4.1 Compute the determinant using the Laplace expansion (using the first row) and the Sarrus Rule for

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}.$$

$$|A| = (-1)^{1+1} \cdot 1 \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2} \cdot 3 \begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + (-1)^{1+3} \cdot 5 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= 4 - 24 + 20 = 0$$

4.2 Compute the following determinant efficiently:

$$A = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$P_{45}(-1) = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 2 \\ -2 & 0 & 2 & -3 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_{51}(-2) = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -4 & 1 & 2 & -1 & 2 \\ -6 & 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{41}(-2) = \begin{bmatrix} -2 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -2 & 1 & 2 & -1 & 2 \\ 0 & 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_{43}\left(\frac{2}{3}\right) = \begin{bmatrix} -2 & 0 & -\frac{1}{3} & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -2 & 1 & \frac{4}{3} & -1 & 2 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{32}\left(-\frac{3}{4}\right) = \begin{bmatrix} -2 & \frac{1}{4} & -\frac{1}{3} & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -2 & 0 & \frac{4}{3} & -1 & 2 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_{31}\left(\frac{3}{2}\right) = \begin{bmatrix} -\frac{5}{2} & \frac{1}{4} & -\frac{1}{3} & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & \frac{4}{3} & -1 & 2 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore |A| = -\frac{5}{2} \times -1 \times \frac{4}{3} \times -3 \times 1$$

$$= -10$$

4.3 Compute the eigenspaces of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.

i) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad (1-\lambda)^2 = 0$

$\lambda = 1$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \therefore E_1 = \text{span}[\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$

ii) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad (-2-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 + \lambda - 2 - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda + 3)(\lambda - 2) = 0$

$\lambda = -3$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$ $\quad \therefore E_{-3} = \text{span}[\begin{bmatrix} 2 \\ -1 \end{bmatrix}]$

$\lambda = 2$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-4x_1 + 2x_2 = 0$

$2x_1 - x_1 = 0$ $\quad \therefore E_2 = \text{span}[\begin{bmatrix} 1 \\ 2 \end{bmatrix}]$

4.4 Compute all eigenspaces of

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

$$\begin{vmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 1-\lambda & -2 & 3 \\ 2 & -1 & -\lambda & 0 \\ 1 & -1 & 1 & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & -1 & 1 & 0 \\ -1 & 1-\lambda & -2 & 5 \\ 2 & -1 & -\lambda & \lambda \\ 1 & -1 & 1 & -\lambda-1 \end{vmatrix} = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ -1 & 1-\lambda & -1-\lambda & 5 \\ 2 & -1 & -\lambda-1 & \lambda \\ 1 & -1 & 0 & -\lambda-1 \end{vmatrix}$$

$$\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ -1 & 1-\lambda & -1-\lambda & 5 \\ 2 & -1 & -\lambda-1 & \lambda \\ 1 & -1 & 0 & -\lambda-1 \end{vmatrix} \quad \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ -1 & 1-\lambda & -1-\lambda & 5 \\ 2 & -1 & -\lambda-1 & \lambda \\ 1 & -1 & 0 & -\lambda-1 \end{vmatrix}$$

$$-\lambda \begin{vmatrix} 1-\lambda & -1-\lambda & 5 \\ -1 & -\lambda-1 & \lambda \\ -1 & 0 & -\lambda-1 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1-\lambda & 5 \\ 2 & -\lambda-1 & \lambda \\ 1 & 0 & -\lambda-1 \end{vmatrix}$$

$$-\lambda \{ (1-\lambda)(\lambda+1)^2 - (\lambda+1)((\lambda+1)+\lambda) - 5(\lambda+1) \}$$

$$+ \{ (-1)(\lambda+1)^2 - (\lambda+1)(2(\lambda+1)+\lambda) + 5(\lambda+1) \}$$

4.5 Diagonalizability of a matrix is unrelated to its invertibility. Determine for the following four matrices whether they are diagonalizable and/or invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1$

$E_1 = \text{span}[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}]$

$\det(A) = 1$

∴ 대각화 가능, 정칙행렬

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $-(1-\lambda)\lambda = 0$

$\lambda = 0$ $E_0 = \text{span}[\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$

$\lambda = 1$ $E_1 = \text{span}[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]$

$\det(A) = 0$

∴ 대각화 가능, 비정칙행렬

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $(1-\lambda)^2 = 0$

$\lambda = 1$

$E_1 = \text{span}[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]$

$\det(A) = 1$

∴ 대각화 불가능, 정칙행렬

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda = 0$

$E_0 = \text{span}[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]$

$\det(A) = 0$

∴ 대각화 불가능, 비정칙행렬

4.6 Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

a.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

a. $\begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$= (2-\lambda)(4-\lambda)(1-\lambda) - 3(1-\lambda)$

$= (\lambda^2 - 6\lambda + 8)(1-\lambda) - 3(1-\lambda)$

$= (1-\lambda)(\lambda^2 - 6\lambda + 5)$

$= (1-\lambda)(\lambda-1)(\lambda-5)$

$= -(\lambda-1)^2(\lambda-5)$

i) $\lambda = 1$ $\begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 3x_2 = 0$

$x_1 = -3x_2$

$E_1 = \text{span}[\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}]$

ii) $\lambda = 5$ $\begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 3 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$E_5 = \text{span}[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}]$

$\Rightarrow$ 대각화 불가능

b.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_1 = \text{span}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]$$

$\lambda = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_0 = \text{span}\left[\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right]$$

모르겠음

$\Rightarrow$ 대각화 가능

4.7 Are the following matrices diagonalizable? If yes, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal. If no, give reasons why they are not diagonalizable.

a.

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

a. $\begin{bmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{bmatrix}$

$\lambda(\lambda-4)+8=0$

$\lambda^2-4\lambda+8=0$

$\lambda=2+2i,\ \lambda=2-2i$ → 모두 단일근이므로 대각화 가능한 것

i) $2+2i$ $\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(-2-2i)x_1+x_2=0$ $\times(2-2i)$

$-8x_1+(2-2i)x_2=0$

$$\begin{aligned} &(-2-2i)(2-2i)x_1+(2-2i)x_2=0 \\ -\big(&\quad -8x_1+(2-2i)x_2=0 \\ \hline &(-2-2i)(2-2i)x_1+8x_1=0 \end{aligned}$$

$(-4+4i-4i-4)x_1+8x_1=0$

$E_{2+2i}=\text{span}\left[\begin{bmatrix} 1 \\ -2-2i \end{bmatrix}\right]$

ii) $2-2i$ $\begin{bmatrix} -2+2i & 1 \\ -8 & 2+2i \end{bmatrix}$

$E_{2-2i}=\text{span}\left[\begin{bmatrix} 1 \\ -2+2i \end{bmatrix}\right]$

$\therefore A=\begin{bmatrix} 1 & 1 \\ -2-2i & -2+2i \end{bmatrix}\begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2-2i & -2+2i \end{bmatrix}^{-1}$

4.8 Find the SVD of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

4.9 Find the singular value decomposition of

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}.$$

4.10 Find the best rank-1 approximation of

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

4.11 Show that for any $A \in \mathbb{R}^{m \times n}$ the matrices $A^\top A$ and $AA^\top$ possess the same nonzero eigenvalues.

$$|A^T A - \lambda I| = |(A^T A - \lambda I)^T|$$
$$= |AA^T - \lambda I|$$

4.12 Show that for $x \neq 0$ Theorem 4.24 holds, i.e., show that

$$\max_{x} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1 ,$$

where σ_1 is the largest singular value of $A \in \mathbb{R}^{m \times n}$.