2 Linear Algebra

· linear algebra : study of vectors and rules to manipulate vectors

to solve the linear equation problem

1) geometric vectors (most familiar)

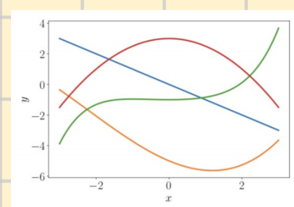
⇒ can be added or multiplied by scalar. $\vec{x}+\vec{y}$ / $\lambda\vec{x}$

we can present our intuition about "direction", "magnitude"

2) polynomials 다항식, ex)

A polynomial of degree n is described as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$



⇒ can be added or multiplied by scalar

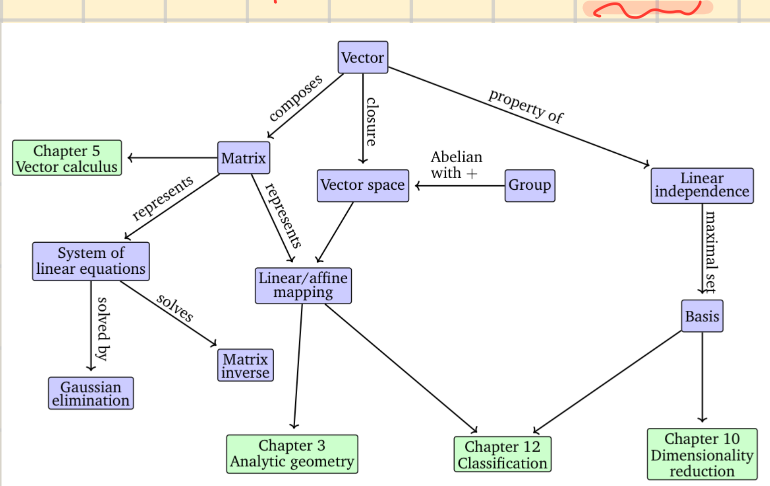
3) Elements of $\mathbb{R}^n$

= array of number in computer

ex) $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in R^3$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in R^3$ ⇒ $a+b \in R^3$, $2a \in R^3$

R^n = tuples of n real number

= n-dimensional real coordinate space (n차 실좌표공간)



2.1 Systems of Linear Equations = 선형 연립방정식

- Linear algebra gives us tools for solving systems of linear equations
- 해가 없거나 / unique / infinite!

Linear regression (OLS 와 MLE 활용)

matrix inverse 와 가우스 소거법

to find solution

a total of

$$a_{i1}x_1 + \cdots + a_{in}x_n \tag{2.2}$$

many units of resource R_i. An optimal production plan $(x_1, \ldots, x_n) \in \mathbb{R}^n$, therefore, has to satisfy the following system of equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} , \tag{2.3}$$

where $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$.

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{2.9}$$

$$\iff \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} . \tag{2.10}$$

2.2 Matrices

$$
\begin{aligned}
2x_1 + 3x_2 + 5x_3 &= 1 \\
4x_1 - 2x_2 - 7x_3 &= 8 \\
9x_1 + 5x_2 - 3x_3 &= 2
\end{aligned}
\tag{2.35}
$$

and use the rules for matrix multiplication, we can write this equation system in a more compact form as

$$
\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}}_{b} . \tag{2.36}
$$

⇒ 1) represent systems of linear equations
2) represent linear functions (linear mappings)

- $A \in \mathbb{R}^{m \times n}$ (matrix) ⇒ set of n column vectors which is element of $\mathbb{R}^m$ or which is tuple of m real numbers

- identity matrix ⇒ $n \times n$ matrix containing 1 on the diagonal and 0 everywhere else

- matrix inverse
⇒ square matrix 에만 가능. 그리고, 그 inverse matrix는 unique 하다.

$$
\begin{aligned}
AA^{-1} &= I = A^{-1}A && (2.26) \\
(AB)^{-1} &= B^{-1}A^{-1} && (2.27) \\
(A + B)^{-1} &\neq A^{-1} + B^{-1} && (2.28) \\
(A^\top)^\top &= A && (2.29) \\
(A + B)^\top &= A^\top + B^\top && (2.30) \\
(AB)^\top &= B^\top A^\top && (2.31)
\end{aligned}
$$

2.3 Solving Systems of Linear Equations

(연습장) → 가우스 소거법 연습

For $a \in \mathbb{R}$, we seek all solutions of the following system of equations:

$$\begin{array}{rcrcrcrcrcr} -2x_1 & + & 4x_2 & - & 2x_3 & - & x_4 & + & 4x_5 & = & -3 \\ 4x_1 & - & 8x_2 & + & 3x_3 & - & 3x_4 & + & x_5 & = & 2 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & x_5 & = & 0 \\ x_1 & - & 2x_2 & & & - & 3x_4 & + & 4x_5 & = & a \end{array} . \quad (2.44)$$

$$\left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

계수행렬 — 상수

$$\xrightarrow{①\leftrightarrow③} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

$$\downarrow ② = ② - ① \times 4$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

$$\xleftarrow{③ = ③ + ① \times 2} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

$$\downarrow ④ = ④ - ①$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{array}\right] \xrightarrow{④ = ④ - ②} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & -3 & 6 & a-2 \end{array}\right]$$

$$③ = ③ \times \tfrac{-1}{3} \quad \downarrow \quad ④ = ④ - ③$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & \cancel{-3}\ 1 & \cancel{6}\ -2 & \cancel{-3}\ 1 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array}\right]$$

$$\therefore a = -1$$

$$\downarrow ① = ① + ②$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow ② = ② + ③$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

REF Row echelon form

x_1 x_3 x_4 : pivot variable!

x_2 x_5 : free variable : 임의의 실수

$$\begin{array}{rrrrrl} x_1 & -2x_2 & +0 & +0 & -2x_5 & = 2 \\ 0 & 0 & -x_3 & 0 & -x_5 & = 1 \\ 0 & 0 & 0 & +x_4 & -2x_5 & = 1 \end{array}$$

$$\Downarrow$$

$$x_1 = 2x_2 + 2x_5 + 2$$
$$x_3 = -x_5 - 1$$
$$x_4 = x_5 + 1$$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{a} + x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b} + x_5 \underbrace{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}}_{c}$$ = 벡터 a,b,c 의 선형결합

임의의 실수

2.4 Vector Spaces

$\mathbb{R}^n$

"group" [closure, associativity, neutral element, inverse

1. Vector space : $V = (V, +, \cdot)$ is a set of vector V with two operations $(+ : V \times V \to V,\ \cdot : \mathbb{R} \times V \to V)$

선형결합 연산

Definition 2.9 (Vector Space). A real-valued *vector space* $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

vector space

scaling

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \quad (2.62)$$

$$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V} \quad (2.63)$$

$\forall v \in V,\ \forall \lambda, \psi \in \mathbb{R}$

1. closure : $v + v \in V$, $\lambda v \in V$
2. associativity : $(\lambda \cdot \psi) x = \lambda(\psi \cdot x)$
3. neutral element : $v + 0 = v$ or $v \cdot 1 = v$
4. inverse : $v + (-v) = 0$, $v \cdot v^{-1} = 1$

where

1. $(\mathcal{V}, +)$ is an Abelian group $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Distributivity:
   1. $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y}$
   2. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$
3. Associativity (outer operation): $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : \lambda \cdot (\psi \cdot \boldsymbol{x}) = (\lambda\psi) \cdot \boldsymbol{x}$
4. Neutral element with respect to the outer operation: $\forall \boldsymbol{x} \in \mathcal{V} : 1 \cdot \boldsymbol{x} = \boldsymbol{x}$

ex) $\mathbb{R}^n$: n-dimensional real coordinate space
[all n-dimensional real vectors with n components
"$(x_1, \dots x_n)$ 으로 정의되는 모든 벡터가 정의되는 공간"
(0 벡터 포함)

Vector space
$G(V, +, \cdot)$
⊗ $+ : V \times V \to V$ addition
⊗ $\cdot : \mathbb{R} \times V \to V$ multiplication by scalars
* ⊗ : operation
× : operation indicated $(+, \cdot)$

Example 2.11 (Vector Spaces)
Let us have a look at some important examples:

- $\mathcal{V} = \mathbb{R}^n, n \in \mathbb{N}$ is a vector space with operations defined as follows:
  - Addition: $\boldsymbol{x} + \boldsymbol{y} = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
  - Multiplication by scalars: $\lambda \boldsymbol{x} = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$ for all $\lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^n$
- $\mathcal{V} = \mathbb{R}^{m \times n}, m, n \in \mathbb{N}$ is a vector space with
  - Addition: $\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$ is defined elementwise for all $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$
  - Multiplication by scalars: $\lambda \boldsymbol{A} = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}$ as defined in Section 2.2. Remember that $\mathbb{R}^{m \times n}$ is equivalent to $\mathbb{R}^{mn}$.
- $\mathcal{V} = \mathbb{C}$, with the standard definition of addition of complex numbers.

2. Vector Subspace

- 0 벡터 포함
- $\mathbb{R}^n$내 벡터가 선형결합 연산에 닫힌 벡터공간

$V = (v, +, \cdot)$, $U \subseteq V$ 일 때,
$U(u, +, \cdot)$는 vector subspace

to use vector subspaces for dimensionality reduction.

vector space

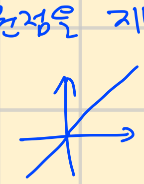
set → operation (addition, multiplication by scalar)

Definition 2.10 (Vector Subspace). Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{U} \subseteq \mathcal{V}, \mathcal{U} \neq \emptyset$. Then $U = (\mathcal{U}, +, \cdot)$ is called *vector subspace* of V (or *linear subspace*) if U is a vector space with the vector space operations $+$ and $\cdot$ restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$. We write $U \subseteq V$ to denote a subspace U of V.

vect
line

[Vector subspace]
- 0 벡터 포함
- 선형결합 연산에 닫힌 벡터공간

ex) in $\mathbb{R}^2$, 원점을 지나는 직선



[Column space] = matrix subspace
임의의 행렬 A에 대해
→ column들의 선형결합 연산에 닫힌 subspace

2.5 Linear Independence

[Linear Combination]

$V(v, +, \cdot)$ 내에 v 벡터들의 addition, multiplication by scalar 연산으로 얻어낸 조합 ($v \in V$. $v = \lambda_1 x_1 + \cdots + \lambda_n x_n$)

[Linear independency]

$(k \in N)$

in R^n, n-dimensional vector space, let us consider k vectors(v) and k λ ($\lambda \in R$). then we can get the linear combination like this : $0 = \sum_{i=1}^{k} v_i \lambda_i$. and if we cannot get the solution of this equation with at least one $\lambda_i \neq 0$. then $V(v_1 \dots v_k)$ is linearly dependent.

if there is only unique solution with $\lambda_1 = \cdots = \lambda_k = 0$, then, $V(v_1 \dots v_k)$ is linearly independent.

[선형독립 확인 방법]

1. Gaussian Elimination 통해 row Echelon form의 pivot variable 확인. (free variable 존재하면 종속)

2. $m \begin{bmatrix} n \\ \ \end{bmatrix}$ if $m > n$ 이면, 선형종속.

Why? $m > n$ 이면. 해의 개수가 무수히 많아짐, 즉, $[\ ]$의 solution set이 '0'벡터 외에도 더 많다는 의미로, 선형 종속일 수 밖에 없다!

[선형 종속의 예시]

$3x_2 = x_1 + x_3$
$\Rightarrow$ linearly dependent
$(x_1 - x_3)$

$\Rightarrow x_1 - 3x_2 + x_3 = 0$

$\lambda_1, \lambda_2, \lambda_3$
$= 1, -3, 1$

2.6 Basis and Rank

1. generating sets : Sets of vectors that span vector space or vector subspace

2. span of A : Set of all linear combination of vectors in vector space (A) or subspace (A)

3. basis : A basis is a minimal generating set and a maximal linearly independent set of vectors.

· dimension : the number of basis vector

4. rank : 선형독립인 column vector or row vector 갯수

5. *square matrix라도, full-rank 여야 가역적

6. * full-rank, m×n matrix A
$\text{rank}(A) = \min(m, n)$

1. Linear mapping

if, $\forall x, y \in V$, $\forall \lambda, \psi \in \mathbb{R}$,
$\Phi(\lambda x + \psi y) = \lambda \Phi(x) + \psi \Phi(y)$ 라면,
$\Phi : V \to W$, (V, W are vector spaces) is linear mapping

2. matrix
① linear mapping
② collection of vectors

3. injective, surjective, bijective
단사(1:1) unique solution 전사(모든 y) 전단사 (for every y, x has unique solution)

$\Phi : V \to W$. linear mapping Φ

① injective : $\forall x, y \in V$, $\Phi(x) = \Phi(y) \Rightarrow x = y$.
② Surjective : $\Phi(V) = W$
"Codomain is equal to its image"
공역 치역
③ bijective : injective and surjective

4. inversible

'f' is inversible if and only if there is unique solution of x, for every y
injective
surjective

즉, bijective 해야 inversible 한 function이 될수 있다.

[inversible 여부 판별]

$T: \underset{(V)}{R^n} \to \underset{(W)}{R^m}$, $T(\vec{x}) = \underset{(m\times n)}{A}\underset{(n\times 1)}{\vec{x}}$ 일때,

T, A는 inversible ?

Ans) NO, A, T is not inversible because it is not bijective.

① Surjective 한가?

* surjective: if, $\Phi(V) = W$, Φ is surjective mapping

Surjective라면, A의 REF 상 pivot variable이 m개 (span of V =W 이어야 함)

즉, $rank(A) = m$ 이면, A, T는 surjective mapping

② injective 한가?

* injective: $\Phi(x) = \Phi(y) \Rightarrow x = y$, Φ is injective mapping

injective 라면, $N(A) = \{\phi\}$, 즉 선형독립

$\therefore$ $rank(\underset{m\times n}{A}) = n$ 이면, 선형독립 $\longrightarrow$ Φ is injective mapping.

③ $\therefore$ $rank(A) = m = n$, A가 square matrix 여야 bijective 하고, inversible!

$\therefore$ A는 not inversible

※ 왜 injective 이면 선형독립 $(N(A) = \{\phi\})$ 인가?

injective 라면, $\Phi(x) = \Phi(y) \longrightarrow x = y \iff \Phi(x) - \Phi(y) = 0 = x - y$

$\iff$

$\iff \Phi(x) - \Phi(y) = 0 = x - y \iff \Phi(x-y) = 0$ ($\because$ Φ is linear mapping)

$\iff$ $\iff \Phi(x-y) = 0 = x-y$ 일때, $x-y$를 'v'라는 벡터 ($v \in V$)로 치환.

$\iff \Phi(v) = 0 = v$

$\iff \Phi(\Sigma c_i v_i) = 0 = \Sigma c_i v_i$

$*N(A) = ?$ $A\vec{x} = \vec{0}$, $N(A)$: all $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$

※ 왜 injective 이면 선형독립 ($N(A) = \{\phi\}$) 인가?

injective 라면, $\Phi(x) = \Phi(y) \longrightarrow x = y \iff \Phi(x) - \Phi(y) = 0 = x - y$

$\iff$

$\iff \Phi(x) - \Phi(y) = 0 = x - y \iff \Phi(x-y) = 0$ ($\because \Phi$ is linear mapping)

$\iff$ $(\Rightarrow)$ $\Phi(x-y) = 0 = x-y$ 일때, $x-y$를 'v'라는 벡터 ($v \in V$)로 치환.

$(\Rightarrow)$ $\Phi(v) = 0 = v$

$(\Rightarrow)$ $\Phi(\Sigma c_i v_i) = 0 = \Sigma c_i v_i$,

5. Morphism

① isomorphism : $\Phi : V \rightarrow W$, linear and bijective 동형

$*$homomorphism : $\Phi : V \rightarrow W$, linear and surjective or injective 준동형

② Endomorphism : $\Phi : V \rightarrow V$, linear 자기

③ Automorphism : $\Phi : V \rightarrow V$, linear and bijective 자기동형사상

$*$ space V and W is isomorphic if and only if $\dim(V) = \dim(W)$

(Φ should be linear and bijective, T is square)

($\Phi : V \rightarrow W$, $AT = B$ ($A \in V$, $B \in W$))

6. Coordinates (좌표)

: 어떤 벡터를 나타내기 위해, 기저 벡터의 선형조합시, 선형조합의 계수벡터

7. Transformation matrix

Definition 2.19 (Transformation Matrix). Consider vector spaces V, W with corresponding (ordered) bases $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m)$. Moreover, we consider a linear mapping $\Phi : V \to W$. For $j \in \{1, \dots, n\}$,

$$\Phi(\boldsymbol{b}_j) = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m = \sum_{i=1}^{m} \alpha_{ij}\boldsymbol{c}_i \tag{2.92}$$

is the unique representation of $\Phi(\boldsymbol{b}_j)$ with respect to C. Then, we call the $m \times n$-matrix $\boldsymbol{A}_\Phi$, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij}, \tag{2.93}$$

the *transformation matrix* of Φ (with respect to the ordered bases B of V and C of W).

transformation matrix

$$\underbrace{\begin{bmatrix} c \end{bmatrix}}_{\text{basis of } W} = \underbrace{\begin{bmatrix} A_\Phi \end{bmatrix}}_{\text{변환행렬}} \underbrace{\begin{bmatrix} b \end{bmatrix}}_{\text{basis of } V}$$

8. basis change

① Coordinate의 변환 (기저변환에 따른)

좌표계
트좌표계

A basis effectively defines a coordinate system. We are familiar with the Cartesian coordinate system in two dimensions, which is spanned by the canonical basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$. In this coordinate system, a vector $\boldsymbol{x} \in \mathbb{R}^2$ has a representation that tells us how to linearly combine $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ to obtain $\boldsymbol{x}$. However, any basis of $\mathbb{R}^2$ defines a valid coordinate system, and the same vector $\boldsymbol{x}$ from before may have a different coordinate representation in the $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ basis. In Figure 2.8, the coordinates of $\boldsymbol{x}$ with respect to the standard basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is $[2, 2]^\top$. However, with respect to the basis $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ the same vector $\boldsymbol{x}$ is represented as $[1.09, 0.72]^\top$, i.e., $\boldsymbol{x} = 1.09\boldsymbol{b}_1 + 0.72\boldsymbol{b}_2$. In the following sections, we will discover how to obtain this representation.

[좌표변환]

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix}\begin{bmatrix} x_{e_1} \\ x_{e_2} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix}$$

$$\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

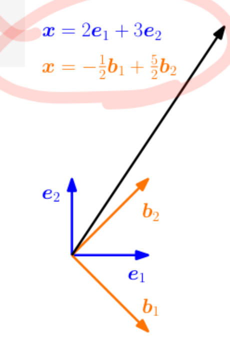
$$= \frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

예 —

Example 2.20

Let us have a look at a geometric vector $\boldsymbol{x} \in \mathbb{R}^2$ with coordinates $[2, 3]^\top$ with respect to the standard basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of $\mathbb{R}^2$. This means, we can write $\boldsymbol{x} = 2\boldsymbol{e}_1 + 3\boldsymbol{e}_2$. However, we do not have to choose the standard basis to represent this vector. If we use the basis vectors $\boldsymbol{b}_1 = [1, -1]^\top, \boldsymbol{b}_2 = [1, 1]^\top$ we will obtain the coordinates $\frac{1}{2}[-1, 5]^\top$ to represent the same vector with respect to $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ (see Figure 2.9).

Figure 2.9 Different coordinate representations of a vector $\boldsymbol{x}$, depending on the choice of basis.



기저벡터에 따라, 좌표는 달라진다.

Remark. For an n-dimensional vector space V and an ordered basis B of V, the mapping $\Phi : \mathbb{R}^n \to V$, $\Phi(\boldsymbol{e}_i) = \boldsymbol{b}_i$, $i = 1, \dots, n$, is linear (and because of Theorem 2.17 an isomorphism), where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ is the standard basis of $\mathbb{R}^n$. ♢

Now we are ready to make an explicit connection between matrices and linear mappings between finite-dimensional vector spaces.

② Transformation matrix의 변환 (기저벡터의 변환에 따른)

$$\tilde{A}_\Phi = T^{-1} A_\Phi S$$

Theorem 2.20 (Basis Change)**.** *For a linear mapping $\Phi : V \to W$, ordered bases*

$$\begin{tikzcd} B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n), \arrow[d, "A_\Phi"'] & \tilde{B} = (\tilde{\boldsymbol{b}}_1, \dots, \tilde{\boldsymbol{b}}_n) \arrow[l, bend right, "S"'] \arrow[d, "\tilde{A}_\Phi"] \\ C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m), & \tilde{C} = (\tilde{\boldsymbol{c}}_1, \dots, \tilde{\boldsymbol{c}}_m) \arrow[l, bend right, "T"'] \end{tikzcd}$$

$\tilde{B}$ onto B V (2.103)

$\tilde{C}$ onto C W (2.104)

of V and *of W, and a transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to B and C, the corresponding transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ with respect to the bases $\tilde{B}$ and $\tilde{C}$ is given as*

$$\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{T}^{-1}\boldsymbol{A}_\Phi\boldsymbol{S}\,. \tag{2.105}$$

Here, $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ is the transformation matrix of id_V that maps coordinates with respect to $\tilde{B}$ onto coordinates with respect to B, and $\boldsymbol{T} \in \mathbb{R}^{m\times m}$ is the transformation matrix of id_W that maps coordinates with respect to $\tilde{C}$ onto coordinates with respect to C.

Example 2.24 (Basis Change)
Consider a linear mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^4$ whose transformation matrix is

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & 7 & 1 \\ -1 & 2 & 4 \end{bmatrix} \tag{2.117}$$

with respect to the standard bases

$$B = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right), \quad C = \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right). \tag{2.118}$$

We seek the transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ of Φ with respect to the new bases

$$\tilde{B} = \left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right) \in \mathbb{R}^3, \quad \tilde{C} = \left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right). \tag{2.119}$$

Then,

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.120}$$

where the ith column of $\boldsymbol{S}$ is the coordinate representation of $\tilde{\boldsymbol{b}}_i$ in terms of the basis vectors of B. Since B is the standard basis, the coordinate representation is straightforward to find. For a general basis B, we would need to solve a linear equation system to find the λ_i such that

B의 관점으로 $\tilde{B}$를 표현 ($B \xleftarrow{S} \tilde{B}$)

$$\begin{tikzcd} B \arrow[d, "A_\Phi"'] & \tilde{B} \arrow[l, bend right, "S(\tilde{B} \text{ onto } B)"'] \arrow[d, "\tilde{A}_\Phi"] \\ C & \tilde{C} \arrow[l, bend left, "T(\tilde{C} \text{ onto } C)"] \end{tikzcd}$$

$\tilde{A}_\Phi = T^{-1} A S$

Example 2.23 (Basis Change)
Consider a transformation matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{2.100}$$

based on $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

with respect to the canonical basis in $\mathbb{R}^2$. If we define a new basis

$$B = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right) \tag{2.101}$$

A는 어떻게 변하는가?

we obtain a diagonal transformation matrix

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = BAB^{-1} \tag{2.102}$$

(not $B^{-1}AB$)

with respect to B, which is easier to work with than $\boldsymbol{A}$.

$$\begin{tikzcd} E = (e_1, e_2) \arrow[d, "\begin{bmatrix}2&1\\1&2\end{bmatrix} = A"'] \arrow[rr, bend left=60, phantom, ""] & & B = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right) \arrow[ll, bend right, "B^{-1}"'] \arrow[d, "\tilde{A} = \begin{bmatrix}3&0\\0&1\end{bmatrix}"] \\ {} \arrow[rr, "B"] & & {} \end{tikzcd}$$

* A mapping is $V \to V$,
∴ Transformation matrix 그대로 사용가능.

So, $\boxed{\tilde{A} = BAB^{-1}}$