Section 2.3 Homework

In Exercises 1–5, determine if the vector **v** is a linear combination of the remaining vectors.

1. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_1+R_2=R_1} \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & -2 \end{array}\right]$$

The system has a Solution, so v a linear Combination of u_1 and u_2.

2. $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\mathbf{u}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$x\begin{bmatrix} 4 \\ -2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 2 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{2R_2+R_1=R_1} \left[\begin{array}{cc|c} 0 & 0 & 4 \\ -2 & 1 & 1 \end{array}\right] \rightarrow$$

Can't happen, so no Solution.
v is not a linear Combination of the Remaining vectors.

3. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2-R_1]{R_2:} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

$\downarrow$

Can't happen.

So, v is not a linear Combination of u_1 and u_2.

4. $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$; $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow[R_2 - R_1]{R_2:} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

$\vec{v}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$

5. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xleftarrow[R_2 - R_1]{R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow[R_3 - R_2]{R_3:} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$\vec{v}$ is a linear combination of the remaining vectors.

In Exercises 7 and 8, determine if the vector $\mathbf{b}$ is in the span of the columns of the matrix A.

7. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \xrightarrow{R_1 : (3R_1 - R_2)} \left[\begin{array}{cc|c} 0 & 2 & 9 \\ 3 & 4 & 6 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{cc|c} 3 & 4 & 6 \\ 0 & 2 & 9 \end{array}\right]$$

has a Solution.

So, b is in the span of columns of A

8. $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow{R_1 : 5R_1 - R_2} \left[\begin{array}{ccc|c} 0 & 4 & 8 & 12 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow[R_2 : R_3 - R_2]{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow{R_2 : 9R_2 - R_3} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow{R_2 : R_1 + R_2} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

$$\xrightarrow[R_2 \longleftrightarrow R_3]{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|c} 9 & 10 & 11 & 12 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ infinitely many Solutions exist, So, b is in the Span of Columns of A.

9. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & a\\1 & -1 & b\end{array}\right] \xrightarrow[R_2 - R_1]{R_2:} \left[\begin{array}{cc|c}1 & 1 & a\\0 & -2 & b-a\end{array}\right] \xrightarrow[2R_1 + R_2]{R_1:} \left[\begin{array}{cc|c}2 & 0 & a+b\\0 & -2 & b-a\end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{2}R_1} \left[\begin{array}{cc|c}1 & 0 & \frac{a+b}{2}\\0 & -1 & \frac{b-a}{2}\end{array}\right]$$

$$\left(\frac{a+b}{2}\right)\begin{bmatrix}1\\1\end{bmatrix} + \left(\frac{b-a}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

System has unique Solutions, So the vectors Span $\mathbb{R}^2$

10. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$

$$x\begin{bmatrix}3\\-2\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & 0 & a\\-2 & 1 & b\end{array}\right] \xrightarrow{\frac{R_1}{3}} \left[\begin{array}{cc|c}1 & 0 & \frac{a}{3}\\-2 & 1 & b\end{array}\right] \xrightarrow[2R_1 + R_2]{R_2:} \left[\begin{array}{cc|c}1 & 0 & \frac{a}{3}\\0 & 1 & \frac{2a}{3} + b\end{array}\right]$$

$$\left(\frac{a}{3}\right)\begin{bmatrix}3\\-2\end{bmatrix} + \left(\frac{2a}{3} + b\right)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

System has unique Solutions, So vectors span $\mathbb{R}^2$

11. Show that $\mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow[R_3-R_1]{R_3:} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & 1 & c-a \end{array}\right] \xrightarrow[R_3+R_2]{R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & c-a+b \end{array}\right]$$

System has unique solutions, so vectors span R^3.

12. Show that $\mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 2 & 1 & b \\ 0 & 3 & -1 & c \end{array}\right] \xrightarrow[R_2-R_1]{R_2:} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 3 & -1 & c \end{array}\right] \xrightarrow[3R_2-R_3]{R_3:} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & -2 & 3(b-a)-c \end{array}\right]$$

System has unique solutions, so vectors span R^3.

Use the method of Example 2.23 and Theorem 2.6 to determine if the sets of vectors in Exercises 22–30 are linearly independent. If, for any of these, the answer can be determined by inspection (i.e., without calculation), state why. For any sets that are linearly dependent, find a dependence relationship among the vectors.

22. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$2x = -1 \rightarrow x = -\frac{1}{2}$

$-x = 2 \rightarrow x = -2$

linearly independent.

if the sets were linearly dependent, then there would exist 'x' such that

$x \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

23. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

no obvious relation.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_3:\ R_3 - R_1]{R_2:\ R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{2R_1 - R_3} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3:\ R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_3]{} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2:\ R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2:\ R_2 + R_3} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1:\ R_1 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 = c_2 = c_3 = \varnothing$ linearly independent

24. $\begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-5\\2\end{bmatrix}$

$$\left[\begin{array}{ccc|c}2&3&1&0\\2&1&-5&0\\1&2&2&0\end{array}\right] \xrightarrow[R_1-R_3]{R_1:} \left[\begin{array}{ccc|c}1&1&-1&0\\2&1&-5&0\\1&2&2&0\end{array}\right] \xrightarrow[R_3-R_1]{R_3} \left[\begin{array}{ccc|c}1&1&-1&0\\2&1&-5&0\\0&1&3&0\end{array}\right] \xrightarrow[R_2:\ R_2+R_3]{R_1:\ 3R_1+R_3} \left[\begin{array}{ccc|c}3&4&0&0\\2&2&-2&0\\0&1&3&0\end{array}\right]$$

$$\xrightarrow[R_1-R_2]{R_1:} \left[\begin{array}{ccc|c}2&3&1&0\\1&1&-1&0\\0&1&3&0\end{array}\right] \xrightarrow{\frac{1}{2}(R_1-R_3)} \left[\begin{array}{ccc|c}1&1&-1&0\\1&1&-1&0\\0&1&3&0\end{array}\right] \longrightarrow \text{NVM}$$

use this:

$$R_1 \longleftrightarrow R_3 \quad \left[\begin{array}{ccc|c}1&2&2&0\\2&1&-5&0\\2&3&1&0\end{array}\right] \xrightarrow[R_3:\ -2R_1+R_3]{R_2:\ -2R_1+R_2} \left[\begin{array}{ccc|c}1&2&2&0\\0&-3&-9&0\\0&-1&-3&0\end{array}\right] \xrightarrow{-\frac{1}{3}R_2 \to R_3} \left[\begin{array}{ccc|c}1&2&2&0\\0&-3&-9&0\\0&0&0&0\end{array}\right]$$

does not have a unique solution, so linearly dependent.

25. $\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$

By observing the vectors we see that:

$$\begin{bmatrix}0\\1\\2\end{bmatrix} + \begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$ So, the vectors are linearly dependent.

26. $\begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$

we have more vectors than the dimension of space.
So linearly dependent.

27. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

zero vector can be represented as linear combination of other vectors, so:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 0 \cdot \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

Linearly dependent since the zero vector is a linear combination of the other vectors in the set.

28. $\begin{bmatrix}-1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\2\\2\\4\end{bmatrix}, \begin{bmatrix}2\\3\\1\\-1\end{bmatrix}$

$$\left[\begin{array}{ccc|c}-1&3&2&0\\1&2&3&0\\2&2&1&0\\1&4&-1&0\end{array}\right] \xrightarrow[R_2+R_1]{R_2:} \left[\begin{array}{ccc|c}-1&3&2&0\\0&5&5&0\\2&2&1&0\\1&4&-1&0\end{array}\right] \xrightarrow[R_4+R_1]{R_4:} \left[\begin{array}{ccc|c}-1&3&2&0\\0&5&5&0\\2&2&1&0\\0&7&1&0\end{array}\right] \xrightarrow[R_3+2R_1]{R_3:} \left[\begin{array}{ccc|c}-1&3&2&0\\0&5&5&0\\0&8&5&0\\0&7&1&0\end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&8&5&0\\0&7&1&0\end{array}\right] \xrightarrow[R_3-8R_2]{R_3} \left[\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&0&-3&0\\0&7&1&0\end{array}\right] \xrightarrow{R_4-7R_2} \left[\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&0&-3&0\\0&0&-6&0\end{array}\right]$$

$C_1 = C_1 = C_3 = 0$

linearly independent.

29. $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{R_2:} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_3:} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_4+R_2]{R_4:} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_4-R_3]{R_4:} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$c_1 = c_2 = c_3 = c_4 = 0$

linearly independent.

30. $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\3\\2\\1\end{bmatrix}, \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

$c_1 = c_2 = c_3 = c_4 = 0$

linearly independent.