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(a) Truthfully write the phrase ***"I have read and understand the course policies."***

Solution: I have read and understand the course policies. ■

(b) Prove that the composition of two PL homeomorphisms of the plane is another PL homeomorphism.

Solution: Let ψ and ϕ be any two PL homeomorphisms. Let $\Delta_{\mathbb{H}}$ denote a triangulation of a PL homeomorphism $\mathbb{H}$ such that the number of triangles in it is equal to the complexity of $\mathbb{H}$.

In order to show $\theta = \psi \circ \phi$ is also a PL homeomorphism, we construct a triangulation Δ of the plane such that the restriction of θ to any triangle in Δ is affine.

First let $\Delta = \Delta_\psi$. Since Δ_ψ and Δ_ϕ are trigulations of the plane, every point of the plane is both in a unique triangle of Δ_ψ and a unique triangle of Δ_ϕ, and every triangle of Δ overlaps with at least one triangle of Δ_ϕ.

For each triangle T of Δ, for every triangle T' of Δ_ϕ such that T and T' overlap, if the overlap of T and T' is not a line segment, it must be an n-gon where $n \in \{3,4,5,6\}$, we add this n-gon to Δ and create a frugal trigulation for this n-gon in Δ if $n > 3$.

It is not hard to see that now Δ is also a triangulation of the plane. Let $f|_R$ denote the restriction of function f to a region R. Let T be any triangle of Δ. Then T is either whole or part of the overlap of some triangle T' of Δ_ψ and a triangle T'' of Δ_ϕ. Therefore, $\Theta|_T$ to this triangle is the composition of two affine maps $\psi|_{T'}$ and $\phi|_{T''}$, i.e., $\theta|_T = \psi_{T'} \circ \phi|_{T''}$. Since the composition of two affine maps are also affine, Δ is a triangulation such that the restriction of $\psi \circ \phi$ to any triangle of it is affine. The composition of two PL homeomorphisms of the plane is another PL homeomorphism.

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(c) Suppose ϕ is a PL homeomorphism with complexity x and ψ is a PL homeomorphism with complexity y. What can you say about the complexity of the PL homeomorphism $\psi \circ \phi$?

Solution: We reuse some notations in Solution(b). Given the procedure we used to build a triangulation for the PL homeomorphism $\psi \circ \phi$ in Solution(b), let $|\Delta|$ denote the number of triangles in triangulation Δ, we observe that the complexity of $\psi \circ \phi$ will be ***at most***

$$\min_{\Delta_\psi, \Delta_\phi} |\Delta|$$

where $\Delta_\psi / \Delta_\phi$ is any triangulation of the plane such that the restriction of ψ/ϕ to every triangle in it is affine, and Δ is the triangulation we build using the procedure in Solution(b) with Δ_ψ and Δ_ϕ as inputs.

We could find an upper bound for $|\Delta|$ given $|\Delta_\psi|$ and $|\Delta_\phi|$. For every triangle T in Δ_ψ, at most $|\Delta_\phi|$ triangles of Δ_ϕ overlap with T. Every overlap corresponds to at most 4 triangles in Δ because the overlap part is at most a simple 6-gon. Therefore, we have

$$|\Delta| \leq 4 \cdot |\Delta_\psi| \cdot |\Delta_\phi|.$$

and because there exist at least one triangulation Δ_ψ with $|\Delta_\psi| = x$ and one triangulation Δ_ϕ with $|\Delta_\phi| = y$, we can see the ***complexity of*** $\psi \circ \phi$ is ***at most***

$$\min_{\Delta_\psi, \Delta_\phi} |\Delta| \leq 4 \cdot x \cdot y$$

On the other hand, we think we cannot say anything about the lower bound of the complexity of $\psi \circ \phi$. It could be only 2 while the affine map of $\psi \circ \phi$ could be the same at every point on the plane which depends on ψ and ϕ. ■

(d) Prove that for any simple n-gon P, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n)$ that maps the polygon P to a triangle.

Solution: ■

(e) Prove that for any two simple n-gons P and Q, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n^2)$ such that $\phi(P) = Q$.

Solution: From Problem(d), we know there exist two PL homeomorphism ϕ_P, ϕ_Q with complexity $O(n)$ that map P and Q to two triangles.

Let $\phi_P(P) = T_P$ and $\phi_Q(Q) = T_Q$. Let $\{v_{p1}, v_{p2}, v_{p3}\}$ be vertices of T_P and $\{v_{q1}, v_{q2}, v_{q3}\}$ be vertices of T_Q.

For $i \in \{1, 2, 3\}$, fix an extra point v_i close to the middle point of edge $\{v_{pi}, v_{p(i+1)}\}$ and outside T_p. Let T_i denote the triangle with vertices $\{v_i, v_{pi}, v_{p(i+1)}\}$. We build a mapping ψ_i that maps $v_{pi}, v_{p(i+1)}$ to $v_{qi}, v_{q(i+1)}$ (let $v_{p4} = v_{p1}, v_{q4} = v_{q1}$). Then we extend ψ_i across triangle T_i and keep it as identity map outside this triangle.

It is easy to see $\forall i$, ψ_i is a PL homeomorphism of $O(1)$ complexity that maps edge $\{v_{pi}, v_{p(i+1)}\}$ to edge $\{v_{qi}, v_{q(i+1)}\}$ because we can build a triangulation of the plane by use triangle T_i and triangulate $\mathbb{R}^2 \backslash T_i$ with 6 triangles such that the restriction of ψ_i to any triangle in it is affine. Therefore, we now have $\psi_1 \circ \psi_2 \circ \psi_3(T_P) = T_Q$.

Let ϕ_Q^{-1} be the inverse of ϕ_Q. The complexity of ϕ_Q^{-1} is same as ϕ_Q which is $O(n)$ and $\phi_Q^{-1}(T_Q) = Q$. Put them together we get $\phi_Q^{-1} \circ \psi_1 \circ \psi_2 \circ \psi_3 \circ \phi_P(P) = Q$.

Let $\psi = \phi_Q^{-1} \circ \psi_1 \circ \psi_2 \circ \psi_3 \circ \phi_P$. From Solution(b) we know ψ is a PL homeomorphism. Let $|\phi|$ denote the complexity of a PL homeomorphism ϕ. From Solution(c) we know the complexity of ψ is at most

$$4^4 \cdot |\phi_Q^{-1}| \cdot |\psi_1| \cdot |\psi_2| \cdot |\psi_3| \cdot \phi_P \leq O(n^2)$$

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Homework 1 Problem 2 Jiashuai Lu (jxl173630)

(a) Prove that every connected plane graph has either a vertex with degree at most 3 or a face with degree at most 3.

Solution: We suppose there is a plane graph G of n vertices, m edges and f faces with every vertex has degree at least 4 and every face has degree at least 4.

Every edge can be shared by at most two faces and at most two vertices in G. So we know

$$4n \leq 2m,\ 4f \leq 2m \implies n + f - m \leq 0$$

According to Euler Formula, $n + f - m = 2$ as G is a connected plane graph. A contradiction. Therefore, every connected plane graph has either a vertex with degree at most 3 or a grace with degree at most 3. ■

(b) Prove that every simple bipartite planar graph has at most $2n-4$ edges.

Solution: We assume $n \geq 3$ since if $n = 2$ the bipartite planar graph could have $1 \geq 2 \cdot 2 - 4$ edge.

Without loss of generality, we assume the graph is connected. If the graph has more than one connected components, for every connected component, suppose the number of vertices in it is n_i, if the number of edges is at most $2n_i - 4$, then the number of edges in the graph is at most $2n - 4$.

We prove the argument by considering the following two cases:

1) The graph only has one face.
In this case, $n + m - f = 2 \implies m = n - 1 \implies m \leq 2n - 4, \forall n \geq 3$
2) The graph has more than one face. In this case, every cycle in this graph is of even size, which means every face has degree ≥ 4 including the outer face. Every edge is shared by at most two faces. So $4f \leq 2e \implies e \geq 2f$.
According to Euler Formula, $2n + 2f - 2e = 4 \implies 2n - 4 = 2e - 2f \geq e$. Therefore, every simple bipartite planar graph has at most $2n - 4$ edges if $n \geq 3$.

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Homework 1 Problem 3 Jiashuai Lu (jxl173630)

Let G be an arbitrary plane graph, let T be an arbitrary spanning tree of G, and let e be an arbitrary edge of T. Color the vertices in one component of $T \setminus e$ red and the vertices in the other component blue. Prove that any face of G is incident to either zero or two edges that have one red endpoint and one blue endpoint.

Solution: We say an edge *bicolor* if it have one red endpoint and one blue endpoint.

First we notice that no edge in $T\backslash e$ is bicolor. All dual bicolor edges are in $G^*\backslash T^* \cup \{e^*\}$. Let C^* be the unique dual cycle in $G^*\backslash T^* \cup \{e^*\}$. To prove the proposition, we show that all dual bicolor edges are in C^* and all dual edges in C^* are bicolor.

Notice that C^* separates G^* into two components. Let R^*, B^* be those two components in $G^*\backslash C^*$ such that R^* contains the red dual face adjacent to the dual edge e^* and B^* contains the blud dual face adjacent to e^*.

Because red vertices in G are all in one component of $T\backslash e$, they are still connected in $G\backslash C$. So red dual faces in $G^*\backslash C^*$ are also connected. Similarly, we can see blue dual faces in $G^*\backslash C^*$ are connected.

Since we know there is at least one red dual face in R^* and at least one blue dual face in B^*, from the connectivity argument above, we can also infer that no red dual face is in B^* and no blue dual faces is in R^*. Plus the fact that every vertex is either red or blue, it is easy to see every dual face in R^* is red and every dual face in B^* is blue.

According to the way we define bicolor, all dual edges in R^* and B^* are not bicolor because every dual edge in R^* and B^* is shared by two dual faces with same color. Moreover, every dual edge in C^* is shared by one dual face in R^* and one dual face in B^*. So we proved that no dual edge in $G^*\backslash C^*$ is bicolor and all dual edges in C^* are bicolor.

Every dual vertex in C^* is incident to exact two dual edges in C^*, its corresponding primal face is incident to two bicolor edges. Every dual vertex in $G^*\backslash C^*$ is incident no dual edge in C^*, its corresponding primal face is incident to zero bicolor edge. The proposition is proved.

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