2.1,, ⓐ Ans) ∴ $a \circledast b = b \circledast a$ (a.b is real numbers Which is in R arbitrarily except for -1)

ⓑ Ans) ∴ $x = 1$ or -3

2.2,,

2.3,, So, Group $g:(G,\cdot)$ is a group.
and it is not Abelian group. because $\forall A, X \in G,\ A\cdot X \neq X\cdot A$

2.4,,

ⓐ $\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ = We can't do matrix multiplication because of dimension

3×2 3×3

ⓑ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$

ⓒ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$

ⓓ $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2+2+10 & 3+(-2)+1+4 \\ 1+(-2)+(-20) & 12+(-1)+(-1)+(-8) \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ -21 & 2 \end{bmatrix}$

ⓔ $\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$

2.5,, ⓐ 해가 존재하지 않음

ⓑ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \cancel{x_5}\,\alpha \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $(\alpha \in R)$

2.6,, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \cancel{x_5}\,\alpha \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $(\alpha \in R)$

2.7,, $(x_1, x_2, x_3) = (-\frac{7}{3}, 2, \frac{4}{3})$

2.8 (a) A matrix is not inversible
($\because$ $C_1 + C_3 = 2C_2$, Linear dependent)

(b) A^{-1}
$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$ inversible

2.9 // ⓐ, ⓑ, ⓓ are subspaces.

2.10 // ⓐ linearly dependent.
ⓑ linearly independent

2.11 // $-6x_1 + 3x_2 + 2x_3$

2.12 // $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

2.13 // ⓐ $\mathrm{Dim}(U_1) = \mathrm{Dim}(U_2) = 1$
ⓑ bases of U_1 and U_2 = $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$
ⓒ $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

2.14 // ⓐ : $\mathrm{Dim}(U_1) = \mathrm{Dim}(U_2) = 2$
ⓑ bases of U_1 and U_2 = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
ⓒ basis of $U_1 \cap U_2$ = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

2.15 // ⓐ 풀이 참조
ⓑ $F \cap G$를 어떻게 계산?
ⓒ F와 G의 basis 어떻게...?

2.16 // ⓐ, ⓑ, ⓓ linear mapping
(c, e는 삼각함수 이해 부족으로 시도해보지 못함..)

2.17

ⓐ $\therefore T = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

ⓐ Ans)

ⓑ Ans) $\mathrm{rank}(\Phi) = 3$

ⓒ Ans) $\dim(\ker(\Phi)) = 0$ (zero-vector)

$\dim(\mathrm{image}(\Phi)) = 3$

2.18

잘 모르겠음....

2.19

$\begin{bmatrix} 1 & -2 & 4 \\ 0 & -4 & 7 \\ 0 & -2 & 4 \end{bmatrix} = \tilde{A}_\Phi$

2.20

ⓐ 1) B is bases of R^2

$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \xRightarrow{② = ② - ① \times \frac{1}{2}} \begin{bmatrix} 2 & -1 \\ 0 & \cancel{\frac{3}{2}}\ 1 \end{bmatrix} \xRightarrow{① = ① + ②} \begin{bmatrix} \cancel{2}\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

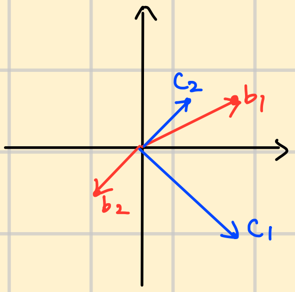
⇒ b_1, b_2는 서로 linear independent 하기 때문에, R^2의 basis vector가 될 수 있다.

2) B' is bases of R^2

$B' = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \xRightarrow{② = ② + ①} \begin{bmatrix} 2 & 1 \\ 0 & \cancel{2}\ 1 \end{bmatrix} \xRightarrow{① = ① - ②} \begin{bmatrix} \cancel{2}\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

⇒ c_1, c_2는 b_1, b_2와 마찬가지로 linear independent 해서 basis vector가 될 수 있다.

3) draw



P_1

ⓑ $\frac{1}{4} \begin{bmatrix} 0 & -4 \\ 3 & 1 \end{bmatrix}$

2.20

(c) (i) Determinant of $C = 4 \neq 0$

So, c_1, c_2, c_3 are linearly independent.
We can say C is basis of R^3.

(ii) $P_2 = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}$

(d) $A_\Phi = \frac{1}{3}\begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}$

(e) $A'_\Phi = \frac{1}{4}\begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 0 & -4 \end{bmatrix}$

(f) (i) $\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ = the Coordinates of x in B

(ii) $\begin{bmatrix} x_{c_1} \\ x_{c_2} \\ x_{c_3} \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$

(iii) $\begin{bmatrix} x_{c'_1} \\ x_{c'_2} \\ x_{c'_3} \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ -18 \end{bmatrix}$

(iv) $A'B'x_{b'} = C'x_{c'}$

2.1

if, $\forall x, y \in$ group, group : $x \otimes y = y \otimes x$,
then group = (group, $\otimes$) is Abelian group

※참고, $R \setminus \{-1\} = R - \{-1\}$ = Set of all real numbers except -1

ⓐ Let us consider group 'g'. $g = (R \setminus \{-1\}, \circledast)$
and $a, b \in g$.
we have to show that g is Abelian group.

so, if $a \circledast b = b \circledast a$, then 'g' is Abelian group.

- $a \circledast b = ab + a + b$,
- $b \circledast a = ba + b + a$.

Ans) $\therefore a \circledast b = b \circledast a$ (a.b is real numbers which is in R arbitrarily except for -1)

ⓑ We have to solve '$3 \circledast x \circledast x = 15$' in the Abelian group.
according to the definition of operation '$\circledast$' in Abelian group.

- $3 \circledast x = 3x + 3 + x$
- and component of 'g' is real number, so the operation '$\circledast$' can be associated

$$\therefore 3 \circledast x \circledast x = (3 \circledast x) \circledast x = (3x+3+x)x + 3x+3+x + x$$
$$= 3x^2 + 3x + x^2 + 5x + 3 = 4x^2 + 8x + 3$$

so, if we solve the equation, we can get the 'x'

$$4x^2 + 8x + 3 = 15. \iff 4x^2 + 8x - 12 = 0$$
$$\iff x^2 + 2x - 3 = 0$$
$$\iff (x-1)(x+3) = 0$$

Ans) $\therefore x = 1$ or -3

2.2

2.3

to be a group,
there are 4 features to be followed.

※ 참고,

set, operation

Definition 2.7 (Group). Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a *group* if the following hold:

exist, for all

1. *Closure* of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. *Associativity:* $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. *Neutral element:* $\exists e \in \mathcal{G}\, \forall x \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. *Inverse element:* $\forall x \in \mathcal{G}\, \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$, where e is the neutral element. We often write x^{-1} to denote the inverse element of x.

closure
결합법칙
항등원
역원
⇒ Group 구성요건

1. It should be closed under the operation. $\forall x, y \in G$, $x \cdot y \in G$
2. Associativity : $\forall x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
3. Neutral element : $\exists e \in G$ $\forall x \in G$: $x \cdot e = x$ and $e \cdot x = x$
4. Inverse element : $\forall x \in G$ $\exists y \in G$: $x \cdot y = e$ and $y \cdot x = e$ ('e' is neutral element)

if, $\forall x, y \in G$: $x \cdot y = y \cdot x$, then g is Abelian group.

$$g = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

① Closure

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \quad A, X \in g.$$

$$A \cdot X = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix}$$

$x+a,\ y+az+b,\ z+c \in \mathbb{R}$ ($\because a, b, c, x, y, z \in \mathbb{R}$)

$\therefore$ 'g' is closed under operation '$\cdot$'

② Associativity

Let us consider matrix $B = \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$

$$(A \cdot X) \cdot B = \begin{bmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (e+x+a) & f+g(x+a)+y+az+b \\ 0 & 1 & g+(z+c) \\ 0 & 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \quad A, X \in g$

Let us consider matrix $B = \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$

$$A \cdot (X \cdot B) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \left(\underbrace{\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}} \right)$$

$$= \begin{bmatrix} 1 & (e+x) & f+gx+y \\ 0 & 1 & g+z \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (e+x) & f+gx+y \\ 0 & 1 & g+z \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (e+x)+a & (f+gx+y) + a(g+z) + b \\ 0 & 1 & g+z+c \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A \cdot X) \cdot B = \begin{bmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (e+x+a) & f+g(x+a)+y+az+b \\ 0 & 1 & g+(z+c) \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ g is associative

③ Neutral Element

$\exists\, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in g, \quad \forall A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in g$

then, $A \cdot E = A$ and $E \cdot A = A$ (?)

$$A \cdot E = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$E \cdot A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ E is neutral element

④ inverse Element.

$\exists Y \in g\ \forall X \in g$: $X \cdot Y = e$ and $X \cdot Y = e$ (e is neutral element)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \xRightarrow{\text{Numpy}} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x & c \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 10 & 20 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \xRightarrow{\text{Numpy}} \begin{bmatrix} 1 & -10 & -280 \\ 0 & 1 & -30 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & c \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & c - xz + y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore\ c = xz - y$$

$$\therefore\ Y = \begin{bmatrix} 1 & -x & xz-y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$$ inverse element exists

So, Group g : $(G, \cdot)$ is a group.

and it is not Abelian group. because $\forall A, X \in G$, $A \cdot X \neq X \cdot A$

2.4

ⓐ $\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ = we can't do matrix multiplication because of dimension

$3\times 2 \qquad 3\times 3$

ⓑ $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

ⓒ $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

ⓓ $$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2+2+10 & 3+(-2)+1+4 \\ 1+(-2)+(-20) & 12+(-1)+(-1)+(-8) \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ -21 & 2 \end{bmatrix}$$

ⓔ $$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

2.5.

ⓐ square matrix A, Unique solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \overset{②=②-③}{=} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 6 & -8 & -8 & -6 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 6 & -8 & -8 & -6 \\ 0 & -3 & 3 & 5 & 2 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \xleftarrow{③=③-2\times①}$$

$$\xRightarrow{④=④-5\times①} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 6 & -8 & -8 & -6 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right]$$

$$\Downarrow \; ④=④-③$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 6 & -8 & -8 & -6 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & 0 & -2 & 2 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -4 & -4 & -3 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 2 & -1 \end{array}\right] \xleftarrow[③=②+③]{②=②\times\frac{1}{2}}$$

$\Downarrow$

?

해가 존재하지 않음 ... Ans

ⓑ fat matrix, A, (식 개수 < 미지수 개수), infinite number of solutions

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \overset{②=②+④}{=} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & -5 & -1 & 5 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right]$$

$$\Downarrow \; ④=④+①$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & -5 & -1 & 5 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \xLeftarrow{③=③-2\times①} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & -5 & -1 & 5 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & -5 & -1 & 5 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \xRightarrow{④ = ④ - ③} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & -5 & -1 & 5 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right]$$

$$\Downarrow \quad ② = ② - 3 \times ③$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \xLeftarrow[④ = ④ \times \frac{1}{3}]{② = ② \times \frac{1}{8}} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -8 & 8 & 8 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right]$$

$$\Downarrow \quad ④ = ④ - ②$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow{① = ① + ③} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Downarrow \quad ② = ② + ③$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xLeftarrow{③ = ③ - ②} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & & & & & 0 \end{array}\right]$$

$$\Downarrow \quad ① = ① - ③$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{aligned} x_1 - x_5 &= 3 \\ x_2 - 2x_5 &= 0 \\ x_4 - x_5 &= -1 \end{aligned} \iff \begin{aligned} x_1 &= 3 + x_5 \\ x_2 &= 2x_5 \\ x_4 &= -1 + x_5 \end{aligned}$$

Ans) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \cancel{x_5}\,\alpha \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (\alpha \in \mathbb{R})$$

2.6

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \xRightarrow{③=③-①} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Downarrow$$

$$x_2 + x_5 = 2 \iff x_2 = 2 - x_5$$
$$x_4 + x_5 = -1 \iff x_4 = -1 - x_5$$
$$x_6 = 1 \iff x_6 = 1$$

ANS)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \cancel{x_5}\,\alpha \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad (\alpha \in \mathbb{R})$$

2.7

$\sum x_i = 1 \iff x_i =$ 가중치! $\iff Ax = 12x =$ 가중평균

$Ax = 12x$, $(A-12E)x = 0$, $N(A-12E)$를 찾아라!

$$A - 12E = \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix}$$

<REF>

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xRightarrow{②=②+①} \left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 0 & -8 & 12 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xRightarrow{③=③+②} \left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 0 & -8 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Downarrow \quad ①=①\times -\tfrac{1}{6},\ ②=②\times -\tfrac{1}{8}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{4}{6} & -\frac{3}{6} & 0 \\ 0 & 1 & -\frac{12}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xLeftarrow{①=①+②\times\frac{6}{4}} \ \ldots$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{4} & 0 \\ 0 & 1 & -\frac{12}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + \tfrac{7}{4}x_3 = 0, \quad x_2 - \tfrac{12}{8}x_3 = 0$$
$$x_1 = -\tfrac{7}{4}x_3, \quad x_2 = \tfrac{12}{8}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{x_3}\,\alpha \begin{bmatrix} -\frac{7}{4} \\ \frac{12}{8} \\ 1 \end{bmatrix} \qquad (\alpha \in \mathbb{R})$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{x_3}\,\alpha \begin{bmatrix} -\frac{7}{4} \\ \frac{12}{8} \\ 1 \end{bmatrix} \qquad \begin{aligned} x_1 + \tfrac{7}{4}x_3 &= 0 \\ x_2 - \tfrac{12}{8}x_3 &= 0 \\ x_1 &= -\tfrac{7}{4}x_3 \\ x_2 &= \tfrac{12}{8}x_3 \end{aligned}$$

$(\alpha \in \mathbb{R})$

$$\alpha\left(-\frac{7}{4} + \frac{12}{8} + 1\right) = 1 = \alpha\left(\frac{8-14+12}{8}\right) = \alpha\,\frac{6}{8} = \frac{3}{4}\alpha$$

$$\therefore \alpha = \frac{4}{3}$$

$$\therefore (x_1, x_2, x_3) = \left(\frac{4}{3} \times -\frac{7}{4},\ \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{12}}{\cancel{8}},\ \frac{4}{3} \times 1\right)$$

$$(x_1, x_2, x_3) = \left(-\frac{7}{3},\ 2,\ \frac{4}{3}\right)$$ Ans)

2.8

a. A matrix is not inversible
($\because$ $C_1 + C_3 = 2C_2$, Linear dependent)

b.

$$\begin{bmatrix} 1&0&1&0\\0&1&1&0\\1&1&0&1\\1&1&1&0 \end{bmatrix} \xrightarrow{④=④-③} \begin{bmatrix} 1&0&1&0\\0&1&1&0\\1&1&0&1\\0&0&1&-1 \end{bmatrix} \xrightarrow{③=③-①} \begin{bmatrix} 1&0&1&0\\0&1&1&0\\0&1&-1&1\\0&0&1&-1 \end{bmatrix}$$

$$\xrightarrow{③=③-②} \begin{bmatrix} 1&0&1&0\\0&1&1&0\\0&0&-2&1\\0&0&1&-1 \end{bmatrix} \xrightarrow{③=③\times-\frac{1}{2}} \begin{bmatrix} 1&0&1&0\\0&1&1&0\\0&0&1&-\frac{1}{2}\\0&0&1&-1 \end{bmatrix} \xrightarrow{④=④-③} \begin{bmatrix} 1&0&1&0\\0&1&1&0\\0&0&1&-\frac{1}{2}\\0&0&0&-\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{②=②-③} \begin{bmatrix} 1&0&1&0\\0&1&0&\frac{1}{2}\\0&0&1&-\frac{1}{2}\\0&0&0&-\frac{1}{2} \end{bmatrix} \xrightarrow{②=②+④} \begin{bmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&-\frac{1}{2}\\0&0&0&-\frac{1}{2} \end{bmatrix} \xrightarrow{①=①-③} \begin{bmatrix} 1&0&0&\frac{1}{2}\\0&1&0&0\\0&0&1&-\frac{1}{2}\\0&0&0&-\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{①=①+④,\ ③=③-④} \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-\frac{1}{2} \end{bmatrix} \xrightarrow{④=④\times(-2)} \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0&-1&0&1\\-1&0&0&1\\1&1&0&-1\\1&1&1&-2 \end{bmatrix}$$ inversible

2.9. Which sets are subspaces of $\mathbb{R}^3$ among 'a ~ d' ?

Let us consider $V = (V, +, \cdot)$ = Vectorspace. and $S \subseteq V$ and $S \neq \varnothing$.

then we can say $S = (S, +, \cdot)$ is subspace of V. $(S \ni \varnothing)$

* · Contain zero vector
- closed under 'linear combination operation'

ⓐ

- $A \ni \varnothing$
- closure (inner production, +)
- closure (outer production, ·)

$\alpha(a, a+b^3, a-b^3) + \beta(c, c+d^3, c-d^3) \in \mathbb{R}^3$

ⓑ

- $B \ni \varnothing$
- closure (inner production, +)
- closure (outer ″ , ·)

$\alpha(a^2, -a^2, 0) + \beta(b^2, -b^2, 0)$

$= (\alpha a^2 + \beta b^2, -(\alpha a^2 + \beta b^2), 0) \in \mathbb{R}^3$

~~ⓒ~~

- $C \not\ni \varnothing$ (X) 만약, $\gamma \neq 0$ 이라면, (0,0,0)은 C에 포함되지 않는다. (zero-vector가 없다면, 부분공간으로 정의할 수 없다)
- closure (inner production, +)
- closure (outer ″ , ·)

ⓓ

- $D \ni \varnothing$
- closure (inner production, +)
- closure (outer ″ , ·)

$\alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \in \mathbb{R}^3$

2.10. Determine linear independence.

ⓐ $X = [\,x_1 \;\; x_2 \;\; x_3\,] = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad C_1 \;\; C_2 \;\; C_3$

< Row Echelon Form of X >

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \xRightarrow{③=③+②\times3} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \xRightarrow{①=①+②} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Downarrow ②=①+②$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xLeftarrow{①=①-③\times2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xLeftarrow{②=②-③\times3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$C_1 = \frac{1}{2}(C_2 + C_3)$ $\therefore$ Vectors of 'a' is linearly dependent.

ⓑ $X = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ skinny matrix. $\Rightarrow$ 식의 갯수 > 미지수 갯수

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xRightarrow{④=④-③} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{①=①-④} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow ③=③-①$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xLeftarrow{④=④-②} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xLeftarrow{②=②-①\times2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2.11

$$X = [x_1 \;\; x_2 \;\; x_3]$$

$$y = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

$$Xw = y, \quad \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x_1 w_1 + x_2 w_2 + x_3 w_3 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

<REF>

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \xRightarrow{③=③-②} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & 7 \end{array}\right] \xRightarrow{①=①-③} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

$$\Downarrow \; ②=②-③\times 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & -3 & -3 \\ 0 & 1 & 2 & 7 \end{array}\right] \xLeftarrow{②=②+③} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 0 & -5 & -10 \\ 0 & 1 & 2 & 7 \end{array}\right] \xLeftarrow{②=②-①} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 1 & 0 & -5 & -16 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

$$\Downarrow \; ③=③-②$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & \cancel{5}\,1 & \cancel{10}\,2 \end{array}\right] \xRightarrow{②=②+③\times 3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\therefore \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$$

2.12

basis of U_1

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xRightarrow[\textcircled{4}=\textcircled{4}-\textcircled{1}]{\textcircled{2}=\textcircled{2}-\textcircled{1}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ -3 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xRightarrow{\textcircled{3}=\textcircled{3}+\textcircled{1}\times 3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Big\Downarrow \ \textcircled{3}=\textcircled{3}+\textcircled{2}\times 2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xLeftarrow{\textcircled{1}=\textcircled{1}+\textcircled{2}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xLeftarrow{\textcircled{2}=\textcircled{2}+\textcircled{4}\times 2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xLeftarrow{\textcircled{2}=\textcircled{2}-\textcircled{4}\times 3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Big\Downarrow \ \textcircled{2}=\textcircled{2}+\textcircled{4}\times(-2)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xRightarrow{\textcircled{2}=\textcircled{2}+-1\times\textcircled{4}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xRightarrow{\textcircled{4}=\textcircled{4}+\textcircled{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

pivot column ⇒ linear independent, "basis"

basis of U_2

$$\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \xRightarrow{\textcircled{3}=\textcircled{3}-\textcircled{4}\times 2} \begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\Big\Downarrow \ \textcircled{2}=\textcircled{2}+\textcircled{1}\times -2$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -4 \end{bmatrix} \xLeftarrow{\textcircled{2}=\textcircled{2}+\textcircled{4}\times 3} \begin{bmatrix} -1 & 2 & -3 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \\ 0 & 2 & -4 \end{bmatrix} \xLeftarrow{\textcircled{4}=\textcircled{4}+\textcircled{1}} \begin{bmatrix} -1 & 2 & -3 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

pivot variable...

basis of $U_1 \cap U_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

2.13, $U_1 = N(A_1)$, $U_2 = N(A_2)$

ⓐ $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \xRightarrow{④=④-①} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{③=③-②\times 2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$\Downarrow$ ②=②−①

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xLeftarrow{③=③+②\times\frac{5}{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

ⓐ $U_1 = N(A_1) =$ span of $\left(\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right)$

ⓐ $\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \xRightarrow{④=④-①} \begin{bmatrix} \cancel{3}\,1 & \cancel{-3}\,-1 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xRightarrow[③=③-①\times 7]{②=②-①} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$\Downarrow$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xLeftarrow{①=①+②} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xLeftarrow[④=④-②]{③=③-②} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

ⓐ $\mathrm{Dim}(U_1) = \mathrm{Dim}(U_2) = 1$

ⓑ $U_1 = N(A_1) =$ span of $\left(\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right)$

$U_2 = N(A_2) =$ span of $\left(\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right)$

ⓒ $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

2.14,, U_1, U_2 is $C(A)$.

ㄴ, 각 column의 독립성이 중요!

ⓐ $N(A_1) = U_1 = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}$ span

$N(A_2) = U_2 = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}$ span

ⓐ : $\mathrm{Dim}(U_1) = \mathrm{Dim}(U_2) = 2$

ⓑ bases of U_1 and U_2 $= \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}$

ⓒ basis of $U_1 \cap U_2$ $= \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}$

2.15,,

ⓐ to be a Subspace,
Subspaces $f(\subseteq F)$, $g(\subseteq G)$ should be closed under linear Combination operation
and contain zero-vector

For f,

$\alpha(x+a) + \beta(y+b) + r(-z-c) \in R$,

For g,

$\alpha((x-y)+(a-b)) + \beta((x+y)+(a+b)) + r((x-3y)+(a-3b)) \in R$

both f and g Contain zero-vector.
So, f and g are Subspaces.

ⓑ ?

ⓒ ?

2.16

To be linear for mapping...

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V \ \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y}) \,. \qquad (2.87)$$

then, Φ is linear mapping...

ⓐ $a, b \in \mathbb{R}$ $\quad L^1([a,b]) \longrightarrow \mathbb{R}$

$\Phi \Rightarrow f(x)$의 정적분 mapping...

$\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ (?)

$\alpha \int_a^b f\,dx + \beta \int_a^b g\,dx$

$\int_a^b (\alpha f + \beta g)\,dx = \int_a^b \alpha f\,dx + \int_a^b \beta g\,dx$

$= \alpha \int_a^b f\,dx + \beta \int_a^b g\,dx$

$\therefore$ Φ is linear mapping

ⓑ $\Phi \Rightarrow$ 미분!

$\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ (?)

$\alpha \times f' + \beta \times g'$

$(\alpha f + \beta g)' = (\alpha f)' + (\beta g)' = \alpha f' + \beta g'$

$\therefore$ Φ is linear mapping

ⓒ $\Phi : R \to R$,

↳ cos 함수 ⟹ not linear mapping

ⓓ $\Phi : R^3 \to R^2$

Φ matrix $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}$

Matrix represents linear mapping.

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

↳ $\Phi(\alpha x + \beta y) = \alpha \Phi(x) + \beta \Phi(y)$ (?)

넣어보면 동일!

ⓔ ? . . . .

2.17

* kernel = null space

dim(kernel) = free-variable 의 수

image = Column space

dim(image) = pivot-variable 의 수

ⓐ Transformation matrix $A_\Phi = T$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\therefore T = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

ⓐ Ans)

ⓑ $rk(A_\Phi)$?

* rank = the number of linear independent vector.

[REF of T]

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[③=③-②]{①=①-④} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{④=④-②\times2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{②=②-①} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Downarrow ③=③+②$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow[①=①+②]{④=④+②} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & \cancel{3}\,1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xleftarrow{①=①+④} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xleftarrow{②=②-④\times2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

pivot

* C(T) = image

$$= \left\{ \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \right\}$$

* N(T) = kernel

$= \{\emptyset\}$ $\because$ $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

linear independent

$\therefore$ ⓑ Ans) $rank(\Phi) = 3$

ⓒ Ans) $\dim(ker(\Phi)) = 0$ (zero-vector)

$\dim(image(\Phi)) = 3$

2.18

$*$ automorphism : $V \rightarrow V$, linear and bijective

- $f : E \rightarrow E$, linear, bijective
- $g : E \rightarrow E$, 〃
- $f \circ g = id_E$
- $\ker(f) \Longleftrightarrow$ f mapping 의 null space.

$\ker(g \circ f) \Longleftrightarrow$ $g \circ f$ mapping 의 null space

–. . . ?

2.19

ⓐ $T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

- $\ker(\Phi) = \{\emptyset\}$
- $Im(\Phi) = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

[REF]

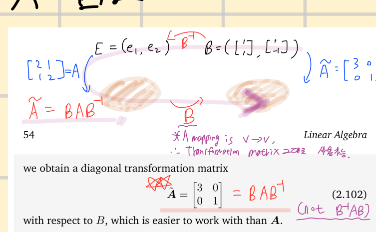
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xRightarrow{③ = ③ - ①} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xRightarrow{② = ② - ①} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \cancel{-2}\ 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xRightarrow{① = ① - ②} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ⓑ basis change

$$\begin{tikzcd} E \arrow[loop left, "A_\Phi"] & B \arrow[l, bend right, "B^{-1}"'] \arrow[loop right, "\tilde{A}_\Phi"] \end{tikzcd}$$

$*$ endomorphism : $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, linear

$*$ 참고


$E = (e_1, e_2)$ B^{-1} $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$ $\tilde{A} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
$\tilde{A} = BAB^{-1}$ B
54 *Linear Algebra*
$*$ A mapping is $V \rightarrow V$,
∴ Transformation matrix 그대로 사용하는.

we obtain a diagonal transformation matrix

$$\tilde{A} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = BAB^{-1} \quad (2.102)$$

(not $B^{-1}AB$)

with respect to B, which is easier to work with than A.

$\tilde{A}_\Phi = BA_\Phi B^{-1}$,

$$\tilde{A}_\Phi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -4 & 7 \\ 0 & -2 & 4 \end{bmatrix} = \tilde{A}_\Phi$$

2.20

$$B = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right), \quad B' = \left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

R²의 bases   R²의 bases

ⓐ 1) B is bases of R^2

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \xRightarrow{②=②-①\times\frac{1}{2}} \begin{bmatrix} 2 & -1 \\ 0 & \cancel{\frac{3}{2}}\ 1 \end{bmatrix} \xRightarrow{①=①+②} \begin{bmatrix} \cancel{2}\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

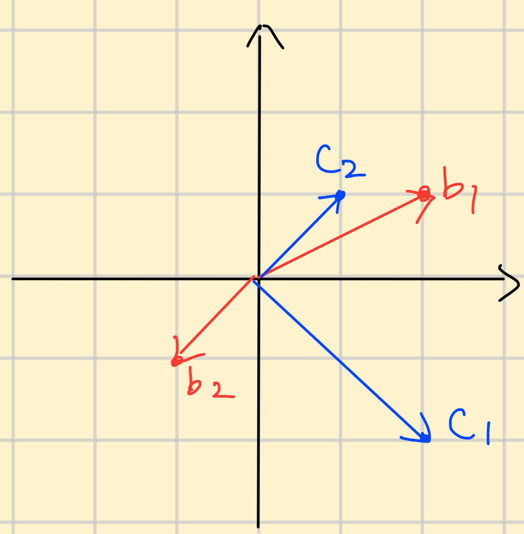
⇒ b_1, b_2는 서로 linear independent 하기 때문에, R^2의 basis vector가 될 수 있다.

2) B' is bases of R^2

$$B' = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \xRightarrow{②=②+①} \begin{bmatrix} 2 & 1 \\ 0 & \cancel{2}\ 1 \end{bmatrix} \xRightarrow{①=①-②} \begin{bmatrix} \cancel{2}\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⇒ c_1, c_2는 b_1, b_2와 마찬가지로 linear independent해서 basis vector가 될 수 있다.

3) draw



ⓑ b. Compute the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.

$$\begin{tikzcd} B \arrow[d, "A_\Phi"'] & B' \arrow[l, bend right, "P_1"'] \arrow[d, "A'"] & R^2 \\ C \arrow[r, "P_2"] & C' & R^3 \end{tikzcd}$$

$C' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = PB'$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = P_1 \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{4} \begin{bmatrix} 0 & -4 \\ 3 & 1 \end{bmatrix}$$

ⓒ if determinants $\neq 0$,
then matrix is inversible and column vectors are linearly independent each other. full rank matrix.

(i) $C = [c_1, c_2, c_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

Determinant of C = $4 \neq 0$

So, c_1, c_2, c_3 are linearly independent.
we can say C is basis of R^3.

(ii)

$C' = P_2 C$, $C' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_2 = C' C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C^{-1} = C^{-1}$

$P_2 = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}$

ⓓ $A_\Phi = ?$

$$\begin{tikzcd} B_{2\times2} \arrow[d, "A_\Phi"'] & B' \arrow[l, bend right, "P_1"'] \arrow[d, "A'"] & R^2 \\ C_{3\times3} \arrow[r, "P_2"'] & C' & R^3 \end{tikzcd}$$

$x = [b_1 \ b_2]\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_b} = [c_1 \ c_2 \ c_3]\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{x_c}$

$y = [b_1 \ b_2]\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x_b} = [c_1 \ c_2 \ c_3]\underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{x_c}$

$A_\Phi = \frac{1}{3}\begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}$

*풀이과정은 뒷페이지

[A_Φ 구하는 과정]

$x = [e_1 \ e_2] x_e = [g_1 \ g_2] x_g$

$* \ [g_1 \ g_2] = [e_1 \ e_2] A$

$x = \cancel{[e_1 \ e_2]} x_e = \cancel{[e_1 \ e_2]} A x_g$

$x_e = A x_g$

$x_g = A^{-1} x_e$, A^{-1} = 변환행렬

$$\underset{3\times 2}{A} \ \overset{2\times 2}{[b_1 \ b_2]} \overset{2\times 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \overset{3\times 3}{[c_1 \ c_2 \ c_3]} \overset{3\times 1}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$A \ [b_1 \ b_2] \overset{2\times 1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = [c_1 \ c_2 \ c_3] \overset{3\times 1}{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}$$

$\Downarrow$

$$A \ [b_1 \ b_2] \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underset{3\times 3}{[c_1 \ c_2 \ c_3]} \underset{3\times 2}{\begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}}$$

$$A \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = [c_1 \ c_2 \ c_3] \overset{3\times 2}{\begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 5 \\ 1 & -7 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 5 \\ 1 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 \\ -1 & 5 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}$$

ⓔ $A'_{\Phi} = ?$

$$\begin{tikzcd} B \arrow[d, "A_{\Phi}"'] & B' \arrow[l, bend right, "P_1"'] \arrow[d, "A'"] & R^2 \\ C \arrow[r, bend right, "P_2"'] & C' & R^3 \end{tikzcd}$$

$\cdot \; A' = P_2 A P_1$

$$A' = \left(\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}\right)\left(\frac{1}{4}\begin{bmatrix} 0 & -4 \\ 3 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{48}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 0 & -4 \end{bmatrix}$$

ⓕ $B' = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$, $x = 2\begin{bmatrix} 2 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$* \; x_{B'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_{b_1'} \\ x_{b_2'} \end{bmatrix}$

(i) $[b_1' \; b_2']\begin{bmatrix} 2 \\ 3 \end{bmatrix} = [b_1 \; b_2]\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \text{the Coordinates of } x \text{ in } B$$

(ii) $\underset{3\times 2}{A}\,\underset{2\times 2}{[b_1 \; b_2]}\,\underset{2\times 1}{\begin{bmatrix} 2 \\ -3 \end{bmatrix}} = \underset{3\times 3}{[c_1 \; c_2 \; c_3]}\,\underset{3\times 1}{\begin{bmatrix} x_{c_1} \\ x_{c_2} \\ x_{c_3} \end{bmatrix}}$

$$[c_1 \; c_2 \; c_3]^{-1}\,\frac{1}{3}\begin{bmatrix} 5 & -1 \\ 5 & -4 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x_{c_1} \\ x_{c_2} \\ x_{c_3} \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}^{-1}$$

(iii)

$$\underset{3\times 2}{A}\,\underset{2\times 2}{[b_1\ b_2]}\,\underset{2\times 1}{\begin{bmatrix}2\\-3\end{bmatrix}} = \underset{3\times 3}{[c_1\ c_2\ c_3]}\,\underset{3\times 1}{\begin{bmatrix}x_{c_1}\\x_{c_2}\\x_{c_3}\end{bmatrix}} \iff ABx_b = Cx_c$$

$$\Downarrow$$

$$\underset{3\times 2}{A}\,\underset{2\times 2}{[b_1\ b_2]}\,\underset{2\times 1}{\begin{bmatrix}2\\-3\end{bmatrix}} = \underset{3\times 3}{[\overset{c_1'}{\cancel{c_1}}\ \overset{c_2'}{\cancel{c_2}}\ \overset{c_3'}{\cancel{c_3}}]}\,\underset{3\times 1}{\begin{bmatrix}x_{c_1'}\\x_{c_2'}\\x_{c_3'}\end{bmatrix}} \iff ABx_b = C'x_{c'}$$

$$\Downarrow$$

$$[c_1'\ c_2'\ c_3']^{-1}\,\frac{1}{3}\begin{bmatrix}5 & -1\\5 & -4\\-7 & 5\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}x_{c_1'}\\x_{c_2'}\\x_{c_3'}\end{bmatrix} = \begin{bmatrix}12\\13\\-18\end{bmatrix}$$

(iv)

$$x = [b_1'\ b_2']\begin{bmatrix}x_{b_1'}\\x_{b_2'}\end{bmatrix} = B'x_{b'}$$

$$\begin{tikzcd}
B \arrow[d, "A"'] & B' \arrow[l, bend right, "P_1"'] \arrow[d, "A'"] & R^2 \\
C \arrow[r, bend right, "P_2"'] & C' & R^3
\end{tikzcd}$$

$$A\underbrace{Bx_b}_{x} = C'x_{c'} \quad \text{(iii의 식)}$$

$$\Rightarrow\ A'B'x_{b'} = C'x_{c'}$$