3.1 Norms

Norm : norm ($\|\cdot\|$) is a fuction that x ($x \in V$, V is vectorspace) maps to $\|x\|$ which represents the length of the vector x.

$$\|\cdot\| : V \rightarrow R$$
$$\vec{x} \longmapsto \|x\| \qquad (\|x\| \in R)$$

'Norm' function should follow below :
$(\lambda \in R,\ x, y \in V)$

1) absolutely homogeneous $\quad \|\lambda x\| = |\lambda| \, \|x\|$
2) triangle inequality $\quad \|x+y\| \le \|x\| + \|y\|$
3) positive definite $\quad \|x\| \ge 0$
$(\|x\| = 0 \iff x = 0)$

· L1 norm (manhattan norm)

$$\|x\|_1 = \sum |x_i|$$

· L2 norm (Euclidean norm)

$$\|x\|_2 = \left(\sum (x_i)^2\right)^{\frac{1}{2}}$$

⟨행렬 Norm 정의⟩

- 놈의 값은 0이상이다. 영행렬일 때만 놈의 값이 0이 된다.

$$\|A\| \ge 0 \quad \Longleftrightarrow A = 0 \qquad \text{positive definite} \qquad (2.3.12)$$

- 행렬에 스칼라를 곱하면 놈의 값도 그 스칼라의 절대값을 곱한 것과 같다.

$$\|\alpha A\| = |\alpha| \|A\| \qquad \text{homogeneous} \qquad (2.3.13)$$

- 행렬의 합의 놈은 각 행렬의 놈의 합보다 작거나 같다.

$$\|A + B\| \le \|A\| + \|B\| \qquad \text{triangle inequality} \qquad (2.3.14)$$

- 정방행렬의 곱의 놈은 각 정방행렬의 놈의 곱보다 작거나 같다.

$$\|AB\| \le \|A\|\|B\| \qquad (2.3.15)$$

absolutely homogeneous
triangle inequality
positive definite
$\|AB\| \le \|A\| \, \|B\|$ (A, B = square matrix)

3.2 Inner Products

- Inner products $\supset$ dot product "$\sum x_i y_i$"
- Inner products (bilinear, symmetric, positive definite)

  Ω is a bilinear mapping that takes two vectors and maps them onto Real number.

  $\Omega : V \times V \to R$ (V is vector space)

  if, Ω is 'symmetric' and 'positive definite',
  Ω is called "inner products"
  so, $\Omega(x, y) = \langle x, y \rangle$

* 'symmetric'
if, $\forall x, y \in V$, $\Omega(x, y) = \Omega(y, x)$
then Ω is 'symmetric'

* 'positive definite'
if, $\forall x \in V \setminus \{0\}$, $\Omega(x, x) > 0$, $\Omega(0, 0) = 0$
then Ω is 'positive definite'

* Inner products but not dot product
$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + (x_1 y_2 - x_2 y_1)$

• Symmetric, Positive definite matrix

if, matrix A is symmetric and

$\forall x \in V \setminus \{0\}$, $x^T A x > 0$ then,

matrix A is symmetric and positive definite

* Properties of Symmetric, Positive definite matrix A

$N(A) = \{\emptyset\}$

① $N(A) = \{\emptyset\}$, because $\forall x \in V \setminus \{\emptyset\}$, $x^T A x > 0$.
so, $Ax \neq 0$ if $x \neq 0$. ($x = 0 \longrightarrow Ax = 0$)

대각성분>0

② Diagonal Elements of A, a_{ii} are positive value.
because $a_{ii} = e_i^T A e_i > 0$ (e_i is standard basis of R^n)

3.3 Lengths and Distances

$\|x\|$ = norm (lengths)
$= \sqrt{\langle x, x\rangle}$

$\|x\|$
- absolutely homo-
- pos-def
- triangle inequality

L_1 norm / L_2 norm
using dot product as inner products

$\|x-y\|$ = distance
$= \sqrt{\langle x-y, x-y\rangle}$

$d(x,y)$
- symmetric
- pos-def
- triangle inequality

other / Euclidean distance
using dot product as inner products

$\langle x, y\rangle$ inner products
= bilinear mapping
(symmetric, pos-def)

< dot product
other

· Lengths and distance vary depending on types of inner products

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· Length : $\|x\| = \sqrt{\langle x, x\rangle}$
- $\sqrt{x_1^2 + \cdots}$ (dot product) $\Rightarrow$ length of vectors (dot product as inner products)
- $\sqrt{\langle x, x\rangle}$ (other inner product)

· distance : $\|x-y\| = \sqrt{\langle x-y, x-y\rangle}$
- $\sqrt{(x_1-y_1)^2 + \cdots}$ $\Rightarrow$ 유클리드 거리 (dot product as inner products)
- $\sqrt{\langle x-y, x-y\rangle}$

· 'metric' mapping (function)

$d : V \times V \to R$
$(x, y) \mapsto d(x, y)$

· 'inner product' mapping

$\langle \cdot, \cdot \rangle : V \times V \to R$
$(x, y) \mapsto \langle x, y\rangle$

opposite direction

metric 'd' satisfies followings

1) symmetric $d(x,y) = d(y,x)$
2) pos-def $d(x,y) > 0$, $d(0,0) = 0$
3) triangle inequality $d(x,z) \le d(x,y) + d(y,z)$

$\langle \cdot, \cdot \rangle$ satisfies followings

1) symmetric $\langle x, y\rangle = \langle y, x\rangle$
2) pos-def $\langle x, x\rangle > 0$, $\langle 0, 0\rangle = 0$

$* \ \forall x \in V \setminus \{0\}$

('norm' mapping)
$\|x\|$ satisfies followings

1) absolutely homogeneous $\|\alpha x\| = |\alpha| \, \|x\|$
2) pos-def $\|x\| > 0$, $\|0\| = 0$ $(\forall x \in V \setminus \{0\})$
3) triangle inequality $\|x+y\| \le \|x\| + \|y\|$

* very similar vectors x, y
$\langle x, y\rangle \uparrow\uparrow$, $d(x,y) \downarrow\downarrow$
opposite direction mappings

3.4 Angles and Orthogonality

- **Angles 'w'**. $W = \dfrac{\langle x, y \rangle}{\|x\| \, \|y\|}$

angles changes depending on the inner products type.

*angles ⇒ "how similar their orientations are"

- **orthogonality, orthonormality**

$x \perp y \iff \langle x, y \rangle = 0$

angles changes depending on the inner products type.

x, y are orthonormal $\iff \langle x, y \rangle = 0$ and $\|x\| = \|y\| = 1$

- **orthogonal Matrix**

square matrix A ($A \in \mathbb{R}^{n \times n}$) is orthogonal matrix if and only if its columns are orthonormal so that

$$A^T A = I = A A^T \quad (A^T = A^{-1})$$

- **Transformation with orthogonal matrix**

It preserves angles and distance.

* angles. ($\frac{\langle x, y \rangle}{\|x\| \, \|y\|}$) with dot product,

$$\frac{(Ax)^T Ay}{\|Ax\| \, \|Ay\|} = \frac{x^T A^T A y}{\|Ax\| \, \|Ay\|} = \frac{x^T y}{\sqrt{\langle Ax, Ax \rangle} \sqrt{\langle Ay, Ay \rangle}} = \frac{x^T y}{\sqrt{(Ax)^T Ax} \sqrt{(Ay)^T Ay}}$$

$$= \frac{x^T y}{\sqrt{x^T x} \sqrt{y^T y}} = \frac{x^T y}{\|x\| \, \|y\|}$$

angles of Ax, Ay = angles of x, y

* distances, $\|x - y\| = \sqrt{\langle x-y, x-y \rangle}$ with dot product

$$\|Ax - Ay\| = \sqrt{(Ax - Ay)^T (Ax - Ay)} = \sqrt{x^T A^T A x - x^T A^T A y - y^T A^T A x + y^T A^T A y}$$

$$= \sqrt{x^T x - x^T y - y^T x + y^T y}$$

$$\|x - y\| = \sqrt{(x-y)^T (x-y)}$$

$$= \sqrt{x^T x - x^T y - y^T x + y^T y}$$

distance between Ax, Ay = 〃 x, y

3.5 Orthonormal Basis

Let us consider basis $B = \{b_1, \dots b_n\}$ of V ($V \in \mathbb{R}^n$, vector space)

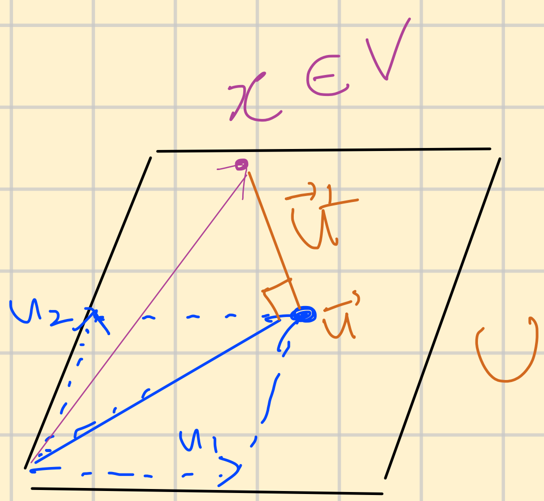
if basis follows below, then b_i ($i=1 \sim n$) is called "orthonormal basis"

$$\langle b_i, b_j \rangle = 0 \quad (i \neq j)$$
$$\langle b_i, b_i \rangle = 1$$

3.6 Orthogonal Complement

↳ normal vector!

- normal vector : vectors which is perpendicular to a given object such as line, plane.



- normal vectors of $\underbrace{U}_{\text{plane}}$ = $u^{\perp}$ ($u^{\perp} \in U^{\perp}$)

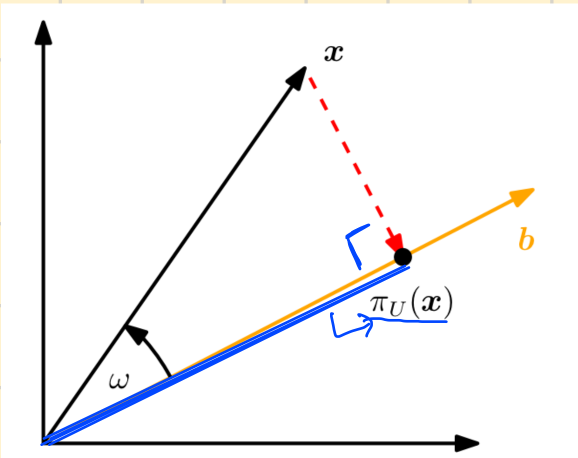
- $u^{\perp}$ can be used to describe x ($x \in V$)

x = span of $[u_1, u_2]$ + span of $[u^{\perp}]$

3.8 Orthogonal Projections

1. Projection definition

Let there be Vector spaces V, U. $(U \subseteq V)$.
And linear mapping $\pi : V \to U$ is called 'projection' if $\pi^2 = \pi \circ \pi = \pi$.

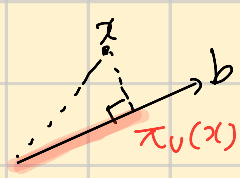


(a) Projection of $x \in \mathbb{R}^2$ onto a subspace U with basis vector b.

- Projection $\pi_U(x)$ is closest to 'x'.

"closest" means that $\|x - \pi_U(x)\|$ is minimal.

1. $\pi_U(x)$ =



$\pi_U(x) = \lambda b$

2. λ 구하기.

$\langle x - \pi_U(x), b \rangle = 0$

bilinearity $\iff \langle x, b \rangle - \langle \pi_U(x), b \rangle = 0$

$\pi_U(x) = \lambda b$ $\iff \langle x, b \rangle - \lambda \langle b, b \rangle = 0$

$\therefore \lambda \langle b, b \rangle = \langle x, b \rangle$

$\lambda = \frac{\langle x, b \rangle}{\langle b, b \rangle} = \frac{x^T b}{\|b\|^2}$

(using dotproduct)

3. $\pi_U(x)$ 구하기

$\pi_U(x) = \lambda b = \frac{x^T b}{\|b\|^2} b = \frac{b b^T}{b^T b} x$

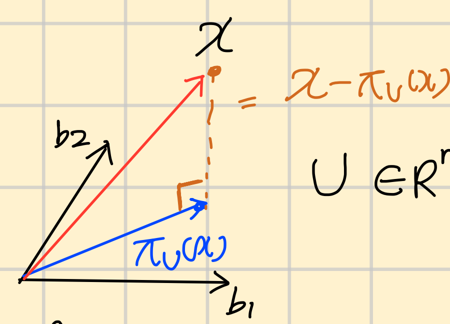
4. P_π 구하기 (Projection matrix of x to rank-1 subspace)

$\pi_U(x) = \frac{x^T b}{\|b\|^2} b = \frac{b^T x}{\|b\|^2} b$

$= b \frac{b^T x}{\|b\|^2} = \frac{b b^T}{\|b\|^2} x = P_\pi x$

$P_\pi = \frac{b b^T}{\|b\|^2}$

1. $\pi_U(x)$ =



$U \in R^m$, $\pi_U(x) \in R^n$

$= \sum_{i=1}^{n} \lambda_i b_i$ (∵ $\pi_U(x)$ is element of U. So, $\pi_U(x)$ can be represented span of $\{b_i\}$. (b_i is basis vector of U))

$= B\lambda$

$* B = [b_1 \cdots b_m]$ $(n \times m)$, $\lambda = [\lambda_1 \cdots \lambda_m]^T$ $(m \times 1)$

2. λ 구하기

$\langle b_1, x - \pi_U(x) \rangle$ ⊥ $\quad b_1^T(x - \pi_U(x)) = 0$

$\langle b_2, x - \pi_U(x) \rangle$ ⊥ $\longrightarrow \quad b_2^T(x - \pi_U(x)) = 0$

$\vdots \qquad\qquad \iff \qquad\qquad \vdots$

$\langle b_n, x - \pi_U(x) \rangle \quad b_n^T(x - \pi_U(x)) = 0$

$\therefore \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} \begin{bmatrix} x - \pi_U(x) \end{bmatrix} = B^T(x - B\lambda) = 0$

$\therefore B^T x = B^T B \lambda$ (=normal equation)

$\therefore \lambda = (B^T B)^{-1} B^T x$

5. verify

① $x - \pi_U(x)$ 와 U의 subspace 가 orthogonal 확인.

② $P_\pi[\pi_U(x)] = \pi_U(x)$

(∵ projection linear mapping means $\pi^2 = \pi \circ \pi = \pi$ by definition)

(∵ $\pi_U(x) = P_\pi x$)

3. $\pi_U(x)$ 구하기

$$\underset{n \times 1}{\pi_U(x)} = \underset{n \times m}{B}\,\underset{m \times 1}{\lambda} = B(B^T B)^{-1} B^T x$$

$* \lambda = (B^T B)^{-1} B^T x$

4. P_π 구하기 (rank-n 공간에 x가 projection 한 endpoint로 transformation)

$* n =$ 'b', basis vector 개수

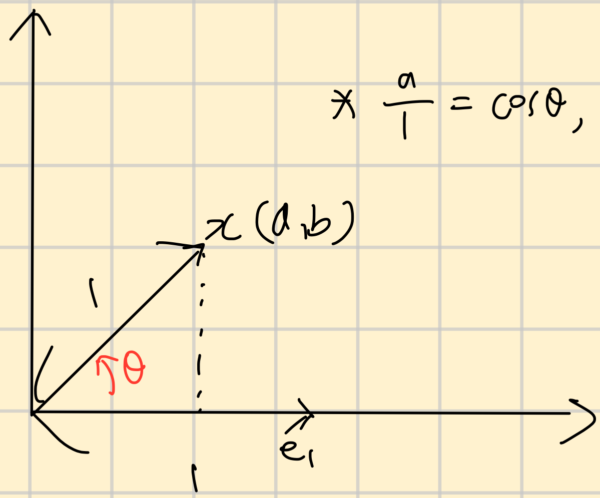
$$\underset{n \times 1}{\pi_U(x)} = \underset{n \times m}{B}\,\underset{m \times 1}{\lambda}$$

$* \lambda = (B^T B)^{-1} B^T x$

$$= B(B^T B)^{-1} B^T x$$

$$\therefore P_\pi = B(B^T B)^{-1} B^T$$

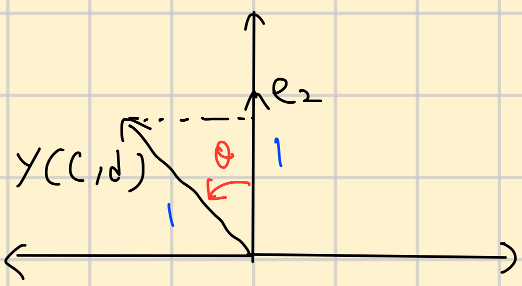
3.9 Rotations



$Rot_\theta(\vec{x}) = ? = A\vec{x}$

$* \ \frac{a}{1} = \cos\theta, \ \frac{b}{1} = \sin\theta$

$x = Rot_\theta(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



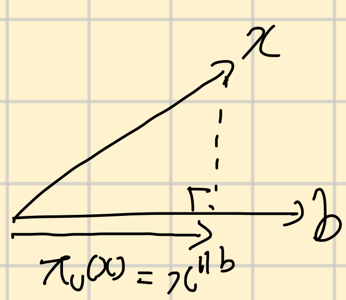
$y = Rot_\theta(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$* \ \frac{d}{1} = \cos\theta, \ \frac{-c}{1} = \sin\theta$

(ex) $\begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \begin{bmatrix} -s \\ c \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Rot_\theta(\vec{x}) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\vec{x}$$

< projection >



$\pi_U(x)$

$x^{\parallel b} = b \cdot c \quad (c \in \mathbb{R}^1)$

$cb^T(x - cb) = 0$

$cb^Tx - c^2b^Tb = 0$

$b^Tx = cb^Tb$

$c = \frac{b^Tx}{b^Tb}$ $\quad \therefore \ x^{\parallel b} = \pi_U(x) = \frac{b^Tx}{\|b\|^2} b$