Homework 1

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September 5, 2018

1 Problem 1

Prove by mathematical induction that

$$\sum_{i=1}^{p} i^3 = \frac{1}{4}p^2(p+1)^2 \tag{1.1}$$

We will be using induction on p

1.1 Base Case

Proving holds for $p = 1$

$$\sum_{i=1}^{1} i^3 = 1 \tag{1.2}$$

$$\frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4}(2)^2 = \frac{4}{4} = 1 \tag{1.3}$$

1.2 Induction Hypothesis

We will be using weak induction. Assume that 1.1 holds for $p = n$, thus

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2 \tag{1.4}$$

1.3 Induction Step

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 && \text{By Induction Hypothesis} \\
&= (n+1)^2(\frac{1}{4}n^2 + n + 1)
\end{aligned}$$

Which is the same as showing that

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \frac{1}{4}(n+1)^2(n+2)^2 \\
&= (n+1)^2\frac{1}{4}(n^2+4n+4) \\
&= (n+1)^2(\frac{1}{4}n^2+n+1))
\end{aligned} \tag{1.5}$$

1.4 Conclusion

Since 1.1 holds true for the base case, and 1.1 is also true for $p = n$, it follows that (1.1) is true for all $p \geq 1$

2 Problem 2

Consider the sequence of real numbers defined by the relations

$$R_1 = 1 \tag{2.1}$$

$$R_x = \sqrt{1 + 2R_{x-1}} \qquad \text{for all } x > 1$$

Prove by mathematical induction that for $R_x < 4$ for all $x \geq 1$

2.1 Base Case

$$R_1 = 1 < 4 \tag{2.2}$$

Thus the base case holds and we can continue using weak induction

2.2 Induction Hypothesis

Assume that $R_x < 4$ holds for $x = n$, therefore

$$R_n < 4 \tag{2.3}$$

2.3 Induction Step

$$\begin{aligned} R_{n+1} &= \sqrt{1+2R_n} \\ &< \sqrt{1+2(4)} && \text{By Induction Hypothesis} \\ &< \sqrt{9} \\ &< 3 \end{aligned} \tag{2.4}$$

Thus

$$R_{n+1} < 3 < 4 \tag{2.5}$$

2.4 Conclusion

Since $R_x < 4$ holds for the base case of $x = 1$ and we have proven that $R_n < 4$ for all $x = n$, it follows that $R_x < 4$ is true for all $x > 1$

3 Problem 3

Consider the following definition of a non-empty binary tree:

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;; a Non-Empty Binary Tree [NEBT] is either
;; - a leaf l, with no internal structure or children, or
;; - (make-node left[NEBT] right[NEBT]) an internal node with two children,
;; each of which is a NEBT.
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Now consider this NEBT as a graph: each leaf or internal node is a VERTEX in V, and every internal node has two unique EDGES in E linking the internal node to two other NEBTs. Prove that

$$|V| = |E| + 1 \tag{3.1}$$

3.1 Base Case

A leaf l has no internal structure or children, so thus it has one vertex and no edges. Therefore $1 = 0 + 1$. The base case is true so we can continue.

3.2 Induction Hypothesis

Let us take three NEBTs, denoted n, l, r. Assume that 3.1 holds for these three NEBTs, therefore

$$|V_n| = |E_n| + 1 \tag{3.2}$$

$$|V_l| = |E_l| + 1 \tag{3.3}$$

$$|V_r| = |E_r| + 1 \tag{3.4}$$

3.3 Induction Step

Let us take NEBT n and create a new NEBT by calling (make-node r l) on an external node of n. This new NEBT created shall be denoted m. We would like to show that

$$|V_m| = |E_m| + 1 \tag{3.5}$$

We know that m will hold the following properties due to the nature of binary trees

$$|V_m| = |V_n| + |V_r| + |V_l| \tag{3.6}$$

$$|E_m| = |E_n| + |E_r| + |E_l| + 2 \tag{3.7}$$

We can now continue with the induction

$$\begin{aligned}
|V_m| &= |V_n| + |V_r| + |V_l| && (3.8)\\
&= (|E_n| + 1) + (|E_r| + 1) + (|E_l| + 1) && \text{By Induction Hypothesis}\\
&= (|E_n| + |E_r| + |E_l| + 2) + 1 \\
&= |E_m| + 1 && \text{By 3.7}\\
& && (3.9)
\end{aligned}$$

3.4 Conclusion

Since 3.1 holds for the base case, and 3.1 also holds for any arbitrary structure, we've proven we can construct any other structure where 3.1 holds.

4 Problem 4

Consider the alphabet $\sum = \{0, 1\}$ and the following definition of the set S of strings:
$1 \in S$
$100 \in S$
if $s \in S$ then $11s \in S$ (denoted $A(s)$)
if $s \in S$ then $00s \in S$ (denoted $B(s)$)

Prove by structural induction that every string $s \in S$ has an odd number of 1's and an even number of 0's

4.1 Base Case

Two base cases:
1 has an odd number of 1's and an even number of 0's, so thus the proposition holds.
100 has an odd number of 1's and an even number of 0's, so thus the proposition holds.
We can now continue with the induction.

4.2 Induction Hypothesis

Assume that ω is a string that has an odd number of 1's and an even number of 0's, and that $\omega \in S$

4.3 Induction Step

Case 1:
Using the first transformation function, $A(\omega) = 11\omega$. Since ω had an odd number of 1's to begin with (by the induction hypothesis), adding two more 1's still result in an odd number of 1's. Also since ω had an even 0's to begin with (by the induction hypothesis), no more 0's are added so the number of 0's is still even. Thus the proposition holds that $A(\omega) \in S$ and it has an odd number of 1's and an even number of 0's

Case 2:
Using the second transformation function, $B(\omega) = 00\omega$. Since ω had an odd number of 1's to begin with (by the induction hypothesis), adding no 1's still result in an odd number of 1's. Also since ω had an even 0's to begin with (by the induction hypothesis), two more 0's are added so thus number of 0's is still even. Thus the proposition holds that $B(\omega) \in S$ and it has an odd number of 1's and an even number of 0's

4.4 Conclusion

Since we have shown that given an arbitrary string $s \in S$, we can construct a new string that is also in S, and we have proven the base cases. We now know that any string constructed using the rules will have an oddn umber of 1's and an even number of 0's.