CS 7301.003 Fall 2020	Dongpeng Liu (dxl200000)
Homework 1 Problem 1	Jiashuai Lu (jxl173630)

1. Let Σ be a combinatorial 2-manifold, where each corner x of each face of Σ is assigned a positive real number $\angle x$, called the ***angle*** at x. Let corners(v) or corners(f) denote the set of corners incident to a vertex v or a face f, respectively. We define the ***curvature*** of each vertex and face as follows:[1]

$$\kappa(v) := 1 - \sum_{x \in \text{corners}(v)} \angle x \qquad \kappa(f) := 1 - \sum_{x \in \text{corners}(f)} (1/2 - \angle x).$$

Prove the ***combinatorial Gauß-Bonnet theorem***:

$$\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) = \chi(\Sigma).$$

Solution: Let $V(\Sigma), E(\Sigma), F(\Sigma)$ denote the vertex, edge, and face sets of a 2-manifold Σ.

Observe that

$$\begin{aligned} \sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) &= \sum_{\text{vertex } v \text{ of } \Sigma} (1 - \sum_{x \in \text{corners}(v)} \angle x) \\ &= |V(\Sigma)| - \sum_{\text{corner } x \text{ of } \Sigma} \angle x \end{aligned}$$

and since every edge contributes to two faces, we have

$$\begin{aligned} \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) &= \sum_{\text{face } f \text{ of } \Sigma} (1 - \sum_{x \in \text{corners}(f)} (1/2 - \angle x)) \\ &= |F(\Sigma)| - 2|E(\Sigma)|/2 + \sum_{\text{corner } x \text{ of } \Sigma} \angle x \end{aligned}$$

together we get

$$\begin{aligned} \sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) &= |V(\Sigma)| + |F(\Sigma)| - |E(\Sigma)| \\ &= \chi(\Sigma) \end{aligned}$$

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2. Suppose every face of Σ is a triangle. Prove the following special case of the combinatorial Gauß-Bonnet theorem:

$$\sum_{\text{vertex } v \text{ of } \Sigma} (6 - |\text{corners}(v)|) = 6\chi(\Sigma).$$

[1] The definitions use individual passes around a *circle* as the unit of angular measurements.

Solution: While every face of Σ is a triangle, we know $3|F(\Sigma)| = 2|E(\Sigma)|$. And it is easy to see that,

$$\begin{aligned}
\sum_{\text{vertex } v \text{ of } \Sigma} (6 - |\operatorname{corners}(v)|) &= 6|V(\Sigma)| - \sum_{\text{vertex } v \text{ of } \Sigma} |\operatorname{corners}(v)| \\
&= 6|V(\Sigma)| - 2|E(\Sigma)| \\
&= 6|V(\Sigma)| - 3|F(\Sigma)| \\
&= 6|V(\Sigma)| + 6|F(\Sigma)| - 9|F(\Sigma)| \\
&= 6|V(\Sigma)| + 6|F(\Sigma)| - 6|E(\Sigma)| \\
&= 6\chi(\Sigma)
\end{aligned}$$

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3. Now suppose Σ has boundary components. Let $\chi(v)$ denote the number of edges incident to v, counting loops twice, minus the number of corners incident to v. We now redefine the curvature of a vertex v as follows:

$$\kappa(v) := 1 - \frac{\chi(v)}{2} - \sum_{x \in \operatorname{corners}(v)} \angle x.$$

Prove that the combinatorial Gauß-Bonnet theorem holds in this more general setting.

Solution: Let $|E(v)|$ denote the number of edges incident to v, counting loops twice. Then we get

$$\begin{aligned}
\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) &= |V(\Sigma)| - \sum_{\text{vertex } v \text{ of } \Sigma} \frac{\chi(v)}{2} - \sum_{\text{corner } x \text{ of } \Sigma} \angle x \\
&= |V(\Sigma)| - \sum_{\text{vertex } v \text{ of } \Sigma} \frac{|E(v)| - |\operatorname{corners}(v)|}{2} - \sum_{\text{corner } x \text{ of } \Sigma} \angle x
\end{aligned}$$

combining this with the result in question 1.1, we have,

$$\begin{aligned}
&\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) \\
&= |V(\Sigma)| - \sum_{\text{vertex } v \text{ of } \Sigma} \frac{|E(v)| - |\operatorname{corners}(v)|}{2} - \sum_{\text{corner } x \text{ of } \Sigma} \angle x + |F(\Sigma)| \\
&- \sum_{\text{face } v \text{ of } \Sigma} \frac{|\operatorname{corners}(f)|}{2} + \sum_{\text{corner } x \text{ of } \Sigma} \angle x \\
&= |V(\Sigma)| + |F(\Sigma)| - \sum_{\text{vertex } v \text{ of } \Sigma} \frac{|E(v)|}{2} \\
&= |V(\Sigma)| + |F(\Sigma)| - |E(\Sigma)| = \chi(\Sigma)
\end{aligned}$$

The second equation is from $\sum_{\text{face } v \text{ of } \Sigma} |\operatorname{corners}(f)| = \sum_{\text{vertex } v \text{ of } \Sigma} |\operatorname{corners}(v)|$, and the fact that every edge contributes to the number of incident edges for two vertices implies the last equation.

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4. Suppose the surface Σ' is homeomorphic to a disk, and every face and *interior* vertex of Σ' has curvature at most 0. Prove that at least three boundary vertices of Σ' have strictly positive curvature.

   Solution: We know $\chi(\Sigma') = 1$, therefore we have

   $$\sum_{\text{vertex } v \text{ of } \Sigma'} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma'} \kappa(f) = 1$$

   Let V_B be the set of boundary vertices of Σ'. Then we get

   $$\sum_{v \in V_B} \kappa(v) + \sum_{v \in V(\Sigma') - V_B} \kappa(v) + \sum_{f \in F(\Sigma')} \kappa(f) = 1$$

   We know $\kappa(v) \le 0, \forall v \in V(\Sigma') - V_B$ and $\kappa(f) \le 0, \forall f \in F(\Sigma')$. So $\sum_{v \in V_B} \kappa(v) \ge 1$. From the definition of curvature of vertex v in question 1.3, we know if $v \in V_B$,

   $$\begin{aligned}
   \kappa(v) &= 1 - \frac{|E(v)| - |\operatorname{corners}(v)|}{2} - \sum_{x \in \operatorname{corners}(v)} \angle x \\
   &= 1 - \frac{1}{2} - \sum_{x \in \operatorname{corners}(v)} \angle x \\
   &= \frac{1}{2} - \sum_{x \in \operatorname{corners}(v)} \angle x
   \end{aligned}$$

   where the second equation is from the number of edges incident to a boundary vertex is one more than the number of corners incident to the vertex. Since the angle of every corner is positive. Every boundary vertex has curvature strictly less than $1/2$. Therefore, to make $\sum_{v \in V_B} \kappa(v) \ge 1$, we need at least $\lfloor \frac{1}{1/2} \rfloor + 1 = 3$ boundary vertices has positve curvature. ■

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Homework 3 Problem 2	Jiashuai Lu (jxl173630)

Let G be a cellularly embedded (i.e. every face is a disk) graph on a surface Σ *with boundary*. Recall, a ***cut graph*** is a subgraph H of G such that the closure of $\Sigma \setminus H$ is a disk. A cut graph is minimal if no proper subgraph is a cut graph. For example, a minimal cut graph of an annulus is a path from one boundary to the other.

A ***pair of pants*** is a sphere minus three open disks. Let G be a graph with non-negatively weighted edges, cellulary embedded in a pair of pants Σ. Describe an algorithm to find the minimum-length cut graph in G in $O(n \log n)$ time. *[Hint: What does a minimal cut graph of a pair of pants look like?]*

Solution: Let D denote the set of three open disks minus from the sphere.

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MINIMUM-LENGTH CUT GRAPH (G, D):
  For every disk d ∈ D
      Let d_1, d_2 be two disks in D\{d}.
      Compute a minimum-length path σ_{u,v} for every pair of points u, v, u ∈ d_1, v ∈ d_2.
      Let σ_{min_1} be the minimum-length path among all paths computed from the step above.
      Compute a minimum-length path σ_{u,v} for every pair of points u, v, u ∈ σ_{min_1} ∪ d_1 ∪ d_2, v ∈ d.
      Let σ_{min_2} be the minimum-length path among all paths computed from the step above.
      Let H be the minimum-length cut-graph computed in this iteration
  Return the cut-graph with minimum-length among the cut-graphs we get in three iterations.
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