Homework 1

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1 Problem 1

Prove by mathematical induction that

$$\sum_{i=1}^{p} i^3 = \frac{1}{4}p^2(p+1)^2 \tag{1.1}$$

We will be using induction on p

1.1 Base Case

Proving holds for $p = 1$

$$\sum_{i=1}^{1} i^3 = 1 \tag{1.2}$$

$$\frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4}(2)^2 = \frac{4}{4} = 1 \tag{1.3}$$

1.2 Induction Hypothesis

We will be using weak induction. Assume that 1.1 holds for $p = n$, thus

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2 \tag{1.4}$$

1.3 Induction Step

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 && \text{By Induction Hypothesis} \\
&= (n+1)^2(\frac{1}{4}n^2 + n + 1)
\end{aligned}$$

Which is the same as showing that

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \frac{1}{4}(n+1)^2(n+2)^2 \\
&= (n+1)^2\frac{1}{4}(n^2+4n+4) \\
&= (n+1)^2(\frac{1}{4}n^2+n+1))
\end{aligned} \tag{1.5}$$

1.4 Conclusion

Since 1.1 holds true for the base case, and 1.1 is also true for $p = n$, it follows that (1.1) is true for all $p \geq 1$

2 Problem 2

Consider the sequence of real numbers defined by the relations

$$R_1 = 1 \tag{2.1}$$

$$R_x = \sqrt{1 + 2R_{x-1}} \qquad \text{for all } x > 1$$

Prove by mathematical induction that for $R_x < 4$ for all $x \geq 1$

2.1 Base Case

$$R_1 = 1 < 4 \tag{2.2}$$

Thus the base case holds and we can continue using weak induction

2.2 Induction Hypothesis

Assume that $R_x < 4$ holds for $x = n$, therefore

$$R_n < 4 \tag{2.3}$$

2.3 Induction Step

$$
\begin{aligned}
R_{n+1} =& \sqrt{1+2R_n} \\
<& \sqrt{1+2(4)} && \text{By Induction Hypothesis} \\
<& \sqrt{9} \\
<& 3
\end{aligned}
\tag{2.4}
$$

Thus

$$R_{n+1} < 3 < 4 \tag{2.5}$$

2.4 Conclusion

Since $R_x < 4$ holds for the base case of $x = 1$ and we have proven that $R_n < 4$ for all $x = n$, it follows that $R_x < 4$ is true for all $x > 1$

3 Problem 3

Consider the following definition of a non-empty binary tree:

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;; a Non-Empty Binary Tree [NEBT] is either
;; - a leaf l, with no internal structure or children, or
;; - (make-node left[NEBT] right[NEBT]) an internal node with two children,
;; each of which is a NEBT.
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Now consider this NEBT as a graph: each leaf or internal node is a VERTEX in V, and every internal node has two unique EDGES in E linking the internal node to two other NEBTs. Prove that

$$\mid V \mid =\mid E \mid +1 \tag{3.1}$$

3.1 Base Case

A leaf l has no internal structure or children, so thus it has one vertex and no edges. Therefore $1 = 0 + 1$. The base case is true so we can continue.

3.2 Induction Hypothesis

Let us take three NEBTs, denoted n, l, r. Assume that 3.1 holds for these three NEBTs, therefore

$$\mid V_n \mid =\mid E_n \mid +1 \tag{3.2}$$

$$\mid V_l \mid =\mid E_l \mid +1 \tag{3.3}$$

$$\mid V_r \mid =\mid E_r \mid +1 \tag{3.4}$$

3.3 Induction Step

Let us take NEBT n and create a new NEBT by calling (make-node r l) on an external node of n. This new NEBT created shall be denoted m. We would like to show that

$$|V_m| = |E_m| + 1 \tag{3.5}$$

We know that m will hold the following properties due to the nature of binary trees

$$|V_m| = |V_n| + |V_r| + |V_l| \tag{3.6}$$

$$|E_m| = |E_n| + |E_r| + |E_l| + 2 \tag{3.7}$$

We can now continue with the induction

$$\begin{aligned}
|V_m| &= |V_n| + |V_r| + |V_l| && (3.8)\\
&= (|E_n| + 1) + (|E_r| + 1) + (|E_l| + 1) && \text{By Induction Hypothesis}\\
&= (|E_n| + |E_r| + |E_l| + 2) + 1 &&\\
&= |E_m| + 1 && \text{By 3.7}\\
& && (3.9)
\end{aligned}$$

3.4 Conclusion

Since 3.1 holds for the base case, and 3.1 also holds for any arbitrary structure, we've proven we can construct any other structure where 3.1 holds.

4 Problem 4

4.1 Base Case

4.2 Induction Hypothesis

4.3 Induction Step

4.4 Conclusion