Discrete Math – Example Problem Set 1

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✠ Sub-Experience Two: *Binomial Coefficients and Fibonacci Numbers*

Number of shortest paths using the *city block metric*. Starting at Center and Main, you decide to walk to X North and Y East. The distance is measured in blocks and so the distance from X_1 North and Y_1 East to X_2 North and Y_2 East is $|X_2 - X_1| + |Y_2 - Y_1|$.

SE 2.1. For yuks, determine π in the Logan metric. Remember that π is defined to be the ratio of the circumference to the diameter of a circle. And that a circle is the set of points equidistant from a given point (that distance is typically called the *radius* of the circle and the point is called its *center*.

Calculations:

In the Logan metric, a circle centered on the origin of a graph would be a square centered at the origin and rotated 45 degrees about the origin.

To calculate π in the Logan metric, we can consider a circle of radius, r, 1.
A circle of radius 1 would have a diameter of 2 units and a circumference of $2 + 2 + 2 + 2 = 8$ units.
Therefore π in the Logan metric would be:
$\pi = 8/2 = 4$.

SE 2.2. Count the number of shortest paths from Center and Main to X North and Y East. A more erudite way to pose this problem is as follows: count paths from $(0, 0)$ to (X, Y) using steps of the form $R : (x, y) \mapsto (x + 1, y)$ or $U : (x, y) \mapsto (x, y + 1)$. So paths trace rectilinear "curves" in the first quadrant of the Cartesian coordinate system that visit only integer-valued coordinates.

Finding the number of shortest paths

A shortest path from $(0, 0)$ to a point (x, y) is any path where the distance of the path equals $|x| + |y|$.
For any point (x, y) in the first quadrant of the Cartesian coordinate system, the number of shortest paths from the origin to the point is equal to the number of shortest paths from the point 1 unit to its left + the number of shortest paths from the point 1 unit below.

To more clearly illustrate this concept consider the recursive function $S_p(x, y)$ defined to be the number of shortest paths from the origin to the point (x, y).:

$$
\begin{aligned}
S_p(0,0) &= 1\\
S_p(0,y) &= 1\\
S_p(x,0) &= 1\\
\\
S_p(x,y) &= S_p(x-1,y) + S_p(x,y-1)
\end{aligned}
$$

In the graphic below, the value for S_p at each drawn coordinate is shown. If we ponder on the recurrence relation $S_p(x, y)$ we realize that it is identical to the recurrence relation of Pascal's Identity illustrated in the famous Pascal's triangle.

SE-2.2-Data.jpg

The figure above clearly illustrates that plotting the values of S_p yields Pascal's triangle. Since all points along either axis have 1 shortest path from the origin, this relationship is guaranteed to hold.

To simplify the counting of shortest paths, we can create a function using the relationship described above. We define $P(x, y)$ to be the number of shortest paths from $(0, 0)$ to (x, y).
$P(x, y) = \binom{x+y}{x}$.

SE 2.3. Please prove the following identity:

$$\text{For } n, r, s \in \mathbb{N},\ \sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}.$$

Claim:
For $n, r, s \in \mathbb{N}$; $\displaystyle\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}$.

Proof:
The right hand side of the identity counts the number of ways to create a set of n object from $r + s$ objects.

Now consider the left hand side of the identity:
Suppose we were to first choose which objects we will take from r. There are $\binom{r}{i}$ total ways to choose i objects from r objects.
Then we choose the remaining $n - i$ items from s. $\binom{s}{n-i}$.
Therefore, for some i, $\binom{r}{i}\binom{s}{n-i}$ is the number of ways to choose n objects from $r + s$ objects where i objects are guaranteed to be from r.
Evaluating this expression for every $i, 0 \leq i \leq n$ would count the total possible ways to choose n objects from $r + s$ with any number of those objects being from r.

Therefore, for $n, r, s \in \mathbb{N}$; $\displaystyle\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}$. ■

SE 2.4. Please prove the following identity in which F_k denotes the k^{th} Fibonacci Number:

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

Claim:
$\displaystyle F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$.

Proof: I will argue by way of induction.

If n=0, then the relationship is true:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{0}{2} \rfloor} \binom{0-i}{i} \\
&= \sum_{i=0}^{0} \binom{0-i}{i} \\
&= \binom{0}{0} \\
&= 1 \\
&= F_1.
\end{aligned}$$

I have established the basis for the induction. I will assume that the relationship holds for some $k \geq 0$; that is,
$F_{k+1} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i}$

To show that this relationship implies $F_{k+2} = \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i}$, consider $\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i}$:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} \\
&= \binom{k+1}{0} + \binom{k}{1} + \binom{k-1}{2} + \binom{k-2}{3} + \binom{k-3}{4} + \ldots + \underbrace{\binom{k+1-(\frac{k+1}{2}-1)}{\frac{k+1}{2}-1}}_{\binom{\frac{k+3}{2}}{\frac{k-1}{2}}} + \binom{\frac{k+1}{2}}{\frac{k+1}{2}} \\
&= \underbrace{\binom{k}{-1}}_{0} + \binom{k}{0} + \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-2}{1} + \binom{k-2}{2} + \ldots + \binom{\frac{k+1}{2}}{\frac{k-3}{2}} + \underbrace{\binom{\frac{k+1}{2}}{\frac{k+1}{2}}}_{1} + \underbrace{\binom{\frac{k-1}{2}}{\frac{k-1}{2}}}_{1} + \underbrace{\binom{\frac{k-1}{2}}{\frac{k+1}{2}}}_{0} \text{ By Pascal's Identity} \\
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left[\binom{k-i}{i}\right] + \binom{k-1}{0} + \binom{k-2}{1} + \ldots + 1 \\
&= F_{k+1} + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left[\binom{k-1-i}{i}\right] \\
&= F_{k+1} + F_k \\
&= F_{k+2}.
\end{aligned}$$

It has been shown that the relationship $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ is true for $n = 0$, and that the truthfulness of the relationship for $n =$ some $k \geq 0$, implies the truthfulness of the relationship for $n = k+1$.
Therefore, the relationship is true for all non-negative integers by the Principle of Mathematical Induction. ■

✠ Sub-Experience Four: *Game Time!*

SE 4.1: Modified Towers of Hanoi. The *Towers of Hanoi* are mythical diamond needles, three of them, with 64 gold discs impaled on one of the needles. God told the monks of Brahma to transfer all the discs to another needle with the constraints that no larger disc ever be placed atop a smaller disc, and only one disc at a time is to be moved. Once the monks finish moving the discs the World will end. Evidently the monks have not yet finished.

Label the needles of the Towers of Hanoi L, M, and R, for the left, middle, and right, respectively. Consider the original Towers of Hanoi game, but with the additional constraint that a disk can only be moved to an adjacent peg; that is, a disk can only be moved to M from L or R, and can only be moved to L or R from M. Assume all discs are initially on L.

Determine the minimum number of moves (where a move *is defined to be the transfer of one disc from one needle to another) required transfer* n *discs from* L *to* R*.*

Solution:

First: The recursive formula: $H_n = 3H_{n-1} + 2$

Proof:

Consider the movement of n discs from L to R in the modified towers of Hanoi game:

1. To move n discs from the left tower to the right tower, we first must move $n-1$ discs from the left tower to the right tower, requiring H_{n-1} moves, making our current total move count: H_{n-1}.
2. We then must move the nth disc from the left tower to the middle tower, adding 1 move to the total count, now $H_{n-1} + 1$.
3. A large disc cannot be placed on top of a smaller disc, so in order to move the nth disc to the right tower, we must first move the $n-1$ discs on the right tower to the left tower. This is accomplished by inversing the moves required to move $n-1$ discs from L to R and then doing them in the reverse order. This adds H_{n-1} more moves to our current total of moves, now $2H_{n-1} + 1$.
4. Now we move the nth disc from M to R, increasing the total of moves by 1, now $2H_{n-1} + 2$.
5. We finish by moving the $n-1$ discs from L to R, requiring H_{n-1} moves. The total move count is now $3H_{n-1}+2$.

Therefore, the total moves necessary to move n discs from L to R, H_n, is equal to $3H_{n-1} + 2$.

Now the Explicit Formula:

The explicit (closed) formula is easily obtained by iterating the recursion and solving the resulting summation:

$$
\begin{aligned}
H_n &= 3H_{n-1} + 2 \\
&= 3(3H_{n-2} + 2) + 2 = 9H_{n-2} + 6 + 2 \\
&= 9(3H_{n-3} + 2) + 6 + 2 = 27H_{n-3} + 18 + 6 + 2 \\
&= 27(3H_{n-4} + 2) + 18 + 6 + 2 = 81H_{n-4} + 54 + 18 + 6 + 2 \\
&. \\
&. \\
&. \\
&= 3^n \cdot H_0 + 2 \cdot \sum_{k=1}^{n} 3^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
S_j = \sum_{k=1}^{j} 3^{k-1} &= 3^0 + 3^1 + 3^2 + ... + 3^{j-1} \\
3 \cdot S_j &= 3 + 3^{1+1} + 3^{2+1} + .. + 3^j \\
&= S_j + (3^j - 1) \\
3 \cdot S_j &= S_j + 3^j - 1 \\
3S_j - S_j &= 3^j - 1 \\
2S_j &= 3^j - 1 \\
S_j &= \frac{3^j - 1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= 3^n(0) + 2 \cdot \sum_{k=1}^{n} 3^{k-1} \\
&= 2\left[\frac{3^n - 1}{2}\right] \\
&= 3^n - 1.
\end{aligned}
$$

Therefore, the minimum number of moves required to transfer n discs from L to R, is equal to $3^n - 1$.

How Long? Assume the monks of Brahma were given this game with 64 discs, and can move a disc at a rate of one per second. How long in centuries will it take for the monks to complete their task?
$(3^64 - 1)/3.1556953 \cdot 10^9 = 108809108696318721930\text{3centuries}.$

Sources:

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