Discrete Math Examples – Problem Set 2

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✠ Sub-Experience Two: An Optimization Problem

Let H denote an arbitrary graph. Recall that the distance between vertices in H is the length of the shortest path that has those vertices as its endpoints. Denote by $d_{\max}(H)$ the maximum distance among all pairs of vertices of H. Recall that $\Delta(H)$ denotes the maximum degree among all vertices of H.

Define the function $N(d, k)$ to be the maximum number of vertices among all graphs H with $d_{\max}(H) = d$ and $\Delta(H) = k$.

One. *Determine* $N(n, 2)$.

Two. *Determine* $N(2, 3)$.

Three. *Verify that the graph* G *drawn to the right has* $d_{\max}(G) = 2$.

Four. *Determine* $N(2, 4)$.

BrownGraph.pdf

Potential 4-regular graph with diameter 2 on 15 vertices.

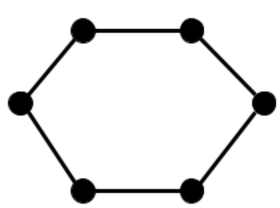
Sub-Experience 2 Answers:

1. **Claim:** $N(n, 2) = 2 \cdot n + 1$.

Proof:

Let G be a graph, where $\Delta(G) = 2$, and $d_{max}(G) = n$.
There are two possible structures for the graph G:

1. An $(n + 1)$-cycle. See figure below (example with 6 vertices).



2. A tree with two vertices of degree 1 and $n - 1$ vertices of degree 2. See figure below (example with 6 vertices).

Consider a cycle on q_c vertices. It clear that the maximum distance between any pair of vertices is $\lfloor \frac{q_c}{2} \rfloor$. Therefore,

$$\left\lfloor \frac{q_c}{2} \right\rfloor = d_{max}$$
$$\Longrightarrow$$
$$q_c = 2 \cdot d_{max}$$
$$\text{or}$$
$$q_c = 2 \cdot d_{max} + 1.$$

Consider a tree with q_t vertices (arranged linearly), where two vertices (v_s and v_e), have degree 1 and the other $q_t - 2$ vertices between them have degree 2. The maximum distance between any pair of vertices is $q_t - 1$ (the distance between v_s and v_e).
Therefore,

$$q_t - 1 = d_{max}$$
$$q_t = d_{max} + 1.$$

One can clearly see that the maximum value of q_t is less than the maximum value of q_c for any value of d_{max}, $2 \leq d_{max}$.

$$q_t < q_c$$
$$d_{max} + 1 < 2 \cdot d_{max} + 1$$

Therefore, the maximum number of vertices in a graph where no vertex has degree > 2, is equal to $2 \cdot$ (the maximum distance among all pairs of vertices in the graph) $+ 1$.
This proves that $N(n, 2) = 2 \cdot n + 1$). ■

2. Claim: $N(2,3) = 10$.

Proof:

To make this proof clearer to the reader, we begin by pointing out a few observations.

Observations:

- It is clear that every connected graph has at least one spanning tree.
- By the definition of a spanning tree, the number of vertices in a spanning tree T, of a graph H, is equal to the number of vertices in H. ($|V(T)| = |V(H)|$)

We will prove that $N(2,3) = 10$ by constructing a tree with N vertices, and from that tree, construct a graph with diameter of 2 and maximum degree of 3. (Graph diameter is defined in source 2.1, see citations)

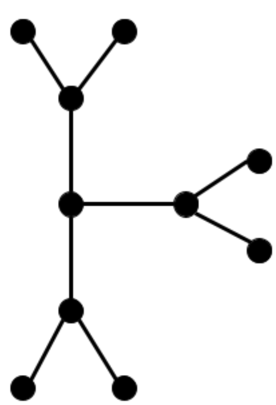
Let T_G be a spanning tree of a graph G, where $d_{max}(G) = 2$ (diameter of G is 2), $\Delta(G) = 3$, and $|V(G)| = N(2,3)$. Let T_G have diameter of 2 (this will be useful when using T_G to construct G).

We can attempt to construct the graph G from a tree T_G defined to have $d_{max}(T_G) = 2$.

First, we must construct a possible spanning tree for G (T_G). $|V(T_G)|$ must equal $N(2,3)$ (the maximum number of vertices in a graph with diameter of 2 and a maximum degree of 3). So we want T_G to have the maximum number of vertices possible.
We do so using the following steps:

1. Draw a vertex to start from.
2. Construct the rest of the tree, giving every vertex degree 3 ($\Delta(G)$) until it reaches distance 2 from the start vertex. (If you drew a vertex any more that distance 2 from the start it would violate the stipulation that T_G has $d_{max}(T_G) = 2$, making this construction technique not work).

Below is the tree constructed using this method:

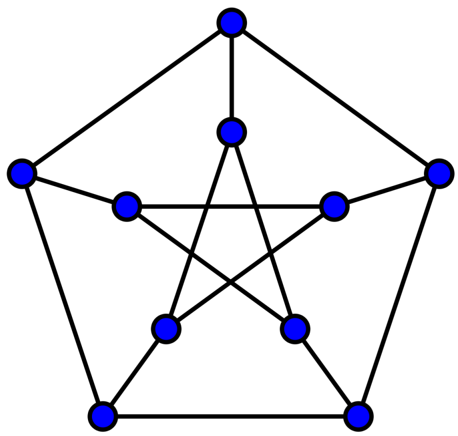


The tree above has 10 vertices. Effectively, we have just placed an upper bound on $N(2,3)$.
We can say that $N(2,3) \leq 10$.

To find $N(2,3)$, we will take the following steps:

1. Attempt to construct the graph G by trying various configurations of edges between the vertices of T_G with degree 1. If a graph G is found that has $d_{max}(G) = 2$ and $\Delta(G) = 3$, then it is proved that $N(2,3) = 10$.
2. If no such graph exists on 10 vertices, then delete one of the vertices and try again, repeating this process until we find a graph G for which $d_{max}(G) = 2$ and $\Delta(G) = 3$. Then $N(2,3) = |V(G)|$.

The Petersen graph (which has 10 vertices), satisfies these conditions (diameter of 2 and maximum degree of 3).



Peterson Graph *(Image above taken from source 2.2)*

We have proven that $N(2,3) \leq 10$ and that there is a graph on 10 vertices for which $d_{max} = 2$ and the maximum degree is 3.
Therefore, it is proven that $N(2,3) = 10$. ■

3. Claim: The graph G drawn above has $d_{max}(G) = 2$.

Proof:

Below is the the adjacency matrix, A, of the graph G. Any non-zero entries in this matrix mean that there is a path of length 1 between the corresponding vertices (By definition of an adjacency matrix of a graph).
There are some non-zero entries, which means the graph is not complete (every vertex is not adjacent to every other vertex).

0	1	0	0	0	0	0	0	1	0	0	0	0	1	1
1	0	1	0	0	1	0	0	0	0	0	1	0	0	0
0	1	0	1	0	0	0	1	0	0	0	0	1	0	0
0	0	1	0	1	0	0	0	0	0	0	1	0	1	0
0	0	0	1	0	1	0	0	1	0	0	0	1	0	0
0	1	0	0	1	0	1	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0	1	0	0	1	0
0	0	1	0	0	0	1	0	1	1	0	0	0	0	0
1	0	0	0	1	0	0	1	0	0	1	0	0	0	0
0	0	0	0	0	1	0	1	0	0	0	1	0	0	1
0	0	0	0	0	0	1	0	1	0	0	1	1	0	0
0	1	0	1	0	0	0	0	0	1	1	0	0	0	0
0	0	1	0	1	0	0	0	0	0	1	0	0	0	1
1	0	0	1	0	0	1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1	0	0	1	1	0

Below is the adjacency matrix of G, squared (A^2). Any non-zero entries in this matrix mean that there exists a walk of length 2 between the corresponding vertices (By Theorem A, see Notes 34). A walk of length 2 between two vertices x and y, where $x \neq y$, implies that the walk is a path of length 2, because the only way for an edge to be repeated in a walk of length 2 is if the starting and ending vertices are the same.

4	0	1	1	1	1	1	1	0	1	1	1	1	1	1
0	4	0	2	1	0	1	1	1	2	1	0	1	1	1
1	0	4	0	2	1	1	0	1	1	1	2	0	1	1
1	2	0	4	0	1	1	1	1	1	1	0	2	0	1
1	1	2	0	4	0	1	1	0	1	2	1	0	1	1
1	0	1	1	0	4	0	2	1	0	1	2	1	1	1
1	1	1	1	1	0	4	0	2	2	0	1	1	0	1
1	1	0	1	1	2	0	4	0	0	2	1	1	1	1
0	1	1	1	0	1	2	0	4	1	0	1	2	1	1
1	2	1	1	1	0	2	0	1	4	1	0	1	1	0
1	1	1	1	2	1	0	2	0	1	4	0	0	1	1
1	0	2	0	1	2	1	1	1	0	0	4	1	1	1
1	1	0	2	0	1	1	1	2	1	0	1	4	1	0
1	1	1	0	1	1	0	1	1	1	1	1	1	4	1
1	1	1	1	1	1	1	1	1	0	1	1	0	1	4

By looking at the matrices, one can see that there is a path of length 1 or 2 between every vertex pair in the graph G.
To visualize this more clearly, consider the overlay of the non-zero entries of the A^2 matrix on top of the adjacency matrix (A) below.

4	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	4	1	2	1	1	1	1	1	2	1	1	1	1	1
1	1	4	1	2	1	1	1	1	1	1	2	1	1	1
1	2	1	4	1	1	1	1	1	1	1	1	2	1	1
1	1	2	1	4	1	1	1	1	1	2	1	1	1	1
1	1	1	1	1	4	1	2	1	1	1	2	1	1	1
1	1	1	1	1	1	4	1	2	2	1	1	1	1	1
1	1	1	1	1	2	1	4	1	1	2	1	1	1	1
1	1	1	1	1	1	2	1	4	1	1	1	2	1	1
1	2	1	1	1	1	2	1	1	4	1	1	1	1	1
1	1	1	1	2	1	1	2	1	1	4	1	1	1	1
1	1	2	1	1	2	1	1	1	1	1	4	1	1	1
1	1	1	2	1	1	1	1	2	1	1	1	4	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	4	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	4

It is clearly seen that there are no "0" entries in this "overlaid-matrix", meaning that every vertex in G can be reached by traveling a distance of ≤ 2 from any other vertex in G.

Therefore, the graph G has $d_{max}(G) = 2$. ■

4. Claim: $N(2,4) = 15$.

Proof:

We can begin by establishing a lower bound. We can say that $N(2,4) \geq 15$ because the graph G in Sub-Experience 2 part 2 has 15 vertices, $d_m ax(G) = 2$, and $\Delta(G) = 4$.

To aid in the explanation of our proof, we define a *leaf vertex* to be a vertex with degree 1.

To continue our proof, we will follow the process outlined in part two of Sub-Experience 2 (see part two for explanation of logic behind this approach), namely:

1. Construct a potential spanning tree T_G with diameter 2, of a graph G, where $d_{max}(G) = 2$ (diameter of G is 2), $\Delta(G) = 4$, and $|V(G)| = N(2,4)$.

   (a) Draw a vertex to start from.

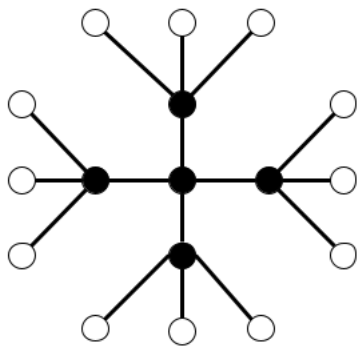
   (b) Construct the rest of the tree, giving every vertex degree 4 ($\Delta(G)$) until it reaches distance 2 from the start vertex.

2. The number of vertices in T_G will be an upper bound for $N(2,4)$.

3. Attempt to construct the graph G by trying various configurations of edges between the leaf vertices of T_G. If a graph G is found that has $d_{max}(G) = 2$ and $\Delta(G) = 4$, then it is proved that $N(2,4) = |V(T_G)|$.

4. If no such graph exists on $|V(T_G)|$ vertices, then delete one of the vertices and try again, repeating this process until we find a graph G for which $d_{max}(G) = 2$ and $\Delta(G) = 4$.

We begin following these steps by constructing T_G. (See figure below).



T_G, *leaf vertices are white-filled*

The tree T_G has 17 vertices. Thus, we can say that $15 \leq N(2,4) \leq 17$.

Sub-claim (a): There is no graph G on 17 vertices, where $d_{max} = 2$ and $\Delta(G) = 4$.

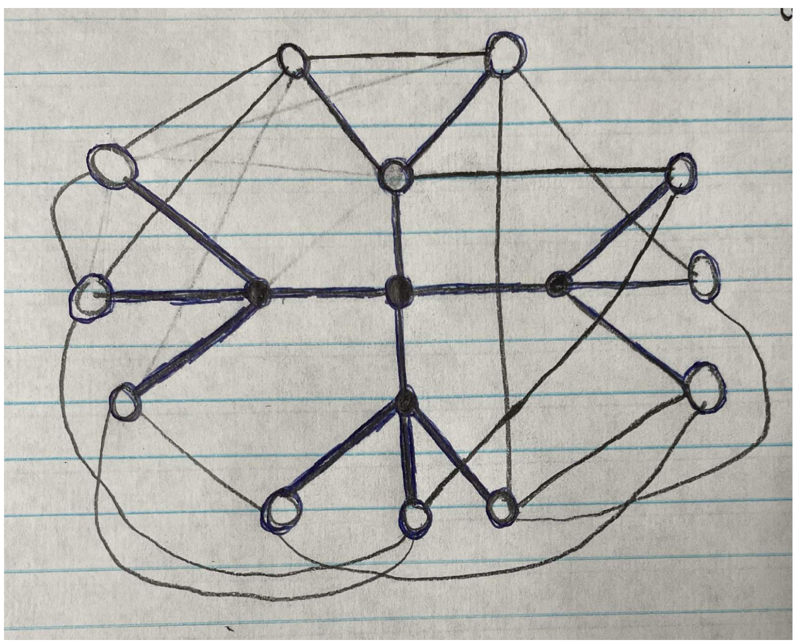
Sub-proof (a):

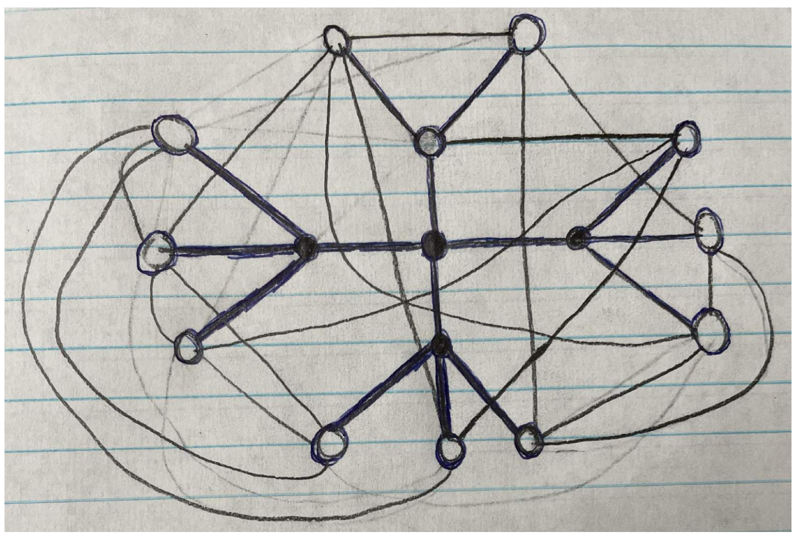
It can be seen clearly, that when attempting to construct the graph G from T_G, in order for any leaf vertex, v in the tree T_G to have eccentricity ≤ 2 (see source 2.1 for definition of eccentricity), it must be adjacent to all leaf vertices in T_G that are not adjacent to the same vertex as v. Therefore, every leaf vertex in T_G must be made adjacent to 9 other leaf vertices (10 adjacent vertices in total), to satisfy the condition that $d_{max}(G) = 2$. This however contradicts our condition that $\Delta(G) = 4$. Therefore, it has been proven impossible to construct the graph G on 17 vertices, giving us a new upper-bound for $N(2,4)$. ($N(2,4) \leq 16$)

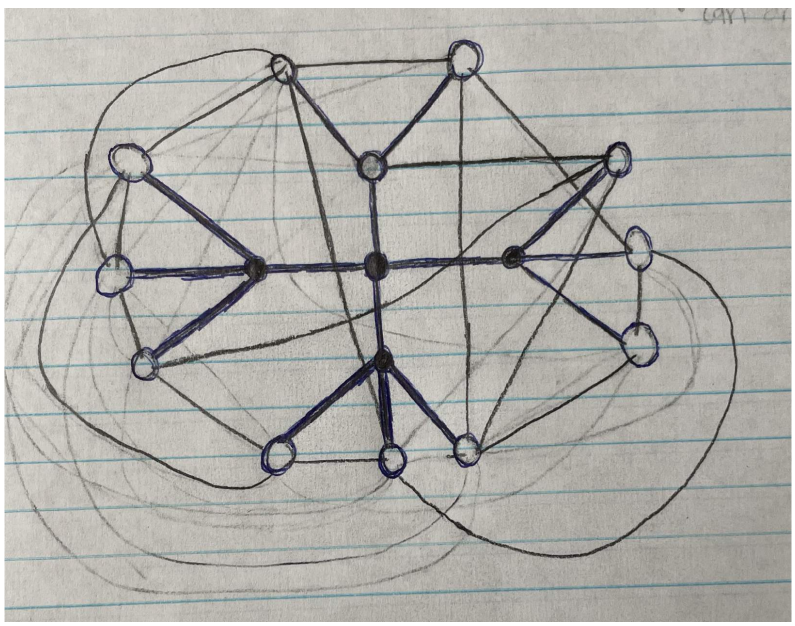
End Sub-proof (a) -

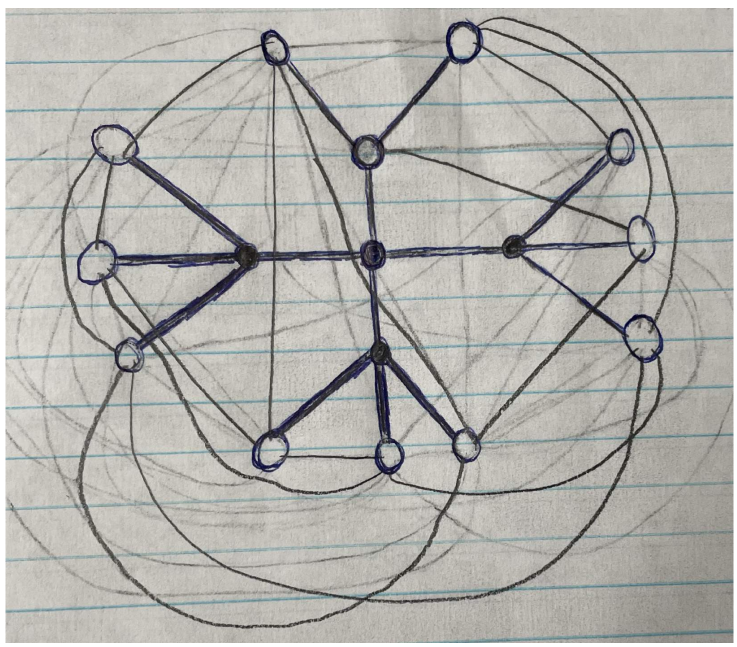
Sub-claim (b): There is no graph G on 16 vertices, where $d_{max} = 2$ and $\Delta(G) = 4$.

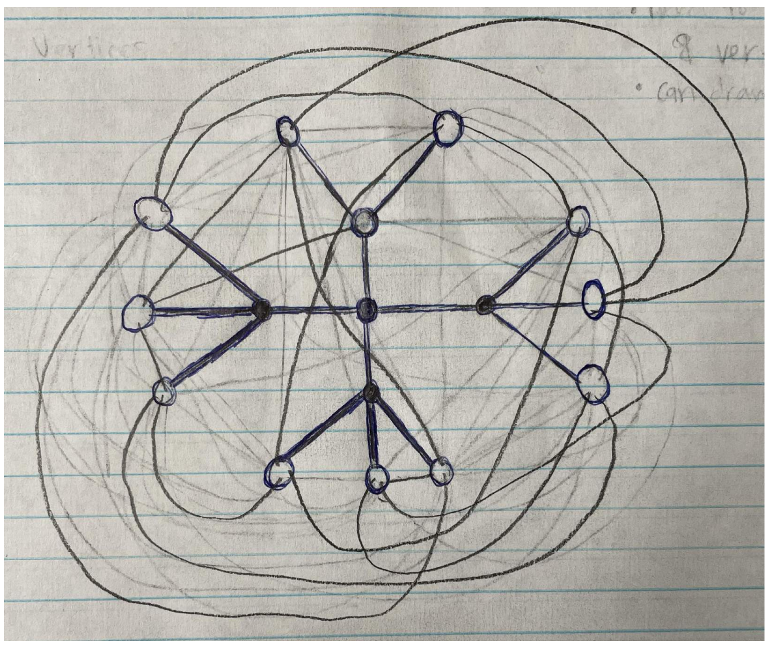
Sub-proof (b):
See following images.

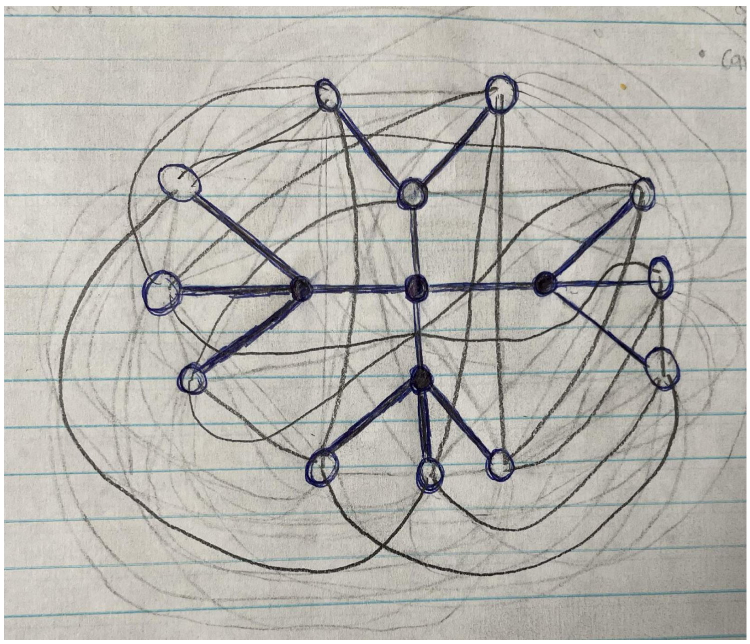


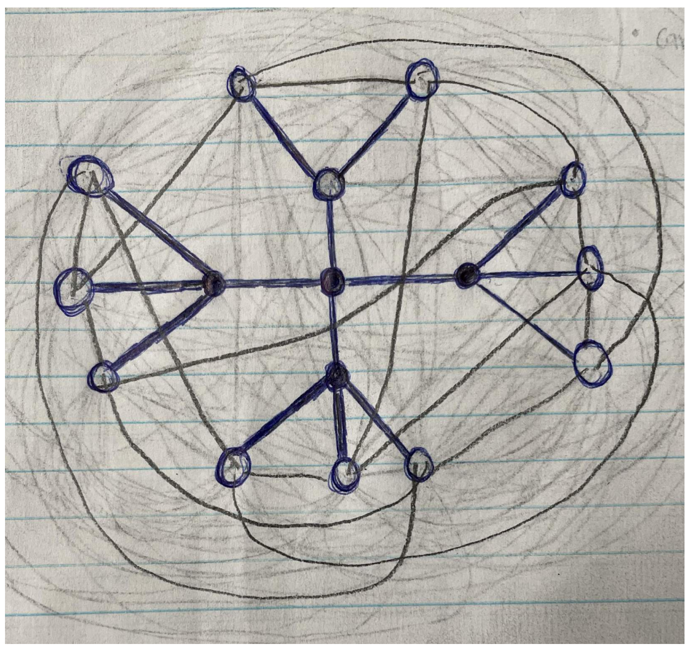


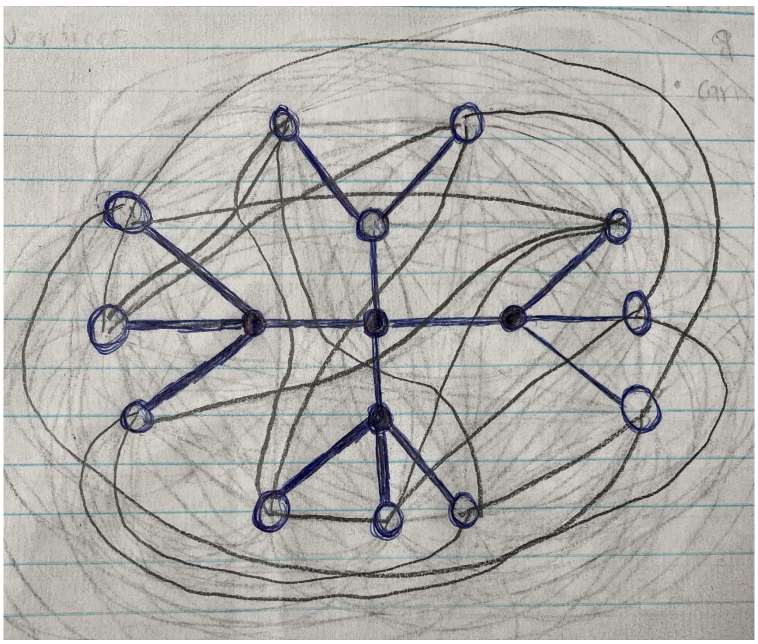


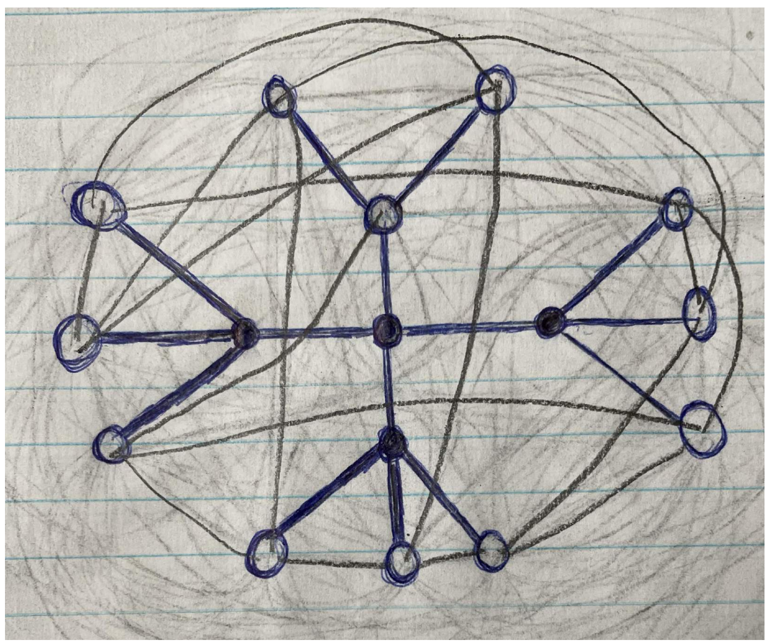


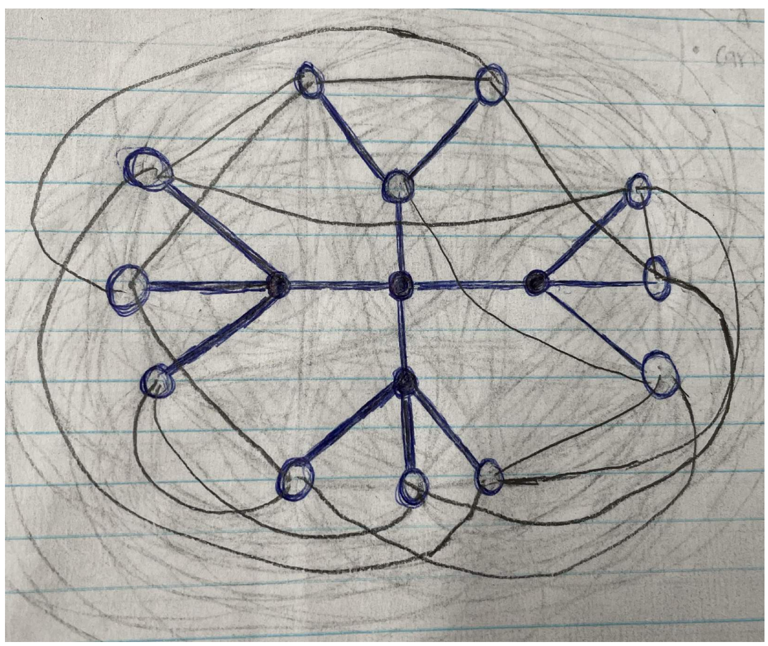

Vertices

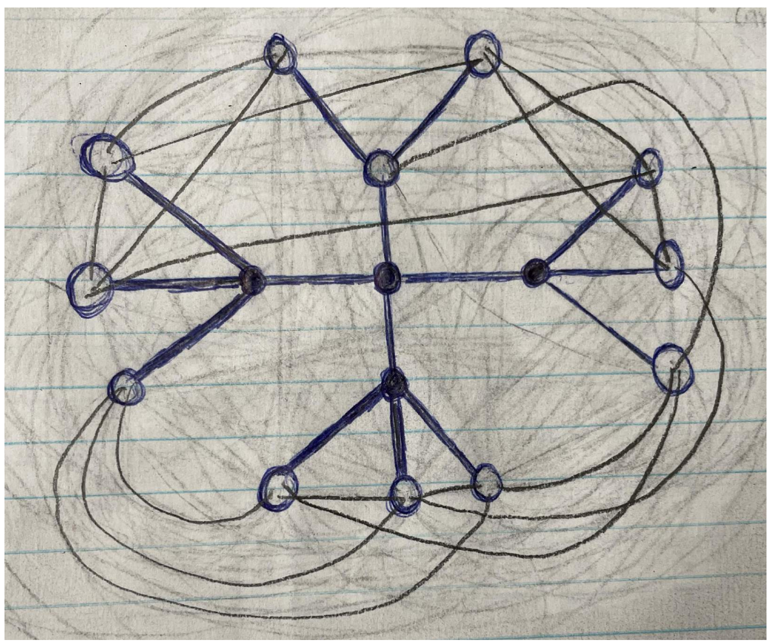


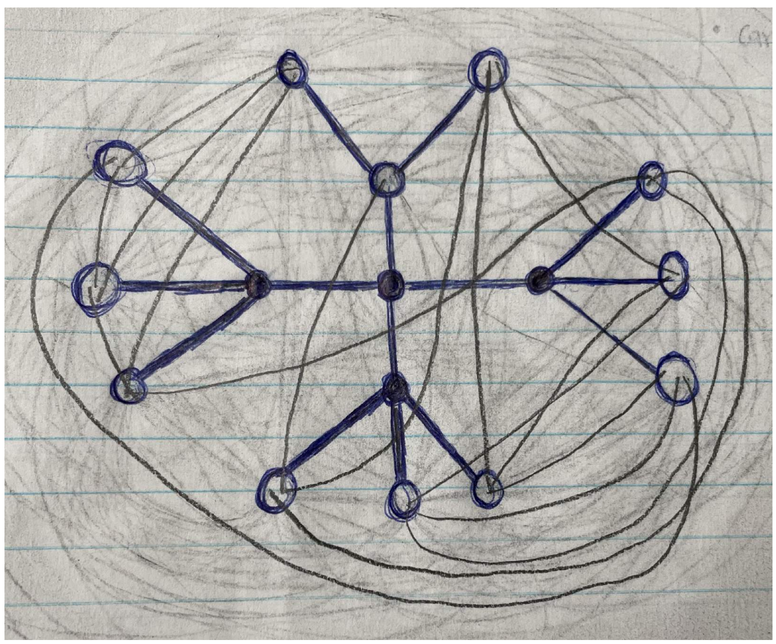


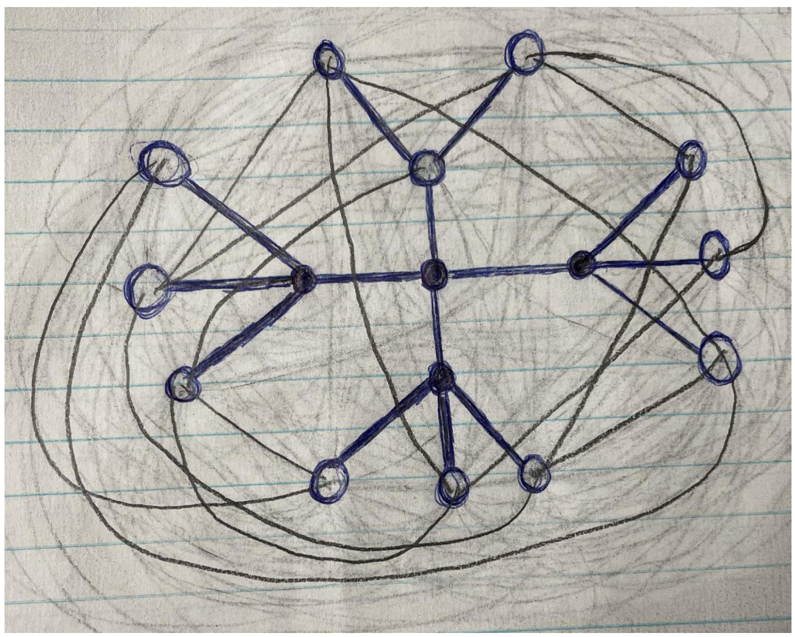


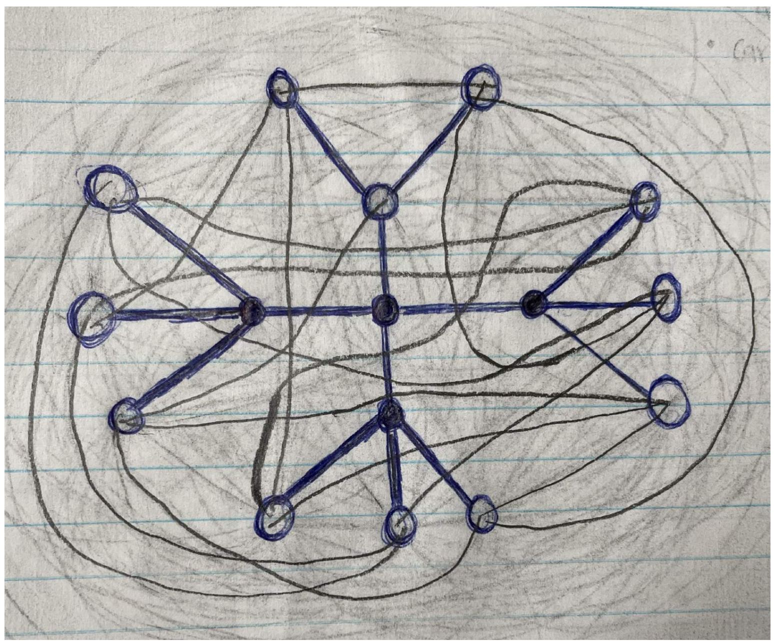


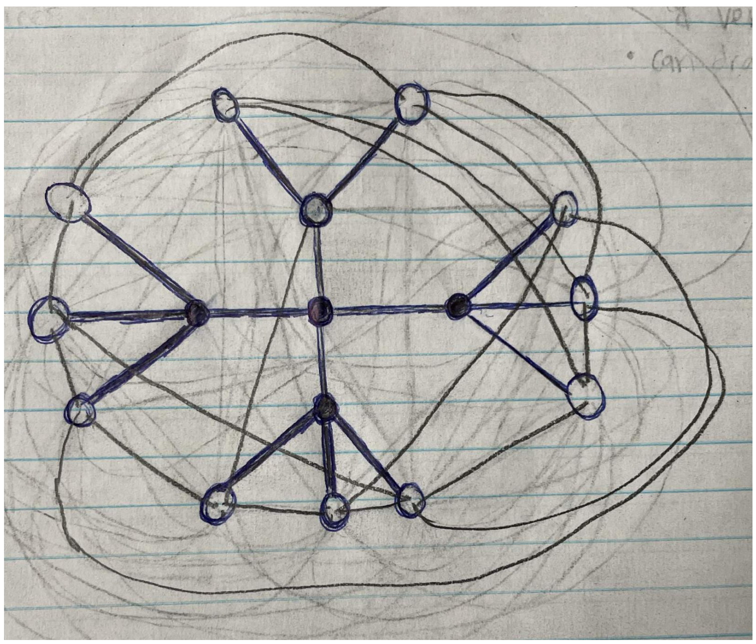


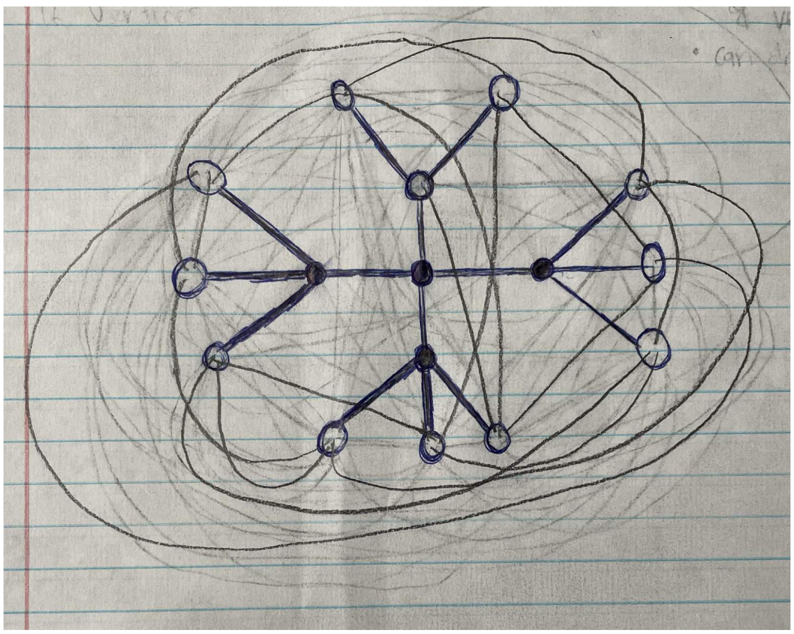


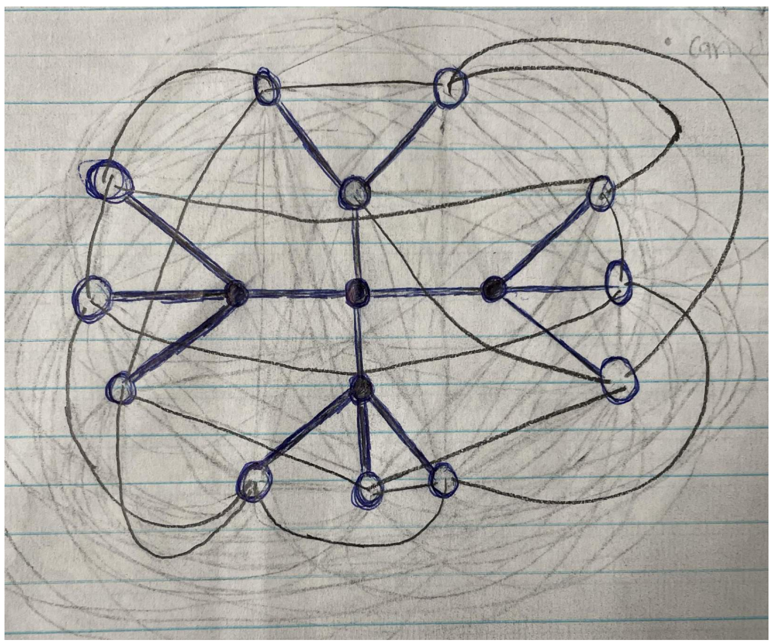


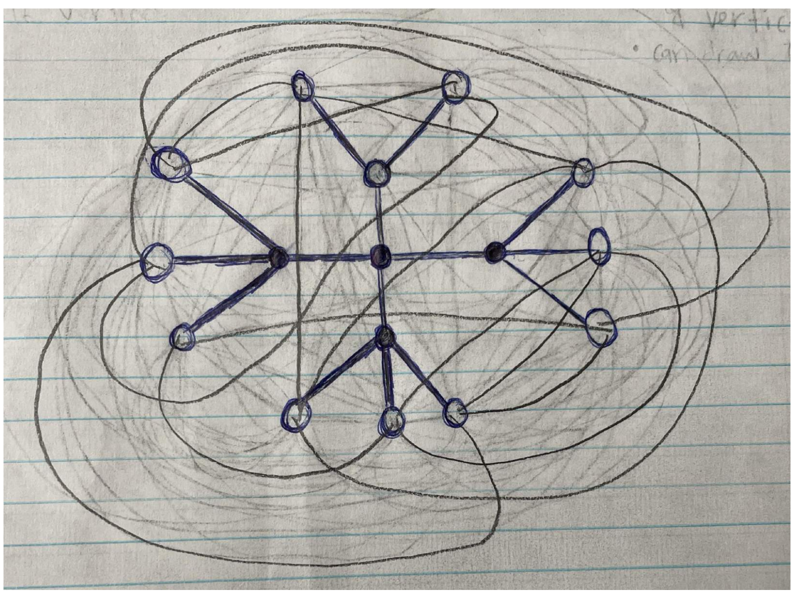








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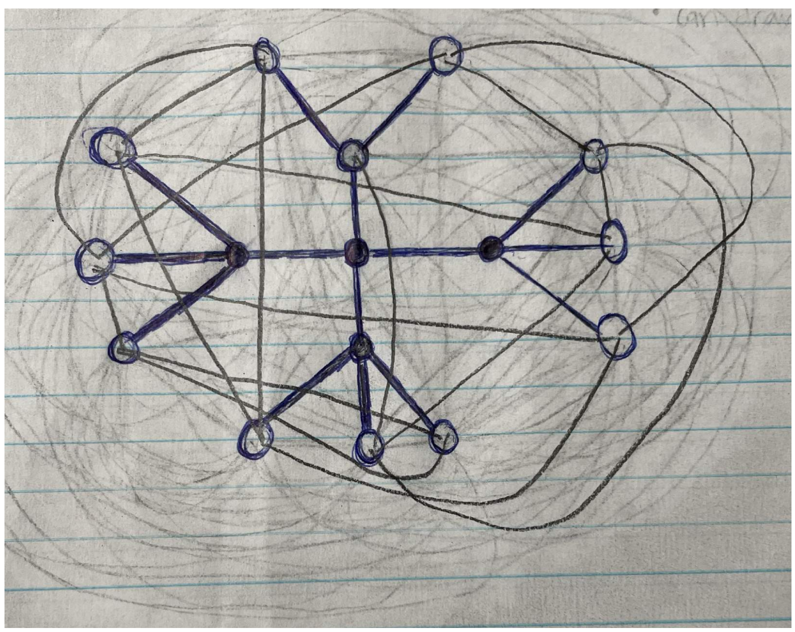

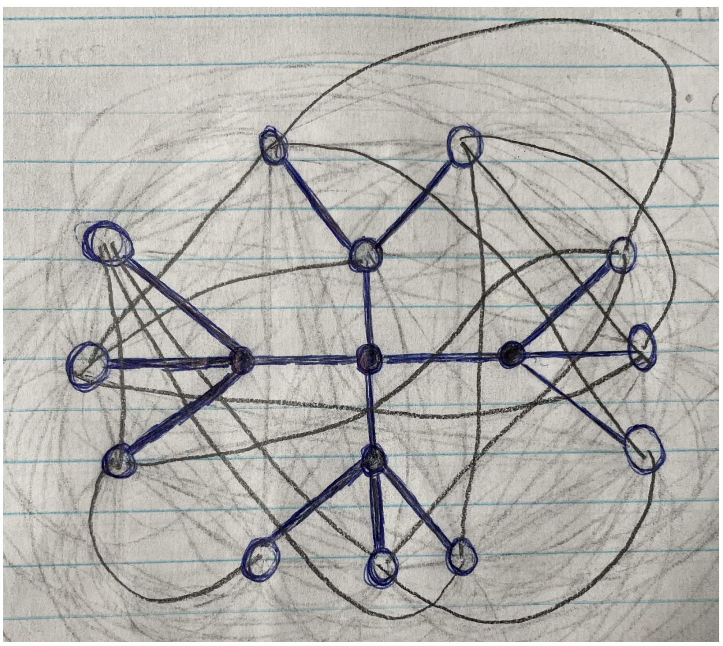

None of the above attempted constructions of G have diameter of 2. Therefore, there is no graph G on 16 vertices, where $d_{max} = 2$ and $\Delta(G) = 4$.

End Sub-proof (b) -

We can now say that $15 \leq N(2,4) \leq 15$.
Therefore, it has been proven that $N(2,4) = 15$. ■

Sub-Experience 2 Citations

2.1 "Graph Theory - Basic Properties." Tutorials Point, https://www.tutorialspoint.com/graph_theory/graph_theory_basic_properties.htm.

2.2 "Petersen Graph." Wikipedia, Wikimedia Foundation, 29 Nov. 2021, https://en.wikipedia.org/wiki/Petersen_graph.