4.1 Determinant and Trace

- Determinants are only defined for square matrix.

- A is inversible if and only if $Det(A) \neq 0$.

  if and only if A is full-rank

  eg) A^{-1} is defined for $AA^{-1} = I$. but if $Det(A) = 0$,

  then $A^{-1} = \frac{1}{Det(A)} A' = 0$.

  then $AA^{-1} = 0 \neq I$.

- determinants act as a function that measures signed volume formed by column vectors composed in a matrix

- Laplace Expansion ($n \times n$ matrix determinants)

$$\sum_{k=1}^{n} \underbrace{(-1)^{k+j} \cdot a_{kj}}_{\text{Cofactor}} \cdot \underbrace{\det(A_{k,j})}_{\text{Minor}}$$

*j = column..

4.2 Eigenvalues and Eigenvectors

Definition 4.6. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an *eigenvalue* of $\boldsymbol{A}$ and $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$ is the corresponding *eigenvector* of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}\,. \tag{4.25}$$

We call (4.25) the *eigenvalue equation*.

- Eigen space (E_λ) is the nullspace of $A - \lambda I_n$

  $(A-\lambda I_n)x = 0$, $x \in \ker(A-\lambda I_n)$

- Similar matrices possess the same eigenvalues.
  they are invariant under the basis change.

- Geometric multiplicity : the dimension of Eigen space → null space of $(A-\lambda I)$ $= \ker(A-\lambda I)$ $= (A-\lambda I)x=0$, x
  "the number of linearly independent eigen-vectors"

- Algebraic multiplicity : eigen value 의 중복도.

- if $\det(A) = 1$, then, transformation of A preserves its area.

- 고유벡터라고 무조건 서로 직교하는 건 X.
  (characteristic polynomial → λ 구하기 → v 구하기)
  ↳ $v_1 \perp v_3$, $v_1 \not\perp v_2$
  그람슈미트 과정으로 직교하는 고유벡터 찾기 가능!

- Cholesky decomposition $A = LL^T$
  양의 정부호 대칭행렬을 제곱근 분해!

  ⟹ 대칭인 pos-def 행렬의 연산에 활용↑
  공분산 행렬
  ↳ determinants 계산!
  ☆ $A = LL^T$, $\det(A) = \det(L)\det(L^T)$
  $= \det(L)^2$

  $\times \det(L) = \prod_{i=1}^{n} L_{ii}$
  ↳ triangle matrix

- Eigen decompsition

① λ, v using characteristic polynomial

② Check whether matrix can be diagonalized
(eigen vectors are othogonal ?)

③ Diagonalize
$$A = V \Lambda V^{-1}$$

1) we know 'V' at '①'
2) $\Lambda = V^{-1} A V$, we can get matrix Λ
3) then we can diagonalize matrix A ($A = V \Lambda V^{-1}$)

- A (symmetric)

1) $A^k = V \Lambda^k V^{-1}$, Computing Λ^k is easy. $\because$ Λ is diagonal matrix.

2) $Det(A) = Det(\Lambda) = \prod_{i=1}^{n} \lambda_i$ (Λ_{ii})

$\because Det(A) = Det(V \Lambda V^{-1}) = \cancel{Det(V)} Det(\Lambda) \cancel{Det(V^{-1})}$
$= \underbrace{Det(\Lambda)}_{\text{diagonal matrix}} = \prod_{i=1}^{n} \Lambda_{ii} = \prod_{i=1}^{n} \lambda_i$