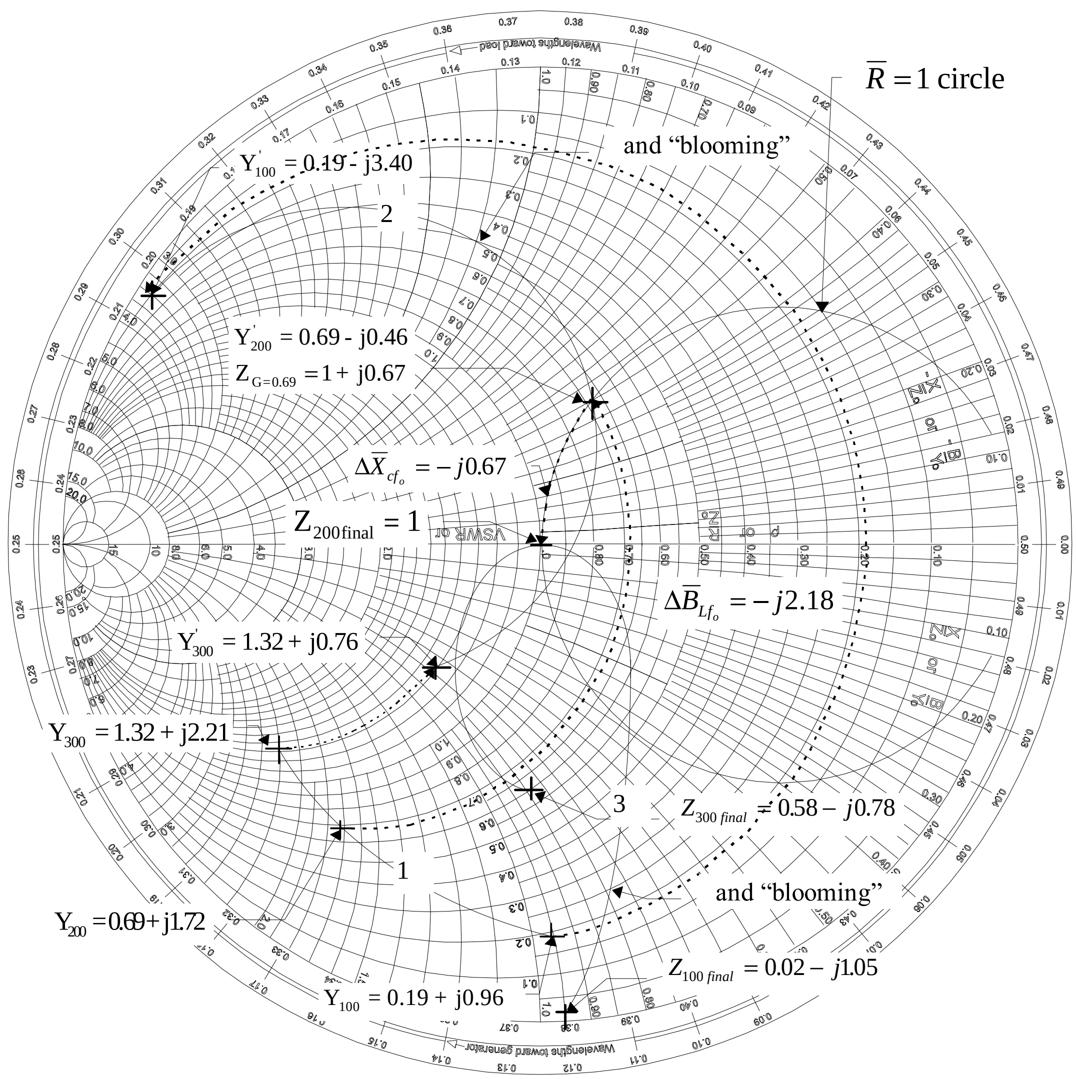
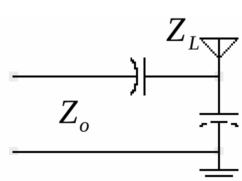



**Figure 82** **Example 12, Matching with shunt inductance and series capacitance**

The VSWR is found to be 50. This was found by taking the value of normalized resistance of the furthest out impedance point $Z_{100\,final}$, and taking the inverse.

The value of capacitance is found from $\Delta\overline{X}_{Cf_o} = 1/j\omega_o\overline{C} = 1/j2\pi f_o\overline{C}$ and as $\overline{X}_{Cf_o} = -j0.67$, we find C= 1.188$Z_o$ nF.

gain note the “blooming” of the impedance band. t is wider than the original unma tched band. .



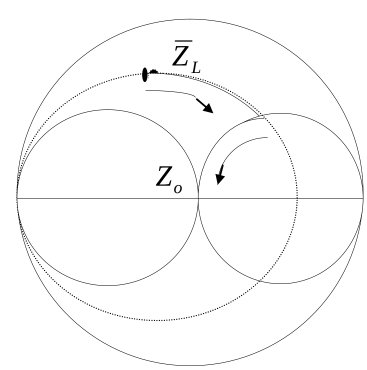




**Figure 34** **L-type circuit number [5], (CSCSH)**

Referring to Figures 30 and 34, as the capacitor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle clockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_{cf_o}$ , around the admittance chart will permit the value of the shunt capacitance to be determined, as $\Delta\overline{B}_{cf_o} = j\omega_o C = j2\pi f_o C$ . From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add series capacitance" circles. he amount of rotation on the Smith chart permits the determination of the series capacitance, from $\Delta\overline{X}_{Cf_o} = 1/j\omega_o C = 1/j2\pi f_o C$ .

EXAMPLE 13

A normalized load impedance is given as 0.2 +j0.5 at a centre frequency of 200 MHz. Determine the load impedances at the extremes of the band, which are 100 MHz and 300 MHz and plot these points on a Smith chart with their corresponding admittance points. Using an overlay admittance chart, rotate the admittance points until the centre band impedance point (which are admittance points on the admittance chart) cuts the unity resistance circle at the lower point , as shown in Figure 34, and determine the value of the shunt capacitance required. By rotating the centre frequency impedance along the unity resistance circle until the centre of the Smith chart is reached, determine the value of the series capacitance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

As the load is inductive

From equation 4-10,

$$X_{L1}(100) = X_{L1}(200)\frac{100}{200} = +j0.5\frac{100}{200} = j0.25 \text{ so } Z_{100} = 0.2 + j0.25$$

From equation 4-10,

$$X_{L1}(300) = X_{L1}(200)\frac{300}{200} = j0.5\frac{300}{200} = j0.75 \text{ so } Z_{300} = 0.2 + j0.75$$

The values of the calculated admittances are:

$$Y_{100} = 1.951 - j2.439\ ,\ Y_{200} = 0.69 - j1.724\ ,\ Y_{300} = 0.332 - j1.245$$

The rotation of . $Y_{200} = 0.69 - j1.724$ along a circle of constant conductance $\overline{G} = 0.69$ on the admittance chart, until it reaches the $\overline{R} = 1$ circle on the Smith chart permits the value of capacitive susceptance to be determined. This value of capacitive susceptance at the centre frequency can be translated to values of capacitive susceptance at the other two frequencies using equation 4-17 which are then plotted on the Smith chart. The admittance chart overlayed on the Smith chart together with the plotted impedance (corresponding admittance points) and the rotation clockwise along the $\overline{G} = 0.69$ circle is shown on Figure 83 From Figure 83, it can be seen that the intersection of the $\overline{G} = 0.69$ circle with the $\overline{R} = 1$ circle occurs at the admittance point

$Y_{R=1} = 0.69 - j0.46$ .

The value of $\Delta\overline{B}_{cf_o}$ is found by subtracting $Y_{200} = 0.69 - j1.724$ from $Y_{R=1} = 0.69 - j0.46$ giving $\Delta\overline{B}_{cf_o} = j1.26$ . The load capacitive susceptances at 100 MHz and 300 MHz move around their respective conductance circles by an amount which is related to $\Delta\overline{B}_{cf_o}$ . The relationship to $\Delta\overline{B}_{cf_o}$ is given by equation 4.17. Hence,

$B_C(100) = \Delta\overline{B}_{cf_o}\dfrac{100}{200} = j0.63$, and so

$Y'_{100} = Y_{100} + \Delta\overline{B}_{cf_o} = 1.951 - j2.439 + j0.63 = 1.951 - j1.809$

$B_C(300) = \Delta\overline{B}_{cf_o}\dfrac{300}{200} = j1.89$, and so

$Y'_{300} = Y_{300} + \Delta\overline{B}_{cf_o} = 0.332 - j1.245 + j1.89 = 0.332 + j0.645$

The value of shunt capacitance is found from the relation $\Delta\overline{B}_{cf_o} = j\omega_o\overline{C} = j2\pi f_o\overline{C}$, so at the band centre of 200 MHz, with $\Delta\overline{B}_{cf_o} = j1.26$, and C = 1.003$Z_o$ nF.

As the centre frequency admittance $Y_{R=1} = 0.69 - j0.46$, is to be changed to its equivalent impedance for matching at the centre of the Smith chart, we find that the impedance to be, $Z_{G=0.69} = 1 + j0.67$. This means that an capacitive reactance of $\overline{X}_{Cf_o} = -j0.67$ is required to be added in series with $Z_{G=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in an anti- clockwise direction of $Z_{G=0.69}$ to the centre of the Smith chart is shown in Figure 83. The outer band frequencies however, do not rotate anti-clockwise through a reactance of j0.67, but have to be modified because of their different frequencies.

The values of the outer band impedances are obtained from the Smith/Admittance chart as,

$Z'_{100} = 0.28 + j0.26$ and $Z'_{300} = 0.63 - j1.23$

Equation 4-3 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$X_C(100) = -j0.67\dfrac{200}{100} = -j1.34$, so

$Z_{100\,final} = 0.28 + j0.26 - j1.34 = 0.28 - j1.08$

Converting this to an admittance for ease in plotting on the admittance chart

$Y_{100\,final} = 0.23 + j0.87$

$X_C(300) = -j0.67\dfrac{200}{300} = -j0.447$, so

$Z_{300\,final} = 0.63 - j1.226 - j0.447 = 0.63 - j1.67$

Again converting this to an admittance for ease in plotting,

$Y_{300\,final} = 0.20 + j0.52$

These admittance points are shown plotted as impedances on Figure 83.

The VSWR is found to be approximately 7.5. This was found be drawing a circle around the impedance point which was the furthest out from the centre of the Smith chart, $Z_{300\,final}$, and reading the VSWR where the circle crossed the horizontal axis. Note that this VSWR is worse at 100 MHz than the VSWR of the original load impedance point $Z_{100}$, where the VSWR = . . Note the "blooming" of the impedance band. t is wider than the original unmatched band. However at the higher frequency, there is some small improvement in VSWR, and at the centre frequency of 200 MHz we have perfect matching.

Again, if the impedance or admittance bandwidth is small to begin with, the blooming will not be so prominent.

The value of capacitance is found from $\Delta\overline{X}_{Cf_o} = 1/j\omega_o C = 1/j2\pi f_o C$ and as $\overline{X}_{cf_o} = -j0.67$, we find C= 1.19$Z_o$ nF.

As the problem involves a shunt component closest to the load, an admittance chart with a unity normalized resistance circle from the Smith chart is used.

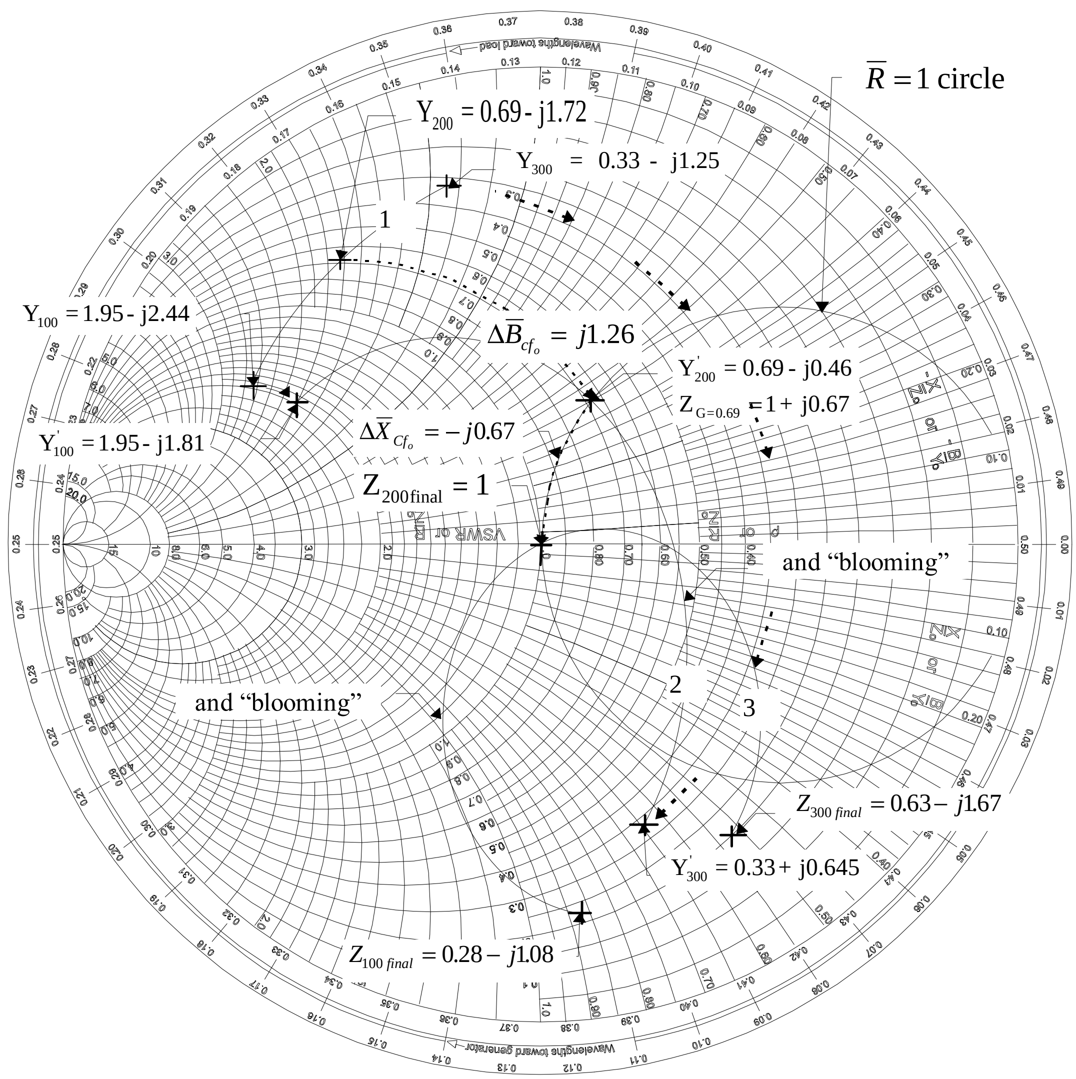
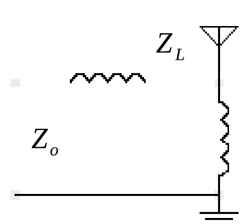



**Figure 83 Example 13, Matching with shunt capacitance and series capacitance**



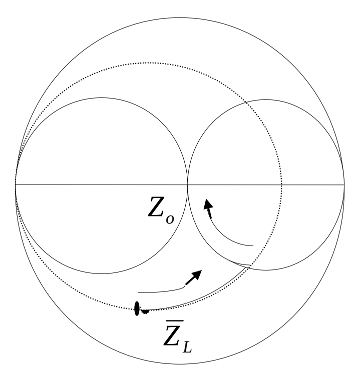




**Figure 35 L-type circuit number [7], (LSLSH)**

Referring to Figures 30 and 35, as the inductor is in shunt with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and using an overlay admittance chart, follow the constant G circle anticlockwise until we reach the $\overline{R} = 1$ circle on the Smith chart. The total change in the value of susceptance $\Delta\overline{B}_{Lf_o}$ , around the admittance chart will permit the value of the shunt inductance to be determined, as $\Delta\overline{B}_{Lf_o} = 1/\omega_o L = 1/2\pi f_o L$ . From intersection of the constant G circle with the $\overline{R} = 1$ circle, we then follow the $\overline{R} = 1$ circle clockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add series inductance" circles. he amount of rotation on the mith chart permits the determination of the series inductance, from $\Delta X_{Lf_o} = \omega_o L = 2\pi f_o L$ .

EXAMPLE 14
A normalized load impedance is given as 0.2 -j0.5 at a centre frequency of 200 MHz. Determine the load impedances at the extremes of the band, which are 100 MHz and 300 MHz and plot these points on a Smith chart with their corresponding admittance points. Using an overlay admittance chart, rotate the admittance points until the centre band impedance point (which are admittance points on the admittance chart) cuts the unity resistance circle at the lower point , as shown in Figure 35, and determine the value of the shunt inductance required. By rotating the centre frequency impedance clockwise along the unity resistance circle until the centre of the Smith chart is reached, determine the value of the series inductance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION
From equation 4-3, $X_{c1}(100) = X_{c1}(200)\dfrac{200}{100} = -j0.5\dfrac{200}{100} = -j1.0$ so $Z_{100} = 0.2 - j1.0$

From equation 4-3, $X_{c1}(300) = X_{c1}(200)\dfrac{200}{300} = -j0.5\dfrac{200}{300} = -j0.33$ so $Z_{300} = 0.2 - j0.33$

The values of the calculated admittances are:

$Y_{100} = 0.192 + j0.962$ , $Y_{200} = 0.69 + j1.724$ , $Y_{300} = 1.324 + j2.206$

The rotation of $Y_{200} = 0.69 + j1.724$ along a circle of constant conductance $\overline{G} = 0.69$ on the admittance chart, until it reaches the lower portion of the $\overline{R} = 1$ circle on the Smith chart permits the value of inductive susceptance to be determined. This value of inductive susceptance at the centre frequency can be translated to values of inductive susceptance at the other two frequencies using equation 4-24 which are then plotted on the Smith chart. The admittance chart overlayed on the Smith chart together with the plotted impedance (corresponding admittance points) and the rotation anti-clockwise along the $\overline{G} = 0.69$ circle is shown on Figure 84. From Figure 84, it can be seen that the intersection of the $\overline{G} = 0.69$ circle with the $\overline{R} = 1$ circle occurs at the admittance point
$Y_{R=1} = 0.69 + j0.46$ .
The value of $\Delta\overline{B}_{Lf_o}$ is found by subtracting $Y_{200} = 0.69 + j1.724$ from
$Y_{R=1} = 0.69 + j0.46$ giving $\Delta\overline{B}_{Lf_o} = -j1.26$ . The load inductive susceptances at 100 MHz and 300 MHz move around their respective conductance circles by an amount which is related to $\Delta\overline{B}_{Lf_o}$ . The relationship to $\Delta\overline{B}_{Lf_o}$ is given by equation 4.24. Hence,

$B_L(100) = \Delta\overline{B}_{Lf_o}\dfrac{200}{100} = -j2.52$ , and so

$Y_{100}^{'} = Y_{100} + \Delta\overline{B}_{Lf_o} = 0.192 + j0.962 - j2.52 = 0.192 - j1.558$

$B_C(300) = \Delta\overline{B}_{Lf_o}\dfrac{200}{300} = -j0.84$ , and so

$Y_{300}^{'} = Y_{300} + \Delta\overline{B}_{Lf_o} = 1.324 + j2.206 - j0.84 = 1.324 + j1.366$

The value of shunt inductance is found from the relation $\Delta\overline{B}_{Lf_o} = 1/j\omega_o L = 1/j2\pi f_o L$ , so at the band centre of 200 MHz, with $\Delta\overline{B}_{Lf_o} = -j1.26$ , and L = 0.632$Z_o$ nH.

As the centre frequency admittance $Y_{R=1} = 0.69 + j0.46$ , is to be changed to its equivalent impedance for matching at the centre of the Smith chart, we find that the impedance to be, $Z_{G=0.69} = 1 - j0.67$ . This means that an inductive reactance of $\overline{X}_{Lf_o} = j0.67$ is required to be added in series with $Z_{G=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$ . The rotation in a anti-clockwise direction of $Z_{G=0.69}$ to the centre of the Smith chart is shown in Figure 84. The outer band frequencies however, do not rotate clockwise through a reactance of j0.67, but have to be modified because of their different frequencies.

The values of the outer band impedances are obtained from the Smith/Admittance chart as,

$Z'_{100} = 0.08 + j0.632$ and $Z'_{300} = 0.366 - j0.377$

Equation 4-10 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$X_L(100) = j0.67\dfrac{100}{200} = j0.335$, so

$Z_{100\,final} = 0.08 + j0.632 + j0.335 = 0.08 + j0.967 \approx 0.08 + j0.97$

Converting this to an admittance for ease in plotting on the admittance chart

$Y_{100\,final} = 0.085 - j1.027$

$X_L(300) = j0.67\dfrac{300}{200} = j1.005$, so $Z_{300\,final} = 0.366 - j0.377 + j1.005 \approx 0.37 + j0.63$

Again converting this to an admittance for ease in plotting,

$Y_{300\,final} = 0.69 - j1.19$

These admittance points are shown plotted as impedances on Figure 84.

The VSWR is found to be approximately 17. This was found by taking the value of normalized resistance of the furthest out impedance point $Z_{100\,final}$ , and drawing a circle through it, with the centre at the centre of the admittance chart, and then reading the value of VSWR on the B=0 axis.

The value of series inductance is found from $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ and as $\overline{X}_{Lf_o} = j0.67$ , we find L= 0.533$Z_o$ nH.

gain note the “blooming” of the impedance band. t is wider than the original unmatched band.

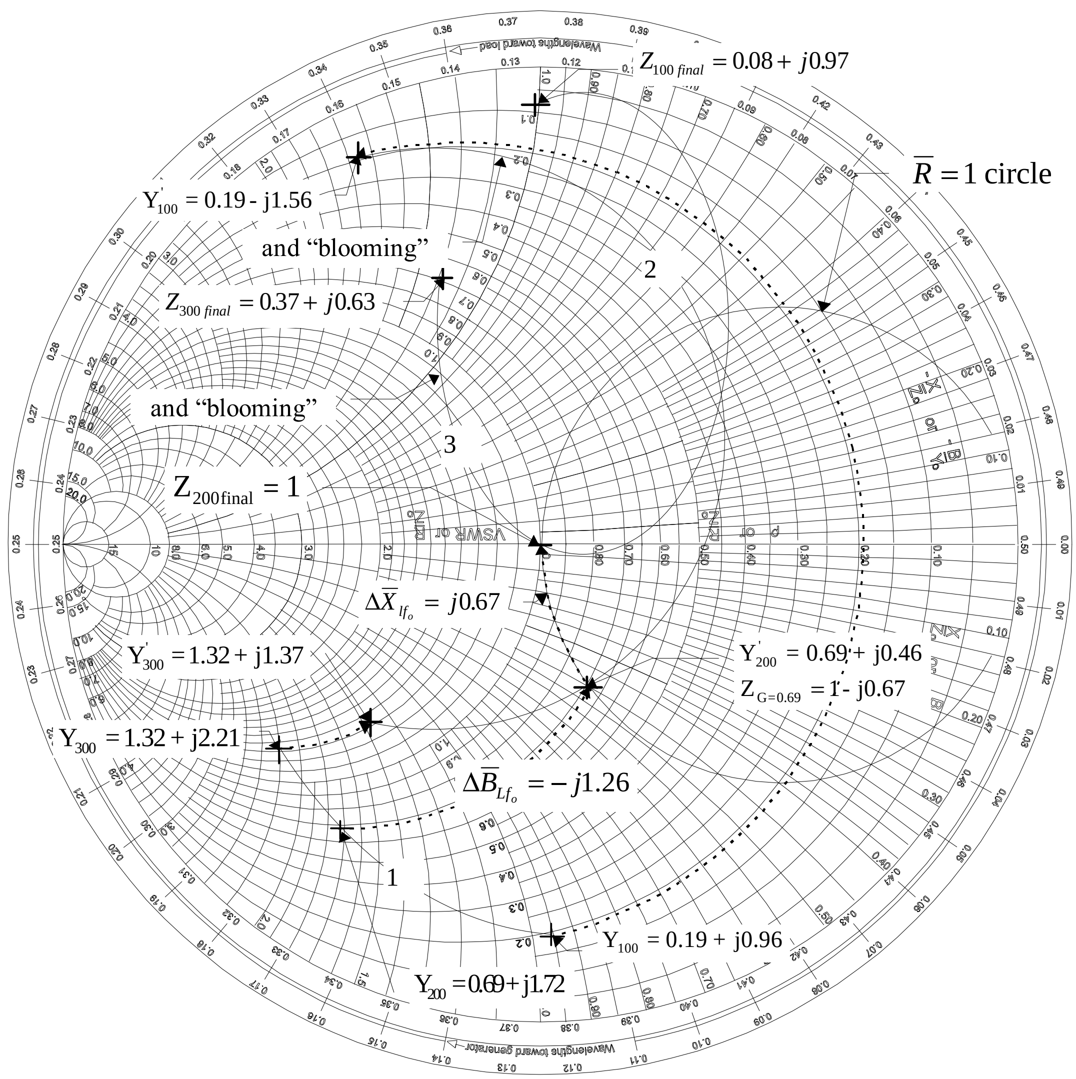
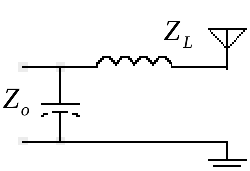



**Figure 84 Example 14, Matching with shunt inductance and series inductance**

### *4.3.1.2 Constant resistance circles*



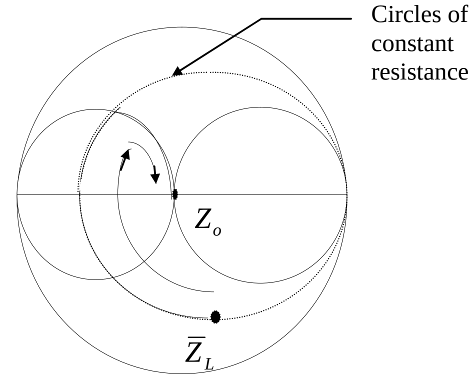




**Figure 36 L-type circuit number [2], (CSHLS)**

Referring to Figures 30 and 36, as the inductor is in series with the load, we take the plotted load impedance $\overline{Z}_{Lf}$ on the Smith chart and follow the constant R circle clockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_{Lf_o}$, around the Smith chart will permit the value of the series inductance to be determined, as $\Delta\overline{X}_{Lf_o} = \omega_o L = 2\pi f_o L$ . From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle clockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add shunt capacitance" circles. he amount of clockwise rotation on the admittance chart permits the determination of the shunt capacitance, from $\Delta\overline{B}_{Cf_o} = \omega_o C = 2\pi f_o C$ .

EXAMPLE 15

Normalized load impedance are given as $Z_{100} = 1.951 - j2.439$, $Z_{200} = 0.69 - j1.724$, $Z_{300} = 0.332 - j1.245$ at frequencies of 100 MHz, 200 MHz and 300 MHz.. Using an overlay admittance chart, rotate clockwise the impedance points until the centre band impedance cuts the unity conductance circle at the lower point , as shown in Figure 36, and determine the value of the series inductance required. By rotating the centre frequency impedance (which is converted into an admittance) along the unity conductance circle until the centre of the Smith chart is reached, determine the value of the shunt capacitance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

The rotation of . $Z_{200} = 0.69 - j1.724$ along a circle of constant resistance $\overline{R} = 0.69$ on the Smith chart, until it reaches the $\overline{G} = 1$ circle on the admittance chart permits the value of inductive reactance to be determined. This value of inductive reactance at the centre frequency can be translated to values of inductive reactance at the other two frequencies using equation 4-10 which are then plotted on the Smith chart. The Smith chart overlayed on the admittance chart together with the plotted impedances and the rotation clockwise along the $\overline{R} = 0.69$ circle is shown on Figure 85. From Figure 85, it can be seen that the intersection of the $\overline{R} = 0.69$ circle with the $\overline{G} = 1$ circle occurs at the impedance point $Z_{G=1} = 0.69 + j0.46$.

The value of $\Delta\overline{X}_{Lf_o}$ is found by subtracting $Z_{200} = 0.69 - j1.724$ from $Z_{G=1} = 0.69 + j0.46$ giving $\Delta\overline{X}_{Lf_o} = j2.18$. The load inductive reactances at 100 MHz and 300 MHz move around their respective resistance circles by an amount which is related to $\Delta\overline{X}_{Lf_o}$. The relationship to $\Delta\overline{X}_{Lf_o}$ is given by equation 4.10. Hence,

$X_C(100) = \Delta\overline{X}_{Lf_o}\dfrac{100}{200} = j1.09$, and so

$$Z'_{100} = Z_{100} + \Delta\overline{X}_{Lf_o} = 1.951 - j2.439 + j1.09 = 1.951 - j1.349$$

$X_C(300) = \Delta\overline{X}_{Lf_o}\dfrac{300}{200} = j3.27$, and so

$$Z'_{300} = Z_{300} + \Delta\overline{X}_{Lf_o} = 0.332 - j1.245 + j3.27 = 0.332 + j2.025$$

The value of series inductance is found from the relation $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ , so at the band centre of 200 MHz, with $\Delta\overline{X}_{Lf_o} = j2.18$, and L = 1.735$Z_o$ nH.

As the centre frequency admittance $Z_{R=1} = 0.69 + j0.46$, is to be changed to its equivalent admittance for matching at the centre of the Smith chart, we find the admittance to be, $Y_{R=0.69} = 1 - j0.67$. This means that a capacitive susceptance of $\overline{B}_{Cf_o} = j0.67$ is required to be added in shunt with $Y_{R=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in a clockwise direction of $Y_{R=0.69}$ to the centre of the Smith chart is shown in Figure 85. The outer band frequencies however, do not rotate clockwise through a susceptance of j0.67, but have to be modified because of their different frequencies.
The values of the outer band susceptances are obtained from the Smith/Admittance chart as,

$Y_{100}^{'} = 0.35 + j0.24$ and $Y_{300}^{'} = 0.08 - j0.48$

Equation 4-17 provides the means of modifying the centre frequency susceptance to give the outer band susceptance values. That is,

$B_C(100) = j0.67 \dfrac{100}{200} = j0.335$, so

$Y_{100\,final} = 0.35 + j0.24 + j0.335 = 0.35 + j0.575 \approx 0.35 + j0.58$

Converting this to an impedance for ease in plotting on the Smith chart

$Z_{100\,final} = 0.772 - j1.269$

$B_C(300) = j0.67 \dfrac{300}{200} = j1.005$, so

$Y_{300\,final} = 0.0788 - j0.481 + j1.005 = 0.08 + j0.524 \approx 0.08 + j0.52$

Again converting this to an impedance for ease in plotting,

$Z_{300\,final} = 0.236 - j1.536$

These impedance points are shown plotted on Figure 85.

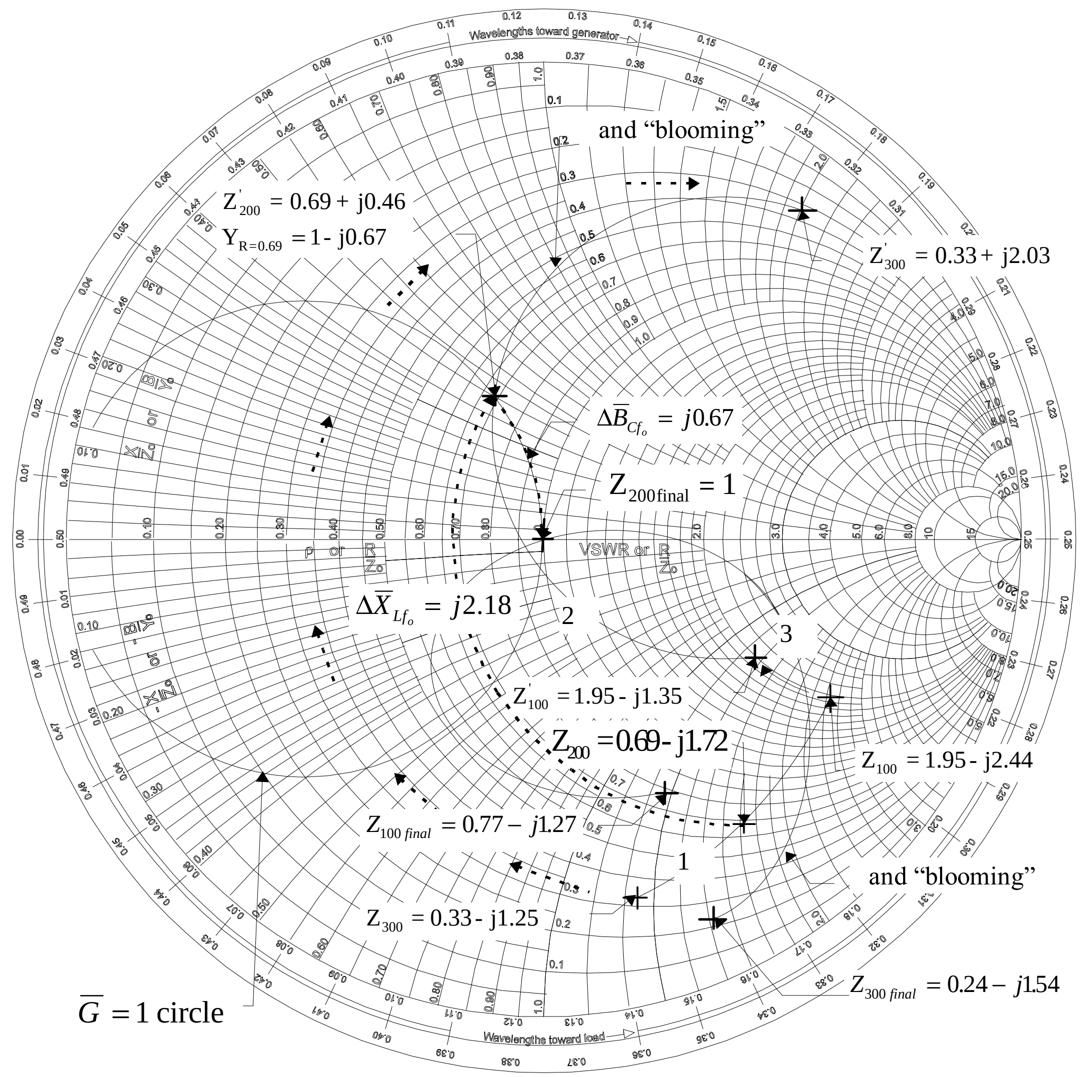



**Figure 85** **Example 15, Matching with series inductance and shunt capacitance**

The VSWR is found to be 12.5. This was found be drawing a circle around the impedance point which was the furthest out from the centre of the Smith chart, $Z_{300\,final}$ , and reading the VSWR
Note that this VSWR is worse at 300 MHz than the VSWR of the original load impedance point $Z_{300}$, where the . . Note the “blooming” of the impedance band. t is wider than the original unmatched band. However at the lower frequency, there is some improvement in VSWR, and at the centre frequency of 200 MHz we have perfect matching. If the impedance or admittance bandwidth is small to begin with, the blooming will not be so prominent.

The value of capacitance is found from $\Delta\overline{B}_{Cf_o} = j\omega_o\overline{C} = j2\pi f_o\overline{C}$ and as $\overline{B}_{Cf_o} = j0.67$, we find C= 0.533$Z_o$ nF.
Overlaying the admittance chart onto the Smith chart produces a chart which is exceptionally complex and difficult to work with. In these matching cases, this problem is alleviated by using only the unit circle of the admittance with the Smith chart. That is, if the problem involves a series component closest to the load, then use the Smith chart with a unity normalized conductance circle from the admittance chart.

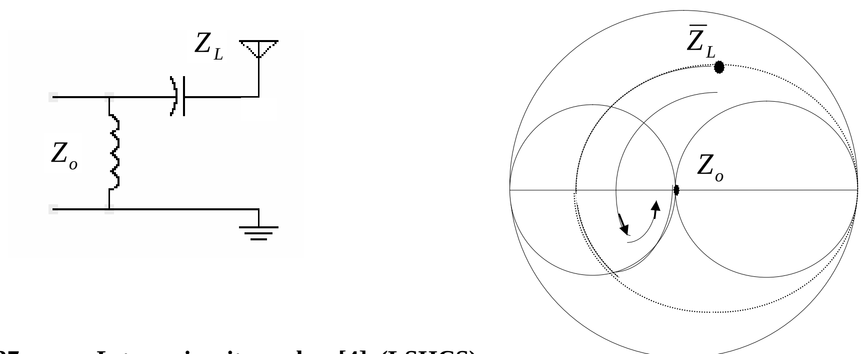



**Figure 37 L-type circuit number [4], (LSHCS)**

Referring to Figures 30 and 37, as the capacitor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle anticlockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_{Cf_o}$, around the Smith chart will permit the value of the series capacitance to be determined, as $\Delta\overline{X}_{Cf_o} = 1/\omega_o\overline{C} = 1/2\pi f_o\overline{C}$ . From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the “add shunt inductance” circles. he amount of anticlockwise rotation on the admittance chart permits the determination of the shunt inductance, from $\Delta\overline{B}_{Lf_o} = 1/\omega_o\overline{L} = 1/2\pi f_o\overline{L}$ .

EXAMPLE 16
Normalized load impedance are given as $Z_{100} = 0.192 + j0.962$, $Z_{200} = 0.690 + j1.724$, $Z_{300} = 1.324 + j2.206$ at frequencies of 100 MHz, 200 MHz and 300 MHz respectively. Using an overlay admittance chart, rotate anti-clockwise the impedance points until the centre band impedance cuts the unity conductance circle at the lower point , as shown in Figure 37, and determine the value of the series capacitance required. By rotating anti-clockwise the centre frequency impedance (which is converted into an admittance) along the unity conductance circle until the centre of the Smith chart is reached, determine the value of the shunt inductance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

The rotation of $Z_{200} = 0.69 + j1.724$ along a circle of constant resistance $\overline{R} = 0.69$ on the Smith chart, until it reaches the $\overline{G} = 1$ circle on the Smith chart permits the value of capacitive reactance to be determined. This value of capacitive reactance at the centre frequency can be translated to values of capacitive reactance at the other two frequencies using equation 4-3 which are then plotted on the Smith chart. The admittance chart $\overline{G} = 1$ circle overlayed on the Smith chart together with the plotted impedances and the rotation anti-clockwise along the $\overline{R} = 0.69$ circle is shown on Figure 86. From Figure 86, it can be seen that the intersection of the $\overline{R} = 0.69$ circle with the $\overline{G} = 1$ circle occurs at the impedance point $Z_{G=1\,f=200} = 0.69 - j0.46 = Z'_{200}$.

The value of $\Delta\overline{X}_{Cf_o}$ is found by subtracting $Z_{200} = 0.69 + j1.724$ from $Z'_{200} = 0.69 - j0.46$ giving $\Delta\overline{X}_{Cf_o} = -j2.18$. The load capacitive reactances at 100 MHz and 300 MHz move around their respective resistance circles by an amount which is related to $\Delta X_{Cf_o}$. The relationship to $\Delta X_{Cf_o}$ is given by equation 4.3. Hence,

$X_C(100) = \Delta\overline{X}_{Cf_o}\dfrac{200}{100} = -j4.36$, and so

$$Z'_{100} = Z_{100} + \Delta\overline{X}_{Cf_o} = 0.192 + j0.962 - j4.36 = 0.192 - j3.398$$

$X_C(300) = \Delta\overline{X}_{Cf_o}\dfrac{200}{300} = -j1.453$, and so

$$Z'_{300} = Z_{300} + \Delta\overline{X}_{Cf_o} = 1.324 + j2.206 - j1.453 = 1.324 + j0.753$$

The value of series capacitance is found from the relation $\Delta\overline{X}_{Cf_o} = 1/j\omega_o\overline{C} = 1/j2\pi f_o\overline{C}$, so at the band centre of 200 MHz, with $\Delta\overline{X}_{Cf_o} = -j2.18$, and C = 0.365$Z_o$ nF

As the centre frequency impedance $Z_{G=1} = 0.69 - j0.46$, is to be changed to its equivalent admittance for matching at the centre of the Smith chart, we find that the admittance to be, $\overline{Y}_{R=0.69} = 1 + j0.67$. This means that an inductive susceptance of $\overline{B}_{Lf_o} = -j0.67$ is required to be added in series with $Y_{R=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in a anti-clockwise direction of $Y_{R=0.69}$ to the centre of the Smith chart is shown in Figure 86. The outer band frequencies however, do not rotate anti-clockwise through a susceptance of -j0.67, but have to be modified because of their different frequencies.

The values of the outer band admittances are obtained from the Smith/Admittance chart as,

$Y'_{100} = 0.02 + j0.29$ and $Y'_{300} = 0.58 - j0.33$

Equation 4-24 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$Y_L(100) = -j0.67\dfrac{200}{100} = -j1.34$, so $Y_{100\,final} = 0.02 + j0.29 - j1.34 = 0.02 - j1.05$

Converting this to an impedance for ease in plotting on the admittance chart

$Z_{100\,final} = 0.02 + j0.95$

$B_L(300) = -j0.67\dfrac{200}{300} = -j0.447$, so

$Y_{300\,final} = 0.58 - j0.33 - j0.447 = 0.58 - j0.78$

Again converting this to an admittance for ease in plotting,

$Z_{300\,final} = 0.61 + j0.83$

These impedance points are shown plotted on Figure 86.

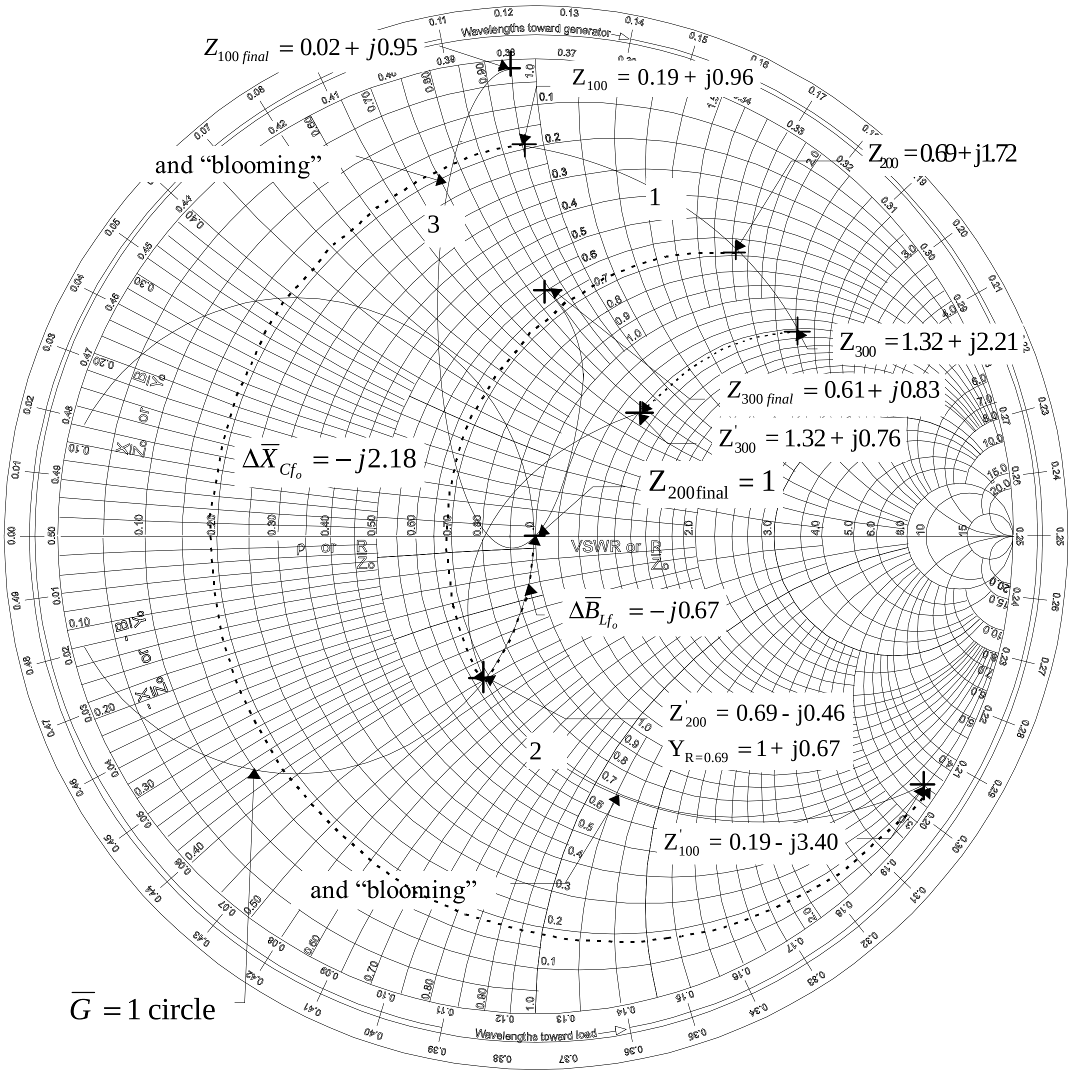
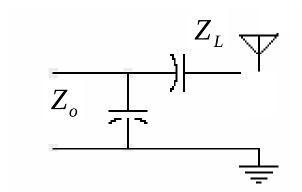



**Figure 86** **Example 16, Matching with a series capacitance and shunt inductance**

The VSWR is found to be 50. This was found by taking the value of normalized resistance of the furthest out impedance point $Z_{100\,final}$, and drawing a circle through this point with the centre of the Smith chart as the centre of the circle and then reading off the VSWR.

The value of shunt inductance is found from $\Delta\overline{B}_{Lf_o} = 1\big/ j\omega_o\overline{L} = 1\big/ j2\pi f_o\overline{L}$ and as $\overline{B}_{Lf_o} = -j0.67$, we find L= 1.188Z$_o$ nH.

gain note the "blooming" of the impedance band. t is wider than the original unmatched band.



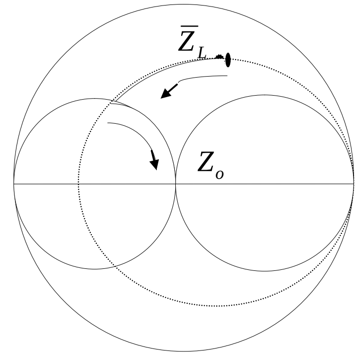




**Figure 38 L-type circuit number [6], (CSHCS)**

Referring to Figures 30 and 38, as the capacitor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle anticlockwise on the Smith chart until we reach the $\overline{G}=1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_{Cf_o}$, around the Smith chart will permit the value of the series capacitance to be determined, as $\Delta\overline{X}_{Cf_o}=1/\omega_o\overline{C}=1/2\pi f_o\overline{C}$. From the intersection of the constant R circle with the $\overline{G}=1$ circle, we then follow the $\overline{G}=1$ circle clockwise around to the $R/Z_o=1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add shunt capacitance" circles. he amount of clockwise rotation on the admittance chart permits the determination of the shunt capacitance, from $\Delta\overline{B}_{Cf_o}=\omega_o\overline{C}=2\pi f_o\overline{C}$ .

EXAMPLE 17

Normalized load impedances are given as $\overline{Z}_{100}=0.192+j0.962$, $\overline{Z}_{200}=0.69+j1.724$, $\overline{Z}_{300}=1.324+j2.206$ at frequencies of 100, 200 and 300 MHz respectively. Using an overlay admittance chart unity conductance circle, rotate the impedance points anti-clockwise until the centre band impedance point cuts the unity conductance circle at the upper point , as shown in Figure 38, and determine the value of the series capacitance required. By rotating the centre frequency admittance anti-clockwise along the unity conductance circle until the centre of the Smith chart is reached, determine the value of the shunt capacitance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

The anti-clockwise rotation of $Z_{200}=0.69+j1.724$ along a circle of constant resistance $\overline{R}=0.69$ on the Smith chart until it reaches the upper portion of the $\overline{G}=1$ circle on the admittance chart permits the value of capacitive reactance to be determined. This value of capacitive reactance at the centre frequency can be translated to values of capacitive reactance at the other two frequencies using equation 4-3 which are then plotted on the Smith chart. The admittance chart $\overline{G}=1$ circle overlayed on the Smith chart together with the plotted impedance points and the rotation anti-clockwise along the $\overline{R}=0.69$ circle is shown on Figure 87. From Figure 87, it can be seen that the intersection of the $\overline{R}=0.69$ circle with the $\overline{G}=1$ circle occurs at the impedance point $Z_{G=1}=0.69+j0.46$.

The value of $\Delta\overline{X}_{Cf_o}$ is found by subtracting $Z_{200}=0.69+j1.724$ from $Z_{G=1}=0.69+j0.46=Z'_{200}$ giving $\Delta\overline{X}_{Cf_o}=-j1.26$. The load capacitive reactances at 100 MHz and 300 MHz move around their respective reisitance circles by an amount which is related to $\Delta\overline{X}_{Cf_o}$. The relationship to $\Delta\overline{X}_{Cf_o}$ is given by equation 4.3. Hence,

$$X_C(100)=\Delta\overline{X}_{Cf_o}\frac{200}{100}=-j2.52\text{, and so}$$

$$Z'_{100}=Z_{100}+\Delta\overline{X}_{Cf_o}=0.192+j0.962-j2.52=0.192-j1.558$$

$$X_C(300) = \Delta\overline{X}_{Cf_o}\frac{200}{300} = -j0.84$$ , and so

$$Z'_{300} = Z_{300} + \Delta\overline{X}_{Cf_o} = 1.324 + j2.206 - j0.84 = 1.324 + j1.366$$

The value of series capacitance is found from the relation $\Delta\overline{X}_{Cf_o} = 1/j\omega_o\overline{C} = 1/j2\pi f_o\overline{C}$ , so at the band centre of 200 MHz, with $\Delta\overline{X}_{Cf_o} = -j1.26$, and C = 0.632$Z_o$ nF.

As the centre frequency impedance $Z_{G=1} = Z'_{200} = 0.69 + j0.46$, is to be changed to its equivalent admittance for matching at the centre of the Smith chart, we find the admittance to be, $Y_{R=0.69} = 1 - j0.67$. This means that a capacitive susceptance of $\overline{B}_{Cf_o} = j0.67$ is required to be added in shunt with $Y_{G=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in a clockwise direction of $Y_{G=0.69}$ along the $\overline{G} = 1$ circle to the centre of the Smith chart is shown in Figure 87. The outer band frequencies however, do not rotate clockwise through a susceptance of j0.67, but have to be modified because of their different frequencies.

The values of the outer band impedances are obtained from the Smith/Admittance chart as,

$Y'_{100} = 0.08 + j0.632$ and $Y'_{300} = 0.366 - j0.377$

Equation 4-17 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$$\overline{B}_C(100) = j0.67\frac{100}{200} = j0.335$$ , so

$$Y_{100\,final} = 0.08 + j0.632 + j0.335 = 0.08 + j0.967 \approx 0.08 + j0.97$$

Converting this to an impedance for ease in plotting on the Smith chart

$$Z_{100\,final} = 0.085 - j1.024$$

$$\overline{B}_L(300) = j0.67\frac{300}{200} = j1.005$$ , so $Y_{300\,final} = 0.366 - j0.377 + j1.005 \approx 0.37 + j0.63$

Again converting this to an admittance for ease in plotting,

$$Z_{300\,final} = 0.69 - j1.18$$

These impedance points are shown plotted on Figure 87.

The VSWR is found to be approximately 13. This was found by taking the value of normalized resistance of the furthest out impedance point $Z_{100\,final}$, and drawing a circle through it, with the centre at the centre of the Smith chart, and then reading the value of VSWR on the X=0 axis.

The value of shunt capacitance is found from $\Delta\overline{B}_{Cf_o} = j\omega_o\overline{C} = j2\pi f_o\overline{C}$ and as $\overline{B}_{Cf_o} = j0.67$, we find C= 0.533$Z_o$ nF.

Again note the “blooming” of the impedance band. t is wider than the original unmatched band.

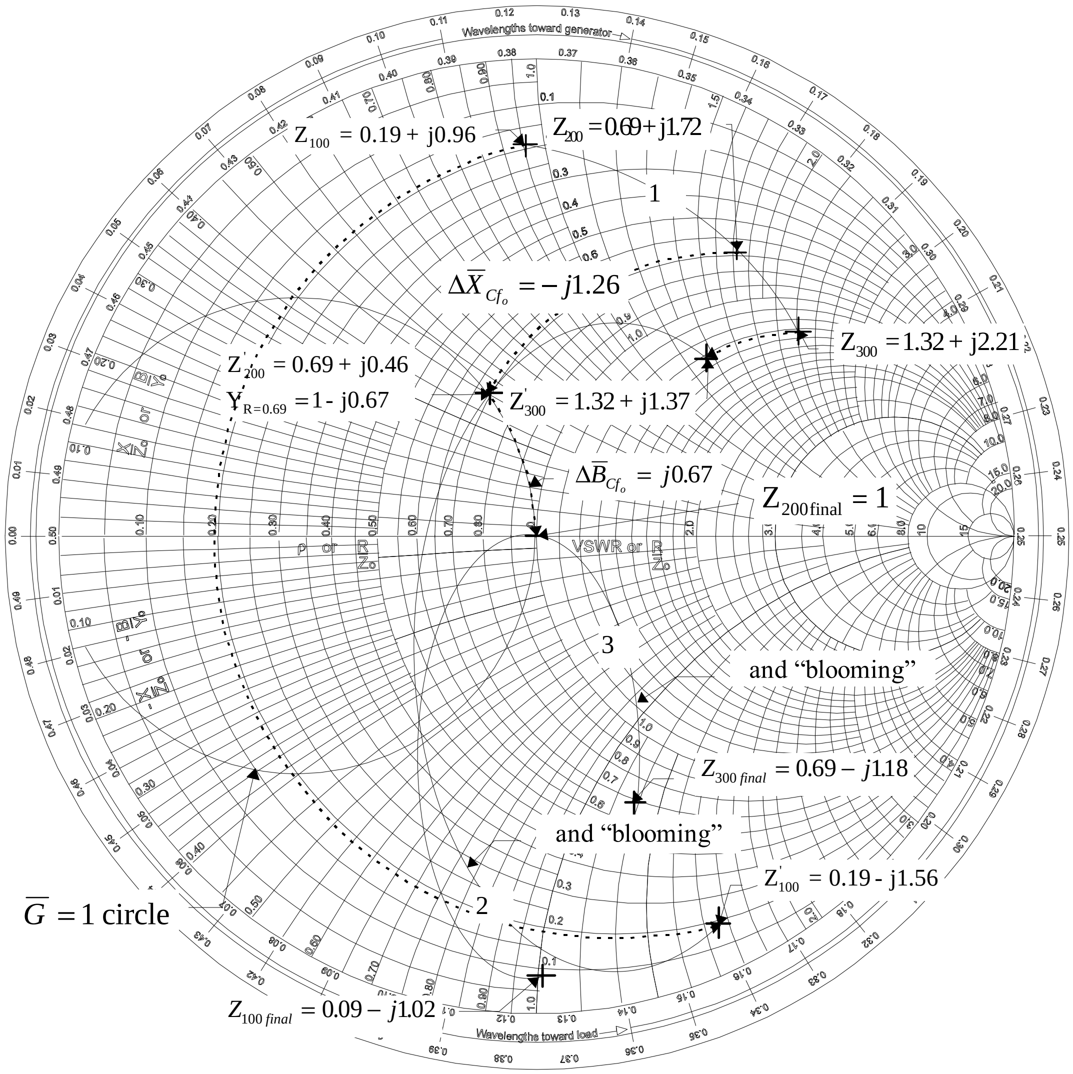
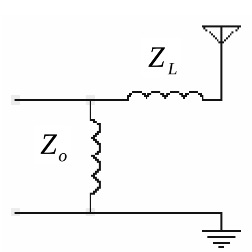



**Figure 87** **Example 17, Matching with shunt capacitance and series capacitance**



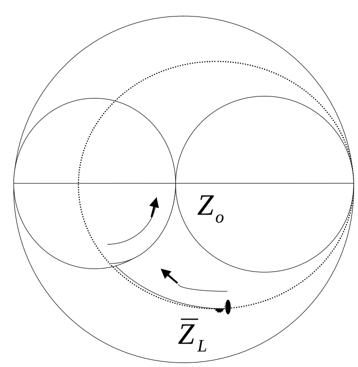




**Figure 39** **L-type circuit number [8], (LSHLS)**

Referring to Figures 30 and 39, as the inductor is in series with the load, we take the plotted load impedance $\overline{Z}_L$ on the Smith chart and follow the constant R circle clockwise on the Smith chart until we reach the $\overline{G} = 1$ circle by using an overlay admittance chart. The total change in the value of reactance $\Delta\overline{X}_{Lf_o}$, around the Smith chart will permit the value of the series inductance to be determined, as $\Delta\overline{X}_{Lf_o} = \omega_o\overline{L} = 2\pi f_o\overline{L}$. From the intersection of the constant R circle with the $\overline{G} = 1$ circle, we then follow the $\overline{G} = 1$ circle anticlockwise around to the $R/Z_o = 1$ point, or centre of the Smith chart. From Figure 30, we see that we are following one of the "add shunt inductance" circles. he amount of anticlockwise rotation on the admittance chart permits the determination of the shunt inductance, from $\Delta\overline{B}_{Lf} = 1/\omega_o\overline{L} = 1/2\pi f_o\overline{L}$.

EXAMPLE 18

A normalized load admittance is given as 0.2 +j0.5 at a centre frequency of 200 MHz. Determine the load admittances at the extremes of the band, which are 100 MHz and 300 MHz and then determine the impedance points. Plot these points on a Smith chart. Using an overlay unity conductance circle from an admittance chart, rotate the impedance points clockwise until the centre band impedance point cuts the unity conductance circle at the lower point , as shown in Figure 88, and determine the value of the series inductance required. By rotating the centre frequency admittance anticlockwise along the unity conductance circle until the centre of the Smith chart is reached, determine the value of the shunt inductance required for perfect matching at the centre frequency. Draw the impedance bands and determine the final worst VSWR of the impedance band.

SOLUTION

As the load is capacitive

From equation 4-17,

$$B_{C1}(100) = B_{C1}(200)\frac{100}{200} = +j0.5\frac{100}{200} = j0.25 \text{ so } Y_{100} = 0.2 + j0.25$$

From equation 4-17,

$$B_{C1}(300) = B_{C1}(200)\frac{300}{200} = j0.5\frac{300}{200} = j0.75 \text{ so } Y_{300} = 0.2 + j0.75$$

The values of the calculated impedances are:

$$Z_{100} = 1.951 - j2.439,\ Z_{200} = 0.69 - j1.724,\ Z_{300} = 0.332 - j1.245$$

The rotation of . $Z_{200} = 0.69 - j1.724$ , clockwise, along a circle of constant resistance $\overline{R} = 0.69$ on the Smith chart, until it reaches the $\overline{G} = 1$ circle on the Smith chart permits the value of inductive reactance to be determined. This value of inductive reactance at the centre frequency can be translated to values of inductive reactances at the other two frequencies using equation 4-10 which are then plotted on the Smith chart. The admittance chart $\overline{G} = 1$ circle overlayed on the Smith chart together with the plotted impedance points and the rotation clockwise along the $\overline{R} = 0.69$ circle is shown on Figure 88. From Figure 88, it can be seen that the intersection of the $\overline{R} = 0.69$ circle with the $\overline{G} = 1$ circle occurs at the impedance point $Z_{G=1} = Z'_{200} = 0.69 - j0.46$.

The value of $\Delta\overline{X}_{Lf_o}$ is found by subtracting $Z_{200} = 0.69 - j1.724$ from $Z_{G=1} = Z'_{200} = 0.69 - j0.46$ giving $\Delta\overline{X}_{Lf_o} = j1.26$. The load inductive reactances at 100 MHz and 300 MHz move around their respective resistance circles by an amount which is related to $\Delta\overline{X}_{Lf_o}$. The relationship to $\Delta\overline{X}_{Lf_o}$ is given by equation 4.10. Hence,

$$X_L(100) = \Delta\overline{X}_{Lf_o}\frac{100}{200} = j0.63, \text{ and so}$$

$$Z'_{100} = Z_{100} + \Delta\overline{X}_{Lf_o} = 1.951 - j2.439 + j0.63 = 1.951 - j1.809$$

$$X_L(300) = \Delta\overline{X}_{Lf_o}\frac{300}{200} = j1.89, \text{ and so}$$

$$Z'_{300} = Z_{300} + \Delta\overline{X}_{Lf_o} = 0.332 - j1.245 + j1.89 = 0.332 + j0.645$$

The value of series inductance is found from the relation $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ , so at the band centre of 200 MHz, with $\Delta\overline{X}_{Lf_o} = j1.26$, and L = 1.003Z$_o$ nH.

As the centre frequency impedance $Z_{G=1} = 0.69 - j0.46$, is to be changed to its equivalent admittance for matching at the centre of the Smith chart, we find the admittance to be, $Y_{R=0.69} = 1 + j0.67$. This means that an inductive susceptance of $\overline{B}_{Lf_o} = -j0.67$ is required to be added in shunt with $Y_{R=0.69}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in an anti-clockwise direction of $Y_{R=0.69}$ to the centre of the Smith chart is shown in Figure 88. The outer band frequencies however, do not rotate anti-clockwise through a reactance of j0.67, but have to be modified because of their different frequencies.

The values of the outer band admittances are obtained from the Smith/Admittance chart or by inverting $Z'_{100}$ and $Z'_{300}$ obtained above, as,

$Y'_{100} = 0.28 + j0.26$ and $Y'_{300} = 0.63 - j1.23$

Equation 4-24 provides the means of modifying the centre frequency susceptance to give the outer band susceptance values. That is,

$B_L(100) = -j0.67\dfrac{200}{100} = -j1.34$ , so

$Y_{100\,final} = 0.28 + j0.26 - j1.34 = 0.28 - j1.08$

Converting this to an impedance for ease in plotting on the Smith chart

$Z_{100\,final} = 0.22 + j0.87$

$B_L(300) = -j0.67\dfrac{200}{300} = -j0.447$ , so

$Y_{300\,final} = 0.63 - j1.226 - j0.447 = 0.63 - j1.67$

Again, converting this to an impedance for ease in plotting on the Smith chart

$Z_{300\,final} = 0.19 + j0.50$

These impedance points are shown plotted on Figure 88

The VSWR is found to be approximately 7.8. This was found be drawing a circle around the impedance point which was the furthest out from the centre of the Smith chart, $Z_{100\,final}$ , and reading the VSWR where the circle crossed the horizontal axis. Note that this VSWR is worse at 100 MHz than the VSWR of the original load impedance point $Z_{100}$ , where the . . gain note the "blooming" of the impedance band. t is wider than the original unmatched band. However at the higher frequency, there is some small improvement in VSWR, and at the centre frequency of 200 MHz we have perfect matching.

Again, if the impedance or admittance bandwidth is small to begin with, the blooming will not be so prominent.

The value of shunt inductance is found from $\Delta\overline{B}_{Lf_o} = 1/j\omega_o\overline{L} = 1/j2\pi f_o\overline{L}$ and as $\overline{B}_{Lf_o} = -j0.67$, we find L= 1.19Z$_o$ nH.

As the problem involves a series component closest to the load, a Smith chart with a unity normalized conductance circle from the admittance chart is used.

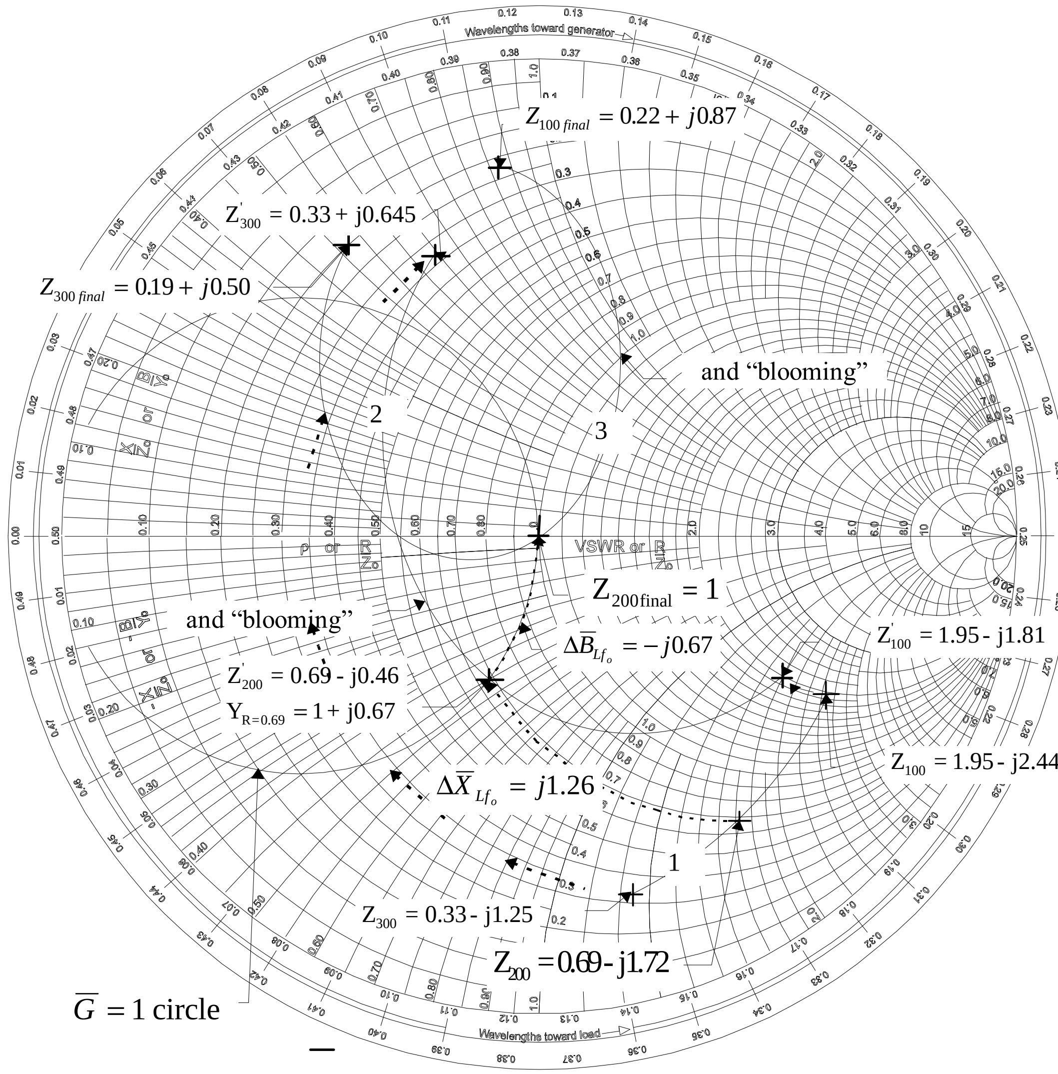



**Figure 88** **Example 18, Matching with shunt inductance and series inductance**

Examples 11 through to 18 show how the Smith chart can show what is happening when a band of frequencies is presented to a load and a simple L-type circuit is used to match at the centre frequency of the band. The values of the components of the L-type circuit for matching at the centre of the band can also be determined with the aid of the Smith chart in conjunction with the admittance chart. As seen, the VSWR at the band outer frequencies is worse than the original unmatched impedance at the outer band frequencies, although this would not be the case if the band is around 5% of the centre frequency rather than the 100% which was taken in these examples. The next section shows how the VSWR can be brought under control at the band outer frequencies. This is at the expense of perfect matching at the centre frequency, by attempting to match the band of frequencies into a predefined VSWR circle.

## *4.4. Matching to within a specified VSWR*

In this section, we will consider two cases of matching to a specified VSWR. The first case being for narrow-band systems and the other for wide-band, or more strictly, **non** narrow-band systems. The concepts developed in the narrow-band system will assist in explaining and using the concepts for matching a wide-system to a specific VSWR.
To match a wide-band or a narrow-band network to a small VSWR circle, two or more matching networks may be required, or, in the difficult case, the load to be matched may have to be modified.

The rough rule of thumb which is used to limit the signal distortion to be within the limit of practical equalizers, is that the fractional bandwidth should be kept within the range of:

$$1\% < \frac{100BW}{f_{RF}} < 10\% \qquad \text{(narrow-band)}$$

From this, it can be seen that, to obtain large bandwidths there is a requirement for high carrier frequencies. For the purposes of Smith chart design methods, a narrow-band system is defined as a system that is given by the fractional frequency limit of equation 4-31, that is,

$$\frac{100BW}{f_{RF}} \leq 10\% \qquad \text{(narrow-band)} \qquad \textbf{(4-31)}$$

and a wide-band system is defined as a non-narrow-band system, where,

$$\frac{100BW}{f_{RF}} > 10\% \qquad \text{(wide-band)} \qquad \textbf{(4-32)}$$

The terms narrow-band and wide-band used in this context is not the same as the definitions provided in angle modulation theory, where the modulation index is used.

### 4.4.1. Narrow-band systems

As mentioned above, for the purposes of using the Smith chart in design, a narrow-band system is defined as a system that is given by the fractional frequency limit of equation 4-32, that is,

$$\frac{100BW}{f_{RF}} \leq 10\% \qquad \text{(narrow-band)} \qquad \textbf{(4-31)}$$

The philosophy behind matching a narrow-band network to a load, is to first determined where the load band lies on the Smith chart, and then from Section 2.6.1 determine the best L-type impedance transformation circuit to be used or from Chapter 3, determine the best component that can be used, in order to bring the load band in towards the centre of centre of the chart. The adjustment of component values is then made to ensure that the complete band lies within a specified VSWR circle. This philosophy will become apparent as specific examples are used to demonstrate matching procedures.

In practice, it is better to measure the load impedance at various frequencies across the band, to obtain a representation of the actual impedance band, rather than calculating the various impedances from a single measurement. This is because, in practice, the resistive component will change with frequency, whereas in the simplified theory used, it will not. When multiple reflections are set up in a transmission line, coaxial or waveguide, propagation of the wave around obstacles or around imperfections in the line introduce loss. With higher VSWR a larger mismatch is required to reduce the VSWR to unity, which means that a larger percentage of power is reflected from the load and the matching circuit. This multiple reflected power traverses the distance between the discontinuities until it is dissipated in the waveguide walls or cable dielectric as heat, thus producing a larger total loss. Due to the skin effect the loss or attenuation becomes frequency dependent, and more so at higher VSWR. Once, this impedance band has been obtained by measurement it is converted to its normalized impedance or normalized admittance and plotted on the Smith chart. The specified VSWR is also drawn on the Smith chart and the process of solving the matching problem commences. To solve the problem at hand, the component resources, already described in this book, are considered so that the best circuit, component, or combination of circuits and components can be employed. The way in which the uses of the various resources are employed to solve matching problems are best described by examples as shown below.

### 4.4.1.1.Use of L-type circuits in matching

These circuits permit a band which is near the edge of the Smith chart to be moved towards the centre of the chart. This is done at the expense of increasing the size of the band as it moves towards the centre. The L-type circuits of Figure 32 and 36, which use a series inductance and shunt capacitance behaves the same as a short-section of transmission line, as discussed in section 1.2.

EXAMPLE 19
Match a 50 Ohm transmission line to the load impedance points given below, so that a VSWR of 1.60 or less is obtained. The measured load impedance points across the band are given as,

| **Frequency (MHz)** | **Measured impedance (Ω)** | **Normalized impedance** |
|---|---|---|
| 800 | $Z_{800} = 10 - j57$ | $\overline{Z}_{800} = 0.20 - j1.14$ |
| 830 | $Z_{830} = 16 - j55$ | $\overline{Z}_{830} = 0.32 - j1.10$ |
| 860 | $Z_{860} = 20 - j53$ | $\overline{Z}_{860} = 0.40 - j1.06$ |

SOLUTION
Figure 89 shows the normalized impedance points plotted on the Smith chart, together with the specified VSWR. Referring to section 2.6.1.1., the L-type circuits which could be used to match a normalized impedance such as this, are Figures 33, 35, 36, and 39. We will for this problem choose the L-type circuit of Figure 35, as shown below.

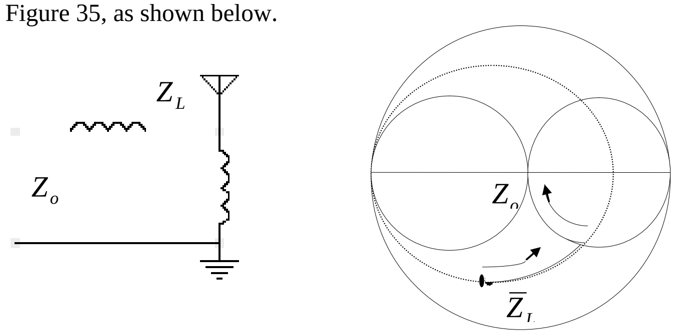



**Figure 35 L-type circuit number [7], (LSLSH)**

The majority of the work involved in the solution of this problem is similar to the work involved in solving the problem of Example 14. Working through the same steps as that of Example 14, we get, The values of the calculated admittances are:

$$Y_{800} = 0.149 + j0.851,\ Y_{830} = 0.244 + j0.838,\ Y_{860} = 0.312 + j0.826$$

**Step 1 Matching the centre frequency impedance**

The rotation of $Y_{830} = 0.244 + j0.838$ along a circle of constant conductance $\overline{G} = 0.244$ on the admittance chart, until it reaches the lower portion of the $\overline{R} = 1$ circle on the Smith chart permits the value of inductive susceptance to be determined. This value of inductive susceptance at the centre frequency can be translated to values of inductive susceptance at the other two frequencies using equation 4-24 which are then plotted on the Smith chart. The admittance chart overlayed on the Smith chart together with the plotted impedance (corresponding admittance points) and the rotation anti-clockwise along the $\overline{G} = 0.244$ circle is shown on Figure 89. From Figure 89, it can be seen that the intersection of the $\overline{G} = 0.244$ circle with the $\overline{R} = 1$ circle occurs at the admittance point $Y_{R=1} = Y'_{830} = 0.244 + j0.428$.

The value of $\Delta\overline{B}_{Lf_o}$ is found by subtracting $Y_{830} = 0.244 + j0.838$ from $Y_{R=1} = Y'_{830} = 0.244 + j0.428$ giving $\Delta\overline{B}_{Lf_o} = -j0.41$. The load inductive susceptances at 860 MHz and 800 MHz move around their respective conductance circles by an amount which is related to $\Delta\overline{B}_{Lf_o}$. The relationship to $\Delta\overline{B}_{Lf_o}$ is given by equation 4.24. Hence,

$B_L(800) = \Delta\overline{B}_{Lf_o}\dfrac{830}{800} = -j0.425$, and so

$Y'_{800} = Y_{800} + \Delta\overline{B}_{Lf_o} = 0.149 + j0.851 - j0.425 = 0.149 + j0.426$

$B_C(860) = \Delta\overline{B}_{Lf_o}\dfrac{830}{860} = -j0.396$, and so

$Y'_{860} = Y_{860} + \Delta\overline{B}_{Lf_o} = 0.312 + j0.826 - j0.396 = 0.312 + j0.43$

The value of shunt inductance is found from the relation $\Delta\overline{B}_{Lf_o} = 1/j\omega_o L = 1/j2\pi f_o L$, so at the band centre of 830 MHz, with $\Delta\overline{B}_{Lf_o} = -j0.425$, L = 0.467$Z_o$ nH.

As the centre frequency admittance $Y_{R=1} = Y'_{830} = 0.244 + j0.428$, is to be changed to its equivalent impedance for matching at the centre of the Smith chart, we find that the impedance to be, $Z_{G=0.24} = 1 - j1.77$. This means that an inductive reactance of $\overline{X}_{Lf_o} = j1.77$ is required to be added in series with $Z_{G=0.24}$ to make the input impedance to the circuit $\overline{Z}_{in} = 1$. The rotation in a anti-clockwise direction of $Z_{G=0.24}$ to the centre of the Smith chart is shown in Figure 89. The outer band frequencies however, do not rotate clockwise through a reactance of j1.77, but have to be modified because of their different frequencies.
The values of the outer band impedances are obtained from the Smith/Admittance chart as,

$Z'_{800} = 0.732 - j2.092$ and $Z'_{860} = 1.105 - j1.524$

Equation 4-10 provides the means of modifying the centre frequency reactance to give the outer band reactance values. That is,

$X_L(800) = j1.77\dfrac{800}{830} = j1.706$, so

$Z_{800\,final} = 0.732 - j2.092 + j1.706 = 0.732 - j0.386 \approx 0.73 - j0.34$

Converting this to an admittance for ease in plotting on the admittance chart

$Y_{800\,final} = 1.07 + j0.50$

$X_L(860) = j1.77\dfrac{860}{830} = j1.834$, so $Z_{860\,final} = 1.105 - j1.524 + j1.834 = 1.105 + j0.31$

Again converting this to an admittance for ease in plotting,

$Y_{860\,final} = 0.839 - j0.235 \approx 0.84 - j0.24$

These admittance points are shown plotted as impedances on Figure 89.
The VSWR is found to be approximately 1.65. This was found by taking the value of normalized resistance of the furthest out impedance point $Z_{800\,final}$, and drawing a circle through it, with the centre at the centre of the admittance chart, and then reading the value of VSWR on the B=0 axis.
The value of series inductance is found from $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ and as $\overline{X}_{Lf_o} = j1.77$, we find L= 0.339$Z_o$ nH.
gain note the “blooming” of the impedance band which is worse at Hz. t is wider than the original unmatched band. The 800 MHz final impedance is the impedance which is causing the band to lie outside of the specified VSWR of 1.60. If we increase the value of the shunt inductance so that the mid-point of the admittance band intercepts the unity resistance circle of the Smith chart, we should be able to bring the final impedance band within the specified VSWR of 1.60.

**Step 2 Matching to within the specified VSWR circle**

The additional of an inductive reactance added in series with the final impedance at 860 MHz, $Z_{860\,final} = 1.105 + j0.31$ will move this final impedance clockwise along a line of constant resistance $\overline{R} = 1.105$. When the movement of the final impedance along the line of constant resistance is such that it intersects the VSWR circle, or lies just within the VSWR circle, the amount of additional inductive reactance is determined. Referring to Figure 89, and remembering that this is an admittance chart, it can be seen that just underneath the intersection point of the VSWR=1.60

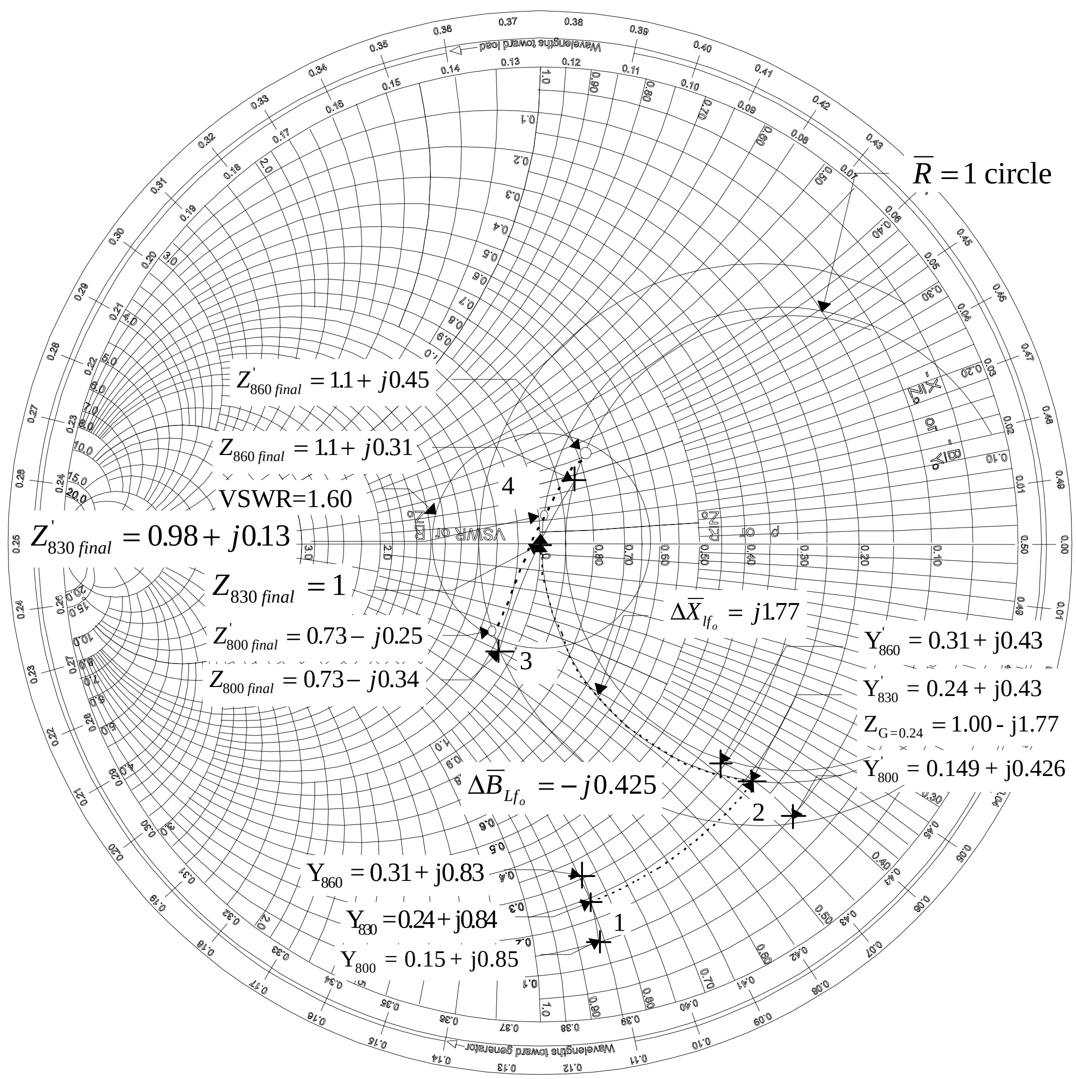



**Figure 89** **Example 19, Matching with shunt inductance and series inductance**

circle the admittance is $Y'_{860\,final} = 0.775 - j0.315$. Converting this to an impedance, as we are working on the Smith chart, due to the series inductive reactance added in series with the final impedance at 860 MHz, we find $Z'_{860\,final} = 1.105 + j0.450$. The added series inductive reactance at 860 MHz is the difference between $Z'_{860\,final} = 1.105 + j0.450$ and $Z_{860\,final} = 1.105 + j0.310$, which is $\Delta\overline{X}_{L860} = j0.140$.

Equation 4-10 provides the means of modifying the 860 MHz frequency reactance to give other outer band and band centre frequency reactance values. That is,

$X_L(800) = j0140\dfrac{800}{860} = j0.130$, so

$Z'_{800\,final} = 0.732 - j0.386 + j0.130 = 0.732 - j0.256$

Converting this to an admittance for ease in plotting on the admittance chart

$Y'_{800\,final} = 1.217 + j0.426$

$X_L(830) = j0.140\dfrac{830}{860} = j0.135$, so $Z'_{830\,final} = 1.00 + j0.135$

Again converting this to an admittance for ease in plotting,

$Y'_{830\,final} = 0.982 - j0.133$

These admittance points are shown plotted as impedances on Figure 89.
The value of added series inductance is found from $\Delta\overline{X}_{Lf_o} = j\omega_o\overline{L} = j2\pi f_o\overline{L}$ and as $\overline{X}_{Lf_o} = j0.135$, we find L= 0.026$Z_o$ nH. This means that the final series inductance is the addition of L= 0.339$Z_o$ nH and 0.026$Z_o$ nH which equals 0.365$Z_o$ nH
From the Smith chart, the final VSWR is found to be 1.55, which is within the specified VSWR of 1.60.
Note that each of the points on the final shifted impedance band moved from the band outside of the VSWR=1.60 circle along circles of constant resistance, and because of the different frequencies, each point moved a different distance along its respective constant resistance circle. For an inductance, we would expect that the movement along the constant resistance circle of the reactance would be greater for the higher frequencies, than for the lower. Because of the difference in these relative movements, the upper frequency impedance point was the point chosen to move to the VSWR circle edge in preference to the lower frequency point. If the lower frequency point could not be placed inside of the VSWR circle, then additional circuits are required. This is discussed later in this chapter when considering wide-band matching.
A similar procedure is adopted for the other seven types of L-type circuits, for matching a narrow-band mismatched load. Step 1 in example 19, has been discussed for the other seven types of circuits, and step 2 is the additional procedure required to obtain a solution.

#### 4.4.1.2.The use of a short-transformer in matching

Section 3.2.1 outlined the theory behind the short transformer. In this section, we will consider the use of the short transformer for matching an impedance band. This component is a simple and effective means of moving an impedance band to a different region of the Smith chart. It becomes useful when the impedance band is rounded in shape and not located near the centre of the chart, because, for a band such as this, the short-transformer can move the band across the chart and in some cases directly into the specified VSWR circle.
Figure 66 shows a diagram which represents the short transformer. This diagram indicates the short transformer has an input impedance $Z_{in}$ looking from the transmission line which has a characteristic impedance $Z_o$ and is terminated in a single load at a particular frequency. The corresponding Smith chart representation, is shown in Figure 90. In this diagram, the normalized impedance used for the Smith chart is $Z_o$.

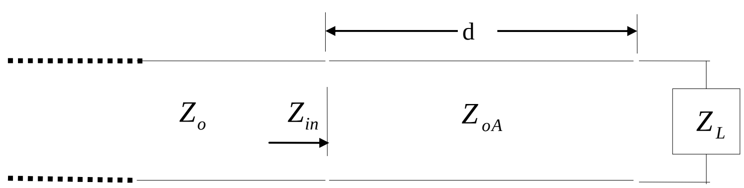



**Figure 66 A short transformer**

In order for a solution to exist,

$$\overline{R} > \frac{\overline{X}^2}{1-\overline{R}},\ \text{giving } \overline{R}(1-\overline{R}) > \overline{X}^2 \text{ and } 1 \neq \overline{R},\ \text{but } \overline{R} < 1 \text{ or } \overline{R} > 1 \qquad \textbf{(3-23)}$$

The boundary circles enclosing permissible values of load resistance and reactance for a solution to exist are shown on Figure 90. To use the short transformer for matching a load, the load points at the different frequencies when plotted on the Smith chart should straddle or lie within one of the two boundary circles. Any impedance which lies outside of the pertinent boundary circle will not be able to be matched to the centre of the chart, but may however, lie within a specified VSWR circle.
Because there is a band of frequencies being considered, in the short transformer, there will be different wavelengths operating . Each impedance in the band will produce a different standing wave.
As we traverse the short transformer away from the load, the various load impedance will change relative to each other due to these differences in wavelengths. The amount of relative change of each wavelength for each different frequency can be determined from the following calculation.
The phase velocity $v_p$, of a signal in a dielecric medium is assumed to be constant and is given by

$$v_p = f_1\lambda_1 = f_2\lambda_2 = \ldots = f_n\lambda_n \text{ where } n \to 1,\ldots\infty$$

From this the wavelength $\lambda_{f_x}$ at some frequency $f_x$, knowing a reference wavelength $\lambda_{f_x}$ and frequency $f_r$ is given by,

$$\lambda_{f_x} = \lambda_{f_r} \frac{f_r}{f_x} \qquad \textbf{(4-34)}$$

PROCEDURE FOR MATCHING

The procedure to be used, for matching a narrow-band load using a short transformer, is as follows:

- Determine the characteristic impedance of the short transformer. Two methods are discussed here, the first is based in the work in section 3. . , and the other is the "traditional " approach.

<u>*Finding the characteristic impedance of the transformer, based on Section 3.2.1*</u>

The graphical construction is made by first determining the centre of the short transformer circle by having equal distances from the centre of the Smith chart to the centre of the transformer circle and from the load band impedance to the centre of the transformer circle. The short transformer locus can then be drawn and from the diameter point furthers from the centre of the chart the value of $\overline{Z}_{oA}^2$ or $\frac{1}{\overline{Z}_{oA}^2}$ can be read off. From this the value of the short transformer characteristic impedance $\overline{Z}_{oA}$ can be determined. The point on the load locus where the circle intersects is arbitrary as all that is being done is to fix the characteristic impedance of the short transformer. Once this value is fixed, the length of the transformer becomes the variable in determining the solution to the problem.

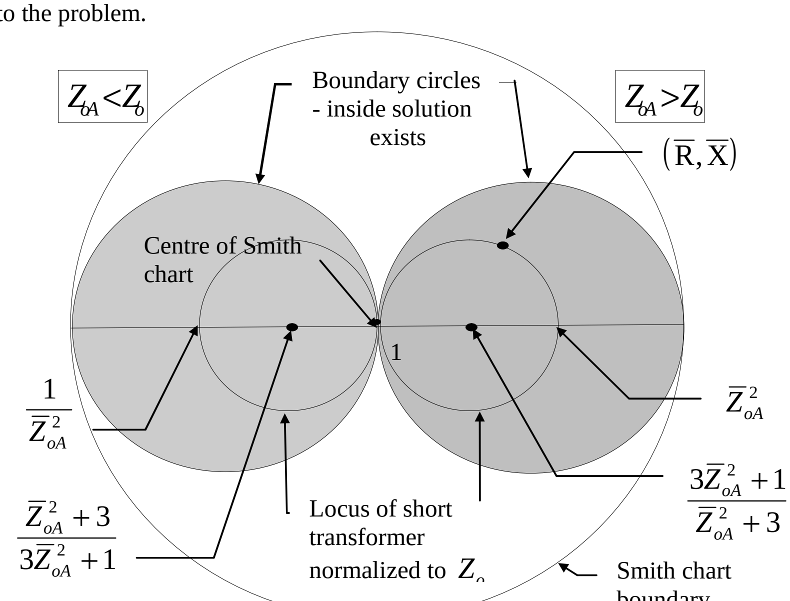



**Figure 90 A short transformer locus on the Smith chart with boundary circles**

<u>*The "traditional" approach to finding the characteristic impedance of the transformer*</u>

This method determines the geometric mean resistance of the impedance band by drawing a "modified" boundary circle, with the smallest possible radius, so that it encompasses all the points of the impedance band, with the centre of the circle located on the X=0 axis. The value of the characteristic impedance is determined by finding the geometric mean of the resistance circles which cut the diameters of this "modified" oundar circle on the axis. hat is

$$\overline{Z}_{oA} = \sqrt{\overline{R}_{d1}\overline{R}_{d2}} \qquad \textbf{(4-33)}$$

The VSWR performance can be estimated by drawing a VSWR circle on the interior of, and tangent to, the "modified" boundary circle and reading its axis intercept.

- The specified VSWR circle is then drawn on the Smith chart.
- The load impedance points and the VSWR circle are then mapped onto the Smith chart which is normalized to the characteristic impedance of the short transformer. This allows us to work within the short transformer itself and determine the length of the transformer for moving the impedance points into the translated VSWR circle.
- Noting that the lower the frequency, the greater the rotation, the load point with the lowest frequency, or the load point which may present the greatest problem fitting into

the translated Smith chart, is used to initially determine the length of the short transformer. The difference in wavelengths between the position of this unrotated load point and the final rotated position when placed inside the translated VSWR circle, is the length of the short transformer. he "traditional" approach is to rotate the band centre frequency point to the nearest X = 0 axis. However, this may present problems if applied without consideration of the remaining points in the band.

- Once all load points are satisfactorily placed inside the translated VSWR circle, using the frequency/wavelength ratio equation 4-34, as described above, the VSWR circle and final load points are translated back to the original Smith chart, which is normalized to the transmission line. If all of the load points cannot be satisfactorily placed inside the translated VSWR circle after equation 4-34 has been used, then the load point causing the problem may have to be used to determine the length of the short transformer as described in the procedural point above.

EXAMPLE 20
Match a transmission line to a VSWR of 1.5 or better, using a short transformer, given the following normalized mismatched load impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

SOLUTION
The VSWR circle and the impedance points are shown plotted on Figure 91.
In this problem, the locus of the short transformer has been chosen to cut the band impedance between the 4th and 5th point. The value of $\overline{Z}_{oA}^2$ is read from the Smith chart as 2.7, giving the value of the characteristic impedance of the short transformer $\overline{Z}_{oA}$ = 1.643.
Each of the load impedances and the VSWR circle are transformed into a Smith chart which is normalized to the short transformer characteristic impedance, $\overline{Z}_{oA}$ = 1.643. This allows the length of the transformer to be determined. By dividing the load impedances and VSWR circle by 1.643, the transformed values become,

| **Frequency (GHz)** | **Transformed Normalized impedance to Short Transformer Smith chart** |
|---|---|
| 1.00 | $\tilde{Z}_1 = 2.008 - j0.487$ |
| 1.02 | $\tilde{Z}_2 = 1.583 - j0.852$ |
| 1.04 | $\tilde{Z}_3 = 0.974 - j0.791$ |
| 1.06 | $\tilde{Z}_4 = 0.791 - j0.487$ |
| 1.08 | $\tilde{Z}_5 = 0.852 - j0.244$ |

and the VSWR circle becomes
Diameters: 0.406, 0.913
Centre 0.609
These points are shown plotted on Figure 92.

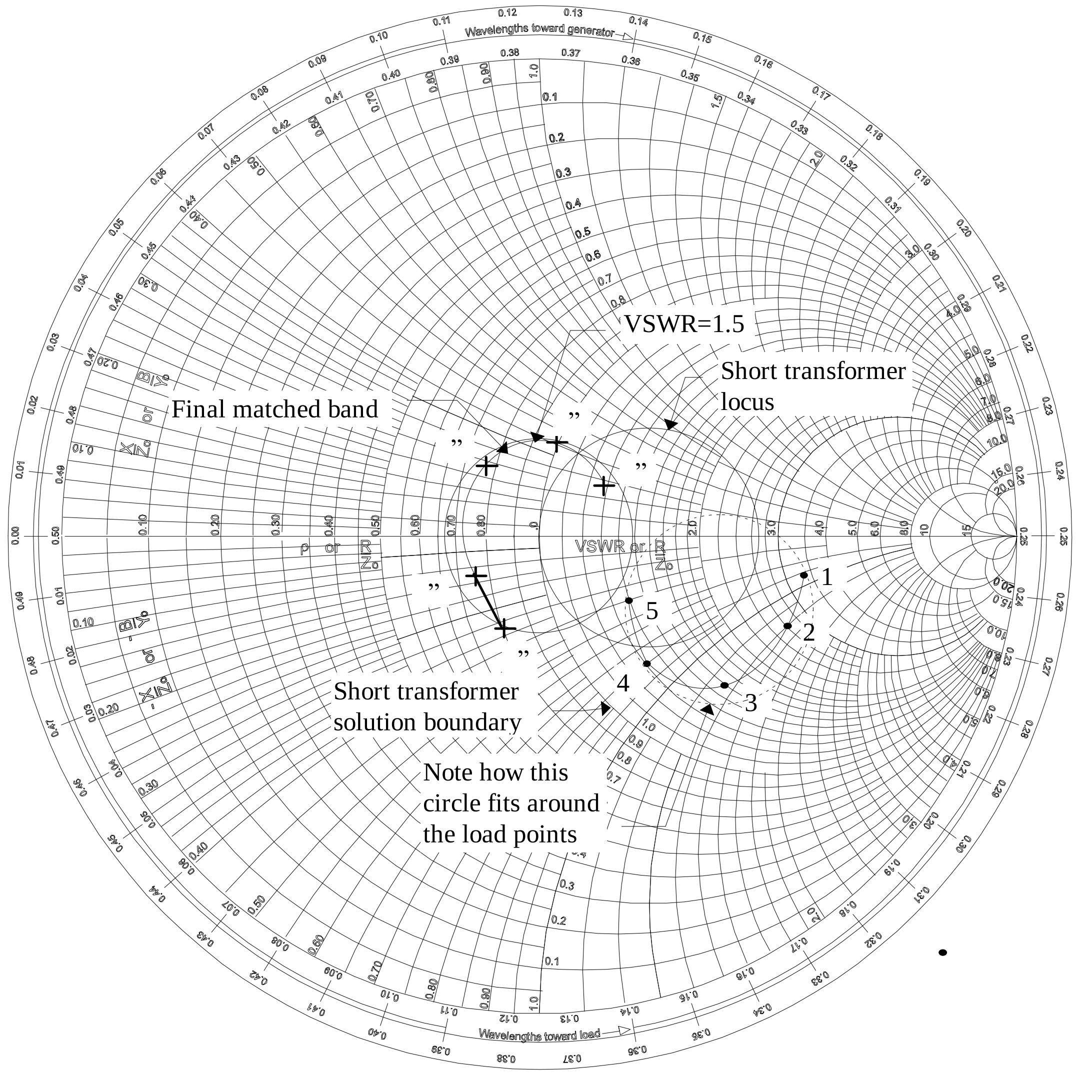



**Figure 91** **Example 20 - Short transformer normalized to transmission line**

Once the transformed VSWR circle and load impedance points have been plotted onto the Smith chart normalized to the short transformer characteristic impedance, the length of the transformer is determined from the angle (in wavelengths) between line A, which is extended from the centre of the Smith chart through the lowest frequency point (longest wavelength) and line B, which is a line extended from the centre of the Smith chart through a point on the VSWR chart which intersects the rotated lowest fre uency point point . his is shown in Figure . ine reads . λ and line B reads 0.459λ on the edge of the Smith chart. The difference between these two readings is 0.187 λ, which is the length of the transformer.

The rotation of the other transformed load impedance points (points 2 to 5) must be each be rotated through a different angle, as each operates at a different frequency. Below is a table of the angle each is to be rotated, using equation 4-34, $\lambda_{f_x} = \lambda_{f_r}\dfrac{f_r}{f_x} = 0.187\dfrac{1}{f_x}$.

| **Frequency (GHz)** | **Smith chart readings (λ)** | **Wavelength $\lambda_{f_x}$** |
|---|---|---|
| 1.00 | 0.459 - 0.272 = 0.187 | 0.187 |
| 1.02 | 0.301 + 0.183 =0.484 | 0.183 |
| 1.04 | 0.348 + 0.180 = 0.528 | 0.180 |
| 1.06 | 0.387 + 0.176 = 0.563 | 0.176 |
| 1.08 | 0.410 + 0.173 = 0.583 | 0.173 |

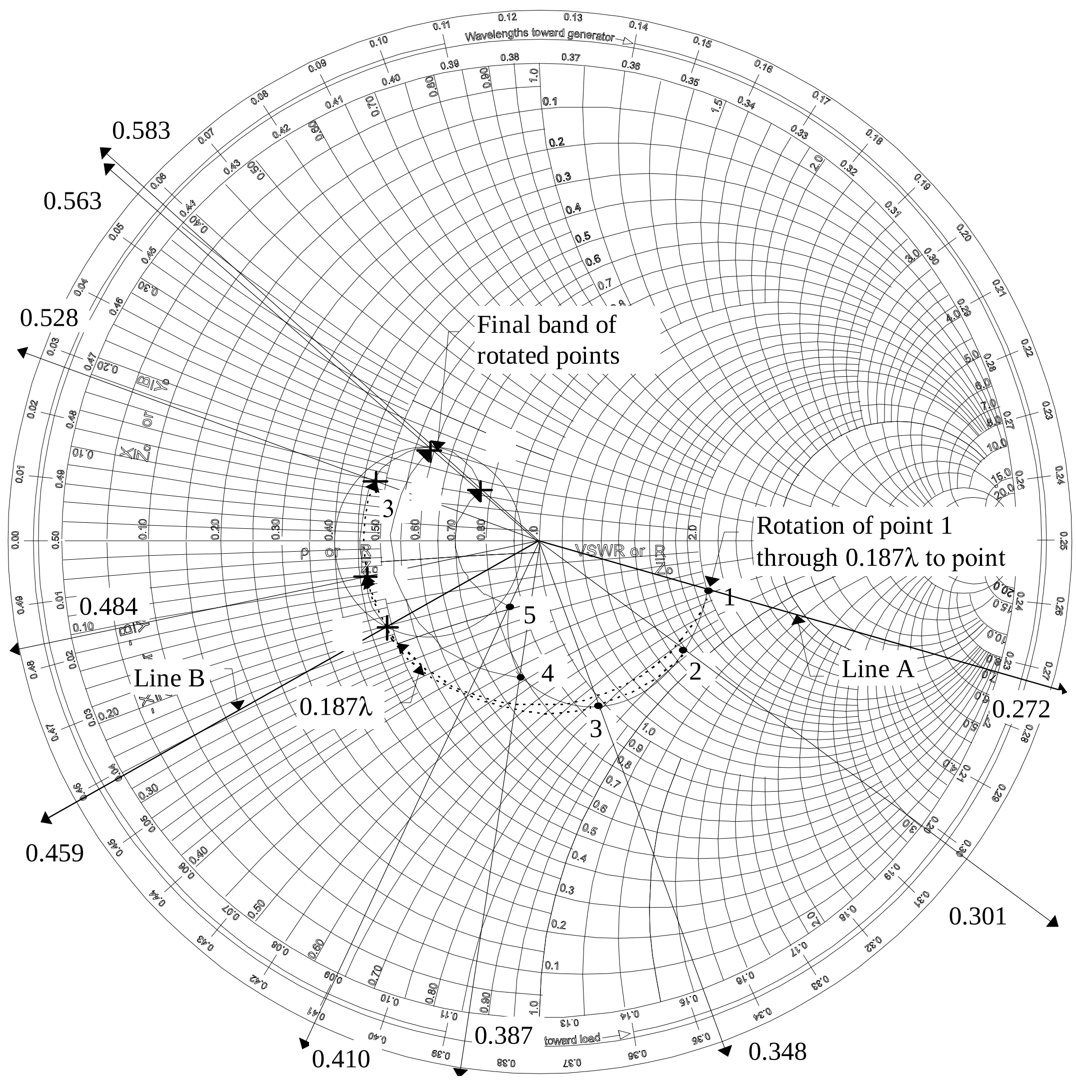



**Figure 92** **Example 20 - Short transformer normalized to short transformer $\overline{Z}_{oA}$**

The final band of load points at the different frequencies shown on Figure 92, are converted to the transmission line Smith chart, by multiplying each point by , $\overline{Z}_{oA}$ = 1.643. These points are then plotted on the Smith chart normalized to $\overline{Z}_o$ as shown on Figure 91. The final band impedances shown on Figure 92, together with the converted impedances back to the transmission line band impedances are given below.

| **Frequency (GHz)** | **Point Number** | **Short transformer final band impedances ( ’, ’, ’, ’, ’)** | **Transmission line final band impedances (1”,2”,3”,4”,5”) Fig. 91.** |
|---|---|---|---|
| 1.00 | 1 | 0.49 - j0.20 | 0.80 - j0.33 |
| 1.02 | 2 | 0.46 - j0.08 | 0.76 - j0.13 |
| 1.04 | 3 | 0.47 + j0.138 | 0.77 +j0.23 |
| 1.06 | 4 | 0.60 + j0.25 | 0.99 +j0.41 |
| 1.08 | 5 | 0.78 + j0.16 | 1.28 +j0.26 |

Some points about the solution to this type of problem may be worth noting. The first is, if the a circle is drawn which circles all of the load impedance points, then this circle will be close to the diameter of the minimum VSWR circle which can be obtained. This circle can be moved so that its centre is the centre of the Smith chart and then compared with the original VSWR specification. If it is the same size or smaller, then the short transformer can be used as a solution to this matching problem. If it is larger, then an alternative method for matching must be found. he "dashed" circle enclosing the load impedance points is shown in Figure 91. It can be seen that this equals the specified VSWR circle and therefore a matching solution exists. The second point is, the determination of the value of the short transformer characteristic impedance does not require any rigid procedures to be followed. This is because, once a value has been chosen, the length of the transformer becomes the important variable in the matching process. The choice of the characteristic impedance or the drawing of the short transformer locus as is shown in Figure 91, must be such that it is in the same half of the Smith chart as the impedance points, and that it does not extend out to the extremes of the chart. If a manufacturer supplies only a set number of characteristic impedances for short transformers, then the choice would be to select that characteristic impedance which allows the locus to pass through the impedance band as shown in Figure 91. If that is not possible, then the choice would be the characteristic impedance which is allows the short transformer locus to be near to the furthest out impedance point. Finally, the solution of the matching problem does not depend on knowing the characteristic impedance of the transmission line. All impedances are normalized and all transformations are made with normalized parameters. If the actual value of the short transmission line characteristic impedance is to be determined, then the characteristic impedance of the transmission line must be known.
A natural consequence of section 4.4.1.2 is to extend the short transformer into two short transformers in tandem and use this as a tool for matching. The preliminary work on two short transformers in tandem was given in section 3.2.3.
However, before considering this matching element, it is worthwhile considering the short-circuit and open-circuit stub because, individually, these can be used in conjunction with other matching elements, as will be shown in Example 21.

### 4.4.1.3. The use of the short-circuit and open circuit stub in matching

The basic rule for use is that the lower frequency impedance in an impedance band must exist in the top semicircle of the Smith chart and the higher frequency impedance in the lower semicircle. The purpose of these stubs is to close the "mouth" of the band, aiding the matching process.

#### 4.4.1.3.1.The short-circuit quarter-wavelength stub

Equation 1-93 showed that for a short-circuit transmission line, the normalized impedance is,

$$\overline{Z} = \frac{Z}{Z_o} = j\,tan\frac{2\pi d}{\lambda} \qquad \textbf{(1-93)}$$

Where $Z_o$ is the characteristic impedance of the transmission line used for the stub. If this characteristic impedance of the stub is renamed to $Z_{ch}$, in order to show that it may be different from the characteristic impedance $Z_o$, of the main transmission line, then equation 1-93 can be modified to give the impedance looking into the non-short circuit end of the stub as,

$$\overline{Z} = \frac{Z}{Z_{ch}} = \frac{\frac{Z}{Z_o}}{\frac{Z_{ch}}{Z_o}} = j\,tan\frac{2\pi d}{\lambda} \ , \quad \text{giving,}$$

$$\overline{Z} = \frac{Z}{Z_o} = \frac{Z_{ch}}{Z_o}\,j\,tan\frac{2\pi d}{\lambda} \qquad \textbf{(4-35)}$$

Equation 4-35 shows that the short-circuit stub may be of a different characteristic impedance from the main transmission line, and yet can be plotted on a Smith chart normalized to the main transmission line impedance $Z_o$. What is useful about equation 4-35, is that is permits the short-circuit stub susceptance values at different values of wavelengths $\lambda$, to be modified by a factor $Z_{ch}/Z_o$ over that where the characteristic impedance of the stub is the same as the main transmission line. Example 21 shows how this factor is used.

For a quarter-wavelength line $\lambda/4$, the stub transmission line behaves in a similar manner to a parallel tuned circuit with $\overline{Z} \to \infty$, as shown in Figure 41. Because it behaves as a parallel tuned circuit it may be used to modify admittance circuits by virtue of its properties at frequencies about the resonant frequency.

One application is to change the curvature of an admittance band of frequencies so that the band is shaped for more easy final manipulation into a specified VSWR circle. This is achieved by placing the quarter-wavelength stub in parallel with a load. The load may comprise an initial load admittance band and circuits which already modify its band or a load admittance band which has not yet been modified. The frequency chosen to be the resonant frequency of the stub is one of the load admittance band frequencies. The particular load frequency which is chosen to be the resonant frequency of the stub is the frequency of an admittance which lies on the X=0 axis of the Smith chart. This is because at resonance, the quarter-wave tuned stub has a susceptance equal to zero, whereas at frequencies above or below the resonant frequency, a pure susceptance, other than zero, is obtained. As will be shown, for wavelengths greater than the quarter-wavelength stub resonant frequency wavelength $\lambda_o$, or frequencies less than the stub resonant frequency $f_o$, the susceptance is negative and for wavelengths less than the resonant frequency wavelength $\lambda_o$, or frequencies greater than the resonant frequency $f_o$, the susceptance is positive.

The admittance of the quarter-wavelength stub of characteristic impedance $Z_{ch}$ and which is normalized to the characteristic of the main transmission line is given by,

$$\overline{Y} = \frac{Z_o}{Z} = -\frac{Z_o}{Z_{ch}}\, j\cot\frac{2\pi d}{\lambda_{f_x}} \qquad \textbf{(4-36)}$$

From equation 4-34, we find that inside of the stub, at some frequency $f_x$, other than the stub resonant frequency $f_o$, the corresponding wavelength $\lambda_{f_x}$, changes from $\lambda_o$, to become,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

Therefore, for a fixed length d of the stub equal to $\lambda_o/4$, equation 4-36 becomes,

$$\overline{Y} = -\frac{Z_o}{Z_{ch}}\, j\cot\frac{\dfrac{2\pi\lambda_o}{4}}{\dfrac{\lambda_o f_o}{f_x}} = -\frac{Z_o}{Z_{ch}}\, j\cot\frac{\pi}{2}\frac{f_x}{f_o} \qquad \textbf{(4-37)}$$

Hence, for $f_x > f_o$, we find $\overline{Y}$ is a positive susceptance, and for $f_x < f_o$, $\overline{Y}$ is a negative susceptance. The same results are obtained for an open-circuit half-wavelength stub.

Equation 4-37 contains all of the information required to modify a band of frequencies. If it is assumed that a band of frequencies exists which has the admittance points of its lower frequencies in the upper half of the Smith chart, and the admittance points of its upper frequencies in the lower half of the Smith chart, then by adding a short-circuit stub, with its resonant frequency at the frequency of the admittance band point which on the horizontal dividing axis of the Smith chart, the lower frequency load susceptance points will be brought down closer to the horizontal dividing axis (X=0 axis) of the Smith chart and the high frequency load susceptance points will be raised up closer to the X=0 axis of the Smith chart. That is the admittance band will become closed as shown in Figure 93.

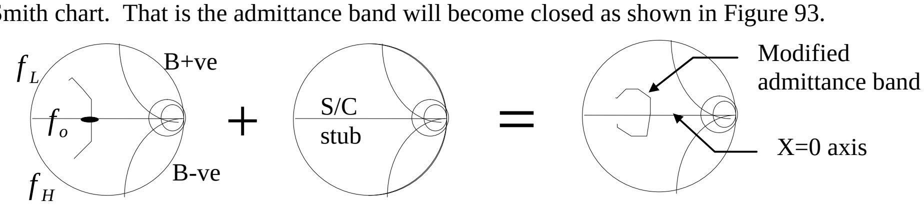



**Figure 93** **Effect of short-circuit stub on admittance band**

EXAMPLE 21

A load impedance band is given from the solution to Example 20. This impedance band is required to have a VSWR of 1.4 or less.

SOLUTION
To reduce the band to fit into a VSWR circle of 1.4 or less, it is possible to use a short-circuit quarter-wavelength transformer because, as can be seen from Figure 91, the final band of Example 20 can be further closed and the low frequency impedance lies in the lower half of the Smith chart. This means that the low frequency admittance point will lie in the upper half of the Smith chart, which is one of the conditions for using a short-circuit quarter-wa elength stu . o close the and ore, the oints ” and ” will rise along their res ecti e constant resistance circles, and the oints ”, ” and ” will e lowered along their respective constant resistance circles. Therefore, it is possible that a smaller VSWR circle will accommodate all points. The first thing that must be done is to convert all of the impedance points to admittance points so that the quarter-wavelength stub can be added to the load band in parallel. Below is listed the load impedance band points and their frequencies, together with the converted admittances at these frequencies. The admittances are plotted on Figure 94.

| **Frequency (GHz)** | **Point Number** | **Transmission line load band impedances (1”,2”,3”,4”,5”) Fig. 91.** | **Transmission line load band admittances (1, 2, 3, 4, 5) Figure 94** |
|---|---|---|---|
| 1.00 | 1 | 0.80 - j0.33 | 1.068 + j0.441 |
| 1.02 | 2 | 0.76 - j0.13 | 1.278 + j0.219 |
| 1.04 | 3 | 0.77 +j0.23 | 1.192 - j0.356 |
| 1.06 | 4 | 0.99 +j0.41 | 0.862 - j0.357 |
| 1.08 | 5 | 1.28 +j0.26 | 0.750 - j0.152 |

Using equation 4-37, with the centre frequency estimated to be 1.027 GHz, the following table is obtained. As the high frequency admittance point (point 5) will move the most, a susceptance point on the Smith chart constant resistance circle of point 5, is found which will give a VSWR of 1.4 or better. This turns out to be j0.100. From this value and the original value, it is determined that the susceptance must move from -j0.152 to +j0.100, or a distance of +j0.252. As column 3 provides equation 4-37 without the $Z_o/Z_{ch}$ factor, column 4 divided by column 3, gives the value of $Z_o/Z_{ch}$. This value is 3.1, therefore the characteristic impedance of the quarter-wave short transformer is to be $Z_{ch} = \frac{Z_o}{3.1}$. Figure 94 shows the final admittance band laying within a VSWR = 1.38 circle, which is better than the specified VSWR of 1.4.

| Column 1 | 2 | 3 | 4 (Eqn. 4-37) | 5 | 6 |
|---|---|---|---|---|---|
| **Frequency (GHz)** $f_x$ | **Point No.** | $-j\cot\frac{\pi}{2}\frac{f_x}{f_o}$ | $-j\frac{Z_o}{Z_{ch}}\cot\frac{\pi}{2}\frac{f_x}{f_o}$ For $Z_o = 3.1Z_{ch}$ | **Original Band admittances (1, 2, 3, 4, 5) Fig. 94** | **Final Band admittances (1”, 2”, 3”, 4”, 5”) Fig. 94 Col. (5 + 4)** |
| 1.00 | 1 | - j0.0413 | -j0.128 | 1.068 + j0.441 | 1.068 + j0.313 |
| 1.02 | 2 | - j0.0107 | -j0.033 | 1.278 + j0.219 | 1.278 + j0.186 |
| 1.027 ( $f_o$ ) | | 0 | 0 | | |
| 1.04 | 3 | + j0.020 | +j0.062 | 1.192 - j0.356 | 1.192 - j0.294 |
| 1.06 | 4 | +j 0.0505 | +j0.157 | 0.862 - j0.357 | 0.862 - j0.200 |
| 1.08 | 5 | +j0.0812 | +j0.252 | 0.750 - j0.152 | 0.750 + j0.100 |

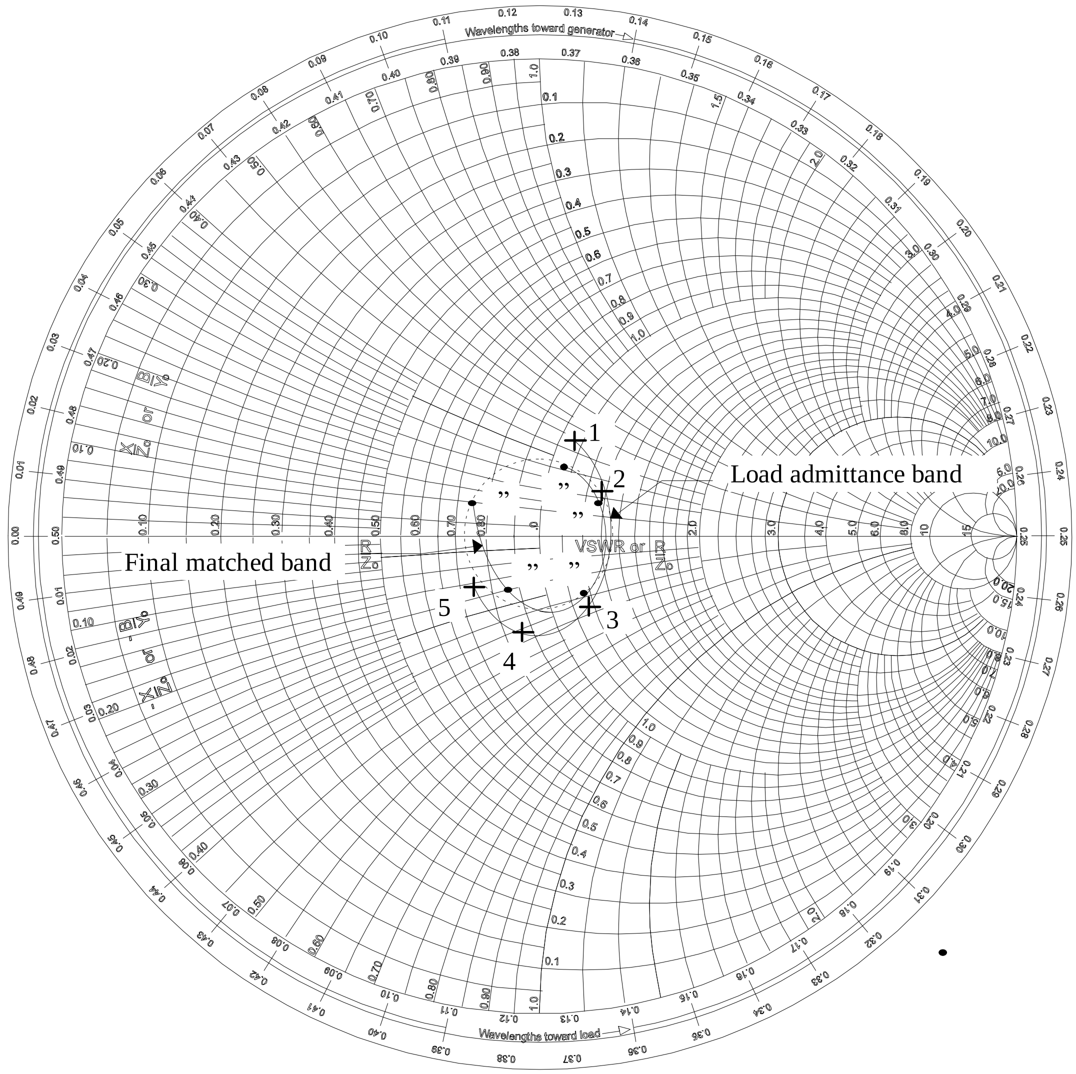



**Figure 94 Example 21 - Short circuit quarter wavelength stub effect on admittance band**

### 4.4.1.3.2.The open-circuit quarter-wavelength stub

Equation 1-98 showed that for an open-circuit transmission line, the normalized impedance is given by,

$$\overline{Z} = \frac{Z}{Z_o} = -j\cot\frac{2\pi d}{\lambda} \qquad \textbf{(1-98)}$$

Where $Z_o$ is the characteristic impedance of the transmission line used for the stub. If this characteristic impedance of the stub is renamed to $Z_{ch}$, in order to show that it may be different from the characteristic impedance $Z_o$, of the main transmission line, then equation 1-98 can be modified to give the impedance looking into the stub from the generator end, as,

$$\overline{Z} = \frac{Z}{Z_{ch}} = \frac{\dfrac{Z}{Z_o}}{\dfrac{Z_{ch}}{Z_o}} = -j\cot\frac{2\pi d}{\lambda},$$

giving,

$$\overline{Z} = \frac{Z}{Z_o} = -j\frac{Z_{ch}}{Z_o}\cot\frac{2\pi d}{\lambda} \qquad \textbf{(4-38)}$$

Equation 4-38 shows that the open-circuit stub may be of a different characteristic impedance from the main transmission line, and yet can be plotted on a Smith chart normalized to the main transmission line impedance $Z_o$. This permits the open-circuit stub susceptance values at different values of wavelengths λ, to be modified by a factor $Z_{ch}/Z_o$ over that where the characteristic impedance of the stub is the same as the main transmission line.
For a quarter-wavelength line λ/4, the open-circuit stub transmission line behaves in a similar manner to a series tuned circuit with $\overline{Z} = 0$, as shown in Figure 45. Because it behaves as a series tuned circuit it may be used to modify impedance circuits by virtue of its properties at frequencies about the resonant frequency.
Similar to the short-circuit stub, one application is to change the curvature of an impedance band of frequencies so that the band is shaped for more easy final manipulation into a specified VSWR circle. This is achieved by placing the open-circuit quarter-wavelength stub in series with a load. The frequency chosen to be the resonant frequency of the stub is one of the load impedance band frequencies. The particular load frequency which is chosen to be the resonant frequency of the stub is the frequency at which a resistance occurs, that is where the band crosses the X=0 axis of the Smith chart. This is because at resonance, the quarter-wave stub has a reactance equal to zero, whereas at frequencies above or below the resonant frequency, a pure reactance, other than zero, is obtained. As will be shown, for wavelengths greater than the quarter-wavelength stub resonant frequency wavelength $\lambda_o$, or frequencies less than the stub resonant frequency $f_o$, the reactance is negative and for wavelengths less than the resonant frequency wavelength $\lambda_o$, or frequencies greater than the resonant frequency $f_o$, the reactance is positive. This is the same set of results as that obtained with the quarter-wavelength short-circuit stub. However, here it must be remembered that impedances, reactances and series circuits are being considered.
The impedance of the open-circuit quarter-wavelength stub of characteristic impedance $Z_{ch}$ and which is normalized to the characteristic impedance of the main transmission line is given by equation 4-38.
From equation 4-34, we find that inside of the stub, at some frequency $f_x$, other than the stub resonant frequency $f_o$, the corresponding wavelength $\lambda_{f_x}$, changes from $\lambda_o$, to become,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

Therefore, for a fixed length d of the stub equal to $\lambda_o/4$, equation 4-38 becomes,

$$\overline{Z} = \frac{Z}{Z_o} = -j\frac{Z_{ch}}{Z_o}\cot\frac{2\pi\frac{\lambda_o}{4}}{\lambda_o\frac{f_o}{f_x}} = -j\frac{Z_{ch}}{Z_o}\cot\frac{\pi}{2}\frac{f_x}{f_o} \qquad \textbf{(4-39)}$$

Hence, for $f_x > f_o$, we find $\overline{Z}$ is a positive reactance, and for $f_x < f_o$, $\overline{Z}$ is a negative reactance. The same results are obtained for a short-circuit half-wavelength stub.
Equation 4-39 contains all of the information required to modify a band of frequencies. If it is assumed that a band of frequencies exists which has the impedance points of its lower frequencies in the upper half of the Smith chart, and the impedance points of its upper frequencies in the lower half of the Smith chart, then by adding a series open-circuit stub, with its resonant frequency at the frequency of the impedance band point which on X=0 axis of the Smith chart, the lower frequency load reactance points will be brought down closer to the X=0 axis of the Smith chart and the upper frequency load reactance points will be raised up closer to the X=0 axis of the Smith chart. That is the impedance band will become closed as shown in Figure 95.

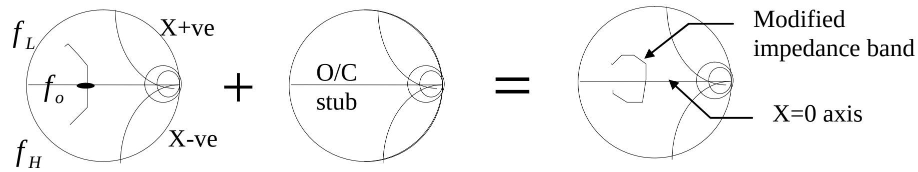



**Figure 95 Effect of an open-circuit stub on impedance band**

### 4.4.1.4. The use of two quarter-wavelength transformers in tandem in matching

Quarter-wavelength transformers have a finite bandwidth which are determined by the maximum VSWR and the ratio of the characteristic impedances. Typical values are 17% for a single section and 61% for two quarter-wavelength transformers in tandem. Section 3.2.2 outlined the theory behind the quarter wavelength transformers and showed that the characteristic impedance of the transformer $Z_{oA}$, is given by,

$$Z_{oA} = \sqrt{Z_o Z_L} \tag{1-105}$$

where $Z_o$ is the characteristic impedance of the main transmission line, and $Z_L$ is the load impedance. It is important to note at this point that the characteristic impedance is the geometric mean of the characteristic impedance of the main transmission line and the load impedance. This property will be used to develop the equations behind the two quarter-wavelength transformers in tandem.

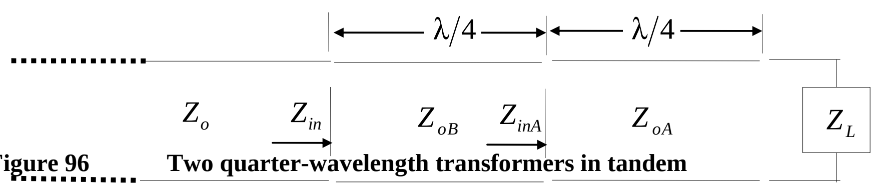



**Figure 96 Two quarter-wavelength transformers in tandem**

Considering Figure 96, the junction between the two quarter wavelength transformers at the point $Z_{inA}$ looks in two directions. One direction being towards the load $Z_L$ and the other being towards the main transmission line $Z_o$. This is shown in Figure 97, together with the values of $Z_{inA}$ seen from each direction.

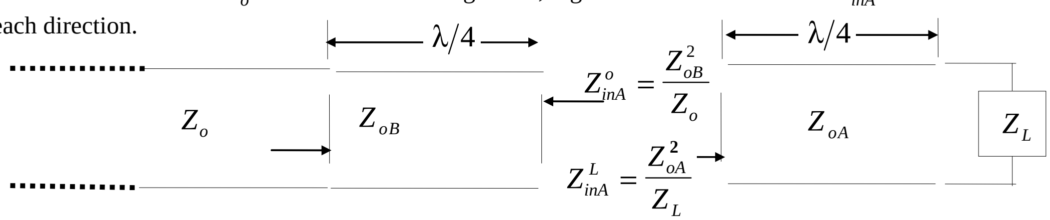



**Figure 97 Splitting two quarter-wavelength transformers in tandem**

Both $Z_{inA}^o$ and $Z_{inA}^L$ are equal at the junction to avoid an infinite discontinuity and if we let the impedance at the junction be expressed as the geometric mean of the source and the load, which is consistent with the theory of small reflections[i], then,

$$\frac{Z_{oB}^2}{Z_o} = \frac{Z_{oA}^2}{Z_L} = \sqrt{Z_o Z_L} \tag{4-40}$$

Thus,

$$\frac{Z_{oB}^2}{Z_o} = \sqrt{Z_o Z_L} \quad \text{and} \quad \frac{Z_{oA}^2}{Z_L} = \sqrt{Z_o Z_L}$$

and

$$Z_{oB} = \left(Z_L Z_o^3\right)^{1/4} \tag{4-41}$$

$$Z_{oA} = \left(Z_o Z_L^3\right)^{1/4} \tag{4-42}$$

To use two quarter-wave transformers in tandem to match a band, the band must be centred around the X=0 axis. If it is not centred around the X=0 axis then the first thing to be done is to use a component to centre it. The procedure for using the two quarter-wave transformers in tandem is to first determine the geometric mean impedance of the band using equation 4-33. The resistance of this geometric mean value is then chosen to be the resistance of the load for determining the values of the transformers characteristic impedances, using equations 4-41 and 4-42. By placing the centre of the modified boundary circle on the real axis and drawing a circle of minimum radius around the band impedance points, the values of the modified boundary circle diameters $\overline{R}_{d1}$ and $\overline{R}_{d2}$, can be found, and from equation 4-33, the value of the geometric mean of the load impedances can be determined.

$$\overline{Z}_L = \sqrt{\overline{R}_{d1}\overline{R}_{d2}} \qquad \textbf{(4-33)}$$

The better alternative to use, is to find the average value of the impedance band. This is because, if the average value of the impedance contains a reactive component, then an inductor or capacitor may be used to cancel it out, whereas, using the modified boundary circle, the value of the geometric mean of the load impedance may be greater than really required. After the load and then the characteristic impedances of the lines have been determined a frequency is chosen. This is the frequency of a given or assumed point in the band which is the geometric mean of the highest and lowest frequency in the band, that is $f_o = \sqrt{f_L f_H}$ . At this frequency the length of the transformer closest to the load is set to $\lambda/4$. Equation 4-34 allows the wavelengths at the other frequencies to be determined. The Smith chart is now normalized to the characteristic impedance of the first transformer, which is the transformer closest to the load. On this Smith chart the renormalized band impedances are plotted and the band impedances rotated through their respective wavelengths (towards the generator) to their new position. The new band of load impedances are then renormalized to the second transformer using the characteristic impedance of that transformer and plotted on the Smith chart normalized to the second transformer. As this second transformer is also $\lambda/4$ long, the same wavelengths determined from equation 4-34 are used to rotate each of the impedances through their respective impedances. Finally, the newly rotated band is renormalized to the characteristic impedance of the main transmission line and plotted on the Smith chart which normalized to this characteristic impedance. As the two quarter-wavelength transformers are designed to match a single load to the main transmission line characteristic impedance, the final band will reside within a VSWR circle. The value of the VSWR obtained will be close to that of a test VSWR. To do this, the original band is tested by placing a circle around the points and transferring this circle to the centre of the chart, making it a VSWR circle. The VSWR read will be close to the final value. To ensure that the VSWR is reduced before being processed by the transformers, a short-circuit stub or some other component can preprocess the band first. The value of this double transformer is in its bandwidth. It has a larger useful bandwith than a single-section quarter-wavelength transformer.
Below is an example of the use of the two quarter-wavelength transformers in tandem for matching a band of frequencies.

EXAMPLE 22
Match a transmission line to a VSWR of 1.6 or better, using two quarter-wave transformers in tandem, given the following normalized mismatched load impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

SOLUTION
First we test these points to see if it is possible to obtain a VSWR of 1.6. Figure 91 shows a dotted circle drawn around the points. This circle when transferred to the centre of the chart reads a VSWR of 1.5. Thus, it is possible to achieve a VSWR of 1.6 with the double transformer. The specified VSWR circle and the impedance points are shown plotted on Figure 98. Because the band opening is not centred around the X=0 axis, an inductor may be placed in series before the double transformer is used, in order centre the band, by rotating it clock-wise around circles of constant resistance. The estimated frequency which is closest to the geometric mean of the highest and lowest frequency in the

band , that is, $f_o = \sqrt{f_L f_H}$ , is $\sqrt{1.00x1.08} = 1.0392$ , is 1.04 GHz. The band impedance at 1.04 GHz will be used to determine the value of the inductor, since this inductor must cancel the average value of the reactance to rotate the average value of the impedance onto the X=0 axis of the Smith chart. The average normalized impedance value is found from the given normalized load impedance points as 2.04 - j 0.94. The value of +j0.94 is used to cancel the -j0.94 given as the reactance of the impedance given in the table above.

As $j\omega L = 2\pi f_o L = j0.94$ , the value of L=143.9 $Z_o$ pH

The values of the reactances at the different frequencies are found from equation 4-10,

$X(f_x)_x = X(f_o)\frac{f_x}{f_o}$. The values of the reactances at the different frequencies are given in the table below and are shown plotted on Figure 98,

**Addition of inductance to rotate the band about the X = 0 axis**

| **Frequency (GHz)** | **Reactance (L = 143.9 $Z_o$ pH)** | **New value of impedance** |
|---|---|---|
| 1.00 | $\overline{X}_1 = j0.904$ | $\overline{Z}_1 = 3.3 + j0.104$ |
| 1.02 | $\overline{X}_2 = j0.922$ | $\overline{Z}_2 = 2.6 - j0.478$ |
| 1.04 | $\overline{X}_3 = j0.94$ | $\overline{Z}_3 = 1.6 - j0.360$ |
| 1.06 | $\overline{X}_4 = j0.958$ | $\overline{Z}_4 = 1.3 + j0.158$ |
| 1.08 | $\overline{X}_5 = j0.976$ | $\overline{Z}_5 = 1.4 + j0.576$ |

The average value, using the Smith chart non-linear scale, of the load resistance, $\overline{Z}_L$ , found from the table above, is $\overline{Z}_L$ = 2.04. This is used in to determine the characteristic impedances of the quarter-wave transformers. Had the modified boundary circle and equation 4-33 been used **after** the band had been rotated onto the real axis, the value of $\overline{Z}_L$ would be $\overline{Z}_L = \sqrt{1.15x3.8} = 2.09$. This is to be expected to be the same, as the band has already been centralized about the X=0 axis. As the value of the characteristic impedance of the main transmission line $Z_o$ , is unknown, equations 4-41 and 4-42 must be normalized to $Z_o$ , that is

$$\overline{Z}_{oB} = \left(\overline{Z}_L\right)^{1/4} \qquad \textbf{(4-43)}$$

$$\overline{Z}_{oA} = \left(\overline{Z}_L^3\right)^{1/4} \qquad \textbf{(4-44)}$$

From equations 4.43 and 4-44, $\overline{Z}_{oB} = 1.1951$ and $\overline{Z}_{oA} = 1.7070$

The impedances normalized to the quarter-wave transformer closest to the load is found by dividing the normalized impedances after the inductor has been added. given in the table above, by $\overline{Z}_{oA} = 1.7070$ . These renormalized impedances are given below,

**Renormalization to transformer closest to the load ( $\overline{Z}_{oA} = 1.7070$ )**

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 1.9332 + j0.061$ |
| 1.02 | $\overline{Z}_2 = 1.5231 - j0.280$ |
| 1.04 | $\overline{Z}_3 = 0.9373 - j0.211$ |
| 1.06 | $\overline{Z}_4 = 0.7616 + j0.093$ |
| 1.08 | $\overline{Z}_5 = 0.8202 + j0.337$ |

These impedance points are shown plotted on Figure 99.

The estimated frequency which is closest to the geometric mean of the highest and lowest frequency in the band , that is, $f_o = \sqrt{f_L f_H}$ , is $\sqrt{1.00x1.08} = 1.0392$ , is 1.04 GHz. This impedance point at 1.04 GHz is rotated through λ/4. The rotation of the other transformed load impedance points (points 1,2,4 and 5) must be each be rotated through a different angle, as each operates at a different frequency. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance point.

$$\lambda_{f_x} = \lambda_{f_r}\frac{f_r}{f_x} = 0.250\frac{1.04}{f_x}.$$

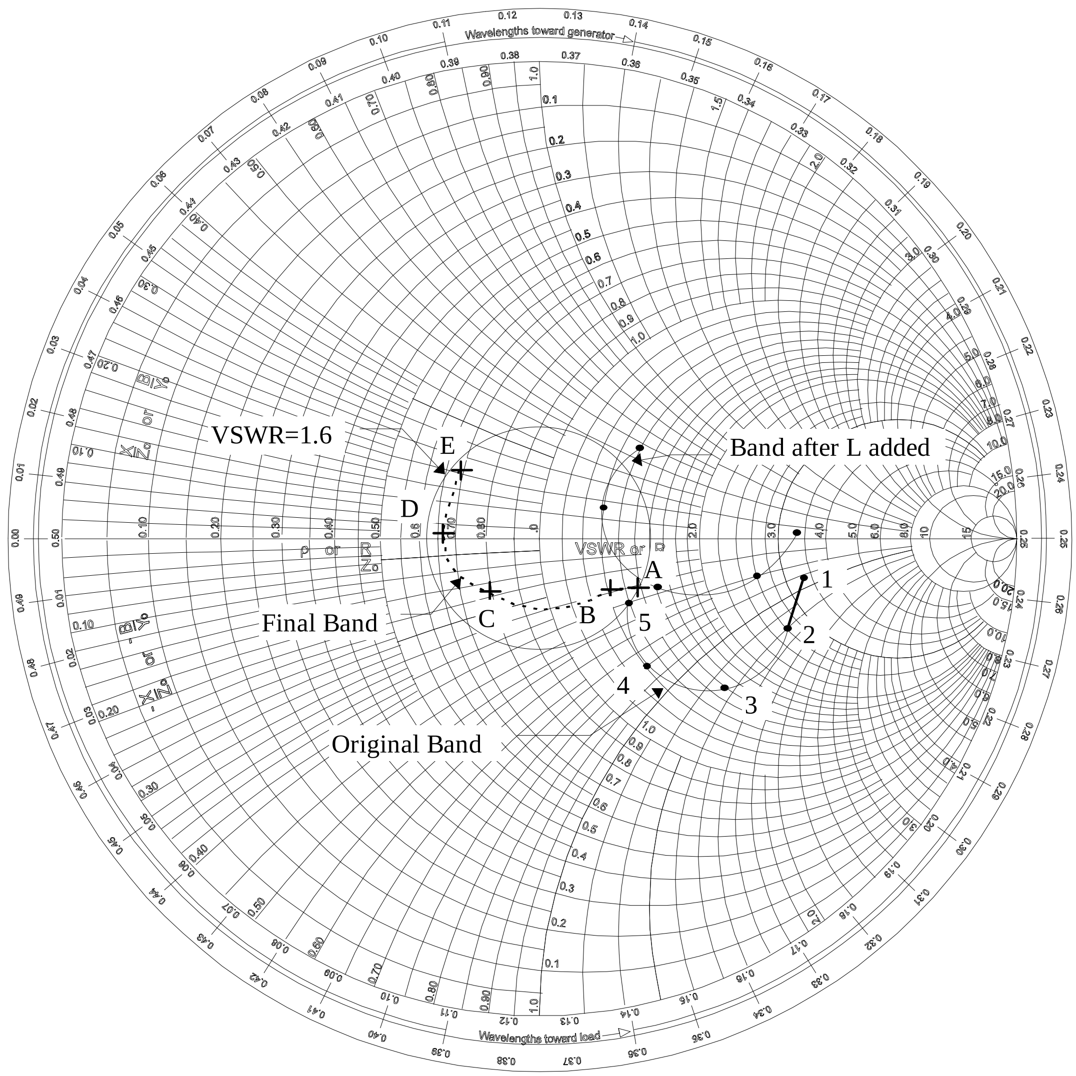



**Figure 98 Example 22 - Two quarter-wavelength transformer matching problem**

**Transformed band impedance points in transformer closest to the load (** $\bar{Z}_{oA} = 1.7070$ )

| **Frequency (GHz)** | **Initial impedance position (λ), points 1,2,3,4,5** | **Wavelength $\lambda_{f_x}$** | **Final impedance position (λ) , points ’, ’, ’, ’, ’** | **Final Impedance points of transformer closest to the load (Not shown on Fig.99)** |
|---|---|---|---|---|
| 1.00 | 0.246 | 0.2600 | 0.006 | $\bar{Z}_1 = 0.52 + j0.080$ |
| 1.02 | 0.279 | 0.2549 | 0.034 | $\bar{Z}_2 = 0.61 + j0.125$ |
| 1.04 | 0.390 | 0.2500 | 0.140 | $\bar{Z}_3 = 1.02 + j0.230$ |
| 1.06 | 0.038 | 0.2453 | 0.283 | $\bar{Z}_4 = 1.33 - j0.225$ |
| 1.08 | 0.11 | 0.2407 | 0.351 | $\bar{Z}_5 = 1.04 - j0.420$ |

Normalizing the impedance points of the transformer closest to the load back to that of the transmission line and then renormalizing to the transformer closest to the transmission line, so that each impedance of the band is presented on the Smith chart belonging to the transformer closest to the transmission line. To do this, the impedance points on the transformer closest to the load would be multiplied by $\overline{Z}_{oA} = 1.7070$ and then divided by $\overline{Z}_{oB} = 1.1951$. The table below shows the con ersion and igure shows these oints lotted as ", ", ", " and ".

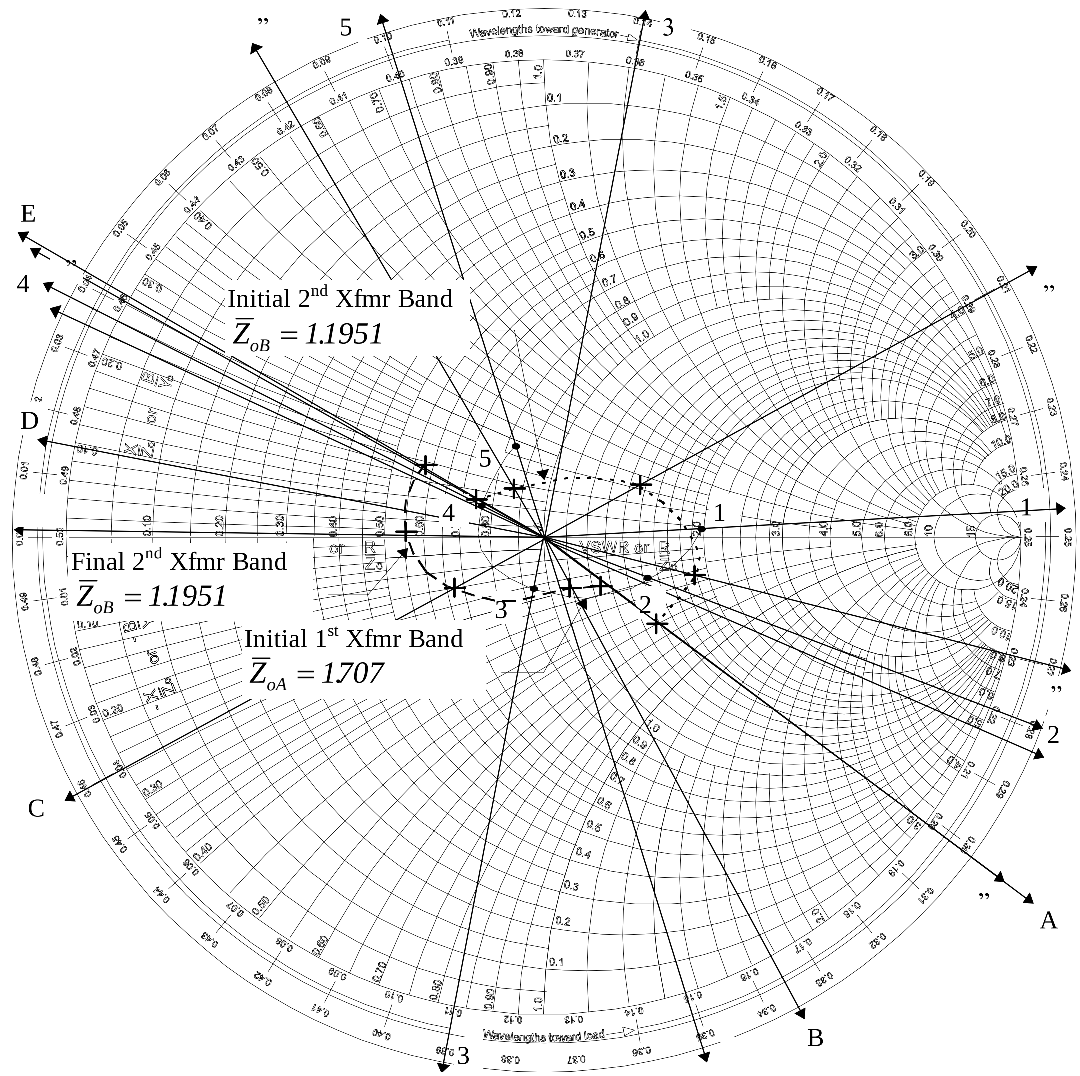



**Figure 99 Example 22 - Two quarter-wavelength transformer matching problem**

**Renormalization to transformer furthest from the load ($\overline{Z}_{oB} = 1.1951$)**

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the load (Not shown on Fig.99)** | **Renormalized impedance points for processing by transformer furthest from the load. (1",2",3",4",5")** |
|---|---|---|
| 1.00 | $\overline{Z}_1 = 0.52 + j0.080$ | $\overline{Z}_1 = 0.743 + j0.114$ |
| 1.02 | $\overline{Z}_2 = 0.61 + j0.125$ | $\overline{Z}_2 = 0.871 + j0.179$ |
| 1.04 | $\overline{Z}_3 = 1.02 + j0.230$ | $\overline{Z}_3 = 1.475 + j0.329$ |
| 1.06 | $\overline{Z}_4 = 1.33 - j0.225$ | $\overline{Z}_4 = 1.900 - j0.321$ |
| 1.08 | $\overline{Z}_5 = 1.04 - j0.420$ | $\overline{Z}_5 = 1.485 - j0.600$ |

Similar to the first transformer the impedance points renormalized to the second transformer must be rotated. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance points exiting the first transformer and renormalized to the beginning of the transformer which is closest to the transmission line.

$$\lambda_{f_x} = \lambda_{f_r}\frac{f_r}{f_x} = 0.250\frac{1.04}{f_x}.$$

**Calculation of final band in transformer furthest from the load**

| **Frequency (GHz)** | **Initial impedance position (λ), points 1”,2”,3”,4”,5”** | **Wavelength $\lambda_{f_x}$** | **Final impedance position (λ) , points A,B,C,D,E** | **Final Impedance points of transformer closest to the transmission line** |
|---|---|---|---|---|
| 1.00 | 0.0410 | 0.2600 | 0.301 | $\overline{Z}_A = 1.25 - j0.26$ |
| 1.02 | 0.0815 | 0.2549 | 0.336 | $\overline{Z}_B = 1.10 - j0.24$ |
| 1.04 | 0.210 | 0.2500 | 0.46 | $\overline{Z}_C = 0.67 - j0.15$ |
| 1.06 | 0.270 | 0.2453 | 0.015 | $\overline{Z}_D = 0.55 + j0.07$ |
| 1.08 | 0.301 | 0.2407 | 0.0417 | $\overline{Z}_E = 0.57 + j0.18$ |

Figure 99 shows the final second quarter-wave transformer impedance band still in normalized to the second quarter wave transformer. The multiplication of the impedance points by $\overline{Z}_{oB} = 1.1951$ will renormalize them to the transmission line characteristic impedance. The table below shows these renormalized impedance points and Figure 98 shows the final plotted band.
From Figure 98, the VSWR is seen to be 1.6, which is the VSWR specified.

**Renormalization to main transmission line**

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the transmission line** | **Final Impedance points renormalized to transmission line $Z_o$** |
|---|---|---|
| 1.00 | $\overline{Z}_A = 1.25 - j0.26$ | $\overline{Z}_A = 1.49 - j0.31$ |
| 1.02 | $\overline{Z}_B = 1.10 - j0.24$ | $\overline{Z}_B = 1.31 - j0.29$ |
| 1.04 | $\overline{Z}_C = 0.67 - j0.15$ | $\overline{Z}_C = 0.80 - j0.18$ |
| 1.06 | $\overline{Z}_D = 0.55 + j0.07$ | $\overline{Z}_D = 0.66 + j0.08$ |
| 1.08 | $\overline{Z}_E = 0.57 + j0.18$ | $\overline{Z}_E = 0.68 + j0.22$ |

The final circuit is given in Figure 100.

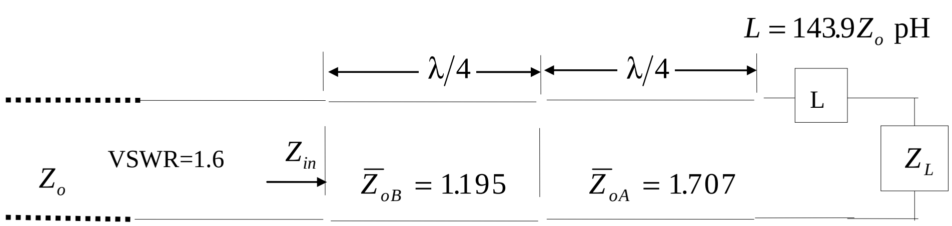



**Figure 100** **Example 22 - Two quarter-wavelength transformer matching problem**

[i] Robert E. Collin., *Foundations for Microwave Engineering,* McGraw-Hill, Inc., Singapore, 1992.

### 4.4.1.5 The use of the various components to match a band of impedances

In this section two problems will posed (Examples 23 and 24) and the various components already described, used to solve them. There may be a better method, than used in this section, to solve these particular problems. However, the methods chosen here will demonstrate when and how to use the available tools and the effect which each has on the preceding impedance band. During the course of the solutions, alternative circuits are discussed and some basic rules given.

#### 4.4.1.5.1 Capacitors and inductors

These components are used to move impedance points in a band of frequencies along circles of constant resistance or conductance. On the Smith chart, series capacitors will move the band in an anti-clockwise direction and series inductors will move the band clockwise. Shunt capacitors will move the band in a clockwise direction and shunt inductors will move the band anti-clockwise.

##### *4.4.1.5.1.1 Series capacitors and inductors*

As a series capacitor or inductor is being used, any admittance band must be converted to an impedance band. This section will consider how an impedance band can be balanced about the X = 0 axis of the Smith chart by using these components.

Capacitors

The addition of a series capacitor would rotate the band impedance points in an anti-clockwise direction along their respective constant resistance circles and in some cases may be used balance the band about the X=0 axis in preparation for the next process.

The amount of negative reactance required to balance the band can be estimated from the end points of an impedance band, say, points D and E where point D is the lower frequency impedance point in the upper half of the Smith chart and point E is the upper frequency in the lower half. The new reactance of points D and E must be equal and opposite. From equation 4-3 , we find

$$X_{c1}(f_1) = X_{c1}(f_2)\frac{f_2}{f_1} \qquad \textbf{(4-3)}$$

Let the reactance point at point E be $X_h\left(f_h\right)$ and at point D be $X_l\left(f_l\right)$, then as the reactances at each point must be equal and opposite in sign,

$$X_l\left(f_l\right) + X_c\left(f_l\right) = -\left(X_h\left(f_h\right) + X_c\left(f_h\right)\right)$$

$$X_l\left(f_l\right) + X_h\left(f_h\right) = -X_c\left(f_h\right) - X_c\left(f_l\right) = -X_c\left(f_l\right)\left(1 + \frac{f_l}{f_h}\right)$$

giving,

$$X_c\left(f_l\right) = -\frac{X_l\left(f_l\right) + X_h\left(f_h\right)}{\left(1 + \dfrac{f_l}{f_h}\right)} \qquad \textbf{(4-45)}$$

Equation 4-45 is the amount of negative reactance required to balance the band.

Inductors

The addition of a series inductor would rotate the band impedance points in a clockwise direction along their respective constant resistance circles and in some cases may be used balance the band about the X=0 axis in preparation for the next process.

The amount of positive reactance required to balance the band can again be estimated from the end points of an impedance band, say, points D and E where point D is the lower frequency impedance point in the upper half of the Smith chart and point E is the upper frequency in the lower half. The new reactance of points D and E must be equal and opposite. From equation 4-10 , we find

$$X_{L1}(f_2) = X_{L1}(f_1)\frac{f_2}{f_1} \qquad \textbf{(4-10)}$$

Let the reactance point at point E be $X_h\left(f_h\right)$ and at point D be $X_l\left(f_l\right)$, then as the reactances at each point must be equal and opposite in sign,

$$X_l\left(f_l\right) + X_L\left(f_l\right) = -\left(X_h\left(f_h\right) + X_L\left(f_h\right)\right)$$

$$X_l\left(f_l\right) + X_h\left(f_h\right) = -X_L\left(f_h\right) - X_L\left(f_l\right) = -X_L\left(f_l\right)\left(1+\frac{f_h}{f_l}\right)$$

giving,

$$X_L\left(f_l\right) = -\frac{X_l\left(f_l\right) + X_h\left(f_h\right)}{\left(1+\frac{f_h}{f_l}\right)} \qquad \textbf{(4-46)}$$

Equation 4-46 is the amount of positive reactance required to balance the band. The negative sign comes about due to the negative reactance being greater than the positive reactance if an inductor is to be used, so that a positive reactance is obtained.

#### *4.4.1.5.1.2 Shunt capacitors and inductors*

If a shunt capacitor or inductor is used, any impedance band must be converted to an admittance band. This section will consider how an admittance band can be balanced about the X = 0 axis of the Smith chart by using these components. The Smith chart rather than the admittance chart is used because of the convenience of using one chart only.

Capacitors

The addition of a shunt capacitor on the Smith chart will rotate the band admittance points in a clockwise direction along their respective constant conductance circles and in some cases may be used balance the band about the X=0 axis in preparation for the next process.

The amount of positive susceptance required to balance the band can again be estimated from the end points of an admittance band, say, points D and E where point D is the lower frequency admittance point in the upper half of the Smith chart and point E is the upper frequency in the lower half. The new susceptance of points D and E must be equal and opposite. From equation 4-17 , we find

$$B_{C1}(f_2) = B_{C1}(f_1)\frac{f_2}{f_1} \qquad \textbf{(4-17)}$$

Let the susceptance point at point E be $B_h\left(f_h\right)$ and at point D be $B_l\left(f_l\right)$, then as the susceptances at each point must be equal and opposite in sign,

$$B_l\left(f_l\right) + B_C\left(f_l\right) = -\left(B_h\left(f_h\right) + B_C\left(f_h\right)\right)$$

$$B_l\left(f_l\right) + B_h\left(f_h\right) = -B_C\left(f_h\right) - B_C\left(f_l\right) = -B_C\left(f_l\right)\left(1+\frac{f_h}{f_l}\right)$$

giving,

$$B_C(f_l) = -\frac{B_l(f_l) + B_h(f_h)}{\left(1 + \frac{f_h}{f_l}\right)} \qquad \textbf{(4-47)}$$

Equation 4-47 is the amount of positive susceptance required to balance the admittance band. The negative sign comes about due to the negative susceptance being greater than the positive susceptance if a capacitor is to be used, so that a positive susceptance is obtained.

Inductors

The addition of a shunt inductor would rotate the band impedance points in an anti-clockwise direction along their respective constant conductance circles and in some cases may be used balance the band about the X=0 axis in preparation for the next process.

The amount of negative susceptance required to balance the band can be estimated from the end points of an admittance band, say, points D and E where point D is the lower frequency admittance point in the upper half of the Smith chart and point E is the upper frequency in the lower half. The new susceptance of points D and E must be equal and opposite. From equation 4-24 , we find

$$B_{L1}(f_1) = B_{L1}(f_2)\frac{f_2}{f_1} \qquad \textbf{(4-24)}$$

Let the susceptance point at point E be $B_h(f_h)$ and at point D be $B_l(f_l)$, then as the susceptances at each point must be equal and opposite in sign,

$$B_l(f_l) + B_L(f_l) = -\left(B_h(f_h) + B_L(f_h)\right)$$

$$B_l(f_l) + B_h(f_h) = -B_L(f_h) - B_L(f_l) = -B_L(f_l)\left(1 + \frac{f_l}{f_h}\right)$$

giving,

$$B_L(f_l) = -\frac{B_l(f_l) + B_h(f_h)}{\left(1 + \frac{f_l}{f_h}\right)} \qquad \textbf{(4-48)}$$

Equation 4-48 is the amount of negative susceptance required to balance the admittance band.

#### 4.4.1.5.2 The series and parallel tuned circuit

##### *4.4.1.5.2.1 The series tuned circuit*

The half-wavelength short-circuit stub and the quarter-wavelength open-circuit stub behaves like a series tuned circuit. If a series tuned circuit was to be considered, then the following analysis would apply.

The impedance $Z_s$, of a series tuned circuit with no series resistance is given by

$$Z_s = j\omega_o L + \frac{1}{j\omega_o C}$$

where L is the inductance, C is the capacitance and $\omega_o = 2\pi f_o$ is the resonant frequency.

At resonance $C = \frac{1}{\omega_o^2 L}$.

At some other value of $\omega_x$, the impedance $Z_x$, is given by

$$Z_x = j\omega_x L\left(1 - \frac{1}{\omega_x^2 LC}\right)$$

However, $C = \dfrac{1}{\omega_o^2 L}$, and the impedance $Z_x$ is a pure reactance which is equal to $jX_x$

so $Z_x = j\omega_x L\left(\dfrac{f_x^{\,2} - f_o^{\,2}}{f_x^{\,2}}\right) = j2\pi L\left(\dfrac{f_x^{\,2} - f_o^{\,2}}{f_x}\right) = \begin{array}{c} jX_x \, for \, f_x > f_o \\ -jX_x \, for \, f_x < f_o \end{array}$

giving,

$$L = \frac{X_x f_x}{2\pi\left(f_x^2 - f_o^2\right)} \qquad \textbf{(4-49)}$$

and

$$C = \frac{1}{2\pi f_x X_x}\left[\left(\frac{f_x}{f_o}\right)^2 - 1\right] \qquad \textbf{(4-50)}$$

Equations 4-49 and 4-50 can be used to determine the value of inductance and capacitance required to provide a reactance of $jX_x$ at a frequency $f_x > f_o$, which will cancel out a reactance of $-jX_x$ at a frequency of $f_x$. Similar to the quarter or half-wavelength stubs, if the lower frequency impedance point has a positive reactance and the higher frequency impedance point has a negative reactance, the series tuned circuit will reduce the band impedances and thus close the mouth of the impedance band.

###### *4.4.1.5.2.2 The parallel tuned circuit*

The half-wavelength open-circuit stub and the quarter-wavelength short-circuit stub behaves like a parallel tuned circuit. If a parallel tuned circuit was to be considered, then the analysis of a parallel tuned circuit with a perfect inductor and capacitor, that is with no equivalent parallel resistance across the tank circuit, would be similar to that of the series tuned circuit, where impedance is replaced by admittance and reactance by susceptance. The relevant equations are,

so $Y_x = j\omega_x L\left(\dfrac{f_x^{\,2} - f_o^{\,2}}{f_x^{\,2}}\right) = j2\pi L\left(\dfrac{f_x^{\,2} - f_o^{\,2}}{f_x}\right) = \begin{array}{c} jB_x \, for \, f_x > f_o \\ -jB_x \, for \, f_x < f_o \end{array}$

giving,

$$L = \frac{B_x f_x}{2\pi\left(f_x^2 - f_o^2\right)} \qquad \textbf{(4-51)}$$

and

$$C = \frac{1}{2\pi f_x B_x}\left[\left(\frac{f_x}{f_o}\right)^2 - 1\right] \qquad \textbf{(4-52)}$$

EXAMPLE 23
Match a transmission line to a VSWR of 1.9 or better, given the following normalized mismatched load measured impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 6.00 | $\overline{Z}_1 = 0.250 - j2.40$ |
| 6.02 | $\overline{Z}_2 = 0.230 - j2.30$ |
| 6.04 | $\overline{Z}_3 = 0.215 - j2.10$ |
| 6.06 | $\overline{Z}_4 = 0.205 - j1.95$ |
| 6.08 | $\overline{Z}_5 = 0.200 - j1.80$ |

SOLUTION
The measured impedance points are plotted on Figure as band " ".

PART 1
The first consideration is to bring the measured impedance band in towards the centre of the Smith chart. This is achieved by using an L-type circuit, as discussed in section 4.3.1.2. The circuit chosen is a series inductor and shunt inductor, and the circuit and the Smith chart procedure for a single frequency is shown in Figure 39.

RULE 1
Each impedance point in a band must lie outside of its smaller special conductance circle, shown in Figure 101, and within the larger conductance circle, or have impedance points which lie on circles of constant reactance which pass into the larger special conductance circle, if matching is to be easily attained. The same applies for admittance bands.

If the impedance points are lying outside of the smaller special conductance circle and within the larger special conductance circle, then the equivalent admittance band points when following circles of constant reactance will enter the VSWR circle.
It usually happens that if a band is not to have any impedance point lying inside the smaller special conductance circle, one or more impedance points in the band may lie outside of the larger special conductance circle. If this happens, then as long as the circles of constant reactance that these impedance points lie on enter the larger conductance circle a matching solution usually can be found. If the impedance points do not lie on circles of constant reactance which enter the larger special conductance circle, then to find a solution the specified VSWR must be increased to accommodate these lines of constant reactance.

Determination of the impedance band centre-frequency
For clarity in the following description, the impedance at the frequency which is the geometric mean of the frequency band will be called the band centre impedance. The method to be used, if rule 1 is not to be broken, is to move the impedance band along the larger constant resistance circle using an inductor, until the band centre impedance intersects with a special constant conductance circle. This constant conductance circle is constructed with the aid of the specified VSWR circle. When the band centre admittance $\overline{Y}_{Lf_o}$, moves from the intersection of this special constant conductance circle with its constant resistance circle, along the conductance circle it comes to a point on the B=0 axis. This point is the outer diameter of the VSWR circle. Figure 102, is a diagram which may be used to visualize the process just described.
The reason for placing the band centre admittance at the extremity of the VSWR circle, is that at this point the admittance is a pure conductance only and is within specifications. The further processing which may be required is mainly to close the extremities of the band in such a way that these extremities fall within the VSWR circle, without moving the band centre frequency admittance (or impedance).
Comparing this to the single frequency case in section 2.6.1, the strategy in this case is matching the band centre frequency impedance to the circumference, or just inside, the given VSWR circle rather than to the centre of the Smith chart, so that it maximizes the area in the VSWR circle to accommodate the remainder

of the band. If the centre of the chart was to be used there would only be half of the area of the VSWR circle to accommodate the band.
The band centre frequency, or the geometric mean of the band frequencies is given by

$$f_o = \sqrt{f_L f_H} \qquad \textbf{(4-53)}$$

RULE 2
The band centre frequency is chosen using equation 4-53.

If the frequency chosen is different due to other constraints, then after further processing the band will be lopsided and some of the points may not enter the specified VSWR circle.

From the table of frequencies given in the Example 23, the band centre frequency is found to be,

$$f_o = \sqrt{f_L f_H} = \sqrt{6.00x6.08} \approx 6.04\ MHz\,.$$

The value of VSWR we are going to use is 1.7. This is tighter than the specified VSWR of 1.9, so that we can see the effects of the 6 GHz impedance in impedance band “ ” contravening ule . e are aiming high and expect to fall short.

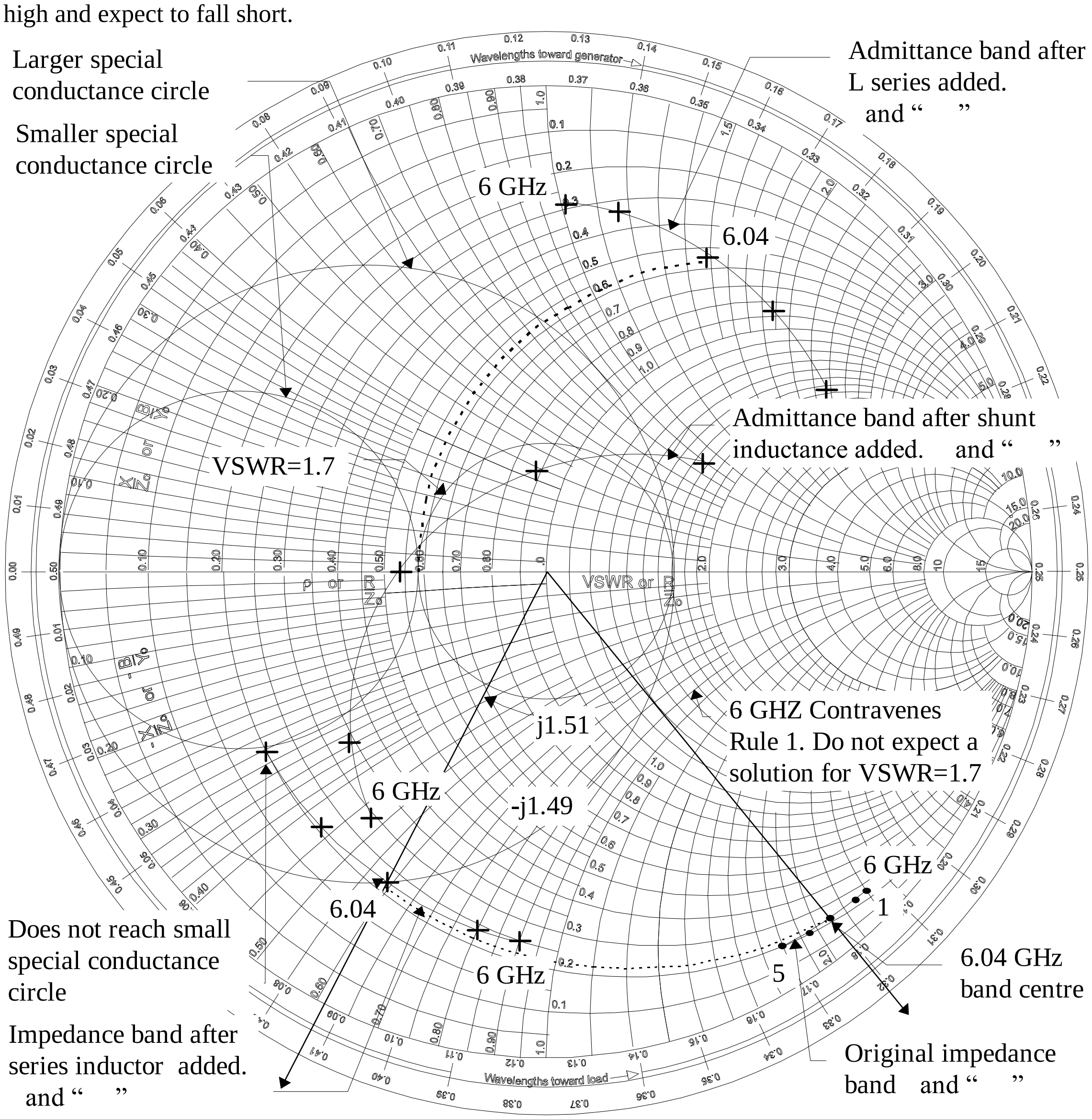



**Figure 101** **Example 23 - Part 1 - L-type circuit implementation**

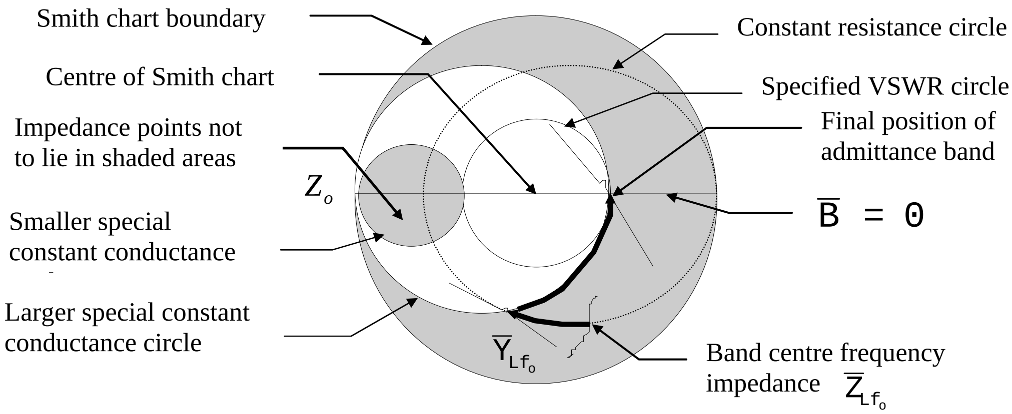



**Figure 102 Example 23 - Part 1 - Procedure using in matching problem**

The value of $\Delta\overline{X}_{Lf_o}$ is found from subtracting the 6.04 GHz impedance in the original band with the impedance found from the intersection of the constant resistance circle of this impedance with the large special conductance circle. This is estimated to be,

$$\Delta\overline{X}_{Lf_o} = \overline{Z}_{int\,er\,sec\,tion} - \overline{Z}_{6.04(orig)} = 1.51$$

where $\overline{Z}_{6.04(orig)} = 0.21 - j2.10$ and $\overline{Z}_{int\,er\,sec\,tion} = 0.21 - j0.59$

As the relationship of the other band reactances to $\Delta\overline{X}_{Lf_o}$ is given by equation 4.10. We find,

$$\overline{X}_L(f) = \Delta\overline{X}_{Lf_o}\frac{f}{f_o} = 1.51\frac{f}{6.04} = 0.25f$$

**Movement of ban s i s in t n t n , i**

| **Frequency (GHz)** | **Initial impedance** | **Addition of series reactance** | **Impedance points after series inductive reactance added** | **Admittance points at n , i** |
|---|---|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.250 - j2.40$ | j1.500 | $\overline{Z}_1 = 0.250 - j0.900$ | $\overline{Y}_1 = 0.287 + j1.032$ |
| 6.02 | $\overline{Z}_2 = 0.230 - j2.30$ | j1.505 | $\overline{Z}_2 = 0.230 - j0.795$ | $\overline{Y}_2 = 0.336 + j1.161$ |
| 6.04 | $\overline{Z}_3 = 0.215 - j2.10$ | j1.510 | $\overline{Z}_3 = 0.215 - j0.590$ | $\overline{Y}_3 = 0.545 + j1.496$ |
| 6.06 | $\overline{Z}_4 = 0.205 - j1.95$ | j1.515 | $\overline{Z}_4 = 0.205 - j0.435$ | $\overline{Y}_4 = 0.886 + j1.881$ |
| 6.08 | $\overline{Z}_5 = 0.200 - j1.80$ | j1.520 | $\overline{Z}_5 = 0.200 - j0.280$ | $\overline{Y}_5 = 1.689 + j2.365$ |

From $\Delta\overline{X}_{Lf_o} = 2\pi f_o L = 1.51$, the value of series inductance is found to be 39.79 $Z_o$ pH

As the centre frequency admittance point (6.04 GHz) is to cut the B=0 axis, the distance traversed around the special conductance circle is estimated from the Smith chart or table above, as (0 - j1.496) = -j1.496. Equation 4-24 provides the means of modifying the centre frequency susceptance to give the outer band susceptance values. That is,

$$B_L(f) = -j1.496\frac{6.04}{f} = -j\frac{9.036}{f}$$, so

**v nt n s nt in t n t n , i**

| **Freq. (GHz)** | **Initial admittance n , i** | **Addition of susceptance** | **Admittance points after inductive susceptance ( n )** | **Equivalent impedance ints t n , Fig. 103** |
|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 0.287 + j1.032$ | -j1.5060 | $\overline{Y}_1 = 0.287 - j0.474$ | $\overline{Z}_1 = 0.935 + j1.544$ |
| 6.02 | $\overline{Y}_2 = 0.336 + j1.161$ | -j1.5010 | $\overline{Y}_2 = 0.336 - j0.340$ | $\overline{Z}_2 = 1.470 + j1.488$ |
| 6.04 | $\overline{Y}_3 = 0.545 + j1.496$ | -j1.4960 | $\overline{Y}_3 = 0.545$ | $\overline{Z}_3 = 1.835$ |
| 6.06 | $\overline{Y}_4 = 0.886 + j1.881$ | -j1.4911 | $\overline{Y}_4 = 0.886 + j0.390$ | $\overline{Z}_4 = 0.945 - j0.416$ |
| 6.08 | $\overline{Y}_5 = 1.689 + j2.365$ | -j1.4862 | $\overline{Y}_5 = 1.689 + j0.879$ | $\overline{Z}_5 = 0.466 - j0.242$ |

As inductive susceptance at the centre frequency is given by

$\overline{B}_L\left(f_o\right) = \frac{1}{j\omega L} = -j\frac{1}{2\pi f_o L} = -j1.496$, the value of shunt inductance is found to be 17.61 $Z_o$ pH.

n Figure , note the band blooming of band " " with centre frequency at the edge of the VSWR circle after the L-type circuit has been added. Using L-type circuits, it is usually possible to place the band where you want it to go, but at the expense of an increase in the size of the band.

PART 2

The next part of the solution is concerned with rotating the band " " along circles of constant , using a length of main transmission line between the L-type circuit and the next component. In other words, a length of transmission line will rotate the band around the centre of the chart. The amount of rotation depends on the length of the main transmission line.

As the impedance or admittance points each have their own VSWR, each follows its own VSWR circle, in a clockwise direction, on the Smith chart as the distance from the load increases. Each impedance or admittance point does not return to its original value each half-wavelength traversed from the load, that is, one revolution around the Smith chart, because each of the wavelengths associated with a particular impedance point frequency scaled to the Smith chart wavelength has a different value. This means that as the distance from the load increases, the impedance or admittance band is continually changing shape and may never become the same shape again as that seen at the load. The Smith chart wavelength, although normally not given a number value, is selected when the centre frequency is chosen. It is used in much the same way as the normalized impedance is used. Equation 4-34 provides the means of determining the value of each scaled wavelength with reference to the Smith chart wavelength $\lambda_o$.

he rotation of the impedance band along it s impedance points constant circles would permit the 6.08 GHz point, which lies on a 0.466 circle of constant resistance, which is outside of the specified VSWR circle, to be brought onto a constant resistance circle which passes through the specified VSWR circle. f care is not taken, moving the Hz point along it s circle too far, will take it outside of the specified circle. Noting the wavelength marker of the Hz point on band " ", Figure 3 as 0.175λ, and then moving the 6 GHz impedance point in a clock-wise direction around the chart, that is "towards the generator" on it s circle until it crosses the . circle. t this intersection the wavelength reading is 0.345λ. However, this point places the low frequency in the lower part of the Smith chart and makes it difficult for further processing to close the band s mouth. further .3 3λ must be traversed to bring the Hz impedance point into the upper region of the mith chart where it s circle cuts the 0.6 constant resistance circle. Therefore the distance that the 6 GHZ point has moved is 0.652λ - 0.175λ = 0.477λ. As the 6 GHz frequency impedance point has been used as the reference for determining the impedance change and thus, the wavelength change, for scaling purposes all other frequencies will be based on the 6 GHz frequency and 0.477λ.

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.477\frac{6}{f_x} = \frac{2.862}{f_x}$$

v nt n t ns issi n in t n , i

| Freq. (GHz) | Initial wavelength (λ) on Smith chart | Change in wavelength (λ) on Smith chart | Final wavelength (λ) on Smith chart | Equivalent impedance points t n , Fig. 103 |
|---|---|---|---|---|
| 6.00 | 0.175 | 0.477 | 0.152 | $\overline{Z}_1 = 0.62 + j1.21$ |
| 6.02 | 0.192 | 0.475 | 0.167 | $\overline{Z}_2 = 0.95 + j1.26$ |
| 6.04 | 0.250 | 0.474 | 0.224 | $\overline{Z}_3 = 1.73 + j0.33$ |
| 6.06 | 0.366 | 0.472 | 0.338 | $\overline{Z}_4 = 1.11 - j0.42$ |
| 6.08 | 0.451 | 0.471 | 0.422 | $\overline{Z}_5 = 0.54 - j0.41$ |

he band " " which is the band after the rotation from band " " is shown on Figure 3. The length of the main transmission line used is 0.224λ at the centre frequency of 6.04 GHz.

PART 3

he band " " shows that all i edance oints lie on the constant resistance circles etween . and 1.73 and with the proper changes in reactance, can be bought into the specified VSWR circle. As the 6.0 GHz impedance point lies in the upper semicircle of the Smith chart a series half-wavelength short-circuit stub will be used. A series quarter-wavelength open-circuit stub could have been used, but it is easier to practically realize a short-circuit than an open-circuit. Also at 6 GHz, the length of a half-wavelength stub is not a problem.

From equations 4-34 and 4-35, for a half-wavelength stub,

$$\overline{Z} = \frac{Z}{Z_o} = \frac{Z_{ch}}{Z_o}\, j\, tan\frac{2\pi d}{\lambda} \qquad \textbf{(4-35)}$$

$$\lambda_{f_x} = \lambda_o\frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

the impedance looking into the series stub from the transmission line side, is given by,

$$\overline{Z} = \frac{Z}{Z_o} = \frac{Z_{ch}}{Z_o}\, j\, tan\frac{2\pi\frac{\lambda_o}{2}}{\lambda_o\frac{f_o}{f_x}} = \frac{Z_{ch}}{Z_o}\, j\, tan\frac{\pi f_x}{f_o} \qquad \textbf{(4-54)}$$

Choosing somewhere near the mid point where the 6 GHz and 6.08 GHz impedance points come close to meeting on the same circle of constant reactance, and allowing some overlap between them, we find a change in reactance , $|\Delta\overline{X}| = 0.864$ to be satisfactory.

This means that the reactance of the 6 GHz impedance will come from j1.21 to approximately j0.35, and the reactance of the 6.08 GHz impedance will rise clockwise from -j0.41 to approximately j0.46.

The value of -j0.864 will be used in equation 4-54 to determine the value of the characteristic impedance of the stub $\overline{Z}_{ch}$ .

Taking the 6 GHz point as the impedance point which moves, from equation 4-54, we find

$$\overline{Z} = \frac{Z_{ch}}{Z_o} j \tan \frac{\pi f_x}{f_o} = -j0.864 = \frac{Z_{ch}}{Z_o} j \tan \frac{\pi 6}{6.04} = -\frac{Z_{ch}}{Z_o} j0.0208$$

from which $\overline{Z}_{ch} = 41.53$.

Thus, equation 4-54 becomes, $\overline{Z} = 41.53 j \tan 0.5201 f_x$. Below is a table of impedances before the short-circuit half-wavelength stub and after it. Figure 103 shows the resulting band after the stub is serially inserted.

| Column 1 | 2 | 3 (Eqn. 4-54) | 4 |
|---|---|---|---|
| **Frequency (GHz)** $f_x$ | **i in n impedances Fig. 103** | $\overline{Z} = 41.53 j \tan 0.5201 f_x$ | **in n impedances Fig. 103 Col. (2 + 3)** |
| 6.00 | 0.62 + j1.21 | -j0.8643 | 0.62 + j0.35 |
| 6.02 | 0.95 + j1.26 | -j0.4321 | 0.95 + j0.83 |
| 6.04 ( $f_o$ ) | 1.73 + j0.33 | 0 | 1.73 + j0.33 |
| 6.06 | 1.11 - j0.42 | +j0.4321 | 1.11 + j0.01 |
| 6.08 | 0.54 - j0.41 | +j0.8643 | 0.54 + j0.45 |

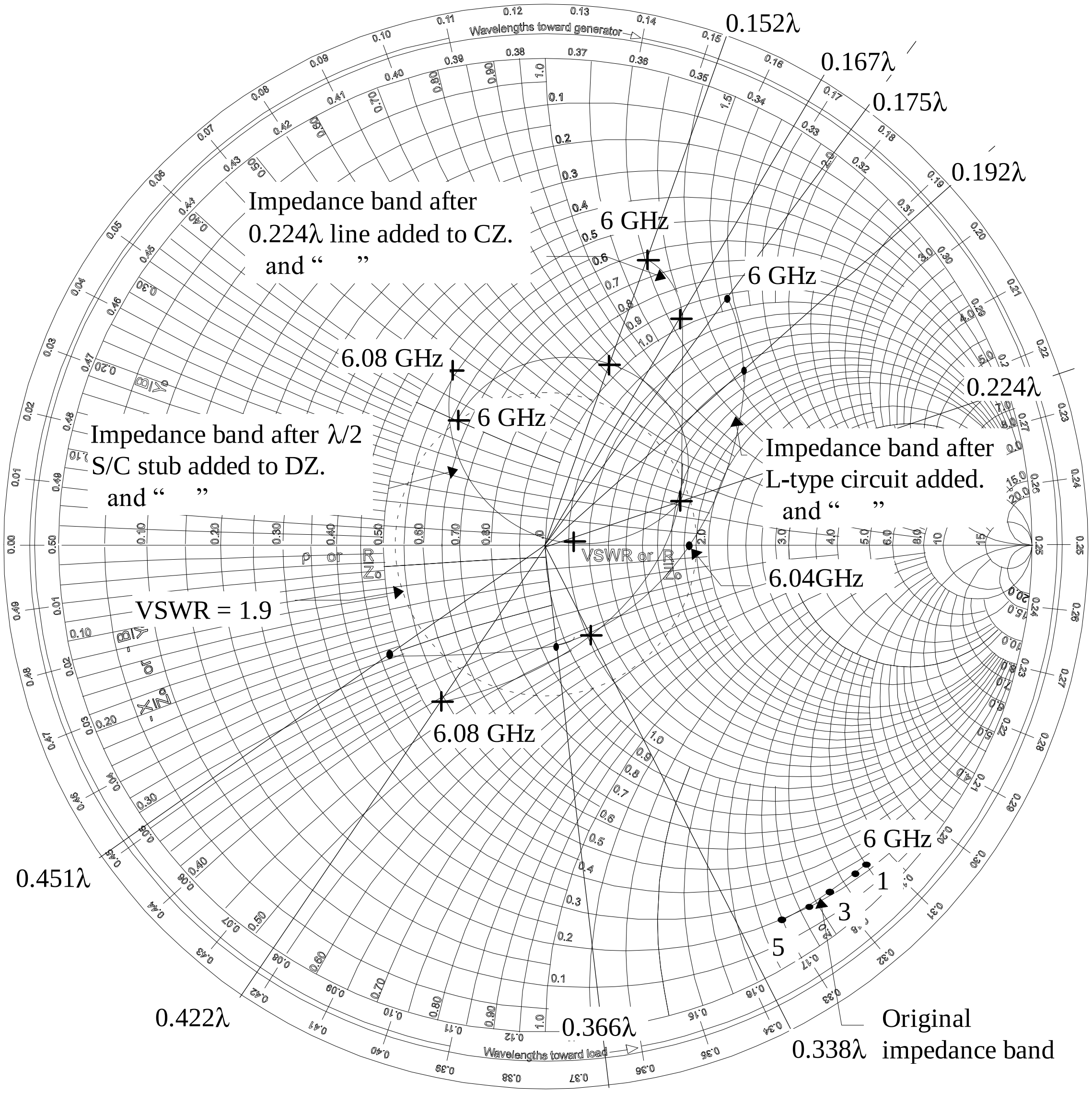



**Figure 103 Example 23 - Part 2 and 3 -The use of a length of transmission line and s/c stub**

**An alternative to the half-wavelength short-circuit stub**

Before proceeding to the last part of the solution, an alternative solution to the use of the half-wavelength short-circuit series stub is the series tuned circuit, described in 4.4.1.5.2.1. From equations 4-49 and 4-50, the value of inductance L, and capacitance C, required to provide a reactance of $-j\overline{X}_x = -j0.864$ at a frequency of $f_x = 6GHz$, given that the centre frequency $f_o = 6.04GHz$, is, L = 1.713 $Z_o$ nH, C = 0.405 $Z_o$ pF.

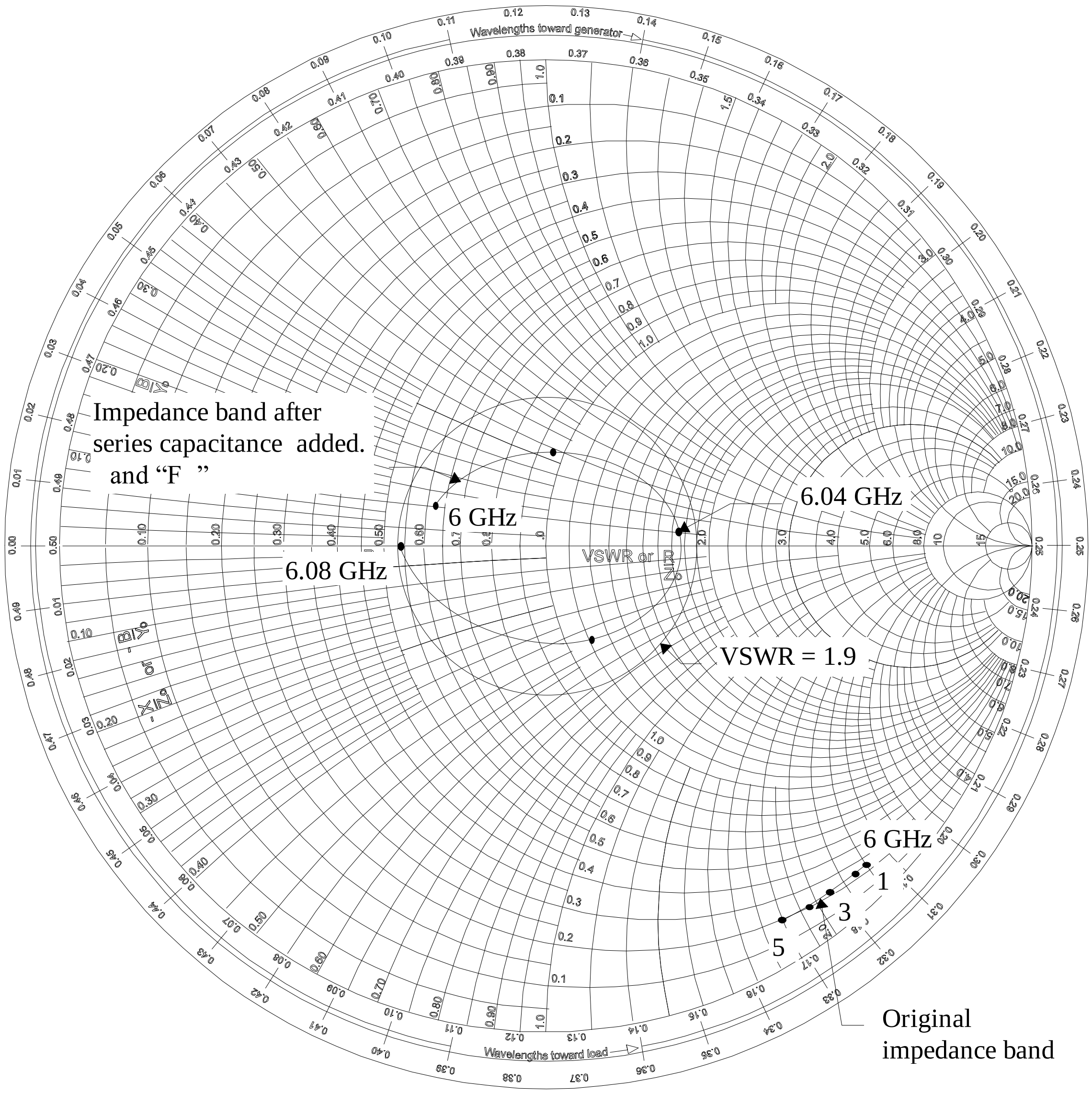



**Figure 104** **Example 23 -Part 4 - The use of series capacitance**

PART 4

onsidering the impedance band “ after the half-wavelength short-circuit stub has been added, as shown in Figure 103. The addition of a series capacitor would rotate the band impedance points in an anti-clockwise direction along their respective constant resistance circles. The extent of the rotation is determined by the most critical impedance point in the band, which in this case is the 6.08 GHz impedance point. This point is critical because it determines the final VSWR more than any other point in the band. If this point is rotated anti-clockwise to the X = 0 line of the Smith chart, all of the other points will enter the specified VSWR circle, and the VSWR of the band is at a minimum. Therefore, taking a reactance of - . at . Hz, which will cancel the positive reactance of the . Hz impedance point in band “ ” , namely, $\overline{Z}_{EZ6.08GHz} = 0.54 + j0.45$, we find from equation 4-3, the reactances at the other frequencies.

$$X_c(f_x) = X_c(f_o)\frac{f_o}{f_x} = -0.45\frac{6.08}{f_x} = -\frac{2.736}{f_x}$$

**iti n s i s it t i n n**

| **Frequency (GHz)** | **in n impedances Fig. 103** | **Addition of reactance** $-\frac{2.736}{f_x}$ | **Impedance points after capacitive reactance added ( n , i )** |
|---|---|---|---|
| 6.00 | 0.62 + j0.35 | -j0.4560 | 0.62 - j0.11 |
| 6.02 | 0.95 + j0.83 | -j0.4545 | 0.95 + j0.38 |
| 6.04 | 1.73 + j0.33 | -j0.4530 | 1.73 - j0.12 |
| 6.06 | 1.11 + j0.01 | -j0.4515 | 1.11 - j0.44 |
| 6.08 | 0.54 + j0.45 | -j0.4500 | 0.54 |

The value of the series capacitance is found from,

$$\overline{X}_c\left(f_o\right) = -0.45 = -\frac{1}{2\pi f_o C} = -\frac{10^{-9}}{2\pi 6.08C}$$, which gives C = 58.171 $Z_o$ pF.

Figure shows the final band “F ” together with the specified circle of . . s can be seen the impedance band lies inside of the specified VSWR circle.

Note that in this example, apart from the L-type circuit, only series circuits and thus impedances were used. This is not usually the case. Usually, there is a mixture of shunt and series, that is, admittance and impedance type circuits. Another point worth noting, is that as the number of components increases, the accuracy of the impedance or admittance points decreases. This is due to the difficulty of accurately determining the admittances or impedances from the Smith chart for each process. Apart from economic reasons, the least number of components to place a band within a specified VSWR the more accurate the design will be. The value of VSWR = 1.7 was not obtained as expected, due to the contravention of Rule 1. Had the VSWR been drawn as 1.9 on Figure 103, it would have shown that a solution was possible using Rule 1.
The final circuit is shown in Figure 105.

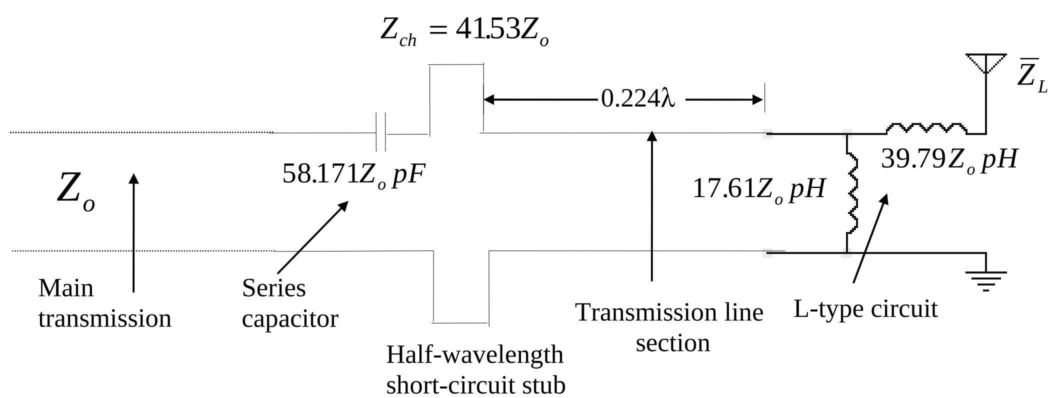



**Figure 105** **Components used in Example 23**

RULE 3
When plotted on a Smith chart, the normalized impedance or admittance load bands must have a clockwise direction with increasing frequency if a matching solution is to be found.

#### 4.4.1.5.3 Summary of short-circuit and open-circuit quarter and half-wavelength stubs for matching

The various types of stubs are fairly commonplace in matching problems. Because of this, below is presented in table form the various types of stubs and the values of admittance and impedance at the low and high frequencies, as well as the type of resonant circuit (series or parallel). Given are the relevant equations and the equation numbers where applicable. It can be noticed that the quarter-wavelength short-circuit admittance is equivalent to the half-wavelength open-circuit admittance in terms of the sign values and infinity at resonance, except that the half-wavelength stub will produce a greater change for a given angle difference. In other words, the half-wavelength stub has a higher “ ” factor. For example, cot tan 181, where for cot 91 there is a one degree change in 90 degrees and for tan 181 there is a one degree change in 180 degrees. Similarly, the impedance for a half-wavelength short-circuit stub is equivalent in terms of the sign values of the impedance at the different frequencies as the quarter-wavelength open-circuit stub. However, the impedance change for a given angle change is greater for the half-wavelength stub than for the quarter-wavelength stub.

### *4.4.1.5.3.1 short-circuit stub*

| | λ/4 | | λ/2 | |
|---|---|---|---|---|
| Freq. | Impedance | Admittance | Impedance | Admittance |
| | | Eqn. 4-37 | Eqn. 4-54 | |
| es t | Parallel | Parallel | Series | Series |
| | $\overline{Z} = j\frac{Z_{ch}}{Z_o}\tan\frac{\pi}{2}\frac{f_x}{f_o}$ | $\overline{Y} = -j\frac{Z_o}{Z_{ch}}\cot\frac{\pi}{2}\frac{f_x}{f_o}$ | $\overline{Z} = j\frac{Z_{ch}}{Z_o}\tan\frac{\pi f_x}{f_o}$ | $\overline{Y} = -j\frac{Z_o}{Z_{ch}}\cot\frac{\pi f_x}{f_o}$ |
| $f_L$ | + | – | – | + |
| $f_o$ | ∞ | 0 | 0 | ∞ |
| $f_H$ | – | + | + | – |

### *4.4.1.5.3.2 Open-circuit stub*

| | λ/4 | | λ/2 | |
|---|---|---|---|---|
| Freq. | Impedance | Admittance | Impedance | Admittance |
| | Eqn. 4-39 | | | |
| es t | Series | Series | Parallel | Parallel |
| | $\overline{Z} = -j\frac{Z_{ch}}{Z_o}\cot\frac{\pi}{2}\frac{f_x}{f_o}$ | $\overline{Y} = j\frac{Z_o}{Z_{ch}}\tan\frac{\pi}{2}\frac{f_x}{f_o}$ | $\overline{Z} = -j\frac{Z_{ch}}{Z_o}\cot\frac{\pi f_x}{f_o}$ | $\overline{Y} = j\frac{Z_o}{Z_{ch}}\tan\frac{\pi f_x}{f_o}$ |
| $f_L$ | – | + | + | – |
| $f_o$ | 0 | ∞ | ∞ | 0 |
| $f_H$ | + | – | – | + |

EXAMPLE 24

Match a transmission line to a VSWR of 1.9 or better, given the following normalized mismatched load measured impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 6.00 | $\overline{Z}_1 = 0.250 + j1.80$ |
| 6.02 | $\overline{Z}_2 = 0.230 + j1.95$ |
| 6.04 | $\overline{Z}_3 = 0.215 + j2.15$ |
| 6.06 | $\overline{Z}_4 = 0.205 + j2.30$ |
| 6.08 | $\overline{Z}_5 = 0.200 + j2.40$ |

SOLUTION

he measured impedance points are plotted on Figure as band “ ”.

PART 1
The first consideration is to bring the measured impedance band in towards the centre of the Smith chart. This is achieved by using an L-type circuit, as discussed in section 4.3.1.2. The circuit chosen is a series capacitor and shunt capacitor, and the circuit and the Smith chart procedure for a single frequency is shown in Figure 38.
From the table of frequencies given in the Example 23, the band centre frequency is found to be,

$$f_o = \sqrt{f_L f_H} = \sqrt{6.00x6.08} \approx 6.04\ MHz\,.$$

The value of VSWR we are going to use for this example is the specified VSWR of 1.9. This is so that we can make a comparison with the aiming high and expecting to fall short philosophy of Example 23 and seeing the effects of complying with Rule 1.
Rather than move the band using a series capacitance until the centre frequency impedance point intersects the large special conductance circle, the 6 GHz impedance point of the band is moved close to the boundary of the smaller special conductance circle. In order to prevent Rule 1 being broken, the band cannot enter into the smaller special conductance circle. It should be noted that the further away from the small special conductance circle the 6 GHz is placed the better. However, too far away would make the 6.08 GHz point not lie on a circle of constant reactance which enters the larger conductance circle. One solution is shown on Figure 106. The placement of the band as shown indicates that the specified VSWR of 1.9 should just be attained.
The value of $\Delta\overline{X}_{Cf_o}$ is found from subtracting the 6.00 GHz impedance in the original band with the impedance found near the intersection point of the smaller of the special constant reactance circles.
This is estimated to be,

$$\Delta\overline{X}_{Cf_o} = \overline{Z}_{near-intersection} - \overline{Z}_{6.00(orig)} = -1.48$$

where $\overline{Z}_{6.00(orig)} = 0.25 + j1.80$ and $\overline{Z}_{near-intersection} = 0.25 + j0.32$

As the relationship of the other band reactances to $\Delta\overline{X}_{Cf_o}$ is given by equation 4.3. We find,

$$\overline{X}_C(f) = \Delta\overline{X}_{Cf_o}\frac{f_o}{f} = -1.48\frac{6.00}{f} = -\frac{8.88}{f}$$

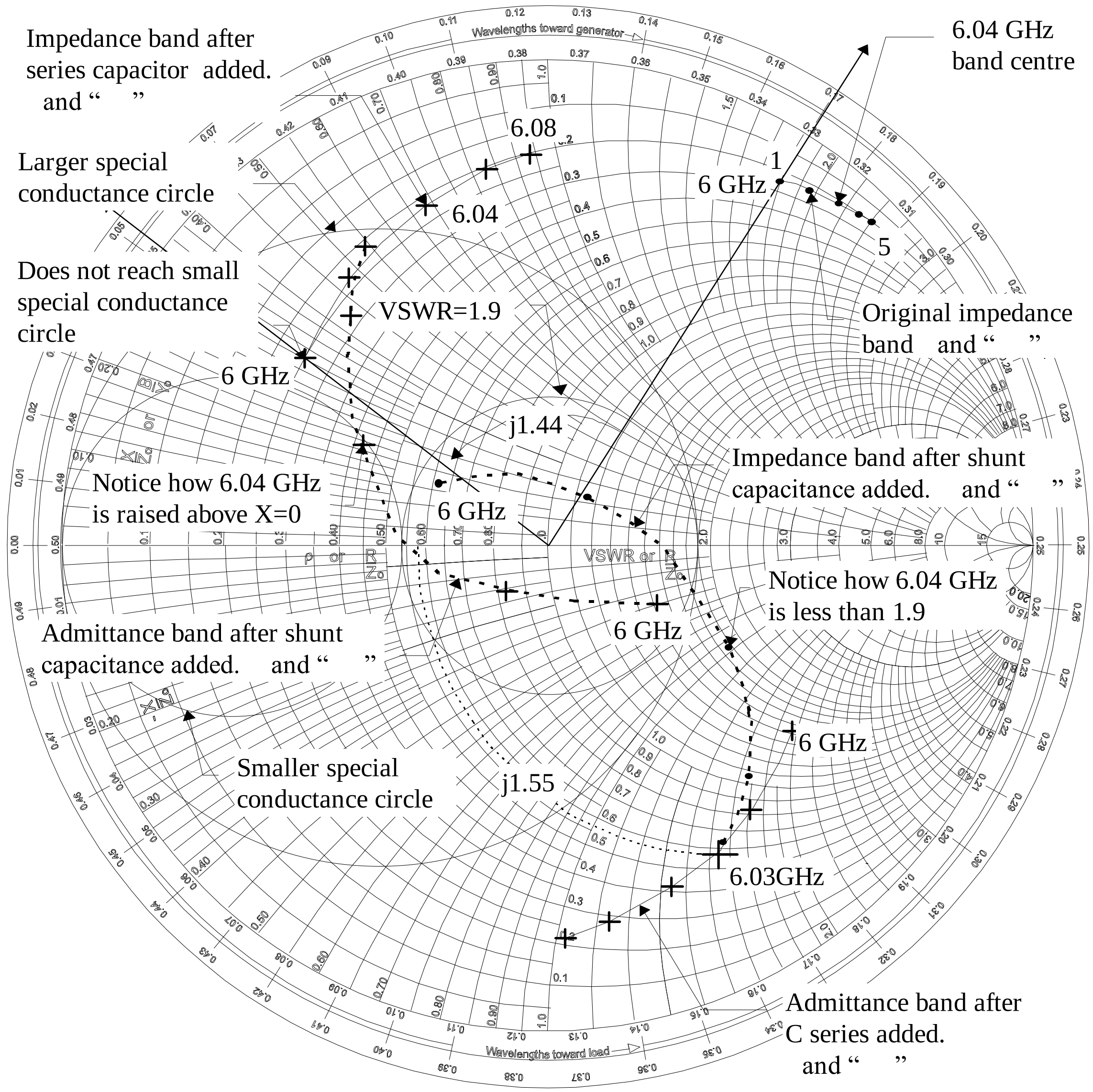



**Figure 106** **Example 24 - Part 1 - L-type circuit implementation**

**v nt n s i s it n t n , i**

| **Frequency (GHz)** | **Initial impedance** | **Addition of series reactance** | **Impedance points after series inductive reactance added** | **Admittance points at n , i** |
|---|---|---|---|---|
| 6.00 | $\bar{Z}_1 = 0.250 + j1.80$ | -j1.4800 | $\bar{Z}_1 = 0.250 + j0.320$ | $\bar{Y}_1 = 1.516 - j1.941$ |
| 6.02 | $\bar{Z}_2 = 0.230 + j1.95$ | -j1.4751 | $\bar{Z}_2 = 0.230 + j0.475$ | $\bar{Y}_2 = 0.826 - j1.705$ |
| 6.04 | $\bar{Z}_3 = 0.215 + j2.15$ | -j1.4702 | $\bar{Z}_3 = 0.215 + j0.680$ | $\bar{Y}_3 = 0.423 - j1.337$ |
| 6.06 | $\bar{Z}_4 = 0.205 + j2.30$ | -j1.4653 | $\bar{Z}_4 = 0.205 + j0.835$ | $\bar{Y}_4 = 0.277 - j1.130$ |
| 6.08 | $\bar{Z}_5 = 0.200 + j2.40$ | -j1.4605 | $\bar{Z}_5 = 0.200 + j0.940$ | $\bar{Y}_5 = 0.217 - j1.018$ |

From $\Delta\overline{X}_{Cf_o} = \dfrac{1}{2\pi f_o C} = 1.48$, the value of series capacitance is found to be 17.923 $Z_o$ pF

Considering the admittance band “ ”, shown on Figure . llowing the admittance point which lies on the 0.57 constant conductance circle, and which has an estimated frequency of 6.03 GHz to move to the B=0 axis should place all of the final impedance on circles of constant resistance which pass through the VSWR circle. The determination of the choice of admittance is made by considering the admittance points conductance. Those admittances which have conductances outside of the VSWR circle should be raised well above the X=0 axis in order to allow their impedance points to enter into the VSWR circle. From band “ ” on Figure , it can be seen that the centre fre uency admittance point has a conductance whose conductance circle is outside of the VSWR circle. The 6.03 GHz admittance point however, has an estimated conductance of 0.59, which lies within the 0.526 boundary conductance circle of the VSWR circle. Choosing the 6.03 GHz admittance point to cut the X=0 axis will raise the rotated 6.04 GHz admittance above the axis and allow it s impedance point to lie on a resistance circle which passes through the VSWR circle.

To estimate accurately the admittance point in the band and its frequency which can rotate on the special admittance circle is difficult and may not always bring other impedance band points into the final VSWR circle. It is easier, and usually more effective, to choose an admittance and its frequency which lies on a conductance circle which enters into the VSWR circle, such as done by choosing the 6.03 GHz admittance point (0.59 - . . t is at this uncture it can be noticed that the use of the mith chart becomes an “ rt” rather than a science.

The distance traversed by the susceptance around the conductance circle by the chosen admittance point (0.59 - j1.55) is estimated to be (0 - [-j1.550]) = j1.550 at an estimated frequency of 6.03 GHz..

Knowing the estimated frequency equation 4-24 provides the means of modifying this susceptance to give the other susceptance values in the band. That is,

$$B_C\left(f\right) = 1.550\frac{f}{6.03} = 0.2570f$$ , so

**v nt n s nt in t n t n , i**

| Freq. (GHz) | Admittance points at n , i | Addition of susceptance | Admittance points after inductive susceptance ( n ) | Equivalent impedance ints t n , Fig. 106 |
|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 1.516 - j1.941$ | j1.5423 | $\overline{Y}_1 = 1.516 - j0.399$ | $\overline{Z}_1 = 0.617 + j0.162$ |
| 6.02 | $\overline{Y}_2 = 0.826 - j1.705$ | j1.5474 | $\overline{Y}_2 = 0.826 - j0.158$ | $\overline{Z}_2 = 1.168 + j0.223$ |
| 6.04 | $\overline{Y}_3 = 0.423 - j1.337$ | j1.5526 | $\overline{Y}_3 = 0.423 + j0.216$ | $\overline{Z}_3 = 1.875 - j0.958$ |
| 6.06 | $\overline{Y}_4 = 0.277 - j1.130$ | j1.5577 | $\overline{Y}_4 = 0.277 + j0.428$ | $\overline{Z}_4 = 1.066 - j1.647$ |
| 6.08 | $\overline{Y}_5 = 0.217 - j1.018$ | j1.5629 | $\overline{Y}_5 = 0.217 + j0.545$ | $\overline{Z}_5 = 0.631 - j1.584$ |

Figure 106 shows that all of the impedances in band “ ” lie on circles of constant resistance which pass through the specified VSWR circle. This means that closing the band mouth using a quarter-wavelength stub will possibly bring all of the impedance points within the specified VSWR circle.

As capacitive susceptance at the frequency of 6.03 GHz is given by

$\overline{B}_C\left(f_o\right) = j\omega C = j2\pi f_{6.03}C = j1.55$, the value of shunt capacitance is found to be 40.910 $Z_o$ pF.

PART 2

his part is concerned with closing the “mouth” of band “ ” or band “ ”. o do this a stub or a tuned circuit may be used. For this example a series tuned circuit will be used. However, if a stub were to be used, then, referring to section 4.4.1.3, the lower frequency of the band must be in the upper half of the Smith chart. From Figure , it can be noticed that only the impedance band “ ” has its lower frequency impedance in the upper half of the Smith chart. Therefore, a series stub would be used with zero impedance at the chosen resonant frequency of 6.03 GHz. The length of the stub is determined from section 4.4.1.5.3, as either a half-wavelength short-circuit or a quarter-wavelength open-circuit stub.

Because, in practice, it is easier to implement a short-circuit stub, a half-wavelength stub normally would be chosen.

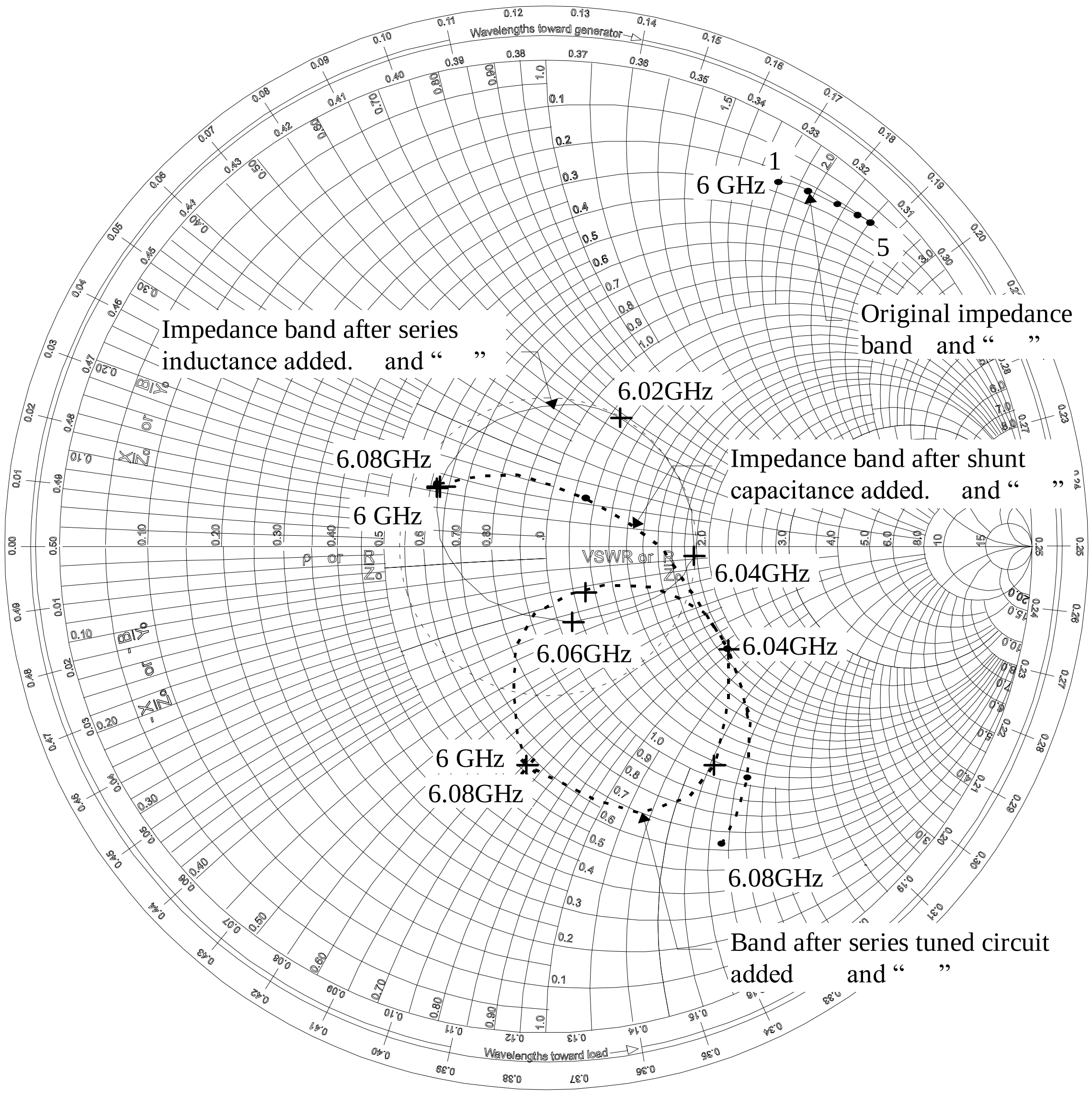



**Figure 107 Example 24 - Part 2 and 3 - Stub and inductance implementation**

The theory behind the series tuned circuit for matching purposes was given in section 4.4.1.5.2.1. If the band “ ” was closed, there would still be a problem because some of the band impedance points would still lie outside of the VSWR circle. The closed band would have to be rotated clockwise along circles of constant resistance using a series inductor, in order to bring the closed band within the specified VSWR circle. Part 3 will determine the amount of series inductance required to rotate the band. An alternative approach is to balance the band using a series inductor and the theory outlined in section 4.4.1.5.1.1 and then use the series tuned circuit to close the band “mouth”.

As there is a total of two series inductors involved for both processes, the two circuits will fuse and become a single series inductor and capacitor. This is a component count advantage over using the half-wavelength short-circuit stub.

Equations 4-49 and 4-50 can be used to determine the value of inductance and capacitance required to provide a reactance of $jX_x$ at a frequency $f_x > f_o$, which will cancel out a reactance of $-jX_x$ at a frequency of $f_x$.

The object is to close the band mouth completely. To do this the reactance at the extremes of the band, that is, 6 GHz (+j0.162) must be changed to a value x, where

x = j0.162 - $j\overline{X}_x$

and 6.08 GHz(-j1.584) must also be changed to the same value x, where

x = -j1.584 + $j\overline{X}_x$

giving,

$$j\overline{X}_x = j\frac{0.162-(-1.584)}{2} = j0.873 \ .$$

giving x = -j0.7110.

The frequency $f_o$, is the centre frequency of the impedance band, that is $f_o = \sqrt{6x6.08}$

At $f_x$ = 6.08 GHz

$$L = \frac{X_x f_x}{2\pi\left(f_x^2 - f_o^2\right)} \qquad \textbf{(4-49)}$$

$$L = 1.7368 Z_o nH$$

and

$$C = \frac{1}{2\pi f_x X_x}\left[\left(\frac{f_x}{f_o}\right)^2 - 1\right] \qquad \textbf{(4-50)}$$

$$C = 0.3998 Z_o pF$$

The reactance is given by,

$$Z_x = j\omega_x L\left(\frac{f_x^2 - f_o^2}{f_x^2}\right) = j2\pi L\left(\frac{f_x^2 - f_o^2}{f_x}\right) = \begin{matrix} jX_x \, for \, f_x > f_o \\ -jX_x \, for \, f_x < f_o \end{matrix}$$

That is,

so $\overline{X}_x = j10.9126\left(\frac{f_x^2 - 36.4800}{f_x}\right)$

he modified impedances due to the addition of this reactance to the band “ ” are,

**sin n s i s t n i it t iv n , i**

| **Freq. (GHz)** | **Equivalent impedance ints t n , Fig. 107** | **Addition of reactance $\overline{X}_x$** | **Equivalent impedance ints t n , Fig. 107** |
|---|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.617 + j0.162$ | -j0.8730 | $\overline{Z}_1 = 0.617 - j0.711$ |
| 6.02 | $\overline{Z}_2 = 1.168 + j0.223$ | -j0.4343 | $\overline{Z}_2 = 1.168 - j0.211$ |
| 6.04 | $\overline{Z}_3 = 1.875 - j0.958$ | +0.0029 | $\overline{Z}_3 = 1.875 - j0.955$ |
| 6.06 | $\overline{Z}_4 = 1.066 - j1.647$ | +j0.4387 | $\overline{Z}_4 = 1.066 - j1.208$ |
| 6.08 | $\overline{Z}_5 = 0.631 - j1.584$ | +j0.8730 | $\overline{Z}_5 = 0.631 - j0.711$ |

he closed band “ ” is shown on Figure .

PART 3

This part is concerned with using a series inductor to rotate band “ ” on Figure into the specified VSWR circle.

The critical impedance point is the 6.04 GHz point, as this must move into the circle near the perimeter of the circle. It would have to move from -j0.995 to -j0.12 or through a reactance of j0.875. The 6.02 GHz point can move to the edge of the VSWR circle from -j0.211 to +j0.66, a distance of j0.871. Taking account of the slight increase in inductive reactance with frequency, the choice is to take the 6.02 GHz point as the reference as it s distance is slightly less than that re uired by the . Hz point. he remaining points are not so critical, because, for example, the 6.08 GHz would move onto the X=0 axis a distance of 0.711 and can move a distance ranging from j0.36 up to j1.06.

The value of inductive reactance required is j0.871 at 6.02 GHz. Below is a table of the values of inductive reactance at the other frequencies using equation 4-10 and values of the new impedance points after this reactance has been added.

$$\overline{X}_L(f_x) = \overline{X}_L(f_o)\frac{f_x}{f_o} = j0.871\frac{f_x}{6.02} = 0.1447 f_x$$

**t tin n s i s in t t iv n , i**

| Freq. (GHz) | Equivalent impedance ints t n , Fig. 107 | Addition of reactance $\overline{X}_L(f_x)$ | Equivalent impedance ints t n , Fig. 107 |
|---|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.617 - j0.711$ | +j0.8681 | $\overline{Z}_1 = 0.617 + j0.157$ |
| 6.02 | $\overline{Z}_2 = 1.168 - j0.211$ | +j0.8710 | $\overline{Z}_2 = 1.168 + j0.660$ |
| 6.04 | $\overline{Z}_3 = 1.875 - j0.955$ | +0.8739 | $\overline{Z}_3 = 1.875 - j0.081$ |
| 6.06 | $\overline{Z}_4 = 1.066 - j1.208$ | +j0.8768 | $\overline{Z}_4 = 1.066 - j0.331$ |
| 6.08 | $\overline{Z}_5 = 0.631 - j0.711$ | +j0.8797 | $\overline{Z}_5 = 0.631 + j0.1687$ |

he final band “ ” is shown plotted in Figure and as can be seen, lies within the specified circle of 1.9.

From $\overline{X}_L = 2\pi f\overline{L} = 0.871 = 2\pi(6.02x10^9)\overline{L}$, the value of $L = 23.0272 Z_o pH$. Adding this to the series resonant inductance $L = 1736.8 Z_o pH$, the total series inductance becomes

$L_{tot} = 1,759.8 Z_o pH$

Figure 108 shows the final circuit.

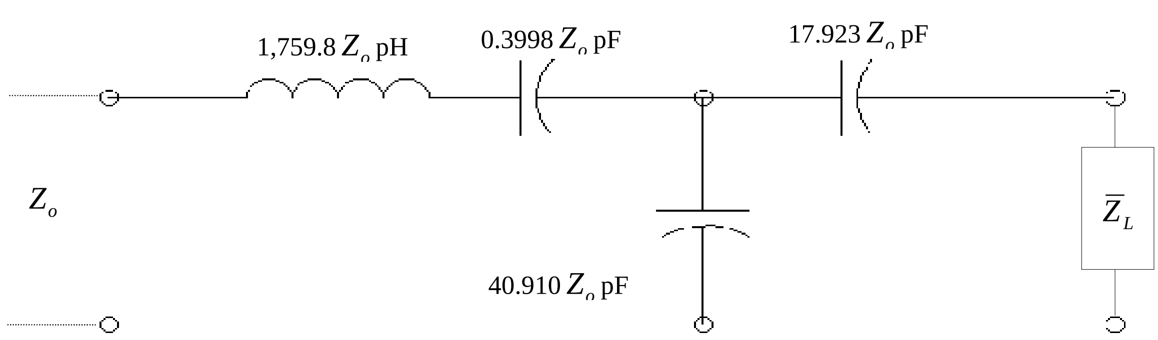



**Figure 108 Final circuit for Example 24**

### 4.4.1.6 The use of the single, double and triple stubs in matching

#### 4.4.1.6.1 Single stub matching

The procedure used in single stub matching is similar to that used for the single frequency case, given in section 3.3.2.1. The difference is, the centre frequency of the admittance band does not have to be rotated until it reaches the unit circle. The centre frequency is rotated onto a constant conductance circle which lies in the left hand semi-circle of the specified VSWR circle. This is done, in order to permit the band extremities to be closed, by further processing, without the band extremities lying on conductance circles which are outside of the specified VSWR circle. An example will be given in which the impedance band of Example 24 is used.

EXAMPLE 25
Match a transmission line to a VSWR of 1.9 or better, using a single shunt stub and any other components required, given the following normalized mismatched load measured impedance points,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 6.00 | $\overline{Z}_1 = 0.250 + j1.80$ |
| 6.02 | $\overline{Z}_2 = 0.230 + j1.95$ |
| 6.04 | $\overline{Z}_3 = 0.215 + j2.15$ |
| 6.06 | $\overline{Z}_4 = 0.205 + j2.30$ |
| 6.08 | $\overline{Z}_5 = 0.200 + j2.40$ |

SOLUTION
he circle is plotted on Figure together with the measured impedance points band “ ” .

PART 1 - Determining values of the single stub
After converting the impedance band into an admittance band, because shunt stubs are involved, and plotting on Figure as band “ ”, the band is rotated clockwise about constant circles, as we move along the transmission line, away from the load. The amount of rotation towards the generator of the admittance band “ ” is determined by the intersection points of the admittance band circles and the circles of constant conductance which pass through the specified VSWR circle.

**Conversion of load impedance to load admittance**

| Frequency (GHz) | Normalized load impedance | Normalized load admittance |
|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.250 + j1.80$ | $\overline{Y}_1 = 0.076 - j0.545$ |
| 6.02 | $\overline{Z}_2 = 0.230 + j1.95$ | $\overline{Y}_2 = 0.060 - j0.506$ |
| 6.04 | $\overline{Z}_3 = 0.215 + j2.15$ | $\overline{Y}_3 = 0.046 - j0.461$ |
| 6.06 | $\overline{Z}_4 = 0.205 + j2.30$ | $\overline{Y}_4 = 0.038 - j0.431$ |
| 6.08 | $\overline{Z}_5 = 0.200 + j2.40$ | $\overline{Y}_5 = 0.034 - j0.414$ |

From the table of frequencies given in the Example 24, the band centre frequency is found to be,

$$f_o = \sqrt{f_L f_H} = \sqrt{6.00x6.08} \approx 6.04\ MHz\,.$$

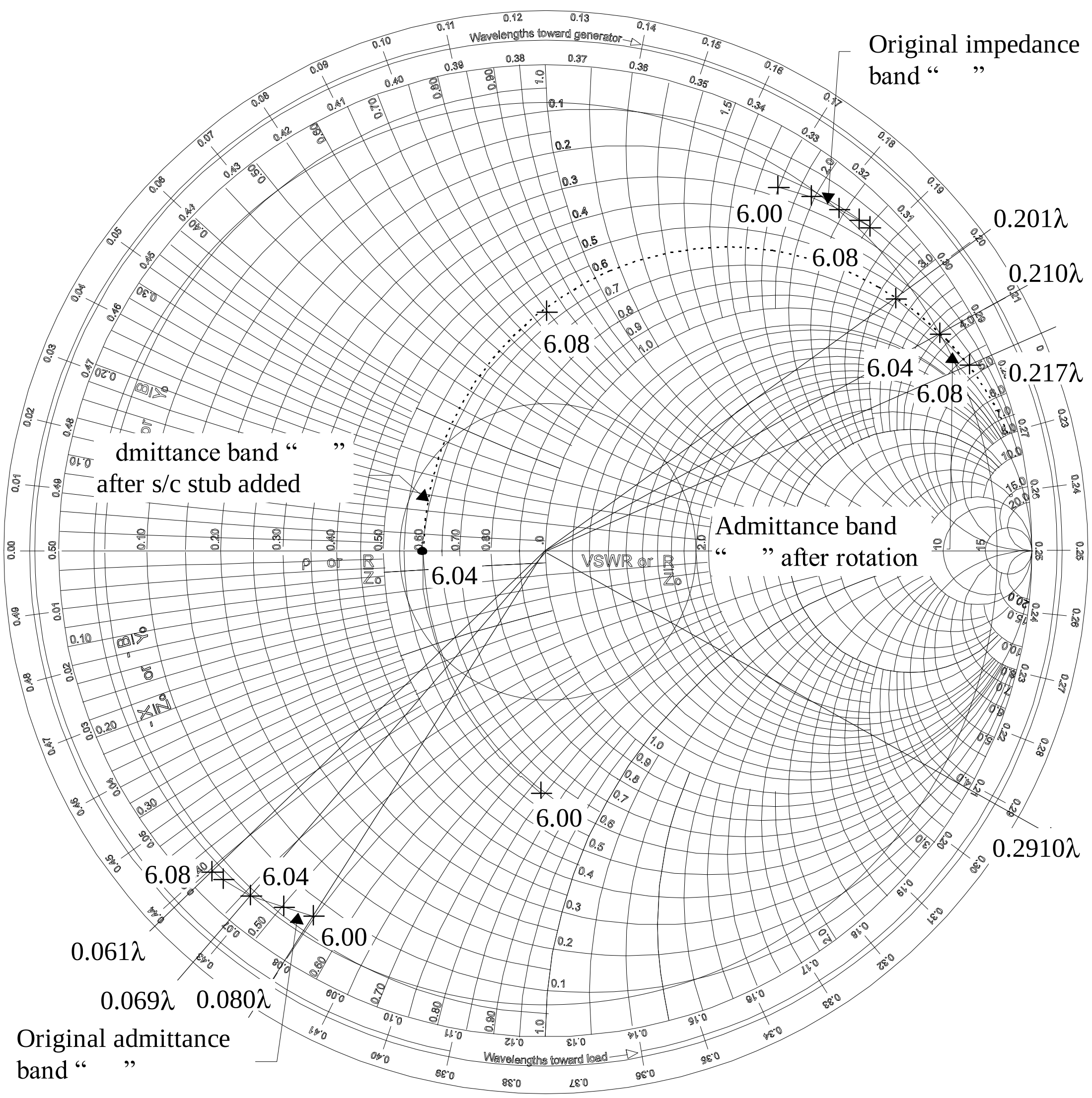



**Figure 109 Example 25 Single stub matching of an impedance band**

The centre frequency is rotated onto the 0.6 conductance circle, in order to keep it close to the left-hand edge of the specified VSWR circle, and away from the high values of reactance. Rotating into the high values of reactance will lead to high inaccuracies when the band is next rotated into the specified VSWR circle.

The amount of rotation is (0.069 + 0.210) λ = 0.279λ. The length of transmission line between the load and the first stub is 0.279λ.

Using equation 4-34, the remaining rotated band admittance points can be calculated.

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.279 \frac{6.04}{f_x} = \frac{1.685}{f_x}$$

**v nt n t ns issi n in t n , i**

| **Freq. (GHz)** | **Initial wavelength (λ) on Smith chart** | **Change in wavelength (λ) on Smith chart** | **Final wavelength (λ) on Smith chart** | **Equivalent admittance points t n , Fig. 109** |
|---|---|---|---|---|
| 6.00 | 0.0800 | 0.2809 | 0.2009 | $\overline{Y}_1 = 0.62 + j3.00$ |
| 6.02 | 0.0745 | 0.2799 | 0.2054 | - |
| 6.04 | 0.0690 | 0.2790 | 0.2100 | $\overline{Y}_3 = 0.60 + j3.80$ |
| 6.06 | 0.0620 | 0.2781 | 0.2161 | - |
| 6.08 | 0.0605 | 0.2772 | 0.2167 | $\overline{Y}_5 = 0.59 + j4.60$ |

The length of the first stub is determined by the susceptance required to cancel the susceptive component of the centre frequency admittance point $\overline{Y}_3$. That is -j3.80. The distance between the short circuit admittance point on the Smith chart and the -j3.80 susceptance point is measured from the Smith chart as, (0.291 - 0.25)λ = 0.041λ. Thus , the length of the short-circuit stub is 0.041λ at the centre frequency of 6.04 GHz.
The equivalent lengths of the admittances at the other frequencies are determined by equation 4-34 as,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.041\frac{6.04}{f_x} = \frac{0.24764}{f_x}$$

**v nt n s t-circuit stub to b n , i**

| **Freq. (GHz)** | **Equivalent admittance points at n , Figure 109** | **Susceptance wavelength (λ) (actual)** | **Susceptance wavelength (λ) on Smith chart** | **Equivalent susceptance** | **Admittance band t susceptance added Figure 109** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 0.62 + j3.00$ | 0.0413 | 0.2913 | -j3.79 | $\overline{Y}_1 = 0.62 - j0.79$ |
| 6.04 | $\overline{Y}_3 = 0.60 + j3.80$ | 0.0410 | 0.2910 | -j3.80 | $\overline{Y}_3 = 0.60$ |
| 6.08 | $\overline{Y}_5 = 0.59 + j4.60$ | 0.0407 | 0.2907 | -j3.81 | $\overline{Y}_5 = 0.59 + j0.79$ |

PART 2 - losing the “mouth” of band “ ”
From section 4.4.1.5.3, a half-wavelength series stub will add a negative reactance to the lower frequencies and add a positive reactance to the higher frequencies. In this case, the impedance band will be reduced to nearly a line on the real axis, as the extremities of the band are evenly distributed about the real axis (X=0 axis) and the extremity band resistances are separated from the centre frequency resistance ( $\overline{R}_{6.04} = 1.667$ ).
Figure shows the admittance band “ ”, plotted as an impedance band “ ”.
From equation 4-54

$$\overline{Z} = \frac{Z}{Z_o} = \frac{Z_{ch}}{Z_o} j \tan\frac{\pi f_x}{f_o} \qquad \textbf{(4-54)}$$

Choosing the centre band frequency, 6.04GHz, the impedance points come close to meeting on the same circle of constant reactance (j0.80). A change in reactance , $|\Delta\overline{X}| = 0.800$ should produce a satifactory match.
This means that the reactance of the 6 GHz impedance will come from j0.78 to approximately -j0.02, and the reactance of the 6.08 GHz impedance will rise clockwise from -j0.81 to approximately -j0.1.

The value of -j0.800 will be used in equation 4-54 to determine the value of the characteristic impedance of the stub $\overline{Z}_{ch}$ .
Taking the 6 GHz point as the impedance point which moves, from equation 4-54, we find

$$\overline{Z} = \frac{Z_{ch}}{Z_o} j \tan \frac{\pi f_x}{f_o} = -j0.800 = \frac{Z_{ch}}{Z_o} j \tan \frac{\pi 6}{6.04} = -\frac{Z_{ch}}{Z_o} j 0.0208$$

from which $\overline{Z}_{ch} = 38.4615$.

Thus, equation 4-54 becomes, $\overline{Z} = 38.4615 j \tan 3.121 f_x$. Below is a table of impedances before the short-circuit half-wavelength stub and after it.
Figure 111 shows the resulting band after the stub is serially inserted. As expected the impedance band becomes a line along the real axis, between 0.62 and 1.67. The band VSWR is 1.67, which lies within the specified VSWR of 1.9. The VSWR could be more improved by rotating the original admittance band “ ”, shown in Figure , until the centre frequency was closer to the unity resistance circle, that is band “ ” is rotated further clockwise. However, the band would be crowded into the high reactance area and small errors in this area would be magnified when the band is rotated to the real axis. The band centre of band “ ” would be shifted closer to the centre of the chart, but the extremities of the band would not easily be determined. In principle it is possible to match the band, in this case, to a point at the centre of the chart.

**Mov nt n s i s -wavelength short- i it st t n**

| **Freq. (GHz)** | **Admittance band t susceptance added Figure 109** | **Equivalent impedance points at n , Figure 111** | **Addition of reactance** $\overline{Z} = 38.46 j \tan 3.121 f_x$ | **Final Ban impedances Fig. 111 Col. (3 + 4)** |
|---|---|---|---|---|
| Col.1 | 2 | 3 | 4 | 5 |
| 6.00 | $\overline{Y}_1 = 0.62 - j0.79$ | $\overline{Z}_1 = 0.62 + j0.78$ | -j0.8003 | 0.62 - j0.02 |
| 6.04 | $\overline{Y}_3 = 0.60$ | $\overline{Z}_3 = 1.67$ | 0 | 1.67 |
| 6.08 | $\overline{Y}_5 = 0.59 + j0.79$ | $\overline{Z}_5 = 0.61 - j0.81$ | +j0.8003 | 0.61 - j0.01 |

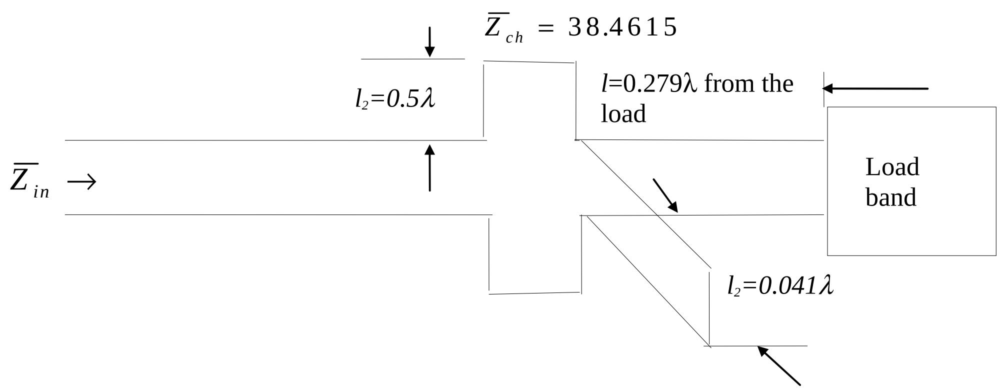



**Figure 110 Example 25 Final schematic of single stub matching network**

The closer the original band is to the perimeter of the Smith chart, the more difficult it is to match. From this Rule 4 can be developed, that is,

RULE 4
Matching cannot be achieved for a pure reactance or susceptance. That is a band which lies on the perimeter of the Smith chart cannot be matched. The closer a band lies to the perimeter of the Smith chart, the more difficult it is to find a matching solution.

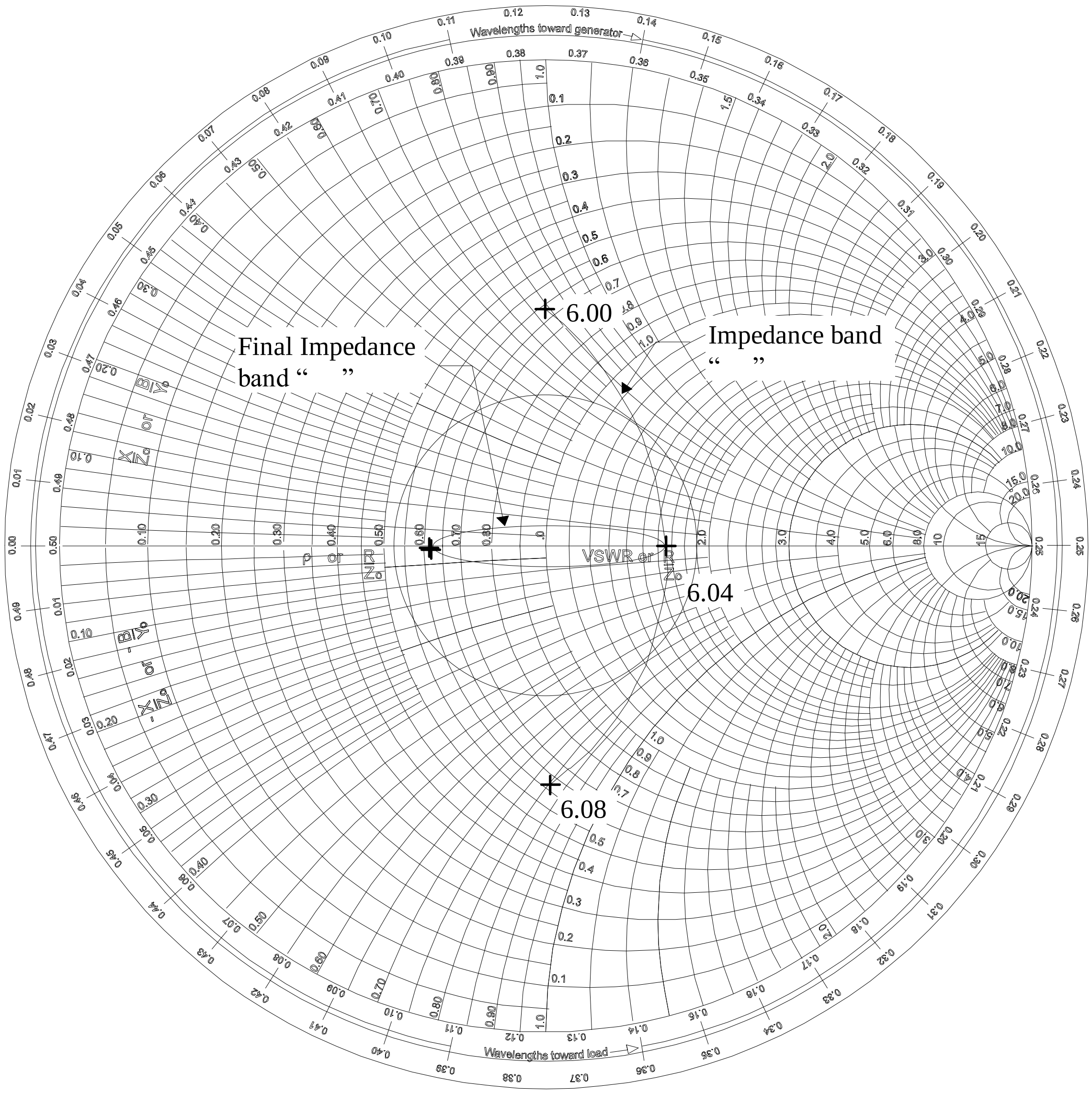



**Figure 111 Example 25 Addition of series half- v n t st t n**

#### 4.4.1.6.2 Double stub matching

Section 3.1.3.2.2 considered the case of the double stub in matching a single frequency to the centre of the Smith chart. In this section, the double stub is used to match an impedance band to a specified VSWR. At the higher microwave frequencies due to the smallness of the wavelength, it is preferred that the distance between the stubs be fixed at 0.375λ, rather than the closer 0.125λ. The use of a separation of 0.5λ is not recommended due to the networks being frequency sensitive. Figure 112a) shows the 0.375λ fixed separation between the stubs, and the variable distance xλ of the first stub from the impedance band of frequencies.

If a VSWR circle is drawn on the Smith chart, then the circles of constant resistance which are drawn tangent to this circle, that is, the large and small VSWR boundary circles, are the circles which the band has to fit between, in the second last stage of the matching processes. This is similar to the single frequency case where the unity constant resistance circle was the circle which was used to perform the last processing stage of the matching to the centre of the chart.

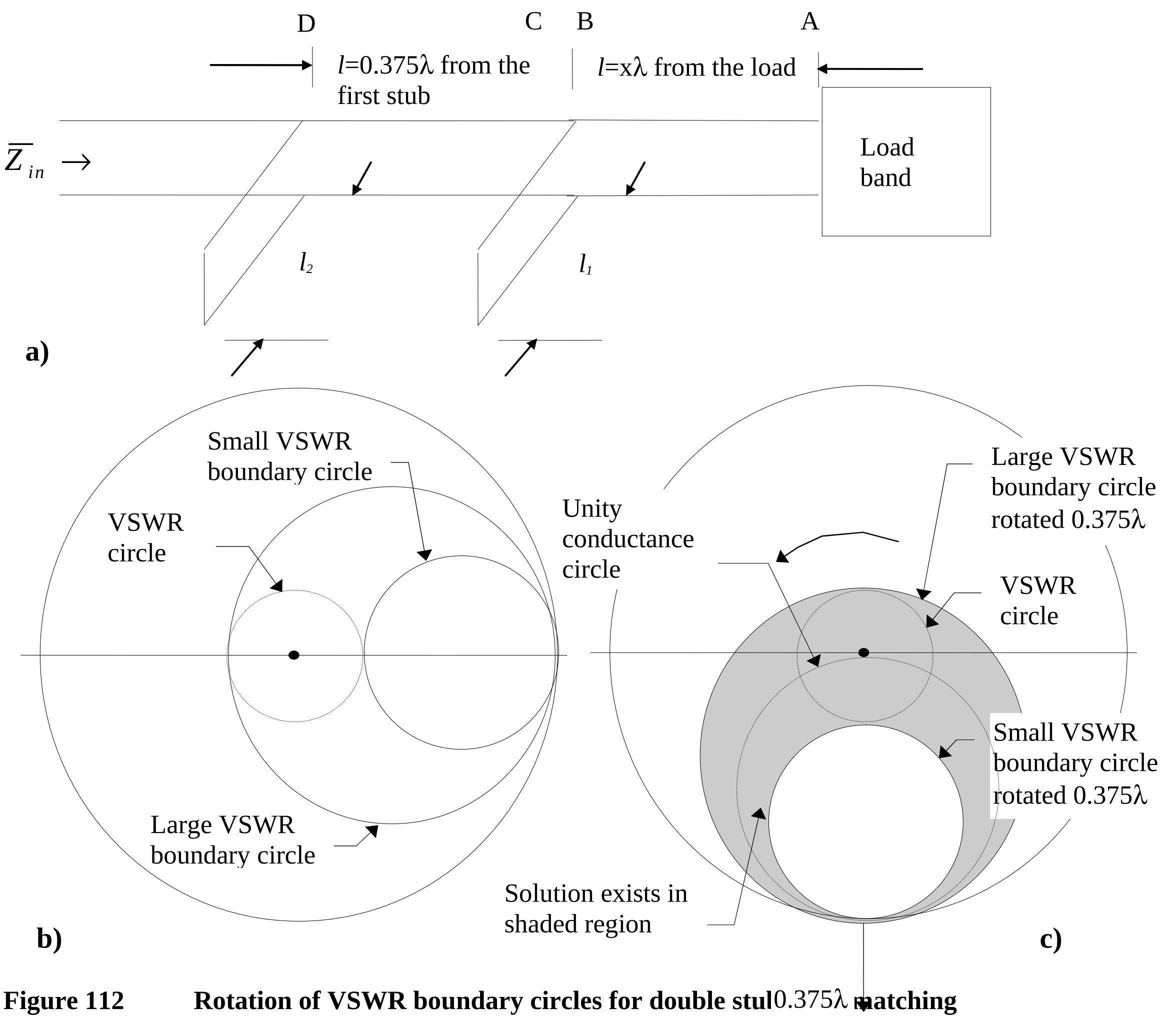



**Figure 112 Rotation of VSWR boundary circles for double stub matching**
**a) Schematic of double stub b) VSWR and boundary circles**
**c) Rotated boundary circles.**

A procedure for matching to a specified VSWR using the double stub is,

1. Draw the specified VSWR circle on the Smith chart
2. Draw the boundary constant resistance circles tangent to the VSWR circle. Figure 112b) shows how the boundary circles are formed.
3. Rotate in an anti-clockwise direction the boundary circles and unity conductance circle about the centre of the chart, through an angle which is the distance between the stubs. This permits the band to be placed correctly between the boundary circles (step 6) in readiness for step 8.
4. Pre-process the original impedance band so that it becomes an admittance band which is placed in a position which is suitable for the further processing steps given below.
5. Place the first stub at a distance xλ, from the load admittance band or the pre-processed admittance band (step 4). The first stub, in practice, cannot be placed directly across the original load admittance band or perhaps, the pre-processed original band, so it is placed at some distance xλ from it. The band admittance is rotated along circles of constant VSWR to a new band with one of the band frequencies rotated through xλ, and the others according to equation 4-34. It is important to ensure that the band lies outside of and below the forbidden region, shown on Figure 113 and, if possible, away from the crowded reactance region so that reasonable accuracy is obtained reading admittance values from the Smith chart. The centre frequency should lie on a constant conductance circle which is close to the centre of the chart, and if possible to the left of centre of the Smith chart.

6. Move the centre frequency of the rotated admittance band, determined in step 5, clockwise along its circle of constant conductance, until it is placed into position on the unity conductance circle and within the specified VSWR circle. The amount of susceptance between the admittance value of the centre frequency in its band position (step 5) and the intersection with the unity conductance circle must be cancelled by the short-circuit stub.
7. Plot the negative of the value of susceptance determined in step 6 on the perimeter of the Smith chart. Determine the length of the first stub $l_1$ by measuring the angle in wavelengths between the short-circuit admittance point (the right-hand side of the chart where $\overline{Y} \to \infty$ ) and the angle at the susceptance point. This angle is used in equation 4-34 to determine the equivalent length of the stub at the different frequencies. For each of the different frequencies, plot the equivalent length of the stub and determine its susceptance from the perimeter of the chart. Add each value of susceptance found to the admittance band determined in step 5. This will produce an admittance band at the point where the first stub is inserted. Figure 112c) shows the region between the boundary circles where the new admittance points of the band are to lie if a solution is to be found.
8. Rotate clockwise the boundary circles back to their original positions, together with the band from step 7, which is contained between them, again using equation 4-34 for the different values of frequencies in the band. That is, the boundary circles have moved anticlockwise, secured the band and returned to their original position with the band in place. The centre frequency of the band will now lie on the unity conductance circle ready to be rotated to the centre of the chart.
9. The second stub is determined by the value of susceptance that must be added to the centre frequency admittance of the admittance band determined in step 8 so that the band is balanced about the real axis. Plot the negative of the value of susceptance required to bring the centre frequency admittance up towards the centre of the chart, on the perimeter of the Smith chart. Determine the length of the second stub $l_2$ by measuring the angle in wavelengths between the short-circuit admittance point (the right-hand side of the chart where $\overline{Y} \to \infty$ ) and the angle at the perimeter susceptance point. This angle is used in equation 4-34 to determine the equivalent length of the stub at the different frequencies. For each of the different frequencies, plot the equivalent length of the stub and determine its susceptance from the perimeter of the chart. Add each value of susceptance found to the admittance band determined in step 8. This will produce an admittance band at the point where the second stub is inserted.
10. lose the "mouth" of the admittance band found in step so that all band points are enclosed by the specified VSWR circle. To do this, use a quarter-wavelength short-circuit stub and equation 4-37.

EXAMPLE 26

Match a transmission line to a VSWR of 1.9 or better, using a double stub and any other components required, given the following normalized mismatched load measured impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 6.00 | $\overline{Z}_1 = 0.076 - j0.545$ |
| 6.02 | $\overline{Z}_2 = 0.060 - j0.506$ |
| 6.04 | $\overline{Z}_3 = 0.046 - j0.461$ |
| 6.06 | $\overline{Z}_4 = 0.038 - j0.431$ |
| 6.08 | $\overline{Z}_5 = 0.034 - j0.414$ |

SOLUTION

he measured impedance points and admittance are plotted on Figure 3 as band " " and " ".

PART 1
This part will determine the admittance band to the first stub, and will determine the value of the length of the first stub.

**Conversion of load impedance to load admittance**

| **Frequency (GHz)** | **Normalized load impedance** | **Normalized load admittance** |
|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.076 - j0.545$ | $\overline{Y}_1 = 0.250 + j1.80$ |
| 6.02 | $\overline{Z}_2 = 0.060 - j0.506$ | $\overline{Y}_2 = 0.230 + j1.95$ |
| 6.04 | $\overline{Z}_3 = 0.046 - j0.461$ | $\overline{Y}_3 = 0.215 + j2.15$ |
| 6.06 | $\overline{Z}_4 = 0.038 - j0.431$ | $\overline{Y}_4 = 0.205 + j2.30$ |
| 6.08 | $\overline{Z}_5 = 0.034 - j0.414$ | $\overline{Y}_5 = 0.200 + j2.40$ |

From the table of frequencies given in the Example 24, the band centre frequency is found to be,

$$f_o = \sqrt{f_L f_H} = \sqrt{6.00x6.08} \approx 6.04\ MHz\,.$$

The 6 GHz frequency admittance point is rotated onto the 1.8 conductance circle, in order to keep it inside the specified VSWR circle of 1.9 and away from the high values of reactance. Rotating into the high values of reactance will lead to high inaccuracies when the band is next rotated into the specified VSWR circle.
The amount of rotation is (0.2782-0.170) λ = 0.1082λ. The length of transmission line between the load and the first stub is 0.1082λ.
Using equation 4-34, the remaining rotated band admittance points can be calculated.

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \tag{4-34}$$

$$\lambda_{f_x} = 0.1082\frac{6.00}{f_x} = \frac{0.6492}{f_x}$$

**v nt n t ns issi n in t n , i**

| **Freq. (GHz)** | **Initial wavelength (λ) on Smith chart** | **Change in wavelength (λ) on Smith chart** | **Final wavelength (λ) on Smith chart** | **Equivalent admittance points n , Figure 113** |
|---|---|---|---|---|
| 6.00 | 0.1700 | 0.1082 | 0.2782 | $\overline{Y}_1 = 1.80 - j5.10$ |
| 6.04 | 0.1820 | 0.1075 | 0.2895 | $\overline{Y}_3 = 0.60 - j3.90$ |
| 6.08 | 0.1875 | 0.1068 | 0.2943 | $\overline{Y}_5 = 0.40 - j3.50$ |

The length of the first stub is determined by the susceptance required to cancel the susceptive component of the centre frequency admittance point $\overline{Y}_3$, that is -j3.90 and bring the centre frequency admittance point to the intersection of the unity conductance circle (0.6 - j0.09). To obtain a cancelling susceptance of (0.6 - j0.09) - (0.6 - j3.90) =+j3.81, the length of the stub is found from the distance between the short circuit admittance point on the Smith chart (right-hand side of the chart, where $\overline{Y} \to \infty$) and the j3.81 susceptance point. This is measured from the Smith chart as, (0.250 + 0.2095)λ = 0.4595λ. Thus , the length of the short-circuit stub is 0.4595λ at the centre frequency of 6.04 GHz.

The equivalent lengths of the admittances at the other frequencies are determined by first determining the equivalent length of the stub using equation 4-34 and then converting these into susceptances. That is,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.4595 \frac{6.04}{f_x} = \frac{2.7754}{f_x}$$

**v nt n s t- i it st t n , i**

| **Freq. (GHz)** | **Equivalent admittance points n , Figure 113** | **Susceptance wavelength (λ) (calculated)** | **Susceptance wavelength (λ) on Smith chart** | **Equivalent susceptance** | **Admittance band t susceptance added Figure 113** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y_1} = 1.80 - j5.10$ | 0.4626 | 0.2126 | j4.15 | $\overline{Y_1} = 1.80 - j0.95$ |
| 6.04 | $\overline{Y_3} = 0.60 - j3.90$ | 0.4595 | 0.2095 | j3.81 | $\overline{Y_3} = 0.60 - j0.09$ |
| 6.08 | $\overline{Y_5} = 0.40 - j3.50$ | 0.4565 | 0.2065 | j3.51 | $\overline{Y_5} = 0.4 + j0.01$ |

he admittance band “ ” is the band after the short-circuit stub of length 0.4595λ has been added. This band is next rotated towards the generator by the distance between the two stubs, that is 0.375λ.

PART 2

This part will determine the band after rotation 0.375λ along the transmission line, and the length of the second stub to bring the centre frequency admittance point up towards the centre of the Smith chart, permitting the band to be balanced about the real axis.

gain because of the different fre uencies in the admittance band “ ”, the centre fre uency is chosen to be rotated by 0.375λ. The other admittance points rotate according to,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.375 \frac{6.04}{f_x} = \frac{2..265}{f_x}$$

**v nt n nt λ t ns issi n in t n , i**

| **Freq. (GHz)** | **Equivalent admittance points n , Figure 114** | **n wavelength (λ) initial position (col. 2)** | **Rotational amount (λ) on Smith chart (col. 3)** | **n position after rotation (col. 2 + 3)** | **Admittance band t t ti n Figure 114** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y_1} = 1.80 - j0.95$ | 0.291 | 0.3775 | 0.1685 | 1.10 + j1.10 |
| 6.04 | $\overline{Y_3} = 0.60 - j0.09$ | 0 | 0.3750 | 0.3750 | 1.00 - j0.50 |
| 6.08 | $\overline{Y_5} = 0.4 + j0.01$ | 0.019 | 0.3725 | 0.3915 | 0.70 - j0.73 |

he rotated band is shown on Figure as band “ ”.

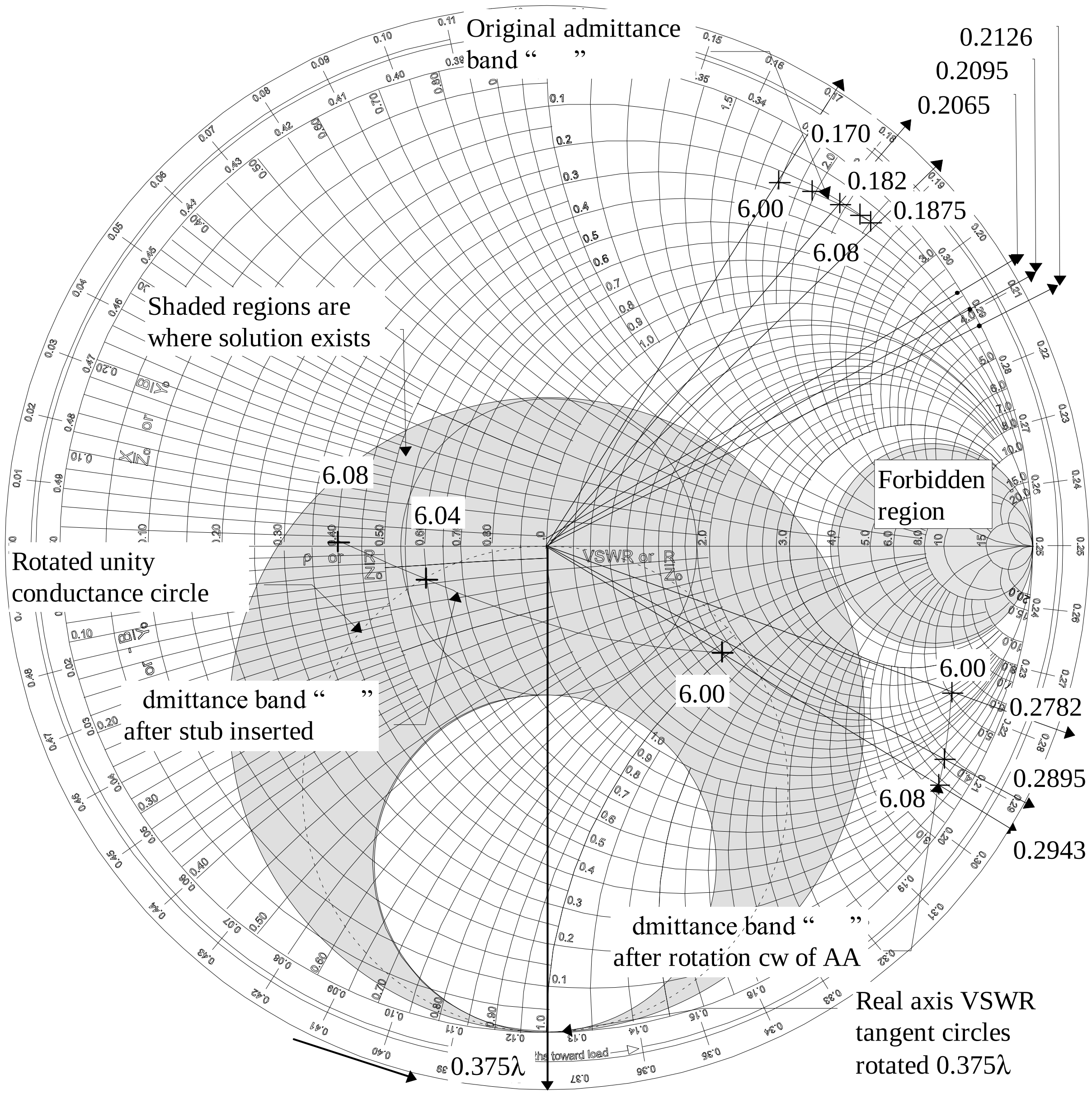



**Figure 113 Example 26 - Part 1 - Distance from the load of first stub and first stub length**

eferring to Figure , the band “ ” is already balanced about the real axis and a second stub is not required.

For the sake of completion of this part of the problem, we will consider what happens if the centre frequency admittance point is brought to the centre of the chart. The length of the second stub is determined by the susceptance required to cancel the susceptive component of the centre frequency admittance point $\overline{Y}_3$, that is -j0.50 and bring the centre frequency admittance point to the centre of the Smith chart along the unity conductance circle. To obtain a cancelling susceptance of +j0.5, the length of the stub is found from the distance between the short circuit admittance point on the Smith chart (right-hand side of the chart, where $\overline{Y} \rightarrow \infty$) and the j0.50 susceptance point. This is measured from the Smith chart as, $(0.250 + 0.074)\lambda = 0.324\lambda$. Thus, the length of the short-circuit stub is $0.324\lambda$ at the centre frequency of 6.04 GHz.

The equivalent lengths of the admittances at the other frequencies are determined by first determining the equivalent length of the stub using equation 4-34 and then converting these into susceptances. That is,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.324 \frac{6.04}{f_x} = \frac{1.9570}{f_x}$$

**v nt n s t- i it st t n , i**

| **Freq. (GHz)** | **Equivalent admittance points n , Figure 114** | **Susceptance wavelength (λ) (calculated)** | **Susceptance wavelength (λ) on Smith chart** | **Equivalent susceptance** | **Admittance band t susceptance added Figure 114** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y_1} = 1.10 + j1.10$ | 0.3262 | 0.0762 | j0.52 | $\overline{Y_1} = 1.10 + j1.62$ |
| 6.04 | $\overline{Y_3} = 1.00 - j0.50$ | 0.3240 | 0.0740 | j0.5 | $\overline{Y_3} = 1.00$ |
| 6.08 | $\overline{Y_5} = 0.70 - j0.73$ | 0.3219 | 0.0719 | j0.485 | $\overline{Y_5} = 0.70 - j0.245$ |

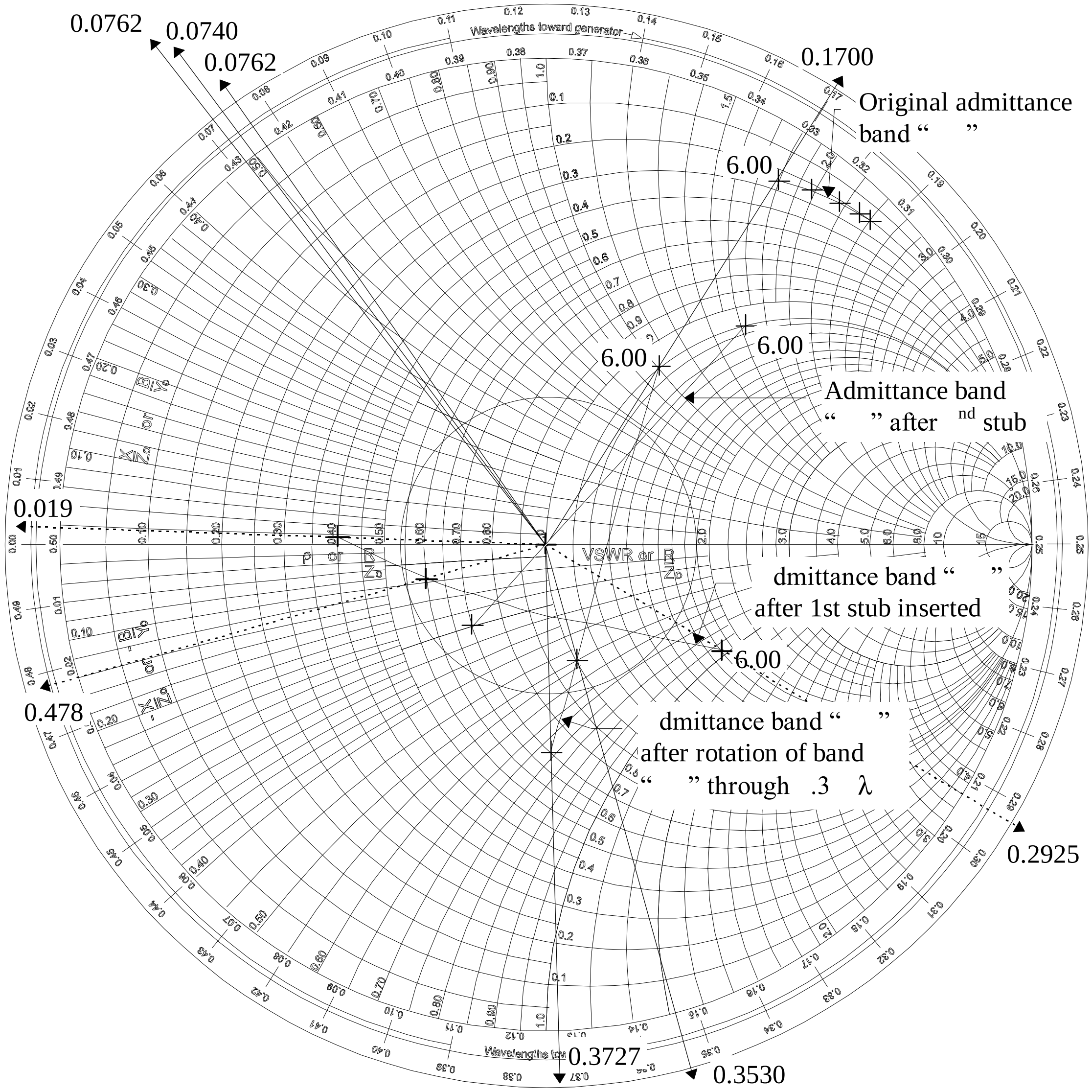



**Figure 114 Example 26 Part 2 - Rotation of band through 0.3175λ and second stub**

s can be seen from Figure , band “ ” has been raised up and rotated clockwise approximately 0.015λ. The 6.08 GHz point entering the specified VSWR circle and the 6.00 GHz admittance point further out from the VSWR circle. The overall VSWR is worse than that where there was no second stub.

PART 3
This part will determine the characteristic impedance of the quarter-wavelength stub to bring the impedance band “ ” into the specified circle. he final impedance band is also calculated. Figure shows the admittance band “ ” which is the band .3 λ down the transmission line, towards the generator, from the insertion point of the first stub.
From Section 4.4.1.5.3, it can be seen that for an admittance band, a quarter-wavelength stub is suitable to close the “mouth” of band “ ”.

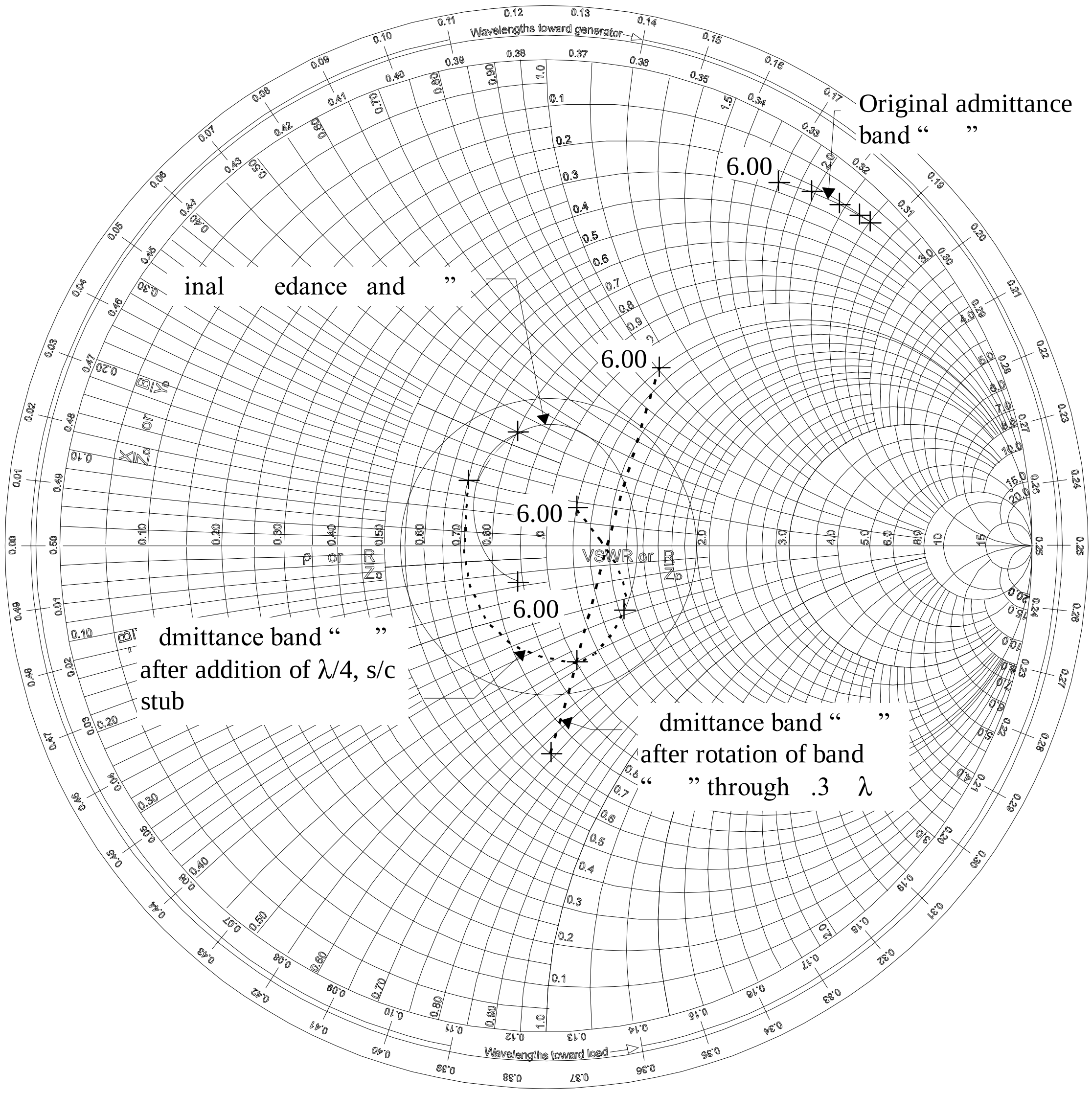



**Figure 115 Example 26 Part 3 - sin n sin -wavelength stub**

Therefore,

$$\overline{Y} = -\frac{Z_o}{Z_{ch}}\, j \cot \frac{\pi}{2} \frac{f_x}{f_o} \qquad \textbf{(4-37)}$$

Using equation 4-37, with the centre frequency at 6.04 GHz, the following table is obtained. As the 6.08 GHz frequency admittance point will move the most, a susceptance point inside the specified VSWR circle of j0.200 on the constant conductance circle of 0.7 should minimize the VSWR. From this value and the original value, it is determined that the susceptance must move from -j0.730 to +j0.200, or a distance of +0.930. As column 2 provides equation 4-37 without the $Z_o/Z_{ch}$ factor, column 3 divided by column 2, gives the value of $Z_o/Z_{ch}$. This value is 89.397, therefore the characteristic impedance of the quarter-wave short transformer is to be $Z_{ch} = \dfrac{Z_o}{89.397}$. Figure 115 shows the final admittance band laying on a VSWR = 1.6 circle, which is better than the specified VSWR of 1.9.

| Column 1 | 2 | 3 (Eqn. 4-37) | 4 | 5 | **6** |
|---|---|---|---|---|---|
| **Frequency (GHz)** $f_x$ | $-j\cot\frac{\pi}{2}\frac{f_x}{f_o}$ | $-j\frac{Z_o}{Z_{ch}}\cot\frac{\pi}{2}\frac{f_x}{f_o}$ For $Z_o = 89.397 Z_{ch}$ | **Original Band admittances Figure 115 n** | **Final Band admittances Figure 115 Col. (3 + 4)** | **Final Band i n s Figure 115** |
| 6.00 | | -j0.93 | $\overline{Y}_1 = 1.10 + j1.10$ | $\overline{Y}_1 = 1.10 + j0.17$ | $\overline{Z}_1 = 0.89 - j0.137$ |
| 6.04 | | 0 | $\overline{Y}_3 = 1.00 - j0.50$ | $\overline{Y}_3 = 1.00 - j0.50$ | $\overline{Z}_3 = 0.80 + j0.40$ |
| 6.08 | j0.0104 | j0.93 | $\overline{Y}_5 = 0.70 - j0.73$ | $\overline{Y}_5 = 0.70 + j0.20$ | $\overline{Z}_5 = 1.32 - j0.378$ |

The final schematic is shown in Figure 116.

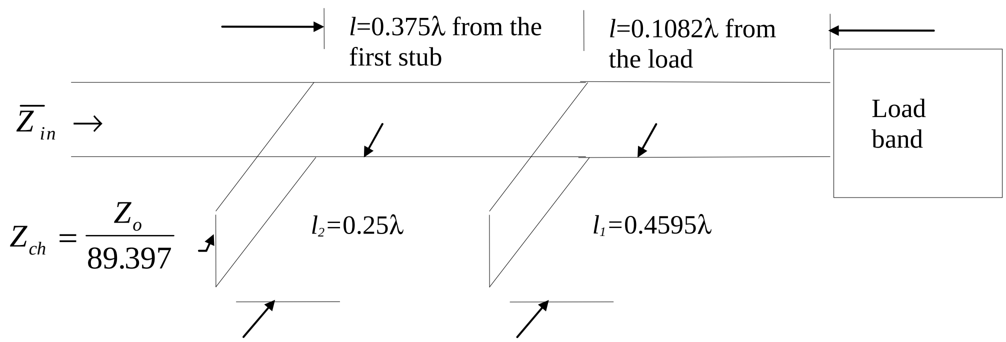



**Figure 116 Example 26 Final schematic diagram**

### 4.4.1.6.2. Triple Stub matching

Section 3.1.3.2.3 considered the case if the triple stub in matching a single frequency to the centre of the Smith chart. In this section, the emphasis will be placed on determining the value of the VSWR circle that a band of frequencies lies within, when the centre of the band is matched to the centre of the Smith chart using the single frequency triple stub matching. The triple stub may be employed over the double stub, to ensure matching can occur for all impedances which may be encountered, as discussed in section 3.1.3.2.3. Similar to the double stub, at the higher microwave frequencies due to the small wavelengths involved, the distance between the stubs may be fixed at 0.375λ. Figure 117 shows a schematic of the triple stub with the 0.375λ fixed separation between the stubs, and the variable distance xλ of the first stub from the impedance band of frequencies.

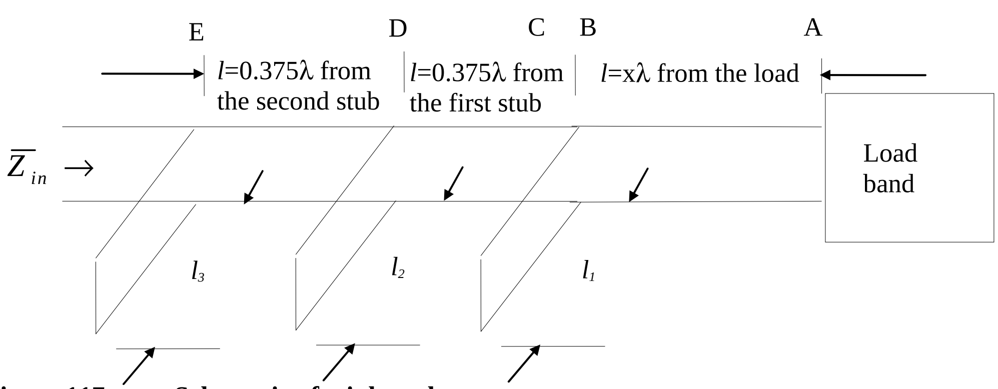



**Figure 117 Schematic of triple stub**

EXAMPLE 27

Given the following normalized mismatched load measured impedance points below, design a shunt triple-stub tuner to match the band so that it lies in a VSWR circle less or equal to 1.7. Determine the maximum value of the VSWR which results from the VSWR circle which encloses the band. The distance between the stubs is to be 0.375λ, and the distance from the load of the first stub is to be 0.250λ.

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 6.00 | $\overline{Z}_1 = 0.534 - j1.414$ |
| 6.04 | $\overline{Z}_2 = 0.517 - j1.11$ |
| 6.08 | $\overline{Z}_3 = 0.476 - j0.950$ |

SOLUTION

he measured impedance points and admittance are plotted on Figure as band “ ” and “ ”.

PART 1

This part will determine the admittance band to the first stub, and will determine the value of the length of the first stub.

From the table of frequencies given in the Example 27, the band centre frequency is found to be,

$$f_o = \sqrt{f_L f_H} = \sqrt{6.00x6.08} \approx 6.04\ MHz\,.$$

By rotating the load admittance $\overline{Y}_2 = 0.345 + j0.740$ clockwise though a distance of 0.250λ, plot the resulting normalized load admittance $\overline{Y}_{LA} = 0.517 - j1.11$ , on the Smith chart, at point A, as shown in Figure 118.

Draw a forbidden region circle from equation 3-12, that is $\overline{G} > \dfrac{1}{\sin^2 2\pi d/\lambda}$ where d is the distance between the stubs, which is 0.375λ . This gives $\overline{G} > 2$

**Conversion of load impedance to load admittance**

| **Frequency (GHz)** | **Normalized load impedance** | **Normalized load admittance band , i** |
|---|---|---|
| 6.00 | $\overline{Z}_1 = 0.534 - j1.414$ | $\overline{Y}_1 = 0.234 + j0.619$ |
| 6.04 | $\overline{Z}_2 = 0.517 - j1.11$ | $\overline{Y}_2 = 0.345 + j0.740$ |
| 6.08 | $\overline{Z}_3 = 0.476 - j0.950$ | $\overline{Y}_3 = 0.422 + j0.841$ |

The movement of the outer band frequencies do not move by 0.250λ as does the 6.04 GHz band centre frequency, but by an amount dependent on their frequency, as determined by equation 4-34.

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.250\frac{6.04}{f_x} = \frac{1.51}{f_x}$$

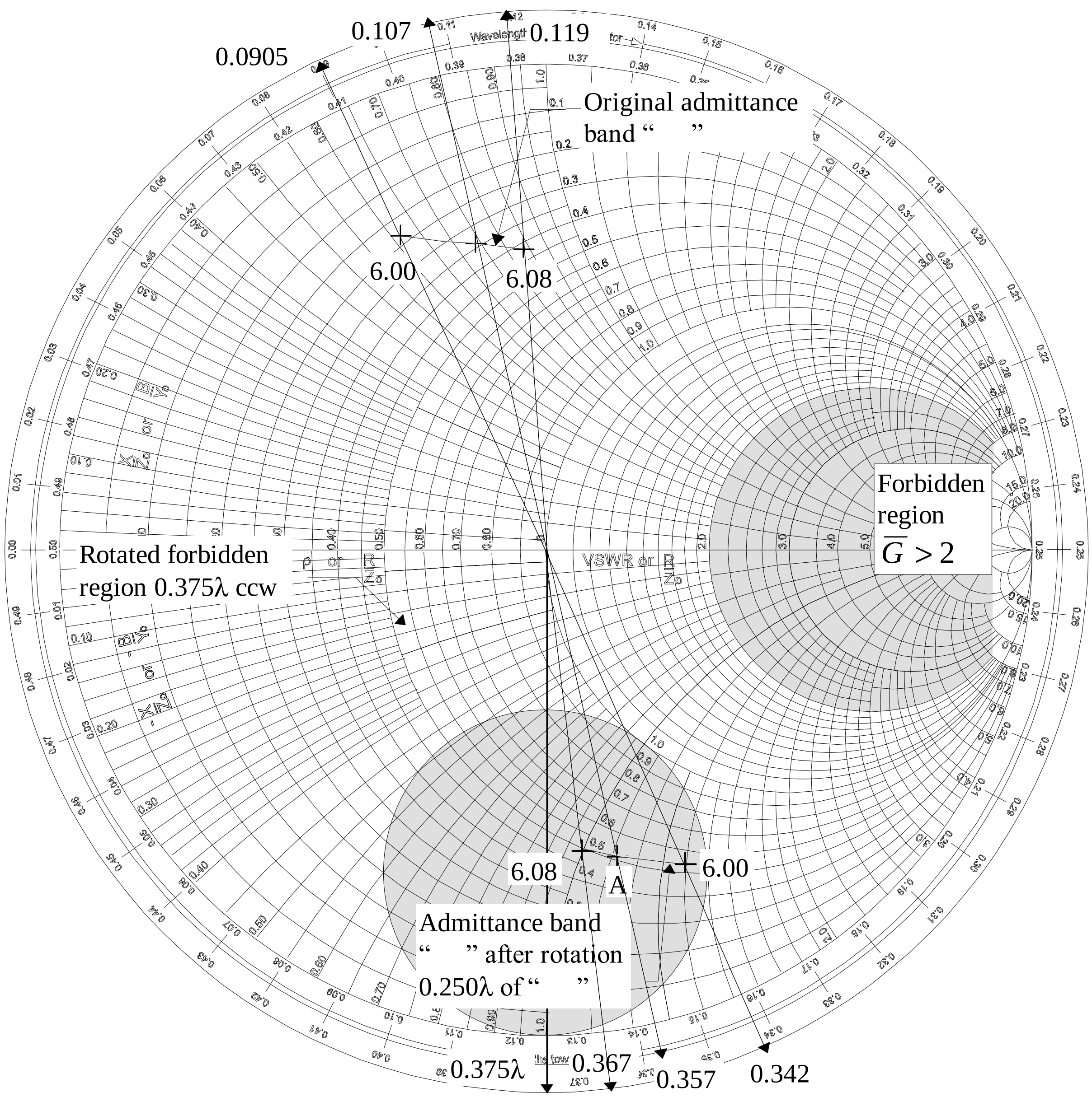



**Figure 118 Example 27 - Part 1 - Distance from the load of first stub and first stub length**

**v nt n λ line between load and first short-circuit stub to**
**n , i**

| **Freq. (GHz)** | **Admittance points of n , Figure 118** | **Line length (λ) (calculated)** | **Original position (λ) on Smith chart** | **Final position (λ) on Smith chart** | **Admittance band t tation Figure 118** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 0.234 + j0.619$ | 0.2517 | 0.0905 | 0.3422 | $\overline{Y}_1 = 0.53 - j1.40$ |
| 6.04 | $\overline{Y}_2 = 0.345 + j0.740$ | 0.2500 | 0.107 | 0.357 | $\overline{Y}_2 = 0.517 - j1.11$ |
| 6.08 | $\overline{Y}_3 = 0.422 + j0.841$ | 0.2484 | 0.119 | 0.367 | $\overline{Y}_3 = 0.49 - j0.98$ |

The admittances of band “ ”, shown on Figure , are determined by drawing a circle through each of the admittance points of band “ ” so that each individual circle intersects its final distance position line extended from the centre of the Smith chart to its wavelength position on the edge of the chart. In order to prevent cluttering. these circles are not shown on Figure 118.
In order to rotate anti-clockwise the band “ ” completely out of the rotated forbidden region, a negative susceptance must be added. The value of this susceptance is found from the amount of anti-clockwise rotation of the 6.08 GHz admittance point required to place it outside of the rotated forbidden region. This value is the difference between the final 6.08 GHz admittance point, $\overline{Y}_3^{CA} = 0.49 - j1.50$, in band “ ”, and its value in band “ ” $\overline{Y}_3 = 0.49 - j0.98$. That is, -j0.52. This value determines the length of the first short circuit stub, which is from the Smith chart 0.174λ.
The equivalent lengths of the admittances at the other frequencies are determined by equation 4-34 as,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.174 \frac{6.08}{f_x} = \frac{1.05792}{f_x}$$

**v nt n s t- i it st t n , i**

| Freq. (GHz) | Equivalent admittance points t n , Figure 118 | Susceptance wavelength (λ) (actual) | Susceptance wavelength (λ) on Smith chart | Equivalent susceptance | Admittance band t susceptance added Figure 119 |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 0.53 - j1.40$ | 0.1763 | 0.4263 | -j0.500 | $\overline{Y}_1 = 0.53 - j1.90$ |
| 6.04 | $\overline{Y}_2 = 0.517 - j1.11$ | 0.1752 | 0.4252 | -j0.510 | $\overline{Y}_2 = 0.517 - j1.621$ |
| 6.08 | $\overline{Y}_3 = 0.49 - j0.98$ | 0.1740 | 0.4240 | -j0.520 | $\overline{Y}_3 = 0.49 - j1.50$ |

PART 2
his part will determine the band “ ” after the band “ ” has traversed .3 λ along the transmission line. It will also determine the length of the second stub.
he rotation of band “ ” clockwise through .3 λ to band “ ” is shown on Figure . gain each of the admittance points of band “ ” will rotate by different amounts according to

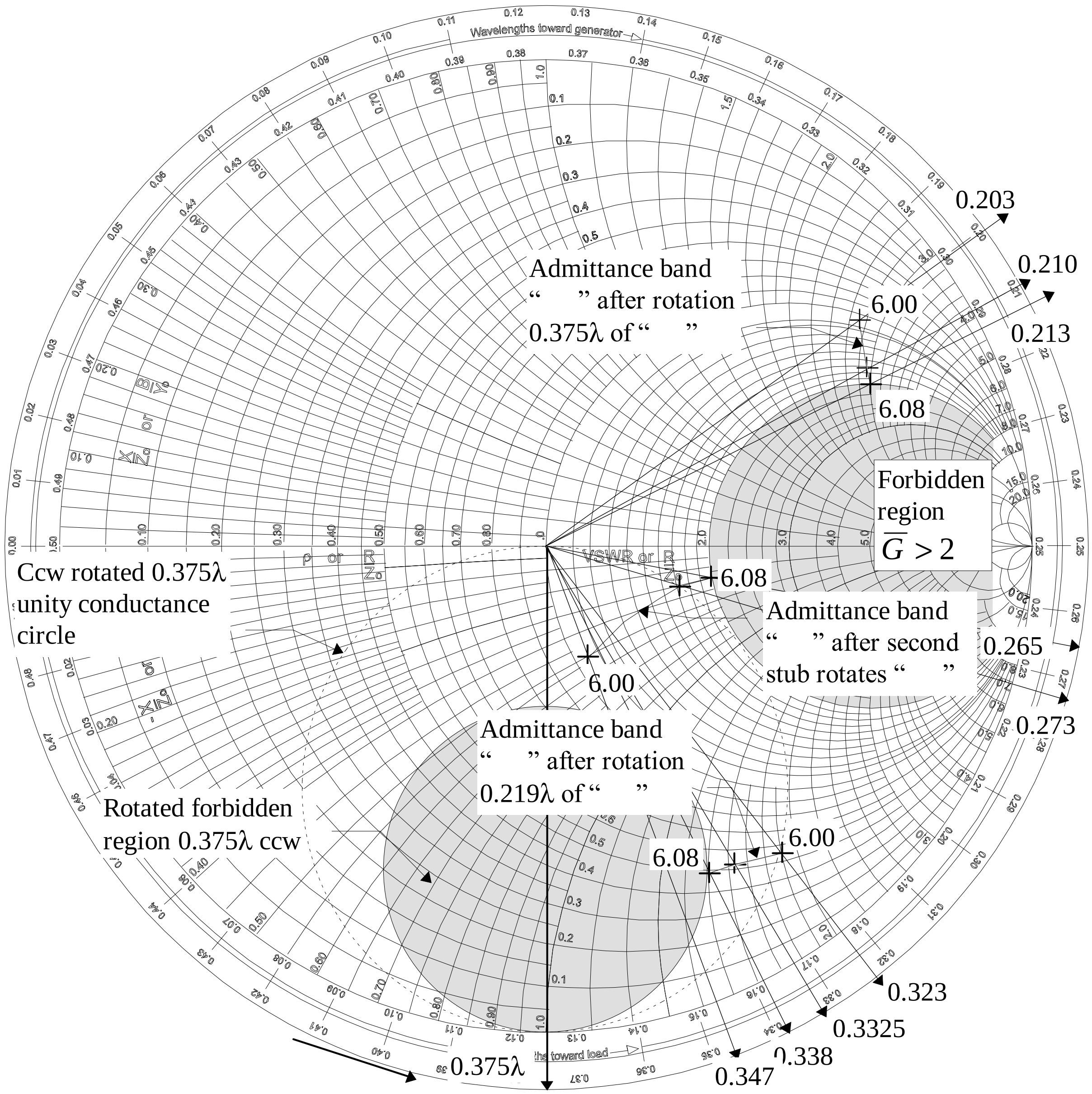



**Figure 119 Example 27 - Part 2 - Distance from first stub and second stub length**

equation 4-34 . Choosing the 6.08 GHz frequency admittance as the admittance point which rotates by 0.375λ, as it is the critical admittance point, gives the following table for determining the band “ ”.

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.375\frac{6.08}{f_x} = \frac{2.28}{f_x}$$

**v nt n λ t ns issi n in t n , i**

| Freq. (GHz) | Equivalent admittance points t n , Figure 119 | Admittance points of n wavelength (λ) (actual) | Calculated distance (λ) to add to column 3 | Final wavelength points (λ) | Admittance band Figure 119 |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 0.53 - j1.90$ | 0.323 | 0.38 | 0.203 | $\overline{Y}_1 = 1.10 + j2.70$ |
| 6.04 | $\overline{Y}_2 = 0.517 - j1.621$ | 0.3325 | 0.3775 | 0.210 | $\overline{Y}_2 = 1.72 + j2.92$ |
| 6.08 | $\overline{Y}_3 = 0.49 - j1.50$ | 0.338 | 0.3750 | 0.213 | $\overline{Y}_3 = 2.00 - j3.00$ |

The problem now becomes that of a double stub. We want the centre frequency admittance point to rotate anticlockwise along its 1.72 conductance circle until it intersects the anticlockwise rotated unity conductance circle. From Figure 119, this can be seen to be at the admittance 1.72 - j0.33. The length of the second stub is to be the difference in the final admittance and the centre frequency admittance in the band " ", that is the difference between $\overline{Y}_2 = 1.72 - j0.33$ and $\overline{Y}_2 = 1.72 + j2.92$, which is -j3.25, which from the Smith chart gives a stub length of 0.048λ.
The equivalent lengths of the admittances at the other frequencies are determined by equation 4-34 as,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \tag{4-34}$$

$$\lambda_{f_x} = 0.048 \frac{6.04}{f_x} = \frac{0.28992}{f_x}$$

**v nt n s n s t- i it st t n , i**

| Freq. (GHz) | Equivalent admittance points t n , Figure 118 | Susceptance wavelength (λ) (actual) | Susceptance wavelength (λ) on Smith chart | Equivalent susceptance | Admittance band t susceptance added Figure 119 |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 1.10 + j2.70$ | 0.0483 | 0.2983 | -j3.20 | $\overline{Y}_1 = 1.10 - j0.5$ |
| 6.04 | $\overline{Y}_2 = 1.72 + j2.92$ | 0.0480 | 0.2980 | -j3.25 | $\overline{Y}_2 = 1.72 - j0.33$ |
| 6.08 | $\overline{Y}_3 = 2.00 - j3.00$ | 0.0477 | 0.2977 | -j3.30 | $\overline{Y}_3 = 2.00 - j0.30$ |

PART 3
his part will determine the band "F " after the band " " has traversed .3 λ along the transmission line. It will also determine the length of the third stub.
he rotation of band " " clockwise through .3 λ to band "F " is shown on Figure . gain each of the admittance points of band " " will rotate by different amounts according to e uation 4-34 . Choosing the centre frequency admittance as the admittance point which rotates by 0.375λ, as it is to lie on the unity conductance circle, gives the following table for determining the band " ".

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \tag{4-34}$$

$$\lambda_{f_x} = 0.375 \frac{6.04}{f_x} = \frac{2.265}{f_x}$$

**v nt n 375λ t ns issi n in t n , i**

| **Freq. (GHz)** | **Equivalent admittance points t n , Figure 120** | **Admittance points of n wavelength (λ) (actual)** | **Calculated distance (λ) to add to column 3** | **Final wavelength points (λ)** | **Admittance band Figure 120** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y}_1 = 1.10 - j0.5$ | 0.347 | 0.3775 | 0.2245 | $\overline{Y}_1 = 0.86 + j0.68$ |
| 6.04 | $\overline{Y}_2 = 1.72 - j0.33$ | 0.273 | 0.3750 | 0.1480 | $\overline{Y}_2 = 1.00 + j0.60$ |
| 6.08 | $\overline{Y}_3 = 2.00 - j0.30$ | 0.265 | 0.3725 | 0.1375 | $\overline{Y}_3 = 1.60 + j0.25$ |

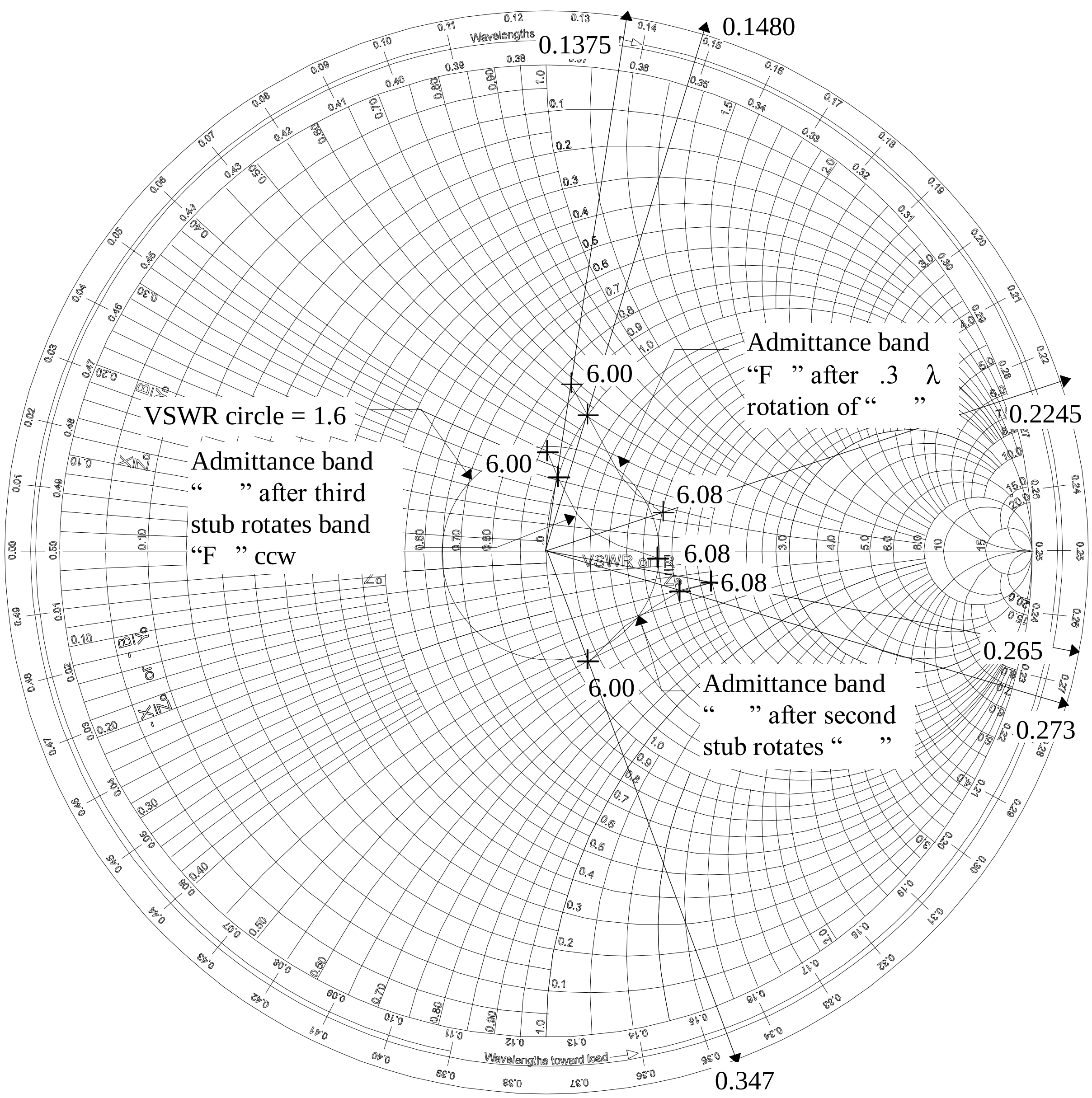



**Figure 120 Example 27 - Part 3 - Distance from second stub and third stub length**

Finally, to determine the length of the third stub, the . Hz fre uency admittance point of band “F ” is rotated along the unity conductance circle until it is just below the real axis, so that the band is balanced about the real axis. That is, a short-circuit stub of susceptance value -j0.3 is required at a frequency of 6.08 GHz. From the Smith chart for a susceptance of -j0.3, a short-circuit stub of length 0.204λ is required.
The equivalent lengths of the admittances at the other frequencies are determined by equation 4-34 as,

$$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \qquad \textbf{(4-34)}$$

$$\lambda_{f_x} = 0.204 \frac{6.08}{f_x} = \frac{1.24032}{f_x}$$

**v nt n t i s t- i it st t n , Figure 120.**

| **Freq. (GHz)** | **Equivalent admittance points t n , Figure 120** | **Susceptance wavelength (λ) (actual)** | **Susceptanc e wavelength (λ) on Smith chart** | **Equivalent susceptance** | **Admittance band t susceptance added Figure 120** |
|---|---|---|---|---|---|
| 6.00 | $\overline{Y_1} = 0.86 + j0.68$ | 0.2067 | 0.4567 | -j0.28 | $\overline{Y_1} = 0.86 + j0.40$ |
| 6.04 | $\overline{Y_2} = 1.00 + j0.60$ | 0.2054 | 0.4554 | -j0.29 | $\overline{Y_2} = 1.00 + j0.31$ |
| 6.08 | $\overline{Y_3} = 1.60 + j0.25$ | 0.2040 | 0.4540 | -j0.30 | $\overline{Y_3} = 1.60 - j0.05$ |

The final VSWR is found to be approximately 1.6.

Figure 121 shows the final circuit schematic.

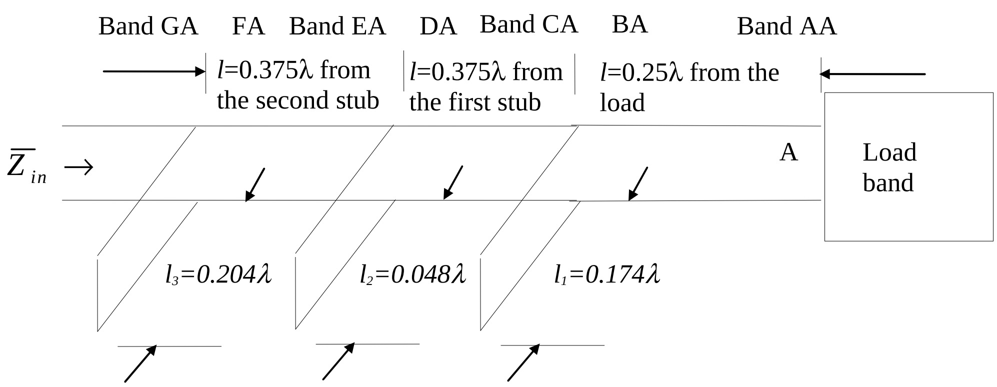



**Figure 121 Example 27 -Final schematic of triple stub**

### 4.4.1.7. Velocity factor

So far, in the examples provided in chapters 3 and 4, the answers for stub lengths, etc., have been given in terms of the wavelengths at some particular frequency, where the medium is assumed to be that of free space. The physical length of the stub, etc., is not usually determined from the wavelength of the free-space electromagnetic radiation, but from the wavelength in a dielectric medium or a waveguide. This means that the physical length of the stub is shorter than it would be if the speed of light was used to determine its length from the value of the wavelength calculated. Section 1.7 discussed the velocity of propagation of a signal in a transmission line. The ratio of the phase velocity of the signal in the medium $v_p$ to the speed of light c, is the velocity factor VF, that is,

$$VF = \frac{v_p}{c} \qquad 0 < VF \leq 1 \tag{4-55}$$

From equations 1-33 and 1-32 and since $v_p = f\lambda_{medium}$ and $c = f\lambda_{vacuum}$

$$VF = \frac{v_p}{c} = \frac{2\pi f}{c\beta} = \frac{2\pi}{\beta\lambda_{vacuum}} = \frac{\lambda_{medium}}{\lambda_{vaccum}} \tag{4-56}$$

where $\beta$ is the phase change coefficient.

Thus, the wavelength in a medium $\lambda_{medium}$, can be expressed in terms of the velocity factor VF, and the wavelength in free-space $\lambda_{vacuum}$, by,

$$\lambda_{medium} = (VF)\lambda_{vacuum} \tag{4-57}$$

and from an approximation of equation 1-36, for a coaxial cable transmission line,

$$VF \approx \frac{1}{\sqrt{\mu_r \varepsilon_r}} \tag{4-58}$$

If no magnetic material is present, then $\mu_r = 1$, and equation 4-58 reduces to

$$VF \approx \frac{1}{\sqrt{\varepsilon_r}} \tag{4-59}$$

For a waveguide, the wavelength in the guide, $\lambda_g$, filled with a dielectric of relative permittivity $\varepsilon_r$, is given by,

$$\lambda_g = \frac{\lambda_{vauum}}{\sqrt{\varepsilon_r - \left(\frac{\lambda_{vacuum}}{\lambda_{cutoff}}\right)^2}} \tag{4-60}$$

where $\lambda_{cutoff}$ is the cutoff wavelength of the empty waveguide.

The velocity factor for a waveguide is found to be from equations 4-57 and 4-60, as,

$$VF = \frac{1}{\sqrt{\varepsilon_r - \left(\frac{\lambda_{vacuum}}{\lambda_{cutoff}}\right)^2}} \tag{4-61}$$

EXAMPLE 28

Example 27 showed that the length of a short-circuit stub was found to be 0.174λ at 6.08 GHz. Determine the physical length of the stub, if the transmission line used has a dielectric constant of 2.5.

SOLUTION
From equation 4-59, the velocity factor is found to be approximately,

$$VF \approx \frac{1}{\sqrt{\varepsilon_r}} = 0.6325$$

From equation 4-57.

$$\lambda_{medium} = (VF)\lambda_{vacuum} = 0.6325\lambda_{vacuum}$$

At 6.08 GHz, one wavelength in a vacuum is found to be 0.0493m, therefore one wavelength in the medium is 0.6325 of this value. That is, 0.0312m. For a stub of length 0.174λ, the length of the transmission line stub is 0.0312 x 0.174 = 5.426 mm.

## 4.4.2. Wide-band systems

As mentioned in section 4.4, for the purposes of using the Smith chart in design, a wide-band system is defined as a non-narrow-band system, where the fractional frequency is given by equation 4-32,
that is,

$$\frac{100BW}{f_{RF}} > 10\% \qquad \text{(wide-band)} \qquad \textbf{(4-32)}$$

The terms narrow-band and wide-band used in this context are not the same as the definitions provided in angle modulation theory, where the modulation index is used.
The work covered in the matching of narrow band systems, section 4.4.1, is directly applicable in this section. This section can be considered an extension of the principles covered in the narrow band systems section and differs only by, perhaps, the additional circuits required in order to find a matching solution.
In this section a set of guide lines will be established to assist in finding a matching solution to a wide-band problem. These guidelines have also been used for narrow band matching and may be considered as a summary of this chapter.

GUIDLINES

- RULE 1
  From a specified VSWR circle, two boundary circles are drawn tangent to this VSWR circle.
  Each impedance point in a band must lie outside of its smaller boundary circle, shown in Figure 122, and within its larger boundary circle, or have impedance points which lie on circles of constant reactance which pass into the larger special conductance circle, if matching is to be easily attained.
  The same applies for admittance bands.

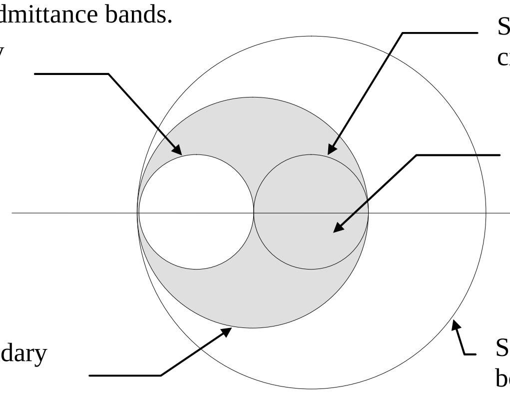



**Figure 122 Shaded solution region determined by boundary circles derived from VSWR circle**

If an impedance band lies outside of the shaded boundary region, the use of an inductor or capacitor, shunt or series, must be made to bring the band into this region so that a solution can be found.

- Series capacitors and inductors will move a band around circles of constant resistance, according to equations 4-3 and 4-10, whereas, shunt capacitors and inductors will move a band of frequencies around circles of constant conductance, according to equations 4-17 and 4-24.
- RULE 2
  The band centre frequency, or the geometric mean of the band frequencies is given by
  $$f_o = \sqrt{f_L f_H} \tag{4-53}$$
  The frequency chosen for movement of the band may often not be the centre frequency, due to the placement of the band edges in a more advantageous position. However, if the frequency chosen is different due to other constraints, then after further processing the band may be lopsided and some of the points may not enter the specified VSWR circle.
- A band may be rotated clockwise around the centre of the Smith chart along circles of constant VSWR using a length of transmission line. However, each admittance or impedance point in the band, which is at a different frequency, will be rotated by a different amount. A particular frequency in the band is chosen as the reference frequency and the length of the transmission line is based on the reference wavelength at this frequency. The amount of wavelength rotation of the other points in the band, from their initial positions, is based on equation 4-34.
  $$\lambda_{f_x} = \lambda_o \frac{f_o}{f_x} \tag{4-34}$$
  Notice that the wavelength scale on the edge of the Smith chart must be used to determine the rotation of each impedance or admittance point in the band.
- A short-circuited or open-circuited stub may be used to "close" the mouth of the band. he different stubs are discussed in section 4.4.1.5.3. This is useful, when a band passes through a specified VSWR circle but the extremities of the band lie outside of it. The short- or open-circuit stub can bring the band extremities into the VSWR circle. The band requirement is that the lower frequency must lie in the top half of the Smith chart and the higher frequency in the lower. This is discussed in section 4.4.1.3.
- The series and parallel tuned circuit discussed in section 4.4.1.5.2 may also be used to close the mouth of the band similar to the open- or short-circuit stub.
- A short-transformer may be used to shift a band from one region of the Smith chart to another. Ideally the band is compact and not spread out before this tool becomes useful. Section 4.4.1.2. discussed its application.
- The two quarter-wavelength transformers in tandem may also be used to shift a band from one position on the Smith chart to another position. Section 4.4.1.4 discusses this tool.
- RULE 3
  When plotted on a Smith chart, the normalized impedance or admittance load bands must have a clockwise direction with increasing frequency if a matching solution is to be found.
- RULE 4
  Matching cannot be achieved for a pure reactance or susceptance. That is a band which lies on the perimeter of the Smith chart cannot be matched. The closer a band lies to the perimeter of the Smith chart, the more difficult it is to find a matching solution.
- What appears to be a small band on the right side of the Smith chart becomes a large band on the left side of the Smith chart. This is because the scales of reactance or susceptance are larger for smaller angles on the right side than on the left.

**EXERCISES**

QUESTION 1

A measured load impedance band which is converted into a normalized impedance band is given below. It is required that this impedance band be modified so that it is matched to a VSWR of 1.25 or less. Draw the impedance and admittance band on the Smith chart and determine the VSWR of the unmatched band. Using a short-circuit quarter-wavelength stub, determine the impedance of the stub to match the band, and draw the final admittance band on the Smith chart, ensuring that it is within the specified VSWR.

| Number | Frequency (GHz) | Normalized impedance |
|---|---|---|
| 1 | 5.50 | $\overline{Z}_1 = 0.50 + j0.55$ |
| 2 | 5.40 | $\overline{Z}_2 = 0.80 + j0.4$ |
| 3 | 5.30 | $\overline{Z}_3 = 0.90$ |
| 4 | 5.10 | $\overline{Z}_4 = 0.70 - j0.45$ |
| 5 | 5.00 | $\overline{Z}_5 = 0.40 - j0.55$ |

Suggested answer - The VSWR of the load band is found from the Smith chart to be approximately 3.3 and the characteristic impedance of the quarter-wave short transformer is to be $Z_{ch} = \frac{Z_o}{15}$.

QUESTION 2

What would be the minimum VSWR of a band of frequencies given below if the quarter-wavelength frequency of two quarter-wave transformers in tandem was taken as 0.8 GHz?. The following normalized mismatched load impedance points are given as,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

Suggested answer - With $\overline{Z}_{oA} = 1.7070$ and $\overline{Z}_{oB} = 1.1951$, the final VSWR is found to be 2.00.

QUESTION 3

What would be the minimum VSWR of a band of frequencies given below if the quarter-wavelength frequency of two quarter-wave transformers in tandem was taken as the geometric mean of the upper and lower band frequencies?. The following normalized mismatched load impedance points are given as,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

Suggested Answer - With $\overline{Z}_{oB} = 1.1951$ and $\overline{Z}_{oA} = 1.7070$ and the centre frequency at approximately 1.04 GHz, the final VSWR is found to be 2.35.

QUESTION 4
ework xample using the "traditional" approach when using a short-transformer to match a transmission line. Determine the best VSWR that you can obtain, given the following normalized mismatched load impedance points,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

Make comments on how you could improve the VSWR obtained.

Suggested Answer - With $\overline{Z}_{oA}$ = 2.0794, the VSWR = 2.08 which is not acceptable.
Some points about the solution to this type of problem may be worth noting. The first is, the traditional approach is not always the best approach. The selection of the characteristic impedance is not in dispute in this case, it is the rotation of the centre of the band to the real axis which presents the problem. This problem needs to be reworked with the rotation of the lower band frequency to the edge of the VSWR=1.5 circle (the lower band rotates more according to equation 4-34). If the higher band frequency does not fit into the a VSWR circle of 1.5 after this, then the impedance of the transformer needs to be altered. The final VSWR circle falls short of what was estimated by the "traditional" method . .

# 5. MICROWAVE AMPLIFIER DESIGN

At microwave frequencies, the transistor is described by its two-port scattering matrix, because its scattering matrix parameters $S_{ij}$, can be measured by inserting it into a test circuit with 50$\Omega$ input and output lines, applying the appropriate bias voltages and currents and making the measurements, assuming that 50Ω is the characteristic impedance of the line. Impedance and admittance parameters cannot be measured directly and open- and short-circuit measurements, required by other descriptions, are difficult to obtain. In addition, no tuning or adjustment of the electrical length of the line between the 50 Ω and the circuit is required with a characteristic impedance termination. An open- or short-circuit requires that there be an adjustment of the electrical length between it and the circuit to place it into its correct position. Wide-band swept-frequency measurements are possible over a broad frequency range. This is not the case if an open- or short-circuit is used because retuning is necessary after a narrow band of frequencies has been covered. Devices when tested must remain stable. Stability can be ensured if it is terminated in its characteristic impedance. Tunnel diodes, and other negative resistance devices tend to oscillate under different loads.
Usually, the manufacturer of the transistor provides the scattering matrix parameters in his data sheets, however, the $S_{ij}$ parameters can vary with bias conditions and temperature as well as from transistor to transistor of the same type, so in design some allowance is made for these variations.
The scattering matrix is sufficient to enable the complete design of an amplifier or multi-stage amplifier. The criteria used for amplifier design is to have a the largest power gain that can be obtained for a stable circuit over a specific frequency band, an input and output VSWR close to unity with a minimum noise figure of the first stage using the optimum source impedance, and a phase response that is a linear function of frequency. Since the parameter pairs, gain and stability, and VSWR and optimum source impedance, are at variance, the design must be a trade off between the conflicting parameters according to the application.
This chapter will discuss these criteria and use the Smith chart to assist in the design of an amplifier or amplifier stages.

## *5.1. The scattering matrix*

The direct measurement of voltages and currents in a waveguide are not possible because in a uniform waveguide, for a given mode and frequency, a travelling wave is characterized by the electromagnetic field distribution in a transverse cross-section and by a propagation constant γ. The field in any other cross-section, at a distance z in the direction of propagation, has the same pattern but is multiplied by $e^{-j\gamma z}$. Therefore, the impedance can be considered as a virtual parameter rather than a real one, as it is a derived quantity. Similarly, voltages and currents can be considered as derived quantities. This applies to many microwave circuits where only the relative field strength using a probe inserted into the coaxial cable or waveguide, such as a slotted line, can be measured. Using this probe, the direct measurement of power can be made as well as the direct measurement of the voltage standing wave ratio (VSWR) from the relative **values** of the minimum and maximum field strength, and the relative **position** of either the minimum or maximum field strength. From the VSWR, the reflection coefficient can be determined as given by equations 1-63 and 1-53, and from the power, the absolute value of the field can be determined. Another directly measurable parameter is the transmission coefficient through a circuit or boundary. It is the relative measurement of the amplitude and phase of the transmitted wave **out** of a circuit or boundary when compared against the amplitude and phase of the forward or incident wave **into** the circuit or boundary. The transmitted wave out of a circuit or boundary may be considered as the reflected or scattered wave. To define accurately the waves incident on a waveguide junction and those scatttered from it, reference locations must be chosen in the waveguide. These locations are called the ports of the junction. In a waveguide that can support several propagating modes, there is a port for every mode and each port may be physically located in a different place in the waveguide. Because the field equations are linear for incident and reflected waves and because the incident and reflected waves in microwave devices are treated as linearly related, the scattered wave amplitudes are treated as being linearly related to the incident wave amplitudes. The matrix which describes this linear relationship is called the scattering matrix.
The components of the scattering matrix of an N-port junction are a set of quantities that relate the forward and reflected, or newly expressed, the incident and scattered waves at each of the ports with the other ports terminated. The coefficients along the main diagonal $S_{nn}$, are the reflection coefficients and the remaining coefficients $S_{mn}$ of the matrix are the transmission coefficients. Each

individual circuit stage in a chain can have its own scattering matrix. In this chapter, only two-port scattering matrices will be considered, since most microwave circuits have only two-ports. The two-port junction includes the case of an obstacle or discontinuity which may be placed in a waveguide or the case where two different waveguides are connected together. Also, scattering matrices are not restricted to waveguides, but may be used for transmission lines and lower frequency networks.
For convenience in dealing with the scattering matrix at a junction, virtual voltages and currents will be used to describe the incident wave and scattered wave at a junction. In many texts, the use of $a_1$, and $a_2$, are used to describe the wave entering and leaving the two port network from left to right and $b_1$, and $b_2$, for the wave entering and leaving the two port network from right to left.
If an incident voltage wave $V_1^+$ impinges on the junction of port 1 of a two-port network, the reflected portion of the incident wave is given by $S_{11}V_1^+$, where $S_{11}$ is the reflection coefficient, or scattering coefficient for port 1. A portion of the incident wave will also be transmitted through the two-port network to port 2, and will have an amplitude directly proportional to $V_1^+$, that is $S_{21}V_1^+$. A fraction of the transmitted wave will then be reflected back from the load and become incident upon port 2. This wave is given by $V_2^+$. At port 2 the incident wave $V_2^+$, impinging on the junction is again reflected. The reflected portion is given by $S_{22}V_2^+$, where $S_{22}$ is the reflection coefficient, or scattering coefficient for port 2. A portion of this incident wave will also be transmitted through the two-port network to port 1, and will have an amplitude directly proportional to $V_2^+$, that is $S_{12}V_2^+$.
The total wave coming out of port 1, $V_1^-$, comprises the original incident wave reflected portion $S_{11}V_1^+$, and that portion which has been transmitted back from port 2 after reflection from the load $S_{12}V_2^+$. That is,

$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+ \tag{5-1a}$$

Similarly, the total wave coming out of port 2, $V_2^-$, comprises that which was transmitted through the two port network from port 1, $S_{21}V_1^+$, and that which was reflected back at port 2, from the wave coming back from the load, $S_{22}V_2^+$. That is,

$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ \tag{5-1b}$$

The above description of the incident and scattered wave process is shown in Figure 123.
The scattering matrix is formed from equations 5-1a and 5-1b, to give,

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \tag{5-2}$$

where,
$S_{11} =$ the input reflection coefficient, looking into port 1,
$S_{12} =$ the reverse transmission coefficient,
$S_{21} =$ the forward transmission coefficient,
$S_{22} =$ the output reflection coefficient, looking into port 2.

The quantity $-20\log_{10}|S_{12}|$ is the insertion loss from port 2 to port 1. Similarly, $-20\log_{10}|S_{21}|$ is the insertion loss from port 1 to port 2.

The advantage that scattering coefficients have over the other two port network parameters, such as h-, z-, or y- parameters, is that a short- or open-circuit is **not** required to obtain their values.
To obtain the values of the scattering coefficients either the two-port network is terminated with a load impedance equal to that of the characteristic impedance of the transmission line or the input impedance of the two port network is made equal to the characteristic impedance of the transmission line.
Note that, with a termination equal to that of the characteristic impedance, the reflection coefficient **of that termination** is zero (matching to the centre of the Smith chart). If the termination was open circuited its reflection coefficient would be +1 (far right-hand side of the Smith chart), and for a short-circuited termination, its reflection coefficient is -1 (far left-hand side of the Smith chart). Another

point worth noting at this point, is that, at a standing wave maximum or minimum, the phase of the reflection coefficient is zero and the reflection coefficient is a real quantity.

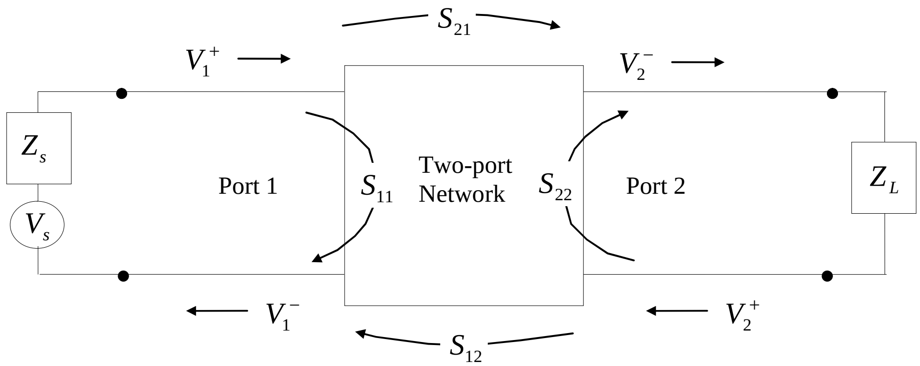



**Figure 123 Scattering matrix formation for a Two-port network**

With the two-port network terminated with a load impedance equal to that of the characteristic impedance of the line, that is $Z_L = Z_o$, no reflections will occur at the load, and so $V_2^+ = 0$, although reflections will still be produced at the input to the two-port network, that is port 1, by its input impedance not being equal to $Z_o$.

With the generator impedance of the two-port network equal to the characteristic impedance of the line, and the generator turned off, no reflections will be produced at the junction of port 1, and so $V_1^+ = 0$. However, the a wave must be generated to enter port 2 of the network, with its generator impedance $Z_L$, not equal to the characteristic impedance of the line, $Z_o$.

These two conditions are summarized in the definition of each of the scattering coefficients which are derived from equations 5-1a and 5-1b, that is,

$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \frac{1}{\rho_s} \qquad \textbf{(5-3a)}$$

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} \qquad \textbf{(5-3b)}$$

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0} \qquad \textbf{(5-3c)}$$

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0} = \frac{1}{\rho_L} \qquad \textbf{(5-3d)}$$

The various reflection coefficients for different terminating conditions can be found as follows,

**<u>Port 1 input impedance and load impedance not matched</u>**

From equation 5-1a, 5.3d and 5-1b

$$V_1^- - S_{11}V_1^+ = S_{12}V_2^+ \qquad = S_{12}\rho_L V_2^-$$
$$-S_{21}V_1^+ = S_{22}V_2^+ - V_2^- = \left(S_{22}\rho_L - 1\right)V_2^-$$

Solving for $\dfrac{V_1^-}{V_1^+}$, by eliminating $V_2^-$ gives,

$$\frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} = \rho_{in} \qquad \textbf{(5-4)}$$

**The reflection coefficient at the load impedance** $\rho_L$ is given by,

$$\rho_L = \left.\frac{V_2^+}{V_2^-}\right|_{V_1^+=0} = \frac{1}{S_{22}} \qquad \textbf{(5-5)}$$

Equation 5-4 shows that the reflection coefficient $\rho_{in}$, at the input to the two-port network is affected by the load not being matched.

**Port 1 input impedance not matched and load impedance matched**

If the load was matched, the input reflection coefficient would be given by $S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0}$

This equation can be obtained by substituting $\rho_L$ = 0 into equation 5-4.

**Port 2 input impedance and generator impedance not matched**

Similarly, looking into port 2, from equations 5-1a, 5-3a and 5-1b, when port 1 is terminated with a generator impedance $Z_s$, which is not equal to the characteristic impedance $Z_o$, of the transmission line,

$$\frac{V_2^-}{V_2^+} = S_{22} + \frac{S_{12}S_{21}\rho_s}{1 - S_{11}\rho_s} = \rho_{out} \qquad \textbf{(5-6)}$$

**The reflection coefficient at the generator impedance** $\rho_s$

is given by,

$$\rho_s = \left.\frac{V_1^+}{V_1^-}\right|_{V_2^+=0} = \frac{1}{S_{11}} \qquad \textbf{(5-7)}$$

Equation 5-6 shows that the reflection coefficient $\rho_{out}$, at the output of the two-port network is affected by the generator impedance not being matched.

**Port 2 input impedance not matched and generator impedance matched**

If the generator impedance was matched, that is $\rho_s$ =0, the output reflection coefficient, which is the reflection coefficient looking into port 2, would be given by $S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_2^+=0}$. This equation can also be obtained by substituting $\rho_s$ =0 into equation 5-6.

The measurements of the scattering parameters over a band of frequencies is described in Reference 1, using a network analyser system[i].

## 5.1.1. Properties of S parameters for lossless passive junctions[ii,iii]

Some of the more important properties of S parameters are discussed below,

### *5.1.1.1. Symmetry*

If a microwave junction satisfies the condition of reciprocity and there are no active solid-state devices at the junction, then the junction is a linear passive device in which the S parameters matrix **S,** equals its transpose $\mathbf{S^T}$. That is,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \mathbf{S} = \mathbf{S^T} = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \qquad \textbf{(5-8)}$$

Which shows,

$$S_{12} = S_{21} \qquad \textbf{(5-9)}$$

As is usual for matrix notations, boldface Roman letters are used to represent matrix quantities.
The proof of equation 5-8 may be found in References 2 and 3. Equation 5-8 restricts the number of independent elements of the **S** matrix of the order *n* to $\left(n^2 + n\right)/2$. For the scattering matrix, $n = 2$, giving the number of independent elements as three. As each element has amplitude and phase, the number of independent parameters is six.

### *5.1.1.2. Unitary matrix identity*

The **S** matrix of a lossless network multiplied by its transpose complex conjugate matrix $\left(\mathbf{S}^*\right)^T$, is the unitary matrix **I.** That is, if $\left(\mathbf{S}^*\right)^T = \mathbf{S}^+$,

$$\mathbf{SS}^+ = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \qquad \textbf{(5-10)}$$

From this property the following can be derived,

#### 5.1.1.2.1. Unit Property

The sum of the products of each term of any one row or any one column of the S matrix multiplied by its complex conjugate is unity. That is,

$$\sum_i^n S_{ij} S_{ij}^* = 1 \quad \text{for } j = 1,2,3,\ldots,n \qquad \textbf{(5-11)}$$

This provides two equations,

$$S_{11}S_{11}^* + S_{12}S_{12}^* = 1 \qquad \textbf{(5-12a)}$$

$$S_{22}S_{22}^* + S_{21}S_{21}^* = 1 \qquad \textbf{(5-12b)}$$

However, from equations 5-11a, 5-11b and 5-9,

$$\left|S_{11}\right| = \left|S_{22}\right| \qquad \textbf{(5-13)}$$

which shows that the reflection coefficients of the junctions of port 1 and port 2 are equal in magnitude for a lossless passive junction. Also, from equations 5-12,

$$\left|S_{12}\right| = \sqrt{1 - \left|S_{11}\right|^2} \qquad \textbf{(5-14)}$$

#### 5.1.1.2.2. Zero Property

The sum of the products of each term of any row (or any column) multiplied by the complex conjugate of the corresponding terms of any other row (or column) is zero. That is,

$$\sum_i^n S_{ik} S_{ij}^* = 0 \quad \text{for } k \neq j, \quad \begin{matrix} k = 1,2,3,\ldots,n \\ j = 1,2,3,\ldots,n \end{matrix} \qquad \textbf{(5-15)}$$

This provides one equation, since $S_{12} = S_{21}$, that is

$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0 \qquad \textbf{(5-16)}$$

If ,

$$S_{11} = \left|S_{11}\right| e^{j\theta_{11}}, \qquad \textbf{(5-17a)}$$

$$S_{22} = |S_{11}|e^{j\theta_{22}}, \qquad \textbf{(5-17b)}$$

$$S_{12} = S_{21} = |S_{12}|e^{j\theta_{12}} = \sqrt{1-|S_{11}|^2}\, e^{j\theta_{12}}, \qquad \textbf{(5-18)}$$

then from equation 5-15,

$$|S_{11}|\sqrt{1-|S_{11}|^2}\left\{e^{j(\theta_{11}-\theta_{12})} + e^{j(\theta_{12}-\theta_{22})}\right\} = 0$$

giving.

$$\theta_{11} + \theta_{22} = 2\theta_{12} - \pi \pm 2n\pi, \quad \text{for } n = 1,2,3,\ldots,\infty$$

and therefore,

$$\theta_{12} = \frac{\theta_{11} + \theta_{22} + \pi(1 \mp 2n)}{2}, \quad \text{for } n = 1,2,3,\ldots,\infty \qquad \textbf{(5-19)}$$

Equations 5-18 and 5-19 permit the transmission coefficient $S_{12}$ or $S_{21}$ to be determined completely from the reflection coefficients $S_{11}$ and $S_{22}$.

#### *5.1.1.3. Phase shift*

If any of the reference terminal planes of the k-th port is moved away from the junction by an electrical distance $\beta_k l_k$, where k = 1 or 2 for the two-port network, then each of the coefficients of the scattering matrix will be multiplied by the factor $e^{-j\beta_k l_k}$ .
A change in the specific location of the terminal planes of a junction will affect only the phase of the scattering coefficients of the junction. That is, for the new scattering matrix **'**, after the change,

$$\mathbf{S}' = \phi\mathbf{S}\phi = \begin{bmatrix}\phi_{11} & 0 \\ 0 & \phi_{22}\end{bmatrix}\begin{bmatrix}S_{11} & S_{12} \\ S_{21} & S_{22}\end{bmatrix}\begin{bmatrix}\phi_{11} & 0 \\ 0 & \phi_{22}\end{bmatrix} \qquad \textbf{(5-20)}$$

where

$$\phi_{11} = \phi_{22} = e^{-j\beta_k l_k} \qquad \textbf{(5-21)}$$

As $S_{11}$ and $S_{22}$ involves the product of $e^{-j\beta l}$ twice in equation 5-20, the resulting scattering matrix **'** will be changed by a factor of $e^{-j2\beta l}$ .

### 5.1.2. Two-port Power-Gain equations for active networks

There are three different definitions used for the power gain of an amplifier comprised of a two-port network . In this section, these definitions will be discussed and equations presented in terms of the relevant two port scattering parameters.
The three different power gain definitions are derived from a single stage amplifier with matching networks shown in Figure 124, are,

- The power gain $G_p$,
- The transducer gain $G_t$,
- The available power gain $G_a$.

The derivations of these different gains are found in reference 1 and 2.

#### *5.1.2.1. The power gain $G_p$,*

The Power gain $G_p = \dfrac{\text{power delivered to the load}}{\text{input power to the amplifier}} = \dfrac{P_L}{P_{in}}$ **(5-22)**

Following equation 1-68, the input power is

$$P_{in} = \frac{|V_1^+|^2\left(1-|\rho_{in}|^2\right)}{2Z_o} \qquad \textbf{(5-23)}$$

The power delivered to the load, derived in Appendix A, is given as

$$P_L = \frac{1}{2Z_o}\left(1-\left|\rho_L\right|^2\right)\frac{\left|S_{21}\right|^2}{\left|1-S_{22}\rho_L\right|^2}\left|V_1^+\right|^2 \tag{5-24}$$

In terms of the S parameters the power gain is found to be

$$G_p = \frac{P_L}{P_{in}} = \frac{1}{1-\left|\rho_{in}\right|^2}\left|S_{21}\right|^2 \frac{1-\left|\rho_L\right|^2}{\left|1-S_{22}\rho_L\right|^2} \tag{5-22b}$$

where from equation 5-4, $\rho_{in} = \frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1-S_{22}\rho_L}$, giving

$$G_p = \frac{P_L}{P_{in}} = \frac{1-\left|\rho_L\right|^2}{\left|1-S_{22}\rho_L\right|^2 - \left|S_{11}-\Delta\rho_L\right|^2}\left|S_{21}\right|^2 \tag{5-22c}$$

where

$$\Delta = S_{11}S_{22} - S_{12}S_{21} \tag{5-25}$$

#### 5.1.2.1.1. Perfect matching

When the input and output networks are perfectly matched to resistive source and load impedances, respectively, the reflection coefficient at the source and at the load is zero, that is, $\rho_s = \rho_L = 0$ and equation 5-22c reduces to

$$G_p\Big|_{\rho_L,\rho_s=0} = \left|S_{21}\right|^2 \tag{5-22d}$$

#### 5.1.2.1.2. Conjugate matching

Referring to Figure 124, when $\overline{Z}_s = \overline{Z}_{in}^*$, $\overline{Z}_L = \overline{Z}_{out}^*$, $\rho_s = \rho_{in}^*$, $\rho_L = \rho_{out}^*$, equation 5-22b can be expressed as,

$$G_p\Big|_{conjmatch} = \frac{1}{1-\left|\rho_s\right|^2}\left|S_{21}\right|^2 \frac{1-\left|\rho_L\right|^2}{\left|1-S_{22}\rho_L\right|^2} \tag{5-22e}$$

### *5.1.2.2. The transducer gain $G_t$,*

Transducer gain $G_t = \frac{\text{power delivered to the load}}{\text{available input power from the source}} = \frac{P_L}{P_{avs}}$ **(5-26a)**

The available input power from the source, as derived in Appendix A, is given by

$$P_{avs} = \frac{1}{2Z_o}\frac{\left|1-\rho_{in}\rho_s\right|^2}{1-\left|\rho_s\right|^2}\left|V_1^+\right|^2 \tag{5-27}$$

From equation 5-26 and 5-24, the transducer gain is given by

$$G_t = \frac{P_L}{P_{avs}} = \frac{1-\left|\rho_s\right|^2}{\left|1-\rho_{in}\rho_s\right|^2}\left|S_{21}\right|^2 \frac{1-\left|\rho_L\right|^2}{\left|1-S_{22}\rho_L\right|^2} \tag{5-26b}$$

An alternative form, in terms of output parameters is given by

$$G_t = \frac{P_L}{P_{avs}} = \frac{1-|\rho_L|^2}{|1-\rho_{out}\rho_L|^2}|S_{21}|^2 \frac{1-|\rho_s|^2}{|1-S_{11}\rho_s|^2} \qquad \textbf{(5-26c)}$$

where,

$$\rho_{out} = \frac{S_{22}-\Delta\rho_s}{1-S_{11}\rho_s} \qquad \textbf{(5-6)}$$

Another alternative form, in terms of parameters not involving the input reflection coefficient $\rho_{in}$, or output reflection coefficient $\rho_{out}$, is given by

$$G_t = \frac{P_L}{P_{avs}} = \frac{\left(1-|\rho_s|^2\right)\left(1-|\rho_L|^2\right)|S_{21}|^2}{\left|(1-S_{22}\rho_L)(1-S_{11}\rho_s)-S_{12}S_{21}\rho_s\rho_L\right|^2} \qquad \textbf{(5-26d)}$$

#### 5.1.2.2.1. Perfect matching

When the input and output networks are perfectly matched to resistive source and load impedances, respectively, the reflection coefficient at the source and at the load is zero, that is, $\rho_s = \rho_L = 0$ and equation 5-26d reduces to

$$G_t\big|_{\rho_L,\rho_s=0} = |S_{21}|^2 \qquad \textbf{(5-26e)}$$

#### 5.1.2.2.2. Conjugate matching

Again, referring to Figure 124, when $\overline{Z}_s = \overline{Z}_{in}^*$, $\overline{Z}_L = \overline{Z}_{out}^*$, $\rho_s = \rho_{in}^*$, $\rho_L = \rho_{out}^*$, equation 5-26b can be expressed as,

$$G_t\big|_{conjmatch} = \frac{1}{1-|\rho_s|^2}|S_{21}|^2 \frac{1-|\rho_L|^2}{|1-S_{22}\rho_L|^2} \qquad \textbf{(5-26f)}$$

where $\rho_s\rho_s^* = |\rho_s|^2$, giving the same equation as the conjugate matching of the power gain expression, equation 5-22b.

#### 5.1.2.2.3. Maximum unilateral transducer power gain[2]

The conditions for the maximum unilateral transducer power gain are

$\rho_s = S_{11}^*$, $\rho_L = S_{22}^*$ and $|S_{12}|^2 = 0$.

Using these conditions, equation 5-26d becomes,

$$G_t\big|_{uni\max} = \frac{\left(1-|S_{11}|^2\right)\left(1-|S_{22}|^2\right)|S_{21}|^2}{\left|\left(1-|S_{22}|^2\right)\left(1-|S_{11}|^2\right)-S_{12}S_{21}\rho_s\rho_L\right|^2} = \frac{|S_{21}|^2}{\left(1-|S_{22}|^2\right)\left(1-|S_{11}|^2\right)} \qquad \textbf{(5-26g)}$$

### ***5.1.2.3. The available power gain*** $G_a$,

Available power gain $G_a = \dfrac{\text{available power from the network}}{\text{available input power from the source}} = \dfrac{P_{avnw}}{P_{avs}}$ **(5-27a)**

The available power from the network, as derived in Appendix A, is given by,

$$P_{avnw} = \frac{\left|1-\rho_{in}\rho_s\right|^2}{\left|1-S_{11}\rho_s\right|^2}\frac{\left|S_{21}\right|^2}{2Z_o}\frac{\left|V_1^+\right|^2}{1-\left|\rho_{out}\right|^2} \qquad \textbf{(5-28)}$$

From equation 5-27 and 5-28, the available power gain is given by

$$G_a = \frac{P_{avnw}}{P_{avs}} = \frac{1-\left|\rho_s\right|^2}{\left|1-S_{11}\rho_s\right|^2}\left|S_{21}\right|^2\frac{1}{1-\left|\rho_{out}\right|^2} \qquad \textbf{(5-27b)}$$

Substituting equation 5-6 for $\rho_{out}$ into equation 5-27b,

$$G_a = \frac{P_{avnw}}{P_{avs}} = \frac{1-\left|\rho_s\right|^2}{\left|1-S_{11}\rho_s\right|^2-\left|S_{22}-\Delta\rho_s\right|^2}\left|S_{21}\right|^2 \qquad \textbf{(5-27c)}$$

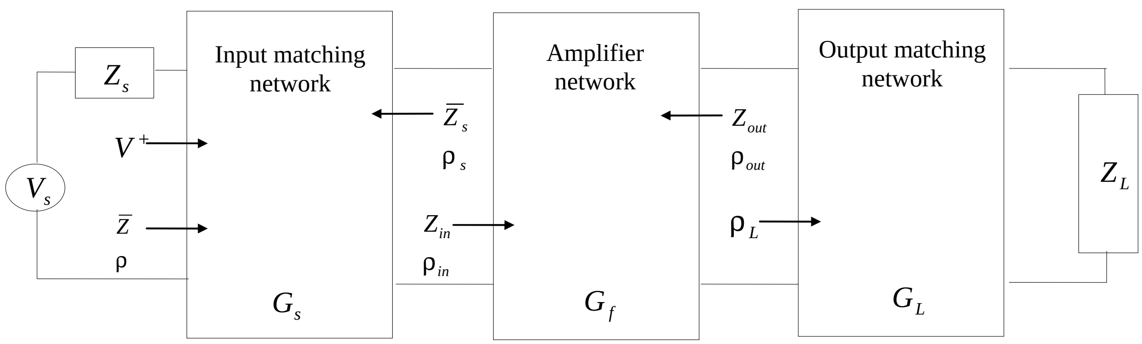



**Figure 124 Gain definitions network**

### *5.1.2.4. The maximum power gain using conjugate impedance matching*

Conjugate impedance matching at the input and output is only used if the amplifier or device is unconditionally stable. This form of matching permits the maximum power gain for the device to be obtained. Referring to Figure 124, conjugate impedance matching occurs when $\overline{Z}_s = \overline{Z}_{in}^*,\ \overline{Z}_L = \overline{Z}_{out}^*,\ \rho_s = \rho_{in}^*,\ \rho_L = \rho_{out}^*$. Using these conditions, with equation 5-27d, the maximum power gain for the device, as shown derived by Collin (Reference 3), can be expressed as,

$$G_{a\max} = \left|\frac{S_{21}}{S_{12}}\right|\left(K \pm \sqrt{K^2-1}\right) \qquad \textbf{(5-29)}$$

where the stability factor K, in terms of the scattering parameters is given as

$$K = \frac{1-\left|S_{11}\right|^2-\left|S_{22}\right|^2+\left|\Delta\right|^2}{2\left|S_{12}S_{21}\right|} \qquad \textbf{(5-30)}$$

and the determinant of the scattering matrix is given by,

$$\Delta = S_{11}S_{22}-S_{12}S_{21} \qquad \textbf{(5-31)}$$

In equation 5-29, the condition for using the negative sign is when the computed value of $A_i > 0$, in equations 5-32, and the condition for using the positive sign is when $A_i < 0$.

$$A_1 = 1+\left|S_{11}\right|^2-\left|S_{22}\right|^2-\left|\Delta\right|^2 \qquad \textbf{(5-32a)}$$

$$A_2 = 1 - |S_{11}|^2 + |S_{22}|^2 - |\Delta|^2 \quad \textbf{(5-32b)}$$

For an absolutely stable device $A_i > 0$, and K > 1.

## *5.2. Amplifier stability*

A primary concern in the design of a microwave amplifier is that it does not oscillate. The designer must be aware of the conditions that cause oscillation and those which prevent them. These conditions can be considered as stability conditions, of which there are two;

1. *Unconditional stability*
   An amplifier is unconditionally stable if the **real part** of its input and output impedances are positive for **any** passive terminations at a specific frequency, or if

$$\begin{aligned} |\rho_{in}| &< 1 \qquad \text{for all } |\rho_L| < 1 \\ |\rho_{out}| &< 1 \qquad \text{for all } |\rho_s| < 1 \end{aligned} \quad \textbf{(5-33)}$$

   where $|\rho_{in}|$ and $|\rho_{out}|$ are defined by equations 5-4 and 5-6.
   When these conditions hold, conjugate matching can be used

2. *Conditional stability*
   An amplifier is conditionally stable if the **real part** of its input and output impedances are positive for **some** passive terminations at a specific frequency, or if

$$\begin{aligned} |\rho_{in}| &< 1 \qquad \text{for some } |\rho_L| < 1 \\ |\rho_{out}| &< 1 \qquad \text{for some } |\rho_s| < 1 \end{aligned} \quad \textbf{(5-34)}$$

   where $|\rho_{in}|$ and $|\rho_{out}|$ are defined by equations 5-4 and 5-6.

### 5.2.1. Stability circles

The boundary conditions for stability are given by values of $\rho_L$ or $\rho_s$ where,

$$\begin{aligned} |\rho_{in}| &= 1 \\ |\rho_{out}| &= 1 \end{aligned} \quad \textbf{(5-35)}$$

The mapping of equations 5-35 into the $\rho_L$ or $\rho_s$ plane corresponds to circles defining the boundary between stable and unstable regions. The region either inside or outside of the circle can represent the stable region of a device or amplifier. To determine which region represents the stable operating condition a test point is chosen and its stability determined using the stability conditions given by equations 5-33 and 5-34. It is these regions and what the conditions are for stability, which will be considered next.

#### *5.2.1.1. Load reflection coefficient stability circle*

The load reflection coefficient stability circle is found from equation 5-35, by finding the equation of a circle in terms of $\rho_L$. This circle can be mapped directly onto the Smith chart plane, and may lie within, intersect with or lie completely outside of the Smith chart.
Considering equations 5-4, and 5-35 we find,

$$|\rho_{in}| = \left| S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} \right| = \left| \frac{\Delta\rho_L - S_{11}}{S_{22}\rho_L - 1} \right| = 1 \quad \textbf{(5-36)}$$

that is,

$$|\Delta\rho_L - S_{11}|^2 = |S_{22}\rho_L - 1|^2$$

which is equivalent to

$(S_{22}\rho_L - 1)(S_{22}^*\rho_L^* - 1) = (\Delta\rho_L - S_{11})(\Delta^*\rho_L^* - S_{11}^*)$ .

By expanding and collecting terms, using

$$B_2 = S_{22}^* - \Delta^* S_{11} \qquad \textbf{(5-37)}$$

$$\left[\rho_L - \frac{B_2}{\left(|S_{22}|^2 - |\Delta|^2\right)}\right]\left[\rho_L^* - \frac{B_2^*}{\left(|S_{22}|^2 - |\Delta|^2\right)}\right] = \frac{B_2 B_2^* + \left(|S_{11}|^2 - 1\right)\left(|S_{22}|^2 - |\Delta|^2\right)}{\left(|S_{22}|^2 - |\Delta|^2\right)^2}$$

which reduces to

$$\left|\rho_L - \frac{B_2}{\left(|S_{22}|^2 - |\Delta|^2\right)}\right| = \pm\frac{|S_{12}S_{21}|}{\left(|S_{22}|^2 - |\Delta|^2\right)} \qquad \textbf{(5-38)}$$

Equation 5-38 is the equation of a circle,

of centre
$$c_L = \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2} \qquad \textbf{(5-39)}$$

and radius

$$r_L = \frac{|S_{12}S_{21}|}{\left||S_{22}|^2 - |\Delta|^2\right|} \qquad \textbf{(5-40)}$$

Figure 124 shows a typical plot of $c_L$ and $r_L$ on the $\rho_L$ plane with the Smith chart. The position of $c_L$ is given in polar coordinates relative to the centre of the Smith chart. In this diagram, the value of the polar distance $|c_L|$, is greater than the Smith chart radius (1) plus the radius of the stability circle $r_L$, that is, $|c_L| > 1 + r_L$ .

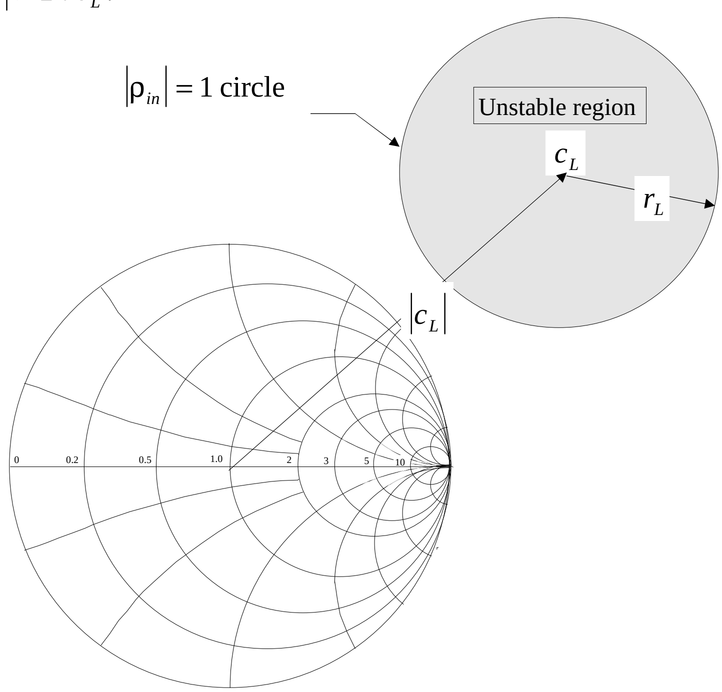



**Figure 125** **Definition of $c_L$ and $r_L$ on the $\rho_L$ plane and condition 1 implementation of unconditional stability**

The stability criteria is given by,

*1. Stable:*

where K and $\Delta$ are defined by equations 5-30 and 5-31.

a) ***Unconditionally:***

*condition 1*

$$|c_L| - r_L > 1$$

$$\mathrm{K} > 1, \quad |S_{11}| < 1, \quad |S_{22}| > |\Delta|$$

As shown in Figure 125 this is represented by the $|\rho_{in}| = 1$ circle drawn with its centre far enough outside of the Smith chart boundary, the boundary being $\rho_L = 1$ so that it does not touch or intersect with the Smith chart. The unstable region of $\rho_L$, that is the region which does not include for perfect matching the centre of the Smith chart, $\rho_L = 0$, is represented by the region inside the $|\rho_{in}| = 1$ circle. This region is where $|\rho_{in}| > 1$. All regions **outside** of the $|\rho_{in}| = 1$ circle, including the Smith chart, represent **regions of stability** of $\rho_L$, that is regions where $|\rho_{in}| < 1$. These regions are stable because they include the $\rho_L = 0$ point..

*condition 2*

$$r_L - |c_L| > 1$$

$$K > 1 + \frac{|\Delta|^2 - 1}{2|S_{12}S_{21}|}, \quad |S_{11}| < 1, \quad |S_{22}| < |\Delta|$$

As shown in Figure 126 this is represented by the $|\rho_{in}| = 1$ circle drawn outside of the Smith chart boundary, the boundary being $\rho_L = 1$, but which includes the complete Smith chart. As this circle includes the centre of the Smith chart $\rho_L = 0$, any point **inside** this $|\rho_{in}| = 1$ circle represents a region of $\rho_L$ that is **stable,** that is, a region where $|\rho_{in}| < 1$. The region **outside** of the $|\rho_{in}| = 1$ circle does not include the $\rho_L = 0$ point and is where $|\rho_{in}| > 1$. It represents the **region of instability** of $\rho_L$.

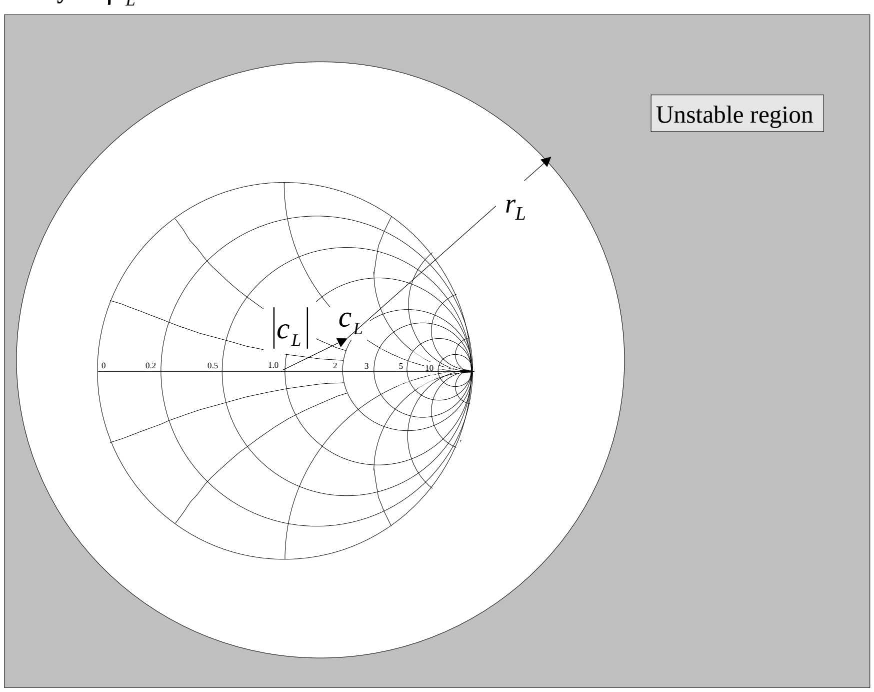



**Figure 126** **Condition 2 implementation of unconditional stability**

***b) Conditionally:***

<u>*condition 1*</u>

$$r_L + |c_L| < 1 \text{ and } |c_L| < 1$$
$$\text{K} > 1, \quad |S_{11}|^2 > 1 - (2K-1)|S_{12}S_{21}|, \quad |S_{22}| < |\Delta|$$

The load stability circle may lie entirely inside the Smith chart, as shown in Figure 127. The region within the load stability circle is absolutely stable.

<u>*condition 2*</u>

$$|c_L|^2 < |1 - r_L|^2 \text{ and } |c_L| < 1$$
$$\text{K} < \text{-1}, \quad |S_{11}|^2 < 1 - (2K-1)|S_{12}S_{21}|, \quad |S_{22}| > |\Delta|$$

The load stability circle may lie entirely inside the Smith chart, as shown in Figure 127. As devices with K < -1 are not likely to be encountered in practice, this case will possibly never occur.

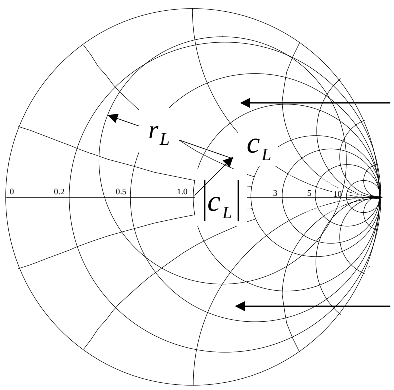



**Figure 127 Condition 1 and 2 implementation of conditional stability**

<u>*condition 3*</u>

$$r_L + |c_L| > 1 \text{ and } |c_L| < 1$$
$$|S_{11}| < 1, \quad |S_{22}| < |\Delta|$$

The load stability circle $|\rho_{in}| = 1$, intersects the Smith chart at two points and encloses the origin, as shown in Figure 128. The origin is enclosed only when $|S_{11}| < 1$ and $|\Delta| > |S_{22}|$. When this happens, the interior values of the $|\rho_{in}| = 1$ circle are stable values of $\rho_L$.

<u>*condition 4*</u>

$$r_L + |c_L| > 1 \text{ and } |c_L| < 1$$
$$|S_{11}| > 1, \quad |S_{22}| > |\Delta|$$

The load stability circle $|\rho_{in}| = 1$, intersects the Smith chart at two points and encloses the origin, as shown in Figure 128. The origin is enclosed only when $|S_{11}| > 1$ and $|\Delta| < |S_{22}|$. When this happens, the exterior values of the $|\rho_{in}| = 1$ circle are stable values of $\rho_L$.

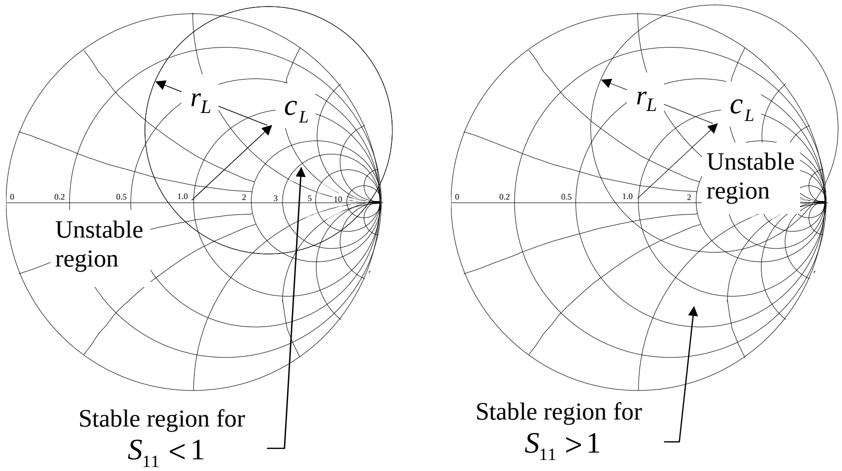



**Figure 128** $\left|\rho_{in}\right| = 1$ **circles the origin -Condition 3 and 4 implementation of conditional stability**

*condition 5*

$$r_L + \left|c_L\right| > 1 \text{ and } \left|c_L\right| > 1$$
$$\left|S_{11}\right| < 1, \quad \left|S_{22}\right| > \left|\Delta\right|$$

The load stability circle $\left|\rho_{in}\right| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 129. The origin is not enclosed when $\left|S_{11}\right| < 1$ and $\left|\Delta\right| < \left|S_{22}\right|$. When this happens, the **exterior** values of the $\left|\rho_{in}\right| = 1$ circle are stable values of $\rho_L$.

*condition 6*

$$r_L + \left|c_L\right| > 1 \text{ and } \left|c_L\right| > 1$$
$$\left|S_{11}\right| > 1, \quad \left|S_{22}\right| < \left|\Delta\right|$$

The load stability circle $\left|\rho_{in}\right| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 129. The origin is not enclosed when $\left|S_{11}\right| > 1$ and $\left|\Delta\right| > \left|S_{22}\right|$. When this happens, the **interior** values of the $\left|\rho_{in}\right| = 1$ circle are stable values of $\rho_L$.

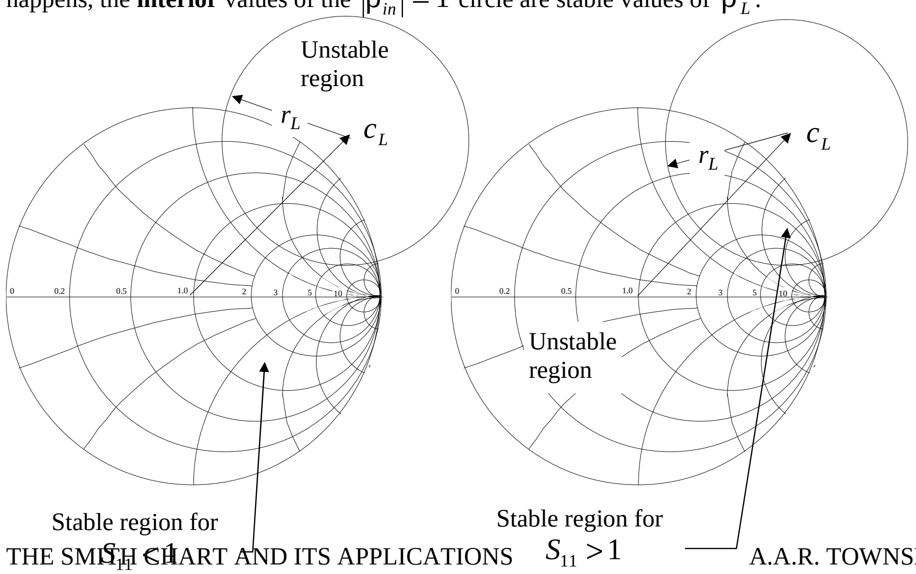


**Figure 129** $\left|\rho_{in}\right| = 1$ **does not circle the origin -Condition 5 and 6 implementation of conditional stability**

*2. Unstable or potentially unstable:*

*condition 1*

$$\left|c_L\right| - r_L > 1$$

$$\text{K} < -1, \quad \left|S_{11}\right|^2 > 1, \quad \left|S_{22}\right| < \left|\Delta\right|$$

The load stability circle is placed entirely outside of the Smith chart as shown in Figure 125. However, all values of $\rho_L$ on the Smith chart are unstable values.

*condition 2*

$$r_L - \left|c_L\right| > 1$$

$$\text{K} < -1, \quad \left|S_{11}\right| > 1, \quad \left|S_{22}\right| > \left|\Delta\right|$$

The load stability circle may enclose the Smith chart, as shown in Figure 126, however, all values of $\rho_L$ on the Smith chart are unstable values.

### *5.2.1.2. Source reflection coefficient stability circle*

From,

$$\left|\rho_{out}\right| = \left|S_{22} + \frac{S_{12}S_{21}\rho_s}{1 - S_{11}\rho_s}\right| = \left|\frac{\Delta\rho_s - S_{22}}{S_{11}\rho_s - 1}\right| = 1 \qquad \textbf{(5-41)}$$

and the method used in section 5.2.1.1, the equation of the source stability circle, which is the circle of source reflection coefficient $\rho_s$ values that make $\left|\rho_{out}\right| = 1$, is found to be a circle,

of centre

$$c_s = \frac{S_{11}^* - \Delta^* S_{22}}{\left|S_{11}\right|^2 - \left|\Delta\right|^2} \qquad \textbf{(5-42)}$$

and radius

$$r_s = \frac{\left|S_{12}S_{21}\right|}{\left|\left|S_{11}\right|^2 - \left|\Delta\right|^2\right|} \qquad \textbf{(5-43)}$$

These equations are the same as equations 5-39 and 5-40, with $S_{11}$ and $S_{22}$ interchanged.
The stability criteria is similar to that for the load reflection coefficient stability circle, the difference being that $S_{11}$ and $S_{22}$ are interchanged throughout. That is,

*1. Stable:*

where K and $\Delta$ are defined by equations 5-30 and 5-31.
Unconditionally:

condition 1

$$\left|c_s\right| - r_s > 1$$

$$\mathrm{K} > 1, \quad |S_{22}| < 1, \quad |S_{11}| > |\Delta|$$

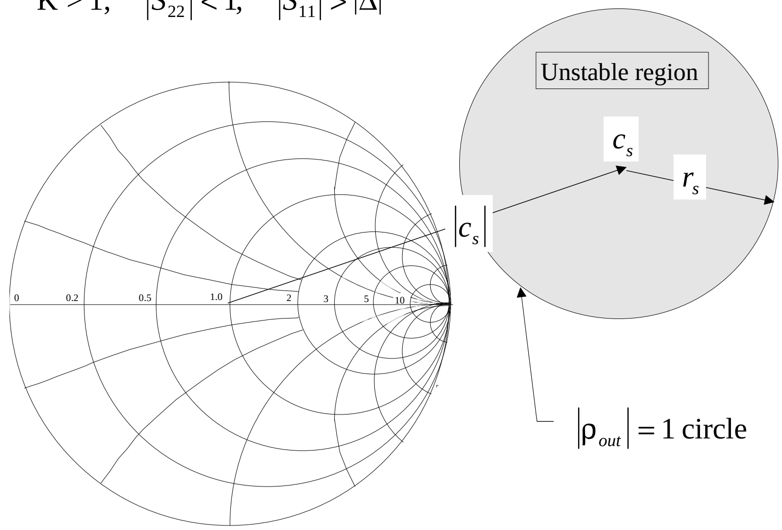



**Figure 130** **Definition of $c_s$ and $r_s$ on the $\rho_s$ plane and condition 1 implementation of unconditional stability**

As shown in Figure 130 this is represented by the $|\rho_{out}| = 1$ circle drawn with its centre far enough outside of the Smith chart boundary, the boundary being $\rho_s = 1$ so that it does not touch or intersect with the Smith chart.
In this diagram, the value of the polar distance $|c_s|$, is greater than the Smith chart radius (1) plus the radius of the stability circle $r_s$, that is, $|c_s| > 1 + r_s$.

The unstable region of $\rho_s$, that is the region which does not include for perfect matching the centre of the Smith chart, $\rho_s = 0$, is represented by the region inside the $|\rho_{out}| = 1$ circle. This region is where $|\rho_{out}| > 1$. All regions **outside** of the $|\rho_{iout}| = 1$ circle, including the Smith chart, represent **regions of stability** of $\rho_s$, that is regions where $|\rho_{out}| < 1$. These regions are stable because they include the $\rho_s = 0$ point..

<u>*condition 2*</u>

$$r_s - |c_s| > 1$$

$$K > 1 + \frac{|\Delta|^2 - 1}{2|S_{12}S_{21}|}, \quad |S_{22}| < 1, \quad |S_{11}| < |\Delta|$$

As shown in Figure 131 this is represented by the $|\rho_{out}| = 1$ circle drawn outside of the Smith chart boundary, the boundary being $\rho_s = 1$, but which includes the complete Smith chart. As this circle includes the centre of the Smith chart $\rho_s = 0$, any point **inside** this $|\rho_{out}| = 1$ circle represents a region of $\rho_s$ that is **stable,** that is, a region where $|\rho_{out}| < 1$. The region **outside** of the $|\rho_{out}| = 1$ circle does not include the $\rho_s = 0$ point and is where $|\rho_{out}| > 1$. It represents the **region of instability** of $\rho_s$.

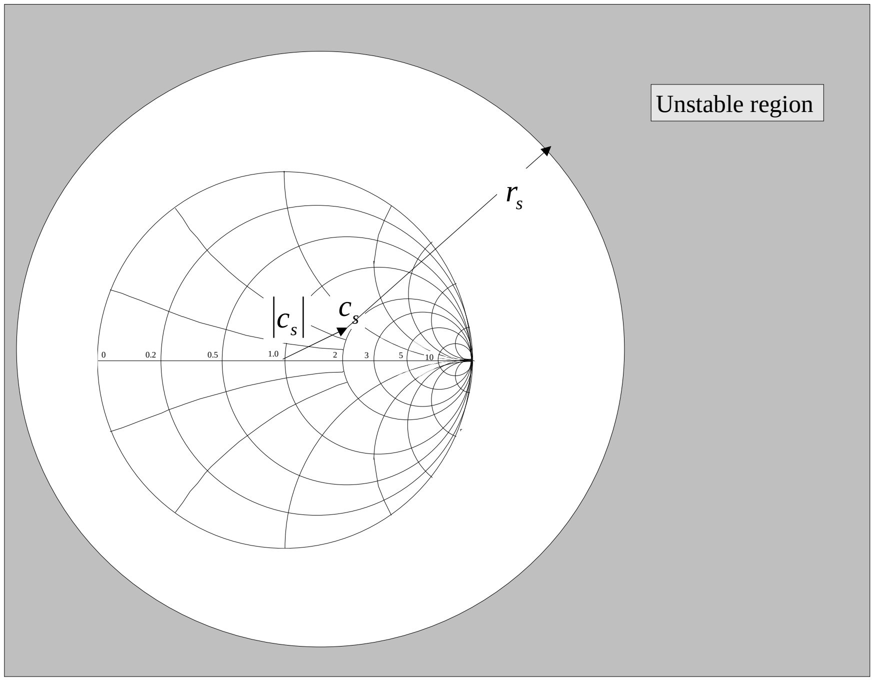



**Figure 131 Condition 2 implementation of unconditional stability**

***b) Conditionally:***

<u>*condition 1*</u>

$$r_s + |c_s| < 1 \text{ and } |c_s| < 1$$

$$\text{K} > 1, \quad |S_{22}|^2 > 1-(2K-1)|S_{12}S_{21}|, \quad |S_{11}| < |\Delta|$$

The load stability circle may lie entirely inside the Smith chart, as shown in Figure 132. The region within the load stability circle is absolutely stable.

<u>*condition 2*</u>

$$|c_s|^2 < |1-r_s|^2 \text{ and } |c_s| < 1$$

$$\text{K} < -1, \quad |S_{22}|^2 < 1-(2K-1)|S_{12}S_{21}|, \quad |S_{11}| > |\Delta|$$

The load stability circle may lie entirely inside the Smith chart, as shown in Figure 132. As devices with K < -1 are not likely to be encountered in practice, this case will possibly never occur.

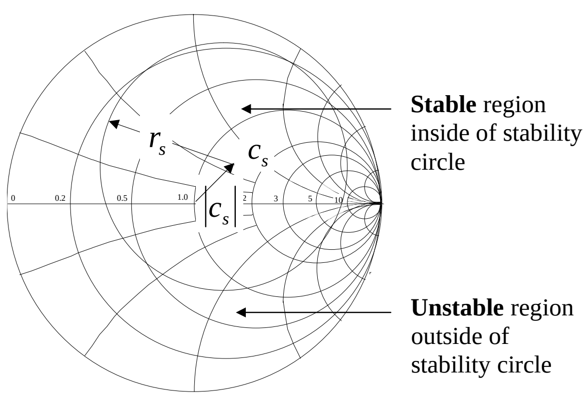



**Figure 132 Condition 1 and 2 implementation of conditional stability**

*condition 3*

$$r_s + |c_s| > 1 \text{ and } |c_s| < 1$$
$$|S_{22}| < 1, \quad |S_{11}| < |\Delta|$$

The load stability circle $|\rho_{out}| = 1$, intersects the Smith chart at two points and encloses the origin, as shown in Figure 133. The origin is enclosed only when $|S_{22}| < 1$ and $|\Delta| > |S_{11}|$. When this happens, the interior values of the $|\rho_{out}| = 1$ circle are stable values of $\rho_s$.

*condition 4*

$$r_s + |c_s| > 1 \text{ and } |c_s| < 1$$
$$|S_{22}| > 1, \quad |S_{11}| > |\Delta|$$

The load stability circle $|\rho_{out}| = 1$, intersects the Smith chart at two points and encloses the origin, as shown in Figure 133. The origin is enclosed only when $|S_{22}| > 1$ and $|\Delta| < |S_{11}|$. When this happens, the exterior values of the $|\rho_{out}| = 1$ circle are stable values of $\rho_s$.

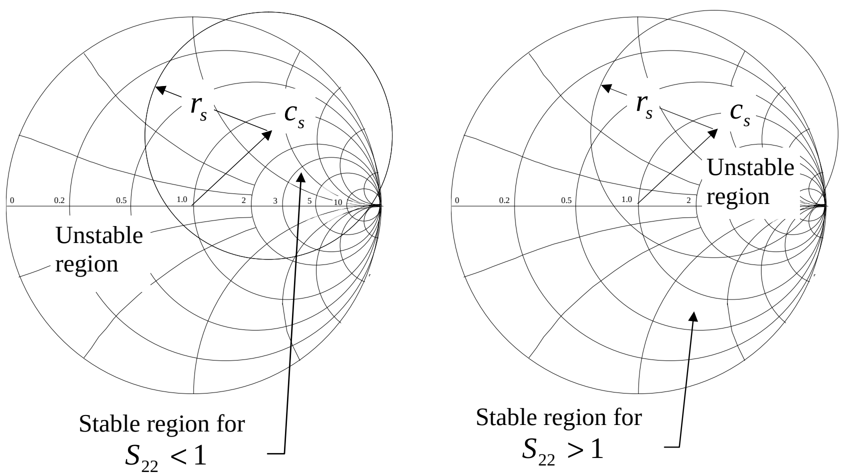



**Figure 133** $|\rho_{out}| = 1$ **circles the origin -Condition 3 and 4 implementation of conditional stability**

*condition 5*

$$r_s + |c_s| > 1 \text{ and } |c_s| > 1$$
$$|S_{22}| < 1, \quad |S_{11}| > |\Delta|$$

The load stability circle $|\rho_{out}| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 134. The origin is not enclosed when $|S_{22}| < 1$ and $|\Delta| < |S_{11}|$. When this happens, the **exterior** values of the $|\rho_{out}| = 1$ circle are stable values of $\rho_s$.

*condition 6*

$$r_s + |c_s| > 1 \text{ and } |c_s| > 1$$
$$|S_{22}| > 1, \quad |S_{11}| < |\Delta|$$

The load stability circle $|\rho_{out}| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 134. The origin is not enclosed when $|S_{22}| > 1$ and $|\Delta| > |S_{11}|$. When this happens, the **interior** values of the $|\rho_{out}| = 1$ circle are stable values of $\rho_s$.

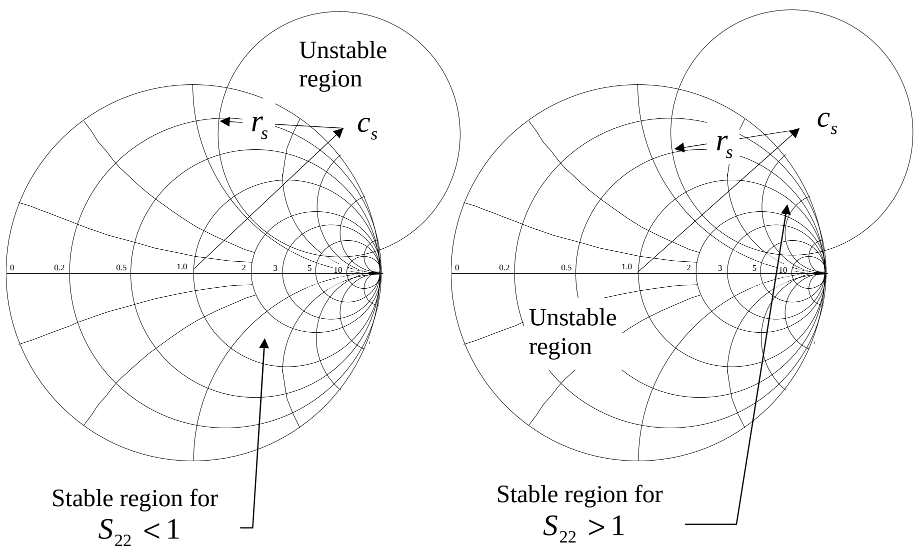



**Figure 134** $|\rho_{out}| = 1$ **does not circle the origin -Condition 5 and 6 implementation of conditional stability**

2. *Unstable or potentially unstable:*

*condition 1*

$$|c_s| - r_s > 1$$
$$K < -1, \quad |S_{22}|^2 > 1, \quad |S_{11}| < |\Delta|$$

The load stability circle is placed entirely outside of the Smith chart as shown in Figure 130. However, all values of $\rho_s$ on the Smith chart are unstable values.

*condition 2*

$$r_s - |c_s| > 1$$
$$K < -1, \quad |S_{22}| > 1, \quad |S_{11}| > |\Delta|$$

The load stability circle may enclose the Smith chart, as shown in Figure 131, however, all values of $\rho_s$ on the Smith chart are unstable values.

### *5.2.1.3. Stability design strategy*

To design an amplifier, **both** the load reflection coefficient stability circle and the source reflection coefficient stability circle must be used together in order to ensure that the amplifier is stable. The use of one stability circle by itself will not ensure that stability will be achieved.

### *5.2.1.4. S-parameters and the transistor data sheet*[iv]

Before proceeding with an example on stability circles, it may be beneficial to consider some points which are relevant to microwave transistor data sheets. he measurement of the transistor s - parameters may be given in both the chip form and package form. Differences will arise because the package form will have additional inductance and capacitance which the chip form will not have. When comparing S-parameter information, it is important to ensure that the bias conditions for the transistors being compared are the same. Also, if the device is to be used in an amplifier designed with S-parameters of a given bias conditions, the bias conditions of the constructed amplifier should be the same in order that the design be valid. This is because the S-parameters can vary widely with different biases. Should the bias conditions of the constructed amplifier be different from those conditions specified for the S-parameters, then the S-parameters are required to be determined for those conditions. This can be achieved using specialized test equipment, described in References 1 and 4. The values of the S-parameters will vary also with frequency. The packaged transistor frequency band having a larger reactance spread on the Smith chart than the chip transistor form. Figure 135 show a typical spread for the $S_{11}$ and $S_{22}$ parameters in both chip and packaged form for a bipolar transistor with a fixed $V_{cb}$ and $I_c$.

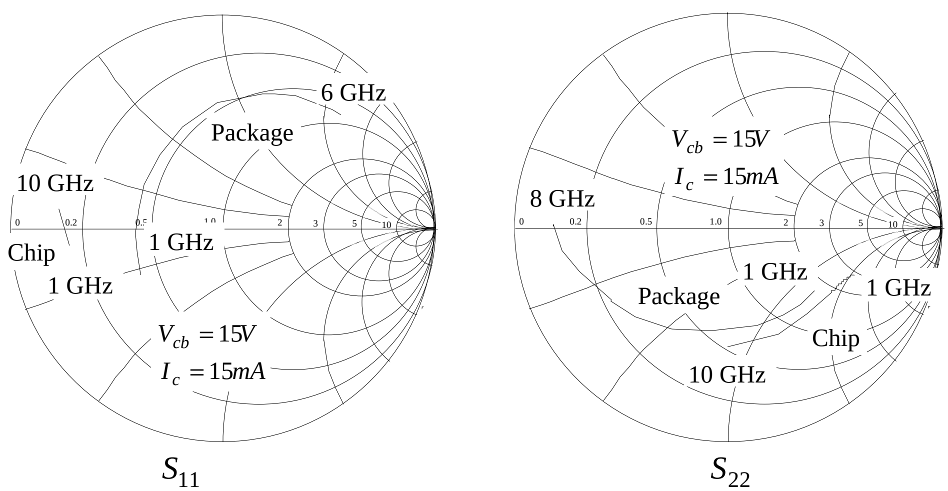



**Figure 135 Typical bipolar transistor $S_{11}$ and $S_{22}$ parameters**

Figure 136 show a typical spread for the $S_{21}$ and $S_{12}$ parameters in both chip and packaged form for a bipolar transistor with a fixed $V_{cb}$ and $I_c$.

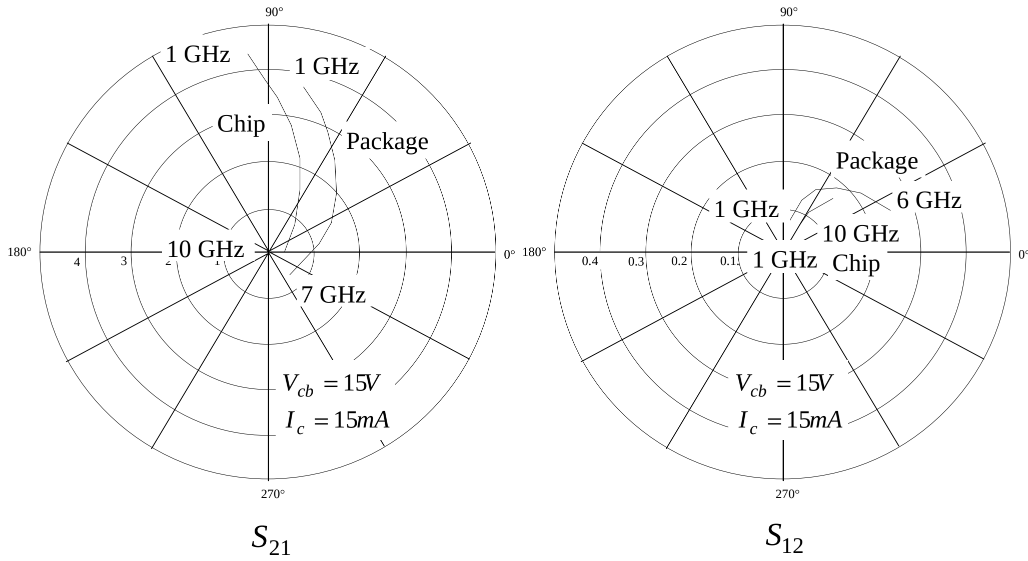



**Figure 136** **Typical bipolar transistor $S_{21}$ and $S_{12}$ parameters**

For bipolar transistors in the common emitter configuration (as shown in Figures 135 and 136), $|S_{11}|$ and $|S_{22}|$ are usually less than unity. This applies to GaAs MESFETs in the common source configuration.
For bipolar transistors in the common base configuration, $|S_{11}|$ and $|S_{22}|$ are usually greater than unity. This applies to GaAs MESFETs in the common gate configuration.

In design, it is not uncommon to select the upper, lower and midband frequencies of the anticipated band of frequencies and from the S-parameters at these frequencies, plot the source and load stability circles. This will permit the determination of the stability of the circuit over the required band.

## 5.2.1.5. Some definitions of frequencies used in data sheets

### 5.2.1.5.1. Current Gain Bandwidth Product ( $f_t$ )

Associated with microwave bipolar transistors only, the current gain bandwidth product $f_t$, is defined as the frequency at which the short-circuit current gain or forward current transfer ratio $h_{fe}$, becomes unity. Because of the difficulty in obtaining a true short-circuit at microwave frequencies, $h_{fe}$ is usually derived from the S-parameters and is found to be the lowest of the other upper frequency limits of the device. The other frequency limits being defined below.

### 5.2.1.5.2. Power Gain Bandwidth Product ( $f_s$ )

The power gain bandwidth product is defined as the frequency at which the transducer power gain for perfect matching $G_t|_{\rho_L,\rho_s=0} = |S_{21}|^2$, equals unity. The parameter $|S_{21}|^2$ appears on data sheets because of its frequent use by circuit designers. The power gain bandwidth product is higher in frequency than the current gain bandwidth product.

### 5.2.1.5.3. Maximum Unilateral Transducer Power Gain Bandwidth Product ( $f_{\max}$ )

When the reverse power gain in a feedback amplifier is set to zero, that is $|S_{12}|^2 = 0$, by adjusting a lossless reciprocal feedback network connected around the microwave amplifier device and the source reflection coefficient $\rho_s = S_{11}^*$ and $\rho_L = S_{22}^*$, the maximum unilateral transducer gain is obtained, that is

$$G_t\big|_{uni\,\max} = \frac{|S_{21}|^2}{\left(1-|S_{22}|^2\right)\left(1-|S_{11}|^2\right)} \quad \textbf{(5-26g)}$$

The frequency at which this gain is unity, is the maximum unilateral transducer power gain bandwidth product $f_{\max}$. This frequency is higher than that of the power gain bandwidth product.

Example 28 demonstrates the use of stability circles using S-parameter data.

Having completed the discussion on stability, the next section will deal with gain circles so that the maximum achievable gain for a stable circuit can be determined.

EXAMPLE 27
A manufacture supplies the following S-parameter data for a microwave bipolar transistor,

**Table 5-1 S-Parameter Data - Magnitude and angle**

$V_{CE} = 6V$, $I_C = 4mA$

| ***Frequency*** | $S_{11}$ | | $S_{21}$ | | $S_{12}$ | | $S_{22}$ | |
|---|---|---|---|---|---|---|---|---|
| **GHz** | $\lvert S_{11}\rvert$ | $\angle\theta$ | $\lvert S_{21}\rvert$ | $\angle\theta$ | $\lvert S_{12}\rvert$ | $\angle\theta$ | $\lvert S_{22}\rvert$ | $\angle\theta$ |
| **1** | 0.69 | -83 | 7.03 | 115 | 0.07 | 45 | 0.74 | -35 |
| **2** | 0.51 | -131 | 4.14 | 82 | 0.08 | 31 | 0.57 | -35 |
| **3** | 0.45 | -165 | 2.97 | 60 | 0.10 | 26 | 0.50 | -65 |
| **4** | 0.47 | 166 | 2.26 | 40 | 0.11 | 18 | 0.45 | -83 |
| **5** | 0.52 | 150 | 1.80 | 21 | 0.12 | 13 | 0.44 | -103 |
| **6** | 0.53 | 143 | 1.48 | 6 | 0.12 | 9 | 0.48 | -123 |
| **7** | 0.53 | 138 | 1.23 | -8 | 0.13 | 6 | 0.56 | -139 |
| **8** | 0.50 | 135 | 1.07 | -19 | 0.14 | 4 | 0.62 | -147 |

$V_{CE} = 6V$, $I_C = 6mA$

| ***Frequency*** | $S_{11}$ | | $S_{21}$ | | $S_{12}$ | | $S_{22}$ | |
|---|---|---|---|---|---|---|---|---|
| **GHz** | $\lvert S_{11}\rvert$ | $\angle\theta$ | $\lvert S_{21}\rvert$ | $\angle\theta$ | $\lvert S_{12}\rvert$ | $\angle\theta$ | $\lvert S_{22}\rvert$ | $\angle\theta$ |
| **1** | 0.61 | -97 | 8.25 | 109 | 0.06 | 42 | 0.66 | -40 |
| **2** | 0.48 | -144 | 4.68 | 79 | 0.08 | 32 | 0.49 | -40 |
| **3** | 0.45 | -176 | 3.25 | 58 | 0.09 | 28 | 0.42 | -67 |
| **4** | 0.48 | 159 | 2.44 | 40 | 0.10 | 23 | 0.40 | -84 |
| **5** | 0.52 | 145 | 1.94 | 21 | 0.11 | 19 | 0.40 | -104 |
| **6** | 0.53 | 139 | 1.61 | 7 | 0.12 | 15 | 0.44 | -124 |
| **7** | 0.53 | 134 | 1.34 | -7 | 0.13 | 11 | 0.52 | -139 |
| **8** | 0.49 | 131 | 1.17 | -18 | 0.14 | 7 | 0.58 | -147 |

$V_{CE} = 6V$, $I_C = 10mA$

| ***Frequency*** | $S_{11}$ | | $S_{21}$ | | $S_{12}$ | | $S_{22}$ | |
|---|---|---|---|---|---|---|---|---|
| **GHz** | $\lvert S_{11}\rvert$ | $\angle\theta$ | $\lvert S_{21}\rvert$ | $\angle\theta$ | $\lvert S_{12}\rvert$ | $\angle\theta$ | $\lvert S_{22}\rvert$ | $\angle\theta$ |
| **1** | 0.54 | -113 | 9.34 | 103 | 0.05 | 41 | 0.58 | -42 |
| **2** | 0.46 | -157 | 5.09 | 75 | 0.07 | 37 | 0.44 | -53 |
| **3** | 0.45 | -174 | 3.48 | 56 | 0.08 | 35 | 0.38 | -66 |
| **4** | 0.48 | 152 | 2.59 | 38 | 0.10 | 30 | 0.36 | -83 |
| **5** | 0.53 | 140 | 2.06 | 21 | 0.11 | 25 | 0.37 | -104 |
| **6** | 0.54 | 134 | 1.71 | 7 | 0.12 | 20 | 0.41 | -124 |
| **7** | 0.53 | 130 | 1.43 | -6 | 0.13 | 15 | 0.49 | -140 |
| **8** | 0.49 | 127 | 1.25 | -18 | 0.15 | 11 | 0.56 | -147 |

Determine the stability factor K, the maximum stable gain, and the source and load stability circles at 1GHz, 4GHz and 8 GHz, for a collector current of 10mA.

SOLUTION

PART 1 STABILITY FACTOR K

The S-parameters to be used are shaded in the table of S-parameters given above.
From equation 5-30, the stability factor K is given by,

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} \quad \textbf{(5-30)}$$

where the determinant of the scattering matrix is given by,

$$\Delta = S_{11}S_{22} - S_{12}S_{21} \quad \textbf{(5-31)}$$

Using equations 5-30 and 5-31, the value of K, the stability factor, and $\Delta$, are calculated to be as shown in Table 5-2, below.

**Table 5-2 Calculated values of $\Delta$ and K for different frequencies**

| ***GHz*** | $\lvert S_{11}\rvert^2$ | $\lvert S_{22}\rvert^2$ | $\lvert S_{21}S_{12}\rvert$ | $\Delta$ | $\lvert\Delta\rvert^2$ | $K$ |
|---|---|---|---|---|---|---|
| **1** | 0.292 | 0.336 | 0.467 | 0.418∠-77 | 0.174 | 0.585 |
| **4** | 0.230 | 0.130 | 0.259 | 0.086∠-114 | 7.44E-3 | 1.250 |
| **8** | 0.240 | 0.314 | 0.188 | 0.101∠-44.7 | 0.010 | 1.213 |

PART 2 LOAD AND SOURCE STABILITY CIRCLES

For the load and source stability circles, discussed in section 5.2.1.1 and 5.2.1.2, the centres and radii are found from equations 5-39 and 5-40 for the load stability circle and 5-40 and 5-43 for the source stability circle, that is,

centre
$$c_L = \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2} \quad \textbf{(5-39)}$$

radius
$$r_L = \frac{|S_{12}S_{21}|}{\left||S_{22}|^2 - |\Delta|^2\right|} \quad \textbf{(5-40)}$$

centre
$$c_s = \frac{S_{11}^* - \Delta^* S_{22}}{|S_{11}|^2 - |\Delta|^2} \quad \textbf{(5-42)}$$

radius
$$r_s = \frac{|S_{12}S_{21}|}{\left||S_{11}|^2 - |\Delta|^2\right|} \quad \textbf{(5-43)}$$

Table 5-3 shows the calculated centre and radius values for load and source circles for the three different frequencies.

**Table 5-3 Calculated values of c and r for different frequencies**

| ***GHz*** | $c_L$ | $r_L$ | $c_s$ | $r_s$ |
|---|---|---|---|---|
| **1** | 3.562∠64.5<br>1.533+j3.215 | 2.883 | 4.640∠138.8<br>-3.492+j3.055 | 3.983 |
| **4** | 3.274∠83.3<br>0.382+j3.251 | 2.113 | 2.296∠-151.8<br>-2.024-j1.084 | 1.164 |
| **8** | 1.696∠144.7<br>-1.384+j0.980 | 0.618 | 1.910∠-130.08<br>-1.23-j1.461 | 0.817 |

***Load reflection coefficient stability circle***

A summary of the stability parameters derived from the calculations so far, which are used for determining if the device is unconditionally or conditionally stable for varying load reflection coefficients, is shown in Table 5-4.

**Table 5-4 Load reflection coefficient stability parameter conditions**

| *GHz* | $K$ | $K$ ***conditions*** | $\lvert S_{11}\rvert$ | $\lvert S_{22}\rvert$ | $\lvert\Delta\rvert$ | $\lvert S_{11}\rvert$ ***conditions*** | $\lvert S_{22}\rvert$ ***conditions*** |
|---|---|---|---|---|---|---|---|
| **1** | 0.585 | $K<1$ | 0.540 | 0.580 | 0.417 | $\lvert S_{11}\rvert<1$ | $\lvert S_{22}\rvert>\lvert\Delta\rvert$ |
| **4** | 1.250 | $K>1$ | 0.480 | 0.361 | 0.086 | $\lvert S_{11}\rvert<1$ | $\lvert S_{22}\rvert>\lvert\Delta\rvert$ |
| **8** | 1.213 | $K>1$ | 0.490 | 0.560 | 0.100 | $\lvert S_{11}\rvert<1$ | $\lvert S_{22}\rvert>\lvert\Delta\rvert$ |

An analysis of Tables 5-3 and 5-4, shows that at 1 GHz, the bipolar transistor is conditionally stable, because for condition 5 of conditional stability discussed in section 5.2.1.1, that is,

$$r_L + \lvert c_L\rvert > 1 \text{ and } \lvert c_L\rvert > 1$$

$$\lvert S_{11}\rvert < 1, \quad \lvert S_{22}\rvert > \lvert\Delta\rvert$$

the conditions are satisfied and the load stability circle intersects with the Smith chart . The load stability circle does not enclose the origin and so values of $\rho_L < 1$ which jointly lie between the Smith chart and the stability circle represent the region of instability. At 4GHz, the stability circle lies outside of the Smith chart and so a condition of absolute stability exists within the Smith chart, since the unstable region lies within the stability circle, and since K > 1 and $\lvert S_{11}\rvert < 1$. Similarly, at 8 GHz. The 4 GHz and 8 GHz stability circles when drawn would be similar to that shown in Figure 125. Figure 137 shows the load reflection coefficient stability circle at 1 GHz together with the region of instability.

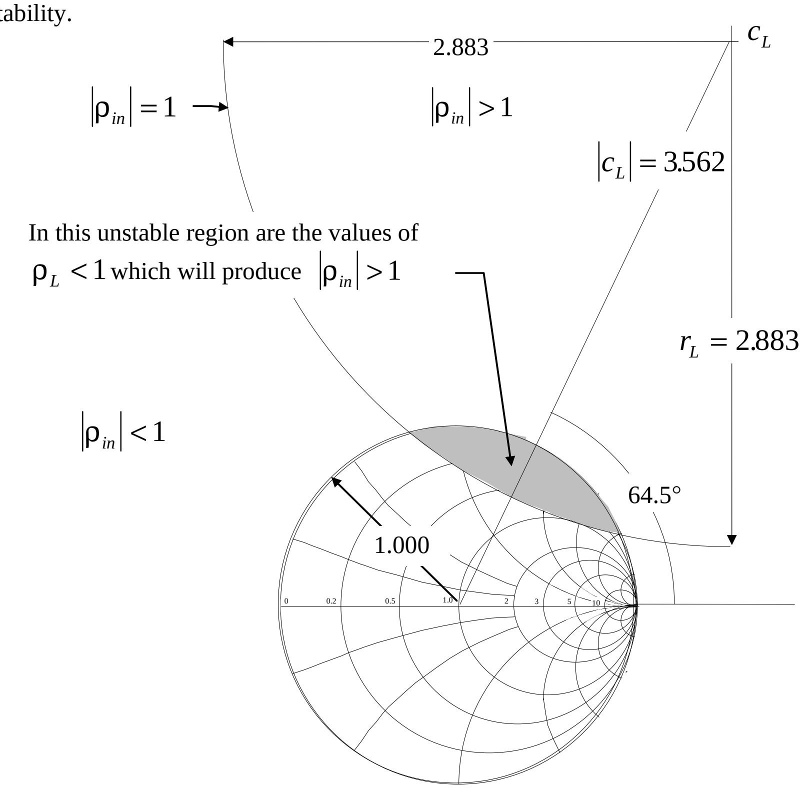



**Figure 137 Load stability circle at 1 GHz showing conditional stability**

***Source reflection coefficient stability circle***

A summary of the stability parameters derived from the calculations so far, which are used for determining if the device is unconditionally or conditionally stable for varying source reflection coefficients, is shown in Table 5-5.

**Table 5-5 Source reflection coefficient stability conditions**

| GHz | $K$ | $K$ conditions | $\|S_{11}\|$ | $\|S_{22}\|$ | $\|\Delta\|$ | $\|S_{22}\|$ conditions | $\|S_{11}\|$ conditions |
|---|---|---|---|---|---|---|---|
| **1** | 0.585 | $K<1$ | 0.540 | 0.580 | 0.417 | $\|S_{22}\|<1$ | $\|S_{11}\|>\|\Delta\|$ |
| **4** | 1.250 | $K>1$ | 0.480 | 0.361 | 0.086 | $\|S_{22}\|<1$ | $\|S_{11}\|>\|\Delta\|$ |
| **8** | 1.213 | $K>1$ | 0.490 | 0.560 | 0.100 | $\|S_{22}\|<1$ | $\|S_{11}\|>\|\Delta\|$ |

An analysis of Tables 5-3 and 5-5, shows that at 1 GHz, the bipolar transistor is conditionally stable, because for condition 5 of conditional stability discussed in section 5.2.1.2, that is,

$$r_s + |c_s| > 1 \text{ and } |c_s| > 1$$
$$|S_{22}| < 1, \quad |S_{11}| > |\Delta|$$

the conditions are satisfied and the load stability circle intersects with the Smith chart . The load stability circle does not enclose the origin and so values of $\rho_s < 1$ which jointly lie between the Smith chart and the stability circle represent the region of instability. At 4GHz, the stability circle lies outside of the Smith chart and so a condition of absolute stability exists within the Smith chart, since the unstable region lies within the stability circle, and since K > 1 and $|S_{22}| < 1$. Similarly, at 8 GHz. The 4 GHz and 8 GHz stability circles when drawn would be similar to that shown in Figure 130.

Figure 138 shows the source reflection coefficient stability circle at 1 GHz together with the region of instability.

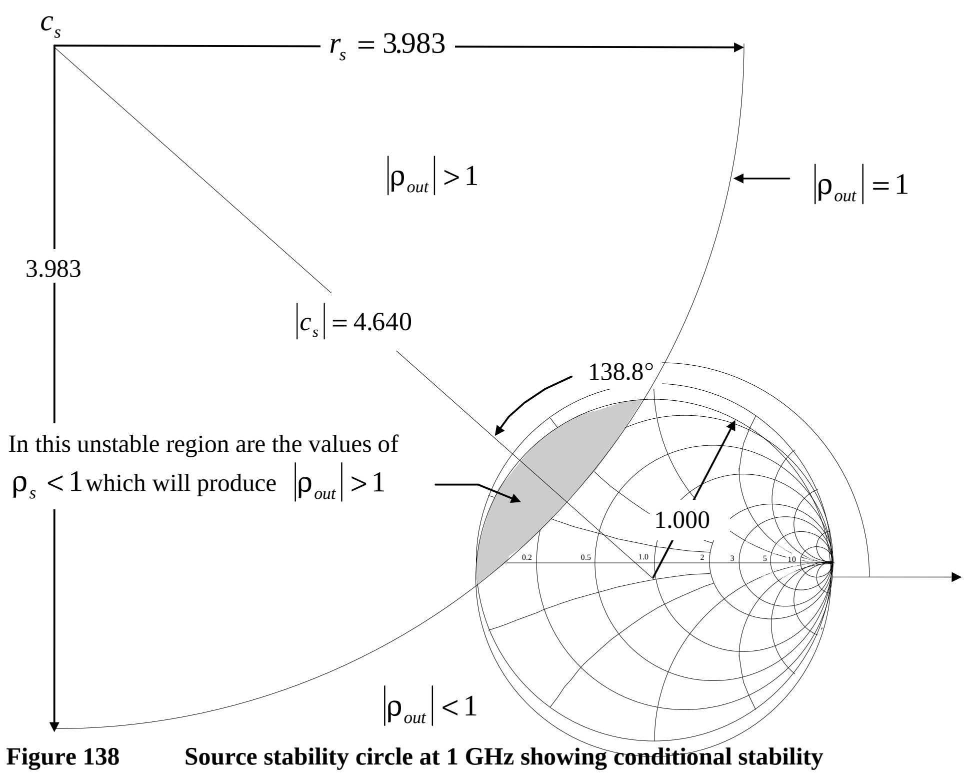



**Figure 138 Source stability circle at 1 GHz showing conditional stability**

PART 3 MAXIMUM STABLE GAIN

For frequencies where the amplifier or device is unconditionally stable, equation 5-29 is used to determine the maximum power gain using conjugate impedance matching, that is,

$$G_{a\,\max} = \left|\frac{S_{21}}{S_{12}}\right|\left(K \pm \sqrt{K^2 - 1}\right) \quad \textbf{(5-29)}$$

From table 5-1 and table 5-4, and using equation 5-29, the values of the maximum stable gain at 4 and 8 GHz are calculated. Table 5-6 presents the results together with the values of the parameters used. Since $A_i > 0$ of equation 5-32, the negative sign in front of the radical of equation 5-29 is used.

**Table 5-6 Maximum power gain using conjugate impedance matching**

| ***GHz*** | $A_1$ | $A_2$ | ***K*** | $\lvert S_{21}\rvert$ | $\lvert S_{12}\rvert$ | $\left\lvert\frac{S_{21}}{S_{12}}\right\rvert$ | $G_{a\,max}$ |
|---|---|---|---|---|---|---|---|
| **4** | 1.0926 | 0.8926 | 1.250 | 2.59 | 0.10 | 25.90 | 12.95 |
| **8** | 0.916 | 1.064 | 1.213 | 1.25 | 0.15 | 8.33 | 4.385 |

For frequencies where the amplifier or device is only conditionally stable, conjugate matching cannot be used. This is the case at the frequency of 1 GHz. To determine what the source and load impedances are and what the maximum power gain is to be, is discussed in the next section.

## 5.3. *Constant power gain circles*

This section will describe and then apply constant power gain circles in the design of an amplifier. As a conditionally stable amplifier cannot safely use conjugate matching at the input and output to obtain the maximum power gain, as given by equation 5-29, some method must be found which permits complex source and load impedances to be determined for stability and then from these impedances, the determination of the power gain which will be obtained for the amplifier. Constant power gain circles reverse this process, by permitting the load and source impedances to be determined after a particular power gain has been found. The particular power gain is first determined from stability considerations using the stability circles of a conditionally stable amplifier.

In the description of the constant power gain circles, for completeness and to aid in the understanding of them, constant gain circles for stable devices will first be considered.

### 5.3.1. Constant power gain circle equations

It is convenient to normalize the power gain circles to the maximum power gain that can be obtained from a device, that is, the power gain for perfect matching where the source and load impedances are resistive and which reduce the source and load reflection coefficients to zero. That is, the normalizing factor, from equation 5-26e, is $\lvert S_{21}\rvert^2$.

Usually, for a stable device, the normalized constant power gain circles are plotted in steps of 1dB, less than the maximum power gain of equation 5-29 which has been normalized. The maximum normalized power gain is given by,

$$g_{a\,\max} = \left|\frac{S_{21}}{S_{12}}\right|\frac{\left(K \pm \sqrt{K^2 - 1}\right)}{\lvert S_{21}\rvert^2} = \frac{\left(K \pm \sqrt{K^2 - 1}\right)}{\lvert S_{12}S_{21}\rvert} \quad \textbf{(5-44)}$$

For an unstable device, the normalized constant power gain circles are plotted in steps of 1dB, less than the normalized “Figure of erit” gain.

he “Figure of erit” gain $G_{FOM}$, is the maximum stable gain that can be obtained for an amplifier, that is when K = 1 in equation 5- . he normalized “Figure of erit” gain $g_{FOM}$, is given as,

$$g_{FOM} = \left|\frac{S_{21}}{S_{12}}\right|\frac{1}{\lvert S_{21}\rvert^2} = \frac{1}{\lvert S_{12}S_{21}\rvert} \quad \textbf{(5-45)}$$

The normalized available power gain $g_a$, for an amplifier is given by equation 5-27d divided by equation 5-26e, that is,

$$g_a = \frac{1 - \lvert\rho_L\rvert^2}{\lvert 1 - S_{22}\rho_L\rvert^2 - \lvert S_{11} - \Delta\rho_L\rvert^2} \quad \textbf{(5-46)}$$

It is this equation from which the constant power gain circles are derived. By rearranging (Ref.3), an equation of a circle in the $\rho_L$ plane, in terms of the normalized available power gain and S-parameters, can be obtained. Being a circle in the $\rho_L$ plane, means that it can be plotted on the Smith chart and can be used with the load stability circle, which is also plotted on the $\rho_L$ plane.
The centre of the constant power gain circle is given by,

of centre $$c_g = \frac{\left(S_{22}^* - \Delta^* S_{11}\right) g_a}{\left(\left|S_{22}\right|^2 - |\Delta|^2\right) g_a + 1} \qquad \textbf{(5-47)}$$

and radius $$r_g = \frac{\sqrt{\left(1 - 2Kg_a\left|S_{12}S_{21}\right| + g_a^2\left|S_{12}S_{21}\right|^2\right)}}{\left|\left(\left|S_{22}\right|^2 - |\Delta|^2\right) g_a + 1\right|} \qquad \textbf{(5-48)}$$

Comparing equation 5-47 with equation 5-39, it may be noticed that the complex portion of the numerator in each equation is the same. This means that both the constant power gain circle and the load stability circle lie on the same radial from the centre of the Smith chart because their polar angle is the same, although their centres are at different positions along this radial.
By taking the limit of equation 5-47, as $g_a \to \infty$ it can be seen that the centre becomes that of the stability circle centre, equation 5-39. Similarly, equation 5-48 becomes the radius of the stability circle given by equation 5-40. This means that the load stability circle is the constant power gain circle when the power gain is infinite.
When the power gain $g_a$ = 0, the centre becomes the centre of the Smith chart and the radius becomes unity. That is, the power gain circle becomes the boundary of the Smith chart, corresponding to a load impedance which is pure reactance and which cannot absorb any power.

## 5.3.2. Constant power gain circles for stable amplifiers

There are two cases where an amplifier may be unconditionally or absolutely stable, as described in section 5.2.1.1. The first is where the stability circle lies outside of the Smith chart and does not enclose the Smith chart. The second case is where the stability circle encloses the Smith chart. Both of these cases will be briefly discussed in this section.

### 5.3.2.1. Stability circle not enclosing the Smith chart

Three cases are considered. These are:

- constant power gain circles which lie within the Smith chart,
- constant power gain circles which lie within the stability circle,
- constant power gain circles which lie in the region between the Smith chart and the stability circle,

Of these three cases, only the first is of any importance as it is the case where there is a real positive power gain and the region is stable. The other two cases are where there is instability or negative power gain.

#### 5.3.2.1.1. Constant power gain circles which lie within the Smith chart

As mentioned in section 5.3.1, when the power gain $g_a$ = 0, the centre becomes the centre of the Smith chart and the power gain circle becomes the boundary of the Smith chart. From equation 5-46, it can be seen that as $\rho_L$ decreases from unity, the power gain will start to increases from zero up to a maximum of

$$g_{a\,\max} = \frac{\left(K - \sqrt{K^2 - 1}\right)}{\left|S_{12}S_{21}\right|}$$

where the radius of the maximum constant power gain circle must become zero, as verified by substituting the value of $g_{a\,\max}$ given above, into equation 5-48. The centre point of the maximum constant power gain circle, however, will not be the centre of the Smith chart, but at a value found by

substituting $g_{a\max}$ into equation 5-47. The substitution of $g_{a\max}$ with the negative sign before the radical into equation 5-47 produces a centre which lies within the Smith chart. This centre when plotted on the Smith chart will allow the normalized load impedance to be determined.
The outcome is, that for maximum power gain where conjugate matching is employed, a value of normalized load impedance can be found which produces that power gain. For any power gain less than the maximum, where non-conjugate matching is used, a normalized load impedance can be chosen which lies on that constant power gain circle inside the Smith chart.
If the value of K is unity, which is the limit of stability, the implication is that perfect matching has been achieved. Figure 139 shows the constant power gain circles lying inside the Smith chart.

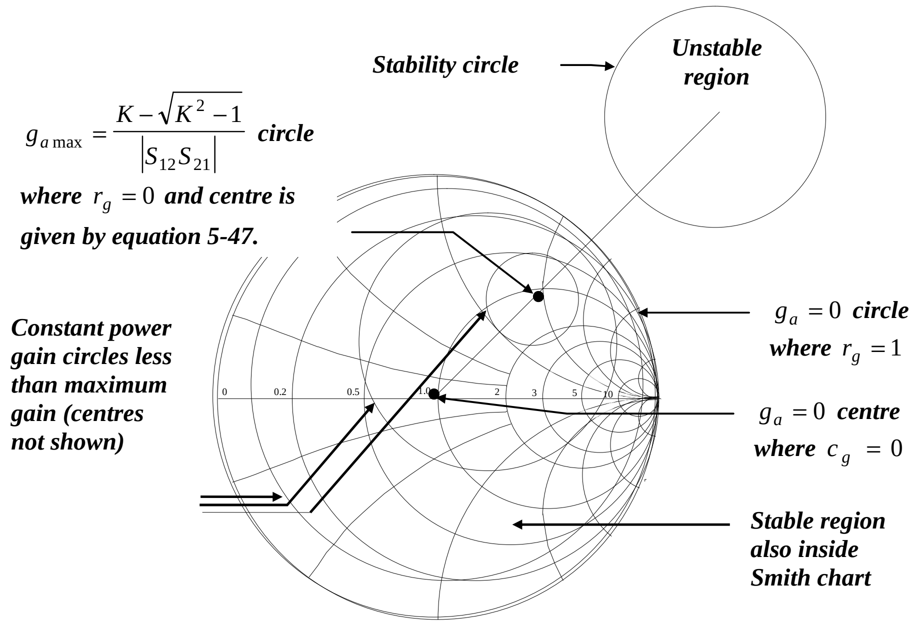



**Figure 139 Constant power gain circles which lie within the Smith chart**

### 5.3.2.1.2. Constant power gain circles which lie within the stability circle

If the value of $g_{a\max} = \frac{\left(K + \sqrt{K^2 - 1}\right)}{|S_{12}S_{21}|}$, that is a positive sign before the radical, is substituted into equation 5-47, the centre of the constant power gain circle lies within the stability circle. However, this region is the unstable region of $\rho_L$ values, which results in $\rho_{in} > 1$.

### 5.3.2.1.3. Constant power gain circles which lie between the Smith chart and the stability circle

In the region outside of the Smith chart, $|\rho_L| > 1$. However, the region between the boundary of the Smith chart and the load stability circle is a stable region, where each value of $\rho_L$ produces a $\rho_{in}$ value where $|\rho_{in}| < 1$. Although a stable region, $|\rho_L| > 1$. This means that the load reflects more power than is incident on it and so the power gain is negative. A negative power gain is of no use in normal amplifier design.

### 5.3.2.2. Stability circle enclosing the Smith chart

The conditions for the load stability circle to enclose the Smith chart were given in section 5.2.1.1. as:

$$r_L - |c_L| > 1$$

$$K > 1 + \frac{|\Delta|^2 - 1}{2|S_{12}S_{21}|}, \quad |S_{11}| < 1, \quad |S_{22}| < |\Delta|$$

Similar to the case where the stability circle does not enclose the Smith chart, the power gain circles have a maximum inside the Smith chart at

$$g_{a\max} = \frac{\left(K - \sqrt{K^2 - 1}\right)}{|S_{12}S_{21}|}$$

where the radius of the maximum constant power gain circle must become zero.
The centre point of the maximum constant power gain circle, however, will not be the centre of the Smith chart, but at a value found by substituting $g_{a\max}$ into equation 5-47. The substitution of $g_{a\max}$ with the negative sign before the radical into equation 5-47 produces a centre which lies within the Smith chart.
As the constant power circles increase in radii there is a decrease in power gain $g_a$, until at the Smith chart boundary where $|\rho_L| = 1$, the power gain drops to zero.
Beyond the Smith chart boundary and still within the load stability circle, $|\rho_L| > 1$, $|\rho_{in}| < 1$ and the power gain becomes negative in value where it approaches minus infinity at the load stability circle boundary. This region is of no use for designers of normal amplifiers.
Outside of the load stability circle boundary the region is unstable, and again is of no use to the designers of normal amplifiers.
Figure 140 shows the region where the power gain circles lie within the Smith chart for stability and positive gain.

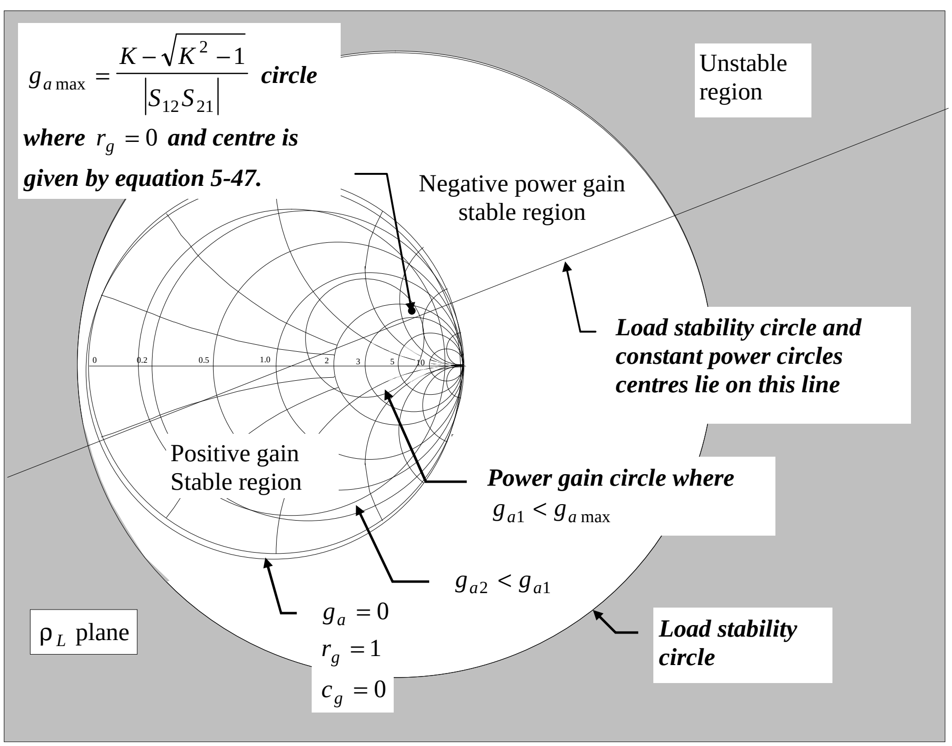



**Figure 140 Constant Power gain circles where stability circle encloses the Smith chart**

[i] Adam, S.F., Hewlett Packard, *Microwave Theory and Applications,* (Prentice-Hall, 1969).
[ii] Liao, S.Y., *Microwave Circuit Analysis and Amplifier Design,* (Prentice-Hall, 1987).

[iii] Collin, R.E., *Foundations for Microwave Engineering,* (McGraw-Hill, 1992).
[iv] Laverghetta, T.S., *Practical Microwaves,* (Howard W. Sams, 1984).

### 5.3.3. Constant power gain circles for conditionally stable amplifiers

As seen in section 5.2.1.1., for a conditionally stable device, the load stability circle may lie within the Smith chart or intersect the Smith chart and for both cases may or may not enclose the origin.
In this section the four cases where the Smith chart is intersected at two points and where the origin is and is not enclosed by the stability circle as shown in Figures 133 and 134 will be discussed. The case where the stability circle lies within the Smith chart provides a region of stability within the load stability circle and a region of instability outside, that is, between the boundary of the load stability circle and the Smith chart boundary. The method used in this case is similar to that shown in Figure 139, except for the above mentioned provisos.

#### *5.3.3.1. Intersection of load stability circle with the Smith chart*

For many cases of conditional stability the load stability circle will cut the Smith chart boundary in two places. Another surprise is that these two points are invariant points to all constant power gain circles. That is, all constant power gain circles, with different gains, will also have their circles intersected at these two places on the Smith chart boundary. This is shown in Figure 141.

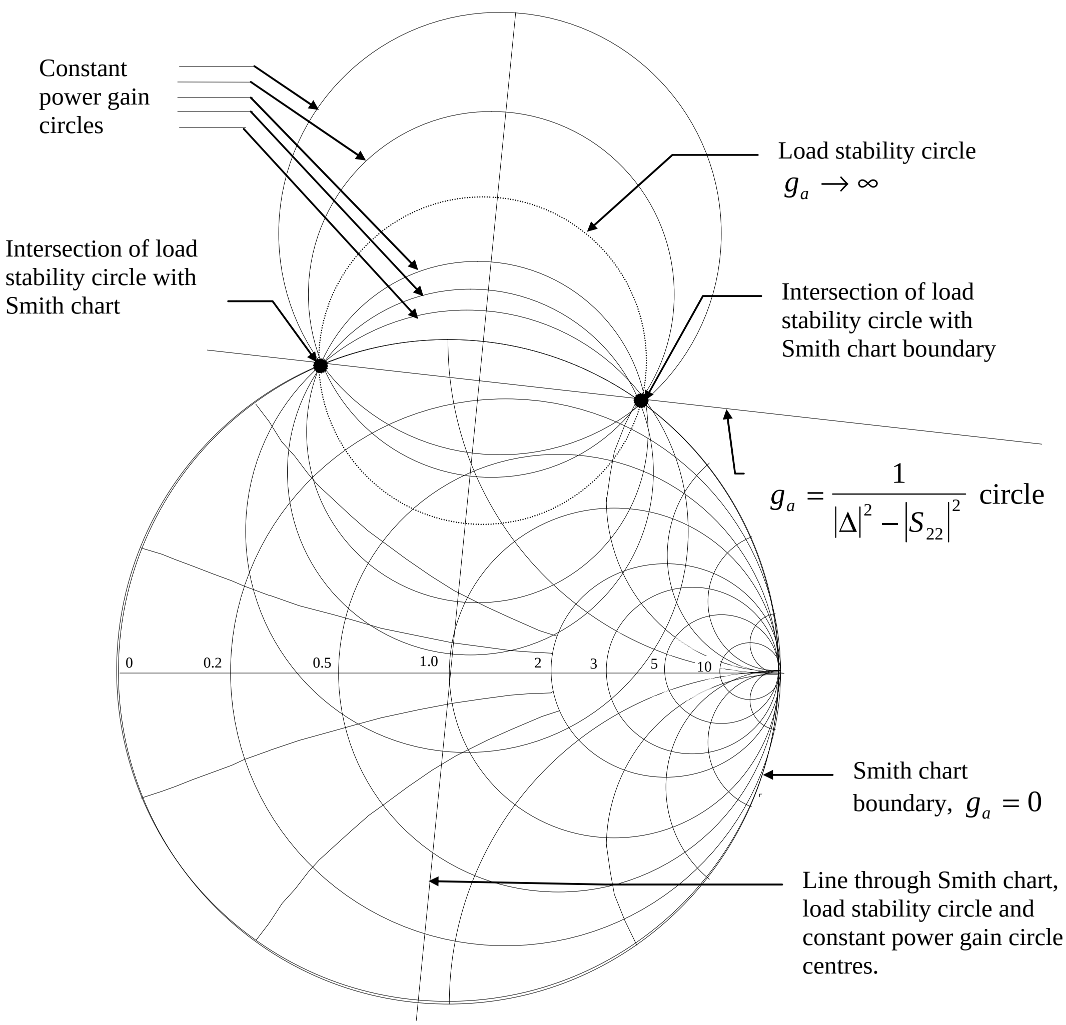



**Figure 141 Constant power gain circles and load circle for a conditionally stable device**

The four situations of conditional stability, labelled condition 1 through to condition 4, where the load stability circle cuts the Smith chart boundary, as discussed in section 5.2.1.1. will now be considered with reference to the constant gain circles.

#### 5.3.3.1.1. Constant gain circles where stability circle encloses the origin $|S_{11}| < 1$

The origin is enclosed when $|S_{11}| < 1$ and $|\Delta| > |S_{22}|$. As shown in Figure 142, the interior values of the $|\rho_{in}| = 1$ circle, that is the region inside the load stability circle, are the stable values of $\rho_L$. Thus, for a stable amplifier design, $\rho_L$ must be such that is lies on a constant gain circle in this region. For an amplifier which is stable and which has a positive gain, the values of $\rho_L$ must lie in the region which is both within the Smith chart and within the load stability circle. This is because the edge of the Smith chart is the $g_a = 0$ boundary. Outside of the Smith chart $\rho_L > 1$, which gives rise to a negative value of constant power gain. On the load stability circle the gain $g_a \to -\infty$ facing the centre of the load stability circle and $g_a \to +\infty$ facing away from the circle.

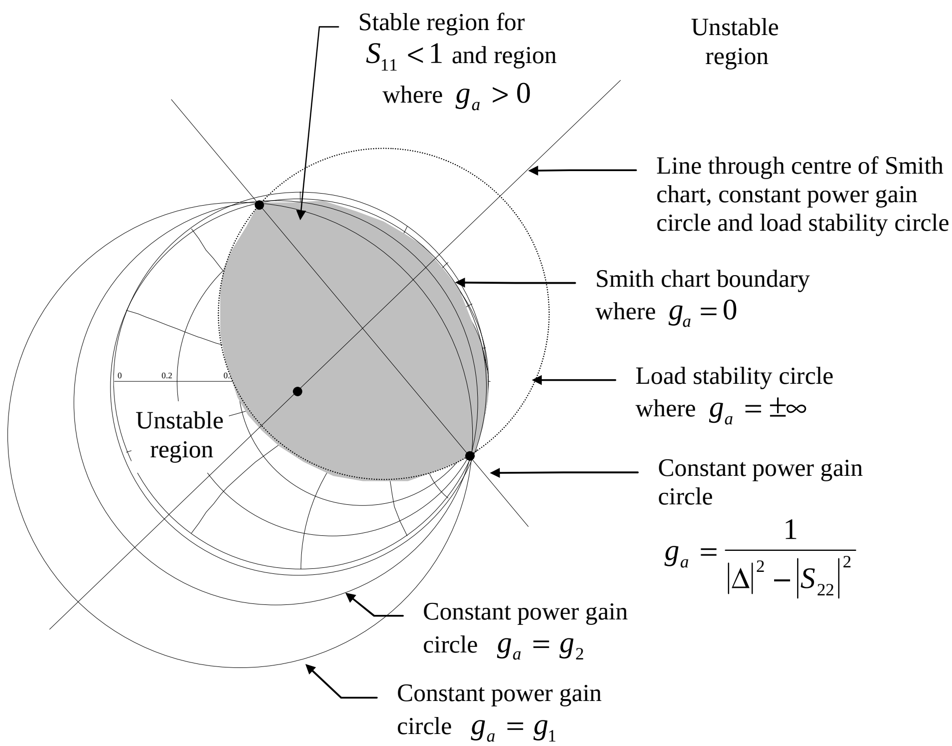



**Figure 142 Constant gain circles where stability circle encloses the origin $|S_{11}| < 1$**

#### 5.3.3.1.2. Constant gain circles where stability circle encloses the origin $|S_{11}| > 1$

The origin is enclosed when $|S_{11}| > 1$ and $|S_{22}| > |\Delta|$. As shown in Figure 143, the exterior values of the $|\rho_{in}| = 1$ circle, that is the region outside the load stability circle, are the stable values of $\rho_L$. Thus, for a stable amplifier design, $\rho_L$ must be such that is lies on a constant gain circle outside of this region. For an amplifier which is stable and which has a positive gain, the values of $\rho_L$ must lie in the region which is both within the Smith chart and outside of the load stability circle boundary, that is the region which does not contain the origin (the Smith chart centre). This is because the edge of the Smith chart is the $g_a = 0$ boundary. Outside of the Smith chart $\rho_L > 1$, which gives rise to a negative value of constant power gain.

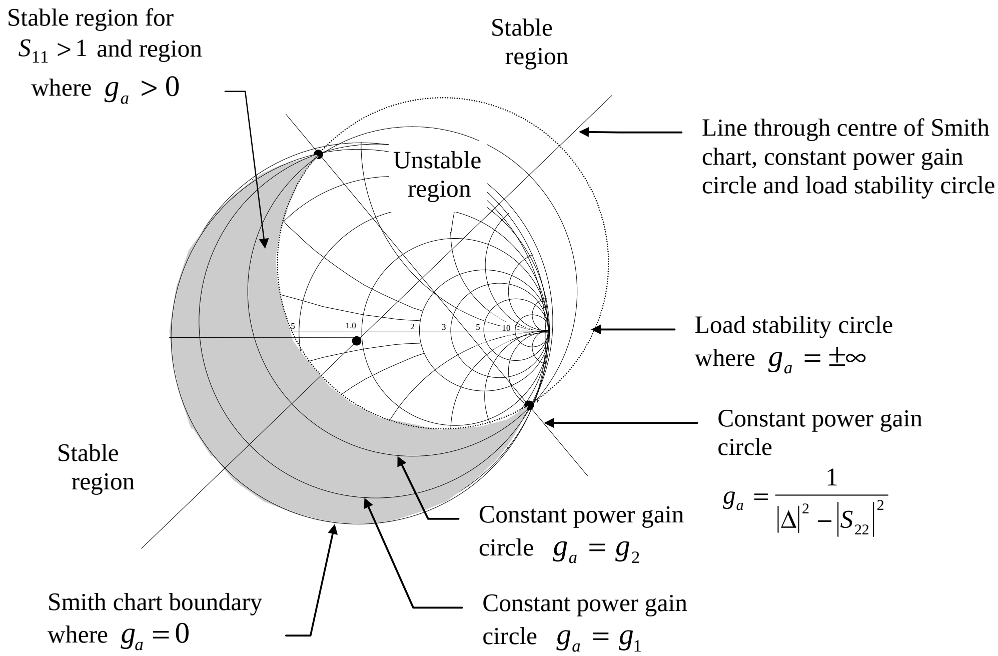



**Figure 143 Constant gain circles where stability circle encloses the origin** $|S_{11}| > 1$

### 5.3.3.1.3. Constant gain circles where stability circle does not enclose the origin $|S_{11}| < 1$

The load stability circle $|\rho_{in}| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 144. The origin is not enclosed when $|S_{11}| < 1$ and $|\Delta| < |S_{22}|$. When this happens, the **exterior** values of the $|\rho_{in}| = 1$ circle are stable values of $\rho_L$.

When the origin is not enclosed, for stable amplifier design, the constant gain circles lie in the region outside of the stability circle and within the boundary of the Smith chart. The load reflection coefficient $\rho_L$, may be chosen to lie on any one of these constant gain circles. In the mutually inclusive region inside the Smith chart and inside the load stability circle, the amplifier is unstable for all values of $\rho_L$ in this region and the gain varies from $g_a \to +\infty$ at the edge of the load stability circle to zero at the Smith chart boundary. The power gain is negative outside of both circles, with $g_a \to -\infty$ on the edge of the load stability circle.

Figure 144 shows the stable region shaded.

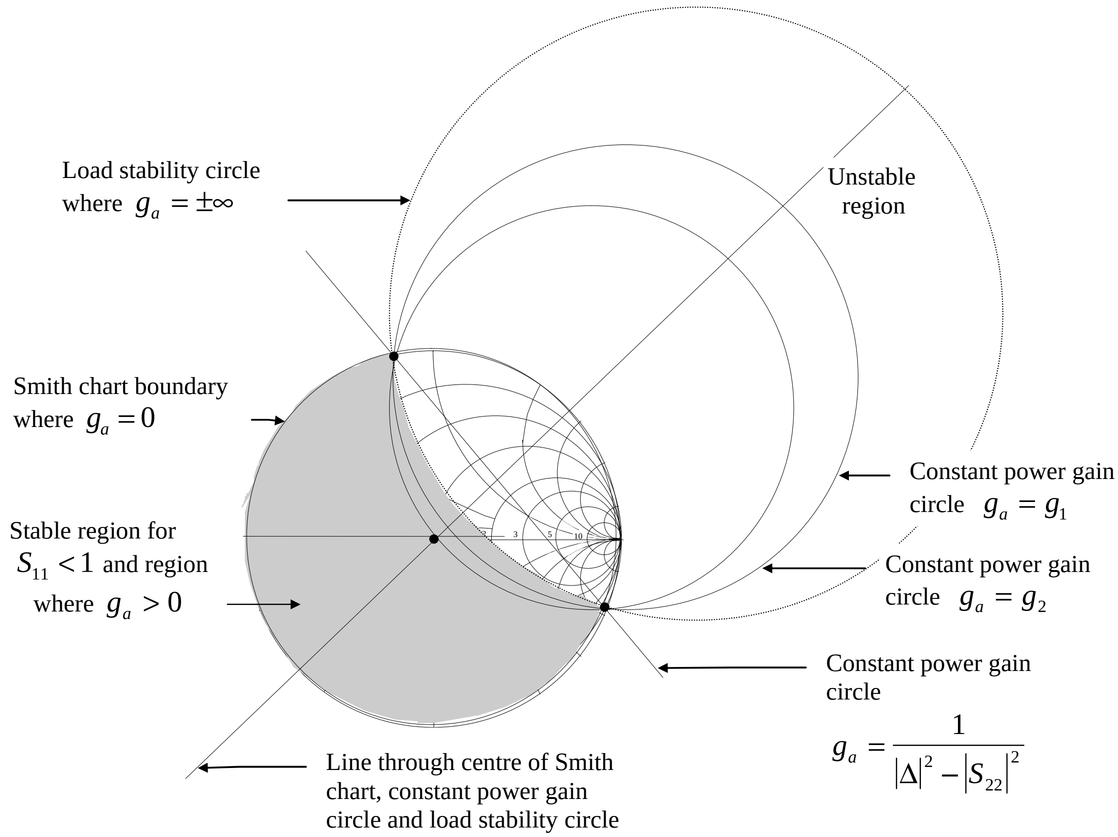



**Figure 144 Constant gain circles where stability circle does not enclose the origin** $|S_{11}| < 1$

### 5.3.3.1.4. Constant gain circles where stability circle does not enclose the origin $|S_{11}| > 1$

The load stability circle $|\rho_{in}| = 1$, intersects the Smith chart at two points and does **not** enclose the origin, as shown in Figure 145. The origin is not enclosed when $|S_{11}| > 1$ and $|\Delta| > |S_{22}|$. When this happens, the **interior** values of the $|\rho_{in}| = 1$ circle are stable values of $\rho_L$.

When the origin is not enclosed, for stable amplifier design, the constant gain circles lie in the mutually inclusive region inside of the stability circle and within the boundary of the Smith chart. The load reflection coefficient $\rho_L$, may be chosen to lie on any one of these constant gain circles. In the region outside the Smith chart and inside the load stability circle, the amplifier is stable for all values of $\rho_L$ in this region but the gain is negative. In the region outside of the load stability circle the amplifier will be unstable for all values of $\rho_L$. On the load stability circle, in the mutually inclusive region, the gain varies from $g_a \to +\infty$ at the edge of the load stability circle to zero at the Smith chart boundary. Outside of the Smith chart, but still inside the load stability circle, the power gain is negative and ranges from $g_a \to -\infty$ on the edge of the load stability circle to zero on the boundary of the Smith chart.

The stable region is shown as the shaded region in Figure 145.

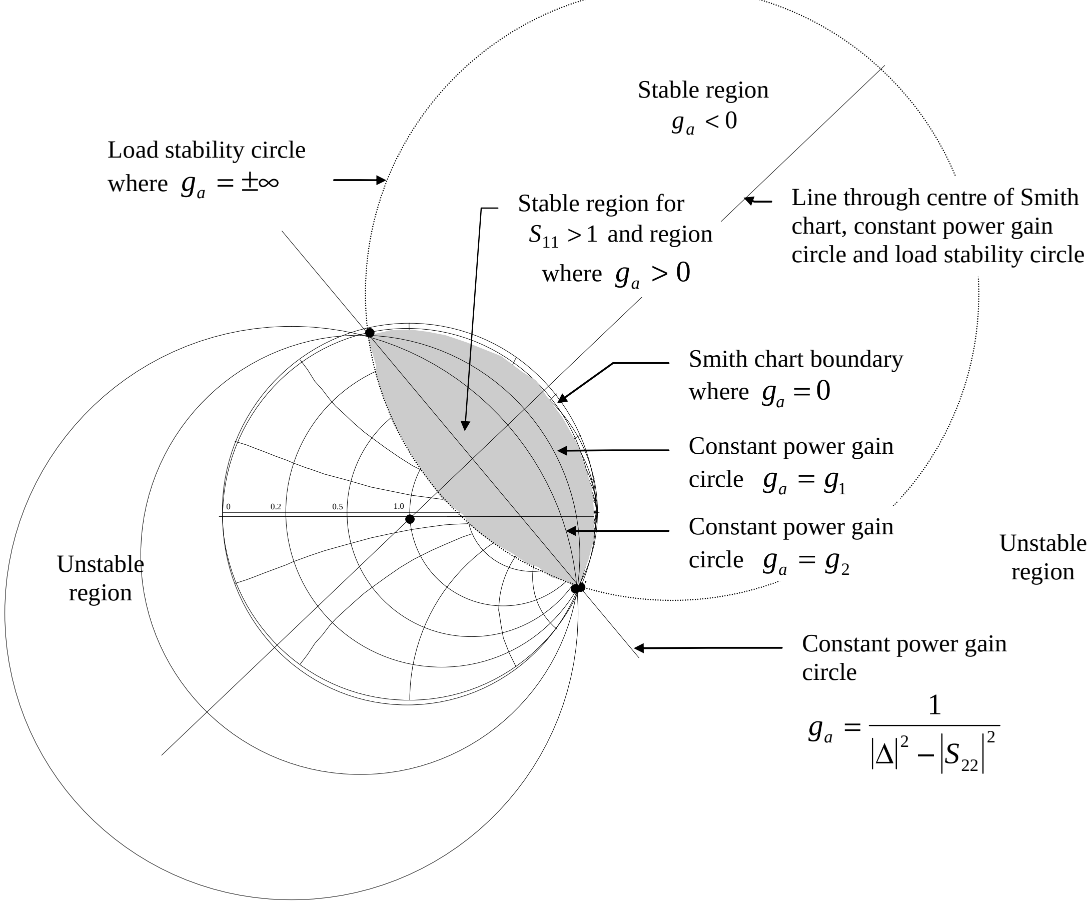



**Figure 145 Constant gain circles where stability circle does not enclose the origin** $|S_{11}| > 1$

Having completed the four cases which can occur for conditional stability, we are now in a position to continue with Example 27 where the bipolar transistor at 1 MHz was shown to be conditionally stable.

EXAMPLE 28
sing the manufacturer s data for the bipolar transistor for a collector current of m , given in Example 27, determine the normalized Figure of Merit, and draw the relevant constant power gain circles on Figures 137 and 138 combined. Also determine the power gain for conjugate matching, $\rho_L$, $\rho_{in}$, $\rho_s$, $\rho_{out}$ and the input and output VWSR for a stable amplifier.

SOLUTION - PART 1 Determination of normalized Figure of merit and the centre and radius of each of the constant power gain circles .

he normalized “Figure of erit” gain $g_{FOM}$ , is given as,

$$g_{FOM} = \left|\frac{S_{21}}{S_{12}}\right| \frac{1}{|S_{21}|^2} = \frac{1}{|S_{12}S_{21}|} \qquad \textbf{(5-45)}$$

From Table 5-2,

**Table 5-2 Calculated values of $\Delta$ and K for different frequencies**

| *GHz* | $\lvert S_{11}\rvert^2$ | $\lvert S_{22}\rvert^2$ | $\lvert S_{21}S_{12}\rvert$ | $\Delta$ | $\lvert\Delta\rvert^2$ | $K$ |
|---|---|---|---|---|---|---|
| **1** | 0.292 | 0.336 | 0.467 | 0.418∠-77 | 0.174 | 0.585 |

the normalized figure of merit is found to be 2.141 or 3.3 dB.
Figure 146 shows the combined drawing of Figures 137 and 138.
As $|S_{11}| < 1$ and $|\Delta| < |S_{22}|$, the constant power circles shown in Figure 144 are applicable.

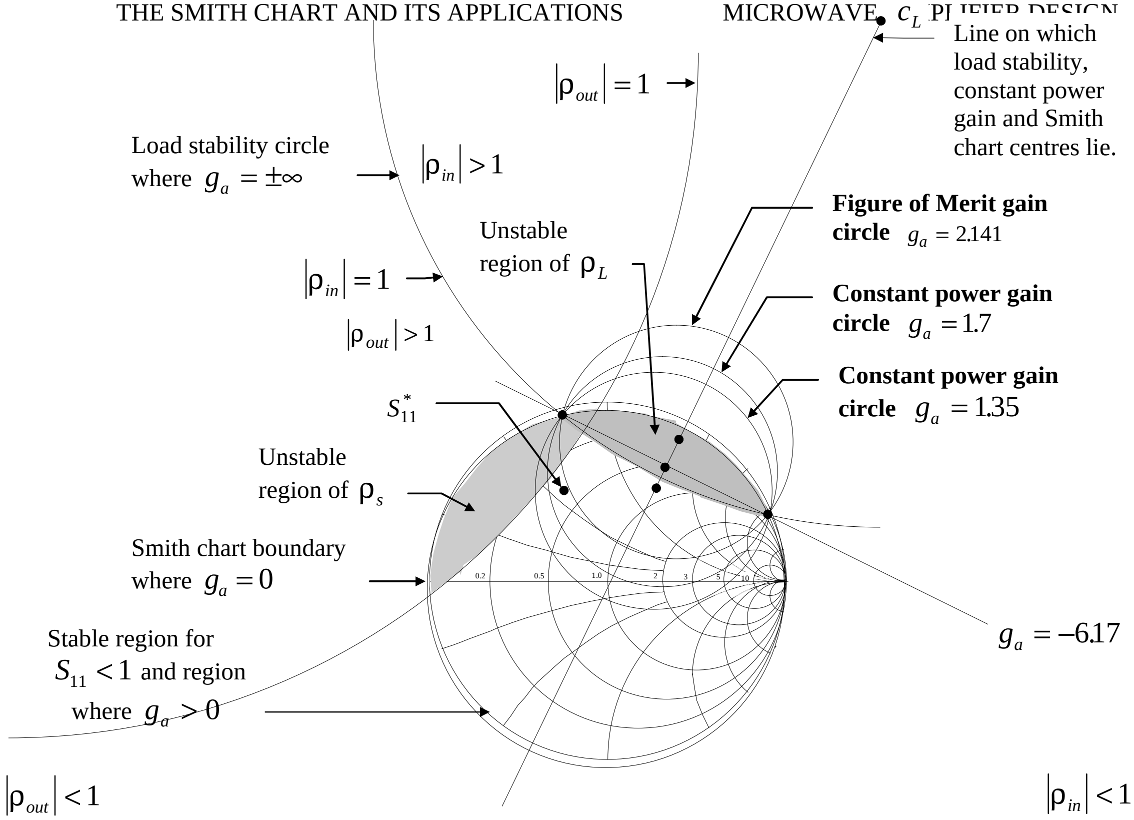



**Figure 146** **Example 28 - Load and source stability circles with constant power gain circles at 1 GHz**

The two constant power gain circles, shown on Figure 146, are for normalized gains which are 1 dB and 2 dB less than the normalized Figure of Merit, that is, 2.3 dB and 1.3 dB, giving the non-logarithmic normalized power gains as 1.7 and 1.35. From equation 5-47 and 5-48, the centre and radius of each constant power gain circle can be determined.

Centre
$$c_g = \frac{\left(S_{22}^* - \Delta^* S_{11}\right) g_a}{\left(|S_{22}|^2 - |\Delta|^2\right) g_a + 1} \qquad \textbf{(5-47)}$$

Radius
$$r_g = \frac{\sqrt{\left(1 - 2K g_a |S_{12}S_{21}| + g_a^2 |S_{12}S_{21}|^2\right)}}{\left|\left(|S_{22}|^2 - |\Delta|^2\right) g_a + 1\right|} \qquad \textbf{(5-48)}$$

From the manufacturer s data and from able . , the centres and radii of the constant power circles with normalized power gains 1.7 and 1.35 are shown in Table 5-7.

**Table 5-7** **Calculated values of the centre and radius of two constant power gain circles of gain 1.7 and 1.35**

| $g_a$ | $\left(\lvert S_{22}\rvert^2 - \lvert\Delta\rvert^2\right) g_a + 1$ | $\sqrt{\left(1 - 2K g_a \lvert S_{12}S_{21}\rvert + g_a^2 \lvert S_{12}S_{21}\rvert^2\right)}$ | $r_g$ | $c_g$ |
|---|---|---|---|---|
| **1.7** | 1.2754 | 0.8375 | 0.6567 | $0.7692\angle 64.5$ |
| **1.35** | 1.2187 | 0.8123 | 0.6665 | $0.6392\angle 64.5$ |
| **2.141** | 1.3469 | 0.9110 | 0.6764 | $0.9171\angle 64.5$ |

PART 2 The power gain for conjugate matching, $\rho_L$, $\rho_{in}$, $\rho_s$, $\rho_{out}$ and the input and output VWSR for a stable amplifier.

Figure 146 shows that the origin is a stable point. The unstable regions are those shown shaded and are the regions between the boundary of the load and source stability circles and the Smith chart boundary.

*Conjugate matching*

When $\rho_L = 0$, that is at the centre of the Smith chart, equation 5-46 shows that the normalized available power gain

$$g_a = \frac{1-|\rho_L|^2}{|1-S_{22}\rho_L|^2 - |S_{11}-\Delta\rho_L|^2} \qquad \textbf{(5-46)}$$

becomes ,

$$g_a\big|_{\rho_L=0} = \frac{1}{1-|S_{11}|^2}$$

which equals 1.412. This represents a true power gain of $G_a\big|_{\rho_L=0} = 1.412|S_{21}|^2 = (1.412)(9.34)^2 = 123.2$, or 20.9 dB which is considered adequate. Usually, designers are looking at figure around 10 dB or more. The advantage of using $\rho_L = 0$ is that no separate output matching network is required between the output of the amplifier and the transmission line, to match the amplifier output impedance to the characteristic impedance of the line. However, the gain is nearly 2 dB less than the 2.141 figure for the Figure of Merit gain. The gain of the amplifier should be no greater than the Figure of Merit gain if stability is to be maintained and VSWR at the input and output are to be kept low.

For the moment we will continue with $\rho_L = 0$ in order to determine what happens with the input circuit conjugately matched. From equation 5-4, with $\rho_L = 0$, we find $\rho_{in} = S_{11}$. For conjugate impedance matching at the input, $\rho_s = \rho_{in}^* = S_{11}^*$. From equation 5- and the manufacturer s data,

$$\rho_{out} = S_{22} + \frac{S_{12}S_{21}\rho_s}{1-S_{11}\rho_s} = \frac{S_{22}-\Delta\rho_s}{1-S_{11}\rho_s} = \frac{S_{22}-\Delta S_{11}^*}{1-|S_{11}|^2} = 0.3509 - j0.7356 = 0.815\angle -64.5$$

The magnitude of the output reflection coefficient is 0.815. Since $\rho_L = 0$ the output VSWR is given by equation 1-63 as 9.81 which is quite high and may require a matching network, between this amplifier and a second amplifier, as shown in Figure 124, if a second amplifier is to be added.

The point $\rho_s = S_{11}^* = 0.54\angle 113$ , shown on Figure 146, lies outside of the source stability circle and hence represents a stable point. The source impedance, normalized to the transmission line characteristic impedance, can be measured directly from the Smith chart when $\rho_s$ is plotted. This is found to be $\overline{Z}_s = 0.43 + j0.57$ . The output impedance of the amplifier can also be determined by plotting $\rho_{out} = 0.815\angle -64.5$ on the Smith chart. This gives a normalized output impedance of $\overline{Z}_o = 0.36 - j1.52$ . The normalized input impedance from the plot of $\rho_{in} = S_{11} = 0.54\angle -113$ on the Smith chart, is found to be $\overline{Z}_{in} = 0.43 - j0.57$ and the VSWR is found to be 3.4. The input circuit to the amplifier requires a matching network to match to the characteristic impedance of the transmission line and to reduce the VSWR to unity. Again, the position of the input matching network relative to the amplifier is shown in Figure 124.

*Non-conjugate matching*

As mentioned above, usually designers do not exceed the Figure of Merit gain for stability reasons. In the next example an amplifier with a power gain to be equal to the normalized Figure of Merit gain of 2.141, in order to obtain a larger gain than with conjugate matching, will be considered. Using the Figure of Merit gain will give a true power gain equal to $2.141|S_{21}|^2 = (2.141)(9.34)^2 = 186.77$ . Using $g_a = 2.141$ the centre and radius of the constant power gain circle are found and given in Table 5.7. This circle is plotted on Figure 146. Before the next example is considered, method of determining VSWR and a value of source reflection coefficient for a good stability margin is considered.

## *5.4. Conjugate input matching circles*

The use of conjugate impedance matching at the input for conditionally stable amplifiers where the load reflection coefficient $\rho_L = 0$, may not give a stable or sufficiently stable amplifier output circuit, either because the source reflection coefficient $\rho_s$, is too close to the source stability circle, or lies inside the unstable region of the source stability circle, that is $|\rho_{out}| > 1$. The use of the conjugate input matching circle permits $\rho_s$ to be selected so that there exists a good stability margin on the output circuit and a low input VSWR, for a specified normalized power gain. This is achieved at the expense of perfect matching of the load to the amplifier, as $\rho_L \neq 0$.

The method used to construct the conjugate input matching circle, is to take a chosen constant power gain circle, which is a circle in the $\rho_L$ plane, and map this circle into a circle of conjugate input reflection coefficient values, in the $\rho^*_{in}$ plane. The conjugate of the input reflection coefficient is used in order to conjugately match the input circuit of the amplifier to the source, that is $\rho_s = \rho^*_{in}$. If a specific input VSWR is required to be satisfied, then for a chosen value of $\rho_L$, there will be a circle of values of $\rho_s$ from which to choose that VSWR. The value of $\rho_s$ chosen on the circle of values will be such that it lies in a stable region on the Smith chart. From the chosen value of $\rho_s$ the normalized value of the source impedance $\overline{Z}_s$, can be directly read from the Smith chart.

In order to understand how the mapping of one circle into another can occur, Appendix B gives a brief description of the bilinear transform and the derivation of the conjugate input matching circles given by equations 5-49 and 5-50.

The centre and radius of the conjugate input matching circles in the $\rho^*_{in}$ plane are given by,

$$c_{\rho^*_{in}} = \frac{r_g^2 S_{22}\Delta^* + \left(1 - S_{22}c_g\right)\left(\Delta^* c_g^* - S_{11}^*\right)}{\left|r_g S_{22}\right|^2 - \left|S_{22}c_g - 1\right|^2} \qquad \textbf{(5-49)}$$

$$r_{\rho^*_{in}} = \frac{\left|S_{12}S_{21}\right| r_g}{\left|\left|r_g S_{22}\right|^2 - \left|S_{22}c_g - 1\right|^2\right|} \qquad \textbf{(5-50)}$$

### 5.4.1. A brief overview of the reflection coefficient domains

At this juncture, it may be worthwhile noting that there are four planes which may be considered. These are the $\rho_L$, $\rho_s$, $\rho_{in}$, and $\rho_{out}$ planes. For each plane there is associated a Smith chart which has its reflection coefficient pertinent to that plane, that is for the $\rho_L$ plane the reflection coefficient on the Smith chart is $\rho_L$, for the $\rho_{in}$ plane the reflection coefficient on the Smith chart is $\rho_{in}$, etc.

The load stability circle and the power gain circles lie in the $\rho_L$ plane, the source stability circle lies in the $\rho_s$ plane. Each of the four planes are like rooms in a house which are separated by walls. That is you cannot mix different reflection coefficients with different planes, such as using $\rho_L$ in the $\rho_s$ plane or $\rho_{in}$ in the $\rho_L$ plane. You can, however, draw source stability circles in the $\rho_L$ plane over load stability and power gain circles as long as it is understood that the source stability circle refers only to $\rho_s$ and not to any circle or process that involves $\rho_L$, and also that the Smith chart is an overlay of two Smith charts, one for the $\rho_L$ plane and the other for the $\rho_s$ plane. The movement between the different planes is achieved by mapping, and the mapping in the cases presented in this chapter are by the use of bilinear transformations. The source stability circle in the $\rho_s$ plane was derived from a mapping to the unit circle in the $\rho_{out}$ plane, the load stability circle was derived from a mapping to the unit circle in the $\rho_{in}$ plane, and recently the conjugate input reflection coefficient circle in the $\rho_{in}$ plane was derived from a mapping of the constant power gain circle in the $\rho_L$ plane. To consider the properties of the conjugate input reflection coefficient circle, the source stability circle in the $\rho_s$ plane must be mapped to the $\rho_{in}$ plane, but because of the condition that $\rho_s = \rho_{in}^*$, in order that the input reflection coefficient be conjugately matched, the mapping of the source stability circle to the $\rho_{in}$ plane is direct, that is the source stability circle in the $\rho_s$ plane becomes the conjugate source stability circle in the $\rho_{in}$ plane. With the constant power gain circles and the source stability circle mapped into the $\rho_{in}$ plane together with the Smith chart which uses $\rho_{in}$ as its reflection coefficient, we are now in a position to determine relationships between these three circles.

#### 5.4.1.1. Properties of the conjugate input reflection coefficient circle $\rho_{in}^*$

Substituting equation 5-47 for $c_g$ and equation 5-48 for $r_g$ into equation 5-58 to expand $c_{\rho_{in}^*}$ (see Appendix B), we find that the centre of the $\rho_{in}^*$ circle lies on the same line which connects the centre of the Smith chart with the source stability circle $c_s$, as shown by the numerator of equation 5-42.

$$c_s = \frac{S_{11}^* - \Delta^* S_{22}}{|S_{11}|^2 - |\Delta|^2} \qquad \textbf{(5-42)}$$

Furthermore, the $\rho_{in}^*$ circle cuts the Smith chart at the same two points as does the source stability circle, in a similar fashion to the constant power gain circle and the load stability circle.
A plot of the conjugate input reflection coefficient circle and the source stability circle together with the Smith chart with a reflection coefficient $\rho_{in}^*$, is shown on Figure 147.

The source stability circle in the $\rho_{in}^*$ plane is the same as in the $\rho_L$ plane because $\rho_s = \rho_{in}^*$. A value of $\rho_s$ is chosen to lie as far away from the unstable region of the source stability circle as possible. This in some cases may be achieved by choosing a value of $\rho_s$ on the $\rho_{in}^*$ circle which intersects the radial from the centres of the source stability and the $\rho_{in}^*$ circles to the centre of the Smith chart with the $\rho_{in}^*$ circle.

Once a value of $\rho_s$ has been chosen, by taking its conjugate to form $\rho_s^*$, we can use the inverse transform of equation 5-04 to determine $\rho_L$. That is, the inverse bilinear transform of equation 5-04 is given by

$$\rho_L = \frac{\rho_{in} - S_{11}}{S_{22}\rho_{in} - \Delta} = \frac{\rho_s^* - S_{11}}{S_{22}\rho_s^* - \Delta} \qquad \textbf{(5-51)}$$

The value of $\rho_L$ determined must be checked to ensure that it lies in the stable region of the Smith chart in the $\rho_L$ plane, which encloses the Smith chart boundary and the load stability circle. Also from the value of $\rho_L$ plotted back into the $\rho_L$ plane permits the value of the output VSWR to be determined, either from the Smith chart or from equation 1-63.

$$VSWR = \frac{1+|\rho_L|}{1-|\rho_L|} \qquad \textbf{(1-63)}$$

From the plot of the value of $\rho_L$, the load impedance can be read off directly from the Smith chart or calculated using equation 1-57.

$$Z_L = Z_0 \frac{1+\rho_L}{1-\rho_L} \qquad \textbf{(1-57)}$$

As $\rho_s$ has been chosen, there will exist a fixed value of $\rho_{out}$ which can be found using equation 5-41.

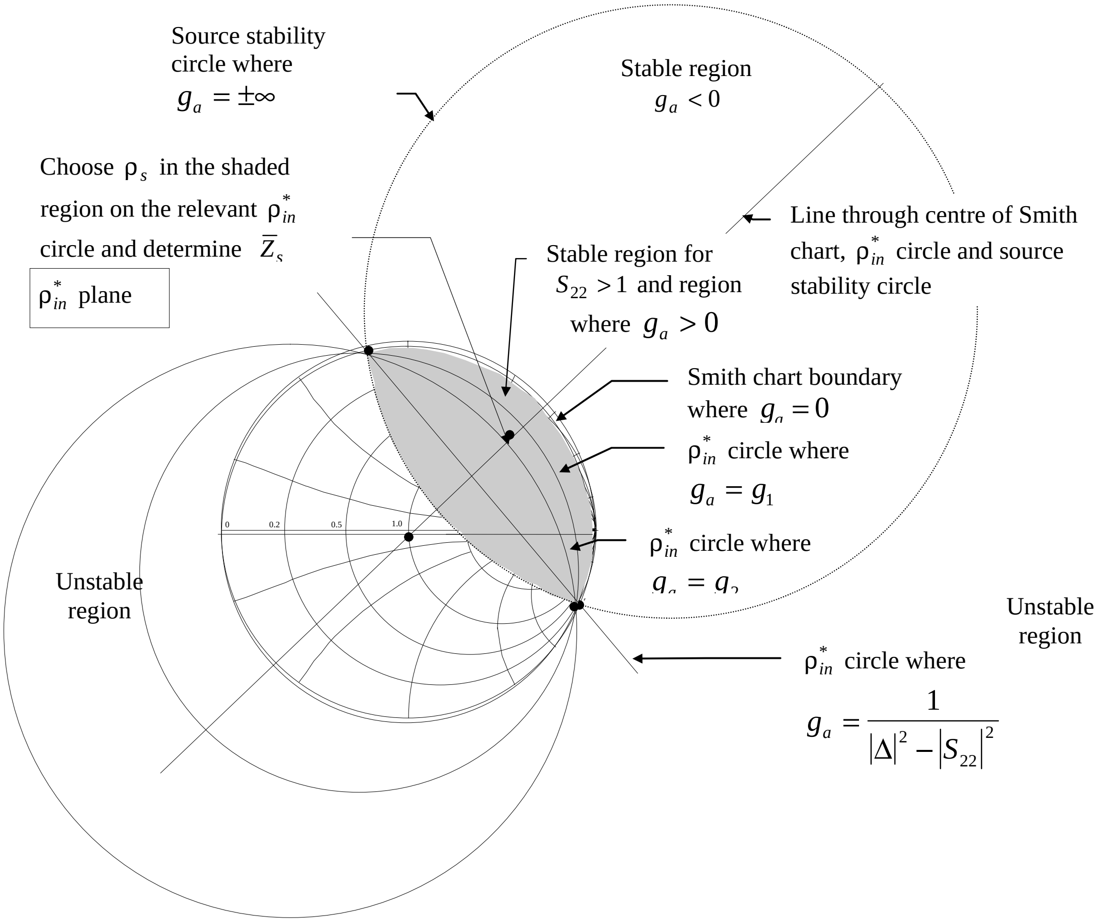



**Figure 147** $\rho_{in}^*$ **circle where source stability circle does not enclose the origin** $|S_{22}| > 1$

EXAMPLE 29

sing the manufacturer s data for the bipolar transistor for a collector current of 10mA, given in Example 27, draw the $\rho_{in}^*$ circle for the “Figure of erit” gain and the stability circle and then determine a value for $\rho_s$ for complex conjugate input matching. Determine the value of $\rho_{out}$ , $\rho_L$ and the values of the input and output VSWRs. Compare the answers with that of Example 28.

SOLUTION

From xample , the normalized “Figure of erit” gain was found to be . , that is $g_a = 2.141$, and the centre $c_g$ , and radius $r_g$ , of the constant power circle for this gain, given in Table 5-7, was found to be, $r_g = 0.6162$, and $c_g = 0.9173\angle 64.5^o$.

PART 1 Determination of the centre and radius of $\rho_{in}^*$ circle for “Figure of erit” gain and drawing the source stability and $\rho_{in}^*$ circle

sing the manufacturer s data, as presented in able -8, together with equations 5-49 and 5-50, the centre $c_{\rho_{in}^*}$ and radius $r_{\rho_{in}^*}$ of the $\rho_{in}^*$ circle is found to be that given in Table 5-9.

**Table 5-8 n t ’s t n v i s n**

| $g_a$ | $r_g$ | $c_g$ | $S_{22}$ | $S_{11}$ | $\|S_{21}S_{12}\|$ | $\Delta^*$ |
|---|---|---|---|---|---|---|
| **2.141** | 0.6736 | 0.9171∠64.5 | 0.58∠-42 | 0.54∠-113 | 0.467 | 0.418∠77 |

**Table 5-9 Values of the centre and radius of $\rho_{in}^*$ i i e it" gai**

| $\left\|r_g S_{22}\right\|^2 - \left\|S_{22} c_g - 1\right\|^2$ | $r_g^2 S_{22} \Delta^* + \left(1 - S_{22} c_g\right)\left(\Delta^* c_g^* - S_{11}^*\right)$ | $c_{\rho_{in}^*}$ | $r_{\rho_{in}^*}$ |
|---|---|---|---|
| -0.1460 | 0.4039∠-41.18 | 2.7658∠138.82 | 2.1632 |

Figure 148 shows the plot of the stability circle and the $\rho_{in}^*$ circle for "Figure of erit" gain.

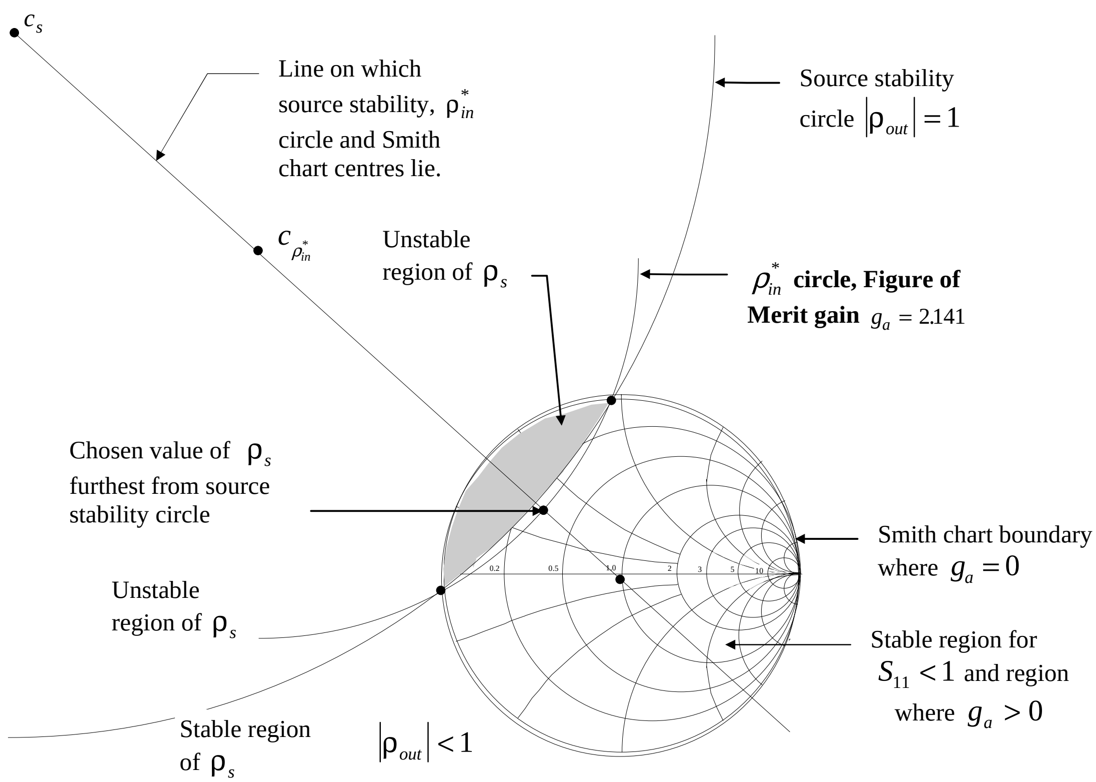



**Figure 148 Example 29 - Source stability and $\rho_{in}^*$ circle for a normalized constant power gain of 2.141 at 1 GHz .**

PART 2 Determination of the value of $\rho_s$ and the value of $Z_s$, and the input VSWR

From the Smith chart shown on Figure 148, the value of $\rho_s$ is found to be 0.588∠138.82. This corresponds to a value of source impedance normalized to the main transmission line of $\overline{Z}_s = 0.29 + j0.34$ having a VSWR =3.9. To match this source impedance to the transmission line characteristic impedance, the use of the Smith chart covered in Chapter 4 is employed to determine the input matching network shown in Figure 124.

PART 3 Determination of the value of $\rho_{out}$ , $\rho_L$ and the values of the output VSWR.

Once a value of $\rho_s$ has been chosen, by taking its conjugate to form $\rho_s^*$, we can use the inverse transform of equation 5-04 to determine $\rho_L$ . That is, the inverse bilinear transform of equation 5-04 is given by

$$\rho_L = \frac{\rho_{in} - S_{11}}{S_{22}\rho_{in} - \Delta} = \frac{\rho_s^* - S_{11}}{S_{22}\rho_s^* - \Delta} \qquad \textbf{(5-51)}$$

The value of $\rho_L$ is found to be, $\rho_L = 0.428\angle 18.04^o$. When plotted on the $\rho_L$ plane shown in Figure 146, it is seen that $\rho_L$ lies on the "Figure of erit" circle, as expected, and is in the stable region.
The value of output VSWR is determined from the plot of $\rho_L$ on the Smith chart in $\rho_L$ plane as VSWR = 2.50. Also the load impedance is determined from this plot as $\overline{Z}_L = 2.2 + j0.72$ or from

$$Z_L = Z_0 \frac{1+\rho_L}{1-\rho_L} \qquad \textbf{(1-57)}$$

as $\overline{Z}_L = 2.212 + j0.718$

To match the load impedance to the transmission line characteristic impedance, the use of the Smith chart covered in Chapter 4 is employed to determine the output matching network whose position relative to the amplifier is shown in Figure 124

From equation 5-41

$$\rho_{out} = \frac{\Delta\rho_s - S_{22}}{S_{11}\rho_s - 1} \qquad \textbf{(5-41)}$$

the value of the amplifier output reflection coefficient is found as, $\rho_{out} = 0.937\angle -51.5^o$.

Comparison with Example 28 shows that if the use of the conjugate input matching circle had not be made, then the particular choice of the value of $\rho_L$ which lies on the "Figure of erit" constant power gain circle, shown in Figure 146, would not ensure that the input would be stable. The conjugate input matching circle technique has permitted a normalized power gain of 2.141, or an actual value of power gain of $2.141|S_{21}|^2 = 2.141(9.34)^2 = 186.77 \equiv 22.7dB$. The addition 2 dB gain obtained in this Example over that of Example 28, where $\rho_L$ =0 was considered at the expense of an additional output matching network and a higher input VSWR (3.9 in this case against 3.4 for Example 28).

o use a power gain higher than the "Figure of erit" gain would possibly lead to instability in the output circuit, because as shown in Figure 148, the chosen value of $\rho_s$, although at its furthest point from the source stability circle, is still very close to it. Raising the gain would bring $\rho_s$ closer to the source stability circle, so that any small deviations from the calculated value may cause it to enter the unstable region.

As mentioned in the beginning of this chapter, the criteria used for amplifier design is to have the largest power gain that can be obtained for a stable circuit over a specific frequency band, an input and output VSWR close to unity with a minimum noise figure of the first stage using the optimum source impedance, and a phase response that is a linear function of frequency. To some extent the discussion has dealt with the parameters associated with obtaining the largest power gain for a stable circuit. Not yet considered is the determination of parameters for a specific input and output VSWR, although the input and output VSWR can be calculated, once the reflection coefficients for a certain power have been determined. Also yet to be considered is the minimization of the noise figure at the expense of the gain and stability as well as that of the VSWR. The next section will develop the constant noise figure circles which are used to determine, from a specific noise figure, the source impedance for a specific power. Thus, with these circles, noise figure may enter the juggling act with power gain, stability, and VSWR.

## *5.5. Constant noise figure circles*

Noise figure circles are plotted on the source reflection coefficient plane $\rho_s$ and as derived in Appendix C, are given as ,

centre

$$c_{nf} = \frac{\rho_m}{1+N} \qquad \textbf{(C1-33)}$$

and radius,

$$r = \frac{\sqrt{N^2 + N\left(1-|\rho_m|^2\right)}}{(1+N)} \qquad \textbf{(C1-34)}$$

where the following parameters used in equations C1-33 and 34 are defined as,

$\rho_m$ optimum source reflection coefficient

$$N = \frac{\left(F_{spot} - F_{\min}\right)\left|1-\rho_m\right|^2}{4\overline{G}_e} = \frac{\left(F_{spot} - F_{\min}\right)\left|1+\rho_m\right|^2}{4\overline{R}_e} \qquad \textbf{(C1-30)}$$

$\overline{R}_e$ — equivalent noise resistance normalized to the transmission line characteristic impedance $Z_o$, that is $\overline{R}_e = R_e / Z_o$ .

$\overline{G}_e$ — equivalent noise conductance normalized to the transmission line characteristic admittance $Y_o$, that is $\overline{G}_e = G_e Z_o = G_e / Y_o$ .

$F_m$ — minimum value of noise factor, when

$Z_m = R_m + jX_m$ — the optimum source impedance or

$\rho_m$ — the optimum source reflection coefficient is used with $G_e$ .

$F_{spot}$ — The value of the spot noise figure chosen to form the constant noise figure circle

Note: anufacturer s usually provide the minimum noise factor $F_m$, the optimum source impedance $Z_m$ or the optimum source reflection coefficient $\rho_m$, and the noise conductance $G_e$ to allow the determination of $F_{spot}$, for a specific source impedance $Z_s = R_s + jX_s$, using equation C1-21.

$$F_{spot} = F_{\min} + \frac{G_e}{R_s}\left[\left(R_s - R_m\right)^2 + \left(X_s - X_m\right)^2\right] \qquad \textbf{(C1-21)}$$

If the noise resistance $R_e$ is provided by the manufacturer instead of $G_e$, then equation C1-27, permits the value of noise conductance $G_e$, to be found. That is,

$$G_e = \frac{R_e}{R_m^2 + X_m^2} = \frac{R_e}{\left|Z_m\right|^2} = \frac{R_e\left|1-\rho_m\right|^2}{\left|1+\rho_m\right|^2} \qquad \textbf{(C1-27)}$$

The plotting of circles with centre given by equation C1-33 and radius given by equation C1-34 for a chosen value of $F_{spot}$ shows the values of $\rho_s$ that can be used on this circle in order to obtain a noise factor of $F_{spot}$ . The minimum noise figure is obtained when N=0, which is no longer a circle, but a point $\rho_m$ . If this point is a stable point, then choosing $\rho_s$ to be equal to it will permit the network to exhibit a minimum noise figure as shown by equation C1-26. The use of $\rho_s = \rho_m$ may produce an input VSWR which is too high. In this case, to reduce the VSWR to a tolerable value, the value of $\rho_s$ will have to be chosen to lie on a $F_{spot}$ circle where $F_{spot} > F_{\min}$ .

EXAMPLE 30

sing the manufacturer s data for the bipolar transistor for a collector current of m , given in Example 27 and the noise data given below, draw the $\rho_F$ circle for $F_m + 0.1$ dB, $F_m + 1$ dB, the $\rho_{in}^*$ circle for the “Figure of erit” gain and the stability circle and then determine the best value for $\rho_s$ . Determine the value of $\rho_{out}$ , $\rho_L$ and the values of the input and output VSWRs. Compare the answers with that of Example 29.

**Noise Data**

At 1 GHz, $V_{CE} = 6V$, $I_C = 10mA$ with a 50Ω resistance, the following noise figure parameters are measured:

$F_m = 2.1\, dB$, $\rho_m = 0.48\angle 155^o$, $R_e = 3.6\Omega$

SOLUTION

Example 29 determined that for a conditionally stable device the “Figure of erit” gain could be used with a source impedance which was conjugately matched to the input of the amplifier. As shown in Figure 148, the source reflection coefficient was chosen so that it was at its furthest distance from the source stability circle in order to ensure that the amplifier remained stable for small changes in the source impedance. n this problem, a value of source impedance, which again lies on the “figure of erit” gain $\rho_{in}^*$ circle in order to maintain the “figure of merit” gain, will be chosen. However, this time the impedance is that which gives the lowest value of noise factor.

PART 1 - PLOTTING OF NOISE FACTOR CIRCLES

$F_m = 2.1\ dB$, $\rho_{\mathrm{m}} = 0.48\angle 155^o$, $\mathrm{R_e} = 3.6\Omega$

The value of $F_m$ expressed as a factor is 1.6218.

The values of $F_{spot}$ for $F_m$ + 0.1 and $F_m$ + 1.0 dB are respectively, 1.66 and 2.0417.

The value of $\overline{R}_e$ = 3.6/50 = 0.072.

The polar form of $\rho_{\mathrm{m}} = 0.48\angle 155^o$ is found to be -0.43503 + j0.202857.

The value of N in equation C1-30 is found to be, $N_{2.2} = 0.0482$ and $N_{3.1} = 0.525374$.

From these values, using equations C1-33 and C1-34, the centres and radii of the circles are found to be,

$c_{nf\,2.2} = 0.458\angle 155^o$ and $c_{nf\,3.1} = 0.3147\angle 155^o$

$r_{nf\,2.2} = 0.1894$ and $r_{nf\,3.1} = 0.5407$.

These points are shown plotted on Figure 149, together with the source stability and $\rho_{in}^*$ circles.

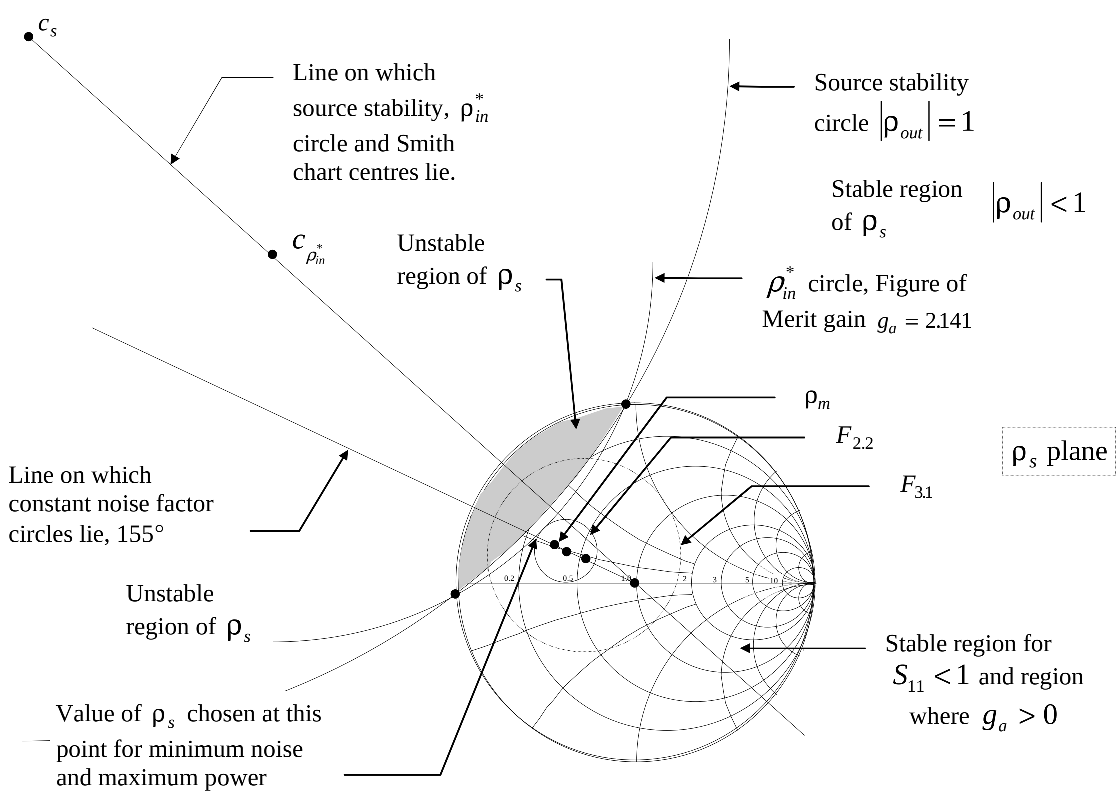



**Figure 149 Example 30 - Constant noise factor circles together with source stability and $\rho_{in}^*$ circle for a normalized constant power gain of 2.141 at 1 GHz.**

The source reflection coefficient is found to be at the intersection of the $F_{spot}$ =2.2 dB circle and the $\rho_{in}^*$ circle for a normalized constant power gain of . at Hz "Figure of erit" gain . his value of source reflection coefficient $\rho_s = 0.62\angle 155^o$ , permits the gain to remain maximized and the noise figure to be only 0.1 dB above the minimum. The source reflection coefficient also still remains, by a tolerable amount, in the stable region. The value of normalized source resistance for $\rho_s = 0.62\angle 155^o$ is found from the Smith chart to be $\overline{Z}_s = 0.25 + j0.21$. There is however, an increase in input VSWR from 3.9 to 4.5.

PART 2 Determination of the value of $\rho_{out}$ , $\rho_L$ and the values of the output VSWR.
From the conjugate of $\rho_s$ substituted into equation 5-51, together with the value of the S-parameters from table 5.8, the value of $\rho_L$ can be determined. That is, from

$$\rho_L = \frac{\rho_{in} - S_{11}}{S_{22}\rho_{in} - \Delta} = \frac{\rho_s^* - S_{11}}{S_{22}\rho_s^* - \Delta} \qquad \textbf{(5-51)}$$

The value of $\rho_L$ is found to be, $\rho_L = 0.6267\angle 15.65^o$. When plotted on the $\rho_L$ plane shown in Figure 146, it is seen that $\rho_L$ lies on the "Figure of erit" circle, as expected, and is in the stable region.
The value of load VSWR is determined from the plot of $\rho_L$ on the Smith chart in $\rho_L$ plane as VSWR = 4.4. This is an increase from the 2.50 obtained in Example 29. Also the load impedance is determined from this plot as $\overline{Z}_L = 3.25 + j1.82$ or from

$$Z_L = Z_0 \frac{1 + \rho_L}{1 - \rho_L} \qquad \textbf{(1-57)}$$

as $\overline{Z}_L = 3.267 + j1.820$
To match the load impedance to the transmission line characteristic impedance, the use of the Smith chart covered in Chapter 4 is employed to determine the output matching network whose position relative to the amplifier is shown in Figure 124
From equation 5-41

$$\rho_{out} = \frac{\Delta\rho_s - S_{22}}{S_{11}\rho_s - 1} \qquad \textbf{(5-41)}$$

the value of the amplifier output reflection coefficient is found as, $\rho_{out} = 0.949\angle -43^o$.
Finally, to conclude this chapter is the determination of parameters for a specific input and output VSWR. By reducing the VSWR at the input or output, the power gain will decrease and the noise will possibly increase. These are the considerations of the next section where the input and output matching circles are considered.

### *5.6. Constant Input Mismatch Circles*

The constant input mismatch circle is used when an input VSWR is specified. By plotting this circle, which is in the $\rho_s$ plane, the source stability circle, the conjugate input impedance circle and the constant noise factor circles can be taken into consideration in order to obtain the best possible noise figure and power gain whilst still remaining stable.
For a given load reflection coefficient $\rho_L$, there will be a fixed input reflection coefficient $\rho_{in}$, as given by equation 5-57. That is,

$$\rho_{in} = \frac{\Delta\rho_L - S_{11}}{S_{22}\rho_L - 1} \qquad \textbf{(5-57)}$$

Also, for a given input VSWR ($VSWR_{in}$), there will be a fixed value of $M_s$ as shown by equation D1-9.

$$M_s = \frac{4VSWR_{in}}{\left(VSWR_{in} + 1\right)^2} \qquad \textbf{(D1-9)}$$

As derived in Appendix D, the constant input mismatch circles have a centre and radius given by,

centre
$$c_{cim} = \frac{M_s \rho_{in}^*}{1-|\rho_{in}|^2(1-M_s)} \qquad \textbf{(D1-11)}$$

radius
$$r_{cim} = \frac{\sqrt{1-M_s}\left(1-|\rho_{in}|^2\right)}{1-|\rho_{in}|^2(1-M_s)} \qquad \textbf{(D1-12)}$$

The method used to match an amplifier or network so that it satisfies a specified input VSWR and provides the largest gain and smallest noise figure and still remains stable, is to first select the load reflection coefficient. This is done by drawing the constant power gain circles and load stability circle and determining the value of $\rho_L$ which lies on the required constant power gain circle and is in the stable region, and perhaps, which produces a low load VSWR. Once $\rho_L$ has been determined the input reflection coefficient can be calculated using equation 5-57. With the specified input VSWR and the value of $\rho_{in}$ known, the mismatch factor $M_s$ can be calculated using equation D1-9, and from this, the centre and radius of the constant input mismatch circle using equations D1-11 and D1-12. These circles are drawn on the $\rho_s$ plane together with the constant noise factor circles, conjugate input matching circles and source stability circles. The choice of $\rho_s$ which must be made from a point lying on the constant input VSWR circle will depend on its position on this circle. If it is chosen to lie on the line joining the centre of the VSWR circle with the noise figure circle, then the choice of $\rho_s$ will be for the minimization of the noise figure.

EXAMPLE 31

An amplifier is to be designed with a VSWR no greater than 2.2. sing the manufacturer s data for the bipolar transistor for a collector current of 10mA at 1 GHz, given in Example 27 and the noise data given in Example 30, determine a value for $\rho_L$ and draw the constant input mismatch circle in the $\rho_s$ plane for the specified VSWR. Determine the value of available power gain for the device. Select the source impedance such that the input noise is minimized and determine the noise factor. Compare the answers with that of Example 29.

SOLUTION

PART 1 Determination of $\rho_L$, load VSWR, and $\rho_{in}$

From Figure 146, of Example 28, it can be seen by looking at the constant power gain circle for the "Figure of erit" gain, where $g_a = 2.141$ that the load reflection coefficient $\rho_L$ is furthest from the load stability circle if it lies along the line joining the centres of the load stability circle and the constant power gain circle. The choice of $\rho_L$ is made by maintaining the "Figure of erit" gain and maximizing its distance from the load stability circle. The value of $\rho_L$ is found to be:

$\rho_L = 0.24\angle 64.5^o$. The load VSWR is found to be approximately 1.6.

From equation 5-57, the input reflection coefficient is found to be,

$$\rho_{in} = \frac{S_{11} - \Delta\rho_L}{1 - S_{22}\rho_L} = 0.6494\angle -119.5^o$$

PART 2 Determination of $M_s$, $c_{cim}$, and $r_{cim}$

With a specified input VSWR of 2.2, the value of the mismatch factor $M_s$ is found from equation D1-9 to be,

$$M_s = \frac{4VSWR_{in}}{\left(VSWR_{in}+1\right)^2} = 0.8594$$

Using the value of $\rho_{in}$ and $M_s$ in equations D1-10 and D1-11, the centre and radius of the constant mismatch circle can be found. That is,

centre
$$c_{cim} = \frac{M_s \rho_{in}^*}{1-|\rho_{in}|^2(1-M_s)} = 0.5932\angle 119.5°$$

radius $$r_{cim} = \frac{\sqrt{1-M_s}\left(1-|\rho_{in}|^2\right)}{1-|\rho_{in}|^2(1-M_s)} = 0.2305$$

PART 3 Plotting of constant input mismatch circle, selection of the source impedance such that the input noise is minimized and the determination of the noise factor..

From the values of the centre and the radius, the constant mismatch circle for an input VSWR of 2.2 is shown plotted on Figure 149, together with the source stability, $\rho_{in}^*$ circle, and 2.2dB noise figure circle.

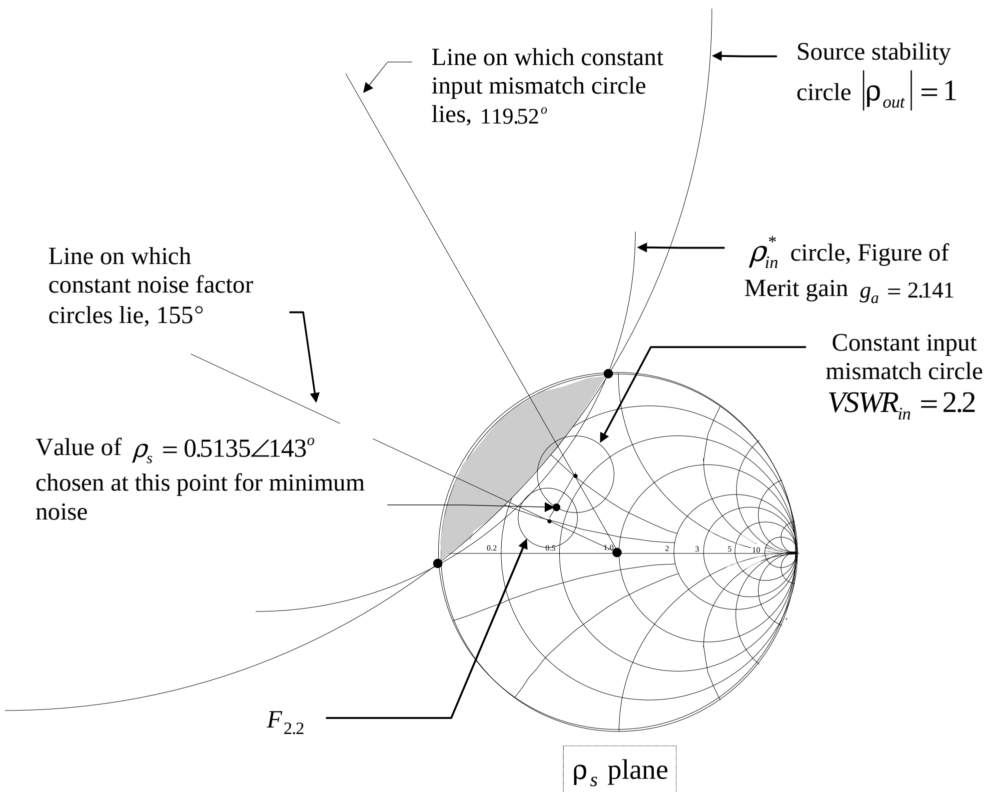



**Figure 150** **Example 31 - Constant input mismatch circle together with noise factor circle (2.2 dB) source stability and $\rho_{in}^*$ circle for a normalized constant power gain of 2.141 at 1 GHz.**

The value of $\rho_s$ for the minimization of the noise figure, as shown in Figure 150, is found to be, $\rho_s = 0.5135\angle 43°$ . This value of $\rho_s$ is chosen with a power gain e ual to the “Figure of erit” gain, $g_a = 2.141$, an input VSWR = 2.2, a load VSWR = 1.6, and a noise figure of less than 2.2 dB. As the minimum noise figure is 2.1 dB, it is anticipated from Figure 150, that this noise figure is 2.15 dB.

Comparing these results with Example 30, it can be seen that the input VSWR has been reduced from 4.5 to 2.2, the load VSWR has been reduced from 4.4 to 1.6, the noise figure reduced from 2.2 to 2.15 and the normalized power gain has been maintained at the “figure of merit” value of . , and the input and output circuits are stable.

## 5.7. Constant Output Mismatch Circles

The constant output mismatch circle is used when an output VSWR is specified (not a load VSWR), such as when the input of the second stage of a two-stage amplifier requires a specific source impedance from the output of the first stage. By plotting this circle, which is in the $\rho_L$ plane, the load stability circle and the constant power circles can be taken into consideration in order to obtain the best possible power gain whilst still remaining stable.

For a given source reflection coefficient $\rho_s$, there will be a fixed output reflection coefficient $\rho_{out}$, as given by equation 5-41. That is,

$$\rho_{out} = \frac{\Delta\rho_s - S_{22}}{S_{11}\rho_s - 1} \qquad \textbf{(5-41)}$$

Also, for a given output VSWR ($VSWR_{out}$), there will be a fixed value of $M_{out}$ as shown by equation D1-20.

$$M_{out} = \frac{4VSWR_{out}}{\left(VSWR_{out}+1\right)^2} \qquad \textbf{(D1-20)}$$

As derived in Appendix D, the constant output mismatch circles have a centre and radius given by,

centre $$c_{com} = \frac{M_{out}\rho_{out}^*}{1-\left|\rho_{out}\right|^2\left(1-M_{out}\right)} \qquad \textbf{(D1-21)}$$

radius $$r_{com} = \frac{\sqrt{1-M_{out}}\left(1-\left|\rho_{out}\right|^2\right)}{1-\left|\rho_{out}\right|^2\left(1-M_{out}\right)} \qquad \textbf{(D1-22)}$$

The method used to match an amplifier or network output so that it satisfies a specified VSWR when looking back into the amplifier and provides the largest gain whilst still remaining stable, is to first select the source reflection coefficient. This is done by drawing the conjugate input reflection coefficient circle for a chosen power gain, the source stability circle and the chosen noise figure circle and from these circles determining the value of $\rho_s$ which is satisfactory for the designer s requirements of power gain, input VSWR and noise figure.

Once $\rho_s$ has been determined the output reflection coefficient can be calculated using equation 5-41. With the specified output VSWR and the value of $\rho_s$ known, the mismatch factor $M_{out}$ can be calculated using equation D1-20, and from this, the centre and radius of the constant input mismatch circle using equations D1-21 and D1-22. These circles are drawn on the $\rho_L$ plane together with the constant power gain circles and load stability circle. If there is no restrictions placed on $\rho_L$ by the second stage of a two-stage amplifier, the choice of $\rho_L$ is made from a point which lies on the intersection of the constant output VSWR circle and the chosen constant power gain circle, or if this is not possible, then, which lies on the intersection of a line which joins the constant power gain circle with the output VSWR circle and the output VSWR circle itself.

EXAMPLE 32

n amplifier is to be designed with an output no greater than . . sing the manufacturer s data for the bipolar transistor for a collector current of 10mA at 1 GHz, given in Example 27 and the noise data given in Example 30, determine a value for $\rho_s$ and draw the constant input mismatch circle in the $\rho_L$ plane for the specified VSWR. Determine the value of available power gain for the device, the noise figure for a stable amplifier, the input VSWR and the load VSWR.

SOLUTION

PART 1 Determination of $\rho_s$, source VSWR, and $\rho_{out}$

From Example 31, $\rho_s$ was chosen to be $\rho_s = 0.5135\angle 43^o$ to give a noise figure less than 2.2 dB and an input VSWR of 2.2. The same value of $\rho_s$ will be chosen, and with the specified output VSWR being no greater than 1.2, the power gain, source VSWR and output reflection coefficient will be determined.

With $\rho_s = 0.5135\angle 43^o$, the source VSWR is found to be approximately 1.95.

From equation 5-41, the input reflection coefficient is found to be,

$$\rho_{out} = \frac{\Delta\rho_s - S_{22}}{S_{11}\rho_s - 1} = 0.385\angle 62.7°$$

PART 2 Determination of $M_{out}$, $c_{com}$, and $r_{com}$

With a specified output VSWR of 1.2, the value of the mismatch factor $M_{out}$ is found from equation D1-20 to be,

$$M_{out} = \frac{4VSWR_{out}}{\left(VSWR_{out}+1\right)^2} = 0.9722$$

Using the value of $\rho_s$ and $M_{out}$ in equations D1-21 and D1-22, the centre and radius of the constant output mismatch circle can be found. That is,

centre $$c_{com} = \frac{M_{out}\rho_{out}^*}{1-\left|\rho_{out}\right|^2\left(1-M_{out}\right)} = 0.3821\angle 62.7°$$

radius $$r_{com} = \frac{\sqrt{1-M_{out}}\left(1-\left|\rho_{out}\right|^2\right)}{1-\left|\rho_{out}\right|^2\left(1-M_{out}\right)} = 0.9722$$

PART 3 Plotting of constant output mismatch circle, selection of the load impedance such that the available power gain is maximized and determination of the noise factor..

From the values of the centre and the radius, the constant mismatch circle for an output VSWR of 1.4 is shown plotted on Figure 150, together with the load stability circle, and constant power gain circles.

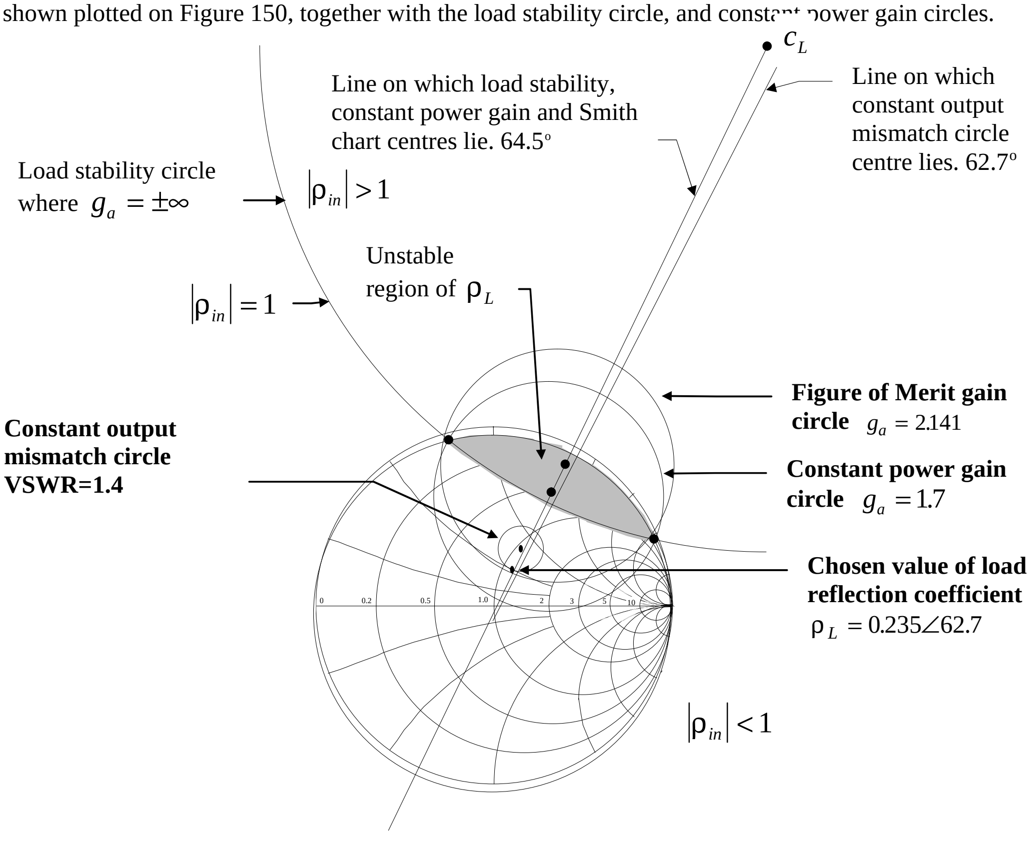



**Figure 151 Example 32 - Constant output mismatch circle with load stability and constant power gain circles at 1 GHz**

The value of load reflection coefficient is chosen to be $\rho_L = 0.235\angle 62.7°$, which gives a load VSWR of 1.6, and a normalized power gain which is still at the "Figure of erit" value. he output circuit of the amplifier is also stable. The noise figure is less than 2.2 dB, as the source reflection coefficient was chosen to be $\rho_s = 0.5135\angle 43^o$. This value of source reflection coefficient also gives a source VSWR of 2.2 and the input is stable.

# 1. Worked solutions of questions given in Chapter 4

QUESTION 1 - Chapter 4

A measured load impedance band which is converted into a normalized impedance band is given below. It is required that this impedance band be modified so that it is matched to a VSWR of 1.25 or less. Draw the impedance and admittance band on the Smith chart and determine the VSWR of the unmatched band. Using a short-circuit quarter-wavelength stub, determine the impedance of the stub to match the band, and draw the final admittance band on the Smith chart, ensuring that it is within the specified VSWR.

| Number | Frequency (GHz) | Normalized impedance |
|---|---|---|
| 1 | 5.50 | $\overline{Z}_1 = 0.50 + j0.55$ |
| 2 | 5.40 | $\overline{Z}_2 = 0.80 + j0.4$ |
| 3 | 5.30 | $\overline{Z}_3 = 0.90$ |
| 4 | 5.10 | $\overline{Z}_4 = 0.70 - j0.45$ |
| 5 | 5.00 | $\overline{Z}_5 = 0.40 - j0.55$ |

SOLUTION

Note how the highest frequency has its impedance point in the upper half of the Smith chart and the lowest frequency impedance point lies in the lower half of the Smith chart. This means that the highest frequency admittance will lay in the lower half of the Smith chart and the lowest frequency admittance will lay in the upper half of the Smith chart. The band also lies around the X=0 axis. These two events imply that a quarter-wavelength short-circuit stub may be used to close the band.

Because we wish to place an short-circuit stub in parallel with the load, we must work in admittances. The normalized load impedances converted into admittances are given below and are plotted on the Smith chart shown in Figure “ uestion ”. **The VSWR of the load band is found from the Smith chart to be approximately 3.3.**

| Number | Frequency (GHz) | Normalized admittance |
|---|---|---|
| | 5.50 | $\overline{Y}_1 = 0.905 - j0.995$ |
| | 5.40 | $\overline{Y}_2 = 1.000 - j0.500$ |
| 3 | 5.30 | $\overline{Y}_3 = 1.111$ |
| | 5.10 | $\overline{Y}_4 = 1.011 + j0.650$ |
| | 5.00 | $\overline{Y}_5 = 0.865 + j1.190$ |

Using equation 4-37, with the centre frequency 5.30 GHz, the following table is obtained. As the high frequency admittance point (point 1) will move the most, a susceptance point
on the Smith chart constant resistance circle of point 1, is found which will give a VSWR of 1.25 or better. This turns out to be -j0.106. From this value and the original value, it is determined that the susceptance must move from -j0.995 to -j0.106, or a distance of +j0.889. As column 3 provides equation 4-37 without the $Z_o/Z_{ch}$ factor, column 4 divided by column 3, gives the value of $Z_o/Z_{ch}$. This value is 15, **therefore the characteristic impedance of the quarter-wave short transformer is to be** $Z_{ch} = \dfrac{Z_o}{15}$**.** Figure “ uestion ” shows the final admittance band laying within a . circle, which is the VSWR specified.

| Column 1 | 2 | 3 | 4 (Eqn. 4-37) | 5 | 6 |
|---|---|---|---|---|---|
| **Frequency (GHz)** $f_x$ | **Point No.** | $-j\cot\frac{\pi}{2}\frac{f_x}{f_o}$ | $-j\frac{Z_o}{Z_{ch}}cot\frac{\pi}{2}\frac{f_x}{f_o}$ For $Z_o = 15Z_{ch}$ | **Original Band admittances (1, 2, 3, 4, 5) Fig. 96** | **Final Band admittances (1", 2", 3", 4", 5") Fig. 96 Col. (5 + 4)** |
| 5.50 | 1 | + j0.0593 | +0.8895 | 0.905 - j0.995 | 0.905 - j0.106 |
| 5.40 | 2 | + j0.0296 | +j0.444 | 1.000 - j0.500 | 1.000 - j0.056 |
| 5.30 ( $f_o$ ) | 3 | 0 | 0 | 1.111 | 1.111 |
| 5.10 | 4 | - j0.0593 | -j0.8895 | 1.011 + j0.650 | 1.011 - j0.239 |
| 5.00 | 5 | - j 0.0891 | -j1.3365 | 0.865 + j1.190 | 0.865 - j0.147 |

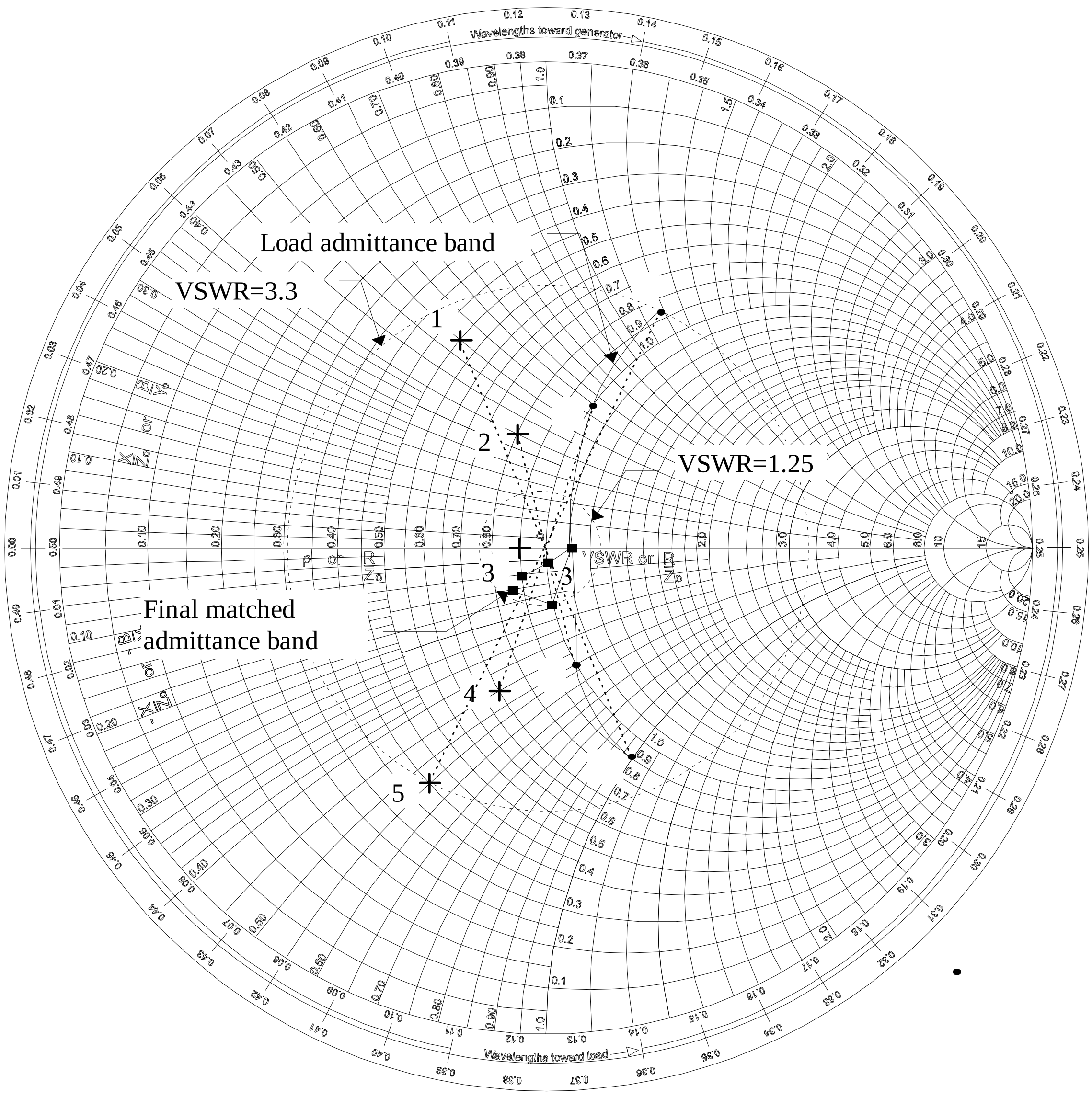



**Figure Q 1 Question 1 - Short-circuit quarter-wavelength stub effect on admittance band**

QUESTION 2 - Chapter 4
What would be the minimum VSWR of a band of frequencies given below if the quarter-wavelength frequency of two quarter-wave transformers in tandem was taken as 0.8 GHz?. The following normalized mismatched load impedance points are given as,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

SOLUTION
The VSWR circle and the impedance points are shown plotted on Figure 'Q2a'. As the band opening is not centred around the X=0 axis, there is no point in using short-circuit stub to try and close it more. A short circuit stub may be worthwhile considering after the transformers have processed the band, if the VSWR is not within specifications.
The average normalized impedance value is found from the given normalized load impedance points as 2.04 - j 0.94. The value of the load resistance $\overline{Z}_L$, is taken as $\overline{Z}_L$ = 2.04, for use in determining the characteristic impedances of the quarter-wave transformers. As the value of the characteristic impedance of the main transmission line $Z_o$, is unknown, equations 4-41 and 4-42 must be normalized to $Z_o$, that is

$$\overline{Z}_{oB} = \left(\overline{Z}_L\right)^{1/4} \qquad \textbf{(4-43)}$$

$$\overline{Z}_{oA} = \left(\overline{Z}_L^3\right)^{1/4} \qquad \textbf{(4-44)}$$

From equations 4.43 and 4-44, $\overline{Z}_{oB} = 1.1951$ and $\overline{Z}_{oA} = 1.7070$
The impedances normalized to the quarter-wave transformer closest to the load is found by dividing the transmission line normalized impedances given in the Example description, divided by $\overline{Z}_{oA} = 1.7070$.
These renormalized impedances are given below,

| Frequency (GHz) | Normalized impedance |
|---|---|
| 1.00 | $\overline{Z}_1 = 1.9332 - j0.4687$ |
| 1.02 | $\overline{Z}_2 = 1.5231 - j0.8202$ |
| 1.04 | $\overline{Z}_3 = 0.9373 - j0.7616$ |
| 1.06 | $\overline{Z}_4 = 0.7616 - j0.4687$ |
| 1.08 | $\overline{Z}_5 = 0.8202 - j0.2343$ |

These impedance points are shown plotted on Figure 'Q2b'.
The given frequency at which the two quarter-wavelength transformers are λ/4 is 0.8 GHz. As this point, if it existed, would be rotated through λ/4, the rotation of the other transformed load impedance points (points 1 to 5) must be each be rotated through a different angle, as each operates at a different frequency. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance point.

$$\lambda_{f_x} = \lambda_{f_r}\frac{f_r}{f_x} = 0.250\frac{0.8}{f_x}.$$

| **Frequency (GHz)** | **Initial impedance position (λ), points 1,2,3,4,5** | **Wavelength $\lambda_{f_x}$** | **Final impedance position (λ) , points ’, ’, ’, ’, ’** | **Final Impedance points of transformer closest to the load** |
|---|---|---|---|---|
| 1.00 | 0.273 | 0.2000 | 0.4730 | $\overline{Z}_1 = 0.49 - j0.13$ |
| 1.02 | 0.305 | 0.1961 | 0.5011 | $\overline{Z}_2 = 0.47$ |
| 1.04 | 0.3495 | 0.1923 | 0.5418 | $\overline{Z}_3 = 0.495 + j0.21$ |
| 1.06 | 0.391 | 0.1887 | 0.5797 | $\overline{Z}_4 = 0.66 + j0.34$ |
| 1.08 | 0.416 | 0.1852 | 0.6012 | $\overline{Z}_5 = 0.86 + j0.27$ |

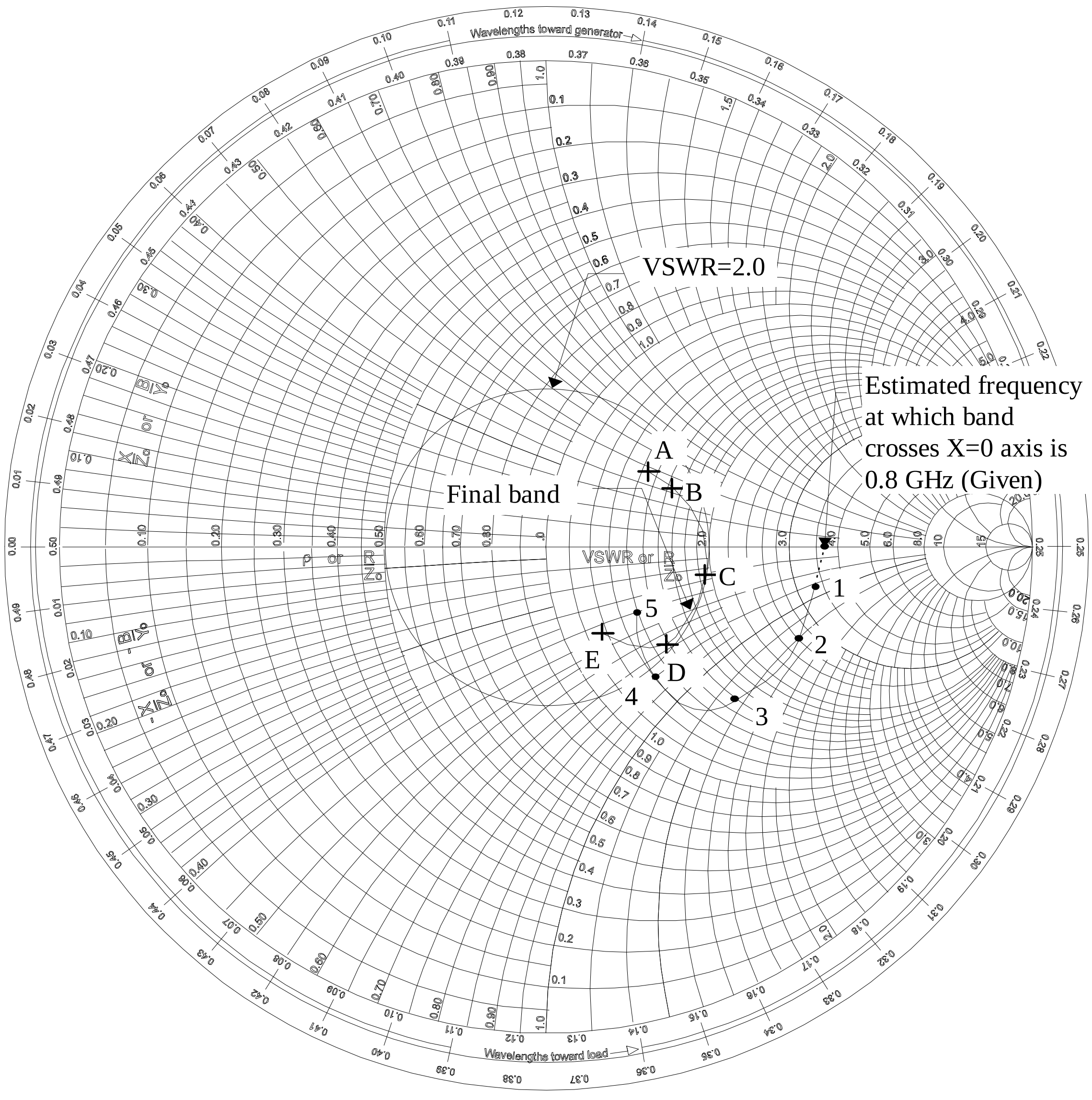



**Figure Q2a** **Question 2 - Two quarter-wavelength transformer matching problem**

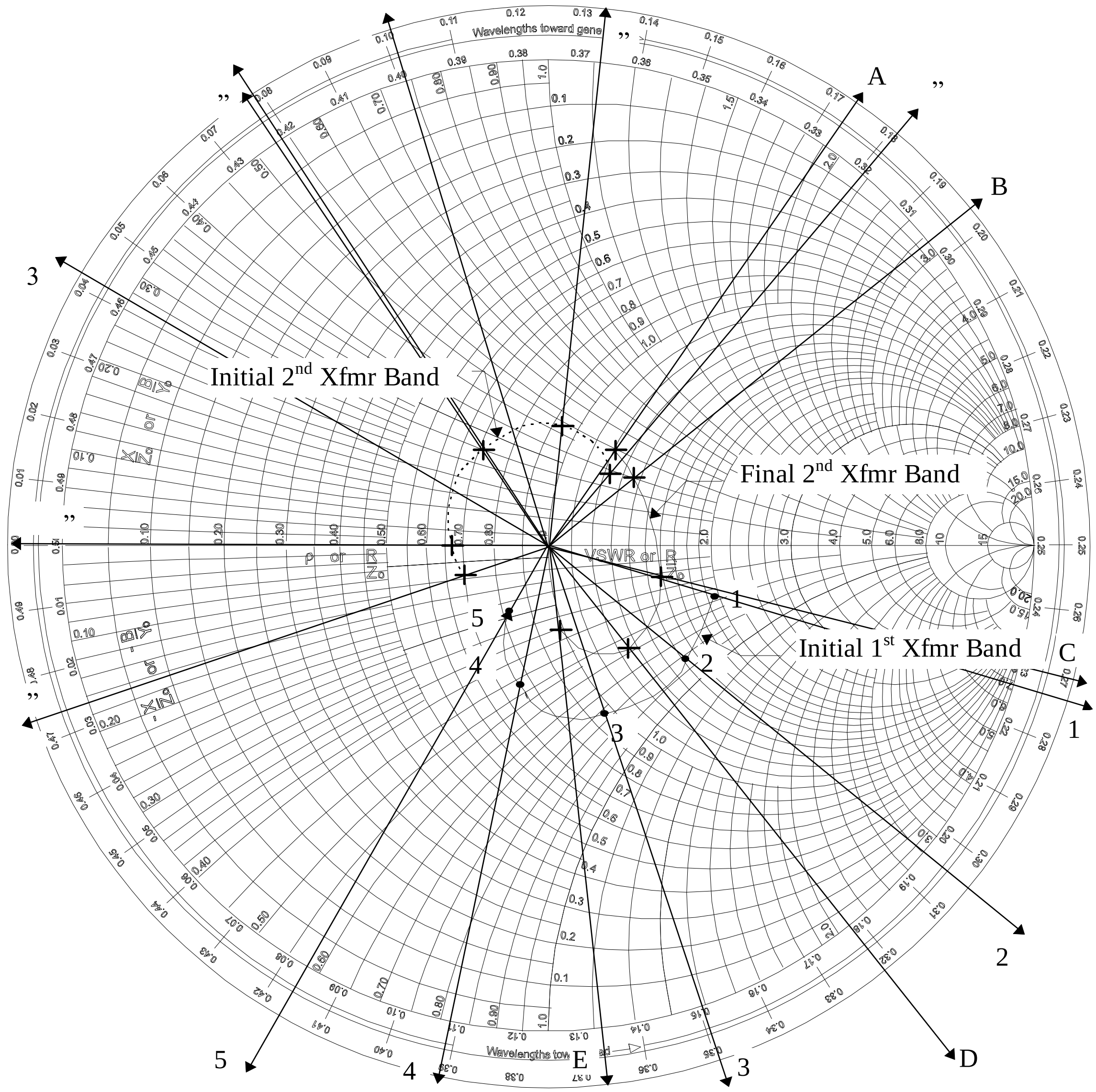



**Figure Q2b** **Question 2 - Two quarter-wavelength transformer matching problem**

Normalizing the impedance points of the transformer closest to the load back to that of the transmission line and then renormalizing to the transformer closest to the transmission line, so that each impedance of the band is presented on the Smith chart belonging to the transformer closest to the transmission line. To do this, the impedance points on the transformer closest to the load would be multiplied by $\overline{Z}_{oA} = 1.7070$ and then divided by $\overline{Z}_{oB} = 1.1951$. The table below shows the conversion and Figure 'Q2b' shows these points plotted as 1",2",3",4" and 5".

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the load** | **Renormalized impedance points for processing by transformer furthest from the load. (1”,2”,3”,4”,5”)** |
|---|---|---|
| 1.00 | $\overline{Z}_1 = 0.49 - j0.13$ | $\overline{Z}_1 = 0.700 - j0.186$ |
| 1.02 | $\overline{Z}_2 = 0.47$ | $\overline{Z}_2 = 0.671$ |
| 1.04 | $\overline{Z}_3 = 0.495 + j0.21$ | $\overline{Z}_3 = 0.707 + j0.300$ |
| 1.06 | $\overline{Z}_4 = 0.66 + j0.34$ | $\overline{Z}_4 = 0.943 + j0.486$ |
| 1.08 | $\overline{Z}_5 = 0.86 + j0.27$ | $\overline{Z}_5 = 1.228 + j0.386$ |

Similar to the first transformer the impedance points renormalized to the second transformer must be rotated. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance points exiting the first transformer and renormalized to the beginning of the second transformer, or that transformer which is closest to the transmission line.

$$\lambda_{f_x} = \lambda_{f_r}\frac{f_r}{f_x} = 0.250\frac{0.8}{f_x}.$$

| **Frequency (GHz)** | **Initial impedance position (λ), points 1”,2”,3”,4”,5”** | **Wavelength** $\lambda_{f_x}$ | **Final impedance position (λ) , points A,B,C,D,E** | **Final Impedance points of transformer closest to the transmission line** |
|---|---|---|---|---|
| 1.00 | 0.473 | 0.2000 | 0.1730 | $\overline{Z}_A = 1.2 + j0.4$ |
| 1.02 | 0.500 | 0.1961 | 0.1961 | $\overline{Z}_B = 1.35 + j0.35$ |
| 1.04 | 0.078 | 0.1923 | 0.2703 | $\overline{Z}_C = 1.6 - j0.22$ |
| 1.06 | 0.133 | 0.1887 | 0.3217 | $\overline{Z}_D = 1.25 - j0.475$ |
| 1.08 | 0.181 | 0.1852 | 0.3662 | $\overline{Z}_E = 0.99 - j0.36$ |

Figure ‘Q2b’ shows the final second quarter-wave transformer impedance band still in normalized to the second quarter wave transformer. The multiplication of the impedance points by $\overline{Z}_{oB} = 1.1951$ will renormalize them to the transmission line characteristic impedance. The table below shows these renormalized impedance points and Figure ‘Q2a’ shows the final plotted band.

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the transmission line** | **Final Impedance points renormalized to transmission line** $Z_o$ |
|---|---|---|
| 1.00 | $\overline{Z}_A = 1.2 + j0.4$ | $\overline{Z}_A = 1.43 + j0.48$ |
| 1.02 | $\overline{Z}_B = 1.35 + j0.35$ | $\overline{Z}_B = 1.61 + j0.42$ |
| 1.04 | $\overline{Z}_C = 1.6 - j0.22$ | $\overline{Z}_C = 1.91 - j0.26$ |
| 1.06 | $\overline{Z}_D = 1.25 - j0.475$ | $\overline{Z}_D = 1.49 - j0.57$ |
| 1.08 | $\overline{Z}_E = 0.99 - j0.36$ | $\overline{Z}_E = 1.18 - j0.43$ |

The final VSWR is found to be 2.00

QUESTION 3 - Chapter 4
What would be the minimum VSWR of a band of frequencies given below if the quarter-wavelength frequency of two quarter-wave transformers in tandem was taken as the geometric mean of the upper and lower band frequencies?. The following normalized mismatched load impedance points are given as,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

SOLUTION
The VSWR circle and the impedance points are shown plotted on Figure Q3a.
The average normalized impedance value is found from the given normalized load impedance points as 2.04 - j 0.94. The value of the load resistance $\overline{Z}_L$, is taken as $\overline{Z}_L$ = 2.04, for use in determining the characteristic impedances of the quarter-wave transformers. As the value of the characteristic impedance of the main transmission line $Z_o$, is unknown, equations 4-41 and 4-42 must be normalized to $Z_o$, that is

$$\overline{Z}_{oB} = \left(\overline{Z}_L\right)^{1/4} \qquad \textbf{(4-43)}$$

$$\overline{Z}_{oA} = \left(\overline{Z}_L^3\right)^{1/4} \qquad \textbf{(4-44)}$$

From equations 4.43 and 4-44, $\overline{Z}_{oB} = 1.1951$ and $\overline{Z}_{oA} = 1.7070$
The impedances normalized to the quarter-wave transformer closest to the load is found by dividing the transmission line normalized impedances given in the Example description, divided by $\overline{Z}_{oA} = 1.7070$.
These renormalized impedances are given below,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 1.9332 - j0.4687$ |
| 1.02 | $\overline{Z}_2 = 1.5231 - j0.8202$ |
| 1.04 | $\overline{Z}_3 = 0.9373 - j0.7616$ |
| 1.06 | $\overline{Z}_4 = 0.7616 - j0.4687$ |
| 1.08 | $\overline{Z}_5 = 0.8202 - j0.2343$ |

These impedance points are shown plotted on Figure Q3b.
The estimated frequency which is closest to the geometric mean of the highest and lowest frequency in the band , that is, $f_o = \sqrt{f_L f_H}$, is $\sqrt{1.00 x 1.08} = 1.0392$, is 1.04 GHz. This impedance point at 1.04 GHz is rotated through λ/4. The rotation of the other transformed load impedance points (points 1,2,4 and 5) must be each be rotated through a different angle, as each operates at a different frequency. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance point.

$$\lambda_{f_x} = \lambda_{f_r}\frac{f_r}{f_x} = 0.250\frac{1.04}{f_x}.$$

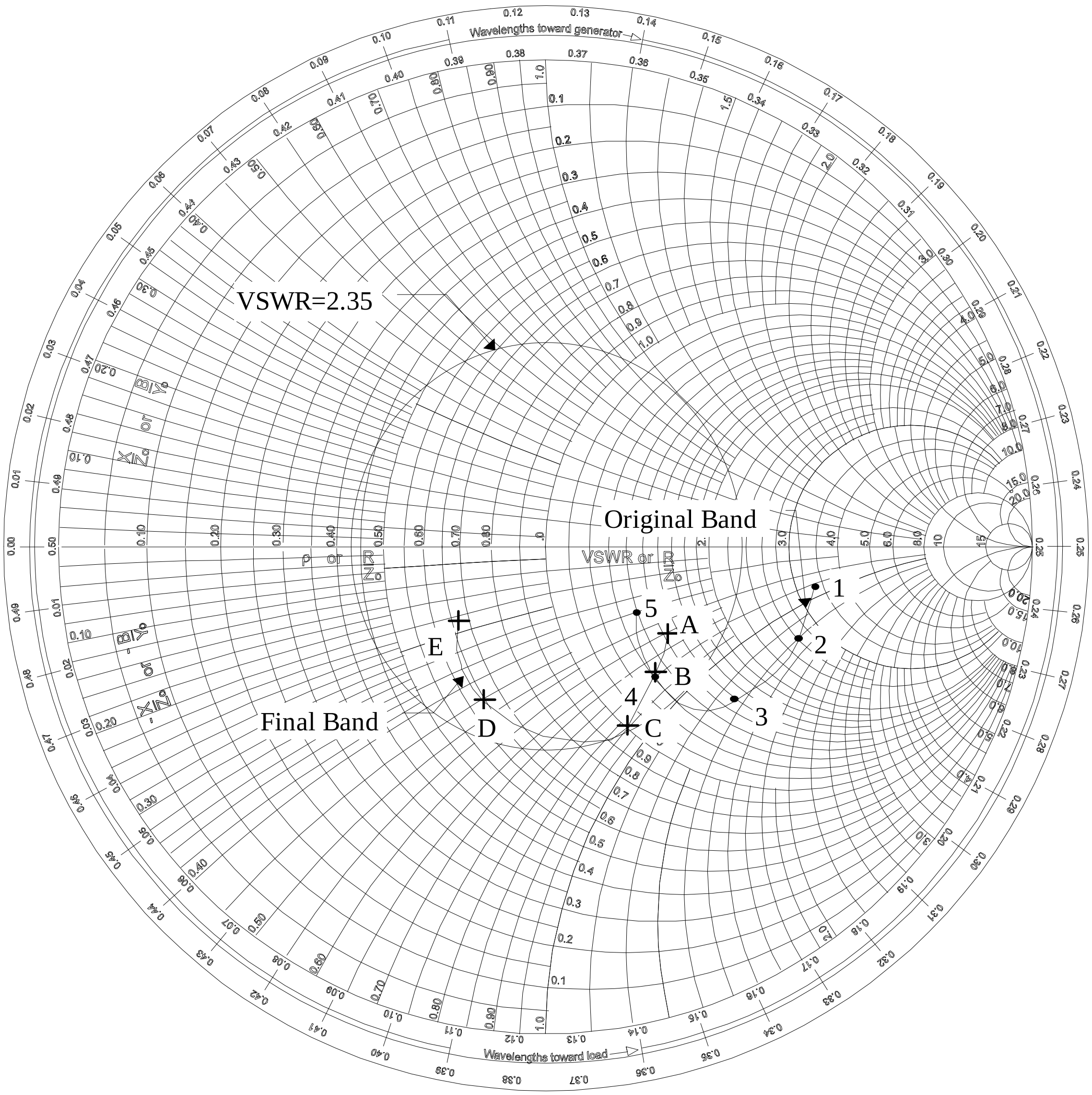



**Figure Q3a** **Question 3 - Two quarter-wavelength transformer matching problem**

| **Frequency (GHz)** | **Initial impedance position (λ), points 1,2,3,4,5** | **Wavelength** $\lambda_{f_x}$ | **Final impedance position (λ) , points ', ', ', ', '** | **Final Impedance points of transformer closest to the load (Not shown on Fg.99)** |
|---|---|---|---|---|
| 1.00 | 0.273 | 0.2600 | 0.0330 | $\overline{Z}_1 = 0.47 + j0.149$ |
| 1.02 | 0.305 | 0.2549 | 0.0599 | $\overline{Z}_2 = 0.47 + j0.250$ |
| 1.04 | 0.3495 | 0.2500 | 0.0995 | $\overline{Z}_3 = 0.47 + j0.390$ |
| 1.06 | 0.391 | 0.2453 | 0.1363 | $\overline{Z}_4 = 0.91 + j0.590$ |
| 1.08 | 0.416 | 0.2407 | 0.1567 | $\overline{Z}_5 = 1.09 + j0.370$ |

Normalizing the impedance points of the transformer closest to the load back to that of the transmission line and then renormalizing to the transformer closest to the transmission line, so that each impedance of the band is presented on the Smith chart belonging to the transformer closest to the transmission line. To do this, the impedance points on the transformer closest to the load would be multiplied by $\overline{Z}_{oA} = 1.7070$ and then divided by $\overline{Z}_{oB} = 1.1951$. The table below shows the conversion and Figure shows these oints lotted as ", ", ", " and ".

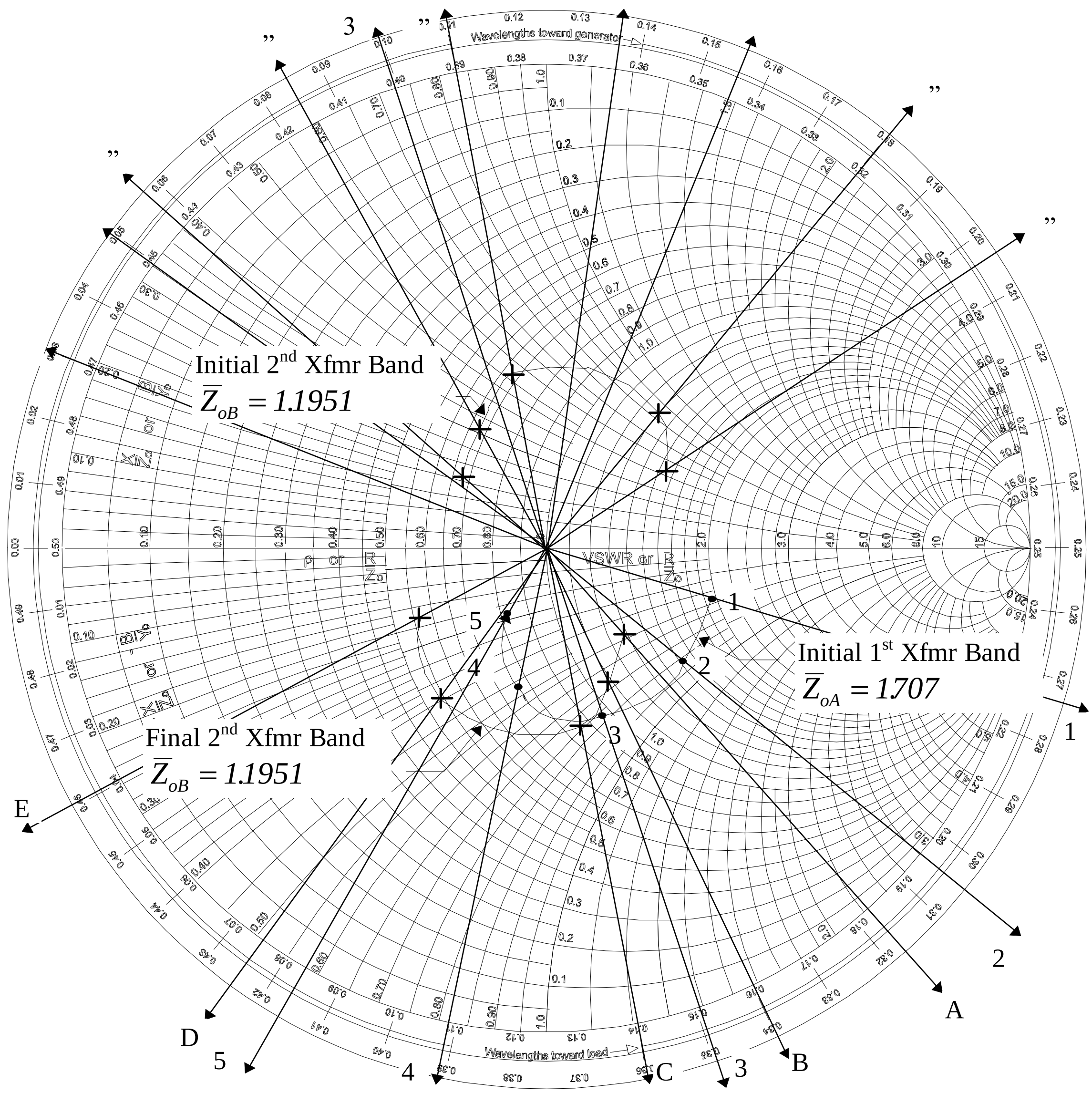



**Figure Q3b Question 3 - Two quarter-wavelength transformer matching problem**

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the load (Not shown on Fig.3b)** | **Renormalized impedance points for processing by transformer furthest from the load. (1”,2”,3”,4”,5”)** |
|---|---|---|
| 1.00 | $\overline{Z}_1 = 0.47 + j0.149$ | $\overline{Z}_1 = 0.671 + j0.213$ |
| 1.02 | $\overline{Z}_2 = 0.47 + j0.250$ | $\overline{Z}_2 = 0.671 + j0.357$ |
| 1.04 | $\overline{Z}_3 = 0.47 + j0.390$ | $\overline{Z}_3 = 0.671 + j0.557$ |
| 1.06 | $\overline{Z}_4 = 0.91 + j0.590$ | $\overline{Z}_4 = 1.300 + j0.843$ |
| 1.08 | $\overline{Z}_5 = 1.09 + j0.370$ | $\overline{Z}_5 = 1.557 + j0.529$ |

Similar to the first transformer the impedance points renormalized to the second transformer must be rotated. Below is a table of the angle each is to be rotated, using equation 4-34, and the initial and final position in wavelengths of the impedance points exiting the first transformer and renormalized to the beginning of the transformer which is closest to the transmission line.

$$\lambda_{f_x} = \lambda_{f_r} \frac{f_r}{f_x} = 0.250 \frac{1.04}{f_x}.$$

| **Frequency (GHz)** | **Initial impedance position (λ), points 1”,2”,3”,4”,5”** | **Wavelength $\lambda_{f_x}$** | **Final impedance position (λ) , points A,B,C,D,E** | **Final Impedance points of transformer closest to the transmission line** |
|---|---|---|---|---|
| 1.00 | 0.058 | 0.2600 | 0.3180 | $\overline{Z}_A = 1.29 - j0.48$ |
| 1.02 | 0.085 | 0.2549 | 0.3399 | $\overline{Z}_B = 1.1 - j0.65$ |
| 1.04 | 0.110 | 0.2500 | 0.3600 | $\overline{Z}_C = 0.85 - j0.74$ |
| 1.06 | 0.180 | 0.2453 | 0.4253 | $\overline{Z}_D = 0.54 - j0.38$ |
| 1.08 | 0.24 | 0.2407 | 0.4807 | $\overline{Z}_E = 0.56 - j0.18$ |

Figure Q3b shows the final second quarter-wave transformer impedance band still in normalized to the second quarter wave transformer. The multiplication of the impedance points by $\overline{Z}_{oB} = 1.1951$ will renormalize them to the transmission line characteristic impedance. The table below shows these renormalized impedance points and Figure Q3a shows the final plotted band.

| **Frequency (GHz)** | **Final Impedance points of transformer closest to the transmission line** | **Final Impedance points renormalized to transmission line $Z_o$** |
|---|---|---|
| 1.00 | $\overline{Z}_A = 1.29 - j0.48$ | $\overline{Z}_A = 1.54 - j0.57$ |
| 1.02 | $\overline{Z}_B = 1.1 - j0.65$ | $\overline{Z}_B = 1.31 - j0.78$ |
| 1.04 | $\overline{Z}_C = 0.85 - j0.74$ | $\overline{Z}_C = 1.02 - j0.88$ |
| 1.06 | $\overline{Z}_D = 0.54 - j0.38$ | $\overline{Z}_D = 0.65 - j0.45$ |
| 1.08 | $\overline{Z}_E = 0.56 - j0.18$ | $\overline{Z}_E = 0.67 - j0.22$ |

The final VSWR is shown to be 2.35.

QUESTION 4 - Chapter 4
ework xample using the “traditional” approach when using a short-transformer to match a transmission line. Determine the best VSWR that you can obtain, given the following normalized mismatched load impedance points,

| **Frequency (GHz)** | **Normalized impedance** |
|---|---|
| 1.00 | $\overline{Z}_1 = 3.3 - j0.8$ |
| 1.02 | $\overline{Z}_2 = 2.6 - j1.4$ |
| 1.04 | $\overline{Z}_3 = 1.6 - j1.3$ |
| 1.06 | $\overline{Z}_4 = 1.3 - j0.8$ |
| 1.08 | $\overline{Z}_5 = 1.4 - j0.4$ |

Make comments on how you could improve the VSWR obtained.

SOLUTION
The VSWR circle and the impedance points are shown plotted on Figure Q4.
In this problem, the modified boundary circle cuts the X=0 axis at 0.94 and 4.6, from equation 4-33, thr value of $\overline{Z}_{oA}^2 = 0.94 \text{ x } 4.6$ , giving the value of the characteristic impedance of the short transformer $\overline{Z}_{oA}$ = 2.0794.
Each of the load impedances and the VSWR circle are transformed into a Smith chart which is normalized to the short transformer characteristic impedance, $\overline{Z}_{oA}$ = 2.0794. This allows the length of the transformer to be determined. By dividing the load impedances and VSWR circle by 2.0794, the transformed values become,

| **Frequency (GHz)** | **Transformed Normalized impedance to Short Transformer Smith chart** |
|---|---|
| 1.00 | $\tilde{\overline{Z}}_1 = 1.5870 - j0.3847$ |
| 1.02 | $\tilde{\overline{Z}}_2 = 1.2504 - j0.6733$ |
| 1.04 | $\tilde{\overline{Z}}_3 = 0.7695 - j0.6252$ |
| 1.06 | $\tilde{\overline{Z}}_4 = 0.6252 - j0.3847$ |
| 1.08 | $\tilde{\overline{Z}}_5 = 0.6733 - j0.1924$ |

and the VSWR circle becomes
Diameters: 0.3206, 0.7214
Centre 0.4809
These points are shown plotted on Figure Q4/b.
Once the transformed VSWR circle and load impedance points have been plotted onto the Smith chart normalized to the short transformer characteristic impedance, the length of the transformer is determined from the angle (in wavelengths) between the centre frequency impedance point (point 3 - 1.04 GHz), and the nearest real axis (X=0 axis- point . his is shown in Figure 4/b. The centre frequency impedance point reads 0.376λ and the real axis reads 0.500λ on the edge of the Smith chart. The difference between these two readings is 0.124λ, which is the length of the transformer.

The rotation of the other transformed load impedance points (points 1,2,4, and 5) must be each be rotated through a different angle, as each operates at a different frequency. Below is a table of the angle each is to be rotated, using equation 4-34, $\lambda_{f_x} = \lambda_{f_r} \dfrac{f_r}{f_x} = 0.124 \dfrac{1.04}{f_x} = \dfrac{0.129}{f_x}$ .

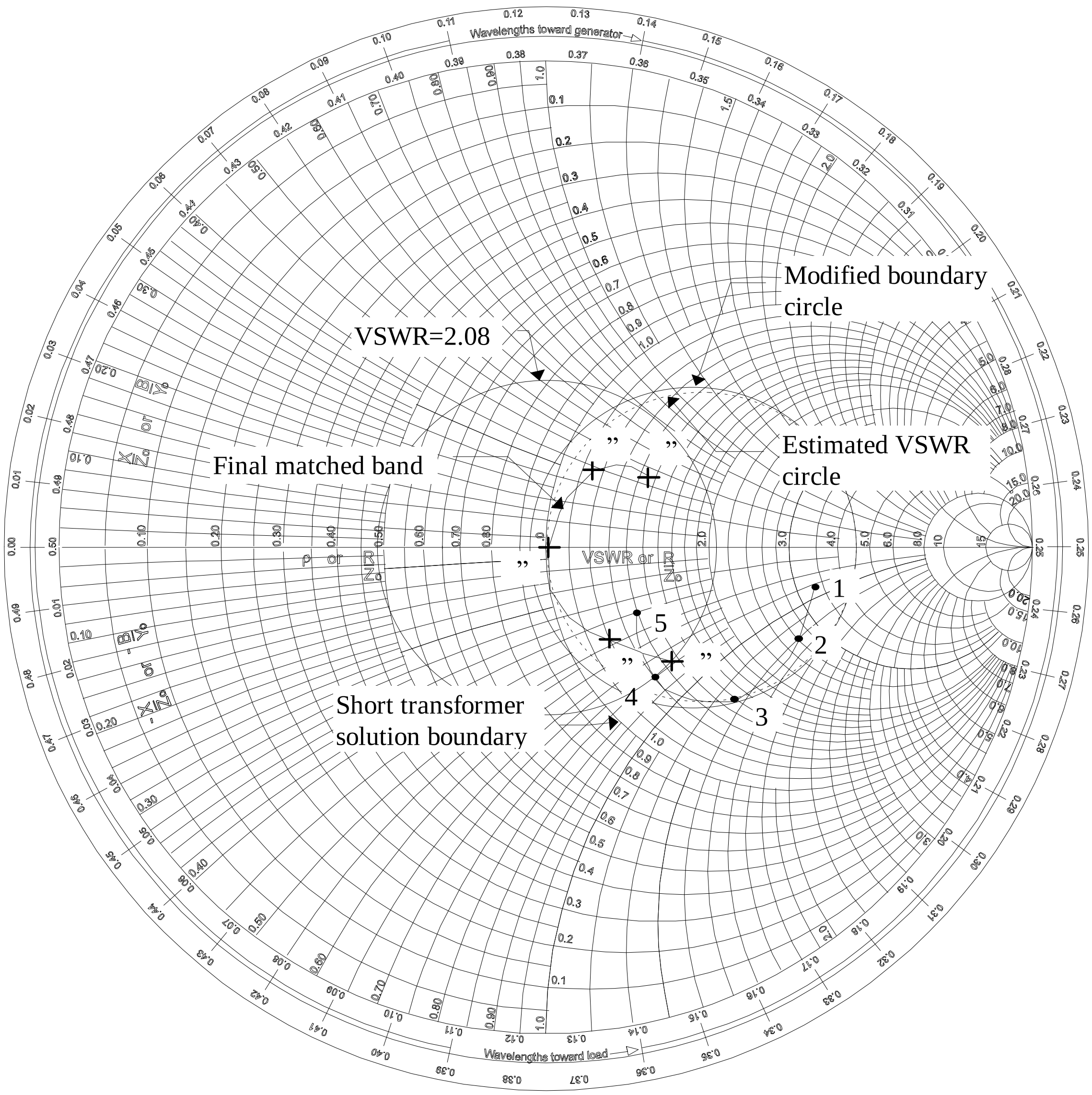



**Figure Q4a Question 4 - Short transformer normalized to transmission line**

| **Frequency (GHz)** | **Original Smith chart readings (λ)** | **Wavelength** $\lambda_{f_x}$ | **Final Smith chart readings (λ)** |
|---|---|---|---|
| Col.1 | Col.2 | Col.3 | Col.2 + Col.3 |
| 1.00 | 0.283 | 0.1290 | 0.4120 |
| 1.02 | 0.323 | 0.1264 | 0.4494 |
| 1.04 | 0.376 | 0.1240 | 0.5000 |
| 1.06 | 0.419 | 0.1217 | 0.5407 |
| 1.08 | 0.448 | 0.1194 | 0.5674 |

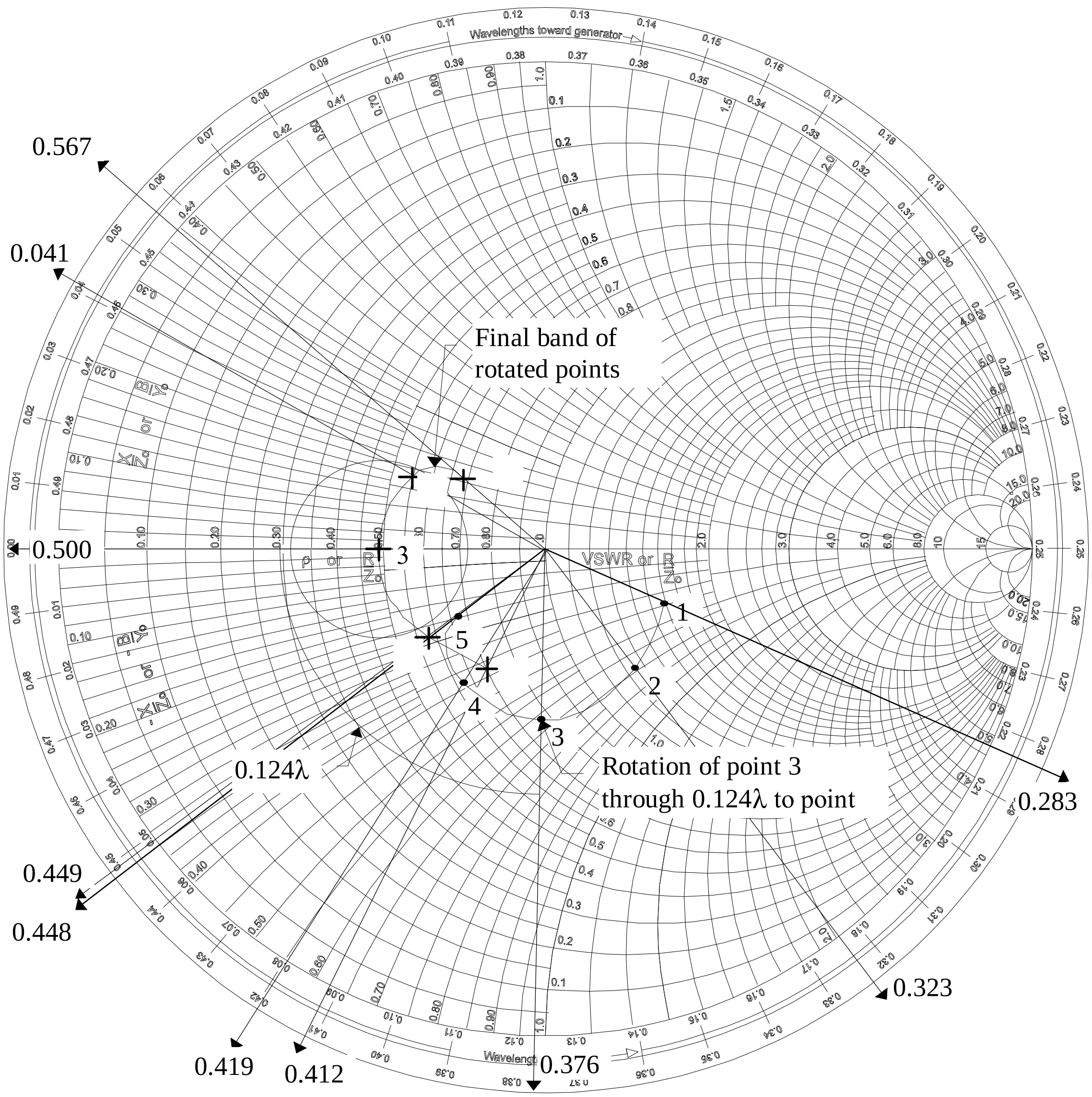



**Figure Q4b** **Question 4 - Short transformer normalized to short transformer**

The final band of load points at the different frequencies shown on Figure Q4/b, are converted to the transmission line Smith chart, by multiplying each point by , $\overline{Z}_{oA}$ = 2.0794. These points are then plotted on the Smith chart normalized to $\overline{Z}_o$ as shown on Figure Q4/a. The final band impedances shown on Figure Q4/a, together with the converted impedances back to the transmission line band impedances are given below.

| **Frequency (GHz)** | **Point Number** | **Short transformer final band impedances ( ’, ’, ’, ’, ’)** | **Transmission line final band impedances (1”,2”,3”,4”,5”) Fig. 91.** |
|---|---|---|---|
| 1.00 | 1 | 0.705 - j0.380 | 1.466 - j0.790 |
| 1.02 | 2 | 0.575 - j0.225 | 1.196 - j0.468 |
| 1.04 | 3 | 0.490 | 1.012 |
| 1.06 | 4 | 0.55 + j0.180 | 1.144 + j0.374 |
| 1.08 | 5 | 0.69 + j0.210 | 1.435 + j0.437 |

Some points about the solution to this type of problem may be worth noting. The first is, the traditional approach is not always the best approach. The selection of the characteristic impedance is not in dispute in this case, it is the rotation of the centre of the band to the real axis which presents the problem. This problem needs to be reworked with the rotation of the lower band frequency to the edge of the VSWR=1.5 circle (the lower band rotates more according to equation 4-34). If the higher band frequency does not fit into the a VSWR circle of 1.5 after this, then the impedance of the transformer needs to be altered. The final circle falls short of what was estimated by the "traditional" method . .

The VSWR = 2.08 is not acceptable, as it can easily be improved as stated above.

# APPENDIX A

## Derivation of various power equations

### *Input power to the amplifier* $P_{in}$

Given a two port network as shown in Figure A-1, the power dissipated in the input to the two port network is the difference between the forward power from the source and the reflected power back from Port 1 of the two port network, where the forward power is given by $P_f = \dfrac{|V_1^+|^2}{2Z_o}$ and the reflected power is given by $P_r = \dfrac{|V_1^-|^2}{2Z_o}$.

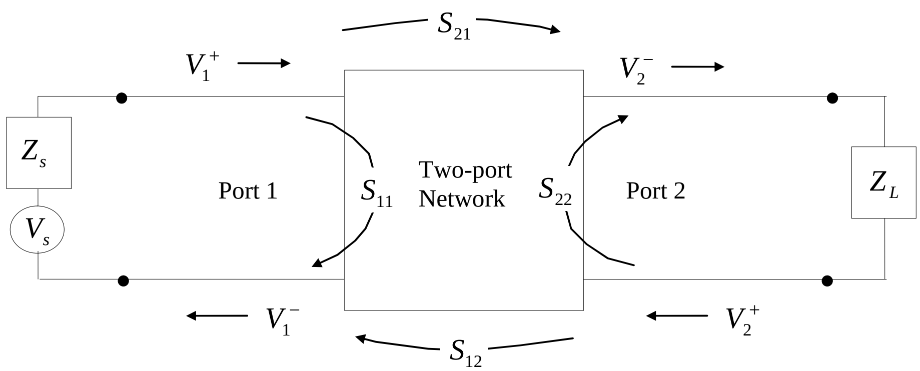



**Figure A-1 Scattering matrix formation for a Two-port network**

That is,

$$P_{in} = \frac{|V_1^+|^2}{2Z_0} - \frac{|V_1^-|^2}{2Z_0} \qquad \textbf{(A-1)}$$

As the reflected voltage $V_1^-$, can be related to the input reflection coefficient $\rho_{in}$ to the two port network and the forward voltage $V_1^+$ by,

$$V_1^- = \rho_{in} V_1^+ \qquad \textbf{(A-2)}$$

The power dissipated in the two port network, port 1, $P_{in}$, can be expressed as,

$$P_{in} = \frac{|V_1^+|^2}{2Z_0} - \frac{|\rho_{in}|^2 |V_1^+|^2}{2Z_0} = \frac{|V_1^+|^2}{2Z_0}\left(1 - |\rho_{in}|^2\right) \qquad \textbf{(A-3)}$$

which is that expression given in equation 5-23

### *Power delivered to the load*

A similar expression for the power dissipated in the load can be found, which is,

$$P_L = \frac{|V_2^-|^2}{2Z_0} - \frac{|\rho_L|^2 |V_2^-|^2}{2Z_0} = \frac{|V_2^-|^2}{2Z_0}\left(1 - |\rho_L|^2\right) \qquad \textbf{(A-4)}$$

where $\rho_L$ is the load reflection coefficient, and forward and reflected voltage waves are shown in Figure A-1.

Using equation 5-1b, the expression for the voltage wave exiting the two port network at Port 2, is given by,

$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ \qquad \textbf{(A-5)}$$

As

$$V_2^+ = \rho_L V_2^- \qquad \textbf{(A-6)}$$

equation A-5 can be expressed as,

$$V_2^- = S_{21}V_1^+ + S_{22}\rho_L V_2^- \qquad \textbf{(A-7)}$$

which gives

$$V_2^- = \frac{S_{21}}{1 - S_{22}\rho_L}V_1^+ \qquad \textbf{(A-8)}$$

or

$$\left|V_2^-\right|^2 = \frac{\left|S_{21}\right|^2}{\left|1 - S_{22}\rho_L\right|^2}\left|V_1^+\right|^2 \qquad \textbf{(A-9)}$$

Substituting equation A-4 into equation A1-9, gives

$$P_L = \frac{1}{2Z_o}\left(1 - \left|\rho_L\right|^2\right)\frac{\left|S_{21}\right|^2}{\left|1 - S_{22}\rho_L\right|^2}\left|V_1^+\right|^2 \qquad \textbf{(A-10)}$$

which is that given as equation 5-24.

## *Available input power from the source* $P_{avs}$

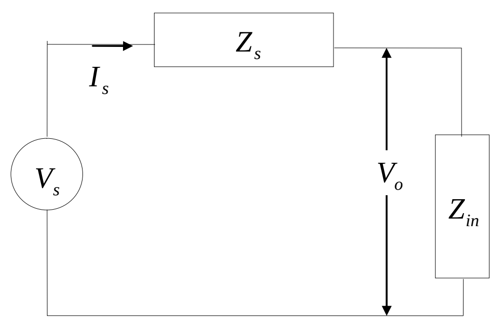



**Figure A-2** **Schematic used in derivation of** $P_{avs}$

Referring to Figure A-2, for a general derivation of the power dissipated in an impedance $Z_{in}$ ,

$$I_s^* = \frac{V_s^*}{Z_s^* + Z_{in}^*} \qquad \textbf{(A-11)}$$

and

$$V_o = I_s Z_{in} = \frac{V_s Z_{in}}{Z_s + Z_{in}} \qquad \textbf{(A-12)}$$

As the power dissipated in $Z_{in}$ is given by $P_{Z_{in}} = \mathrm{Re}\left(\frac{V_0 I_s^*}{2}\right)$,

$$P_{Z_{in}} = \mathrm{Re}\left(\frac{V_0 I_s^*}{2}\right) = \frac{\left|V_s\right|^2 R_{in}}{2\left|Z_s + Z_{in}\right|^2} \qquad \textbf{(A-13)}$$

When the source impedance is resistive and equals the input impedance, the maximum available power derived from the source is obtained, that is,

$$P_{avs} = \frac{|V_s|^2 Z_{in}}{2|Z_s + Z_{in}|^2} = \frac{|V_s|^2 R_s}{2|R_s + R_s|^2} = \frac{|V_s|^2}{8R_s} \qquad \textbf{(A-14)}$$

From equation A-13, the power dissipated in the input port of a two port network, becomes,

$$P_{in} = \frac{|V_1^+|^2 R_{in}}{2|Z_s + Z_{in}|^2} = \frac{|V_1^+|^2}{8R_s} \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} = P_{avs} \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} \qquad \textbf{(A-15)}$$

using the relationships,

$Z = Z_o \dfrac{1+\rho}{1-\rho}$ and $2R = Z + Z^*$

$$R = Z_o \frac{1-|\rho|^2}{|1-\rho|^2} \qquad \textbf{(A-16)}$$

From which,

$$\frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} = \frac{\left(1-|\rho_s|^2\right)\left(1-|\rho_{in}|^2\right)}{|1-\rho_s \rho_{in}|^2} \qquad \textbf{(A-17)}$$

From equations A-3, A-15 and A-17, equation A-15 becomes,

$$P_{avs} = \frac{1}{2Z_o} \frac{|1-\rho_{in}\rho_s|^2}{1-|\rho_s|^2} |V_1^+|^2 \qquad \textbf{(5-27)}$$

## Available power from the network $P_{avnw}$

When the load is resistive and equals the output resistance, the maximum available power from the network is obtained, that is,

$$P_{avsnw} = \frac{|V_2^-|^2}{8R_{out}} \qquad \textbf{(A-18)}$$

where,

$$R_{out} = 1/2\left(Z_{out} + Z_{out}^*\right) \qquad \textbf{(A-19)}$$

Considering Figure A-2, and the resulting equation A-13, the power dissipated in the load from a 2-port network is given by,

$$P_L = \frac{|V_2^-|^2 R_L}{2|Z_{out} + Z_L|^2} = \frac{|V_2^-|^2}{8R_{out}} \frac{4R_{out} R_L}{|Z_{out} + Z_L|^2} = P_{avsnw} \frac{4R_{out} R_L}{|Z_{out} + Z_L|^2} \qquad \textbf{(A-20)}$$

As,

$$Z_{out} = \frac{1+\rho_{out}}{1-\rho_{out}} \qquad \textbf{(A-21)}$$

using equation A-19, equation A-20 can be rearranged to give,

$$P_{avsnw} = \frac{|V_2^-|^2}{2Z_o} \frac{|1-\rho_L \rho_{out}|^2}{1-|\rho_{out}|^2} \qquad \textbf{(A-22)}$$

From equation A-8,

$$P_{avsnw} = \frac{|V_2^+|^2}{2Z_o} \frac{|S_{21}|^2 |1-\rho_L \rho_{out}|^2}{\left(1-|\rho_{out}|^2\right)|1-S_{22}\rho_L|^2} \qquad \textbf{(A-23)}$$

Using equations 5-41 and 5-57, the following relationship is found,

$$\left|1-\rho_s\rho_{in}\right|^2\left|S_{22}\rho_L-1\right|^2=\left|1-\rho_L\rho_{out}\right|^2\left|S_{11}\rho_s-1\right|^2 \qquad \textbf{(A-24)}$$

Thus,

$$P_{avsnw}=\frac{\left|V_2^+\right|^2}{2Z_o}\frac{\left|S_{21}\right|^2\left|1-\rho_s\rho_{in}\right|^2}{\left(1-\left|\rho_{out}\right|^2\right)\left|1-S_{11}\rho_s\right|^2} \qquad \textbf{(5-28)}$$

# APPENDIX B

## *Bilinear Transforms*

### Finding the circle in the Z plane which produces a given circle in the W plane

In this case, we are given a circle in the W plane and want to know what circle produced it in the Z plane.
Given the general equation of a circle,

$$(x - x_o)^2 + (y - y_o)^2 = r^2$$

If $Z = x + jy$ and $Z_o = x_o + jy_o$ then the equation of the circle can be expressed in the Z plane as, $\left|Z - Z_o\right|^2 = r^2 = (Z - Z_o)(Z^* - Z_o^*)$, that is,

$$ZZ^* - ZZ_o^* - Z^* Z_o + Z_o Z_o^* = r^2 \qquad \textbf{(B-1)}$$

Similarly, a circle which is in the W plane can be written as
$\left|W - W_o\right|^2 = m^2 = (W - W_o)(W^* - W_o^*)$
Given a bilinear transformation which maps circles from the Z plane to the W plane, where by circles it is meant to include circles of infinite radius which are straight lines,

$$W = \frac{AZ + B}{CZ + D} \qquad \textbf{(B-2)}$$

from which

$$W - W_o = \frac{AZ + B}{CZ + D} - W_o = \frac{(A - CW_o)Z + (B - DW_o)}{CZ + D} = \frac{A'Z + B'}{CZ + D} \qquad \textbf{(B-3)}$$

where

$$A' = A - CW_o \text{ and } B' = B - DW_o \qquad \textbf{(B-4)}$$

the circle in the W plane can be written as

$$m^2 = (W - W_o)(W^* - W_o^*) = \left(\frac{A'Z + B'}{CZ + D}\right)\left(\frac{A'^* Z^* + B'^*}{C^* Z^* + D^*}\right)$$

which upon expanding and collecting terms, gives,

$$ZZ^*\left(A'A'^* - m^2 CC^*\right) - Z\left(m^2 CD^* - A'B'^*\right) - Z^*\left(m^2 C^* D - A'^* B'\right) + B'B'^* - m^2 DD^* = 0$$

which gives,

$$ZZ^* - Z\frac{\left(m^2 CD^* - A'B'^*\right)}{\left(A'A'^* - m^2 CC^*\right)} - Z^* \frac{\left(m^2 C^* D - A'^* B'\right)}{\left(A'A'^* - m^2 CC^*\right)} + \frac{B'B'^* - m^2 DD^*}{\left(A'A'^* - m^2 CC^*\right)} = 0 \qquad \textbf{(B-5)}$$

comparing this equation with equation B-1, which is the general equation in the Z plane, the centre and the radius of the circle in the Z plane which produces the circle in the W plane, can be found. In other words, from the Z plane a circle has been mapped into the W plane using the bilinear transformation, where A, B, C, and D are given. What is required to be known is the centre and radius of the circle in the Z plane which produced the given circle in the W plane. The centre and radius is found to be,

$$Z_o = \frac{\left(m^2 C^* D - A'^* B'\right)}{\left(\left|A'\right|^2 - m^2 \left|C\right|^2\right)} \qquad \textbf{(B-6)}$$

And as $Z_o Z_o^* - r^2 = \dfrac{B'B'^* - m^2 DD^*}{\left(A'A'^* - m^2 CC^*\right)}$ , rearranging gives,

$$\frac{\left(m^2 C^* D - A'^* B'\right)\left(m^2 CD^* - A'B'^*\right) - \left(|A'|^2 - m^2|C|^2\right)\left(B'B'^* - m^2 DD^*\right)}{\left(|A'|^2 - m^2|C|^2\right)\left(|A'|^2 - m^2|C|^2\right)} = r^2$$

from which,

$$r = \frac{m|A'D - B'C|}{\left(|A'|^2 - m^2|C|^2\right)} \qquad \textbf{(B-7)}$$

*The Load stability circle*
Given an equation of a circle of unit radius in the $\rho_{in}$ plane (the W plane),

$$|\rho_{in}|^2 = 1 \qquad \textbf{(B-8)}$$

determine what the circle is in the $\rho_L$ plane (Z plane) which produced it. The bilinear transform is given by

$$\rho_{in} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} = \frac{\Delta\rho_L - S_{11}}{S_{22}\rho_L - 1} \qquad \textbf{(B-9)}$$

That is, comparing equation B-9 with equation B-2, $A = \Delta$, $B = -S_{11}$, $C = S_{22}$, $D = -1$.
As the centre of the $\rho_{in}$ circle is zero, that is, $W_o = 0$ and its radius, given by equation B-8, is $m = 1$, using equation B-4, the variables of the bilinear transform are found to be
$A' = A = \Delta$, $B' = B = -S_{11}$, $C = S_{22}$, and $D = -1$
Hence, using the properties of the bilinear transform, equations B-6 and B-7 give the circle properties in the $\rho_L$ plane as,

$$c_L = \frac{\Delta^* S_{11} - S_{22}^*}{|\Delta|^2 - |S_{22}|^2} \qquad \textbf{(5-39)}$$

$$r_L = \frac{|S_{12}S_{21}|}{\left||\Delta|^2 - |S_{22}|^2\right|} \qquad \textbf{(5-40)}$$

The source stability circles given by equations 5-42 and 5-43 can be found in a similar fashion.

## Finding the circle in the W plane which is mapped from a given circle in the Z plane

Given the general equation of a circle,

$$(x - x_o)^2 + (y - y_o)^2 = r^2$$

If $W = x + jy$ and $W_o = x_o + jy_o$ then the equation of the circle can be expressed in the W plane as,
$|W - W_o|^2 = q^2 = (W - W_o)(W^* - W_o^*)$
that is,

$$WW^* - WW_o^* - W^*W_o + W_oW_o^* = q^2 \qquad \textbf{(B-11)}$$

Similarly, a circle which is in the Z plane can be written as

$$\left|Z - Z_o\right|^2 = n^2 = \left(Z - Z_o\right)\left(Z^* - Z_o^*\right)$$

Given a bilinear transformation which maps circles from the Z plane to the W plane, where by circles it is meant to include circles of infinite radius which are straight lines,

$$W = \frac{AZ + B}{CZ + D} \qquad \textbf{(B-2)}$$

The inverse transformation will be

$$Z = \frac{-WD + B}{CW - A} \qquad \textbf{(B-12)}$$

from which

$$Z - Z_o = \frac{-WD + B}{CW - A} - Z_o = \frac{-\left(D + CZ_o\right)W + \left(B + AZ_o\right)}{CW - A} = \frac{-WD' + B'}{CW - A} \qquad \textbf{(B-13)}$$

where

$$D' = D + CZ_o \text{ and } B' = B + AZ_o \qquad \textbf{(B-14)}$$

the circle in the Z plane can be written as

$$n^2 = \left(Z - Z_o\right)\left(Z^* - Z_o^*\right) = \left(\frac{-D'W + B'}{CW - A}\right)\left(\frac{-D'^*W^* + B'^*}{C^*W^* - A^*}\right)$$

which upon expanding and collecting terms, gives,

$$WW^*\left(n^2CC^* - D'D'^*\right) - W\left(n^2CA^* - D'B'\right) - W^*\left(n^2AC^* - D'^*B'\right) + n^2AA^* - B'B'^* = 0$$

which gives,

$$WW^* - W\frac{\left(n^2CA^* - D'B'^*\right)}{\left(n^2CC^* - D'D'^*\right)} - W^*\frac{\left(n^2AC^* - D'^*B'\right)}{\left(n^2CC^* - D'D'^*\right)} + \frac{n^2AA^* - B'B'^*}{\left(n^2CC^* - D'D'^*\right)} = 0 \qquad \textbf{(B-15)}$$

comparing equation B-15 with equation B-11, which is the general equation in the W plane, the centre and the radius of a conjugate $\rho_{in}$ circle in the W plane can be found which will produced from a centre and radius of a circle in the Z plane. In other words, you give me the centre and radius of a circle in the Z plane which you want to map into the W plane using a bilinear transformation whose constants are A, B, C, and D, and I can tell you what the equation of the circle is in the W plane by using the inverse of this bilinear transform. Furthermore, I can tell you what the conjugate of this circle is in the W-plane. The centre and radius of the conjugate circle is found to be,

$$W_o^* = \frac{\left(n^2CA^* - D'B'^*\right)}{\left(n^2|C|^2 - |D'|^2\right)}$$

Which on substituting equation B-5, $D' = D + CZ_o$ and $B' = B + AZ_o$ gives,

$$W_o^* = \frac{\left(n^2CA^* - \left(D + CZ_o\right)\left(B^* + A^*Z_o^*\right)\right)}{\left(n^2|C|^2 - |D + CZ_o|^2\right)} \qquad \textbf{(B-16)}$$

And as $W_oW_o^* - q^2 = \dfrac{n^2AA^* - B'B'^*}{\left(n^2CC^* - D'D'^*\right)}$ , rearranging gives,

from which,

$$q = \frac{n|AD' - B'C|}{\left(n^2|C|^2 - |D'|^2\right)}$$

Which on substituting equation B-14, $D' = D + CZ_o$ and $B' = B + AZ_o$ gives,

$$q = \frac{n|AD - BC|}{\left(n^2|C|^2 - |D + CZ_o|^2\right)} \qquad \textbf{(B-17)}$$

From equation 5-4, the complex conjugate of input reflection coefficient is found to be,

$$\rho_{in} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} = \frac{\Delta\rho_L - S_{11}}{S_{22}\rho_L - 1} \qquad \textbf{(B-8)}$$

That is, comparing equation B-8 with the equation of the bilinear transform, which moves from the Z plane $(\rho_L \text{ plane})$ to the W plane $(\rho_{in} \text{ plane})$, that is,

$$W = \frac{AZ + B}{CZ + D} \qquad \textbf{(B-2)}$$

The constants are, $A = \Delta,\ B = -S_{11},\ C = S_{22},\ D = -1$.

As the centre of the constant power gain circle, given by equation 5-47, to be mapped is $c_g$, that is, $Z_o = c_g$ and its radius, given by equation 5-48, is $n = r_g$, using the properties of the inverse bilinear transform, where the constant power gain circle is mapped into the $\rho_{in}^*$ circle, the centre and radius of the $\rho_{in}^*$ circle is found by substituting the values of A, B, C, and D into equations B-16 and B-17 to give,

$$c_{\rho_{in}^*} = \frac{r_g^2 S_{22}\Delta^* + \left(1 - S_{22}c_g\right)\left(\Delta^* c_g^* - S_{11}^*\right)}{\left|r_g S_{22}\right|^2 - \left|S_{22}c_g - 1\right|^2} \qquad \text{(B-}\textbf{18)}$$

$$r_{\rho_{in}^*} = \frac{\left|S_{12}S_{21}\right| r_g}{\left|\left|r_g S_{22}\right|^2 - \left|S_{22}c_g - 1\right|^2\right|} \qquad \textbf{(B-19)}$$

**Intersection of constant power gain circle with the Smith chart boundary on the** $\rho_L$ plane

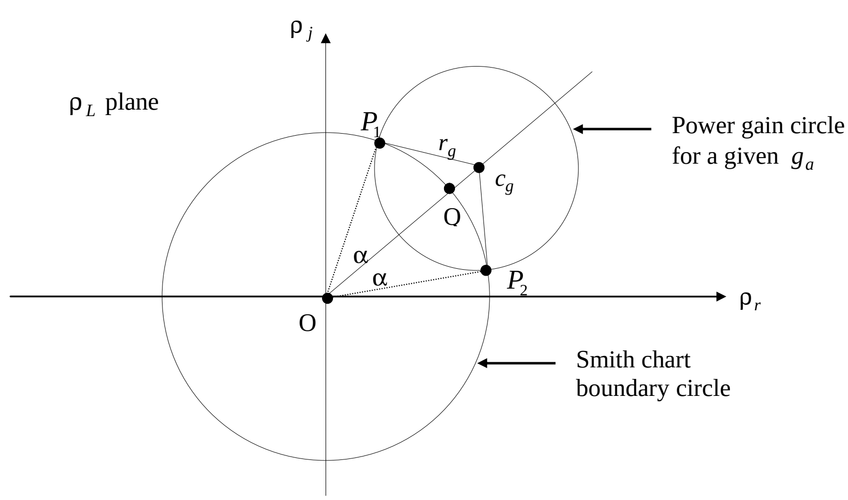



**Figure B-1 Diagram to aid intersection of constant power gain circle with Smith chart**

Referring to Figure B-1, if the power gain circle intersects the boundary of the Smith chart then it does so in two points, namely, $P_1$ and $P_2$.

As point $Q = \frac{c_g}{|c_g|}$ , then $P_1 = \frac{c_g}{|c_g|} e^{j\alpha}$ and $P_2 = \frac{c_g}{|c_g|} e^{-j\alpha}$ .

In order for the intersection to occur, it is required from the cosine rule, that,

$$\cos\alpha = \frac{|c_g|^2 + 1 - r_g^2}{2|c_g|} \quad \textbf{(B-20)}$$

As each of the constant power gain circles are at different values of $g_a$ , if the points $P_1$ and $P_2$ are to be invariant for all of the constant power gain circles it is required that equation B-20 be independent of $g_a$ .

From equation 5-47, the centre of the constant power gain circle is given by,

$$c_g = \frac{\left(S_{22}^* - \Delta^* S_{11}\right) g_a}{\left(|S_{22}|^2 - |\Delta|^2\right) g_a + 1} \quad \textbf{(5-47)}$$

and the radius is given by,

$$r_g = \frac{\sqrt{\left(1 - 2Kg_a |S_{12}S_{21}| + g_a^2 |S_{12}S_{21}|^2\right)}}{\left|\left(|S_{22}|^2 - |\Delta|^2\right) g_a + 1\right|} \quad \textbf{(5-48)}$$

Let

$$A = |S_{22}|^2 - |\Delta|^2 \quad \textbf{(B-21)}$$

after substituting equations 5-47 and 5-48 into equation B-20, the denominator of equation B-20 becomes, $2|S_{22}^* - \Delta^* S_{11}| g_a (Ag_a + 1)$. If it can be shown that the numerator of equation B-20 is of the form $g_a (Ag_a + 1)\mu$ , where $\mu$ is a constant independent of $g_a$ , then equation B-20 is independent of $g_a$ , as required.

The numerator of equation B-20, after simplification is found to be,

$$\left(|S_{22}|^2 - |\Delta|^2 + 1 - |S_{11}|^2\right) g_a (Ag_a + 1)$$

giving,

$$\cos\alpha = \frac{|S_{22}|^2 - |\Delta|^2 + 1 - |S_{11}|^2}{2|S_{22}^* - \Delta^* S_{11}|} \quad \textbf{(B-22)}$$

which is independent of $g_a$ . If any constant power gain circle intersects the Smith chart boundary then it intersects at the same place as any other.

## Co-linearity of $\rho_{in}^*$ circle centre with the source stability circle centre in the $\rho_{in}$ plane

The centre of the $\rho_{in}^*$ circle is given by equation B-18 as,

$$c_{\rho_{in}^*} = \frac{r_g^2 S_{22} \Delta^* + \left(1 - S_{22} c_g\right)\left(\Delta^* c_g^* - S_{11}^*\right)}{|r_g S_{22}|^2 - |S_{22} c_g - 1|^2} \quad \textbf{(B-23)}$$

and the centre of the source stability circle is given by,

$$c_s = \frac{S_{11}^* - \Delta^* S_{22}}{|S_{11}|^2 - |\Delta|^2} \quad \textbf{(5-42)}$$

As the denominators of equations B-23 and 5-42 are real, the two centres will be colinear if the complex portions of the numerators are shown to be the same.
Taking the numerator of B-23 ( $N_{23}$ ), expanding and collecting terms, gives

$$N_{23} = S_{22}\Delta^*\left(r_g^2 - \left|c_g\right|^2\right) - S_{11}^* + \Delta^* c_g^* + S_{22} S_{11}^* c_g \qquad \textbf{(B-24)}$$

The following expressions are developed so that equation B-24 can be put into a form which is a constant multiplied by $S_{11}^* - \Delta^* S_{22}$ .
From equation 5-47 and B-21

$$\left|c_g\right| = \left|S_{22}^* - \Delta^* S_{11}\right| \frac{g_a}{Ag_a + 1} \qquad \textbf{(B-25)}$$

By equating equations B-20 and B-22 and making use of equation B-25

$$\left|c_g\right|^2 + 1 - r_g^2 = \left(\left|S_{22}\right|^2 - |\Delta|^2 + 1 - \left|S_{11}\right|^2\right) \frac{g_a}{Ag_a + 1} \qquad \textbf{(B-26)}$$

Multiplying the conjugate of equation 5-47 by $\Delta^*$ gives

$$\Delta^* c_g^* = \Delta^*\left(S_{22} - \Delta S_{11}^*\right) \frac{g_a}{Ag_a + 1} \qquad \textbf{(B-27)}$$

Using equation 5-47,

$$S_{22} S_{11}^* c_g - S_{11}^* = S_{11}^* \left( \frac{\left(\left|S_{22}\right|^2 - \Delta^* S_{11} S_{22}\right) g_a}{Ag_a + 1} - 1 \right) \qquad \textbf{(B-28)}$$

Substituting expressions B-25 to B-28 into the numerator expression, B-24 and simplifying, the numerator becomes,

$$N_{23} = \left( \frac{S_{11}^* - S_{22}\Delta^*}{-\left(Ag_a + 1\right)} \right) \qquad \textbf{(B-29)}$$

showing that the centre of the $\rho_{in}^*$ circle is colinear with the source stability circle.
An alternative expression for the centre of the $\rho_{in}^*$ circle, which involves the constant power gain $g_a$ , is then given by,

$$c_{\rho_{in}^*} = \frac{\left(S_{11}^* - S_{22}\Delta^*\right)}{\left(\left|r_g S_{22}\right|^2 - \left|S_{22} c_g - 1\right|^2\right)\left(\left(|\Delta|^2 - \left|S_{22}\right|^2\right) g_a - 1\right)} \qquad \textbf{(B-30)}$$

# APPENDIX C

## Derivation of constant spot noise factor circles

### *Dependent noise sources*

In an active device, to completely characterise the thermal noise performance of the device, the actual network can be modelled as a noise-free device with two noise generators, $e_n(t)$ and $i_n(t)$ as shown in Figure C-1. Two generators are required to account for the fact that noise does appear at the output if the source impedance $Z_s$ is short-circuited or open circuited. When $Z_s$ is short-circuited the noise current generator $i_n(t)$ is inoperative. Similarly, if the source impedance $Z_s$ is open-circuited, the noise voltage generator $e_n(t)$ is inoperative and cannot produce any noise at the output . Because the thermal noise may arise from the same mechanism within the active device, these two generators are not independent. Because of this, there exists a reaction between the two, which produces another noise source.

anufacturer s data sheets seldom provide the cross-correlation function between $e_n(t)$ and $i_n(t)$, and because the spread of values of these generators for a device normally overshadows the effect of the noise source which is produced by the interaction effect, it is common practice to assume the correlation coefficient is equal to zero. In this section the correlation coefficient is not assumed to be zero.

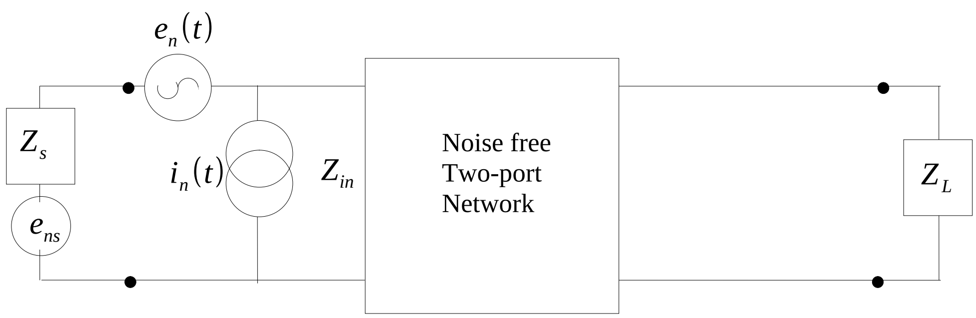



**Figure C-1 Equivalent noise representation of a Two-port network**

Consider the general case of a voltage source V and current source I, as shown in Figure C-2. An expression for the input power is to be first developed, and then this expression used for the case where the voltage and current sources are dependent noise sources..

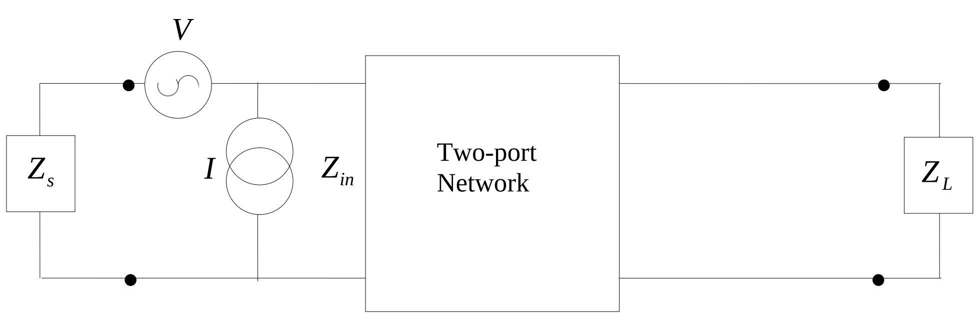



**Figure C-2 Dependent voltage and current sources at the input to a two-port network**

The current entering the input of the two port network is given by,

$$I_{in} = \frac{V + IZ_s}{Z_s + Z_{in}} \qquad \textbf{(C-1)}$$

As the average input power is found from ,

$$P_{in} = \mathrm{Re}\left(\frac{I_{in}I_{in}^*}{2} Z_{in}\right) \qquad \textbf{(C-2)}$$

$$P_{in} = \mathrm{Re}\left(\frac{(V + IZ_s)(V^* + I^*Z_s^*)}{2|Z_s + Z_{in}|^2} Z_{in}\right) = \frac{|V|^2}{8R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{|I|^2|Z_s|^2}{8R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \mathrm{Re}\frac{(VI^*Z_s^* + IV^*Z_s)Z_{in}}{2|Z_s + Z_{in}|^2}$$

Due to the interaction of the voltage and current sources, there exists the third term in the expression above. If $e^{j\theta} = \gamma = \gamma_r + j\gamma_j$ , then the last term in the above equation can be rewritten using the following results,

$$VI^*Z_s^* = |VI|e^{j\theta}Z_s^* = |VI|(Z_s^*\gamma) = |VI|\big((R_s - jX_s)(\gamma_r + j\gamma_j)\big) = |VI|(R_s\gamma_r + X_s\gamma_j) + j|VI|(R_s\gamma_j - X_s\gamma_r)$$

$$V^*IZ_s = |VI|e^{-j\theta}Z_s = |VI|(Z_s\gamma^*) = |VI|\big((R_s + jX_s)(\gamma_r - j\gamma_j)\big) = |VI|(R_s\gamma_r + X_s\gamma_j) - j|VI|(R_s\gamma_j - X_s\gamma_r)$$

Adding these expressions,

$$VI^*Z^* + V^*IZ = |VI|(Z_s^*\gamma + Z_s\gamma^*) = 2|VI|(R_s\gamma_r + X_s\gamma_j) = 2\mathrm{Re}(\mathrm{VI}^*Z_s^*) = 2|VI|\,\mathrm{Re}(\gamma Z_s^*)$$

allowing $P_{in}$ to be expressed in terms of ratios of the available power from each of the sources, as

$$P_{in} = \frac{|V|^2}{8R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{|I|^2|Z_s|^2}{8R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{|VI|}{4R_s}\frac{4R_sR_{in}}{|Z_s + Z_{in}|^2}\mathrm{Re}(\gamma Z_s^*) \qquad \textbf{(C-3)}$$

The term $(Z_s^*\gamma)$ can be considered to represent the result of a dimensionless phase factor of unit modulus γ, multiplied by the conjugate source impedance $Z_s^*$.

Substituting the rms counterparts of the peak voltage V, and current I, into equation C-3, then equation C-3 can be rewritten as

$$P_{in} = \frac{|V_{rms}|^2}{4R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{|I_{rms}|^2|Z_s|^2}{4R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{|V_{rms}||I_{rms}|(R_s\gamma_r + X_s\gamma_j)}{2R_s}\frac{4R_sR_{in}}{|Z_s + Z_{in}|^2}$$

**(C-4)**

Considering that the time dependent thermal noise voltage and current sources, shown in Figure C-1, can be converted into uniform power spectral densities in the frequency domain, using a 1Ω resistor, the noise power in a small bandwidth $\Delta f$ centred around a chosen frequency can be found by multiplying this spectral density by the bandwidth. From this power, the rms values of the noise sources at a chosen single frequency can be determined. The rms noise voltage at the chosen frequency, is given by,

$$|e_n| = \sqrt{4kT\Delta fR_e} \qquad \textbf{(C-5)}$$

and the rms noise current, also in a bandwidth $\Delta f$ , is given by,

$$|i_n| = \sqrt{4kT\Delta fG_e} \qquad \textbf{(C-6)}$$

where; $R_e$ is the equivalent noise resistance, $G_e$ is the equivalent noise conductance, T is the absolute temperature (K), and $k$ is oltzmann s constant $1.38x10^{-23}$ J/K).

For independent noise sources, the third term in equation C1-4 would be zero, due averaging to zero of the interaction power by the random phase of the noise waveforms. In this case, the noise powers from each source add. For the dependent noise sources, this interaction term does not reduce to zero.

he input power at a single fre uency, or “spot” fre uency, due to the noise contributions from the network is given by substituting equations C-5 and C-6, into C-4, where $|V_{rms}| = |e_n|$ and $|I_{rms}| = |i_n|$ , to give,

$$P_{in} = \frac{kT\Delta fR_e}{R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + \frac{kT\Delta f|Z_s|^2G_e}{R_s}\frac{4R_{in}R_s}{|Z_s + Z_{in}|^2} + 2kT\Delta f\,\frac{R_s\gamma_r + X_s\gamma_j}{R_s}\frac{4R_sR_{in}}{|Z_s + Z_{in}|^2} \qquad \textbf{(C-7)}$$

There is also a noise power contribution from the noise generated in the source resistance. This noise power is an independent noise source which has nothing to do with the noise generated by the active device.
As it is independent, the noise power given by equation C-7 and this noise power will add.
For a rms noise voltage $|e_{sn}|$ and using the results of equation A-13,

$$P_{e_{sn}} = \mathrm{Re}\left(E_{sn} I_{sn}^{*}\right) = \frac{|e_{sn}|^2 R_{in}}{|Z_s + Z_{in}|^2} = \frac{|e_{sn}|^2}{4R_s} \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} \qquad \textbf{(C-8)}$$

For a thermal rms noise voltage $|e_{sn}|$

$$|e_{sn}| = \sqrt{4kT\Delta f R_s} \qquad \textbf{(C-9)}$$

which produces,

$$P_{e_{sn}} = kT\Delta f \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} \qquad \textbf{(C-10)}$$

The total noise input to the noise-free two-port network from dependent and independent sources in a bandwidth $\Delta f$ , at a specific frequency, is given by the sum of equation C-7 and C-10, that is,

$$P_{in_{total}} = kT\Delta f \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} + \frac{kT\Delta f R_e}{R_s} \frac{4R_{in} R_s}{|Z_s + Z_{in}|^2} + \frac{kT\Delta f |Z_s|^2 G_e}{R_s} \frac{4R_{in} R_s}{|Z_s + Z_{in}|^2}$$

$$+ 2kT\Delta f \frac{R_s \gamma_r + X_s \gamma_j}{R_s} \frac{4R_s R_{in}}{|Z_s + Z_{in}|^2} \qquad \textbf{(C-11)}$$

## *Noise Figure*

The noise factor F, for an individual network, such as shown in Figure A-1, is defined as;

$$F = \frac{\text{Signal-to-Noise ratio at the input}}{\text{Signal-to-Noise ratio at the output}} = \frac{(S/N)_{in}}{(S/N)_{out}} \qquad \textbf{(C-12)}$$

The standard definition requires that the source impedance be conjugately matched to the input impedance of the network, that is $Z_s = Z_{in}^{*}$ , and that the source resistance $R_s$ be at the standard temperature $T_o = 290K$ . The noise figure is ten times the logarithm to the base 10, of the noise factor, that is,

$$\text{Noise Figure (FdB)} = 10\log(NoiseFactor) \qquad \textbf{(C-13)}$$

*Operating noise factor*
If the source resistance is not at 290 K and/or the source impedance is not matched to the input impedance of the network, then the noise factor becomes the operating noise factor.

*Spot noise factor*
If the noise factor F, is determined at a single frequency f, and is based on the noise power in a 1 Hz band, then the noise factor is called the spot noise factor. In practice a 1 Hz band is not usually used, and the noise is measured in a small band $\Delta f$ . When all of the noise sources are referred to the input as equivalent noise sources, as shown in Figure C-1, then the spot noise factor can be defined as,

$$F = \frac{\text{total input noise power to the noise free network}}{\text{thermal noise input power from the source resistance}} \qquad \textbf{(C-14)}$$

It is the spot noise factor which is applicable to equation C-11, as this equation is only valid at a specific frequency. As the thermal noise input power from the source resistance can be given by equation C-10, the spot noise factor F may be found by dividing equation C-11 by C-10 to give,

$$F_{spot} = 1 + \frac{R_e}{R_s} + \frac{|Z_s|^2 G_e}{R_s} + 2\,\frac{R_s\gamma_r + X_s\gamma_j}{R_s} = 1 + \frac{R_e}{R_s} + \frac{G_e}{G_s} + 2\,\frac{R_s\gamma_r + X_s\gamma_j}{R_s} \qquad \textbf{(C-15)}$$

where

$$G_s = \frac{R_s}{R_s^2 + X_s^2} \qquad \textbf{(C-16)}$$

showing that the spot noise factor depends upon the source resistance as well as on the noise parameters, $R_e$, $G_e$, $\gamma_r$, and $\gamma_j$, but not upon bandwidth $\Delta f$ ,over which the noise at a particular frequency is measured.

*Minimum value of spot noise factor* $F_m$

By differentiating $F_{spot}$ in equation C-15, with respect to the source reactance $X_s$, and then letting the partial differential equal to zero, and then repeating the process for the source resistance $R_s$, the resulting expressions for $X_s$ and $R_s$ when substituted back into equation C-15, will permit the value of the minimum noise factor to be found.

The values of $X_s$ and $R_s$ which make $F_{spot}$ a minimum are found to be,

$$X_m = -\frac{\gamma_j}{G_e} \qquad \textbf{(C-17)}$$

and from,

$- R_e + G_e R_s^2 - X_s^2 G_e - 2X_s\gamma_j = 0$ which gives

$$R_s^2 + X_s^2 = R_e / G_e \qquad \textbf{(C-18)}$$

From which

$$R_m = \sqrt{\frac{R_e}{G_e} - \frac{\gamma_j^2}{G_e^2}} = \sqrt{\frac{R_e}{G_e} - X_m^2} \qquad \textbf{(C-19)}$$

Thus, the minimum spot noise factor becomes,

$$F_m = 1 + 2\gamma_r + 2R_m G_e \qquad \textbf{(C-20)}$$

*Spot noise factor equation*

In terms of the spot noise factor, the following relationship $F_{spot} - F_m$, is considered, using equation C-15 and C-20

$$F_{spot} - F_{\min} = \frac{R_e}{R_s} + \frac{\left(X_s^2 + R_s^2\right)G_e}{R_s} - \frac{2X_s X_m G_e}{R_s} - 2R_m G_e$$

giving,

$$F_{spot} - F_{\min} = \frac{R_e}{R_s} + \frac{\left(X_s^2 - 2X_s X_m + X_m^2 + R_s^2 - 2R_m R_s + R_m^2\right)G_e}{R_s} - \frac{X_m^2 G_e}{R_s} - \frac{R_m^2 G_e}{R_s}$$

From equation C-18, $R_e - G_e\left(R_s^2 + X_s^2\right) = 0$, giving,

$$F_{spot} - F_{\min} = \frac{\left(X_s^2 - 2X_sX_m + X_m^2 + R_s^2 - 2R_mR_s + R_m^2\right)G_e}{R_s}$$

Therefore,

$$F_{spot} = F_{\min} + \frac{G_e}{R_s}\left[\left(R_s - R_m\right)^2 + \left(X_s - X_m\right)^2\right] \qquad \textbf{(C-21)}$$

Equation C-21 permits a spot noise factor to be expressed in terms of the minimum value of noise factor $F_m$, when the optimum source impedance $Z_m = R_m + jX_m$ is used with $G_e$. anufacturer s usually provide the minimum noise factor $F_m$, the optimum source impedance $Z_m$ or the optimum source reflection coefficient $\rho_m$, and the noise conductance $G_e$, allowing equation C-21 to determine the value of the spot noise factor . If the noise resistance $R_e$ only is provided by the manufacturer, then equation C-18, permits the value of noise conductance $G_e$, to be found.

## *Constant noise figure circles*

Noise figure circles are plotted on the source reflection coefficient plane $\rho_s$, because part of the definition of noise figure, as discussed in the noise figure section of this Appendix, requires that the source impedance be the conjugate of the input resistance of the network under consideration. That is, from equation 1-60, as

$$\rho_s = \frac{\overline{Z}_s - 1}{\overline{Z}_s - 1} \qquad \textbf{(C-22)}$$

and $\overline{Z}_s = \overline{Z}_{in}^*$, we find $\rho_s = \frac{\overline{Z}_s - 1}{\overline{Z}_s - 1} = \frac{\overline{Z}_{in}^* - 1}{\overline{Z}_{in}^* - 1} = \rho_{in}^*$

giving $\rho_s = \rho_{in}^*$ which shows that the constant noise figure circles are plotted on the source reflection coefficient plane $\rho_s$, if the source impedance is conjugately matched.
Using the general form of the equation of a circle in the $\rho_s$ plane,

$$\left(\overline{R}_s - \overline{R}_m\right)^2 + \left(\overline{X}_s - \overline{X}_m\right)^2 = \left|\overline{Z}_s - \overline{Z}_m\right|^2 \qquad \textbf{(C-23)}$$

together with the impedance form of equation C-22 for $\rho_s$ and $\rho_m$, that is,

$$\overline{Z}_s = \frac{\rho_s + 1}{\rho_s - 1} \qquad \textbf{(C-24)}$$

and

$$\overline{Z}_m = \frac{\rho_m + 1}{\rho_m - 1} \qquad \textbf{(C-25)}$$

the spot noise figure equation, equation C-21, can be rewritten as,

$$\left(F_{spot} - F_{\min}\right) = \frac{\overline{G}_e}{\overline{R}_s}\left[\left(\overline{R}_s - \overline{R}_m\right)^2 + \left(\overline{X}_s - \overline{X}_m\right)^2\right] \qquad \textbf{(C-26)}$$

where the resistances and reactances are normalized to the characteristic impedance of the transmission line $Z_o$ and $\overline{G}_e = G_e / Y_o = G_e Z_o$ .
From equations C-23 and C-26,

$$\left(F_{spot} - F_{\min}\right) = \frac{\overline{G}_e}{\overline{R}_s}\left|\overline{Z}_s - \overline{Z}_m\right|^2 = \frac{\overline{G}_e}{\overline{R}_s}\frac{4\left|\rho_m - \rho_s\right|^2}{\left|1-\rho_m\right|^2\left|1-\rho_s\right|^2}$$

or from equation C-18, where,

$$\overline{G}_e = \frac{\overline{R}_e}{\overline{R}_m^2 + \overline{X}_m^2} = \frac{\overline{R}_e}{\left|\overline{Z}_m\right|^2} = \frac{\overline{R}_e\left|1-\rho_m\right|^2}{\left|1+\rho_m\right|^2} \tag{C-27}$$

and from equations C-21 and C-23

$$\left(F_{spot} - F_{\min}\right) = \frac{\overline{R}_e}{\overline{R}_s}\frac{\left|1-\rho_m\right|^2}{\left|1+\rho_m\right|^2}\left|\overline{Z}_s - \overline{Z}_m\right|^2 = \frac{\overline{R}_e}{\overline{R}_s}\frac{4\left|\rho_m - \rho_s\right|^2}{\left|1+\rho_m\right|^2\left|1-\rho_s\right|^2}$$

From equation A-16,

$$\overline{R}_s = \frac{1-\left|\rho_s\right|^2}{\left|1-\rho_s\right|^2} \tag{C-28}$$

and so,

$$\left(F_{spot} - F_{\min}\right) = 4\overline{G}_e\frac{\left|\rho_m - \rho_s\right|^2}{\left|1-\rho_m\right|^2\left(1-\left|\rho_s\right|^2\right)} \tag{C-29a}$$

$$\left(F_{spot} - F_{\min}\right) = 4\overline{R}_e\frac{\left|\rho_m - \rho_s\right|^2}{\left|1+\rho_m\right|^2\left(1-\left|\rho_s\right|^2\right)} \tag{C-29b}$$

As $\rho_s$ is the only variable in equations C-29, any part of equations C-29 which does not involve $\rho_s$ can be grouped.
Letting,

$$N = \frac{\left(F_{spot} - F_{\min}\right)\left|1-\rho_m\right|^2}{4\overline{G}_e} = \frac{\left(F_{spot} - F_{\min}\right)\left|1+\rho_m\right|^2}{4\overline{R}_e} = \frac{\left|\rho_m - \rho_s\right|^2}{\left(1-\left|\rho_s\right|^2\right)} \tag{C-30}$$

From which,

$$N = \frac{\left|\rho_m - \rho_s\right|^2}{\left(1-\left|\rho_s\right|^2\right)} = \frac{\left(\rho_m - \rho_s\right)\left(\rho_m^* - \rho_s^*\right)}{1-\rho_s\rho_s^*} = \frac{\rho_m\rho_m^* - \rho_m\rho_s^* - \rho_s\rho_m^* + \rho_s\rho_s^*}{1-\rho_s\rho_s^*} \tag{C-31}$$

by rearranging equation C-31,

$$\rho_s\rho_s^* - \rho_s\frac{\rho_m^*}{1+N} - \rho_s^*\frac{\rho_m}{1+N} + \frac{\left|\rho_m\right|^2 - N}{1+N} = 0 \tag{C-32}$$

Equation C-32 can be recognized from equation B-1 as a circle in the $\rho_s$ plane, with centre

$$c_{nf} = \frac{\rho_m}{1+N} \tag{C-33}$$

and radius, from $\dfrac{\left|\rho_m\right|^2}{\left(1+N\right)^2} - r^2 = \dfrac{\left|\rho_m\right|^2 - N}{1+N}$

$$r^2 = \frac{|\rho_m|^2 - (1+N)\left(|\rho_m|^2 - N\right)}{(1+N)^2} = \frac{N(N+1) - N|\rho_m|^2}{(1+N)^2}$$

which gives the radius as,

$$r = \frac{\sqrt{N^2 + N\left(1 - |\rho_m|^2\right)}}{(1+N)} \qquad \textbf{(C-34)}$$

## APPENDIX D

# Derivation of constant Mismatch circles

## *The input mismatch circle*

### *The mismatch parameter M*

Without a matching circuit, from the transmission line, which has a characteristic impedance $Z_o$, the input circuit to the amplifier will appear as shown in Figure D-1.

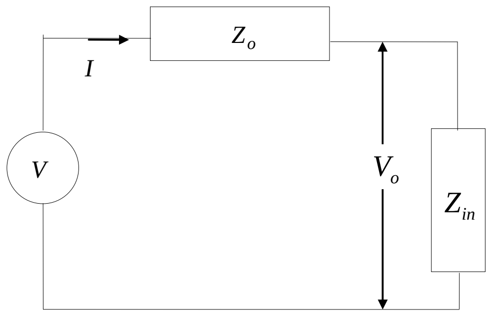



**Figure D-1 Schematic used in derivation of M**

Using the result of A-15,

$$P_{in} = \frac{4Z_o R_{in}}{\left|Z_o + Z_{in}\right|^2} P_{avs} = MP_{avs} \qquad \textbf{(D-1)}$$

where the input mismatch factor M is defined as,

$$M = \frac{4Z_o R_{in}}{\left|Z_o + Z_{in}\right|^2} = \frac{4\overline{R}_{in}}{\left|1+\overline{Z}_{in}\right|^2} \qquad \textbf{(D-2)}$$

From

$\overline{Z}_{in} = \dfrac{1+\rho_{in}}{1-\rho_{in}}$ and $2\overline{R}_{in} = \overline{Z}_{in} + \overline{Z}_{in}^*$

$$\overline{R}_{in} = \frac{1-\left|\rho_{in}\right|^2}{\left|1-\rho_{in}\right|^2} \qquad \textbf{(D-3)}$$

and

$$1+\overline{Z}_{in} = \frac{2}{1-\rho_{in}} \qquad \textbf{(D-4)}$$

the following result is obtained for M,

$$M = 1-\left|\rho_{in}\right|^2 \qquad \textbf{(D-5)}$$

As VSWR is defined by equation 1-63, as,

$$VSWR = \frac{1+|\rho|}{1-|\rho|} \qquad \textbf{(1-63)}$$

the input VSWR in terms of the input mismatch factor M, is found from equation D-5 as,

$$VSWR_{in} = \frac{1+\sqrt{1-M}}{1-\sqrt{1-M}} \qquad \textbf{(D-6)}$$

Consider now the case where there is an input matching circuit between the transmission line and the input to the amplifier or network, as shown in Figure 124. The VSWR looking from the transmission line into the matching network will be unity, but the VSWR looking from the amplifier into the matching network will be the same as that when there was no matching network. However, there will be a source impedance $Z_s$ other than the characteristic impedance of the transmission line as seen by the amplifier looking into the matching network. The mismatch factor now becomes that given by equation A-17, that is,

$$M_s = \frac{4R_s R_{in}}{\left|Z_s + Z_{in}\right|^2} = \frac{\left(1-\left|\rho_s\right|^2\right)\left(1-\left|\rho_{in}\right|^2\right)}{\left|1-\rho_s\rho_{in}\right|^2} \qquad \textbf{(D-7)}$$

As the VSWRs are the same,

$$VSWR_{in} = \frac{1+\sqrt{1-M}}{1-\sqrt{1-M}} = \frac{1+\sqrt{1-M_s}}{1-\sqrt{1-M_s}} \qquad \textbf{(D-8)}$$

and therefore,

$$M_s = \frac{4VSWR_{in}}{\left(VSWR_{in}+1\right)^2} \qquad \textbf{(D-9)}$$

### The constant input mismatch circle

For a given load reflection coefficient $\rho_L$, there will be a fixed input reflection coefficient $\rho_{in}$, as given by equation 5-57. Also, for a given input VSWR, there will be a fixed value of $M_s$ as shown by equation D-9. Equation D-7 can be expressed as the equation of a circle with the variable $\rho_s$ in the $\rho_s$ plane. This means that the constant input mismatch circle will be in the same plane as the source stability circle, the conjugate input impedance circle and the constant noise factor circles. Being in the one plane, the effect caused by varying one parameter on the other parameters can be easily seen.

From equation D-7,

$$M_s = \frac{\left(1-\left|\rho_s\right|^2\right)\left(1-\left|\rho_{in}\right|^2\right)}{\left|1-\rho_s\rho_{in}\right|^2} = \frac{\left(1-\rho_s\rho_s^*\right)\left(1-\left|\rho_{in}\right|^2\right)}{\left(1-\rho_s\rho_{in}\right)\left(1-\rho_s^*\rho_{in}^*\right)}$$

which gives,

$$\rho_s\rho_s^* - \rho_s^*\frac{M_s\rho_{in}^*}{1-\left|\rho_{in}\right|^2\left(1-M_s\right)} - \rho_s\frac{M_s\rho_{in}}{1-\left|\rho_{in}\right|^2\left(1-M_s\right)} + \frac{\left|\rho_{in}\right|^2+M_s-1}{1-\left|\rho_{in}\right|^2\left(1-M_s\right)} = 0$$

comparing this with the equation of a circle given by equation B-1, it can be seen that this circle has a centre and radius given by,

centre
$$c_{cim} = \frac{M_s\rho_{in}^*}{1-\left|\rho_{in}\right|^2\left(1-M_s\right)} \qquad \textbf{(D-10)}$$

radius
$$r_{cim} = \frac{\sqrt{1-M_s}\left(1-\left|\rho_{in}\right|^2\right)}{1-\left|\rho_{in}\right|^2\left(1-M_s\right)} \qquad \textbf{(D-11)}$$

## The output mismatch circle

### The mismatch parameter $M_o$

Without a matching circuit, from the transmission line, which has a characteristic impedance $Z_o$, the output circuit of the amplifier will appear as shown in Figure D-2.
Using the result of A-15, where the input is replaced by the output,

$$P_{out} = \frac{4Z_o R_{out}}{\left|Z_o + Z_{out}\right|^2} P_{avs} = M_o P_{avso} \qquad \textbf{(D-12)}$$

where the output mismatch factor $M_o$ is defined as,

$$M_o = \frac{4Z_o R_{out}}{\left|Z_o + Z_{out}\right|^2} = \frac{4\overline{R}_{out}}{\left|1+\overline{Z}_{out}\right|^2} \quad \textbf{(D-13)}$$

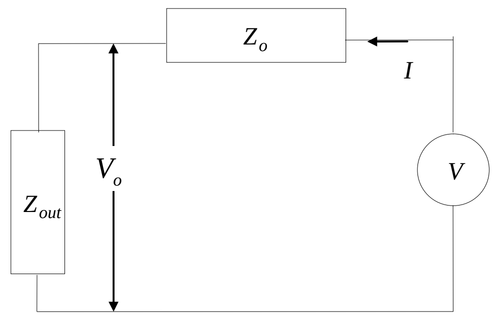



**Figure D-2 Schematic used in derivation of** $M_o$

From

$\overline{Z}_{out} = \frac{1+\rho_{out}}{1-\rho_{out}}$ and $2\overline{R}_{out} = \overline{Z}_{out} + \overline{Z}^*_{out}$

$$\overline{R}_{out} = \frac{1-\left|\rho_{out}\right|^2}{\left|1-\rho_{out}\right|^2} \quad \textbf{(D-14)}$$

and

$$1+\overline{Z}_{out} = \frac{2}{1-\rho_{out}} \quad \textbf{(D-15)}$$

the following result is obtained for $M_o$,

$$M_o = 1-\left|\rho_{out}\right|^2 \quad \textbf{(D-16)}$$

As VSWR is defined by equation 1-63, as,

$$VSWR = \frac{1+|\rho|}{1-|\rho|} \quad \textbf{(1-63)}$$

the output VSWR in terms of the input mismatch factor M, is found from equation D-16 as,

$$VSWR_{out} = \frac{1+\sqrt{1-M_o}}{1-\sqrt{1-M_o}} \quad \textbf{(D-17)}$$

Consider now the case where there is an output matching circuit between the transmission line and the output from the amplifier or network, as shown in Figure 124. The VSWR looking from the transmission line into the output matching network will be unity, but the VSWR looking from the matching network into the amplifier output will be the same as that when there was no matching network. However, there will be a load impedance $Z_L$ other than the characteristic impedance of the transmission line as seen by the amplifier looking into the output matching network. The mismatch factor now becomes that given by equation A-17, that is,

$$M_{out} = \frac{4R_L R_{out}}{\left|Z_L + Z_{out}\right|^2} = \frac{\left(1-\left|\rho_L\right|^2\right)\left(1-\left|\rho_{out}\right|^2\right)}{\left|1-\rho_L\rho_{out}\right|^2} \quad \textbf{(D-18)}$$

As the VSWRs are the same,

$$VSWR_{out} = \frac{1+\sqrt{1-M_o}}{1-\sqrt{1-M_o}} = \frac{1+\sqrt{1-M_{out}}}{1-\sqrt{1-M_{out}}} \quad \textbf{(D-19)}$$

and therefore,

$$M_{out} = \frac{4VSWR_{out}}{\left(VSWR_{out}+1\right)^2} \quad \textbf{(D-20)}$$

### *The constant output mismatch circle*

For a given source reflection coefficient $\rho_s$, there will be a fixed output reflection coefficient $\rho_{out}$, as given by equation 5-41. Also, for a given output VSWR, there will be a fixed value of $M_{out}$ as shown by equation D-20. Equation D-18 can be expressed as the equation of a circle with the variable $\rho_L$ in the $\rho_L$ plane. This means that the constant output mismatch circle will be in the same plane as the load stability circle and the constant power gain circle. Being in the one plane, the effect caused by varying one parameter on the other parameters can be easily seen.

From equation D-18,

$$M_{out} = \frac{\left(1-|\rho_L|^2\right)\left(1-|\rho_{out}|^2\right)}{|1-\rho_L\rho_{out}|^2} = \frac{\left(1-\rho_L\rho_L^*\right)\left(1-|\rho_{out}|^2\right)}{\left(1-\rho_L\rho_{out}\right)\left(1-\rho_L^*\rho_{out}^*\right)}$$

which gives,

$$\rho_L\rho_L^* - \rho_L^*\frac{M_{out}\rho_{out}^*}{1-|\rho_{out}|^2(1-M_{out})} - \rho_L\frac{M_{out}\rho_{out}}{1-|\rho_{out}|^2(1-M_{out})} + \frac{|\rho_{out}|^2 + M_{out} - 1}{1-|\rho_{out}|^2(1-M_{out})} = 0$$

comparing this with the equation of a circle given by equation B-1, it can be seen that this circle has a centre and radius given by,

centre $$c_{com} = \frac{M_{out}\rho_{out}^*}{1-|\rho_{out}|^2(1-M_{out})} \qquad \textbf{(D-21)}$$

radius $$r_{com} = \frac{\sqrt{1-M_{out}}\left(1-|\rho_{out}|^2\right)}{1-|\rho_{out}|^2(1-M_{out})} \qquad \textbf{(D-22)}$$