3.1 풀이 참고

3.2 풀이 참고 $\langle \,,\, \rangle$ is not "inner product".

3.3 ⓐ $d(x,y) = \sqrt{\begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}} = \sqrt{4+9+9} = \sqrt{22}$

ⓑ $d(x,y) = \sqrt{\langle x-y, x-y \rangle} = \sqrt{47}$

3.4 ⓐ angle $= -0.95$

ⓑ angle $= -0.98$

3.5

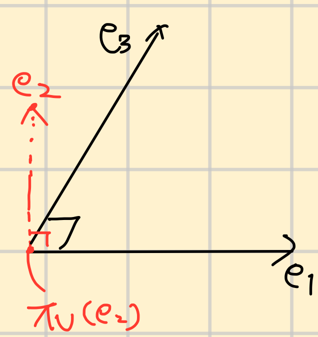
```
1  # a
2  print(projection)
[[ 3.49971198]
 [-9.60973502]
 [ 4.41676267]
 [-3.05040323]
 [ 6.56854839]]

1  # b
2  print(x-projection)
[[-4.49971198]
 [ 0.60973502]
 [-5.41676267]
 [ 7.05040323]
 [-5.56854839]]
```

3.6 ⓐ O

ⓑ $d(e_2, U) = \|e_2 - \pi_U(e_2)\|$

$= \|e_2\| = 1$

ⓒ



3.7 ??

3.8 ONB $C = \left(\sqrt{\frac{5}{14}} \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}, \frac{1}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right)$

3.9

3.10 $x_1' = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c-3s \\ 2s+3c \end{bmatrix} = \begin{bmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3\sqrt{3}}{2} \end{bmatrix}$

$x_2' = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} s \\ -c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$

3.1 Prove that given $\langle , \rangle$ function is inner product.

Let V is a vector space, and $\Omega : V \times V \to R$ be a bilinear mapping that takes two vectors to a real number.

and if this bilinear mapping (Ω) is 'symmetric' and 'positive definite', we can call this bilinear mapping 'inner product'

we can say bilinear mapping is symmetric when below statement is satisfied:

$$\forall x, y \in V, \quad \Omega(x, y) = \Omega(y, x)$$

Ω is called 'positive definite' if

$$\forall x \in V \setminus \{0\}, \quad \Omega(x, x) > 0, \quad \Omega(0, 0) = 0$$

So, if bilinear mapping $\langle \cdot , \cdot \rangle$ is 'symmetric' and 'positive definite',

then we can say $\langle \cdot , \cdot \rangle$ is inner product.

① symmetric

$\langle x, y \rangle = x_1 y_1 - (x_2 y_2 + x_2 y_1) + 2(x_2 y_2)$

$\langle y, x \rangle = y_1 x_1 - (y_2 x_2 + y_1 x_2) + 2(y_2 x_2)$

and $x, y \in R^2$, so x_1, x_2, y_1, y_2 are real number.

so, $\langle x, y \rangle = \langle y, x \rangle$

② positive definite

$$\langle x, x \rangle = x_1^2 - (x_1 x_2 + x_2 x_1) + 2(x_2)^2 = x_1^2 - 2x_1 x_2 + 2x_2^2$$
$$= (x_1 - x_2)^2 + x_2^2 > 0 \quad (\because x_1, x_2 \neq 0)$$

So, $\langle \cdot , \cdot \rangle$ is 'inner product'

3.2 1.

$\langle \cdot , \cdot \rangle$: $x, y \in \mathbb{R}^2$, $V \times V \to \mathbb{R}$ is bilinear mapping.

① Whether $\langle \cdot , \cdot \rangle$ is 'symmetric' bilinear mapping?

$$\langle x, y \rangle = [x_1 \ x_2] \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [2x_1 + x_2 \quad 2x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2x_1y_1 + x_2y_1 + 2x_2y_2$$

$$\langle y, x \rangle = [y_1 \ y_2] \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2y_1 + y_2 \quad 2y_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= 2x_1y_1 + x_1y_2 + 2x_2y_2$$

if $x_2y_1 = x_1y_2$, then $\langle \cdot , \cdot \rangle$ is symmetric, but it is not guaranteed, so $\langle \cdot , \cdot \rangle$ is not 'symmetric'.

② 'positive definite'? (?)

$$\langle x, x \rangle = [x_1 \ x_2] \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2x_1 + x_2 \quad 2x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= 2x_1^2 + x_1x_2 + 2x_2^2$$
$$= 2(x_1 + x_2)^2 - 3x_1x_2$$
$$= (x_1 - x_2)^2 + 3x_1x_2 + x_1^2 + x_2^2$$
$$= (x_1 - x_2)^2 + (x_1 + x_2)^2 + x_1x_2$$

* $2x_1^2 + 2x_2^2 + x_1x_2 > 0$ (?)

So, $\langle , \rangle$ is not 'inner product'.

33.

distance is measured by the concept of 'Norm' which uses 'inner product' bilinear mapping.

So, the distance between two vectors (x, y) is

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

ⓐ $\langle x, y \rangle = x^T y$ (dot product)

$$x-y = \begin{bmatrix} 1-(-1) \\ 2-(-1) \\ 3-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \quad \langle x-y, x-y \rangle = (x-y)^T (x-y)$$

$$d(x,y) = \sqrt{[2\ 3\ 3]\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}} = \sqrt{4+9+9} = \sqrt{22}$$

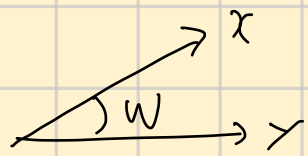
ⓑ $\langle x, y \rangle = x^T A y$, $\langle x-y, x-y \rangle = (x-y)^T A (x-y)$

$$\langle x-y, x-y \rangle = [2\ 3\ 3]\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = [7\ 8\ 3]\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$= 14+24+9 = 47$$

$$d(x,y) = \sqrt{\langle x-y, x-y \rangle} = \sqrt{47}$$

3.4



"cos(w)" represents the angle between vectors(x, y) and it is measured by the concept of inner product.

$$\cos(w) = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}, \quad \|x\| = \sqrt{\langle x, x\rangle}, \quad \|y\| = \sqrt{\langle y, y\rangle}$$

ⓐ $\langle x, y\rangle = x^T y$, $\cos(w) = \dfrac{x^T y}{\|x\|\,\|y\|}$

```
In [36]:
# 3.4 calculation
# a
x = np.array([[1],[2]])
y = np.array([[-1],[-1]])
x_y_inner = x.T@y
x_norm = np.sqrt(x.T@x)
y_norm = np.sqrt(y.T@y)
angle = x_y_inner[0][0] / (x_norm[0][0]*y_norm[0][0])

In [37]:
angle

Out[37]: -0.9486832980505138
```

angle = −0.95

ⓑ $\langle x, y\rangle = x^T B y$, $\cos(w) = \dfrac{x^T B y}{\|x\|\,\|y\|}$

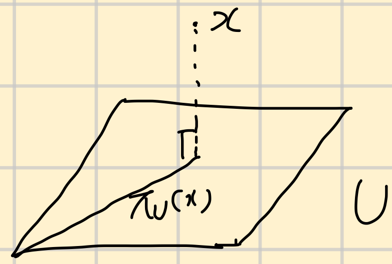
```
In [38]:
# 3.4 calculation
# b
x = np.array([[1],[2]])
y = np.array([[-1],[-1]])
B = np.array([[2,1],[1,3]])
x_y_inner = x.T@B@y
x_norm = np.sqrt(x.T@B@x)
y_norm = np.sqrt(y.T@B@y)
angle = x_y_inner[0][0] / (x_norm[0][0]*y_norm[0][0])

In [39]:
angle

Out[39]: -0.9799578870122228
```

angle = −0.98

3.5

· "Euclidean vector space" $\Rightarrow$ use dot product as inner product

ⓐ $\pi_U(x) = x^{\parallel U}$



$$\pi_U(x) = \frac{x^T u}{\|u\|} \cdot \frac{u}{\|u\|}$$

$$= \sum_{i=1}^{4} \pi_{u_i}(x)$$

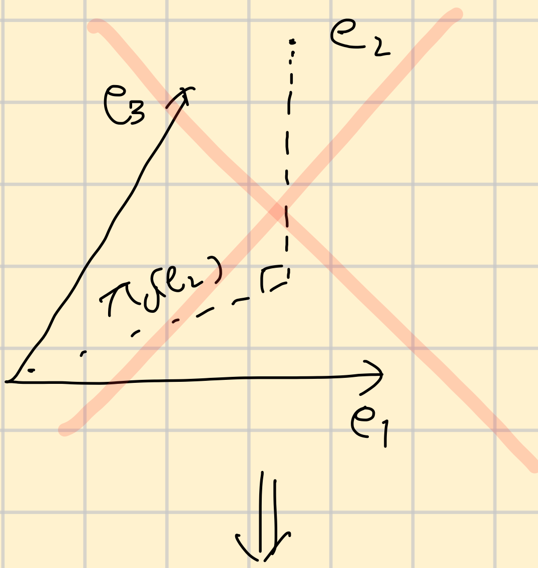
ⓑ $x - \pi_U(x)$

```
In [48]:
u1 = np.array([[0],[-1],[2],[0],[2]])
u2 = np.array([[1],[-3],[1],[-1],[2]])
u3 = np.array([[-3],[4],[1],[2],[1]])
u4 = np.array([[-1],[-3],[5],[0],[7]])
x = np.array([[-1],[-9],[-1],[4],[1]])

norm1 = np.linalg.norm(u1)
norm2 = np.linalg.norm(u2)
norm3 = np.linalg.norm(u3)
norm4 = np.linalg.norm(u4)

unit1 = u1/norm1
unit2 = u2/norm2
unit3 = u3/norm3
unit4 = u4/norm4

dot1 = (x.T@unit1)[0][0]
dot2 = (x.T@unit2)[0][0]
dot3 = (x.T@unit3)[0][0]
dot4 = (x.T@unit4)[0][0]

projection1 = dot1*unit1
projection2 = dot2*unit2
projection3 = dot3*unit3
projection4 = dot4*unit4
projection = projection1+projection2+projection3+projection4
```

```
In [60]:
# a
print(projection)

[[ 3.49971198]
 [-9.60973502]
 [ 4.41676267]
 [-3.05040323]
 [ 6.56854839]]
```

```
In [64]:
# b
print(x-projection)

[[-4.49971198]
 [ 0.60973502]
 [-5.41676267]
 [ 7.05040323]
 [-5.56854839]]
```

3.6.



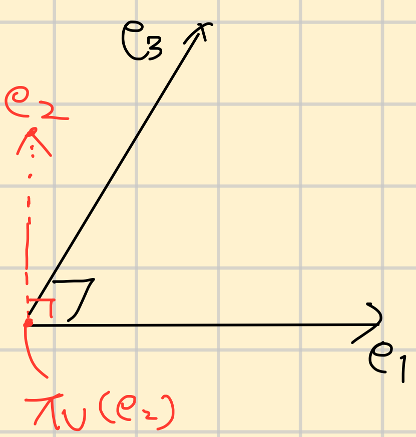
ⓐ $\pi_U(e_2) = e_2^{\parallel U} = e_2^{\parallel U_1} + e_2^{\parallel U_2}$

$= \langle e_2, e_1 \rangle \cdot 1 + \langle e_2, e_3 \rangle \cdot 1$

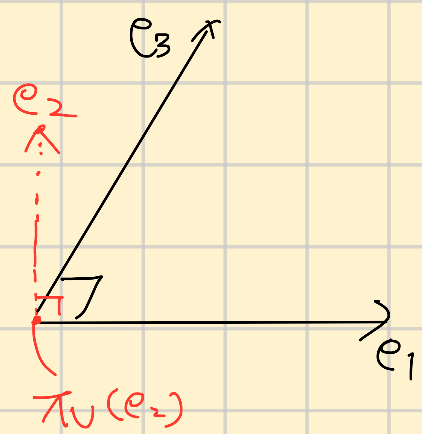
$= 0$

ⓑ $d(e_2, U) = \| e_2 - \pi_U(e_2) \|$

$= \| e_2 \| = 1$



ⓒ



3.7.

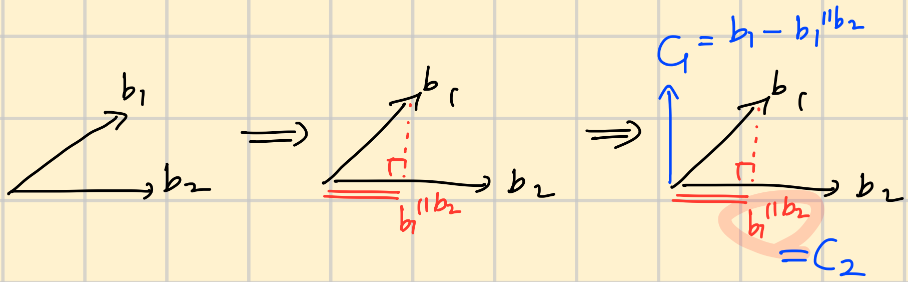
$V \Leftarrow$ vector space,

$\pi \Leftarrow$ endomorphism of V. $\pi : V \to V$. linear

- *Isomorphism:* $\Phi : V \to W$ linear and bijective (bijective function scalar
- *Endomorphism:* $\Phi : V \to V$ linear
- *Automorphism:* $\Phi : V \to V$ linear and bijective
- We define $\mathrm{id}_V : V \to V$, $\boldsymbol{x} \mapsto \boldsymbol{x}$ as the *identity mapping* or *identity automorphism* in V.

"maps to"

3.8 Gram-schmidt method



① get c_2

*assuming dotproduct as innerproduct.

$$c_2 = \pi_{b_2}(b_1) = \frac{b_1^T b_2}{\|b_2\|} \cdot \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}} \times \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = c_2$$

$$* \; b_1^T b_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1+2+0 = 1$$

$$* \; \|b_2\| = \sqrt{1+4+0} = \sqrt{5}$$

② get c_1

$$\alpha_1 = b_1 - c_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{5} \\ 1-\frac{2}{5} \\ 1-0 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

$$\|\alpha_1\| = \sqrt{\frac{36}{25} + \frac{9}{25} + 1} = \sqrt{\frac{70}{25}} = \sqrt{\frac{14}{5}}$$

$$c_1 = \frac{\alpha_1}{\|\alpha_1\|} = \sqrt{\frac{5}{14}}\begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

$$\therefore \text{ONB } C = \left(\sqrt{\frac{5}{14}}\begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right)$$

3.9

Cauchy - schwarz inequality

: 내적 공간 위에 성립하는 부등식.

$$|\langle x, y \rangle|^2 \le \langle x, x \rangle \langle y, y \rangle$$

ⓐ $x_1^2 + \cdots + x_n^2 \ge \frac{1}{n}$ (?)

$x_1 + \cdots + x_n = 1$

$|\langle x, x \rangle|^2 = x_1^2 + \cdots + x_n^2$

$\langle x, x \rangle = \sqrt{x_1^2 + \cdots + x_n^2}$

3.10. Rotate the vectors

$$Rot_\theta(\vec{x}) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \vec{x}$$

$$x_1' = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c - 3s \\ 2s + 3c \end{bmatrix} = \begin{bmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$x_2' = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} s \\ -c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\left(\begin{aligned} \cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{6} &= \frac{1}{2} \end{aligned} \right)$$