CS 7301.003 Fall 2020 Author A. One (aa0123456)
Homework 1 Problem 1 Jiashuai Lu (jxl173630)

(a) Truthfully write the phrase ***"I have read and understand the course policies."***

Solution: I have read and understand the course policies. ■

(b) Prove that the composition of two PL homeomorphisms of the plane is another PL homeomorphism.

Solution: Let ψ and ϕ be any two PL homeomorphisms. Let $\Delta_{\mathbb{H}}$ denote a triangulation of a PL homeomorphism $\mathbb{H}$ such that the number of triangles in it is the complexity of $\mathbb{H}$.

In order to show $\theta = \psi \circ \phi$ is also a PL homeomorphism, we construct a triangulation Δ of the plane such that the restriction of θ to any triangle in Δ is affine.

First let $\Delta = \Delta_\psi$. Since Δ_ψ and Δ_ϕ are trigulations of the plane, every point of the plane is both in a unique triangle of Δ_ψ and a unique triangle of Δ_ϕ, and every triangle of Δ overlaps with one or more triangles of Δ_ϕ.

For each triangle T of Δ, for every triangle T' of Δ_ϕ such that T and T' overlap, if the overlap of T and T' is a triangle, we add vertices of this triangle to Δ (if a vertex is already in Δ, we skip it) and add edge between any two of those three vertices if the edge not already exists in Δ. Otherwise, the overlap is a n-gon with $4 \le n \le 6$. Similarly, we add vertices and edges of this n-gon to Δ. Then we create a frugal trigulation of this n-gon in Δ.

It is not hard to see that now Δ is also a triangulation of the plane. Let $f|_R$ denote the restriction of function f to a region R. Let T be any triangle of Δ. Then T is either whole or part of the overlap of some triangle T' of Δ_ψ and a triangle T'' of Δ_ϕ, $\Theta|_T$ to this triangle is the composition of two affine maps $\psi|_{T'}$ and $\phi|_{T''}$, i.e., $\theta|_T = \psi_{T'} \circ \phi|_{T''}$. The composition of two affine maps are also affine. Therefore, Δ is a triangulation such that the restriction of $\psi \circ \phi$ to any triangle of it is affine. The composition of two PL homeomorphisms of the plane is another PL homeomorphism.

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(c) Suppose ϕ is a PL homeomorphism with complexity x and ψ is a PL homeomorphism with complexity y. What can you say about the complexity of the PL homeomorphism $\psi \circ \phi$?

Solution: We reuse notations in Solution(b). Given the procedure we used to build a triangulation for the PL homeomorphism $\psi \circ \phi$ in Solution(b), let $|\Delta|$ denote the number of triangles in triangulation Δ, we observe that the complexity of $\psi \circ \phi$ will be at most

$$\min_{\Delta_\psi, \Delta_\phi} |\Delta|$$

where $\Delta_\psi / \Delta_\phi$ is any triangulation of the plane such that the restriction of ψ/ϕ to every triangle in it is affine, and Δ is the triangulation we build using the procedure in Solution(b) with Δ_ψ and Δ_ϕ as inputs.

We can find an upper bound for $|\Delta|$ given $|\Delta_\psi|$ and $|\Delta_\phi|$. For every triangle T in Δ_ψ, at most $|\Delta_\phi|$ triangles of Δ_ϕ overlap with T. Every overlap corresponds to at most 4 triangles in Δ because the overlap part is at most a 6-gon. Therefore, we have

$$|\Delta| \leq 4 \cdot |\Delta_\psi| \cdot |\Delta_\phi|.$$

and because there exist at least one triangulation $|\Delta_\psi| = x$ and one triangulation $|Delta_\phi| = y$, we can infer the complexity of $\psi \circ \phi$ is at most

$$\min_{\Delta_\psi, \Delta_\phi} |\Delta| \leq 4 \cdot x \cdot y$$

On the other hand, we think we cannot say anything about the lower bound of the complexity of $\psi \circ \phi$. It could be only 2 while the affine map of $\psi \circ \phi$ could be the same at every point on the plane which depends on ψ and ϕ. ■

(d) Prove that for any simple n-gon P, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n)$ that maps the polygon P to a triangle.

Solution: ■

(e) Prove that for any two simple n-gons P and Q, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n^2)$ such that $\phi(P) = Q$.

Solution: From Problem(d), we know there exists two PL homeomorphism ϕ_P, ϕ_Q with complexity $O(n)$ that map P and Q to triangles.

Let $\phi_P(P) = T_P$ and $\phi_Q(Q) = T_Q$. Let $\{v_{p1}, v_{p2}, v_{p3}\}$ be vertices of T_P and $\{v_{q1}, v_{q2}, v_{q3}\}$ be edges of T_Q.

For $i \in \{1, 2, 3\}$, fix an extra point v_i close to the middle point of edge $\{v_{pi}, v_{p(i+1)}\}$ and outside T_p. Let T_i denote the triangle with vertices $\{v_i, v_{pi}, v_{p(i+1)}\}$. We build a mapping ψ_i that maps $v_{pi}, v_{p(i+1)}$ to $v_{qi}, v_{q(i+1)}$ (let $v_{p4} = v_{p1}, v_{q4} = v_{q1}$). Then we extend ψ_i across triangle T_i and keep it as identity map outside this triangle.

It is easy to see $\forall i$, ψ_i is a PL homeomorphism of $O(1)$ complexity that maps edge $\{v_{pi}, v_{p(i+1)}\}$ to edge $\{v_{qi}, v_{q(i+1)}\}$ because we can build a triangulation of the plane by use triangle T_i and triangulate $\mathbb{R}^2 \backslash T_i$ with 6 triangles such that the restriction of ψ_i to any triangle in it is affine. Therefore, we now have $\psi_1 \circ \psi_2 \circ \psi_3(T_P) = T_Q$.

Let ϕ_Q^{-1} be the reverse of ϕ_Q. The complexity of ϕ_Q^{-1} is same as ϕ_Q which is $O(n)$ and $\phi_Q^{-1}(T_Q) = Q$. Put them together we get $\phi_Q^{-1} \circ \psi_1 \circ \psi_2 \circ \psi_3 \circ \phi_P(P) = Q$.

Let $\psi = \phi_Q^{-1} \circ \psi_1 \circ \psi_2 \circ \psi_3 \circ \phi_P$. From Solution(b) we know ψ is a PL homeomorphism. Let $|\phi|$ denote the complexity of a PL homeomorphism ϕ. From Solution(c) we know the complexity of ψ is at most

$$4^4 \cdot |\phi_Q^{-1}| \cdot |\psi_1| \cdot |\psi_2| \cdot |\psi_3| \cdot \phi_P \leq O(n^2)$$

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Homework 1 Problem 2	Jiashuai Lu (jxl173630)

(a) Prove that every connected plane graph has either a vertex with degree at most 3 or a face with degree at most 3.

Solution: We suppose there is a plane graph G of n vertices, m edges and f faces with every vertex has degree at least 4 and every face has degree at least 4.

Every edge can be shared by at most two faces and at most two vertices in G. So we know

$$4n \leq 2m,\ 4f \leq 2m \implies n + f - m \leq 0$$

According to Euler Formula, $n + f - m = 2$ as G is a connected plane graph. A contradiction. Therefore, every connected plane graph has either a vertex with degree at most 3 or a grace with degree at most 3. ■

(b) Prove that every simple bipartite planar graph has at most $2n - 4$ edges.

Solution: We assume $n \geq 3$ since if $n = 2$ the bipartite planer graph with $1 \geq 2 \cdot 2 - 4$ edge.

We prove the argument by considering the following two cases:

1) The graph only has one face.
   In this case, $n + m - f = 2 \implies m = n - 1 \implies m \leq 2n - 4, \forall n \geq 3$
2) The graph has more than one face. In this case, every cycle in this graph is even size, which means every face has degree ≥ 4 including the outer face. Every edge is shared by at most two faces. So $4f \leq 2e \implies e \geq 2f$.
   According to Euler Formula, $2n + 2f - 2e = 4 \implies 2n - 4 = 2e - 2f \geq e$. Therefore, every simple bipartite planar graph as at most $2n - 4$ edges if $n \geq 3$.

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Homework 1 Problem 3	Jiashuai Lu (jxl173630)

Let G be an arbitrary plane graph, let T be an arbitrary spanning tree of G, and let e be an arbitrary edge of T. Color the vertices in one component of $T \setminus e$ red and the vertices in the other component blue. Prove that any face of G is incident to either zero or two edges that have one red endpoint and one blue endpoint.

Solution: We say an edge *bicolor* if it have one red endpoint and one blue endpoint.

First we notice that no edge in $T\backslash e$ is bicolor. All dual bicolor edges are in $G^*\backslash T^* \cup \{e^*\}$. Let C^* be the unique dual cycle in $G^*\backslash T^* \cup \{e^*\}$. To prove the proposition, we show that all dual bicolor edges are in C^* and all dual edges in C^* are bicolor.

Notice that C^* separates G^* into two components. Let R^*, B^* be those two components in $G^*\backslash C^*$ such that R^* contains the red dual face adjacent to the dual edge e^* and B^* contains the blud dual face adjacent to e^*.

Because red vertices in G are all in one component of $T\backslash e$, they are still connected in $G\backslash C$. So red dual faces in $G^*\backslash C^*$ are also connected. Similarly, we can see blue dual faces in $G^*\backslash C^*$ are connected.

Since we know there is at least one red dual face in R^* and at least one blue dual face in B^*, from the connectivity argument above, we can also infer that no red dual face is in B^* and no blue dual faces is in R^*. Plus the fact that every vertex is either red or blue, it is easy to see every dual face in R^* is red and every dual face in B^* is blue.

According to the way we define bicolor, all dual edges in R^* and B^* are not bicolor because every dual edge in R^* and B^* is shared by two dual faces with same color. Moreover, every dual edge in C^* is shared one dual face in R^* and one dual face in B^*. So we proved that no dual edge in $G^*\backslash C^*$ is bicolor and all dual edges in C^* are bicolor.

Every dual vertex in C^* is incident to exact two dual edges in C^*, its corresponding primal face is incident to two bicolor edges. Every dual vertex in $G^*\backslash C^*$ is incident no dual edge in C^*, its corresponding primal face is incident to zero bicolor edge. The proposition is proved.

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